Operads and Maurer–Cartan spaces

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Operads and Maurer–Cartan spaces

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The math is dark and full of functors.

Damien Calaque

Chapter 1

Introduction

1.1 Préamble (version française)

Pour paraphraser Adrien Douady [Dou66], le but de ce travail est de munir son auteur du titre de docteur en mathématiques et plusieurs différents espaces de structures d'algèbres de Lie à homotopie près. Nous en étudierons les propriétés et nous les appliquerons à la théorie de la déformation et à l'homotopie rationnelle.

1.1.1 Opérades

Notre outil de travail principal sont les opérades algébriques. Une opérade est une structure algébrique qui code un "type d'algèbres". Par exemple, il y a une opérade qui code les algèbres associatives, une autre les algèbres commutatives, une troisième les algèbres de Lie, etc. Un point de vue intuitif sur ces objets est le suivant. La plupart des types d'algèbres qu'on rencontre sont définis comme un espace vectoriel muni d'opérations avec un certain nombre d'entrées — leur *arité* — et une sortie, qui satisfont certaines relations entre elles. Par exemple, une algèbre associative est un espace vectoriel A avec une opération binaire

$$m : A \otimes A \longrightarrow A$$

qui est associative :

$$m(m(a,b),c) = m(a,m(b,c))$$

pour tout $a, b, c \in A$. La relation d'associtivité peut être exprimée sans faire référence aux éléments de l'espace vectoriel sous-jacent comme

$$m(m \otimes 1_A) = m(1_A \otimes m) \, .$$

L'idée des opérades est d'oublier l'espace vectoriel sous-jacent et de se concentrer sur les opérations. Une opérade $\mathscr{P}$ est une collection d'espaces vectoriels $\mathscr{P}(n)$, pour $n \geq 0$, qui codent les opérations d'arité n, munie d'une règle pour composer de telles opérations. Dans l'exemple précédent on a $m \in \mathscr{P}(2)$, et une règle de composition qui nous dit comme on peut composer deux copies de m de deux façons différentes pour obtenir au final la même opération en $\mathscr{P}(3)$. Les opérades nous permettent de donner des énoncés généraux sur le types possibles d'algèbres ainsi que sur les relations entre algèbres de types différents de façon propre et catégorique. Dualement, il y a une notion de coopérade, qui code les cogèbres.

1.1.2 Algèbre homotopique

La théorie des opérades est profondément liée à l'algèbre homotopique, c'est-à-dire l'étude des structures algébriques où l'on n'identifie pas deux algèbres seulement si elles sont isomorphes, mais aussi quand elles sont reliées que par un quasi-isomorphisme, c'est-à-dire un morphisme qui induit un isomorphisme en homologie. Cette construction peut être formalisée comme une structure de modèles sur la catégorie des algèbres. Faire de l'algèbre homotopique nous permet d'utiliser certains outils puissants. Ceux que l'on va manipuler le plus souvent sont les ∞-morphismes d'algèbres et le théorème de transfert homotopique. Les premiers sont une version relachée de la notion de morphisme d'algèbres qui peut être interprétée comme des applications qui sont des morphismes d'algèbres "à homotopie près". Le deuxième est un théorème qui dit que si l'on a deux complexes de chaînes homotopiquement équivalents et une structure algébrique sur un des deux, alors on peut mettre une structure algébrique "à homotopie près" sur le second complexe de chaînes de façon telle que les deux algèbres soient homotopiquement équivalentes.

1.1.3 Algèbres de convolution

Donnons-nous un certain type d'algèbres, codé par une opérade $\mathscr{P}$, un type de cogèbres, codé par une coopérade $\mathscr{C}$ et une relation entre $\mathscr{C}$ et $\mathscr{P}$, donnée par ce qui est appelé un morphisme tordant $\alpha : \mathscr{C} \to \mathscr{P}$. Dans cette situation, on peut mettre une structure naturelle d'algèbre de Lie à homotopie près sur l'espace des applications linéaires entre $\mathscr{C}$-cogèbres et $\mathscr{P}$-algèbres. Plus précisement, si C est une $\mathscr{C}$-cogèbre et A est une $\mathscr{P}$-algèbre alors on a un complexe de chaînes d'applications linéaires $\hom(C, A)$. En ayant fixé un morphisme tordant α, on a une structure naturelle d'algèbre de Lie à homotopie près sur $\hom(C, A)$. On note $\hom^{\alpha}(C, A)$ l'algèbre que l'on obtient de cette façon et on l'appelle l'algèbre de convolution de C et A (relative à α). Le premier objectif de cette thèse est de développer la théorie des algèbres de convolution et de comprendre comment elles se comportent par rapport aux outils de l'algèbre homotopique. Au cours de cette thèse, nous allons montrer les résultats suivants.

1. Les algèbres de convolution sont partiellement compatibles avec les ∞-morphismes. Plus précisement, la notion d'algèbre de convolution définit un bifoncteur des $\mathscr{C}$-cogèbres et des $\mathscr{P}$-algèbres vers les algèbres de Lie à homotopie près. Il est possible d'étendre ce bifoncteur soit en un bifoncteur qui accepte les ∞-morphismes de $\mathscr{C}$-cogèbres dans sa première entrée ou en un bifoncteur qui accepte les ∞-morphismes de $\mathscr{P}$-algèbres dans sa seconde entrée. Par contre il n'est pas possible de l'étendre ultérieurement en un bifoncteur qui accepte des ∞-morphismes dans ses deux entrées en même temps.

2. Si l'on fait des suppositions ultérieures sur le morphisme tordant α — notamment, en demandant que α soit de Koszul — alors on peut étendre le bifoncteur des algèbres de convolution de façon à ce qu'il accepte des ∞-morphismes dans ses deux entrées en même temps, mais le bifoncteur qu'on obtient de cette façon n'est défini qu'à homotopie près.

3. Les algèbres de convolution sont complètement compatibles avec le théorème de transfert homotopique.

Par la suite, on va appliquer la théorie des algèbres de convolution à la théorie de la déformation dérivée et à la théorie de l'homotopie rationnelle.

1.1.4 Théorie de la déformation

Un problème de déformation est donné en fixant un objet sous-jacent, comme un espace vectoriel, et un type de structure qu'on aimerait comprendre mieux sur cet objet, par exemple les structures d'algèbre associative sur l'espace vectoriel. Une fois donnée une telle structure, on veut la perturber de façon à obtenir une autre structure du même type. Ce processus peut nous donner des informations utiles sur la structure que l'on était en train d'étudier. Il y a une myriade d'exemples de problèmes de déformation : on peut considérer les déformations des structures d'algèbre d'un certain type sur un espace vectoriel, mais aussi les déformations de structures complexes analytiques sur une variété fixée, les déformations de points ou de sous-schémas fermés dans un schéma sous-jacent, les déformations de connexions plates sur un fibré vectoriel, et beaucoup d'autres encore. L'exemple le plus célèbre d'un résultat provenant de la théorie de la déformation est probablement la quantification par déformation des variétés de Poisson, due à Konstevich.

Un principe heuristique énoncé par Deligne affirme que chaque problème de déformation (en charactéristique zéro) est équivalent à l'étude de l'espace des éléments de Maurer–Cartan d'une algèbre de Lie différentielle graduée ou plus généralement d'une algèbre de Lie à homotopie près. Cet espace d'éléments de Maurer–Cartan a été exprimé par Hinich comme un ensemble simplicial $\mathrm{MC}_\bullet(\mathfrak{g})$ qui peut être fonctoriellement associé à chaque algèbre de Lie à homotopie près $\mathfrak{g}$. Récemment ce dernier a été étudié en détail par Getzler. La première application qu'on donnera des algèbres de convolution est la construction explicite d'une algèbre de Lie à homotopie près cosimpliciale $\mathfrak{mc}_\bullet^\infty$ qui représente l'espace de Maurer–Cartan à homotopie près. Plus précisément, pour chaque algèbre de Lie à homotopie près $\mathfrak{g}$ qui satisfait certaines conditions de complétude, on a une équivalence homotopique naturelle d'ensembles simpliciaux

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \hom_{\mathscr{L}_\infty\text{-alg}}(\mathfrak{mc}_\bullet^\infty, \mathfrak{g}) \,.$$

On étudiera cette algèbre $\mathfrak{mc}_\bullet^\infty$ et on l'utilisera pour comprendre certaines propriétés de $\mathrm{MC}_\bullet(\mathfrak{g})$ et d'autres ensembles simpliciaux associés.

1.1.5 Homotopie rationnelle

En homotopie rationnelle on s'intéresse à un invariant des espaces simplement connexes qui est plus faible que leur type d'homotopie mais qui a l'avantage d'être calculable dans beaucoup de cas intéressants. Soit X un espace simplement connexe. Ses groupes d'homotopie sont tous abéliens et donc on peut définir ses groupes d'homotopie rationnelle par

$$\pi_\bullet(X, x) \otimes_{\mathbb{Z}} \mathbb{Q} \,,$$

où $x \in X$ est un point base. On dit qu'un morphisme est une équivalence rationnelle s'il induit un isomorphisme entre les groupes d'homotopie rationnelle. Par un résultat du travail fondateur de Quillen, l'étude de l'homotopie rationnelle, c'est-à-dire l'étude de la catégorie des espaces simplement connexes modulo les équivalences rationnelles, est la même chose que l'étude de la catégorie des algèbres de Lie différentielles graduées concentrées en degré supérieur ou égal à 1 modulo les quasi-isomorphismes, qui est elle-même équivalente à l'étude de la catégorie des cogèbres cocommutatives concentrées en degré supérieur ou égal à 2 modulo les quasi-isomorphismes. Ceci nous dit qu'on peut modéliser les espaces par des algèbres de Lie ou des cogèbres cocommutatives pour avoir des outils calculatoires explicites qui nous permettent de calculer le type d'homotopie rationnelle d'un espace. Les travaux de Sullivan nous donnent une autre approche encore, en modélisant les espaces par des algèbres commutatives différentielles graduées.

On peut aussi considérer des types de modèles plus généraux, comme des modèles en algèbres de Lie à homotopie près et des modèles en cogèbres cocommutatives à homotopie près. Un théorème de Berglund nous dit comment construire un modèle en algèbres de Lie à homotopie près pour l'espaces des morphismes entre deux espaces en partant d'un modèle commutatif de l'espace de départ et d'un modèle Lie à homotopie près de l'espace d'arrivée. On généralise légèrement ce résultat en démontrant qu'on peut prendre des modèles cocommutatifs à homotopie près pour l'espace de départ et on exprime le modèle résultant pour l'espaces d'applications entre les deux espaces comme une algèbre de convolution.

1.1.6 Invariance homotopique des espaces de Maurer–Cartan

Le dernier chapitre de cette thèse est dédié à des résultats seulement partiellement reliés aux algèbres de convolution. Deux résultats importants sur les espaces de Maurer–Cartan sont le théorème de Goldman–Millson et sa généralisation plus récente, le théorème de Dolgushev–Rogers. Ces résultats nous disent que certains morphismes entre algèbres de Lie à homotopie près — les ∞-quasi-isomorphismes filtrés — induisent des équivalences en homotopie entre les espaces de Maurer–Cartan respectifs. Ce résultat a un parfum homotopique : il nous dit qu'une certaine classe de morphismes d'algèbres de Lie à homotopie près est envoyée sur les équivalences faibles d'ensembles simpliciaux par un certain foncteur. Par contre sa preuve n'est pas faite en n'utilisant que de la théorie de l'homotopie et se base sur une démonstration par récurrence sur certaines filtrations, en travaillant directement sur les algèbres de Lie à homotopie près en jeu. On donne une nouvelle approche à la preuve de ce théorème qui n'utilise que de la théorie de l'homotopie. Même si l'on ne récupère pas complètement le théorème de Dolgushev–Rogers, on réussit tout de même à en obtenir une version plus faible et l'on croise des constructions intéressantes le long du chemin. Par exemple cette approche nous fait redécouvrir l'algèbre de Lie cosimpliciale qui représente les espaces de Maurer–Cartan.

1.1.7 Résultats ultérieurs

Cachées dans les rappels, il y a deux résultats originaux supplémentaires dans cette thèse. Le premier est une charactérisation complète des équivalences faibles dans la structure de modèles construite sur la catégorie des cogèbres sur une coopérade par Vallette. Le degré d'originalité de ce résultat n'est pas énorme, car tous les ingrédients dont on a besoin pour sa démonstration étaient déja présents dans les travaux de Vallette. Le second est une démonstration du fait que le dual d'un modèle cocommutatif pour un espace simplement connexe est un modèle commutatif pour le même espaces et *vice versa*, sous certaines hypothèses peu contraignantes de finitude. Ce dernier résultat était certainement bien connu par les experts, mais nous n'avons pas su en trouver une démonstration dans la litterature.

1.1.8 Remarques d'ordre général

Cette thèse veut être le plus autonome possible. Pour cette raison, on consacre une importante première partie à l'exposition de plusieurs notions préliminaires dont on a besoin dans le reste du texte : la théorie de opérades, les catégories de modèles, l'homotopie simpliciale, l'homotopie rationnelle, les espaces de Maurer–Cartan et la théorie de la déformation. On suppose que le lecteur possède des notions de base d'algèbre homologique, de théorie des catégories et de topologie algébrique.

Les résultats nouveaux contenus dans cette thèse sont extraits des articles [RN17a], [RN17b], [RNW17], [RNW], [RN18] et de l'esquisse [RNV].

1.2 Preamble

To paraphrase Adrien Douady [Dou66], the goal of this work is to endow the author with the degree of Doctor in Mathematics, and many different spaces with structures of homotopy Lie algebras, as well as studying their properties and apply the developed theory to various fields, such as deformation theory and rational homotopy theory.

1.2.1 Operads

Our main working tool is algebraic operads. An operad is an algebraic structure that codes a "type of algebra". For example, there is an operad coding associative algebras, one coding commutative algebras, one for Lie algebras, and so on. One possible intuitive point of view is the following one. Most types of algebras are defined as a vector space endowed with certain operations with a certain number of entries — their *arity* — and one exit, which satisfy certain relations between them. For example, an associative algebra is a vector space A together with a binary multiplication

$$m : A \otimes A \longrightarrow A$$

which is associative:

$$m(m(a,b),c) = m(a,m(b,c))$$

for all $a, b, c \in A$. The associativity relation can be expressed without referring to elements of the underlying vector space as

$$m(m \otimes 1_A) = m(1_A \otimes m) \ .$$

The idea of operads is to forget the vector spaces and concentrate on the operations. An operad $\mathscr{P}$ is a sequence of vector spaces $\mathscr{P}(n)$, for $n \geq 0$, encoding the operations of arity n, together with a rule for composing such operations. In the example above, we have $m \in \mathscr{P}(2)$, and a composition rule telling us how composing two copies of m in two different ways gives us the same operation in $\mathscr{P}(3)$. Operads allow us to give general statements about types of algebras and relations between algebras of different types in a clean and elegant way. Dually, one also has cooperads, which are a similar object encoding coalgebras.

1.2.2 Homotopical algebra

Operad theory is closely linked to homotopical algebra, i.e. the study of algebraic structures where one does not identify two algebras only if they are isomorphic, but also when they are related in a more flexible way. Namely, in homotopical algebra one studies algebras up to quasi-isomorphisms, that is up to morphisms inducing isomorphisms in homology. This construction can be formalized as a model category of algebras. Doing homotopical algebra allows us to use some powerful instruments. The ones we will manipulate the most are ∞-morphisms of algebras and the homotopy transfer theorem. The former are a "relaxed" version of morphisms of algebras, which can be interpreted as maps that are morphisms of algebras, but only "up to homotopy". The latter is a theorem that essentially says that if we have two chain complexes that are homotopically the same, and some algebraic structure on one of the two, then we can put an algebraic structure on the other chain complex so that the resulting *algebras* are homotopically the same.

1.2.3 Convolution algebras

Given a certain type of algebra, encoded by an operad $\mathscr{P}$, a certain type of coalgebras, encoded by a cooperad $\mathscr{C}$, and a relation between $\mathscr{C}$ and $\mathscr{P}$ given by what is called a twisting morphisms $\alpha : \mathscr{C} \to \mathscr{P}$, then one can give natural homotopy Lie algebra structures on the spaces of linear maps between $\mathscr{C}$-coalgebras and $\mathscr{P}$-algebras. Namely, if C is a $\mathscr{C}$-coalgebra, and A is a $\mathscr{P}$-algebra, then we have the chain complex of linear maps $\hom(C, A)$. Given the twisting morphism α, there is a natural homotopy Lie algebra on $\hom(C, A)$. We denote the resulting algebra by $\hom^{\alpha}(C, A)$, and call it the convolution algebra of C and A (relative to α). The first goal of the present work is to develop the theory of convolution algebras, and to understand how they behave with respect to the tools of homotopical algebra. We will show what follows.

1. Convolution algebras are partially compatible with ∞-morphisms. Namely, taking convolution algebras defines a bifunctor from $\mathscr{C}$-coalgebras and $\mathscr{P}$-algebras to homotopy Lie algebras. It is possible to extend this bifunctor either to a bifunctor also accepting ∞-morphisms of $\mathscr{C}$-coalgebras in the first slot, or accepting ∞-morphisms of $\mathscr{P}$-algebras in the second slot. However, it is not possible to further extend it into a bifunctor accepting ∞-morphisms in both slots simultaneously.

2. If one makes further assumptions about the twisting morphism α — namely if one asks that α is Koszul — then one can extend the convolution algebra bifunctor to a bifunctor accepting ∞-morphisms in both slots at the same time, but the resulting bifunctor is only defined up to homotopy.

3. Convolution algebras are completely compatible with the homotopy transfer theorem.

We will then apply the theory of convolution algebras to derived deformation theory and rational homotopy theory.

1.2.4 Deformation theory

A deformation problem is given by a fixed underlying object, such as a vector space, and a type of structure on that object one wants to understand, for example the possible associative algebra structures on the vector space. Given one structure, one wants to perturb it in such a way as to obtain another structure of the same type. This process can yield useful information on the structure one is studying. Examples of deformation problems are legion: one can consider deformations of algebraic structures of a certain type on a vector space, but also deformations of complex analytic structures on a fixed underlying manifold, deformations of points or closed sub-schemes in an underlying scheme, deformations of flat connections on a vector bundle, and many others. Perhaps one of the most celebrated results of deformation theoretical nature is Kontsevich's deformation quantization of Poisson manifolds.

It is a heuristic principle due to Deligne that any deformation problem (in characteristic zero) is equivalent to the study of the space of Maurer–Cartan elements of a differential graded Lie algebra, or more generally of a homotopy Lie algebra. This space of Maurer–Cartan elements was expressed by Hinich as a simplicial set $\mathrm{MC}_{\bullet}(\mathfrak{g})$ functorially associated to any homotopy Lie algebra $\mathfrak{g}$, and then studied in depth by Getzler in more recent years. The first application of convolution algebras we will give is the explicit construction of a universal cosimplicial homotopy Lie algebra $\mathfrak{mc}_{\bullet}^{\infty}$ representing the Maurer–Cartan space up to homotopy. To be more precise, for any homotopy Lie algebra $\mathfrak{g}$ satisfying some completeness condition, we have a natural homotopy equivalence of simplicial sets

$$\mathrm{MC}_{\bullet}(\mathfrak{g}) \simeq \hom_{\mathscr{L}_{\infty}\text{-alg}}(\mathfrak{mc}_{\bullet}^{\infty}, \mathfrak{g}) \, .$$

We will study the cosimplicial homotopy Lie algebra $\mathfrak{mc}_\bullet^\infty$, and use it to derive some properties of $\mathrm{MC}_\bullet(\mathfrak{g})$ and other related simplicial sets.

1.2.5 Rational homotopy theory

In rational homotopy theory, one is interested in an invariant of simply connected spaces that is weaker than their homotopy type, but which has the advantage of being computable in many interesting cases: its rational homotopy type. Let X be a simply connected space. Then its homotopy groups are all abelian, so that one defines its rational homotopy groups as

$$\pi_\bullet(X, x) \otimes_{\mathbb{Z}} \mathbb{Q} \,,$$

where $x \in X$ is a basepoint. One says that a morphism of spaces is a rational homotopy equivalence if it induces isomorphisms on the rational homotopy groups. It is a result of the seminal work of Quillen that rational homotopy theory, i.e. the study of the category of simply connected spaces modulo rational equivalences, is the same as the study of the category of differential graded Lie algebra concentrated in degree greater or equal than 1 modulo quasi-isomorphisms, which is itself equivalent to the study of the category of differential graded cocommutative coalgebras concentrated in degree greater or equal than 2 modulo quasi-isomorphisms. This tells us that we can model spaces by Lie algebras or cocommutative coalgebras in order to have explicit computational tools to find the rational homotopy type of a space. Sullivan took another approach, modeling spaces with differential graded commutative algebras.

One can also consider more general models, such as homotopy Lie algebra models, and homotopy cocommutative coalgebra models. A theorem of Berglund tells us how one can construct a homotopy Lie algebra rational model for the mapping space of two spaces starting from a commutative model of the source space and a homotopy Lie model of the target space. We slightly generalize this result by proving that one can in fact allow homotopy cocommutative models for the source space, and express the resulting model for the mapping space as a convolution algebra.

1.2.6 Homotopy invariance of Maurer–Cartan spaces

The last chapter of this thesis is dedicated to some material only tangentially related to convolution algebras. Two important result on Maurer–Cartan spaces are the Goldman–Millson theorem, and its modern generalization, the Dolgushev–Rogers theorem. They tell us that certain maps between homotopy Lie algebras, namely filtered ∞-quasi-isomorphisms, induce homotopy equivalences between the respective Maurer–Cartan spaces. This result has a strong homotopy theoretical flavor: it tells us that a certain class of maps of homotopy Lie algebras is sent to weak equivalences of simplicial sets under a certain functor. However, its proof is not completely homotopy theoretical, and relies on a proof by induction on certain filtrations, working directly with the homotopy Lie algebras in play. We give a new approach to the proof of this theorem, relying purely on homotopy theory. Although we do not recover the full strength of the Dolgushev–Rogers theorem, we are still able to obtain a weaker version, and incur in interesting constructions along the way. For example, this approach allows us to recover the cosimplicial Lie algebra representing Maurer–Cartan spaces.

1.2.7 Other results

There are two further original results in the present work, hiding in the recollections. The first result is a complete characterization of the weak equivalences in the model structure put on coalgebras over a cooperad by Vallette. The degree of originality of this result is not enormous, as all the ingredients needed in the proof were already present in Vallette's work. The second one is a proof of the fact that the dual of a cocommutative rational model for a simply connected space is a commutative model for the same space, and *vice versa*, under some slight finiteness assumptions the dual of a commutative rational model is a cocommutative rational model. This last result was folklore and certainly well-known to experts. However, we were not able to find a proof in the existing literature.

1.2.8 General remarks

Our goal is to be as self-contained as possible. For this reason, we indulge in a lengthy exposition of preliminary notions in the first part of the thesis, explaining the necessary notions of operad theory, model categories, simplicial homotopy theory, rational homotopy theory, Maurer–Cartan spaces, and deformation theory. Still, we assume that the reader is acquainted with the basic notions of homological algebra, category theory, and algebraic topology.

All the original material contained in this thesis is extracted from the articles [RN17a], [RN17b], [RNW17], [RNW], [RN18], and from the draft [RNV].

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1.4 Structure of this work

This thesis is divided in two parts. The first part is an overview of prerequisites for the second part, which is composed of the original results of this work.

The recollections begin in Chapter 2 with an overview of operad theory. We recall all of the necessary notions of operads, cooperads, algebras and coalgebras over them, Koszul duality, twisting morphisms, bar and cobar constructions for (co)operads and (co)algebras, and homotopy theory of algebras up to homotopy. In Chapter 3, we recall the basic notions of model categories, which is the framework we will use to do homotopy theory. We give various examples of model categories in Section 3.3. This chapter contains an original result, Theorem 3.3.15, which gives a complete characterization of the weak equivalences in the Vallette model structure on conilpotent coalgebras over a cooperad. Chapter 4 is a review of simplicial homotopy theory. Chapter 5 is a recollection on rational homotopy theory. It contains an original result, Theorem 5.3.10, stating that one can dualize cocommutative rational models to obtain commutative rational models, and *vice versa*. Chapter 6 covers in detail the notions of differential graded Lie algebras, their Maurer–Cartan elements and the equivalence relations between them, the analogous notions for homotopy Lie algebras, and some modern results: the Dolgushev–Rogers theorem and the formal Kuranishi theorem. We conclude the first part of the thesis with a review of deformation theory in Chapter 7. This chapter is mostly a motivation of why one is interested in studying the space of Maurer–Cartan elements on homotopy Lie algebras.

The second part starts with the introduction and the study of basic properties of ∞-morphisms of algebras and coalgebras relative to a twisting morphism, in Chapter 8. We continue with Chapter 9, where we introduce convolution homotopy algebras and study them in depth. They are a central object in this thesis, and we give applications of their theory in the subsequent two chapters. In Chapter 10, we apply it to construct a universal cosimplicial (homotopy) Lie algebra representing the Maurer–Cartan space functor. In Chapter 11, we apply it to rational homotopy theory and use it to generalize a theorem due to Berglund. The last chapter of this part is Chapter 12. There, we give a completely homotopical approach to the proof of the Goldman–Millson theorem and of the Dolgushev–Rogers theorem.

Finally, there are three appendices containing auxiliary notions we need here and there. In Appendix A, we give the definitions and notations we use for trees (rooted and planar). In Appendix B, we explain the notions of filtered chain complexes and filtered algebras over operads.

In Appendix C, we define formal fixed-point equations and formal differential equations, and prove some existence and uniqueness results for their solutions.

1.5 Notations and conventions

Before really starting with the main body of this work, we fix some notations and conventions we will use throughout the text.

1.5.1 General conventions

We work over a field of characteristic 0 unless stated otherwise. The symbol $\mathbb{S}_n$ is reserved for the symmetric group on n elements.

All operads, cooperads, algebras and coalgebras are always differential graded, unless explicitly stated otherwise. We usually work over chain complexes, with some exceptions where it is more natural to use cochain complexes, for example in Chapter 7.

We reserve the symbol $\circ$ for the composite product of $\mathbb{S}$-modules and related concepts, see Definition 2.1.8, and omit compositions whenever talking about functions, writing fg for the composition of two functions f and g.

1.5.2 Categories into play

Here are the notations we will use for some of the most frequently appearing categories.

Δ	The ordinal number category.
Ch	The category of chain complexes and chain maps.
$\mathscr{C}$-cog	The category of conilpotent coalgebras over a cooperad $\mathscr{C}$.
coOp	The category of (conilpotent) cooperads.
$\mathfrak{s}\mathscr{L}_\infty$-alg	The category of shifted homotopy Lie algebras.
$\mathbb{S}$-mod	The category of $\mathbb{S}$-modules.
Op	The category of differential graded operads.
$\mathscr{P}$-alg	The category of algebras over an operad $\mathscr{P}$.
$\widehat{\mathscr{P}}$-alg	The category of proper complete algebras over an operad $\mathscr{P}$.
sSets	The category of simplicial sets.
Top	The category of topological spaces.

Moreover, we will denote by either $\mathsf{C}(x, y)$ or by $\hom_{\mathsf{C}}(x, y)$ the set of morphisms in the category C with source $x \in \mathsf{C}$ and target $y \in \mathsf{C}$.

1.5.3 Chain complexes

We will mainly work over the category Ch of chain complexes over some field $\mathbb{K}$. We fix from the start the notations and conventions we will use in what follows. We assume that the reader is already familiar with the notion of chain complex.

We reserve the letter s to denote a formal element of degree 1. If V is a chain complex, the chain complex sV is the suspension of V, that is the chain complex with $(sV)_n \cong V_{n-1}$ and differential

$$d_{sV}(sv) \coloneqq -sd_V v$$

for $v \in V$, coherently with the Koszul sign rule of Subsection 1.5.4. We denote by s^{-1} the dual of s.

Let V and W be two chain complexes. We denote by $V \otimes W$ the tensor product of V and W. Its element of degree n are those in

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q \ ,$$

and its differential is given by

$$d_{V\otimes W}(v \otimes w) \coloneqq d_V v \otimes w + (-1)^v v \otimes d_W w \ ,$$

which, once again, is coherent with the Koszul sign rule.

Let V and W be two chain complexes. By $\hom(V, W)$ we will always mean the inner hom of V and W, i.e. the chain complex having in degree n the linear maps $V \to W$ of degree n, or equivalently the degree zero linear maps $s^n V \to W$. Its differential is given by

$$\partial(f) \coloneqq d_W f - (-1)^f f d_V$$

on $f \in \hom(V, W)$. Notice that the closed elements of degree 0 are exactly the chain maps, while the exact elements of degree 0 are chain maps that are homotopic to the zero map.

Let V be a chain complex. We denote by $V^\vee \coloneqq \hom(V, \mathbb{K})$ its dual chain complex, where $\mathbb{K}$ is seen as a chain complex concentrated in degree 0.

A chain complex V is said to be *of finite type* if V_n is finite dimensional for all $n \in \mathbb{Z}$.

1.5.4 The Koszul sign rule

The Koszul sign rule is a sign convention that is put on the switching maps in the (symmetric monoidal) category of graded vector spaces. Namely, if V, W are two graded vector spaces, then the isomorphism

$$V \otimes W \longrightarrow W \otimes V$$

is given by

$$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v$$

on homogeneous elements. This gives an automatic way of obtaining the correct signs in computations. An example of application of the Koszul sign rule is the following. Let V_1, V_2, W_1, W_2 be graded vector spaces, and let $f_i : V_i \to W_i$ be linear maps of homogeneous degree. Then the map $f_1 \otimes f_2$ is given by

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|f_2||v_1|} f_1(v_1) \otimes f_2(v_2) \ .$$

In particular, notice that the dual to $s^n \coloneqq s \otimes \cdots \otimes s$ is

$$(-1)^{\frac{n(n-1)}{2}} s^{-n} \coloneqq (-1)^{\frac{n(n-1)}{2}} s^{-1} \otimes \cdots \otimes s^{-1},$$

and not simply s^{-n} as one might naively expect.

1.5.5 Shuffles

Shuffles are a special type of permutation of a finite set. One can think of them as taking a deck of cards, cutting it into two, and then shuffling the two parts once — hence the name. The order of both parts remains unchanged, while the order of the whole deck is not the original one anymore. The formal definition is as follows.

Definition 1.5.1. *Let $p, q \geq 0$, and let $n \coloneqq p + q$. A (p, q)-*shuffle *is a permutation $\sigma \in \mathbb{S}_n$ such that*

$$\sigma(1) < \cdots < \sigma(p) \quad \textit{and} \quad \sigma(p+1) < \cdots < \sigma(n) \, .$$

Equivalently, it is a partition $I \sqcup J = \{1, \ldots, n\}$ such that $|I| = p$ and $|J| = q$. The set of all (p, q)-shuffles is denoted by $\sqcup\!\sqcup(p, q)$. A shuffle is trivial *if either $p = 0$ or $q = 0$.*

1.5.6 Invariants and coinvariants

Let V be a vector space with a left action by a finite group G. The *invariants* of V are

$$V^G \coloneqq \{v \in V \mid \forall g \in G : g \cdot v = v\} \subseteq V \, .$$

The *coinvariants* of V are

$$V_G \coloneqq V/(v - g \cdot v)_{v \in V, g \in G} \, .$$

It is well know and easy to check that the coinvariants of V are the dual space to the invariants of V. Moreover, in characteristic 0, we have an isomorphism

$$V^G \xrightarrow{\cong} V_G$$

given by sending an invariant element of V to its equivalence class. The inverse is given by sending a class $[v] \in V_G$ with representative $v \in V$ to

$$\frac{1}{|G|} \sum_{g \in G} g \cdot v \in V^G.$$

We will sometimes implicitly use this identification in the main body of the text.

Part I

Prerequisites

Chapter 2

Operad theory

The main tool used throughout the present thesis is algebraic operads. These objects give us an effective way to encode "types of algebras" and to study their properties. They were first introduced in the context of algebraic topology in the works of J. M. Boardman and R. M. Vogt [BV73] and J. P. May [May06] – the name “operad" is due to J. P. May himself, coming from the two words “operation" and “monad" — but have since then found many applications in various domains of mathematics, in the works of many authors. It would be impossible to give a complete survey of the history and applications of operads throughout mathematics, and besides, it would be outside the scope of this thesis, so we will leave it at that.

In this chapter, we give a short overview on the subject of operads — more precisely, algebraic operads, meaning operads in the symmetric monoidal category of chain complexes — and review the notions we will need in subsequent chapters. Our main reference is the book [LV12]. We will try to stay as close as possible to the notations and conventions used there, and to stress whenever we choose to do things differently.

Here and throughout the whole thesis, unless otherwise explicitly stated, we will always work in the differential graded context. We work in the category Ch of chain complexes over a field $\mathbb{K}$ of characteristic 0. The symbol $\mathbb{S}_n$ is reserved to denote the symmetric groups. All operads and cooperads we will consider are *reduced*, meaning that they are trivial in arity 0, and that their arity 1 component is spanned by the identity operation.

2.1 (Co)operads and (co)algebras over them

In this first section, we introduce the basic objects of operad theory: $\mathbb{S}$-modules, operads, cooperads, algebras and coalgebras over operads and cooperads respectively.

2.1.1 $\mathbb{S}$-modules

The underlying object of all operads and cooperads is the $\mathbb{S}$-module. For further details on the material presented here, see [LV12, Sect. 5.1].

Definition 2.1.1. *An $\mathbb{S}$-module M is a sequence of chain complexes*

$$M = (M(0), M(1), M(2), \dots) ,$$

where for each $n \geq 0$, the chain complex $M(n)$ — called the component of M of arity n *— is a right $\mathbb{K}[\mathbb{S}_n]$-module. A* morphism of $\mathbb{S}$-modules $f : M \longrightarrow N$ *is a collection of $\mathbb{S}_n$-equivariant chain maps $f(n) : M(n) \to N(n)$. The category of $\mathbb{S}$-modules and their morphisms is denoted by* $\mathbb{S}$-mod.

Example 2.1.2. *The following two $\mathbb{S}$-modules play a central role in the theory of operads and algebras over them.*

1. *The* unit $\mathbb{S}$-module *is*
$$I \coloneqq (0, \mathbb{K}, 0, 0, \ldots)$$
with the trivial $\mathbb{S}$-action. It is the unit for the composite product of $\mathbb{S}$-modules described below in Definition 2.1.8.

2. *To any chain complex V we can associate a canonical $\mathbb{S}$-module End_V, the* endomorphism $\mathbb{S}$-module, *defined by*
$$\mathrm{End}_V(n) \coloneqq \hom(V^{\otimes n}, V)$$
with the $\mathbb{S}_n$-action given by permuting the n starting copies of V.

Example 2.1.3. *Given a chain complex V, one can see it as an $\mathbb{S}$-module concentrated in arity 0, i.e. $(V, 0, 0, \ldots)$ with trivial $\mathbb{S}$-action.*

There are various natural operations that one would like do be able to do on $\mathbb{S}$-modules, such as direct sums and tensor products. There is a good way to define such operations, which we review here.

Definition 2.1.4. *Let M and N be two $\mathbb{S}$-modules.The* direct sum $M \oplus N$ *of M and N is the $\mathbb{S}$-module given by*
$$(M \oplus N)(k) \coloneqq M(k) \oplus N(k)$$
for $k \geq 0$, with the obvious $\mathbb{S}_k$-action.

Definition 2.1.5. *Let M and N be two $\mathbb{S}$-modules. The* (Hadamard) tensor product $M \otimes N$ *of M and N is the $\mathbb{S}$-module*
$$(M \otimes N)(k) \coloneqq M(k) \otimes N(k)$$
for $k \geq 0$, with the diagonal $\mathbb{S}_k$-action.

Remark 2.1.6. *In the literature, one sometimes finds specific symbols for the Hadamard tensor product, in order to distinguish it from other notions of tensor product. Since in this thesis a tensor product between $\mathbb{S}$-modules or (co)operads is always meant as a Hadamard tensor product, no confusion is possible and we will simply use the symbol $\otimes$.*

Definition 2.1.7. *Let M and N be two $\mathbb{S}$-modules. The* inner hom $\hom(M, N)$ *of M and N is the $\mathbb{S}$-module*
$$\hom(M, N)(k) \coloneqq \hom(M(k), N(k))$$
for $k \geq 0$, and the $\mathbb{S}_k$ action given by $(f^\sigma)(x) \coloneqq f(x^{\sigma^{-1}})^\sigma$.

Definition 2.1.8. *Let M and N be two $\mathbb{S}$-modules. The* composite $M \circ N$ *of M and N is the $\mathbb{S}$-module*
$$(M \circ N)(k) \coloneqq \bigoplus_{i \geq 0} M(i) \otimes_{\mathbb{S}_i} \left(\bigoplus_{j_1 + \cdots j_i = k} \mathrm{Ind}_{\mathbb{S}_{j_1} \times \cdots \times \mathbb{S}_{j_i}}^{\mathbb{S}_k} (N(j_1) \otimes \cdots \otimes N(j_i)) \right)$$
for $k \geq 0$, and the $\mathbb{S}_k$-action induced by the $\mathbb{S}$-actions on M and N. The $\mathbb{S}_i$-action on
$$\mathrm{Ind}_{\mathbb{S}_{j_1} \times \cdots \times \mathbb{S}_{j_i}}^{\mathbb{S}_k} (N(j_1) \otimes \cdots \otimes N(j_i))$$
is given by permutations of the tuple $(j_1, \ldots, j_i)$.

The spaces $(M \circ N)(k)$ are spanned by the equivalence classes of the elements

$$(\mu; \nu_1, \dots, \nu_i; \sigma)$$

under the action of $\mathbb{S}_i$, where $\mu \in M(i)$, $\nu_a \in N(j_a)$, and $\sigma \in \mathrm{Sh}(j_1, \dots, j_i)$ is a shuffle. If σ is the identity permutation, then we will denote the element simply by

$$(\mu; \nu_1, \dots, \nu_i) = \mu \circ (\nu_1, \dots, \nu_i) \, .$$

If $f : M \to M'$ and $g : N \to N'$ are two morphisms of $\mathbb{S}$-modules, then there is an obvious induced morphism of $\mathbb{S}$-modules

$$f \circ g : M \circ N \longrightarrow M' \circ N'$$

given by

$$(f \circ g)(\mu; \nu_1, \dots, \nu_i) \coloneqq (f(\mu); g(\nu_1), \dots, g(\nu_i)) \, .$$

Notice that the (arity-wise) homology of an $\mathbb{S}$-module is again an $\mathbb{S}$-module in a natural way. There is the following important result relating the homology of the composite of two $\mathbb{S}$-modules with the composite of the homologies.

Theorem 2.1.9 (Operadic Künneth formula)**.** *Suppose M is an $\mathbb{S}$-module and assume that the base field $\mathbb{K}$ has characteristic 0. Then we have*

$$H_\bullet(M \circ N) \cong H_\bullet(M) \circ H_\bullet(N) \, .$$

One can also consider the following version of the composite product.

Definition 2.1.10. *Let M and N be two $\mathbb{S}$-modules. We define*

$$(M \bar{\circ} N)(k) \coloneqq \bigoplus_{i \geq 0} \left(M(i) \otimes \left(\bigoplus_{j_1 + \cdots j_i = k} \mathrm{Ind}_{\mathbb{S}_{j_1} \times \cdots \times \mathbb{S}_{j_i}}^{\mathbb{S}_k} \left(N(j_1) \otimes \cdots \otimes N(j_i) \right) \right) \right)^{\mathbb{S}_i} .$$

Remark 2.1.11. *When working over a field $\mathbb{K}$ of characteristic 0, one has an identification between invariants and coinvariants, as explained in Section 1.5.6, so that there is a natural isomorphism $M \circ N \cong M \bar{\circ} N$. Since we sill never work in positive characteristic when using symmetric (co)operads, we will implicitly use this identification throughout the rest of the present work.*

As it is often the case, one desires a nice monoidal structure on the category of $\mathbb{S}$-modules. Almost surprisingly, the correct monoidal product one is led to consider if one wants to use $\mathbb{S}$-modules to encode algebraic structures is not the Hadamard tensor product, but the composite product.

Proposition 2.1.12. *The category of $\mathbb{S}$-modules $(\mathbb{S}\text{-}\mathsf{mod}, \circ, I)$ endowed with the composite product of $\mathbb{S}$-modules and the unit $\mathbb{S}$-module is a monoidal category. The same is true by replacing $\circ$ by $\bar{\circ}$.*

Given an $\mathbb{S}$-module, one can associate an endofunctor of chain complexes to it.

Definition 2.1.13. *Let M be an $\mathbb{S}$-module. The* Schur functor

$$M : \mathsf{Ch} \longrightarrow \mathsf{Ch}$$

associated to M is defined by

$$M(V) \coloneqq \bigoplus_{n \geq 0} M(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

with $\mathbb{S}_n$ acting on $V^{\otimes n}$ by permuting the n copies of V.

Notice that the notation is coherent: the Schur functor associated to the composite of two $\mathbb{S}$-modules is the composite of the two Schur functors. Also, morphisms of $\mathbb{S}$-modules induce natural transformations between the associated Schur functors.

2.1.2 Operads and cooperads

We go on by defining the central objects of this chapter — and indeed the central objects of this whole thesis: operads and cooperads. A possible point of view to interpret and understand operads is the following. A type of algebra — such as associative algebras, commutative algebras, Lie algebras, and so on — is usually defined as a vector space endowed with some operations respecting some relations. For example, a commutative algebra is a vector space A together with a commutative binary operation

$$\mu : A^{\otimes 2} \longrightarrow A$$

which is associative, i.e. $\mu(\mu \otimes 1_A) = \mu(1_A \otimes \mu)$. An operad is an object encoding the operations of a type of algebra, as well as the relations between them. This gives us a way to speak and to prove theorems about all algebras of the same type at once, and to prove general relations between different types of algebras. Cooperads are the analogue notion for coalgebras, and are almost dual to operads. For more details, see e.g. [LV12, Sect. 5.2–8].

We begin with the definition of an operad.

Definition 2.1.14. *An* operad $\mathscr{P}$ *is an* $\mathbb{S}$*-module* $\mathscr{P}$ *endowed with two morphisms of* $\mathbb{S}$*-modules*

$$\gamma_{\mathscr{P}} : \mathscr{P} \circ \mathscr{P} \longrightarrow \mathscr{P} ,$$

called the composition map *of* $\mathscr{P}$*, and*

$$\eta_{\mathscr{P}} : I \longrightarrow \mathscr{P} ,$$

the unit map *of* $\mathscr{P}$*, making* $\mathscr{P}$ *into a monoid. Explicitly, that means that we have natural isomorphisms*

$$\gamma_{\mathscr{P}}(\gamma_{\mathscr{P}} \circ 1_{\mathscr{P}}) \cong \gamma_{\mathscr{P}}(1_{\mathscr{P}} \circ \gamma_{\mathscr{P}}), \qquad \gamma_{\mathscr{P}}(\eta_{\mathscr{P}} \circ 1_{\mathscr{P}}) \cong \gamma_{\mathscr{P}} \cong \gamma_{\mathscr{P}}(1_{\mathscr{P}} \circ \eta_{\mathscr{P}}) .$$

A morphism $f : \mathscr{P} \to \mathscr{Q}$ *of operads is a morphism commuting with the compositions and the unit maps of the source and of the target. Explicitly, one asks that*

$$f\gamma_{\mathscr{P}} = \gamma_{\mathscr{Q}}(f \circ f), \qquad f\eta_{\mathscr{P}} = \eta_{\mathscr{Q}}$$

on $\mathscr{P} \circ \mathscr{P}$ *and* I *respectively. The category of operads and their morphisms is denoted by* Op.

This means that for any $k \geq 0$ and $n_1, \ldots, n_k \geq 0$ we can compose

$$\mathscr{P}(k) \otimes_{\mathbb{S}_n} \mathscr{P}(n_1) \otimes \cdots \otimes \mathscr{P}(n_k) \longrightarrow \mathscr{P}(n_1 + \cdots + n_k) ,$$

which can be seen as taking an operation of arity k of an algebra, and k other operations of arity $n_1, \ldots, n_k$, and putting the latter operations in the k slots of the former operation, thus obtaining an operation of arity $n_1 + \cdots + n_k$. The original definition by P. May [May06] was given in term of these composition maps.

Another equivalent way to define an operad is to give just the *partial composition maps*

$$\circ_j : \mathscr{P}(k) \otimes \mathscr{P}(n) \longrightarrow \mathscr{P}(k + n - 1)$$

for $1 \leq j \leq k$, corresponding to composing an arity n operation in the jth slot of an arity k operation, without touching the other slots. Those maps must satisfy certain "associativity" axioms, see e.g. [LV12, Sect. 5.3.4].

An operad is *augmented* if there is a morphism $\varepsilon_{\mathscr{P}} : \mathscr{P} \to I$ such that $\varepsilon_{\mathscr{P}}\eta_{\mathscr{P}} = 1_I$. If $\mathscr{P}$ is augmented, the kernel $\ker(\varepsilon_{\mathscr{P}})$ of $\varepsilon_{\mathscr{P}}$ is called the *augmentation ideal* of $\mathscr{P}$ and is denoted by $\overline{\mathscr{P}}$.

An operad is *reduced*[1] if $\mathscr{P}(0) = 0$ and $\mathscr{P}(1) = \mathbb{K}$. Notice that if $\mathscr{P}$ is reduced, then the unit map $\eta_{\mathscr{P}}$ is invertible, and its inverse gives a canonical augmentation $\varepsilon_{\mathscr{P}}$ of $\mathscr{P}$.

Example 2.1.15. *We look back at the $\mathbb{S}$-modules of Example 2.1.2. Both can be made into an operad with very natural composition and unit map.*

1. *The unit $\mathbb{S}$-module I becomes an operad with both maps*

$$\gamma_I : I \circ I = I \xrightarrow{1_I} I, \qquad \eta_I : I \xrightarrow{1_I} I$$

given by the identity map.

2. *Given a chain complex V, the endomorphism $\mathbb{S}$-module End_V becomes an operad — the* endomorphism operad *of V — by defining*

$$\gamma_{\mathrm{End}_V}(\mu; \nu_1, \dots, \nu_k; \sigma) \coloneqq \mu(\nu_1, \dots, \nu_k)^\sigma$$

given by composing the operations $\nu_1, \dots, \nu_k$ into the k slots of $\mu : V^{\otimes k} \to V$, and

$$\eta_{\mathrm{End}_V}(1) \coloneqq 1_V$$

given by sending $1 \in \mathbb{K}$ to the identity map of V.

Given two operads $\mathscr{P}$ and $\mathscr{Q}$ one can endow the tensor product of the underlying $\mathbb{S}$-modules with an operad structure, thus constructing the *(Hadamard) tensor product* $\mathscr{P} \otimes \mathscr{Q}$. Details are given e.g. in [LV12, Sect. 5.3.2].

A very important construction is the free operad over a given $\mathbb{S}$-module. We give an explicit construction and some properties of this object, but omit all of the proofs. See [LV12, Sect. 5.5] for a complete treatment.

Definition 2.1.16. *Let M be an $\mathbb{S}$-module. The* tree module $\mathcal{T}(M)$ *over M is defined by the recursion*

$$\mathcal{T}_0 M \coloneqq I, \qquad \mathcal{T}_{n+1} M \coloneqq I \oplus (M \circ \mathcal{T}_n M)$$

and by setting $\mathcal{T}(M) \coloneqq \mathrm{colim}_n \mathcal{T}_n M$.

Elements of $\mathcal{T}(M)$ can be visualized as rooted trees with vertices of arity n labeled by an element of $M(n)$. Then one defines the composition map

$$\gamma_{\mathcal{T}(M)} : \mathcal{T}(M) \circ \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$$

simply as grafting of trees, and the unit map

$$\eta_{\mathcal{T}(M)} : I \longrightarrow \mathcal{T}(M)$$

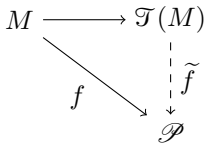
by sending $1 \in \mathbb{K} = I(1)$ to the trivial tree of arity one without any vertex. Moreover, there is a natural inclusion of $\mathbb{S}$-modules

$$M \longrightarrow \mathcal{T}(M)$$

given by sending an element $\mu \in M(n)$ to the n-corolla with its single vertex labeled by μ.

[1] This convention differs slightly from the one of [LV12], where a reduced operad is only required to satisfy $\mathscr{P}(0) = 0$.

Theorem 2.1.17. *The triple $\mathcal{T}(M) \coloneqq (\mathcal{T}(M), \gamma_{\mathcal{T}(M)}, \eta_{\mathcal{T}(M)})$ forms an operad, called the* free operad *over M. It satisfies the following universal property. For any morphism of $\mathbb{S}$-modules $M \to \mathscr{P}$, where $\mathscr{P}$ is an operad, there exists a unique morphism of operads $\widetilde{f} : \mathcal{T}(M) \to \mathscr{P}$ making the diagram*



commute in $\mathbb{S}$-mod.

Notice that all operads satisfying the universal property in the Theorem above are isomorphic. Moreover, the universal property gives a bijection

$$\hom_{\mathsf{Op}}(\mathcal{T}(M), \mathscr{P}) \cong \hom_{\mathbb{S}\text{-}\mathsf{mod}}(M, \mathscr{P}) \,,$$

that is to say that the free operad functor is left adjoint to the forgetful functor from operads to $\mathbb{S}$-modules.

Almost dual to the notion of operad is the notion of cooperad.

Definition 2.1.18. *A* cooperad *$\mathscr{C}$ is an $\mathbb{S}$-module $\mathscr{C}$ together with two morphisms of $\mathbb{S}$-modules*

$$\Delta_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C} \bar{\circ} \mathscr{C} \,,$$

called the decomposition map *of $\mathscr{C}$, and*

$$\varepsilon_{\mathscr{C}} : \mathscr{C} \longrightarrow I \,,$$

the counit map *of $\mathscr{C}$, making $\mathscr{C}$ into a comonoid. A morphism of cooperads is a morphism commuting with the decomposition maps and the counit maps of the source and of the target. The category of cooperads and their morphisms is denoted by* coOp.

Warning 2.1.19. *The notion defined above is often called* conilpotent cooperad *in the literature, e.g. in [LV12]. We will never use non-conilpotent cooperads.*

A cooperad $\mathscr{C}$ is *coaugmented* if there is a morphism $\eta_{\mathscr{C}} : I \to \mathscr{C}$ such that $\varepsilon_{\mathscr{C}} \eta_{\mathscr{C}} = 1_I$. The image of $1 \in \mathbb{K} = I(1)$ under $\eta_{\mathscr{C}}$ is denoted by $\mathrm{id} \in \mathscr{C}(1)$ and is called the *identity cooperation*.

A cooperad $\mathscr{C}$ is *reduced* if $\mathscr{C}(0) = 0$ and $\mathscr{C}(1) = \mathbb{K}$. Again, a reduced cooperad is automatically coaugmented.

Similarly to what happens with operads, given an $\mathbb{S}$-module M one can build a cofree cooperad by taking once again the tree module $\mathcal{T}(M)$ as underlying $\mathbb{S}$-module, defining the decomposition map

$$\Delta_{\mathcal{T}(M)} : \mathcal{T}(M) \longrightarrow \mathcal{T}(M) \circ \mathcal{T}(M)$$

by sending a tree to the sum of all its possible decompositions in 2-leveled trees by cutting the edges of the original tree, and defining the counit

$$\varepsilon_{\mathcal{T}(M)} : \mathcal{T}(M) \longrightarrow I$$

as the projection. Moreover, we consider the natural projection of $\mathbb{S}$-modules

$$\mathcal{T}(M) \longrightarrow M$$

More details are found in [LV12, Sect. 5.8.6–7].

Theorem 2.1.20. *The triple $\mathcal{T}^c(M) \coloneqq (\mathcal{T}(M), \Delta_{\mathcal{T}(M)}, \varepsilon_{\mathcal{T}(M)})$ forms a cooperad, called the* cofree cooperad *over M. It satisfies the following universal property, dual to the one of the free operad. For any morphism of $\mathbb{S}$-modules $f : \mathscr{C} \to M$, where $\mathscr{C}$ is a cooperad, there exists a unique morphism of cooperads making the diagram*

$$\begin{array}{ccc} \mathscr{C} & \overset{\widetilde{f}}{\dashrightarrow} & \mathcal{T}^c(M) \\ & \searrow^{f} & \downarrow \\ & & M \end{array}$$

commute in $\mathbb{S}$-mod.

Observe that all cooperads satisfying the universal property in the Theorem above are isomorphic. Moreover, the universal property gives a natural bijection

$$\hom_{\mathsf{coOp}}(\mathscr{C}, \mathcal{T}^c(M)) \cong \hom_{\mathbb{S}\text{-}\mathsf{mod}}(\mathscr{C}, M) \, ,$$

that is to say that the cofree cooperad functor is right adjoint to the forgetful functor from cooperads to $\mathbb{S}$-modules.

Given a cooperad $\mathscr{C}$, one can iterate the decomposition map $\Delta_{\mathscr{C}}$ in order to obtain a *monadic decomposition map*

$$\Delta^{\mathrm{mon}}_{\mathscr{C}} : \overline{\mathscr{C}} \longrightarrow \mathcal{T}^c(\overline{\mathscr{C}})$$

sending an element of $\overline{\mathscr{C}}$ to all of its possible decompositions not including the identity element $\mathrm{id} \in \mathscr{C}(1)$. Formally, we define the *reduced decomposition map* $\overline{\Delta}_{\mathscr{C}}$ of $\mathscr{C}$ as

$$\overline{\Delta}_{\mathscr{C}}(c) \coloneqq \Delta_{\mathscr{C}} - \mathrm{id} \circ c - c \circ \mathrm{id}^{\otimes n}$$

for $c \in \mathscr{C}(n)$. Now we define iteratively

$$\widetilde{\Delta}_1 \coloneqq 1_{\mathscr{C}} : \overline{\mathscr{C}} \longrightarrow \overline{\mathscr{C}} \, ,$$

and

$$\widetilde{\Delta}_n \coloneqq 1_{\mathscr{C}} + (1_{\mathscr{C}} \circ \widetilde{\Delta}_{n-1})\overline{\Delta}_{\mathscr{C}} : \overline{\mathscr{C}} \longrightarrow \mathcal{T}_n(\overline{\mathscr{C}})$$

for $n \geq 2$. Then, we set

$$\Delta^{\mathrm{mon}}_{\mathscr{C}} \coloneqq \operatorname*{colim}_n \widetilde{\Delta}_n : \overline{\mathscr{C}} \longrightarrow \mathcal{T}(\overline{\mathscr{C}}) \, .$$

One can always dualize a cooperad to obtain an operad, while the converse is not always true. This works as follows. Let $\mathscr{C}$ be a cooperad, then for $n \geq 0$ the chain complex $\mathscr{C}(n)^\vee$ is a left $\mathbb{S}_n$-module. We make it into a right $\mathbb{S}_n$-module by $c^\sigma \coloneqq \sigma \cdot c$ for $c \in \mathscr{C}(n)^\vee$. The dual of the decomposition map $\Delta_{\mathscr{C}}$ gives the composition map $\gamma_{\mathscr{C}^\vee}$, and the dual of the counit map gives the unit. The problem when trying to do the same thing with an operad $\mathscr{P}$ is that the dual of the composition map $\gamma_{\mathscr{P}}$ could give us infinite sums, and thus not land in $\mathscr{P}^\vee \circ \mathscr{P}^\vee$. However, if we assume that $\mathscr{P}$ is reduced and that either

- for all $n \geq 2$, the degrees in which $\mathscr{P}(n)$ is non-zero are bounded below, and $\mathscr{P}(n)$ is finite-dimensional in every degree, or that
- for all $n \geq 2$, the degrees in which $\mathscr{P}(n)$ is non-zero are bounded above, and $\mathscr{P}(n)$ is finite-dimensional in every degree,

then everything works out and the dual of an operad gives a cooperad. Notice that The conditions we gave in order to be able to dualize an operad and obtain a cooperad are sufficient, but not necessary. In particular, if an operad is finite dimensional in every arity, then it can be dualized to obtain a cooperad.

2.1.3 (Co)algebras over (co)operads

As already mentioned, the principal use of (co)operads is to encode (co)algebras.

Definition 2.1.21. *Let $\mathscr{P}$ be an operad. A $\mathscr{P}$-*algebra *A is a chain complex A together with a composition map*

$$\gamma_A : \mathscr{P}(A) \longrightarrow A$$

satisfying the relations

$$\gamma_A(1_{\mathscr{P}} \circ \gamma_A) = \gamma_A(\gamma_{\mathscr{P}} \circ 1_A)$$

on $\mathscr{P}(\mathscr{P}(A))$, and

$$\gamma_A(\eta_{\mathscr{P}} \circ 1_A) = \gamma_A$$

on $I(A) \cong A$. A morphism of $\mathscr{P}$-algebras *$f : A \to B$ is a chain map commuting with the respective composition maps, that is to say*

$$f\gamma_A = \gamma_B(1_{\mathscr{P}} \circ f)$$

on $\mathscr{P}(A)$. The category of $\mathscr{P}$-algebras with their morphisms is denoted by $\mathscr{P}$-alg.

The elements of the operad can be interpreted as operations on the algebra, as is made evident by the following result.

Proposition 2.1.22. *The structure of a $\mathscr{P}$-algebra on a chain complex A is equivalent to a map of operads*

$$\rho_A : \mathscr{P} \longrightarrow \mathrm{End}_A \, .$$

Proof. Given γ_A, one obtains ρ_A by

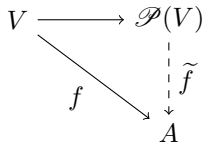
$$\rho_A(p)(a_1, \ldots, a_n) \coloneqq \gamma_A(p \otimes a_1 \otimes \cdots \otimes a_n) \, ,$$

and *vice versa*. The details are left to the reader. □

Proposition 2.1.23. *Let $\mathscr{P}$ be an operad and V be a chain complex. Then the map*

$$\gamma_{\mathscr{P}(V)} \coloneqq \gamma_{\mathscr{P}} \circ 1_V : \mathscr{P}(\mathscr{P}(V)) \cong (\mathscr{P} \circ \mathscr{P})(V) \longrightarrow \mathscr{P}(V)$$

makes $\mathscr{P}(V)$ into a $\mathscr{P}$-algebra. It is called the free $\mathscr{P}$-algebra *over V, and it satisfied the following universal property. For any morphism of chain complexes $f : V \to A$, where A is a $\mathscr{P}$-algebra, there exists a unique morphism of $\mathscr{P}$-algebras $\widetilde{f} : \mathscr{P}(V) \to A$ making the diagram*



commute in Ch*, where $V \to \mathscr{P}(V)$ is the canonical inclusion.*

Once again, the free $\mathscr{P}$-algebra functor is left adjoint to the forgetful functor from $\mathscr{P}$-algebras to chain complexes.

Lemma 2.1.24. *Let A, B be two $\mathscr{P}$-algebras, and let $f : A \to B$ be a morphism of $\mathscr{P}$-algebras. Suppose that f is an isomorphism of chain complexes. Then f is an isomorphism of $\mathscr{P}$-algebras as well.*

Proof. Write $g : B \to A$ for the inverse of f in chain complexes. We prove that g is a morphism of $\mathscr{P}$-algebras. We have

$$\begin{aligned} g\gamma_B &= g\gamma_B(1_{\mathscr{P}} \circ fg) \\ &= gf\gamma_B(1_{\mathscr{P}} \circ g) \\ &= \gamma_B(1_{\mathscr{P}} \circ g) \end{aligned}$$

as maps $\mathscr{P}(B) \to A$, where in the second line we used the fact that f is a morphism of $\mathscr{P}$-algebras. □

Suppose we have a morphism of operads $f : \mathscr{Q} \to \mathscr{P}$. Then it is obvious from Proposition 2.1.22 that every $\mathscr{P}$-algebra A is also a $\mathscr{Q}$-algebra, which we denote by f^*A, by $\rho_{f^*A} = f^*\rho_A = \rho_A f$. This defines a functor from $\mathscr{P}$-algebras to $\mathscr{Q}$-algebras, called *restriction of structure*. This functor is right adjoint to a functor $f_!$, called *extension of structure*, i.e.

$$f_! \colon \mathscr{Q}\text{-alg} \rightleftarrows \mathscr{P}\text{-alg} : f^*.$$

The $\mathscr{P}$-algebra $f_!A$ associated to a $\mathscr{Q}$-algebra A can be described as a reflexive coequalizer, see [Fre09a, Sect. 3.3.5] for details. These functors define a Quillen adjunction between the respective categories of algebras. For these model categorical aspects, we invite the interested reader to consult [Fre09a, Sect. 16].

Almost dually to the notion of algebra over an operad is the notion of coalgebra over a cooperad.

Definition 2.1.25. *Let $\mathscr{C}$ be a cooperad. A* $\mathscr{C}$-coalgebra *C is a chain complex C together with a decomposition map*

$$\Delta_C : C \longrightarrow \widehat{\mathscr{C}}(C) := \prod_{n\geq 0} \left(\mathscr{C}(n) \otimes C^{\otimes n}\right)^{\mathbb{S}_n}$$

such that

$$(\Delta_{\mathscr{C}} \circ 1_C)\Delta_C = (1_{\mathscr{C}} \circ \Delta_C)\Delta_C, \qquad (\varepsilon_{\mathscr{C}} \circ 1_C)\Delta_C = \Delta_C$$

on C. A morphism of $\mathscr{C}$-coalgebras $f : C \to D$ is a chain map commuting with the respective decomposition maps, that is

$$(1_{\mathscr{C}} \circ f)\Delta_C = \Delta_D f$$

on C.

There is no analogue to Proposition 2.1.22 since it is not possible to define an "endomorphism cooperad" playing the dual role of End_V.

Let $\mathscr{C}$ be a reduced, coaugmented cooperad, and let C be a $\mathscr{C}$-coalgebra. We denote the image of $x \in C$ under the decomposition map Δ_C by

$$\Delta_C(x) = (x_1, x_2, \ldots) \in \prod_{n\geq 1} \left(\mathscr{C}(n) \otimes C^{\otimes n}\right)^{\mathbb{S}_n}.$$

There is a canonical ascending filtration on C given by

$$\mathscr{F}^0_{\mathscr{C}}C := 0\,, \quad \text{and} \quad \mathscr{F}^n_{\mathscr{C}}C := \{x \in C \mid x_k = 0 \text{ for all } k > n\}$$

for $n \geq 1$. It is called the *coradical filtration* of the coalgebra.

Remark 2.1.26. *If one works more in general with connected weight graded cooperads instead of reduced cooperads, then one must be a little more careful when defining the coradical filtration, see e.g. [Val14, Def. 2.2]. What one desires in the end, is that the coproduct Δ_C is a morphism of connected weighted cooperads from C to $\mathscr{C}(C)$ endowed with the induced connected weight grading.*

Definition 2.1.27. *A $\mathscr{C}$-coalgebra C is* conilpotent *if the canonical filtration is exhaustive, that is*

$$C = \operatorname*{colim}_n \mathscr{F}^n_{\mathscr{C}} C$$

*as $\mathscr{C}$-coalgebras. Conilpotent $\mathscr{C}$-coalgebras form a full subcategory of the category of $\mathscr{C}$-coalgebras and their morphisms, which we will denote by $\mathscr{C}$-*cog*.*

Remark 2.1.28. *In the rest of this work, we will always work with conilpotent coalgebras, and never with non-conilpotent ones. For this reason, we will sometimes omit the adjective conilpotent when talking about coalgebras, and simply say $\mathscr{C}$-coalgebras when we really mean conilpotent $\mathscr{C}$-coalgebras.*

The coradical filtration is well-behaved with respect to the coproduct.

Lemma 2.1.29. *Let C be a conilpotent $\mathscr{C}$-coalgebra. For each $n \geq 0$, we have*

$$\Delta_C(\mathscr{F}^n_{\mathscr{C}} C) \subseteq \bigoplus_{\substack{k \geq 1 \\ n_1 + \cdots + n_k = n}} \left(\mathscr{C}(k) \otimes \mathscr{F}^{n_1}_{\mathscr{C}} C \otimes \cdots \otimes \mathscr{F}^{n_k}_{\mathscr{C}} C\right)^{\mathbb{S}_k} .$$

Proof. The fact that we land in invariants is automatic, so that we can work in the non-symmetric case without loss of generality. The case $n = 0$ is trivial. Let $n \geq 1$ and $x \in \mathscr{F}^n_{\mathscr{C}} C$. The relation $(\Delta_{\mathscr{C}} \circ 1_C)\Delta_C = (1_{\mathscr{C}} \circ \Delta_C)\Delta_C$ implies that, for every $m > n$, we have

$$\begin{aligned} 0 &= (\Delta_{\mathscr{C}} \circ 1_C)\Delta^m_C(x) \\ &= \sum_{\substack{k \geq 1 \\ n_1 + \cdots + n_k = m}} (1_{\mathscr{C}} \circ (\Delta^{n_1}_C, \ldots, \Delta^{n_k}_C))\Delta^k_C(x) . \end{aligned}$$

Since C is conilpotent, the last sum is finite. Every term has a different underlying tree, so that we must have

$$(1_{\mathscr{C}} \circ (\Delta^{n_1}_C, \ldots, \Delta^{n_k}_C))\Delta^k_C(x) = 0 \tag{2.1}$$

for each $k \geq 1$ and $n_1 + \cdots + n_k = m$. Now fix $k \geq 1$ and suppose that $\Delta^k_C(x)$ has a term which is not in the correct space. Since C is conilpotent, that term lives in

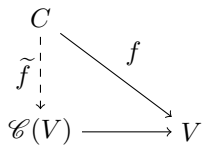
$$\mathscr{C}(k) \otimes \mathscr{F}^{\widetilde{n}_1}_{\mathscr{C}} C \otimes \cdots \otimes \mathscr{F}^{\widetilde{n}_k}_{\mathscr{C}} C$$

for some $\widetilde{n}_1 + \cdots + \widetilde{n}_k > n$, which contradicts relation (2.1). □

Proposition 2.1.30. *Let $\mathscr{C}$ be an operad and V be a chain complex. Then the map*

$$\Delta_{\mathscr{C}(V)} \coloneqq \Delta_{\mathscr{C}} \circ 1_V : \mathscr{C}(V) \longrightarrow (\mathscr{C} \circ \mathscr{C})(V) \cong \mathscr{C}(\mathscr{C}(V))$$

makes $\mathscr{C}(V)$ into a conilpotent $\mathscr{C}$-coalgebra. It is called the cofree conilpotent $\mathscr{C}$-coalgebra *over V, and it satisfies the following universal property. For any morphism of chain complexes $g : C \to V$, where C is a conilpotent $\mathscr{C}$-coalgebra, there exists a unique morphism of $\mathscr{C}$-coalgebras $\widetilde{g} : C \to \mathscr{C}(V)$ making the diagram*



commute in Ch, *where the map* $\mathscr{C}(V) \to V$ *is the canonical projection.*

Lemma 2.1.31. *Let* C, D *be two conilpotent* $\mathscr{C}$*-coalgebras, and let* $f : C \to D$ *be a morphism of* $\mathscr{C}$*-coalgebras. Suppose that* f *is an isomorphism of chain complexes. Then* f *is an isomorphism of* $\mathscr{C}$*-coalgebras as well.*

Proof. The proof is dual to the proof of Lemma 2.1.24 and left to the reader. □

Remark 2.1.32. *One can also define coalgebras over an operad — see [LV12, Sect. 5.2.15] — and algebras over a cooperad. For a modern treatment of these notions and the study of their homotopical behavior, see e.g. [LGL18].*

2.1.4 Limits and colimits of (co)algebras

The categories of algebras over operads and conilpotent coalgebras over cooperads are complete and cocomplete.

Theorem 2.1.33 ([GJ94, Sect. 1.6–7])**.** *Let* $\mathscr{P}$ *be an operad, and let* $\mathscr{C}$ *be a cooperad.*

1. *The category* $\mathscr{P}$-alg *of* $\mathscr{P}$*-algebras admits all limits and colimits.*
2. *The category* $\mathscr{C}$-cog *of conilpotent* $\mathscr{C}$*-coalgebras admits all limits and colimits.*

Remark 2.1.34. *The category of* $\mathscr{P}$*-algebras is always complete and cocomplete, even when working over a field of positive characteristic. For conilpotent* $\mathscr{C}$*-coalgebras, one gets all limits and colimits provided one supposes that the cooperad* $\mathscr{C}$ *satisfies some additional properties.*

Here are some easily described limits and colimits of (co)algebras.

Proposition 2.1.23 tells us that we have a natural bijection

$$\hom_{\mathsf{Ch}}(V, A^{\#}) \cong \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\mathscr{P}(V), A) \,,$$

where $(-)^{\#}$ is the forgetful functor from $\mathscr{P}$-algebras to chain complexes. In other words, we have an adjunction

$$\mathscr{P} : \mathsf{Ch} \rightleftarrows \mathscr{P}\text{-}\mathsf{alg} : (-)^{\#} \,.$$

In particular, the forgetful functor preserves limits, since it is right adjoint.

Proposition 2.1.35. *Let* A, B *be* $\mathscr{P}$*-algebras. The product* $A \times B$ *of* A *and* B *has the product of the chain complexes* A *and* B *as underlying chain complex, which we identify with the direct sum* $A \oplus B$*, and the* $\mathscr{P}$*-algebra structure is given by the composite*

$$\gamma_{A\times B} \coloneqq \Big(\mathscr{P}(A \oplus B) \xrightarrow{\mathrm{proj}_A \oplus \mathrm{proj}_B} \mathscr{P}(A) \oplus \mathscr{P}(B) \xrightarrow{\gamma_A \oplus \gamma_B} A \oplus B \Big) \,.$$

Here $\mathrm{proj}_A : \mathscr{P}(A \oplus B) \to \mathscr{P}(A)$ *is the canonical projection, and similarly for* proj_B*.*

In other words, the product of A and B is given by $A \oplus B$ with the $\mathscr{P}$-algebra structures of A and B on the two factors, and nothing mixing them, i.e. if we put some elements of A and some elements of B in some operation, we get 0.

The description of coproducts is only slightly more complicated.

Proposition 2.1.36. *Let A, B be $\mathscr{P}$-algebras. The coproduct $A \sqcup B$ of A and B is*

$$A \sqcup B \coloneqq \mathscr{P}(A \oplus B)/\sim ,$$

where the equivalence relation $\sim$ is generated by

$$p \otimes_{\mathbb{S}_n} (a_1 \otimes \cdots \otimes a_n) \sim \gamma_A(p \otimes_{\mathbb{S}_n} (a_1 \otimes \cdots \otimes a_n)) ,$$
$$p \otimes_{\mathbb{S}_n} (b_1 \otimes \cdots \otimes b_n) \sim \gamma_B(p \otimes_{\mathbb{S}_n} (b_1 \otimes \cdots \otimes b_n))$$

for any $p \in \mathscr{P}(n)$, $a_i \in A$, and $b_i \in B$.

In other words, the coproduct of A and B is given by a copy of A, one copy of B, plus all the elements generated freely by mixing elements of A and B and acting with operations from the operad $\mathscr{P}$. Another useful way to express the coproduct of two algebras is as the coequalizer in chain complexes

$$(\mathscr{P} \circ_{(1)} \mathscr{P})(A \oplus B) \underset{1_{\mathscr{P}} \circ_{(1)} (\gamma_A + \gamma_B)}{\overset{\gamma_{\mathscr{P}} \circ 1_{A \oplus B}}{\rightrightarrows}} \mathscr{P}(A \oplus B) \longrightarrow A \sqcup B$$

where

$$\gamma_A + \gamma_B : \mathscr{P}(A \oplus B) \longrightarrow A \oplus B$$

is given by γ_A on $\mathscr{P}(A)$, by γ_B on $\mathscr{P}(B)$, and by zero on anything containing mixed terms. The $\mathscr{P}$-algebra structure is the one induced by the $\mathscr{P}$-algebra structure of $\mathscr{P}(A \oplus B)$.

The situation for cooperads is essentially dual. By Proposition 2.1.30, we have the adjunction

$$(-)^{\#} : \mathscr{C}\text{-cog} \rightleftarrows \mathsf{Ch} : \mathscr{C}$$

between the forgetful functor $(-)^{\#}$ and the cofree conilpotent coalgebra functor $\mathscr{C}$. In particular, the forgetful functor preserves colimits.

Proposition 2.1.37. *Let C, D be conilpotent $\mathscr{C}$-coalgebras. The coproduct $C \sqcup D$ of C and D is the conilpotent $\mathscr{C}$-coalgebra*

$$C \sqcup D \coloneqq C \oplus D$$

with structure map

$$\Delta_{C \sqcup D} \coloneqq \left(C \oplus D \xrightarrow{\Delta_C \oplus \Delta_D} \mathscr{C}(C) \oplus \mathscr{C}(D) \hookrightarrow \mathscr{C}(C \oplus D) \right) .$$

The product of two conilpotent $\mathscr{C}$-coalgebras C and D is a sub-coalgebra of $\mathscr{C}(C \oplus D)$, which we can express as an equalizer.

Proposition 2.1.38. *Let C, D be conilpotent $\mathscr{C}$-coalgebras. The product $C \times D$ of C and D is the conilpotent $\mathscr{C}$-coalgebra given by the equalizer*

$$C \times D \longrightarrow \mathscr{C}(C \oplus D) \underset{1_{\mathscr{C}} \circ (\Delta_C + \Delta_D)}{\overset{\Delta_{(1)} \circ 1_{C \oplus D}}{\rightrightarrows}} (\mathscr{C} \circ_{(1)} \mathscr{C})(C \oplus D)$$

where

$$\Delta_C + \Delta_D : C \oplus D \xrightarrow{\Delta_C \oplus \Delta_D} \mathscr{C}(C) \oplus \mathscr{C}(D) \hookrightarrow \mathscr{C}(C \oplus D) ,$$

with the $\mathscr{C}$-coalgebra structure induced by the $\mathscr{C}$-coalgebra structure of $\mathscr{C}(C \oplus D)$.

2.1.5 Quadratic (co)operads

Many classically arising operads and cooperads can be presented by generators and relations in the way we expose now. More details can be found in [LV12, Sect. 7.1].

Definition 2.1.39. *An* operadic quadratic data (E, R) *is a graded $\mathbb{S}$-module E (i.e. an $\mathbb{S}$-module where all the differentials are trivial) together with a graded sub-$\mathbb{S}$-module $R \subseteq \mathcal{T}(E)^{(2)}$ of the graded sub-$\mathbb{S}$-module $\mathcal{T}(E)^{(2)}$ of the tree module $\mathcal{T}(E)$ given by trees with 2 vertices. The elements of E are called the* generating operations*, and the elements of R the* relations*. A* morphism *of quadratic data*

$$f : (E, R) \longrightarrow (F, S)$$

is a morphism $f : E \to F$ of $\mathbb{S}$-modules such that $\mathcal{T}(f)(R) \subseteq S$.

One can associate an operad $\mathscr{P}(E, R)$ to an operadic quadratic data (E, R) by

$$\mathscr{P}(E, R) \coloneqq \mathcal{T}(E)/(R) \,,$$

where (R) is the smallest operadic ideal of $\mathcal{T}(E)$ containing R. Equivalently, $\mathscr{P}(E, R)$ is universal among the quotient operads $\mathscr{P}$ of $\mathcal{T}(E)$ such that the composite

$$(R) \xrightarrow{\text{incl.}} \mathcal{T}(E) \xrightarrow{\text{proj.}} \mathscr{P}$$

is zero. An operad obtained in this way is called a *quadratic operad*. If moreover E is concentrated in arity 2, one speaks of a *binary quadratic operad*.

Dually, one can associate a cooperad $\mathscr{C}(E, R)$ to an operadic quadratic data (E, R). It is defined through an universal property dual to the one given above, but it doesn't have a nice presentation such as $\mathscr{P}(E, R)$. Explicitly, $\mathscr{C}(E, R)$ is such that for any sub-cooperad $\mathscr{C}$ of $\mathcal{T}^c(E)$ such that the composite

$$\mathscr{C} \xrightarrow{\text{incl.}} \mathcal{T}^c(E) \xrightarrow{\text{proj.}} \mathcal{T}^c(E)^{(2)}/(R)$$

is zero, there exists a unique morphism of cooperads $\mathscr{C} \to \mathscr{C}(E, R)$ such that

$$\mathscr{C} \longrightarrow \mathscr{C}(E, R) \xrightarrow{\text{incl.}} \mathcal{T}^c(E) = \mathscr{C} \xrightarrow{\text{incl.}} \mathcal{T}^c(E) \,.$$

Any morphism of quadratic data induces a morphism between the associated operads, respectively cooperads.

2.1.6 Some classical (co)operads and (co)algebras over them

We will now give some examples of operads and cooperads encoding some classical types of algebras and coalgebras. All of them are induced by quadratic data. We begin by the *three graces*, the operads Com, Ass, and Lie.

To define the operad Com, we take the $\mathbb{S}$-module E with $E(2) = \mathbb{K}\mu$ and $E(n) = 0$ for all $n \neq 2$, with $|\mu| = 0$ and the trivial action of $\mathbb{S}_2$ on μ, and the relations R spanned by $\mu \circ (\mathrm{id} \otimes \mu) - \mu \circ (\mu \otimes \mathrm{id})$. We set

$$\mathrm{Com} \coloneqq \mathscr{P}(E, R) \,.$$

Notice that, because of the relations, we have $\mathrm{Com}(n) = \mathbb{K}\mu_n$ for $n \geq 2$, where μ_n is any composition of copies of μ giving an operation of arity n. All such compositions are equal in Com because of the relations. The algebras over Com are exactly the commutative (not necessarily

unital) algebras, i.e. vector spaces A together with a commutative, associative multiplication. Commutativity is encoded in the fact that the $\mathbb{S}_2$-action on μ is trivial, while associativity is given by R.

For the operad Ass, we take E with $E(2) = \mathbb{K}[\mathbb{S}_2] = \mathbb{K}a_e \oplus \mathbb{K}a_{(12)}$ with $|a_e| = |a_{(12)}| = 0$ and $\mathbb{S}$-action given by $a_e^{(12)} = a_{(12)}$, and R spanned by $a_e \circ (\mathrm{id} \otimes a_e) - a_e \circ (a_e \otimes \mathrm{id})$ and the elements of $\mathscr{T}(E)^{(2)}$ that can be obtained by it by $\mathbb{S}_3$-action. We set

$$\mathrm{Ass} \coloneqq \mathscr{P}(E, R) \ .$$

The operad Ass is spanned by operations $\{a_\sigma \mid \sigma \in \mathbb{S}_n\}$ in arity $n \geq 2$, corresponding to the multiplication

$$(x_1, \ldots, x_n) \longmapsto x_{\sigma(1)} \cdots x_{\sigma(n)}$$

in an associative (not necessarily unital) algebra.

There is a non-symmetric version of Ass, which is denoted by As. It is also a binary quadratic ns operad. It is generated by a single binary operation μ_2 in arity 2 and has only one relation, given by $m \circ (m \otimes \mathrm{id}) - m \circ (\mathrm{id} \otimes m)$. The resulting operad is 1-dimensional in every arity $n \geq 1$, where it is spanned by the operation μ_n multiplying n elements in the order they are given. We recover Ass from As by tensoring by the regular representation $\mathbb{K}[\mathbb{S}]$ of the symmetric groups.

The last one of the three graces is the operad Lie. In order to define it, we define E by $E(2) = \mathbb{K}b$ with $|b| = 0$ and the sign representation of $\mathbb{S}_2$, and R spanned by $b \circ (b \otimes \mathrm{id}) + b \circ (b \otimes \mathrm{id})^{(123)} + b \circ (b \otimes \mathrm{id})^{(321)}$ and the other elements obtained from this one by $\mathbb{S}_3$-action. An algebra over Lie is the usual notion of (differential graded) Lie algebra: the operation b gives the Lie bracket, which is antisymmetric since b carries the sign representation, and R gives the Jacobi rule.

The three graces are related by morphisms of operads

$$\mathrm{Lie} \xrightarrow{a} \mathrm{Ass} \xrightarrow{u} \mathrm{Com} \ ,$$

all of them coming from morphisms of operadic quadratic data. The morphism $a : \mathrm{Lie} \to \mathrm{Ass}$ is the *antisymmetrization morphism*, and it is given by sending $b \in \mathrm{Lie}(2)$ to $a_e - a_{(12)} \in \mathrm{Ass}(2)$. This corresponds to the fact that if A is an associative algebra, then we can see it as a Lie algebra by $[x, y] \coloneqq xy - (-1)^{xy} yx$. The morphism $u : \mathrm{Ass} \to \mathrm{Com}$ is induced by sending both a_e and $a_{(12)}$ to μ, and correspond to the fact that every commutative algebra is trivially an associative algebra.

One can also define operads uCom and uAss encoding unital commutative, respectively associative algebras simply by setting $E(0) = \mathbb{K}u$ with $|u| = 0$ in both cases, and adding the relations $\mu \circ (u \otimes \mathrm{id}) = \mathrm{id}$, respectively $a_e \circ (u \otimes \mathrm{id})$ and $a_{(12)} \circ (u \otimes \mathrm{id})$ to R. Notice that those relations are now contained in $\mathscr{T}(E)^{(1)} \oplus \mathscr{T}(E)^{(2)}$, so we are not in the quadratic case anymore.

On the coalgebra side, we give the example of conilpotent coassociative coalgebras in the non-symmetric setting. These coalgebras are encoded by the cooperad coAs which is dual to As. In other words, take $E = E(2) = \mathbb{K}a^\vee$, and $R = a^\vee \circ (\mathrm{id} \otimes a^\vee) - a^\vee \circ (a^\vee \otimes \mathrm{id})$. We set

$$\mathrm{coAs} \coloneqq \mathscr{C}(E, R) \ .$$

A conilpotent coalgebra over this cooperad is a vector space[2] C together with a decomposition map

$$\Delta : C \longrightarrow \mathscr{C} \otimes C \ ,$$

[2]Or graded vector space, or chain complex. Here we work over vector spaces to avoid signs.

corresponding to the binary part of the whole coproduct Δ_C, such that

$$(\Delta \otimes 1_C)\Delta(x) = (1_C \otimes \Delta)\Delta(x)$$

and such that any iteration of Δ on any element eventually terminates (which corresponds to the fact of being conilpotent).

2.1.7 Non-symmetric (co)algebras and (co)operads

There is a version of (co)operads which codes (co)algebras without symmetries (such as commutativity of a multiplication, and so on). It is called *non-symmetric*, or *ns (co)operads*. It is given by replacing $\mathbb{S}$-modules by sequences of chain complexes without any group action — also called *arity graded chain complexes* — and by forgetting all of the group actions in the definitions. For example the composite product of M and N in this context becomes

$$(M \circ N)(k) \coloneqq \bigoplus_{\substack{i \geq 0 \\ j_1 + \cdots + j_i = k}} M(i) \otimes N(j_1) \otimes \cdots \otimes N(j_i) \ ,$$

and similarly for all the rest. More details are given in [LV12, Sect. 5.9].

One can always pass from the symmetric world to the ns world by forgetting the $\mathbb{S}$-actions, and go the other way around by tensoring by the regular representation of the symmetric groups. This gives a pair of adjoint functors.

It is often true that results holding in the symmetric world are also true in the ns world, while results that are true in the ns world hold in the symmetric world *in characteristic* 0. This is because e.g. when making operads and cooperads (or algebras and coalgebras) interact, in the symmetric world one has to identify invariants and coinvariants, which one can only do in characteristic 0. One possible way to go around this restriction is to work with operads with divided powers [LV12, Sect. 5.2.9].

2.2 Operadic homological algebra

In order to go on, we have to introduce the constructions and notations allowing us to do homological algebra in the operadic context, such as the correct notion of suspension, the infinitesimal composites corresponding in some sense to "derivatives" of the composite product, and so on.

2.2.1 Operadic suspension

Notice that, given an operad $\mathscr{P}$, there is no natural way to put an operad structure on $s\mathscr{P}$. However, there is a good way to suspend and desuspend (co)operads. We present it here, following [LV12, Sect. 7.2.2].

Let $\mathscr{S} \coloneqq \mathrm{End}_{s\mathbb{K}}$ be the endomorphism operad of the chain complex $s\mathbb{K}$, which is of dimension 1 and concentrated in degree 1. We have

$$\mathscr{S}(n) = \mathbb{K}\mathscr{S}_n \ ,$$

where $\mathscr{S}_n$ is the linear map of degree $1-n$ sending s^n to s and carries the sign representation. We have

$$\mathscr{S}_n \circ_1 \mathscr{S}_m = \mathscr{S}_{n+m-1}$$

and thus

$$\mathscr{S}_n \circ_j \mathscr{S}_m = (-1)^{(j-1)(1-m)} \mathscr{S}_{n+m-1} \,.$$

This determines most of the signs in operad theory, for example in minimal models for Koszul operads, see Section 2.3.

Definition 2.2.1. *Let $\mathscr{P}$ be an operad. The* (operadic) suspension *of $\mathscr{P}$ is the operad $\mathscr{S} \otimes \mathscr{P}$.*

In order to desuspend operads, we consider instead $\mathscr{S}^{-1} \coloneqq \mathrm{End}_{s^{-1}\mathbb{K}}$. It behaves similarly to $\mathscr{S}$, and we have an isomorphism of operads

$$\mathscr{S}^{-1} \otimes \mathscr{S} \cong \mathrm{End}_{\mathbb{K}} \tag{2.2}$$

with the unit for the Hadamard tensor product given by

$$\mathscr{S}_n^{-1} \otimes \mathscr{S}_n \longmapsto (-1)^{\frac{n(n-1)}{2}} m_n \,,$$

where $m_n \in \mathrm{End}_{\mathbb{K}}(n)$ is the map sending $1_{\mathbb{K}}^{\otimes n}$ to $1_{\mathbb{K}}$. The sign can be found by

$$\begin{aligned}
(-1)^{\frac{n(n-1)}{2}} &= s^{-n} s^n \\
&= (-1)^{\epsilon} \mathscr{S}_n^{-1} \mathscr{S}_n s^{-n} s^n \\
&= (-1)^{\epsilon + (1-n)(-n)} \mathscr{S}_n^{-1} s^{-n} \mathscr{S}_n s^n \\
&= (-1)^{\epsilon} s^{-1} s^n \\
&= (-1)^{\epsilon} \,,
\end{aligned}$$

with the additional sign appearing in the third line comes from the Koszul sign rule. Explicitly, the composition in $\mathscr{S}^{-1}$ is given by

$$\mathscr{S}_n^{-1} \circ_j \mathscr{S}_m^{-1} = (-1)^{(j-1)(1-m)} \mathscr{S}_{n+m-1}^{-1} \,.$$

A straightforward computation shows that this is compatible with the isomorphism (2.2).

Definition 2.2.2. *Let $\mathscr{P}$ be an operad. The* (operadic) desuspension *of $\mathscr{P}$ is the operad $\mathscr{S}^{-1} \otimes \mathscr{P}$.*

Proposition 2.2.3. *The structure of a $\mathscr{P}$-algebra on a chain complex A is equivalent to the structure of a $\mathscr{S} \otimes \mathscr{P}$-algebra on sA, respectively to the structure of a $\mathscr{S}^{-1} \otimes \mathscr{P}$-algebra on $s^{-1}A$.*

Proof. The structure on sA is given by

$$\gamma_{sA}(\mathscr{S}_n p \otimes sa_1 \otimes sa_n) \coloneqq (-1)^{\epsilon} s \gamma_A(p \otimes a_1 \otimes \cdots \otimes a_n)$$

with

$$\epsilon = np + \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_j$$

coming from the Koszul sign rule. We leave the rest of the proof to the reader. □

For cooperads, one can endow the $\mathbb{S}$-module $\mathrm{End}_{s\mathbb{K}}$ with the cooperad structure dual to the operad structure of $\mathrm{End}_{s^{-1}\mathbb{K}}$ to obtain a cooperad $\mathscr{S}^c$, and similarly a cooperad $(\mathscr{S}^{-1})^c$, defining cooperadic suspension and desuspension.

Definition 2.2.4. *The cooperads $\mathscr{S}^c$ and $(\mathscr{S}^{-1})^c$ are defined by*

$$(\mathscr{S}^c)^\vee \coloneqq \mathscr{S}^{-1}, \qquad \textit{and} \qquad (\mathscr{S}^{-1})^c \coloneqq \mathscr{S}^\vee.$$

One can explicitly describe the decomposition map by finding the signs in

$$\Delta_{(\mathscr{S}^{-1})^c}(\mathscr{S}_n^{-1}) = \sum_{\substack{k\geq 1 \\ n_1+\cdots+n_k=n}} (-1)^{\epsilon(k,n_1,\ldots,n_k)} \mathscr{S}_k^{-1} \otimes \mathscr{S}_{n_1}^{-1} \otimes \cdots \mathscr{S}_{n_k}^{-1}$$

by computing

$$\begin{aligned}(-1)^{\epsilon(k,n_1,\ldots,n_k)} &= \left\langle \Delta_{(\mathscr{S}^{-1})^c}(\mathscr{S}_n^{-1}), \mathscr{S}_k \otimes \mathscr{S}_{n_1} \otimes \cdots \mathscr{S}_{n_k} \right\rangle \\ &= \left\langle \mathscr{S}_n^{-1}, \gamma_{\mathscr{S}}(\mathscr{S}_k \otimes \mathscr{S}_{n_1} \otimes \cdots \mathscr{S}_{n_k}) \right\rangle \ .\end{aligned}$$

2.2.2 Connected weight gradings

Sometimes, in order to make certain homological arguments, one needs an additional grading on (co)operads. This is encoded in the notion of *connected weight graded* (co)operads, see e.g. [Val14, Sect. 1.7]. When working with reduced (co)operads, one can usually consider the grading given by putting in weight w the elements of arity $w+1$. We will always implicitly consider this canonical additional grading on reduced (co)operads. Many results presented in this thesis for reduced (co)operads also hold in the context of connected weight graded (co)operads.

2.2.3 Infinitesimal composites

The composite product of $\mathbb{S}$-modules is linear in the first slot, i.e.

$$(M_1 \oplus M_2) \circ N \cong (M_1 \circ N) \oplus (M_2 \circ N)$$

for $\mathbb{S}$-modules M_1, M_2 and N, but not in the second one. In order to do homological algebra, we need a version of the composite product which is linear also on the right. This is the material of [LV12, Sect. 6.1].

Let M, N_1 and N_2 be three $\mathbb{S}$-modules. Then we can consider the sub-module $M \circ (N_1; N_2)$ of $M \circ (N_1 \oplus N_2)$ where N_2 appears exactly once in each summand. This construction is linear in both M and N_2. Notice that $M \circ (N; N)$ is not isomorphic to $M \circ N$, because in $M \circ (N_1; N_2)$ we always remember the position of the copy of N_2. However, we have an obvious forgetful map $M \circ (N; N) \to M \circ N$.

Definition 2.2.5. *Let $f : M_1 \to M_2$ and $g : N_1 \to N_2$ be two morphisms of $\mathbb{S}$-modules. The* infinitesimal composite of morphisms *is defined as*

$$f \circ' g : M_1 \circ N_1 \longrightarrow M_2 \circ (N_1; N_2)$$

by

$$(f \circ' g)(\mu; \nu_1, \ldots, \nu_n) = \sum_{i=1}^{n} (f(\mu); \nu_1, \ldots, g(\nu_i), \ldots, \nu_n)$$

for $(\mu; \nu_1, \ldots, \nu_n) \in (M_1 \circ N_1)(n)$.

Example 2.2.6. *Let M and N be two $\mathbb{S}$-modules, then the differential of $M \circ N$ can be seen as*

$$d_{M\circ N} = d_M \circ 1_N + 1_M \circ' d_N$$

after applying the canonical forgetful map $M \circ (N; N) \to M \circ N$ to the last term.

Definition 2.2.7. *Let M and N be two $\mathbb{S}$-modules. The* infinitesimal composite *of M and N is the $\mathbb{S}$-module*

$$M \circ_{(1)} N \coloneqq M \circ (I; N) .$$

If $f : M_1 \to M_2$ and $g : N_1 \to N_2$ are two morphisms of $\mathbb{S}$-modules, we define

$$f \circ_{(1)} g : M_1 \circ_{(1)} N_1 \longrightarrow M_2 \circ_{(1)} N_2$$

is defined by

$$(f \circ_{(1)} g)(\mu; \mathrm{id}, \ldots, \nu, \ldots, \mathrm{id}) = (f(\mu); \mathrm{id}, \ldots, g(\nu), \ldots, \mathrm{id})$$

for $\mu \in M_1$ and $\nu \in N_1$.

Notice that we always have

$$M \circ_{(1)} M \cong \mathcal{T}(M)^{(2)}.$$

Lemma 2.2.8. *The infinitesimal composite product $\circ_{(1)}$ is linear in both variables.*

If $\mathscr{P}$ is an operad, we define the *infinitesimal composition map*

$$\gamma_{(1)} : \mathscr{P} \circ_{(1)} \mathscr{P} \longrightarrow \mathscr{P}$$

as the composite

$$\mathscr{P} \circ_{(1)} \mathscr{P} = \mathscr{P} \circ (I; \mathscr{P}) \longrightarrow \mathscr{P} \circ (I \oplus \mathscr{P}) \xrightarrow{1_{\mathscr{P}} \circ (\eta_{\mathscr{P}} + 1_{\mathscr{P}})} \mathscr{P} \circ \mathscr{P} \xrightarrow{\gamma_{\mathscr{P}}} \mathscr{P} .$$

It is simply the composition map $\gamma_{\mathscr{P}}$ restricted to the composition of two operations in $\mathscr{P}$.

Dually, if $\mathscr{C}$ is a cooperad we define its *infinitesimal decomposition map*

$$\Delta_{(1)} : \mathscr{C} \longrightarrow \mathscr{C} \circ_{(1)} \mathscr{C}$$

by

$$\mathscr{C} \xrightarrow{\Delta_{\mathscr{C}}} \mathscr{C} \circ \mathscr{C} \xrightarrow{1_{\mathscr{C}} \circ' 1_{\mathscr{C}}} \mathscr{C} \circ (\mathscr{C}; \mathscr{C}) \xrightarrow{1_{\mathscr{C}} \circ (\varepsilon_{\mathscr{C}}; 1_{\mathscr{C}})} \mathscr{C} \circ (I; \mathscr{C}) = \mathscr{C} \circ_{(1)} \mathscr{C} .$$

It can be seen as the decomposition of elements of $\mathscr{C}$ into two parts.

2.2.4 Convolution operads

Given a cooperad $\mathscr{C}$ and an operad $\mathscr{P}$, one can give a natural operad structure to $\hom(\mathscr{C}, \mathscr{P})$. This was first proven in [BM03, p. 3] and is detailed in [LV12, Sect. 6.4.1].

The composition map on $\hom(\mathscr{C}, \mathscr{P})$ is defined as follows. Let $f \in \hom(\mathscr{C}(k), \mathscr{P}(k))$, and let $g_i \in \hom(\mathscr{C}(n_i), \mathscr{P}(n_i))$ for $1 \leq i \leq k$. Then $\gamma_{\hom(\mathscr{C},\mathscr{P})}(f; g_1, \ldots, g_k)$ is given by the composite

$$\begin{aligned} \mathscr{C}(n) &\xrightarrow{\Delta_{\mathscr{C}}} (\mathscr{C} \bar{\circ} \mathscr{C})(n) \xrightarrow{\text{proj.}} \mathscr{C}(k) \otimes \mathscr{C}(n_1) \otimes \cdots \otimes \mathscr{C}(n_k) \otimes \mathbb{K}[\mathbb{S}_n] \\ &\xrightarrow{f \otimes g_1 \otimes \cdots \otimes g_k \otimes 1_{\mathbb{K}[\mathbb{S}_n]}} \mathscr{P}(k) \otimes \mathscr{P}(n_1) \otimes \cdots \otimes \mathscr{P}(n_k) \otimes \mathbb{K}[\mathbb{S}_n] \\ &\xrightarrow{\text{incl.}} (\mathscr{P} \circ \mathscr{P})(n) \xrightarrow{\gamma_{\mathscr{P}}} \mathscr{P}(n) , \end{aligned}$$

where $n = n_1 + \cdots + n_k$. The projection map sends an element of $(\mathscr{C} \bar{\circ} \mathscr{C})(n)$ to all its components that live in $\mathscr{C}(k) \otimes \mathscr{C}(n_1) \otimes \cdots \otimes \mathscr{C}(n_k)$, with possibly a permutation of the arguments encoded by an $(n_1, \ldots, n_k)$-shuffle. Moreover, the composite

$$\mathscr{C} \xrightarrow{\varepsilon_{\mathscr{C}}} I \xrightarrow{\eta_{\mathscr{P}}} \mathscr{P}$$

defines. the unit map $\eta_{\hom(\mathscr{C},\mathscr{P})}$.

Theorem 2.2.9. *Let $\mathscr{C}$ be a cooperad, and let $\mathscr{P}$ be an operad. With the maps described above, the $\mathbb{S}$-module $\hom(\mathscr{C},\mathscr{P})$ becomes an operad, which is called the* convolution operad *of $\mathscr{C}$ and $\mathscr{P}$.*

Proof. By inspection and left to the reader. □

Notice that if $\mathscr{P}$ is reduced, then so is $\hom(\mathscr{C},\mathscr{P})$.

Now suppose that C is a conilpotent $\mathscr{C}$-coalgebra, and that A is a $\mathscr{P}$-algebra. Then we can endow the chain complex $\hom(C,A)$ with a $\hom(\mathscr{C},\mathscr{P})$-algebra structure as follows. Let $\phi \in \hom(\mathscr{C},\mathscr{P})(n)$, and let $f_1,\dots,f_n \in \hom(C,A)$, then $\gamma_{\hom(C,A)}(\phi \otimes_{\mathbb{S}_n} (f_1\otimes\cdots\otimes f_n))$ is given by the composite

$$C \xrightarrow{\Delta_C^n} \big(\mathscr{C}(n)\otimes C^{\otimes n}\big)^{\mathbb{S}_n} \hookrightarrow \mathscr{C}(n)\otimes C^{\otimes n} \xrightarrow{\sum_{\sigma\in\mathbb{S}_n}(-1)^\epsilon \phi\otimes f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)}} \mathscr{P}(n)\otimes A^{\otimes n}$$
$$\twoheadrightarrow \mathscr{P}(n)\otimes_{\mathbb{S}_n} A^{\otimes n} \xrightarrow{\gamma_A} A\,,$$

where ϵ is the Koszul sign appearing from the permutation of the f_i.

Theorem 2.2.10. *Let $\mathscr{C}$ be a cooperad, let $\mathscr{P}$ be an operad, let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. With the structure described above, $\hom(C,A)$ is a $\hom(\mathscr{C},\mathscr{P})$-algebra, which we call the* convolution algebra *of C and A.*

Proof. By inspection and left to the reader. □

2.2.5 Twisting morphisms

A central notion for the theory of operads is that of twisting morphisms. More details on the subject can be found in [LV12, Sect. 6.4].

To any reduced operad $\mathscr{P}$ it is possible to associate a (pre-)Lie algebra structure on the chain complex $\prod_{n\geq 0}\mathscr{P}(n)$, see [LV12, Sect. 5.4.3]. If the operad is a convolution operad $\hom(\mathscr{C},\mathscr{P})$ with $\mathscr{C}$ and $\mathscr{P}$ reduced[3], then the pre-Lie product is simply given by the composite

$$f\star g = \big(\mathscr{C} \xrightarrow{\Delta_{(1)}} \mathscr{C}\circ_{(1)}\mathscr{C} \xrightarrow{f\circ_{(1)}g} \mathscr{P}\circ_{(1)}\mathscr{P} \xrightarrow{\gamma_{(1)}} \mathscr{P}\big)$$

for $f,g\in\prod_{n\geq 0}\hom(\mathscr{C},\mathscr{P})(n)$, and it preserves the subspace of $\mathbb{S}$-equivariant maps, i.e. (possibly graded) morphisms of $\mathbb{S}$-modules.

Definition 2.2.11. *Let $\mathscr{C}$ be a reduced cooperad, and let $\mathscr{P}$ be a reduced operad. An* (operadic) twisting morphism *$\alpha:\mathscr{C}\to\mathscr{P}$ from $\mathscr{C}$ to $\mathscr{P}$ is a morphism of $\mathbb{S}$-modules of degree -1 satisfying the Maurer–Cartan equation*

$$\partial(\alpha)+\alpha\star\alpha = 0\,.$$

We denote the set of twisting morphisms from $\mathscr{C}$ to $\mathscr{P}$ by $\mathrm{Tw}(\mathscr{C},\mathscr{P})$.

Given a morphism of $\mathbb{S}$-modules $\alpha:\mathscr{C}\to\mathscr{P}$ of degree -1, we can consider the unique derivation d^r_α on $\mathscr{C}\circ\mathscr{P}$ extending

$$\mathscr{C} \xrightarrow{\Delta_{(1)}} \mathscr{C}\circ_{(1)}\mathscr{C} \xrightarrow{1_{\mathscr{C}}\circ_{(1)}\alpha} \mathscr{C}\circ_{(1)}\mathscr{P} \longrightarrow \mathscr{C}\circ\mathscr{P}\,,$$

cf. [LV12, Prop. 6.3.9]. Explicitly, it is given by $d^r_\alpha = (1_{\mathscr{C}}\circ\gamma_{\mathscr{P}})(((1_{\mathscr{C}}\circ\alpha)\Delta_{(1)})\circ 1_{\mathscr{P}})$. Define $d_\alpha := d_{\mathscr{C}\circ\mathscr{P}} + d^r_\alpha$ on $\mathscr{C}\circ\mathscr{P}$. Dually, on $\mathscr{P}\circ\mathscr{C}$ we consider the unique derivation d^ℓ_α extending

$$\mathscr{C} \xrightarrow{\Delta_{\mathscr{C}}} \mathscr{C}\circ\mathscr{C} \xrightarrow{\alpha\circ 1_{\mathscr{C}}} \mathscr{P}\circ\mathscr{C}$$

and define $d_\alpha := d_{\mathscr{P}\circ\mathscr{C}} + d^\ell_\alpha$.

[3] So that $\hom(\mathscr{C},\mathscr{P})$ is also reduced.

Lemma 2.2.12. *On $\mathscr{C}\circ\mathscr{P}$ we have $d^2_\alpha = d^r_{\partial(\alpha)+\alpha\star\alpha'}$, and on $\mathscr{P}\circ\mathscr{C}$ we have $d^2_\alpha = d^\ell_{\partial(\alpha)+\alpha\star\alpha}$. Therefore, in both cases we have that $\alpha\in\mathrm{Tw}(\mathscr{C},\mathscr{P})$ if, and only if $d^2_\alpha = 0$.*

Thus, given a twisting morphism $\alpha\in\mathrm{Tw}(\mathscr{C},\mathscr{P})$, one can define two chain complexes

$$\mathscr{C}\circ_\alpha\mathscr{P} \coloneqq \big(\mathscr{C}\circ\mathscr{P}, d_\alpha\big) \qquad \text{and} \qquad \mathscr{P}\circ_\alpha\mathscr{C} \coloneqq \big(\mathscr{P}\circ\mathscr{C}, d_\alpha\big)\,,$$

called the *left* and *right twisted composite product* of $\mathscr{C}$ and $\mathscr{P}$ respectively. These constructions are functorial in the sense that if we have a commutative square

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{f} & \mathscr{C}' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathscr{P} & \xrightarrow{g} & \mathscr{P}' \end{array}$$

where α,α' are twisting morphisms and f,g are a morphism of cooperads and a morphism of operad respectively, then

$$f\circ g : \mathscr{C}\circ_\alpha\mathscr{P} \longrightarrow \mathscr{C}'\circ_{\alpha'}\mathscr{P}'$$

is a chain map, and similarly for $g\circ f$.

Theorem 2.2.13 (Comparison lemma for twisted composite products)**.** *Suppose that we are in the situation above.*

1. *If two among f, g and $f\circ g : \mathscr{C}\circ_\alpha\mathscr{P}\longrightarrow\mathscr{C}'\circ_\alpha\mathscr{P}'$ are quasi-isomorphisms, then so is the third.*
2. *Dually, if two among f, g and $g\circ f : \mathscr{P}\circ_\alpha\mathscr{C}\longrightarrow\mathscr{P}'\circ_\alpha\mathscr{C}'$ are quasi-isomorphisms, then so is the third.*

The proof of this theorem uses a spectral sequence argument. It is found in [LV12, Sect. 6.7] and it is based on [Fre04, Sect. 2].

Remark 2.2.14. *This result also holds for (co)operads that are not reduced, but in that case it is crucial to assume that everything is connected weight graded.*

2.2.6 Operadic bar and cobar construction

It is natural ask whether the functors $\mathrm{Tw}(\mathscr{C},-)$ and $\mathrm{Tw}(-,\mathscr{P})$ are representable or not, and whether one can find canonical resolutions of operads and cooperads in general or not. Both questions are answered in the positive by the two constructions we present here, following [LV12, Sect. 6.5]. The argument will be completed in Sections 2.2.7 and 2.2.8.

We start by defining the *operadic bar construction*, which is a functor

$$\mathrm{B} : \mathsf{aug.\ Op}\longrightarrow\mathsf{coaug.\ coOp}\,.$$

Let $\mathscr{P}$ be an augmented operad. The functor B associates to $\mathscr{P}$ the quasi-free cooperad

$$\mathrm{B}\mathscr{P} \coloneqq \big(\mathcal{T}^c(s\overline{\mathscr{P}}), d\coloneqq d_1+d_2\big)\,,$$

where d_1 is the natural differential induced on $\mathcal{T}^c(s\overline{\mathscr{P}})$ by the differential $d_{\mathscr{P}}$ of $\mathscr{P}$, and d_2 is defined as the unique coderivation extending the composite

$$\mathcal{T}^c(s\overline{\mathscr{P}}) \xrightarrow{\text{proj.}} \mathcal{T}^c(s\overline{\mathscr{P}})^{(2)} \cong (\mathbb{K}s\otimes\overline{\mathscr{P}})\circ_{(1)}(\mathbb{K}s\otimes\overline{\mathscr{P}})$$

$$\cong (\mathbb{K}s \otimes \mathbb{K}s) \otimes (\overline{\mathscr{P}} \circ_{(1)} \overline{\mathscr{P}}) \xrightarrow{\mathscr{S}_2 \otimes \gamma_{(1)}} s\overline{\mathscr{P}} .$$

Here we see $\mathscr{S}_2$ as the operation sending s^2 to s. It is straightforward to prove that d_2 squares to zero and anticommutes with d_1, so that also $d_1 + d_2$ squares to zero.

Proposition 2.2.15. *Suppose the characteristic of the base field $\mathbb{K}$ is 0. The operadic bar construction B preserves quasi-isomorphisms.*

Dually, the *operadic cobar construction* is the functor

$$\Omega : \mathsf{coaug.\ coOp} \longrightarrow \mathsf{aug.\ Op}$$

defined as

$$\Omega\mathscr{C} \coloneqq \left(\mathcal{T}(s^{-1}\overline{\mathscr{C}}), d \coloneqq d_1 + d_2\right) ,$$

where again d_1 is induced by $d_{\mathscr{C}}$, and d_2 is the unique derivation extending the composite

$$\begin{aligned} s^{-1}\overline{\mathscr{C}} &\xrightarrow{\Delta_s \otimes \Delta_{(1)}} (\mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}) \otimes (\overline{\mathscr{C}} \circ_{(1)} \overline{\mathscr{C}}) \\ &\cong (\mathbb{K}s^{-1} \otimes \overline{\mathscr{C}}) \circ_{(1)} (\mathbb{K}s^{-1} \otimes \overline{\mathscr{C}}) \cong \mathcal{T}(s^{-1}\overline{\mathscr{C}})^{(2)} \xrightarrow{\text{incl.}} \mathcal{T}(s^{-1}\overline{\mathscr{C}}) . \end{aligned}$$

Here, $\Delta_{s^{-1}}$ is the dual of $\mathscr{S}_2$, and is given by sending s^{-1} to $-s^{-1} \otimes s^{-1}$. Once again, one easily proves that d_2 squares to zero and anticommutes with d_1, so that $d_1 + d_2$ also squares to zero. We also have a compatibility with quasi-isomorphisms, but with rather more stringent assumptions.

Proposition 2.2.16. *The operadic cobar construction Ω preserves quasi-isomorphisms between cooperads that are non-negatively graded, I in degree 0, and 0 in degree 1.*

The operadic bar and cobar constructions form a pair of adjoint functors.

Theorem 2.2.17. *Let $\mathscr{P}$ be an augmented operad, and let $\mathscr{C}$ be a coaugmented cooperad. There are natural isomorphisms*

$$\hom_{\mathsf{Op}}(\Omega\mathscr{C}, \mathscr{P}) \cong \mathrm{Tw}(\mathscr{C}, \mathscr{P}) \cong \hom_{\mathsf{coOp}}(\mathscr{C}, \mathrm{B}\mathscr{P}) .$$

In particular, the functors B and Ω form an adjoint pair.

The unit and the counit of the adjunction give rise to canonical twisting morphisms. Namely, let $\mathscr{C}$ be a cooperad, then the unit of the adjunction is a map

$$\mathscr{C} \longrightarrow \mathrm{B}\Omega\mathscr{C}$$

of cooperads and as such it corresponds to a canonical twisting morphism

$$\iota : \mathscr{C} \longrightarrow \Omega\mathscr{C} .$$

It is given by the composite

$$\iota = \left(\mathscr{C} \xrightarrow{s^{-1}} s^{-1}\mathscr{C} \xrightarrow{\text{incl.}} \Omega\mathscr{C}\right) .$$

Dually, for $\mathscr{P}$ an operad, the counit of the adjunction is a map

$$\Omega\mathrm{B}\mathscr{P} \longrightarrow \mathscr{P}$$

of operads, and gives us a canonical twisting morphism

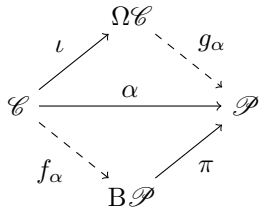
$$\pi : \mathrm{B}\mathscr{P} \longrightarrow \mathscr{P}$$

given by the composite

$$\pi = \left(\mathrm{B}\mathscr{P} \xrightarrow{\text{proj.}} s\mathscr{P} \xrightarrow{s^{-1}} \mathscr{P} \right).$$

These two twisting morphisms are universal, in the sense that every twisting morphism splits through them.

Theorem 2.2.18. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism. There exist a unique morphism of cooperads $f_\alpha : \mathscr{C} \to \mathrm{B}\mathscr{P}$ and a unique morphism of operads $g_\alpha : \Omega\mathscr{C} \to \mathscr{P}$ such that the diagram*



commutes.

Notation 2.2.19. *From now on, we reserve the Greek letters ι and π for the canonical twisting morphisms we just described.*

2.2.7 Koszul twisting morphisms

Certain twisting morphisms behave especially well with respect to the homotopy theory of (co)operads and (co)algebras over them. The archetype for such twisting morphisms are the universal twisting morphisms ι and π.

Lemma 2.2.20. *Let $\mathscr{C}$ be a coaugmented cooperad and let $\mathscr{P}$ be an augmented operad. The chain complexes*

$$\mathscr{C} \circ_\iota \Omega\mathscr{C}\,, \quad \Omega\mathscr{C} \circ \iota\mathscr{C}\,, \quad \mathscr{P} \circ_\pi \mathrm{B}\mathscr{P} \quad \text{and} \quad \mathrm{B}\mathscr{P} \circ_\pi \mathscr{P}$$

are acyclic.

Proof. This is [LV12, Lemma 6.5.9]. □

Using this fact, one can prove the following important fact [LV12, Thm. 6.6.1].

Theorem 2.2.21. *Let $\mathscr{C}$ be a connected weight graded cooperad, let $\mathscr{P}$ be a connected weight graded operad, and let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism. The following are equivalent.*

1. *The right twisted composite product $\mathscr{C} \circ_\alpha \mathscr{P}$ is acyclic.*
2. *The left twisted composite product $\mathscr{P} \circ_\alpha \mathscr{C}$ is acyclic.*
3. *The morphism of cooperads $f_\alpha : \mathscr{C} \to \mathrm{B}\mathscr{P}$ of Theorem 2.2.18 is a quasi-isomorphism.*
4. *The morphism of operads $g_\alpha : \mathrm{B}\mathscr{P} \to \mathscr{C}$ of Theorem 2.2.18 is a quasi-isomorphism.*

Definition 2.2.22. *Let $\mathscr{C}$ be a connected weight graded cooperad and let $\mathscr{P}$ be a connected weight graded operad. A twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$ is* Koszul *if any of the equivalent conditions of Theorem 2.2.21 is satisfied.*

Example 2.2.23. *By Lemma 2.2.20, the universal twisting morphisms π and ι are Koszul.*

2.2.8 Bar-cobar resolution

Finally, we can prove that the bar-cobar adjunction gives a canonical resolution of operads and cooperads, following [LV12, Thm. 6.6.3].

Theorem 2.2.24. *Let $\mathscr{C}$ be a cooperad. The unit*

$$\mathscr{C} \longrightarrow \mathrm{B}\Omega\mathscr{C}$$

of the bar-cobar adjunction is a quasi-isomorphism of cooperads. Dually, let $\mathscr{P}$ be an operad. The counit

$$\Omega\mathrm{B}\mathscr{P} \longrightarrow \mathscr{P}$$

of the bar-cobar adjunction is a quasi-isomorphism of operads.

Proof. This is a direct corollary of Theorem 2.2.21 and Lemma 2.2.20 in the connected weight graded case. The more general case can be found in [Fre04]. □

Remark 2.2.25. *In particular, $\Omega\mathrm{B}\mathscr{P}$ provides a functorial cofibrant resolution of $\mathscr{P}$, see Corollary 3.3.6. One should notice that if $\mathscr{C}$ is a cooperad, then in general $\Omega\mathscr{C}$ is not cofibrant. However, this is the case if one has a connected weight graded cooperad, and in particular $\Omega\mathscr{C}$ is always cofibrant if $\mathscr{C}$ is a reduced cooperad.*

2.3 Minimal models and Koszul duality

The bar-cobar resolution of operads is very useful, but almost always really big. Fortunately, in some cases it is possible to find smaller resolutions for operads, such as minimal models. One way to do that is Koszul duality, which recall briefly in this section. It is a theory originally developed by V. Ginzburg and M. Kapranov [GK94] for operads. Koszul duality for associative and Lie algebras existed prior to it, and other versions for related concepts have been developed since, for example in [Val07] for props.

2.3.1 Minimal models

Let $\mathscr{P}$ be an operad. A *model* for $\mathscr{P}$ is an operad $\mathscr{Q}$ together with a surjective quasi-isomorphism of operads $\mathscr{Q} \to \mathscr{P}$.

An operad $\mathscr{Q}$ is *minimal* if it is *quasi-free*, i.e. of the form $\mathscr{Q} = (\mathcal{T}(M), d)$ for some graded $\mathbb{S}$-module M (that is, for each $n \geq 0$ the object $M(n)$ is just a graded vector space, not a chain complex), and such that the differential satisfies the following two conditions.

1. The differential d of $\mathscr{Q}$ is *decomposable*, that is $d(E) \subseteq \mathcal{T}(E)^{(\geq 2)}$.
2. The graded $\mathbb{S}$-module E admits a decomposition

$$E = \bigoplus_{k \geq 1} E^{(k)}$$

such that

$$d(E^{(k+1)}) \subseteq \mathcal{T}\left(\bigoplus_{i=1}^{k} E^{(i)}\right)$$

for any $k \geq 0$.

Definition 2.3.1. *A* minimal model *for an operad $\mathscr{P}$ is a model $\mathscr{Q} \to \mathscr{P}$ such that $\mathscr{Q}$ is a minimal operad.*

Theorem 2.3.2 ([DCV13, Prop. 3.7])**.** *If an operad $\mathscr{P}$ admits a minimal model, then the model is unique up to (non-unique) isomorphism.*

2.3.2 Koszul duality

For the rest of this section, we work with quadratic (co)operads. We refer the reader to [LV12, Sect. 7.2 and 7.4] for more details.

Definition 2.3.3. *Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad. The* Koszul dual cooperad *of $\mathscr{P}$ is the quadratic cooperad*

$$\mathscr{P}^{\text{¡}} := \mathscr{C}(sE, s^2R) .$$

The Koszul dual operad *op $\mathscr{P}$ is the operad*

$$\mathscr{P}^{!} := (\mathscr{S}^c \otimes \mathscr{P}^{\text{¡}})^\vee .$$

The Koszul dual cooperad of an operad is of greater theoretical importance than the Koszul dual operad. However, the latter has the advantage of being often easy to describe explicitly, as the following result explains.

Proposition 2.3.4. *Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad generated by an $\mathbb{S}$-module E which is reduced and finite dimensional in every arity. Then the Koszul dual operad of $\mathscr{P}$ is quadratic with presentation*

$$\mathscr{P}^{!} = \mathscr{P}(s^{-1}\mathscr{S}^{-1} \otimes E^\vee, R^\perp) ,$$

where $R^\perp$ is the subspace of $\mathcal{T}(s^{-1}\mathscr{S}^{-1} \otimes E^\vee)^{(2)}$ obtained by taking the subspace orthogonal to s^2R in $\mathcal{T}(sE)^{(2)}$ and desuspending its elements in the obvious way. Moreover, we have

$$(\mathscr{P}^{!})^{!} = \mathscr{P} .$$

Proof. This is [LV12, Prop. 7.2.1 and 7.2.2]. □

Let (E, R) be a quadratic data. Then both $\mathscr{P}(E, R)^{(1)}$ and $\mathscr{C}(E, R)^{(1)}$ are given by E, and thus we can define the map

$$\kappa : \mathscr{C}(sE, s^2R) \xrightarrow{\text{proj}} sE \xrightarrow{s^{-1}} E \xrightarrow{\text{incl.}} \mathscr{P}(E, R) .$$

Lemma 2.3.5. *The map κ described above is a twisting morphism.*

Proof. This is [LV12, Lemma 7.4.1]. □

Notation 2.3.6. *From now on, we reserve the Greek letter κ as notation for the canonical twisting morphism defined above.*

Definition 2.3.7. *A quadratic operad $\mathscr{P}$ is* Koszul *if the canonical twisting morphism*

$$\kappa : \mathscr{P}^{\text{¡}} \longrightarrow \mathscr{P}$$

is Koszul.

Consider the canonical inclusion

$$i \coloneqq f_\kappa : \mathscr{P}^{\text{¡}} \longrightarrow \mathrm{B}\mathscr{P}$$

and the canonical projection

$$p \coloneqq g_\kappa : \Omega\mathscr{P}^{\text{¡}} \longrightarrow \mathscr{P}$$

given by applying Theorem 2.2.18 to κ. We have the following version of Theorem 2.2.21 for κ. Notice that we don't need a connected weight grading for it to hold. In particular, it also works without restrictions even if the quadratic data is not reduced.

Theorem 2.3.8. *Let $\mathscr{P} = \mathscr{P}(E, R)$ a quadratic operad. The following are equivalent.*

1. *The* right Koszul complex $\mathscr{P}^{¡} \circ_\kappa \mathscr{P}$ *is acyclic.*
2. *The* left Koszul complex $\mathscr{P} \circ_\kappa \mathscr{P}^{¡}$ *is acyclic.*
3. *The canonical inclusion $i : \mathscr{P}^{¡} \to \mathrm{B}\mathscr{P}$ is a quasi-isomorphism.*
4. *The canonical projection $p : \Omega\mathscr{P}^{¡} \to \mathscr{P}$ is a quasi-isomorphism.*

Proof. This is [LV12, Thm. 7.4.2]. □

Corollary 2.3.9. *Suppose $\mathscr{P}$ is a Koszul operad. Then the operad $\Omega\mathscr{P}^{¡}$ is the minimal model of $\mathscr{P}$. It will often be denoted by $\mathscr{P}_\infty$.*

Therefore, we have a minimal model for any Koszul operad. Checking that an operad is Koszul looks like a difficult problem, but one has various methods of doing it which do not involve checking if the right or left Koszul complex are acyclic. This is the topic of [LV12, Ch. 8]. It is outside the scope of the present work, and we will not mention this problematic again.

2.3.3 Homotopy algebras and homotopy morphisms

Given an arbitrary operad $\mathscr{P}$, often the category of $\mathscr{P}$-algebras does not have very good homotopical properties. However, whenever $\mathscr{P}$ is Koszul there is a notion of $\mathscr{P}$-algebra up to homotopy which is much better behaved. The material presented here is contained in [LV12, Ch. 10].

For this section, we fix a Koszul operad $\mathscr{P}$.

Definition 2.3.10. *A* homotopy $\mathscr{P}$-algebra *is an algebra over the operad $\mathscr{P}_\infty = \Omega\mathscr{P}^{¡}$. We also use the name $\mathscr{P}_\infty$-*algebras *for homotopy $\mathscr{P}$-algebras.*

Notice that every $\mathscr{P}$ algebra is also a $\mathscr{P}_\infty$-algebra via restriction of structure along the projection $\mathscr{P}_\infty \to \mathscr{P}$. The following point of view on $\mathscr{P}_\infty$-algebras is often useful.

Proposition 2.3.11. *A structure of $\mathscr{P}_\infty$-algebra on a chain complex A is equivalent to a twisting morphism $\varphi_A \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$.*

Proof. This is an immediate consequence of Proposition 2.1.22 and Theorem 2.2.17. □

Proposition 2.3.12. *A $\mathscr{P}_\infty$-algebra A is a $\mathscr{P}$-algebra if, and only if the twisting morphism φ_A is concentrated in weight* 1.

Proof. This is [LV12, Prop. 10.1.4]. □

Proposition 2.3.13. *Let A be a graded vector space. A structure of $\mathscr{P}_\infty$-algebra on A is equivalent to a square-zero coderivation on the (non-differential) graded cofree $\mathscr{P}^{\text{¡}}$-coalgebra $\mathscr{P}^{\text{¡}}(A)$.*

Proof. This is [LV12, Prop. 10.1.11] □

One can of course consider the category of $\mathscr{P}_\infty$-algebras with the morphisms of $\mathscr{P}_\infty$-algebras between them. However, there is a notion of morphism of $\mathscr{P}_\infty$-algebras "relaxed up to homotopy" which has better homotopical properties.

Definition 2.3.14. *Let A and A' be two $\mathscr{P}_\infty$-algebras. An* ∞-morphism *Ψ of $\mathscr{P}_\infty$-algebras from A to A', denoted by $\Psi : A \rightsquigarrow B$, is a morphism of $\mathscr{P}^{\text{¡}}$-coalgebras*

$$\Psi : \mathscr{P}^{\text{¡}}(A) \longrightarrow \mathscr{P}^{\text{¡}}(A')$$

between the $\mathscr{P}^{\text{¡}}$-coalgebras associated to A and A' through Proposition 2.3.13. The composition of ∞-morphisms is induced by the usual composition of morphisms of $\mathscr{P}^{\text{¡}}$-coalgebras. The category of $\mathscr{P}_\infty$-algebras with ∞-morphisms as morphisms is denoted by ∞-$\mathscr{P}_\infty$-alg.

Since an ∞-morphism Ψ of $\mathscr{P}_\infty$-algebras is a morphism between cofree coalgebras, it is completely determined by its projection

$$\mathscr{P}^{\text{¡}}(A) \longrightarrow \mathscr{P}^{\text{¡}}(A') \xrightarrow{\text{proj.}} A',$$

which we will again denote by Ψ, abusing notation. Therefore, such an ∞-morphism is equivalent to a collection of maps

$$\psi_n : \mathscr{P}^{\text{¡}}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \longrightarrow A'$$

satisfying certain relations.

From now on, we suppose that the quadratic data (E, R) describing $\mathscr{P}$ is such that $E(0) = E(1) = 0$. Then the cooperad $\mathscr{P}^{\text{¡}}$ is reduced, and ψ_1 is a chain map

$$\psi_1 : A \longrightarrow A'.$$

There is an analogous point of view for general Koszul operads treated in [LV12, p.373], where the map ψ_1 above corresponds to $\psi_{(0)}$, but we will not use it.

Notice that every morphism of $\mathscr{P}_\infty$-algebras $\psi : A \to A'$ is an ∞-morphism by $\Psi := \mathscr{P}^{\text{¡}}(\psi)$. For the converse direction, we have the following.

Proposition 2.3.15. *An ∞-morphism $\Psi : A \rightsquigarrow A'$ of $\mathscr{P}_\infty$-algebras is a strict morphism of $\mathscr{P}_\infty$-algebras if, and only if $\psi_n = 0$ for all $n \geq 2$.*

Proof. This is [LV12, Prop. 10.2.5]. □

Finally, we introduce the important notion of ∞-quasi-isomorphism.

Definition 2.3.16. *An ∞-morphism of $\mathscr{P}_\infty$-algebras $\Psi : A \rightsquigarrow A'$ is an* ∞-quasi-isomorphism *if $\psi_1 : A \to A'$ is a quasi-isomorphism.*

The notions of ∞-morphisms in general and ∞-quasi-isomorphisms in particular will be studied some more later, in Section 2.5, as well as in Chapter 8.

2.3.4 The homotopy transfer theorem

The homotopy transfer theorem is an important result on the homotopical behavior of algebras over (cofibrant) operads. It tells us that if we have a homotopy retraction between two chain complexes — which in particular implies being homotopically the same — and an algebra structure over one of them, then we can produce in a very natural way an algebra structure on the other one which has the same homotopical information.

The first notions we need is those of homotopy retractions and contractions of chain complexes.

Definition 2.3.17. *Let V, W be two chain complexes. A* homotopy retraction *of V into W is three maps*

$$h \circlearrowright V \underset{i}{\overset{p}{\rightleftarrows}} W$$

such that $pi = 1_W$ and $1 - ip = d_V h + h d_V$. It is a contraction *if the* side conditions

$$h^2 = 0\,, \qquad ph = 0\,, \qquad \text{and} \qquad hi = 0$$

are satisfied.

Remark 2.3.18. *It is possible to construct a sensible category whose objects are the homotopy retractions, cf. Section 6.5.1.*

Let $\mathscr{P}$ be a Koszul operad, let A be a $\mathscr{P}_\infty$-algebra, and suppose that we have a contraction of chain complexes

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

The *transfer problem* asks under what conditions it is possible to produce a $\mathscr{P}_\infty$-algebra structure on B such that i extends to an ∞-morphism of $\mathscr{P}_\infty$-algebras. It turns out that this is always true. This result is known as the *homotopy transfer theorem*, and has been developed throughout the years by many authors. It has been known for a long time for $\mathscr{A}_\infty$-algebras — see [Kad80, Thm. 1], and e.g. [KS00, Sect. 6.4] for an explicit formula in terms of trees — and $\mathscr{L}_\infty$-algebras. It was proven for $\mathscr{P}_\infty$-algebras — and more generally for algebras over $\Omega\mathscr{C}$ for $\mathscr{C}$ a reduced cooperad[4] — in [Ber14a, Thm. 1.5] with explicit formulæ, but the existence part was already known by [BM03] and [Fre09b]. Another approach using pre-Lie deformation theory was recently given in [DSV16, Sect. 8]. For a more extensive review of the literature on this result, we refer the reader to the introduction of [LV12, Sect. 10.3] and to the survey [Sta10], which also provides a nice historical perspective, as well as links with theoretical physics. The formulæ we will present here are found in [LV12, Sect. 10.3].

The main tool we need to give explicit formulæ for the transfered structure and the induced ∞-morphisms is the *van der Laan morphism* [VdL03] associated to the contraction. It is a morphism of cooperads

$$\mathrm{VdL}_B : \mathrm{B}(\mathrm{End}_A) \longrightarrow \mathrm{B}(\mathrm{End}_B)$$

defined as follows. An element of $\mathrm{B}(\mathrm{End}_A) = \mathscr{T}^c(s\mathrm{End}_A)$ is given by a rooted tree $\tau \in \mathrm{RT}$ with vertices labeled by elements of $s\mathrm{End}_A$. Using the notations of Appendix A, we write $\tau(f)$ for such a tree, where $f : V \to s\mathrm{End}_A$ is a function from the set of vertices of τ to $s\mathrm{End}_A$. Given

[4] This slightly more general case will be explained later, in Section 9.3.1.

such a tree $\tau(f)$ with $\tau \neq \emptyset$, we denote by $\tau^h(f)$ the element of End_A defined as follows. If $\tau = c_n$ is the n-corolla, then

$$c_n(f) \coloneqq f(*) \,.$$

Else, we write $\tau = c_k \circ (\tau_1, \dots, \tau_k)$, where $\tau_i \in \mathrm{RT}_{n_i}$ are possibly empty, and define

$$\tau^h(f) \coloneqq \gamma_{\mathrm{End}_A}\big(f(*) \circ (hs^{-1}\tau_1^h(f|_{V_1}), \dots, hs^{-1}\tau_k^h(f|_{V_k})\big) \,,$$

where $*$ is the unique vertex of c_k, V_i is the set of vertices of τ_i, and where we formally set $\emptyset^h = h^{-1}$. Then we define

$$\mathrm{VdL}_B(\tau(f)) \coloneqq s(p\tau^h(f)i^{\otimes n})$$

to obtain a map

$$\mathrm{VdL}_B : \mathrm{B}(\mathrm{End}_A) \longrightarrow s\mathrm{End}_B \,.$$

The drawing in [LV12, p. 378] might prove illuminating to the reader confused by the exposition above. By universal property of the bar construction, this map extends to give a morphism

$$\mathrm{B}(\mathrm{End}_A) \longrightarrow \mathrm{B}(\mathrm{End}_B)$$

of cooperads in graded vector spaces, which we denote again by VdL_B, abusing notation.

Theorem 2.3.19 ([VdL03, Thm. 5.2])**.** *The map*

$$\mathrm{VdL}_B : \mathrm{B}(\mathrm{End}_A) \longrightarrow \mathrm{B}(\mathrm{End}_B)$$

described above is a morphism of cooperads.

A proof of this result can also be found in [LV12, Prop. 10.3.2]. The transfered structure on B can now be expressed as follows. The $\mathscr{P}_\infty$-algebra structure on A is given by a twisting morphism $\varphi_A \in \mathrm{Tw}(\mathscr{P}^{\mathsf{i}}, \mathrm{End}_A)$. This is equivalent to a morphism of cooperads

$$f_A : \mathscr{P}^{\mathsf{i}} \longrightarrow \mathrm{B}(\mathrm{End}_A)$$

by Theorem 2.2.17. We compose this with VdL_B to get a morphism of cooperads

$$\mathrm{VdL}_B f_A : \mathscr{P}^{\mathsf{i}} \longrightarrow \mathrm{B}(\mathrm{End}_B) \,,$$

which again is equivalent to a twisting morphism $\varphi_B \in \mathrm{Tw}(\mathscr{P}^{\mathsf{i}}, \mathrm{End}_B)$ defining a $\mathscr{P}_\infty$-algebra structure on B. Of course, this is nothing else than $\varphi_B = \mathrm{VdL}_B f_A$. Explicitly, the structure is given by

$$\varphi_B = \left(\mathscr{P}^{\mathsf{i}} \xrightarrow{\Delta^{\mathrm{monadic}}_{\mathscr{P}^{\mathsf{i}}}} \mathcal{T}^c\big(\overline{\mathscr{P}^{\mathsf{i}}}\big) \xrightarrow{\mathcal{T}^c(s\varphi_A)} \mathrm{B}(\mathrm{End}_A) \xrightarrow{\mathrm{VdL}_B} \mathrm{End}_B\right) \,.$$

This explicit formulation for the transferred structure first appeared in [GCTV12].

Next, one defines a map

$$i_\infty : \mathscr{P}^{\mathsf{i}}(B) \longrightarrow A$$

by

$$i_\infty \coloneqq \left(\mathscr{P}^{\mathsf{i}} \circ B \xrightarrow{\Delta^{\mathrm{monadic}}_{\mathscr{P}^{\mathsf{i}}} \circ 1_B} \mathcal{T}^c\big(\overline{\mathscr{P}^{\mathsf{i}}}\big) \circ B \xrightarrow{\mathcal{T}^c(s\varphi_A)\circ 1_B} \mathrm{B}(\mathrm{End}_A) \circ B \xrightarrow{\mathrm{VdL}^i_B \circ 1_B} \mathrm{End}^B_A \circ B \longrightarrow A\right) \,,$$

where End^B_A is the $\mathbb{S}$-module given by $\mathrm{End}^B_A(n) \coloneqq \hom_{\mathsf{Ch}}(B^{\otimes n}, A)$, the map

$$\mathrm{VdL}^i_B : \mathrm{B}(\mathrm{End}_A) \longrightarrow \mathrm{End}^B_A$$

is given on an element of $\mathrm{B}(\mathrm{End}_A)(n)$ by

$$\mathrm{VdL}_B^i(\tau(f)) \coloneqq h\tau^h(f)i^{\otimes n},$$

and where the last arrow is the obvious evaluation map.

Theorem 2.3.20 (Homotopy transfer theorem)**.** *The map i_∞ defines an ∞-quasi-isomorphism from B with the $\mathscr{P}_\infty$-algebra structure defined above to A.*

Proof. E.g. [LV12, Thm. 10.3.6]. □

The following fact is also important and will prove useful later on.

Proposition 2.3.21. *The map $p : A \to B$ can also be extended to an ∞-quasi isomorphism $p_\infty : A \rightsquigarrow B$. Moreover, it can be take such that the composite*

$$p_\infty i_\infty = 1_B$$

is the identity on B.

An explicit formula for p_∞, together with a proof of this result, can be found in [LV12, Prop. 10.3.9] and [DSV16, Thm. 5].

We will now give some examples of algebras up to homotopy and their homotopy transfer theorems.

2.3.5 Homotopy associative algebras

Our first example is that of homotopy associative algebras, i.e. $\mathscr{A}_\infty$-algebras. This kind of algebra has a long history in the literature. For example, they appear naturally when one considers the Massey products of topological spaces, cf. Section 2.3.9. For this subsection, we work in the non-symmetric setting, so that by associative algebra we mean an algebra over the operad As, cf. Section 2.1.6.

A straightforward computation using Proposition 2.3.4 shows that $\mathrm{As}^! = \mathrm{As}$. Then, we have

$$\mathrm{As}^{\mathrm{i}} \cong (\mathscr{S}^{-1})^c \otimes (\mathscr{P}^!)^\vee = (\mathscr{S}^{-1})^c \otimes \mathrm{As}^\vee.$$

This cooperad is spanned in arity $n \geq 2$ by $\mathscr{S}_n^{-1}\mu_n^\vee$. The operad As is Koszul, see e.g. [LV12, Thm. 9.15], and thus, the operad $\mathscr{A}_\infty \coloneqq \Omega\mathrm{As}^{\mathrm{i}}$ is a minimal model for As. It is freely generated by the operations

$$m_n \coloneqq s^{-1}\mathscr{S}_n^{-1}\mu_n^\vee\,.$$

The only thing left to determine in order to understand $\mathscr{A}_\infty$-algebras is the differential of the operad $\mathscr{A}_\infty$. As defined in Section 2.2.6, this is given on m_n by

$$d_{\mathscr{A}_\infty}(m_n) = d_1(m_n) + d_2(m_n)\,,$$

where $d_1 = 0$ since the differential on As is trivial, and where $d_2(m_n)$ is obtained by

$$\begin{aligned} m_n \longmapsto & -(s^{-1}\otimes s^{-1}) \otimes \sum_{\substack{n_1+n_2=n+1\\ 1\leq j\leq n_1}} (-1)^{(j-1)(1-n_2)+(n_2-1)(1-n_1)} \mathscr{S}_{n_1}^{-1}\mu_{n_1}^\vee \otimes_j \mathscr{S}_{n_2}^{-1}\mu_{n_2}^\vee \\ = & \sum_{\substack{n_1+n_2=n+1\\ 1\leq j\leq n_1}} (-1)^{(1-n_2)(j+n_1)+(1-n_1)} s^{-1}\mathscr{S}_{n_1}^{-1}\mu_{n_1}^\vee \otimes_j s^{-1}\mathscr{S}_{n_2}^{-1}\mu_{n_2}^\vee \end{aligned}$$

$$= \sum_{\substack{n_1+n_2=n+1 \\ 1\leq j\leq n_1}} (-1)^{n_2(n_1-j)+j+1} m_{n_1} \otimes_j m_{n_2} \ .$$

where the signs come from the Koszul sign rule[5]. Summarizing, an $\mathscr{A}_\infty$-algebra is defined as follows.

Definition 2.3.22. *An $\mathscr{A}_\infty$-algebra A is a chain complex together with operations*

$$m_n : A^{\otimes n} \longrightarrow A \qquad \textit{for } n \geq 2$$

satisfying

$$\partial(m_n) = \sum_{\substack{n_1+n_2=n+1 \\ 1\leq j\leq n_1}} (-1)^{n_2(n_1-j)+j+1} m_{n_1} \otimes_j m_{n_2}$$

for all $n \geq 2$.

If we write explicitly what that means for $n = 2, 3$, we get that

$$\partial(m_2) = 0 \ ,$$

i.e. $d_A m_2(x, y) = m_2(d_A x, y) + (-1)^x m_2(x, d_A y)$, and

$$\partial(m_3) = m_2 \otimes_1 m_2 - m_2 \otimes_2 m_2 \ ,$$

which tells us that the binary operation m_2 is not associative, but it is up to a homotopy given by the ternary operation m_3. The relations for $n \geq 4$ are higher compatibilities between the operations.

If A and B are $\mathscr{A}_\infty$-algebras, an ∞-morphism $\Psi : A \rightsquigarrow B$ between them is a collection

$$\psi_n : A^{\otimes n} \longrightarrow B$$

of linear maps of degrees $|\psi_n| = n - 1$ which put together form a coherent morphism of $\mathrm{As}^!$-coalgebras

$$\Psi : \mathrm{As}^!(A) \longrightarrow \mathrm{As}^!(B) \ ,$$

where the differential of $\mathrm{As}^!(A)$ and $\mathrm{As}^!(B)$ is the one given by Proposition 2.3.13. Explicitly, let $n \geq 2$ and $a_1, \ldots, a_n \in A$, and for brevity write $a \coloneqq a_1 \otimes \cdots \otimes a_n$. Then

$$\Psi(\mathscr{S}_n^{-1}\mu_n^\vee \otimes a) = \sum_{\substack{k\geq 2 \\ m_1+\cdots+m_k=n}} \pm \mathscr{S}_k^{-1}\mu_k^\vee \otimes \psi_{m_1}(a_1, \ldots, a_{m_1}) \otimes \cdots \otimes \psi_{m_k}(a_{m_k+1}, \ldots, a_n) \ ,$$

where the sign comes both from the decomposition of $\mathscr{S}_n^{-1}$, and from a Koszul sign coming from the fact that we have to switch the maps ψ_{m_i}, which have degree $1 - m_i$, and some of the elements of A. The differential of $\mathrm{As}^!(A)$ is explicitly given by

$$d_{\mathrm{As}^!(A)}(\mathscr{S}_n^{-1}\mu_n^\vee \otimes a) =$$

[5] In particular, to compute the sign associated to the decomposition of $\mathscr{S}_n^{-1}$, one knows that the part with underlying tree $c_{n_1} \circ_j c_{n_2}$ is of the form $(-1)^\epsilon \mathscr{S}_{n_1}^{-1} \otimes_j \mathscr{S}_{n_2}^{-1}$. To find ϵ, one computes on one side $\langle \Delta_{(1)}(\mathscr{S}_n^{-1}), \mathscr{S}_{n_1} \otimes_j \mathscr{S}_{n_2} \rangle = (-1)^\epsilon \langle \mathscr{S}_{n_1}^{-1} \otimes_j \mathscr{S}_{n_2}^{-1}, \mathscr{S}_{n_1} \otimes_j \mathscr{S}_{n_2} \rangle = (-1)^{\epsilon+(n_2-1)(1-n_1)}$, and on the other side $\langle \Delta_{(1)}(\mathscr{S}_n^{-1}), \mathscr{S}_{n_1} \otimes_j \mathscr{S}_{n_2} \rangle = \langle \mathscr{S}_n^{-1}, (-1)^{(j-1)(1-n_2)} \mathscr{S}_n \rangle = (-1)^{(j-1)(1-n_2)}$.

$$= (-1)^{n-1}\mathscr{S}_n^{-1}\mu_n^\vee \otimes d_{A^{\otimes n}}(a) + \\ + \sum_{\substack{n_1+n_2=n+1 \\ 1\leq j\leq n_1}} \pm \mathscr{S}_{n_1}^{-1}\mu_{n_1}^\vee \otimes a_1 \otimes \cdots a_{j-1} \otimes m_{n_2}^A(a_j,\ldots,a_{j+n_2}) \otimes a_{j+n_2+1} \otimes \cdots \otimes a_n \ ,$$

where again the sign comes both from the decomposition of $\mathscr{S}_n^{-1}$, and from the Koszul rule. Imposing

$$d_{\mathrm{As}^{\text{¡}}(B)}\Psi = \Psi d_{\mathrm{As}^{\text{¡}}(A)}$$

and projecting on A, we obtain the relations

$$\partial(\psi_n) + \sum_{\substack{k\geq 2 \\ m_1+\cdots+m_k=n}} \pm m_k^B \circ (\psi_{m_1},\ldots,\psi_{m_k}) = \sum_{\substack{n_1+n_2=n+1 \\ 1\leq j\leq n_1}} \pm \psi_{n_1} \circ_j m_{n_2}^A \ .$$

The homotopy transfer theorem is given as follows. Let

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

be a contraction, and suppose that A is an $\mathscr{A}_\infty$-algebra with operations m_n^A for $n \geq 2$. The monadic decomposition map in $\mathrm{As}^\vee$ is given by

$$\Delta^{\mathrm{mon}}_{\mathrm{As}^\vee}(\mu_n^\vee) = \sum_{t\in \mathrm{PT}_n} t(v \mapsto \mu_{|v|}^\vee) \ ,$$

and thus the transfered structure on B is given by

$$m_n^B = \sum_{t\in \mathrm{PT}_n} \pm pt^h(v \mapsto m_{|v|}^A) i^{\otimes n},$$

where the signs come from the monadic decomposition of $\mathscr{S}_n^{-1}$.

2.3.6 Homotopy commutative algebras

Another type of algebra up to homotopy which we will encounter later, e.g. in Section 10.3, are commutative algebras up to homotopy, often called $\mathscr{C}_\infty$-algebras. These also appeared in the literature long before the theory of Koszul duality for operads, see [Kad88].

An easy computation shows that we have

$$\mathrm{Com}^! = \mathrm{Lie} \quad \text{and} \quad \mathrm{Lie}^! = \mathrm{Com} \ ,$$

see e.g. [LV12, Sect. 7.6.4], while the methods of [LV12, Ch. 8] show that both operads Com and Lie are Koszul. Therefore, its minimal model is given by

$$\mathscr{C}_\infty \coloneqq \Omega \mathrm{Com}^{\text{¡}} \ ,$$

where

$$\mathrm{Com}^{\text{¡}} \cong (\mathscr{S}^{-1})^c \otimes \mathrm{Lie}^\vee.$$

However, the operad Lie is complicated and difficult to treat, due to the Jacobi relation. Thus, one has to go another way in order to understand $\mathscr{C}_\infty$-algebras a bit better.

Let A be a chain complex, and let $p, q \geq 1$. A *non-trivial* (p,q)*-shuffle* of elements of A is an element of $A^{\otimes(p+q)}$ of the form

$$\sum_{\sigma \in \sqcup\!\sqcup(p,q)} (-1)^{\epsilon} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(p+q)}$$

for $a_1, \ldots, a_{p+q} \in A$, where ϵ is the Koszul sign. In other words, it is the sum over all the ways of shuffling $a_1 \otimes \cdots \otimes a_p$ and $a_{p+1} \otimes \cdots \otimes a_{p+q}$.

Proposition 2.3.23 ([LV12, Prop. 13.1.6])**.** *A $\mathscr{C}_\infty$-algebra is an $\mathscr{A}_\infty$-algebra such that each one of the generating operations m_n vanishes on all non-trivial (p,q)-shuffle of elements of A, for $p+q=n$.*

The ∞-morphisms of $\mathscr{C}_\infty$-algebras are similarly characterized.

Proposition 2.3.24. *Let A, B be two $\mathscr{C}_\infty$-algebras. An ∞-morphism $\Psi : A \rightsquigarrow B$ of $\mathscr{C}_\infty$-algebras is an ∞-morphism of $\mathscr{A}_\infty$-algebras such that each of its components*

$$\psi_n : A^{\otimes n} \longrightarrow B$$

vanishes over all non-trivial (p,q)-shuffle of elements of A, for $p+q=n$.

One may also wonder if the homotopy transfer theorem for $\mathscr{C}_\infty$-algebra structures can be recovered from the one for $\mathscr{A}_\infty$-algebras. The answer is positive.

Theorem 2.3.25 ([CG08, Thm. 12])**.** *Suppose we obtain an $\mathscr{A}_\infty$-algebra structure on a chain complex B by homotopy transfer from an $\mathscr{A}_\infty$-algebra A. If A were a $\mathscr{C}_\infty$-algebra, then so is B with the transfered $\mathscr{A}_\infty$-algebra structure, and this structure corresponds with the one obtained by homotopy transfer for $\mathscr{C}_\infty$-algebras.*

2.3.7 Homotopy algebras over the dual numbers and spectral sequences

One can recover the spectral sequence associated to a bicomplex via an application of the homotopy transfer theorem. This example is extracted from [DSV15, Sect. 1].

The operad of *dual numbers* — which we will denote by $\mathscr{D}$ in this section, but which will not make any other appearances in the rest of the present work — is the quadratic operad

$$\mathscr{D} := \mathscr{P}(\mathbb{K}\Delta, \Delta \circ \Delta) \,,$$

where Δ is an arity 1 element of degree 1. An algebra over $\mathscr{D}$ is a chain complex A together with an operation

$$\Delta : A \longrightarrow A$$

such that $\Delta^2 = 0$ and $d\Delta + \Delta d = 0$. In other words, a $\mathscr{D}$-algebra is nothing else than a bicomplex.

It can be shown that the operad $\mathscr{D}$ is Koszul. Thanks to Proposition 2.3.4, one sees that its Koszul dual operad is

$$\mathscr{D}^! = \mathscr{P}(s^{-1}\Delta^\vee) \,,$$

with no relations, and thus its Koszul dual cooperad is concentrated in arity 1, where it is given by

$$\mathscr{D}^{\mathrm{i}}(1) = \bigoplus_{n \geq 1} \mathbb{K}\delta_n$$

with δ_n corresponding to the dual of $(s^{-1}\Delta^\vee)^n$. We have $|\delta_n| = 2n$ and

$$\Delta_{(1)}(\delta_n) = \sum_{n_1+n_2=n} \delta_{n_1} \circ \delta_{n_2} \ .$$

It follows that the minimal model for $\mathscr{D}$ is the operad $\mathscr{D}_\infty$ freely generated by $\Delta_1, \Delta_2, \ldots$, where Δ_n corresponds to $s^{-1}\delta_n$ in $\Omega\mathscr{D}^{\text{¡}}$ and has degree $2n-1$. These operators satisfy

$$d\Delta_n + \Delta_n d = - \sum_{n_1+n_2=n} \Delta_{n_1} \circ \Delta_{n_2} \ ,$$

or, writing $\Delta_0 \coloneqq d$,

$$\sum_{n_1+n_2=n} \Delta_{n_1} \circ \Delta_{n_2} = 0 \ .$$

A $\mathscr{D}_\infty$-algebra is also known as a *multicomplex*.

Let A be a bicomplex. Since we work over a field, one can always choose a contraction

$$_h\circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} H(A)$$

from A to its homology $H(A)$. The homotopy transfer theorem endows $H(A)$ with a multicomplex structure, where

$$\Delta_n \coloneqq p \underbrace{\Delta h \Delta h \cdots h \Delta}_{n \text{ copies of } \Delta} i \ .$$

The operator Δ_n is essentially the differential of the nth page of the spectral sequence associated to the bicomplex A.

2.3.8 Homotopy Lie algebras

This last example is the most important for the original results of this work. Homotopy Lie algebras — also known as strong homotopy Lie algebras, or $\mathscr{L}_\infty$-algebras — have a long history in the literature, where they appeared in a multitude of subjects. For example, they arise naturally in Kontsevich's proof of deformation quantization of Poisson manifolds [Kon03], in string field theory [Zwi93], in derived deformation theory [Pri10], [SS12] and others, as algebras of symmetries for conformal field theories [BFT17], in symplectic topology [Kon95], [Sei08], [FOOO09] and others, as rational models for mapping spaces [Ber15], and in many other places.

As already stated before, we have

$$\mathrm{Lie}^! = \mathrm{Com} \ ,$$

and the usual methods prove that Lie is Koszul. Therefore, its minimal model is given by

$$\mathscr{L}_\infty \coloneqq \Omega\mathrm{Lie}^{\text{¡}} \ ,$$

and

$$\mathrm{Lie}^{\text{¡}} \cong (\mathscr{S}^{-1})^c \otimes \mathrm{Com}^\vee .$$

Since Com is 1-dimensional in every arity, $\mathscr{L}_\infty$ is freely generated by the operations

$$\ell_n \coloneqq s^{-1}\mathscr{S}_n^{-1}\mu_n^\vee \in \mathscr{L}_\infty(n), \qquad n \geq 2 \ ,$$

which have degree $|\ell_n| = n-2$. The only thing left to study is the differential, which is given by

$$d_{\mathscr{L}_\infty}(\ell_n) = d_1(\ell_n) + d_2(\ell_n)\ ,$$

where $d_1 = 0$ since the differential of Com is trivial. The $d_2(\ell_n)$ is given by

$$\begin{aligned}\ell_n \mapsto\ & -(s^{-1}\otimes s^{-1})\otimes \sum_{\substack{n_1+n_2=n+1\\ \sigma\in \sqcup\!\sqcup(n_1,n_2-1)}} (-1)^{(n_2-1)(1-n_1)+\sigma}(\mathscr{S}^{-1}_{n_1}\mu^\vee_{n_1}\otimes_1 \mathscr{S}^{-1}_{n_2}\mu^\vee_{n_2})^\sigma \\ &= \sum_{\substack{n_1+n_2=n+1\\ \sigma\in \sqcup\!\sqcup(n_1,n_2-1)}} (-1)^{n_2(n_1-1)+\sigma+1}(\ell_{n_1}\otimes_1 \ell_{n_2})^\sigma.\end{aligned}$$

In summary, an $\mathscr{L}_\infty$-algebra is defined as follows.

Definition 2.3.26. *An $\mathscr{L}_\infty$-algebra $\mathfrak{g}$ is a chain complex together with graded antisymmetric operations (also called brackets)*

$$\ell_n : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g} \qquad \textit{for } n\geq 2$$

satisfying

$$\partial(\ell_n) = \sum_{\substack{n_1+n_2=n+1\\ \sigma\in \sqcup\!\sqcup(n_1-1,n_2)}} (-1)^{n_2(n_1-1)+\sigma+1}(\ell_{n_1}\otimes_1 \ell_{n_2})^\sigma$$

for all $n\geq 2$.

Proceeding as we did in Section 2.3.5, we obtain that an ∞-morphism of $\mathscr{L}_\infty$-algebras $\Psi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is a collection of linear maps

$$\psi_n : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{h}$$

of degree $|\psi_n| = n-1$ that are antisymmetric and satisfy

$$\partial(\psi_n) + \sum_{\substack{k\geq 2\\ m_1+\cdots+m_k=n\\ \sigma\in \sqcup\!\sqcup(m_1,\ldots,m_k)}} \pm\, \ell^B_k\circ(\psi_{m_1},\ldots,\psi_{m_k})^\sigma = \sum_{\substack{n_1+n_2=n+1\\ \theta\in \sqcup\!\sqcup(n_1,n_2-1)}} \pm\,(\psi_{n_1}\circ_1 \ell^A_{n_2})^\theta.$$

Finally, for the homotopy transfer theorem, if

$$h\circlearrowright \mathfrak{g} \underset{i}{\overset{p}{\rightleftarrows}} \mathfrak{h}$$

is a contraction and $\mathfrak{g}$ is an $\mathscr{L}_\infty$-algebra, then the transferred $\mathscr{L}_\infty$-algebra structure on $\mathfrak{h}$ is given by

$$\ell^{\mathfrak{h}}_n = \sum_{\tau\in \mathrm{RT}_n} \pm\, p\tau^h(v\mapsto \ell^{\mathfrak{g}}_{|v|})i^{\otimes n}.$$

As usual, the signs come from the decomposition map of $(\mathscr{S}^{-1})^c$.

2.3.9 Operadic Massey products

Given a chain complex V, one can always obtain a (non-canonical) contraction

$$h\circlearrowright V \underset{i}{\overset{p}{\rightleftarrows}} H(V)$$

from V to its homology $H(V)$. It can be done as follows. For each $n \in \mathbb{Z}$, choose a complement of the subspace of cycles $\mathscr{Z}_n(V)$ and notice that it is isomorphic to the boundaries $\mathscr{B}_{n-1}(V)$ via the differential. Thus,

$$V_n \cong \mathscr{Z}_n(V) \oplus \mathscr{B}_{n-1}(V) .$$

Now choose a complement of $\mathscr{B}_n(V)$ in $\mathscr{Z}_n(V)$ and notice that it is isomorphic to $H_n(V)$. Therefore, we have

$$V_n \cong \mathscr{B}_n(V) \oplus H_n(V) \oplus \mathscr{B}_{n-1}(V) .$$

Define i by sending $H(V)$ to the chosen copy of $H(V)$ in V, p by projecting onto the copy of $H(V)$ in V, and h on V_n by first projecting onto $\mathscr{B}_n(V)$ and then identifying it with the copy of $\mathscr{B}_n(V)$ contained in V_{n+1}.

Lemma 2.3.27. *The three maps described above form a contraction from V to $H(V)$.*

Proof. By inspection. □

Let A be an $\mathscr{A}_\infty$-algebra. For example, one can take the the singular cochain complex[6] of a topological space X with coefficients in the field $\mathbb{K}$ together with the cup product — an associative algebra. Then the choice of a contraction as above gives an $\mathscr{A}_\infty$-algebra structure on the homology $H(A)$ of A.

There is a well known classical construction of higher products — the Massey products — on the homology of an $\mathscr{A}_\infty$-algebra due to Massey [Mas58] and May [May69]. As one might expect, the induced $\mathscr{A}_\infty$-algebra structures on $H(A)$ are strictly related to the Massey product. The exact relationship has been studied in [MF17].

2.4 Bar and cobar construction for (co)algebras

Given an operadic twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$, it is possible to define a bar-cobar adjunction relating the category of conilpotent $\mathscr{C}$-coalgebras and the category of $\mathscr{P}$-algebras. This helps for example to give a cleaner definition of the notion of ∞-morphisms, and will be the base for a generalization of the notion which will be exposed in Chapter 8. One also has a notion of twisting morphism relative to α giving a result analogous to Theorem 2.2.17 for (co)algebras. The material presented here comes from [LV12, Sect. 11.1–3].

2.4.1 Bar and cobar construction relative to a twisting morphism

We begin with the definition of the bar and cobar construction relative to a twisting morphism. For the rest of this section, fix a twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$.

Let A be a $\mathscr{P}$ algebra, then we define a conilpotent $\mathscr{C}$-coalgebra by

$$\mathrm{B}_\alpha A \coloneqq (\mathscr{C}(A), d_{\mathrm{B}_\alpha A} \coloneqq d_1 + d_2) ,$$

where $d_1 \coloneqq d_{\mathscr{C}} \circ 1_A + 1_{\mathscr{C}} \circ' d_A$, and d_2 is the unique coderivation extending the composite

$$\mathscr{C}(A) \xrightarrow{\alpha \circ 1_A} \mathscr{P}(A) \xrightarrow{\gamma_A} A .$$

That is to say, the full expression for d_2 is given by the composite

$$\mathscr{C}(A) \xrightarrow{\Delta_{(1)} \circ 1_A} (\mathscr{C} \circ_{(1)} \mathscr{C})(A) \xrightarrow{(1_{\mathscr{C}} \circ_{(1)} \alpha) \circ 1_A} (\mathscr{C} \circ_{(1)} \mathscr{P})(A) \cong \mathscr{C} \circ (A; \mathscr{P}(A)) \xrightarrow{1_{\mathscr{C}} \circ (1_A; \gamma_A)} \mathscr{C}(A) .$$

[6] The theory works exactly the same if we exchange chain and cochain complexes, of course.

The coderivation $d_{\mathrm{B}_\alpha A}$ squares to zero by [LV12, Lemma 11.2.1]. Given a morphism $f : A \to A'$ of $\mathscr{P}$-algebras, we obtain a morphism of conilpotent $\mathscr{C}$-coalgebras by

$$\mathrm{B}_\alpha f \coloneqq \mathscr{C}(f) : \mathrm{B}_\alpha A \longrightarrow \mathrm{B}_\alpha A' \,,$$

and this assignment is functorial.

Definition 2.4.1. *The functor*

$$\mathrm{B}_\alpha : \mathscr{P}\text{-alg} \longrightarrow \mathsf{conil.}\,\mathscr{C}\text{-cog}$$

defined above is called the bar construction (relative to α).

Dually, let C be a conilpotent $\mathscr{C}$-coalgebra. We define a $\mathscr{P}$-algebra by

$$\Omega_\alpha C \coloneqq (\mathscr{P}(C), d_{\Omega_\alpha C} \coloneqq d_1 + d_2) \,,$$

where $d_1 \coloneqq d_{\mathscr{P}} \circ 1_D + 1_{\mathscr{P}} \circ' d_D$ and $-d_2$ is the unique derivation extending the composite

$$C \xrightarrow{\Delta_C} \mathscr{C}(C) \xrightarrow{\alpha \circ 1_C} \mathscr{P}(C) \,.$$

Similarly to the previous case, the full expression for $-d_2$ is given by the composite

$$\begin{aligned} \mathscr{P}(C) &\xrightarrow{1_{\mathscr{P}} \circ' \Delta_C} \mathscr{P} \circ (C; \mathscr{C}(C)) \\ &\xrightarrow{1_{\mathscr{P}} \circ (1_C; \alpha \circ 1_C)} \mathscr{P} \circ (C; \mathscr{P}(C)) \cong (\mathscr{P} \circ_{(1)} \mathscr{P})(C) \xrightarrow{\gamma_{(1)} \circ 1_C} \mathscr{P}(C) \,. \end{aligned}$$

Once again, the derivation $d_{\Omega_\alpha C}$ squares to zero, see [LV12, Lemma 11.2.4]. Given a morphism $g : C' \to C$ of $\mathscr{C}$-coalgebras, we obtain a morphism of $\mathscr{P}$-algebras by

$$\Omega_\alpha g \coloneqq \mathscr{P}(g) : \Omega_\alpha C' \longrightarrow \Omega_\alpha C \,,$$

and this assignment is functorial.

Definition 2.4.2. *The functor*

$$\Omega_\alpha : \mathsf{conil.}\,\mathscr{C}\text{-cog} \longrightarrow \mathscr{P}\text{-alg}$$

defined above is called the cobar construction (relative to α).

The relative bar construction gives us a cleaner way to define $\mathscr{P}_\infty$-algebras for $\mathscr{P}$ a Koszul operad and ∞-morphisms of $\mathscr{P}_\infty$-algebras.

Proposition 2.4.3 ([LV12, Prop. 11.4.1])**.** *Let $\mathscr{P}$ be a Koszul operad. The bar construction B_ι associated to the twisting morphism $\iota : \mathscr{P}^{\mathsf{i}} \to \Omega\mathscr{P}^{\mathsf{i}}$ extends to an isomorphism of categories*

$$\mathrm{B}_\iota : \infty\text{-}\mathscr{P}_\infty\text{-alg} \longrightarrow \text{quasi-free } \mathscr{P}^{\mathsf{i}}\text{-cog}$$

from the category of $\mathscr{P}_\infty$-algebras with their ∞-morphisms to the full subcategory of the conilpotent $\mathscr{P}^{\mathsf{i}}$-coalgebras given by the quasi-free coalgebras, i.e. the ones whose underlying $\mathscr{P}^{\mathsf{i}}$-coalgebra in graded vector spaces is of the form $\mathscr{P}^{\mathsf{i}}(V)$.

In other words, if A is a $\mathscr{P}_\infty$-algebra, then $\mathrm{B}_\iota A$ is the $\mathscr{P}^{\mathsf{i}}$-coalgebra given by Proposition 2.3.13, and an ∞-morphism of $\mathscr{P}_\infty$-algebras is a morphism of coalgebras between the bar constructions.

2.4.2 Relative bar and cobar constructions and quasi-isomorphisms

Quasi-isomorphisms behave really well with respect to the bar construction.

Proposition 2.4.4 ([LV12, Prop. 11.2.3]). *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism, and let $f : A \to A'$ be a quasi-isomorphism of $\mathscr{P}$-algebras. Then*

$$\mathrm{B}_\alpha f : \mathrm{B}_\alpha A \longrightarrow \mathrm{B}_\alpha A'$$

is a quasi-isomorphism.

They behave a bit worse with respect to the cobar construction.

Proposition 2.4.5 ([LV12, Prop. 11.2.6]). *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism, and let $g : C' \to C$ be a quasi-isomorphism between connected conilpotent $\mathscr{C}$-coalgebras, i.e. $\mathscr{C}$-coalgebras that are 0 in degrees smaller or equal to 0. Then*

$$\Omega_\alpha g \coloneqq \mathscr{P}(g) : \Omega_\alpha C' \longrightarrow \Omega_\alpha C$$

is a quasi-isomorphism.

There are quasi-isomorphisms between conilpotent $\mathscr{C}$-coalgebras that are not sent to quasi-isomorphisms by the cobar construction Ω_α, see e.g. [LV12, Prop. 2.4.3].

2.4.3 Relative twisting morphisms

Let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. We consider the operator $\star_\alpha$ of degree -1 acting on the chain complex $\hom(C, A)$ given by the composite

$$\star_\alpha(\varphi) \coloneqq \left(C \xrightarrow{\Delta_C} \mathscr{C} \circ C \xrightarrow{\alpha \circ \varphi} \mathscr{P} \circ A \xrightarrow{\gamma_A} A \right)$$

for $\varphi \in \hom(C, A)$.

Definition 2.4.6. *A* twisting morphism relative to α *is an element $\varphi \in \hom(C, A)$ of degree 0 satisfying the Maurer–Cartan equation*

$$\partial(\varphi) + \star_\alpha(\varphi) = 0 \,. \tag{2.3}$$

We denote the set of all such relative twisting morphisms by $\mathrm{Tw}_\alpha(C, A)$.

2.4.4 Bar-cobar adjunction for (co)algebras

Similarly to the operadic case, the bar and cobar functors form an adjoint pair, cf. Theorem 2.2.17.

Theorem 2.4.7. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism, let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. There are bijections*

$$\hom_{\mathscr{P}\text{-alg}}(\Omega_\alpha C, A) \cong \mathrm{Tw}_\alpha(C, A) \cong \hom_{\mathscr{C}\text{-cog}}(C, \mathrm{B}_\alpha A) \,,$$

natural both in C and A. In particular, Ω_α and B_α form an adjoint pair.

Proof. We prove the first bijection, the second one being dual. A morphism of $\mathscr{P}$-algebras $f : \Omega_\alpha C \to A$ is in particular a morphism of algebras in graded vector spaces from $\mathscr{P}(C) \to A$, and is therefore completely determined by its restriction $\varphi \coloneqq f|_C$ to C. In the other direction, given φ we recover f as

$$f = \gamma_A(1_{\mathscr{P}} \circ \varphi) .$$

We only need to show that f commuting with the differentials is equivalent to φ satisfying the Maurer–Cartan equation (2.3). The restriction to C of the relation

$$d_A f = f d_{\Omega_\alpha C}$$

gives

$$\begin{aligned} d_A\varphi = & -\varphi d_C - f(\alpha \circ 1_C)\Delta_C \\ = & -\varphi d_C - \gamma_A(1_{\mathscr{P}} \circ \varphi)(\alpha \circ 1_C)\Delta_C \\ = & -\varphi d_C - \gamma_A(\alpha \circ \varphi)\Delta_C \\ = & -\varphi d_C - \star_\alpha(\varphi) . \end{aligned}$$

Therefore, if f commutes with the differentials, then φ satisfies the Maurer–Cartan equation. The other direction also follows from this computation and from the fact that the restriction to C determines the whole morphism. □

2.4.5 Bar-cobar resolutions of (co)algebras

Now we would like to use the bar-cobar adjunction for (co)algebras to give (functorial) resolutions of the same, as we did for operads in Section 2.2.8. This is indeed possible, but we have to further require that α is Koszul.

Theorem 2.4.8. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism. The following are equivalent.*

1. *α is a Koszul morphism.*
2. *The counit*
$$\epsilon_A : \Omega_\alpha \mathrm{B}_\alpha A \longrightarrow A$$
of the bar-cobar adjunction is a quasi-isomorphism for any $\mathscr{P}$-algebra A.

Remark 2.4.9. *In particular, the bar-cobar adjunction provides a cofibrant resolution for $\mathscr{P}$-algebras, cf. Section 3.3.6.*

Proof. This is [LV12, Thm. 11.3.3]. □

Remark 2.4.10. *This is true as stated because we supposed that all of our (co)operads are reduced, and thus are canonically connected weight graded by the arity. In more generality, one has to assume that $\mathscr{C}$ and $\mathscr{P}$ are connected weight graded, and that α preserves this additional grading.*

A similar statement is true for coalgebras.

Theorem 2.4.11. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism. The following are equivalent.*

1. *α is a Koszul morphism.*
2. *The unit*
$$\eta_C : C \longrightarrow \mathrm{B}_\alpha \Omega_\alpha C$$
is a quasi-isomorphism for any conilpotent $\mathscr{C}$-coalgebra C.

Remark 2.4.12. *A refinement version of the implication* (1) $\Longrightarrow$ (2) *has been given in [Val14, Thm. 2.6(2)][7]. We will present it in more details in Section 3.3.7, see Corollary 3.3.19.*

Proof. This is [LV12, Thm. 11.3.4]. □

2.5 Homotopy theory of homotopy algebras

In this section, we give some results on the homotopy theory of $\mathscr{P}_\infty$-algebras, for $\mathscr{P}$ a Koszul operad. The homotopy theory of homotopy algebras, and more generally of (co)algebras related by an operadic Koszul morphisms will be studied again in more detail in Section 3.3.7 and Section 8.2. The material presented here is extracted from [LV12, Sect. 11.4]. For this section, fix a Koszul operad $\mathscr{P}$, and as usual let $\kappa : \mathscr{P}^{\text{¡}} \to \mathscr{P}$ be the Koszul morphism given by Koszul duality, and let $\iota : \mathscr{P}^{\text{¡}} \to \mathscr{P}_\infty$ be the canonical twisting morphism, which is also Koszul.

2.5.1 Rectification of $\mathscr{P}_\infty$-algebras

Every $\mathscr{P}$-algebra is in particular a $\mathscr{P}_\infty$-algebra. This can be done as follows. Let $g_\kappa : \mathscr{P}_\infty \to \mathscr{P}$ be the morphism of operads given by Theorem 2.2.18 applied to $\alpha = \kappa$. Notice that it is a quasi-isomorphism by Theorem 2.2.21. Then the restriction of structure g_κ^* gives the desired functor

$$g_\kappa^* : \mathscr{P}\text{-alg} \longrightarrow \mathscr{P}_\infty\text{-alg} .$$

Denote by

$$i : \mathscr{P}\text{-alg} \longrightarrow \infty\text{-}\mathscr{P}_\infty\text{-alg}$$

the composite of g_κ^* with the inclusion of $\mathscr{P}_\infty$-alg into ∞-$\mathscr{P}_\infty$-alg. On the other hand, we have the functor

$$\Omega_\kappa \mathrm{B}_\iota : \infty\text{-}\mathscr{P}_\infty\text{-alg} \longrightarrow \mathscr{P}\text{-alg} ,$$

where B_ι is the extension of the bar construction of Proposition 2.4.3.

Proposition 2.5.1. *The functors described above form an adjoint pair*

$$\Omega_\kappa \mathrm{B}_\iota : \infty\text{-}\mathscr{P}_\infty\text{-alg} \rightleftarrows \mathscr{P}\text{-alg} : i .$$

Proof. This is [LV12, Prop. 11.4.3]. □

Theorem 2.5.2. *Let A be a $\mathscr{P}_\infty$-algebra, then there is an ∞-quasi-isomorphism*

$$A \overset{\sim}{\rightsquigarrow} \Omega_\kappa \mathrm{B}_\iota A ,$$

natural in A. Moreover, the $\mathscr{P}$-algebra $\Omega_\kappa \mathrm{B}_\iota A$ is unique, up to isomorphism, with respect to the universal property that any ∞-morphism with A as domain and a $\mathscr{P}$-algebra as target factors into the ∞-quasi-isomorphism above followed by a strict morphism of $\mathscr{P}$-algebras.

Proof. This result is [LV12, Thm. 11.4.4, and Prop. 11.4.5 and 11.4.6]. □

The functor $\Omega_\kappa \mathrm{B}_\iota$ is called the *rectification functor* for $\mathscr{P}_\infty$-algebras. In fact, it gives an equivalence between the homotopy categories of ∞-$\mathscr{P}_\infty$-alg and $\mathscr{P}$-alg, where in ∞-$\mathscr{P}_\infty$-alg the weak equivalences are the ∞-quasi-isomorphism. See [LV12, Sect. 11.4] for details.

[7] It is stated there only for the twisting morphism κ given by Koszul duality, but the result holds in general for twisting morphisms between connected weight graded (co)operads. In particular, it always hold in our setting.

2.5.2 Relation between quasi-isomorphisms and ∞-quasi-isomorphisms

Quasi-isomorphisms and ∞-quasi-isomorphisms are closely related. The following result is of fundamental importance, for example in interpreting certain classical definitions in rational homotopy theory, cf. Chapter 5.

Theorem 2.5.3. *Let A and B be two $\mathscr{P}_\infty$-algebras. The following are equivalent.*

1. *There is a zig-zag of quasi-isomorphisms of $\mathscr{P}_\infty$-algebras*
$$A \xleftarrow{\sim} \bullet \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \cdots \bullet \xrightarrow{\sim} B\,.$$
2. *There are two quasi-isomorphisms of $\mathscr{P}_\infty$-algebras*
$$A \xleftarrow{\sim} \bullet \xrightarrow{\sim} B\,.$$
3. *There exists an ∞-quasi-isomorphism of $\mathscr{P}_\infty$-algebras*
$$A \overset{\sim}{\rightsquigarrow} B\,.$$

Proof. The case where A and B are $\mathscr{P}$-algebras is proven in [LV12, Thm. 11.4.9]. The more general case of $\mathscr{P}_\infty$-algebras is done similarly, but in order to prove (1) $\implies$ (3) one has to use a model categorical argument, cf. the case for coalgebras of Theorem 8.2.6. □

An immediate consequence of this result is the following one.

Corollary 2.5.4. *Let A and B be two $\mathscr{P}_\infty$-algebras. If there is an ∞-quasi-isomorphism*
$$A \overset{\sim}{\rightsquigarrow} B\,,$$
then there exists an ∞-quasi-isomorphism in the other direction
$$B \overset{\sim}{\rightsquigarrow} A\,.$$

Chapter 3

Model categories

Model categories were introduced by Quillen in [Qui67] in order to do "non-linear homological algebra", also known as homotopical algebra. They give a generalized framework in which to study homotopy theory in some category, which consists into taking a category and formally inverting some morphisms that one would like to consider as equivalences. Two motivating examples are:

1. One considers the category of topological spaces and wants to study their homotopy groups. Therefore, one wants to formally invert continuous maps that induce isomorphisms on all homotopy groups, so that the isomorphisms classes of the new category correspond to the existing homotopy types of topological spaces.
2. One considers the category of chain complexes and wants to study their homology. Therefore, one wants to invert quasi-isomorphisms, i.e. the chain maps inducing isomorphisms in homology. Trying to do this, one essentially recovers classical homological algebra.

In this chapter, we will give a fast introduction to model categories and the concepts surrounding them, and then give examples, some for motivation, and some because we will need them later on. Our main references are the books [Hov99], and [GJ09, Ch. II], but the reader should be aware that there are many good items on this subject in the literature, such as the already mentioned seminal work [Qui67], and [DS95]. One should also mention that model categories have been somewhat superseded by ∞-categories[1] in the recent years. The topic is outside the scope of this work, but the interested reader should have no problems finding references on the subject, e.g. starting with [Lur09].

3.1 Model categories

We begin by giving the basic definitions of model categories and explaining the notion of a homotopy between morphisms in a model category.

3.1.1 Definitions

Without further ado, we give the definition of a model category.

Definition 3.1.1. *A* model category[2] *is a category* C*, together with three classes of arrows* W*,* F *and*

[1] More precisely, $(\infty, 1)$-categories, of which one model are quasicategories.
[2] Sometimes, this is called a *closed* model category.

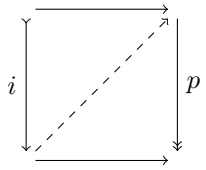
C — called respectively weak equivalences, fibrations, *and* cofibrations — *satisfying the following properties.*

M1 *The category* C *has all finite limits and colimits.*

M2 *Given two arrows* f, g *in* C *such that the target of* f *is the domain of* g*, if any two of* f, g *and* gf *are weak equivalences, then so is the third. We say that* W *satisfies the* 2-out-of-3 property.

M3 *The classes of weak equivalences, fibrations, and cofibrations are all closed under retracts.*

M4 *Suppose we are given the following solid arrow diagram*



where i *is a cofibration, and* p *is a fibration. If either one of* i *or* p *is also a weak equivalence, then there exist a diagonal filler (the dashed arrow in the diagram).*

M5 *Every arrow* f *in* C *can be factored both as* $f = pi$ *with* i *a cofibration which is also a weak equivalence and* p *a fibration, and as* $f = qj$ *with* j *a cofibration, and* q *a fibration that is also a weak equivalence.*

We often abuse of notation and just talk of the model category C *when no confusion is possible about the three classes of maps.*

Remark 3.1.2. *There are many slight variations on the definition of a model category in the literature. For example, [Qui67] asks that all limits and colimits exist in* C*, and [Hov99] moreover requires that the factorizations of (**M5**) be functorial. As a rule of thumb, essentially all results that can be proven with one version of the definition of a model category hold for all other sensible versions of the definition, with at most minor changes if needed.*

We will often emphasize the fact that an arrow is a cofibration by adding a tail to it, that it is a fibration by giving it a double head, and that it is a weak equivalence by writing a $\sim$ next to it.

It is usual to call *trivial fibrations*, respectively *trivial cofibrations*, the fibration, resp. cofibrations, that are also weak equivalences. One often rephrases the axiom (**M4**) by saying that trivial cofibrations have the *left lifting property* with respect to fibrations, or dually, that fibrations have the *right lifting property* with respect to trivial cofibrations, and the same for the relation between cofibrations and trivial fibrations.

Given a model category, one can characterize the fibrations and cofibrations.

Lemma 3.1.3. *Let* C *be a model category.*

1. *A morphism if* C *is a fibration if, and only if it has the right lifting property with respect to all trivial cofibrations.*
2. *A morphism if* C *is a trivial fibration if, and only if it has the right lifting property with respect to all cofibrations.*
3. *A morphism if* C *is a cofibration if, and only if it has the left lifting property with respect to all trivial fibrations.*

4. *A morphism if* C *is a trivial cofibration if, and only if it has the left lifting property with respect to all fibrations.*

Corollary 3.1.4. *Let* C *be a model category.*

1. *Cofibrations and trivial cofibrations are closed under pushouts and compositions, and all isomorphisms are cofibrations.*
2. *Fibrations and trivial fibrations are closed under pullbacks and compositions, and all isomorphisms are fibrations.*

One can also say something about weak equivalences.

Lemma 3.1.5. *Let* C *be a model category. All isomorphisms are weak equivalences.*

There are two sets of objects that play special roles in model categories.

Definition 3.1.6. *Let* C *be a model category.*

1. *An object* $X \in \mathsf{C}$ *is* cofibrant *if the unique map*
$$\emptyset \longrightarrow X$$
from the initial object to X *is a cofibration.*
2. *An object* $X \in \mathsf{C}$ *is* fibrant *if the unique map*
$$X \longrightarrow *$$
from X *to the final object is a fibration.*

We call an object bifibrant *if it is both fibrant and cofibrant.*

Notice that using (**M5**), for any object $X \in \mathsf{C}$ one can find another object $\widetilde{X}$ which is cofibrant and weakly equivalent to X, or fibrant and weakly equivalent to X. These new objects are called *cofibrant*, resp. *fibrant*, *replacements* of X.

A very useful result about model categories is Ken Brown's lemma.

Lemma 3.1.7 (Ken Brown's lemma)**.** *Let* C *be a model category, and let* C' *be a category with a class of weak equivalences which satisfies (**M2**). Let* $F : \mathsf{C} \to \mathsf{C}'$ *be a functor.*

1. *If* F *takes trivial cofibrations between cofibrant objects to weak equivalences, then* F *takes all weak equivalences between cofibrant objects to weak equivalences.*
2. *Dually, if* F *takes trivial fibrations between fibrant objects to weak equivalences, then* F *takes all weak equivalences between fibrant objects to weak equivalences.*

3.1.2 Duality

Fibrations and cofibrations of a model category play dual roles, in a way which is made precise by the following result.

Proposition 3.1.8. *Suppose that* C *is a model category. Then the opposite category* C^{op} *is also a model category with the same weak equivalences, the fibrations of* C *as cofibrations, and the cofibrations of* C *as fibrations.*

Therefore, it is usually only necessary to prove a result for fibrations in order to have a dual result for cofibrations, and *vice versa*.

3.1.3 Path objects, cylinder objects, and homotopies

Let C category with finite limits and colimits. If $X \in \mathsf{C}$ is an object, then the identity induces two canonical maps. The *fold map*

$$\nabla : X \sqcup X \longrightarrow X \ ,$$

and the *diagonal map*

$$\Delta : X \longrightarrow X \times X \ .$$

Using these maps, one can define two notions of homotopy between morphisms in a model category.

Definition 3.1.9. *Let C be a model category, let $X, Y \in \mathsf{C}$ be two objects, and let $f, g : X \to Y$ be two morphisms.*

1. *A* cylinder object *for X is a commutative diagram*

$$\begin{array}{ccccc} & & \overset{\nabla}{\frown} & & \\ X \sqcup X & \overset{i}{\rightarrowtail} & \mathrm{Cyl}(X) & \overset{w}{\xrightarrow{\sim}} & X \end{array}$$

   with i a cofibration, and w a weak equivalence. We often abuse of notation and speak of the cylinder object $\mathrm{Cyl}(X)$.

2. *The two morphisms f and g are* left homotopic *if there exists a cylinder object $\mathrm{Cyl}(X)$ for X and a morphism $H : \mathrm{Cyl}(X) \to Y$, called a* left homotopy *between f and g, such that*

$$\begin{array}{ccccc} & & \overset{f \sqcup g}{\frown} & & \\ X \sqcup X & \overset{i}{\rightarrowtail} & \mathrm{Cyl}(X) & \overset{H}{\longrightarrow} & Y \end{array}$$

   We write $f \sim_\ell g$.

3. *A* path object *for Y is a commutative diagram*

$$\begin{array}{ccccc} & & \overset{\Delta}{\frown} & & \\ Y & \overset{w}{\xrightarrow{\sim}} & \mathrm{Path}(Y) & \overset{p}{\twoheadrightarrow} & Y \times Y \end{array}$$

   with p a fibration, and w a weak equivalence. We often abuse of notation and speak of the path object $\mathrm{Path}(X)$.

4. *The two morphisms f and g are* right homotopic *if there exists a path object $\mathrm{Path}(Y)$ for Y and a morphism $H : X \to \mathrm{Path}(Y)$, called a* right homotopy *between f and g, such that*

$$\begin{array}{ccccc} & & \overset{f \times g}{\frown} & & \\ X & \overset{H}{\longrightarrow} & \mathrm{Path}(Y) & \overset{p}{\twoheadrightarrow} & Y \times Y \end{array}$$

We write $f \sim_r g$.

Remark 3.1.10. *The notions of cylinder and path objects are inspired by topology. Indeed, there is a model structure on the category of topological spaces for which a choice of cylinder and path objects for a space X are the usual ones, namely $X \times I$ gives a cylinder object, and X^I a path object, cf. Section 3.3.1. This way, one recovers the usual notions of homotopy between continuous maps.*

The following result tells us that the notions of left and right homotopy given above are well defined and independent from the choice of cylinder, resp. path object.

Lemma 3.1.11. *Let* C *be a model category, and let* $X \in \mathsf{C}$ *be an object.*

1. *A cylinder object for X always exists. Any two cylinder objects for X are weakly equivalent. If two morphisms are left homotopic with respect to a cylinder object, then they are for all cylinder objects.*
2. *A path object for X always exists. Any two path objects for X are weakly equivalent. If two morphisms are right homotopic for a path object, then they are for all path objects.*

Proof. We will only prove the first result, the second one being dual. Existence of a cylinder object is guaranteed by (**M5**). Applying it to the fold map, we factor it into

$$\begin{array}{ccccc} & & \overset{\nabla}{\curvearrowright} & & \\ X \sqcup X & \overset{i}{\rightarrowtail} & \mathrm{Cyl}(X) & \overset{w}{\xrightarrow{\sim}} & X \end{array}$$

with w being not only a weak equivalence, but a trivial fibration. Suppose we are given a second cylinder object

$$\begin{array}{ccccc} & & \overset{\nabla}{\curvearrowright} & & \\ X \sqcup X & \overset{i'}{\rightarrowtail} & \mathrm{Cyl}(X)' & \overset{w'}{\xrightarrow{\sim}} & X \end{array}$$

Then we have the following commutative diagram.

$$\begin{array}{ccc} X \sqcup X & \overset{i}{\rightarrowtail} & \mathrm{Cyl}(X) \\ {\scriptstyle i'}\downarrow & \overset{\sim}{\nearrow} & \downarrow{\scriptstyle w} \\ \mathrm{Cyl}(X)' & \underset{w'}{\xrightarrow{\sim}} & X \end{array}$$

The dashed arrow exists by (**M4**), and it is a weak equivalence by (**M2**). So any two cylinder objects are weakly equivalent. The last statement follows in a straightforward manner. □

If the domain of the arrows we consider is cofibrant, and if the target is fibrant, then the homotopy relations are very well behaved.

Proposition 3.1.12. *Let* C *be a model category, and let* $X, Y \in \mathsf{C}$ *be two objects.*

1. *If X is cofibrant, then being left homotopic is an equivalence relation on* $\hom_{\mathsf{C}}(X, Y)$.

2. *If Y is fibrant, then being right homotopic is an equivalence relation on $\hom_{\mathsf{C}}(X,Y)$.*
3. *If X is cofibrant, and Y is fibrant, then two maps $X \to Y$ in C are left homotopic if, and only if they are right homotopic. In this case, we simply say that the two maps are* homotopic.

One can now give an analogue of the Whitehead theorem in the general context of model categories.

Definition 3.1.13. *Let C be a model category, and let $X, Y \in \mathsf{C}$ be two bifibrant objects. We say that $f : X \to Y$ is a* homotopy equivalence *if there exists a morphism $g : Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.*

Theorem 3.1.14 (Whitehead)**.** *Let C be a model category, and let $X, Y \in \mathsf{C}$ be two bifibrant objects. If $f : X \to Y$ is a weak equivalence, then it is a homotopy equivalence.*

3.2 Homotopy categories

As explained at the beginning of the chapter, one wants to formally invert some class of maps to obtain the "homotopy category" of the original category. We explain the naive way to do that and the problems in which one might incur, and then expose how one can do the desired process if one wants to invert the class of weak equivalences in a model category.

3.2.1 Localization at a class of maps

Suppose that C is a category, and that W is a class of morphisms in C that one wishes to formally invert.

Definition 3.2.1. *The* localization of C at W*, if it exists, is a category $\mathsf{C}[W^{-1}]$ together with a functor $F : \mathsf{C} \to \mathsf{C}[W^{-1}]$ satisfying the following universal property. If C' is another category, and $G : \mathsf{C} \to \mathsf{C}'$ is a functor, then G factors through F if, and only if every morphism in W is sent into an isomorphism by G.*

Remark 3.2.2. *This is very closely related with localizations in rings and modules over rings.*

One can try to construct the localized category $\mathsf{C}[W^{-1}]$ by taking the category with the same objects as C, and as morphisms the words formed by composable morphisms of C and formal inverses of morphisms in W, and then identifying the word fg with the composite of f and g whenever f, g are in C, the words ww^{-1} and $w^{-1}w$ with the identity whenever $w \in W$ and w^{-1} is its formal inverse, and the letter w^{-1} with the inverse of w whenever $w \in W$ is invertible. The problem with this process is that it adds a potentially huge amount of morphisms to C. So many, in fact, that one easily incurs in set theoretical issues that make it so that the structure $\mathsf{C}[W^{-1}]$ defined this way is no longer a category.

3.2.2 The homotopy category of a model category

Now suppose that C is a model category. Then one can localize C at the weak equivalences, and moreover one can give a relatively small equivalent model for the resulting category.

Definition 3.2.3. *The* homotopy category *of C is the category $\mathrm{Ho}(\mathsf{C})$ which has as objects the bifibrant objects of C, and if $X, Y \in \mathsf{C}$ are two such objects, then*

$$\hom_{\mathrm{Ho}(\mathsf{C})}(X,Y) \coloneqq \hom_{\mathsf{C}}(X,Y)/\sim$$

is the quotient of the set of morphisms from X to Y by the homotopy relation.

Theorem 3.2.4. *Suppose that* C *is a model category. Then the localization* $\mathsf{C}[W^{-1}]$ *of* C *at the class of weak equivalences exists, and moreover there is an equivalence of categories*

$$\mathrm{Ho}(\mathsf{C}) \longrightarrow \mathsf{C}[W^{-1}]$$

between the homotopy category of C *and this localization.*

In particular, we have the following result.

Lemma 3.2.5. *Suppose* $X_1, X_2, Y \in \mathsf{C}$ *are bifibrant, and let* $\phi : X_1 \to X_2$ *be a weak equivalence. Then the maps*

$$\phi^* : \hom_{\mathrm{Ho}(\mathsf{C})}(X_2, Y) \longrightarrow \hom_{\mathrm{Ho}(\mathsf{C})}(X_1, Y) \quad \textit{and} \quad \phi_* : \hom_{\mathrm{Ho}(\mathsf{C})}(Y, X_1) \longrightarrow \hom_{\mathrm{Ho}(\mathsf{C})}(Y, X_2)$$

are bijections.

Proof. This looks trivial with the statement to Theorem 3.2.4, but it is in fact a step of the proof of that result. See [Hov99, Prop. 1.2.5(iv)] for a proof. □

3.2.3 Quillen functors and Quillen equivalences

One is now interested in understanding under what conditions a functor between two model categories induces a functor on the homotopy categories, and in particular when two categories have equivalent homotopy categories[3]. The correct notions giving an answer to these questions are *Quillen functors*, *Quillen adjunctions*, and *Quillen equivalences*. We will not use these concepts a lot, and so we refer the reader to [Hov99, Sect. 1.3] for details.

3.3 Examples

Examples of model categories abound. In this section, we give a few ones. Some of them are standard and we write them down for completeness, while others are less well known. The model structures on chain complexes, operads, cooperads, algebras over operads, and coalgebras over cooperads will be of interest in the rest of the present work.

3.3.1 Topological spaces

The first example we give, which is the one on which model categories themselves are modeled upon, is the category of topological spaces.

Theorem 3.3.1 (Quillen)**.** *The following three classes of continuous maps make* Top *into a model category.*

1. *A continuous map* $f : X \to Y$ *is a* weak equivalence *if all of the induced maps on the homotopy groups*

$$\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y) \qquad \textit{and} \qquad \pi_n(f, x) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

are isomorphisms, for all $x \in X$ *and all* $n \geq 1$.

[3] This can happen even when the two model categories are not equivalent themselves.

2. *A continuous map is a* fibration[4] *if it has the right lifting property with respect to all the inclusions* $D^n \to D^n \times I$ *of the* n*-disk into the product of the* n*-disk with an interval given by sending* $x \in D^n$ *to* $(x, 0) \in D^n \times I$*.*

3. *A continuous map is a* cofibration *if it has the left lifting property with respect to all trivial fibrations.*

As already mentioned in Remark 3.1.10, a cylinder object for a topological space X is given by $X \times I$, and a path object is given by X^I.

3.3.2 Simplicial sets

Simplicial sets are certain combinatorial objects whose homotopy theory is equivalent to the one of topological spaces. Namely, the category sSets of simplicial sets admits a model structure[5] and a Quillen equivalence with the model category of topological spaces. This will be discussed in more detail in Chapter 4.

3.3.3 Chain complexes

Since model categories give us "non-linear homological algebra[6]", one would like to recover the usual homological algebra when treating chain complexes. This is indeed possible, by considering the following model structure on Ch.

Theorem 3.3.2. *The following three classes of chain maps make the category* Ch *of chain complexes*[7] *into a model category.*

1. *A chain map* $f : V \to W$ *is a* weak equivalence *if it is a quasi-isomorphism, i.e. if all the induced maps*
$$H_n(f) : H_n(V) \to H_n(W)$$
are isomorphisms, for all $n \in \mathbb{Z}$*.*

2. *A chain map is a* fibration *if it is surjective in every degree.*

3. *A chain map is a* cofibration *if it has the left lifting property with respect to all trivial fibrations.*

In particular, all chain complexes are fibrant.

Since we are working over a field, we have even more.

Proposition 3.3.3. *All chain complexes are cofibrant.*

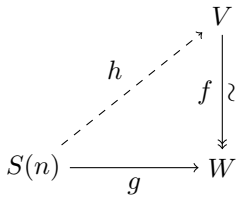
Proof. For $n \in \mathbb{Z}$, denote by
$$D(n) \coloneqq \mathbb{K}x_n \oplus \mathbb{K}y_{n-1} \qquad \text{and} \qquad S(n) \coloneqq \mathbb{K}z_n$$
the chain complexes with $|x_n| = |z_n| = n$, $|y_n| = n$, and $dx_n = y_{n-1}$. Since $\mathbb{K}$ is a field, every chain complex can be written as a colimit of copies of $D(n)$ and $S(n)$, with varying n. Since the colimit of cofibrant objects is cofibrant, it is enough to show that $D(n)$ and $S(n)$ are cofibrant. Fix any trivial fibration $f : V \to W$, i.e. a surjective chain map which induces an isomorphism in homology. Given

[4] Also known as a *Serre fibration* in the literature.

[5] In fact, it admits more than one. Here, we mean the classical — or Quillen — model structure, as opposed to the Joyal model structure, which is very important in the world of ∞-categories.

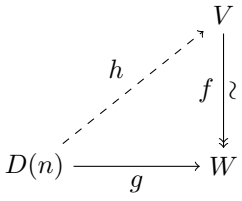
[6] See the very beginning of the introduction to [Qui67].

[7] Chain complexes do not have any boundedness assumption throughout this work, unless explicitly stated.



we have the existence of the dashed lift h. Indeed, $g(z_n)$ represents a class in the homology of W. Since f is bijective in homology, there exists a closed element $v \in V$ such that $f(v) = g(z_n) + dw$, for some $w \in W_{n+1}$. Let $\widetilde{w} \in V_{n+1}$ be any preimage of w under f, whose existence is guaranteed by the fact that f is surjective. Then $h(z_n) \coloneqq v - d\widetilde{w}$ is a well-defined chain map lifting g.

Similarly, given



the dashed lift h exists. To see this, simply take any preimage v of $g(x_n)$ under f, and set $h(x_n) \coloneqq v$, and $h(y_n) \coloneqq dv$. □

For details about this model structure, as well as references to the original literature, we invite the reader to take a look at [Hov99, Sect. 2.3].

3.3.4 Operads

In his article [Hin97b, Sect. 6], Hinich introduced a model structure on the category of operads in chain complexes. It is given as follows.

Theorem 3.3.4 ([Hin97b, Thm. 6.1.1])**.** *There is a closed model structure on the category* Op *of operads with*

- *the arity-wise quasi-isomorphisms as weak equivalences, and*
- *the arity-wise surjections as fibrations.*

The cofibrations are the morphisms that have the left lifting property with respect to the class of trivial fibrations. In particular, all operads are fibrant in this model structure.

We call this model structure the *Hinich model structure* on operads.

Proposition 3.3.5 ([MV09, Prop. 38[8]])**.** *The cofibrant objects in the category of operads are the retracts of quasi-free operads[9] whose generating $\mathbb{S}$-module M admits an exhaustive filtration*

$$M_0 \coloneqq \{0\} \subseteq \mathscr{F}^1 M \subseteq \mathscr{F}^2 M \subseteq \cdots \subseteq M = \operatorname*{colim}_n \mathscr{F}^n M$$

such that $d(\mathscr{F}^i M) \subseteq \mathscr{T}(\mathscr{F}^{i-1} M)$ and such that the inclusions $\mathscr{F}^i M \hookrightarrow \mathscr{F}^{i+1} M$ are split monomorphisms whose cokernel is isomorphic to a free $\mathbb{S}$-module.

[8] This is Proposition 95 in the arXiv version of the article.

[9] Recall that a quasi-free operad is an operad of the form $\mathscr{T}(M)$ for some $\mathbb{S}$-module M, endowed with some differential.

The following corollary motivates various constructions seen in Chapter 2.

Corollary 3.3.6. *Let $\mathscr{P}$ be an operad. The bar-cobar resolution $\Omega \mathrm{B} \mathscr{P}$ is a functorial cofibrant resolution of $\mathscr{P}$. Moreover, if $\mathscr{P}$ is Koszul, then the minimal resolution $\mathscr{P}_\infty$ of $\mathscr{P}$ also is a cofibrant resolution of $\mathscr{P}$.*

3.3.5 Cooperads

There are various model structures one can put on the category of cooperads. A good example, in the same spirit as the Vallette model structure that we will see in Section 3.3.7, can be found in [LG17, Sect. 3].

3.3.6 Algebras over operads

A model structure on algebras over an operad was also given by Hinich in [Hin97b, Sect. 4]. Notice that, since our base field $\mathbb{K}$ has characteristic 0, every operad is $\mathbb{S}$-split. Fix an operad $\mathscr{P}$.

Theorem 3.3.7 ([Hin97b, Thm. 4.1.1])**.** *There is a closed model structure on the category $\mathscr{P}$-alg of $\mathscr{P}$-algebras with*

- *the quasi-isomorphisms of $\mathscr{P}$-algebras as weak equivalences, and*
- *the surjective morphisms of $\mathscr{P}$-algebras as fibrations.*

The cofibrations are the morphisms that have the left lifting property with respect to the class of trivial fibrations. In particular, all $\mathscr{P}$-algebras are fibrant in this model structure.

We call this model structure the *Hinich model structure* on $\mathscr{P}$-algebras.

Lemma 3.3.8. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a Koszul twisting morphism. Then every $\mathscr{P}$-algebra of the form $\Omega_\alpha C$, where C is a conilpotent $\mathscr{C}$-coalgebra, is cofibrant.*

Proof. This follows immediately from Theorem 3.3.9. □

3.3.7 The Vallette model structure for coalgebras over a cooperad

Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. The Hinich model structure on $\mathscr{P}$-algebra can be transfered along the relative cobar functor Ω_α to give a closed model structure on $\mathscr{C}$-coalgebras. This was done by Vallette [Val14] in the case where $\mathscr{P}$ is a Koszul operad and $\alpha = \kappa$ is the twisting morphism given by Koszul duality, and then generalized by Drummond-Cole–Hirsch [DCH16] and Le Grignou [LG16].

The main results are the following ones.

Theorem 3.3.9. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism.*

1. [Val14, Thm. 2.1(1)] and [LG16, Thm. 10 and Prop. 26] *There is a closed model structure on the category* conil.$\mathscr{C}$-cog *of conilpotent $\mathscr{C}$-coalgebras with*
   - *the morphisms $g : C \to D$ of $\mathscr{C}$-coalgebras such that $\Omega_\alpha g$ is a quasi-isomorphism as weak equivalences, and*
   - *the injective morphisms of $\mathscr{C}$-coalgebras as cofibrations.*

The fibrations are the morphisms that have the right lifting property with respect to the class of trivial cofibrations. In particular, all $\mathscr{C}$-coalgebras are cofibrant in this model structure.

2. [LG16, Prop. 32] *If $\alpha' = f\alpha$ with $f : \mathscr{P} \to \mathscr{P}'$ a quasi-isomorphism of operads, then the model structure induced by α' coincides with the model structure induced by α.*

3. [Val14, Thm. 2.1(2)] and [LG16, Thm. 14] *Let $\iota : \mathscr{C} \to \Omega\mathscr{C}$ be the canonical twisting morphism. If α is Koszul, then the fibrant objects are exactly the conilpotent $\mathscr{C}$-coalgebras isomorphic to a $\mathscr{C}$-coalgebra of the form $\mathrm{B}_\iota A$ for some $\Omega\mathscr{C}$-algebra A.*

4. *Every $\mathscr{C}$-coalgebra of the form $\mathrm{B}_\alpha A$ for A any $\mathscr{P}$-algebra is fibrant.*

5. [Val14, Thm. 2.1(3)] and [LG16, Thm. 13] *The bar-cobar adjunction relative to $\iota : \mathscr{C} \to \Omega\mathscr{C}$*

$$\Omega_\iota : \mathsf{conil}.\ \mathscr{C}\text{-}\mathsf{cog} \rightleftarrows \Omega\mathscr{C}\text{-}\mathsf{alg} : \mathrm{B}_\iota$$

is a Quillen equivalence.

Proof. The only point which might not be immediately clear from the references given above is the point (4). By Theorem 2.2.21, the twisting morphism α splits into $\alpha = g_\alpha \iota$. Then one has

$$\mathrm{B}_\alpha A = \mathrm{B}_\iota(g_\alpha^* A)$$

by Lemma 8.1.3, and we conclude by point (3). □

We call this model structure the *Vallette model structure* on conilpotent $\mathscr{C}$-coalgebras. When α is Koszul[10], we can also completely characterize the class of weak equivalences. All the accessory results we need were proven[11] in [Val14], but the conclusion is original work.

Definition 3.3.10. *Let C be a conilpotent $\mathscr{C}$-coalgebra. A* cofiltration *on C is a sequence of sub-chain complexes*

$$0 = \mathscr{F}^0 C \subseteq \mathscr{F}^1 C \subseteq \mathscr{F}^2 C \subseteq \ldots \subseteq C$$

satisfying [Val14, Prop. 2.2]. Namely, we require that

1. *it respects the coproduct, that is*

$$\overline{\Delta}_C(\mathscr{F}^n C) \subseteq \bigoplus_{\substack{k \geq 1 \\ n_1 + \cdots + n_k = n}} (\mathscr{C}(k) \otimes \mathscr{F}^{n_1} C \otimes \cdots \otimes \mathscr{F}^{n_k} C)^{\mathbb{S}_k},$$

   where $\overline{\Delta}_C(c) \coloneqq \Delta_C(c) - c$, and

2. *it is preserved by the differential: $d_C(\mathscr{F}^n C) \subseteq \mathscr{F}^n C$.*

Moreover, if

3. *the cofiltration is exhaustive: $\mathrm{colim}_n \mathscr{F}^n C \cong C$,*

then we say that C is a cocomplete coalgebra.

Remark 3.3.11. *This is also called a filtered coalgebra in the literature. To attempt to have more clarity, we decided to call filtrations the descending filtrations and cofiltrations the ascending filtrations.*

[10] In which case, by Theorem 2.2.21 and Theorem 3.3.9(2), reduces to studying the model structure induced by the Koszul morphism $\iota : \mathscr{C} \to \Omega\mathscr{C}$.

[11] For $\mathscr{P}$ a Koszul morphism and $\alpha = \kappa : \mathscr{P}^{\mathrm{i}} \to \mathscr{P}_\infty$, but the proofs readily generalize to the slightly more general case we desire.

Example 3.3.12. *A cofiltration that exists for any conilpotent $\mathscr{P}^{\mathrm{i}}$-coalgebra C is the coradical filtration introduced in Section 2.1.3.*

Definition 3.3.13. *Let C_1, C_2 be two conilpotent $\mathscr{C}$-coalgebras, and let $\phi : C_1 \to C_2$ be a morphism of $\mathscr{C}$-coalgebras.*

1. *Let $\mathscr{F}^\bullet C_1$ and $\mathscr{F}^\bullet C_2$ be filtrations. The morphism ϕ is* cofiltered *with respect to the cofiltrations if for each $n \geq 0$ we have $\phi(\mathscr{F}^n C_1) \subseteq \mathscr{F}^n C_2$.*
2. *The morphism ϕ is a* cofiltered quasi-isomorphism *if it is a quasi-isomorphism and if there are cocomplete cofiltrations $\mathscr{F}^\bullet C_1$ and $\mathscr{F}^\bullet C_2$ such that ϕ is cofiltered with respect to the cofiltrations, and such that for each $n \geq 0$, the morphism ϕ induces a quasi-isomorphism*

$$\mathscr{F}^n C_1/\mathscr{F}^{n-1} C_1 \longrightarrow \mathscr{F}^n C_2/\mathscr{F}^{n-1} C_2 \ .$$

Remark 3.3.14. *Notice that the coradical filtration is final, in the sense that if we put the coradical filtration on C_1 and any cofiltration on C_2 then any morphism of $\mathscr{C}$-coalgebras will be cofiltered.*

The following result was proven in [RN18], even though all the ingredient of the proof were already present in [Val14].

Theorem 3.3.15 ([RN18, Thm. 4.9])**.** *The class W of weak equivalence is the smallest class of arrows of* conil. $\mathscr{C}$-cog *containing all cofiltered quasi-isomorphisms and which is closed under the 2-out-of-3 property.*

The proof of this theorem is similar to what found in [Pos11, Sect. 9.3]. Before going on, we need a couple of preliminary results, all of which come from [Val14].

Lemma 3.3.16. *Let f be a cofiltered quasi-isomorphism of conilpotent $\mathscr{C}$-coalgebras. Then the morphism of $\Omega\mathscr{C}$-algebras $\Omega_\iota f$ is a quasi-isomorphism.*

Proof. This follows from the proof of [Val14, Prop. 2.3]. □

Lemma 3.3.17. *Let f be a quasi-isomorphism of $\Omega\mathscr{C}$-algebras. Then $\mathrm{B}_\iota f$ is a cofiltered quasi-isomorphism of conilpotent $\mathscr{C}$-coalgebras.*

Proof. This is [Val14, Prop. 2.4]. □

Lemma 3.3.18. *Let C be a conilpotent $\mathscr{C}$-coalgebra. Then the unit map*

$$\eta_C : \mathrm{B}_\iota \Omega_\iota C \longrightarrow C$$

is a cofiltered quasi-isomorphism.

Proof. This is a consequence of the proof of [Val14, Thm. 2.6]. □

Proof of Theorem 3.3.15. We denote by Fqi the smallest class of arrows in conil. $\mathscr{C}$-cog which contains all cofiltered quasi-isomorphisms and which is closed under the 2-out-of-3 property.

Lemma 3.3.16 implies that that $\mathrm{Fqi} \subseteq W$. To prove the other inclusion, let

$$f : C \longrightarrow D$$

be a morphism of conilpotent $\mathscr{C}$-coalgebras such that $\Omega_\iota f$ is a quasi-isomorphism, that is to say $f \in W$. We consider the diagram

$$\begin{array}{ccc} \mathrm{B}_\iota\Omega_\iota C & \xrightarrow{\mathrm{B}_\iota\Omega_\iota f} & \mathrm{B}_\iota\Omega_\iota D \\ \downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_D} \\ C & \xrightarrow{f} & D \end{array}$$

where by Lemma 3.3.17 the arrow $\mathrm{B}_\iota\Omega_\iota f$ is a filtered quasi-isomorphism, and by Lemma 3.3.18 both vertical arrows are also cofiltered quasi-isomorphisms. A double application of the 2-out-of-3 property proves that $f \in \mathrm{Fqi}$, concluding the proof. □

The following result is a refinement of one direction of Theorem 2.4.11, and provides the "good" dual version to Theorem 2.4.8.

Corollary 3.3.19. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a Koszul morphism. Then the unit*

$$\eta_C : C \longrightarrow \mathrm{B}_\alpha\Omega_\alpha C$$

of the bar-cobar adjunction relative to α is a weak equivalence in the Vallette model structure, for any conilpotent $\mathscr{C}$-coalgebra C. In particular, the bar-cobar adjunction provides a fibrant resolution for conilpotent $\mathscr{C}$-coalgebras.

Proof. The proof of [Val14, Thm. 2.6] works in this case and shows that the unit η_C is a cofiltered quasi-isomorphism. Theorem 3.3.15 concludes the proof. □

3.3.8 Algebras with ∞-morphisms

Let $\mathscr{P}$ be a Koszul operad, and let $\mathscr{P}_\infty$ be its minimal model. The category of $\mathscr{P}_\infty$-algebra and their ∞-morphisms can be identified as the full subcategory of the category of conilpotent $\mathscr{P}^{\text{¡}}$-coalgebras given by quasi-free $\mathscr{P}^{\text{¡}}$-coalgebras via the bar functor B_ι, cf. Proposition 2.4.3. This way, one immediately sees that the category of $\mathscr{P}_\infty$-algebra and their ∞-morphisms cannot be a model category: it is not complete since for example products of quasi-free $\mathscr{P}^{\text{¡}}$-coalgebras are not necessarily quasi-free, cf. Proposition 2.1.38. However, one can use the Vallette model structure on coalgebras to speak of the homotopy theory of ∞-morphisms of $\mathscr{P}_\infty$-algebras. One simply defines two ∞-morphisms to be homotopic if they are when seen as morphisms of $\mathscr{P}^{\text{¡}}$-coalgebras. More details of this theory are given in [Val14, Sect. 3].

3.4 Framings

The idea of framings is to give a simplicial or cosimplicial resolution of an object in a category, i.e. a "nice" simplicial, resp. cosimplicial, object in that category whose 0-simplices are given by the original object we wanted to study. We present here the theory of simplicial framings, as it is what we will need later. Cosimplicial framings are the dual concept. We assume some knowledge about simplicial sets, see also Chapter 4. The material presented here is extracted from [Hov99, Ch. 5].

3.4.1 The model structures on simplicial objects

Let C be a category. Recall that a simplicial object is a functor

$$X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathsf{C}$$

from the ordinal number category to C. Cosimplicial objects in C form a category sC, whose morphisms are given by the natural transformations.

Definition 3.4.1. *Let C be a category with all finite limits and colimits. Let $X_\bullet \in \mathsf{sC}$ be a simplicial object in C, and let $n \geq 0$ be an integer.*

1. *The n*th latching object *$L_n X_\bullet \in \mathsf{C}$ of $X_\bullet$ is the union of all the degenerate n-cells of $X_\bullet$. In particular, $L_0 X_\bullet$ is the initial object, and $L_1 X_\bullet = K_0$.*

2. *The n*th matching object *$M_n X_\bullet \in \mathsf{C}$ of $X_\bullet$ is $M_n X_\bullet := K_\bullet^{\partial\Delta[n]}$, i.e. the limit of $X_\bullet$ over the diagram of all simplices of $\partial\Delta[n]$. In particular, $M_0 X_\bullet$ is the terminal object, and $L_1 X_\bullet = X_0 \times X_0$.*

For each $n \geq 0$, we have natural maps $L_n X_\bullet \to X_n \to M_n X_\bullet$. Moreover, any morphism of simplicial objects induces natural morphisms between the latching and matching objects.

One can use latching and matching objects to define a model structure on the category of simplicial objects in a model category.

Theorem 3.4.2. *Let C be a model category. The following classes of maps define a model structure — the* Reedy model structure *— on the category sC of simplicial objects in C.*

- *The weak equivalences are the level-wise weak equivalences. That is to say that a morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence in C for every $n \geq 0$.*
- *The (trivial) cofibrations are the morphisms $f_\bullet : X_\bullet \to Y_\bullet$ such that the induced morphisms*

  $$X_i \sqcup_{L_i X_\bullet} L_i Y_\bullet \longrightarrow Y_i$$

  are (trivial) cofibrations for all $n \geq 0$.
- *The (trivial) fibrations are the morphisms $f_\bullet : X_\bullet \to Y_\bullet$ such that the induced morphisms*

  $$X_i \longrightarrow Y_i \times_{M_i Y_\bullet} M_i X_\bullet$$

  are (trivial) fibrations for all $n \geq 0$.

3.4.2 Simplicial framings

Let C be a model category. Suppose we are given an object $x \in \mathsf{C}$, which we would like to see as X_0 for a simplicial object $X_\bullet \in \mathsf{sC}$. There are two natural choices for the whole object $X_\bullet$.

1. One defines $\ell_\bullet x \in \mathsf{sC}$ by

   $$\ell_n x := x$$

   for all n, with the identity map for all boundary and degeneracy maps.

2. One defines $r_\bullet x \in \mathsf{sC}$ by

   $$r_n x := x \times \cdots \times x \ ,$$

   the product of $n+1$ copies of x. The face maps are given by the projections forgetting one of the factors, and the degeneracy maps are induced by the diagonal map $x \to x \times x$.

Definition 3.4.3. *A* simplicial frame $X_\bullet$ *for* $x \in \mathsf{C}$ *is a factorization*

$$\ell_\bullet x \xrightarrow{\sim} X_\bullet \twoheadrightarrow r_\bullet x$$

of the natural map $\ell_\bullet x \to r_\bullet x$ *into a weak equivalence followed by a fibration.*

Notice that simplicial frames always exist by (**M5**). We will need the following result in Chapter 12.

Proposition 3.4.4. *Suppose* x *is a fibrant object in* C *and let* $X_\bullet$ *be a simplicial frame on* x*. Then the functor*

$$\hom_{\mathsf{C}}(-, X_\bullet) : \mathsf{C}^{op} \longrightarrow \mathsf{sSets}$$

preserves fibrations, trivial fibrations, and weak equivalences between fibrant objects.

Proof. The fact that the functor preserves fibrations and trivial fibrations is proven in [Hov99, Cor. 5.4.4(2)]. The fact that it preserves weak equivalences between fibrant objects then follows by Ken Brown's lemma, Lemma 3.1.7. □

Chapter 4

Simplicial homotopy theory

Simplicial sets are combinatorial objects that can be used to study the homotopy theory of topological spaces. In fact, they carry a model structure with which they are Quillen equivalent to topological spaces. Therefore, in modern algebraic topology they are often used as models for spaces instead of topological spaces themselves.

In this chapter, we begin by giving a rapid definition of the category of simplicial sets. Then we present the model structure on simplicial sets and the Quillen equivalence between simplicial sets and topological spaces. Finally, we give a rapid overview of the Dold–Kan correspondence.

The material of this chapter is extracted from [GJ09], and in particular Chapters I and III in *op. cit.*

4.1 Simplicial sets

We begin by giving an introduction to simplicial sets as the category of presheaves of sets, i.e. the category of contravariant functors from the ordinal number category Δ to the category Sets of sets.

4.1.1 The ordinal number category

The most basic object encoding all of the combinatorial information of simplicial sets is the following category.

Definition 4.1.1. *The* ordinal number category Δ *is the category whose objects are the ordered sets*

$$[n] \coloneqq \{0 < 1 < \cdots < n\} ,$$

for $n \in \mathbb{N}$, and whose morphisms are the order preserving[1] *maps.*

There are two classes of special morphisms in the ordinal number category.

Definition 4.1.2. *Let $n \geq 1$. The i*th coface map*, $0 \leq i \leq n$, is the unique morphism*

$$d^i : [n-1] \longrightarrow [n]$$

[1] Not strictly, meaning that if $i < j$, then we only require that such a map ϕ satisfies $\phi(i) \leq \phi(j)$.

*which is injective and skips i. The j*th codegeneracy map, *$0 \leq j \leq n$, is the unique morphism*

$$s^j : [n] \longrightarrow [n+1]$$

which is surjective and takes the value j twice.

The cofaces and codegeneracies generate all morphisms in Δ, in the sense that any morphism in Δ can be written in terms of compositions of cofaces and codegeneracies. They satisfy certain relations between them, commonly called the *cosimplicial identities*, see [GJ09, p. 4].

4.1.2 Simplicial objects in a category and simplicial sets

Functors from the ordinal number category Δ and its opposite category to any category have a nice combinatorial structure.

Definition 4.1.3. *Let* C *be a category. A* simplicial object *in* C *is a functor*

$$\Delta^{\mathrm{op}} \longrightarrow \mathsf{C} \, .$$

The category sC *of simplicial objects in* C *is the category whose objects are simplicial objects in* C*, and whose morphisms are the natural transformations between simplicial objects.*

Definition 4.1.4. *Dually, a* cosimplicial object *in* C *is a functor*

$$\Delta \longrightarrow \mathsf{C} \, .$$

The case of interest to us for the moment is when we take $\mathsf{C} = \mathsf{Sets}$.

Definition 4.1.5. *The category* sSets *of* simplicial sets *is the category of simplicial objects in the category of sets.*

Let $K_\bullet \in \mathsf{sSets}$ be a simplicial set. We denote by

$$K_n \coloneqq K_\bullet([n])$$

the *set of n-simplices* of $K_\bullet$. The cofaces and codegeneracies of the category Δ naturally induce maps between the various sets of simplices of a simplicial set, called the *face maps* d_i and *degeneracy maps* s_j respectively. A simplex in K_n is *degenerate* if it is in the image of a degeneracy map.

4.1.3 Limits and colimits

Limits and colimits are well behaved in the category of simplicial sets.

Proposition 4.1.6. *The category* sSets *of simplicial sets is complete and cocomplete, and limits and colimits are taken level-wise.*

In other words, if I is an index category, and $L : I \to \mathsf{sSets}$ is a functor, then

$$\left(\lim_{i \in I} L(i) \right)_n = \lim_{i \in I} L(i)_n \, ,$$

and similarly for colimits. In particular, the initial object in the category of simplicial sets is the empty simplicial set $\emptyset$, and the final object is the point $* \coloneqq \Delta[0]$.

4.1.4 The Yoneda embedding, boundaries and horns

The Yoneda embedding embeds Δ as a subcategory of sSets.

Definition 4.1.7. *We define a functor*

$$\Delta[-] : \Delta \longrightarrow \mathsf{sSets}$$

by

$$\Delta[n] \coloneqq \hom_{\Delta}(-, [n]) .$$

The simplicial sets $\Delta[n]$ are the "building blocks" of all simplicial sets.

Lemma 4.1.8. *Let $K_\bullet$ be a simplicial set. Then*

$$K_n \cong \hom_{\mathsf{sSets}}(\Delta[n], K_\bullet) .$$

Proof. This follows immediately from the Yoneda lemma, e.g. [ML70, p. 61]. □

There are some other very important simplicial sets that can be built from the $\Delta[n]$.

Definition 4.1.9. *The* boundary *$\partial\Delta[n]$ of the simplicial set $\Delta[n]$ is the simplicial set obtained by gluing all of the non-degenerate $(n-1)$-simplices of $\Delta[n]$ along of their faces, whenever the faces of two such $(n-1)$-simplices coincide. In other words, it is given as the coequalizer*

$$\coprod_{[n-2]\to[n] \text{ inj.}} \Delta[n-2] \rightrightarrows \coprod_{[n-1]\to[n] \text{ inj.}} \Delta[n-1] \longrightarrow \partial\Delta[n] .$$

Definition 4.1.10. *The i*th horn *$\Lambda^i[n]$ of $\Delta[n]$ is the simplicial set defined the same way as $\partial\Delta[n]$ but without the $(n-1)$-simplex opposite to the ith 0-simplex of $\Delta[n]$. In other words, it is given as the coequalizer*

$$\coprod_{\substack{[n-2]\to[n] \text{ inj.} \\ i \text{ in the image}}} \Delta[n-2] \rightrightarrows \coprod_{\substack{[n-1]\to[n] \text{ inj.} \\ i \text{ in the image}}} \Delta[n-1] \longrightarrow \Lambda^i[n] .$$

4.2 The model structure on simplicial sets

In this section, we present the model structure on simplicial sets. The main theorem (which we will not prove here) is the following.

Theorem 4.2.1. *The classes of cofibrations, fibrations, and weak equivalences that will be defined in Definitions 4.2.2, 4.2.3 and 4.2.7 define a closed model structure on the category* sSets *of simplicial sets.*

4.2.1 Cofibrations

Cofibrations are easy.

Definition 4.2.2. *A morphism $f : K_\bullet \to L_\bullet$ of simplicial sets is a* cofibration *if it is level-wise injective, i.e. if all of the maps of sets $f_n : K_n \to L_n$ are injective.*

In particular, all simplicial sets are cofibrant.

4.2.2 Fibrations and Kan complexes

Fibrations are defined by right lifting property with respect to a certain set of generating cofibrations, namely the inclusions of horns.

Definition 4.2.3. *A morphism $f : K \to L$ of simplicial sets is a* fibration *if it satisfies the right lifting property with respect to all the natural inclusions $\Lambda^i[n] \hookrightarrow \Delta[n]$ for $0 \leq i \leq n$ and all $n \geq 2$.*

Definition 4.2.4. *Fibrant simplicial sets are called* Kan complexes.

Many naturally arising simplicial sets are Kan. Examples are the nerve of a group, the underlying simplicial set of a simplicial abelian group, and the singular complex of a topological space — which we will see in more detail later. This last fact motivates the heuristic that Kan complex correspond to "spaces".

4.2.3 Realization and weak equivalences

Let CGHaus be the category of compactly generated Hausdorff[2] topological spaces. Homotopically speaking, the category CGHaus is essentially the same as the whole category Top of topological spaces, since CW-complexes are in CGHaus. However, notice that for example products in CGHaus are not the same as products in Top.

Definition 4.2.5. *For $n \geq 0$, the* geometric n-simplex *is the topological space*

$$\Delta^n \coloneqq \left\{ (x_0, \ldots, x_n) \;\middle|\; \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0 \; \forall 0 \leq i \leq n \right\} \subset \mathbb{R}^{n+1} .$$

Notice that $\Delta^n \in$ CGHaus. The topological space Δ^n is the geometric analogue of the simplicial set $\Delta[n]$, and since all simplicial sets can be constructed by gluing spaces of the form $\Delta[n]$ together, we try to recover topological spaces by gluing geometric simplices together.

Definition 4.2.6. *The* geometric realization *is the functor*

$$|-| : \mathsf{sSets} \longrightarrow \mathsf{CGHaus}$$

given by sending a simplicial set $K \in$ sSets to the topological space

$$|K| \coloneqq \operatorname*{colim}_{\Delta[n] \to K} \Delta^n.$$

In other words, given $K \in$ sSets we take one copy of Δ^n for each n-simplex in K, and then glue all those geometric simplices together according to the face maps in K.

Definition 4.2.7. *A morphism $f : K \to L$ of simplicial sets is a* weak equivalence *if its geometric realization*

$$|f| : |K| \longrightarrow |L|$$

is a weak equivalence of simplicial sets, i.e. if it induces an isomorphism on all homotopy groups.

[2] Also known as T_2.

4.2.4 Quillen equivalence between simplicial sets and topological spaces

The geometric realization functor has a right adjoint, the *singular set* functor

$$S_\bullet : \mathsf{CGHaus} \longrightarrow \mathsf{sSets} ,$$

which is defined as

$$S_\bullet(X) := \hom_{\mathsf{Top}}(\Delta^\bullet, X) ,$$

the simplicial set of singular simplices of a topological space $X \in \mathsf{CGHaus}$.

Proposition 4.2.8. *Let $K \in \mathsf{sSets}$ and $X \in \mathsf{CGHaus}$. There is a natural isomorphism*

$$\hom_{\mathsf{Top}}(|K|, X) \cong \hom_{\mathsf{sSets}}(K, S_\bullet(X)) .$$

There is more. This adjunction induces an equivalence of categories between the homotopy categories of simplicial sets and compactly generated, Hausdorff space, showing that their homotopy theory is "the same".

Theorem 4.2.9. *The adjunction given by the geometric realization and the singular set functor is a Quillen equivalence.*

Proof. See e.g. [Hov99, Thm. 3.6.7]. □

This motivates the fact that one can usually replace topological spaces by simplicial sets when doing homotopy theory, in order to work with objects that have a nicer combinatorial behavior.

4.2.5 Homotopy groups of simplicial sets

Let K be a Kan complex, i.e. a fibrant simplicial set. Define $\pi_0(K)$ to be the set of homotopy classes of vertices of K. Let $x \in K_0$ be a vertex of K. For $n \geq 1$, we define $\pi_n(K, x)$ to be the set of homotopy classes rel $\partial\Delta[n]$ of maps $\alpha : \Delta[n] \to K$ fitting into the diagram

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \Delta[0] \\ \downarrow & & \downarrow x \\ \Delta[n] & \xrightarrow[\alpha]{} & K \end{array}$$

In other words, α must map the boundary $\partial\Delta[n]$ of $\Delta[n]$ to the basepoint x. The sets $\pi_n(K, x)$, $n \geq 1$, are endowed with a group structure as follows. Let $\alpha, \beta : \Delta[n] \to K$ be as above. Then one constructs a morphism of simplicial sets

$$(\alpha, \beta) : \Lambda^n[n+1] \longrightarrow K$$

by setting it to be constant with value x on all faces, except the $(n-1)$th face, where it has value α, and the $(n+1)$th, where it has value β. Since K is Kan, we have a lift

$$\begin{array}{ccc} \Lambda^n[n+1] & \xrightarrow{(\alpha,\beta)} & K \\ \downarrow & \nearrow_{\omega} & \\ \Delta[n+1] & & \end{array}$$

It is straightforward to check that $d_n\omega$, the "missing" n-face of $\Lambda^n[n+1]$ recovered by filling, is constant on its boundary with value x, and thus defines an element of $\pi_n(K,x)$, which we will denote by $\alpha * \beta$. The following lemma tells us that $\alpha * \beta$ is well defined as an element of $\pi_n(K,x)$.

Lemma 4.2.10. *The homotopy class of $\alpha * \beta$ is independent of the choice of the lift ω.*

We also define $e \in \pi_n(K,x)$ to be the constant map with value x.

Theorem 4.2.11. *Taking $*$ as multiplication, and e as identity element, the $\pi_n(K,x)$ are groups for all $n \geq 1$, and $\pi_n(K,x)$ are abelian for every $n \geq 2$. We call $\pi_n(K,x)$ the n*th homotopy group of K *with basepoint x.*

We have the following very useful fact.

Theorem 4.2.12. *There is a canonical isomorphism*

$$\pi_0(K) \cong \pi_0(|K|)\,, \qquad \textit{and} \qquad \pi_n(K,x) \cong \pi_n(|K|,x)$$

for all $n \geq 1$. In other words, the homotopy groups of a Kan complex are the same as the homotopy groups of its geometric realization.

4.2.6 The long exact sequence associated to a fibration

It is possible to associate a long exact sequence of homotopy groups to a fibration of simplicial sets.

Definition 4.2.13. *Let $p : K \to L$ be a fibration of simplicial sets. The* fibre *of p over a point $x \in L_0$ is $i : F \to K$ defined by the pullback*

$$\begin{array}{ccc} F & \xrightarrow{\;i\;} & K \\ \downarrow & & \downarrow p \\ \Delta[0] & \xrightarrow{\;x\;} & L \end{array}$$

Suppose that $p : K \to L$ is a fibration with fibre $i : F \to K$ over $x \in L_0$. Fix a point $y \in F_0$, and by abuse of notation denote $i(y) \in K_0$ also by y. Let $\alpha : \Delta[n] \to L$ represent a homotopy class $[\alpha] \in \pi_n(L,x)$, and define an n-horn $\Lambda^0[n] \to L$ by sending everything to y. We have a lift

$$\begin{array}{ccc} \Lambda^0[n] & \longrightarrow & K \\ \wr\downarrow & \overset{\theta}{\nearrow} & \downarrow p \\ \Delta[n] & \xrightarrow{\;\alpha\;} & L \end{array}$$

and one can check that the element $\partial(\alpha) \coloneqq [d_0\theta] \in \pi_{n-1}(F,y)$ does not depend on the choice of the particular lift. Thus, we obtain a map

$$\partial : \pi_n(L, x = f(y)) \longrightarrow \pi_{n-1}(F,y)\,,$$

called the *boundary map*.

Theorem 4.2.14. *For all $n \geq 1$, the maps*

$$\partial : \pi_n(L, x) \longrightarrow \pi_{n-1}(F, y)$$

are group homomorphisms. They fit in a long sequence

$$\cdots \xrightarrow{\partial} \pi_n(F, y) \xrightarrow{i} \pi_n(K, y) \xrightarrow{p} \pi_n(L, x) \xrightarrow{\partial} \pi_{n-1}(F, y) \xrightarrow{i} \cdots$$
$$\cdots \xrightarrow{p} \pi_1(L, x) \xrightarrow{\partial} \pi_0(F, y) \xrightarrow{i} \pi_0(K, y) \xrightarrow{p} \pi_0(L, x)$$

which is exact in the sense that kernel equals image everywhere. Furthermore, $\pi_1(L, x)$ acts on $\pi_0(F)$, and two elements of $\pi_0(F)$ have the same image under i if, and only if they are in the same orbit of the $\pi_1(L, x)$-action.

4.3 The Dold–Kan correspondence

The celebrated Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and non-negatively graded chain complexes. In this section, we give a rapid overview of this very important result.

4.3.1 The normalized chain complex and the Moore complex of a simplicial abelian group

We start by reminding an important fact about simplicial groups, due to [Moo55, Thm. 3].

Theorem 4.3.1 (Moore)**.** *Let G be a simplicial group. Then the underlying simplicial set of G is a Kan complex.*

For the rest this section, A will always denote a simplicial abelian group.

Definition 4.3.2. *The* normalized chain complex *NA of a simplicial abelian group A is the non-negatively graded chain complex which is given in degree $n \geq 0$ by*

$$NA_n \coloneqq \bigcap_{i=0}^{n-1} \ker(d_i : A_n \to A_{n-1}) \subseteq A_n \ ,$$

and whose differential is given by d_n.

Remark 4.3.3. *For this section only, "chain complex" means chain complex of $\mathbb{Z}$-modules. Later, we will apply these constructions to simplicial $\mathbb{K}$-vector spaces instead of just simplicial abelian groups, thus obtaining non-negatively graded $\mathbb{K}$-chain complexes.*

It is straightforward to check that NA is a chain complex, i.e. that its differential squares to zero. This assignment is functorial, giving

$$N : \mathsf{sAb} \longrightarrow \mathsf{Ch}_+$$

from simplicial abelian groups to non-negatively graded chain complexes.

Another closely related construction is the Moore chain complex.

Definition 4.3.4. *The* Moore complex $\mathscr{M}(A)$ *of a simplicial abelian group* A *is the non-negatively graded chain complex given in degree* $n \geq 0$ *by*

$$\mathscr{M}(A)_n \coloneqq A_n \ ,$$

and whose differential is given by $\sum_{i=0}^{n}(-1)^i d_i$.

The relation between these constructions is given by the following result.

Theorem 4.3.5. *The natural inclusion*

$$i : NA \longrightarrow \mathscr{M}(A)$$

is a chain homotopy equivalence.

Therefore, the normalized chain complex and the Moore complex can be used interchangeably when doing homotopy theory. Moreover, these constructions allow one to readily compute the homotopy groups of A.

Theorem 4.3.6. *Let* A *be a simplicial abelian group. There are natural isomorphisms of groups*

$$\pi_0(A) \cong H_0(\mathscr{M}(A)) \ , \qquad \textit{and} \qquad \pi_n(A,0) \cong H_n(\mathscr{M}(A))$$

for all $n \geq 1$.

4.3.2 The Dold–Kan correspondence

This chapter would not be complete without at least mentioning the Dold–Kan correspondence.

There is a functor

$$\Gamma : \mathsf{Ch}_+ \longrightarrow \mathsf{sAb}$$

defined by

$$\Gamma_n(C) \coloneqq \bigoplus_{[n]\to[k] \text{ surj.}} C_k$$

and endowed with certain natural simplicial structure maps.

The following celebrated result is due to A. Dold and D. Kan, and is widely known as the Dold–Kan correspondence.

Theorem 4.3.7 (Dold–Kan)**.** *The functors*

$$N : \mathsf{sAb} \longrightarrow \mathsf{Ch}_+ \qquad \textit{and} \qquad \Gamma : \mathsf{Ch}_+ \longrightarrow \mathsf{sAb}$$

form an equivalence of categories.

Chapter 5

Rational homotopy theory

Studying the homotopy type of topological spaces is a very difficult problem — even for well-behaved ones: suffices to say that at the present time we don't know all the homotopy groups of spheres. The idea of rational homotopy theory is to simplify the problem by studying the *rational* homotopy groups of (nice) spaces. If X is a simply connected space, then all of its homotopy groups are abelian (since the fundamental group is trivial), so one considers

$$\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for} \quad n \geq 2 \,.$$

These groups are much easier to compute and to study than the homotopy groups $\pi_n(X)$. They are closely related to the rational homotopy groups $H_n(X;\mathbb{Q})$ of the space, the relation being stricter than the one between the usual homotopy groups and integer homology, and it is possible to give algebraic models for the spaces that completely encode their rational homotopy theory.

In other words, rational homotopy theory is "the study of the rational homotopy category, that is the category obtained from the category of 1-connected pointed spaces by localizing with respect to the family of those maps which are isomorphisms modulo the class in the sense of Serre of torsion abelian groups" — [Qui69, p. 205].

The author does not make any claims of being an expert on the domain of rational homotopy theory. This section is a naive introduction to the subject, based on the reference books [Qui69] and [FHT01].

5.1 Conventions and basic definitions

For the rest of this chapter, the base field will be the field of rational numbers $\mathbb{K} = \mathbb{Q}$. All vector spaces, chain complexes, algebras, etc. will always be over this field.

5.1.1 The rational homotopy category

We place ourselves in the category $\mathsf{Top}_{*,1}$ of simply-connected, pointed topological spaces and continuous pointed maps[1].

[1] This category is denoted by Top_2 in [Qui69], for "spaces beginning in dimension 2".

Definition 5.1.1. *Let $X \in \mathsf{Top}_{*,1}$. The* rational homotopy *of X is the graded $\mathbb{Q}$-vector space $\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ starting in degree 2. The* rational homology *of X is the graded $\mathbb{Q}$-vector space*[2] *$H_\bullet(X;\mathbb{Q})$.*

In this category, maps inducing isomorphisms in rational homotopy and maps inducing isomorphisms in rational homology are the same, as proven by Serre in [Ser53, Thm. 3].

Theorem 5.1.2 (Serre)**.** *Let $f : X \to Y$ be a map in $\mathsf{Top}_{*,1}$. The following are equivalent.*

1. *The map f induces an isomorphism*

$$\pi_\bullet(f) \otimes_{\mathbb{Z}} \mathbb{Q} : \pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \pi_\bullet(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

   in rational homotopy.

2. *The map f induces an isomorphism*

$$H_\bullet(f;\mathbb{Q}) : H_\bullet(X;\mathbb{Q}) \longrightarrow H_\bullet(Y;\mathbb{Q})$$

   in rational homology.

Definition 5.1.3. *A map f in $\mathsf{Top}_{*,1}$ inducing an isomorphism in rational homotopy or, equivalently, in rational homology is called a* rational equivalence*. We denote by RH — the* rational homotopy category *— the localization of $\mathsf{Top}_{*,1}$ at the rational equivalences.*

The category $\mathsf{Top}_{*,1}$ cannot be made into a model category for a trivial reason: it is not complete. However, if one takes the set of rational equivalences as "weak equivalences" and define appropriate sets of "fibrations" and "cofibrations", then the resulting category behaves very similarly to a model category, see [Qui69, Thm. II.6.1(a)]. In particular, RH is a well defined category.

One similarly defines a category sSets_1 given by the full subcategory of sSets spanned by 2-reduced simplicial sets, i.e. the simplicial sets that have a single 0-simplex and a single 1-simplex. This category admits a model structure where the weak equivalences are the rational equivalences, making it into a model category, see [Qui69, Thm. II.2.2]. The homotopy theory of this model category is then equivalent to the homotopy theory of $\mathsf{Top}_{*,1}$, with one of the equivalences being given by geometric realization of simplicial sets, see [Qui69, Thm. II.6.1(b)].

Remark 5.1.4. *Because of this equivalence, we will often speak of "spaces" without specifying if we mean topological spaces or simplicial sets.*

The category RH has the same objects as $\mathsf{Top}_{*,1}$, but morphisms behave a bit differently. For example, if two morphisms $f, g : X \to Y$ are homotopic in $\mathsf{Top}_{*,1}$, then they are the same morphism in RH. However, it is not true that two maps $f, g : X \to Y$ in $\mathsf{Top}_{*,1}$ inducing the same maps in rational homotopy or rational homology are identified in RH.

5.1.2 Conventions on cocommutative coalgebras and Lie algebras

We will mainly be dealing with algebras and coalgebras representing the singular chain complexes of topological spaces. Therefore, we consider the category $\mathsf{Ch}_{\geq 0}$ of chain complexes concentrated in degree greater or equal than 0 instead of the category Ch.

For Lie algebras and cocommutative coalgebras, we consider

- the category $\mathsf{dgLie}_{\geq 1}$ of Lie algebras concentrated in degrees greater or equal than 1, and

[2] Also starting in degree 2 by the Hurewicz theorem.

- the category $\mathsf{coCom}_{\geq 2}$ of cocommutative coalgebras concentrated in degrees greater or equal than 2.

Remark 5.1.5. *In [Qui69], one considers* counital, coaugmented *cocommutative coalgebras, but directly looking at non-counital cocommutative coalgebras is equivalent.*

By the Künneth formula, the homology of an object in $\mathsf{dgLie}_{\geq 1}$, respectively $\mathsf{coCom}_{\geq 2}$, is again in $\mathsf{dgLie}_{\geq 1}$, respectively $\mathsf{coCom}_{\geq 2}$, albeit with trivial differential. A morphism in $\mathsf{dgLie}_{\geq 1}$ or $\mathsf{coCom}_{\geq 2}$ is a *weak equivalence* if it is a quasi-isomorphism. One can also define classes of fibrations and cofibrations, making both categories into model categories, cf. [Qui69, Thm. 5.1 and 5.2].

Remark 5.1.6. *Notice that this is perfectly coherent with the model structures on algebras and coalgebras we looked at in Chapter 3. The quasi-isomorphisms are exactly the weak equivalences in the category algebras over an operad, as seen in Section 3.3.6, and while they are not the same as the weak equivalences for coalgebras in general, cf. Section 3.3.7, they are indeed for coalgebras concentrated in degree ≥ 2 by Proposition 2.4.5.*

5.1.3 Quillen's main theorem

Let $X \in \mathsf{Top}_{*,1}$, then the rational homotopy of X can be made into a graded[3] Lie algebra $\pi(X) \in \mathsf{grLie}_{\geq 1}$ by

$$\pi(X)_n \coloneqq \pi_{n+1}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \,,$$

i.e. $\pi(X) \coloneqq s^{-1}\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a chain complex, and using the Whitehead product to define the Lie bracket. This yields a functor

$$\pi : \mathsf{Top}_{*,1} \longrightarrow \mathsf{grLie}_{\geq 1} \,.$$

By the definition of RH, the functors $X \mapsto \pi(X)$ and $X \mapsto H_\bullet(X;\mathbb{Q})$ from $\mathsf{Top}_{*,1}$ to graded Lie algebras and graded cocommutative algebras extend uniquely to functors with RH as domain.

The main theorem of [Qui69], Theorem I.1 in loc. cit., is the following.

Theorem 5.1.7 (Quillen)**.** *There are equivalences of categories*

$$\mathsf{RH} \xrightarrow{\lambda} \mathrm{Ho}(\mathsf{dgLie}_{\geq 1}) \xrightarrow{s\mathrm{B}_\kappa} \mathrm{Ho}(\mathsf{coCom}_{\geq 2}) \,,$$

where the second functor is the suspension of the usual bar construction with respect to the twisting morphism

$$\kappa : \mathscr{S}^{-1} \otimes \mathrm{Com}^\vee = \mathrm{Lie}^{\mathrm{i}} \longrightarrow \mathrm{Lie} \,.$$

Moreover, there are isomorphisms of functors

$$\pi(X) \longrightarrow H_\bullet(\lambda(X)) \quad \textit{and} \quad H_\bullet(X) \longrightarrow H_\bullet(s\mathrm{B}_\kappa\lambda(X))$$

from RH *to graded Lie algebras and graded cocommutative coalgebras respectively.*

The functor $\lambda : \mathsf{RH} \to \mathrm{Ho}(\mathsf{dgLie}_{\geq 1})$ comes from a sequence of adjunctions leading from $\mathsf{Top}_{*,1}$ to $\mathsf{dgLie}_{\geq 1}$, see [Qui69, p. 211]. A full description of this functor is out of the scope of the present work.

Corollary 5.1.8. *If $\mathfrak{g}$ is a graded Lie algebra concentrated in degrees ≥ 1, then $\mathfrak{g} \simeq \pi(X)$ for some $X \in \mathsf{Top}_{*,1}$. If $\mathscr{C}$ is a graded cocommutative coalgebra concentrated in degrees ≥ 2, then $C \simeq H_\bullet(X;\mathbb{Q})$ for some $X \in \mathsf{Top}_{*,1}$.*

This is a first hint to the fact that, in rational homotopy theory, one should be able to model spaces by Lie algebras and cocommutative coalgebras.

[3] By which we mean differential graded with trivial differential.

5.1.4 Rationalization

This part is based on [FHT01, Ch. 9].

Definition 5.1.9. *An abelian group G is called* rational *if multiplication by k is an automorphism for each $k \in \mathbb{Z}\backslash\{0\}$.*

A rational abelian group is therefore canonically a $\mathbb{Q}$-vector space. Notice that for any abelian group G, the abelian group $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is rational. It is called the *rationalization* of G. If G was rational to start with, then

$$G \otimes_{\mathbb{Z}} \mathbb{Q} \cong G$$

canonically via the obvious map induced by $g \otimes 1 \mapsto g$.

Something analogous can be done with spaces.

Theorem 5.1.10 ([FHT01, Thm. 9.3])**.** *Let $X \in \mathsf{Top}_{*,1}$. The following are equivalent.*

1. *$\pi_\bullet(X)$ is rational.*
2. *$H_\bullet(X, \mathrm{pt}; \mathbb{Z})$ is rational.*

Definition 5.1.11. *A space $X \in \mathsf{Top}_{*,1}$ is* rational *if $\pi_\bullet(X)$ is rational, or equivalently if $H_\bullet(X, \mathrm{pt}; \mathbb{Z})$ is rational.*

Let $f : X \to Y$ be a morphism in $\mathsf{Top}_{*,1}$, and suppose that Y is rational. Then the morphism $\pi_\bullet(f)$ of abelian groups canonically extends to a morphism

$$\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \pi_\bullet(Y)$$

of $\mathbb{Q}$-vector spaces.

Definition 5.1.12. *A* rationalization *of a space $X \in \mathsf{Top}_{*,1}$ is a morphism $f : X \to X_{\mathbb{Q}}$ to a rational space $X_{\mathbb{Q}}$ such that the induced map*

$$\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \pi_\bullet(X_{\mathbb{Q}})$$

is an isomorphism.

Theorem 5.1.13 ([FHT01, Thm. 9.7])**.** *For every $X \in \mathsf{Top}_{*,1}$ there exists a relative CW complex $(X_{\mathbb{Q}}, X)$ with no 0-cells and no 1-cells such that the inclusion*

$$X \longrightarrow X_{\mathbb{Q}}$$

is a rationalization. This rationalization is unique up to homotopy equivalence $\mathrm{rel}\, X$.

In particular, the weak homotopy type of $X_{\mathbb{Q}}$ is the rational homotopy type of X.

5.2 Sullivan's approach and the rational de Rham theorem

Sullivan's [Sul77] approach to rational homotopy theory focuses on rational cohomology rather than rational homotopy or homology. This way, one can use commutative algebras as models for spaces, instead of cocommutative algebras or Lie algebras.

In this section, we work with cochain complexes instead of chain complexes, as it is more conventional to work with differentials of degree 1 on differential forms.

Most of the material present in this section works over an arbitrary field $\mathbb{K}$ of characteristic 0. In particular, PL differential forms and Dupont's contraction do not need the base field to be $\mathbb{Q}$. What we present here is extracted from [FHT01, Sect. 10] and [Dup76].

5.2.1 PL differential forms and rational cohomology

Instead of focusing on rational homotopy or rational homology, one can look at rational cohomology. The fact that this approach gives the same information as the other ones follows from the following result.

Proposition 5.2.1. *A map $f : X \to Y$ in $\mathsf{Top}_{*,1}$ is a rational equivalence if, and only if it induces an isomorphism*

$$H^\bullet(f;\mathbb{Q}) : H^\bullet(Y;\mathbb{Q}) \longrightarrow H^\bullet(X;\mathbb{Q})$$

in rational cohomology.

Proof. This is an immediate consequence of Theorem 5.1.2 and of the fact that

$$H^n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Q})^\vee,$$

since $\mathbb{Q}$ is a field. □

One now tries to approach cohomology in an algebraic way, using commutative algebras[4]. We want to work combinatorially, so we will use simplicial sets rather than topological spaces. One defines a good commutative algebra of *polynomial differential forms* — more commonly, *PL differential forms* — on the basic simplices, before extending this construction to arbitrary simplicial sets.

Definition 5.2.2. *Let $n \geq 0$. One defines a commutative algebra*

$$\mathrm{A_{PL}}(\Delta[n]) \coloneqq \frac{\mathbb{K}[t_0,\dots,t_n,dt_0,\dots,dt_n]}{(\sum_{i=0}^n t_i - 1, \sum_{i=0}^n dt_i)},$$

where $|t_i| = 0$ and $|dt_i| = 1$, with differential $d(t_i) \coloneqq dt_i$. The collection of the $\mathrm{A_{PL}}(\Delta[n])$ forms a simplicial commutative algebra $\Omega_\bullet$ by

$$\Omega_n \coloneqq \mathrm{A_{PL}}(\Delta[n])$$

with face maps

$$\partial_i : \mathrm{A_{PL}}(\Delta[n+1]) \longrightarrow \mathrm{A_{PL}}(\Delta[n]), \qquad t_i \longmapsto \begin{cases} t_k & \text{if } k < i\,, \\ 0 & \text{if } k = i\,, \\ t_{k-1} & \text{if } k > i\,, \end{cases} \qquad \text{for} \quad 0 \leq i \leq n\,,$$

and degeneracy maps

$$s_j : \mathrm{A_{PL}}(\Delta[n]) \longrightarrow \mathrm{A_{PL}}(\Delta[n+1]), \qquad t_i \longmapsto \begin{cases} t_k & \text{if } k < j\,, \\ t_k + t_{k+1} & \text{if } k = j\,, \\ t_{k+1} & \text{if } k > j\,, \end{cases} \qquad \text{for} \quad 0 \leq j \leq n\,.$$

The simplicial commutative algebra $\Omega_\bullet$ is called the Sullivan algebra.

[4] One should notice that singular cochains $C^\bullet(X;\mathbb{Q})$ are not a commutative algebra, but only an $\mathbb{E}_\infty$-algebra. However, since the characteristic of the base field is 0, the category of $\mathbb{E}_\infty$-algebras is equivalent to the category of commutative algebras, so that we can look for an equivalent strictly commutative model for the cochain algebra.

One now extends this assignment to a functor

$$\mathrm{A_{PL}} : \mathsf{sSets} \longrightarrow \mathsf{Com}$$

from simplicial sets to commutative algebras by

$$\mathrm{A_{PL}}(K) \coloneqq \operatorname*{colim}_{\Delta[n]\to K} \mathrm{A_{PL}}(\Delta[n]).$$

In other words, we take a copy of $\mathrm{A_{PL}}(\Delta[n])$ for every n-simplex of K, and then glue them together according to the face maps of K. Notice that this is very similar to the construction of the geometric realization functor.

We will not see how $\mathrm{A_{PL}}$ gives the same cohomology as the usual one.

5.2.2 Dupont's contraction and the PL de Rham theorem

In [Dup76], Dupont describes a contraction

$$h_\bullet \circlearrowright \Omega_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C_\bullet$$

from the Sullivan algebra to a subcomplex $C_\bullet$. A consequence of the existence of such a contraction is the "de Rham theorem" telling us that the cohomology of $\mathrm{A_{PL}}$ is the same as the usual cohomology. We will now describe in detail the elements appearing in this contraction, and state the theorem.

Other good reference for Dupont's contraction are [Get09] and [CG08].

Fix $n \geq 0$, let $k \geq 1$, and let $0 \leq i_0, \ldots, i_k \leq n$ be pairwise different. One defines a differential form

$$\omega_{i_0 \ldots i_k} \coloneqq k! \sum_{j=0}^{k} (-1)^j t_{i_j} dt_{i_0} \cdots \widehat{dt_{i_j}} \cdots dt_{i_k} \in \Omega_n \ ,$$

where the hat means that we omit the term. Then C_n is defined as the span of all forms appearing this way, that is

$$C_n \coloneqq \mathrm{span}_{\mathbb{K}} \{ \omega_{i_0 \ldots i_k} \mid k \geq 1 \text{ and } 0 \leq i_0 < i_1 < \cdots < i_k \leq n \} \ .$$

The fact that $C_n \subseteq \Omega_n$ is a subcomplex is implied by the following lemma, while the fact that it gives rise to a simplicial cochain complex $C_\bullet$ is straightforward.

Lemma 5.2.3. *Let $k \geq 1$, and let $0 \leq i_0, \ldots, i_k \leq n$ be pairwise different. Then*

$$d(\omega_{i_0 \ldots i_k}) = \sum_{i=0}^{n} \omega_{i i_0 \ldots i_k} \ ,$$

where $\omega_{i i_0 \ldots i_k} = 0$ whenever $i = i_j$ for some $0 \leq j \leq k$.

The easiest map of the contraction is the morphism $i_\bullet : C_\bullet \to \Omega_\bullet$. It is simply given by the inclusion of $C_\bullet$ into $\Omega_\bullet$.

Next in line is the morphism $p_\bullet : \Omega_\bullet \to C_\bullet$, which is given by integration. Namely, if $\omega \in \Omega_n$ is a polynomial differential form, one defines

$$p_n(\omega) \coloneqq \sum_{\substack{0\le i_0<\cdots<i_k\le n \\ f:\{i_0,\dots,i_k\}\hookrightarrow[n]}} \left(\int_{\Delta[p]} f^*\omega\right) \omega_{i_0\dots i_k} \ .$$

Finally, the contracting homotopy $h_\bullet : \Omega_\bullet \to \Omega_\bullet$ is defined as follows. Let

$$\varphi_i : [0,1]\times\Delta^n \longrightarrow \Delta^n$$

be defined by

$$\varphi_i(u,t_0,\dots,t_n) \coloneqq ((1-u)t_0,\dots,(1-u)t_i+u,\dots(1-u)t_n) \ .$$

Geometrically, it is the map contracting the standard geometric n-simplex to its ith vertex. For $0\le i\le n$, define

$$h_{(i)} : \Omega_n \longrightarrow \Omega_n$$

as the map taking a form in Ω_n, pulling it back by φ_i, and then integrating the resulting form along the fiber of φ_i. Then the map h_n is defined by

$$h_n \coloneqq \sum_{\substack{0\le k\le n-1 \\ 0\le i_0<\cdots<i_k\le n}} \omega_{i_0\dots i_k} h_{(i_k)}\cdots h_{(i_0)} \ .$$

Proposition 5.2.4. *The maps defined above are all simplicial, and they form a contraction from $\Omega_\bullet$ to $C_\bullet$.*

Similarly to what done for $\mathrm{A_{PL}}$, one extends the assignment $\Delta[n]\mapsto C_n$ to a functor

$$\mathrm{C_{PL}} : \mathsf{sSets} \longrightarrow \mathsf{coCh}$$

from simplicial sets to cochain complexes.

To conclude, we have the following theorem.

Theorem 5.2.5. *Let $X\in\mathsf{sSets}$. There is a natural quasi-isomorphism of cochain complexes*

$$\mathrm{A_{PL}}(X) \xrightarrow{\simeq} \mathrm{C_{PL}}(X) \ ,$$

where the first map is induced by $p_\bullet$. In particular,

$$H^\bullet(\mathrm{A_{PL}}(X)) \cong H^\bullet(X;\mathbb{K}) \ .$$

5.3 Algebraic models for spaces

Given a space X, one models its rational homotopy, homology and cohomology by the Lie algebra $\lambda(X)$, the cocommutative coalgebra $s\mathrm{B}_\kappa\lambda(X)$, and the commutative algebra $\mathrm{A_{PL}}(X)$ respectively. However, one could also consider other algebraic models for a space, by taking (co)algebras having the same properties as the previous three. Such objects are called *rational models* for the space X. We review their definitions and some basic existence results in this section.

The material of this section is extracted from [FHT01] and [Maj00, Ch. 4], with the exception of Section 5.3.4, which is original work.

5.3.1 Commutative models

We begin with commutative models. The idea is that $\mathrm{A_{PL}}(X)$ for a space X is huge, and has a relatively complicated algebraic structure, so one tries to replace it by a smaller, simpler commutative algebra.

Definition 5.3.1. *Let $X \in \mathsf{sSets}_1$ be a 1-reduced simplicial set. A* commutative rational model *for X is an augmented unital*[5] *commutative algebra A linked to $\mathrm{A_{PL}}(X)$ by a zig-zag of quasi-isomorphisms of commutative algebras*

$$A \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathrm{A_{PL}}(X) \,.$$

Equivalently, it is a commutative algebra A together with an ∞-morphism[6] *from A to $\mathrm{A_{PL}}(X)$, cf. Theorem 2.5.3.*

There is a particular kind of commutative models that are of special interest in rational homotopy, namely Sullivan models, and more specifically minimal Sullivan models.

Definition 5.3.2. *A* Sullivan model *for a space X is a rational commutative model which is a quasi-free commutative algebra $(\mathrm{Com}(V), d)$, where V is a graded vector space, such that*

$$V = \bigcup_{k=0}^{\infty} V(k) \,,$$

where $V(0) \subseteq V(1) \subseteq \cdots$ is an increasing sequence of graded subspaces of V, and the differential satisfies

$$d(V(0)) = 0 \,, \quad \textit{and} \quad d(V(k)) \subseteq \mathrm{Com}(V(k-1)) \quad \textit{for} \quad k \geq 1 \,.$$

A Sullivan model is minimal *if moreover we have*

$$\mathrm{Im}(d) \subseteq \overline{\mathrm{Com}(V)} \cdot \overline{\mathrm{Com}(V)} \,.$$

Minimal Sullivan models for spaces exist under some not too restrictive assumptions.

Definition 5.3.3. *A simply connected topological space $X \in \mathsf{Top}_{*,1}$ is* of finite type *if $H_i(X; \mathbb{Q})$ is finite dimensional for all i.*

Theorem 5.3.4 ([FHT01, p. 146])**.** *Let $X \in \mathsf{Top}_{*,1}$ be of finite type. Then X admits a minimal Sullivan model*

$$M_X \coloneqq (\mathrm{Com}(V), d) \xrightarrow{\sim} \mathrm{A_{PL}}(X)$$

such that V is concentrated in degree ≥ 2 and finite dimensional in every degree. It is unique up to (non-canonical) isomorphism.

5.3.2 Lie models

A similar idea comes into play for Lie algebras. Quillen's theorem gives us a canonical candidate for a Lie algebra modeling a simply connected space $X \in \mathsf{Top}_{*,1}$, namely the Lie algebra $\lambda(X) \in \mathsf{dgLie}_{\geq 1}$.

[5] As already mentioned previously, because of the augmentation hypothesis, one can equivalently work with non-unital commutative algebras.

[6] By which we mean an ∞-morphism or $\mathscr{C}_\infty$-algebras, cf. Section 2.3.6.

Definition 5.3.5. *A Lie algebra $\mathfrak{g} \in \mathsf{dgLie}_{\geq 1}$ is a* Lie rational model *for X if there is a zig-zag of quasi-isomorphisms of Lie algebras*

$$\mathfrak{g} \xleftarrow{\sim} \bullet \xrightarrow{\sim} \lambda(X) \, .$$

Equivalently, it is a Lie algebra $\mathfrak{g}$ together with an ∞-morphism from $\mathfrak{g}$ to $\lambda(X)$.

Lie models have many nice properties. We invite the interested reader to consult [FHT01, Part IV] for them.

Remark 5.3.6. *In [FHT01, p. 322], Lie models are defined as those Lie algebras $\mathfrak{g} \in \mathsf{dgLie}_{\geq 1}$ such that the commutative algebra $(s\mathrm{B}_\kappa(\mathfrak{g}))^\vee$ is a commutative rational model. By Theorem 5.3.10, this definition is equivalent to the one we gave in this section.*

A very important property is the following result, due to Berglund.

Theorem 5.3.7 ([Ber15, Prop. 6.1])**.** *Let $X \in \mathsf{Top}_{*,1}$, and let $\mathfrak{g}$ be a Lie model for X. Then*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq X_{\mathbb{Q}}.$$

In other words, the Maurer–Cartan space of $\mathfrak{g}$ is rationally homotopic to X.

5.3.3 Cocommutative models

Once again with the same ideas in mind, the definition of a cocommutative rational model for a space is as follows.

Definition 5.3.8. *Let $X \in \mathsf{Top}_{*,1}$ be a simply connected topological space. A cocommutative coalgebra $C \in \mathsf{coCom}_{\geq 2}$ is a* cocommutative rational model *for X if there is a zig-zag of weak equivalences (in the Vallette model structure on coalgebras)*

$$C \xrightarrow{\sim} \bullet \xleftarrow{\sim} s\mathrm{B}_\kappa \lambda(X) \, .$$

We will see in Theorem 8.2.6 that this condition is also equivalent to the existence of an ∞-morphism of coalgebras from C to $s\mathrm{B}_\kappa\lambda(X)$.

5.3.4 Dualizing (co)commutative models

Commutative and cocommutative models are strictly related, as one would expect. We will now provide a proof of the fact that the dual of a cocommutative model is a commutative model, and *vice versa* that the dual of a commutative model of finite type is a cocommutative model. This result is certainly well-known to experts and part of the folklore, but we haven't been able to find a proof in the literature.

We begin by recalling the following theorem, due to Majewski, which relates commutative and Lie models.

Theorem 5.3.9 ([Maj00, Thm. 4.90])**.** *Let $X \in \mathsf{sSets}_1$ be a simply-connected space. Let*

$$\ell_X : M_X \xrightarrow{\sim} A^*_{PL}(X)$$

be a simply-connected commutative model of finite type for X, and let

$$\mu_X : \mathfrak{g}_X \xrightarrow{\sim} \lambda(X)$$

be a free suspended Lie model for X. There exists a canonical homotopy class of quasi-isomorphisms of Lie algebras

$$\alpha_X : \mathfrak{g}_X \xrightarrow{\sim} \Omega_\kappa(M_X^\vee) \, .$$

Using this, we can prove the result we wished.

Theorem 5.3.10. *Let X be a simply-connected space of finite $\mathbb{Q}$-type.*

1. *Let A be a commutative model of finite type for X. Then its dual $A^\vee$ is a cocommutative model for X.*
2. *Dually, let C be a cocommutative model for X. Then its linear dual $C^\vee$ is a commutative model for X.*

Notice that we do not have any finiteness assumption on our cocommutative models.

Proof. We begin by proving (1). Let A be a simply-connected commutative model of finite type for X. Every simply-connected space X of finite $\mathbb{Q}$-type admits a minimal commutative model M_X, which in particular is simply-connected and of finite type, see Theorem 5.3.4. It follows that we have a zig-zag of quasi-isomorphisms

$$A \longleftarrow \bullet \longrightarrow M_X$$

by Theorem 2.5.3. Inspecting the proof in loc. cit. we notice that we can take $\Omega_\iota \mathrm{B}_\iota M_X$ as intermediate algebra, which is again simply-connected and of finite type. Dualizing linearly, we obtain a zig-zag of quasi-isomorphisms

$$A^\vee \longrightarrow \bullet \longleftarrow M_X^\vee \,, \tag{5.1}$$

where all terms are well-defined coalgebras thanks to the fact that they are of finite type. Finally, we obtain a zig-zag

$$A^\vee \longrightarrow \mathrm{B}_\pi\Omega_\pi A^\vee \longrightarrow \mathrm{B}_\pi\Omega_\pi(\bullet) \longleftarrow \mathrm{B}_\pi\Omega_\pi M_X^\vee \longleftarrow \mathrm{B}_\kappa\lambda(X) \,,$$

where the first arrow is the unit of the bar-cobar adjunction and is a weak equivalence by [Val14, Thm. 2.6(2)], the second and third arrows are obtained by the arrows of the zig-zag (5.1) by applying $\mathrm{B}_\pi\Omega_\pi$ and are also weak equivalences. The last arrow is obtained as follows. The Lie algebra $\lambda(X)$ is a Lie model for X. Therefore, by Theorem 5.3.9 there is a quasi-isomorphism

$$\alpha_X : \lambda(X) \xrightarrow{\sim} \Omega_\pi(M_X^\vee) \,.$$

Applying the bar construction we obtain the desired weak equivalence

$$\mathrm{B}_\pi\alpha_X : \mathrm{B}_\pi\lambda(X) \longrightarrow \mathrm{B}_\pi\Omega_\pi(M_X^\vee) \,,$$

concluding the first part of the proof.

For point (2), let C be a cocommutative model for X. By point (1), we have in particular that the dual $M_X^\vee$ of the minimal commutative model M_X is a cocommutative model for X. Therefore, we have a zig-zag of weak equivalences

$$C \longrightarrow \bullet \longleftarrow M_X^\vee \,,$$

which in particular are quasi-isomorphisms. Dualizing linearly, we obtain a zig-zag of quasi-isomorphisms

$$C^\vee \longleftarrow \bullet \longrightarrow M_X^{\vee\vee} \cong M_X \,,$$

where the last isomorphism holds because M_X is of finite type. Therefore, the commutative algebra $C^\vee$ is a commutative model for X. □

Chapter 6

Maurer–Cartan spaces

Maurer–Cartan elements of Lie algebras, and more generally homotopy Lie algebras, appear naturally throughout the whole of mathematics. The reason of this fact comes from deformation theory, and we will try to explain it later, namely in Chapter 7. Since the study of the spaces of Maurer–Cartan elements will be one of the central topics of the present thesis, we dedicate this chapter to the definition of these objects and of all the notions surrounding them, as well as the statement and the study of some central theorems in this area.

6.1 Lie algebras: the Deligne groupoid and the Goldman–Millson theorem

In this section, we introduce the Maurer–Cartan set of a Lie algebra and some of its properties. We look at the Baker–Campbell–Hausdorff formula, which makes it into a groupoid – the Deligne groupoid. This leads us naturally to consider complete Lie algebras in the sense of Appendix B.2. Finally, we present the Goldman–Millson theorem, which is the precursor of the Dolgushev–Rogers theorem.

6.1.1 The Maurer–Cartan set of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra, that is a chain complex together with a bracket

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

of degree 0 which

- is antisymmetric, i.e.
$$[x,y] = (-1)^{|x||y|+1}[y,x] \,,$$
and
- satisfies the Jacobi rule, that is
$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0 \,.$$

Definition 6.1.1. *A* Maurer–Cartan element *of a Lie algebra* $\mathfrak{g}$ *is an element* $x \in \mathfrak{g}_{-1}$ *of degree* -1 *satisfying the* Maurer–Cartan equation

$$dx + \frac{1}{2}[x, x] = 0 \,. \tag{6.1}$$

The set of all Maurer–Cartan elements of $\mathfrak{g}$ *will be denoted by* $\mathrm{MC}(\mathfrak{g})$.

Example 6.1.2. *In any Lie algebra* $\mathfrak{g}$*, the element* 0 *is always a Maurer–Cartan element.*

Notice that whenever $\mathfrak{g}$ is finite dimensional, then $\mathrm{MC}(\mathfrak{g})$ is an algebraic variety. More specifically, it is an intersection of quadrics in $\mathfrak{g}_{-1}$.

6.1.2 Gauges between Maurer–Cartan elements

Let $\mathfrak{g}$ be a Lie algebra, and let $\lambda \in \mathfrak{g}_0$ be a degree 0 element. One can consider the "vector field" given by

$$x \in \mathfrak{g} \longmapsto d\lambda + [x, \lambda] \in \mathfrak{g} \,.$$

In the finite dimensional case, it makes sense to identify the target copy of $\mathfrak{g}$ with $T_x\mathfrak{g}$, and to take the flow of such a vector field, but in general we have to proceed formally. One considers the differential equation in $\mathfrak{g}[[t]] \coloneqq \mathfrak{g} \otimes \mathbb{K}[[t]]$ given by

$$\frac{d}{dt}x(t) = d\lambda + [x(t), \lambda] \,, \tag{6.2}$$

where $\lambda \in \mathfrak{g}_0$ is seen as a constant element in $\mathfrak{g}[[t]]$.

Lemma 6.1.3. *Let* $x_0 \in \mathfrak{g}$*, then the unique solution of equation (6.2) with initial value* $x(0) = x_0$ *is given by*

$$x(t) = \frac{e^{t\,\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) + e^{t\,\mathrm{ad}_\lambda}(x_0) \,,$$

where $\mathrm{ad}_\lambda(x) \coloneqq [x, \lambda]$*, and the exponential has to be understood as a formal power series.*

A conceptual way to derive the formula above is presented in [DSV16, Sect. 1], cf. also the differential trick exposed in Section 6.1.4. Another way is provided by Proposition C.2.5, cf. Section 6.2.4.

Proof. Obviously, we have that $x(0) = x_0$. Differentiating formally, we have

$$\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}\left(\sum_{n\geq 1}\frac{t^n}{n!}\,\mathrm{ad}_\lambda^{n-1}(d\lambda) + \sum_{n\geq 0}\frac{t^n}{n!}\,\mathrm{ad}_\lambda^{n}(x_0)\right) \\
&= \sum_{n\geq 1}\frac{t^{n-1}}{(n-1)!}\,\mathrm{ad}_\lambda^{n-1}(d\lambda) + \sum_{n\geq 0}\frac{t^{n-1}}{(n-1)!}\,\mathrm{ad}_\lambda^{n}(x_0) \\
&= d\lambda + [x(t), \lambda] \,,
\end{aligned}$$

where $\mathrm{ad}_\lambda^n \coloneqq \mathrm{ad}_\lambda\,\mathrm{ad}_\lambda^{n-1}$ for $n \geq 1$, and $\mathrm{ad}_\lambda^0 = \mathrm{id}$. □

We would like to evaluate this formula at a $t \in \mathbb{K}$, in order that the solution of (6.1) makes sense in $\mathfrak{g}$, and not just formally in $\mathfrak{g}[[t]]$. However, the formula contains infinite sums, which do not make sense in a mere chain complex. Therefore, one has to ask that the Lie algebra is proper complete[1], so that $\lambda \in \mathscr{F}_1\mathfrak{g}$ and the iterated brackets with λ live in increasing degrees of the filtration, and the infinite sums make sense.

Lemma 6.1.4. *Suppose $\mathfrak{g}$ is a proper complete Lie algebra, and let $x_0 \in \mathrm{MC}(\mathfrak{g})$ be a Maurer–Cartan element. Then the evaluation of the solution to equation (6.2) always exists for any time $t \in \mathbb{K}$, and it remains in $\mathrm{MC}(\mathfrak{g})$.*

Proof. Existence is trivial. To see that the solution remains in the Maurer–Cartan set, we differentiate the Maurer–Cartan equation applied to the solution:

$$\begin{aligned}\frac{d}{dt}\left(dx(t) + \frac{1}{2}[x(t), x(t)]\right) &= d\dot{x}(t) + \frac{1}{2}([\dot{x}(t), x(t)] + [x(t), \dot{x}(t)]) \\ &= d(d\lambda + [x(t), \lambda]) + \frac{1}{2}([d\lambda + [x(t), \lambda], x(t)] + [x(t), d\lambda + [x(t), \lambda]]) \\ &= 0\,,\end{aligned}$$

where one has to use the Jacobi rule to conclude. □

Definition 6.1.5. *Let $\mathfrak{g}$ be a proper complete Lie algebra. Two Maurer–Cartan elements $x_0, x_1 \in \mathrm{MC}(\mathfrak{g})$ of $\mathfrak{g}$ are* gauge equivalent *if there exists $\lambda \in \mathfrak{g}_0$ such that the solution $x(t)$ of equation (6.1) with initial value $x(0) = x_0$ satisfies $x(1) = x_1$. We write $x_0 \sim_{\mathrm{g}} x_1$, and say that λ is a* gauge *between x_0 and x_1.*

Remark 6.1.6. *Since the gauge equation (6.2) is autonomous, any two Maurer–Cartan elements on a solution of the equation are gauge equivalent.*

By taking $\lambda = 0$, one sees that for any $x \in \mathrm{MC}(\mathfrak{g})$ we have $x \sim_{\mathrm{g}} x$, and if λ is a gauge from x_0 to x_1, then $-\lambda$ is a gauge from x_1 to x_0. Next, we will show that the gauge relation is transitive, effectively making it into an equivalence relation.

6.1.3 The Lawrence–Sullivan algebra

In [LS14], R. Lawrence and D. Sullivan defined a Lie algebra representing the interval.

Definition 6.1.7. *The Lawrence–Sullivan algebra is the free complete Lie algebra on elements x_0, x_1 of degree -1 and λ of degree 0 with*

$$\begin{aligned}dx_i &= -\frac{1}{2}[x_i, x_i]\,, \\ d\lambda &= \sum_{n\geq 0} \frac{B_n}{n!}\,\mathrm{ad}_\lambda^n(x_1 - x_0) - \mathrm{ad}_\lambda(x_0)\,,\end{aligned}$$

where B_n is the nth Bernoulli number.

This Lie algebra is the free complete Lie algebra over two Maurer–Cartan elements x_0 and x_1, representing the endpoints of the interval, and a gauge λ from x_0 to x_1, which corresponds to the interval itself. It is easily derived from Lemma 6.1.3. Indeed, we want

$$x_1 = \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) + e^{\mathrm{ad}_\lambda}(x_0) = \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda + \mathrm{ad}_\lambda(x_0)) + x_0\,.$$

[1] One can also drop the properness assumption and only consider those Maurer–Cartan elements and λ that are in $(\mathscr{F}_1\mathfrak{g})_0$.

Thus, we have

$$d\lambda = \left(\frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}\right)^{-1} (x_1 - x_0) - \mathrm{ad}_\lambda(x_0) \,,$$

and using the formal power series

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} t^n \,,$$

we find that we must have

$$d\lambda = \sum_{n \geq 0} \frac{B_n}{n!} \mathrm{ad}_\lambda^n (x_1 - x_0) - \mathrm{ad}_\lambda(x_0) \,.$$

6.1.4 The Baker–Campbell–Hausdorff formula

Let $\mathfrak{g}$ be a Lie algebra. The Baker–Campbell–Hausdorff formula

$$\mathrm{BCH} : \mathfrak{g}_0 \times \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$$

is a formula making $\mathfrak{g}_0$ into a group, and the gauge action into a group action. It has a long history, going back to F. Schur [Sch90], with the first explicit formula is attributed to E. Dynkin.

It can be approached with what is called the *differential trick*, which we learned from [DSV16, Sect. 2]. We extend the Lie algebra $\mathfrak{g}$ by an element δ of degree -1, defining

$$\mathfrak{g}^+ \coloneqq \mathfrak{g} \oplus \mathbb{K}\delta \,,$$

with the original differential and bracket on $\mathfrak{g}$, and imposing

$$d(\delta) = 0 \,, \qquad [\delta, \delta] = 0 \,, \qquad \text{and} \qquad [\delta, x] = dx$$

for any $x \in \mathfrak{g}$. Then we can look at the Maurer–Cartan elements $x \in \mathrm{MC}(\mathfrak{g})$ of $\mathfrak{g}$ as

$$\overline{x} \coloneqq \delta + x \in \mathfrak{g}^+,$$

with the Maurer–Cartan equation becoming just

$$[\overline{x}, \overline{x}] = 0 \,.$$

If we have $\lambda \in \mathfrak{g}_0$, then the differential equation (6.2) defining the gauge action simply becomes

$$\dot{\overline{x}}(t) = \mathrm{ad}_\lambda(\overline{x}(t)) \,,$$

which is solved by

$$\overline{x}(t) = e^{t\,\mathrm{ad}_\lambda}(\overline{x_0}) \,.$$

This immediately recovers Lemma 6.1.3. Moreover, we can derive the Baker–Campbell–Hausdorff formula from it. Let $\lambda, \mu \in \mathfrak{g}_0$. We want an element $\mathrm{BCH}(\lambda, \mu) \in \mathfrak{g}_0$ such that

$$e^{\mathrm{ad}_{\mathrm{BCH}(\lambda,\mu)}}(\overline{x}) = e^{\mathrm{ad}_\lambda}(e^{\mathrm{ad}_\mu}(\overline{x})) \,.$$

It is not obvious that such an element $\mathrm{BCH}(\lambda, \mu)$ exists. However, one can then proceed as in [Hal15, Sect. 5.3–6] to obtain that

$$\mathrm{BCH}(\lambda, \mu) = \lambda + \int_0^1 g(e^{\mathrm{ad}_\lambda} e^{t\,\mathrm{ad}_\mu})(\mu) dt \,,$$

where

$$g(z) \coloneqq \frac{\log z}{1-\frac{1}{z}}$$

is a holomorphic function on the disk (and thus we consider its formal power series to perform the integration). If one performs the integration, one obtains for the first terms

$$\mathrm{BCH}(\lambda,\mu) = \lambda + \mu + \frac{1}{2}[\lambda,\mu] + \frac{1}{12}\left([\lambda,[\lambda,\mu]] + [\mu,[\mu,\lambda]]\right) + \cdots$$

With this definition for BCH, we immediately have the following result.

Proposition 6.1.8. *Let $\mathfrak{g}$ be a proper complete Lie algebra, and let $x_0, x_1, x_2 \in \mathrm{MC}(\mathfrak{g})$ be three Maurer–Cartan elements. If $\lambda \in \mathfrak{g}_0$ is a gauge from x_0 to x_1, and $\mu \in \mathfrak{g}_0$ is a gauge from x_1 to x_2, then $\mathrm{BCH}(\lambda,\mu)$ is a gauge from x_0 to x_2. In particular, being gauge equivalent is an equivalence relation.*

Definition 6.1.9. *Let $\mathfrak{g}$ be a Lie algebra. The* Deligne groupoid $\mathrm{Del}(\mathfrak{g})$ *of $\mathfrak{g}$ is the groupoid with the Maurer–Cartan elements $\mathrm{MC}(\mathfrak{g})$ as objects and the gauges as morphisms.*

Remark 6.1.10. *The Deligne groupoid admits an extension to a 2-groupoid. This was first done in a letter [Del94] sent by P. Deligne to L. Breen in 1994. See [Yek12, Sect. 6] or [BGNT15, Sect. 3.3] for a clean definition.*

An object of interest in deformation theory is the quotient of the set of Maurer–Cartan elements by the gauge equivalence relation.

Definition 6.1.11. *The* moduli space of Maurer–Cartan elements $\overline{\mathrm{MC}}(\mathfrak{g})$ *of $\mathfrak{g}$ is the quotient*

$$\overline{\mathrm{MC}}(\mathfrak{g}) \coloneqq \mathrm{MC}(\mathfrak{g})/\sim_{\mathrm{g}}$$

of the set of Maurer–Cartan elements of $\mathfrak{g}$ by the gauge equivalence relation.

6.1.5 The Goldman–Millson theorem

The Goldman–Millson theorem[2] [GM88] tells us that, under some conditions, quasi-isomorphic Lie algebras give rise to equivalent Deligne groupoids.

The original statement of the result is as follows.

Theorem 6.1.12 (Goldman–Millson)**.** *Let $(A,\mathfrak{m})$ be an Artinian local $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras such that*

$$H_i(\phi) : H_i(\mathfrak{g}) \longrightarrow H_i(\mathfrak{h})$$

is an isomorphism for $i = 0, 1$, and is injective for $i = 2$. Then the map

$$\mathrm{Del}(\phi \otimes 1) : \mathrm{Del}(\mathfrak{g} \otimes \mathfrak{m}) \longrightarrow \mathrm{Del}(\mathfrak{h} \otimes \mathfrak{m})$$

is an equivalence of categories. In particular, it induces a bijection between the respective moduli spaces of Maurer–Cartan elements.

The tensorization by the maximal ideal of an Artinian local $\mathbb{K}$-algebra corresponds to some kind of localization. This result has been generalized in various ways during the years, e.g. by Yekutieli [Yek12], the most general version being given by the Dolgushev–Rogers theorem, which we will treat in detail in Section 6.4. The proof is done by "Artinian induction". We will not treat it here, but we will see similar ideas appear in the proof of the Dolgushev–Robers theorem, and then again in Chapter 10.

[2] Which W. M. Goldman and J. J. Millson attribute to Deligne and Schlessinger–Stasheff, see [GM88, p.46].

6.2 Homotopy Lie algebras: the deformation ∞-groupoid

Since $\mathscr{L}_\infty$-algebras are a natural generalization of Lie algebras, it is natural to wonder what is the correct generalization of the theory treated above in this context. The answer was given by V. Hinich [Hin97a] with the introduction of the deformation ∞-groupoid[3], which was further studied by various authors, such as E. Getzler [Get09], and V. A. Dolgushev and C. L. Rogers [DR15].

6.2.1 Maurer–Cartan elements in a homotopy Lie algebras

Recall from Section 2.3.8 that an $\mathscr{L}_\infty$-algebra $\mathfrak{g}$ is a chain complex equipped with graded anti-symmetric operations

$$\ell_n : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}$$

of degree $n-2$, for all $n \geq 2$, satisfying the relations

$$\sum_{\substack{n_1+n_2=n+1 \\ \sigma \in \sqcup(n_1-1,n_2)}} (-1)^{n_2(n_1-1)} (\ell_{n_1} \circ_1 \ell_{n_2})^\sigma = 0 \tag{6.3}$$

for all $n \geq 1$, where we use the short-hand notation $\ell_1 \coloneqq d$.

Let's explore these relations a bit. For $n = 1$, equation (6.3) becomes

$$\ell_1^2 = 0 \,,$$

which is the same as to say that $d_\mathfrak{g}$ squares to zero, i.e. that $\mathfrak{g}$ is a chain complex. For $n = 2$, we have

$$d\ell_2(x,y) = \ell_2(dx,y) + (-1)^{|x|}\ell_2(x,dy) \,,$$

i.e. d is a derivation with respect to ℓ_2. For $n = 3$, we obtain

$$\begin{aligned} \partial(\ell_3)(x,y,z) &= -\ell_2(\ell_2(x,y),z) - (-1)^{|z|(|x|+|y|)}\ell_2(\ell_2(z,x),y) - (-1)^{|x|(|y|+|z|)}\ell_2(\ell_2(y,z),x) \\ &= (-1)^{|x||z|+1}\Big((-1)^{|x||z|}\ell_2(\ell_2(x,y),z) + (-1)^{|z||y|}\ell_2(\ell_2(z,x),y) + (-1)^{|y||x|}\ell_2(\ell_2(y,z),x)\Big) \,, \end{aligned}$$

which tells us that the Jacobi rule is satisfied up to a homotopy given by the ternary bracket ℓ_3. For $n \geq 4$, we have higher compatibility relations between the brackets.

Definition 6.2.1. *A* Maurer–Cartan element *of a proper complete $\mathscr{L}_\infty$-algebra is an element $x \in \mathfrak{g}_{-1}$ of degree -1 satisfying the Maurer–Cartan equation*

$$dx + \sum_{n\geq 2} \frac{1}{n!}\ell_n(x,\dots,x) = 0 \,. \tag{6.4}$$

The set of all Maurer–Cartan elements of $\mathfrak{g}$ will be denoted by $\mathrm{MC}(\mathfrak{g})$.

Remark 6.2.2. *If $\mathfrak{g}$ is a Lie algebra, then the Maurer–Cartan equation (6.4) reduces to the Maurer–Cartan equation for Lie algebras (6.1).*

Remark 6.2.3. *While in the case of Lie algebras we only needed the algebras to be proper complete in order for the gauge relation to be well-defined, in the case of $\mathscr{L}_\infty$-algebras we already need this condition for the Maurer–Cartan equation to make sense.*

[3] Called the *contents* of an $\mathscr{L}_\infty$-algebra in *op. cit.*

6.2.2 The deformation ∞-groupoid

One now would like a notion of gauge equivalence between Maurer–Cartan elements of an $\mathscr{L}_\infty$-algebra. As a matter of fact, one obtains a whole hierarchy of relations thanks to the following object defined by V. Hinich [Hin97a, Def. 2.1.1].

Definition 6.2.4. *Let $\mathfrak{g}$ be a proper complete $\mathscr{L}_\infty$-algebra. The* deformation ∞-groupoid *of $\mathfrak{g}$ — also known as its* Maurer–Cartan space*, or its* Deligne–Hinich–Getzler ∞-groupoid *— is the simplicial set*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \coloneqq \lim_n \mathrm{MC}\big(\mathfrak{g}/\mathscr{F}_n\mathfrak{g} \otimes \Omega_\bullet\big) \in \mathsf{sSets}\ ,$$

where $\Omega_\bullet$ is the Sullivan algebra, cf. Definition 5.2.2, with inverted degrees (so that it is a chain complex instead of a cochain complex).

The elements of $\mathrm{MC}_1(\mathfrak{g})$ are the analogue of the gauges between Maurer–Cartan elements in this context. We will look at them in more detail in Section 6.2.3.

The simplicial set $\mathrm{MC}_\bullet(\mathfrak{g})$ has a lot of nice properties. It is extends to a functor

$$\mathrm{MC}_\bullet : \infty\text{-}\widehat{\mathscr{L}}_\infty\text{-}\mathsf{alg} \longrightarrow \mathsf{sSets}$$

from proper complete $\mathscr{L}_\infty$-algebras with filtered ∞-morphisms to simplicial sets. Its action on filtered ∞-morphisms is induced by the following fact.

Lemma 6.2.5. *If $\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is a filtered ∞-morphism between proper complete $\mathscr{L}_\infty$-algebras, and $x \in \mathrm{MC}(\mathfrak{g})$ is a Maurer–Cartan element of $\mathfrak{g}$, then*

$$\mathrm{MC}(\Phi)(x) \coloneqq \sum_{n\geq 1} \frac{1}{n!}\phi_n(x,\dots,x)$$

is a Maurer–Cartan element of $\mathfrak{h}$. Moreover, if $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\Psi : \mathfrak{g}_2 \to \mathfrak{g}_3$ are two such filtered ∞-morphisms, then

$$\mathrm{MC}(\Psi)\mathrm{MC}(\Phi) = \mathrm{MC}(\Psi\Phi)\ .$$

Proof. This is just developing the Maurer–Cartan equation for $\mathrm{MC}(\Phi)(x)$ and using the explicit relations satisfied by the components of an ∞-morphism that we wrote down in Section 2.3.8 to recover the Maurer–Cartan equation for x. Proving that $\mathrm{MC}(-)$ respects compositions is also straightforward. □

We have the following result, motivating the name "∞-groupoid" and "space".

Theorem 6.2.6. *Let either:*

- *[Get09, Prop. 4.7]: $\mathfrak{g},\mathfrak{h}$ be nilpotent $\mathscr{L}_\infty$-algebras and $\Phi : \mathfrak{g} \to \mathfrak{h}$ be a surjective strict morphism of $\mathscr{L}_\infty$-algebras, or*
- *[Rog16, Thm. 2]: $\mathfrak{g},\mathfrak{h}$ be proper complete $\mathscr{L}_\infty$-algebras and $\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ be a filtered ∞-morphism that induces a surjection at every level of the filtrations.*

Then

$$\mathrm{MC}_\bullet(\Phi) : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h})$$

is a fibration of simplicial sets. In particular, for any proper complete $\mathscr{L}_\infty$-algebra $\mathfrak{g}$, the simplicial set $\mathrm{MC}_\bullet(\mathfrak{g})$ is a Kan complex.

This result was originally proven by Hinich [Hin97a, Th. 2.2.3] for strict surjections between nilpotent Lie algebras concentrated in positive degrees, and then generalized by E. Getzler and by C. L. Rogers to the versions stated above. A precursor to the version of [Rog16] is [Yal16, Prop. 3.1].

6.2.3 Homotopies and gauges

There are two possible — and equivalent, as we will see — definitions of equivalence between Maurer–Cartan elements of an $\mathscr{L}_\infty$-algebra.

Definition 6.2.7. *Let $\mathfrak{g}$ be a proper complete $\mathscr{L}_\infty$-algebra. Two Maurer–Cartan elements $x_0, x_1 \in \mathrm{MC}(\mathfrak{g}) = \mathrm{MC}_0(\mathfrak{g})$ are* homotopy equivalent *if there is an element $\alpha \in \mathrm{MC}_1(\mathfrak{g})$ such that*

$$d_0(\alpha) = x_0, \quad \textit{and} \quad d_1(\alpha) = x_1 \,.$$

Let's write down explicitly what it means for two Maurer–Cartan elements to be homotopy equivalent. For simplicity, assume that the $\mathscr{L}_\infty$-algebra $\mathfrak{g}$ is nilpotent. The general case works similarly.

Under the isomorphism $\Omega_1 \cong \mathbb{K}[t, dt]$ of unital commutative algebras given by sending t_1 to t, a generic element of $(\mathfrak{g} \otimes \Omega_1)_{-1}$ is given by

$$\alpha = x(t) + \lambda(t)dt \,,$$

where $x(t) \in \mathfrak{g}_{-1}[t]$, and $\lambda(t) \in \mathfrak{g}_0[t]$. The Maurer–Cartan equation (6.4) becomes

$$\begin{aligned}
0 &= d\alpha + \sum_{n\geq 2} \frac{1}{n!}\ell_n(\alpha, \dots, \alpha) \\
&= d(x(t) + \lambda(t)dt) + \sum_{n\geq 2}\left(\frac{1}{n!}\ell_n(x(t), \dots, x(t)) + \frac{1}{(n-1)!}\ell_n(x(t), \dots, x(t), \lambda(t))\right) \\
&= d_{\mathfrak{g}}x(t) + \sum_{n\geq 2}\frac{1}{n!}\ell_n(x(t), \dots, x(t)) + \\
&\qquad + \left(-\frac{d}{dt}x(t) + d_{\mathfrak{g}}\lambda(t) + \sum_{n\geq 2}\frac{1}{(n-1)!}\ell_n(x(t), \dots, x(t), \lambda(t))\right)dt \,.
\end{aligned}$$

To do this computation, we used the following facts:

- Since $|dt| = -1$, we have that $(dt)^2 = 0$. In particular, if the term $\lambda(t)dt$ appears twice in a bracket, then the bracket gives 0.
- For $1 \leq i \leq n$, we have

$$\begin{aligned}
\ell_n(\underbrace{x(t), \dots, x(t)}_{i}, \lambda(t)dt, \underbrace{x(t), \dots, x(t)}_{n-(i+1)}) &= \ell_n(x(t), \dots, x(t), \lambda(t)dt) \\
&= \ell_n(x(t), \dots, x(t), \lambda(t))dt \,.
\end{aligned}$$

- Let $x \in \mathfrak{g}_{-1}$ and let $k \geq 0$. Then

$$d(xt^k) = d_{\mathfrak{g}}(x)t^k - kxt^{k-1}dt \,.$$

Therefore, α is a Maurer–Cartan element if, and only if we have:

1. $x(t) \in \mathrm{MC}(\mathfrak{g})$ for all times $t \in \mathbb{K}$, and

2. $x(t)$ satisfies the differential equation

$$\frac{d}{dt}x(t) = d_{\mathfrak{g}}\lambda(t) + \sum_{n\geq 2}\frac{1}{(n-1)!}\ell_n(x(t),\dots,x(t),\lambda(t))\ .$$

The two boundaries are given by evaluation of $x(t)$ at $t=0$ and $t=1$, that is

$$\partial_0(\alpha) = x(0) \quad \text{and} \quad \partial_1(\alpha) = x(1)\ .$$

Lemma 6.2.8. *Being homotopy equivalent is an equivalence relation on the set of Maurer–Cartan elements of a proper complete $\mathscr{L}_\infty$-algebra.*

Proof. This is an immediate consequence of the fact that $\mathrm{MC}_\bullet(\mathfrak{g})$ is a Kan complex for any proper complete $\mathscr{L}_\infty$-algebra. □

By taking the quotient of the set of Maurer–Cartan elements by the homotopy equivalence relation, we obtain the zeroth homotopy group $\pi_0\mathrm{MC}_\bullet(\mathfrak{g})$ of the deformation ∞-groupoid of $\mathfrak{g}$.

By keeping $\lambda(t)$ constant, we get a second notion of equivalence of Maurer–Cartan elements.

Definition 6.2.9. *Let $\mathfrak{g}$ be a proper complete $\mathscr{L}_\infty$-algebra. Two Maurer–Cartan elements $x_0, x_1 \in \mathrm{MC}(\mathfrak{g})$ are* gauge equivalent *if there exists a $\lambda \in \mathfrak{g}_0$ such that the solution $x(t) \in \mathfrak{g}[[t]]$ of the formal differential equation*

$$\frac{d}{dt}x(t) = d\lambda + \sum_{n\geq 2}\frac{1}{(n-1)!}\ell_n(x(t),\dots,x(t),\lambda) \tag{6.5}$$

with initial value $x(0) = x_0$ is such that $x(1) = x_1$. The element $\lambda \in \mathfrak{g}_0$ is called a gauge *from x_0 to x_1.*

If $\lambda \in \mathfrak{g}_0$ is a gauge from x_0 to x_1, then one immediately obtains a homotopy equivalence between the two Maurer–Cartan elements by taking

$$\alpha \coloneqq x(t) + \lambda dt\ ,$$

where $x(t)$ is the solution of the gauge equation (6.5). Therefore, if two elements are gauge equivalent, then they are homotopy equivalent. The other direction is also known in the literature, see e.g. [DP16, Prop. 9]. In particular, gauge equivalence is an equivalence relation. We will give a new proof of this fact and explicit formulæ to obtain a gauge equivalence from a homotopy equivalence in Section 10.2.1.

6.2.4 An explicit formula for gauges

We can use Proposition C.2.5 to give an explicit formula for the action of a gauge λ on a Maurer–Cartan element x_0. For each $n \geq 1$, we fix

$$f_{n,1}(y_1,\dots,y_n) \coloneqq \frac{1}{n!}\ell_{n+1}(y_1,\dots,y_n,\lambda)\ ,$$

and $f_{0,1}(1) \coloneqq d\lambda$. We set $f_{n,k} = 0$ for all $k \geq 2$. Then equation (C.2) becomes equation (6.5), and Proposition C.2.5 gives us what we wanted. A similar formula is already present in [Get09, Prop. 5.7].

The combinatorics of planar trees is rather complicated, especially since we allow vertices of valence 0 and 1. Therefore, we do not know of any way to express the resulting formula in a simpler way than the formula already given. However, if we work with a Lie algebra instead of an $\mathscr{L}_\infty$-algebra, then things simplify a lot, and we recover the formula we gave in Lemma 6.1.3. This gives an alternative proof of that result.

Proof of Lemma 6.1.3 (alternative version). The only non-vanishing operators are $f_{0,1} = d\lambda$, and $f_{1,1} = \mathrm{ad}_\lambda$. Therefore, we are only have to work with trees that are a composition of 0-corollas and 1-corollas, and which have the weight 1 at each vertex. They all fall in the following two categories.

1. A linear composition of 1-corollas. We denote such trees by a_n, with
$$a_0 \coloneqq \emptyset, \quad \text{and} \quad a_n \coloneqq c_1 \circ a_{n-1}$$
for $n \geq 1$.
2. A linear composition of 1-corollas with a 0-corolla instead of the free leaf. We denote such trees by b_n, with
$$b_1 \coloneqq c_0, \quad \text{and} \quad b_n \coloneqq c_1 \circ b_{n-1}$$
for $n \geq 2$.

One easily computes the coefficients of those trees to be

$$F(a_n) = F(b_n) = n! \,,$$

and

$$a_n(x_0) = t^n \,\mathrm{ad}_\lambda^n(x_0) \,, \quad b_n(x_0) = t^n \,\mathrm{ad}_\lambda^{n-1}(d\lambda) \,.$$

Thus, the solution of the gauge equation is given by

$$\begin{aligned} x(t) &= \sum_{n\geq 0} \frac{1}{F(a_n)} a_n(x_0) + \sum_{n\geq 1} \frac{1}{F(b_n)} b_n(x_0) \\ &= \sum_{n\geq 0} \frac{t^n}{n!} \,\mathrm{ad}_\lambda^n(x_0) + \sum_{n\geq 1} \frac{t^n}{n!} \,\mathrm{ad}_\lambda^{n-1}(d\lambda) \\ &= e^{t\,\mathrm{ad}_\lambda}(x_0) + \frac{e^{t\,\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) \,, \end{aligned}$$

concluding the proof. □

6.2.5 Getzler's functor

In his article [Get09], E. Getzler introduced an object which is smaller, but homotopically equal to the Maurer–Cartan space of an $\mathscr{L}_\infty$-algebra.

Recall Dupont's contraction

$$h_\bullet \circlearrowright \Omega_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C_\bullet$$

from Section 5.2.2.

Definition 6.2.10. Getzler's functor *is the functor*

$$\gamma_\bullet : \mathscr{L}_\infty\text{-alg} \longrightarrow \mathsf{sSets}$$

given by sending an $\mathscr{L}_\infty$-algebra $\mathfrak{g}$ to the simplicial set

$$\gamma_\bullet(\mathfrak{g}) \coloneqq \mathrm{MC}_\bullet(\mathfrak{g}) \cap \ker(1_\mathfrak{g} \otimes h_\bullet) \,,$$

where $h_\bullet$ comes from the Dupont contraction, cf. Section 5.2.2. The action on morphisms[4] *is given by*

$$\gamma_\bullet(\phi)(x) = \phi(x)$$

for $\phi : \mathfrak{g} \to \mathfrak{h}$ a morphism of $\mathscr{L}_\infty$-algebras.

Theorem 6.2.11 ([Get09, Thm. 5.8])**.** *Let $\mathfrak{g}, \mathfrak{h}$ be proper complete $\mathscr{L}_\infty$-algebras, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a filtered morphism that induces surjections at every level of the filtration. Then the morphism*

$$\gamma_\bullet(\phi) : \gamma_\bullet(\mathfrak{g}) \longrightarrow \gamma_\bullet(\mathfrak{h})$$

is a fibration of simplicial sets. In particular, the simplicial set $\gamma_\bullet(\mathfrak{g})$ is a Kan complex for any $\mathscr{L}_\infty$-algebra $\mathfrak{g}$.

Remark 6.2.12. *This was originally stated for surjections between nilpotent $\mathscr{L}_\infty$-algebras. The more general statement above follows from a straightforward limit argument.*

Theorem 6.2.13 ([Get09, Thm. 5.9])**.** *Let $\mathfrak{g}$ be a proper complete $\mathscr{L}_\infty$-algebra. The inclusion*

$$\gamma_\bullet(\mathfrak{g}) \longhookrightarrow \mathrm{MC}_\bullet(\mathfrak{g})$$

is a homotopy equivalence of simplicial sets.

In fact, the simplicial set $\gamma_\bullet(\mathfrak{g})$ has a property which is stronger than simply being a Kan complex.

Definition 6.2.14. *An n-simplex $x \in \gamma_n(\mathfrak{g})$ is* thin *if*

$$\int_{\Delta[n]} x = 0 \,,$$

where integration acts on the part of x living in Ω_n.

Theorem 6.2.15 ([Get09, Thm. 5.4])**.** *Let $\mathfrak{g}$ be a nilpotent $\mathscr{L}_\infty$-algebra. Then $\gamma_\bullet(\mathfrak{g})$ has the following properties.*

1. *Every degenerate simplex is thin.*
2. *Every horn has a unique thin simplex filling it.*

Remark 6.2.16. *In [Get09], an ∞-groupoid is a Kan complex which moreover has a set of thin simplices which satisfies Theorem 6.2.15. We adopt a more ∞-categorical point of view and use the terms Kan complex and ∞-groupoid interchangeably.*

6.3 Paradigm change: shifted homotopy Lie algebras

The operadic suspension functor

$$\mathscr{S} \otimes - : \mathsf{Op} \longrightarrow \mathsf{Op}$$

is an automorphism of categories. Moreover, for any chain complex V there is a canonical isomorphism

$$\mathscr{S} \otimes \mathrm{End}_V \cong \mathrm{End}_{sV} \,,$$

[4] Here, only strict morphisms are considered. An extension to an action of ∞-morphisms is possible using the results of Section 9.2 and Section 10.2.

which implies that if $\mathscr{P}$ is an operad, then a $\mathscr{S} \otimes \mathscr{P}$-algebra is equivalent to a $\mathscr{P}$-algebra via a suspension, and *vice versa*.

Relying on these facts, we will often work with $\mathfrak{s}\mathscr{L}_\infty$-algebras — shifted homotopy Lie algebras — instead of $\mathscr{L}_\infty$-algebras in what follows. This gives the same exact theory, but has the great advantage to substantially reduce the amount of signs appearing.

Here are some basic, useful facts. A $\mathfrak{s}\mathscr{L}_\infty$-algebra is an algebra over

$$\mathfrak{s}\mathscr{L}_\infty \coloneqq \mathscr{S} \otimes \mathscr{L}_\infty \cong \Omega \mathrm{Com}^\vee \, .$$

The operations on it are generated by graded *symmetric* arity n brackets

$$\ell_n \coloneqq s^{-1} \mu_n^\vee \, ,$$

for all $n \geq 2$, all of which have degree -1. They satisfy the relations

$$\sum_{\substack{n_1+n_2=n+1 \\ \sigma \in \sqcup\!\sqcup(n_1,n_2-1)}} (\ell_{n_2} \circ_1 \ell_{n_1})^\sigma = 0 \, .$$

A Maurer–Cartan element of a proper complete $\mathfrak{s}\mathscr{L}_\infty$-algebra $\mathfrak{g}$ is a degree 0 element $x \in \mathfrak{g}_0$ satisfying the equation

$$dx + \sum_{n \geq 2} \frac{1}{n!} \ell_n(x, \dots, x) = 0 \, .$$

6.4 The Dolgushev–Rogers theorem

In this section, we will present the Dolgushev–Rogers theorem [DR15, Thm. 2.2] in some detail. The ideas of the proof will return in some applications, most notably in Chapter 10. The proof we present here is very close to the original one, with some slight variations in the presentation, namely by putting more stress on the obstruction theoretical aspects of one step of the demonstration, see Section 6.4.4.

The main theorem of this section is the following one.

Theorem 6.4.1 (Dolgushev–Rogers)**.** *Let $\mathfrak{g}, \mathfrak{h}$ be two complete proper $\mathfrak{s}\mathscr{L}_\infty$-algebras, and let*

$$\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$$

be a filtered quasi-isomorphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras. Then the induced map

$$\mathrm{MC}_\bullet(\Phi) : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h})$$

is a weak equivalence of simplicial sets.

This generalizes the Goldman–Millson in the following sense. Suppose that $\mathfrak{g}, \mathfrak{h}$ are any two Lie algebras, and let $(A, \mathfrak{m})$ be an Artinian local $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}$. Then $\mathfrak{g} \otimes \mathfrak{m}$ has a natural filtration given by

$$\mathscr{F}_n(\mathfrak{g} \otimes \mathfrak{m}) \coloneqq \mathfrak{g} \otimes \mathfrak{m}^n,$$

making it into a proper complete Lie algebra[5]. The same thing holds for $\mathfrak{h}$. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of Lie algebras, then the induced morphism

$$\phi \otimes 1_\mathfrak{m} : \mathfrak{g} \otimes \mathfrak{m} \longrightarrow \mathfrak{h} \otimes \mathfrak{m}$$

[5]Since A is Artinian, there is an N such that $\mathfrak{m}^N = 0$. Therefore, $\mathfrak{g} \otimes \mathfrak{m}$ is in fact nilpotent, and the filtration only has finitely many levels.

is a filtered (strict) quasi-isomorphism. By applying Theorem 6.4.1 to this situation and looking at the 0th homotopy group of the spaces of Maurer–Cartan elements, we get the natural bijection

$$\overline{\mathrm{MC}}(\phi) : \overline{\mathrm{MC}}(\mathfrak{g}) \longrightarrow \overline{\mathrm{MC}}(\mathfrak{h})$$

between the moduli spaces of Maurer–Cartan elements of $\mathfrak{g}$ and $\mathfrak{h}$.

The strategy of the proof of Theorem 6.4.1 is the following. First one proves that the statement holds for abelian $\mathfrak{sL}_\infty$-algebras with a certain trivial filtration and strict quasi-isomorphisms, and the statement holds in full generality at the level of the 0th homotopy groups. The following step is to prove that we can always twist the $\mathfrak{sL}_\infty$-algebras into play to set the basepoint of the higher homotopy groups at 0. Then one uses these statements as the base case for an induction proving that the theorem holds in full generality for complete proper $\mathfrak{sL}_\infty$-algebras whose filtration terminates at a finite filtration degree — and which are therefore necessarily nilpotent. Finally, a limit argument concludes the proof.

6.4.1 The case of abelian $\mathfrak{sL}_\infty$-algebras

The first step is to prove that the theorem holds in the case of abelian $\mathfrak{sL}_\infty$-algebras, i.e. $\mathfrak{sL}_\infty$-algebras in which all brackets are constantly zero. Let $\mathfrak{g}$ be such an $\mathfrak{sL}_\infty$-algebra, and equip it with the filtration

$$\mathcal{F}_1\mathfrak{g} \coloneqq \mathfrak{g} \supseteq \mathcal{F}_2\mathfrak{g} = \mathcal{F}_3\mathfrak{g} = \ldots \coloneqq 0 \tag{6.6}$$

making it into a proper complete $\mathfrak{sL}_\infty$-algebra. A Maurer–Cartan element is nothing else than a 0-cycle:

$$\mathrm{MC}_n(\mathfrak{g}) = \mathfrak{L}_0(\mathfrak{g} \otimes \Omega_n) \,,$$

and in particular it follows that $\mathrm{MC}_\bullet(\mathfrak{g})$ is a simplicial vector space.

Proposition 6.4.2 ([DR15, Prop. 2.4])**.** *Let $\mathfrak{g}, \mathfrak{h}$ be two abelian $\mathfrak{sL}_\infty$-algebras endowed with filtrations as in (6.6), and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a quasi-isomorphism. Then the induced map*

$$\mathrm{MC}_\bullet(\phi) : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h})$$

is a weak equivalence of simplicial sets.

Proof. By Theorem 4.3.6, we know that the homotopy groups of a simplicial vector space $V_\bullet$ are given by the homology of the Moore complex:

$$\pi_0(V_\bullet) = H_0(\mathcal{M}(V_\bullet)) \quad \text{and} \quad \pi_k(V_\bullet, 0) = H_k(\mathcal{M}(V_\bullet))$$

for $k \geq 1$. It follows that a map $f : V_\bullet \to W_\bullet$ of simplicial vector spaces is a weak equivalence if, and only if the induced map

$$\mathcal{M}(f) : \mathcal{M}(V_\bullet) \longrightarrow \mathcal{M}(W_\bullet)$$

is a quasi-isomorphism of chain complexes.

Recall Dupont's contraction from Section 5.2.2

$$h_\bullet \circlearrowright \Omega_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C_\bullet$$

We extend it to a contraction

$$1_{\mathfrak{g}} \otimes h_\bullet \circlearrowright \mathfrak{g} \otimes \Omega_\bullet \underset{1_{\mathfrak{g}} \otimes i_\bullet}{\overset{1_{\mathfrak{g}} \otimes p_\bullet}{\rightleftarrows}} \mathfrak{g} \otimes C_\bullet$$

We claim that the following sequence is exact:

$$0 \longrightarrow d(1_{\mathfrak{g}} \otimes h_\bullet)(\mathfrak{g} \otimes \Omega_\bullet)_0 \hookrightarrow \mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet) \xrightarrow{1_{\mathfrak{g}} \otimes p_\bullet} \mathfrak{T}_0(\mathfrak{g} \otimes C_\bullet) \longrightarrow 0 \,,$$

where d denotes the differential of $\mathfrak{g} \otimes \Omega_1$. The only thing that is not immediately obvious is the fact that

$$\ker(1_{\mathfrak{g}} \otimes p_\bullet)|_{\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)} = d(1_{\mathfrak{g}} \otimes h_\bullet)(\mathfrak{g} \otimes \Omega_\bullet)_0 \,. \tag{6.7}$$

We have

$$\begin{aligned} \ker(1_{\mathfrak{g}} \otimes p_\bullet)|_{\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)} &= \ker(1 - d(1_{\mathfrak{g}} \otimes h_\bullet) - (1_{\mathfrak{g}} \otimes h_\bullet)d)|_{\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)} \\ &= \ker(1 - d(1_{\mathfrak{g}} \otimes h_\bullet))|_{\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)} \\ &= \{x \in \mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet) \mid d(1_{\mathfrak{g}} \otimes h_\bullet)(x) = x\} \,. \end{aligned}$$

An easy algebraic manipulation using the fact that we are working with a contraction shows that $(d(1_{\mathfrak{g}} \otimes h_\bullet))^2 = d(1_{\mathfrak{g}} \otimes h_\bullet)$ on $\mathfrak{g} \otimes \Omega_\bullet$. Together with the fact that $d(1_{\mathfrak{g}} \otimes h_\bullet)$ takes image in $\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)$, this implies (6.7).

Now we show that the first space of the short exact sequence is acyclic. Recall [Get09, Lemma 3.2], which states that the simplicial set $(\Omega_\bullet)_k$ is contractible for all k. If follows that $(\mathfrak{g} \otimes \Omega_\bullet)_k$ is also contractible for all k, and in particular for $k = 0$. Since

$$d(1_{\mathfrak{g}} \otimes h_\bullet) : (\mathfrak{g} \otimes \Omega_\bullet)_0 \longrightarrow d(1_{\mathfrak{g}} \otimes h_\bullet)(\mathfrak{g} \otimes \Omega_\bullet)_0$$

is a retraction, it follows that $d(1_{\mathfrak{g}} \otimes h_\bullet)(\mathfrak{g} \otimes \Omega_\bullet)_0$ is contractible, and thus that the Moore complex

$$\mathscr{M}(d(1_{\mathfrak{g}} \otimes h_\bullet)(\mathfrak{g} \otimes \Omega_\bullet)_0)$$

is acyclic. By applying the Moore complex to the short exact sequence defined above, we obtain that

$$\mathscr{M}(1_{\mathfrak{g}} \otimes p_\bullet) : \mathscr{M}(\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)) \longrightarrow \mathscr{M}(\mathfrak{T}_0(\mathfrak{g} \otimes C_\bullet))$$

is a quasi-isomorphism.

To conclude, consider the commutative square

$$\begin{array}{ccc} \mathscr{M}(\mathfrak{T}_0(\mathfrak{g} \otimes \Omega_\bullet)) & \xrightarrow{\mathscr{M}(1_{\mathfrak{g}} \otimes p_\bullet)} & \mathscr{M}(\mathfrak{T}_0(\mathfrak{g} \otimes C_\bullet)) \\ \downarrow{\scriptstyle \mathscr{M}(\mathrm{MC}_\bullet(\phi))} & & \downarrow{\scriptstyle \mathscr{M}(\phi \otimes 1_{C_\bullet})} \\ \mathscr{M}(\mathfrak{T}_0(\mathfrak{h} \otimes \Omega_\bullet)) & \xrightarrow{\mathscr{M}(1_{\mathfrak{h}} \otimes p_\bullet)} & \mathscr{M}(\mathfrak{T}_0(\mathfrak{h} \otimes C_\bullet)) \end{array}$$

We already know that the horizontal arrows are quasi-isomorphisms. The right vertical arrow is also a quasi-isomorphisms. Indeed, the simplicial vector space $\mathfrak{T}_0(\mathfrak{g} \otimes C_\bullet)$ is the result of applying the Dold–Kan functor Γ of Section 4.3.2 to the truncation

$$\cdots \xrightarrow{d} \mathfrak{g}_{-2} \xrightarrow{d} \mathfrak{g}_{-1} \xrightarrow{d} \mathfrak{T}_0(\mathfrak{g}) \,.$$

By the Dold–Kan correspondence — Theorem 4.3.7 — we have natural isomorphisms

$$\mathfrak{g} \otimes C_\bullet \cong \mathscr{M}(\mathfrak{X}_0(\mathfrak{g} \otimes C_\bullet)) \, .$$

Since ϕ is a quasi-isomorphism, it induces a quasi-isomorphism between the respective truncations of $\mathfrak{g}$ and $\mathfrak{h}$. It follows that $\mathscr{M}(\phi \otimes 1_{C_\bullet})$ is a quasi-isomorphism. Therefore, the morphism $\mathscr{M}(\mathrm{MC}_\bullet(\phi))$ must also be a quasi-isomorphism, which implies that the morphism $\mathrm{MC}_\bullet(\phi)$ itself is a weak equivalence by Theorem 4.3.6. □

6.4.2 The theorem holds for the zeroth homotopy group

One now proves that the statement holds for the 0th homotopy group, without any further assumption on the algebras or on the morphism than the ones of the theorem.

Theorem 6.4.3 ([DR15, Sect. 3]). *Let $\mathfrak{g}, \mathfrak{h}$ and $\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ be as in the statement of Theorem 6.4.1. The induced map*

$$\pi_0(\mathrm{MC}_\bullet(\Phi)) : \pi_0(\mathrm{MC}_\bullet(\mathfrak{g})) \longrightarrow \pi_0(\mathrm{MC}_\bullet(\mathfrak{h}))$$

is bijective.

The proof of this result is technical and not so important for the rest of the present work, and will therefore be omitted.

Remark 6.4.4. *Proposition 10.2.3 provides a clean alternative to the proof of [DR15, Lemma B.2], a technical lemma of fundamental importance for the proof of Theorem 6.4.3, partially simplifying the proof of this result. Notice that the proof of Proposition 10.2.3 is independent of Theorem 6.4.3.*

6.4.3 Setting the basepoint to zero

In order to avoid having to worry about the basepoints, one shows that these can always be set to be 0. This step can probably be avoided, but it has the advantage of making the rest of the proof much cleaner.

Lemma 6.4.5. *Let $\alpha \in \mathrm{MC}(\mathfrak{g})$, and let $\mathfrak{g}^\alpha$ be the $\mathfrak{sL}_\infty$-algebra obtained by twisting $\mathfrak{g}$ by α, that is the $\mathfrak{sL}_\infty$-algebra with the same underlying graded vector space, but with differential*

$$d^\alpha(x) \coloneqq dx + \sum_{n \geq 2} \frac{1}{(n-1)!} \ell_n(\alpha, \dots, \alpha, x)$$

and brackets

$$\ell^\alpha(x_1, \dots, x_m) \coloneqq \sum_{n \geq m} \frac{1}{(n-m)!} \ell_n(\alpha, \dots, \alpha, x_1, \dots, x_m) \, .$$

Let

$$\mathrm{Shift}_\alpha : \mathrm{MC}_\bullet(\mathfrak{g}^\alpha) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{g})$$

be the isomorphism of simplicial sets induced by the morphism of $\mathfrak{sL}_\infty$-algebras given by

$$\beta \in \mathfrak{g}^\alpha \longmapsto \alpha + \beta \in \mathfrak{g} \, .$$

Then the following diagram commutes

$$\begin{array}{ccc} \mathrm{MC}_\bullet(\mathfrak{g}^\alpha) & \xrightarrow{\mathrm{Shift}_\alpha} & \mathrm{MC}_\bullet(\mathfrak{g}) \\ \downarrow{\scriptstyle \Phi^\alpha} & & \downarrow{\scriptstyle \Phi} \\ \mathrm{MC}_\bullet(\mathfrak{h}^\alpha) & \xrightarrow{\mathrm{Shift}_{\mathrm{MC}(\Phi)(\alpha)}} & \mathrm{MC}_\bullet(\mathfrak{h}) \end{array}$$

where

$$\mathrm{MC}_\bullet(\Phi^\alpha)(\beta) \coloneqq \sum_{k\geq 1} \phi_k^\alpha(\beta^{\otimes k})$$

and

$$\phi_k^\alpha(\beta_1 \otimes \cdots \otimes \beta_k) \coloneqq \sum_{j\geq 0} \frac{1}{j!}\phi_{k+j}(\alpha^{\otimes j} \otimes \beta_1 \otimes \ldots \otimes \beta_k)$$

is the twist of Φ by the Maurer–Cartan element α. Here, we identified $\alpha \in \mathfrak{g}$ with $\alpha \otimes 1 \in \mathfrak{g} \otimes \Omega_\bullet$.

The proof is a straightforward unwinding of definitions, and will be omitted here. A conceptual approach to this kind of twisting procedures is given in [DSV18]. Given $\mathfrak{sL}_\infty$-algebras $\mathfrak{g}, \mathfrak{h}$ with basepoints $\alpha \in \mathrm{MC}(\mathfrak{g})$ and $\mathrm{MC}(\Phi)(\alpha) \in \mathrm{MC}(\mathfrak{h})$, then the shifts in the lemma above change the basepoints to $0 \in \mathfrak{g}$ and $0 \in \mathfrak{h}$.

6.4.4 The induction step

We are now set to do the induction step. Let $n \geq 2$, then we denote

$$\mathfrak{g}^{(n)} \coloneqq \mathfrak{g}/\mathscr{F}_n\mathfrak{g}\,,$$

and similarly for $\mathfrak{h}$, and also

$$\Phi^{(n)} : \mathfrak{g}^{(n)} \rightsquigarrow \mathfrak{h}^{(n)}$$

the induced map, which is well defined because Φ is filtered.

Proposition 6.4.6. *For each $n \geq 2$, the filtered quasi-isomorphism Φ induces a weak equivalence of simplicial sets*

$$\mathrm{MC}_\bullet(\Phi^{(n)}) : \mathrm{MC}_\bullet(\mathfrak{g}^{(n)}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h}^{(n)})\,.$$

Proof. The statement is true for $n = 2$ by Proposition 6.4.2, because the $SLoo$-algebras $\mathfrak{g}^{(2)}$ and $\mathfrak{h}^{(2)}$ are abelian. Now suppose that the statement is true up to some n. We will prove that then it also holds for $n + 1$.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_n\mathfrak{g}/\mathscr{F}_{n+1}\mathfrak{g} & \longrightarrow & \mathfrak{g}^{(n+1)} & \longrightarrow & \mathfrak{g}^{(n)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Phi^{(n+1)}} & & \downarrow{\scriptstyle \Phi^{(n)}} & & \\ 0 & \longrightarrow & \mathscr{F}_n\mathfrak{h}/\mathscr{F}_{n+1}\mathfrak{h} & \longrightarrow & \mathfrak{h}^{(n+1)} & \longrightarrow & \mathfrak{h}^{(n)} & \longrightarrow & 0 \end{array}$$

where the horizontal maps are the natural inclusions and projections, and the vertical ones are all induced by Φ. Notice that the leftmost vertical arrow is in fact a strict morphism, induced by the linear part ϕ_1 of Φ. In particular, it is a quasi-isomorphism between abelian $\flat\mathscr{L}_\infty$-algebras, and thus induces a weak equivalence of simplicial sets between the respective Maurer–Cartan spaces by Proposition 6.4.2. The ∞-morphism $\Phi^{(n)}$ also induces a weak equivalence between Maurer–Cartan spaces by induction hypothesis. Therefore, applying the Maurer–Cartan functor we have

$$\begin{array}{ccccc}
\mathrm{MC}_\bullet(\mathscr{F}_n\mathfrak{g}/\mathscr{F}_{n+1}\mathfrak{g}) & \longrightarrow & \mathrm{MC}_\bullet(\mathfrak{g}^{(n+1)}) & \longrightarrow & \mathrm{MC}_\bullet(\mathfrak{g}^{(n)}) \\
\downarrow\wr & & \downarrow & & \downarrow\wr \\
\mathrm{MC}_\bullet(\mathscr{F}_n\mathfrak{h}/\mathscr{F}_{n+1}\mathfrak{h}) & \longrightarrow & \mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)}) & \longrightarrow & \mathrm{MC}_\bullet(\mathfrak{h}^{(n)})
\end{array}$$

Since the canonical projection $\mathfrak{g}^{(n+1)} \to \mathfrak{g}^{(n)}$ is surjective, the induced map on the Maurer–Cartan spaces is a fibration of simplicial sets by Theorem 6.2.6, and the term on the left in the diagram above is easily checked to be the fibre above 0. The same thing holds for $\mathfrak{h}$. We pass to the long exact sequences of homotopy groups — which is in fact exact everywhere except at π_0 — and using the 5-lemma conclude that $\mathrm{MC}_\bullet(\Phi^{(n+1)})$ induces bijections between the respective kth homotopy groups, for all $k \geq 2$. The case $k = 0$ was covered by Theorem 6.4.3. We are left to check the case $k = 1$. The relevant part of the long sequence is

$$\begin{array}{ccccccccc}
\pi_2\mathrm{MC}_\bullet(\mathfrak{g}^{(n)}) & \xrightarrow{\partial} & \pi_1\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{g}}{\mathscr{F}_{n+1}\mathfrak{g}}\right) & \longrightarrow & \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n+1)}) & \longrightarrow & \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n)}) & \xrightarrow{\partial} & \pi_0\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{g}}{\mathscr{F}_{n+1}\mathfrak{g}}\right) \\
\downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
\pi_2\mathrm{MC}_\bullet(\mathfrak{h}^{(n)}) & \xrightarrow{\partial} & \pi_1\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{h}}{\mathscr{F}_{n+1}\mathfrak{h}}\right) & \longrightarrow & \pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)}) & \longrightarrow & \pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n)}) & \xrightarrow{\partial} & \pi_0\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{h}}{\mathscr{F}_{n+1}\mathfrak{h}}\right)
\end{array}$$

We need to prove that the central map is an isomorphism, but we cannot directly use the 5-lemma to do it as the rightmost terms are just sets, not groups, and the map between them is only a bijection of sets. The proof, however, is an obstruction theoretical argument similar to the proof of the 5-lemma.

The map is surjective. Let $y \in \pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)})$. Denote by $\overline{y}$ its image in $\pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)})$. Since the relevant vertical map is an isomorphism, there exists an $\overline{x} \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n)})$ mapping to $\overline{y}$. Moreover, since $\overline{y}$ maps to $0 \in \pi_0\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{h}}{\mathscr{F}_{n+1}\mathfrak{h}}\right)$, then $\overline{x}$ maps to $0 \in \pi_0\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{g}}{\mathscr{F}_{n+1}\mathfrak{g}}\right)$. By "exactness" of the sequence, there exists an $x \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n+1)})$ mapping to $\overline{x}$. Denote by y' the image of x in $\pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)})$. It is not necessarily equal to y. However, we have that $y'y^{-1}$ maps to $0 \in \pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n)})$, and thus there exists an element $z \in \pi_1\mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{g}}{\mathscr{F}_{n+1}\mathfrak{g}}\right)$ mapping to it by doing first the vertical arrow, and then the horizontal arrow. Denote by $x' \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n+1)})$ the image of z under the horizontal arrow. Then $(x')^{-1}x$ maps to y, proving that the middle vertical map is surjective.

The map is injective. Suppose $x \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n+1)})$ maps to $0 \in \pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)})$. Then, since the map $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n)}) \to \pi_1\mathrm{MC}_\bullet(\mathfrak{h}^{(n)})$ is an isomorphism, it follows that x maps to $0 \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(n)})$,

and thus that there exists $y \in \pi_1 \mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{g}}{\mathscr{F}_{n+1}\mathfrak{g}}\right)$ mapping to x. Let $\overline{y} \in \pi_1 \mathrm{MC}_\bullet\left(\frac{\mathscr{F}_n\mathfrak{g}}{\mathscr{F}_{n+1}\mathfrak{g}}\right)$ be the image of y. Then $\overline{y}$ maps to $0 \in \pi_1 \mathrm{MC}_\bullet(\mathfrak{h}^{(n+1)})$, and it follows that there exists $\overline{z} \in \pi_2 \mathrm{MC}_\bullet(\mathfrak{h}^{(n)})$ mapping to it. Let $z \in \pi_2 \mathrm{MC}_\bullet(\mathfrak{g}^{(n)})$ be the preimage of $\overline{z}$. Then z maps to y, and by exactness of the long sequence, it follows that $x = 0$, concluding the proof. □

6.4.5 Conclusion of the proof

The only step missing for the proof of Theorem 6.4.1 is a passage to the limit.

Proposition 6.4.6, together with all we have said before, shows that $\mathrm{MC}_\bullet(\Phi^{(n)})$ is a weak equivalence for all $n \geq 2$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{MC}_\bullet(\mathfrak{g}^{(4)}) & \xrightarrow{\sim} & \mathrm{MC}_\bullet(\mathfrak{h}^{(4)}) \\
\downarrow & & \downarrow \\
\mathrm{MC}_\bullet(\mathfrak{g}^{(3)}) & \xrightarrow{\sim} & \mathrm{MC}_\bullet(\mathfrak{h}^{(3)}) \\
\downarrow & & \downarrow \\
\mathrm{MC}_\bullet(\mathfrak{g}^{(2)}) & \xrightarrow{\sim} & \mathrm{MC}_\bullet(\mathfrak{h}^{(2)})
\end{array}$$

where all objects are Kan complexes, all horizontal arrows are weak equivalences, and all vertical arrows are fibrations of simplicial sets by Theorem 6.2.6. It follows that the collection of horizontal arrows defines a weak equivalence between fibrant objects in the model category of tower of simplicial sets, see [GJ09, Sect. VI.1]. The functor from towers of simplicial sets to simplicial sets given by taking the limit is right adjoint to the constant tower functor, which trivially preserves cofibrations and weak equivalences. Thus, the constant tower functor is a left Quillen functor, and it follows that the limit functor is a right Quillen functor. In particular, it preserves weak equivalences between fibrant objects. Applying this to the diagram above proves that $\mathrm{MC}_\bullet(\Phi)$ is a weak equivalence.

6.5 The formal Kuranishi theorem

In this section, we present some results of Bandiera [Ban14], [Ban17], which in particular give another construction of Getzler's ∞-groupoid $\gamma_\bullet(\mathfrak{g})$ for a $\mathfrak{s}\mathscr{L}_\infty$-algebra $\mathfrak{g}$. In particular, the results presented here will give a very important link between the results of Chapter 10 and the theory developed in the present chapter.

6.5.1 Complete contractions and the homotopy transfer theorem

The first results of [Ban17] we present give a categorical point of view on the homotopy transfer theorem. They were originally stated for $\mathscr{L}_\infty$-algebras, but we give a slightly more general

version, using the generalized homotopy transfer theorem, see Theorem 9.3.1.

We place ourselves in the context of proper complete chain complexes and algebras, cf. Appendix B.

Definition 6.5.1. *A* complete contraction *is a contraction of chain complexes*

$$h \circlearrowright V \underset{i}{\overset{p}{\rightleftarrows}} W$$

such that $(V, \mathscr{F}_\bullet V)$ is a proper complete chain complex, and the chain maps h and ip are filtered.

Given such a complete contraction, one defines the filtration

$$\mathscr{F}_n W \coloneqq i^{-1}(\mathscr{F}_n V)$$

on W.

Lemma 6.5.2. *The filtration $\mathscr{F}_\bullet W$ is unique with respect to the property that both p and i are filtered morphisms. The filtered chain complex $(W, \mathscr{F}_\bullet W)$ is proper complete.*

Proof. The fact that i, p are bot continuous with respect to $\mathscr{F}_\bullet W$ is obvious. Conversely, let $\mathscr{F}_\bullet W$ be any filtration of W such that i, p are filtered. Then for every $x \in \mathscr{F}_n W$ we must have $i(x) \in \mathscr{F}_n V$. Conversely, if $x \in i^{-1}(\mathscr{F}_n V)$, then $i(x) \in \mathscr{F}_n V$, and thus

$$x = pi(x) \in \mathscr{F}_n W \ .$$

The first statement follows.

To show that W is complete, notice that since they are filtered, the maps i and p induce chain maps

$$i : W/\mathscr{F}_n W \longrightarrow V/\mathscr{F}_n V \qquad \text{and} \qquad p : V/\mathscr{F}_n V \longrightarrow W/\mathscr{F}_n W$$

at every level of the filtration, which still satisfy $pi = 1_{W/\mathscr{F}_n W}$. In particular, p is surjective at all levels of the filtration. Then

$$W = p(V) \cong p\left(\lim_n V/\mathscr{F}_n V\right) \cong \lim_n W/\mathscr{F}_n W \ ,$$

where in the last identification we used the fact that p is filtered to switch it over the limit. □

Definition 6.5.3. *A* morphism ϕ between complete contractions *is*

$$\begin{array}{ccc} h \circlearrowright V & \underset{i}{\overset{p}{\rightleftarrows}} & W \\ \Big\downarrow \phi & & \\ h' \circlearrowright V' & \underset{i'}{\overset{p'}{\rightleftarrows}} & W' \end{array}$$

such that $\phi : V \to V'$ is filtered, and such that $h'\phi = \phi h$. The composite of two such morphisms is simply given by composing the vertical maps of the diagram above. This notion of morphism makes complete contractions into a category, which we denote by cCtr.

There are two obvious “projection" functors

$$\mathrm{pr}_{1,2} : \mathsf{cCtr} \longrightarrow \mathsf{cCh}$$

giving back V and W respectively. Their action on morphisms of complete contractions are given by

$$\mathrm{pr}_1(\phi) \coloneqq \phi : V \longrightarrow V', \quad \text{and} \quad \mathrm{pr}_2(\phi) \coloneqq p'\phi i : W \to W'$$

respectively. The fact that pr_2 commutes with compositions follows from the fact that in a contraction we have $pi = 1_W$.

The homotopy transfer theorem is well behaved with respect to complete contractions.

Theorem 6.5.4. *Let $\mathscr{C}$ be a cooperad, and suppose we have a complete contraction*

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

where A is a proper complete $\Omega\mathscr{C}$-algebra with respect to the given filtration on A. Then the transfered structure makes B in a proper complete $\Omega\mathscr{C}$-algebra, and the induced ∞-quasi-isomorphisms i_∞ and p_∞ are filtered ∞-quasi-isomorphisms. Moreover, if we have two such contractions and a morphism ϕ of complete contractions which is a strict morphism of $\Omega\mathscr{C}$-algebras, then $\mathrm{pr}_2(\phi)$ also is a strict morphism of $\Omega\mathscr{C}$-algebras.

A couple further compatibility results can be found in [Ban17, Sect. 1]. These imply the following result. Consider the filtered product $\widehat{\Omega\mathscr{C}}\text{-}\mathsf{alg} \times_{\mathsf{cCh}} \mathsf{cCtr}$ given by

$$\begin{array}{ccc} \widehat{\Omega\mathscr{C}}\text{-}\mathsf{alg} \times_{\mathsf{cCh}} \mathsf{cCtr} & \longrightarrow & \mathsf{cCtr} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \widehat{\Omega\mathscr{C}}\text{-}\mathsf{alg} & \xrightarrow{(-)^{\#}} & \mathsf{cCh} \end{array}$$

Proposition 6.5.5. *The homotopy transfer theorem gives a functor*

$$\widehat{\Omega\mathscr{C}}\text{-}\mathsf{alg} \times_{\mathsf{cCh}} \mathsf{cCtr} \longrightarrow \widehat{\Omega\mathscr{C}}\text{-}\mathsf{alg}\ .$$

6.5.2 The formal Kuranishi theorem

We specialize now to $\mathfrak{sL}_\infty$-algebras. Suppose we have a complete contraction

$$h \circlearrowright \mathfrak{g} \underset{i}{\overset{p}{\rightleftarrows}} \mathfrak{h}$$

with $\mathfrak{g}$ a $\mathfrak{sL}_\infty$-algebra, and consider the induced $\mathfrak{sL}_\infty$-algebra structure on $\mathfrak{h}$.

Theorem 6.5.6 ([Ban17, Thm. 1.13])**.** *The map of sets*

$$\mathrm{MC}(\mathfrak{g}) \longrightarrow \mathrm{MC}(\mathfrak{h}) \times h(\mathfrak{g}_0)$$

given by

$$x \longmapsto (\mathrm{MC}(p_\infty)(x), h(x))$$

is bijective.

The proof is done by considering and solving recursively certain fixed point equations in nilpotent $\mathfrak{sL}_\infty$-algebras, and then concluding by passing to the limit. We will not do do it here, and we refer the reader to the original references [Ban17, Thm. 1.13] and [Ban14, Thm. 2.3.3].

Remark 6.5.7. *The reason behind the name of this theorem is the fact that it generalizes a result by Kuranishi [Kur62]. See [Get18, p. 3] for a nice exposition.*

We are mostly interested by the following consequence. Consider the contraction

$$1_{\mathfrak{g}} \otimes h_\bullet \circlearrowright \mathfrak{g} \otimes \Omega_\bullet \underset{1_{\mathfrak{g}} \otimes i_\bullet}{\overset{1_{\mathfrak{g}} \otimes p_\bullet}{\rightleftarrows}} \mathfrak{g} \otimes C_\bullet$$

induced by Dupont's contraction. Theorem 6.5.6 implies the following.

Corollary 6.5.8. *The map*

$$\gamma_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$$

given by

$$x \longmapsto \mathrm{MC}((1_{\mathfrak{g}} \otimes p)_\infty)(x)$$

is an isomorphism of simplicial sets.

Proof. The result is a direct consequence of applying Theorem 6.5.6 to the contraction above, and considering the restriction of the bijection to $\gamma_\bullet(\mathfrak{g}) = \mathrm{MC}_\bullet(\mathfrak{g}) \cap \ker(1_{\mathfrak{g}} \otimes h_\bullet)$. □

Chapter 7

Deformation theory

The goal of this chapter is to motivate the interest in the space of Maurer–Cartan elements of Lie and homotopy Lie algebras. The archetypal deformation problem goes like this. One is given some fixed object, say a vector space V, or a manifold M, and a structure on this object, such as an associative algebra structure on V, or a complex structure on M. Then the goal is to understand how one can "deform" the given structure in such a way as to obtain another structure of the same kind on the same base object, and to understand which deformations give us structures that are isomorphic to the original one, and which ones don't. This way, one can obtain information about the original structure.

Lie algebras, their Maurer–Cartan elements, and gauge equivalences appear naturally in deformation theory, and provide a generalized framework to study deformation problems. In fact, it is a "philosophical principle" attributed to Deligne [Del87] that, in characteristic zero, every deformation problem corresponds to the study of Maurer–Cartan elements and the gauges between them in a (possibly homotopy) Lie algebra. This principle has been formalized in recent years by Pridham [Pri10] and Lurie [Lur11] to give a theorem in the context of ∞-categories.

Deformation theory is a deep and beautiful subject with a long history and plenty of applications. Its study goes back to the works of Grothendieck, Artin, Quillen and many others, up until the more recent works of Kontsevich in deformation quantization and mirror symmetry, and the already mentioned formalization of Deligne's principle by Pridham and Lurie.

We will introduce deformation theory by the means of an algebraic example: deformations of associative algebra structures on a vector space. We will begin by talking about infinitesimal deformations, before passing to general deformations, introducing the deformation complex, and state the fundamental principle of deformation theory. We will then conclude the section by presenting various other examples of deformation problems, both algebraic and geometrical, and the Lie algebras governing them.

In this section, we work in cochain complexes. Some good introductory references for deformation theory are e.g. the notes of Kontsevich [Kon94] and [Sze99].

7.1 Deformation theory through an example

In this section, we attempt to give an overview of deformation theory through the archetypal example of deformations of associative algebra structures on a vector space. We begin by explaining what an infinitesimal deformation is, before passing to more general deformations, and

then presenting the differential graded Lie algebra governing this specific deformation problem. We conclude the section by presenting the fundamental principle of deformation theory, due to P. Deligne, which relates deformation problems in characteristic 0 to Maurer–Cartan elements of (homotopy) Lie algebras.

The exposition given here is heavily based on the introductory note to deformation theory [Sze99] by B. Szendrői.

7.1.1 Infinitesimal deformations

Let A be an associative algebra, not differential graded. In other words, A is a finite dimensional vector space together with a linear map

$$m : A \otimes A \longrightarrow A$$

satisfying the associativity condition

$$m(m(a,b),c) = m(a,m(b,c))$$

for all $a, b, c \in A$.

One wants now to "infinitesimally deform" m in order to obtain another associative algebra structure on A. In order to do this, one considers structures of the form $\widetilde{m} \coloneqq m + \epsilon f$ with $f \in \hom(A \otimes A, A)$ and ϵ a formal parameter satisfying $\epsilon^2 = 0$. This can be formalized by saying that we are looking at elements of

$$\hom(A \otimes A, A) \otimes \mathbb{K}[\epsilon]/(\epsilon^2) \,,$$

which corresponds to look at the "tangent space" of our space of structures[1]. The associativity condition now reads

$$\begin{aligned} 0 &= \widetilde{m}(\widetilde{m}(a,b),c) - \widetilde{m}(a,\widetilde{m}(b,c)) \\ &= m(m(a,b),c) - m(a,m(b,c)) + \epsilon\big(f(m(a,b),c) - f(a,m(b,c)) + m(f(a,b),c) - m(a,f(b,c)\big) \\ &= \epsilon\big(f(m(a,b),c) - f(a,m(b,c)) + m(f(a,b),c) - m(a,f(b,c)\big) \end{aligned}$$

where in the second line we used the fact that $\epsilon^2 = 0$, and in the third line the fact that m was associative to begin with. Thus, $\widetilde{m}$ being associative is equivalent to

$$f(m(a,b),c) - f(a,m(b,c)) + m(f(a,b),c) - m(a,f(b,c) = 0 \,.$$

One also wants to understand when a deformed structure is isomorphic to the original one. In order to do this, one considers automorphisms of A of the form $T \coloneqq 1_A + \epsilon g$ for $g \in \hom(A,A)$. Notice that such a morphism is indeed invertible, with its inverse being given by $T^{-1} = 1_A - \epsilon g$. Then one looks at the pullback of the multiplication m by T:

$$Tm(T^{-1}x, T^{-1}y) = m(x,y) + \epsilon\big(\underbrace{g(m(x,y)) - m(g(x),y) - m(x,g(y))}_{f_T \coloneqq}\big) \,.$$

This tells us that a deformed multiplication $\widetilde{m} = m + \epsilon f$ is isomorphic to the original multiplication m if, and only if

$$f = g(m(x,y)) - m(g(x),y) - m(x,g(y))$$

[1] From an algebro-geometrical point of view, $\mathbb{K}[\epsilon]/(\epsilon^2)$ models tangent vectors.

for some $g \in \hom(A, A)$.

This can be wrapped up together nicely as the first levels of a homology theory. Indeed, one considers the Hochschild cochain complex

$$\mathrm{Hoch}(A, m) \coloneqq \left(\hom(\mathbb{K}, A) \xrightarrow{d} \hom(A, A) \xrightarrow{d} \hom(A^{\otimes 2}, A) \xrightarrow{d} \hom(A^{\otimes 2}, A) \xrightarrow{d} \cdots \right) ,$$

where we see $\hom(A^{\otimes n}, A)$ as living in degree $n-1$, with the differential being given by

$$\begin{aligned} d(f)(a_1, \ldots, a_{n+1}) &= \\ &= m(a_1, f(a_2, \ldots, a_{n+1})) + \sum_{k=1}^{n} (-1)^k f(a_1, \ldots, m(a_k, a_{k+1}), \ldots, a_{n+1}) + \\ &\quad + (-1)^{n+1} m(f(a_1, \ldots, a_n), a_{n+1}) \, . \end{aligned}$$

This complex was introduced by G. Hochschild in [Hoc45]. It is an easy exercise to check that this differential squares to zero. Then the infinitesimally deformed associative structures are exactly the cycles in $\mathrm{Hoch}_1(A, m)$, and two such structures are isomorphic if, and only if the respective cycles are cohomologous. Thus, we have

$$\{\text{essentially distinct infinitesimal deformations of } m\} \cong H^1(\mathrm{Hoch}(A, m)) \, .$$

This is what normally happens in general when one tries to deform infinitesimally some kind of structure. One is led to some cochain complex where the cycles at some level correspond to the deformed structures, and which are cohomologous if, and only if the deformed structures are equivalent. This happens because working infinitesimally — i.e. over the ring $\mathbb{K}[\epsilon]/(\epsilon^2)$ — correspond to linearizing the higher structure hiding behind more general deformations: the structure of a Lie algebra, where the cycle equation will be replaced by the Maurer–Cartan equation. One can already see this happening when trying to understand deformations to a higher order, working over $\mathbb{K}[\epsilon]/(\epsilon^n)$, or over an artinian ring in general, and also with formal deformations, i.e. working over the ring of formal power series $\mathbb{K}[[t]]$.

7.1.2 Deformations in general

Now one wants to understand deformations in general. A deformation of the original associative structure m is just

$$\widetilde{m} \coloneqq m + f$$

with $f \in \hom(A \otimes A, A)$. The associativity condition now reads

$$f(m(a,b),c) - f(a,m(b,c)) + m(f(a,b),c) - m(a,f(b,c)) + f(f(a,b),c) - f(a,f(b,c)) = 0 \, ,$$

where now the lat two terms of the left-hand side — which are quadratic in f — do not necessarily vanish. If one writes

$$[f_1, f_2](a,b,c) \coloneqq f_1(f_2(a,b),c) - f_1(a,f_2(b,c)) + f_2(f_1(a,b),c) - f_2(a,f_1(b,c)) \, ,$$

then $[-,-]$ is graded antisymmetric (if one considers f_1, f_2 as elements of degree 1) and satisfies the Jacobi rule. With this in mind, we can write the associativity condition as

$$df + \frac{1}{2}[f,f] = 0 \, ,$$

where d is the differential of the Hochschild complex defined in Section 7.1.1, which can also be written as $d \coloneqq [m, -]$. Thus, assuming that we can extend this bracket to define a Lie bracket on the whole Hochschild complex, deformations of m correspond to Maurer–Cartan elements of $\mathrm{Hoch}_\bullet(A, m)$.

To understand what deformations are equivalent, suppose that A is finite dimensional. To any linear map $\hom(A, A)$ we associate an automorphism of A by

$$e^g = \sum_{n \geq 0} \frac{1}{n!} g^n \in \hom(A, A) \, ,$$

whose inverse is given by e^{-g}. We consider the path $t \mapsto e^{tg}$ and the algebra structure

$$m_t(x, y) \coloneqq e^{tg} m(e^{-tg} x, e^{-tg} y) \, .$$

This can be written as

$$m_t = m + f_t \, ,$$

for $f_t \coloneqq m_t - m \in \hom(A \otimes A, A)$. Then

$$\frac{d}{dt} f_t(x, y) = g m(x, y) + m(g(x), y) + m(x, g(y)) + g f_t(x, y) + f_t(g(x), y) + f_t(x, g(y)) \, .$$

Defining

$$[g, f](x, y) \coloneqq g(f(x, y)) + f(g(x), y) + f(x, g(y))$$

for $f \in \hom(A \otimes A, A)$ and $g \in \hom(A, A)$, we can rewrite this as

$$\frac{d}{dt} f_t = d f_t + [g, f_t] \, ,$$

which one recognizes as the differential equation saying that g is a gauge from m to the pullback $e^g m(e^{-g}, e^{-g})$ of m by e^g.

7.1.3 The deformation complex

Consider the following intrinsic version of the Hochschild complex, corresponding to the trivial algebra structure $m = 0$ on A. The cochain complex is given by

$$\mathrm{Hoch}_n(A) \coloneqq \hom(A^{\otimes (n+1)}, A)$$

for $n \geq -1$, with trivial differential[2]. One defines a Lie bracket on this complex by

$$\begin{aligned} [f, g](a_1, \ldots, a_{n+m-1}) \coloneqq & \sum_{i=1}^{n} (-1)^{im} f(a_1, \ldots, g(a_i, \ldots, a_{i+m-1}), \ldots, a_{n+m-1}) + \\ & + \sum_{j=1}^{n} (-1)^{jn} g(a_1, \ldots, g(a_j, \ldots, a_{j+n-1}), \ldots, a_{n+m-1}) \end{aligned}$$

for $f \in \hom(A^{\otimes n}, A)$ and $g \in \hom(A^{\otimes m}, A)$. This bracket was introduced by M. Gerstenhaber [Ger63], and is therefore usually called the Gerstenhaber bracket. Notice that this recovers the bracket given in Section 7.1.2. The Maurer–Cartan elements of $\mathrm{Hoch}(A)$ are the elements

$$m \in \hom(A \otimes A, A) = \mathrm{Hoch}_1(A)$$

[2] If A is itself a cochain complex, then one takes the differential induced by the differential of A.

such that the Maurer–Cartan equation

$$0 = \frac{1}{2}[m, m] = \frac{1}{2}(m(m \otimes 1_A) - m(1_A \otimes A))$$

is satisfied, i.e. the associative algebra structures on A. Moreover, by what we have seen before, two such algebraic structures are isomorphic if, and only if the corresponding Maurer–Cartan elements are gauge equivalent. Therefore, we have

$$\{\text{isomorphism classes of associative algebra structures on } A\} \cong \overline{\mathrm{MC}}(\mathrm{Hoch}(A)) \ .$$

If one wants to look at deformations of a given associative algebra structure m, i.e. study the elements $f \in \hom(A \otimes A, A)$ such that $m+f$ is again an associative algebra structure on A, then one can do so by means of a *twist*. Namely, one considers the Lie algebra $\mathrm{Hoch}(A, m) \coloneqq \mathrm{Hoch}(A)^m$ which has the same underlying graded vector space, the same Lie bracket, but differential

$$d^m \coloneqq [m, -] \ .$$

Since m is a Maurer–Cartan element, this is again a Lie algebra, and f is a Maurer–Cartan element in $\mathrm{Hoch}(A, m)$ if, and only if $m + f$ is a Maurer–Cartan element in $\mathrm{Hoch}(A)$. Cf. also Lemma 6.4.5.

Finally, if one wants to recover infinitesimal deformations, it suffices to consider the Lie algebra

$$\mathrm{Hoch}(A) \otimes \mathbb{K}[\epsilon]/(\epsilon^2) \ .$$

If m is a Maurer–Cartan element of $\mathrm{Hoch}(A)$, i.e. an associative algebra structure on A, then $m + \epsilon f$ is a Maurer–Cartan element of $\mathrm{Hoch}(A) \otimes \mathbb{K}[\epsilon]/(\epsilon^2)$ if, and only if it is an infinitesimal deformation of m.

7.1.4 The fundamental principle of deformation theory

One might think that the situation above is specific to the case of associative algebras, or maybe of some specific algebraic situations. This is not the case, and a plethora of examples have been found during the years, both in the algebraic setting, in geometrical problems, and even in situations strictly linked with physics. We will expose some of those in Section 7.2. In fact, this situation is so ubiquitous that it led P. Deligne [Del87] to formulate the following heuristic principle, often referred to as the *fundamental principle of deformation theory*:

> "When working over a field of characteristic zero, all deformation problems can be formulated as the study of the Maurer–Cartan elements in a differential graded Lie algebra."

More generally, one might want to consider homotopy Lie algebras instead of strict Lie algebras. To be more precise, the principle above means that for every deformation problem there exists a (homotopy) Lie algebra $\mathfrak{g}$ such that one can establish the following dictionary:

Deformation problem	**Homotopy Lie algebra $\mathfrak{g}$**
Structures of the desired type	Maurer–Cartan elements $\mathrm{MC}(\mathfrak{g})$
Equivalences between structures	Gauges
Infinitesimal deformations	Study of $\mathfrak{g} \otimes \mathbb{K}[\epsilon]/(\epsilon^2)$
Formal deformations	Study of $\mathfrak{g} \otimes \mathbb{K}[[t]]$

And so on. One can also identify the space of infinitesimal deformation of a fixed Maurer–Cartan element α with the (formal) tangent space of $\mathrm{MC}(\mathfrak{g}^\alpha)$, and thus deduce e.g. that the dimension of the moduli space of the structures of the desired type at α is the same as the dimension of the cohomology $H^1(\mathfrak{g}^\alpha)$.

To conclude this section, we should mention that while the principle exposed above is just a heuristic, it has in fact been made into a formal result in the context of ∞-categories by Pridham [Pri10] and Lurie [Lur11]. There, a sensible ∞-category of deformation problems is defined, and it is shown that it is equivalent to the category of differential graded Lie algebras as an ∞-category through what is essentially the Maurer–Cartan functor.

7.2 Other examples and applications

In order to demonstrate the importance of deformation theory in mathematics, we sketch various classical examples and applications.

7.2.1 Lie algebras and commutative algebras

If one is interested in deforming a fixed Lie algebra structure on a vector space $\mathfrak{g}$, then proceeding as for associative algebras one is led to consider the Chevalley–Eilenberg complex

$$\mathrm{CE}_n(\mathfrak{g}) \coloneqq \hom(\Lambda^{n+1}A, A)$$

for $n \geq -1$, where $\Lambda^n A$ denotes the nth exterior power of A, with differential given by

$$\begin{aligned} d(f)(x_0, \dots, x_n) \coloneqq &\sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n) + \\ &+ \sum_{k=1}^{n+1} (-1)^{k+1} [x_k, f(x_0, \dots, \widehat{x}_k, \dots, x_n)] \,, \end{aligned}$$

where the hat denotes omission. This cochain complex was introduced by C. Chevalley and S. Eilenberg in [CE48]. One puts the following bracket on the Chevalley–Eilenberg complex. For $f \in \hom(\Lambda^n A, A)$ and $g \in \hom(\Lambda^m A, A)$, one defines

$$\begin{aligned} [f, g](x_1, \dots, x_{n+m-1}) \coloneqq &\sum_{\sigma \in \sqcup\!\sqcup(m, n-1)} (-1)^\sigma f(g(x_{\sigma(1)}, \dots, x_{\sigma(m)}), x_{\sigma(m+1)}, \dots, x_{\sigma(n+m-1)}) \\ &+ \sum_{\sigma \in \sqcup\!\sqcup(n, m-1)} (-1)^\sigma g(f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), x_{\sigma(n+1)}, \dots, x_{\sigma(n+m-1)}) \,. \end{aligned}$$

This bracket makes the Chevalley–Eilenberg complex into a differential graded Lie algebra, providing the deformation complex for Lie algebra structures on $\mathfrak{g}$.

For commutative algebra structures, the situation is similar. The cochain complex one considers is the Harrison complex, which was defined in [Har62] by D. K. Harrison. Its explicit description is more complicated, and we will not carry it out here.

7.2.2 Algebras over an operad

More generally, let $\mathscr{P} \coloneqq \Omega\mathscr{C}$ be a cofibrant operad, where $\mathscr{C}$ is a reduced cooperad. Fix a chain complex A. Then the deformation complex of $\mathscr{P}$-algebra structures on A is the Lie algebra

$$\mathrm{Def}_{\mathscr{P}}(A) \coloneqq \hom(\mathscr{C}, \mathrm{End}_A) \ ,$$

where the Lie bracket is given by

$$[f, g] \coloneqq f \star g - g \star f \ ,$$

with $\star$ the pre-Lie product described in Section 2.2.5. One can in fact directly work in the pre-Lie setting to obtain powerful results, see [DSV16]. The fact that this is the correct deformation complex is given by the following result, which follows from Theorem 2.2.17 and [DSV16, Thm. 3].

Theorem 7.2.1. *The Maurer–Cartan elements of $\mathrm{Def}_{\mathscr{P}}(A)$ are in bijective correspondence with the possible $\mathscr{P}$-algebra structures on A. Moreover, two Maurer–Cartan elements are gauge equivalent if, and only if the respective $\mathscr{P}$-algebra structures are ∞-isotopic, i.e. if they are linked by an ∞-morphism whose first component is the identity.*

One can recover the Hochschild, Chevalley–Eilenberg, and Harrison complex as subcomplexes of $\mathrm{Def}_{\mathscr{P}}(A)$ if one takes $\mathscr{P} = \mathrm{As}$, Lie, and Com respectively.

7.2.3 Complex analytic structures on a manifold

All the examples we have given until now are algebraic in nature, but deformation theory also works in geometrical situations. One such example is the deformation of complex analytic structures on manifolds, due to K. Kodaira and D. C. Spencer [KS58a], [KS58b]. The deformation complex in this case is given in degree $n \geq 0$ by the cochain complex of holomorphic vector fields tensor $(0, n)$-forms.

A first intuition on this problem goes back to Riemann, who in 1857 famously calculated that the "number of independent parameters" on which the deformations of a complex structure on a closed Riemann surface of genus g, i.e. the dimension of the moduli space of complex structures, is $3g - 3$.

7.2.4 Deformation quantization

A last famous example is due to M. Kontsevich [Kon03], and is closely related to mathematical physics, more precisely to the quantization procedure.

The physical motivation is as follows. One encodes physics via a "space of states" given by a manifold M and "observables", the smooth functions on M. Physics is then given by how the observables interact. If one wants to describe classical mechanics, then one must ask for a Poisson structure on $C^\infty(M)$. For quantum mechanics, one must give a star product on $C^\infty(M)$, which is a deformation of the usual pointwise product of functions as a formal power series in a parameter $\hbar$. One can recover a Poisson bracket in the first order of the expansion of a star product, which is interpreted as the procedure of getting back classical physics from quantum mechanics. The question is how to go the other way: given a Poisson structure on M, is it possible to find a star product recovering the Poisson bracket? In other words, is it always possible to quantize classical mechanics?

To give some details, a star product on $C^\infty(M)$ is an associative, $\mathbb{R}[[\hbar]]$-linear product on the space $C^\infty(M)[[\hbar]]$ of formal power series of smooth functions in $\hbar$ of the form

$$f \star g = fg + \sum_{n\geq 1} B_n(f,g)\hbar^n,$$

where the B_n are bidifferential operators on $C^\infty(M)$. One obtains a Poisson bracket from a star product by

$$\{f,g\} \coloneqq B_1(f,g) - B_1(g,f)\ .$$

Two star product are said to be equivalent if they are related by a certain type of gauge relation. Kontsevich associated a deformation complex to star products, and to Poisson brackets, and then proved that the two complexes are linked by a zig-zag of ∞-quasi-isomorphisms. This implies that equivalence classes of Poisson structures are in bijection with equivalence classes of star products, answering the quantization question in the positive.

Part II

Results

Chapter 8

∞-morphisms relative to twisting morphisms

In Section 2.3.3, a notion of ∞-morphisms between $\mathscr{P}_\infty$-algebras was introduced, where $\mathscr{P}$ is a Koszul operad. Namely, such an ∞-morphism is the same thing as a morphism of conilpotent $\mathscr{P}^{\text{¡}}$-coalgebras between the bar constructions of the algebras relative to the canonical twisting morphism

$$\iota : \mathscr{P}^{\text{¡}} \longrightarrow \mathscr{P}_\infty \, ,$$

cf. Notation 2.2.19. There is a natural, useful generalization of this notion. Namely, one can consider morphisms of coalgebras between the bar construction with respect to *any* twisting morphism

$$\alpha : \mathscr{C} \longrightarrow \mathscr{P} \, .$$

Dually, this gives us a notion of ∞-morphism between conilpotent coalgebras by considering morphisms of algebras between their cobar constructions relative to a twisting morphism. Versions of this idea have already been used more or less explicitly in the literature e.g. in [Mar04], [Ber14a], and [Ber14b], at least for algebras. In this section, we give a precise definition of this generalized notion of ∞-morphisms, and then study some of their properties. In particular, we look in detail at their homotopical behavior.

We expect the theory presented here to work without great changes for morphisms between connected weight graded (co)operads. We give here the case of reduced (co)operads for ease of presentation, and because all the cases of more immediate interest (e.g. associative, commutative, and Lie algebras, as well as their up to homotopy counterparts) are included in this framework.

Most of the material of this chapter is extracted from the article [RNW17].

8.1 Basic definitions and rectifications

In this section, we define the notion of ∞-morphisms of algebras and coalgebras relative to a twisting morphism, and study some of their basic properties.

8.1.1 Basic definitions

The definition of ∞-morphisms of algebras and coalgebras relative to an arbitrary twisting morphism is the following one.

Definition 8.1.1. *Let $\mathscr{C}$ be a cooperad, let $\mathscr{P}$ be an operad, and let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism.*

1. *An* ∞-morphism of $\mathscr{P}$-algebras relative to α, *or an* ∞_α-morphism of $\mathscr{P}$-algebras, *between two $\mathscr{P}$-algebras A and A' is a morphism Ψ of $\mathscr{C}$-coalgebras*

$$\Psi : \mathrm{B}_\alpha A \longrightarrow \mathrm{B}_\alpha A' .$$

Composition of ∞_α-morphisms of $\mathscr{P}$-algebras is given by the standard composition of morphisms of $\mathscr{C}$-coalgebras between the bar constructions. We denote the category of $\mathscr{P}$-algebras with ∞_α-morphisms by ∞_α-$\mathscr{P}$-alg.

2. *An* ∞-morphism of conilpotent $\mathscr{C}$-coalgebras relative to α, *or an* ∞_α-morphism of conilpotent $\mathscr{C}$-algebras, *between two conilpotent $\mathscr{C}$-algebras D' and D is a morphism Φ of $\mathscr{P}$-algebras*

$$\Phi : \Omega_\alpha D' \longrightarrow \Omega_\alpha D .$$

Composition of ∞_α-morphisms of $\mathscr{C}$-coalgebras is given by the standard composition of morphisms of $\mathscr{P}$-algebras between the cobar constructions. We denote the category of conilpotent $\mathscr{C}$-coalgebras with ∞_α-morphisms by ∞_α-$\mathscr{C}$-cog.

Remark 8.1.2. *If $\mathscr{P}$ is a Koszul operad and*

$$\iota : \mathscr{P}^{\mathrm{i}} \longrightarrow \mathscr{P}_\infty$$

is the canonical twisting morphism, then the notion of ∞_ι-morphisms of $\mathscr{P}_\infty$-algebras coincides with the classical one, cf. Proposition 2.4.3.

When confronted with such a definition, it is natural to wonder what happens to the notion of ∞_α-morphism under changes in the twisting morphism α. Here is a first result in that direction.

Lemma 8.1.3. *Let $\mathscr{C}', \mathscr{C}$ be two cooperads and let $\mathscr{P}$ be an operad. Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$, let $f : \mathscr{C}' \to \mathscr{C}$ be a morphism of cooperads, and let D be a conilpotent $\mathscr{C}'$-coalgebra. Then*

$$\Omega_\alpha(f_* C) = \Omega_{f^*\alpha} C .$$

In particular, $\infty_{f^*\alpha}$-morphisms between conilpotent $\mathscr{C}'$-coalgebras are the same as ∞_α-morphisms between the same coalgebras seen as $\mathscr{C}$-coalgebras by pushforward of the structure along f.

Dually, let $\mathscr{C}$ be a cooperad and let $\mathscr{P}, \mathscr{P}'$ be two operads. Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$, let $g : \mathscr{P} \to \mathscr{P}'$ be a morphism of operads, and let A be a $\mathscr{P}'$-algebra. Then

$$\mathrm{B}_\alpha(g^* A) = \mathrm{B}_{g_*\alpha} A .$$

In particular, $\infty_{g_*\alpha}$-morphisms between $\mathscr{P}'$-algebras are the same as ∞_α-morphisms between the same algebras seen as $\mathscr{P}$-algebras by pullback of the structure along g.

Proof. We only prove the first of the two facts, the proof of the second one being dual. As algebras over graded vector spaces, it is clear that we have

$$\Omega_{f^*\alpha} C = \mathscr{P}(C) = \Omega_\alpha(f_* C) ,$$

so that we only have to check that the differentials agree. We have

$$d_{\Omega_{f^*\alpha}C} = d_{\mathscr{P}(C)} + d_2^{f^*\alpha}$$

with $d_2^{f^*\alpha}$ given by the composite

$$\mathscr{P}(C) \xrightarrow{1_{\mathscr{P}} \circ' \Delta_C} \mathscr{P} \circ (C; \mathscr{C}' \circ C) \xrightarrow{1_{\mathscr{P}} \circ (1_C; f^*\alpha \circ 1_C)} \mathscr{P} \circ (C; \mathscr{P} \circ C) \cong (\mathscr{P} \circ_{(1)} \mathscr{P})(C) \xrightarrow{\gamma_{(1)} \circ 1_C} \mathscr{P}(C) .$$

The part $d_{\mathscr{P}(C)}$ is independent of the twisting morphism, and thus of no interest to us. For the other part, we notice that

$$\begin{aligned}(1_{\mathscr{P}} \circ (1_C; f^*\alpha \circ 1_C))(1_{\mathscr{P}} \circ' \Delta_C) &= (1_{\mathscr{P}} \circ (1_C; \alpha f \circ 1_C))(1_{\mathscr{P}} \circ' \Delta_C) \\ &= (1_{\mathscr{P}} \circ (1_C; \alpha \circ 1_C))(1_{\mathscr{P}} \circ' f\Delta_C) \\ &= (1_{\mathscr{P}} \circ (1_C; \alpha_1 \circ 1_C))(1_{\mathscr{P}} \circ' \Delta_{f_*C}) ,\end{aligned}$$

which implies the result. □

8.1.2 Rectifications

Suppose we have a commutative diagram

$$\begin{array}{ccc} \mathscr{C}' & & \\ {\scriptstyle f}\downarrow & \searrow^{f^*\alpha} & \\ \mathscr{C} & \xrightarrow[\alpha]{} & \mathscr{P} \end{array}$$

where α is a twisting morphism, and where f is a morphism of cooperads. Then Lemma 8.1.3 tells us that whenever we are given two conilpotent $\mathscr{C}'$-coalgebras, the $\infty_{f^*\alpha}$-morphisms between them are exactly the same thing as the ∞_α-morphisms between the same coalgebras seen as $\mathscr{C}$-coalgebras by pushing forward their structure along f. Suppose instead that we are given two conilpotent $\mathscr{C}$-coalgebras. Is it possible to go the other way around and understand the ∞_α-morphisms between them in terms of $\infty_{f^*\alpha}$-morphisms? The dual question is asked for ∞-morphisms of algebras. The answer is not so immediate this time, and goes through what we call rectification functors, in analogy to Section 2.5.1.

Let $\mathscr{C}$ and $\mathscr{C}'$ be two cooperads, let $\mathscr{P}$ be an operad, let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism, and let $f : \mathscr{C}' \to \mathscr{C}$ be a morphism of cooperads. We define the *rectification functor*

$$R_{\alpha,f} : \infty_\alpha\text{-}\mathscr{C}\text{-}\mathsf{cog} \longrightarrow \infty_\alpha\text{-}\mathscr{C}\text{-}\mathsf{cog}$$

by

$$R_{\alpha,f}(C) \coloneqq f_* \mathrm{B}_{f^*\alpha} \Omega_\alpha C$$

on conilpotent $\mathscr{C}$-coalgebras, and

$$R_{\alpha,f}(\Phi) \coloneqq f_* \mathrm{B}_{f^*\alpha} \Phi$$

on ∞_α-morphisms of $\mathscr{C}$-coalgebras. The counit of the bar-cobar adjunction relative to $f^*\alpha$ induces a natural transformation

$$E_C : \Omega_\alpha R_{\alpha,f}(C) = \Omega_{f^*\alpha} \mathrm{B}_{f^*\alpha} \Omega_\alpha C \xrightarrow{\epsilon_{\Omega_\alpha C}} \Omega_\alpha C ,$$

where C is a conilpotent $\mathscr{C}$-coalgebra, and where the equality is given by Lemma 8.1.3. Therefore, we have a natural ∞_α-morphism of conilpotent $\mathscr{C}$-coalgebras

$$E_C : R_{\alpha,f}(C) \rightsquigarrow C$$

from the rectification functor to the identity functor.

Remark 8.1.4. *If f is the identity of $\mathscr{C}$, then E is a strict morphism, namely the counit of the bar-cobar adjunction.*

Dually, let $\mathscr{C}$ be a cooperad, let $\mathscr{P}, \mathscr{P}'$ be operads, let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism, and take $g : \mathscr{P} \to \mathscr{P}'$ a morphism of operads. We define the *rectification functor*

$$R^{g,\alpha} : \infty_\alpha\text{-}\mathscr{P}\text{-alg} \longrightarrow \infty_\alpha\text{-}\mathscr{P}\text{-alg}$$

by

$$R^{g,\alpha}(A) \coloneqq g^*\Omega_{g_*\alpha}\mathrm{B}_\alpha A$$

for a $\mathscr{P}$-algebra A, and

$$R^{g,\alpha}(\Psi) \coloneqq g^*\Omega_{g_*\alpha}\Psi$$

on ∞_α-morphisms. There is a natural transformation N from $R^{g,\alpha}$ to the identity of ∞_α-$\mathscr{P}$-alg given by

$$N_A : \mathrm{B}_\alpha A \xrightarrow{\eta_{\mathrm{B}_\alpha A}} \mathrm{B}_{g_*\alpha}\Omega_{g_*\alpha}\mathrm{B}_\alpha A = \mathrm{B}_\alpha R^{g,a}(A)\ ,$$

where η is the unit of the bar-cobar adjunction relative to $g_*\alpha$. Therefore, we have a natural ∞_α-morphism of $\mathscr{P}$-algebras

$$N_A : A \rightsquigarrow R^{g,\alpha}(A)$$

from the identity to the rectification functor.

Remark 8.1.5. *If g is the identity of $\mathscr{P}$, then N is a strict morphism, namely the unit of the bar-cobar adjunction.*

Next, we will see that the rectification functors are homotopically well behaved.

8.2 Homotopy theory of ∞-morphisms of coalgebras

The homotopy theory for classical ∞-morphisms is quite well known, cf. Section 2.5 and Section 3.3.8. Notice that the theory developed there passes without problems to ∞_α-morphisms of algebras when α is a Koszul morphism[1]. In this section, we will develop the analogous results for coalgebras. For this section, we a Koszul morphism $\alpha : \mathscr{C} \to \mathscr{P}$.

We begin with the following definition.

Definition 8.2.1. *Let $\mathscr{C}$ be a cooperad, let $\mathscr{P}$ be an operad, and let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism.*

1. *An ∞_α-morphism of $\mathscr{P}$-algebras $\Psi : A \rightsquigarrow A'$ is an* α-weak equivalence *if the morphism*

$$\Psi : \mathrm{B}_\alpha A \longrightarrow \mathrm{B}_\alpha A'$$

is a weak equivalence of coalgebras in the category of conilpotent $\mathscr{C}$-coalgebras with the Vallette model structure [Val14], i.e. if

$$\Omega_\alpha\Psi : \Omega_\alpha\mathrm{B}_\alpha A \longrightarrow \Omega_\alpha\mathrm{B}_\alpha A'$$

is a quasi-isomorphism.

[1] Here, as usual, we take Koszul morphisms between reduced (co)operads. We expect the theory presented in this chapter to work more generally for Koszul morphisms between connected weight graded (co)operads.

2. *An ∞_α-morphism of $\mathscr{P}$-algebras $\Psi : A \rightsquigarrow A'$ is an* ∞_α-quasi-isomorphism *if the chain map*

$$\psi_1 : A \longrightarrow A'$$

is a quasi-isomorphism.

3. *An ∞_α-morphism of $\mathscr{C}$-coalgebras $\Phi : C' \rightsquigarrow C$ is an* α-weak equivalence *if the morphism*

$$\Phi : \Omega_\alpha C' \longrightarrow \Omega_\alpha C$$

is a quasi-isomorphism, i.e. if it is a weak equivalence in the classical Hinich model structure on the category of $\mathscr{P}$-algebras.

4. *An ∞_α-morphism of $\mathscr{C}$-coalgebras $\Phi : C' \rightsquigarrow C$ is an* ∞_α-quasi-isomorphism *if the chain map*

$$\phi_1 : C' \longrightarrow C$$

is a quasi-isomorphism.

We will now try to understand how these four notions are related to each other. We begin with a classical fact. It was originally stated for $\mathscr{P}$ a Koszul operad and the classical ∞-morphisms of homotopy $\mathscr{P}$-algebras, but the proof readily generalizes to our setting. See also [LG16, Prop. 32].

Theorem 8.2.2 ([LV12, Prop. 11.4.7])**.** *An ∞_α-morphism of $\mathscr{P}$-algebras is an α-weak equivalence if, and only if it is an ∞_α-quasi-isomorphism.*

Thanks to this result, we see that the rectification functors for algebras are naturally ∞_α-quasi-isomorphic to the identity functor.

Lemma 8.2.3. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a Koszul morphism, and let $g : \mathscr{P} \to \mathscr{P}'$ be a quasi-isomorphism of operads. Then the natural ∞_α-morphism*

$$N_A : A \rightsquigarrow R^{g,\alpha}(A)$$

is an ∞_α-quasi-isomorphism for any $\mathscr{P}$-algebra A.

Proof. Since α is Koszul and g is a quasi-isomorphism, it follows that $g_*\alpha$ is also Koszul by Theorem 2.2.21. This is equivalent to $\eta_{\mathrm{B}_\alpha A}$ being a quasi-isomorphism by Theorem 2.4.11. Therefore,

$$N_A : \mathrm{B}_\alpha A \xrightarrow{\eta_{\mathrm{B}_\alpha A}} \mathrm{B}_{g_*\alpha}\Omega_{g_*\alpha}\mathrm{B}_\alpha A = \mathrm{B}_\alpha R^{g,a}(A)\ ,$$

is an α-weak equivalence of $\mathscr{P}$-algebras, and thus an ∞_αquasi-isomorphism by Theorem 8.2.2. □

For coalgebras, we can proceed similarly to prove a slightly weaker statement.

Proposition 8.2.4. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a Koszul morphism, and let $f : \mathscr{C}' \to \mathscr{C}$ be a quasi-isomorphism of cooperads. The α-weak equivalence*

$$E_C : R_{\alpha,f}(C) \rightsquigarrow C$$

is an ∞_α-quasi-isomorphism for any conilpotent $\mathscr{C}$-coalgebra C.

Proof. Denote $\alpha' \coloneqq \alpha f$. We have to prove that the first component

$$e_1 : R_{\alpha,f}(C) \longrightarrow C$$

of E_C is a quasi-isomorphism. We will do this by a spectral sequence argument analogous to the one of [LV12, Thm. 11.3.3 and 11.4.4]. We start by noticing that

$$e_1 : (\mathscr{C}' \circ \mathscr{P})(C) \longrightarrow C$$

is given by the projection onto C. We filter the left-hand side by the number of times that C appears, i.e. by

$$F_p \coloneqq \bigoplus_{k \leq p} (\mathscr{C}' \circ \mathscr{P})(k) \otimes_{\mathbb{S}_k} C^{\otimes k} .$$

This filtration is increasing, bounded below and exhaustive. The page E^0 of the associated spectral sequence equals $(\mathscr{C}' \circ_{\alpha'} \mathscr{P})(C)$, since the only parts of the differential that preserve the weight (that is, the arity) are the internal differential of C and the part coming from the twisting morphism α'. The page E^1 of the spectral sequence is

$$H_\bullet((\mathscr{C}' \circ_{\alpha'} \mathscr{P})(C)) \cong H_\bullet(\mathscr{C}' \circ_{\alpha'} \mathscr{P}) \circ H_\bullet(C) \cong H_\bullet(C)$$

by the operadic Künneth formula, Theorem 2.1.9, and the fact that α' is a Koszul morphism. On the other side, we filter C by $F_pC = C$ for $p \geq 0$ and $F_pC = 0$ otherwise. This filtration is also increasing, bounded below and exhaustive. The map e_1 is a map of spectral sequences, and so the induced map at the page E^1 is $H_\bullet(e_1)$, which induces an isomorphism. Therefore, the chain map e_1 is a quasi-isomorphism. □

Notice that if $f = 1_{\mathscr{C}}$, then the rectification becomes the functor $\mathrm{B}_\alpha\Omega_\alpha$, and the natural ∞_α-morphism E_C is given by the counit ε of the bar-cobar adjunction (seen as an ∞_α-morphism). As a consequence of this result, we have the following.

Theorem 8.2.5. *If an ∞_α-morphism of $\mathscr{C}$-coalgebras is an α-weak equivalence, then it is an ∞_α-quasi-isomorphism.*

Notice that the inverse implication is not true: it is known that there are (strict) quasi-isomorphisms of $\mathscr{C}$-coalgebras that are not sent to quasi-isomorphisms under the cobar construction, cf. [LV12, Prop. 2.4.3].

Proof. The proof is similar to the one of [LV12, Prop. 11.4.7]. Suppose $\Phi : C' \rightsquigarrow C$ is an α-weak equivalence of $\mathscr{C}$-coalgebras. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{B}_\alpha\Omega_\alpha C' & \xrightarrow[\sim]{\mathrm{B}_\alpha\Phi} & \mathrm{B}_\alpha\Omega_\alpha C \\ \varepsilon_{C'} \downsquigarrow & & \downsquigarrow \varepsilon_C \\ C' & \xrightarrow{\quad\Phi\quad} & C \end{array}$$

where the vertical arrows are ∞_α-quasi-isomorphisms by Proposition 8.2.4 and the top arrow is a quasi-isomorphism by Proposition 2.4.4. Restriction to the first component gives us the diagram

$$\begin{array}{ccc} \mathrm{B}_\alpha\Omega_\alpha C' & \xrightarrow[\sim]{\mathrm{B}_\alpha\Phi} & \mathrm{B}_\alpha\Omega_\alpha C \\ {\scriptstyle\varepsilon_{C'}}\downarrow\wr & & \wr\downarrow{\scriptstyle\varepsilon_C} \\ C' & \xrightarrow{\Phi} & C \end{array}$$

from which the statement follows. □

Finally, we can show that α-weak equivalences of coalgebras are equivalent to zig-zags of weak equivalences of coalgebras, which is analogous to Theorem 2.5.3.

Theorem 8.2.6. *Let C and D be two $\mathscr{C}$-coalgebras. The following are equivalent.*

1. *There is a zig-zag of weak equivalences*
$$C \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdots \longleftarrow D$$
of $\mathscr{C}$-coalgebras in the Vallette model structure.
2. *There are two weak equivalences of $\mathscr{C}$-coalgebras forming a zig-zag*
$$C \longrightarrow \bullet \longleftarrow D\,.$$
3. *There is an α-weak equivalence*
$$C \rightsquigarrow D\,.$$

Proof. The fact that (2) implies (1) is obvious.

We prove that (3) implies (2). Suppose we have an α-weak equivalence

$$\Phi : C \rightsquigarrow D\,.$$

Then $\mathrm{B}_\alpha\Phi$ is a weak equivalence, and thus we have the zig-zag

$$C \xrightarrow{\eta_C} \mathrm{B}_\alpha\Omega_\alpha C \xrightarrow{\mathrm{B}_\alpha\Phi} \mathrm{B}_\alpha\Omega_\alpha D \xleftarrow{\eta_D} D\,,$$

where the units of the bar-cobar adjunction are weak equivalences by Corollary 3.3.19.

Finally, we show that (1) implies (3). Every weak equivalence of coalgebras is in particular an α-weak equivalence. Therefore, it is enough to prove that whenever we have a weak equivalence

$$C \xleftarrow{\phi} D$$

of coalgebras, then we have an α-weak equivalence going the other way round. Since ϕ is a weak equivalence, we have that

$$\Omega_\alpha C \xleftarrow{\Omega_\alpha\phi} \Omega_\alpha D$$

is a quasi-isomorphism. Moreover, every $\mathscr{P}$-algebra is fibrant and Ω_α lands in the cofibrant $\mathscr{P}$-algebras by [Val14, Thm. 2.9(1)]. Therefore, we can apply [DS95, Lemma 4.24] to obtain a homotopy inverse

$$\Omega_\alpha C \xrightarrow{\Psi} \Omega_\alpha D\,,$$

which is again a quasi-isomorphism, and thus defines an α-weak equivalence from C to D, as we desired. □

Chapter 9

Convolution homotopy Lie algebras

Independently in [Wie16] and [RN17a], and then jointly in [RNW17] and [RNW], the author and Felix Wierstra showed that operadic twisting morphisms from a cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ are equivalent to morphisms from the operad $\mathscr{L}_\infty$ to the convolution operad $\hom(\mathscr{C}, \mathscr{P})$. It follows that the chain complex of linear maps from a conilpotent $\mathscr{C}$-coalgebra to a $\mathscr{P}$-algebra is an $\mathscr{L}_\infty$-algebra in a functorial way, by restriction of the structure. The $\mathscr{L}_\infty$-algebra structures obtained this way are really well behaved with respect to the tools of homotopical operadic algebra: the homotopy transfer theorem and ∞-morphisms.

The results presented here were developed and generalized progressively during the last few years. We present only the current state of the art in this chapter. In the whole chapter, we work with shifted $\mathscr{L}_\infty$-algebras. This greatly reduces the signs appearing in the proofs, improving readability. One can pass to usual $\mathscr{L}_\infty$-algebras simply by desuspending everything, see Section 6.3.

9.1 Convolution homotopy Lie algebras

We begin by showing how operadic twisting morphisms are equivalent to morphisms from $\mathfrak{s}\mathscr{L}_\infty$ to the convolution operad. We proceed by studying the Maurer–Cartan elements of the $\mathfrak{s}\mathscr{L}_\infty$-algebras obtained that way, and take a look at what happens in other settings, such as non-symmetric operads.

9.1.1 Definitions

We begin with the following remark.

Lemma 9.1.1. *Let $\mathscr{P}$ be an operad. Then we have*

$$\hom(\mathrm{Com}^\vee, \mathscr{P}) \cong \mathscr{P}$$

as operads.

Proof. The statement is straightforwardly true at the level of $\mathbb{S}$-modules, the isomorphism being given by

$$\phi \in \hom(\mathrm{Com}^\vee, \mathscr{P})(n) \longmapsto \phi(\mu_n^\vee) \in \mathscr{P}(n) \, .$$

We only have to prove that the composition maps coincide. Let $\phi \in \hom(\mathrm{Com}^\vee, \mathscr{P})(k)$, and $\psi_i \in \hom(\mathrm{Com}^\vee, \mathscr{P})(n_i)$ for all $1 \leq i \leq k$. Fix $\theta \in \sqcup(n_1, \dots, n_k)$. Then we have

$$\begin{aligned}\gamma_{\hom(\mathrm{Com}^\vee,\mathscr{P})}(\phi \circ (\psi_1, \dots, \psi_k)^\theta)(\mu_n^\vee) = \\ &= \sum_{\sigma \in \sqcup(n_1,\dots,n_k)} \gamma_{\mathscr{P}}(\phi \circ (\psi_1, \dots, \psi_k)^\theta)(\mu_k^\vee \circ (\mu_{n_1}^\vee, \dots, \mu_{n_k}^\vee)^\sigma) \\ &= \gamma_{\mathscr{P}}(\phi(\mu_k^\vee) \circ (\psi_1(\mu_{n_1}^\vee), \dots, \psi_k(\mu_{n_k}^\vee))^\theta) \,,\end{aligned}$$

concluding the proof. □

Thanks to this, we can give a clean proof of the following theorem, on which relies all of the theory of convolution $\mathscr{L}_\infty$-algebras.

Theorem 9.1.2 ([Wie16, Sect. 7] and [RN17a, Thm. 3.1])**.** *Let $\mathscr{C}$ be a cooperad, and let $\mathscr{P}$ be an operad. There is a bijection*

$$\mathrm{Tw}(\mathscr{C}, \mathscr{P}) \cong \hom_{\mathsf{Op}}(\mathfrak{s}\mathscr{L}_\infty, \hom(\mathscr{C}, \mathscr{P})) \,,$$

given by sending $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$ to the morphism W_α sending $\ell_n \coloneqq s^{-1}\mu_n^\vee \in \mathfrak{s}\mathscr{L}_\infty(n)$ to

$$\mathsf{W}_\alpha(s^{-1}\mu_n^\vee) = \alpha(n) \in \hom(\mathscr{C}(n), \mathscr{P}(n)) \,.$$

It is natural in both $\mathscr{C}$ and $\mathscr{P}$ in the following sense. Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$, then for any morphism $f : \mathscr{C}' \to \mathscr{C}$ of cooperads, we have

$$\mathsf{W}_{f^*\alpha} = (f^*)\mathsf{W}_\alpha \,,$$

and for any morphism $g : \mathscr{P} \to \mathscr{P}'$ of operads, we have

$$\mathsf{W}_{g_*\alpha} = (g_*)\mathsf{W}_\alpha \,.$$

Proof. The bijection is given by

$$\begin{aligned}\mathrm{Tw}(\mathscr{C}, \mathscr{P}) &= \mathrm{MC}(\hom(\mathscr{C}, \mathscr{P})) \\ &\cong \mathrm{MC}(\hom(\mathrm{Com}^\vee, \hom(\mathscr{C}, \mathscr{P}))) \\ &= \mathrm{Tw}(\mathrm{Com}^\vee, \hom(\mathscr{C}, \mathscr{P})) \\ &\cong \hom_{\mathsf{Op}}(\Omega\mathrm{Com}^\vee, \hom(\mathscr{C}, \mathscr{P})) \\ &= \hom_{\mathsf{Op}}(\mathfrak{s}\mathscr{L}_\infty, \hom(\mathscr{C}, \mathscr{P})) \,,\end{aligned}$$

where in the first line, $\hom(\mathscr{C}, \mathscr{P})$ denotes the Lie algebra associated to the convolution operad. The other properties are straightforward to check, and left as an exercise to the reader. □

Remark 9.1.3. *If one wants to work with $\mathscr{L}_\infty$-algebras, instead of shifted ones, then one needs to desuspend everything, obtaining a bijection*

$$\mathrm{Tw}(\mathscr{C}, \mathscr{P}) \cong \hom_{\mathsf{Op}}(\mathscr{L}_\infty, \mathscr{S}^{-1} \otimes \hom(\mathscr{C}, \mathscr{P})) \,.$$

Remark 9.1.4. *Some special cases of this result and ideas hinting to it have appeared in the existing literature, for example already in Ginzburg–Kapranov [GK94, Prop. 3.2.18], and then more recently in [BL15, Appendix C], [DHR15], [DP16].*

Remark 9.1.5. *Given a morphism $\Psi : \mathscr{Q} \to \mathscr{P}$ of operads, then one gets a twisting morphism*

$$\psi \coloneqq \left(\mathrm{B}\mathscr{Q} \xrightarrow{\pi} \mathscr{Q} \xrightarrow{\Psi} \mathscr{P}\right),$$

and thus a morphism of operads

$$\mathsf{W}_\psi : \mathfrak{s}\mathscr{L}_\infty \longrightarrow \hom(\mathrm{B}\mathscr{Q}, \mathscr{P}) .$$

This is the special case originally treated in [RN17a].

Suppose now that we have a cooperad $\mathscr{C}$, and operad $\mathscr{P}$, and a twisting morphism $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$. Then, given a conilpotent $\mathscr{C}$-coalgebra C and a $\mathscr{P}$-algebra A, we know by Theorem 2.2.10 that $\hom(C, A)$ is a $\hom(\mathscr{C}, \mathscr{P})$-algebra. We can therefore apply restriction of structure by W_α to obtain a $\mathfrak{s}\mathscr{L}_\infty$-algebra structure on $\hom(C, A)$. We denote the $\mathfrak{s}\mathscr{L}_\infty$-algebra obtained this way by $\hom^\alpha(C, A)$.

Definition 9.1.6. *The algebra $\hom^\alpha(C, A)$ is called the* convolution $\mathfrak{s}\mathscr{L}_\infty$-algebra *of C and A.*

Given a morphism of $\mathscr{C}$-coalgebras, resp. of $\mathscr{P}$-algebras, then we get a morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras by pullback, resp. pushforward. Therefore, the assignment $\hom^\alpha$ defines a bifunctor

$$\hom^\alpha : \mathscr{C}\text{-}\mathsf{cog}^{\mathrm{op}} \times \mathscr{P}\text{-}\mathsf{alg} \longrightarrow \mathfrak{s}\mathscr{L}_\infty\text{-}\mathsf{alg} . \tag{9.1}$$

The rest of this chapter will be dedicated to the study of some of the properties of this bifunctor. Here is a first, straightforward fact. It is a direct consequence of the functoriality of W.

Lemma 9.1.7. *Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$ be a twisting morphism, and suppose $f : \mathscr{C}' \to \mathscr{C}$ is a morphism of cooperads, and that $g : \mathscr{P} \to \mathscr{P}'$ is a morphism of operads.*

1. *Let A be a $\mathscr{P}$-algebra, let D be a conilpotent $\mathscr{C}'$-coalgebra. We have*

$$\hom^{f^*\alpha}(D, A) = \hom^\alpha(f_* D, A)$$

   as $\mathfrak{s}\mathscr{L}_\infty$-algebras.

2. *Dually, let D be a $\mathscr{C}$-coalgebra, let A be a $\mathscr{P}'$-algebra. We have*

$$\hom^{g_*\alpha}(D, A) = \hom^\alpha(D, g^* A)$$

   as $\mathfrak{s}\mathscr{L}_\infty$-algebras.

Proof. We only prove (1), the other case being dual. Let $f_1, \dots, f_n \in \hom(D, A)$, and denote by $F \coloneqq f_1 \otimes \cdots \otimes f_n$. We have

$$\begin{aligned}
\gamma_{\hom^{f^*\alpha}(D,A)}(s^{-1}\mu_n^\vee \otimes_{\mathbb{S}_n} F) &= \gamma_{\hom(D,A)}(\mathsf{W}_{f^*\alpha}(s^{-1}\mu_n^\vee) \otimes F) \\
&= \gamma_{\hom(D,A)}((f^*)\mathsf{W}_\alpha(s^{-1}\mu_n^\vee) \otimes F) \\
&= \gamma_{\hom(f_*D,A)}(\mathsf{W}_\alpha(s^{-1}\mu_n^\vee) \otimes F) \\
&= \gamma_{\hom^\alpha(f_*D,A)}(s^{-1}\mu_n^\vee \otimes_{\mathbb{S}_n} F) ,
\end{aligned}$$

where $\hom(D, A)$ is seen as a $\hom(\mathscr{C}', \mathscr{P})$-algebra and similarly for $\hom(f_* D, A)$. □

9.1.2 Maurer–Cartan elements of convolution $\mathscr{L}_\infty$-algebras

Whenever one has a $\mathfrak{s}\mathscr{L}_\infty$-algebra, it is a very natural question to ask what the Maurer–Cartan elements are. We answer this question for convolution homotopy Lie algebras.

Fix a cooperad $\mathscr{C}$, an operad $\mathscr{P}$, and a twisting morphism $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$.

Lemma 9.1.8. *Let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. Then the ascending filtration*

$$\mathscr{F}_n \hom^\alpha(C, A) \coloneqq \{f \in \hom^\alpha(C, A) \mid \mathscr{F}^n_{\mathscr{C}} C \subseteq \ker(f)\}$$

makes $\hom^\alpha(C, A)$ into a proper complete $\mathfrak{s}\mathscr{L}_\infty$-algebra.

Proof. Let $f \in \mathscr{F}_n \hom^\alpha(C, A)$. Let $x \in \mathscr{F}^n_{\mathscr{C}} C$, then

$$\partial(f)(x) = d_A f(x) - (-1)^f f(d_C x) = 0$$

since $d_C(\mathscr{F}^n_{\mathscr{C}} C) \subseteq \mathscr{F}^n_{\mathscr{C}} C$.

Let $f_i \in \mathscr{F}_{n_i} \hom^\alpha(C, A)$ for $i = 1, \dots, k$, and let $x \in \mathscr{F}^n_{\mathscr{C}} C$, where $n \coloneqq n_1 + \cdots + n_k$. Then

$$\ell_n(f_1, \dots, f_k)(x) = \gamma_A(\alpha \otimes F)\Delta_C(x) = 0 \,,$$

since

$$\Delta_C(\mathscr{F}^n_{\mathscr{C}} C) \subseteq \bigoplus_{\substack{k \geq 0 \\ n'_1 + \cdots + n'_k = n}} \mathscr{C}(k) \otimes \mathscr{F}^{n'_1}_{\mathscr{C}} C \otimes \cdots \otimes \mathscr{F}^{n'_k}_{\mathscr{C}} C \,.$$

Finally, there is a natural identification

$$\frac{\hom^\alpha(C, A)}{\mathscr{F}_n \hom^\alpha(C, A)} \xrightarrow{\cong} \hom^\alpha(\mathscr{F}^n_{\mathscr{C}} C, A) \,,$$

given by taking any representative of an equivalence class of morphisms on the right hand side, and restricting it to $\mathscr{F}^n_{\mathscr{C}} C$. It is injective, because if a morphism from C to A restricts to zero on $\mathscr{F}^n_{\mathscr{C}} C$, then it is in $\mathscr{F}_n \hom^\alpha(C, A)$, and thus it is zero in the quotient. To see that it is surjective, choose a complement V of $\mathscr{F}^n_{\mathscr{C}} C$ in C — as graded vector spaces, the differential plays no role here. An inverse to the map described above is then given by sending a linear map $f : \mathscr{F}^n_{\mathscr{C}} C \to A$ to its extension by 0 on V, and then taking the equivalence class in the quotient. Notice that this is independent of the choice of V. It is straighrforward to check that this isomorphism holds at the level of $\mathfrak{s}\mathscr{L}_\infty$-algebras, and not only as chain complexes. Therefore, we have

$$\begin{aligned} \lim_n \frac{\hom^\alpha(C, A)}{\mathscr{F}_n \hom^\alpha(C, A)} &\cong \lim_n \hom^\alpha(\mathscr{F}^n_{\mathscr{C}} C, A) \\ &\cong \hom^\alpha(\operatorname{colim}_n \mathscr{F}^n_{\mathscr{C}} C, A) \\ &\cong \hom^\alpha(C, A) \,, \end{aligned}$$

concluding the proof. □

Therefore, it makes sense to speak about Maurer–Cartan elements in $\hom^\alpha(C, A)$.

Theorem 9.1.9 ([Wie16, Thm. 7.1] and [RN17a, Thm. 6.3])**.** *Let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. We have*

$$\mathrm{MC}(\hom^\alpha(C, A)) = \mathrm{Tw}_\alpha(C, A) \,,$$

where the set $\mathrm{Tw}_\alpha(C, A)$ of twisting morphisms relative to α was given in Definition 2.4.6.

Proof. Let $\varphi \in \hom(C, A)$. Denote by $\star_\alpha^{(n)}(\varphi)$ the part of $\star_\alpha(\varphi)$ passing through $\mathscr{C}(n) \otimes C^{\otimes n}$. With this notation, we have

$$\begin{aligned}\frac{1}{n!}\ell_n(\varphi, \dots, \varphi) &= \frac{1}{n!}\gamma_{\hom(C,A)}(\mathsf{W}_\alpha(\ell_n) \otimes \varphi^{\otimes n}) \\ &= \gamma_A(\mathsf{W}_\alpha(\ell_n) \circ \varphi)\Delta_C^n \\ &= \gamma_A(\alpha \circ \varphi)\Delta_C^n \\ &= \star_\alpha^{(n)}(\varphi)\,,\end{aligned}$$

where the factor $\frac{1}{n!}$ is eliminated by the fact that we sum over all permutations when applying the $\hom(\mathscr{C}, \mathscr{P})$-algebra structure $\gamma_{\hom(C,A)}$, cf. Theorem 2.2.10. Since

$$\star_\alpha(\varphi) = \sum_{n\geq 2} \star_\alpha^{(n)}(\varphi)\,,$$

we have that

$$\partial(\varphi) + \star_\alpha(\varphi) = \partial(\varphi) + \sum_{n\geq 2} \frac{1}{n!}\ell_n(\varphi, \dots, \varphi)\,.$$

Thus, the two Maurer–Cartan equations coincide[1], concluding the proof. □

Remark 9.1.10. *The result above was already known in the case where $\mathscr{P}$ is a Koszul operad and α is given by $\kappa : \mathscr{P}^¡ \to \mathscr{P}$, the twisting morphism provided by Koszul duality. In this case, $\hom^\kappa(C, A)$ is a Lie algebra, since κ is non-zero only on the binary part of $\mathscr{P}^¡$, cf. its definition in Section 2.3.2. See [LV12, Sect. 11.1.2].*

Corollary 9.1.11. *Let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra.*

1. *We have a natural isomorphism*

$$\mathrm{MC}(\hom^\alpha(C, A)) \cong \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\Omega_\alpha C, A)\,.$$

In particular, if $A = \Omega_\alpha C'$ for some conilpotent $\mathscr{C}$-coalgebra C', then

$$\mathrm{MC}(\hom^\alpha(C, A)) \cong \hom_{\infty_\alpha\text{-}\mathscr{C}\text{-}\mathsf{cog}}(C, C')\,.$$

2. *We have a natural isomorphism*

$$\mathrm{MC}(\hom^\alpha(C, A)) \cong \hom_{\mathscr{C}\text{-}\mathsf{cog}}(C, \mathrm{B}_\alpha A)\,.$$

In particular, if $C = \mathrm{B}_\alpha A'$ for some $\mathscr{P}$-algebra A', then

$$\mathrm{MC}(\hom^\alpha(C, A)) \cong \hom_{\infty_\alpha\text{-}\mathscr{P}\text{-}\mathsf{alg}}(A', A)\,.$$

Proof. This is a direct consequence of Theorem 9.1.9 and Theorem 2.4.7. □

9.1.3 Other settings: tensor products, non-symmetric (co)operads

There are some other contexts where it is natural to try to apply the theory developed above.

[1] Motivating *a posteriori* the name "Maurer–Cartan equation" for the equation $\partial(\varphi) + \star_\alpha(\varphi) = 0$.

Tensor products

First of all, if $\mathscr{C}$ is a cooperad which is finite dimensional in every arity, then for any operad $\mathscr{P}$ we have

$$\hom(\mathscr{C}, \mathscr{P}) \cong \mathscr{C}^\vee \otimes \mathscr{P}$$

with the obvious isomorphism, so that all results pass to tensor products. In order to have everything to pass through without having to do the proofs all over again, one should then only consider dualizable $\mathscr{C}^\vee$-algebras. However, if one is willing to do the effort, then one can prove that the result.

Theorem 9.1.12. *Let $\mathscr{C}$ be a cooperad which is finite dimensional in every arity, and let $\mathscr{P}$ be an operad. There is a bijection*

$$\mathrm{Tw}(\mathscr{C}, \mathscr{P}) \cong \hom_{\mathsf{Op}}(\mathfrak{s}\mathscr{L}_\infty, \mathscr{P} \otimes \mathscr{C}^\vee) \ ,$$

given by sending $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$ to the morphism M_α sending $s^{-1}\mu_n^\vee \in \mathfrak{s}\mathscr{L}_\infty(n)$ to

$$\mathsf{M}_\alpha(s^{-1}\mu_n^\vee) = \sum_i \alpha(c_i) \otimes c_i^\vee \ ,$$

where $\{c_i\}_i$ is a basis of $\mathscr{C}(n)$, and where $\{c_i^\vee\}_i$ is the dual basis. It is natural in both $\mathscr{C}$ and $\mathscr{P}$ in the following sense. Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$, then for any morphism $f : \mathscr{C}' \to \mathscr{C}$ of cooperads, we have

$$\mathsf{M}_{f^*\alpha} = (1 \otimes f^\vee)\mathsf{M}_\alpha \ ,$$

and for any morphism $g : \mathscr{P} \to \mathscr{P}'$ of operads, we have

$$\mathsf{M}_{g_*\alpha} = (g \otimes 1)\mathsf{M}_\alpha \ .$$

Given a $\mathscr{P}$-algebra A and a $\mathscr{C}^\vee$-algebra D, then one obtains an $\mathfrak{s}\mathscr{L}_\infty$-algebra $A \otimes^\alpha D$ by pulling back the natural $\mathscr{P} \otimes \mathscr{C}^\vee$-algebra structure of $A \otimes D$ by M_α. This assignment is compatible with morphisms of algebras in both slots, and thus defines a bifunctor

$$- \otimes^\alpha - : \mathscr{P}\text{-}\mathsf{alg} \times \mathscr{C}^\vee\text{-}\mathsf{alg} \longrightarrow \mathfrak{s}\mathscr{L}_\infty\text{-}\mathsf{alg} \ .$$

This is compatible with $\hom^\alpha(-,-)$, in the sense that if A is a $\mathscr{P}$-algebra and C is a $\mathscr{C}$-coalgebra, then the natural morphism

$$A \otimes C^\vee \longrightarrow \hom(C, A)$$

is a morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras

$$A \otimes^\alpha C^\vee \longrightarrow \hom^\alpha(C, A) \ ,$$

which is an isomorphism if C is dualizable.

Theorem 9.1.13. *Let D be a dualizable $\mathscr{C}^\vee$-algebra, and let A be a $\mathscr{P}$-algebra. We have*

$$\mathrm{MC}(A \otimes^\alpha D) \cong \mathrm{Tw}_\alpha(D^\vee, A) \ .$$

Non-symmetric (co)operads

If we consider non-symmetric (co)operads, then everything goes through with exactly the same proofs[2] simply by changing the operad $\mathscr{L}_\infty$ with the ns operad $\mathscr{A}_\infty$. Moreover, since we don't need to identify invariants and coinvariants, we can work over any field, without restrictions on the characteristic.

[2] Removing all symmetric group actions, of course.

Theorem 9.1.14. *Let $\mathscr{C}$ be an ns cooperad, and let $\mathscr{P}$ be an ns operad. There is a bijection*

$$\mathrm{Tw}(\mathscr{C}, \mathscr{P}) \cong \hom_{\mathsf{Op}}(\mathfrak{s}\mathscr{A}_\infty, \hom(\mathscr{C}, \mathscr{P})) ,$$

given by sending $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$ to the morphism W_α sending $s^{-1}\mu_n^\vee \in \mathfrak{s}\mathscr{A}_\infty(n)$ to

$$\mathsf{W}_\alpha(s^{-1}\mu_n^\vee) = \alpha(n) \in \hom(\mathscr{C}(n), \mathscr{P}(n)) .$$

It is natural in both $\mathscr{C}$ and $\mathscr{P}$ in the following sense. Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$, then for any morphism $f : \mathscr{C}' \to \mathscr{C}$ of cooperads, we have

$$\mathsf{W}_{f^*\alpha} = (f^*)\mathsf{W}_\alpha ,$$

and for any morphism $g : \mathscr{P} \to \mathscr{P}'$ of operads, we have

$$\mathsf{W}_{g_*\alpha} = (g_*)\mathsf{W}_\alpha .$$

Theorem 9.1.15. *Let C be a conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. We have*

$$\mathrm{MC}(\hom^\alpha(C, A)) = \mathrm{Tw}_\alpha(C, A) .$$

Here, the Maurer–Cartan elements of an $\mathfrak{s}\mathscr{A}_\infty$-algebra A are the elements $x \in A_0$ of degree 0 satisfying the non-symmetric Maurer–Cartan equation

$$dx + \sum_{n \geq 2} m_n(x, \ldots, x) = 0 .$$

The results of the rest of the present chapter all have versions for tensor products and non-symmetric operads. We will only mention those that we will need in the rest of this work.

9.1.4 The binary quadratic case and Manin products

Here, we restrict to the binary quadratic case, and study convolution tensor products in this context. We start by recalling the notion of what we call the Manin morphisms, which are morphisms arising from maps between operads via the adjunction between the black and white Manin products. We go on to prove that convolution tensor products directly generalize Manin morphisms.

In the category of operads given by binary quadratic data and morphisms induced by morphisms of quadratic data, one can define two operations, called the *white* and *black Manin products* and denoted by $\circ$ and $\bullet$ respectively, both taking two binary quadratic operads and giving back another one. These objects first appeared in the context of algebras in [Man87] and [Man88], and then in [GK94] in relation to operads. For a more conceptual treatment, see [Val08] or [LV12, Sect. 8.8].

Proposition 9.1.16. *Fix a binary quadratic operad $\mathscr{Q}$. Then there is a natural isomorphism*

$$\hom_{\text{bin. quad. op.}}(\mathscr{R} \bullet \mathscr{Q}, \mathscr{P}) \cong \hom_{\text{bin. quad. op.}}(\mathscr{R}, \mathscr{P} \circ \mathscr{Q}^!) .$$

That is to say, the functors $- \bullet \mathscr{Q}$ and $- \circ \mathscr{Q}^!$ are adjoint. Moreover, the operad Lie *is a unit for the black product.*

Therefore, any morphism

$$\Psi : \mathrm{Lie} \bullet \mathscr{Q} \cong \mathscr{Q} \longrightarrow \mathscr{P}$$

coming from a quadratic data is equivalent to a morphism

$$\mathrm{Lie} \longrightarrow \mathscr{P} \circ \mathscr{Q}^! .$$

As explained in [Val08, Sect. 3.2], the white product is the best binary quadratic approximation of the Hadamard product, and there is a canonical morphism

$$\mathscr{P} \circ \mathscr{Q}^! \longrightarrow \mathscr{P} \otimes \mathscr{Q}^! .$$

Definition 9.1.17. *We call the composite*

$$\mathsf{m}_\Psi \coloneqq \left(\mathrm{Lie} \longrightarrow \mathscr{P} \circ \mathscr{Q}^! \longrightarrow \mathscr{P} \otimes \mathscr{Q}^!\right)$$

the Manin morphism *associated to* Ψ.

The Manin morphism m_Ψ has the following explicit description. Assume $\mathscr{P} = \mathscr{P}(E, R)$ and $\mathscr{Q} = \mathscr{P}(F, S)$, fix a basis $f_1, \ldots, f_k$ of F, and let $e_1, \ldots, e_k \in E$ be the images of the f_i under Ψ. Then m_Ψ is the unique morphism of operads extending

$$\mathsf{m}_\Psi(b) \coloneqq \sum_{i=1}^{k} e_i \otimes s^{-1}\mathscr{S}_2^{-1} f_i^\vee,$$

where $b \in \mathrm{Lie}(2)$ is the Lie bracket.

For any quadratic binary operad $\mathscr{Q}$, there is a canonical Manin morphism, namely the one associated to the identity of $\mathscr{Q}$, giving

$$\mathsf{m}_\mathscr{Q} \coloneqq \mathsf{m}_{\mathrm{id}_\mathscr{Q}} : \mathrm{Lie} \longrightarrow \mathscr{Q} \otimes \mathscr{Q}^! .$$

It is easy to see that

$$\mathsf{m}_\Psi = (\Psi \otimes 1)\mathsf{m}_\mathscr{Q} ,$$

so that it is only necessary to know $\mathsf{m}_\mathscr{Q}$ to compute m_Ψ.

We fix two binary quadratic data (E, R) and (F, S) and denote by $\mathscr{P} = \mathscr{P}(E, R)$ and $\mathscr{Q} = \mathscr{P}(F, S)$ the two associated operads. Furthermore, we assume that F is finite dimensional. We fix a morphism

$$\Psi : \mathscr{Q} \longrightarrow \mathscr{P}$$

in the category of binary quadratic operads, and we consider the associated twisting morphism

$$\psi \coloneqq \left(\mathrm{B}\mathscr{Q} \xrightarrow{\pi} \mathscr{Q} \xrightarrow{\Psi} \mathscr{P}\right).$$

The morphism of operads associated to this twisting morphism by Theorem 9.1.12 is

$$\mathsf{M}_\Psi : \mathscr{L}_\infty \longrightarrow \mathscr{P} \otimes \mathscr{Q}^!_\infty ,$$

after a suspension. Here, we used the fact that

$$(\mathscr{S} \otimes \mathrm{B}\mathscr{Q})^\vee \cong \Omega((\mathscr{S}^{-1})^c \otimes \mathscr{Q}^\vee) \cong \Omega\left(\left(\mathscr{Q}^!\right)^{\mathrm{i}}\right) \cong \mathscr{Q}^!_\infty ,$$

because, since F is finite dimensional, $\mathscr{Q}(n)$ is finite dimensional for all $n \geq 0$. The following proposition shows that M_Ψ gives a direct generalization of m_Ψ.

Proposition 9.1.18. *The following square*

$$\begin{array}{ccc} \mathscr{L}_\infty & \xrightarrow{\mathsf{M}_\Psi} & \mathscr{P} \otimes \mathscr{Q}^!_\infty \\ \downarrow & & \downarrow \\ \mathrm{Lie} & \xrightarrow{\mathsf{m}_\psi} & \mathscr{P} \otimes \mathscr{Q}^! \end{array}$$

where the vertical maps are the canonical ones coming from the resolutions, is commutative.

Proof. The left vertical arrow sends ℓ_2 to b and ℓ_n to 0 for all $n \geq 3$. Therefore, the south–west composite is the map sends

$$\ell_2 \longmapsto \sum_{i=1}^{k} e_i \otimes s^{-1}\mathscr{S}_2^{-1} f_i^\vee$$

and all the higher ℓ_n to zero. On the other hand, the right vertical arrow is given by tensoring the identity of $\mathscr{P}$ with the canonical resolution map

$$\mathscr{Q}^!_\infty \longrightarrow \mathscr{Q}^! \,,$$

which is defined on the generators $s^{-1}(\mathscr{Q}^!)^{\mathsf{i}} \cong s^{-1}(\mathscr{S}^{-1})^c \otimes \mathscr{Q}^\vee$ as being the identity on arity 2 and zero on all higher arities, this because $\mathscr{Q}$ is quadratic. By definition, the morphism W_Ψ sends ℓ_n to an element of $\mathscr{P}(n) \otimes s^{-1}(\mathscr{Q}^!)^{\mathsf{i}}(n)$. Therefore, the north–east composite gives zero on ℓ_n, for $n \geq 3$ and sends

$$\ell_2 \longmapsto \sum_{i=1}^{k} e_i \otimes s^{-1}\mathscr{S}_2^{-1} f_i^\vee$$

just like the other map. □

9.1.5 Non-conilpotent coalgebras

In some applications, such as the one we will see in Chapter 10, we are naturally lead to consider non-conilpotent coalgebras. This can be done, but in order to have the results that we desire we need to take proper complete algebras on the other side.

For the rest of this section, fix a cooperad $\mathscr{C}$, an operad $\mathscr{P}$ and a twisting morphism α. Given a $\mathscr{C}$-coalgebra[3] C, one defines a $\mathscr{P}$-algebra by

$$\widehat{\Omega}_\alpha C \coloneqq \left(\widehat{\mathscr{P}}(C) \coloneqq \prod_{n \geq 0} \mathscr{P}(n) \otimes_{\mathbb{S}_n} C^{\otimes n}, d_{\widehat{\Omega}_\alpha C} \coloneqq d_1 + d_2 \right),$$

where d_1 and d_2 are defined exactly as for the usual cobar construction relative to α, but this time d_2 passes through $\mathscr{C}\widehat{\circ}C$, instead of $\mathscr{C} \circ C$, cf. Section 2.4.1. The filtration

$$\mathscr{F}_n \coloneqq \prod_{k \geq n} \mathscr{P}(k) \otimes_{\mathbb{S}_k} C^{\otimes k}$$

makes it into a proper complete $\mathscr{P}$-algebra.

[3] Remember: non-conilpotent!

Definition 9.1.19. *The proper complete $\mathscr{P}$-algebra $(\widehat{\Omega}_\alpha C, \mathscr{F}_\bullet)$ is called the* complete cobar construction relative to α.

Lemma 9.1.20. *Let V be a chain complex, seen as a proper complete chain complex by imposing the filtration $\mathscr{F}_1 V = V$ and $\mathscr{F}_n V = 0$ for all $n \geq 2$. Let A be a proper complete $\mathscr{P}$-algebra. We have a natural bijection*

$$\hom_{\mathsf{fCh}}(V, A) \cong \hom_{\widehat{\mathscr{P}}\text{-}\mathsf{alg}}(\widehat{\mathscr{P}}(V), A) .$$

Proof. Notice that the left hand side is also equal to the set of all morphisms of chain complexes from V to A, forgetting the filtrations. Given such a chain map $\phi : V \to A$, we get a map of algebras by

$$\widehat{\mathscr{P}}(V) \xrightarrow{\widehat{\mathscr{P}}(\phi)} \widehat{\mathscr{P}}(A) \xrightarrow{\gamma_A} A .$$

It is straightforward to check that it is well-defined and that it respects the filtrations. The other direction is given by restricting a morphism of algebras $\widehat{\mathscr{P}}(V) \to A$ to V. □

Remark 9.1.21. *For the result above, it is important that $\mathscr{P}$ is reduced, else we might get infinite sums at every level of the filtration, resulting in the morphisms in the proof not being defined.*

Proposition 9.1.22. *Let $\alpha \in \mathrm{Tw}(\mathscr{C}, \mathscr{P})$ be a twisting morphism, let C be a $\mathscr{C}$-coalgebra, and let A be a proper complete $\mathscr{P}$-algebra. There is a natural bijection*

$$\hom_{\widehat{\mathscr{P}}\text{-}\mathsf{alg}}(\widehat{\Omega}_\alpha C, A) \cong \mathrm{Tw}_\alpha(C, A) .$$

Proof. The proof is analogous to the one for the similar bijection in Theorem 2.4.7. One uses Lemma 9.1.20. □

9.1.6 Examples

We give some examples of $\mathscr{L}_\infty$-algebra structures arising through Theorem 9.1.2. We put ourselves in the situation of Remark 9.1.5 and in the tensor product setting. These examples are all extracted from [RN17a, Sect. 8].

We will study the $\mathscr{L}_\infty$-algebra structures obtained from Theorem 9.1.12 for some canonical morphisms between the three most often appearing operads: the three graces Com, Lie and Ass. Namely, we will study the identities of these operads and the sequence of morphisms

$$\mathrm{Lie} \xrightarrow{a} \mathrm{Ass} \xrightarrow{u} \mathrm{Com} ,$$

where the first morphism corresponds to the antisymmetrization of the multiplication of an associative algebra, and where the second one corresponds to forgetting that the multiplication of a commutative algebra is commutative to get an associative algebra.

Many more examples of less common, but still very interesting operads, both in the symmetric and in the ns case, as well as various morphisms relating them, can be found in [LV12, Sect. 13].

Notations

We will denote by $b \in \mathrm{Lie}(2)$ the generating operation of Lie, i.e. the Lie bracket. The operad Ass is the symmetric version the non-symmetric operad As coding associative algebras. It is

given by $\mathrm{Ass}(n) = \mathbb{K}[\mathbb{S}_n]$. We denote the canonical basis of $\mathrm{Ass}(n)$ by $\{m_\sigma\}_{\sigma\in\mathbb{S}_n}$. The element $m_\sigma \in \mathrm{Ass}(n)$ corresponds to the operation

$$(a_1, \ldots, a_n) \longmapsto a_{\sigma^{-1}(1)} \cdots a_{\sigma^{-1}(n)}$$

at the level of associative algebras. The action of the symmetric group is of course given by $(m_\sigma)^\tau = m_{\sigma\tau}$. As before, we denote by $\mu_n \in \mathrm{Com}(n)$ the canonical element. The morphism $a : \mathrm{Lie} \to \mathrm{Ass}$ is given by sending b to $m_{\mathrm{id}} - m_{(12)}$, and corresponds to antisymmetrization at the level of algebras. The morphism $u : \mathrm{Ass} \to \mathrm{Com}$ is given by sending both m_{id} and $m_{(12)}$ to μ_2. For the homotopy counterparts of the operads mentioned above: as before we denote by $\ell_n \in \mathscr{L}_\infty(n)$ the element of $\mathscr{L}_\infty(n)$ corresponding to the n-ary bracket. We have $\mathrm{Ass}^{\text{¡}} \cong (\mathscr{S}^{-1})^c \otimes \mathrm{Ass}^\vee$, thus in each arity $n \geq 2$ the operad Ass_∞ has $n!$ generators

$$\overline{m}_\sigma \coloneqq s^{-1}\mathscr{S}_n^{-1} m_\sigma^\vee \in \mathrm{Ass}_\infty \ , \qquad \sigma \in \mathbb{S}_n \ .$$

The action of the symmetric group on these generators is given by $(\overline{m}_\sigma)^\tau = (-1)^\tau \overline{m}_{\sigma\tau}$. Finally, the operad $\mathscr{C}_\infty$ coding homotopy commutative algebras the same thing as an $\mathscr{A}_\infty$-algebra that vanishes on the sum of all non-trivial shuffles, see Section 2.3.6.

The identity $\mathrm{Com} \longrightarrow \mathrm{Com}$

This is the simplest example. The identity of Com induces the morphism

$$\mathsf{M}_{\mathrm{Com}} : \mathscr{L}_\infty \longrightarrow \mathrm{Com} \otimes \mathscr{L}_\infty$$

which sends the element ℓ_n to

$$\mu_n \otimes s^{-1}\mathscr{S}_n \mu_n^\vee = \mu_n \otimes \ell_n \ .$$

Therefore, it is the canonical isomorphism

$$\mathscr{L}_\infty \cong \mathrm{Com} \otimes \mathscr{L}_\infty \ .$$

If A is a commutative algebra and C is an $\mathscr{L}_\infty$-algebra, then the operations on $A \otimes C$ are given by

$$\ell_n(a_1 \otimes c_1, \ldots, a_n \otimes c_n) = (-1)^\epsilon \mu_n(a_1, \ldots, a_n) \otimes \ell_n(c_1, \ldots, c_n) \ ,$$

where $(-1)^\epsilon$ is the sign obtained by commuting the a_i's and the c_i's.

The identity $\mathrm{Ass} \longrightarrow \mathrm{Ass}$

Since the operad Ass satisfies $\mathrm{Ass}^! = \mathrm{Ass}$, the induced morphism is

$$\mathsf{M}_{\mathrm{Ass}} : \mathscr{L}_\infty \longrightarrow \mathrm{Ass} \otimes \mathrm{Ass}_\infty \ .$$

It sends ℓ_n to

$$\sum_{\sigma\in\mathbb{S}_n} m_\sigma \otimes s^{-1}\mathscr{S}_n m_\sigma^\vee = \sum_{\sigma\in\mathbb{S}_n} m_\sigma \otimes \overline{m}_\sigma = \sum_{\sigma\in\mathbb{S}_n} (-1)^\sigma (m_{\mathrm{id}} \otimes \overline{m}_{\mathrm{id}}) \ .$$

If A is an associative algebra and C is an Ass_∞-algebra, then the $\mathscr{L}_\infty$ operations on $A \otimes C$ are given by

$$\ell_n(a_1 \otimes c_1, \ldots, a_n \otimes c_n) = \sum_{\sigma\in\mathbb{S}_n} (-1)^{\sigma+\epsilon} m_e(a_{\sigma^{-1}(1)}, \ldots a_{\sigma^{-1}(n)}) \otimes \overline{m}_e(c_{\sigma^{-1}(1)}, \ldots c_{\sigma^{-1}(n)}) \ ,$$

where ϵ is the sign obtained by switching the a_i's and the c_i's, and correspond therefore to a kind of antisymmetrization of Ass_∞.

The identity $\mathrm{Lie} \longrightarrow \mathrm{Lie}$

The last identity we have to look at is the identity of the operad Lie. It gives rise to a morphism of operads

$$\mathsf{M}_{\mathrm{Lie}} : \mathscr{L}_\infty \longrightarrow \mathrm{Lie} \otimes \mathscr{C}_\infty \, .$$

It is of more complicated description, but comparing formulæ we see that it is the same structure that is used in a fundamental way in the article [TW15, pp.19–20] on Hochschild–Pirashvili homology.

The forgetful morphism $u : \mathrm{Ass} \longrightarrow \mathrm{Com}$

This morphism is given by sending

$$m_\sigma \longmapsto \mu_n$$

for all $\sigma \in \mathbb{S}_n$. The corresponding morphism

$$\mathsf{M}_u : \mathscr{L}_\infty \longrightarrow \mathrm{Com} \otimes \mathrm{Ass}_\infty \cong \mathrm{Ass}_\infty$$

is given by

$$\mathsf{M}_u(\ell_n) = \sum_{\sigma \in \mathbb{S}_n} \mu_n \otimes s^{-1} \mathscr{S}_n m_\sigma^\vee = \sum_{\sigma \in \mathbb{S}_n} \mu_n \otimes \overline{m}_\sigma = \mu_n \otimes \sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma (\overline{m}_e)^\sigma .$$

Therefore, under the canonical identification $\mathrm{Com} \otimes \mathrm{Ass}_\infty \cong \mathrm{Ass}_\infty$, it is the standard antisymmetrization of an Ass_∞-algebra structure giving an $\mathscr{L}_\infty$-algebra structure.

The antisymmetrization morphism $a : \mathrm{Lie} \longrightarrow \mathrm{Ass}$

The induced morphism is a morphism of dg operads

$$\mathsf{M}_a : \mathscr{L}_\infty \longrightarrow \mathrm{Ass} \otimes \mathscr{C}_\infty \, .$$

It can be interpreted as follows: a $\mathscr{C}_\infty$-algebra can be seen as an Ass_∞-algebra vanishing on the sum of all non-trivial shuffles, that is, we have a natural morphism of operads

$$i : \mathrm{Ass}_\infty \longrightarrow \mathscr{C}_\infty \, ,$$

which is in fact given by $\Omega((\mathscr{S}^{-1})^c \otimes a^\vee)$. Now Theorem 9.1.12 tells us that

$$\mathsf{M}_a = \mathsf{M}_{1_{\mathrm{Ass}} a} = (1 \otimes i) \mathsf{M}_{1_{\mathrm{Ass}}} \, .$$

Therefore, the $\mathscr{L}_\infty$-algebra structure on the tensor product of an associative and a $\mathscr{C}_\infty$-algebra is given by first looking at the $\mathscr{C}_\infty$-algebra as an Ass_∞-algebra, and then antisymmetrizing the resulting $(\mathrm{Ass} \otimes \mathrm{Ass}_\infty)$-algebra as already done above.

The non-symmetric case

The analogues to the three graces in the non-symmetric setting are the operad As encoding associative algebras, which we already know well, the operad Dend of *dendriform algebras* ([LV12, Sect. 13.6.5]), and the operad Dias encoding *diassociative algebras* ([LV12, Sect. 13.6.7]). They fit into a sequence of morphisms

$$\mathrm{Dias} \longrightarrow \mathrm{As} \longrightarrow \mathrm{Dend} \, .$$

We get induced morphisms from $\mathscr{A}_\infty$ to

$$\mathrm{Dias} \otimes \mathrm{Dend}_\infty \, , \quad \mathscr{A}_\infty \, , \quad \mathrm{Dend} \otimes \mathrm{Dias}_\infty \, , \quad \mathrm{Dend}_\infty \quad \text{and} \quad \mathrm{Dend} \otimes \mathscr{A}_\infty \, .$$

We leave their explicit computation to the interested reader.

9.1.7 Compatibility with (co)limits of (co)algebras

To complete this section, we prove a first compatibility result between (co)algebras and the induced convolution homotopy algebras. As before, let $\mathscr{C}$ be a cooperad, let $\mathscr{P}$ be an operad, and let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism.

Proposition 9.1.23. *Let I be a small category.*

1. *Let $A : I \to \mathscr{P}$-alg be a functor, suppose $\lim_{i\in I} A(i)$ exists, and let C be a conilpotent $\mathscr{C}$-coalgebra. Then*
$$\lim_{i\in I} \hom^{\alpha}(C, A(i)) \cong \hom^{\alpha}\left(C, \lim_{i\in I} A(i)\right)$$
as $\mathfrak{s}\mathscr{L}_\infty$-algebras through the canonical morphism.

2. *Dually, let $C : I \to \mathscr{C}$-cog be a functor, suppose $\operatorname{colim}_{i\in I} C(i)$ exists, and let A be a $\mathscr{P}$-algebra. Then*
$$\lim_{i\in I} \hom^{\alpha}(C(i), A) \cong \hom^{\alpha}\left(\operatorname*{colim}_{i\in I} C(i), A\right)$$
as $\mathfrak{s}\mathscr{L}_\infty$-algebras through the canonical morphism.

Proof. The isomorphisms hold at the level of chain complexes, and the canonical maps are morphisms of $\mathfrak{s}\mathscr{L}_\infty$-algebras. One concludes by Lemma 2.1.24, resp. Lemma 2.1.31. □

We also have a "linear maps-tensor products" duality analogue to what happens for chain complexes.

Proposition 9.1.24. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism, and suppose that $\mathscr{C}$ is finite dimensional in every arity. Let C be a finite-dimensional, conilpotent $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra. Then*
$$\hom^{\alpha}(C, A) \cong A \otimes^{\alpha} C^{\vee}$$
as $\mathfrak{s}\mathscr{L}_\infty$-algebras via the canonical isomorphism of chain complexes
$$f \in \hom(C, A) \longmapsto \sum_{j=1}^{N} f(x_j) \otimes x_j^{\vee} ,$$
where $\{x_j\}_{j=1}^n$ is a basis C.

Proof. In this proof, we will avoid writing down any explicit signs, and work in the non-symmetric setting. The symmetric case works in exactly the same way, but one has to write down the correct permutations occurring in the formulæ.

Since the described morphism is an isomorphism of chain complexes, by Lemma 2.1.24 it is enough to show that it is a morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras. In order to do this, let $f_1, \dots, f_n \in \hom(C, A)$, and write $F \coloneqq f_1 \otimes \cdots \otimes f_n$. Fix a basis $\{c_k\}_k$ of $\mathscr{C}(n)$. In $\hom^{\alpha}(C, A)$, we have
$$\ell_n(F) = \gamma_A(\alpha \otimes F)\Delta_C \in \hom(C, A) .$$
This is sent to the element of $A \otimes C^{\vee}$ given by
$$\sum_j (\gamma_A(\alpha \otimes F)\Delta_C^n(x_j)) \otimes x_j^{\vee} =$$

$$
\begin{aligned}
&= \sum_{k,j,i_1,\dots,i_n} \pm\langle c_k^\vee \otimes x_{i_1}^\vee \otimes \cdots \otimes x_{i_n}^\vee, \Delta_C(x_j)\rangle \left(\gamma_A(\alpha(c_k) \otimes f_1(x_{i_1}) \otimes \cdots \otimes f_n(x_{i_n}))\right) \otimes x_j^\vee \\
&= \sum_{k,i_1,\dots,i_n} \left(\gamma_A(\alpha(c_k) \otimes f_1(x_{i_1}) \otimes \cdots \otimes f_n(x_{i_n}))\right) \otimes \sum_j \pm\langle \gamma_{C^\vee}(c_k^\vee \otimes x_{i_1}^\vee \otimes \cdots \otimes x_{i_n}^\vee), x_j\rangle x_j^\vee \\
&= \sum_{k,i_1,\dots,i_n} \left(\gamma_A(\alpha(c_k) \otimes f_1(x_{i_1}) \otimes \cdots \otimes f_n(x_{i_n}))\right) \otimes \sum_j \pm\gamma_{C^\vee}(c_k^\vee \otimes x_{i_1}^\vee \otimes \cdots \otimes x_{i_n}^\vee) \\
&= \ell_n\left(\sum_{i_1} f_1(x_{i_1}) \otimes x_{i_1}^\vee, \dots, \sum_{i_n} f_n(x_{i_n}) \otimes x_{i_n}^\vee\right),
\end{aligned}
$$

which concludes the proof. □

9.2 Compatibility with ∞-morphisms

In this section, we study the compatibility between convolution homotopy Lie algebras and ∞-morphisms of algebras and coalgebras. More precisely, we will show that the bifunctor $\hom^\alpha(-,-)$ can be extended two natural ways to take either ∞_α-morphisms in the first slot, or in the second one. We prove that those two extension do not admit a further common extension to a bifunctor accepting ∞_α-morphisms in both slots. We conclude the section by proving that however, this last extension is possible if one accepts to work up to homotopy.

9.2.1 The fundamental theorem of convolution homotopy algebras

Fix a cooperad $\mathscr{C}$, an operad $\mathscr{P}$, and a twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$. Let C be a conilpotent $\mathscr{C}$-coalgebra, and let A, A' be two $\mathscr{P}$-algebras. Given an element

$$x \in \hom^\alpha(\mathrm{B}_\alpha A, A') ,$$

we define an element

$$\hom^\alpha(1, x) \in \hom^\iota(\mathrm{B}_\iota \hom^\alpha(C, A), \hom^\alpha(C, A')$$

as follows. Let $f_1, \dots, f_n \in \hom(C, A)$ and denote $F \coloneqq f_1 \otimes \cdots \otimes f_n$, then $x_*(\mu_n^\vee \otimes F)$ is given by the composite

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta_C^n} & \mathscr{C}(n) \otimes_{\mathbb{S}_n} C^{\otimes n} \\
{\scriptstyle \hom^\alpha(1,x)(\mu_n^\vee \otimes F)}\ \vdots & & \Big\downarrow {\scriptstyle F} \\
A' & \xleftarrow{\quad x \quad} & \mathscr{C}(A)
\end{array}
$$

where F acts on $\mathscr{C}(n) \otimes_{\mathbb{S}_n} C^{\otimes n}$ by

$$F(c \otimes y_1 \otimes \cdots y_n) \coloneqq \sum_{\sigma \in \mathbb{S}_n} (-1)^\epsilon c \otimes f_{\sigma(1)}(y_1) \otimes \cdots \otimes f_{\sigma(n)}(y_n) \in \mathscr{C}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} .$$

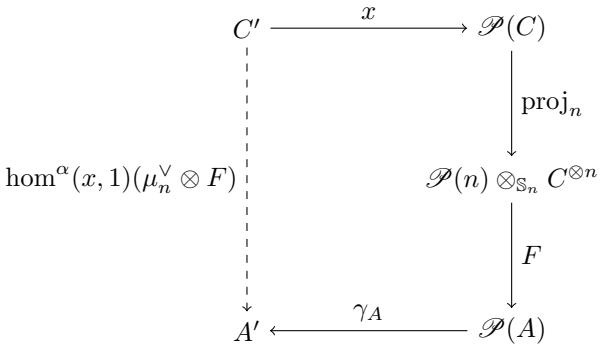
Dually, let C, C' be two conilpotent $\mathscr{C}$-coalgebras, and let A be a $\mathscr{P}$-algebra. Given

$$x \in \hom^\alpha(C', \Omega_\alpha C) ,$$

we define an element

$$\hom^{\alpha}(x,1) \in \hom^{\iota}(\mathrm{B}_{\iota}\hom^{\alpha}(C,A), \hom^{\alpha}(C',A))$$

as follows. Let $f_1,\dots,f_n \in \hom(C,A)$, then $x^*(\mu_n^{\vee}\otimes F)$ is given by the composite



with the action of F similar to the one defined above.

Theorem 9.2.1 ([RNW, Thm. 3.1])**.** *Let $\mathscr{C}$ be a cooperad, let $\mathscr{P}$ be an operad, and let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism. Let C, C' be two conilpotent $\mathscr{C}$-coalgebras, and let A, A' be two $\mathscr{P}$-algebras.*

1. *The map*

$$\hom^{\alpha}(1,-) : \hom^{\alpha}(\mathrm{B}_{\alpha}A, A') \longrightarrow \hom^{\iota}(\mathrm{B}_{\iota}\hom^{\alpha}(C,A), \hom^{\alpha}(C,A'))$$

   defined above is a morphism of $\mathfrak{s}\mathscr{L}_{\infty}$-algebras.

2. *The map*

$$\hom^{\alpha}(-,1) : \hom^{\alpha}(C', \Omega_{\alpha}C) \longrightarrow \hom^{\iota}(\mathrm{B}_{\iota}\hom^{\alpha}(C,A), \hom^{\alpha}(C',A))$$

   defined above is a morphism of $\mathfrak{s}\mathscr{L}_{\infty}$-algebras.

This theorem can reasonably be considered one of the fundamental results about convolution homotopy algebras — at least as of the time of writing of this thesis. It should be considered a refinement of [RN17a, Prop. 4.4] and [RNW17, Thm. 5.1], and indeed we will see these results appear as corollaries of the statement above later on.

For the proof, we begin with two technical lemmas.

Lemma 9.2.2. *Let $k, n_1, \dots, n_k \geq 0$, let M be an $\mathbb{S}$-module, let V, W be two chain complex. Denote $n \coloneqq k_1 + \cdots + n_k$, let $f_1,\dots,f_n \in \hom(V,W)$, and as usual denote $F \coloneqq f_1 \otimes \cdots \otimes f_n$. Then, under the isomorphism*

$$\left(M(k)\otimes\bigotimes_{i=1}^{k} M(n_i)\right)\otimes V^{\otimes n} \cong M(k)\otimes\bigotimes_{i=1}^{k}\left(M(n_i)\otimes V^{\otimes n_i}\right),$$

we have

$$F = \sum_{S_1\sqcup\cdots\sqcup S_k=[n]} 1_M \circ \left(F^{S_1}\otimes\cdots\otimes F^{S_k}\right),$$

where $F^{S_i} \coloneqq \bigotimes_{j\in S_i} f_j$, with the elements of S_i in ascending order.

Proof. This is a straightforward computation — if messy at the level of signs. It is left to the reader. □

Lemma 9.2.3. *Let $\mathscr{C}$ be a cooperad, and let C be a conilpotent $\mathscr{C}$-coalgebra. Then*

$$(\Delta_{(1)} \circ 1_C)\Delta_C^n = \sum_{\substack{n_1+n_2=n+1 \\ 1\leq i\leq n_1}} (1_{\mathscr{C}} \circ (1_C^{\otimes(i-1)} \otimes \Delta_C^{n_2} \otimes 1_C^{\otimes(n-i)}))\Delta_C^{n_1} .$$

Moreover, let $f_1, \dots, f_n \in \hom(C, V)$ for V a chain complex. Under the canonical inclusion

$$\bigoplus_{\substack{n_1+n_2=n+1 \\ 1\leq i\leq n_1}} \mathscr{C}(n_1) \circ (C^{\otimes(i-1)}(\mathscr{C}(n_2) \otimes C^{\otimes n_2}) \otimes C^{\otimes(n-i)}) \longhookrightarrow (\mathscr{C} \circ \mathscr{C})(n) \otimes C^{\otimes n}$$

we have

$$F(\Delta_{(1)} \circ 1_C)\Delta_C^n = \sum_{S_1 \sqcup S_2 = [n]} ((F^{S_1}\Delta_C^{n_1}) \otimes F^{S_2})\Delta_C^{n_2} ,$$

where $n_1 = |S_1|$ and $n_2 = |S_2| + 1$.

Proof. For the first identity, one considers the equality

$$(\Delta_{\mathscr{C}} \circ 1_C)\Delta_C = (1_{\mathscr{C}} \circ \Delta_C)\Delta_C$$

and then projects on the subspace

$$(\mathscr{C} \circ_{(1)} \mathscr{C})(C) \cong \mathscr{C} \circ (C; \mathscr{C}(C)) .$$

We leave the details to the reader. The second statement then follows in a straightforward way. □

Proof of Theorem 9.2.1. We prove the first case, the second one being dual. Let $x_1, \dots, x_k \in \hom^\alpha(\mathrm{B}_\alpha A, A')$. Then we have

$$\ell_k(x_1, \dots, x_k) = \gamma_{A'}(\alpha \otimes X)(\Delta_{\mathscr{C}}^k \circ 1_A) ,$$

and thus for $f_1, \dots, f_n \in \hom^\alpha(C, A)$

$$\begin{aligned}
&\hom^\alpha(1, \ell_k(X))(\mu_n^\vee \otimes F) = \\
&\qquad = \ell_k(X)F\Delta_C^n \\
&\qquad = \gamma_{A'}(\alpha \otimes X)(\Delta_{\mathscr{C}}^k \circ 1_A)F\Delta_C^n \\
&\qquad = \gamma_{A'}(\alpha \otimes X)F(\Delta_{\mathscr{C}}^k \circ 1_C)\Delta_C^n \\
&\qquad = \sum_{n_1+\dots+n_k=n} \gamma_{A'}(\alpha \otimes X)F(1_{\mathscr{C}} \circ (\Delta_C^{n_1} \otimes \dots \otimes \Delta_C^{n_k}))\Delta_C^k \\
&\qquad = \sum_{S_1 \sqcup \dots \sqcup S_k = [n]} (-1)^{\epsilon_1}\gamma_{A'}(\alpha \otimes X)(1_{\mathscr{C}} \circ (F^{S_1}\Delta_C^{n_1} \otimes \dots \otimes F^{S_k}\Delta_C^{n_k}))\Delta_C^k \\
&\qquad = \sum_{\substack{S_1 \sqcup \dots \sqcup S_k = [n] \\ \sigma \in \mathbb{S}_k}} (-1)^{\epsilon_1+\epsilon_2}\gamma_{A'}(\alpha \circ 1_{A'})(1_{\mathscr{C}} \circ (x_{\sigma(1)}F^{S_1}\Delta_C^{n_1} \otimes \dots \otimes \Phi_{\sigma(k)}F^{S_k}\Delta_C^{n_k}))\Delta_C^k
\end{aligned}$$

where the fourth line follows from $(\Delta_{\mathscr{C}} \circ C)\Delta_C = (1_{\mathscr{C}} \circ \Delta_C)\Delta_C$, and in the fifth line we used Lemma 9.2.2 and denoted $n_i \coloneqq |S_i|$. The Koszul signs are

$$\epsilon_1 = \sum_{i=1}^{k} \sum_{s \in S_i} |f_s| \sum_{j<i} \sum_{p \in S_j \text{ s.t. } p>s} |f_p| \,,$$

obtained by shuffling the f_is, and ϵ_2, which is similarly obtained by permuting the x_is and making them jump over the f_js.

On the other hand, we have

$$\begin{aligned}
\ell_k(\hom^\alpha(1,X))(\mu_n^\vee \otimes F) = \\
&= \big(\gamma_{\hom^\alpha(C,A')}(\iota \otimes \hom^\alpha(1,X))(\Delta^k_{\mathrm{Com}^\vee} \otimes 1_{\hom(C,A)})\big)(\mu_n^\vee \otimes F) \\
&= \big(\gamma_{\hom^\alpha(C,A')}(\iota \otimes \hom^\alpha(1,X))\big) \left(\sum_{S_1 \sqcup \ldots \sqcup S_k = [n]} (-1)^{\epsilon_1} \mu_k^\vee \otimes \bigotimes_{i=1}^{k} (\mu_{n_i}^\vee \otimes F^{S_i}) \right) \\
&= \gamma_{\hom^\alpha(C,A')} \left(\sum_{\substack{S_1 \sqcup \ldots \sqcup S_k = [n] \\ \sigma \in \mathbb{S}_k}} (-1)^{\epsilon_1+\epsilon_2} s^{-1} \mu_k^\vee \otimes \bigotimes_{i=1}^{k} \hom^\alpha(1, x_{\sigma(i)})(\mu_{n_i}^\vee \otimes F^{S_i}) \right) \\
&= \sum_{\substack{S_1 \sqcup \ldots \sqcup S_k = [n] \\ \sigma \in \mathbb{S}_k}} (-1)^{\epsilon_1+\epsilon_2} \gamma_{A'} \left(\alpha \otimes \bigotimes_{i=1}^{k} \hom^\alpha(1, x_{\sigma(i)})(\mu_{n_i}^\vee \otimes F^{S_i}) \right) \Delta_C^k \\
&= \sum_{\substack{S_1 \sqcup \ldots \sqcup S_k = [n] \\ \sigma \in \mathbb{S}_k}} (-1)^{\epsilon_1+\epsilon_2} \gamma_{A'} \left(\alpha \otimes \bigotimes_{i=1}^{k} x_{\sigma(i)} F^{S_i} \Delta_C^{n_i} \right) \Delta_C^k \,.
\end{aligned}$$

The reader might have the impression that a sum over permutations coming from $\gamma_{\hom^\alpha(C,A')}$ has been forgotten in the fourth line. This is not the case, because the term

$$\sum_{\substack{S_1 \sqcup \ldots \sqcup S_k = [n] \\ \sigma \in \mathbb{S}_k}} (-1)^{\epsilon_1+\epsilon_2} s^{-1} \mu_k^\vee \otimes \bigotimes_{i=1}^{k} \hom^\alpha(1, x_{\sigma(i)})(\mu_{n_i}^\vee \otimes F^{S_i})$$

in the third line naturally lives in invariants, not coinvariants.

In conclusion, we have

$$\hom^\alpha(1, \ell_k(X)) = \ell_k(\hom^\alpha(1,X)) \,.$$

We are left to prove that the morphism commutes with the differentials. Let $x \in \hom^\alpha(\mathrm{B}_\alpha A, A')$, and let $f_1, \ldots, f_n \in \hom^\alpha(C, A)$. On one hand, we have

$$\begin{aligned}
d(x) &= d_{A'}x - (-1)^{|x|} x d_{\mathrm{B}_\alpha A} \\
&= d_{A'}x - (-1)^{|x|} x d_{\mathscr{C}\circ A} - (-1)^{|x|} x (1_{\mathscr{C}} \circ (1_A; \gamma_A))((1_{\mathscr{C}} \circ_{(1)} \alpha) \circ 1_A)(\Delta_{(1)} \circ 1_A) \,,
\end{aligned}$$

and thus

$$\hom^\alpha(1, d(x))(\mu_n^\vee \otimes F) = d(x) F \Delta_C^n$$

$$
\begin{aligned}
&= d_{A'} x F\Delta_C^n - (-1)^{|x|} x d_{\mathscr{C}\circ A} F\Delta_C^n \\
&\quad - (-1)^{|x|} x(1_{\mathscr{C}} \circ (1_A;\gamma_A))((1_{\mathscr{C}} \circ_{(1)} \alpha)\circ 1_A)(\Delta_{(1)}\circ 1_A)F\Delta_C^n \,.
\end{aligned}
$$

The second term in the second line equals

$$
\begin{aligned}
x d_{\mathscr{C}\circ A} F\Delta_C^n &= x(d_{\mathscr{C}}\circ 1_A)F\Delta_C^n + x(1_{\mathscr{C}} \circ' d_A)F\Delta_C^n \\
&= (-1)^{|F|} x F(d_{\mathscr{C}}\circ 1_C)\Delta_C^n + x\partial(F)\Delta_C^n + (-1)^{|F|} F x(1_{\mathscr{C}} \circ' d_C)\Delta_C^n \\
&= (-1)^{|F|} x F\Delta_C^n d_C + x\partial(F)\Delta_C^n \,, \qquad \text{(T1)}
\end{aligned}
$$

while the term in the last line gives

$$
\begin{aligned}
&x(1_{\mathscr{C}} \circ (1_A;\gamma_A))((1_{\mathscr{C}} \circ_{(1)} \alpha)\circ 1_A)(\Delta_{(1)}\circ 1_A)F\Delta_C^n = \\
&\qquad = x(1_{\mathscr{C}} \circ (1_A;\gamma_A))((1_{\mathscr{C}} \circ_{(1)} \alpha)\circ 1_A)F(\Delta_{(1)}\circ 1_C)\Delta_C^n \\
&\qquad = \sum_{S_1\sqcup S_2=[n]} (-1)^{\epsilon} x(1_{\mathscr{C}} \circ (1_A;\gamma_A))((1_{\mathscr{C}} \circ_{(1)} \alpha)\circ 1_A)((F^{S_1}\Delta_C^{n_1})\otimes F^{S_2})\Delta_C^{n_2} \,, \qquad \text{(T2)}
\end{aligned}
$$

where in the third line we used Lemma 9.2.3. On the other hand,

$$
d(\hom^\alpha(1,x)) = d_{\hom^\alpha(C,A')}\hom^\alpha(1,x) - (-1)^{|\hom^\alpha(1,x)|}\hom^\alpha(1,x) d_{\mathrm{B}_\iota \hom^\alpha(C,A)} \,.
$$

We notice that $(-1)^{|\hom^\alpha(1,x)|} = (-1)^{|x|}$. We apply this to $\mu_n^\vee \otimes F$ and obtain

$$
\begin{aligned}
d(\hom^\alpha(1,x))(\mu_n^\vee\otimes F) &= d_{A'}\hom^\alpha(1,x) - (-1)^{|x|+|F|}\hom^\alpha(1,x)(\mu_n^\vee\otimes F)d_C \\
&\quad - (-1)^{|x|}\hom^\alpha(1,x)(d_{\mathrm{B}_\iota\hom^\alpha(C,A)}(\mu_n^\vee\otimes F)) \,.
\end{aligned}
$$

The first term equals $d_{A'}\Phi F\Delta_C^n$ and cancels with the first term of $\hom^\alpha(1,d(\Phi))(\mu_n^\vee\otimes F)$, and the second term equals the first term of (T1). For the third term, we have

$$
\begin{aligned}
&\hom^\alpha(1,\Phi)(d_{\mathrm{B}_\iota\hom^\alpha(C,A)}(\mu_n^\vee\otimes F)) = \\
&\qquad = \hom^\alpha(1,\Phi)(\mu_n^\vee\otimes\partial(F)) \\
&\qquad\quad + \hom^\alpha(1,\Phi)\left(\sum_{S_1\sqcup S_2=[n]} \mu_{n_2}^\vee\otimes(\ell_{n_1}(F^{S_1})\otimes F^{S_2})\right) .
\end{aligned}
$$

The first term of this expression cancels the remaining term of (T1). Therefore, we are left to show that the second term equals (T2). We have

$$
\begin{aligned}
&\hom^\alpha(1,x)\left(\sum_{S_1\sqcup S_2=[n]} (-1)^{\epsilon}\mu_{n_2}^\vee\otimes(\ell_{n_1}(F^{S_1})\otimes F^{S_2})\right) = \\
&\quad = \sum_{S_1\sqcup S_2=[n]} (-1)^{\epsilon} x(\ell_{n_1}(F^{S_1})\otimes F^{S_2})\Delta_C^{n_2} \\
&\quad = \sum_{S_1\sqcup S_2=[n]} (-1)^{\epsilon} x((\gamma_A(\alpha\otimes F^{S_1})\Delta_C^{n_1})\otimes F^{S_2})\Delta_C^{n_2} \\
&\quad = \sum_{S_1\sqcup S_2=[n]} (-1)^{\epsilon} x(1_{\mathscr{C}}\circ(\gamma_A(\alpha\circ 1_A)\otimes 1_A^{\otimes(n_2-1)}))(1_{\mathscr{C}}\circ(F^{S_1}\Delta_C^{n_1}\otimes F^{S_2}))\Delta_C^{n_2} \\
&\quad = (\text{T2}) \,,
\end{aligned}
$$

where $n_1 = |S_1|$, and $n_2 = |S_2| + 1$. This concludes the proof. □

9.2.2 Compatibility with ∞-morphisms

The first application of Theorem 9.2.1 is the fact that convolution $\mathscr{L}_\infty$-algebras are compatible with ∞-morphisms (relative to the twisting morphism α under consideration).

Let $\mathscr{C}$ be a cooperad, let $\mathscr{P}$ be an operad, and let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism. Take two $\mathscr{P}$-algebras A, A', as well as a conilpotent $\mathscr{C}$-coalgebra C. By Theorem 9.2.1, we have a morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras

$$\hom^\alpha(1,-) : \hom^\alpha(\mathrm{B}_\alpha A, A') \longrightarrow \hom^\iota(\mathrm{B}_\iota \hom^\alpha(C,A), \hom^\alpha(C,A')) \,.$$

Let's look at what happens to Maurer–Cartan elements. By Theorem 9.1.9, the Maurer–Cartan elements of $\hom^\alpha(\mathrm{B}_\alpha A, A')$ are exactly the ∞_α-morphisms $A \rightsquigarrow A'$. Now, since $\hom^\alpha(1,-)$ is a morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras, it preserves Maurer–Cartan elements. Thus, an ∞_α-morphism $\Psi : A \rightsquigarrow A'$ is sent to a Maurer–Cartan element of $\hom^\iota(\mathrm{B}_\iota \hom^\alpha(C,A), \hom^\alpha(C,A'))$. But again by Theorem 9.1.9, these are exactly the ∞-morphisms $\hom^\alpha(C,A) \rightsquigarrow \hom^\alpha(C,A')$. The same thing is true for the dual case. In conclusion, we have the following.

Corollary 9.2.4 ([RNW17, Thm. 5.1])**.** *We place ourselves in the setting of Theorem 9.2.1.*

1. *Suppose $\Psi : A \rightsquigarrow A'$ is an ∞_α-morphism of $\mathscr{P}$-algebras. Then*

$$\hom^\alpha(1,\Psi) : \hom^\alpha(C,A) \rightsquigarrow \hom^\alpha(C,A')$$

   is an ∞-morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras.

2. *Dually, suppose that $\Phi : C' \rightsquigarrow C$ is an ∞_α-morphism of $\mathscr{C}$-coalgebras. Then*

$$\hom^\alpha(\Phi,1) : \hom^\alpha(C,A) \rightsquigarrow \hom^\alpha(C',A)$$

   is an ∞-morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras.

But now, in the situation above, $\hom^\alpha(1,\Psi)$ also preserves Maurer–Cartan elements. Therefore, it sends morphisms $C \to \mathrm{B}_\alpha A$ of $\mathscr{C}$-coalgebras to morphisms $C \to \mathrm{B}_\alpha A'$ of $\mathscr{C}$-coalgebras.

Lemma 9.2.5. *We place ourselves in the situation of Theorem 9.2.1 and Corollary 9.2.4.*

1. *Suppose $\Psi : A \rightsquigarrow A'$ is an ∞_α-morphism of $\mathscr{P}$-algebras, and let $f : C \to \mathrm{B}_\alpha A$ be a morphism of $\mathscr{C}$-coalgebras. Then*

$$\mathrm{MC}(\hom^\alpha(1,\Psi))(f) = \Psi f = \left(C \xrightarrow{f} \mathrm{B}_\alpha A \xrightarrow{\Psi} \mathrm{B}_\alpha A'\right).$$

2. *Suppose $\Phi : C' \rightsquigarrow C$ is an ∞_α-morphism of $\mathscr{C}$-coalgebras, and let $g : \Omega_\alpha C \to A$ be a morphism of $\mathscr{P}$-algebras. Then*

$$\mathrm{MC}(\hom^\alpha(\Phi,1))(g) = g\Phi = \left(\Omega_\alpha C' \xrightarrow{\Phi} \Omega_\alpha C \xrightarrow{g} A\right).$$

Proof. In order to give a clear proof, we will write $f \in \mathrm{MC}(\hom^\alpha(C,A))$ for the element f seen as a linear map $C \to A$, and $\widetilde{f}$ for the equivalent map of $\mathscr{C}$-coalgebras $C \to \mathrm{B}_\alpha A$. We pass from the former to the latter by

$$\widetilde{f} = (1_{\mathscr{C}} \circ f)\Delta_C \,.$$

When writing Ψ, we will mean the map of $\mathscr{C}$-coalgebras $\mathrm{B}_\alpha A \to \mathrm{B}_\alpha A'$, and the associated Maurer–Cartan element is

$$\widetilde{\Psi} \coloneqq \mathrm{proj}_{A'} \Psi \in \mathrm{MC}(\hom^\alpha(\mathrm{B}_\alpha A, A')) \,.$$

With this notation, we have

$$\begin{aligned}
\mathrm{MC}(\hom^\alpha(1, \widetilde{\Psi}))(f) &= \sum_{n\geq 1} \frac{1}{n!} \hom^\alpha(1, \widetilde{\Psi})(\mu_n^\vee \otimes f^{\otimes n}) \\
&= \sum_{n\geq 1} \frac{1}{n!} \widetilde{\Psi} f^{\otimes n} \Delta_C^n \\
&= \sum_{n\geq 1} \mathrm{proj}_{A'} \Psi(1_{\mathscr{C}} \circ f) \Delta_C^n \\
&= \mathrm{proj}_{A'} \Psi(1_{\mathscr{C}} \circ f) \Delta_C \\
&= \mathrm{proj}_{A'} \Psi \widetilde{f} \,,
\end{aligned}$$

where $\mu_1^\vee = \mathrm{id}$. The other case is similar, and left to the reader. □

Remark 9.2.6. *If Ψ is a strict morphism of $\mathscr{P}$-algebras, then the ∞-morphism $\mathrm{MC}(\hom^\alpha(1, \Psi))$ is actually a strict morphism, and it is just given by $\hom^\alpha(1, \Psi)$, the action of the bifunctor (9.1) on morphisms.*

Another powerful corollary of Theorem 9.2.1 is the following.

Corollary 9.2.7. *We place ourselves in the setting of Theorem 9.2.1.*

1. *Suppose $\Psi, \Psi' : A \rightsquigarrow A'$ are two ∞_α-morphisms of $\mathscr{P}$-algebras. If Ψ and Ψ' are homotopic as Maurer–Cartan elements, then so are $\hom^\alpha(1, \Psi)$ and $\hom^\alpha(1, \Psi')$.*
2. *Dually, suppose $\Phi, \Phi' : C' \rightsquigarrow C$ are two ∞_α-morphisms of $\mathscr{C}$-coalgebras. If Φ and Φ' are homotopic as Maurer–Cartan elements, then so are $\hom^\alpha(\Phi, 1)$ and $\hom^\alpha(\Phi', 1)$.*

This is especially important when combined with the following result.

Theorem 9.2.8. *We place ourselves in the setting of Theorem 9.2.1.*

1. *Two ∞_α-morphisms $A \rightsquigarrow A'$ of $\mathscr{P}$-algebras are homotopic if, and only if they are gauge equivalent when seen as Maurer–Cartan elements of $\hom^\alpha(\mathrm{B}_\alpha A, A')$.*
2. *Two ∞_α-morphisms $C' \rightsquigarrow C$ of $\mathscr{C}$-coalgebras are homotopic if, and only if they are gauge equivalent when seen as Maurer–Cartan elements of $\hom^\alpha(C', \Omega_\alpha C)$.*

Proof. We only give a sketch of the proof. The reader can find a detailed version in [RNW, Thm. 2.4].

The proof relies on the fact that there is a natural homotopy equivalence

$$\mathrm{MC}_\bullet(\hom^\alpha(C, A)) \simeq \mathrm{MC}(\hom^\alpha(C, A \otimes \Omega_\bullet))$$

induced by the inclusion of $\hom^\alpha(C, A) \otimes \Omega_\bullet$ into $\hom^\alpha(C, A \otimes \Omega_\bullet)$. Assuming that this is true for a second, the statement follows by looking at the 0th homotopy groups of the two simplicial sets — corresponding to Maurer–Cartan elements modulo gauges — and noticing that a good

path object for the $\mathscr{C}$-coalgebra $\mathrm{B}_\alpha A'$ is given by $\mathrm{B}_\alpha(A' \otimes \Omega_1)$, and a good path object for the $\mathscr{P}$-algebra $\Omega_\alpha C$ is given by $\Omega_\alpha(C) \otimes \Omega_1$.

To prove that we have the aforementioned homotopy equivalence, one proceeds as in Section 10.1 on both $\hom^\alpha(C, A) \otimes \Omega_\bullet$ and $\hom^\alpha(C, A \otimes \Omega_\bullet)$, showing that their Maurer–Cartan spaces are homotopically equivalent to the Maurer–Cartan spaces of $\hom^\alpha(C, A) \otimes C_\bullet$ and $\hom(C, A \otimes C_\bullet)$, both with the $\mathfrak{s}\mathscr{L}_\infty$-algebra structures obtained by homotopy transfer theorem applied on the contraction induced by Dupont's contraction. These two last algebras are easily checked to be isomorphic, concluding the proof. □

9.2.3 Two bifunctors

The results of Section 9.2.2 give us two natural ways to extend the bifunctor

$$\hom^\alpha : \mathscr{C}\text{-cog}^{\mathrm{op}} \times \mathscr{P}\text{-alg} \longrightarrow \mathfrak{s}\mathscr{L}_\infty\text{-alg}$$

to categories of (co)algebras with ∞-morphisms. Namely, we define

$$\hom^\alpha_\ell : \infty_\alpha\text{-}\mathscr{C}\text{-cog}^{\mathrm{op}} \times \mathscr{P}\text{-alg} \longrightarrow \infty\text{-}\mathfrak{s}\mathscr{L}_\infty\text{-alg}$$

by

$$\hom^\alpha_\ell(\Phi, f) \coloneqq \mathrm{MC}(\hom^\alpha(\Phi, 1))\, \hom^\alpha(1, f)$$

for a morphism f of $\mathscr{P}$-algebras and an ∞_α-morphism Φ of $\mathscr{C}$-coalgebras. Dually, we define

$$\hom^\alpha_r : \mathscr{C}\text{-cog}^{\mathrm{op}} \times \infty_\alpha\text{-}\mathscr{P}\text{-alg} \longrightarrow \infty\text{-}\mathfrak{s}\mathscr{L}_\infty\text{-alg}$$

by

$$\hom^\alpha_r(g, \Psi) \coloneqq \hom^\alpha(g, 1)\mathrm{MC}(\hom^\alpha(1, \Psi))$$

for a morphism g of $\mathscr{C}$-coalgebras and an ∞_α-morphism Ψ of $\mathscr{P}$-algebras.

Theorem 9.2.9 ([RNW17, Cor. 5.4])**.** *The two assignments* $\hom^\alpha_\ell$ *and* $\hom^\alpha_r$ *defined above are bifunctors.*

Proof. This follows immediately from Lemma 9.2.5. □

After such a statement, it is very natural to ask if one can extend the original bifunctor $\hom^\alpha$ to take ∞_α-morphisms in both slots, and not just one, in such a way as to agree with $\hom^\alpha_\ell$ and $\hom^\alpha_r$ in the obvious subcategories. In Section 9.2.4, we will show that this is in fact not possible. However, it is possible to do this *up to homotopy*, as will be shown in Section 9.2.6.

9.2.4 Failure of the extension of the two bifunctors

We will work over a field of characteristic 0 and in the non-symmetric setting (see Section 9.1.3). If there were such a bifunctor, then we would necessarily have

$$\hom^\alpha_\ell(\Phi, 1)\, \hom^\alpha_r(1, \Psi) = \hom^\alpha(\Phi, \Psi) = \hom^\alpha_r(1, \Psi)\, \hom^\alpha_\ell(\Phi, 1)$$

for any couple of ∞_α-morphisms. We will give an explicit example where this is not the case. For reference, notice that the two composites are given by the diagrams

$$\begin{array}{ccccc}
D' & \xrightarrow{\Phi} & \mathscr{P}(D) & \xrightarrow{\mathscr{P}(\Delta_D)} & \mathscr{P}(\mathscr{C}(D)) \\
 & & & & \downarrow{\scriptstyle \mathrm{proj}_n} \\
\hom_\ell^\alpha(\Phi,1)\hom_r^\alpha(1,\Psi)(\mu_n^\vee\otimes F) & \vdots & & & (\mathscr{P}\circ\mathscr{C})(n)\otimes_{\mathbb{S}_n} D^{\otimes n} \\
 & & & & \downarrow{\scriptstyle F} \\
A' & \xleftarrow{\gamma_{A'}} & \mathscr{P}(A') & \xleftarrow{\mathscr{P}(\Psi)} & \mathscr{P}(\mathscr{C}(A))
\end{array}$$

and

$$\begin{array}{ccccc}
D' & \xrightarrow{\Delta_{D'}} & \mathscr{C}(D') & \xrightarrow{\mathscr{C}(\Phi)} & \mathscr{C}(\mathscr{P}(D)) \\
 & & & & \downarrow{\scriptstyle \mathrm{proj}_n} \\
\hom_r^\alpha(1,\Psi)\hom_\ell^\alpha(\Phi,1)(\mu_n^\vee\otimes F) & \vdots & & & (\mathscr{C}\circ\mathscr{P})(n)\otimes_{\mathbb{S}_n} D^{\otimes n} \\
 & & & & \downarrow{\scriptstyle F} \\
A' & \xleftarrow{\Psi} & \mathscr{C}(A) & \xleftarrow{\mathscr{C}(\gamma_A)} & \mathscr{C}(\mathscr{P}(A))
\end{array}$$

respectively, when applied to $\mu_n^\vee \otimes F \in \mathrm{B}_\iota(\hom(D, A))$.

We will work with *non-symmetric* associative algebras and (suspended) coassociative coalgebras. Since $\mathrm{As}(n) \cong \mathbb{K}$ for each $n \geq 1$, for any associative algebra A we will implicitly identify $\mathrm{As}(n) \otimes A^{\otimes n}$ with $A^{\otimes n}$ in some places, and similarly for coassociative coalgebras.

The families A^n and H^n

We define A^n for $n \geq 1$ as the commutative algebra

$$A^n \coloneqq \overline{\mathbb{K}[x, y]}$$

seen as an associative algebra. The overline means that we take the augmentation ideal of $\mathbb{K}[x, y]$, i.e. that we only consider polynomials with no constant term. The degrees are $|x| = 0$ and $|y| = 1$ and the differential is given by $dy = x^n$. Notice that $y^2 = 0$. We have

$$d(x^a) = 0\,, \qquad d(x^a y) = x^{a+n}\,.$$

It follows that, as a chain complex,

$$H^n \coloneqq H_\bullet(A^n) \cong \bigoplus_{a=1}^{n-1} \mathbb{K} z_a\,,$$

where $z_a = [x^a]$ is the class of x^a. We have three maps

$$h \circlearrowright A_n \underset{i}{\overset{p}{\rightleftarrows}} H_n$$

given by

1. $i(z_a) = x^a$.
2. $p(x^a) = y_a$ for $a < n$ and zero on all other monomials.
3. $h(x^a) = x^{a-n}y$ for $a \geq n$ and zero on all other monomials.

Lemma 9.2.10. *The maps described above form a contraction.*

Proof. This is a straightforward computation. □

Now we apply the homotopy transfer theorem to obtain an As_∞-algebra structure on H^n and ∞-morphisms between the two algebras.

Lemma 9.2.11. *The algebra A^n is formal, and*

$$H^n \cong \frac{\overline{\mathbb{K}[z]}}{(z^n)}$$

as associative (and As_∞-) algebras.

Proof. The arity 2 operation in H^n is given by

$$m_2(z_a, z_b) = p(i(z_a)i(z_b)) = p(x^{a+b}) = \begin{cases} z_{a+b} & \text{if } a+b < n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the underlying associative algebra is indeed

$$H^n \cong \frac{\overline{\mathbb{K}[z]}}{(z^n)}$$

(keeping in mind that $d = 0$ on H^n, so that associativity is indeed satisfied). For the higher operations, we notice that

$$h(i(z_a)i(z_b)) = \begin{cases} x^{a+b-n}y & \text{if } a+b \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

it follows that if we multiply by any element of A_n and then apply either h or p, we always get 0. Therefore, all higher operations are 0, concluding the proof. □

Lemma 9.2.12. *The ∞-quasi-isomorphism i_∞ of As_∞-algebras extending i is given by $i_1 = i$,*

$$i_2(z^a, z^b) = \begin{cases} x^{a+b-n}y & \textit{if } a+b \geq n, \\ 0 & \textit{otherwise}, \end{cases}$$

and $i_n = 0$ for all $n \geq 3$.

Proof. This is proven with computations analogous to the ones in the proof of Lemma 9.2.11. □

Notice that the ∞-morphism i_∞ is in fact an ∞_κ-morphism of associative algebras.

A coalgebra and an ∞_κ-morphism

A structure of conilpotent dg $\mathrm{As}^!$-coalgebra, that is a shifted coassociative coalgebra, on a graded vector space V is the same thing as a square zero differential d on $\mathrm{As}(V)$ such that

$$d(V) \subseteq V \oplus V^{\otimes 2}.$$

Let

$$V \coloneqq \bigoplus_{i\geq 1} \mathbb{K} v_i$$

with $|v_i| = i$. We define

$$d : \mathrm{As}(V) \longrightarrow \mathrm{As}(V)$$

of degree -1 by

$$d(v_i) = \sum_{j+k=i-1} (-1)^j v_j \otimes v_k \ .$$

Lemma 9.2.13. *The map d squares to 0.*

Proof. This is a straightforward routine computation. □

Thus, we have an $\mathrm{As}^!$-coalgebra V. Notice that, since $d(V) \subseteq V^{\otimes 2}$, the underlying chain complex V of the $\mathrm{As}^!$-coalgebra has the zero differential. We define

$$\Phi : \mathrm{As}(V) \longrightarrow \mathrm{As}(V)$$

by

$$\Phi(v_n) = \sum_{k\geq 1} \sum_{i_1+\cdots+i_k=n} v_{i_1} \otimes \cdots \otimes v_{i_k} \ .$$

Lemma 9.2.14. *The map Φ commutes with the differential, and therefore defines an ∞_κ-morphism $\Phi : V \rightsquigarrow V$.*

Proof. We have to show that $\Phi : V \to \Omega_\kappa V$ satisfies the Maurer–Cartan equation

$$\partial(\Phi) + \star_\kappa(\Phi) = 0 \ .$$

We have

$$\begin{aligned}
\partial(\Phi)(v_n) &= d_{\Omega_\kappa V} \left(\sum_{k\geq 1} \sum_{i_1+\cdots+i_k=n} v_{i_1} \otimes \cdots \otimes v_{i_k} \right) \\
&= \sum_{k\geq 1} \sum_{i_1+\cdots+i_k=n} \sum_{j=0}^{k-1} (-1)^{i_1+\cdots+i_j} v_{i_1} \otimes \cdots \otimes v_{i_j} \otimes \left(\sum_{\alpha+\beta=i_{j+1}-1} (-1)^\alpha v_\alpha \otimes v_\beta \right) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_k} \\
&= \sum_{a,b\geq 1} \sum_{\substack{x_1+\cdots+x_a+ \\ +y_1+\cdots+y_b=n-1}} (-1)^{x_1+\cdots+x_a} v_{x_1} \otimes \cdots \otimes v_{x_a} \otimes v_{y_1} \otimes \cdots \otimes v_{y_b}
\end{aligned}$$

where in the first line we used the fact that $d_V = 0$, and in the last line we substituted $a = j+1, b = k-j+1$, $x_s = i_s$ for $s \leq a$, $x_a = \alpha$, $y_1 = \beta$, and $y_s = i_{j+s}$ for $s \geq 2$. At the same time, we have

$$\star_\kappa(\Phi)(v_n) = (\gamma_{\mathrm{As}} \circ 1)(\kappa \circ \Phi)\Delta_V(v_n)$$

$$
\begin{aligned}
&= -\sum_{i+j=n-1} (-1)^i \Phi(v_i) \otimes \Phi(v_j) \\
&= -\sum_{a,b\geq 1} \sum_{\substack{x_1+\cdots+x_a+\\ +y_1+\cdots+y_b=n-1}} (-1)^{x_1+\cdots+x_a} v_{x_1} \otimes \cdots \otimes v_{x_a} \otimes v_{y_1} \otimes \cdots \otimes v_{y_b} \ .
\end{aligned}
$$

Notice the sign in the second line: it comes from the signs in the definition of the differential d_2 in the cobar construction. This concludes the proof. □

The counterexample

We now prove what claimed at the beginning of the present section by considering $\mathscr{C} = \mathrm{As}^!$, $\mathscr{P} = \mathrm{As}$, the canonical twisting morphism

$$\kappa : \mathrm{As}^! \longrightarrow \mathrm{As} \ ,$$

the associative algebras $A = A^2$, $A' = H^2$, the $\mathrm{As}^!$-coalgebras $D' = D = V$, and the ∞-morphisms $\Psi = i_\infty$ and Φ described above. We take the linear maps $f_1, f_2, f_3 : V \to H^2$ such that $f_i(v_1) = z$ for $i = 1, 2, 3$, $f_1(v_2) = z$, and $f_2(v_2) = f_3(v_2) = 0$. Notice that f_2 and f_3 have degree -1, while f_1 decomposes as the sum of a degree -1 map and a degree -2 map.

We start by computing how $\hom^\kappa_\ell(\Phi, 1)\hom^\kappa_r(1, i_\infty)(\mu_3^\vee \otimes F)$ acts on $v_4 \in V$. We have

$$
\begin{aligned}
\Phi(v_4) = &\,\mathrm{id} \otimes v_4 + \mu_2 \otimes (v_1 \otimes v_3 + v_2 \otimes v_2 + v_3 \otimes v_1) \\
&+ \mu_3 \otimes (v_1 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_1) + \mu_4 \otimes v_1 \otimes v_1 \otimes v_1 \otimes v_1
\end{aligned}
$$

Since we will project on the part with only three copies of V, we don't care about the last term and will omit it in the following step. Notice that the coproduct of V is explicitly given by

$$\Delta_V(v_n) = \mathrm{id} \otimes v_n + \sum_{i_1+i_2=n-1} (-1)^{i_1} \mathscr{S}_2^{-1}\mu_2^\vee \otimes v_{i_1} \otimes v_{i_2} - \sum_{j_1+j_2+j_3=n-2} (-1)^{j_2} \mathscr{S}_3^{-1}\mu_3^\vee \otimes v_{j_1} \otimes v_{j_2} \otimes v_{j_3} + \cdots \ ,$$

where the dots indicate terms with at least 4 copies of V. Applying this to the above, and then using proj_3, we get

$$
\begin{aligned}
&\mathrm{proj}_3 \mathrm{As}(\Delta_V)\Phi(v_4) = \\
&\quad = -\mu_2 \otimes \Big((\mathrm{id} \otimes v_1) \otimes (\mathscr{S}_2^{-1}\mu_2^\vee \otimes v_1 \otimes v_1) + (\mathscr{S}_2^{-1}\mu_2^\vee \otimes v_1 \otimes v_1) \otimes (\mathrm{id} \otimes v_1) \Big) \\
&\qquad + \mu_3 \otimes \Big((\mathrm{id} \otimes v_1) \otimes (\mathrm{id} \otimes v_1) \otimes (\mathrm{id} \otimes v_2) + (\mathrm{id} \otimes v_1) \otimes (\mathrm{id} \otimes v_2) \otimes (\mathrm{id} \otimes v_1) \\
&\qquad\qquad + (\mathrm{id} \otimes v_2) \otimes (\mathrm{id} \otimes v_1) \otimes (\mathrm{id} \otimes v_1) \Big)
\end{aligned}
$$

Applying F gives

$$
\begin{aligned}
&F\mathrm{proj}_3 \mathrm{As}(\Delta_V)\Phi(v_4) = \\
&\qquad = \mu_2 \otimes \Big((\mathrm{id} \otimes z) \otimes (\mathscr{S}_2^{-1}\mu_2^\vee \otimes z \otimes z) - (\mathscr{S}_2^{-1}\mu_2^\vee \otimes z \otimes z) \otimes (\mathrm{id} \otimes z) \Big) \\
&\qquad\quad - \mu_3 \otimes \big((\mathrm{id} \otimes z) \otimes (\mathrm{id} \otimes z) \otimes (\mathrm{id} \otimes z) \big) \ ,
\end{aligned}
$$

and thus

$$\mathrm{As}(i_\infty) F \mathrm{proj}_3 \mathrm{As}(\Delta_V)\Phi(v_4) = \mu_2 \otimes (x \otimes y - y \otimes x) - \mu_3 \otimes x \otimes x \otimes x \ .$$

Finally, we have

$$\hom_\ell^\kappa(\Phi,1)\hom_r^\kappa(1,i_\infty)(\mu_3^\vee\otimes F)(v_4)=\gamma_{A_2}\mathrm{As}(i_\infty)F\mathrm{proj}_3\mathrm{As}(\Delta_V)=-x^3.$$

Now we look at the action of $\hom_r^\kappa(1,i_\infty)\hom_\ell^\kappa(\Phi,1)(\mu_3^\vee\otimes F)$ on v_4. We have

$$\Delta_V(v_4)=\mathrm{id}\otimes v_4+\mathscr{S}_2^{-1}\mu_2^\vee\otimes(-v_1\otimes v_2+v_2\otimes v_1)\,,$$

and thus

$$\begin{aligned}\mathrm{proj}_3\mathrm{As}^!(\Phi)\Delta_V(v_4)&=\\ &=\mathrm{id}\otimes\mu_3\otimes(v_1\otimes v_1\otimes v_2+v_1\otimes v_2\otimes v_1+v_2\otimes v_1\otimes v_1)\\ &\quad+\mathscr{S}_2^{-1}\mu_2^\vee\otimes\big(-(\mathrm{id}\otimes v_1)\otimes(\mu_2\otimes v_1\otimes v_1)+(\mu_2\otimes v_1\otimes v_1)\otimes(\mathrm{id}\otimes v_1)\big)\,.\end{aligned}$$

Applying F we obtain

$$\begin{aligned}F\mathrm{proj}_3\mathrm{As}^!(\Phi)\Delta_V(v_4)&=\\ &=-\mathrm{id}\otimes\mu_3\otimes z\otimes z\otimes z+\mathscr{S}_2^{-1}\mu_2^\vee\otimes\big(-(\mathrm{id}\otimes z)\otimes(\mu_2\otimes z\otimes z)+(\mu_2\otimes z\otimes z)\otimes(\mathrm{id}\otimes z)\big)\,,\end{aligned}$$

and thus

$$\mathrm{As}^!(\gamma_{H^2})F\mathrm{proj}_3\mathrm{As}^!(\Phi)\Delta_V(v_4)=0$$

since $z^2=0$ in H^2 and by Lemma 9.2.11. Therefore,

$$\hom_r^\kappa(1,i_\infty)\hom_\ell^\kappa(\Phi,1)(\mu_3^\vee\otimes F)(v_4)=0\,,$$

showing that

$$\hom_r^\kappa(1,i_\infty)\hom_\ell^\kappa(\Phi,1)\neq\hom_\ell^\kappa(\Phi,1)\hom_r^\kappa(1,i_\infty)$$

as claimed. This implies the result we wanted.

Theorem 9.2.15. *There is no bifunctor*

$$\hom^\alpha:\infty_\alpha\text{-}\mathscr{C}\text{-}\mathsf{cog}^{\mathrm{op}}\times\infty_\alpha\text{-}\mathscr{P}\text{-}\mathsf{alg}\longrightarrow\infty\text{-}\mathfrak{s}\mathscr{L}_\infty.$$

that restricts to the functors

$$\hom_\ell^\alpha:\infty_\alpha\text{-}\mathscr{C}\text{-}\mathsf{cog}^{\mathrm{op}}\times\mathscr{P}\text{-}\mathsf{alg}\longrightarrow\infty\text{-}\mathfrak{s}\mathscr{L}_\infty$$

and

$$\hom_r^\alpha:\mathscr{C}\text{-}\mathsf{cog}^{\mathrm{op}}\times\infty_\alpha\text{-}\mathscr{P}\text{-}\mathsf{alg}\longrightarrow\infty\text{-}\mathfrak{s}\mathscr{L}_\infty$$

defined above in the respective subcategories.

Remark 9.2.16. *The result is true in any characteristic in the non-symmetric case by the same counterexample as above, and in the symmetric case as well, by considering the same counterexample and tensoring the operads by the regular representation of the symmetric groups.*

9.2.5 Compatibility with filtrations and quasi-isomorphisms

We will show that convolution homotopy algebras are compatible with filtrations and quasi-isomorphisms. This is a very useful fact, because it will make it easy to apply the Dolgushev–Rogers theorem to obtain informations about the homotopy type of the homotopy Lie algebras obtained by convolution.

Fix a cooperad $\mathscr{C}$, an operad $\mathscr{P}$, and a twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$. We begin with proving that the convolution homotopy algebra functor $\hom^\alpha$ is compatible with (co)filtrations.

Proposition 9.2.17. *Let C be a $\mathscr{C}$-coalgebra, and let A be a $\mathscr{P}$-algebra.*

1. *Suppose that C is cofiltered with cofiltration $\mathscr{F}^\bullet C$, in the sense of Definition 3.3.10. Then the sequence of subspaces*
$$\mathscr{F}_n \coloneqq \{f \in \hom^\alpha(C, A) \mid \mathscr{F}^n C \subseteq \ker f\}$$
makes the convolution homotopy algebra $\hom^\alpha(C, A)$ into a complete $\mathfrak{s}\mathscr{L}_\infty$-algebra, which is proper if $\mathscr{F}^1 C = 0$.

2. *Dually, suppose that $(A, \mathscr{F}_\bullet A)$ is a complete $\mathscr{P}$-algebra. Then the sequence*
$$\mathscr{F}_n \coloneqq \hom^\alpha(C, \mathscr{F}_n A)$$
makes the convolution homotopy algebra $\hom^\alpha(C, A)$ into a complete $\mathfrak{s}\mathscr{L}_\infty$-algebra, which is proper if A is.

Proof. The first statement was proven in Lemma 9.1.8. The second one is similar. □

Next, we prove that convolution homotopy algebras behave well with respect to ∞_α-quasi-isomorphisms of (co)algebras.

Proposition 9.2.18. *Let C, C' be two conilpotent $\mathscr{C}$-coalgebras, and let A, A' be two $\mathscr{P}$-algebras.*

1. *Suppose $\Phi : C' \rightsquigarrow C$ is an ∞_α-quasi-isomorphism of $\mathscr{C}$-coalgebras. Then*
$$\hom^\alpha(\Phi, 1) : \hom^\alpha(C, A) \rightsquigarrow \hom^\alpha(C', A)$$
is an ∞-quasi-isomorphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras. If C, C' are cofiltered $\mathscr{C}$-coalgebras, and Φ is a cofiltered ∞_α-quasi-isomorphism, in the sense that the restriction of its linear component to every level of the filtration is a quasi-isomorphism, then $\hom^\alpha(\Phi, 1)$ is a filtered ∞-quasi-isomorphism with respect to the induced filtration on the convolution $\mathfrak{s}\mathscr{L}_\infty$-algebras.

2. *Suppose $\Psi : A \rightsquigarrow A'$ is an ∞_α-quasi-isomorphism of $\mathscr{P}$-algebras. Then*
$$\hom^\alpha(1, \Psi) : \hom^\alpha(C, A) \rightsquigarrow \hom^\alpha(C, A')$$
is an ∞-quasi-isomorphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras. If Ψ is a filtered ∞_α-quasi-isomorphism, then $\hom^\alpha(1, \Psi)$ is a filtered ∞-quasi-isomorphism with respect to the induced filtration on the convolution $\mathfrak{s}\mathscr{L}_\infty$-algebras.

Proof. We only prove the first statement, and leave the second one to the reader. We have that the first component
$$\hom^\alpha(\Phi, 1)_1 = \phi_1^*$$
is given by the pullback by ϕ_1. Since ϕ_1 is a quasi-isomorphism, and we are working over a field, this is also a quasi-isomorphism, proving that $\hom^\alpha(\Phi, 1)$ is an ∞-quasi-isomorphism. The same argument applies in the case Φ is a cofiltered ∞_α-quasi-isomorphism to obtain a filtered ∞-morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras. □

Corollary 9.2.19. *Let C, C' be two conilpotent $\mathscr{C}$-coalgebras, and let A, A' be two $\mathscr{P}$-algebras.*

1. *Let $\Phi : C' \rightsquigarrow C$ be an ∞_α-morphism of $\mathscr{C}$-coalgebras, and suppose either*

   - *C', C are cofiltered $\mathscr{C}$-coalgebras, we endow the convolution homotopy algebras with the induced filtrations, and Φ is a cofiltered ∞-quasi-isomorphism, or*
   - *A is a filtered $\mathscr{P}$-algebra, and we endow the convolution homotopy algebras with the filtrations induced by the filtration on A.*

   Then
   $$\mathrm{MC}_\bullet(\hom^\alpha(\Phi, 1)) : \mathrm{MC}_\bullet(\hom^\alpha(C, A)) \longrightarrow \mathrm{MC}_\bullet(\hom^\alpha(C', A))$$
   is a weak equivalence of simplicial sets.

2. *Dually, let $\Psi : A \rightsquigarrow A'$ be an ∞_α-morphism of $\mathscr{P}$-algebras, and suppose either*

   - *A, A' are filtered $\mathscr{P}$-algebras, we endow the convolution homotopy algebras with the induced filtrations, and Ψ is a filtered ∞-quasi-isomorphism, or*
   - *C is a cofiltered $\mathscr{C}$-coalgebra, and we endow the convolution homotopy algebras with the filtrations induced by the cofiltration on C.*

   Then
   $$\mathrm{MC}_\bullet(\hom^\alpha(1, \Psi)) : \mathrm{MC}_\bullet(\hom^\alpha(C, A)) \longrightarrow \mathrm{MC}_\bullet(\hom^\alpha(C, A'))$$
   is a weak equivalence of simplicial sets.

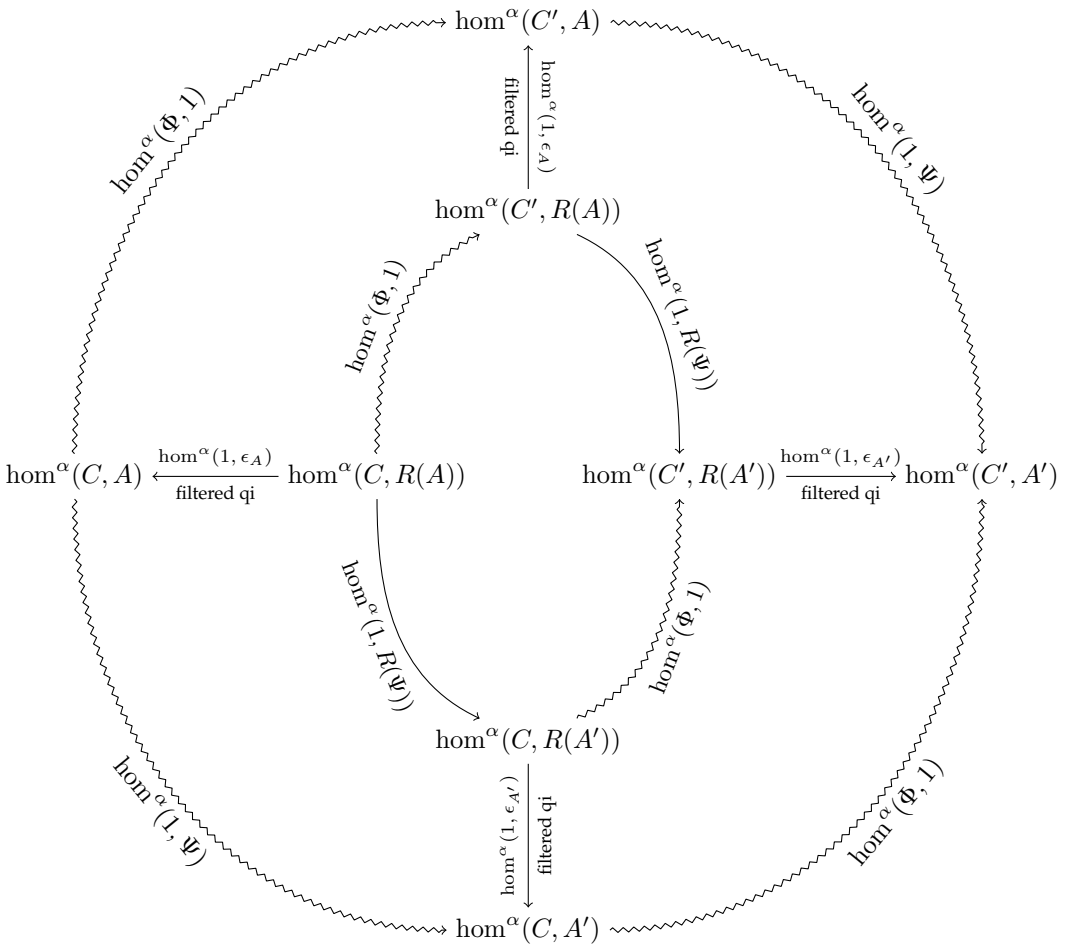
Proof. This follows immediately from Proposition 9.2.18 and the Dolgushev–Rogers theorem. □

9.2.6 Extension of the bifunctor up to homotopy

To conclude the section, we prove that there exists an extension of the convolution homotopy algebra functor accepting ∞-morphisms in both slots, provided we accept to work only up to homotopy.

Theorem 9.2.20 ([RNW])**.** *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a Koszul twisting morphism, let $\Phi : C' \rightsquigarrow C$ be an ∞_α-morphism of $\mathscr{C}$-coalgebras, and let $\Psi : A \rightsquigarrow A'$ be an ∞_α-morphism of $\mathscr{P}$-algebras. The two compositions*
$$\hom^\alpha(\Phi, 1)\hom^\alpha(1, \Psi) \sim \hom^\alpha(1, \Psi)\hom^\alpha(\Phi, 1)$$
are homotopic.

Proof. Denote by $R(A) \coloneqq \Omega_\alpha \mathrm{B}_\alpha A$ the bar-cobar resolution of A, and similarly for A'. Since α is Koszul, the counit of the adjunction
$$\epsilon_A : R(A) \longrightarrow A$$
is a quasi-isomorphism by Theorem 2.4.8. The rectification of the ∞_α-morphism Ψ is given by the strict morphism
$$R(\Psi) : \Omega_\alpha \mathrm{B}_\alpha A \xrightarrow{\Omega_\alpha \Psi} \Omega_\alpha \mathrm{B}_\alpha A' \,.$$
The proof is outlined by the following commutative diagram.



The innermost square is commutative since $R(\Psi)$ is a strict morphism of $\mathscr{P}$-algebras, and the maps passing from the outer rim to the inner one are filtered quasi-isomorphisms. Notice that all squares are commutative, except for the outer one, which fails to be commutative at $\hom^\alpha(C, A)$.

Now consider the morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras

$$\hom^\iota(\mathrm{B}_\iota \hom^\alpha(C, A), \hom^\alpha(C', A')) \xrightarrow{\hom^\iota(\mathrm{B}_\iota \hom^\alpha(1, \epsilon_A), 1)} \hom^\iota(\mathrm{B}_\iota \hom^\alpha(C, R(A)), \hom^\alpha(C', A'))\,.$$

It is a filtered quasi-isomorphism, and it is given on Maurer–Cartan elements by precomposition with $\hom^\alpha(1, \epsilon_A)$. The two compositions

$$\hom^\alpha(\Phi, 1) \hom^\alpha(1, \Psi) \quad \text{and} \quad \hom^\alpha(1, \Psi) \hom^\alpha(\Phi, 1)$$

are naturally elements of $\hom^\iota(\mathrm{B}_\iota \hom^\alpha(C, A), \hom^\alpha(C', A'))$ and are mapped to the same elements, and thus, by the Dolgushev–Rogers theorem, they are homotopic. □

Remark 9.2.21. *The proof above supposes that we are filtering our convolution homotopy algebras with the filtration induced by a filtration on the $\mathscr{C}$-coalgebras — usually the coradical filtration. If one filters them by a filtration induced by filtrations on the $\mathscr{P}$-algebras, then the exact same proof goes through with the sole difference that one has to rectify the ∞_α-morphism Φ instead of Ψ.*

Remark 9.2.22. *The assumption that the twisting morphism is Koszul is necessary for the proof of Theorem 9.2.20. If one removes this assumption, then it is possible to find a counterexample to the conclusion of the result by taking e.g. the zero twisting morphism* $\mathrm{As}^\vee \to \mathrm{As}$ *and some explicit (co)algebras. For one such example, the reader is invited to consult [RNW, Appendix A].*

9.3 Compatibility with the homotopy transfer theorem

Another very powerful tool of homotopical algebra is given by the homotopy transfer theorem, of which we have seen a case in Section 2.3.4. In this section, we will start by giving a more general case of this theorem, where one does homotopy transfer for algebras over an operad of the form $\Omega\mathscr{C}$, and not only for minimal models of a Koszul operad. Then we will explain and prove how the convolution $\mathscr{L}_\infty$-algebra functor is compatible with the homotopy transfer theorem.

9.3.1 A generalized homotopy transfer theorem for algebras

The homotopy transfer theorem holds in a slightly more general situation than the one presented in Section 2.3.4 with the same exact formulæ, as was proven in [Ber14a, Thm. 1.5].

Let $\mathscr{P}$ be an operad of the form $\mathscr{P} = \Omega\mathscr{C}$ for some reduced cooperad $\mathscr{C}$. In particular, $\mathscr{P}$ is a cofibrant operad. Let A be a $\mathscr{P}$-algebra, and suppose that we have a contraction of chain complexes

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

The structure of $\mathscr{P}$-algebra on A is equivalent to a twisting morphism $\varphi_A \in \mathrm{Tw}(\mathscr{C}, \mathrm{End}_A)$. Define

$$\varphi_B = \left(\mathscr{C} \xrightarrow{\Delta^{\mathrm{monadic}}_{\mathscr{C}}} \mathcal{T}^c\left(\overline{\mathscr{C}}\right) \xrightarrow{\mathcal{T}^c(s\varphi_A)} \mathrm{B}(\mathrm{End}_A) \xrightarrow{\mathrm{VdL}_B} \mathrm{End}_B \right),$$

and

$$i_\infty \coloneqq \left(\mathscr{C} \circ B \xrightarrow{\Delta^{\mathrm{monadic}}_{\mathscr{C}} \circ 1_B} \mathcal{T}^c\left(\overline{\mathscr{C}}\right) \circ B \xrightarrow{\mathcal{T}^c(s\varphi_A)\circ 1_B} \mathrm{B}(\mathrm{End}_A) \circ B \xrightarrow{\mathrm{VdL}^i_B \circ 1_B} \mathrm{End}^B_A \circ B \longrightarrow A \right).$$

The same proof as for the case presented in Section 2.3.4 gives the following result.

Theorem 9.3.1 (Homotopy transfer theorem)**.** *The map* $\varphi_B : \mathscr{C} \to \mathrm{End}_B$ *is a twisting morphism in* $\mathrm{Tw}(\mathscr{C}, \mathrm{End}_B)$*, and therefore defines a* $\mathscr{P}$*-algebra structure on* B*. The map* i_∞ *defines an* ∞_ι*-quasi-isomorphism* $B \rightsquigarrow A$ *of* $\mathscr{P}$*-algebras, and the map* p *can also be extended to an* ∞_ι*-quasi-isomorphism* p_∞ *such that* $p_\infty i_\infty = 1_B$*.*

Let $\mathscr{C}, \mathscr{C}'$ be two cooperads, and let $f : \mathscr{C}' \to \mathscr{C}$ be a morphism of cooperads. Let A be a $\Omega\mathscr{C}$-algebra, and suppose we have a contraction

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

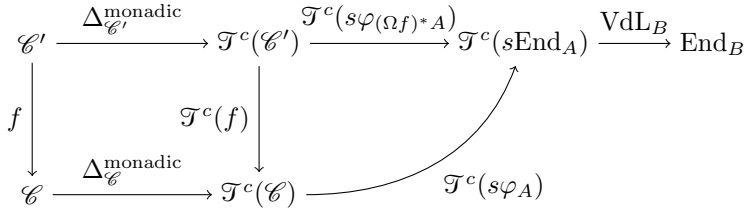
We have two ways of putting a $\Omega\mathscr{C}'$-algebra structure on B:

1. The homotopy transfer theorem applied to the $\Omega\mathscr{C}$-algebra A gives a $\Omega\mathscr{C}$-algebra structure on B. We obtain a $\Omega\mathscr{C}'$-algebra structure on B by pullback by the morphism of operads $\Omega f : \Omega\mathscr{C}' \to \Omega\mathscr{C}$.

2. We obtain a $\Omega\mathscr{C}'$-algebra structure on A by pullback by the morphism of operads $\Omega f : \Omega\mathscr{C}' \to \Omega\mathscr{C}$. Then the homotopy transfer theorem applied to this algebra gives us a $\Omega\mathscr{C}'$-algebra structure on B.

Proposition 9.3.2. *The two $\Omega\mathscr{C}'$-algebra structures thus obtained on B are the same.*

Proof. The proof is given by the following diagram. The two algebra structures on B are the two extremal paths.



The diagram is obviously commutative, concluding the proof. □

9.3.2 Compatibility between convolution algebras and homotopy transfer

We can now prove that taking convolution algebras is compatible with the homotopy transfer theorem.

Let $\mathscr{C}$ be a cooperad, and denote by $\iota : \mathscr{C} \to \Omega\mathscr{C}$ the canonical twisting morphism. Let A be a $\Omega\mathscr{C}$-algebra, let B be a chain complex and suppose we have a contraction

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

Let C be a $\mathscr{C}$-coalgebra. Then, we have two ways to endow the chain complex $\mathrm{hom}(C, B)$ with a $\mathfrak{s}\mathscr{L}_\infty$-algebra structure.

1. We consider the $\mathfrak{s}\mathscr{L}_\infty$-algebra $\mathrm{hom}^\iota(C, A)$. The contraction above induces a contraction

$$h_* \circlearrowright \mathrm{hom}^\iota(C, A) \underset{i_*}{\overset{p_*}{\rightleftarrows}} \mathrm{hom}(C, B)$$

   and thus the homotopy transfer theorem gives a $\mathfrak{s}\mathscr{L}_\infty$-algebra structure on $\mathrm{hom}(C, B)$. We will denote this algebra by $\mathrm{hom}^{HTT}(C, B)$.

2. The generalized homotopy transfer theorem gives a $\Omega\mathscr{C}$-algebra structure on B. Then we take the convolution $\mathfrak{s}\mathscr{L}_\infty$-algebra $\mathrm{hom}^\iota(C, B)$.

Theorem 9.3.3. *The two $\mathfrak{s}\mathscr{L}_\infty$-algebra structures on $\mathrm{hom}(C, B)$ are the same. Moreover, we have that*

$$(i_*)_\infty = \mathrm{hom}^\iota_r(1, i_\infty), \qquad \textit{and} \qquad (p_*)_\infty = \mathrm{hom}^\iota_r(1, p_\infty)\ .$$

The proof of this result is postponed to later in this section.

Corollary 9.3.4. *Let $f : \mathscr{D} \to \mathscr{C}$ be a morphism of cooperads, let*

$$\alpha := f^*\iota = \left(\mathscr{D} \xrightarrow{f} \mathscr{C} \xrightarrow{\iota} \Omega\mathscr{C}\right),$$

and now suppose that C is a conilpotent $\mathscr{D}$-coalgebra. Then the same constructions as for Theorem 9.3.3 can be done to obtain two $\mathfrak{s}\mathscr{L}_\infty$-algebra structures on $\hom(C, B)$. Once again, the two structures are the same.

Proof. By Lemma 9.1.7(1) and Theorem 9.3.3, we have

$$\hom^\alpha(C, B) = \hom^\iota(f_*C, B) = \hom^{HTT}(f_*C, B) .$$

Then for $s^{-1}\mu_n^\vee \otimes F \in s^{-1}\mathrm{Com}^\vee \otimes_{\mathbb{S}_n} \hom(C, B)^{\otimes n}$ we notice that

$$\begin{aligned}\gamma_{\hom^{HTT}(f_*D,B)}(s^{-1}\mu_n^\vee \otimes F) &= \mathrm{VdL}_{\hom(D,B)}\mathcal{T}(\varphi_{\hom^\iota(f_*D,A)})\Delta^{\mathrm{monadic}}_{\mathrm{Com}^\vee}(\mu_n^\vee)\\ &= \mathrm{VdL}_{\hom(D,B)}\mathcal{T}(\varphi_{\hom^\alpha(D,A)})\Delta^{\mathrm{monadic}}_{\mathrm{Com}^\vee}(\mu_n^\vee)\\ &= \gamma_{\hom^{HTT}(D,B)}(s^{-1}\mu_n^\vee \otimes F) ,\end{aligned}$$

where in the second line we used Lemma 9.1.7 once again. Therefore, we have

$$\hom^\alpha(C, B) = \hom^{HTT}(C, B) ,$$

concluding the proof. □

We will now prove Theorem 9.3.3. Let's introduce some notation. Let $\mathscr{C}$ be a cooperad, and let C be a conilpotent $\mathscr{C}$-coalgebra. For every reduced rooted tree $\tau \in \mathrm{rRT}$ we define

$$\Delta^\tau_C : C \longrightarrow \mathcal{T}^c(\mathscr{C}) \circ C$$

recursively as follows. If $\tau = c_n$ is the n-corolla, with $n \geq 2$, then

$$\Delta^{c_n}_C := \Delta^n_C .$$

Else, we have $\tau = c_k \circ (\tau_1, \ldots, \tau_k)$, where $k \geq 2$, the τ_i are allowed to be the empty tree, and we define

$$\Delta^\tau_C := \left(1_{\mathscr{C}} \circ (\Delta^{\tau_1}_C, \ldots, \Delta^{\tau_1}_C)\right)\Delta^k_C ,$$

where $\Delta^\emptyset_C := 1_C$. Notice that $\mathrm{id} \in \mathscr{C}(1)$ will never appear in the image of such an operator.

Lemma 9.3.5. *For any $n \geq 2$, we have*

$$\left(\Delta^{\mathrm{mon}}_{\mathscr{C}} \circ 1_C\right)\Delta^n_C = \sum_{\tau \in RT_n} \Delta^\tau_C ,$$

where the monadic decomposition map $\Delta^{\mathrm{mon}}_{\mathscr{C}}$ was defined at page 21.

Proof. The proof is done by induction on n. For $n = 2$, the statement is trivial, since the cooperad is supposed to be reduced, so that we have that $\Delta^{\mathrm{monadic}}_{\mathscr{C}}$ is the identity on $\mathscr{C}(2)$. For $n > 2$, we have

$$\sum_{\tau \in RT_n} \Delta^\tau_C = \sum_{\substack{k \geq 2,\ n_1 + \cdots + n_k = n \\ \tau_i \in RT_{n_i}\ \forall 1 \leq i \leq k}} \Delta^{c_k \circ (\tau_1, \ldots, \tau_k)}_C$$

$$
\begin{aligned}
&= \sum_{\substack{k\geq 2,\ n_1+\cdots+n_k=n \\ \tau_i\in RT_{n_i}\ \forall 1\leq i\leq k}} \big(1_{\mathscr{C}}\circ(\Delta_C^{\tau_1},\dots,\Delta_C^{\tau_1})\big)\Delta_C^k \\
&= \sum_{\substack{k\geq 2 \\ n_1+\cdots+n_k=n}} \left(1_{\mathscr{C}}\circ\left(\sum_{\tau_1\in RT_{n_1}}\Delta_C^{\tau_1},\dots,\sum_{\tau_k\in RT_{n_k}}\Delta_C^{\tau_k}\right)\right)\Delta_C^k \\
&= \sum_{\substack{k\geq 2 \\ n_1+\cdots+n_k=n}} \big(1_{\mathscr{C}}\circ\big((\Delta_{\mathscr{C}}^{\mathrm{monadic}}\circ 1_C)\Delta_C^{n_1},\dots,(\Delta_{\mathscr{C}}^{\mathrm{monadic}}\circ 1_C)\Delta_C^{n_k}\big)\big)\,\Delta_C^k \\
&= \big(1_{\mathscr{C}}\circ\Delta_{\mathscr{C}}^{\mathrm{monadic}}\circ 1_C\big)\sum_{\substack{k\geq 2 \\ n_1+\cdots+n_k=n}} \big(1_{\mathscr{C}}\circ(\Delta_C^{n_1},\dots,\Delta_C^{n_k})\big)\Delta_C^k \\
&= \big(1_{\mathscr{C}}\circ\Delta_{\mathscr{C}}^{\mathrm{monadic}}\circ 1_C\big)\,(\overline{\Delta}_{\mathscr{C}}\circ 1_C)\Delta_C^n \\
&= \big(\Delta_{\mathscr{C}}^{\mathrm{monadic}}\circ 1_C\big)\Delta_C^n\,,
\end{aligned}
$$

where we consider the empty tree as a rooted tree in order for the first equality to hold, and where in the third line we used the induction hypothesis. □

Recall that the Van der Laan map

$$\mathrm{VdL}_B : \mathcal{T}(s\mathrm{End}_A) \longrightarrow \mathrm{End}_B$$

is given by

$$\mathrm{VdL}_B(\tau(f)) = p\tau^h(f)\imath^{\otimes n},$$

cf. Section 2.3.4, and similarly for the contraction from $\hom(C,A)$ to $\hom(C,B)$.

Every chain complex V is an End_V-algebra in a canonical way, we will denote by

$$\widetilde{\gamma}_V : \mathrm{End}_V\circ V \longrightarrow V$$

its composition map.

Proof of Theorem 9.3.3. We will prove our claims by explicitly comparing the two $\mathfrak{s}\mathscr{L}_\infty$-algebra structures on $\hom(C,B)$. We will denote by

$$\varphi_A\in\mathrm{Tw}(\mathscr{C},\mathrm{End}_A)\,,\qquad \rho_A:\Omega\mathscr{C}\longrightarrow\mathrm{End}_A\,,\qquad\text{and}\qquad \gamma_A:\Omega\mathscr{C}\circ A\longrightarrow A$$

the $\Omega\mathscr{C}$-algebra structure of A, seen in three equivalent ways, and similarly for the other algebras. Notice that we have

$$\varphi_A=\rho_A\iota\,,\qquad\text{and}\qquad \gamma_A=\widetilde{\gamma}_A(\rho_A\circ 1_A)\,,$$

as well as

$$\gamma_A=\widetilde{\gamma}_A(\mathcal{T}(s\varphi_A)\circ 1_A)\,,$$

where we used $\Omega\mathscr{C}=\mathcal{T}(s^{-1}\mathscr{C})$.

Fix $n\geq 2$ and $f_1,\dots,f_n\in\hom(C,B)$. As usual, denote $F\coloneqq f_1\otimes\cdots\otimes f_n$.

We begin by making explicit the structure obtained by taking the convolution algebra between C and B with the $\Omega\mathscr{C}$-algebra structure given by the homotopy transfer theorem. The algebraic structure of B is given by the twisting morphism

$$\varphi_B=\mathrm{VdL}_B\mathcal{T}(s\varphi_A)\Delta_{\mathscr{C}}^{\mathrm{mon}},$$

and thus

$$\gamma_B = \widetilde{\gamma}_B\Big(\mathcal{T}(s\mathrm{VdL}_B\mathcal{T}(s\varphi_A)\Delta^{\mathrm{mon}}_{\mathscr{C}}) \circ 1_B\Big) .$$

For the structure on $\hom(C,B)$, we have

$$\begin{aligned}\varphi_{\hom^\iota(C,B)}(\mu_n^\vee)(F) &= \gamma_B(\iota\otimes F)\Delta^n_C \\ &= \widetilde{\gamma}_B\Big(\mathcal{T}(s\mathrm{VdL}_B\mathcal{T}(s\varphi_A)\Delta^{\mathrm{mon}}_{\mathscr{C}}) \circ 1_B\Big)(\iota\otimes F)\Delta^n_C \\ &= \widetilde{\gamma}_B\Big(s\mathrm{VdL}_B\mathcal{T}(s\varphi_A)\Delta^{\mathrm{mon}}_{\mathscr{C}} \circ 1_B\Big)(\iota\otimes F)\Delta^n_C .\end{aligned}$$

In the third line, we used the fact that the image of $\iota : \mathscr{C} \to \Omega\mathscr{C}$ is $s^{-1}\overline{\mathscr{C}}$. Using that ι is essentially just given by desuspension, we have

$$\Delta^{\mathrm{mon}}_{\mathscr{C}} s\iota = \Delta^{\mathrm{mon}}_{\mathscr{C}} ,$$

and thus

$$\begin{aligned}\varphi_{\hom^\iota(C,B)}(\mu_n^\vee)(F) &= \widetilde{\gamma}_B\Big(\mathrm{VdL}_B\mathcal{T}(s\varphi_A) \circ 1_B\Big)(1_{\mathscr{C}}\otimes F)(\Delta^{\mathrm{mon}}_{\mathscr{C}} \circ 1_C)\Delta^n_C \\ &= \sum_{\tau\in \mathrm{RT}_n} \widetilde{\gamma}_B\Big(\mathrm{VdL}_B\mathcal{T}(s\varphi_A) \circ 1_B\Big)(1_{\mathscr{C}}\otimes F)\Delta^\tau_C \\ &= \sum_{\tau\in \mathrm{RT}_n} \widetilde{\gamma}_B\Big(p\tau^h(\varphi_A)i^{\otimes n} \circ 1_B\Big)(1_{\mathscr{C}}\otimes F)\Delta^\tau_C \\ &= \sum_{\tau\in \mathrm{RT}_n} p\widetilde{\gamma}_A(\tau^h(\varphi_A)\otimes iF)\Delta^\tau_C \qquad \text{(A)}\end{aligned}$$

where in the second line we used Lemma 9.3.5.

The next step is to make as explicit as possible the other $\mathfrak{sL}_\infty$-algebra structure on $\hom(C,B)$, obtained by homotopy transfer theorem between the two hom spaces. We have

$$\varphi_{\hom^{HTT}(C,B)} = \mathrm{VdL}_{\hom(C,B)}\mathcal{T}(s\varphi_{\hom^\iota(C,A)})\Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee} .$$

Therefore, we compute

$$\begin{aligned}\varphi_{\hom^{HTT}(C,B)}(\mu_n^\vee)(F) &= \left(\mathrm{VdL}_{\hom(C,B)}\mathcal{T}(s\varphi_{\hom^\iota(C,A)})\Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee)\right)(F) \\ &= \left(\mathrm{VdL}_{\hom(C,B)}\mathcal{T}(s\varphi_{\hom^\iota(C,A)})\sum_{\tau\in \mathrm{RT}_n}\tau(v\mapsto \mu^\vee_{|v|})\right)(F) \\ &= \sum_{\tau\in \mathrm{RT}_n}\left(\mathrm{VdL}_{\hom(C,B)}\tau\left(v\mapsto s\gamma_A(\iota\otimes -)\Delta^{|v|}_{\mathscr{C}}\right)\right)(F) \\ &= \sum_{\tau\in \mathrm{RT}_n}\left(p_*\tau^{h_*}\left(v\mapsto \gamma_A(\iota\otimes -)\Delta^{|v|}_{\mathscr{C}}\right)i_*^{\otimes n}\right)(F) \\ &= \sum_{\tau\in \mathrm{RT}_n} p\left(\tau^{h_*}\left(v\mapsto \gamma_A(\iota\otimes -)\Delta^{|v|}_{\mathscr{C}}\right)\right)(iF) \\ &= \sum_{\tau\in \mathrm{RT}_n} p\left(\tau^{h_*}\left(v\mapsto \widetilde{\gamma}_A(\varphi_A\otimes -)\Delta^{|v|}_{\mathscr{C}}\right)\right)(iF) , \qquad \text{(B)}\end{aligned}$$

where in the last line we used

$$\gamma_A(\iota\circ 1_A) = \widetilde{\gamma}_A(\rho_A\circ 1_A)(\iota\circ 1_A) = \widetilde{\gamma}_A(\varphi_A\circ 1_A) .$$

The last step is proving that $(A) = (B)$. We will use induction to prove that, for any $n \geq 2$ and $\tau \in \mathrm{RT}_n$, we have

$$\widetilde{\gamma}_A(\tau^h(\varphi_A) \otimes iF)\Delta_C^\tau = \left(\tau^{h_*}\left(v \mapsto \widetilde{\gamma}_A(\varphi_A \otimes -)\Delta_{\mathscr{C}}^{|v|}\right)\right)(iF)\,, \tag{9.2}$$

which implies the claim. If $n = 2$, the only possible tree is the 2-corolla, and one immediately sees that both sides of (9.2) are equal to $p\widetilde{\gamma}_A(\varphi_A \otimes iF)\Delta_C^2$. Similarly, for all the corollas it is straightforward to see that the identity holds. If $\tau \in \mathrm{RT}_n$ is a composite tree, we can write it as

$$\tau = c_k \circ (\tau_1, \dots, \tau_k)\,.$$

Then we have

$$\begin{aligned}
&\left(\tau^{h_*}\left(v \mapsto \widetilde{\gamma}_A(\varphi_A \otimes -)\Delta_{\mathscr{C}}^{|v|}\right)\right)(iF) = \\
&\quad= \sum_{S_1 \sqcup \cdots \sqcup S_k = [n]} \widetilde{\gamma}_A\left(\varphi_A \circ \bigotimes_{j=1}^k h\left(\tau_j^{h_*}\left(v \mapsto \widetilde{\gamma}_A(\varphi_A \otimes -)\Delta_{\mathscr{C}}^{|v|}\right)\right)(iF^{S_j})\right)\Delta_C^k \\
&\quad= \sum_{S_1 \sqcup \cdots \sqcup S_k = [n]} \widetilde{\gamma}_A\left(\varphi_A \circ \bigotimes_{j=1}^k h\widetilde{\gamma}_A(\tau_j^h(\varphi_A) \otimes iF)\Delta_C^{\tau_j}\right)\Delta_C^k \\
&\quad= \widetilde{\gamma}_A(1_{\mathrm{End}_A} \circ \widetilde{\gamma}_A)\left(\sum_{S_1 \sqcup \cdots \sqcup S_k = [n]} \varphi_A \circ \bigotimes_{j=1}^k h(\tau_j^h(\varphi_A) \otimes iF)\right)(1_{\mathscr{C}} \circ (\Delta_C^{\tau_1}), \dots, \Delta_C^{\tau_k})\Delta_C^k \\
&\quad= \widetilde{\gamma}_A(\tau^h(\varphi_A) \otimes iF)\Delta_C^\tau\,,
\end{aligned}$$

as desired.

The fact that

$$(i_*)_\infty = \hom_r^\iota(1, i_\infty)$$

is proven in a completely analogous way, and then we have

$$\hom_r^\iota(1, p_\infty)\hom_r^\iota(1, i_\infty) = \hom_r^\iota(1, p_\infty i_\infty) = 1_{\hom(C,B)}\,,$$

which shows that we can take $(p_*)_\infty = \hom_r^\iota(1, p_\infty)$. □

9.3.3 Compatibility on the coalgebra side

We want to do the same thing as in Section 9.3.2, but this time on the coalgebra side. While we believe that one should be able to write down a sensible version of the homotopy transfer theorem for coalgebras — probably by working on coalgebras over *operads*, see [LV12, Sect. 5.2.15] — for simplicity we will only prove the dual version of the result we think to be true, working with tensor products of algebras.

Let $\mathscr{P}$ be an operad which is finite dimensional in every arity, and as always let $\pi : \mathrm{B}\mathscr{P} \to \mathscr{P}$ be the canonical twisting morphism. Denote by $\mathscr{C} := \mathscr{P}^\vee$ the dual cooperad of $\mathscr{P}$. Suppose that we are given a $\mathscr{P}$-algebra X, a $\Omega\mathscr{C}$-algebra A, and a contraction

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} B$$

from A to B. There are two ways to endow the chain complex $X \otimes B$ with an $\mathfrak{sL}_\infty$-algebra structure.

1. We consider the $\mathfrak{sL}_\infty$-algebra $A \otimes^\iota X$. The contraction above induces a contraction

$$1_X \otimes h \circlearrowright X \otimes^\pi A \underset{1_X \otimes i}{\overset{1_X \otimes p}{\rightleftarrows}} X \otimes B$$

   and thus the homotopy transfer theorem gives a $\mathfrak{sL}_\infty$-algebra structure on $B \otimes X$.

2. The generalized homotopy transfer theorem gives a $\Omega\mathscr{C}$-algebra structure on B. Then we take $B \otimes^\pi X$.

Theorem 9.3.6. *The two $\mathfrak{sL}_\infty$-algebra structures on $X \otimes B$ are equal. Moreover,*

$$(1 \otimes i)_\infty = 1 \otimes^\pi (i_\infty) \qquad \text{and} \qquad (1 \otimes p)_\infty = 1 \otimes^\pi (p_\infty) \, .$$

Remark 9.3.7. *This result of course looks very similar to Theorem 9.3.3, being its version for the "coalgebra side". One should think of the proof as a dual version of Theorem 9.3.3. The above theorem is a direct generalization of [RN17a, Thm. 5.1].*

Proof. We compare the two structures explicitly. In order to do so, let

$$\mathsf{M}_\pi : \mathfrak{sL}_\infty \longrightarrow \mathscr{P} \otimes \Omega\mathscr{C}$$

be the map defined in Theorem 9.1.12. Explicitly, it is given by

$$\mathsf{M}_\pi(s^{-1}\mu_n^\vee) = \sum_{i \in I(n)} (-1)^{c_i} p_i \otimes s^{-1} c_i \, ,$$

where $\{p_i\}_{i \in I(n)}$ is a basis of $\mathscr{P}(n)$, and $c_i \coloneqq p_i^\vee$ is the dual basis. Notice that $(-1)^{c_i} s^{-1} c_i = (sp_i)^\vee$. We encode the $\mathscr{P}$-algebra structure of X by the morphism of operads

$$\rho_X : \mathscr{P} \longrightarrow \mathrm{End}_X \, ,$$

and the $\Omega\mathscr{C}$-algebra structures of A, B (obtained by homotopy transfer theorem), and $X \otimes A$ by the twisting morphisms

$$\varphi_A : \mathscr{C} \longrightarrow \mathrm{End}_A \, ,$$

and similarly for the other algebras. Given two $\mathbb{S}$-modules M and N, the map

$$\Phi : \mathcal{T}(M \otimes N) \longrightarrow \mathcal{T}(M) \otimes \mathcal{T}(N)$$

is given by sending a tree with vertices indexed by pure tensors in $M \otimes N$ to the tensor product of two copies of the underlying tree, the first one with node indexed by the respective elements of M, the second one by the respective elements of N, all multiplied by the appropriate Koszul sign.

The first $\mathfrak{sL}_\infty$-algebra structure on $X \otimes B$ is given by

$$\begin{aligned} \ell_n &= \mathrm{VdL}_{X \otimes B} \mathcal{T}(s\varphi_{X \otimes A}) \Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee) \\ &= (\gamma_{\mathrm{End}_X} \otimes \mathrm{VdL}_B) \, \Phi \mathcal{T}(s\varphi_{X \otimes A}) \Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee) \\ &= (\gamma_{\mathrm{End}_X} \otimes \mathrm{VdL}_B) \, \Phi \mathcal{T}\left((\rho_X \otimes s\varphi_A) s \mathsf{M}_\pi s^{-1}\right) \Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee) \end{aligned}$$

$$
\begin{aligned}
&= \left(\gamma_{\mathrm{End}_X} \otimes \mathrm{VdL}_B\right)\left(\mathcal{T}(\rho_X) \otimes \mathcal{T}(s\varphi_A)\right) \Phi\mathcal{T}(s\mathsf{M}_\pi s^{-1})\Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee) \\
&= \left(\rho_X \otimes \mathrm{VdL}_B\mathcal{T}(s\varphi_A)\right)\left(\gamma_{\mathscr{P}} \otimes 1_{\mathcal{T}(\mathscr{C})}\right) \Phi\mathcal{T}(s\mathsf{M}_\pi s^{-1})\Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee)
\end{aligned}
$$

where in the third line we see $s\mathsf{M}_\pi s^{-1}$ as a map $\mathrm{Com}^\vee \to \mathscr{P} \otimes \mathscr{C}$, given by

$$
s\mathsf{M}_\pi s^{-1}(\mu_n^\vee) = \sum_{i \in I(n)} p_i \otimes c_i \ ,
$$

and in the last line we used the fact that

$$
\gamma_{\mathrm{End}_X}\mathcal{T}(\rho_X) = \rho_X \gamma_{\mathscr{P}} \ .
$$

For the second structure, we have

$$
\begin{aligned}
\ell_n &= \left(\rho_X \otimes \varphi_B\right) s\mathsf{M}_\pi s^{-1}(\mu_n^\vee) \\
&= \left(\rho_X \otimes \mathrm{VdL}_B\mathcal{T}(s\varphi_A)\Delta^{\mathrm{mon}}_{\mathscr{C}}\right) s\mathsf{M}_\pi s^{-1}(\mu_n^\vee) \\
&= \left(\rho_X \otimes \mathrm{VdL}_B\mathcal{T}(s\varphi_A)\right)\left(1_{\mathscr{P}} \otimes \Delta^{\mathrm{mon}}_{\mathscr{C}}\right) s\mathsf{M}_\pi s^{-1}(\mu_n^\vee) \ .
\end{aligned}
$$

Therefore, to conclude we need to show that

$$
\left(\gamma_{\mathscr{P}} \otimes 1_{\mathcal{T}(\mathscr{C})}\right) \Phi\mathcal{T}(s\mathsf{M}_\pi s^{-1})\Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee) = \left(1_{\mathscr{P}} \otimes \Delta^{\mathrm{mon}}_{\mathscr{C}}\right) s\mathsf{M}_\pi s^{-1}(\mu_n^\vee) \ .
$$

Using the notation introduced at the end of Appendix A.1, we have that

$$
\Delta^{\mathrm{mon}}_{\mathrm{Com}^\vee}(\mu_n^\vee) = \sum_{\tau \in \mathrm{RT}_n} \tau(v \mapsto \mu^\vee_{|v|}) \ .
$$

Therefore, the left-hand side above is equal to

$$
\sum_{\tau \in \mathrm{RT}_n} (-1)^\epsilon \sum_{i_v \in I(|v|) \text{ for } v \in V_\tau} \gamma_{\mathscr{P}}(\tau(v \mapsto p_{i_v})) \otimes \tau(v \mapsto c_{i_v}) \ ,
$$

where ϵ is the Koszul sign coming from Φ. At the same time, the right-hand side is given by

$$
\sum_{i \in I(n)} p_i \otimes \Delta^{\mathrm{mon}}_{\mathscr{C}}(c_i) \ .
$$

Both are expression for the map

$$
\gamma_{\mathscr{P}} : \mathcal{T}(\mathscr{P}) \longrightarrow \mathscr{P}
$$

seen as an element of $\mathscr{P} \otimes \mathcal{T}(\mathscr{P})^\vee$, and thus they are equal.

Once again, the statement on the ∞-morphisms is proven in an analogous way. □

Chapter 10

Representation of the deformation ∞-groupoid

We present here a first important application of the theory developed in Chapter 9. It was done in [RN17b], and it was the main goal of the author when developing the material of [RN17a]. In a sense, it should be considered the central result of the present work. Some of the results given here are newer, and are extracted from [RNV].

The idea is the following. A very important object in various areas of mathematics, e.g. deformation theory and rational homotopy theory, is given by the Maurer–Cartan space $\mathrm{MC}_\bullet(\mathfrak{g})$ of a homotopy Lie algebra. However, since the Sullivan algebra $\Omega_\bullet$ is infinite dimensional, this object is always "really big". One has Getzler's ∞-groupoid $\gamma_\bullet(\mathfrak{g})$, which is much smaller, but this object is somewhat complicated — at least in its original presentation — as Dupont's contraction map $h_\bullet$ is. The new idea to obtain a "nice" model for the space of Maurer–Cartan elements[1] is the following. Start with Dupont's contraction

$$h_\bullet \circlearrowright \Omega_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C_\bullet$$

and transfer the simplicial commutative algebra structure on $\Omega_\bullet$ to a simplicial $\mathscr{C}_\infty$-algebra structure on $C_\bullet$. Since $C_\bullet$ is finite dimensional at every simplicial degree, we can take its dual, which is a cosimplicial BLie-coalgebra (up to a suspension). Then we take the complete cobar construction with respect to the canonical twisting morphism $\mathrm{BLie} \to \mathrm{Lie}$ to obtain a cosimplicial Lie algebra

$$\mathfrak{mc}_\bullet \coloneqq \widehat{\Omega}_\pi(sC_\bullet^\vee) \ .$$

Intuitively, this should be a model for Maurer–Cartan elements at cosimplicial degree 0, for gauges at cosimplicial degree 1, and so on. One of the main results of this chapter formalizes this in the form of a natural homotopy equivalence of simplicial sets

$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \simeq \mathrm{MC}_\bullet(\mathfrak{g})$$

when $\mathfrak{g}$ is a Lie algebra. The proof of this result is made in two steps.

[1] Namely, we want Kan complex which is homotopically equivalent to $\mathrm{MC}_\bullet(\mathfrak{g})$ in a natural way.

1. First, we prove that
$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \, ,$$
where $\mathfrak{g} \otimes C_\bullet$ is given a homotopy Lie algebra structure by the homotopy transfer theorem applied to the contraction induced by Dupont's contraction. The proof is similar to the demonstration of the Dolgushev–Rogers theorem, and mainly uses methods of simplicial homotopy theory.

2. Then, we prove that
$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \cong \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \, .$$
This fact will be a straightforward consequence of the results of Chapter 9.

As a nice, immediate conference, we have that taking the space of Maurer–Cartan elements commutes with limits — only up to homotopy, if we take the functor $\mathrm{MC}_\bullet(-)$. We will also study some properties of $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ and of $\mathfrak{mc}_\bullet$. As a corollary, we will obtain an explicit way of "rectifying" homotopy equivalences between Maurer–Cartan elements to gauges between the same elements. We will also present an extension of these results to the case where $\mathfrak{g}$ is a $\mathfrak{s}\mathscr{L}_\infty$-algebra. This was not present in [RN17b], but will be contained in [RNV].

It should be remarked that the author was not aware of Bandiera's results — see Section 6.5 — at the time of publication of [RN17b]. They give another way of proving that $\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$, using very different methods. We believe that the two ways of proceeding are complementary, completing each other to yield a picture of the various relations between the three known models of the space of Maurer–Cartan elements of an $\mathscr{L}_\infty$-algebra.

Contrarily to what we did in most of the rest of this thesis, in this chapter we will work exclusively over cochain complexes. We will also work over shifted Lie and homotopy Lie algebras, but as usual the theory behaves in exactly the same way in the unshifted setting. In particular, Maurer–Cartan elements are in degree 0, and gauges are in degree -1.

10.1 An alternative model for the space of Maurer–Cartan elements

We begin by giving a simplicial set which is smaller, but homotopically equivalent to the space of Maurer–Cartan elements $\mathrm{MC}_\bullet(\mathfrak{g})$. It is given by the Maurer–Cartan elements of the tensorization of the $\mathfrak{s}\mathscr{L}_\infty$-algebra into consideration with the cellular cochains of the geometric simplices.

10.1.1 Statement of the main theorem

Let $\mathfrak{g}$ be a complete $\mathfrak{s}\mathscr{L}_\infty$-algebra. Dupont's contraction induces a contraction

$$1 \otimes h_\bullet \circlearrowright \mathfrak{g} \otimes \Omega_\bullet \underset{1 \otimes i_\bullet}{\overset{1 \otimes p_\bullet}{\rightleftarrows}} \mathfrak{g} \otimes C_\bullet$$

of $\mathfrak{g} \otimes \Omega_\bullet$ onto $\mathfrak{g} \otimes C_\bullet$. Applying the homotopy transfer theorem to this contraction, we obtain a simplicial $\mathfrak{s}\mathscr{L}_\infty$-algebra structure on $\mathfrak{g} \otimes C_\bullet$. We also know that we can extend the maps $1 \otimes p_\bullet$ and $1 \otimes i_\bullet$ to simplicial ∞-morphisms of simplicial $\mathfrak{s}\mathscr{L}_\infty$-algebras $(1 \otimes p_\bullet)_\infty$ and $(1 \otimes i_\bullet)_\infty$. We denote $P_\bullet$ and $I_\bullet$ the induced maps on Maurer–Cartan elements. We will also use the notation

$$(1 \otimes r_\bullet)_\infty \coloneqq (1 \otimes i_\bullet)_\infty (1 \otimes p_\bullet)_\infty \, ,$$

and we dub $\mathrm{Rect}_\bullet$ the map induced by $(1 \otimes r_\bullet)_\infty$ on Maurer–Cartan elements.

Theorem 10.1.1. *Let $\mathfrak{g}$ be a filtered $\mathfrak{sL}_\infty$-algebra. The maps $P_\bullet$ and $I_\bullet$ are inverse one to the other in homotopy, and thus provide a weak equivalence*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$$

of simplicial sets which is natural in $\mathfrak{g}$.

10.1.2 Proof of the main theorem

The rest of this section is dedicated to the proof of this result. We begin with the following lemma.

Lemma 10.1.2. *We have*

$$P_\bullet I_\bullet = \mathrm{id}_{\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)} \ .$$

Proof. This is because $(1 \otimes p_\bullet)_\infty (1 \otimes i_\bullet)_\infty$ is the identity, see e.g. [DSV16, Theorem 5], and the functoriality of the Maurer–Cartan functor MC. □

Therefore, it is enough to prove that the map

$$\mathrm{Rect} = I_\bullet P_\bullet : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{g})$$

is a weak equivalence. The idea is to use the same methods as for the proof of the Dolgushev–Rogers theorem, cf. Section 6.4. The situation is however slightly different, as the map $\mathrm{Rect}_\bullet$ is not of the form $\Phi \otimes 1_{\Omega_\bullet}$, and thus the Dolgushev–Rogers theorem itself cannot be directly applied. The first, easy step is to understand what happens at the level of the zeroth homotopy group.

Lemma 10.1.3. *The map*

$$\pi_0(\mathrm{Rect}) : \pi_0 \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \pi_0 \mathrm{MC}_\bullet(\mathfrak{g})$$

is a bijection.

Proof. We have $\Omega_0 = C_0 = \mathbb{K}$, and the maps i_0 and p_0 both are the identity of $\mathbb{K}$. Therefore, the map R_0 is the identity of $\mathrm{MC}_0(\mathfrak{g})$, and thus obviously induces a bijection on π_0. □

For the higher homotopy groups, we start with a simplified version of Proposition 6.4.2, which gives in some sense the base for an inductive argument. If the $\mathfrak{sL}_\infty$-algebra $\mathfrak{g}$ is abelian, i.e. all of its brackets vanish, then so do the brackets at all levels of $\mathfrak{g} \otimes \Omega_\bullet$. In this case, the Maurer–Cartan elements are exactly the cocycles of the underlying cochain complex, and therefore $\mathrm{MC}_\bullet(\mathfrak{g})$ is a simplicial vector space.

Lemma 10.1.4. *If the $\mathfrak{sL}_\infty$-algebra $\mathfrak{g}$ is abelian, then $\mathrm{Rect}_\bullet$ is a weak equivalence of simplicial vector spaces.*

Proof. Recall that the Moore complex of a simplicial vector space $V_\bullet$ is defined by

$$\mathscr{M}(V_\bullet)_n \coloneqq s^n V_n$$

endowed with the differential

$$\partial \coloneqq \sum_{i=0}^{n} (-1)^i d_i \ ,$$

where the maps d_i are the face maps of the simplicial set $V_\bullet$. It is a standard result that

$$\pi_0(V_\bullet) = H_0(\mathscr{M}(V_\bullet)) \,, \qquad \pi_i(V_\bullet, v) \cong \pi_i(V_\bullet, 0) = H_i(\mathscr{M}(V_\bullet))$$

for all $i \geq 1$ and $v \in V_0$, and that a map of simplicial vector spaces is a weak equivalence if and only if it induces a quasi-isomorphism between the respective Moore complexes [GJ09, Cor. 2.5, Sect. III.2].
In our case,

$$V_\bullet \coloneqq \mathrm{MC}_\bullet(\mathfrak{g}) = \mathfrak{Z}^1(\mathfrak{g} \otimes \Omega_\bullet)$$

is the simplicial vector space of 1-cocycles of $\mathfrak{g} \otimes \Omega_\bullet$. As in the proof of Proposition 6.4.2, it can be proven that the map

$$\mathscr{M}(1 \otimes p_\bullet) : \mathscr{M}(\mathfrak{Z}^1(\mathfrak{g} \otimes \Omega_\bullet)) \longrightarrow \mathscr{M}(\mathfrak{Z}^1(\mathfrak{g} \otimes C_\bullet))$$

is a quasi-isomorphism. But as the bracket vanishes, this is exactly $P_\bullet$. Now

$$\mathscr{M}(1 \otimes p_\bullet)\mathscr{M}(1 \otimes i_\bullet) = 1_{\mathscr{M}(\mathfrak{Z}^1(\mathfrak{g} \otimes \Omega_\bullet))} \,,$$

which implies that $\mathscr{M}(1 \otimes i_\bullet)$ also is a quasi-isomorphism. It follows that $\mathrm{Rect}_\bullet$ is a weak equivalence, concluding the proof. □

Now we basically follow the structure of the proof of Theorem 6.4.1. We define a filtration of $\mathfrak{g} \otimes \Omega_\bullet$ by

$$\mathscr{F}_k(\mathfrak{g} \otimes \Omega_\bullet) \coloneqq (\mathscr{F}_k\mathfrak{g}) \otimes \Omega_\bullet \,.$$

We denote by

$$(\mathfrak{g} \otimes \Omega_\bullet)^{(k)} \coloneqq \mathfrak{g} \otimes \Omega_\bullet / \mathscr{F}_k(\mathfrak{g} \otimes \Omega_\bullet) = \mathfrak{g}^{(k)} \otimes \Omega_\bullet \,.$$

The composite $(1 \otimes i_\bullet)(1 \otimes p_\bullet)$ induces an endomorphism $(1 \otimes i_\bullet)^{(k)}(1 \otimes p_\bullet)^{(k)}$ of $(\mathfrak{g} \otimes \Omega_\bullet)^{(k)}$. All the ∞-morphisms coming into play obviously respect this filtration, and moreover $1 \otimes h_\bullet$ passes to the quotients, so that we have

$$1_{(\mathfrak{g} \otimes \Omega_\bullet)^{(k)}} - (1 \otimes i_\bullet)^{(k)}(1 \otimes p_\bullet)^{(k)} = d(1 \otimes h_\bullet)^{(k)} + (1 \otimes h_\bullet)^{(k)}d$$

for all k, which shows that $(1 \otimes r_\bullet)_\infty$ is a filtered ∞-quasi isomorphism.

The next step is to reduce the study of the homotopy groups with arbitrary basepoint to the study of the homotopy groups with basepoint $0 \in \mathrm{MC}_0(\mathfrak{g})$.

Lemma 10.1.5. *Let $\alpha \in \mathrm{MC}(\mathfrak{g})$, and let $\mathfrak{g}^\alpha$ be the $\mathscr{L}_\infty$-algebra obtained by twisting $\mathfrak{g}$ by α, that is the $\mathscr{L}_\infty$-algebra with the same underlying graded vector space, but with differential*

$$d^\alpha(x) \coloneqq dx + \sum_{n \geq 2} \frac{1}{(n-1)!} \ell_n(\alpha, \ldots, \alpha, x)$$

and brackets

$$\ell^\alpha(x_1, \ldots, x_m) \coloneqq \sum_{n \geq m} \frac{1}{(n-m)!} \ell_n(\alpha, \ldots, \alpha, x_1, \ldots, x_m) \,.$$

Let

$$\mathrm{Shift}_\alpha : \mathrm{MC}_\bullet(\mathfrak{g}^\alpha) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{g})$$

be the isomorphism of simplicial sets induced by the map given by

$$\beta \in \mathfrak{g} \longmapsto \alpha + \beta \in \mathfrak{g}^\alpha \,.$$

Then the following diagram commutes

$$\begin{array}{ccc} \mathrm{MC}_\bullet(\mathfrak{g}^\alpha) & \xrightarrow{\mathrm{Shift}_\alpha} & \mathrm{MC}_\bullet(\mathfrak{g}) \\ \Big\downarrow{\scriptstyle \mathrm{Rect}_\bullet^\alpha} & & \Big\downarrow{\scriptstyle \mathrm{Rect}_\bullet} \\ \mathrm{MC}_\bullet(\mathfrak{g}^\alpha) & \xrightarrow{\mathrm{Shift}_\alpha} & \mathrm{MC}_\bullet(\mathfrak{g}) \end{array}$$

where

$$\mathrm{Rect}^\alpha(\beta) \coloneqq \sum_{k\geq 1} (1\otimes r_\bullet)_k^\alpha(\beta^{\otimes k})$$

and

$$(1\otimes r_\bullet)_k^\alpha(\beta_1\otimes\cdots\otimes\beta_k) \coloneqq \sum_{j\geq 0}\frac{1}{j!}(1\otimes r_\bullet)_{k+j}(\alpha^{\otimes j}\otimes\beta_1\otimes\ldots\otimes\beta_k)$$

is the twist of $(1\otimes r_\bullet)_\infty$ *by the Maurer–Cartan element* α*. Here, we identified* $\alpha\in\mathfrak{g}$ *with* $\alpha\otimes 1\in\mathfrak{g}\otimes\Omega_\bullet$*.*

Proof. The proof of [DR17, Lemma 4.3] goes through *mutatis mutandis*. □

Remark 10.1.6. *The* $\mathscr{L}_\infty$*-algebra* $\mathfrak{g}^\alpha$ *in Lemma 10.1.5 is endowed with the same filtration as* $\mathfrak{g}$*.*

Now we proceed by induction to show that $\mathrm{Rect}^{(k)}$ is a weak equivalence from $\mathrm{MC}_\bullet(\mathfrak{g}^{(k)})$ to itself for all $k\geq 2$. As the $\mathscr{L}_\infty$-algebra $(\mathfrak{g}\otimes\Omega_\bullet)^{(2)}$ is abelian, the base step of the induction is given by Lemma 10.1.4.

Lemma 10.1.7. *Let* $m\geq 2$*. Suppose that*

$$\mathrm{Rect}^{(k)} : \mathrm{MC}(\mathfrak{g}^{(k)}) \longrightarrow \mathrm{MC}(\mathfrak{g}^{(k)})$$

is a weak equivalence for all $2\leq k\leq m$*. Then* $\mathrm{Rect}^{(m+1)}$ *is also a weak equivalence.*

Proof. The zeroth homotopy set π_0 has already been taken care of in Lemma 10.1.3. Thanks to Lemma 10.1.5, it is enough to prove that $\mathrm{Rect}^{(m+1)}$ induces isomorphisms of homotopy groups π_i based at 0, for all $i\geq 1$.

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{\mathscr{F}_m(\mathfrak{g}\otimes\Omega_\bullet)}{\mathscr{F}_{m+1}(\mathfrak{g}\otimes\Omega_\bullet)} & \longrightarrow & (\mathfrak{g}\otimes\Omega_\bullet)^{(m+1)} & \longrightarrow & (\mathfrak{g}\otimes\Omega_\bullet)^{(m)} & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle (1\otimes r_\bullet)_\infty^{(m+1)}} & & \Big\downarrow{\scriptstyle (1\otimes r_\bullet)_\infty^{(m)}} & & \\ 0 & \longrightarrow & \frac{\mathscr{F}_m(\mathfrak{g}\otimes\Omega_\bullet)}{\mathscr{F}_{m+1}(\mathfrak{g}\otimes\Omega_\bullet)} & \longrightarrow & (\mathfrak{g}\otimes\Omega_\bullet)^{(m+1)} & \longrightarrow & (\mathfrak{g}\otimes\Omega_\bullet)^{(m)} & \longrightarrow & 0 \end{array}$$

where the leftmost vertical arrow is given by the linear term $(1\otimes i_\bullet)(1\otimes p_\bullet)$ of $(1\otimes r_\bullet)_\infty$ since all higher terms vanish, as can be seen by the explicit formulæ for the ∞-quasi isomorphisms induced by the homotopy transfer theorem given in [LV12, Sect. 10.3.5–6]. Therefore, it is a weak equivalence as the $\mathfrak{s}\mathscr{L}_\infty$-algebras in question are abelian. The first term in each row is the fibre of the next map, which is surjective. By Theorem 6.2.6, we know that applying the $\mathrm{MC}_\bullet$ functor makes the horizontal maps on the right into fibrations of simplicial sets, while the objects we obtain on the left are easily seen to be the fibres. Taking the long sequence in

homotopy and using the five-lemma, we see that all we are left to do is to prove that $\mathrm{Rect}_\bullet^{(m+1)}$ induces an isomorphism on π_1. Notice that it is necessary to prove this, as the long sequence is exact everywhere except on the level of π_0.

The long exact sequence of homotopy groups (truncated on both sides) reads

$$\pi_2\mathrm{MC}_\bullet(\mathfrak{g}^{(m)}) \xrightarrow{\partial} \pi_1\mathrm{MC}_\bullet\left(\tfrac{F_m\mathfrak{g}}{F_{m+1}\mathfrak{g}}\right) \to \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)}) \to \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)}) \xrightarrow{\partial} \pi_0\mathrm{MC}_\bullet\left(\tfrac{F_m\mathfrak{g}}{F_{m+1}\mathfrak{g}}\right) ,$$

where in the higher homotopy groups we left the basepoint implicit (as it is always 0). The map

$$\partial : \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)}) \longrightarrow \pi_0\mathrm{MC}_\bullet\left(\frac{F_m\mathfrak{g}}{F_{m+1}\mathfrak{g}}\right) = H^1\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right)$$

encodes the obstruction to lifting an element of $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$ to an element of $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ (see e.g. Section 4.2.6).

The map $\pi_1(\mathrm{Rect}^{(m+1)})$ is surjective: Let $y \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ be any element, and denote by $\overline{y}$ its image in $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$. By the induction hypothesis, there exists a unique $\overline{x} \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$ which is mapped to $\overline{y}$ under $\mathrm{Rect}_\bullet^{(m)}$. As $\overline{y}$ is the image of y, we have $\partial(\overline{y}) = 0$, and this implies that $\partial(\overline{x}) = 0$, too. Therefore, there exists $x \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ mapping to $\overline{x}$. Denote by y' the image of x under $\mathrm{Rect}_\bullet^{(m+1)}$. Then $y'y^{-1}$ is in the kernel of the map

$$\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)}) \longrightarrow \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)}) .$$

By exactness of the long sequence, and the fact that $\mathrm{Rect}_\bullet$ induces an automorphism of

$$\pi_1\mathrm{MC}_\bullet\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right) .$$

there exists an element $z \in \pi_1(\mathrm{MC}_\bullet(F_{m+1}\mathfrak{g}/F_m\mathfrak{g}))$ mapping to $y'y^{-1}$ under the composite

$$\pi_1\mathrm{MC}_\bullet\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right) \xrightarrow{\mathrm{Rect}_\bullet} \pi_1\mathrm{MC}_\bullet\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right) \longrightarrow \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)}) .$$

Let x' be the image of z in $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$, then $(x')^{-1}x$ maps to y under $\mathrm{Rect}_\bullet^{(m+1)}$. This proves the surjectivity of the map $\pi_1(\mathrm{Rect}^{(m+1)})$.

The map $\pi_1(\mathrm{Rect}_\bullet^{(m+1)})$ is injective: Assume $x, x' \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ map to the same element under $\mathrm{Rect}^{(m+1)}$. Then $x(x')^{-1}$ maps to the neutral element 0 under $\mathrm{Rect}^{(m+1)}$. It follows that there is a

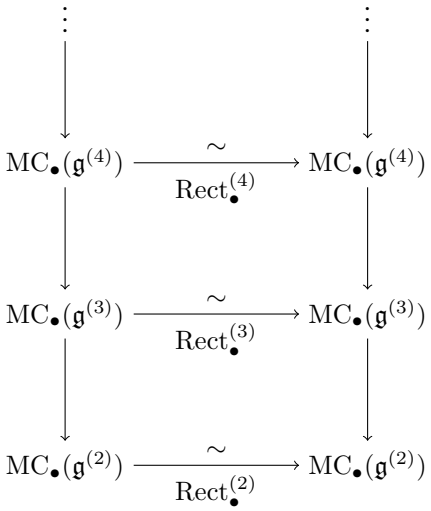
$$z \in \pi_1\mathrm{MC}_\bullet\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right)$$

mapping to $x(x')^{-1}$, which must be such that its image w is itself the image of some element $\widetilde{w} \in \pi_2\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$ under the map ∂. But by the induction hypothesis and the exactness of the long sequence, this implies that z is in the kernel of the next map, and thus that $x(x')^{-1}$ is the identity element. Therefore, the map $\pi_1(\mathrm{Rect}_\bullet^{(m+1)})$ is injective.

This ends the proof of the lemma. □

Finally, we can conclude the proof of Theorem 10.1.1.

Proof of Theorem 10.1.1. Lemma 10.1.7, together with all we have said before, shows that $\mathrm{Rect}_\bullet^{(m)}$ is a weak equivalence for all $m \geq 2$. Therefore, we have the following commutative diagram:



where all objects are Kan complexes, all horizontal arrows are weak equivalences, and all vertical arrows are (Kan) fibrations by Theorem 6.2.6. It follows that the collection of horizontal arrows defines a weak equivalence between fibrant objects in the model category of tower of simplicial sets, see [GJ09, Sect. VI.1]. The functor from towers of simplicial sets to simplicial sets given by taking the limit is right adjoint to the constant tower functor, which trivially preserves cofibrations and weak equivalences. Thus, the constant tower functor is a left Quillen functor, and it follows that the limit functor is a right Quillen functor. In particular, it preserves weak equivalences between fibrant objects. Applying this to the diagram above proves that $\mathrm{Rect}_\bullet$ is a weak equivalence. □

10.2 Properties and comparison with Getzler's functor

Theorem 10.1.1 shows that the simplicial set $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ is a new model for the deformation ∞-groupoid. This section is dedicated to the study of some properties of this object. We start by showing that it is a Kan complex, then we give some conditions on the differential forms representing its simplices. We show how we can use it to rectify cells of the deformation ∞-groupoid, which provides an alternative, simpler proof of [DR15, Lemma B.2]. Finally we compare it with Getzler's functor $\gamma_\bullet$, proving that our model is contained in Getzler's. Independent results by Bandiera [Ban14], [Ban17] imply that the two models are actually isomorphic.

10.2.1 Properties of $\mathrm{MC}_\bullet(\mathfrak{g} \otimes C_\bullet)$

The following proposition is the analogue to Theorem 6.2.6 for our model.

Proposition 10.2.1. *Let $\mathfrak{g}, \mathfrak{h}$ be two complete proper $\mathfrak{s}\mathscr{L}_\infty$-algebras, and suppose that $\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is an ∞-morphism of $\mathscr{L}_\infty$-algebras inducing a fibration of simplicial sets under the functor $\mathrm{MC}_\bullet$, see for example Theorem 6.2.6 for possible sufficient conditions. Then the induced morphism*

$$\mathrm{MC}(\phi \otimes \mathrm{id}_{C_\bullet}) : \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \longrightarrow \mathrm{MC}(\mathfrak{h} \otimes C_\bullet)$$

is also a fibration of simplicial sets. In particular, for any complete proper $\mathfrak{s}\mathscr{L}_\infty$-algebra $\mathfrak{g}$, the simplicial set $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ is a Kan complex.

Proof. By assumption, the morphism

$$\mathrm{MC}_\bullet(\phi) : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{h})$$

is a fibration of simplicial set, and by Lemma 10.1.2 the following diagram exhibits $\mathrm{MC}(\phi \otimes \mathrm{id}_{C_\bullet})$ as a retract of $\mathrm{MC}_\bullet(\phi)$.

$$\begin{array}{ccccc}
\mathrm{MC}(\mathfrak{g}\otimes C_\bullet) & \xrightarrow{I_\bullet} & \mathrm{MC}_\bullet(\mathfrak{g}) & \xrightarrow{P_\bullet} & \mathrm{MC}(\mathfrak{g}\otimes C_\bullet) \\
\downarrow{\scriptstyle \mathrm{MC}(\phi\otimes \mathrm{id}_{C_\bullet})} & & \downarrow{\scriptstyle \mathrm{MC}_\bullet(\phi)} & & \downarrow{\scriptstyle \mathrm{MC}(\phi\otimes \mathrm{id}_{C_\bullet})} \\
\mathrm{MC}(\mathfrak{g}\otimes C_\bullet) & \xrightarrow[I_\bullet]{} & \mathrm{MC}_\bullet(\mathfrak{g}) & \xrightarrow[P_\bullet]{} & \mathrm{MC}(\mathfrak{g}\otimes C_\bullet)
\end{array}$$

As the class of fibrations of a model category is closed under retracts, this concludes the proof. □

We also consider the composite $\mathrm{Rect}_\bullet \coloneqq I_\bullet P_\bullet$, which is not the identity in general.

Definition 10.2.2. *We call the morphism*

$$\mathrm{Rect}_\bullet : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}_\bullet(\mathfrak{g})$$

the rectification map.

The following result is a wide generalization of [DR15, Lemma B.2], as well as a motivation for the name "rectification map" for $\mathrm{Rect}_\bullet$.

Proposition 10.2.3. *We consider an element*

$$\alpha \coloneqq \alpha_1(t_0, \dots, t_n) + \cdots \in \mathrm{MC}_n(\mathfrak{g}) \ ,$$

where the dots indicate terms in $\mathfrak{g}^{1-k} \otimes \Omega_n^k$ with $1 \leq k \leq n$. Then $\beta \coloneqq \mathrm{Rect}_n(\alpha) \in \mathrm{MC}_n(\mathfrak{g})$ is of the form

$$\beta = \beta_1(t_0, \dots, t_n) + \cdots + \xi \otimes \omega_{0\dots n} \ ,$$

where the dots indicate terms in $\mathfrak{g}^{1-k} \otimes \Omega_n^k$ with $1 \leq k \leq n-1$, where ξ is an element of $\mathfrak{g}^{1-n}$, and where α_1 and β_1 agree on the vertices of Δ^n. In particular, if $\alpha \in \mathrm{MC}_1(\mathfrak{g})$, then $\beta = F(\alpha) \in \mathrm{MC}_1(\mathfrak{g})$ is of the form

$$\beta = \beta_1(t) + \lambda dt$$

for some $\lambda \in \mathfrak{g}^0$, and satisfies

$$\beta_1(0) = \alpha_1(0) \qquad \text{and} \qquad \beta_1(1) = \alpha_1(1) \ .$$

Remark 10.2.4. *As $\mathrm{Rect}_\bullet$ is a projector, this proposition in fact gives information on the form of all the elements of $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$.*

Proof. First notice that the map $\mathrm{Rect}_\bullet$ commutes with the face maps and is the identity on 0-simplices, thus evaluation of the part of β in $\mathfrak{g}^1 \otimes \Omega_n^0$ at the vertices gives the same result as evaluation at the vertices of α_1. Next, we notice that β is in the image of $I_\bullet$. We use the explicit formula for $(1 \otimes i_n)_\infty$ of Section 2.3.4: the operator acting on arity $k \geq 2$ is given, up to signs, by the sum over all rooted trees with $1 \otimes i_n$ put at the leaves, the brackets ℓ_n of the corresponding arity at all vertices, and $1 \otimes h$ at the inner edges and at the root. But the $1 \otimes h$ at the root lowers the degree of the part of the form in Ω_n by 1, and thus we cannot get something in $\mathfrak{g}^{1-n} \otimes \Omega_n^n$ from these terms. The only surviving term is therefore the one coming from $(1 \otimes i_n)(P(\alpha))$, given by $\xi \otimes \omega_{0\dots n}$ for some $\xi \in \mathfrak{g}^{1-n}$. □

In particular, take $n = 1$. Then a gauge between two Maurer–Cartan elements is exactly the same as an element $\beta \in \mathrm{MC}_1(\mathfrak{g})$ of the form

$$\beta = \beta_1(t) + \lambda dt$$

such that $\beta_1(t)$ evaluates to the two Maurer–Cartan elements at $t = 0, 1$, cf. Section 6.2.3. With this in our minds, we notice that Proposition 10.2.3 above immediately implies the following two facts.

1. The set $\mathrm{MC}(\mathfrak{g} \otimes C_1)$ is included into $\mathrm{MC}_1(\mathfrak{g})$ as the subset consisting of gauge equivalences via the map I_1.
2. The rectification map Rect_1 gives us an explicit formula to rectify a homotopy between two Maurer–Cartan elements to a gauge between the same elements.

Analogously, Proposition 10.2.3 also tells us that the higher maps Rect_n "rectify" higher relations, making the term of lowest degree in $\mathfrak{g}$ become constant.

10.2.2 Comparison with Getzler's ∞-groupoid

Finally, we compare the simplicial set $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ with Getzler's Kan complex $\gamma_\bullet(\mathfrak{g})$. We start with an easy result that follows directly from our approach, before using Bandiera's results — see Section 6.5.2 — to prove that these two simplicial sets are actually isomorphic.

Lemma 10.2.5. *We have*

$$I_\bullet \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \subseteq \gamma_\bullet(\mathfrak{g}) \ .$$

Proof. We have $h_\bullet i_\bullet = 0$. Therefore, by the explicit formula formula for $(i_\bullet)_\infty$ given in Section 2.3.4, we have $h_\bullet(\beta) = 0$ for any $\beta \in \mathfrak{g} \otimes \Omega_\bullet$ in the image of $I_\bullet$. Thus

$$h_\bullet(\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)) = h_\bullet I_\bullet P_\bullet(\mathrm{MC}_\bullet(\mathfrak{g})) = 0 \ ,$$

which proves the claim. □

An immediate consequence of the formal Kuranishi theorem is the following proposition.

Theorem 10.2.6 ([Ban17, Prop. 2.5])**.** *The map*

$$(P_\bullet, 1 \otimes h_\bullet) : \mathrm{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \times \left(\mathrm{Im}(1 \otimes h_\bullet) \cap (\mathfrak{g} \otimes \Omega_\bullet)^1\right)$$

is bijective. In particular, its restriction to $\gamma_\bullet(\mathfrak{g}) = \ker(1 \otimes h_\bullet) \cap \mathrm{MC}_\bullet(\mathfrak{g})$ *gives a isomorphism of simplicial sets*

$$P_\bullet : \gamma_\bullet(\mathfrak{g}) \longrightarrow \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \ .$$

Proof. The first statement is obtained by applying Theorem 6.5.6 to the contraction

$$1 \otimes h_\bullet \circlearrowright \mathfrak{g} \otimes \Omega_\bullet \underset{1 \otimes i_\bullet}{\overset{1 \otimes p_\bullet}{\rightleftarrows}} \mathfrak{g} \otimes C_\bullet$$

The second statement is a straightforward consequence of the first one, obtained by restricting the map to $\gamma_\bullet(\mathfrak{g}) = \mathrm{MC}_\bullet(\mathfrak{g}) \cap \ker(1 \otimes h_\bullet)$. □

Remark 10.2.7. *Thanks to our approach, we immediately have an inverse for the map $P_\bullet$: it is of course the map $I_\bullet$.*

As a consequence of Bandiera's result and of Proposition 10.2.3, we can partially characterize the thin elements of $\gamma_\bullet(\mathfrak{g})$.

Lemma 10.2.8. *For each $n \geq 1$, the thin elements contained in $\gamma_n(\mathfrak{g})$ are those with no term in $\mathfrak{g}^{1-n} \otimes \Omega^n_n$.*

Proof. By Proposition 10.2.3 and Theorem 10.2.6, we know that if $\alpha \in \gamma_n(\mathfrak{g})$, then α is of the form

$$\alpha = \cdots + \xi \otimes \omega_{0\dots n}$$

for some $\xi \in \mathfrak{g}^{1-n}$, where the dots indicate terms in $\mathfrak{g}^{1-k} \otimes \Omega^k_n$ for $0 \leq k \leq n-1$, which will give zero after integration. Integrating, we get

$$\int_{\Delta^n} \alpha = \xi \otimes \int_{\Delta^n} \omega_{0\dots n} = \xi \otimes 1 \,.$$

Therefore, α is thin if, and only if $\xi = 0$. □

10.3 A model for Maurer–Cartan elements of Lie algebras

Our next goal is to represent the Maurer–Cartan functor $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ by a cosimplicial object. We begin by doing this in the case of Lie algebras, as it was originally done in [RN17b]. The case of $\mathfrak{sL}_\infty$-algebras will be treated in Section 10.4.

Since we are in the shifted setting, we consider shifted Lie algebras, that is algebras over the operad $\mathscr{S} \otimes \mathrm{Lie}$.

10.3.1 Representing $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$

Using the Dupont contraction, the homotopy transfer theorem gives the structure of a simplicial $\mathscr{C}_\infty$-algebra to $C_\bullet$. As the underlying cochain complex C_n is finite dimensional for each n, it follows that its dual is a cosimplicial $\mathrm{B}(\mathscr{S} \otimes \mathrm{Lie})$-coalgebra. Therefore, we can take its complete cobar construction relative to the canonical twisting morphism

$$\pi : \mathrm{B}(\mathscr{S} \otimes \mathrm{Lie}) \longrightarrow \mathscr{S} \otimes \mathrm{Lie}$$

to obtain a shifted Lie algebra.

Definition 10.3.1. *We denote the cosimplicial shifted Lie algebra obtained this way by $\mathfrak{mc}_\bullet := \widehat{\Omega}_\pi(C^\vee_\bullet)$.*

Theorem 10.3.2. *Let $\mathfrak{g}$ be a proper complete shifted Lie algebra. There is a canonical isomorphism*

$$\mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \cong \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \,.$$

It is natural in $\mathfrak{g}$.

Proof. By Theorem 9.3.6, the $\mathfrak{sL}_\infty$-algebra structure we have on $\mathfrak{g} \otimes C_\bullet$ is the same as the structure that we obtain on the tensor product of the shifted Lie algebra $\mathfrak{g}$ with the simplicial $\mathscr{C}_\infty$-algebra $C_\bullet$ by using Theorem 9.1.12 the twisting morphism π. Therefore, we can apply Proposition 9.1.22 and Theorem 9.1.13 to obtain the desired isomorphism. □

With this form for $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$, Theorem 10.1.1 reads as follows.

Corollary 10.3.3. *Let $\mathfrak{g}$ be a proper complete dg Lie algebra. There is a weak equivalence of simplicial sets*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \,,$$

natural in $\mathfrak{g}$.

Remark 10.3.4. *This result was proven independently and simultaneously in [BFMT17, Thm. 0.1].*

We can completely characterize the first levels of the cosimplicial Lie algebra $\mathfrak{mc}_\bullet$. Recall from the Lawrence–Sullivan algebra from Section 6.1.3: it is the unique free complete dg Lie algebra generated by two Maurer–Cartan elements in degree 1 and a single element in degree 0 such that the element in degree 0 is a gauge between the two generating Maurer–Cartan elements.

Proposition 10.3.5. *The first two levels of the cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ are as follows.*

1. *The dg Lie algebra $\mathfrak{mc}_0$ is isomorphic to the free dg Lie algebra with a single Maurer–Cartan element as the only generator.*
2. *The dg Lie algebra $\mathfrak{mc}_1$ is isomorphic to the Lawrence–Sullivan algebra, shifted by 1.*

Proof. For (1), we have $\Omega_0 \cong \mathbb{K} \cong C_0$, both p_0 and i_0 are the identity, and $h_0 = 0$. It follows that, as a complete graded free Lie algebra, $\mathfrak{mc}_0$ is given by

$$\mathfrak{mc}_0 = \widehat{\mathrm{Lie}}(s\mathbb{K}) \,.$$

We denote the generator by $\alpha \coloneqq s1^\vee$. It has degree 1. Let $\mathfrak{g}$ be any complete dg Lie algebra, then a morphism

$$\phi : \mathfrak{mc}_0 \longrightarrow \mathfrak{g}$$

is equivalent to the Maurer–Cartan element

$$\phi(\alpha) \otimes 1 \in \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \cong \mathrm{MC}(\mathfrak{g}) \,.$$

Conversely, through P_0 every Maurer–Cartan element of $\mathfrak{g}$ induces a morphism $\mathfrak{mc}_0 \to \mathfrak{g}$. As this is true for any dg Lie algebra $\mathfrak{g}$, it follows that α is a Maurer–Cartan element.

To prove (2), we start by noticing that

$$C_1 \coloneqq \mathbb{K}\omega_0 \oplus \mathbb{K}\omega_1 \oplus \mathbb{K}\omega_{01}$$

with ω_0, ω_1 of degree 0 and ω_{01} of degree 1. Denoting by $\alpha_i \coloneqq s\omega_i^\vee$ and by $\lambda \coloneqq s\omega_{01}^\vee$, we have

$$\mathfrak{mc}_1 = \widehat{\mathrm{Lie}}(\alpha_0, \alpha_1, \lambda)$$

as a graded Lie algebra. Let $\mathfrak{g}$ be any dg Lie algebra, then a morphism

$$\phi : \mathfrak{mc}_1 \longrightarrow \mathfrak{g}$$

is equivalent to a Maurer–Cartan element

$$\phi(\alpha_0) \otimes \omega_0 + \phi(\alpha_1) \otimes \omega_1 + \phi(\lambda) \otimes \omega_{01} \in \mathrm{MC}(\mathfrak{g} \otimes C_1) \,,$$

see [RN17a, Sect. 6.3–4]. Applying I_1, as in the proof of Proposition 10.2.3 we obtain

$$I_1(\phi(\alpha_0) \otimes \omega_0 + \phi(\alpha_1) \otimes \omega_1 + \phi(\lambda) \otimes \omega_{01}) = a(t_0, t_1) + \phi(\lambda) \otimes \omega_{01} \in \mathrm{MC}_1(\mathfrak{g})$$

with $a(1,0) = \phi(\alpha_0)$ and $a(0,1) = \phi(\alpha_1)$. The Maurer–Cartan equation for $a(t_0,t_1) + \phi(\lambda) \otimes \omega_{01}$ then shows that $\phi(\lambda)$ is a gauge from $\phi(\alpha_0)$ to $\phi(\alpha_1)$. Conversely, if we are given the data of two Maurer–Cartan elements of $\mathfrak{g}$ and a gauge equivalence between them, then this data gives us a Maurer–Cartan element of $\mathfrak{g} \otimes \Omega_1$. Applying P_1 then gives back a non-trivial morphism $\mathfrak{mc}_1 \to \mathfrak{g}$. As this is true for any $\mathfrak{g}$, it follows that $\mathfrak{mc}_1$ is isomorphic to the Lawrence–Sullivan algebra. □

Remark 10.3.6. *Alternatively, one could write down explicitly the differentials for both $\mathfrak{mc}_0$ (which is straightforward) and $\mathfrak{mc}_1$ (with the help of [CG08, Prop. 19]). An explicit description of $\mathfrak{mc}_\bullet$ is made difficult by the fact that one needs to know the whole $\mathscr{C}_\infty$-algebra structure on $C_\bullet$ in order to write down a formula for the differential.*

10.3.2 Relations to rational homotopy theory

The cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ has already made its appearance in the literature not long ago, in the paper [BFMT15], in the context of rational homotopy theory, where it plays the role of a Lie model for the geometric n-simplex. With the goal of simplifying comparison and interaction between our work and theirs, we provide here a short review and a dictionary between our vocabulary and the notations used in [BFMT15].

Notation in the present work	Notation of [BFMT15]
$\mathfrak{mc}_\bullet$	$\mathfrak{L}_\bullet$ or $\mathfrak{L}_{\Delta^\bullet}$
$\Omega_\bullet$	$A_{PL}(\Delta^\bullet)$
B_ι	Quillen functor $\mathscr{C}$
$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, -)$	$\langle - \rangle$
$\hom_{\mathsf{dgCom}}(-, \Omega_\bullet)$	$\langle - \rangle_S$

The following theorem has non-empty intersection with our results. We say a shifted Lie algebra is of *finite type* if it is finite dimensional in every degree and if its degrees are bounded either above or below.

Theorem 10.3.7 ([BFMT15, Th. 8.1])**.** *Let $\mathfrak{g}$ be a dg Lie algebra of finite type with $H^n(\mathfrak{g}, d) = 0$ for all $n > 0$. Then there is a homotopy equivalence of simplicial sets*

$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \simeq \hom_{\mathsf{dgCom}}(\mathrm{B}_\iota(s\mathfrak{g})^\vee, \Omega_\bullet) \,.$$

We can easily recover an analogous result, which works on complete proper shifted Lie algebras of finite type such that $\mathfrak{g}^{-1} = 0$, but without restrictions on the cohomology, using our main theorem and some results of [RN17a].

Proposition 10.3.8. *Let $\mathfrak{g}$ be a complete dg Lie algebra of finite type such that $\mathfrak{g}^{-1} = 0$. Then there is a weak equivalence of simplicial sets*

$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \simeq \hom_{\mathsf{dgCom}}(\mathrm{B}_\iota(s\mathfrak{g})^\vee, \Omega_\bullet) \,.$$

Proof. The proof is given by the sequence of equivalences

$$\begin{aligned} \hom_{\mathsf{dgCom}}(\mathrm{B}_\iota(s\mathfrak{g})^\vee, \Omega_\bullet) &\cong \hom_{\mathsf{dgCom}}\left(\widehat{\Omega}_\pi(s^{-1}\mathfrak{g}^\vee), \Omega_\bullet\right) \\ &\cong \mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet) \\ &\simeq \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}). \end{aligned}$$

In the first line we used the natural isomorphism

$$\mathrm{B}_\iota(s\mathfrak{g})^\vee \cong \widehat{\Omega}_\pi(s^{-1}\mathfrak{g}^\vee)\ .$$

Notice that the assumptions on $\mathfrak{g}$ make it so that $\mathfrak{g}^\vee$ is a $\mathrm{Lie}^\vee$-coalgebra. In the second line we used a slight generalization of [RN17a, Cor. 6.6] for $\mathscr{Q} = \mathscr{P} = \mathrm{Com}$ and Ψ the identity morphism of Com. Notice that here the assumption that $\mathfrak{g}^{-1} = 0$ makes it so that

$$\hom_{\mathsf{dgCom}}\left(\widehat{\Omega}_\pi(s^{-1}\mathfrak{g}^\vee), \Omega_\bullet\right) \cong \hom(s^{-1}\mathfrak{g}^\vee, \Omega_\bullet)^0$$

even though $\Omega_\bullet$ is not complete. Finally, in the third line we used our Corollary 10.3.3. □

10.4 A model for Maurer–Cartan elements of homotopy Lie algebras

One would like to do the same as in Section 10.3 for $\mathfrak{sL}_\infty$-algebras. It is possible to do so, but one critically has to use the generalized homotopy transfer theorem — Theorem 9.3.1 — and consider the operad $\Omega\mathrm{BCom}$ instead of $\mathscr{C}_\infty$ as cofibrant resolution of the operad Com. Doing this, we obtain a new cosimplicial $\mathfrak{sL}_\infty$-algebra $\mathfrak{mc}_\bullet^\infty$ such that

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \hom_{\mathfrak{sL}_\infty\text{-alg}}(\mathfrak{mc}_\bullet^\infty, \mathfrak{g})$$

for any complete proper $\mathfrak{sL}_\infty$-algebra $\mathfrak{g}$. There is a natural morphism

$$\rho : \mathfrak{mc}_\bullet^\infty \longrightarrow \mathfrak{mc}_\bullet$$

through which every morphism of $\mathfrak{sL}_\infty$-algebras from $\mathfrak{mc}_\bullet^\infty$ to a strict Lie algebra splits in a canonical way, showing coherence with the results of Section 10.3. We will explicit $\mathfrak{mc}_1^\infty$, which is a higher analogue of the Lawrence–Sullivan algebra in the context of homotopy Lie algebras. The work presented in this section is extracted from [RNV].

10.4.1 Representing $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$

Let $\mathfrak{g}$ be an $\mathfrak{sL}_\infty$-algebra. If one writes down the first operations for $\mathfrak{g} \otimes C_\bullet$ with the $\mathfrak{sL}_\infty$-algebra structure obtained through homotopy transfer along the contraction

$$1 \otimes h_\bullet \circlearrowright \mathfrak{g} \otimes \Omega_\bullet \underset{1 \otimes i_\bullet}{\overset{1 \otimes p_\bullet}{\rightleftarrows}} \mathfrak{g} \otimes C_\bullet$$

and compares it with the the structure of the operations when $\mathfrak{g}$ is a strict (shifted) Lie algebra, one immediately realizes that there are many more trees appearing in the former case than in the latter. This hints to the fact that we need a finer algebraic structure on $C_\bullet$ than the one of a $\mathscr{C}_\infty$-algebra we obtained by homotopy transfer in Section 10.3.1.

In order to obtain a good model, we consider the resolution of the operad Com given by $\Omega\mathrm{BCom}$, and denote by

$$\pi^\infty : \mathrm{B}(\mathfrak{sL}_\infty) \longrightarrow \mathfrak{sL}_\infty$$

the canonical twisting morphism. Notice that $\mathrm{B}(\mathfrak{sL}_\infty) \cong (\Omega\mathrm{BCom})^\vee$. In this section, we reserve the letter π for the canonical twisting morphism

$$\pi : \mathrm{B}(\mathscr{S} \otimes \mathrm{Lie}) \longrightarrow \mathscr{S} \otimes \mathrm{Lie}\ ,$$

which played the same role in Section 10.3 as π^∞ will in this section.

In Dupont's contraction

$$h_\bullet \circlearrowright \Omega_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C_\bullet$$

we see the simplicial commutative algebra $\Omega_\bullet$ as a simplicial $\Omega\mathrm{BCom}$-algebra. Then the generalized homotopy transfer theorem gives us a $\Omega\mathrm{BCom}$-algebra structure on $C_\bullet$. We will denote the chain complex $C_\bullet$ endowed with this $\Omega\mathrm{BCom}$-algebra structure by $\widetilde{C}_\bullet$. Since — as already mentioned above — we have

$$\mathrm{B}(\mathfrak{s}\mathscr{L}_\infty) \cong (\Omega\mathrm{BCom})^\vee,$$

we can take the cobar construction of $\widetilde{C}_\bullet^\vee$ relative to the twisting morphism π^∞ to obtain a cosimplicial $\mathfrak{s}\mathscr{L}_\infty$-algebra.

Definition 10.4.1. *We denote the cosimplicial shifted homotopy Lie algebra obtained this way by* $\mathfrak{mc}_\bullet^\infty := \widehat{\Omega}_{\pi^\infty}(\widetilde{C}_\bullet^\vee)$.

We immediately recover similar results as in Section 10.3.1.

Theorem 10.4.2. *Let* $\mathfrak{g}$ *be a proper complete* $\mathfrak{s}\mathscr{L}_\infty$*-algebra. There is a canonical isomorphism of simplicial sets*

$$\mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \cong \hom_{\mathfrak{s}\mathscr{L}_\infty\text{-alg}}(\mathfrak{mc}_\bullet^\infty, \mathfrak{g})\ .$$

It is natural in $\mathfrak{g}$.

Proof. Analogous to the proof of Theorem 10.3.2, using π^∞ instead of π. □

Corollary 10.4.3. *Let* $\mathfrak{g}$ *be a proper complete* $\mathfrak{s}\mathscr{L}_\infty$*-algebra. There is a weak equivalence of simplicial sets*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \hom_{\mathfrak{s}\mathscr{L}_\infty\text{-alg}}(\mathfrak{mc}_\bullet^\infty, \mathfrak{g})\ .$$

It is natural in $\mathfrak{g}$.

10.4.2 Compatibility with the strict case

Since Maurer–Cartan elements, gauges and so on for strict shifted Lie algebras are the same as the respective notions as for the same algebras seen as $\mathfrak{s}\mathscr{L}_\infty$-algebras, one expects some compatibility between the cosimplicial algebras $\mathfrak{mc}_\bullet$ and $\mathfrak{mc}_\bullet^\infty$.

There is a canonical morphism of operads

$$g_\kappa : \mathfrak{s}\mathscr{L}_\infty \longrightarrow \mathscr{S} \otimes \mathrm{Lie}\ ,$$

obtained by applying Theorem 2.2.18 to the canonical twisting morphism

$$\kappa : (\mathscr{S} \otimes \mathrm{Lie})^{\mathsf{i}} \cong \mathscr{S}^c \otimes \mathrm{Lie}^{\mathsf{i}} \longrightarrow \mathscr{S} \otimes \mathrm{Lie}$$

given by Koszul duality. It induces a map

$$\rho : \mathfrak{s}\mathscr{L}_\infty \circ C_\bullet^\vee \xrightarrow{g_\kappa \circ 1_{C_\bullet^\vee}} (\mathscr{S} \otimes \mathrm{Lie}) \circ C_\bullet^\vee$$

of chain complexes.

Lemma 10.4.4. *The map described above is a filtered morphism*

$$\rho : \mathfrak{mc}_\bullet^\infty \longrightarrow g_\kappa^* \mathfrak{mc}_\bullet$$

of cosimplicial $\mathfrak{s}\mathscr{L}_\infty$-algebras.

Proof. The fact that the map commutes with the algebraic structure is trivial, as well as the fact that it preserves the filtration, and that it respects the cosimplicial structure. In order to see that it also commutes with differentials, we begin by noticing that by Proposition 9.3.2 we have an equality of $\mathscr{C}_\infty$-algebras

$$C_\bullet = (\Omega f_\kappa)^* \widetilde{C}_\bullet \, ,$$

where

$$f_\kappa : \mathrm{Com}^! \longrightarrow \mathrm{BCom}$$

is the canonical map. The dual of f_κ is the map of operads

$$g_\kappa : \mathfrak{s}\mathscr{L}_\infty \longrightarrow \mathscr{S} \otimes \mathrm{Lie}$$

mentioned above. Therefore, we have the diagram

$$\begin{array}{ccccc}
\widetilde{C}_\bullet^\vee & \xrightarrow{\Delta_{\widetilde{C}_\bullet^\vee}} & \mathrm{B}(\mathfrak{s}\mathscr{L}_\infty) \circ \widetilde{C}_\bullet^\vee & \xrightarrow{\pi^\infty \circ 1} & \mathfrak{s}\mathscr{L}_\infty \circ \widetilde{C}_\bullet^\vee \\
\| & & \downarrow{\scriptstyle \mathrm{B}g_\kappa \circ 1} & & \downarrow{\scriptstyle g_\kappa \circ 1} \\
C_\bullet^\vee & \xrightarrow{\Delta_{C_\bullet^\vee}} & \mathrm{B}(\mathscr{S} \otimes \mathrm{Lie}) \circ C_\bullet^\vee & \xrightarrow{\pi \circ 1} & (\mathscr{S} \otimes \mathrm{Lie}) \circ C_\bullet^\vee
\end{array}$$

which proves that ρ commutes with the differentials of $\mathfrak{mc}_\bullet$ and $\mathfrak{mc}_\bullet^\infty$ (which are completely determined by the horizontal lines of the diagram above). □

Remark 10.4.5. *One would like to show that ρ is a filtered quasi-isomorphism, so that $\mathfrak{mc}_\bullet^\infty$ would be a resolution of $\mathfrak{mc}_\bullet$. We do not have a proof of this fact for the moment.*

Proposition 10.4.6. *Let $\mathfrak{g}$ be a shifted Lie algebra. Every morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras $\mathfrak{mc}_\bullet^\infty \to g_\kappa^* \mathfrak{g}$ splits in a unique way through ρ. In other words, the pullback*

$$\rho^* : \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \cong \hom_{\mathfrak{s}\mathscr{L}_\infty\text{-}\mathsf{alg}}(g_\kappa^* \mathfrak{mc}_\bullet, g_\kappa^* \mathfrak{g}) \longrightarrow \hom_{\mathfrak{s}\mathscr{L}_\infty\text{-}\mathsf{alg}}(\mathfrak{mc}_\bullet^\infty, g_\kappa^* \mathfrak{g})$$

is an isomorphism of simplicial sets (the first isomorphism being given by the fact that shifted Lie algebras form a full subcategory of $\mathfrak{s}\mathscr{L}_\infty$-algebras).

Proof. First, one notice that

$$\mathfrak{g} \otimes^\pi C_\bullet = (g_\kappa^* \mathfrak{g}) \otimes^{\pi^\infty} \widetilde{C}_\bullet$$

by the compatibility with morphisms expressed in Theorem 9.1.12. The proof is then given by the following commutative diagram.

$$\begin{array}{ccc} \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) & \xrightarrow{\sim} & \mathrm{MC}(\mathfrak{g} \otimes^\pi C_\bullet) \\ \downarrow{\scriptstyle \rho^*} & & \| \wr \\ \hom_{\mathfrak{sL}_\infty\text{-}\mathsf{alg}}(\mathfrak{mc}_\bullet^\infty, g_\kappa^*\mathfrak{g}) & \xrightarrow{\sim} & \mathrm{MC}((g_\kappa^*\mathfrak{g}) \otimes^{\pi^\infty} \widetilde{C}_\bullet) \end{array}$$

The horizontal arrows are given by Theorem 10.3.2 and Theorem 10.4.2 respectively, while proving that the vertical arrow is given by pullback by ρ is just a matter of unwinding definitions. □

10.4.3 A higher version of the Lawrence–Sullivan algebra

We want now to give explicit formulæ for the first two levels of $\mathfrak{mc}_\bullet^\infty$. The same arguments as for Proposition 10.3.5 apply, and thus we immediately have the following.

Lemma 10.4.7. *We have*

$$\mathfrak{mc}_0^\infty = \widehat{\mathfrak{sL}_\infty}(\mathbb{K}\alpha)$$

with $|\alpha| = 0$ *and*

$$d\alpha = -\sum_{n\geq 2} \frac{1}{n!} \ell_n(\alpha, \dots, \alpha) \,.$$

In other words, $\mathfrak{mc}_0^\infty$ *is the free* $\mathfrak{sL}_\infty$*-algebra generated by a single Maurer–Cartan element.*

We also know that $\mathfrak{mc}_1^\infty$ is the free $\mathfrak{sL}_\infty$-algebra generated by two Maurer–Cartan elements α_0, α_1 in degree 0, and a gauge λ from α_0 to α_1 in degree 1. In order to translate this into explicit formulæ we will need to do some work. Our starting point is Section 6.2.4. From there, we can deduce the formula

$$\alpha_1 = \sum_{\tau \in \mathrm{PT}} \frac{1}{F(\tau)} \tau(\alpha_0)$$

using Proposition C.2.5 to the formal differential equation with

$$f_{n,1}(y_1, \dots, y_n) \coloneqq \frac{1}{n!} \ell_{n+1}(y_1, \dots, y_n, \lambda)$$

for $n \geq 1$, $f_{0,1} \coloneqq d\lambda$, and no other operators. Notice that the only weight appearing in the trees is 1, so that we simply work over PT. Rearranging terms, we obtain the fixed-point equation

$$d\lambda = \alpha_1 - \alpha_0 - \sum_{\tau \in \mathrm{PT} \setminus \{\emptyset, c_0\}} \frac{1}{F(\tau)} \tau(\alpha_0) \,,$$

where one should notice that $d\lambda$ appears in the right-hand side whenever a tree contains a 0-corolla. The existence and uniqueness of a solution is thus guaranteed by Theorem C.1.3. We can however give an explicit formula for this solution in this case.

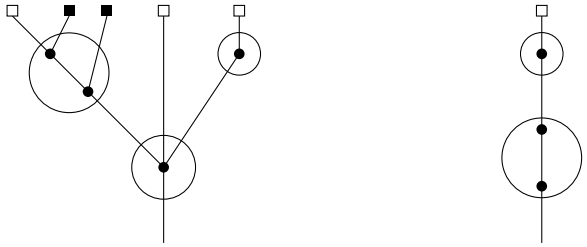
We will denote by $\widetilde{\mathrm{PT}}$ the following set of trees. An element of $\widetilde{\mathrm{PT}}$ is a non-empty[2] planar rooted tree T such that

1. every vertex of T has arity at least 1,

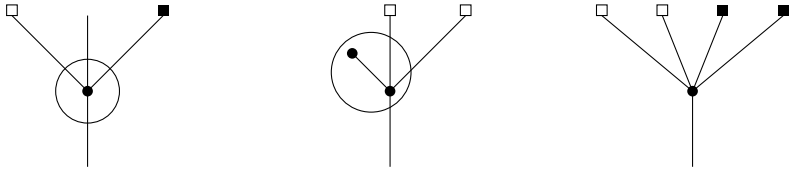
[2] Meaning that their set of vertices is non-empty.

2. every leaf of T is labeled by either “black" or “white", and
3. every vertex v of T, say of arity k, is labeled by a planar tree τ_v of arity k and whose vertices all have arity at least 1.

Here are some examples of such trees:

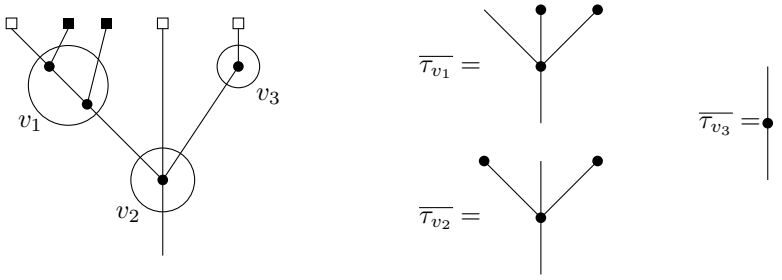


and some non-examples:



The trees given by a single black or white leaf — which we will denote by ▪ and ▫ respectively — are not in $\widetilde{\mathrm{PT}}$, since they don't have any vertex.

Let $T \in \widetilde{\mathrm{PT}}$, and let v be a vertex of T of arity n with associated planar tree τ_v. Then we denote by $\overline{\tau_v}$ the planar tree obtained by taking τ_v and composing a 0-corolla at every leaf that is linked to either another vertex of T or to a black leaf of T. For example,



We define a function

$$G : \widetilde{\mathrm{PT}} \longrightarrow \mathbb{K}$$

by

$$G(T) \coloneqq \prod_{v \in V_T} -\frac{1}{F(\overline{\tau_v})} \,.$$

To a tree $T \in \widetilde{\mathrm{PT}}$ we also associate a function

$$T : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

recursively by

$$T(x, y) \coloneqq \tau_r(T_1(x, y), \ldots, T_k(x, y))$$

whenever $T = \tau_r \circ (T_1, \dots, T_k)$, and by setting $\bar{\bullet}(x, y) \coloneqq x$, and $\bullet(x, y) \coloneqq y$, even though the trees given by a single white or black leaf are not in $\widetilde{\mathrm{PT}}$, strictly speaking.

Proposition 10.4.8. *The $\mathscr{L}_\infty$-algebra $\mathfrak{mc}_1^\infty$ is the free complete $\mathscr{L}_\infty$-algebra generated by two degree 1 elements α_0 and α_1 and a single degree 0 element λ satisfying*

$$d\lambda = \alpha_1 - \alpha_0 + \sum_{T \in \widetilde{\mathrm{PT}}} G(T)\, T(\alpha_0, \alpha_1 - \alpha_0)\,.$$

This $\mathfrak{s}\mathscr{L}_\infty$-algebra is a higher analogue of the Lawrence–Sulliven algebra [LS14], which is a Lie model for the 1-simplex, cf. also Proposition 10.3.5.

Proof. We have to prove that this $d\lambda$ is the unique solution to the fixed-point equation

$$d\lambda = \alpha_1 - \alpha_0 - \sum_{\tau \in \mathrm{PT} \setminus \{\emptyset, c_0\}} \frac{1}{F(\tau)} \tau(\alpha_0)\,,$$

obtained above. Notice that $d\lambda$ appears in the right-hand side whenever there is a 0-corolla in a tree. The fact that the solution exists and is unique is given by Theorem C.1.3.

We have

$$\begin{aligned}
d\lambda &= \alpha_1 - \alpha_0 + \sum_{T \in \widetilde{\mathrm{PT}}} G(T)\, T(\alpha_0, \alpha_1 - \alpha_0) \\
&= \alpha_1 - \alpha_0 - \sum_{\substack{k \geq 1,\ \tau_r \in \mathrm{PT}_k \text{ reduced} \\ T_1, \dots, T_k \in \widetilde{\mathrm{PT}} \cup \{\bar{\bullet}, \bullet\}}} \frac{1}{F(\overline{\tau_r})} \left(\prod_{i=1}^{k} G(T_i) \right) \tau_r(T_1(\alpha_0, \alpha_1 - \alpha_0), \dots, T_k(\alpha_0, \alpha_1 - \alpha_0)) \\
&= \alpha_1 - \alpha_0 - \\
&\quad - \sum_{\substack{k \geq 1, \\ \tau_r \in \widetilde{\mathrm{PT}}_k \text{ red.}}} \frac{1}{F(\overline{\tau_r})} \tau_r \left(\sum_{T_1 \in \widetilde{\mathrm{PT}} \cup \{\bar{\bullet}, \bullet\}} G(T_1)\, T_1(\alpha_0, \alpha_1 - \alpha_0), \dots, \sum_{T_k \in \widetilde{\mathrm{PT}} \cup \{\bar{\bullet}, \bullet\}} G(T_k)\, T_k(\alpha_0, \alpha_1 - \alpha_0) \right) \\
&= \alpha_1 - \alpha_0 - \sum_{\substack{k \geq 1, \\ \tau_r \in \widetilde{\mathrm{PT}}_k \text{ reduced}}} \frac{1}{F(\overline{\tau_r})} \tau_r\,(\alpha_0 + d\lambda, \dots, \alpha_0 + d\lambda) \\
&= \alpha_1 - \alpha_0 - \sum_{\tau \in \mathrm{PT} \setminus \{\emptyset, c_0\}} \frac{1}{F(\tau)} \tau\,(\alpha_0)
\end{aligned}$$

where a tree is *reduced* if it is non-empty and has no arity 0 vertices, in the second line we have $T = \tau_r \circ (T_1, \dots, T_k)$, and $G(\bar{\bullet}) = G(\bullet) \coloneqq 1$. □

10.4.4 Homotopies between ∞-morphisms of $\mathfrak{s}\mathscr{L}_\infty$-algebras

We sketch how, in principle, one can use the formulæ for $\mathfrak{mc}_1^\infty$ given above to describe an explicit notion of homotopy between ∞-morphisms of $\mathfrak{s}\mathscr{L}_\infty$-algebras. More details will appear in [RNV].

Let $\mathfrak{g}$ be a $\mathfrak{s}\mathscr{L}_\infty$-algebra, and consider the contraction

$$1 \otimes h \circlearrowright \mathfrak{g} \otimes \Omega_1 \underset{1 \otimes i}{\overset{1 \otimes p}{\rightleftarrows}} \mathfrak{g} \otimes C_1$$

induced by Dupont's contraction. Endow $\mathfrak{g} \otimes C_1$ with the $\mathfrak{sL}_\infty$-algebra structure obtained via the homotopy transfer theorem.

Proposition 10.4.9 ([Val14, Prop. 3.3])**.** *The $\mathfrak{sL}_\infty$-algebra $\mathfrak{g} \otimes C_1$ is a cylinder for $\mathfrak{g}$ in the category of $\mathfrak{sL}_\infty$-algebras with their ∞-morphisms. In other words, the cocommutative coalgebra $\mathrm{B}_\iota(\mathfrak{g} \otimes C_1)$ is a cylinder for $\mathrm{B}_\iota\mathfrak{g}$.*

But by Theorem 9.3.6, $\mathfrak{g} \otimes C_1$ is the same thing as $\mathfrak{g} \otimes^{\pi^\infty} \widetilde{C}_1$, where $\widetilde{C}_1$ is C_1 the $\Omega\mathrm{BCom}$-algebra structure obtained as in Section 10.4.1. Therefore, if $\Phi, \Psi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ are two ∞-morphisms, then a homotopy between them is an ∞-morphism

$$H : \mathfrak{g} \otimes^{\pi^\infty} \widetilde{C}_1 \rightsquigarrow \mathfrak{h} \, ,$$

which translates into a certain collection of morphisms $\mathfrak{g}^{\otimes n} \to \mathfrak{h}$, of which two extremal subcollections give back Φ and Ψ, and which must respect certain compatibilities encoded by the algebraic structure of $\widetilde{C}_1$. Moreover, one can give an explicit description of the algebraic structure of $\widetilde{C}_1$, which is essentially dual to the $\mathfrak{sL}_\infty$-algebra structure of $\mathfrak{mc}_1^\infty$, using Proposition 10.4.8.

Chapter 11

Rational models for mapping spaces

In this chapter, we present an application of the theory developed in Chapter 9 to rational homotopy theory, using it to generalize a result of Berglund [Ber15]. Most of the material presented here is extracted from [RNW17].

Another application of the methods of Chapter 9 to rational homotopy theory, but which is outside of the scope of the present work, is given in [Wie16] and [Wie17], where F. Wierstra uses them to construct a complete invariant of th real or rational homotopy classes of maps between simply connected manifolds.

In this chapter, we work exclusively over the field $\mathbb{Q}$ of rational numbers.

11.1 Homotopy Lie and homotopy cocommutative models

In this section, we define homotopy Lie and homotopy commutative models for spaces, and define certain additional conditions we will impose on the homotopy Lie models.

11.1.1 Conditions on $\mathfrak{s}\mathscr{L}_\infty$-algebras

Later, we will consider complete proper $\mathfrak{s}\mathscr{L}_\infty$-algebras satisfying some additional finiteness conditions, which we introduce here.

Definition 11.1.1. *A proper complete $\mathfrak{s}\mathscr{L}_\infty$-algebra $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ is* locally finite *if all of the quotients*

$$\mathfrak{g}^{(n)} \coloneqq \mathfrak{g}/\mathscr{F}_n\mathfrak{g}\,, \qquad \text{for } n \geq 2\,,$$

are finite dimensional.

Being locally finite is a good generalization of being finite dimensional, as the following result shows.

Lemma 11.1.2. *Let $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ be a proper complete $\mathscr{L}_\infty$-algebra, and let C be a cocommutative coalgebra. Suppose that either*

1. *the cocommutative coalgebra C is finite dimensional, or*
2. *the proper complete $\mathfrak{s}\mathscr{L}_\infty$-algebra $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ is locally finite.*

Then we have an isomorphism

$$\mathfrak{g}\widehat{\otimes}C^\vee \cong \hom^\iota(C,\mathfrak{g})$$

of $\mathfrak{sL}_\infty$-algebras, where $\iota : \mathrm{Com}^\vee \to \mathfrak{sL}_\infty$ is the canonical twisting morphism.

Here, we use the notation

$$\mathfrak{g}\widehat{\otimes}C^\vee \coloneqq \lim_n \left(\mathfrak{g}/\mathscr{F}_n\mathfrak{g}\otimes C^\vee\right) .$$

It can be seen as a tensor product between filtered algebras, where we have endowed $C^\vee$ with the constant filtration $\mathscr{F}_nC^\vee = C^\vee$.

Proof. The first case is straightforward, so we only give some details for the second one. First begin by considering the case where $\mathfrak{g}$ is finite dimensional. If we fix a homogeneous basis $\{x_i\}_i$ of $\mathfrak{g}$, then we obtain an isomorphism

$$\hom^\iota(C,\mathfrak{g}) \longrightarrow \mathfrak{g}\otimes^\iota C^\vee$$

by sending

$$\phi \longmapsto \sum_i x_i\otimes(x_i^\vee\phi) .$$

It is a straightforward exercise to check that this is independent of the chosen basis, and to see that the isomorphism holds true at the level of $\mathfrak{sL}_\infty$-algebras.

Now if $\mathfrak{g}$ is not necessarily finite dimensional, but only locally finite, we have

$$\begin{aligned}\hom^\iota(C,\mathfrak{g}) &\cong \hom^\iota(C,\lim_n \mathfrak{g}/\mathscr{F}_n\mathfrak{g})\\ &\cong \lim_n \hom^\iota(C,\mathfrak{g}/\mathscr{F}_n\mathfrak{g})\\ &\cong \lim_n \left(\mathfrak{g}/\mathscr{F}_n\mathfrak{g}\otimes C^\vee\right)\\ &= \mathfrak{g}\widehat{\otimes}C^\vee ,\end{aligned}$$

where the fact that the second isomorphism holds at the level of $\mathfrak{sL}_\infty$-algebras is straightforward to check, and in the third line we used the fact that $\mathfrak{g}^{(n)}$ is finite dimensional for all n in order to apply what said above. □

There is another condition we will impose on some of our $\mathfrak{sL}_\infty$-algebras. It was first introduced in [Ber15].

Definition 11.1.3. *Let $(\mathfrak{g},\mathscr{F}_\bullet\mathfrak{g})$ be a proper complete $\mathfrak{sL}_\infty$-algebra. We say that $(\mathfrak{g},\mathscr{F}_\bullet\mathfrak{g})$ is* degree-wise nilpotent *if for any $n\in\mathbb{Z}$ there is a $k\geq 1$ such that $(\mathscr{F}_k\mathfrak{g})_n = 0$.*

The functor $\mathrm{MC}_\bullet$ acts in a very straightforward manner on degree-wise nilpotent filtered $\mathscr{L}_\infty$-algebras satisfying a boundedness condition with respect to the homological degree, as the following result demonstrates.

Proposition 11.1.4. *Let $(\mathfrak{g},\mathscr{F}_\bullet\mathfrak{g})$ be a degree-wise nilpotent proper complete $\mathfrak{sL}_\infty$-algebra, and suppose that the degrees in which $\mathfrak{g}$ is non-zero are bounded below. Then*

$$\mathrm{MC}_\bullet(\mathfrak{g},\mathscr{F}\mathfrak{g}) \cong \mathrm{MC}(\mathfrak{g}\otimes\Omega_\bullet) .$$

In particular, $\mathrm{MC}_\bullet(\mathfrak{g},\mathscr{F}_\bullet\mathfrak{g})$ is independent of the filtration $\mathscr{F}_\bullet\mathfrak{g}$, as long as $(\mathfrak{g},\mathscr{F}_\bullet\mathfrak{g})$ is degree-wise nilpotent.

Proof. Suppose that $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ satisfies the assumptions above. Fix $n \geq 0$, then there exists $k_0 \geq 1$ such that $(\mathscr{F}_{k_0}\mathfrak{g})_m = 0$ for all $m \leq n$, since the degrees of $\mathfrak{g}$ are bounded below. It follows that

$$(\mathscr{F}_k\mathfrak{g} \otimes \Omega_n)_0 = 0 \qquad \text{and} \qquad (\mathscr{F}_k\mathfrak{g} \otimes \Omega_n)_{-1} = 0$$

for all $k \geq k_0$, as Ω_n is concentrated in degrees from $-n$ up to 0. The projection

$$\mathfrak{g} \otimes \Omega_n \longrightarrow \mathfrak{g}/\mathscr{F}_k\mathfrak{g} \otimes \Omega_n$$

has kernel $\mathscr{F}_k\mathfrak{g} \otimes \Omega_n$, and is therefore an isomorphism in degrees 0 and -1. Therefore, the set of Maurer–Cartan elements $\mathrm{MC}(\mathfrak{g}/\mathscr{F}_k\mathfrak{g} \otimes \Omega_n)$ is constant in k for $k \geq k_0$, which implies the statement. □

For any $\mathfrak{sL}_\infty$-algebra $\mathfrak{g}$, one can consider the filtration $\mathscr{F}^{\mathrm{can}}\mathfrak{g}$ which at level n is given by the elements that can be obtained by bracketing at least n elements of $\mathfrak{g}$.

Lemma 11.1.5. *Let $\mathfrak{g}$ be a simply-connected, proper complete $\mathfrak{sL}_\infty$-algebra. Then $(\mathfrak{g}, \mathscr{F}^{\mathrm{can}}\mathfrak{g})$ is degree-wise nilpotent.*

Proof. Suppose we take k elements and we bracket them together. We can use at most $k-1$ brackets (taking only binary brackets), and every element has degree at least 2. It follows that the resulting element has degree at least

$$(1-k) + 2k = k + 1 \,.$$

Thus,

$$\mathscr{F}^{\mathrm{can}}_k\mathfrak{g} \subseteq \mathfrak{g}_{\geq k+1} \,,$$

and the statement follows. □

11.1.2 Homotopy Lie models

One possible generalization of the Lie rational models for spaces of Section 5.3.2 is to use homotopy Lie algebras. For coherence, we work in the shifted setting.

Definition 11.1.6. *Let $K \in \mathsf{sSets}_1$ be a 1-reduced simplicial set. A proper complete $\mathfrak{sL}_\infty$-algebra $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ is a* rational model *for K if there is a homotopy equivalence*

$$\mathrm{MC}_\bullet(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g}) \simeq K_{\mathbb{Q}} \,,$$

where $K_{\mathbb{Q}}$ is the rationalization of K, cf. Theorem 5.1.13.

We will require that our $\mathfrak{sL}_\infty$-models are locally finite and degree-wise nilpotent. This assumption is needed e.g. for Theorem 11.2.1 to hold. This condition is not that strong, as we can model many spaces of interest to us through such algebras.

Proposition 11.1.7. *Every 1-reduced simplicial set K with only finitely many non-degenerate simplices admits a locally finite, degree-wise nilpotent $\mathfrak{sL}_\infty$-model.*

Proof. According to [FHT01, Sect. 24.(e)], every such simplicial set K admits a free Lie model — in the sense of the Definition found at [FHT01, p.322] — of the form $\mathfrak{g} = \mathrm{Lie}(C_\bullet(X))$, where $C_\bullet(X)$ is the complex of simplicial chains of K. Since we supposed that K is finite, $\mathfrak{g}$ is a finitely generated Lie algebra, and thus it satisfies the assumptions of [Ber15, Prop. 6.1]. It follows that we have

$$\mathrm{MC}_\bullet(\mathfrak{g} \otimes \Omega_\bullet) \simeq K_{\mathbb{Q}} \,.$$

An apparent problem is the fact that $\mathfrak{g}$ is not complete. However, we can replace $\mathfrak{g}$ by

$$\widehat{\mathfrak{g}} := \widehat{\mathrm{Lie}}(C_\bullet(X)) = \prod_{n\geq 1} \mathrm{Lie}(n) \otimes_{\mathbb{S}_n} C_\bullet(X)^{\otimes n} .$$

Then Proposition 11.1.4 and Lemma 11.1.5 give

$$\mathrm{MC}_\bullet(\mathfrak{g} \otimes \Omega_\bullet) \cong \mathrm{MC}_\bullet(\widehat{\mathfrak{g}}, \mathscr{F}^{can}\widehat{\mathfrak{g}}) ,$$

so that we may use $(\widehat{\mathfrak{g}}, \mathscr{F}^{\mathrm{can}}\widehat{\mathfrak{g}})$ as a $\mathscr{L}_\infty$-model. This is obviously locally finite, since the operad Lie is finite dimensional in every arity and $C_\bullet(X)$ is finite dimensional, and degree-wise nilpotent e.g. by Lemma 11.1.5. Now suspend $\mathfrak{g}$ to obtain an $\mathfrak{s}\mathscr{L}_\infty$-model satisfying the desired properties. □

11.1.3 Homotopy cocommutative models

By a cocommutative coalgebra up to homotopy, we mean a conilpotent coalgebra over the cooperad $\mathrm{B}\Omega\mathrm{Com}^\vee$. Other resolutions of the cooperad $\mathrm{Com}^\vee$ are possible, but the chosen one has the advantage to provide the canonical commuting diagram

$$\begin{array}{ccc} \mathrm{Com}^\vee & \xrightarrow{\quad\iota\quad} & \\ {\scriptstyle f_\iota}\downarrow & & \\ \mathrm{B}\Omega\mathrm{Com}^\vee & \xrightarrow{\quad\pi\quad} & \mathscr{L}_\infty \end{array}$$

where the quasi-isomorphism f_ι is obtained by Theorem 2.2.18 — it is in fact the unit of the bar-cobar adjunction — and where all twisting morphisms are Koszul.

Definition 11.1.8. *Let X be a simply-connected, pointed topological space. A homotopy cocommutative coalgebra C is a* rational model *for X if there exists a zig-zag of weak equivalences of homotopy cocommutative algebras*

$$(f_\iota)_* s\mathrm{B}_\kappa\lambda(X) \longrightarrow \bullet \longleftarrow \cdots \longrightarrow \bullet \longleftarrow C ,$$

or equivalently (by Theorem 8.2.6) if there exists a π-weak equivalence $(f_\iota)_* s\mathrm{B}_\kappa\lambda(X) \rightsquigarrow C$.

11.2 Rational models for mapping spaces

Given two spaces K and L, a natural question is the following one. Suppose we are given rational models for both K and L. Is it possible to use them to construct a rational model of the mapping space $\mathrm{Map}(K, L)$? A possible answer to this question was given by Berglund [Ber15] in the case when we have a strictly commutative model for the first space, and an $\mathfrak{s}\mathscr{L}_\infty$-model for the second one.

Theorem 11.2.1 ([Ber15] Theorem 6.3)**.** *Let K be a simply-connected simplicial set, let L be a nilpotent space (e.g. a simply-connected space) of finite $\mathbb{Q}$-type and $L_\mathbb{Q}$ the rationalization of L. Let A be a commutative model for K and $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ a degree-wise nilpotent, locally finite $\mathfrak{s}\mathscr{L}_\infty$-model of finite type for L. There is a homotopy equivalence of simplicial sets*

$$\mathrm{Map}(K, L_\mathbb{Q}) \simeq \mathrm{MC}_\bullet(A\widehat{\otimes}\mathfrak{g}),$$

i.e. the $\mathfrak{s}\mathscr{L}_\infty$-algebra $A\widehat{\otimes}\mathfrak{g}$ is a $\mathfrak{s}\mathscr{L}_\infty$-model for the mapping space.

Remark 11.2.2. *In [Ber15], this theorem is stated in terms of the Getzler ∞-groupoid $\gamma_\bullet(\mathfrak{g})$. However, the ∞-groupoid $\gamma_\bullet(\mathfrak{g})$ is homotopy equivalent to $\mathrm{MC}_\bullet(\mathfrak{g})$ by Theorem 6.2.13, and thus the statement above is equivalent to the original one. Also notice that we supposed that $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ is locally finite, and completed the tensor product with respect to the filtration $\mathscr{F}_\bullet\mathfrak{g}$ and not with the degree filtration, as in [Ber15]. An inspection of the original proof reveals that the result still holds in this slightly more general context.*

We will now improve Berglund's Theorem in two ways: we will show that we can take homotopy cocommutative coalgebra models for K instead of just cocommutative ones, and that this model is natural with respect to ∞_π-quasi-isomorphisms of $\mathfrak{s}\mathscr{L}_\infty$-algebras, respectively π-weak equivalences of homotopy cocommutative coalgebras. We will also show that, under certain restrictions on C and $\mathfrak{g}$, this model only depends on the homotopy types of C and $\mathfrak{g}$, i.e. different choices for C and $\mathfrak{g}$ will give homotopy equivalent models for the mapping space. The first result is the following one.

Lemma 11.2.3. *Let K be a simply-connected simplicial set, let L be a nilpotent space (e.g. a simply-connected space) of finite $\mathbb{Q}$-type and $L_\mathbb{Q}$ the $\mathbb{Q}$-localization of L. Let C be a cocommutative model for K and $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ a degree-wise nilpotent, locally finite $\mathfrak{s}\mathscr{L}_\infty$-model of finite type for L. There is a homotopy equivalence of simplicial sets*

$$Map_*(K, L_\mathbb{Q}) \simeq MC_\bullet(\hom^\pi((f_\iota)_*C, \mathfrak{g})),$$

i.e. the convolution $\mathfrak{s}\mathscr{L}_\infty$-algebra $\hom^\pi((f_\iota)_*C, \mathfrak{g})$ is an $\mathscr{L}_\infty$-model for the mapping space.

Proof. By Theorem 5.3.10 and Theorem 11.2.1, we know that $\mathfrak{g} \otimes C^\vee$ is an $\mathscr{L}_\infty$-model for the mapping space. Further, by Lemma 11.1.2 we know that

$$\mathfrak{g}\widehat{\otimes}C^\vee \cong \hom^\iota(C, \mathfrak{g}) = \hom^\pi((f_\iota)_*C, \mathfrak{g}) \,,$$

where the second equality is Lemma 9.1.7. □

Proposition 11.2.4. *Let C be a homotopy cocommutative coalgebra, and let*

$$\Psi : (\mathfrak{h}, \mathscr{F}_\bullet\mathfrak{h}) \rightsquigarrow (\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g}) \,.$$

be a filtered ∞-morphism of $\mathfrak{s}\mathscr{L}_\infty$-algebras. Then there is a weak equivalence of simplicial sets

$$\mathrm{MC}_\bullet(\hom^\pi(C, \mathfrak{h})) \simeq \mathrm{MC}_\bullet(\hom^\pi(C, \mathfrak{g})) \,.$$

Proof. This looks very similar to Corollary 9.2.19 — and indeed, that result is a fundamental ingredient of the proof — but notice that here we have ∞-morphisms of $\mathfrak{s}\mathscr{L}_\infty$-algebras, i.e. ∞_ι-morphisms, instead of ∞_π-morphisms.

The proof is schematized by the following diagram.

$$\begin{array}{ccc}
\hom^\pi(C, \mathfrak{h}) & & \hom^\pi(C, \mathfrak{g}) \\
\uparrow & & \uparrow \\
\hom^\pi(E_D, 1) & & \hom^\pi(E_D, 1) \\
\uparrow & & \uparrow \\
\hom^\pi(R_{\pi,f_\iota}(C), \mathfrak{h}) & & \hom^\pi(R_{\pi,f_\iota}(C), \mathfrak{g}) \\
\| & & \| \\
\hom^\iota(\mathrm{B}_\iota\Omega_\pi C, \mathfrak{h}) & \xrightarrow{\hom^\iota(1,\Psi)} & \hom^\iota(\mathrm{B}_\iota\Omega_\pi C, \mathfrak{g})
\end{array}$$

The vertical equalities are given by Lemma 9.1.7 and the definition of the rectification R_{π,f_ι}. We apply the functor $\mathrm{MC}_\bullet$ on the whole diagram, and all the squiggly arrows become homotopy equivalences of simplicial sets by Corollary 9.2.19 and Proposition 8.2.4. The result follows. □

Our generalization of Theorem 11.2.1 is a direct consequence of this result.

Theorem 11.2.5. *Let K be a simply-connected simplicial set, let L be a nilpotent space (e.g. a simply-connected space) of finite $\mathbb{Q}$-type and $L_\mathbb{Q}$ the $\mathbb{Q}$-localization of L. Let C be a homotopy cocommutative model for K, let $(\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g})$ be a degree-wise nilpotent, locally finite $\mathfrak{sL}_\infty$-model of finite type for L, and let $(\mathfrak{h}, \mathscr{F}_\bullet\mathfrak{h})$ be an $\mathfrak{sL}_\infty$-algebra such that there exists a filtered ∞-quasi-isomorphism*

$$\Psi : (\mathfrak{h}, \mathscr{F}_\bullet\mathfrak{h}) \rightsquigarrow (\mathfrak{g}, \mathscr{F}_\bullet\mathfrak{g}) \ .$$

There is a homotopy equivalence of simplicial sets

$$\mathrm{Map}(K, L_\mathbb{Q}) \simeq \mathrm{MC}_\bullet(\hom^\pi(C, \mathfrak{h})),$$

i.e. the convolution $\mathfrak{sL}_\infty$-algebra $\hom^\pi(C, \mathfrak{h})$ is an $\mathfrak{sL}_\infty$-model for the mapping space.

An example of an application of this theorem is an alternative proof of [BG16, Thm. 3.2], see [Wie16, Cor. 11.1].

Chapter 12

A model structure for the Goldman–Millson theorem

The Goldman–Millson and the Dolgushev–Rogers theorem, Theorems 6.1.12 and 6.4.1, have a distinct homotopical flavor. After all, they state that some class of morphisms, namely filtered (∞-)quasi-isomorphisms, is sent to weak equivalences of simplicial sets under the Maurer–Cartan space functor. However, the proofs of these theorems are not done by purely homotopical methods, but rather by working explicitly with the algebras and performing some induction using the filtrations, as we have seen in Section 6.4. There is a reason behind this: the filtered quasi-isomorphisms are not very well behaved and do not form the class of weak equivalences of any model structure on the category of differential graded Lie algebras that we know of, even after closing them by the 2-out-of-3 property.

The goal of this chapter is to provide a fully homotopical and self-contained approach to the proof of the Goldman–Millson theorem and to the Dolgushev–Rogers theorem. The idea is to consider the Vallette model structure on conilpotent Lie coalgebras. Linear dualization is unfortunately not an equivalence of categories with Lie algebras, but we can work around this fact by using some results of [LG16] to prove that the category of conilpotent Lie coalgebras is equivalent to the category of pro-objects in finite dimensional, nilpotent Lie algebras. This way, we obtain a model structure on this last category. An interplay between this model category and the limit functor allows us to see the gauge relation for Maurer–Cartan elements in a Lie algebra as a homotopy relation between certain morphisms representing the Maurer–Cartan elements. A model categorical argument then immediately gives us a version of the Goldman–Millson theorem. Further, using simplicial framings, we extend the argument to prove a version of the Dolgushev–Rogers theorem for strict morphisms. These results are weaker than the original ones, as they only work on morphisms that are obtained via linear dualization from weak equivalences of conilpotent Lie coalgebras, i.e. essentially duals of filtered quasi-isomorphisms of Lie coalgebras.

The results presented here are those of [RN18]. They have close links with recent works by other authors, which we will try to explain throughout the text. The model structure we produce on the category of pro-objects in finite dimensional, nilpotent Lie algebras starting from the Vallette model structure on conilpotent Lie coalgebras is the same as the model structure described directly on this category of pro-objects by A. Lazarev and M. Markl in [LM15]. The idea of a homotopical approach to the proof of the Goldman–Millson and the Dolgushev–Rogers theorems is already present in the work of U. Buijs, Y. Félix, A. Murillo, and D. Tanré — more specifically

in [BFMT16], where they obtain a result which is strictly stronger than our Theorems 12.2.13 and 12.3.2. However, they need to use a version of the Dolgushev–Rogers theorem in their proof, while our approach is more self-contained, if not as powerful. Moreover, the techniques we present in Section 12.3 can be applied to the results of Buijs–Félix–Murillo–Tanré to give a modest generalization.

In this chapter, we will work over cochain complexes instead of chain complexes, as we did in Chapter 10.

12.1 Some notions of category theory

In this section, we give a reminder of the basic categorical notions we will need later, such as some basic results on equivalences of categories, and ind- and pro-objects in a category (i.e. categories formal colimits and limits).

12.1.1 Equivalences of categories

Given two categories C and D, an *isomorphism of categories* between them is a functor $F : \mathsf{C} \to \mathsf{D}$ such that there exists another functor $F^{-1} : \mathsf{D} \to \mathsf{C}$ satisfying $F^{-1}F = 1_{\mathsf{C}}$ and $FF^{-1} = 1_{\mathsf{D}}$. This notion is far too strict to be really useful. A more sensible notion to compare categories is the following one.

Definition 12.1.1. *An* equivalence of categories *between* C *and* D *is a functor* $F : \mathsf{C} \to \mathsf{D}$ *such that there exists a functor* $G : \mathsf{D} \to \mathsf{C}$ *and two natural isomorphisms* $GF \cong 1_{\mathsf{C}}$ *and* $FG \cong 1_{\mathsf{D}}$. *An* anti-equivalence of categories *between* C *and* D *is an equivalence of categories between* C^{op} *and* D.

There is another notion of equivalence of categories which is at first sight stronger than the previous one.

Definition 12.1.2. *An* adjoint equivalence of categories *between* C *and* D *is an adjoint pair*

$$F\colon \mathsf{C} \rightleftarrows \mathsf{D} : G$$

such that the unit and counit maps are natural isomorphisms.

In fact, the two notions of equivalence of categories are the same.

Theorem 12.1.3. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor. The following statements are equivalent.*

1. *The functor* F *is an equivalence of categories.*
2. *The functor* F *is part of an adjoint equivalence of categories.*
3. *The functor* F *is fully faithful and essentially surjective, i.e. for every object* $d \in \mathsf{D}$ *there exists* $c \in \mathsf{C}$ *such that* $d \cong F(c)$.

For details on these notions, see for example the book [ML70, pp. 92–95].

12.1.2 Categories of ind-objects and categories of pro-objects

Let C be a category. One can then consider the category of "formal colimits" in C (e.g. [AGV72, Sect. 8]).

Definition 12.1.4. *A (non-empty) small category C is* filtered *if*

1. *for every two objects $x, y \in \mathsf{C}$, there exists an object $z \in \mathsf{C}$ and two arrows $z \to x$ and $z \to y$, and*
2. *for every two arrows $f, g : b \to a$ in C, there exists an arrow $h : c \to b$ such that $fh = gh$.*

The category C is cofiltered *if it satisfies the dual properties.*

Definition 12.1.5. *The category $\mathsf{ind}(\mathsf{C})$ of* ind-objects *in C is the category that has as objects all the diagrams $F : \mathscr{D} \to \mathsf{C}$ with $\mathscr{D}$ a small filtered category. If $F : \mathscr{D} \to \mathsf{C}$ and $G : \mathscr{E} \to \mathsf{C}$ are two objects in $\mathsf{ind}(\mathsf{C})$, the set of morphisms between them is*

$$\hom_{\mathsf{ind}(\mathsf{C})}(F, G) \coloneqq \lim_{d \in \mathscr{D}} \operatorname{colim} e \in \mathscr{E} \hom_{\mathsf{C}}(F(d), G(e)) \ ,$$

where the limit and the colimit are taken in Sets.

Dually, one can also consider the category of "formal limits" in C (e.g. [Gro60, Sect. A.2] and [AGV72, Sect. 8]).

Definition 12.1.6. *The category $\mathsf{pro}(\mathsf{C})$ of* pro-objects *in C is the category that has as objects all the diagrams $F : \mathscr{D} \to \mathsf{C}$ with $\mathscr{D}$ a small cofiltered category. If $F : \mathscr{D} \to \mathsf{C}$ and ,$G : \mathscr{E} \to \mathsf{C}$ are two objects in $\mathsf{pro}(\mathsf{C})$, the set of morphisms between them is*

$$\hom_{\mathsf{pro}(\mathsf{C})}(F, G) \coloneqq \lim_{e \in \mathscr{E}} \operatorname{colim} d \in \mathscr{D} \hom_{\mathsf{C}}(F(d), G(e)) \ ,$$

where the limit and the colimit are taken in Sets.

Lemma 12.1.7. *Let C' and C'' be two equivalent or anti-equivalent categories. Then $\mathsf{ind}(\mathsf{C}')$ is equivalent to $\mathsf{ind}(\mathsf{C}'')$, respectively anti-equivalent to $\mathsf{pro}(\mathsf{C}'')$.*

Proof. This is straightforward. □

We recall the definition of a compact object in a category.

Definition 12.1.8. *Let C be a category that admits filtered colimits. An object $c \in \mathsf{C}$ is said to be* compact *if the functor*

$$\hom_{\mathsf{C}}(c, -) : \mathsf{C} \longrightarrow \mathsf{Sets}$$

preserves filtered colimits.

Proposition 12.1.9. *Let C be a category and let C' be a full subcategory of C. Further, assume that*

1. *the category C is cocomplete,*
2. *there exists a functor*

$$\delta : \mathsf{C} \longrightarrow \mathsf{ind}(\mathsf{C}')$$

such that the composite $\operatorname{colim} \delta$ is naturally isomorphic to the identity functor of C, where colim is (a choice for) the functor

$$\operatorname{colim} : \mathsf{ind}(\mathsf{C}') \longrightarrow \mathsf{C}$$

given by taking the colimit in C, and

3. *every object in* C' *is compact in* C.

Then the functors δ *and* colim *exhibit an equivalence of categories between* C *and* $\mathsf{ind}(\mathsf{C}')$.

Proof. Assumption (1) guarantees the existence of a colimit functor. The natural isomorphism

$$\operatorname{colim} \delta \cong \mathrm{id}_{\mathsf{C}}$$

is given by assumption (2). We are left to prove that there exists a natural isomorphism

$$\delta \operatorname{colim} \cong \mathrm{id}_{\mathsf{ind}(\mathsf{C}')} \ .$$

Let $F : \mathscr{D} \to \mathsf{C}'$ and $G : \mathscr{E} \to \mathsf{C}'$ be two objects in $\mathsf{ind}(\mathsf{C}')$. Then we have

$$\begin{aligned}
\hom_{\mathsf{ind}(\mathsf{C}')}(F, G) &= \lim_{d \in \mathscr{D}} \operatorname{colim} e \in \mathscr{E} \hom_{\mathsf{C}'}(F(d), G(e)) \\
&\cong \lim_{d \in \mathscr{D}} \operatorname{colim} e \in \mathscr{E} \hom_{\mathsf{C}}(F(d), G(e)) \\
&\cong \lim_{d \in \mathscr{D}} \hom_{\mathsf{C}}(F(d), \operatorname{colim} G) \\
&\cong \hom_{\mathsf{C}}(\operatorname{colim} F, \operatorname{colim} G) \ .
\end{aligned}$$

In the second line we used the fact that C' is a full subcategory of C, and in the third one the fact that $F(d) \in \mathsf{C}'$ is always a compact object by assumption (3). It follows that for any $F \in \mathsf{ind}(\mathsf{C}')$ we have

$$\begin{aligned}
\hom_{\mathsf{ind}(\mathsf{C}')}(\delta \operatorname{colim} F, F) &\cong \hom_{\mathsf{C}}(\operatorname{colim} \delta \operatorname{colim} F, \operatorname{colim} F) \\
&\cong \hom_{\mathsf{C}}(\operatorname{colim} F, \operatorname{colim} F) \ ,
\end{aligned}$$

where we used the fact that $\operatorname{colim} \delta \cong \mathrm{id}_{\mathsf{C}}$. Then, the identity morphism of $\operatorname{colim} F$ provides a natural isomorphism $\delta \operatorname{colim} \cong \mathrm{id}_{\mathsf{ind}(\mathsf{C}')}$ as we wanted. □

Example 12.1.10. *The easiest example of application of Proposition 12.1.9 is the following one. Let* $\mathsf{C} \coloneqq \mathsf{Sets}$ *be the category of (small) sets and let* $\mathsf{C}' \coloneqq \mathsf{fSets}$ *be the full subcategory of finite sets. Every set is the colimit of its finite subsets, so taking the diagram of all finite subsets gives a functor*

$$\delta : \mathsf{Sets} \longrightarrow \mathsf{ind}(\mathsf{fSets}) \ .$$

As Sets *is cocomplete and finite sets are compact objects in* Sets*, Proposition 12.1.9 tells us that* Sets *is equivalent to* $\mathsf{ind}(\mathsf{fSets})$.

Example 12.1.11. *Another classical example is as follows. Let* $\mathsf{C} \coloneqq \mathsf{Vect}$ *be the category of vector spaces and let* $\mathsf{C}' = \mathsf{C}'' \coloneqq \mathsf{fdVect}$ *be the full subcategory of finite dimensional vector spaces. As every vector space is naturally isomorphic to the colimit of all its finite dimensional subspaces, taking the diagram of all the finite dimensional subspaces of a vector space gives a functor*

$$\delta : \mathsf{Vect} \longrightarrow \mathsf{ind}(\mathsf{fdVect}) \ .$$

It is well-known that Vect *is cocomplete and that the finite dimensional vector spaces are compact objects in* Vect*. Moreover, linear duality gives an anti-equivalence of* fdVect *with itself. Therefore, by Proposition 12.1.9 and Lemma 12.1.7 we have that* Vect *is anti-equivalent to* $\mathsf{pro}(\mathsf{fdVect})$.

Example 12.1.12. *The following example is a result of [GG99]. Let* $\mathsf{C} \coloneqq \mathsf{Cog}$ *be the category of coassociative counital coalgebras, let* $\mathsf{C}' \coloneqq \mathsf{fCog}$ *be the full subcategory of finite dimensional coassociative counital coalgebras, and let* $\mathsf{C}'' \coloneqq \mathsf{fAlg}$ *be the category of finite dimensional unital associative algebras. One can prove that every coassociative counital coalgebra is the colimit of its finite dimensional subcoalgebras By the same arguments as in the previous two examples, Proposition 12.1.9 gives an equivalence of categories between* Cog *and* $\mathsf{ind}(\mathsf{fCog})$*. Linear duality induces an anti-equivalence of categories between* fCog *and* fAlg*. Therefore, by Lemma 12.1.7 we have that* Cog *is anti-equivalent to* $\mathsf{pro}(\mathsf{fAlg})$.

12.2 A model structure for the Goldman–Millson theorem

In this section, we show that a version of the Goldman–Millson theorem [GM88] can be obtained by a purely homotopical argument in the model category of Lie coalgebras.

12.2.1 The model structure on conilpotent Lie coalgebras

We are interested in the study of certain dg Lie algebras, namely the ones obtained dualizing conilpotent Lie coalgebras.

Definition 12.2.1. *A* conilpotent Lie coalgebra *is a conilpotent coalgebra over the cooperad* $\mathrm{coLie} := \mathrm{Lie}^\vee$*, the linear dual of the operad* Lie *encoding Lie algebras.*

Notice that the cooperad coLie is Koszul and that its Koszul dual operad is the operad $\mathscr{S} \otimes \mathrm{Com}$. Both are connected, and we endow them with the canonical weight grading. By what exposed in Section 3.3.7, we obtain a model structure on the category of conilpotent Lie coalgebras by pulling back the Hinich model structure on (suspended) commutative algebras along the Koszul twisting morphism

$$\kappa : \mathrm{coLie} \longrightarrow \mathscr{S} \otimes \mathrm{Com} .$$

We fix the canonical weight grading on coLie. It is given by

$$\mathrm{coLie}^{(w)} := \mathrm{coLie}(w+1) .$$

Therefore, we can change our conventions a bit and we define a *filtration* on a conilpotent Lie coalgebra C to be a sequence of sub-chain complexes

$$0 \subseteq F_1 C \subseteq F_2 C \subseteq F_3 C \subseteq \cdots \subseteq C$$

such that

1. the filtration is exhaustive: $\mathrm{colim}_n F_n C \cong C$,
2. it respects the coproduct, that is

$$\overline{\Delta}_C(F_n C) \subseteq \bigoplus_{\substack{k \geq 1 \\ n_1 + \cdots + n_k = n}} \left(\mathrm{coLie}(k) \otimes F_{n_1} C \otimes \cdots \otimes F_{n_k} C\right)^{\mathbb{S}_k},$$

   and
3. it is preserved by the differential: $d_C(F_n C) \subseteq F_n C$.

Notice that this is equivalent to the notion of cofiltration of Definition 3.3.10 by shifting the indices by 1.

Explicitly, the model structure on the category coLie of conilpotent Lie coalgebras has

- the closure of the class of filtered quasi-isomorphisms under the 2-out-of-3 property as weak equivalences,
- the monomorphisms as cofibrations, and
- the class of morphisms with the right lifting property with respect to trivial cofibrations as fibrations.

Every conilpotent Lie coalgebra is cofibrant, and the fibrant objects are the quasi-free conilpotent Lie coalgebras.

Dualizing a filtered conilpotent Lie coalgebra we obtain a complete Lie algebra.

Lemma 12.2.2. *Let $(C, F_\bullet C)$ be a filtered conilpotent Lie coalgebra. Then the filtration on $\mathfrak{g} \coloneqq C^\vee$ defined by*

$$F_n\mathfrak{g} \coloneqq (F_n C)^\perp$$

makes $\mathfrak{g}$ into a complete Lie algebra. Moreover, the dual of a filtered quasi-isomorphism of conilpotent Lie coalgebras is a filtered quasi-isomorphism of Lie algebras.

Remark 12.2.3. *Not all filtered quasi-isomorphisms of Lie algebras can be obtained by linear dualization. For example, consider abelian (co)Lie (co)algebras, i.e. cochain complexes. For simplicity, take $\mathbb{K} = \mathbb{Q}$. Let*

$$V \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}$$

be concentrated in degree 0, seen as an abelian Lie coalgebra. A quasi-isomorphism from V to itself is just an isomorphism, and we have

$$|\operatorname{Aut}(V)| \leq |\operatorname{End}(V)| = (\dim V)^{|V|} = |\mathbb{N}|^{|\mathbb{N}|}.$$

At the same time, we have $\dim(V^\vee) = |\mathbb{R}|$. Notice that if we fix a basis of $V^\vee$, then bijections from the basis to itself are automorphisms. Therefore, we have

$$|\operatorname{Aut}(V^\vee)| \geq |\operatorname{Sym}(\mathbb{R})| = |\mathbb{R}|^{|\mathbb{R}|} > |\mathbb{N}|^{|\mathbb{N}|}.$$

It follows that there must exist automorphisms of $V^\vee$ that are not given as the dual of an automorphism of V.

There is another twisting morphism we can consider to construct a model structure on conilpotent Lie coalgebras, namely the canonical twisting morphism

$$\iota : \mathrm{coLie} \cong \mathscr{S}^c \otimes \mathrm{Com}^! \longrightarrow \Omega(\mathscr{S}^c \otimes \mathrm{Com}^!) \cong \mathscr{S} \otimes \mathscr{C}_\infty \,. \tag{12.1}$$

Lemma 12.2.4. *The model structure obtained on conilpotent Lie coalgebras by pulling back the Hinich model structure on suspended $\mathscr{C}_\infty$-algebras along Ω_ι is equal to the one obtained by pulling back along Ω_κ.*

Proof. This is an immediate corollary of [LG16, Prop. 32]. □

This fact will be of fundamental importance later on.

12.2.2 Dualization of conilpotent Lie coalgebras

We start by giving the correct framework in which to dualize conilpotent Lie coalgebras.

In the article [LG16], the following result is proven, generalizing what can be found in [GG99, Sect. 1] for coassociative counital coalgebras. See also Example 12.1.12.

Proposition 12.2.5 ([LG16, Lemma 4 and 5 and Prop. 12])**.** *Let $\mathscr{C}$ be a cooperad. The category $\mathscr{C}$-cog of conilpotent $\mathscr{C}$-coalgebras is cocomplete, every conilpotent $\mathscr{C}$-coalgebra is the colimit of the diagram of all its finite-dimensional sub-coalgebras with the relative inclusions, and all finite-dimensional conilpotent $\mathscr{C}$-coalgebras are compact objects in $\mathscr{C}$-cog. In particular, the category $\mathscr{C}$-cog is presentable.*

Denote by fd coLie the full subcategory of coLie of finite dimensional conilpotent Lie coalgebras. As a direct consequence of these facts and of Proposition 12.1.9 in the case where $\mathsf{C} = \mathsf{coLie}$, we have the following.

Corollary 12.2.6. *The categories* coLie *and* ind(fd coLie) *are equivalent. The equivalence is given by the functors*

$$\mathrm{colim} : \mathsf{ind}(\mathsf{fd\ coLie}) \longrightarrow \mathsf{coLie}$$

given by taking a functorial choice of colimit of diagrams and

$$\delta : \mathsf{coLie} \longrightarrow \mathsf{ind}(\mathsf{fd\ coLie})$$

given by associating to a conilpotent Lie coalgebra the diagram of all its finite-dimensional sub-coalgebras with the relative inclusions.

It is always true that the linear dual of a conilpotent Lie coalgebra is a Lie algebra, while the converse holds in the finite dimensional case. Denote by fd nil. Lie the category of finite dimensional nilpotent Lie algebras. As a direct consequence of this fact and of Lemma 12.1.7, we have the following.

Lemma 12.2.7. *Linear duality between finite dimensional Lie coalgebras and finite dimensional Lie algebras induces an anti-equivalence of categories between* ind(fd conil. coLie) *and* pro(fd nil. Lie).

In summary, we have the following commutative diagram, where pronil. Lie is the subcategory of Lie algebras which is the image of linear dualization from conilpotent Lie coalgebras, called the category of *pronilpotent Lie algebras*.

$$\begin{array}{ccc} \mathsf{ind}(\mathsf{fd\ coLie}) & \longleftarrow & \mathsf{pro}(\mathsf{fd\ nil.\ Lie}) \\ \downarrow{\scriptstyle \mathrm{colim}} & & \downarrow{\scriptstyle \mathrm{lim}} \\ \mathsf{coLie} & \longrightarrow & \mathsf{pronil.\ Lie} \end{array}$$

where the horizontal arrows are given by linear dualization and the upper horizontal and left vertical arrows are an anti-equivalence of categories and an equivalence of categories respectively. Notice that all pronilpotent Lie algebras are complete: one can always write them as the dual of a conilpotent Lie coalgebras an take the orthogonal of the coradical filtration.

Remark 12.2.8. *What stated above works in greater generality for conilpotent coalgebras over a cooperad. In a sense, this is the correct framework in which to dualize a conilpotent coalgebra, obtaining not an algebra, but a diagram of finite dimensional algebras.*

We endow the category pro(fd nil. Lie) with the model structure obtained by transporting the Vallette model structure on Lie coalgebras along the two (anti-)equivalences of categories. The two model structures are Quillen equivalent. Notice that fibrations and cofibrations change roles as we pass through an anti-equivalence.

Remark 12.2.9. *One can check that the model structure thus obtained is exactly the one of [LM15, Def. 9.9]. The main difference between our approach and theirs is the fact that we work directly with Lie coalgebras, while they only ever work with pronilpotent Lie algebras. We hope that the point of view we used in the present paper will appear more natural to some readers, as it avoids dualizing every coalgebra in order to only work with algebras.*

12.2.3 The Goldman–Millson theorem

The idea is now to use the model structure given above to obtain statements about pronilpotent Lie algebras. By what we have seen in Section 10.2, we have that the set of Maurer–Cartan elements of a Lie algebra $\mathfrak{g}$ is in natural bijection with the set of morphisms

$$\mathfrak{mc}_0 \longrightarrow \mathfrak{g}$$

of dg Lie algebras, where $\mathfrak{mc}_0$ is the free Lie algebra on a Maurer–Cartan element. Similarly, gauge equivalences between Maurer–Cartan elements are coded by morphisms

$$\mathfrak{mc}_1 \longrightarrow \mathfrak{g} \, .$$

Intuitively, $\mathfrak{mc}_1$ behaves like a cylinder object for $\mathfrak{mc}_0$, so that the gauge relation looks very similar to a left homotopy between Maurer–Cartan elements seen as maps. However, we do not have a good model structure on complete Lie algebras, and that's why we have to lift everything to the category of diagrams.

In order to push down the results we will obtain for diagrams to actual algebras, we will need the following technical lemma.

Lemma 12.2.10. *Let $\mathfrak{g}$ be a Lie algebra which is complete with respect to its canonical filtration, and such that $\mathfrak{g}^{(n)} \coloneqq \mathfrak{g}/F_n^{\mathrm{can}}\mathfrak{g}$ is finite dimensional for all $n \geq 2$. Let $\widetilde{\mathfrak{g}} \in \mathsf{pro}(\mathsf{fd\ nil.\ Lie})$ be the diagram*

$$\widetilde{\mathfrak{g}} = \cdots \longrightarrow \mathfrak{g}^{(4)} \longrightarrow \mathfrak{g}^{(3)} \longrightarrow \mathfrak{g}^{(2)} \longrightarrow 0 \, .$$

Let $H : \mathsf{D} \to \mathsf{fd\ nil.\ Lie}$ be any object of $\mathsf{pro}(\mathsf{fd\ nil.\ Lie})$. Then we have a natural isomorphism

$$\hom_{\mathsf{pro(fd\ nil.\ Lie)}}(\widetilde{\mathfrak{g}}, H) \cong \hom_{\mathsf{dgLie}}(\mathfrak{g}, \lim H) \, .$$

Proof. We have

$$\hom_{\mathsf{dgLie}}(\mathfrak{g}, \lim H) = \lim_{d \in \mathsf{D}} \hom_{\mathsf{dgLie}}(\mathfrak{g}, H(d))$$

with all $H(d)$ finite dimensional. Fix $d \in \mathsf{D}$, then we have the linear diagram

$$\cdots \longleftarrow \hom_{\mathsf{dgLie}}(\mathfrak{g}^{(4)}, H(d)) \longleftarrow \hom_{\mathsf{dgLie}}(\mathfrak{g}^{(3)}, H(d)) \longleftarrow \hom_{\mathsf{dgLie}}(\mathfrak{g}^{(2)}, H(d)) \longleftarrow 0 \, .$$

Since all the maps in the diagram $\widetilde{\mathfrak{g}}$ were surjective, all the maps in this diagram are injective. Let N be the nilpotency degree of $H(d)$. Then every map $\mathfrak{g} \to H(d)$ splits through $\mathfrak{g}^{(n)}$ for all $n \geq N$, so that we have

$$\hom_{\mathsf{dgLie}}(\mathfrak{g}, H(d)) \cong \hom_{\mathsf{dgLie}}(\mathfrak{g}^{(n)}, H(d))$$

for all $n \geq N$. It follows that

$$\hom_{\mathsf{dgLie}}(\mathfrak{g}, H(d)) \cong \operatorname{colim} n \hom_{\mathsf{dgLie}}(\mathfrak{g}^{(n)}, H(d)) \, ,$$

which concludes the proof. □

In particular, this is true for any $\mathfrak{g}$ of the form $\mathfrak{g} \coloneqq \widehat{\mathrm{Lie}}(V)$, for V a finite dimensional graded vector space, with any differential. The examples of main interest to us will be the pronilpotent Lie algebras

$$\mathfrak{mc}_n \coloneqq \widehat{\Omega}_\pi(sC_n^\vee)$$

introduced in Definition 10.3.1.

Lemma 12.2.11. *Let $n \geq 0$. We have*

$$\operatorname{colim} \widetilde{\mathfrak{mc}}_n^{\vee} = \mathrm{B}_\iota(s^{-1}C_n) \, .$$

In particular, $\widetilde{\mathfrak{mc}}_n$ is cofibrant.

Proof. The dual of the quotient

$$\mathfrak{mc}_n^{(k)} = \mathfrak{mc}_n / F_k^{\mathrm{can}} \mathfrak{mc}_n$$

of $\mathfrak{mc}_n$ by the kth space of the canonical filtration is exactly the kth space of the coradical filtration of $\mathrm{B}_\iota(s^{-1}C_n)$, and the projections

$$\mathfrak{mc}_n^{(k+1)} \longrightarrow \mathfrak{mc}_n^{(k)}$$

become the inclusions of the various spaces of the coradical filtration of $\mathrm{B}_\iota(s^{-1}C_n)$. From this, the result follows. □

Lemma 12.2.12. *The diagram $\widetilde{\mathfrak{mc}}_1$ is a cylinder object for $\widetilde{\mathfrak{mc}}_0$. After passing to the limit, the splitting of the codiagonal map becomes*

$$\mathfrak{mc}_0 \sqcup \mathfrak{mc}_0 \xrightarrow{i} \mathfrak{mc}_1 \xrightarrow{t} \mathfrak{mc}_0$$

where the two maps are as follows. The first Lie algebra is the free complete Lie algebra generated by two Maurer–Cartan elements α_0 and α_1, while the second one is the free complete Lie algebra generated by two Maurer–Cartan elements β_1, β_2 and a gauge λ from β_1 to β_2, and the last algebra $\mathfrak{mc}_0$ is the free Lie algebra on a single Maurer–Cartan element α. The first map is determined by $i(\alpha_j) = \beta_j$ for $j = 1, 2$, the second map is determined by $t(\beta_j) = \alpha$ for $j = 1, 2$ and $t(\lambda) = 0$.

Proof. By Lemma 12.2.11, we know that the diagram $\widetilde{\mathfrak{mc}}_n$ corresponds to the conilpotent Lie coalgebra $\mathrm{B}_\iota(s^{-1}C_n)$. Since B_ι is right adjoint, we have

$$\mathrm{B}_\iota(s^{-1}C_0) \times \mathrm{B}_\iota(s^{-1}C_0) \cong \mathrm{B}_\iota(s^{-1}C_0 \times s^{-1}C_0)$$

with the obvious differential in the category of conilpotent Lie coalgebras. The diagonal map

$$C_0 \xrightarrow{\Delta} C_0 \times C_0$$

in the category of $\mathscr{C}_\infty$-algebras splits into

$$C_0 \xrightarrow{T} C_1 \xrightarrow{I} C_0 \times C_0$$

where $T(1) = \omega_0 + \omega_1$, $I(\omega_0) = (1, 0)$, $I(\omega_1) = (0, 1)$, and $I(\omega_{01}) = (0, 0)$. The map T is a quasi-isomorphism as it is induced by the homotopy equivalence $\Delta^1 \to \Delta^0$. Therefore, by a result analogous to Lemma 3.3.17, it is sent to a filtered quasi-isomorphism $\mathrm{B}_\iota(T)$ by the bar construction. The map I is clearly surjective, and thus $\mathrm{B}_\iota(I)$ is a fibration by [Val14, Thm. 2.9(2)]. Notice that the mentioned theorem is stated for the twisting morphism κ, but it also holds for ι by repeating the same proof and using Lemma 12.2.4. It follows that $\mathrm{B}_\iota(s^{-1}C_1)$ is a path object for $\mathrm{B}_\iota(s^{-1}C_0)$, which is equivalent to the statement we wanted to prove. Recovering the exact structure of the maps after passing to the limit is straightforward. □

We are now set for the proof of the Goldman–Millson theorem for pronilpotent Lie algebras.

Theorem 12.2.13. *Let $\mathfrak{g}, \mathfrak{h}$ be two pronilpotent dg Lie algebras, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be the dual of a weak equivalence of conilpotent Lie coalgebras. Then ϕ induces a bijection*

$$\overline{\mathrm{MC}}(\mathfrak{g}) \cong \overline{\mathrm{MC}}(\mathfrak{h}) \, .$$

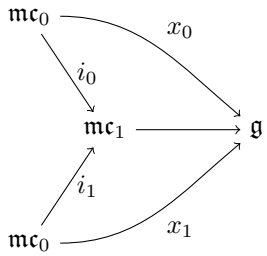
Proof. Let $G, H \in \mathsf{pro(fd\ nil.\ Lie)}$ and

$$\Phi \in \hom_{\mathsf{pro(fd\ nil.\ Lie)}}(G, H)$$

be a weak equivalence such that $\lim G = \mathfrak{g}$, $\lim H = \mathfrak{h}$, and such that Φ corresponds to ϕ after passing to the limits. By definition, the moduli space of Maurer–Cartan elements of a Lie algebra $\mathfrak{g}$ is the quotient

$$\overline{\mathrm{MC}}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g}) / \sim_{\mathrm{gauge}}$$

of the space of Maurer–Cartan elements of $\mathfrak{g}$ by the gauge relation. As explained above, we can see Maurer–Cartan elements in $\mathfrak{g}$ as morphisms $\mathfrak{mc}_0 \to \mathfrak{g}$ and gauge equivalences are coded by morphisms $\mathfrak{mc}_1 \to \mathfrak{g}$. That is to say, two Maurer–Cartan elements $x_0, x_1 \in \mathrm{MC}(\mathfrak{g})$ are gauge equivalent if, and only if there exists a morphism $\mathfrak{mc}_1 \to \mathfrak{g}$ making the following diagram commute.



But by Lemma 12.2.10 and Lemma 12.2.12, this is exactly the left homotopy relation on morphisms $\widetilde{\mathfrak{mc}}_0 \to G$. Therefore,

$$\overline{\mathrm{MC}}(\mathfrak{g}) \cong \hom_{\mathsf{pro(fd\ nil.\ Lie)}}(\widetilde{\mathfrak{mc}}_0, G) / \sim_\ell = \hom_{\mathsf{Ho(pro(fd\ nil.\ Lie))}}(\widetilde{\mathfrak{mc}}_0, G)\ ,$$

where the last equality is given by the fact that all elements of $\mathsf{pro(fd\ nil.\ Lie)}$ are fibrant, as well as the fact that $\widetilde{\mathfrak{mc}}_0$ is cofibrant by Lemma 12.2.11. The same thing is of course true for $\mathfrak{h}$, and since $\phi : \mathfrak{g} \to \mathfrak{h}$ comes from a weak equivalence, it naturally induces a bijection

$$\overline{\mathrm{MC}}(\mathfrak{g}) \cong \overline{\mathrm{MC}}(\mathfrak{h})$$

by Lemma 3.2.5, as desired. □

Although this result is slightly weaker than the full Goldman–Millson theorem (it works only on some algebras, and we don't have all the morphisms we would like, cf. Remark 12.2.3), it has the advantage of having a fully homotopical proof, which is good considering the homotopical flavor of the statement.

12.3 Framings and the Dolgushev–Rogers theorem

The last section can be seen as the 0th level of the Dolgushev–Rogers theorem — it is in fact a statement at the level of the 0th homotopy group of the Maurer–Cartan spaces. Here, we will apply the simplicial framings introduced in Section 3.4 to our model category, showing how the whole cosimplicial Lie algebra $\mathfrak{mc}_\bullet$ appears naturally in this context. We are then able to recover the Dolgushev–Rogers theorem in this context by arguments similar to the ones used for the Goldman–Millson theorem in the last section. We conclude by comparing the results of this chapter with the recent literature.

12.3.1 Using framings to recover the Dolgushev–Rogers theorem

We take for C the category of conilpotent Lie coalgebras and we consider the Lie coalgebra $\mathrm{B}_\iota(s^{-1}C_0)$.

Lemma 12.3.1. *The simplicial conilpotent Lie coalgebra* $\mathrm{B}_\iota(s^{-1}C_\bullet)$ *is a simplicial frame on* $\mathrm{B}_\iota(s^{-1}C_0)$.

Proof. At level 0, both maps are the identity. For $n \geq 0$ the maps are induced by the maps of $\mathscr{C}_\infty$-algebras

$$C_0 \xrightarrow{w} C_n \xrightarrow{p} C_0 \times \cdots \times C_0$$

explicitly given by

$$w(1) = \omega_0 + \cdots + \omega_n$$

and

$$p(\omega_i) = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0), \qquad \text{and} \qquad p(\omega_I) = 0 \quad \forall |I| \geq 2\,.$$

The map w is the pullback by the unique map $\Delta^n \to \Delta^0$, which is a homotopy equivalence. Thus, w is a quasi-isomorphism and we obtain a weak equivalence when we apply the bar construction to it by an analogue to Lemma 3.3.17. Similarly, the map p is surjective, and thus is sent to a fibration by the bar construction, again by [Val14, Thm. 2.9(2)] as in the proof of Lemma 12.2.12. □

This simple fact is enough to recover all the most important results about $\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g})$.

Theorem 12.3.2. *The functor*

$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, -) : \mathsf{pronil.\ Lie} \longrightarrow \mathsf{sSets}$$

sends duals of injections of conilpotent Lie coalgebras to fibrations and duals of weak equivalences of conilpotent Lie coalgebras to weak equivalences. In particular, it has image in the full subcategory of Kan complexes.

Proof. Let $\mathfrak{g}$ be a pronilpotent dg Lie algebra. Then $\mathfrak{g} = \lim G$ for some $G \in \mathsf{pro}(\mathsf{fd\ nil.\ Lie})$. Let $C \coloneqq \operatorname{colim} G^\vee$, so that $C^\vee = \mathfrak{g}$. Then

$$\begin{aligned} \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) &\cong \hom_{\mathsf{pro(fd\ nil.\ Lie)}}(\widetilde{\mathfrak{mc}}_\bullet, G) \\ &\cong \hom_{\mathsf{conil.\ coLie}}(C, \mathrm{B}_\iota(s^{-1}C_\bullet))\,, \end{aligned}$$

where the first isomorphism is given by Lemma 12.2.11, and the second one by the equivalence of categories of Corollary 12.2.6. By Theorem 3.3.9, we know that C is always cofibrant (and thus fibrant in the opposite category), and that $\mathrm{B}_\iota(s^{-1}C_0)$ is fibrant, since it is quasi-free. Proposition 3.4.4 concludes the proof. □

The statement that the functor sends duals of injections to fibrations should be seen analogous to Theorem 6.2.11, while the assertion that duals of weak equivalences of coalgebras are sent to weak equivalences is a weaker version of Theorem 6.4.1, in the same way that Theorem 12.2.13 is a weaker version of Theorem 6.1.12.

12.3.2 Relations with the BFMT model structure on Lie algebras

In the article [BFMT16], Buijs–Félix–Murillo–Tanré introduced a model structure on a slightly larger category: the category of all complete Lie algebras and filtered morphisms. They have the declared goal of developing a way to do rational homotopy theory for all spaces using Lie algebra models. We summarize their results and compare them with the present work.

Let $\mathfrak{g}$ be a complete Lie algebra, and let $x \in \mathrm{MC}(\mathfrak{g})$. Then one can use x to twist the differential to get

$$d^x \coloneqq d + \mathrm{ad}_x \ .$$

This new operator d^x is such that $(\mathfrak{g}, d^x)$ is once again a Lie algebra.

Definition 12.3.3. *The* component of $\mathfrak{g}$ at x *is the Lie algebra obtained by truncating* $(\mathfrak{g}, d^x)$ *in positive degree and taking only* $\ker d^x$ *in degree* 0*, i.e. it is the cochain complex*

$$\cdots \xrightarrow{d^x} \mathfrak{g}^{-2} \xrightarrow{d^x} \mathfrak{g}^{-1} \xrightarrow{d^x} \ker d^x \longrightarrow 0$$

endowed with the Lie algebra structure inherited from $\mathfrak{g}$*.*

One defines the following three classes of maps:

- A filtered morphism is a *fibration* if it is surjective in non-negative degrees.
- A filtered morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ is a *weak equivalence* if $\overline{\mathrm{MC}}(\phi)$ is a bijection and $\phi : \mathfrak{g}^x \to \mathfrak{h}^{\phi(x)}$ is a quasi-isomorphism for all $x \in \mathrm{MC}(\mathfrak{g})$.
- A filtered morphism is a *cofibration* if it has the left lifting property with respect to all trivial cofibrations.

Theorem 12.3.4 ([BFMT16, Thm. 3.1])**.** *These three classes of maps define a model structure on the category of complete Lie algebras and filtered morphisms. We call this model structure the* BFMT model structure *on complete Lie algebras.*

The weak equivalences and fibrations of the model structure considered in the present paper are contained in the weak equivalences and fibrations of the BFMT model structure, as is proven in [BFMT16, Sect. 6].

One also has a Quillen pair

$$\mathfrak{L}\colon \mathsf{sSets} \rightleftarrows \mathsf{cdgLie} : \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, -) \ ,$$

where the functor $\mathfrak{L}$ is obtained by sending the standard n-simplex Δ^n to $\mathfrak{mc}_n$ and then applying the Yoneda lemma to extend it to all simplicial sets. This is proven in [BFMT16, Cor. 3.6]. In particular, both functors preserve weak equivalences. Then one proves:

Proposition 12.3.5 ([BFMT16, Prop. 3.8])**.** *The class of filtered quasi-isomorphisms is contained in the class of weak equivalences.*

Thus, we can give the following alternative proof of the Dolgushev–Rogers theorem.

Proof of Theorem 6.4.1. This is a direct consequence of the two results above and the fact that if $\mathfrak{g}$ is a complete Lie algebra, then there is a canonical homotopy equivalence of simplicial sets

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g})$$

as was proven in [RN17b, Cor. 5.3] (see also [BFMT17]). □

In fact, our argument using framings holds in this model category, too. Indeed, the natural sequence of maps

$$\mathfrak{mc}_0 \sqcup \ldots \sqcup \mathfrak{mc}_0 \longrightarrow \mathfrak{mc}_n \longrightarrow \mathfrak{mc}_0 \ ,$$

which is dual to the one showing that $\mathrm{B}_\iota(s^{-1}C_\bullet)$ is a simplicial frame on $\mathrm{B}_\iota(s^{-1}C_0)$, exhibits $\mathfrak{mc}_\bullet$ as a cosimplicial frame on $\mathfrak{mc}_0$ in the category of complete Lie algebras with the BFMT model structure: the second map is a weak equivalence, since it comes from a weak equivalence in the Vallette model structure, while the first one is a cofibration by [BFMT16, Thm. 4.2]. This gives yet another alternative proof of Theorem 6.4.1. However, one should remark that the proof of Proposition 12.3.5 in *op. cit.* relies on the Dolgushev–Rogers theorem itself, and thus their proof of Theorem 6.4.1 is — in a sense — not self-contained. On the other hand, Theorem 12.2.13 and Theorem 12.3.2 are proved using only our homotopical approach, but they do not recover the full strength of the Dolgushev–Rogers theorem.

Appendices

Appendix A

Trees

It is very natural to use trees to express operadic concepts, and those objects appear — implicitly and explicitly — throughout the whole text of the present work. Here, we give the basic definitions and the notations that we need.

A.1 Rooted trees

We begin with the[1] basic definition of a rooted tree.

Definition A.1.1. *A* rooted tree τ *is a tuple* (V, E, n, N) *where*

1. V *is a (possibly empty) finite set, called the* set of vertices *of* τ*,*
2. E *is a partial order such that*
   - *for every* $v_1, v_2 \in V$*, there exists a* $w \in V$ *such that* $w \leq v_1$ *and* $w \leq v_2$*, and*
   - *every* $v \in V$ *has at most one parent, i.e. an element* $w \in V$ *such that* $w < v$ *and there exists no* $w' \in V$ *such that* $w < w' < v$*,*
3. n *is an integer, and*
4. N *is a function*
$$N : \{1, \dots, n\} \longrightarrow V$$
from the set $\{1, \dots, n\}$ *to the set of vertices.*

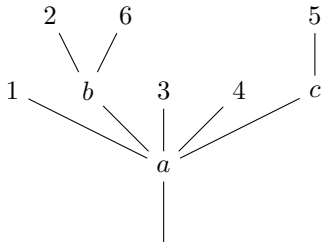
These conditions imply that there is a unique minimal element of V*, which is called the* root *of* τ*. The set of* inner edges *of* τ *— which equivalent to* E*, and thus will also be denoted by* E *— is the set of couples* $(w, v) \in V \times V$ *such that* w *is a parent of* v*, and the set of* leaves *of* τ *is the set*

$$\left\{(v, i) \mid v \in V_{\max} \text{ and } i \in N^{-1}(v)\right\}.$$

For $v \in V$*, the* arity $|v|$ *of* v *is the number of elements of* V *of which* v *is the parent plus* $|N^{-1}(v)|$*. The natural number* n *is called the* arity *of the tree* τ*.*

[1] Better: one. There are many different definitions of the notion of a tree in the literature, most of which are equivalent.

The graphical interpretation we have in mind when working with this definition is illustrated by the following example. Let $V \coloneqq \{a, b, c\}$, take the partial order generated by $a < b$ and $a < c$, let $n = 6$, and take $N : \{1, \dots, 6\} \to \{b, c\}$ be given by sending $\{1, 3, 4\}$ to a, 5 to c, and the rest to b. Then we draw the associated tree as

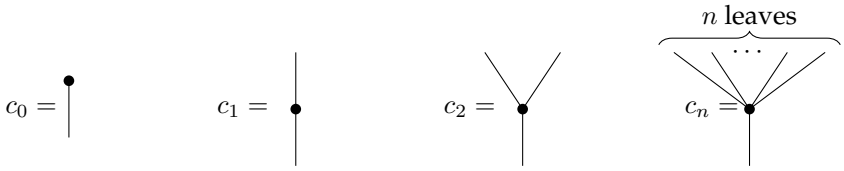


In this example, the arity of the vertex a is 5, and for the other vertices we have $|b| = 2$ and $|c| = 1$.

Definition A.1.2. *The* n-corolla c_n *is the tree given by*

$$c_n \coloneqq (*, \emptyset, n, N : k \mapsto *) .$$

Graphically, the corollas are given by



Let $v_1, v_2 \in V$, then the *first common ancestor* $\mathrm{FCA}(v_1, v_2)$ of v_1 and v_2 is defined by

$$\mathrm{FCA}(v_1, v_2) \coloneqq \max\{v \in V \mid v \leq v_1 \text{ and } v \leq v_2\} .$$

Definition A.1.3. *Let* $\tau_1 = (V_1, E_1, n_1, N_1)$ *and* $\tau_2 = (V_2, E_2, n_2, N_2)$ *be two rooted trees. A* morphism of rooted trees $\phi : \tau_1 \to \tau_2$ *is the data of two maps*

$$\phi_V : V_1 \longrightarrow V_2 \quad \textit{and} \quad \phi_N : \{1, \dots, n_1\} \longrightarrow \{1, \dots, n_2\}$$

such that

1. ϕ_V *is a map of posets, and* ϕ_N *is increasing,*
2. *the diagram*

$$\begin{array}{ccc} \{1, \dots, n_1\} & \xrightarrow{\phi_N} & \{1, \dots, n_2\} \\ {\scriptstyle N_1}\downarrow & & \downarrow{\scriptstyle N_2} \\ V_1 & \xrightarrow{\phi_V} & V_2 \end{array}$$

is commutative,

3. *for all $v' \in \phi_V(V_1)$, if $w' \in V_2$ is such that $w' < v'$, then $w' \in \phi_V(V_1)$, and*
4. *if $v_1, v_2 \in V$ are such that $\phi_V(v_1) = v' = \phi_V(v_2)$, then $\phi_V(\mathrm{FCA}(v_1, v_2)) = v'$.*

Remark A.1.4. *Equivalently to condition (4), one can ask that the preimage of any vertex under the morphism is again a rooted tree.*

Definition A.1.5. *A tree is* reduced *if all of its vertices have arity at least 2. It is* binary *if all of its vertices have arity exactly 2.*

Definition A.1.6. *We denote by RT the set of isomorphism classes of rooted trees, and by rRT the set of isomorphism classes of reduced rooted trees. For $n \geq 0$, we denote by RT_n the arity n elements of RT, and similarly for rRT_n.*

Let M be an $\mathbb{S}$-module, let $\tau = (V, E, n, N)$ be a rooted tree, and let $f : V \to M$ be a function such that $f(v) \in M(|v|)$. Then we denote by $\tau(f) \in \mathcal{T}(M)$ the element with underlying tree τ and the vertices labeled by their image in M under f.

A.2 Planar trees

A planar tree is just a rooted tree with a chosen way of drawing it in the plane. This can be encoded by a total order on the set of children of a vertex, for every vertex of the tree.

Definition A.2.1. *Let $\tau \in \mathrm{RT}$ be a rooted tree, and let v be a vertex of τ. The set $\mathrm{ch}(v)$ of* children of *v is*

$$\mathrm{ch}(v) \coloneqq N^{-1}(v) \sqcup \{w \in V \mid v \text{ is the parent of } w\} .$$

Notice that $|v| = |\mathrm{ch}(v)|$.

Definition A.2.2. *A* planar tree *t is a rooted tree τ together with a total order on $\mathrm{ch}(v)$ for every vertex v of τ.*

Appendix B

Filtered operads and filtered algebras

In order to make sense of certain naturally occurring infinite sums, such as in the Maurer–Cartan equation for $\mathscr{L}_\infty$-algebras, cf. Section 6.2.1, one has to consider additional data on chain complexes and algebras. One convenient way to do this is to look at filtered chain complexes, i.e. chain complexes equipped with a descending filtration. These ideas go back at least to Lazard's PhD thesis [Laz54]. We base ourselves on [DSV18, Sect. 2] for a clean, modern treatment in the context of operad theory.

B.1 Filtered chain complexes

We begin by defining the category of filtered chain complexes and its monoidal structure.

B.1.1 Definitions

Definition B.1.1. *A* filtered chain complex *is a couple $(V, \mathscr{F}_\bullet V)$ where V is a chain complex, and*

$$V = \mathscr{F}_0 V \supseteq \mathscr{F}_1 V \supseteq \mathscr{F}_2 V \supseteq \cdots$$

is a descending chain of sub-chain complexes. We will often omit the filtration from the notation when there is no risk of confusion, and speak of the filtered chain complex V. Moreover, we say that V is proper *if $\mathscr{F}_1 V = V$.*

Any filtered chain complex V automatically carries a first-countable topology of which a basis is given by

$$\{v + \mathscr{F}_k V \mid v \in V \text{ and } k \geq 0\} .$$

With this topology, the chain complex V becomes a Fréchet-Urysohn space, i.e. topological closure and sequential closure in V are the same. Notice that scalar multiplication and sum are continuous with respect to this topology.

Definition B.1.2. *Let V and W be filtered chain complexes. A* filtered morphism *$f : V \to W$ is a morphism of chain complexes from V to W, i.e. a chain map, such that for all $n \geq 0$ we have*

$$f(\mathscr{F}_n V) \subseteq \mathscr{F}_n W .$$

We denote by fCh *the category of filtered chain complexes and filtered morphisms.*

Lemma B.1.3. *Every filtered morphism is continuous with respect to topologies induced by the filtrations.*

Proof. Let $f : V \to W$ be a filtered morphism, and let $n \geq 0$. Then, for any $v \in V$ such that $f(v) \in \mathscr{F}_n W$, we have

$$f(v + \mathscr{F}_n V) = f(v) + f(\mathscr{F}_n V) \subseteq \mathscr{F}_n W \ .$$

Therefore, we have

$$f^{-1}(\mathscr{F}_n W) = \bigcup_{v \in f^{-1}(\mathscr{F}_n W)} v + \mathscr{F}_n V \ ,$$

which implies the claim. We denote by cCh the full subcategory of fCh spanned by complete filtered morphisms. □

B.1.2 Limits and colimits of filtered chain complexes

We can describe all limits and colimits in the category of filtered chain complexes.

Proposition B.1.4. *The category* fCh *of filtered chain complexes is complete and cocomplete, with limits and colimits given as follows.*

1. *The product of a collection $\{(V^i, \mathscr{F}_\bullet V^i)\}_{i \in I}$ of filtered chain complexes is the chain complex $\prod_{i \in I} V^i$ together with the filtration*

$$\mathscr{F}_n \left(\prod_{i \in I} V^i \right) \coloneqq \prod_{i \in I} \mathscr{F}_n V^i .$$

2. *The coproduct of a collection $\{(V^i, \mathscr{F}_\bullet V^i)\}_{i \in I}$ of filtered chain complexes is the chain complex $\bigoplus_{i \in I} V^i$ together with the filtration*

$$\mathscr{F}_n \left(\bigoplus_{i \in I} V^i \right) \coloneqq \bigoplus_{i \in I} \mathscr{F}_n V^i .$$

3. *If $f : (V, \mathscr{F}_\bullet V) \to (W, \mathscr{F}_\bullet W)$ is a filtered chain map, then the kernel of f is the chain complex $\ker f \subseteq V$ endowed with the filtration*

$$\mathscr{F}_n \ker f \coloneqq \ker f \cap \mathscr{F}_n V \ .$$

   Given two filtered chain maps f, g with the same domain and range, their equalizer is $\ker(f - g)$.

4. *If $f : (V, \mathscr{F}_\bullet V) \to (W, \mathscr{F}_\bullet W)$ is a filtered chain map, then the cokernel of f is the chain complex $W/f(V)$ endowed with the filtration*

$$\mathscr{F}_n(W/f(V)) \coloneqq \mathscr{F}_n W/(f(V) \cap \mathscr{F}_n W) \ .$$

   Given two filtered chain maps f, g with the same domain and range, their coequalizer is the cokernel of the difference $f - g$.

In particular, limits and colimits of proper filtered chain complexes are again proper.

B.1.3 The closed symmetric monoidal structure

Let V, W be two filtered chain complexes. On the chain complex $\hom(V, W)$ we put the filtration

$$\mathscr{F}_n \hom(V, W) \coloneqq \{f : V \to W \mid f(\mathscr{F}_k V) \subseteq \mathscr{F}_{k+n} W \text{ for all } k \geq 0\} .$$

This makes $\hom(V, W)$ into a filtered chain complex. Similarly, we endow the chain complex $V \otimes W$ with the filtration

$$\mathscr{F}_n(V \otimes W) \coloneqq \sum_{m+k=n} \mathscr{F}_m V \otimes \mathscr{F}_k W .$$

This makes $V \otimes W$ into a filtered chain complex.

Theorem B.1.5. *The category* fCh *with the tensor product and the inner hom defined as above is a closed symmetric monoidal category.*

B.1.4 Complete chain complexes

If one desires to consider infinite sums in a chain complex, asking for a filtration is not enough. Therefore, one defines a complete chain complex as follows.

Definition B.1.6. *A* complete chain complex V *is a filtered chain complex such that the canonical map*

$$V \longrightarrow \lim_n V/\mathscr{F}_n V$$

is an isomorphism.

In a complete chain complex, an infinite sum $\sum_{k\geq 0} v_k$ of terms $v_k \in V$ makes sense as long as for all $n \geq 0$ there is a k_0 such that $v_k \in \mathscr{F}_n V$ for all $k \geq k_0$.

Lemma B.1.7. *If a filtered chain complex V is complete, then*

$$\bigcap_{n\geq 1} \mathscr{F}_n V = \{0\} .$$

Proof. The kernel of the canonical map

$$V \longrightarrow \lim_n V/\mathscr{F}_n V$$

is the intersection $\bigcap_{n\geq 1} \mathscr{F}_n V$. □

Given a filtered chain complex V, one defines its *completion* as the filtered chain complex

$$\widehat{V} \coloneqq \lim_n V/\mathscr{F}_n V , \qquad \text{and} \qquad \mathscr{F}_n \widehat{V} \coloneqq \lim_k \mathscr{F}_n V/\mathscr{F}_k V = \widehat{\mathscr{F}_n V} .$$

Lemma B.1.8. *Let V be a filtered chain complex. The completion $\widehat{V}$ is complete.*

Proof. The statement follows immediately from the fact that

$$\widehat{V}/\mathscr{F}_n \widehat{V} \cong V/\mathscr{F}_n V .$$

Fix $n \geq 1$. An element of $\widehat{V}$ can be given as a sequence $(v_k)_{k=1}^{\infty}$ with $v_k \in V/\mathscr{F}_k V$, and v_{k+1} mapping to v_k under the canonical projection $\widehat{V}/\mathscr{F}_{k+1}\widehat{V} \to \widehat{V}/\mathscr{F}_k \widehat{V}$. Fix such an element, and

choose representatives in V for the v_k. We will abuse of notation and again write v_k for the chosen representative of v_k. We define a sequence $(w_k)_{k=1}^{\infty}$ by

$$w_k \coloneqq \begin{cases} 0 & \text{if } k \leq n \ , \\ v_k - v_n & \text{if } k \geq n+1 \ , \end{cases}$$

where we see w_k as an element of $V/\mathscr{F}_k V$. It is straightforward to check that $(w_k)_{k=1}^{\infty}$ defines an element of $\mathscr{F}_n \widehat{V}$. Then

$$(v_k)_{k=1}^{\infty} - (w_k)_{k=1}^{\infty} = (v_1, v_2, \ldots, v_n, v_n, v_n, \ldots) \ .$$

Thus, we can map the class of $(v_k)_{k=1}^{\infty}$ in $\widehat{V}/\mathscr{F}_n\widehat{V}$ to $v_n \in V/\mathscr{F}_n V$. This is well defined, and gives us a map

$$\widehat{V}/\mathscr{F}_n\widehat{V} \longrightarrow V/\mathscr{F}_n V \ .$$

This map is an isomorphism, as it has an inverse which is given by sending $v \in V/\mathscr{F}_n V$ to $(v, v, v, v, \ldots) \in \widehat{V}/\mathscr{F}_n\widehat{V}$ (where once again we confound equivalence classes and representatives). □

Morphisms between filtered chain complexes induce morphisms between their completions. With this action, completion defines a functor

$$\widehat{\ }\colon \mathsf{fCh} \longrightarrow \mathsf{cCh}$$

from filtered chain complexes to complete chain complexes.

Lemma B.1.9. *The completion functor is left adjoint to the forgetful functor*

$$(-)^{\#} : \mathsf{cCh} \longrightarrow \mathsf{fCh} \ .$$

Proposition B.1.10. *The category of complete chain complexes is complete and cocomplete. Its limits are the same as the limits taken in the category of filtered chain complexes, and its colimits are the completion of the colimits taken in the category of filtered chain complexes. The subcategory of proper complete chain complexes is also complete and cocomplete, with the same limits and colimits.*

Finally, one defines a tensor product in the category of complete chain complexes as the completion of the tensor product in the category of filtered chain complexes.

B.2 Proper complete algebras over an operad

We are interested in the notion of filtered, proper, and complete algebras over an operad. One can also define a natural notion of filtered operad, but it is not extremely interesting for the applications we have in mind in the rest of the present work.

B.2.1 Filtered algebras over operads and related notions

Let V be a filtered chain complex. We define the *filtered endomorphism operad* by

$$\mathrm{End}_V(n) \coloneqq (\hom(V^{\otimes n}, V), \mathscr{F}_\bullet \hom(V^{\otimes n}, V)) \ .$$

Notice that the composition map is a filtered morphism. Also, if V is a complete chain complex, it is indifferent if we take the completed tensor product or the usual tensor product of filtered chain complexes in the definition.

Let $\mathscr{P}$ be an operad, then a *filtered*, respectively *complete* (and possibly *proper*) $\mathscr{P}$*-algebra* is a filtered, resp. complete (proper) chain complex A together with a morphism of operads

$$\rho_A : \mathscr{P} \longrightarrow \mathrm{End}_A \, .$$

In other words, it is a map

$$\gamma_A : \mathscr{P}(A) \longrightarrow A$$

satisfying the usual axioms for a $\mathscr{P}$-algebra and which is such that

$$\gamma_A(\mathscr{P}(k) \otimes \mathscr{F}_{n_1} A \otimes \cdots \otimes \mathscr{F}_{n_k} A) \subseteq \mathscr{F}_{n_1+\cdots+n_k} A \, .$$

We will sometimes denote the category of proper complete $\mathscr{P}$-algebras with filtered morphisms of $\mathscr{P}$-algebras by $\widehat{\mathscr{P}}$-alg.

B.2.2 Filtered ∞-morphisms

The notion of a filtered morphism of algebra over an operad is immediate. Filtered ∞-morphisms are not much more complicated.

Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism, and let A, B be two filtered $\mathscr{P}$-algebras.

Definition B.2.1. A filtered ∞_α-morphism *is an* ∞_α*-morphism*[1] $\Phi : A \rightsquigarrow B$ *such that*

$$\phi_n(\mathscr{C}(n) \otimes \mathscr{F}_{n_1} A \otimes \cdots \otimes \mathscr{F}_{n_k} A) \subseteq \mathscr{F}_{n_1+\cdots+n_k} B \, .$$

It is a filtered ∞_α-quasi-isomorphism *if for every* $n \geq 1$*, the chain map*

$$\phi_1 : \mathscr{F}_n A \longrightarrow \mathscr{F}_n B$$

is a quasi-isomorphism.

B.2.3 Filtrations and cofiltrations

In Section 3.3.7, we mentioned te notion of a cofiltered coalgebra over a cooperad, see Definition 3.3.10. The relation between filtered algebras and cofiltered coalgebras is strightforward.

Lemma B.2.2. *Let C be a cofiltered $\mathscr{C}$-coalgebra. Then $C^\vee$ is a complete $\mathscr{C}^\vee$-algebra when endowed with the orthogonal filtration*

$$\mathscr{F}_n C^\vee := (\mathscr{F}^n C)^\perp .$$

Proof. Straightforward. □

Notice that in the definition of cofiltered coalgebra we require $\mathrm{colim}_n \mathscr{F}^n C \cong C$, which corresponds to completeness on the algebra side. The condition of being proper corresponds to $\mathscr{F}^1 C = 0$.

[1] See Chapter 8.

Appendix C

Formal fixed-point equations and differential equations

We present some results about fixed-point equations and differential equations in the setting of complete, possibly graded, vector spaces — and in particular, complete chain complexes. We prove that under reasonable assumptions, both fixed-point equations and differential equations admit unique solutions in this context.

C.1 Formal fixed-point equations

We begin by defining formal fixed-point equations in complete graded vector spaces, i.e. equations of the form $x = f(x)$, with some conditions on f. The conditions we impose imply that any such equation admits a unique solution.

C.1.1 Definitions

Let V be a complete graded vector space.

Definition C.1.1. *A* homogeneous polynomial operator *P of (polynomial) degree k on V is a map $P : V \to V$ of the form*

$$P(v) = F(v^{\otimes k})$$

for some linear map

$$F : V^{\otimes k} \longrightarrow V \ .$$

We call F the linear map associated to P. A polynomial operator *on V is a finite sum of homogeneous polynomial operators.*

Definition C.1.2. *A* formal fixed-point equation *on V is an equation of the form*

$$x = P_0 + \sum_{n \geq 1} P_n(x) \ ,$$

in the variable x, where $P_0 \in V$ and for $n \geq 2$, P_n is a polynomial operator on V such that the linear map associated to any homogeneous polynomial composing P_n increases the filtration by n.

C.1.2 Existence and uniqueness of solutions

We show that every formal fixed-point equation admits a unique solution.

Theorem C.1.3. *Let*

$$x = P_0 + \sum_{n\geq 1} P_n(x)\ , \tag{C.1}$$

be a formal fixed-point equation on V. Then there exists a unique element of V solving the equation.

Proof. We begin by constructing a solution recursively. At step $i \geq 0$ we want to have an element $v_i \in \mathscr{F}_i V$ such that $\widetilde{v}_i \coloneqq v0 + v_1 + \cdots + v_i$ solves the equation in $V/\mathscr{F}_{i+1}V$.

For steps 0 and 1, in $V/\mathscr{F}_2 V$ the equation becomes

$$x = P_0 \in V = \mathscr{F}_1 V\ ,$$

as all other terms live at least in $\mathscr{F}_2 V$. Therefore, we fix $v_0 \coloneqq P_0$ and $v_1 = 0$.

Suppose we have our procedure successfully up to step $i \geq 1$. We take equation (C.1), substitute the variable x by $\widetilde{v}_i + y$ and project in $V/\mathscr{F}_{i+2}V$. Under the additional assumption that $y \in \mathscr{F}_{i+1}V$, the resulting equation reads

$$\begin{aligned}\widetilde{v}_i + y &= P_0 + \sum_{n\geq 1} P_n(\widetilde{v}_i + y)\\ &= P_0 + \sum_{n\geq 1} P_n(\widetilde{v}_i)\ ,\end{aligned}$$

since by the definition of a formal fixed-point equation we have

$$P_n(\widetilde{v}_i + y) = P_n(\widetilde{v}_i) + \text{terms in } \mathscr{F}_{i+2}V\ .$$

Therefore, we define

$$v_{i+1} \coloneqq P_0 + \sum_{n\geq 1} P_n(\widetilde{v}_i) - \widetilde{v}_i \in V\ .$$

It is in fact an element of $\mathscr{F}_{i+1}V$ because its projection to $V/\mathscr{F}_{i+1}V$ gives back the equation that we solved at step i.

Passing to the limit for i going to infinity is allowed because we are in a complete graded vector space. Therefore, the process above gives us a solution of the fixed point equation.

For uniqueness, suppose that $v, w \in V$ both solve the fixed-point equation. We show by induction that $v - w \in \mathscr{F}_i V$ for all $i \geq 0$. The cases $i = 0$ and 1 are trivially satisfied. For $i = 2$, in $V/\mathscr{F}_2 V$ we have

$$v = P_0 = w\ .$$

Therefore, we have $v - w \in \mathscr{F}_2 V$.

Suppose the statement is true up to $i \geq 2$. Then the fixed-point equation (C.1) in $V/\mathscr{F}_{i+1}V$ reads

$$\begin{aligned}v &= P_0 + \sum_{n\geq 1} P_n(v)\\ &= P_0 + \sum_{n\geq 1} P_n(w)\end{aligned}$$

$$= w \ ,$$

where in the second line we used the fact that for all $n \geq 1$ we have

$$\begin{aligned} P_n(v) &= P_n(w + (v - w)) \\ &= P_n(w) + \text{terms in } \mathscr{F}_{i+1}V \end{aligned}$$

by induction hypothesis and the definition of a formal fixed-point equation. Therefore, we have $v - w \in \mathscr{F}_{i+1}V$.

We conclude by noticing that this implies that

$$v - w \in \bigcap_{i \geq 1} \mathscr{F}_i V = \{0\} \ ,$$

where we used Lemma B.1.7. □

C.2 Formal differential equations on vector spaces

We define formal differential equations, and prove an existence and uniqueness result for their solution.

C.2.1 Definitions

Let $\mathbb{K}[[t]]$ be the formal power series in the variable t. If V is a (potentially graded) vector space, we denote

$$V[[t]] \coloneqq V \otimes \mathbb{K}[[t]] \ ,$$

the formal power series in t with coefficients in V. We have the "differentiation in t" operator

$$\frac{d}{dt} : V[[t]] \longrightarrow V[[t]]$$

acting by

$$\frac{d}{dt}\left(\sum_{n \geq 0} v_n t^n\right) \coloneqq \sum_{n \geq 1} n v_n t^{n-1} \ .$$

Given a linear map

$$f : V^{\otimes n} \longrightarrow V \ ,$$

there is a canonical extension

$$f : V[[t]]^{\otimes k} \longrightarrow V[[t]] \ .$$

It is given by

$$f\left(\sum_{n_1 \geq 0} v_{n_1,1} t^{n_1}, \ldots, \sum_{n_k \geq 0} v_{n_k,k} t^{n_k}\right) \coloneqq \sum_{n \geq 0} \sum_{n_1 + \cdots + n_k = n} f(v_{n_1,1}, \ldots, v_{n_k,k}) t^n \ .$$

Differentiation in t acts as one expects on such maps.

Definition C.2.1. *Let V be a vector space. For each $k, n \geq 0$ let*

$$f_{n,k} : V^{\otimes n} \longrightarrow V$$

be linear maps, and suppose that for any k, there are finitely many non-zero $f_{n,k}$. A formal differential equation *in V is an equation of the form*

$$\frac{d}{dt}v(t) = \sum_{n,k\geq 0} t^k f_{n,k}(\underbrace{v(t), \dots, v(t)}_{n \text{ times}}) \tag{C.2}$$

in $V[[t]]$.

Remark C.2.2. *The condition that for any k there are finitely many non-zero $f_{n,k}$ is necessary in order for equation (C.2) to make sense. However, it can sometimes be relaxed. For example, if V itself is a proper complete chain complex — see Appendix B — and the $f_{k,n}$ are filtered maps, then we don't have to make any finiteness assumption.*

Notice that for any $v(t) \in V[[t]]$, the value $v(0) \in V$ is well-defined. This in not true for evaluation at any other $t \in \mathbb{K}$ if one does not make any further assumptions.

Lemma C.2.3. *A solution of an equation on the form (C.2) exists and is unique for any fixed initial value at $t = 0$.*

Proof. This is a straightforward induction: one proves that the nth coefficient is completely determined by the coefficients up to the $(n-1)$th, together with the fact that the initial value fixes the 0th coefficient. □

Remark C.2.4. *One can also prove this lemma by imitating formally the proof of the classical Cauchy–Lipschitz theorem*[1] *to obtain a formal fixed-point equation*

$$v(t) = v_0 + \int_0^t \sum_{n,k\geq 0} s^k f_{n,k}(v(s), \dots, v(s))\, ds$$

in the complete graded vector space $V[[t]]$, where the filtration is given by

$$\mathscr{F}_n V[[t]] \coloneqq \{\textit{power series starting at the term } t^n\}\,,$$

and then conclude by Theorem C.1.3.

C.2.2 Solving formal differential equations

We will now exhibit an explicit formula for the solution of a formal differential equation.

Denote by wPT the set of rooted weighted planar trees, that is planar trees with the assignment of an integer weight greater or equal to 1 to each vertex. We include the empty tree $\emptyset$ and the arity 0 and 1 corollas in wPT. Grafting the empty tree to any tree gives back the original tree. We denote by $c_n^{(w)}$ the n-corolla of weight w.

Denoting $\tau = c_n^{(w)} \circ (\tau_1, \dots, \tau_n) \in \text{wPT}$ the grafting of the trees $\tau_1, \dots, \tau_n \in \text{wPT}$ to $c_n^{(w)}$, we construct some functions on wPT recursively. First we have the total weight function

$$W(\emptyset) = 0, \qquad W(c_n^{(w)}) = w, \qquad W(\tau) = w + \sum_{i=1}^{n} W(\tau_i)\,,$$

[1] Or Picard–Lindelöf theorem, depending on who was the reader's lecturer in his or her first course in real analysis.

which gives the sum of all the weights of the tree. Then we have the following coefficient

$$F(\emptyset) = 1, \qquad F(c_n^{(w)}) = w, \qquad F(\tau) = W(\tau)\prod_{i=1}^{n} F(\tau_i)\ .$$

We associate a function

$$\tau : V[[t]]^{\otimes n} \longrightarrow V[[t]]$$

to each tree in wPT by

$$\emptyset \coloneqq \mathrm{id}, \qquad c_n^{(w)} \coloneqq t^w f_{n,w-1}, \qquad \tau \coloneqq t^w f_{n,w-1} \circ (\tau_1, \ldots, \tau_n)\ .$$

Finally, fix $v_0 \in V$. We assign an element of $V[[t]]$ to each tree in wPT by

$$\tau(v_0) \coloneqq \tau(v_0, \ldots, v_0)\ .$$

Proposition C.2.5. *The element of $V[[t]]$ given by*

$$v(t) = \sum_{\tau \in wPT} \frac{1}{F(\tau)}\tau(v_0)$$

is the unique solution for the formal differential equation (C.2) with initial value $v(0) = v_0$.

Proof. Let $\tau \in \mathrm{wPT}$, then notice that the total exponent of the formal variable t in $\tau(v_0)$ is given by the weight $W(\tau)$. Therefore, we have

$$\begin{aligned}
\frac{d}{dt}v(t) &= \frac{d}{dt}\left(\sum_{\tau\in wPT}\frac{1}{F(\tau)}\tau(v_0)\right)\\
&= \frac{d}{dt}\left(v_0 + \sum_{\substack{n\geq 0, w\geq 1\\ \tau_1,\ldots,\tau_n\in wPT}}\frac{1}{F(\tau)}\tau(v_0)\right)\\
&= \sum_{\substack{n\geq 0, w\geq 1\\ \tau_1,\ldots,\tau_n\in wPT}}\frac{1}{F(\tau)}\frac{d}{dt}\tau(v_0)\\
&= \sum_{\substack{n\geq 0, w\geq 1\\ \tau_1,\ldots,\tau_n\in wPT}}\frac{W(\tau)}{F(\tau)}t^{w-1}f_{n,w-1}(\tau_1(v_0),\ldots,\tau_n(v_0))\\
&= \sum_{\substack{n\geq 0, w\geq 1\\ \tau_1,\ldots,\tau_n\in wPT}}\frac{1}{\prod_{i=1}^{n}F(\tau_i)}t^{w-1}f_{n,w-1}(\tau_1(v_0),\ldots,\tau_n(v_0))\\
&= \sum_{n,k\geq 0} t^k f_{n,k}\left(\sum_{\tau_1\in wPT}\frac{1}{F(\tau_1)}\tau_1(v_0),\ldots,\sum_{\tau_n\in wPT}\frac{1}{F(\tau_n)}\tau_n(v_0)\right)\\
&= \sum_{n,k\geq 0} t^k f_{n,k}(v(t),\ldots,v(t))\ .
\end{aligned}$$

In the second line, we wrote the sum over all weighted planar trees as the sum over all possible combinations of a corolla at the root with subtrees grafted at its leaves. Implicitly, we denoted

$\tau = c_n^{(w)} \circ (\tau_1, \dots, \tau_n)$. In the fourth and fifth line, we used the definition of $\tau(v_0)$ and of $F(\tau)$. In the sixth line, we renamed $k = w - 1$ and used the fact that $f_{n,k}$ is multilinear. The fact that we have the correct initial value is obvious from the fact that the only term which doesn't involve the formal variable t is $\emptyset(v_0) = v_0$. □

Remark C.2.6. *The material presented here has close relations with the B-series studied in numerical analysis, cf. e.g. with [But08, Ch. 38].*

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Résumé

Cette thèse s'inscrit dans les thèmes de la théorie des opérades et de l'algèbre homotopique.

Soient donnés un type d'algèbres, un type de cogèbres et une relation entre ces types de structures algébriques (codés respectivement par une opérade, une coopérade et un morphisme tordant). Il est possible alors de mettre une structure naturelle d'algèbre de Lie à homotopie près sur l'espace des application linéaires d'une cogèbre C vers une algèbre A. On appelle l'algèbre de Lie à homotopie près obtenue de cette façon l'algèbre de convolution de A et C.

Dans cette thèse, on étudie la théorie des algèbres de convolution et leur compatibilité avec les instruments de l'algèbre homotopique : les infini-morphismes et le théorème de transfert homotopique. Après avoir fait cela, on applique cette théorie à plusieurs domaines, comme la théorie de la déformation dérivée et la théorie de l'homotopie rationnelle. Dans le premier cas, on utilise les instruments développés en construisant une algèbre de Lie universelle qui représente l'espace des éléments de Maurer-Cartan, un objet fondamental de la théorie de la déformation. Dans le deuxième cas, on donne une généralisation d'un résultat de Berglund sur des modèles rationnels pour les espaces de morphismes entre deux espaces pointés.

Mots Clef

1. Opérades algébriques
2. Éléments de Maurer–Cartan
3. Algèbre homotopique
4. Théorie de la déformation
5. Homotopie rationnelle
6. Algèbres de convolution
7. Algèbres de Lie à homotopie près
8. Catégories de modèles

Summary

This thesis is inscribed in the topics of operad theory and homotopical algebra.

Suppose we are given a type of algebras, a type of coalgebras, and a relationship between those types of algebraic structures (encoded by an operad, a cooperad, and a twisting morphism respectively). Then, it is possible to endow the space of linear maps from a coalgebra C and an algebra A with a natural structure of Lie algebra up to homotopy. We call the resulting homotopy Lie algebra the convolution algebra of A and C.

In this thesis, we study the theory of convolution algebras and their compatibility with the tools of homotopical algebra : infinity morphisms and the homotopy transfer theorem. After doing that, we apply this theory to various domains, such as derived deformation theory and rational homotopy theory. In the first case, we use the tools we developed to construct an universal Lie algebra representing the space of Maurer-Cartan elements, a fundamental object of deformation theory. In the second case, we generalize a result of Berglund on rational models for mapping spaces between pointed topological spaces.

Keywords

1. Algebraic operads
2. Maurer–Cartan elements
3. Homotopical algebra
4. Deformation theory
5. Rational homotopy theory
6. Convolution algebras
7. Homotopy Lie algebras
8. Model categories