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Flots rugueux et inclusions différentielles perturbées

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Antoine Brault. Flots rugueux et inclusions différentielles perturbées. Probabilités [math.PR]. Université Toulouse 3 Paul Sabatier, 2018. Français. NNT: . tel-01958193v1

HAL Id: tel-01958193

<https://hal.science/tel-01958193v1>

Submitted on 17 Dec 2018 (v1), last revised 7 Oct 2019 (v2)

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le 9 octobre 2018 par :

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Flots rugueux et inclusions différentielles perturbées

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Chapter 1

Introduction générale

1.1 Thématique de la thèse

La thématique principale de cette thèse est la théorie des trajectoires rugueuses. Elle a été introduite par T. Lyons [Lyo98] pour étudier des équations différentielles stochastiques dirigées par des bruits très généraux sortant du cadre habituel de la théorie d'Itô. L'objet fondamental de cette théorie est ce que l'on appelle la signature d'une trajectoire. Cela correspond à la famille des intégrales itérées du signal contre lui-même. Son étude débute dans les années cinquante avec les travaux de K. T. Chen [Che54, Che57, Che58] mais se limite à des trajectoires régulières. L'article [Lyo98] montre qu'une connaissance *a priori* des premiers termes de la signature du bruit est suffisante pour résoudre une équation différentielle contrôlée par ce bruit. La puissance de cette théorie repose alors sur une séparation nette entre la construction probabiliste des premiers termes de la signature puis trajectorielle des solutions. De plus la topologie employée dans ce cadre implique la continuité de la solution par rapport au bruit. Cette propriété est particulièrement utile lorsque l'on souhaite étudier des schémas numériques ou des théorèmes de type Wong-Zakai.

Depuis la fin des années quatre-vingt-dix, les domaines de recherche initiés par les idées de T. Lyons sont importants. On peut citer, de manière non exhaustive, les travaux portant sur

- les équations différentielles rugueuses (EDR) [Lyo98, Gub04, Dav07, FV10, CL14, Bai15a] et en particulier celles dirigées par le mouvement brownien fractionnaire [BC07, CQ02];
- les équations différentielles aux dérivées partielles stochastiques (théories des structures de régularité [Hai14], des distributions paracontrôlées [GIP15], équation du transport [BG17, Cat16] et lois de conservation [DGHT16]);
- la caractérisation d'une trajectoire par sa signature [HL10, BGLY16];

- l'apprentissage de signaux temporels [CK16, Gra13];
- la compression de données [LS05].

Ce manuscrit se limite principalement à des travaux en rapport avec les EDR. On trouvera cependant dans le chapitre 3 des éléments de la théorie des structures de régularité appliqués à l'étude des EDR.

1.2 Organisation de la thèse

Cette thèse est composée de trois parties indépendantes contenant des articles acceptés, soumis ou en cours de finalisation. Le lecteur trouvera au début de chacune d'elles, un chapitre introductif en français qui donne les thèmes et les apports de la partie. Chaque partie peut être lue indépendamment. Cependant, nous conseillons à des lecteurs peu familiers avec la théorie des trajectoires rugueuses de commencer par la partie I où nous introduisons des notions nécessaires à la compréhension du reste du manuscrit.

1.2.1 Partie I : Équations différentielles rugueuses et structures de régularité

Cette partie est composée de deux chapitres.

Le chapitre 2 est une brève introduction aux équations différentielles rugueuses. Nous expliquons les principales idées qui sous-tendent cette théorie et de donner un aperçu des différentes approches développées pour résoudre une EDR. Cela nous permettra ensuite de relier certaines idées de la théorie des structures de régularité à celles de la théorie des trajectoires rugueuses. Enfin, nous faisons part des apports de la partie qui se trouvent dans le chapitre suivant.

Le chapitre 3 correspond à l'article [Bra17] accepté au *Séminaire de Probabilités*. On cherche dans celui-ci à redémontrer les théorèmes classiques de la théorie des trajectoires rugueuses (construction de l'aire de Lévy, existence et unicité d'une solution à une équation différentielle rugueuse . . .) via les outils de la théorie des structures de régularité. Il ne contient pas de résultat fondamentalement nouveau mais a une visée pédagogique qui aidera, nous l'espérons, le lecteur à comprendre les liens entre ces deux théories.

1.2.2 Partie II : Flots rugueux

Cette partie, consacrée à l'étude des flots d'EDR, est composée de trois chapitres.

Dans le chapitre 4, nous introduisons la notion de flot que l'on définit comme une famille d'applications $\psi := (\psi_{t,s})_{0 \leq s \leq t \leq T}$ d'un espace métrique dans lui-même vérifiant pour

1.2. ORGANISATION DE LA THÈSE

tout $0 \leq r \leq s \leq t \leq T$,

$$\begin{aligned}\psi_{t,t} &= \text{Id}, \\ \psi_{t,s} \circ \psi_{s,r} &= \psi_{t,r}.\end{aligned}\tag{1.1}$$

On cherche à comparer, dans l'état de l'art, les conditions d'existence et de régularité des flots associés à des équations différentielles ordinaires, stochastiques et rugueuses. Puis on introduit les schémas numériques d'Euler pour les EDO et d'Euler-Maruyama pour les EDS. Ils permettent de construire un flot de solutions à partir d'approximations en temps petit de celui-ci. De telles approximations sont ce que l'on appelle des presque flots. Plus précisément, on dira qu'une famille d'applications $(\phi_{t,s})_{0 \leq s \leq t \leq T}$ est un presque flot si elle vérifie "presque" l'équation (1.1). Cette définition sera au cœur des deux chapitres suivants.

Le chapitre 5 est composé d'un article [BL17a] soumis pour publication et écrit en collaboration avec Antoine Lejay. Il s'inspire des approches par schéma d'Euler [Dav07] et par flot [Bai15a] des EDR. Nous étendons le lemme de la couturière linéaire [FdLP⁺06] sur lequel repose la construction de l'intégrale rugueuse, à la construction de flots de solutions d'EDR à partir de leurs approximations en temps petit (presque flots). Nous définissons ainsi une axiomatique des presque flots, notés couramment ϕ , que nous employons pour montrer l'existence en dimension finie d'un flot mesurable associé à ϕ . Ce résultat est ensuite appliqué à des EDR lorsque la régularité du champ de vecteurs n'est pas suffisante pour assurer l'unicité des solutions. Dans le cas où l'on peut majorer uniformément la norme Lipschitz de l'itération du presque flot sur une subdivision π , noté ϕ^π , on prouve lorsque le pas de π devient nulle que ϕ^π converge vers un flot lipschitz. De plus, s'il existe un flot lipschitz, c'est l'unique flot associé à ϕ . Dans le cas où ϕ est perturbé par un terme additif assez régulier, on montre qu'il conserve ses propriétés. Enfin, on retrouve avec ce formalisme les lemmes de la couturière additif [FdLP⁺06] et multiplicatif [FdLPM08, CL14].

Le chapitre 6 est la suite du travail sur les presque flots du chapitre 5. Il est constitué d'un second article [BL17b] écrit en collaboration avec Antoine Lejay et soumis pour publication. On y étend la définition de presque flot à celle de presque flot stable. Si ϕ est stable alors la norme lipschitz de ϕ^π peut être bornée uniformément en π . D'après les résultats du chapitre 5, cela implique la convergence de ϕ^π vers un flot Lipschitz. On montre par ailleurs que ϕ est stable par inversion. Enfin, on utilise notre cadre pour montrer que les presque flots définis par P. Friz, N. Victoir [FV10] et par I. Bailleul [Bai15a] peuvent être vus comme des perturbations du presque flot dû à A. M. Davie [Dav07].

1.2.3 Partie III : Inclusions différentielles perturbées

Cette partie est composée de deux chapitres.

CHAPTER 1. INTRODUCTION GÉNÉRALE

Le chapitre 7 introduit un problème très étudié dans les années soixante-dix et quatre-vingt qui est celui de l'existence d'une trajectoire $y : [0, T] \rightarrow \mathbb{R}^n$ à une inclusion différentielle

$$\frac{dy_t}{dt} \in F(t, y_t), \quad y_0 = \xi,$$

où

- $T > 0$ est un horizon de temps,
- F est une fonction multivaluée de $[0, T] \times \mathbb{R}^n$ dans les compacts non vides de \mathbb{R}^n ;
- $\xi \in \mathbb{R}^n$ est la condition initiale.

La première stratégie pour résoudre (1.3) consiste à se ramener à une équation différentielle ordinaire en cherchant une fonction f continue telle que $f(t, x) \in F(t, x)$ pour tout $(t, x) \in [0, T] \times \mathbb{R}^n$. Malheureusement, une telle sélection n'existe pas en général, à moins que F soit assez régulière et à images convexes. Une autre stratégie consiste à chercher un point fixe de la fonction multivaluée suivante :

$$y \in B_{\text{Lip}}^\xi \mapsto \left\{ z \in B_{\text{Lip}}^\xi \mid z'(t) \in F(t, y(t)), \forall t \in [0, T] \right\}, \quad (1.2)$$

où $B_{\text{Lip}}^\xi := \{z \in C^0([0, T], \mathbb{R}^n) \mid \|z(t) - z(s)\| \leq M|t - s|, \forall s, t \in [0, T], z(0) = \xi\}$ pour un certain $M > 0$ fixé. Dans certains cas, cela permet alors de se passer de l'hypothèse de convexité sur les images de F . Nous exposons plus précisément dans ce chapitre ces méthodes que nous allons adapter dans le chapitre 8 au cas d'une inclusion perturbée. Enfin un état de l'art sur les résultats d'existence des solutions d'inclusions différentielles stochastiques est fait.

Le chapitre 8 est l'objet d'un travail en cours et en collaboration avec Laure Coutin. On s'intéresse à la résolution d'une inclusion différentielle perturbée par une trajectoire rugueuse. Plus précisément, pour un horizon de temps $T > 0$, on cherche une trajectoire $y : [0, T] \rightarrow \mathbb{R}^n$ telle que

$$dy_t \in F(t, y_t)dt + \sigma(y(t))d\mathbf{X}_t, \quad y_0 = \xi, \quad (1.3)$$

où

- F est une fonction multivaluée de $[0, T] \times \mathbb{R}^n$ dans les compacts non vides de \mathbb{R}^n ;
- $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ est une fonction "assez" régulière;
- \mathbf{X} est une trajectoire rugueuse d -dimensionnelle;
- $\xi \in \mathbb{R}^n$ est la condition initiale.

On définit la notion de solution avant de prouver avec la méthode du point fixe décrite dans le chapitre 7, l'existence de solutions pour deux degrés de régularité de F^1 et sans hypothèse de convexité sur ses images.

¹continue et semi-continue inférieurement

Part I

Équations différentielles rugueuses et structures de régularité

Chapter 2

Introduction aux équations différentielles rugueuses

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2.1 Les équations différentielles rugueuses

Supposons que l'on cherche à étudier un système décrit par une fonction $y : [0, T] \rightarrow \mathbb{R}^n$ dont la dynamique est formellement donnée par

$$y_t = \xi + \int_0^t \sum_{i=1}^d F_i(y_s) dW_s^i, \quad \forall t \in [0, T] \quad (2.1)$$

où $\xi \in \mathbb{R}^n$, $W := (W^1, \dots, W^d) : [0, T] \rightarrow \mathbb{R}^d$ est un signal continu, et $F := (F_1, \dots, F_d)$ est un champ de vecteurs régulier. Nous dirons que y est contrôlée par W .

Dans le cas où W est régulier, on peut donner un sens à la dynamique de y en écrivant (2.1) comme

$$y_t = \xi + \int_0^t F(y_s) \dot{W}_s ds, \quad \forall t \in [0, T],$$

où \dot{W} désigne la dérivée de W . La mécanique de la théorie des équations différentielles ordinaires (EDO) s'applique alors directement et permet d'obtenir l'existence (si F est continu) et l'unicité (si F est lipschitzienne) d'une solution.

La théorie des équations différentielles rugueuses (EDR) commence dès que W n'a plus la régularité escomptée pour faire appel à celle des EDO.

2.2 Approche de Doss-Sussmann

Une première approche qui semble raisonnable consiste à remplacer dans (2.1) le signal W par une fonction régulière $W(\epsilon)$ telle que $W(\epsilon) \rightarrow W$ quand $\epsilon \rightarrow 0$. On prouve alors l'existence et l'unicité d'une solution $y(\epsilon)$. Il ne reste plus qu'à montrer que $y(\epsilon)$ converge vers une fonction y que l'on appellera solution de l'EDR. Cela revient à montrer que l'application d'Itô qui associe à chaque signal d'entrée $W(\epsilon)$ la solution $y(\epsilon)$ est continue pour une certaine topologie.

C'est le programme qui a été entrepris parallèlement par Halim Doss (1977) dans [Dos77] et Héctor J. Sussmann (1978) dans [Sus78] et que nous nommons familièrement procédé de Doss-Sussmann. Ils montrent la continuité de l'application d'Itô pour la topologie uniforme dans le cas où le crochet de Lie $[F_i, F_j] := D(F_i)F_j - D(F_j)F_i = 0$ pour tout $i, j \in \{1, \dots, d\}$, avec D l'opérateur de dérivation classique. On remarque que si $d = 1$, cette condition est toujours remplie.

Donnons une explication heuristique de ce résultat. On suppose F assez régulier pour que les calculs fassent sens. Appliquons la formule de Taylor,

$$\begin{aligned} y_t &= \xi + \sum_{i=1}^d F_i(\xi) \int_0^t dW_s^i(\epsilon) + \sum_{i=1}^d \int_0^t [F_i(y_s) - F_i(\xi)] dW_s^i(\epsilon) \\ &= \xi + \sum_{i=1}^d F_i(\xi) W_{0,t}^i(\epsilon) + \sum_{i=1}^d \sum_{j=1}^d D(F_i)(\xi) \int_0^t \int_0^s F_j(y_r) dW_r^j(\epsilon) dW_s^i(\epsilon) + R_1 \\ &= \xi + \underbrace{\sum_{i=1}^d F_i(\xi) W_{0,t}^i(\epsilon) + \sum_{i=1}^d \sum_{j=1}^d D(F_i)(\xi) F_j(\xi) \int_0^t \int_0^s dW_r^j(\epsilon) dW_s^i(\epsilon)}_{G(W(\epsilon))_t} + R_1 + R_2, \quad (2.2) \end{aligned}$$

où R_1, R_2 sont des restes dépendant de y mais que nous considérons négligeables pour

2.3. INTÉGRALE DE YOUNG

notre propos¹. Dans ce cas simplifié, montrer la continuité de l'application d'Itô revient à montrer celle de G . Or lorsque $[F_i, F_j] = 0$, $G(W)_t$ se récrit

$$G(W(\epsilon))_t = \xi + \sum_{i=1}^d F_i(\xi) W_{0,t}^i(\epsilon) + \sum_{i=1}^d D(F_i)(\xi) F_j(\xi) \int_0^t \int_0^s dW_r^i(\epsilon) dW_s^i(\epsilon) \\ + \sum_{1 \leq i < j \leq d} D(F_i)(\xi) F_j(\xi) \int_0^t \int_0^s [dW_r^j(\epsilon) dW_s^i(\epsilon) + dW_r^i(\epsilon) dW_s^j(\epsilon)]. \quad (2.3)$$

$$= \xi + \sum_{i=1}^d F_i(\xi) W_{0,t}^i(\epsilon) + \sum_{i=1}^d D(F_i)(\xi) F_i(\xi) \frac{W_{0,t}^i(\epsilon) W_{0,t}^i(\epsilon)}{2} \\ + \sum_{1 \leq i < j \leq d} D(F_i)(\xi) F_j(\xi) W_{0,t}^j(\epsilon) W_{0,t}^i(\epsilon). \quad (2.4)$$

On a fait disparaître dans (2.4) les intégrales itérées du signal contre lui-même, susceptibles de diverger lors du passage à la limite $\epsilon \rightarrow 0$. La continuité de G pour la topologie uniforme est alors immédiate.

Malheureusement si la condition $[F_i, F_j] = 0$ n'est plus remplie², on doit directement montrer une convergence des termes

$$\int_0^t \int_0^s dW_r^j(\epsilon) dW_s^i(\epsilon), \quad (2.5)$$

ce qui implique d'avoir un contrôle sur les incréments de W . Donc un passage de la topologie uniforme à celle des espaces Hölder.

2.3 Intégrale de Young

Nous avons vu dans la section précédente que pour traiter (2.1) avec un champ de vecteurs F ne remplissant pas la condition de nullité du crochet de Lie, une piste envisageable était d'exiger une régularité plus forte que la continuité sur W .

Pour $\alpha \in (0, 1]$, notons C^α l'espace des fonctions α -Hölder continues de $[0, T]$ dans \mathbb{R}^d (d un entier), c'est-à-dire l'ensemble des fonctions $f : [0, T] \rightarrow \mathbb{R}^d$ telles que

$$\|f\|_{\alpha, T} := \sup_{s, t \in [0, T], s \neq t} \frac{\|f(t) - f(s)\|}{|t - s|^\alpha} < +\infty,$$

où $\|\cdot\|$ désigne la norme euclidienne de \mathbb{R}^d . Supposons maintenant que W est α -Hölder continue avec $\alpha \in (0, 1]$.

¹Étant donné le gain de régularité à chaque itération, négliger ces termes paraît raisonnable en choisissant l'horizon de temps T assez petit. Typiquement, si W est α -Hölder continu, alors le reste $R_1 + R_2$ sera de l'ordre de $t^{3\alpha}$.

²En fait, la condition optimale est que l'algèbre de Lie générée par les champs de vecteurs soit nilpotent [Yam79]. Mais cela demeure une hypothèse restrictive sur les champs de vecteurs.

En 1936, Young a construit une intégrale [You36] de la forme de (2.5) ayant la propriété de continuité pour la topologie Hölder, justement recherchée ici.

Theorème 2.3.1 (Intégrale de Young). *Soient $\alpha, \beta \in (0, 1]$ tels que $\alpha + \beta > 1$. Soient $(f, g) \in C^\alpha \times C^\beta$. Alors, on peut définir l'intégrale de f contre g indépendamment de toute subdivision P de $[0, t]$ de pas $|P|$, comme*

$$\int_0^t f_s dg_s := \lim_{|P| \rightarrow 0} \sum_{u,v \text{ points successifs de } P} f_u g_{u,v}.$$

De plus, l'application $(f, g) \in C^\alpha \times C^\beta \mapsto \int_0^\cdot f_s dg_s \in C^{\alpha+\beta}$ est continue pour la topologie des espaces considérés.

Ainsi, pour $\alpha \in (1/2, 1]$, on peut donner un sens à (2.5) lorsque ϵ tend vers 0 et l'on résout (2.1). C'est cette construction de l'intégrale qui est utilisée dans [Lyo94] pour résoudre (2.1) avec un théorème de point fixe.

Voyons maintenant à quel type de signal W peut s'appliquer ce résultat.

2.4 Trajectoires de processus stochastiques

Supposons maintenant que W soit la trajectoire d'un processus stochastique.

Dans le cas où W est une semi-martingale, on peut interpréter (2.1) avec la théorie de l'intégrale d'Itô et résoudre le système en supposant le champ de vecteurs lipschitzien. Malheureusement, en pratique bon nombre de processus n'ont pas cette propriété de semi-martingale.

Une des familles de processus la plus célèbre ne répondant pas à ce critère est celle du mouvement brownien fractionnaire. Elle est indexée par un réel $H \in (0, 1)$ appelé exposant de Hurst.

Définition 2.4.1. Le mouvement brownien fractionnaire (mbf) d'indice de Hurst H est l'unique processus gaussien $(B_t^H)_{t \geq 0}$ centré, continu, normalisé tel que $\text{Var}(B_1^H) = 1$ ayant pour fonction de covariance

$$\text{Var}(B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad \forall s, t \in \mathbb{R}_+.$$

Proposition 2.4.1 ([RY13]). *Soient $H \in (0, 1)$ et B^H un mouvement brownien fractionnaire. Alors les trajectoires de B^H sont presque toutes de régularité $(H - \epsilon)$ -Hölder pour tout $\epsilon > 0$. De plus B^H est une semi-martingale si et seulement si $H = 1/2$, si et seulement si B^H est le mouvement brownien standard.*

Remarque 2.4.2. Ce processus trouve de nombreuses applications en finance, voir par exemple [Che03].

2.5. THÉORIE DES TRAJECTOIRES RUGUEUSES

D'après la proposition 2.4.1, le mbf échappe à la théorie d'Itô dès que $H \neq 1/2$. Par contre, si $H > 1/2$, la construction de solutions via l'intégrale de Young que nous avons décrit section 2.3 s'applique. Cette approche est trajectorielle et donc fondamentalement différente de celle probabiliste d'Itô. En particulier le calcul de l'espérance d'une solution de (2.1) n'est pas simple.

La question de l'extension de cette approche trajectorielle aux cas où la régularité Hölder est inférieur à $1/2$ est naturelle. Elle permettrait alors d'une part de donner une alternative à l'approche d'Itô pour le mouvement brownien et de manière plus générale de traiter d'équations différentielles contrôlées par le mbf d'exposant de Hurst quelconque.

2.5 Théorie des trajectoires rugueuses

Dans [Sus78], H. J. Sussman écrit

It seems to us that the search for an appropriate extension of our results to $d > 1$ will require the use of substantially new methods and that, if such an extension is found, it will lead to a significant progress in our understanding of the above mentioned anomalies.

Dans [Lyo98], Terry J. Lyons propose de surmonter ces difficultés en intégrant le problème dans la solution. Il stipule qu'il existe $\mathbb{W} : [0, T]^2 \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ que nous appellerons tenseur d'ordre 2 ayant des propriétés analogues à l'intégrale itérées de W contre lui même. C'est-à-dire pour tout $0 \leq r \leq s \leq t \leq T$ et $i, j \in \{1, \dots, d\}$,

- 1) $\mathbb{W}_{r,t}^{i,j} - \mathbb{W}_{r,s}^{i,j} - \mathbb{W}_{s,t}^{i,j} = W_{r,s}^i W_{s,t}^j$ (Chen's relation),
- 2) $\|\mathbb{W}\|_{2\alpha, T} := \sup_{s \neq t, s, t \in [0, T]} \frac{\|\mathbb{W}_{s,t}\|}{|t-s|^{2\alpha}} < +\infty$.

Lorsque $\alpha \in (1/2, 1]$, on construit via le théorème 2.3.1,

$$\mathbb{W}_{s,t} := \int_s^t W_{s,u} \otimes dW_u,$$

qui vérifie la propriété algébrique 1)³ et analytique 2). Ici \otimes est le produit tensoriel dans \mathbb{R}^d .

Si $\alpha \in (0, 1/2]$, on suppose que \mathbb{W} est construit. T.J. Lyons⁴ a alors montré que si $\alpha > 1/3$, il est possible de construire des solutions trajectoires à (2.1). Comment procède-t-il ?

³Il suffit d'utiliser la relation de Chasles.

⁴Nous nous limitons dans ce chapitre au cas $\alpha \in (1/3, 1]$. Cependant dans [Lyo98] et [FV10] les cas d'une régularité arbitraire ($\alpha \in (0, 1]$) sont traités.

Lorsque qu'on itère N fois l'équation (2.1), comme nous l'avons fait deux fois en (2.2), des termes de la forme

$$\int_{s \leq t_N \leq \dots \leq t_1 \leq t} dW_{t_N} \otimes \dots \otimes dW_{t_1}, \quad (2.6)$$

apparaissent. Pour N fixé, on appelle ce terme, l'intégrale itérée à l'ordre N de W .

Si W est régulier, il suffit de connaître les incréments de W pour construire (2.6) sous la forme d'une somme de Riemann. Par contre, les hypothèses du théorème 2.3.1 ne nous permettent pas d'étendre cela dès que $W \in C^\alpha$, avec $\alpha \in (0, 1/2]$. L'idée importante dans [Lyo98] est que l'on peut construire l'ensemble des intégrales itérées dans le cas $\alpha \in (1/3, 1/2)$ à condition de connaître les incréments de W mais aussi \mathbb{W} .

Introduisons la définition centrale de cette théorie avant donner des précisions.

Définition 2.5.1. Soit $W : [0, T] \rightarrow \mathbb{R}^d$ une fonction de régularité α -Hölderienne pour $\alpha \in (1/3, 1/2]$. On dit que $\mathbf{W} := (W, \mathbb{W})$ est une trajectoire rugueuse α -Hölder continue lorsque \mathbb{W} vérifie 1) et 2). On note \mathcal{C}^α l'espace des trajectoires rugueuses. On munit cet espace de la semi-norme $\|\mathbf{W}\|_{\alpha, T} := \|W\|_{\alpha, T} + \|\mathbb{W}\|_{2\alpha, T}$, où $\|\cdot\|_{\alpha, T}$ est la semi-norme Hölder et $\|\cdot\|_{2\alpha, T}$ est définie en 1).

Le théorème d'extension de Lyons [Lyo98, Théorème 2.2.1] stipule qu'il existe une application bijective qui associe à chaque (W, \mathbb{W}) un élément de l'algèbre tensorielle de \mathbb{R}^d qui correspond à l'intégrale itérée (2.6) pour un ordre N quelconque.

2.6 Petit mot sur la signature

L'ensemble des intégrales itérées pour N parcourant l'ensemble des entiers naturels est ce que l'on appelle la signature de W entre s et t .

Cet objet a été étudié en premier lieu par K. T. Chen dans [Che57]. La signature est à valeurs dans l'algèbre tensorielle de \mathbb{R}^d . Chen a montré que le logarithme de la signature était un élément de Lie, c'est-à-dire qu'il s'exprime comme une combinaison linéaire de crochets de Lie imbriqués d'éléments de \mathbb{R}^d . C'est une généralisation de la formule de Baker-Campbell-Hausdorff-Dynkin qui donne l'expression du logarithme d'un produit d'exponentielles de matrices sous forme de combinaison linéaire de crochet de Lie imbriqués. Cette approche a été utilisée par F. Baudoin pour donner une représentation explicite des flots stochastiques [Bau04]. Cela permet de précisément en étudier les structures locales.

Pour W Lipschitz, il a été montré dans [HL10] que la signature entre 0 et T caractérise à une paramétrisation près le signal tout entier. Récemment ce résultat a été étendu dans [BGLY16] pour les trajectoires rugueuses.⁵

⁵Précisément pour les trajectoires rugueuses faiblement géométriques, c'est-à-dire telles que $\mathbb{W}_{r,t}^{i,j} + \mathbb{W}_{r,t}^{j,i} = W_{r,t}^i W_{r,t}^j$.

2.7. RÉOLUTION D'UNE EDR PAR POINT FIXE

Ces considérations indiquent que la signature est un objet fondamental dont les propriétés dépassent l'étude de (2.1). En effet, elle est utilisée pour faire de la compression [LS05] et trouve des applications en apprentissage automatique pour extraire des caractéristiques de signaux temporels multidimensionnels [CK16]. Elle a été appliquée avec succès dans la reconnaissance d'idéogrammes [Gra13].

2.7 Résolution d'une EDR par point fixe

Les sections précédentes nous ont permis de comprendre l'importance des intégrales itérées du signal W contre lui-même pour la résolution de (2.1).

Dans l'optique de résoudre l'EDR (2.1) de manière rigoureuse dans le cas rugueux, nous souhaiterions utiliser des théorèmes de point fixe qui ont montré leur puissance pour la résolution des équations différentielles ordinaires.

Malheureusement, l'espace des trajectoires rugueuses \mathcal{C}^α n'a pas la structure d'espace vectoriel requise pour l'application des théorèmes de points fixes classiques, à commencer par celui de Schauder.

En 2004, M. Gubinelli a introduit dans [Gub04] la notion d'espaces des chemins contrôlés par W qui sont des espaces de Banach pour une topologie que nous allons définir.

Définition 2.7.1. Étant donné un signal $W \in C^\alpha$ et un entier naturel n , on dit que la trajectoire $y \in C^\alpha$ est contrôlée par W s'il existe $y' \in C^\alpha([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, un reste $R^y : [0, T]^2 \rightarrow \mathbb{R}^n$ tels que pour tout $s, t \in [0, T]$

$$y_{s,t} = y'_s W_{s,t} + R^y_{s,t}, \quad (2.7)$$

avec la condition supplémentaire $\|R^y\|_{2\alpha, T} < +\infty$. On appelle y' la dérivée de Gubinelli. L'ensemble des couples (y, y') forment l'espace vectoriel des chemins contrôlés par W noté $\mathcal{D}_W^{2\alpha}$. Muni de la semi-norme $\|y, y'\|_{2\alpha, T} := \|y'\|_{\alpha, T} + \|R^y\|_{2\alpha, T}$, $\mathcal{D}_W^{2\alpha}$ est un espace de Banach pour la norme $\|y_0\| + \|y'_0\| + \|(y, y')\|_{2\alpha, T}$.

Remarque 2.7.1. Comme les espaces Hölder, l'espace $\mathcal{D}_W^{2\alpha}$ n'est pas séparable pour sa norme associée (voir [FH14]).

On définit ensuite l'intégrale d'un chemin contrôlé par W contre \mathbf{W} .

Theorème 2.7.2. Soient $\alpha \in (1/3, 1/2]$, une trajectoire rugueuse $\mathbf{W} \in \mathcal{C}^\alpha$, un chemin contrôlé $(y, y') \in \mathcal{D}_W^{2\alpha}$. On définit l'intégrale de (y, y') contre \mathbf{W} comme

$$\int_0^t y_s d\mathbf{W}_s := \lim_{|P| \rightarrow 0} \sum_{u,v \text{ points successifs de } P} y_u W_{u,v} + y'_u \mathbb{W}_{u,v}, \quad (2.8)$$

où P est une subdivision quelconque de $[0, t]$ de pas $|P|$.

De plus l'application :

$$I : (y, y') \in \mathcal{D}_W^{2\alpha} \mapsto \left(\int_0^\cdot y_s d\mathbf{W}_s, y \right) \in \mathcal{D}_W^{2\alpha}$$

est continue pour la topologie de $\mathcal{D}_W^{2\alpha}$.

La structure d'espace vectoriel de $\mathcal{D}_W^{2\alpha}$ et la continuité de l'application I permettent de montrer que sous de bonnes conditions de régularité de F , l'application

$$\Phi(y, y') := \left(\xi + \int_0^\cdot F(y_s) d\mathbf{W}_s, F(y_\cdot) \right)$$

a un point fixe. Plus précisément, si $F \in C^{1+\gamma}$ avec $\gamma > 1/\alpha$, on peut appliquer le théorème de point fixe de Schauder sur le compact⁶

$$B_W^{2\alpha, \xi} := \{(y, y') \in \mathcal{D}_W^{2\alpha} : y(0) = \xi, y'(0) = F(\xi), \|y, y'\|_{2\alpha, T} \leq 1\}$$

avec un temps T suffisamment petit. Il existe alors au moins une solution à (2.1). Si $F \in C^{2+\gamma}$, alors on peut montrer que Φ est une contraction sur $B_W^{2\alpha, \xi}$ pour T assez petit et en déduire l'unicité du point fixe et donc de la solution.

2.8 Produit et développement de Taylor

Revenons à la définition 2.7.1 des chemins contrôlés par W . Si $(y, y') \in \mathcal{D}_W^{2\alpha}$, alors l'incrément $y_{s,t}$ est la somme d'un premier terme $y_s W_{s,t}$, α -Hölder et d'un second plus régulier $R_{s,t}^y$, 2α -Hölder. L'expansion (2.7) peut être vu comme un développement de Taylor pour une fonction y non dérivable mais où le temps a été remplacé par un signal irrégulier multidimensionnel.

À partir de ce développement, écrivons formellement l'intégrale d'un chemin contrôlé y contre le signal W . Avec la relation (2.7), pour $s, t \in [0, T]$,

$$\begin{aligned} \int_s^t y_u dW_u &= \int_s^t y_s dW_u + \int_s^t y'_s W_{s,u} dW_u + \int_s^t R_{s,u}^y dW_u \\ &= y_s W_{s,t} + y'_s W_{s,t} + \tilde{R}_{s,t}^y, \end{aligned} \tag{2.9}$$

où $\tilde{R}_{s,t}^y := \int_s^t R_{s,u}^y dW_u$. Il est très important de remarquer que la multiplication de y par un incrément infinitésimal dW a amélioré la régularité de chaque terme dans (2.7) de α . Or en général, la régularité d'un produit de deux fonctions est donnée par celle de la fonction la moins régulière. On remarque donc que si l'on écrit des fonctions sous la forme d'un développement de Taylor classique ou de type (2.7), les régularités de chaque terme du produit vont s'additionner. Ici, on retrouve à partir de (2.9) la définition de l'intégrale rugueuse. En effet, comme $\tilde{R}_{s,t}^y = O(|t-s|^{3\alpha})$ avec $3\alpha > 1$, alors la somme $\sum_{u,v} \text{points successifs de } P \tilde{R}_{s,t}^y \rightarrow 0$ quand $|P| \rightarrow 0$.

⁶En fait $B_W^{2\alpha, \xi}$ n'est pas compacte pour sa norme associée mais l'est pour la norme $\|y_0\| + \|y'_0\| + \|(y, y')\|_{2\beta, T}$ avec $0 < \beta < \alpha$.

2.9 Des développements de Taylor aux structures de régularité

La théorie des structures de régularité a été développée par M. Hairer [Hai14] pour donner un cadre à la résolution d'équations aux dérivées partielles stochastiques (EDPS) comme celle de Kardar-Parisi-Zhang (KPZ)

$$\begin{aligned}\partial_t h(t, x) &= \partial_x^2 h(t, x) + \frac{1}{2}(\partial_x h(t, x))^2 + \xi(t, x), \\ h(0, x) &= h_0(x)\end{aligned}\tag{2.10}$$

où $t \in [0, T]$, $x \in \mathbf{T}$ (où \mathbf{T} est le tore de dimension 1), $h(t, x), h_0(x) \in \mathbb{R}$ et ξ est un bruit blanc spatio-temporel. Formellement, ξ est un processus gaussien centré de covariance $\mathbb{E}(\xi(t, x)\xi(s, x)) = \delta(t - s)\delta(x - y)$ où δ est le dirac en 0. Si l'on fixe l'aléa ω , $\xi(\omega)$ est alors une distribution. On s'attend donc à ce que h soit une distribution aléatoire. Le terme quadratique dans (2.10) n'est a priori pas défini.

Pour définir ce terme, nous devons étudier la régularité de h et donc celle de ξ . On note $\mathcal{B}_{\infty, \infty}^\alpha$ l'espace de Besov classique muni de sa topologie $\|\cdot\|_{\mathcal{B}_{\infty, \infty}^\alpha}$. Nous donnerons Definition 3.3.7, une définition rigoureuse de cet espace et de sa topologie associée. Pour l'instant, il suffit de comprendre que $\mathcal{B}_{\infty, \infty}^\alpha$ est une extension des espaces Hölder qui coïncide avec C^α pour des α positifs et contient des distributions quand $\alpha < 0$. On peut alors montrer que pour tout $\kappa > 0$, $\xi \in \mathcal{B}_{\infty, \infty}^{-\frac{1}{2}-1-\kappa}$ presque sûrement.

L'idée de la théorie des structures de régularité est d'exploiter la forme de l'équation (2.10), pour exprimer la solution h sous la forme d'une expansion similaire à celle de Taylor où le temps est remplacé par des fonctions du bruit ξ .

Pour cela, rappelons le principe de Duhamel ou de variation de la constante. Il permet de résoudre l'équation de la chaleur avec un second membre. On suppose que l'on cherche à résoudre

$$\begin{aligned}\partial_t u(t, x) &= \Delta u(t, x) + f(t, x), \\ u(0, x) &= u_0(x)\end{aligned}$$

où $t > 0$, $x \in \mathbf{T}$, Δ désigne le laplacien, f est une fonction fixée et u_0 la condition initiale. Sous des hypothèses de régularité assez générale de f , la solution prend la forme

$$u(t, x) = \int_0^t \int_{\mathbf{T}} K(t - s, x - y) f(s, y) dy ds + \int_{\mathbf{T}} K(t, x - y) u_0(y) dy,$$

où K est le noyau de la chaleur donné pour $t > 0$ et $x \in \mathbf{T}$ par

$$K(t, x) := \sum_{k \in 2\pi\mathbb{Z}} \frac{1}{(4\pi t)^{1/2}} \exp(-(x - k)^2/t).$$

Afin d'alléger les écritures, on suppose dans ce qui suit que $h_0 = 0$,

$$\begin{aligned} h(t, x) &= (1/2) \int_0^t \int_{\mathbf{T}} K(t-s, x-y) (\partial_y h(s, y))^2 dy ds + \int_0^t \int_{\mathbf{T}} K(t-s, x-y) \xi(s, y) dy ds, \\ &= (1/2) K * (\partial_y h)^2 + K * \xi, \end{aligned} \quad (2.11)$$

où $*$ est le produit de convolution spatio-temporel.

À partir de (2.11), on souhaite appliquer un théorème de point fixe. Pour cela, il faut que les termes soient définis. Les deux théorèmes suivants nous permettent de répondre à cette question.

Le premier résultat quantifie la régularisation du noyau de la chaleur.

Theorème 2.9.1 (Estimée de Schauder [CW15]). *Pour tout $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ et $f \in \mathcal{B}_{\infty, \infty}^\alpha$, alors*

$$\|K * f\|_{\mathcal{B}_{\infty, \infty}^{\alpha+2}} \leq C \|f\|_{\mathcal{B}_{\infty, \infty}^\alpha},$$

où $C \geq 0$ est une constante.

Le second résultat est analogue du théorème 2.3.1 pour les distributions.

Theorème 2.9.2 ([Hai14]). *Soient $\alpha, \beta \in \mathbb{R}$, tels que $\alpha + \beta > 0$, alors il existe une application bilinéaire $B : \mathcal{B}_{\infty, \infty}^\alpha \times \mathcal{B}_{\infty, \infty}^\beta \rightarrow \mathcal{B}_{\infty, \infty}^{\alpha \wedge \beta}$ telle que*

- $B(f, g)$ correspond au produit classique entre deux fonctions quand f, g sont régulières, i.e. $\alpha, \beta \geq 0$.
- Pour tout $f \in \mathcal{B}_{\infty, \infty}^\alpha, g \in \mathcal{B}_{\infty, \infty}^\beta$, il existe une constante $C \geq 0$ telle que

$$\|B(f, g)\|_{\mathcal{B}_{\infty, \infty}^{\alpha \wedge \beta}} \leq C \|f\|_{\mathcal{B}_{\infty, \infty}^\alpha} \|g\|_{\mathcal{B}_{\infty, \infty}^\beta}.$$

Par contre, si $\alpha + \beta \leq 0$, alors il n'existe pas une telle application B .

D'après le théorème 2.9.1, comme pour tout $\kappa > 0, \xi \in \mathcal{B}_{\infty, \infty}^{-3/2-\kappa}$ alors $K * \xi \in \mathcal{B}_{\infty, \infty}^{1/2-\kappa}$. On s'attend donc avec (2.11) à ce que $h \in \mathcal{B}_{\infty, \infty}^{1/2-\kappa}$, mais alors $\partial_x h \in \mathcal{B}_{\infty, \infty}^{-1/2-\kappa}$. Alors d'après le théorème 2.9.2, le terme non linéaire $(\partial_x h)^2$ n'est pas défini. On ne peut donc pas appliquer directement un théorème de point fixe.

Il est cependant intéressant de remarquer que le terme $K * (\partial_x h)^2$ est a priori plus régulier que $K * \xi$.

Tentons alors de régulariser ξ en l'approximation par $\xi_\epsilon := \rho_\epsilon * \xi$, puis d'itérer (2.10). Ici ρ_ϵ est une fonction régularisante. C'est le même procédé que l'on a utilisé pour résoudre (2.1) section 2.2. On obtient alors

$$\begin{aligned} h_\epsilon &= K * \xi_\epsilon + (1/2) K * [\partial_x (K * \xi_\epsilon + (1/2) (\partial_x h)^2)]^2 \\ &= K * \xi_\epsilon + (1/2) K * [\partial (K * \xi_\epsilon)]^2 + R', \end{aligned}$$

2.10. LES OBJECTIFS DE CETTE PARTIE

où R' est un reste qui dépend de h et ξ . En continuant une telle itération, on obtient un développement qui dépend des termes en ξ_ϵ qui ne sont certes pas définis quand $\epsilon \rightarrow 0$ mais dont la régularité augmente. Ainsi en poursuivant ce procédé jusqu'à obtenir des termes de régularité positive, on peut négliger les restes dépendant de h_ϵ . On note

$$\underline{h}_\epsilon := K * \xi_\epsilon + (1/2)K * [\partial_x(K * \xi_\epsilon)]^2 + \dots,$$

où l'on poursuit le développement jusqu'à obtenir des termes de régularité positive et où on néglige les restes dépendant de h_ϵ .

Il faut maintenant caractériser ces termes construits à partir de ξ_ϵ , du produit de convolution avec le noyau de la chaleur K , de la dérivée en espace et enfin du produit. Pour cela, on étudie la vitesse de divergence des termes quand ϵ tend vers 0. On les soustrait ensuite habilement par des constantes C_ϵ pour obtenir des limites finies. C'est ce que l'on appelle la renormalisation de l'équation (2.10). C'est là qu'intervient le théorème de reconstruction (théorème 3.6.5). Il montre que si on a un développement assez précis \underline{h} d'une distribution alors ce développement caractérise complètement cette distribution h que l'on peut reconstruire.

On peut déjà noter ici la forte analogie entre le théorème de reconstruction et le théorème 2.7.2.

2.10 Les objectifs de cette partie

Les objectifs de cette partie sont de donner un sens à une équation différentielle rugueuse et de la résoudre avec la théorie des structures de régularité. Pour cela, nous allons définir un chemin contrôlé à valeurs dans une structure de régularité et utiliser le théorème de reconstruction pour définir l'intégrale rugueuse. Cela nous permettra de préciser l'analogie que nous avons faite section 2.9 entre la théorie de T. Lyons et celle de M. Hairer. Nous utiliserons le théorème de reconstruction dans le cas de non-unicité pour construire le tenseur d'ordre 2 d'une trajectoire irrégulière α -Hölder avec $\alpha \in (1/3, 1/2]$. Enfin, nous procéderons à la résolution d'une EDR par point fixe.

Ces résultats figurent dans le chapitre 3.

CHAPTER 2. INTRODUCTION AUX ÉQUATIONS DIFFÉRENTIELLES
RUGUEUSES

Chapter 3

Solving rough differential equations with the theory of regularity structures

This chapter consists of the article [Bra17] accepted to *Séminaire de Probabilités*.

Abstract

The purpose of this article is to solve rough differential equations with the theory of regularity structures. These new tools recently developed by Martin Hairer for solving semi-linear partial differential stochastic equations were inspired by the rough path theory. We take a pedagogical approach to facilitate the understanding of this new theory. We recover results of the rough path theory with the regularity structure framework. Hence, we show how to formulate a fixed point problem in the abstract space of modelled distributions to solve the rough differential equations. We also give a proof of the existence of a rough path lift with the theory of regularity structure.

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3.1 Introduction

Let $T > 0$ be a finite time horizon. Suppose that we want to solve the following ordinary differential equation

$$\forall t \in [0, T], \quad dy_t = F(y_t)dW_t, \quad y_0 = \xi, \quad (3.1)$$

where $W : [0, T] \rightarrow \mathbb{R}^n$ and $F : \mathbb{R}^d \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ are regular functions. The equation (3.1) can be reformulated as

$$\forall t \in [0, T], \quad y_t = \xi + \int_0^t F(y_u)dW_u. \quad (3.2)$$

When W is smooth, the equation (3.1) is well-defined as

$$y_t = \xi + \int_0^t F(y_u)\dot{W}_u du,$$

where \dot{W} represents the derivative of W . Therefore, it becomes an ordinary differential equation that can be solved by a fixed-point argument.

Unfortunately, there are many natural situations in which we would like to consider the equation of type (3.2) for an irregular path W . This is notably the case when dealing with stochastic processes. For example the paths of the Brownian motion are almost surely nowhere differentiable [KS12]. It is then impossible to interpret (3.1) in a classical sense. Indeed, even if \dot{W} is understood as a distribution, it is not possible in general to define a natural product between distributions, as y is itself to be thought as a distribution.

On the one hand, to overcome this issue, Itô's theory [KS12] was built to define properly an integral against a martingale M (for example the Brownian motion) : $\int_0^t Z_u dM_u$, where Z must have some good properties. The definition is not pathwise as it involves a limit in probability. Moreover, this theory is successful to develop a stochastic calculus with martingales but fails when this property vanishes. This is the case for the fractional Brownian motion, a natural process in modelling. Another bad property is that the map $W \mapsto y$ is not continuous in general with the associated uniform topology [Lyo91].

On the other hand, L.C. Young proved in [You36] that we can define the integral of f against g if f is α -Hölder, g is β -Hölder with $\alpha + \beta > 1$ as

$$\int_0^t f dg = \lim_{|P| \rightarrow 0} \sum_{u,v \text{ successive points in } P} f(u)(g(v) - g(u)),$$

where P is a subdivision of $[0, t]$ and $|P|$ denotes its mesh. This result is sharp, so that it is not possible to extend it to the case $\alpha + \beta \leq 1$ [You36]. If W is α -Hölder it seems natural to think that y is α -Hölder, too. So assuming $\alpha < 1/2$ then $2\alpha < 1$, and Young's

3.1. INTRODUCTION

integral fails to give a meaning to (3.1). The fractional Brownian motion which depends on a parameter H giving its Hölder regularity cannot be dealt with Young's integral as soon as $H \leq 1/2$.

T. Lyons introduced in [Lyo98] the rough path theory which overcomes Young's limitation. The main idea is to construct for $0 \leq s \leq t \leq T$ an object $\mathbb{W}_{s,t}$ which "looks like" $\int_s^t (W_u - W_s) dW_u$ and then define an integral against (W, \mathbb{W}) . This is done with the sewing lemma (Theorem 3.4.15). This theory enabled to solve (3.1) in most of the cases and to define a topology such that the Itô map $(W, \mathbb{W}) \mapsto y$ is continuous. Here, the rough path (W, \mathbb{W}) "encodes" the path W with algebraic operations. It is an extension of the Chen series developed in [Che57] and [Lyo94] to solve controlled differential equations. Since the original article of T. Lyons, other approaches of the rough paths theory were developed in [Dav07], [Gub04] and [Bai15a]. The article [CL14] deals with the linear rough equations with a bounded operators. For monographs about the rough path theory, the reader can refer to [LQ02] or [FV10].

Recently, M. Hairer developed in [Hai14] the theory of regularity structures which can be viewed as a generalisation of the rough path theory. It allows to give a good meaning and to solve singular stochastic partial differential equations (SPDE). One of the main ideas is to build solutions from approximations at different scales. This is done with the reconstruction theorem (Theorem 3.6.5). Another fruitful theory was introduced to solve SPDE in [GIP15] and also studied in [BBF15].

The main goal of this article is to make this new theory understandable to people who are familiar with rough differential equations or ordinary differential equations.

Thus, we propose to solve (3.1) with the theory of regularity structures, when the Hölder regularity of W is in $(1/3, 1/2]$. In particular, we build the rough integral (Theorem 3.4.15) and the tensor of order 2: \mathbb{W} (Theorem 3.4.6) with the reconstruction theorem.

Our approach is very related to [FH14, Chapter 13] where is established the link between rough differential equations and the theory of regularity structures. However, we give here the detailed proofs of Theorem 3.4.15 and Theorem 3.4.15 with the reconstruction theorem. It seems important to make the link between the two theories but is skipped in [FH14].

This article can be read without knowing about rough path or regularity structure theories.

After introducing notation in Section 3.2, we introduce in Section 3.3 the Hölder spaces which allow us to "measure" the regularity of a function. Then, we present the rough path theory in Section 3.4. In the Sections 3.5 and 3.6 we give the framework of the theory of regularity structures and the modelled distributions for solving (3.1). We prove in Sections 3.7 and 3.8 the existence of the controlled rough path integral and the existence of a rough path lift. Finally, after having defined the composition of a function with a modelled distribution in Section 3.9, we solve the rough differential equation (3.1) in Section 3.10.

3.2 Notations

We denote by $\mathcal{L}(A, B)$, the set of linear continuous maps between two vector spaces A and B . Throughout the article, C denotes a positive constant whose value may change. For two functions f and g , we write $f \lesssim g$ if there is a constant C such that $f \leq Cg$. The symbol $:=$ means that the right hand side of the equality defines the left hand side. For a function Z from $[0, T]$ to a vector space, its increment is denoted by $Z_{s,t} := Z_t - Z_s$. If X_1, \dots, X_k are k vectors of a linear space, we denote by $\text{Vect}\langle X_1, \dots, X_k \rangle$ the subspace generated by the linear combinations of these vectors. Let T be a non-negative real, we denote by $[0, T]$ a compact interval of \mathbb{R} . For a continuous function $f : [0, T] \rightarrow E$, where $(E, \|\cdot\|)$ is a Banach space, we denote by $\|f\|_{\infty, T}$ the supremum of $\|f(t)\|$ for $t \in [0, T]$. The tensor product is denoted by \otimes . We denote $[\cdot]$ the floor function.

3.3 Hölder spaces

3.3.1 Classical Hölder spaces with a positive exponent

We introduce Hölder spaces which allow us to characterize the regularity of a non-differentiable function.

Definition 3.3.1. For $0 \leq \alpha < 1$ and $T > 0$, the function $f : [0, T] \rightarrow E$ is α -Hölder if

$$\sup_{s \neq t \in [0, T]} \frac{\|f(t) - f(s)\|}{|t - s|^\alpha} < +\infty.$$

We denote by $C^\alpha(E)$ the *space of α -Hölder functions* equipped with the semi-norm

$$\|f\|_{\alpha, T} := \sup_{s \neq t \in [0, T]} \frac{\|f(t) - f(s)\|}{|t - s|^\alpha}.$$

If $\alpha \geq 1$ such that $\alpha = q + \beta$ where $q \in \mathbb{N}$ and $\beta \in [0, 1)$, we set $f \in C^\alpha(E)$ if f has q derivatives and $f^{(q)}$ is β -Hölder, where $f^{(q)}$ denotes the derivative of order q ($f^{(0)} := f$).

We denote by $C^\alpha = C^\alpha(\mathbb{R}^n)$. For $q \in \mathbb{N}$, we denote by C_b^q the set of all functions $f \in C^q$ such that

$$\|f\|_{C_b^q} := \sum_{k=0}^q \|f^{(k)}\|_{\infty} < +\infty. \quad (3.3)$$

Finally, for $q \in \mathbb{N}$, we define C_0^q the set of functions in C_b^q with a compact support.

Remark 3.3.2. The linear space of α -Hölder functions $C^\alpha(E)$ is a non separable Banach space endowed with one of the two equivalent norms $\|f(0)\| + \|f\|_{\alpha, T}$ or $\|f\|_{\infty, T} + \|f\|_{\alpha, T}$.

Remark 3.3.3. If f is α -Hölder on $[0, T]$, then f is β -Hölder for $\beta < \alpha$, *i.e.* $C^\alpha(E) \subset C^\beta(E)$.

3.3. HÖLDER SPACES

3.3.2 Localised test functions and Hölder spaces with a negative exponent

In equation (3.1), typically W is in C^α with $\alpha \in (0, 1)$. We need to deal with the derivative of W in the sense of distribution which should be of negative Hölder regularity $\alpha - 1 < 0$. We give in this section the definition of the space C^α with $\alpha < 0$. We show in Lemma 3.3.10 that an Hölder function is α -Hölder if and only if the derivative in the sense of distribution is $\alpha - 1$ -Hölder with $\alpha \in (0, 1)$.

For $r > 0$, we denote by B_r the space of all functions in $\eta \in C_b^r$ compactly supported on $[-1, 1]$, such that $\|\eta\|_{C_b^r} \leq 1$.

Definition 3.3.4. For $\lambda > 0$, $s \in \mathbb{R}$ and a test function $\eta \in B_r$, we define the *test function localised* at s by

$$\eta_s^\lambda(t) := \frac{1}{\lambda} \eta\left(\frac{t-s}{\lambda}\right),$$

for all $t \in \mathbb{R}$.

Remark 3.3.5. The lower is λ , the more η_s^λ is localised at s , as can be seen in Figure 3.1.

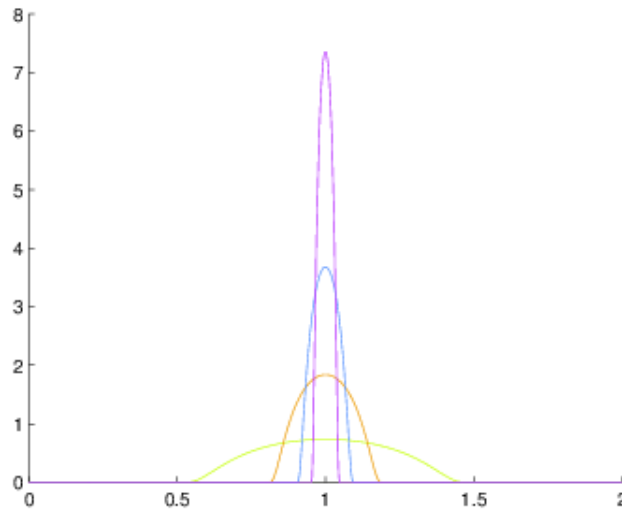


Figure 3.1: Representation of η_s^λ for $s = 1$, $\lambda \in \{0.5, 0.2, 0.1, 0.05\}$ and with $\eta(s) = \exp(-1/(1-s^2))\mathbb{1}_{(-1,1)}(s)$.

Remark 3.3.6. We work here with $t, s \in \mathbb{R}$, because we want to solve stochastic ordinary differential equations. But in the case of stochastic partial differential equations, the parameters t and s belong to \mathbb{R}^e where e is an integer, see [Hai14].

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Definition 3.3.7. For $\alpha < 0$, we define the *Hölder space* C^α as elements in the dual of C_0^r where r is an integer strictly greater than $-\alpha$ and such that for any $\xi \in C^\alpha$ the following estimate holds

$$|\xi(\eta_s^\lambda)| \leq C(T)\lambda^\alpha, \quad (3.4)$$

where $C(T) \geq 0$ is a constant uniform over all $s \in [0, T]$, $\lambda \in (0, 1]$ and $\eta \in B_r$.

We define the semi-norm on C^α as the lowest constant $C(T)$ for a fixed compact $[0, T]$, i.e

$$\xi_{\alpha, T} := \sup_{s \in [0, T]} \sup_{\eta \in B_r} \sup_{\lambda \in (0, 1]} \left| \frac{\xi(\eta_s^\lambda)}{\lambda^\alpha} \right|.$$

Remark 3.3.8. The space C^α does not depend on the choice of r , see for example [FH14] Exercise 13.31, p. 209.

Remark 3.3.9. With Definition 3.3.1, we can give a meaning of an α -Hölder function for $\alpha \in \mathbb{R}$. Moreover it is possible to show that if f is a function in C^α with $\alpha = q + \beta > 0$ where q is an integer and $\beta \in (0, 1)$, then for every $x \in [0, T]$ and localised functions η_x^λ ,

$$|(f - P_x)(\eta_x^\lambda)| \leq C\lambda^\beta,$$

where C is uniform over $x \in [0, T]$, $\lambda \in (0, 1]$ and $\eta \in B_r$ (r a positive integer), P_x is the Taylor expansion of f of the order q in x , and $f - P_x$ is view as the canonical function associated.

Now, when we say that $f \in C^\alpha$ we should distinguish two cases :

- if $\alpha \geq 0$, f is an α -Hölder *function* in the sense of Definition 3.3.1
- if $\alpha < 0$, f is an α -Hölder *distribution* in the sense of Definition 3.3.7.

We give here a characterization of the space C^α for $\alpha \in (-1, 0)$ which is useful to make a link between the rough path and the regularity structures theories.

Lemma 3.3.10. *For any $\beta \in (0, 1)$, the distribution $\xi \in C^{\beta-1}$ if and only if there exist a function $z \in C^\beta$ such that $z(0) = 0$ and*

$$\forall \eta \in C_0^1, \quad \xi(\eta) = -\langle z, \eta' \rangle. \quad (3.5)$$

Which means that $z' = \xi$ in the sense of distribution. Moreover, for all $t \in [0, 1]$,

$$z(t) = \sum_{k \in I_l} \langle \xi, \phi_k^l \rangle \int_0^t \phi_k^l + \sum_{j \geq l} \sum_{k \in I_j} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j,$$

where ϕ, ψ are defined in Theorem 3.3.11 with a compact support in $[-c, c]$ ($c \geq 0$), l is an integer such that $2^{-l}c \leq 1$ and $I_j := [-|c|, 2^j + |c|] \cap \mathbb{Z}$.

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The proof of Lemma 3.3.10 requires to introduce elements of the wavelet theory. The proof of the following theorem can be found in [Mey95].

Theorem 3.3.11. *There exist $\phi, \psi \in C_0^1(\mathbb{R})$ such that for all $n \in \mathbb{N}$*

$$\{\phi_k^i := 2^{i/2} \phi(2^i \cdot -k), k \in \mathbb{Z}\} \cup \{\psi_k^j := 2^{j/2} \psi(2^j \cdot -k), k \in \mathbb{Z}, j \geq i\} \quad (3.6)$$

is an orthonormal basis of $L^2(\mathbb{R})$. This means that for all $f \in L^2(\mathbb{R})$, $i \in \mathbb{N}$ we can write

$$f(t) = \sum_{j \geq i} \sum_{k \in \mathbb{Z}} \langle f, \psi_k^j \rangle \psi_k^j(t) + \sum_{k \in \mathbb{Z}} \langle f, \phi_k^i \rangle \phi_k^i(t), \quad (3.7)$$

where the convergence is in $L^2(\mathbb{R})$. Moreover, we have the very useful property,

$$\int \psi(t) t^k dt = 0, \quad (3.8)$$

for $k \in \{0, 1\}$.

Remark 3.3.12. The notation in Definition 3.3.4 for η_s^λ and in Theorem 3.3.11 for ϕ_k^i, ψ_k^j are similar but the meaning are slightly different.

We now proceed to the proof of Lemma 3.3.10.

Proof of Lemma 3.3.10. The first implication is trivial and does not require the wavelet analysis. If there exists $z \in C^\alpha$ such that for any $\eta \in C_0^1$, $\xi(\eta) = -\langle z, \dot{\eta} \rangle$, then for $\lambda \in (0, 1)$ and $s \in \mathbb{R}$,

$$\begin{aligned} \xi(\eta_s^\lambda) &= -\frac{1}{\lambda^2} \int_{\mathbb{R}} z(u) \dot{\eta} \left(\frac{u-s}{\lambda} \right) du \\ &= -\frac{1}{\lambda^2} \int_{\mathbb{R}} (z(u) - z(-\lambda + s)) \dot{\eta} \left(\frac{u-s}{\lambda} \right) du, \end{aligned}$$

where the last equality holds because η is compactly supported.

With the condition $\eta \in B_1$, $u \mapsto \dot{\eta}((u-s)/\lambda)$ is supported on $[-\lambda + s, \lambda + s]$, which yields to the bound

$$|\xi(\eta_s^\lambda)| \leq 2 \|\eta\|_{C^1} \|z\|_\alpha \lambda^{\alpha-1}, \quad (3.9)$$

and proves that $\xi \in C^{\alpha-1}$.

Now, we prove the converse. Let $\phi, \psi \in C_0^1$ be defined in Theorem 3.3.11. Let $c \geq 0$ be a constant such that supports of ϕ and ψ are in $[-c, c]$. We denote l an integer such that $2^{-l}c \leq 1$. Thus, the support of ϕ_0^l is in $[-1, 1]$ and the support of ψ_k^j is smaller than 2 for $j \geq l$.

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For $\xi \in C^{\alpha-1}$ for $\alpha \in (0, 1)$ we define for $t \in [0, 1]$,

$$z(t) := \sum_{k \in \mathbb{Z}} \langle \xi, \phi_k^l \rangle \int_0^t \phi_k^l + \sum_{j \geq l} \sum_{k \in \mathbb{Z}} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j. \quad (3.10)$$

Noting that for $j \geq l$ and $k \in \mathbb{Z}$, ϕ_k^l and ψ_k^l are compactly supported in $[2^{-j}(k-c), 2^{-j}(k+c)]$, the terms $\int_0^1 \phi_k^j$ and $\int_0^1 \psi_k^j$ vanish when $2^{-j}(k+c) \leq 0$ and $1 \leq 2^{-j}(k-c)$. Thus, we can rewrite (3.10) as

$$z(t) = \sum_{k \in I_l} \langle \xi, \phi_k^l \rangle \int_0^t \phi_k^l + \sum_{j \geq l} \sum_{k \in I_j} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j, \quad (3.11)$$

where $I_j := [-[c], 2^j + [c]] \cap \mathbb{Z}$. The series on the right hand side of (3.10) converges in the sense of distributions. We need to justify that the limit z is in C^α .

We denote for any integer $N \in \mathbb{N}$,

$$S_N^z := \sum_{j=l}^N S_j, \quad (3.12)$$

where $S_j(t) := \sum_{k \in I_l} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j$. According to (3.4), for all $j \geq l$ and $k \in I_j$

$$|\langle \xi, \psi_k^j \rangle| \leq C 2^{j/2-j\alpha}.$$

For $|t-s| \leq 1$, let $j_0 \leq N$ be an integer such that $2^{-j_0} \leq |t-s| < 2^{-j_0+1}$. This is always possible for N large enough. On the one hand, if $l \leq j_0$, for $l \leq j \leq j_0$,

$$\begin{aligned} |S_j(t) - S_j(s)| &\leq \|S_j'\|_\infty |t-s| \\ &\leq |t-s| \sup_{u \in [0,1]} \sum_{I_j} |\langle \xi, \psi_k^j \rangle| \cdot |\psi_k^j(u)| \\ &\leq C 2^{j(1-\alpha)} |t-s|, \end{aligned} \quad (3.13)$$

where we use the fact that $\sum_{k \in I_j} |\psi(2^j t - k)| \leq C$ for a constant $C \geq 0$, because ψ is compactly supported. On the other hand, for $j > \max(j_0, l)$,

$$|S_j(t) - S_j(s)| \leq 2 \|S_j\|_\infty \quad (3.14)$$

$$\leq 2 \sup_{u \in [0,1]} \sum_{k=0}^{2^j-1} |\langle \xi, \psi_k^j \rangle| \cdot |2^{-j} \hat{\psi}_k^j(u)|, \quad (3.15)$$

$$\leq 2C 2^{-j\alpha} \sup_{u \in [0,1]} \sum_{k=0}^{2^j-1} |\hat{\psi}(2^j u - k)|, \quad (3.16)$$

where $\hat{\psi} := \int_0^t \psi$. Because $\int_{\mathbb{R}} \psi = 0$, there is a constant $C' \geq 0$ independent of j such that $\sum_{k=0}^{2^j-1} |\hat{\psi}(2^j u - k) - \hat{\psi}(-k)| < C'$. So finally, for $j > \max(j_0, l)$,

$$|S_j(t) - S_j(s)| \leq C 2^{-j\alpha}. \quad (3.17)$$

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Thus, combining (3.13), (3.17), for $N \geq l$,

$$\begin{aligned} \sum_{j=l}^N |S_j(t) - S_j(s)| &\leq C|t-s| \sum_{j=l}^{j_0} 2^{j(1-\alpha)} + C \sum_{j=j_0+1}^{\infty} 2^{-j\alpha} \\ &\leq C'|t-s|^\alpha, \end{aligned}$$

where C' is a new constant ($\sum_{j=l}^{j_0} 2^{j(1-\alpha)} = 0$ if $j_0 < l$). It follows that $\|S_N^z\|_{\alpha,1}$ is uniformly bounded in N and thus that $z \in C^\alpha$.

Now, we want to check that $\xi = \dot{z}$ in the distribution framework. For any $\eta \in C_0^2$,

$$\begin{aligned} \langle z, \dot{\eta} \rangle &= \sum_{k \in \mathbb{Z}} \langle \xi, \phi_k^l \rangle \left\langle \int_0^t \phi_k^l, \dot{\eta} \right\rangle + \sum_{j \geq l} \sum_{k \in \mathbb{Z}} \langle \xi, \psi_k^j \rangle \left\langle \int_0^t \psi_k^j, \dot{\eta} \right\rangle \\ &= - \sum_{k \in \mathbb{Z}} \langle \xi, \phi_k^l \rangle \langle \phi_k^l, \eta \rangle - \sum_{j \geq l} \sum_{k \in \mathbb{Z}} \langle \xi, \psi_k^j \rangle \langle \psi_k^j, \eta \rangle \\ &= - \langle \xi, \sum_{k \in \mathbb{Z}} \phi_k^l \langle \phi_k^l, \eta \rangle + \sum_{j \geq l} \sum_{k \in \mathbb{Z}} \psi_k^j \langle \psi_k^j, \eta \rangle \rangle \\ &= - \langle \xi, \eta \rangle, \end{aligned}$$

where the commuting of the serie and ξ is justified by the continuity of ξ in C_0^1 and the convergence of the following serie in C_0^1 ,

$$S_N^\eta := \sum_{j=l}^N \sum_{k \in \mathbb{Z}} \psi_k^j \langle \psi_k^j, \eta \rangle. \quad (3.18)$$

Indeed, we have

$$\begin{aligned} |\langle \psi_k^j, \eta \rangle| &\leq 2^{-j/2} \left| \int \psi(x) \eta(2^{-j}(x+k)) dx \right| \\ &\leq 2^{-j/2} \int |\psi(x)| |\eta(2^{-j}(x+k)) - \eta(2^{-j}k) - \eta'(2^{-j}k)(2^j x)| dx \\ &\leq \|\psi\|_\infty \|\eta'\|_\infty 2^{-j/2} 2^{-2j}, \end{aligned}$$

where we use the fact that $\int \psi(t)t^k = 0$ for an integer $k \leq 1$. This implies that

$$\begin{aligned} \sum_{j=l}^N \left\| \sum_{k \in \mathbb{Z}} \psi_k^j \langle \psi_k^j, \eta \rangle \right\|_{C^1} &\leq \sum_{j=l}^N \left\| \sum_{k \in \mathbb{Z}} \psi_k^j \langle \psi_k^j, \eta \rangle \right\|_\infty + \sum_{j=l}^N \left\| \sum_{k \in \mathbb{Z}} 2^j \psi_k^j \langle \psi_k^j, \eta \rangle \right\|_\infty \\ &\leq \left(\sum_{j=0}^N 2^{-2j} + \sum_{j=0}^N 2^{-j} \right) \|\psi\|_\infty \|\eta'\|_\infty, \end{aligned}$$

which proves that S_N^η is absolutely convergent in C_0^1 .

Now by density of C_0^2 in C_0^1 and the continuity of ξ on C_0^1 we conclude that $\langle z, \dot{\eta} \rangle = -\langle \xi, \eta \rangle$ holds for $\eta \in C_0^1$. \blacksquare

3.4 Elements of rough path theory

We introduce here the elements of the rough path theory for solving Equation (3.2). The notions discussed are reformulated in the regularity structure framework in the following sections. For an extensive introduction the reader can refer to [FH14], and for complete monographs to [LQ02, FV10].

3.4.1 The space of rough paths

Let W be a continuous function from $[0, T]$ to \mathbb{R}^n .

We set $\alpha \in (1/3, 1/2]$. Then, (3.2) has not meaning, because the integral term is not defined. The main idea of the rough path theory is to define an object $\mathbb{W}_{s,t}$ which has the same algebraic and analytical properties as $\int_s^t W_{s,u} \otimes dW_u$, the integral of the increment of the path against itself.

The importance of the iterated integrals can be understood with the classical linear differential equations where the solutions are provided with the exponential function. Indeed, if $W : [0, T] \rightarrow \mathbb{R}$ is smooth, the solutions of

$$dy_t = y_t dW_t \tag{3.19}$$

are

$$y_{s,t} = \exp(W_{s,t}) = 1 + \int_s^t dW_{t_1,s} + \int_s^t \int_s^{t_1} dW_{t_2,s} dW_{t_1,s} + \dots \tag{3.20}$$

Definition 3.4.1. An α -Hölder rough path with $\alpha \in (1/3, 1/2]$ is an ordered pair $\mathbf{W} := (W, \mathbb{W})$ of functions, where $W : [0, T] \rightarrow \mathbb{R}^n$ and $\mathbb{W} : [0, T]^2 \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n$ such that

1. For $s, u, t \in [0, T]$, $\mathbb{W}_{s,t} - \mathbb{W}_{s,u} - \mathbb{W}_{u,t} = W_{s,u} \otimes W_{u,t}$ (Chen's relation), i.e., for every $1 \leq i, j \leq n$, $\mathbb{W}_{s,t}^{i,j} - \mathbb{W}_{s,u}^{i,j} - \mathbb{W}_{u,t}^{i,j} = W_{s,u}^i W_{u,t}^j$.
2. The function W is α -Hölder and \mathbb{W} is 2α -Hölder in the sense

$$\|\mathbb{W}\|_{2\alpha, T} := \sup_{s \neq t \in [0, T]} \frac{\|\mathbb{W}_{s,t}\|}{|t - s|^{2\alpha}} < +\infty.$$

One calls \mathbb{W} the *second order process*. We denote by \mathcal{C}^α the *space of α -Hölder rough paths* endowed with the semi-norm

$$\|\mathbf{W}\|_{\alpha, T} = \|W\|_{\alpha, T} + \|\mathbb{W}\|_{2\alpha, T}.$$

Remark 3.4.2. The second order process $\mathbb{W}_{s,t}$ can be thought of as $\int_s^t W_{s,u} \otimes dW_u$.

3.4. ELEMENTS OF ROUGH PATH THEORY

Remark 3.4.3. The first condition which is called Chen's relation represents the algebraic property of $\int_s^t W_{s,u} \otimes dW_u$. Indeed, if W is smooth,

$$\int_s^t W_{s,v}^i \dot{W}_v^j dv - \int_s^u W_{s,v}^i \dot{W}_v^j dv - \int_u^t W_{u,v}^i \dot{W}_v^j dv = W_{s,u}^i W_{u,t}^j$$

for all $1 \leq i, j \leq n$ and $0 \leq s \leq u \leq t$.

Remark 3.4.4. The second condition is also an extension of the analytic property of the smooth case.

Remark 3.4.5. If \mathbb{W} is a second order process of W , for any 2α -Hölder function F taking values in $\mathbb{R}^n \otimes \mathbb{R}^n$, $(s, t) \mapsto \mathbb{W}_{s,t} + F_t - F_s$ satisfies also the two properties of Definition 3.4.1. So if \mathbb{W} exists, it is not unique at all.

Building \mathbb{W} from W is non-trivial as soon as $n \geq 2$.

Theorem 3.4.6. *For any $W \in C^\alpha$ with $\alpha \in (1/3, 1/2]$ there exists a rough path lift \mathbb{W} , i.e. $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha$ in a way that the map $W \mapsto \mathbf{W}$ is continuous for the topology defined in Definition 3.4.1.*

Proof. This result was proved in [LV07]. We prove of this result in the case $\alpha \in (1/3, 1/2]$ in Section 3.8 as an application of the reconstruction theorem (Theorem 3.6.5). ■

3.4.2 Controlled rough paths

The aim of this section is to define an integrand against \mathbf{W} , called a controlled rough path by W . This approach was developed by M. Gubinelli in [Gub04]. We introduce a function with the same regularity as W which is not differentiable with respect to time but with respect to W itself. This is the concept of the Gubinelli's derivative.

Definition 3.4.7. Let W be in C^α , we call a *controlled rough path by W* the pair $(y, y') \in C^\alpha(\mathbb{R}^d) \times C^\alpha(\mathbb{R}^{d \times n})$ such that

$$y_{s,t} = y'_s W_{s,t} + R_{s,t}^y, \tag{3.21}$$

with $\|R^y\|_{2\alpha, T} < +\infty$. The function y' is the *Gubinelli's derivative* of y with respect to W .

We denote $\mathcal{D}_W^{2\alpha}$ the *space of the controlled rough paths* (y, y') driven by W endowed with the semi-norm

$$\|(y, y')\|_{2\alpha, T}^W := \|y'\|_{\alpha, T} + \|R^y\|_{2\alpha, T}. \tag{3.22}$$

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Remark 3.4.8. The identity (3.21) looks like a Taylor expansion of first order

$$f_t = f_s + f'_s(t - s) + O(|t - s|^2),$$

but $(W_t - W_s)$ substitutes the usual polynomial expression $(t - s)$, y'_s the normal derivative and the remainder term is of order 2α whereas order 2. The theory of regularity structures is a deep generalization of this analogy.

Remark 3.4.9. The Gubinelli's derivative y' is matrix-valued which depends on y and W .

Remark 3.4.10. Unlike the rough path space \mathcal{C}^α (see Definition 3.4.1) which is not a linear space, $\mathcal{D}_W^{2\alpha}$ is a Banach space with the norm $\|y_0\| + \|y'_0\| + \|(y, y')\|_{2\alpha, T}^W$ or the norm $\|y\|_{\infty, T} + \|y'\|_{\infty, T} + \|(y, y')\|_{2\alpha, T}^W$. These two norms are equivalent.

Remark 3.4.11. The uniqueness of y' depends on the regularity of W . If W is too smooth, for example in $C^{2\alpha}$, then y is in $C^{2\alpha}$, and every continuous function y' matches with the definition of the Gubinelli's derivative, particularly $y' = 0$. But we can prove that y' is uniquely determined by y when W is irregular enough. The reader can refer to the Chapter 4 of [FH14] for detailed explanations.

3.4.3 Integration against rough paths

If F is a linear operator A , the differential equation (3.1) can be restated on an integral form as

$$y_t = \xi + A \int_0^t y_u dW_u. \quad (3.23)$$

To give a meaning to (3.23) we must define an integral term $\int_0^t y_u dW_u$.

When $W \in C^\alpha$, $y \in C^\beta$ with $\alpha + \beta > 1$, we are able to define (3.23) with Young's integral. Unfortunately, the solution y of (3.23) inherits of the regularity of W . Hence, Young's theory allows us to solve (3.23) only when $\alpha > 1/2$.

When $\alpha \in (1/3, 1/2]$, we need to "improve" the path W in taking into account of \mathbb{W} in the definition of the integral.

3.4.4 Young's integration

Young's integral was developed by Young in [You36] and then used by T. Lyons in [Lyo94] to deal with differential equations driven by a stochastic process.

The integral is defined with a Riemann sum. Let \mathcal{P} be a subdivision of $[s, t]$, we denote by $|\mathcal{P}|$ the mesh of \mathcal{P} . We want to define the integral as follows:

$$\int_s^t y_u dW_u = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{u, v \in \mathcal{P}} y_u W_{u, v},$$

where $u, v \in \mathcal{P}$ denotes successive points of the subdivision.

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Theorem 3.4.12. *If $W \in C^\alpha$ and $y \in C^\beta$ with $\alpha + \beta > 1$, $\sum_{u,v \in \mathcal{P}} y_u W_{u,v}$ converges when $|\mathcal{P}| \rightarrow 0$. The limit is independent of the choice of \mathcal{P} , and it is denoted as $\int_s^t y_u dW_u$. Moreover the bilinear map $(W, y) \rightarrow \int_s^t y_u dW_u$ is continuous from $C^\alpha \times C^\beta$ to C^α .*

Proof. For the original proof cf. [You36]. ■

Some important properties of the classical Riemann integration holds.

Proposition 3.4.13. *1. Chasles' relation holds.*

2. When $t \rightarrow s$ we have the following approximation

$$\int_s^t y_u dW_u = y_s W_{s,t} + O(|t - s|^{\alpha+\beta}). \quad (3.24)$$

3. The map $t \mapsto \int_s^t y_u dW_u$ is α -Hölder continuous.

4. If F is C^1 , $F(y)$ is C^β -Hölder and the Young integral $\int_s^t F(y_u) dW_u$ is well-defined as above.

Remark 3.4.14. Unfortunately with Young's construction, when $\alpha \leq 1/2$, we can find two sequences of smooth functions $W^{1,n}$ and $W^{2,n}$ converging to W in C^α but such that $\int_s^t F(W^{1,n}) dW^{1,n}$ and $\int_s^t F(W^{2,n}) dW^{2,n}$ converge to two different limits for a smooth function F . See for an example the Lejay's area bubbles in [Lej12].

3.4.5 Controlled rough path integration

The rough integral relies on the controlled rough paths introduced previously. Remark 3.4.14 shows that if $y, W \in C^\alpha$, we cannot define a continuous integral such as $\int_s^t y_u dW_u$ looks like $y_s W_{s,t}$ when $t \rightarrow s$. We must use the structure of controlled rough paths to define a "good" integral of y against W . Then, given a rough path $\mathbf{W} \in \mathcal{C}^\alpha$ and considering a controlled rough path $(y, y') \in \mathcal{D}_W^{2\alpha}$ we would like to build an integral $\int_s^t y_u dW_u$ as a good approximation of $y_s W_{s,t} + y'_s \mathbb{W}_{s,t}$ when $t \rightarrow s$.

Theorem 3.4.15. *For $\alpha \in (1/3, 1/2]$, let $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha$ be an α -Hölder rough path. Given a controlled rough path driven by W : $(y, y') \in \mathcal{D}_W^{2\alpha}$ we consider the sum $\sum_{u,v \in \mathcal{P}} y_u W_{u,v} + y'_u \mathbb{W}_{u,v}$ where \mathcal{P} is a subdivision of $[s, t]$ ($s \leq t \in [0, T]$). This sum converges when the mesh of \mathcal{P} goes to 0. We define the integral of y against \mathbf{W} as*

$$\int_s^t y_u d\mathbf{W}_u := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{u,v \in \mathcal{P}} y_u W_{u,v} + y'_u \mathbb{W}_{u,v}.$$

The limit exists and does not depend on the choice of the subdivision. Moreover, the map $(y, y') \rightarrow (t \in [0, T] \mapsto \int_0^t y_u d\mathbf{W}_u, y)$ from $\mathcal{D}_W^{2\alpha}$ into itself is continuous.

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Proof. The classical proof uses the sewing lemma [FH14, Lemma 4.2]. We give a proof with the reconstruction theorem (Theorem 3.6.5) in Section 3.7. ■

To solve (3.1), we need to show that if F is a smooth function, then $F(y_t)$ remains a controlled rough path. The following proposition shows that $(F(y), (F(y))')$ defined by :

$$F(y)_t = F(y_t), \quad F(y)'_t = F'(y_t)y_t, \quad (3.25)$$

is a controlled rough path.

Proposition 3.4.16. *Let $F : \mathbb{R}^d \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ be a function twice continuously differentiable such that F and its derivative are bounded. Given $(y, y') \in \mathcal{D}_W^{2\alpha}$ let $(F(y), F(y)') \in \mathcal{D}_W^{2\alpha}$ defined as above (3.25). Then, there is a constant $C_{\alpha, T}$ depending only on α and T such as*

$$\|F(y), F(y)'\|_{2\alpha, T}^W \leq C_{\alpha, T} \|F\|_{C_b^2} (1 + \|W\|_\alpha)^2 (\|y'_0\| + \|y, y'\|_{2\alpha, T}^W)^2,$$

where $\|F\|_{C_b^2} = \|F\|_\infty + \|F'\|_\infty + \|F''\|_\infty$.

Proof. We can find the proof in [FH14]. This proposition is equivalent to Theorem 3.9.1, which is formulated in the regularity structure framework. ■

3.5 Regularity structures

3.5.1 Definition of a regularity structure

The theory of regularity structures was introduced by Martin Hairer in [Hai14]. The tools developed in this theory allow us to solve a very wide range of semi-linear partial differential equations driven by an irregular noise.

This theory can be viewed as a generalisation of the Taylor expansion theory to irregular functions. The main idea is to describe the regularity of a function at small scales and then to reconstruct this function with the reconstruction operator of Theorem 3.6.5.

First we give the definition of a regularity structure.

Definition 3.5.1. A *regularity structure* is a 3-tuple $\mathcal{T} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$ where

- The *index set* $\mathcal{A} \subset \mathbb{R}$ is bounded from below, locally finite and such that $0 \in \mathcal{A}$.
- The *model space* \mathcal{T} is a graded linear space indexed by $\mathcal{A} : \mathcal{T} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$, where each \mathcal{T}_α is a non empty Banach space. The elements of \mathcal{T}_α are said of *homogeneity* α . For $\tau \in \mathcal{T}$, we denote $\|\tau\|_\alpha$ the norm of the component of τ in \mathcal{T}_α . Furthermore, $\mathcal{T}_0 = \text{Vect}\langle 1 \rangle$ is isomorphic to \mathbb{R} .

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- The set \mathcal{G} is a set of continuous linear operators acting on \mathcal{T} such as for $\Gamma \in \mathcal{G}$, $\Gamma(\mathbb{1}) = \mathbb{1}$ and $\tau \in \mathcal{T}_\alpha$, $\Gamma_\tau - \tau \in \bigoplus_{\beta < \alpha} \mathcal{T}_\beta$. The set \mathcal{G} is called *structure group*.

Remark 3.5.2. We underline the elements of the model space for the sake of clarity.

Remark 3.5.3. We set $m := \min \mathcal{A}$, $\Gamma_\tau = \tau$ for every $\tau \in \mathcal{T}_m$.

Let us explain the motivations of this definition. The classic polynomial Taylor expansion of order $m \in \mathbb{N}$ is given, between 0 and $t \in \mathbb{R}$, where t converges to 0 by

$$f(t) = P(t) + o(t^m), \quad \text{where } P(t) = \sum_{k=0}^m \frac{f^{(k)}(0)}{k!} t^k.$$

In this case the approximation P of f is indexed by integers and the space \mathcal{T} is the polynomial space. For all $h \in \mathbb{R}$, the operator Γ_h associates a Taylor expansion at point t with a Taylor expansion at a point $t + h$. The polynomial $\Gamma_h(P(t)) - P(t)$ is of order less than $m - 1$:

$$\Gamma_h(P(t)) - P(t) := P(t + h) - P(t) = \sum_{k=0}^m \frac{f^{(k)}(0)}{k!} ((t + h)^k - t^k).$$

Moreover we have the structure of group on $(\Gamma_h, h \in \mathbb{R})$:

$$\begin{aligned} \Gamma_{h'} \circ \Gamma_h P(t) &= \Gamma_{h'} \left(\sum_{k=0}^{d-1} \frac{f^k(0)}{k!} (t + h)^k \right) = \sum_{k=0}^{d-1} \frac{f^k(0)}{k!} \Gamma_{h'}((t + h)^k) \\ &= \sum_{k=0}^{d-1} \frac{f^k(0)}{k!} (t + h + h')^k \\ &= \Gamma_{h+h'} P(t). \end{aligned}$$

Hence, we can define the polynomial regularity structure as following.

Definition 3.5.4. We define $\mathcal{S}^1 = (\mathcal{A}^1, \mathcal{T}^1, \mathcal{G}^1)$ the *canonical polynomial regularity structure* as

- $\mathcal{A}^1 = \mathbb{N}$ is the index set.
- For $k \in \mathcal{A}^1$ we define $\mathcal{T}_k^1 = \text{Vect}\langle \underline{X}^k \rangle$. The subspace \mathcal{T}_k^1 contains the monomial of order k . The polynomial model space is $\mathcal{T}^1 = \bigoplus_{k \in \mathcal{A}^1} \mathcal{T}_k^1$.
- For $h \in \mathbb{R}$, $\Gamma_h^1 \in \mathcal{G}^1$ is given by

$$\Gamma_h^1(\underline{X}^k) = (\underline{X} + h\mathbb{1})^k.$$

For $P_k \in \mathcal{T}_k^1$, there is $a_k \in \mathbb{R}$ such that $P_k = a_k \underline{X}^k$. We define the norm on \mathcal{T}_k^1 by $\|P_k\|_k = |a_k|$.

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With the same arguments we define the polynomial regularity structure and its model associated in \mathbb{R}^n .

Definition 3.5.5. We define $\mathcal{T}^p = (\mathcal{A}^p, \mathcal{T}^p, \mathcal{G}^p)$ the *canonical polynomial regularity structure* on \mathbb{R}^n as

- $\mathcal{A}^p = \mathbb{N}$ is the index set.
- For $\delta \in \mathcal{A}^p$, and k a multi-index of \mathbb{N}^n such that $|k| := k_1 + \dots + k_n = \delta$, we define $T_\delta^p = \text{Vect}\langle \underline{X}^k := \prod_{i=1}^n X_i^{k_i}, |k| = \delta \rangle$. This space T_k^p is a linear space of homogeneous polynomial with n variables and of order δ . For $P_\delta \in T_\delta^p$, there are real coefficients $(a_k)_{|k|=\delta}$ such that $P_\delta = \sum_{|k|=\delta} a_k \underline{X}^k$. We chose the norm on T_δ^p such that $\|P_\delta\|_\delta := \sum_{|k|=\delta} |a_k|$.

We define $\mathcal{T}^p = \bigoplus_{\delta \in \mathcal{A}^p} T_\delta^p$ as the polynomial model space.

- For $h \in \mathbb{R}^n$, $\Gamma_h^p \in G^p$ is given by

$$\Gamma_h^p(\underline{X}^k) = \prod_{i=1}^n (X_i + h_i \mathbb{1})^{k_i}.$$

Remark 3.5.6. The polynomial regularity structure is a trivial example of regularity structure which we introduce for a better understanding. But the strength of this theory is to deal with negative degree of homogeneity.

3.5.2 Definition of a model

Definition 3.5.7. Given a regularity structure $\mathcal{T} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$, a *model* $M = (\Pi, \Gamma)$ is two sets of functions such that for any $s, t, u \in \mathbb{R}$

- The operator Π_s is continuous and linear from \mathcal{T} to $\mathcal{D}'(\mathbb{R}, \mathbb{R}^n)$.
- $\Gamma_{t,s}$ belongs to \mathcal{G} , so it is a linear operator acting on \mathcal{T} .
- The following algebraic relations hold: $\Pi_s \Gamma_{s,t} = \Pi_t$ and $\Gamma_{s,t} \Gamma_{t,u} = \Gamma_{s,u}$.
- The following analytic relations hold : for every $\gamma > 0$, $\beta < \alpha \leq \gamma$ with $\alpha, \beta \in \mathcal{A}$ and $\tau \in \mathcal{T}_\alpha$, there is a constant $C(T, \gamma)$ uniform over $s, t \in [0, T]$, $\lambda \in (0, 1]$, $\phi \in B_r$ such that

$$\begin{aligned} |\Pi_s(\tau)(\phi_s^\lambda)| &\leq C(T, \gamma) \lambda^\alpha \|\tau\|_\alpha \\ \text{and } \|\Gamma_{s,t}(\tau)\|_\beta &\leq C(T, \gamma) |t - s|^{\alpha - \beta} \|\tau\|_\alpha. \end{aligned} \tag{3.26}$$

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We denote respectively by $\|\Pi\|_{\gamma,T}$ and $\|\Gamma\|_{\gamma,T}$ the smallest constants such that the bounds (3.26) hold. Namely,

$$\|\Pi\|_{\gamma,T} := \sup_{s \in [0,T]} \sup_{\phi \in B_r} \sup_{\lambda \in (0,1]} \sup_{\alpha < \gamma} \sup_{\tau \in T_\alpha} \frac{|\Pi_s(\tau)(\phi_s^\lambda)|_\alpha}{\lambda^\alpha \|\tau\|_\alpha}$$

and $\|\Gamma\|_{\gamma,T} := \sup_{s \neq t \in [0,T]} \sup_{\beta < \alpha < \gamma} \sup_{\tau \in T_\alpha} \frac{\|\Gamma_{s,t}(\tau)\|_\beta}{|t-s|^{\alpha-\beta} \|\tau\|_\alpha}$.

The two operators $\|\cdot\|_{\gamma,T}$ define semi-norms.

The easiest regularity structure which we can describe is the polynomial one (see Definition 3.5.5). We can now define the model associated to this regularity structure.

Definition 3.5.8. Given that $\mathcal{T}^p = (\mathcal{A}^p, \mathcal{T}^p, \mathcal{G}^p)$ the canonical polynomial regularity structure on \mathbb{R}^n defined in the Definition 3.5.5, we define the *model of the polynomial regularity structure* $M^p = (\Pi^p, \Gamma^p)$ such that for all $x, y \in \mathbb{R}^n$ and k a multi-index of order n ,

$$\begin{aligned} \Pi_x^p(\underline{X}^k)(y) &:= ((y_1 - x_1)^{k_1}, \dots, (y_n - x_n)^{k_n}), \\ \Gamma_{x,y}^p(\underline{X}^k) &:= \Gamma_{x-y}(\underline{X}^k). \end{aligned}$$

Proof. It is straightforward to check that this definition is in accordance with the one of a model (Definition 3.6.1 below). ■

Remark 3.5.9. The operator Π_s which associates to an element of the abstract space a distribution which approximates this element in s . Typically for polynomial regularity structure on \mathbb{R} ,

$$\Pi_s^p(\underline{X}^k) = (t \rightarrow (t-s)^k).$$

Remark 3.5.10. In the model space, the operator $\Gamma_{s,t}$ gives an expansion in a point s , given an expansion in a point t . For example

$$\Gamma_{s,t}^p(\underline{X}^k) = \Gamma_{s-t}^p(\underline{X}^k) = (\underline{X} + (s-t)\mathbf{1})^k. \quad (3.27)$$

Remark 3.5.11. The first algebraic relation means that if a distribution looks like τ near t , the same distribution looks like $\Gamma_{s,t}(\tau)$ near s . In practice, we use this relation to find the suitable operator $\Gamma_{t,s}$. The second algebraic relation is natural. It says that moving an expansion from u to s is the same as moving an expansion from u to t and then from t to s .

Remark 3.5.12. The first analytic relation has to be understood as Π_s approximating $\tau \in T_\alpha$ in s with the precision λ^α . The relation (3.27) shows that the second analytic relation is natural. Indeed,

$$(\underline{X} + (t-s)\mathbf{1})^k = \sum_{i=0}^k \binom{k}{i} (t-s)^{k-i} \underline{X}^i,$$

so for $\ell \leq k$, $\|\Gamma_{s,t}^p(\underline{X}^k)\|_\ell = \binom{k}{i} |t-s|^{k-\ell}$, where $\binom{k}{i} = \frac{k!}{i!(k-i)!}$ are the binomial coefficients.

3.5.3 The rough path regularity structure

We now reformulate the results of Subsections 3.4.1 and 3.4.2 to build up a regularity structure.

In order to find the regularity structure of rough paths, we make some computations for $n = 1$. Then, we give the proof in the general case after Definition 3.5.13.

We fix $\alpha \in (1/3, 1/2]$ and a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha$. We show how to build the regularity structure of rough paths.

Let $(y, y') \in \mathcal{D}_W^{2\alpha}$ be a controlled rough path. According to Definition 3.4.7, $y_t = y_s + y'_s W_{s,t} + O(|t - s|^{2\alpha})$. To describe the expansion of y with the regularity structure framework, we set the symbol $\mathbb{1}$ constant of homogeneity 0 and the symbol \underline{W} of homogeneity α . This leads us to define the elements of the regularity structure of the controlled rough path (y, y') evaluated at time t by

$$\underline{Y}(t) = y_t \mathbb{1} + y'_t \underline{W}.$$

Moreover, we would like to build the rough path integral $\int y d\mathbf{W}$ in the regularity structure context. So we introduce abstract elements \dot{W} and $\dot{\mathbb{W}}$ which “represent” $d\mathbf{W} = d(W, \mathbb{W})$. The function W is α -Hölder, so we define the homogeneity of \dot{W} as $\alpha - 1$. The second order process \mathbb{W} is 2α -Hölder, which leads us to define the homogeneity of $\dot{\mathbb{W}}$ as $2\alpha - 1$.

Finally, with the notation of Definition 3.5.1, the index set $A = \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$ and the model space $\mathcal{T} = \text{Vect}\langle \dot{W}, \dot{\mathbb{W}}, \mathbb{1}, \underline{W} \rangle$. Besides, we order the elements in $\text{Vect}\langle \cdot \rangle$ by homogeneity.

It remains to define \mathcal{G} and an associated model. We start by building the model (Π, Γ) . For $s \in [0, T]$, Π_s should transform the elements of \mathcal{T} to distributions (or functions when it is possible) which approximate this elements at the point s . On the one hand we define

$$\Pi_s(\dot{W})(\phi) := \int \phi(t) dW_t, \quad \Pi_s(\dot{\mathbb{W}})(\phi) := \int \phi(t) d\mathbb{W}_{s,t},$$

where ϕ is a test function. Both integrals are well-defined because ϕ is smooth. The homogeneities of \dot{W} and $\dot{\mathbb{W}}$ are negative, so they are mapped with distributions. On the other hand, $\mathbb{1}$ and \underline{W} have positive homogeneities, so we can approximate them in s with functions as

$$\Pi_s(\mathbb{1})(t) := 1, \quad \Pi_s(\underline{W})(t) := W_{s,t}.$$

Now, we define $\Gamma_{s,t}(\tau)$ for every $\beta \in A$ and $s, t \in [0, T]$ and $\tau \in \mathcal{T}_\beta$. According to Definition 3.5.7 : $\Pi_s \Gamma_{s,t}(\tau)(\phi) = \Pi_t(\tau)(\phi)$. Moreover, following Definition 3.5.1, $\Gamma_{s,t}$ should be a linear combination of elements of homogeneity lower than τ and with the coefficient 1 in front of τ . First, it seems obvious to set $\Gamma_{s,t}(\mathbb{1}) = \mathbb{1}$, because $\mathbb{1}$ represents a constant. Then we look for $\Gamma_{s,t}(\underline{W}) = \underline{W} + a_{s,t} \mathbb{1}$ as a function where $a_{s,t}$ has to be

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determined. If it is not enough, we would look for $\Gamma_{s,t}(\underline{W})$ with more elements of our structure T . By linearity

$$\Pi_s(\underline{W} + a_{s,t}\mathbb{1})(u) = W_{s,u} + a_{s,t},$$

so we want that $W_{s,u} + a_{s,t} = \Pi_t(\underline{W})(u) = W_{t,u}$. Finally, we have to choose $a_{s,t} = W_{t,s}$. Given that \underline{W} has the lowest homogeneity of our structure, we set $\Gamma_{s,t}(\underline{W}) = \underline{W}$ in order to respect the last item of Definition 3.5.1. With the same reason as for \underline{W} and using the Chen's relation of Definition 3.4.1, we find that $\Gamma_{s,t}(\underline{\dot{W}}) = \underline{\dot{W}} + W_{t,s}\underline{W}$ (see the proof of Definition 3.5.13).

All we did here is in one dimension. With the same arguments we can find the regularity structure of a rough path in \mathbb{R}^n .

Definition 3.5.13. For $\alpha \in (1/3, 1/2]$, given a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha$ which take value in $\mathbb{R}^n \oplus (\mathbb{R}^n \otimes \mathbb{R}^n)$. We define the *regularity structure of rough paths* $\mathcal{T}^r = (\mathcal{A}^r, \mathcal{T}^r, \mathcal{G}^r)$ and the model associated $M^r = (\Pi^r, \Gamma^r)$ as

- i) Index set $\mathcal{A}^r := \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$.
- ii) Model space $\mathcal{T}^r := \mathcal{T}_{\alpha-1}^r \oplus \mathcal{T}_{2\alpha-1}^r \oplus \mathcal{T}_0^r \oplus \mathcal{T}_\alpha^r$, with

$$\begin{aligned} \mathcal{T}_{\alpha-1}^r &:= \text{Vect}\langle \underline{\dot{W}}^i, i = 1, \dots, n \rangle, & \mathcal{T}_{2\alpha-1}^r &:= \text{Vect}\langle \underline{\dot{W}}^{i,j}, i, j = 1, \dots, n \rangle, \\ \mathcal{T}_0^r &:= \text{Vect}\langle \mathbb{1} \rangle, & \mathcal{T}_\alpha^r &:= \text{Vect}\langle \underline{W}^i, i = 1, \dots, n \rangle. \end{aligned}$$

- iii) For i, j integers between 1 and n , $h \in \mathbb{R}^n$ and Γ_h^r in the structure group \mathcal{G}^r , the following relations hold

$$\begin{aligned} \Gamma_h^r(\underline{\dot{W}}^i) &:= \underline{\dot{W}}^i, & \Gamma_h^r(\underline{\dot{W}}^{i,j}) &:= \underline{\dot{W}}^{i,j} + h^i \underline{\dot{W}}^j, \\ \Gamma_h^r(\mathbb{1}) &:= \mathbb{1}, & \Gamma_h^r(\underline{W}^i) &:= \underline{W}^i + h^i \mathbb{1}. \end{aligned}$$

- iv) For i, j two integers between 1 and n , for $s, t \in [0, T]$,

$$\begin{aligned} \Pi_s^r(\underline{\dot{W}}^i)(\phi) &:= \int \phi(t) dW_t^i, & \Pi_s^r(\underline{\dot{W}}^{i,j})(\phi) &:= \int \phi(t) d\mathbb{W}_{s,t}^{i,j}, \\ \Pi_s^r(\mathbb{1})(t) &:= 1, & \Pi_s^r(\underline{W}^i)(t) &:= W_{s,t}^i, \end{aligned}$$

where ϕ is a test function.

- v) For $s, t \in \mathbb{R}$, $\Gamma_{s,t}^r := \Gamma_{|h=W_{t,s}}^r$.

Proof. Checking that this definition respects the definitions of a regularity structure (Definition 3.5.1) and of a model (Definition 3.5.7) is straightforward.

Here we only show where Chen's relation of Definition 3.4.1 is fundamental to show that the algebraic condition of Definition 3.5.7 : $\Pi_s^r \Gamma_{s,t}^r \underline{\dot{W}} = \Pi_t^r \underline{\dot{W}}$ holds.

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According to the definition above $\Gamma_{s,t}^r \underline{\dot{W}}^{i,j} = \underline{\dot{W}}^{i,j} + h^i \underline{\dot{W}}^j$. So we have

$$\Pi_s^r(\Gamma_{s,t}^r \underline{\dot{W}}^{i,j})(\phi) = \int \phi(u) d\mathbb{W}_{s,u}^{i,j} + W_{t,s}^i \int \phi(u) dW_{s,u}^j. \quad (3.28)$$

In differentiating Chen's relation $\mathbb{W}_{s,u}^{i,j} = \mathbb{W}_{s,t}^{i,j} + \mathbb{W}_{t,u}^{i,j} + W_{s,t}^i W_{t,u}^j$ with respect to u we get $d\mathbb{W}_{s,u}^{i,j} = d\mathbb{W}_{t,u}^{i,j} + W_{s,t}^i dW_{t,u}^j$. It follows that

$$\Pi_s^r(\Gamma_{s,t}^r \underline{\dot{W}}^{i,j})(\phi) = \int \phi(u) d\mathbb{W}_{t,u}^{i,j} + W_{s,t}^i \int \phi(u) dW_{s,u}^j + W_{t,s}^i \int \phi(u) dW_{s,u}^j. \quad (3.29)$$

Finally $\Pi_s^r(\Gamma_{s,t}^r \underline{\dot{W}}^{i,j})(\phi) = \int \phi(u) d\mathbb{W}_{t,u}^{i,j} = \Pi_t^r \underline{\dot{W}}$, which is the algebraic condition required. ■

3.6 Modelled distributions

3.6.1 Definition and the reconstruction operator

We have defined a regularity structure. We now introduce the space of functions from $[0, T]$ to \mathcal{T} , the *model space* of a regularity structure. These abstract functions should represent at each point of $[0, T]$, a ‘‘Taylor expansion’’ of a real function.

We showed in Section 3.5.3 how to build an abstract function $\underline{Y}(t) = y_t \mathbb{1} + y'_t \underline{W}$ which represents the expansion of a real controlled rough path (y, y') at a point t . The most important result of the theory of regularity structures is to show how to build a real function or distribution from an abstract function. Namely, given an approximation of a function at each time, how to reconstruct ‘‘continuously’’ the function. This is given by the *reconstruction map* theorem.

Definition 3.6.1. Given a regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ and a model $M = (\Pi, \Gamma)$, for $\gamma \in \mathbb{R}$ we define the space D_M^γ of *modelled distributions* as functions $\underline{f} : [0, T] \rightarrow \mathcal{T}_{<\gamma} := \bigoplus_{\beta < \gamma} \mathcal{T}_\beta$ such that for all $s, t \in [0, T]$ and for all $\beta < \gamma$,

$$\|\underline{f}(s) - \Gamma_{s,t}(\underline{f}(t))\|_\beta \leq C(T) |t - s|^{\gamma - \beta},$$

where $C(T)$ is a constant which depends only on T .

Recalling that $\|\cdot\|_\beta$ is the norm of the component in \mathcal{T}_β , we define by

$$\|\underline{f}\|_{\gamma, T} := \sup_{s \neq t \in [0, T]} \sup_{\beta < \gamma} \frac{\|\underline{f}(t) - \Gamma_{t,s}(\underline{f}(s))\|_\beta}{|t - s|^{\gamma - \beta}}$$

a semi-norm on the space D_M^γ . It is also possible to consider the norm

$$\|\underline{f}\|_{\gamma, T}^* := \sup_{t \in [0, T]} \sup_{\beta < \gamma} \|\underline{f}(t)\|_\beta + \|\underline{f}\|_{\gamma, T}.$$

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Moreover $\|\cdot\|_{\gamma,T}^*$ is equivalent to

$$\sup_{\beta < \gamma} \|\underline{f}(0)\|_{\beta} + \|\underline{f}\|_{\gamma,T},$$

so from now we use these two norms without distinction.

Remark 3.6.2. For a fixed model M , the modelled distributions space D_M^{γ} is a Banach space with the norm $\|\cdot\|_{\gamma,T}^*$.

Remark 3.6.3. We choose the same notation for the semi-norm on D_M^{γ} as on \mathcal{D}_W^{γ} (the space of modelled distributions and on C^{α} (the space of Hölder functions or distributions).

So when $\underline{f} \in D_M^{\gamma}$, we have to understand $\|\underline{f}\|_{\gamma,T}$ with Definition 3.6.1 but when $f \in C^{\alpha}$, $\|f\|_{\alpha,T}$ is the Hölder norm of Definition 3.3.1 (for functions $\alpha > 0$) or 3.3.7 (for distributions $\alpha < 0$).

Remark 3.6.4. The modelled distribution space D_M^{γ} can be thought of as abstract γ -Hölder functions. Indeed, for an integer p and $\delta \in [0, 1)$ such that $\gamma = p + \delta$, if f is a smooth function

$$\left| f(x) - \sum_{k=0}^p \frac{f^{(k)}(y)}{k!} (y-x)^k \right| \leq C |t-s|^{\delta},$$

according to the Taylor's inequality. Hence, Definition 3.6.1 of modelled distributions has to be seen as an extension of the Taylor inequality in a no classical way.

Now we are able to outline the main theorem of the theory of regularity structures which given a modelled distribution allows us to build a “real” distribution approximated at each point by the modelled distribution.

Theorem 3.6.5 (Reconstruction map). *Given a regularity structure $\mathcal{T} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$ and a model $M = (\Pi, \Gamma)$, for a real $\gamma > \alpha_* = \min \mathcal{A}$ and an integer $r > |\alpha_*|$ there is a linear continuous map $\mathcal{R} : D_M^{\gamma} \rightarrow C^{\alpha_*}$ such that for all $\underline{f} \in D_M^{\gamma}$,*

$$|[\mathcal{R}(\underline{f}) - \Pi_s(\underline{f}(s))](\phi_s^{\lambda})| \leq C \|\Pi\|_{\gamma,T} \|\underline{f}\|_{\gamma,T}^* \lambda^{\gamma}, \quad (3.30)$$

where C depends uniformly over $\phi \in B_r$, $\lambda \in (0, 1]$, $s \in [0, T]$.

Moreover if $\gamma > 0$, the bound (3.30) defined $\mathcal{R}(\underline{f})$ uniquely.

If $(\tilde{\Pi}, \tilde{\Gamma})$ is an other model for \mathcal{T} and $\tilde{\mathcal{R}}$ the reconstruction map associated to the model, we have the bound

$$\begin{aligned} & |[\mathcal{R}(\underline{f}) - \tilde{\mathcal{R}}(\tilde{\underline{f}}) - \Pi_s(\underline{f}(s)) + \tilde{\Pi}_s(\tilde{\underline{f}}(s))](\eta_s^{\lambda})| \\ & \leq C \left(\|\tilde{\Pi}\|_{\gamma,T} \|\underline{f} - \tilde{\underline{f}}\|_{\gamma,T}^* + \|\Pi - \tilde{\Pi}\|_{\gamma,T} \|\underline{f}\|_{\gamma,T}^* \right) \lambda^{\gamma}, \end{aligned} \quad (3.31)$$

where C depends uniformly over $\phi \in B_r$, $\lambda \in (0, 1]$, $s \in [0, T]$, as above.

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Proof. The proof uses the wavelet analysis in decomposing the function \underline{f} in a smooth wavelet basis. The proof requires many computation. A complete one can be found in [Hai14] and a less exhaustive one is in [FH14]. The construction of $\mathcal{R}(\underline{f})$ is the following. We define a sequence $(\mathcal{R}^j(\underline{f}))_{j \in \mathbb{N}}$ such that

$$\mathcal{R}^j(\underline{f}) := \sum_{k \in \mathbb{Z}} \Pi_{k/2^j}(\underline{f}(k/2^j))(\phi_k^j)\phi_k^j, \quad (3.32)$$

where ϕ_k^j is defined in Definition 3.3.11 with a regularity at almost r . Then, we show that $\mathcal{R}^j(\underline{f})$ converges weakly to a distribution $\mathcal{R}(\underline{f})$ which means that $\mathcal{R}^j(\underline{f})(\eta)$ converges to $\mathcal{R}(\underline{f})(\eta)$ for all $\eta \in C_0^r$. And we show that the bound (3.30) holds. ■

Remark 3.6.6. It can be proved that if for all $s \in [0, T]$ and $\tau \in \mathcal{T}$, $\Pi_s \tau$ is a continuous function then $\mathcal{R}(\underline{f})$ is also a continuous function such that

$$\mathcal{R}(\underline{f})(s) = \Pi_s(\underline{f}(s))(s). \quad (3.33)$$

Corollary 3.6.7. *With the same notation as in Theorem 3.6.5, for every $\gamma > 0$, there is a constant C such as*

$$\|\mathcal{R}(\underline{f})\|_{\alpha, T} \leq C \|\Pi\|_{\gamma, T} \| \underline{f} \|_{\gamma, T}^*.$$

Proof. According to Theorem 3.6.5, for $\phi \in B_r$,

$$\frac{|\mathcal{R}(\underline{f})(\phi_s^\lambda)|}{\lambda^\alpha} \leq \frac{|\Pi_s(\underline{f}(s))(\phi_s^\lambda)|}{\lambda^\alpha} + C \|\Pi\|_{\gamma, T} \| \underline{f} \|_{\gamma, T}^* \lambda^{\gamma-\alpha},$$

and according to the Definition 3.5.7,

$$\frac{|\Pi_s(\underline{f}(s))(\phi_x^\lambda)|}{\lambda^\alpha} \leq \|\Pi\|_{\gamma, T} \| \underline{f} \|_{\gamma, T}.$$

So finally

$$\begin{aligned} \|\mathcal{R}(\underline{f})\|_{\alpha, T} &\leq \|\Pi\|_{\gamma, T} \| \underline{f} \|_{\gamma, T} + C \|\Pi\|_{\gamma, T} \| \underline{f} \|_{\gamma, T}^* \lambda^{\gamma-\alpha} \\ &\leq C \|\Pi\|_{\gamma, T} \| \underline{f} \|_{\gamma, T}^*, \end{aligned}$$

which, by letting λ going to 0 proves the inequality. ■

3.6.2 Modelled distribution of controlled rough paths

We reformulate the definition of a controlled rough path in the regularity structures framework.

Definition 3.6.8. Given $(W, \mathbb{W}) \in \mathcal{C}^\alpha$, $(y, y') \in \mathcal{D}_W^{2\alpha}$, the rough path regularity structure $(\mathcal{A}^r, \mathcal{T}^r, \mathcal{G}^r)$ and $M^r = (\Pi^r, \Gamma^r)$ the model associated (cf. Definition 3.5.13), we define a modelled distribution $\underline{Y} \in D_{M^r}^{2\alpha}$ such that

$$\underline{Y}(t) = y_t \mathbb{1} + y'_t \underline{W}, \quad \forall t \in [0, T].$$

The space $D_{M^r}^{2\alpha}$ is the space of the *modelled distributions of the controlled rough paths*.

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Remark 3.6.9. This definition is a particular case of modelled distributions of Definition 3.6.1.

Proof. Let check that \underline{Y} is in $D_{M^r}^{2\alpha}$. For every $s, t \in [0, T]$,

$$\begin{aligned}\underline{Y}(t) - \Gamma_{t,s}^r(\underline{Y}(s)) &= \underline{Y}(t) - \Gamma_{t,s}^r(y_s \mathbb{1} + y'_s \underline{W}) \\ &= \underline{Y}(t) - (y_s \mathbb{1} + y'_s \underline{W} + y'_s W_{s,t} \mathbb{1}),\end{aligned}$$

using the Definition 3.5.13. Then, we have

$$\|\underline{Y}(t) - \Gamma_{t,s}^r(\underline{Y}(s))\|_0 = \|y(t) - y(s) - y'(s)W_{s,t}\| \leq C|t - s|^{2\alpha},$$

according to the definition 3.4.7 of controlled rough paths. Besides,

$$\|\underline{Y}(t) - \Gamma_{t,s}^r(\underline{Y}(s))\|_\alpha = \|y'(t) - y'(s)\| \leq C|t - s|^\alpha,$$

which proves that $\underline{Y} \in D_{M^r}^{2\alpha}$. ■

Proposition 3.6.10. *With the notations of Definition 3.6.8, the application $(y, y') \in \mathcal{D}_W^{2\alpha} \mapsto \underline{Y} \in D_{M^r}^{2\alpha}$ is an isomorphism and the norms $\|y\|_{\infty, T} + \|y'\|_{\infty, T} + \|(y, y')\|_{2\alpha, T}^W$ and $\|\underline{Y}\|_{2\alpha, T}^*$ are equivalent.*

Proof. We prove the only equivalence between the two norms.

With the notation of Definition 3.4.7, we recall that

$$y_{s,t} = y'_s W_{s,t} + R_{s,t}^y, \tag{3.34}$$

and that $\|(y, y')\|_{2\alpha, T}^W = \|y'\|_{\alpha, T} + \|R^y\|_{2\alpha, T}$. Then according to the previous proof and Definition 3.6.1,

$$\|\underline{Y}\|_{2\alpha, T} = \sup \left\{ \|y'\|_{\alpha, T}, \|R^y\|_{2\alpha, T} \right\}. \tag{3.35}$$

So we have $\|\underline{Y}\|_{2\alpha, T} \leq \|(y, y')\|_{2\alpha, T}^W$ and $\|(y, y')\|_{2\alpha, T}^W \leq 2\|\underline{Y}\|_{2\alpha, T}$. In adding the terms $\|y\|_{\infty, T} + \|y'\|_{\infty, T}$ to each semi-norms, we obtain the result. ■

3.7 Rough path integral with the reconstruction map

The power of the theory of regularity structures is to give a sense in some cases of a product of distributions. Indeed, it is not possible in general to extend the natural product between functions to the space of distributions.

To build the controlled rough path integral of Theorem 3.4.15, with the theory of regularity structures we need to give a meaning to the product between y and \dot{W} , where \dot{W} is a distribution. We start by giving a meaning to the abstract product between \underline{Y} and $\underline{\dot{W}}$. When the product has good properties, we use the reconstruction map (Theorem 3.6.5) to define a “real” multiplication.

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Definition 3.7.1 (Multiplication in the model space). Given a regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$, we say that the continuous bilinear map $\star : \mathcal{T}^2 \rightarrow \mathcal{T}$ defines a *multiplication* (*product*) on the model space \mathcal{T} if

- For all $\tau \in \mathcal{T}$, on has $\mathbb{1} \star \tau = \mathbb{1}$,
- For every $\tau \in T_\alpha$ and $\sigma \in T_\beta$, on has $\tau \star \sigma \in T_{\alpha+\beta}$, if $\alpha + \beta \in \mathcal{A}$ and $\tau \star \sigma = 0$ if $\alpha + \beta \notin \mathcal{A}$.
- For every $\tau \in T_\alpha$, $\sigma \in T_\beta$ and $\Gamma \in \mathcal{G}$, $\Gamma(\tau \star \sigma) = \Gamma(\tau) \star \Gamma(\sigma)$.

We denote by $|\tau|$ the homogeneity α of the symbol τ . The last item of the definition can be rephrased as $|\tau \star \sigma| = |\tau| + |\sigma|$.

Remark 3.7.2. For example in the following Theorem 3.7.3, we define within the regularity structure of rough paths the multiplication described in the table below:

\star	\dot{W}	\ddot{W}	$\mathbb{1}$	\underline{W}
\dot{W}			\underline{W}	\ddot{W}
\ddot{W}			\ddot{W}	
$\mathbb{1}$	\dot{W}	\ddot{W}	$\mathbb{1}$	\underline{W}
\underline{W}	\ddot{W}		\underline{W}	

We are now able to build the rough integral with the reconstruction theorem (Theorem 3.6.5). The operator I corresponding to the integral of a controlled rough path against a rough path.

Theorem 3.7.3. *We set $\alpha \in (1/3, 1/2]$. There is a linear map $I : D_{M^r}^{2\alpha} \rightarrow C^\alpha$ such that for all $\underline{Y} \in D_{M^r}^{2\alpha}$, $I(\underline{Y})(0) = 0$ and such that the map L defined by*

$$\forall t \in [0, T], \quad L(\underline{Y})(t) := I(\underline{Y})(t)\mathbb{1} + \langle \underline{Y}(t), \mathbb{1} \rangle \underline{W}$$

is linear and continuous from $D_{M^r}^{2\alpha}$ into itself. The symbol $\langle \cdot, \mathbb{1} \rangle$ denotes the coordinate along $\mathbb{1}$.

Remark 3.7.4. Recalling that if $\underline{Y} \in D_{M^r}^{2\alpha}$, according to the Definition 3.6.8 there is $(y, y') \in \mathcal{D}_W^{2\alpha}$ such that

$$\underline{Y}(t) = y_t \mathbb{1} + y'_t \underline{W}, \tag{3.36}$$

we show in the proof of the Theorem 3.7.3 that

$$I(\underline{Y})(t) = \int_0^t y_s d\mathbf{W}_s, \tag{3.37}$$

where $\int_0^t y_s d\mathbf{W}_s$ is defined in Theorem 3.4.15. Thus L is the equivalent in the modeled distribution space of the map

$$(y, y') \in \mathcal{D}_W^{2\alpha} \mapsto \left(\int_0^\cdot y_s d\mathbf{W}_s, y \right) \in \mathcal{D}_W^{2\alpha}. \tag{3.38}$$

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Remark 3.7.5. The proof of the existence of I is the same as in Theorem 3.4.15 (classical sewing lemma). But we show how Theorem 3.6.5 (reconstruction map) can be adapted to recover the result.

Proof. For \underline{Y} in $D_{M^r}^{2\alpha}$, we define the point-wise product between \underline{Y} and $\dot{\underline{W}}$ as in Remark 3.7.2, i.e $\underline{Y}(t) \star \dot{\underline{W}} := y_t \dot{\underline{W}} + y'_t \underline{W} \dot{\underline{W}}$, where $\underline{W} \dot{\underline{W}} := \underline{W} \star \dot{\underline{W}} := \dot{\underline{W}} \underline{W}$. We denote this product $\underline{Y} \dot{\underline{W}}(t)$, to simplify the notation. Using the fact that $|\underline{W}| + |\dot{\underline{W}}| = 2\alpha - 1 = |\dot{\underline{W}} \underline{W}|$ it is straightforward to check that the product is consistent with the Definition 3.7.1.

We check now that $\underline{Y} \dot{\underline{W}}$ is in $D_{M^r}^{3\alpha-1}$. According to Definition 3.5.13 item v, we compute

$$\Gamma_{t,s}^r \left(\underline{Y} \dot{\underline{W}}(s) \right) = (y_s + y'_s \underline{W}_{s,t}) \dot{\underline{W}} + y'_s \dot{\underline{W}} \underline{W}_{s,t},$$

since $\underline{Y} \in D_{M^r}^{2\alpha}$ with Definition 3.6.8,

$$\left\| \underline{Y} \dot{\underline{W}}(t) - \Gamma_{t,s}^r \left(\underline{Y} \dot{\underline{W}}(s) \right) \right\|_{\alpha-1} = \|y_{s,t} - y'_s \underline{W}_{s,t}\| \lesssim |t-s|^{2\alpha}, \quad (3.39)$$

$$\left\| \underline{Y} \dot{\underline{W}}(t) - \Gamma_{t,s}^r \left(\underline{Y} \dot{\underline{W}}(s) \right) \right\|_{2\alpha-1} = \|y'_{s,t}\| \lesssim |t-s|^\alpha. \quad (3.40)$$

Thus, by Definition 3.6.1, we get that $\underline{Y} \dot{\underline{W}} \in D_{M^r}^{3\alpha-1}$.

Thus, given that $3\alpha - 1 > 0$, we can apply the reconstruction theorem in the positive case.

So there is a unique distribution $\mathcal{R}(\underline{Y} \dot{\underline{W}})$ in $C^{\alpha-1}$ such that for every $s \in [0, T]$, $\lambda > 0$ and every localized test function η_s^λ of Definition 3.3.4,

$$\left| \mathcal{R}(\underline{Y} \dot{\underline{W}})(\eta_s^\lambda) - y_s \int \eta_s^\lambda(u) dW_u - y'_s \int \eta_s^\lambda(u) d\mathbb{W}_{s,u} \right| \leq C \|\eta\|_{C^1} \lambda^{3\alpha-1}, \quad (3.41)$$

where we use relations of the item iv of Definition 3.5.13.

We define with Lemma 3.3.10 the operator $I : D_{M^r}^{2\alpha} \rightarrow C^\alpha$ such that $I(\underline{Y}) \in C^\alpha$ is associated to $\mathcal{R}(\underline{Y} \dot{\underline{W}})$. It means that $I(\underline{Y})(0) := 0$ and $\langle I(\underline{Y}), \eta' \rangle := -\langle \mathcal{R}(\underline{Y} \dot{\underline{W}}), \eta \rangle$. More precisely, we have for $|t-s| \leq 1$,

$$I(\underline{Y})_{s,t} = \sum_{k \in I_l} \langle \mathcal{R}(\underline{Y} \dot{\underline{W}}), \phi_k^l \rangle \int_s^t \phi_k^l + \sum_{j \geq l} \sum_{k \in I_l} \langle \mathcal{R}(\underline{Y} \dot{\underline{W}}), \psi_k^j \rangle \int_s^t \psi_k^j. \quad (3.42)$$

Moreover, according to Theorem 3.3.11, we can choose the integer l such that $2^{-l} \leq |t-s| < 2^{-l+1}$.

We have

$$I(\underline{Y})_{s,t} - y_s \underline{W}_{s,t} - y'_s \mathbb{W}_{s,t} = \sum_{k \in I_l} \langle \mathcal{R}(\underline{Y} \dot{\underline{W}}) - \Pi_s(\underline{Y} \dot{\underline{W}}(s)), \phi_k^l \rangle \int_s^t \phi_k^l \quad (3.43)$$

$$+ \sum_{j \geq l} \sum_{k \in I_j} \langle \mathcal{R}(\underline{Y} \dot{\underline{W}}) - \Pi_s(\underline{Y} \dot{\underline{W}}(s)), \psi_k^j \rangle \int_s^t \psi_k^j. \quad (3.44)$$

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We have

$$\begin{aligned} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j \rangle &= \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)), \psi_k^j \rangle \\ &\quad + \langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j \rangle. \end{aligned} \quad (3.45)$$

The first term of the right side of (3.45) is bounded by (3.30),

$$|\langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)), \psi_k^j \rangle| \leq C 2^{-j/2} 2^{j(1-3\alpha)}. \quad (3.46)$$

For bounding the second term of the right side of (3.45) we use the algebraic relations between Π and Γ as well as the relations (3.26),

$$\begin{aligned} &\langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j \rangle \\ &= \langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j) - \Gamma_{k/2^j, s} \underline{Y}\dot{\underline{W}}(s)), \psi_k^j \rangle. \end{aligned}$$

Yet $\underline{Y}\dot{\underline{W}} \in D_{M^r}^{3\alpha-1}$, so with (3.39) and (3.40), we have

$$\left\| \underline{Y}\dot{\underline{W}}(k/2^j) - \Gamma_{k/2^j, s} \underline{Y}\dot{\underline{W}}(s) \right\|_{\beta} \leq C |k/2^j - s|^{3\alpha-1-\beta},$$

for $\beta \in \{2\alpha - 1, \alpha - 1\}$. Finally, we obtain with the bounds (3.26),

$$\left| \langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j \rangle \right| \quad (3.47)$$

$$\leq \sum_{\beta \in \{2\alpha-1, \alpha-1\}} 2^{-j\beta-j/2} \left| \frac{k}{2^j} - s \right|^{3\alpha-1-\beta}. \quad (3.48)$$

Moreover, we have $k/2^j \in [-c/2^j - s, c/2^j + t]$ for all terms that are non-vanishing in (3.43) and (3.44). Since $j \geq l$ in the sums and that we assume $2^{-j} \leq 2^{-l} \leq |t-s| < 2^{-l+1}$, we have

$$\left| \frac{k}{2^j} - s \right| \leq C |t - s|, \quad (3.49)$$

for all non-vanishing terms in the sums (3.43) and (3.44).

Firstly we bound (3.43). On the one hand, using (3.46), (3.48), (3.49) and the fact that $|t-s| < 2^{-l+1}$, we obtain

$$\left| \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j \rangle \right| \leq C 2^{-l/2} 2^{-l(3\alpha-1)}. \quad (3.50)$$

On another hand, we have

$$\begin{aligned} \left| \int_s^t \phi_k^l \right| &\leq 2^{l/2} |t-s| \sup_{t \in \mathbb{R}} \|\phi(t)\| \\ &\leq C 2^{-l/2}. \end{aligned} \quad (3.51)$$

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Thus, because there is only a finite number of terms independent on l that contribute to the sum (3.43), we obtain with (3.50) and (3.51) the following bound on (3.43):

$$\left| \sum_{k \in I_l} \langle \mathcal{R}(\underline{Y}\dot{W}) - \Pi_s(\underline{Y}\dot{W}(s)), \phi_k^l \rangle \int_s^t \phi_k^l \right| \leq C 2^{-l3\alpha} \leq C |t-s|^{3\alpha}, \quad (3.52)$$

where C does not depends on l .

Now, we bound (3.44). On the one hand, using (3.46), (3.48), (3.49), we have for $j \geq l$,

$$\begin{aligned} & |\langle \mathcal{R}(\underline{Y}\dot{W}) - \Pi_s(\underline{Y}\dot{W}(s)), \psi_k^j \rangle| \\ & \leq C 2^{-j/2} \left[2^{j(1-3\alpha)} + |t-s|^{2\alpha} 2^{-j(\alpha-1)} + |t-s|^\alpha 2^{-j(2\alpha-1)} \right]. \end{aligned} \quad (3.53)$$

On an other hand, we observe that

$$\left| \sum_{k \in I_j} \int_s^t \psi_k^j \right| \leq C 2^{-j/2}, \quad (3.54)$$

because a primitive of ψ has a compact support and the fact that $\int \psi = 0$. Then, combining (3.53) and (3.54) we obtain,

$$\begin{aligned} & \left| \sum_{j \geq l} \sum_{k \in I_j} \langle \mathcal{R}(\underline{Y}\dot{W}) - \Pi_s(\underline{Y}\dot{W}(s)), \psi_k^j \rangle \int_s^t \psi_k^j \right| \\ & \leq \sum_{j \geq l} 2^{-3j\alpha} + |t-s|^{2\alpha} 2^{-j\alpha} + |t-s|^\alpha 2^{-j2\alpha} \\ & \leq C 2^{-3l\alpha} + |t-s|^{2\alpha} 2^{-l\alpha} + |t-s|^\alpha 2^{-l2\alpha} \\ & \leq C |t-s|^{3\alpha}. \end{aligned} \quad (3.55)$$

With (3.52) and (3.55) we obtain the bound of the left hand side of (3.43),

$$|I(\underline{Y})_{s,t} - y_s W_{s,t} - y'_s \mathbb{W}_{s,t}| \leq C |t-s|^{3\alpha}. \quad (3.56)$$

To show that $L(\underline{Y})$ is in $D_{M^r}^{2\alpha}$, we compute $\Gamma_{t,s}^r(L(\underline{Y})(s)) = (I(\underline{Y})(s) + y_s W_{s,t}) \mathbb{1} + y_s \underline{W}$ and we use the estimation (3.56). Thus, we have

$$\begin{aligned} \|L(\underline{Y})(t) - \Gamma_{t,s}^r(L(\underline{Y})(s))\|_0 &= \|I(\underline{Y})(t) - I(\underline{Y})(s) - y_s W_{s,t}\| \\ &\leq \|y'\|_{\infty, T} \|\mathbb{W}\|_{2\alpha, T} |t-s|^{2\alpha} + C |t-s|^{3\alpha}, \end{aligned} \quad (3.57)$$

$$\text{and } \|L(\underline{Y})(t) - \Gamma_{t,s}(L(\underline{Y})(s))\|_\alpha = \|y_{s,t}\| \leq \|y\|_\alpha |t-s|^\alpha, \quad (3.58)$$

which proves that $L(\underline{Y})$ is in $D_{M^r}^{2\alpha}$.

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It remains to prove the continuity of L . According to (3.30), the constant C in (3.56) is proportional to $\|\underline{Y}\|_{\gamma, T}^*$. So we have,

$$|I(\underline{Y})_{s,t} - y_s W_{s,t}| \leq \|y'\|_{\infty} \|\mathbb{W}\|_{2\alpha, T} |t-s|^{2\alpha} + C \|\underline{Y}\|_{2\alpha, T}^* |t-s|^{3\alpha},$$

which allows with the previous computation (3.57) and (3.58) to bound

$$\|L(\underline{Y})\|_{2\alpha, T}^* \leq C \|\underline{Y}\|_{2\alpha, T}^*. \quad (3.59)$$

This concludes the proof. ■

3.8 Existence of a rough path lift

As an application of the reconstruction operator in the case $\gamma \leq 0$, we prove Theorem 3.4.6 which states that for any $W \in C^\alpha$ ($\alpha \in (1/3, 1/2]$) with values in \mathbb{R}^n , it exists a rough path lift \mathbb{W} and that the map $W \mapsto \mathbf{W}$ is continuous from C^α to \mathcal{C}^α .

Proof (Theorem 3.4.6). We consider the regularity structure $(\mathcal{A}^e, \mathcal{T}^e, \mathcal{G}^e)$ such that $\mathcal{A}^e = \{\alpha - 1, 0\}$, $\mathcal{T}^e = \text{Vect}\langle \dot{W}^i, i = 1, \dots, n \rangle \oplus \text{Vect}\langle 1 \rangle$ and for $\Gamma_h^e \in \mathcal{G}$, $\Gamma_h^e(\dot{W}) = \dot{W}$, $\Gamma_h^e(1) = 1$. We associate the model $M^e = (\Pi^e, \Gamma^e)$ such that for every $s, t \in [0, T]$, $\eta \in B_1$

$$\Pi_s^e(\dot{W})(\eta) := \int \eta(t) dW_t, \quad \Pi_s^e(1)(t) := 1,$$

and $\Gamma_{s,t}^e := \Gamma_{W_{t,s}}^e$.

For $0 \leq s \leq t \leq 1$, and integers $0 \leq i, j \leq n$, the modelled distribution $\dot{\mathbb{W}}$ given by $\dot{\mathbb{W}}^{i,j}(s) := W_s^i \dot{W}^j$ is in $D_M^{2\alpha-1}$. Indeed $\dot{\mathbb{W}}^{i,j}(t) - \Gamma_{t,s}^e(\dot{\mathbb{W}}^{i,j}(s)) = W_t^i \dot{W}^j - W_s^i \dot{W}^j = W_{s,t}^i \dot{W}^j$, then

$$\left\| \dot{\mathbb{W}}^{i,j}(t) - \Gamma_{t,s}^e(\dot{\mathbb{W}}^{i,j}(s)) \right\|_{\alpha-1} \leq |t-s|^\alpha.$$

So, $\gamma - (\alpha - 1) = \alpha$, we have $\gamma = 2\alpha - 1$. We conclude using the Definition 3.6.1.

Given that $\alpha \in (1/3, 1/2]$, we have $2\alpha - 1 \leq 0$. Thus, the uniqueness of the reconstruction map does not hold. But, according to Theorem 3.6.5, there exists $\mathcal{R}(\dot{\mathbb{W}}) \in C^{\alpha-1}$ such that

$$|[\mathcal{R}(\dot{\mathbb{W}}) - \Pi_s^e(\dot{\mathbb{W}})](\eta_s^\lambda)| \leq C \lambda^{2\alpha-1}, \quad (3.60)$$

where $\eta \in B_1$. With Lemma 3.3.10, we define $z \in C^\alpha$ as the primitive of $\mathcal{R}(\dot{\mathbb{W}})$ such that $z(0) = 0$. Moreover, we have for all $s, t \in [0, 1]$,

$$z_{s,t} = \sum_{k \in I_1} \langle \mathcal{R}(\dot{\mathbb{W}}), \phi_k^l \rangle \int_s^t \phi_k^l + \sum_{j \geq 1} \sum_{k \in I_j} \langle \mathcal{R}(\dot{\mathbb{W}}), \psi_k^j \rangle \int_s^t \psi_k^j, \quad (3.61)$$

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and

$$W_{s,t} = \sum_{k \in I_l} \langle \Pi_s^e(\dot{W}), \phi_k^l \rangle \int_s^t \phi_k^l + \sum_{j \geq l} \sum_{k \in I_j} \langle \Pi_s^e(\dot{W}), \psi_k^j \rangle \int_s^t \psi_k^j, \quad (3.62)$$

which yields to

$$W_s \otimes W_{s,t} = \sum_{k \in I_l} \langle \Pi_s^e(\dot{W}(s)), \phi_k^l \rangle \int_s^t \phi_k^l + \sum_{j \geq l} \sum_{k \in I_j} \langle \Pi_s^e(\dot{W}(s)), \psi_k^j \rangle \int_s^t \psi_k^j. \quad (3.63)$$

If there is a constant $C \geq 0$ such that,

$$|z_{s,t} - W_s \otimes W_{s,t}| \leq C|t - s|^{2\alpha}, \quad (3.64)$$

then setting $\mathbb{W}_{s,t} := z_{s,t} - W_s \otimes W_{s,t}$, the pair (W, \mathbb{W}) belongs to \mathcal{C}^α according to the Definition 3.4.1. Let us prove (3.64). We have

$$\begin{aligned} z_{s,t} - W_s \otimes W_{s,t} &= \sum_{k \in I_l} \langle \mathcal{R}(\dot{W}) - \Pi_s^e(\dot{W}(s)), \phi_k^l \rangle \int_s^t \phi_k^l \\ &\quad + \sum_{j \geq l} \sum_{k \in I_j} \langle \mathcal{R}(\dot{W}) - \Pi_s^e(\dot{W}(s)), \psi_k^j \rangle \int_s^t \psi_k^j. \end{aligned} \quad (3.65)$$

From (3.30), we have the bounds

$$|\langle \mathcal{R}(\dot{W}) - \Pi_s^e(\dot{W}(s)), \phi_k^j \rangle| \leq C2^{-j/2-j(2\alpha-1)}, \quad (3.66)$$

and

$$|\langle \mathcal{R}(\dot{W}) - \Pi_s^e(\dot{W}(s)), \psi_k^j \rangle| \leq C2^{-j/2-j(2\alpha-1)}. \quad (3.67)$$

Then, combining (3.65), (3.66) and (3.67), we proceed as in the proof of Lemma 3.3.10 to show (3.64).

It remains to show the continuity. If there is another path $\tilde{W} \in C^\alpha$, we define as for W , a model $(\tilde{\Pi}, \tilde{\Gamma})$, a modelled distribution $\tilde{\mathbb{W}}$, a reconstruction map $\tilde{\mathcal{R}}$ and then $\tilde{\mathbb{W}}$. By denoting

$$\Delta \Pi_{s,k/2^j} := [\Pi(\dot{W}(k/2^j)) - \Pi(\dot{W}(s)) - \tilde{\Pi}(\tilde{\mathbb{W}}(k/2^j)) + \tilde{\Pi}(\tilde{\mathbb{W}}(s))](\psi_k^j), \quad (3.68)$$

we have

$$\begin{aligned} &|\Delta \Pi_{s,k/2^j}| \\ &\leq \|W - \tilde{W}\|_{\alpha,T} \left(\|W\|_{\alpha,T} + \|\tilde{W}\|_{\alpha,T} \right) |s - k/2^j| 2^{j/2(1-\alpha)} 2^{-j/2}. \end{aligned} \quad (3.69)$$

According to the bounds (3.31), (3.69) and in writing

$$\begin{aligned} &[\mathcal{R}(\dot{W}) - \Pi_s(\dot{W}(s)) - \tilde{\mathcal{R}}(\tilde{\mathbb{W}}) + \Pi_s(\tilde{\mathbb{W}}(s))](\psi_k^j) \\ &= \mathcal{R}(\dot{W}) - \Pi_{k/2^j}(\dot{W}(k/2^j)) - \tilde{\mathcal{R}}(\tilde{\mathbb{W}}) + \tilde{\Pi}_{k/2^j}(\tilde{\mathbb{W}}(k/2^j)) + \Delta \Pi_{s,k/2^j}, \end{aligned}$$

we get

$$|\mathbb{W}_{s,t} - \tilde{\mathbb{W}}_{s,t}| \leq C \left[\|\tilde{\Pi}\|_{2\alpha-1,T} \|\dot{\mathbb{W}} - \tilde{\dot{\mathbb{W}}}\|_{2\alpha-1,T}^* + \|\Pi - \tilde{\Pi}\|_{2\alpha-1,T} \|\dot{\mathbb{W}}\|_{2\alpha-1,T}^* \right. \\ \left. + \|W - \tilde{W}\|_{\alpha,T} \left(\|W\|_{\alpha,T} + \|\tilde{W}\|_{\alpha,T} \right) \right] |t - s|^{2\alpha}.$$

Yet we have, $\|\dot{\mathbb{W}} - \tilde{\dot{\mathbb{W}}}\|_{2\alpha-1,T}^* = \|W - \tilde{W}\|_{\infty,T} + \|W - \tilde{W}\|_{\alpha,T}$, and

$$\|\Pi - \tilde{\Pi}\|_{2\alpha-1,T} \leq C \|W - \tilde{W}\|_{\alpha,T}. \quad (3.70)$$

So finally,

$$\|\mathbf{W} - \tilde{\mathbf{W}}\|_{\alpha,T} \leq C \|W - \tilde{W}\|_{\alpha,T}, \quad (3.71)$$

which proves the continuity. ■

Remark 3.8.1. Given that $2\alpha - 1$ is negative, the uniqueness of \mathbb{W} does not hold, which is in accordance with Remark 3.4.5.

3.9 Composition with a smooth function

Before solving the general rough differential equation (3.1) with the theory of regularity structures, we should give a sense of the composition of a modelled distribution with a function. Then we will be able to consider (3.1) in the space of the modelled distributions.

The composition of a modelled distribution $\underline{f} \in D_M^\gamma$ with a smooth function F is developed in [Hai14]. The author gives a general theorem which allows the composition with an arbitrary smooth function F when \underline{f} takes its values in a model space \mathcal{T} such that the smallest index of homogeneity is equal to 0, *i.e.* $\forall t \in \mathbb{R}, \underline{f}(t) \in \text{Vect}\langle \underline{1}, \dots \rangle$. Thus, it is possible to define the composition as a Taylor expansion

$$\hat{F} \circ \underline{f}(t) = \sum_k \frac{F^{(k)}(\bar{\underline{f}}(t))}{k!} (\underline{f}(t) - \bar{\underline{f}}(t)\underline{1})^k, \quad (3.72)$$

where $\bar{\underline{f}}$ is the coordinate of \underline{f} onto $\underline{1}$. The definition above makes no sense if the product between elements of the regularity structure is not defined. We can also find the general definition in [Hai14]. This is not useful here. The idea of the decomposition (3.72) is to compute a Taylor expansion of F in $\bar{\underline{f}}$ the part of \underline{f} which is the first approximation of $\mathcal{R}\underline{f}$.

Here we just prove (what is needed for solving (3.1)) that $\hat{F} \circ \underline{f}$ lives in the same space as \underline{f} and that \hat{F} is Lipschitz in the particular case of modelled distribution of controlled rough paths.

3.9. COMPOSITION WITH A SMOOTH FUNCTION

Theorem 3.9.1. *Let $F \in C_b^2(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$. For $\alpha \in (1/3, 1/2]$, given a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\alpha$, the controlled rough path $(y, y') \in \mathcal{D}_W^{2\alpha}$, for all $\underline{Y} \in D_{M^r}^{2\alpha}$ defined by $\underline{Y}(t) = y_t \mathbb{1} + y'_t \underline{W}$, the map \hat{F} such that*

$$\hat{F} \circ \underline{Y}(t) := F(y_t) \mathbb{1} + F'(y_t) y'_t \underline{W}, \quad (3.73)$$

is in $D_{M^r}^{2\alpha}$. Moreover if $F \in C_b^3$ the function associated \hat{F} is Lipschitz, i.e. for all $\underline{Y}, \tilde{Y} \in D_{M^r}^{2\alpha}$

$$\left\| \hat{F}(\underline{Y}) - \hat{F}(\tilde{Y}) \right\|_{2\alpha, T}^* \leq C \left\| \underline{Y} - \tilde{Y} \right\|_{2\alpha, T}^*, \quad (3.74)$$

where C is a constant.

Remark 3.9.2. This theorem shows that the space $D_{M^r}^{2\alpha}$ is stable by a non linear composition \hat{F} , provided that \hat{F} is regular enough. So with Theorem 3.7.3, we can build the integral

$$I(\hat{F}(\underline{Y})) = \int_0^\cdot F(y_s) d\mathbf{W}_s.$$

Proof. Firstly, let us show that \hat{F} is a map from $D_{M^r}^{2\alpha}$ to $D_{M^r}^{2\alpha}$. A straightforward computation leads us to the two following expressions

$$\begin{aligned} \left\| \hat{F}(\underline{Y})(t) - \Gamma_{t,s}^r \left(\hat{F}(\underline{Y})(s) \right) \right\|_0 &= \|F'(y_t) y'_t - F'(y_s) y'_s\|, \\ \left\| \hat{F}(\underline{Y})(t) - \Gamma_{t,s}^r \left(\hat{F}(\underline{Y})(s) \right) \right\|_\alpha &= \|F(y_t) - F(y_s) - F'(y_s) y'_s W_{s,t}\|. \end{aligned}$$

Let us denote the left-hand of the first equality $\Delta_{s,t}^0$ and of the second one $\Delta_{s,t}^\alpha$. We obtain

$$\begin{aligned} \Delta_{s,t}^0 &\leq \|F'(y_t)\| \|y'_t - y'_s\| + \|y'_s\| \|F'(y_t) - F'(y_s)\| \\ &\leq \|F'\|_{\infty, T} \|y'\|_\alpha |t - s|^\alpha + \|y'\|_{\infty, T} \|F''\|_{\infty, T} \|Y\|_{\alpha, T} |t - s|^\alpha \end{aligned}$$

and

$$\begin{aligned} \Delta_{s,t}^\alpha &= \|F(y_t) - F(y_s) - F'(y_s)(y_{s,t} - R_{s,t}^y)\| \\ &\leq \|F(y_t) - F(y_s) - F'(y_s) y_{s,t}\| + \|F'(y_s) R_{s,t}^y\| \\ &\leq \frac{1}{2} \|F''\|_{\infty, T} \|y_{s,t}\|^2 + \|F'\|_{\infty, T} \|R^y\|_{2\alpha, T} |t - s|^{2\alpha} \\ &\leq \frac{1}{2} \|F''\|_{\infty, T} \|y\|_{\alpha, T}^2 |t - s|^{2\alpha} + \|F'\|_{\infty, T} \|R^y\|_{2\alpha, T} |t - s|^{2\alpha}. \end{aligned}$$

This proves that $\hat{F}(\underline{Y}) \in D_{M^r}^{2\alpha}$.

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We now prove the inequality (3.74). A more general proof can be found in [Hai14]. We define $\tilde{Z} = \underline{Y} - \tilde{Y}$, which is in $D_{Mr}^{2\alpha}$ by linearity. We denote by $Q_{<2\alpha}$ the projection onto $\mathcal{T}_{<2\alpha}$. Using the integration by parts formula, one can check that

$$\hat{F}(\underline{Y}(s)) - \hat{F}(\tilde{Y}(s)) = \sum_{k=0}^1 \int_0^1 F^{(k)}(\tilde{y}_s + uz_s) Q_{<2\alpha} \left[[(\tilde{y}'_s + uz'_s) \underline{W}]^k \tilde{Z}(s) \right] du.$$

Then, we compute the expansion between s and t of $\underline{\Delta}(s) := \hat{F}(\underline{Y}(s)) - \hat{F}(\tilde{Y}(s))$. We denote $\underline{A}_u(s) := \tilde{Y}(s) + u\tilde{Z}(s)$. When u is fixed, \underline{A}_u is in $D_{Mr}^{2\alpha}$. We have

$$\begin{aligned} \Gamma_{t,s} \underline{\Delta}(s) &= \sum_{k=0}^1 \int_0^1 F^{(k)}(\underline{A}_u(s)) \Gamma_{t,s} Q_{<2\alpha} \left([A'_u(s) \underline{W}]^k \tilde{Z}(s) \right) du \\ &= \sum_{k=0}^1 \int_0^1 F^{(k)}(\underline{A}_u(s)) [\Gamma_{t,s}(A'_u(s) \underline{W})]^k \Gamma_{t,s} \tilde{Z}(s) du + \underline{R}(s, t), \end{aligned}$$

where \underline{R} is a remainder such that $\|\underline{R}(s, t)\|_\beta \lesssim |t - s|^{2\alpha - \beta}$ for $\beta \in \{0, \alpha\}$. From now, we denote by \underline{R} all the remainder terms which satisfy this property.

We now shift the last expression from s to t . On the one hand

$$\Gamma_{t,s}^r(A'_u(s) \underline{W}) = \Gamma_{t,s}^r \underline{A}_u(s) - A_u(s) \mathbb{1} = \underline{A}_u(t) - A_u(s) \mathbb{1} + \underline{R}(s, t).$$

On the other hand

$$\Gamma_{t,s}^r \tilde{Z}(s) = \tilde{Z}(t) + \underline{R}(s, t).$$

This yields

$$\Gamma_{t,s}^r \underline{\Delta}(s) = \sum_{k=0}^1 \int_0^1 F^{(k)}(\underline{A}_u(s)) [A'_u(t) \underline{W} + (A_u(t) - A_u(s)) \mathbb{1}]^k \tilde{Z}(s) du + \underline{R}(s, t).$$

It remains to shift $F^{(k)}$ from s to t . With the classical Taylor expansion formula,

$$F^{(k)}(A_u(s)) = \sum_{0 \leq l+k \leq 1} F^{(k+l)}(A_u(t)) (A_u(s) - A_u(t))^l + O(|t - s|^{2\alpha - k\alpha}),$$

because $\|A_u(t) - A_u(s)\| \leq |t - s|^\alpha$. The bound

$$\left\| [A'_u(t) \underline{W} + (A_u(t) - A_u(s)) \mathbb{1}]^k \right\|_\beta \lesssim |t - s|^{k\alpha - \beta}$$

holds. Finally, with the two previous expressions,

$$\begin{aligned} \Gamma_{t,s}^r \underline{\Delta}(s) &= \sum_{0 \leq l+k \leq 1} F^{(k+l)}(A_u(t)) (A_u(s) - A_u(t))^l \\ &\quad \times [A'_u(t) \underline{W} + (A_u(t) - A_u(s)) \mathbb{1}]^k \tilde{Z}(t) + \left\| \tilde{Z} \right\|_{2\alpha, T}^* O(|t - s|^{2\alpha - \beta}) \\ &= \sum_{0 \leq k \leq 1} F^{(k)}(A_u(t)) [A'_u(t) \underline{W}]^k \tilde{Z}(t) + \left\| \tilde{Z} \right\|_{2\alpha, T}^* O(|t - s|^{2\alpha - \beta}) \\ &= \underline{\Delta}(t) + \left\| \tilde{Z} \right\|_{2\alpha, T}^* O|t - s|^{2\alpha - \beta}, \end{aligned}$$

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which proves the inequality. ■

3.10 Solving the rough differential equations

Theorem 3.7.3 combined with Theorem 3.9.1 allow us to solve the rough differential equations in the modelled distribution space $D_{M^r}^{2\alpha}$.

Theorem 3.10.1. *Given $\xi \in \mathbb{R}^d$, $F \in C_b^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$, a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\beta$ with $\beta \in (1/3, 1/2)$, there is a unique modelled distribution $\underline{Y} \in D_{M^r}^{2\beta}$ such that for all $t \in [0, T]$,*

$$\underline{Y}(t) = \xi \mathbf{1} + L(\hat{F}(\underline{Y}))(t), \quad (3.75)$$

where L is defined in Theorem 3.7.3.

Proof. We prove that the operator $N(\underline{Y}) := \xi \mathbf{1} + L(\hat{F}(\underline{Y}))$ where L is defined in Theorem 3.7.3, has a unique fixed point. For this we show that the unit ball of $D_{M^r}^{2\alpha}$ is invariant under the action of N , and then that N is a strict contraction.

These two properties can be obtained by choosing a wise time interval $[0, T]$. We take a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\beta \subset \mathcal{C}^\alpha$ with $1/3 < \alpha < \beta < 1/2$ and $\underline{Y} \in D_{M^r}^{2\alpha}$. This trick allows us to have a $T^{\beta-\alpha}$ in our estimates. Thus, with a T small enough we prove the fixed point property. We start by choosing $T \leq 1$.

According to Theorem 3.9.1 $\hat{F}(\underline{Y}) \in D_{M^r}^{2\alpha}$, thus Theorem 3.7.3 shows that $N(\underline{Y}) \in D_{M^r}^{2\alpha}$. If \underline{Y} is a fixed point of N then $\underline{Y} \in D_{M^r}^{2\beta}$, thanks to the fact that $\mathbf{W} \in \mathcal{C}^\beta$. Indeed,

$$\|\underline{Y}(t) - \Gamma_{t,s}\underline{Y}(s)\|_\beta = \|y_{s,t}\| \leq \|y'\|_{\infty, T} \|W\|_{2\beta, T} |t-s|^{2\beta} + \|R^y\|_{2\alpha, T} |t-s|^{2\alpha},$$

and

$$\|\underline{Y}(t) - \Gamma_{t,s}\underline{Y}(s)\|_0 = \|y_{s,t} - y'_s W_{s,t}\| \leq \|y'\|_{\infty, T} \|\mathbb{W}_{s,t}\| + O(|t-s|^{3\alpha}).$$

As a result of the fixed point property $y' = F(y)$. This proves that $\underline{Y} \in D_{M^r}^{2\beta}$.

We recall that $\|\underline{Y}\|_{2\alpha, T}^* = \sup_{\epsilon \in \{0, \alpha\}} \|\underline{Y}(0)\|_\epsilon + \|\underline{Y}\|_{2\alpha, T}$, where

$$\|\underline{Y}\|_{2\alpha, T} = \sup_{t, s \in [0, T], \epsilon \in \{0, \alpha\}} \frac{\|\underline{Y}(t) - \Gamma_{t,s}\underline{Y}(s)\|_\epsilon}{|t-s|^\epsilon}.$$

It is more convenient to work with the semi-norm $\|\cdot\|_{2\alpha, T}$, so we define the affine ball unit on $[0, T]$

$$B_T = \{\underline{Y} \in D_{M^r}^{2\alpha}, \underline{Y}(0) = \xi \mathbf{1} + f(\xi)\underline{W}, \|\underline{Y}\|_{2\alpha, T} \leq 1\}.$$

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Invariance: For $\underline{Y} \in B_T$, on has $\|\hat{F}(\underline{Y})\|_{2\alpha, T} \leq \|\underline{Y}\|_{2\alpha, T}$ and

$$N(\underline{Y}) = \|L(\hat{F}(\underline{Y}))\|_{2\alpha, T}.$$

On the on hand, according to the reconstruction map,

$$\begin{aligned} & \left\| (I\hat{F}(\underline{Y}))_{s,t} - F(y_s)W_{s,t} \right\| \\ & \leq \|F'(y)y'\|_{\infty, T} \|\mathbb{W}\|_{2\alpha, T} |t-s|^{2\alpha} + C \|\hat{F}(\underline{Y})\|_{2\alpha, T}^* |t-s|^{3\alpha} \\ & \leq \|\hat{F}(\underline{Y})\|_{2\alpha, T}^* \|\mathbb{W}\|_{2\alpha} |t-s|^{2\alpha} + C \|\hat{F}(\underline{Y})\|_{2\alpha, T}^* |t-s|^{3\alpha} \\ & \leq \|F'\|_{\infty} \left[\|\underline{Y}\|_{2\alpha, T}^* \|\mathbb{W}\|_{2\beta, T} T^{\beta-\alpha} |t-s|^{2\alpha} + C \|\underline{Y}\|_{2\alpha, T}^* |t-s|^{2\alpha} T^\alpha \right], \end{aligned}$$

because $\|\cdot\|_{\beta} \leq \|\cdot\|_{\alpha} T^{\beta-\alpha}$. Using the fact that $T^\alpha \leq T^{\beta-\alpha}$ and that $\underline{Y} \in B_T$ we obtain

$$\|N(\underline{Y})\|_0 \leq CT^{\beta-\alpha},$$

where C is independent of \underline{Y} . On the other hand,

$$\begin{aligned} \|y_{s,t}\| & \leq \|y'\|_{\infty, T} \|W\|_{\alpha, T} |t-s|^\alpha + \|R^y\|_{2\alpha, T} |t-s|^{2\alpha} \\ & \leq \|\underline{Y}\|_{2\alpha, T}^* \|W\|_{\beta, T} T^{\beta-\alpha} |t-s|^\alpha + \|R^y\|_{2\alpha, T} T^\alpha |t-s|^\alpha \\ & \leq \|\underline{Y}\|_{2\alpha, T}^* \|W\|_{\beta, T} T^{\beta-\alpha} |t-s|^\alpha + \|\underline{Y}\|_{2\alpha, T}^* T^\alpha |t-s|^\alpha. \end{aligned}$$

Using the last inequality

$$\begin{aligned} \|F(y)\|_{\alpha, T} & \leq \|F'\|_{\infty} \|y\|_{\alpha, T} \\ & \leq \|\underline{Y}\|_{2\alpha, T}^* \|W\|_{\beta, T} T^{\beta-\alpha} |t-s|^\alpha + \|\underline{Y}\|_{2\alpha, T}^* T^\alpha |t-s|^\alpha, \end{aligned}$$

which leads to $\|N(\underline{Y})\|_{\alpha} \leq CT^{\beta-\alpha}$. Finally, we obtain the following estimate

$$\|N(\underline{Y})\|_{2\alpha, T} \leq CT^{\beta-\alpha},$$

where C does not depend on \underline{Y} . By choosing $T = T_0$ small enough, we show that $N(B_{T_0}) \subset B_{T_0}$.

Contraction: For $\underline{Y}, \tilde{\underline{Y}} \in D_{M^r}^{2\alpha}$,

$$\begin{aligned} \|N(\underline{Y}) - N(\tilde{\underline{Y}})\|_{2\alpha, T} & \leq \|N(\underline{Y}) - N(\tilde{\underline{Y}})\|_0 + \|N(\underline{Y}) - N(\tilde{\underline{Y}})\|_{\alpha} \\ & \leq C \|\hat{F}(\underline{Y}) - \hat{F}(\tilde{\underline{Y}})\|_{2\alpha, T}^* T^{\beta-\alpha} + \|F(y) - F(\tilde{y})\|_{\alpha} \\ & \leq C \|\underline{Y} - \tilde{\underline{Y}}\|_{2\alpha, T}^* T^{\beta-\alpha} + \|F'\|_{\infty} \|y - \tilde{y}\|_{\alpha}, \end{aligned}$$

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according to (3.74). Then it is easy to show that

$$\|y - \tilde{y}\|_\alpha \leq CT^{\beta-\alpha} \|\underline{Y} - \tilde{\underline{Y}}\|_{2\alpha, T}.$$

Finally, $\|N(\underline{Y}) - N(\tilde{\underline{Y}})\|_{2\alpha, T} \leq CT^{\beta-\alpha} \|\underline{Y} - \tilde{\underline{Y}}\|_{2\alpha, T}$ where C does not depend on neither \underline{Y} nor $\tilde{\underline{Y}}$. So with T small enough, $N(B_T) \subset B_T$ and N is a strict contraction. So, there is a unique solution $\underline{Y} \in D_{M^r}^{2\alpha}$ to (3.75) on $[0, T]$. As mentioned at the beginning of the proof, \underline{Y} is in $D_{M^r}^{2\beta}$. ■

Corollary 3.10.2. *Given $\xi \in \mathbb{R}^d$, $F \in C_b^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$, a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathcal{C}^\beta$ with $\beta \in (1/3, 1/2)$, there is a unique controlled rough path $(y, y') \in \mathcal{D}_W^{2\beta}$ such that for all $t \in [0, T]$*

$$y(t) = \xi + \int_0^t F(y_u) d\mathbf{W}_u, \quad (3.76)$$

where the integral has to be understood as the controlled rough path integral (Theorem 3.4.15).

Remark 3.10.3. Actually, we can extend this result to $T = +\infty$, because T is chosen uniformly with respect to parameters of the problem.

Proof. It suffices to project Equation (3.75) onto $\underline{\mathbb{1}}$ and onto \underline{W} . ■

CHAPTER 3. SOLVING ROUGH DIFFERENTIAL EQUATIONS WITH THE
THEORY OF REGULARITY STRUCTURES

Notations

C^α	α -Hölder continuous functions
$\mathcal{D}_W^{2\alpha}$	The controlled rough path space
$:=$	Symbol to define an object
\lesssim	Estimation
\mathcal{G}	Structure group of a regularity structure
\mathcal{A}	Index set of a regularity structure
$\langle \cdot, \mathbb{1} \rangle$	The projection onto $\mathbb{1}$
$\mathcal{L}(A, B)$	Linear continuous maps between two vector spaces A and B .
D_M^γ	Modelled distributions of regularity γ
\mathcal{T}	Model space of a regularity structure
$\mathcal{T}_{<\gamma}$	Elements of \mathcal{T} of homogeneity strictly less than γ
\star	Abstract product
$\ f\ _{\gamma, T}^*$	Norm of the modelled distribution space
$\ \tau\ _\alpha$	Norm of the component of τ in \mathcal{T}_α
$\ \cdot\ _{2\alpha, T}^W$	Semi-norm of the controlled rough path space
$\ f\ _{\infty, T}$	Supremum of $\ f(t)\ $ for $t \in [0, T]$
\otimes	Tensor product
$Q_{<2\alpha}$	The projection onto $\mathcal{T}_{<2\alpha}$
\mathcal{R}	The reconstruction operator
\mathbf{W}	Rough path associated to W

CHAPTER 3. SOLVING ROUGH DIFFERENTIAL EQUATIONS WITH THE
THEORY OF REGULARITY STRUCTURES

\mathbb{T}	A regularity structure
\mathbb{T}^p	Model associated to the regularity structure of the rough paths
\mathbb{T}^p	Polynomial regularity structure
\mathbb{T}^p	Regularity structure of the rough paths
$\text{Vect}\langle \cdot \rangle$	Subspace generated by the linear combinations of \cdot .
B_r	The space of all functions in $\phi \in C_b^r$ compactly supported on $[-1, 1]$, such that $\ \eta\ _{C_b^r} \leq 1$.
C_b^q	Bounded functions with all derivative up to order q bounded.
$D_{M^r}^{2\alpha}$	Modelled distributions of the controlled rough paths
M	Model associated to a regularity structure
M^p	Model of the polynomial regularity structure
$Z_{s,t}$	Means $Z_t - Z_s$
\mathcal{P}	A subdivision

Part II

Flots rugueux

Chapter 4

Introduction aux flots

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4.1 Flots des équations différentielles ordinaires

On fixe un horizon de temps $T > 0$ et un réel $r \geq 0$. Supposons qu'il existe une unique solution $y_{\cdot,r}(a) : [0, T] \rightarrow \mathbb{R}^n$ à l'équation différentielle ordinaire (EDO)

$$y_{t,r}(a) = a + \int_r^t f(y_{s,r}(a)) ds, \quad \forall t \in [r, T], \quad (4.1)$$

où $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ est un champ de vecteurs, $a \in \mathbb{R}^n$ la condition initiale au temps r .

On définit le flot associé à l'EDO comme la famille d'applications

$$(\phi_{t,r} : a \mapsto y_{t,r}(a))_{0 \leq r \leq t \leq T}.$$

Étant donné $0 \leq s \leq t \leq T$ et un sous-ensemble non vide $A \subset \mathbb{R}^n$, l'ensemble $\phi_{t,s}(A)$ est une déformation de A sous l'effet du système (4.1) entre le temps r et t (Figure 4.1).

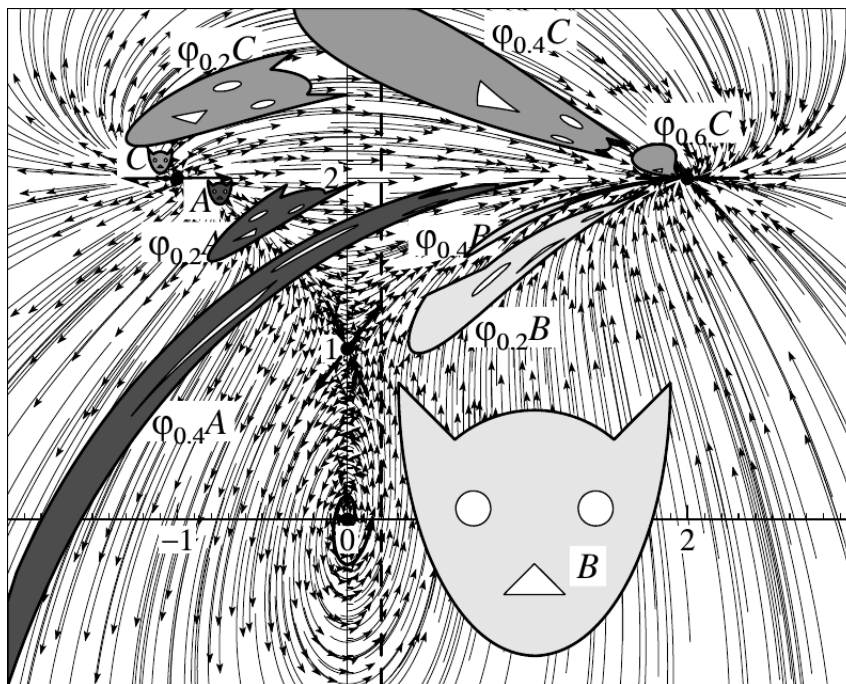


Figure 4.1: On observe ici la déformation des ensembles A, B et C sous l'action du flot ϕ . L'image provient du livre [HNW91]

Pour toute condition initiale $a \in \mathbb{R}^n$, $t \in [s, T] \mapsto \phi_{t,r}(a)$ et $t \in [s, T] \mapsto \phi_{t,s} \circ \phi_{s,r}(a)$ sont solutions de la même EDO (4.1) avec $s = r$. Donc par unicité de la solution, ϕ possède ce que l'on appelle la propriété de flot $\phi_{t,r} = \phi_{t,s} \circ \phi_{s,r}$ pour tout $r \leq s \leq t$ avec $\phi_{r,r}(a) = a$.

Cette propriété étant reliée à l'unicité de la solution, rappelons des conditions suffisantes sur la régularité du champ de vecteurs f pour obtenir l'existence et l'unicité des solutions à (4.1).

Définition 4.1.1. Soient $k \in \mathbb{N}$ et $\gamma \in [0, 1)$, on dit que la fonction $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ appartient à la classe $C_b^{k+\gamma}$ si

- l'application f est k fois continument dérivable,
- pour $\gamma > 0$ la dérivée d'ordre k de f notée $f^{(k)}$ est γ -Hölder continue, c'est-à-dire qu'il existe une constante $H \geq 0$ telle que pour tout $x, y \in \mathbb{R}^n$,

$$\|f^{(k)}(x) - f^{(k)}(y)\| \leq H \|x - y\|^\gamma.$$

- pour $\gamma = 0$, $f^{(k)}$ est seulement continue,
- toutes les dérivées de f jusqu'à l'ordre k sont bornées.

4.1. FLOTS DES ÉQUATIONS DIFFÉRENTIELLES ORDINAIRES

Théorème 4.1.1 (Cauchy-Lipschitz). *Si f est lipschitzienne alors pour tout $r \geq 0$, $a \in \mathbb{R}^n$, il existe une unique solution $y_{\cdot,r}(a) : [r, T] \rightarrow \mathbb{R}^n$ de (4.1).*

La régularité Lipschitzienne de f assure l'existence d'un flot ϕ . On peut alors déterminer la régularité de ce flot par rapport à la condition initiale a . Le théorème suivant montre qu'elle est similaire à celle du champ de vecteurs.

Théorème 4.1.2 ([Mar73]). *Soient $k \geq 1$ et $\gamma \in [0, 1)$. Si $f \in C_b^{k+\gamma}$ alors il existe un flot ϕ associé à (4.1). De plus pour tout $0 \leq r \leq t \leq T$, $\phi_{t,r}$ est un $C^{k+\gamma}$ -difféomorphisme.*

Ces résultats sur l'unicité des solutions ¹ et la régularité du flot ² sont optimaux.

En effet, considérons l'EDO suivante

$$z_t = a^{\frac{1}{1-\alpha}} + \int_0^t |z_u|^\alpha du, \quad (4.2)$$

avec $0 < \alpha < 1$, $a \in \mathbb{R}_+$. Alors l'ensemble des fonctions $(z^{T(a)})_{T(a) \in \mathbb{R}_+}$ telles que

$$z_t^{T(a)} = \begin{cases} (1-\alpha)|t - T(a) - a|^{\frac{1}{1-\alpha}} & \text{si } t \geq T(a) \\ a^{\frac{1}{1-\alpha}} & \text{si } t \in [0, T(a)], \end{cases}$$

sont solutions de (4.2). L'unicité de la solution ne tient plus. La construction d'un flot n'est plus directe.

Afin de rétablir l'unicité, on peut imposer des conditions supplémentaires sur $z^{T(a)}$. Dans notre cas, le fait que des solutions restent constantes sur un intervalle de temps non trivial (si $T(a) > 0$) n'est pas "physique".

Par exemple, si $\alpha = 1/2$ alors $-z^{T(a)}$ la solution de (4.2) modélise l'évolution de la hauteur d'un fluide incompressible et de viscosité nulle dans une cuve percée d'un trou à sa base. Comme la hauteur commence à diminuer dès l'instant initial, la solution "physique" est $-z^0$.

Soit λ la mesure de Lebesgue. On peut montrer que sélectionner la solution avec $T(a) = 0$ est équivalent à supposer que la mesure image de λ par le flot $a \mapsto (1-\alpha)|t - T(a) - a|^{\frac{1}{1-\alpha}}$ est absolument continue par rapport à λ pour tout temps. Cette observation a conduit à la recherche de conditions de régularité sur le champ de vecteurs f , pour obtenir un flot qui conserve la mesure de Lebesgue. Cette théorie sort du cadre de cette thèse. Le lecteur intéressé peut se rapporter à l'article fondateur de R. J. DiPerna et P.L. Lions [DL89] ou à une introduction plus abordable de L. Ambrosio et D. Trevisan [AT15].

Plus récemment, J. E. Cardona et L. Kapitanski ont montré l'existence d'un flot mesurable dans un cas de non-unicité de la solution où le champ de vecteurs f est seulement continu [CK17b, CK17a].

¹théorème 4.1.1

²théorème 4.1.2

Theorem 4.1.3 ([CK17b]). *Si f est continue alors il existe un flot $(\phi_{t,s})_{0 \leq s \leq t \leq T}$ telle que pour tout $0 \leq s \leq t \leq T$, $a \mapsto \phi_{t,s}(a)$ est mesurable.*

Un des résultats principaux de ce chapitre est une extension de ce résultat dans le cadre des équations différentielles rugueuses (Corollaires 5.6.5 et 5.6.8)

4.2 Flots des équations différentielles stochastiques

Comparons maintenant les résultats sur les flots d'EDO de la section précédente à ceux sur les flots associés aux équations différentielles stochastiques (EDS) dirigées par un mouvement brownien.

Soit $(B_t := (B_t^1, \dots, B_t^d))_{t \in [0, T]}$ un mouvement brownien standard d -dimensionnel sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. On note $\mathcal{F}_{r,t}$ la tribu engendrée par l'incrément du brownien entre r et t , c'est-à-dire par $(B_{s,u} := B_u - B_s)_{r \leq s \leq u \leq t}$. Commençons par rappeler la définition de la solution de l'EDS

$$Y_t = a + \int_r^t F(Y_s) dB_s \tag{4.3}$$

$$= a + \sum_{i=1}^d \int_r^t F_i(Y_s) dB_s^i, \tag{4.4}$$

où $F = (F_1, \dots, F_d) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ est un champ de vecteurs. L'intégrale contre B est à comprendre au sens d'Itô.

Définition 4.2.1 (Solution EDS). Pour $r \in [0, T]$, $a \in \mathbb{R}^n$, un processus stochastique continu $(Y_t)_{t \in [r, T]}$ à valeurs dans \mathbb{R}^n est solution de (4.3) avec la condition initiale $Y_r = a$ si

- le processus Y est $\mathcal{F}_{r,t}$ adapté pour tout $t \geq r$,
- le processus Y satisfait (4.3).

La régularité requise pour obtenir l'existence et l'unicité d'une solution à (4.3) est la même que dans le cas des EDO.

Theorem 4.2.1 ([Kun06][Theorem 1.2]). *Si F est lipschitzienne alors il existe une unique solution à (4.3) pour une condition initiale donnée. On la note $(Y_{\cdot, r}(a))_{T \geq t \geq r}$.*

L'unique solution $Y_{\cdot, r}(a)$ définie au théorème précédent a des propriétés analogues à celles des EDO (cf. section 4.1). Définissons le flot associé à (4.3) comme la famille d'applications $(\Phi_{t,r} : a \mapsto Y_{t,r}(a))_{0 \leq r \leq t \leq T}$. Nous allons voir que pour tout $0 \leq r \leq t \leq T$, l'application $a \mapsto \Phi_{t,r}(a)$ hérite de la régularité du champ de vecteurs F avec une petite perte.

4.2. FLOTS DES ÉQUATIONS DIFFÉRENTIELLES STOCHASTIQUES

Theorem 4.2.2 ([Kun06]). *Soient un entier $k \geq 1$ et $\gamma \in (0, 1]$. Si pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^{k+\gamma}(\mathbb{R}^n, \mathbb{R}^n)$ alors il existe une modification $\Phi_{t,r}(a)$ notée à nouveau $\Phi_{t,r}(a)$ telle que*

1. Pour tout $0 \leq r \leq t \leq T$ et $a \in \mathbb{R}^n$, $\Phi_{t,r}(a)$ est $\mathcal{F}_{r,t}$ -mesurable.
2. Pour presque tout $\omega \in \Omega$, $(r, t, a) \mapsto \Phi_{t,r}(a, \omega)$ est continu et $\lim_{t \downarrow r} \Phi_{t,r}(a) = a$.
3. Pour presque tout $\omega \in \Omega$, et $0 \leq r \leq s \leq t \leq T$,

$$\Phi_{t,s}(\Phi_{s,r}(a, \omega), \omega) = \Phi_{t,r}(a, \omega).$$

4. Pour presque tout $\omega \in \Omega$ et $\delta < \gamma$, l'application $\Phi_{t,r}(\cdot, \omega) : \mathbb{R}^n \mapsto \mathbb{R}^n$ est un $C^{k+\delta}$ -difféomorphisme.

Remarque 4.2.3. Contrairement au cas des EDO, on observe ici une légère perte de régularité entre le champ de vecteurs et le flot. Cela est dû à l'utilisation du théorème d'extension continue de Kolmogorov. On ne sait pas si ce résultat est optimal mais nous n'avons pas trouvé de résultat plus précis dans la littérature.

Ici nous avons interprété (4.3) au sens d'Itô. Intéressons-nous maintenant au cas où l'intégrale contre B est à comprendre au sens de Stratonovich :

$$Z_t = a + \int_r^t F(Z_s) \circ dB_s. \quad (4.5)$$

On rappelle le lien entre l'intégrale d'Itô et celle de Stratonovich. Si $(X_t)_{t \in [r, T]}$ est solution de l'EDS (4.3) au sens d'Itô et que $F \in C_b^1$ alors pour tout $0 \leq r \leq t \leq T$,

$$\int_r^t F(X_s) \circ dB_s = \frac{1}{2} \int_r^t F(X_s) F'(X_s) ds + \int_r^t F(X_s) dB_s.$$

Donc si X est solution de l'EDS au sens d'Itô suivante

$$X_t = a + \frac{1}{2} \int_r^t F(X_s) F'(X_s) ds + \int_r^t F(X_s) dB_s,$$

alors X est solution de l'EDS (4.5) au sens de Stratonovich.

On déduit du théorème 4.2.2 le résultat suivant sur la régularité du flot associé à une EDS du type (4.5). Dans le cas où il existe une unique solution à (4.5), on note $\Psi_{t,r} : a \mapsto Z_t$ et on dit que Ψ est le flot associé à (4.5).

Theorem 4.2.4. *Soient un entier $k \geq 1$ et $\gamma \in (0, 1]$. Si pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^{k+1+\gamma}(\mathbb{R}^n, \mathbb{R}^n)$ alors il existe une modification du flot associé à (4.5) que l'on note à nouveau Ψ qui vérifie les points 1, 2, 3, 4 du théorème 4.2.2.*

Remarque 4.2.5. Pour obtenir un flot de régularité k , on a besoin dans le cas d'une EDS au sens de Stratonovich d'un champ de vecteurs ayant une régularité $k + 1 + \gamma$ et seulement d'ordre $k + \gamma$ dans le cas Itô.

Les EDS au sens de Stratonovich sont fortement reliées aux EDO par l'intermédiaire du théorème de Wong-Zakai. L'idée de E. Wong et M. Zakai développée dans [WZ65] est d'approximer le mouvement brownien $(B_t)_{t \in [0, T]}$ par un processus dérivable $(B_t(\epsilon))_{t \in [0, T]}$. On peut alors résoudre, sans utiliser la construction d'une intégrale stochastique, l'EDO

$$Z_t(\epsilon) = a + \int_r^t F(Z_s(\epsilon)) \dot{B}_s(\epsilon) ds, \quad (4.6)$$

où $\dot{B}_s(\epsilon)$ est la dérivée temporelle de $B_s(\epsilon)$. Il est alors montré, en dimension $d = 1$ dans [WZ65], puis en dimension $d \geq 1$ quelconque dans [SV72], que $Z(\epsilon)$ converge vers la solution au sens de Stratonovich Z . Cependant, comme énoncé dans la remarque 4.2.7 ci-dessous, on ne peut pas choisir n'importe quelle approximation $B(\epsilon)$.

Énonçons précisément ce résultat.

Théorème 4.2.6 (Wong-Zakai [IW14]). *Soient $\epsilon > 0$ et $(B_t(\epsilon))_{t \in [0, T]}$ une approximation du brownien $(B_t)_{t \in [0, T]}$ linéaire par morceaux, i.e. pour tout $k \in \{0, \dots, \lfloor \epsilon^{-1} \rfloor - 1\}$ et $t \in [Tk\epsilon, T(k+1)\epsilon)$, $B_t(\epsilon) = B_{Tk\epsilon} + (t - Tk\epsilon)B_{T(k+1)\epsilon}$. On suppose que pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^2$. Alors quand $\epsilon \rightarrow 0$*

- $\mathbb{E}[\sup_{t \in [0, T]} \|B_t(\epsilon) - B_t\|^2] \rightarrow 0$,
- $\mathbb{E}[\sup_{t \in [0, T]} \|Z_t(\epsilon) - Z_t\|^2] \rightarrow 0$ où $Z(\epsilon)$ est solution de (4.6) et Z est solution de l'EDS au sens de Stratonovich (4.5)³.

Remarque 4.2.7. Ici on a choisi une approximation particulière du mouvement brownien B . Le résultat est indépendant de la régularisation si le crochet de Lie $[F_i, F_j] := DF_i F_j - DF_j F_i$ est nul. Dans le cas contraire, suite aux travaux de E. J. McShane [McS72], N. Ikeda et S. Watanabe ont montré dans [IW14] qu'il faut étudier la convergence de l'aire de Levy de $B(\epsilon)$ renormalisée de la façon suivante

$$\frac{\epsilon^{-1}}{2} \int_0^\epsilon [B_{s,u}^i(\epsilon) \dot{B}_u^j(\epsilon) - \dot{B}_u^j(\epsilon) B_{s,u}^i(\epsilon)] du, \quad (i, j) \in \{1, \dots, d\}^2.$$

Pour de nombreuses approximations "naturelles" (linéaire par morceaux, régularisation par convolution, ...), cette quantité tend en moyenne vers 0 et le théorème 4.2.6 reste valide.

Dans le cas contraire, il est nécessaire d'ajouter un terme correctif à l'EDS (4.5) pour obtenir la même convergence qu'au théorème 4.2.6.

³Elles existent d'après le théorème 4.2.4 et les hypothèses sur la régularité de F .

4.3. FLOTS DES ÉQUATIONS DIFFÉRENTIELLES RUGUEUSES

On retrouve l'importance de l'aire de Lévy dans la théorie des trajectoires rugueuses. On trouvera dans [Bau04] une étude fine des propriétés des flots stochastiques basée sur la formule Chen-Strichartz qui permet de représenter un flot stochastique en fonction des termes de la signature du mouvement brownien.

Dans la section suivante nous introduisons des résultats similaires sur la régularité des flots pour les équations différentielles rugueuses.

4.3 Flots des équations différentielles rugueuses

La théorie des trajectoires rugueuses permet de traiter des équations différentielles dirigées par un signal déterministe irrégulier $x : [0, T] \rightarrow \mathbb{R}^d$, comme par exemple la trajectoire d'un processus stochastique. Nous avons introduit cette théorie dans le Chapitre 3.

On appelle une famille $\omega := (\omega_{s,t})_{0 \leq s \leq t \leq T}$ un contrôle si pour tout $0 \leq r \leq s \leq t \leq T$,

- $\omega_{s,t} \in \mathbb{R}_+$,
- $\omega_{r,s} + \omega_{s,t} \leq \omega_{r,t}$ (super-additivité),
- $\lim_{t \rightarrow s} \omega_{s,t} = \omega_{s,s} = 0$ (continuité sur la diagonale).

Pour $p \in [2, 3)$, on note \mathcal{T}^p l'ensemble des p -trajectoires rugueuses contrôlées par ω . La définition de cet ensemble est précisée dans la section 5.6.4. Lorsque $\omega_{s,t} = |t - s|$, \mathcal{T}^p est l'ensemble des trajectoires rugueuses Hölder continues qui a été introduit au chapitre précédent.

Le théorème suivant, dû à L. Coutin et A. Lejay, donne la régularité du flot en fonction de celle du champ de vecteurs.

Théorème 4.3.1 ([CL18, Theorem 3]). *Soient deux réels $p \in [2, 3)$, $\gamma \in (0, 1]$ et une p -trajectoire rugueuse $\mathbf{x} := (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \mathcal{T}^p(\mathbb{R}^d)$. On suppose que pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^{k+1+\gamma}$ avec $k + 1 + \gamma > p$ alors pour tout $(a, r) \in \mathbb{R}^n \times [0, T]$ il existe un unique chemin contrôlé par $\mathbf{x}^{(1)}$ noté $y_{\cdot,r}(a)$ qui est solution de l'EDR*

$$y_{t,r}(a) = a + \int_r^t F(y_{s,r}(a)) d\mathbf{x}_s, \forall t \in [0, T]. \quad (4.7)$$

La famille $(\psi_{t,s} : a \mapsto y_{\cdot,r}(a))_{0 \leq s \leq t \leq T}$ définit alors un flot. De plus pour tout $0 \leq s \leq t \leq T$, $\kappa \in (0, 1)$ avec $2 + \kappa\gamma > p$, l'application $\psi_{t,s}(a) : a \mapsto \psi_{t,s}(a)$ est un $C^{k+(1-\kappa)\gamma}$ -difféomorphisme de \mathbb{R}^n dans \mathbb{R}^n .

Remarque 4.3.2. Ce résultat est en fait un cas particulier de [CL18, théorème 3].

Remarque 4.3.3. Comme dans le cas stochastique (théorème 4.2.4), il y a une petite perte de régularité. Elle s'explique ici par le fait que l'on travaille en norme Hölder et non en norme uniforme. En effet, la boule unité d'un espace α -Hölder n'est pas compacte pour sa norme associée. Cependant, elle le devient pour une norme β -Hölder avec $\beta < \alpha$.

Dans le théorème 4.3.1, la régularité requise sur le champ de vecteurs F pour obtenir l'existence et l'unicité de la solution est presque optimale. A. M. Davie a montré dans [Dav07] un résultat optimal pour obtenir l'unicité de la solution. Malheureusement, la régularité du flot n'est pas étudiée pour le cas limite.

Theorème 4.3.4 ([Dav07][Theorem 3.6]). *Soient deux réels $p \in [2, 3)$, $\gamma \in (0, 1]$ et une p -trajectoire rugueuse $\mathbf{x} := (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \mathcal{T}^p(\mathbb{R}^d)$. On suppose que pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^{2+\gamma}$ avec $2 + \gamma = p$ alors pour tout $(a, r) \in \mathbb{R}^n \times [0, T]$ il existe une unique solution à l'EDR (4.7). Si $2 + \gamma < p$, alors il existe des contre-exemples à l'unicité.*

Appliquons ces résultats au cas du mouvement brownien. On note $\mathbb{B}_{s,t}^{\text{strato}} := \int_s^t B_{s,u} \otimes \circ dB_u$ pour tout $s \leq t$. Le processus $(B_t)_{t \in \mathbb{R}_+}$ ayant une régularité Hölder continue strictement inférieure à $1/2$ presque sûrement, pour tout $\epsilon > 0$ le couple $(B, \mathbb{B}^{\text{strato}})$ est une $(2 + \epsilon)$ -trajectoire rugueuse presque sûrement.

Le théorème 4.3.4 ne nous permet pas de retrouver le résultat sur les EDS au sens de Stratonovich. Ici l'unicité requiert une régularité γ -Hölder ($\gamma > 0$) de la dérivée seconde de F tandis que la continuité suffit pour les EDS.

Comme annoncé précédemment, un des résultats principaux du premier article [BL17a, Corollaire 5.6.8] de ce chapitre montre l'existence d'un flot mesurable dans le cas de non unicité de la solution rugueuse. Pour plus de clarté, nous récrivons ici son énoncé sous la forme d'un théorème.

Theorème 4.3.5. *Soient deux réels $p \in [2, 3)$, $\gamma \in (0, 1]$ et une p -trajectoire rugueuse $\mathbf{x} := (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \in \mathcal{T}^p(\mathbb{R}^d)$. Si l'on suppose que pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^2$ alors il existe un flot $(\phi_{t,s})_{0 \leq s \leq t \leq T}$ associé à (4.7) telle pour tout $0 \leq s \leq t \leq T$, $a \mapsto \phi_{t,s}(a)$ est mesurable.*

4.4 Comparatif EDO, EDS, EDR

Nous récapitulons dans le tableau 4.1 la régularité du flot en fonction de celle du champ de vecteurs. La régularité optimale requise pour obtenir l'existence de solution est donnée dans le tableau 4.2.

4.5 Approximations de flots

Après avoir étudié les propriétés de régularité des flots, intéressons-nous à leurs constructions. Les théorèmes de point fixe sont des outils puissants pour démontrer l'existence ou l'unicité de solution à toutes sortes d'équations différentielles (ordinaires, stochastiques, rugueuses, aux dérivées partielles). Elles présentent cependant l'inconvénient de ne pas donner un moyen de construction de la solution.

4.6. SCHÉMA D'EULER

Régularité	EDO	EDR (cas brownien)
Champ de vecteurs	C_b^0	C_b^2
Flot	mesurable	mesurable

Régularité	EDO	EDS (Itô)	EDS (Strato)	EDR (cas brownien)
Champ de vecteurs	$C_b^{k+\gamma}$	$C_b^{k+\gamma}$	$C_b^{k+1+\gamma}$	$C_b^{k+1+\gamma}$
Flot (difféomorphisme)	$C^{k+\gamma}$	$C^{k+\delta}$	$C^{k+\delta}$	$C^{k+(1-\kappa)\gamma}$

Table 4.1: Régularité du flot en fonction de celle du champ de vecteurs avec $k \geq 1$, $0 < \delta < \gamma \leq 1$, $\kappa \in (0, 1)$, $\kappa\gamma > \epsilon$.

Régularité	EDO	EDS (Itô)	EDS (Strato)	EDR (cas brownien)
Champ de vecteurs	Lipschitz	Lipschitz	C_b^2	$C^{2+\epsilon}$

Table 4.2: Existence et unicité de la solution

Pour cela, nous devons plutôt nous tourner du côté des schémas numériques. Ils ont été employés très tôt pour des études théoriques sur des EDO. Rappelons par exemple que les premières méthodes de Runge-Kutta datent de 1901, bien avant l'avènement de l'informatique. Dans son article originel [Mar55], G. Maruyama introduit une généralisation du schéma d'Euler pour les EDS, qui permet une construction directe de diffusions, sans passer par la définition préalable d'une intégrale stochastique comme celle d'Itô.

Dans cette section, nous présentons les schémas numériques les plus classiques pour les EDO et les EDS. Puis nous introduisons différentes approches développées pour les EDR dans [Dav07, FV10, Bai15a].

Introduisons d'abord quelques notations. Étant donné l'approximation d'un flot⁴

$$(\phi_{t,s}(a))_{0 \leq s \leq t \leq T, a \in \mathbb{R}^n}$$

et une subdivision π de $[0, T]$, on note pour tout $0 \leq s \leq t \leq T$, et $a \in \mathbb{R}^n$,

$$\phi_{t,s}^\pi := \phi_{t,t_j} \circ \cdots \circ \phi_{t_i,s}(a),$$

où $[t_i, t_j]$ est le plus grand intervalle de points de π contenu dans $[s, t]$. On note $|\pi|$ le pas de la subdivision de π .

4.6 Schéma d'Euler

Théorème 4.6.1 (Schéma d'Euler). *Soit $f \in C_b^1$, on définit pour tout $0 \leq s \leq t \leq T$, et $a \in \mathbb{R}^n$, l'approximation de flot*

$$\phi_{t,s}(a) := a + f(a)(t - s). \quad (4.8)$$

⁴Une approximation de flot est ce que nous appellerons ensuite un presque flot (Définition 2).

Alors ϕ^π converge vers un flot lipschitzien ψ uniformément en s, t et en a quand $|\pi|$ tend vers 0. De plus pour tout, $\psi_{\cdot, r}(a)$ est solution de (4.1).

Remarque 4.6.2. La forme de ϕ défini en (4.8) correspond à une approximation en temps petit de (4.1). En effet,

$$\begin{aligned} y_{t,r}(a) &= a + f(a)(t-s) + \int_r^t [f(y_{s,r}(a)) - f(a)] ds \\ &\underset{t \rightarrow s}{=} a + f(a)(t-s) + O((t-s)^2), \end{aligned}$$

puisque $\int_r^t [f(y_{s,r}(a)) - f(a)] ds \leq \|f'\|_\infty \|f\|_\infty |t-s|^2$.

Remarque 4.6.3. Si la régularité de f est plus importante, par exemple de classe C_b^k avec $k \geq 1$. On peut poursuivre le développement de la remarque 4.6.2 à l'ordre k et poser

$$\phi_{t,s}(a) := a + \sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!} (t-s)^i. \quad (4.9)$$

On obtient alors une vitesse de convergence en $O(|\pi|^k)$ où $|\pi|$ est le pas de subdivision de la partition π . La méthode de Runge-Kutta est basée sur le développement (4.9) mais évite soigneusement le calcul des dérivées successives de f .

4.7 Schéma d'Euler-Maruyama

Theorème 4.7.1 (Euler-Maruyama [Mar55]). *On suppose que pour tout $i \in \{1, \dots, d\}$, $F_i \in C_b^1$, on définit pour tout $0 \leq s \leq t \leq T$, et $a \in \mathbb{R}^n$, l'approximation de flot*

$$\phi_{t,s}(a) := a + \sum_{i=1}^d F_i(a) w_{s,t}^i, \quad (4.10)$$

où les $w_{s,t}^i$ sont des gaussiennes indépendantes centrées et de variance $t-s$. Alors,

$$\lim_{|\pi| \rightarrow 0} \mathbb{E}[|\Phi_{t,s}(a) - \phi_{t,s}^\pi(a)|] = 0, \quad (4.11)$$

où Φ est le flot du théorème 4.2.2.

Remarque 4.7.2. Contrairement au cas déterministe, il existe dans le cadre stochastique de nombreux types de convergence. L'équation (4.11) correspond à ce que l'on appelle communément la convergence forte des marginales du processus. Une large littérature que l'on peut retrouver dans [Pla99], a été établie pour étudier différentes formes de convergence.

Remarque 4.7.3. On peut obtenir un meilleur taux de convergence en poussant le développement ϕ à des ordres supérieurs. Cependant, la manipulation est plus délicate que dans le cas déterministe car il faut prendre en compte les intégrales itérées du processus contre lui-même et calculer leurs distributions.

4.8 Les objectifs de cette partie

Dans cette partie, on établit un cadre qui permet l'étude de schémas numériques pour les EDR. Dans [Dav07, FV10, Bai15a] différentes approximations ϕ du flot sont étudiées.

Étant donné une p -trajectoire rugueuse $\mathbf{x} := (\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ avec $p \in [2, 3)$, A. M. Davie a montré dans [Dav07] que l'on pouvait construire des solutions à (4.7) à partir de l'approximation

$$\phi_{t,s}(a) = a + \sum_{i=1}^d F_i(a) \mathbf{x}_{s,t}^i + \sum_{i,j=1}^d DF_i(a) F_j(a) \mathbf{x}_{s,t}^{i,j}.$$

Ce travail a été généralisé par P. Friz et N. Victoir dans [FV10] pour l'ensemble des trajectoires rugueuses géométriques avec une régularité arbitraire.

Dans [Bai15a], I. Bailleul s'intéresse à une autre approximation $\phi_{t,s}(a) := y_{t,r}(1, a)$ où $\alpha \in [0, 1] \mapsto y_{t,s}(\alpha, a)$ est solution de l'EDO⁵

$$y_{t,s}(\alpha, a) = a + \int_0^\alpha \sum_{i=1}^d F_i(y_{t,s}(\beta, a)) \mathbf{x}_{s,t}^{(1)} d\beta + \frac{1}{2} \int_0^\alpha \sum_{i,j=1}^d [F_i, F_j](y_{t,s}(\beta, a)) \mathbf{x}_{s,t}^{i,j} d\beta,$$

où $[F_i, F_j] = D(F_i)F_j - D(F_j)F_i$ est le crochet de Lie de deux champs de vecteurs. Le fait de se ramener au calcul de la solution d'une EDO présente l'avantage de pouvoir utiliser l'ensemble de la littérature sur les schémas numériques des EDO pour construire ϕ .

Dans le chapitre 5, nous avons adapté les travaux sur le lemme de la couturière linéaire introduit par D. Feyel, A. de La Pradelle et G. Mokobodzki dans [FdLPM08] puis généralisé par L. Coutin et A. Lejay dans [CL14], à un cadre non linéaire. Nous donnons des conditions générales sur la forme de l'approximation ϕ , appelé presque flot, pour obtenir une borne uniforme en π sur la norme de ϕ^π . Cela nous permet, en utilisant des résultats obtenus récemment par J. E. Cardona et L. Kapitanski [CK17b, CK17b] de montrer l'existence d'un flot mesurable. Nous appliquons ce résultat dans les cas où le champ de vecteurs d'une EDR n'a pas la régularité requise pour assurer l'unicité des solutions. On prouve ensuite que si la norme Lipschitz de ϕ^π est uniformément bornée en π , alors il existe un unique flot Lipschitz associé à ψ et qu'il correspond à la limite de ϕ^π quand le pas de π tend vers 0. On prouve par ailleurs une formule de perturbation d'un presque flot ϕ par un terme additif. Enfin, on retrouve avec ce formalisme les lemmes de la couturière additif [FdLP⁺06] et multiplicatif [FdLPM08, CL14].

Dans le chapitre 6, on prouve l'existence d'un presque flot ϕ tel que la norme Lipschitz de ϕ^π soit uniformément bornée en π pour appliquer les résultats du chapitre 5 sur l'existence d'un flot Lipschitz. Cela nous mène à définir des presque flots dits *stables* qui vérifient cette propriété à condition d'être plus réguliers que les presque flots du

⁵Cette approximation est généralisée dans [Bai15a] à l'ensemble des trajectoires rugueuses géométriques.

CHAPTER 4. INTRODUCTION AUX FLOTS

chapitre 5. On prouve ensuite des formules de perturbation pour les presque flots stables et l'on montre que les schémas de P. Friz, N Victoir [FV10] et de I. Bailleul [Bai15a] sont des perturbations du presque flot de A. M. Davie.

Chapter 5

Non linear sewing lemma I: weak form

This chapter consists of the preprint [BL17a] written in collaboration with Antoine Lejay and submitted for publication.

Abstract

We introduce a new framework to deal with rough differential equations based on flows and their approximations. Our main result is to prove that measurable flows exist under weak conditions, even solutions to the corresponding rough differential equations are not unique. We show that under additional conditions of the approximation, there exists a unique Lipschitz flow. Then, a perturbation formula is given. Finally, we link our approach to the additive, multiplicative sewing lemmas and the rough Euler scheme.

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5.1 Introduction

5.1.1 Motivations

The theory of rough paths allows one to define the solution to a differential equation of type

$$y_t = a + \int_0^t f(y_s) dx_s, \quad (5.1)$$

for a path x which is irregular, say α -Hölder continuous. Such an equation is then called a *Rough Differential Equation* (RDE) [FV10, LQ02, FH14]. The key point of this theory is to show that such a solution can be defined provided that x is extended to a path \mathbf{x} , called a *rough path*, living in a larger space that depends on the integer part of $1/\alpha$. When $\alpha > 1/2$, no such extension is needed. This case is referred as the *Young case*, as the integrals are constructed in the sense given by L.C. Young [You36, Lyo94]. Provided that one considers a rough path, integrals and differential equations are natural extensions of ordinary ones.

The first proof of existence of a solution to (5.1) from T. Lyons relied on a fixed point [LCL07, LQ02, Lyo98]. It was quickly shown that RDE shares the same properties as ordinary differential equations, including the flow property. In [Dav07], A.M. Davie gives an alternative proof that used an Euler type approximation and gave counter-example to uniqueness. More recently, I. Bailleul gave a direct construction through the flow property [Bai15a, Bai15b, BD15].

A *flow* in a metric space V is a family $\{\psi_{t,s}\}_{0 \leq s \leq t \leq T}$ of maps from V to V such that $\psi_{t,s} \circ \psi_{s,r} = \psi_{t,r}$ for any $0 \leq r \leq s \leq t \leq T$. When $y_t(s, a)$ is a family of solutions to differential equations with $y_s(s, a) = a$, the element $\psi_{t,s}(a)$ can be seen as a map which carries a to $y_t(s, a)$. A flow requires some compatibilities on the family of solutions so that solutions may exist although no flow may hold. Flow are related to dynamical systems. There are a slightly different objects than solutions. One of their interest lies in their characterization as lipeomorphisms (Lipschitz functions with a Lipschitz inverse), diffeomorphisms...

In this article, we develop a generic framework to construct flows from approximations. We do not focus on a particular form of the solutions, so that our construction is a *non-linear sewing lemma*, modelled after the additive and multiplicative sewing lemmas [FdLPM08].

In this first part, we study flows under weak conditions and prove existence of a measurable flow even when the solutions of RDE are not necessarily unique. This is based on a selection theorem [CK17a] due to J.E. Cardona and L. Kapitanski. Such a result is new in the literature where existence of flows were only proved under stronger regularity conditions (the many approaches are summarized in [CL18]). Besides, our approach also

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contains the additive and multiplicative sewing lemmas [FdLPM08, Cou12]. The rough equivalent of the Duhamel formula for solving linear RDE [CL14] with a perturbative terms follows directly from our construction.

In a second part [BL17b], we provide conditions for uniqueness and continuity. Besides, we show that our construction encompass many of the previous approaches or results: A.M. Davie [Dav07], I. Bailleul [Bai15a, Bai14b, Bai14a], P. Friz & N. Victoir [FV10].

Our starting point in the world of classical analysis is the *product formula* which relates how the iterated product of an approximation of a flow converge to the flow [AMR88, CMHM78]. It is important both from the theoretical and numerical point of view.

On a Banach space V , let us consider a family $(\epsilon, a) \in \mathbb{R}_+ \times V \mapsto \Phi(\epsilon, a)$, called an *algorithm* of class \mathcal{C}^1 in (ϵ, a) such that $\Phi(0, a) = a$. The parameter ϵ is related to the quality of the approximation.

For a vector field f , let also ψ be a flow on V , by which we mean that map which associate to each a in V the solution $\psi_t(a) := y_t$ solution to the ODE $\dot{y}_t = f(y_t)$ for $t \geq 0$ with $y_0 = a$.

The algorithm Φ is *consistent* with the vector field when

$$f(a) = \frac{\partial \Phi}{\partial \epsilon}(0, a), \quad \forall a \in V. \quad (5.2)$$

Then

$$\underbrace{\Phi(t/n, \Phi(t/n, \dots \Phi(t/n, a)))}_{n \text{ times}} \text{ converges to } \phi_t(a) \text{ as } n \rightarrow \infty. \quad (5.3)$$

The Euler scheme for solving ODE is the prototypical example of such behavior. Set $\Phi(\epsilon, a) = a + f(a)\epsilon$ so that (5.2) holds. In this case, (5.3) expresses nothing more than the convergence of the Euler scheme.

For example, one recovers easily the proof of Lie's theorem on matrices. For this, if A is a matrix, then $\exp(tA)$ is the solution to $\dot{Y}_t = AY_t$ with $Y_0 = \text{Id}$. The exponential $\exp(tA)$ is easily recovered as

$$\exp(tA) = \lim_{n \rightarrow \infty} \left(\text{Id} + \frac{t}{n}A \right)^n.$$

For two matrices A and B , $\exp(t(A+B))$ is given by

$$\exp(t(A+B)) = \lim_{n \rightarrow \infty} \left(\exp\left(\frac{tA}{n}\right) \exp\left(\frac{tB}{n}\right) \right)^n.$$

For this, we have only to consider $\Phi(\epsilon, a) = \exp(\epsilon A)\exp(\epsilon B)a$ and to verify that $\partial_\epsilon \Phi(0, a) = (A+B)a$ for any matrix a .

For unbounded operators, it is also related to Chernoff and Trotter's results on the approximation of semi-groups [EN00, ZZCW16]. The product formula also justifies the construction of some splitting schemes [BCOR09].

In this article, we consider as an algorithm a family $\{\phi_{t,s}\}_{0 \leq s \leq t \leq T}$ of functions from V to V which is close to the identity map in short time and such that $\phi_{t,s} \circ \phi_{s,r}$ is close to $\phi_{t,r}$ for any time $s \leq r \leq t$. For a path x of finite p -variation, $1 \leq p < 2$ with values in \mathbb{R}^d and a smooth enough function $f : \mathbb{R}^m \rightarrow L(\mathbb{R}^d, \mathbb{R}^m)$, such an example is given by $\phi_{t,s}(a) = a + f(a)x_{s,t}$.

We then study the behavior of the composition ϕ^π of the ϕ_{t_{i+1}, t_i} along a partition $\pi = \{t_i\}_{i=0, \dots, n}$. Clearly, as the mesh of the partition π goes to 0, the limit, when it exists is a candidate to be a flow. In the example given above, it will be the flow associated to the family of Young differential equations $y_t(a) = a + \int_0^t f(y_s(a)) dx_s$.

After having introduced some necessary notations (Section 5.1.2) including the central notion of galaxies (Section 5.1.2) which partition the space of families $\{\phi_{t,s}\}_{0 \leq s \leq t \leq T}$ of functions from a Banach space V to itself according to their proximity, we present the main results of our articles in 5.1.3

5.1.2 Notations, definitions and concepts

The following notations and hypotheses will be constantly used throughout all this article.

Hölder and Lipschitz continuous functions

Let V and W be two metric spaces.

The space of continuous functions from V to W is denoted by $\mathcal{C}(V, W)$.

Let d (resp. d') be a distance on V (resp. W). For $\gamma \in (0, 1]$, we say that a function $f : V \rightarrow W$ is γ -Hölder if

$$\|f\|_\gamma := \sup_{\substack{a, b \in V, \\ a \neq b}} \frac{d'(f(a), f(b))}{d(a, b)^\gamma},$$

whenever this quantity is finite. If $\gamma = 1$ we say that f is *Lipschitz*. We then set $\|f\|_{\text{Lip}} := \|f\|_1$.

For any integer $r \geq 0$ and $\gamma \in (0, 1]$, we denote by $\mathcal{C}^{r+\gamma}(V, W)$ the subspace of functions from V to W whose derivatives $d^k f$ of order $k \leq r$ are continuous and such that $d^r f$ is γ -Hölder.

We denote by $\mathcal{C}_b^{r+\gamma}(V, W)$ the subset of $\mathcal{C}^{r+\gamma}(V, W)$ of bounded functions with bounded derivatives up to order r .

5.1. INTRODUCTION

Controls and remainders

From now, V is a topological, complete metric space with a distance d . A distinguished point of V is denoted by 0 .

We fix $\gamma \in (0, 1]$. Let $N_\gamma : V \rightarrow [1, +\infty)$ be a γ -Hölder continuous function with constant $\|N\|_\gamma$. The indice γ in N_γ is refers to its regularity. If N_γ is Lipschitz continuous ($\gamma = 1$), then we simply write N .

Let us fix a *time horizon* T . We write $\mathbb{T} := [0, T]$ as well as

$$\mathbb{T}_2^+ := \{(s, t) \in \mathbb{T}^2 \mid s \leq t\} \text{ and } \mathbb{T}_3^+ := \{(r, s, t) \in \mathbb{T}^3 \mid r \leq s \leq t\}. \quad (5.4)$$

A *control* $\omega : \mathbb{T}_2^+ \rightarrow \mathbb{R}_+$ is a *super-additive* family, *i.e.*,

$$\omega_{r,s} + \omega_{s,t} \leq \omega_{r,t}, \quad \forall (r, s, t) \in \mathbb{T}_3^+$$

which is continuous close to its diagonal with $\omega_{s,s} = 0$ for all $s \in \mathbb{T}$. A typical example of a control is $\omega_{s,t} = C|t - s|$ where C is a non-negative constant.

For $p \geq 1$, we say that a path $x \in \mathcal{C}(\mathbb{T}, V)$ is a *path of finite p -variation controlled by ω* if

$$\|x\|_p := \sup_{\substack{(s,t) \in \mathbb{T}_2^+ \\ s \neq t}} \frac{d(x_s, x_t)}{\omega_{s,t}^{1/p}} < +\infty.$$

A *remainder* is a function $\varpi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous, increasing function such that for some $0 < \varkappa < 1$,

$$2\varpi\left(\frac{\delta}{2}\right) \leq \varkappa\varpi(\delta), \quad \forall \delta > 0. \quad (5.5)$$

A typical example for a remainder is $\varpi(\delta) = \delta^\theta$ for any $\theta > 1$.

We consider that $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing function with $\lim_{T \rightarrow 0} \delta_T = 0$.

Finally, let $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous, increasing function such that for all $(s, t) \in \mathbb{T}_2^+$,

$$\eta(\omega_{s,t})\varpi(\omega_{s,t})^\gamma \leq \delta_T\varpi(\omega_{s,t}). \quad (5.6)$$

Partitions of \mathbb{T} are customary denoted by $\pi = \{t_i\}_{i=0, \dots, n}$. The *mesh* $|\pi|$ of a partition π is $|\pi| := \max_{i=0, \dots, n} (t_{i+1} - t_i)$. The discrete simplices π_2^+ and π_3^+ are defined similarly to \mathbb{T}_2^+ and \mathbb{T}_3^+ in (5.4).

Galaxies

Notation 1. We denote by $\mathcal{F}(V)$ the set of functions $\{\phi_{t,s}\}_{(s,t) \in \mathbb{T}_2^+}$ from V to V which are continuous in (s, t) , *i.e.* for any $a \in V$, the map $(s, t) \in \mathbb{T}_2^+ \mapsto \phi_{t,s}(a)$ is continuous.

CHAPTER 5. NON LINEAR SEWING LEMMA I: WEAK FORM

Notation 2 (Iterated products). For any $\phi \in \mathcal{F}(V)$, any partition π of \mathbb{T} and any $(s, t) \in \mathbb{T}_2^+$, we write

$$\phi_{t,s}^\pi := \phi_{t,t_j} \circ \phi_{t_j,t_{j-1}} \circ \cdots \circ \phi_{t_{i+1},t_i} \circ \phi_{t_i,s}, \quad (5.7)$$

where $[t_i, t_j]$ is the biggest interval of such kind contained in $[s, t] \subset \mathbb{T}$. If no such interval exists, then $\phi_{t,s}^\pi = \phi_{t,s}$.

Clearly, for any partition, $\phi^\pi \in \mathcal{F}(V)$. A trivial but important remark is that from the very construction,

$$\phi_{t,r}^\pi = \phi_{t,s}^\pi \circ \phi_{s,r}^\pi \text{ for any } s \in \pi, r \leq s \leq t. \quad (5.8)$$

In particular, $\{\phi_{t,s}^\pi\}_{(s,t) \in \pi_+^2}$ enjoys a (semi-)flow property when the times are restricted to the elements of π .

The article is mainly devoted to study the possible limits of ϕ^π as the mesh of π decreases to 0.

Notation 3. From a distance d on V , we define the distance Δ_{N_γ} on the space of functions from V to V by

$$\Delta_{N_\gamma}(f, g) := \sup_{a \in V} \frac{d(f(a), g(a))}{N_\gamma(a)},$$

where N_γ is defined in Section 5.1.2.

This distance is extended on $\mathcal{F}(V)$ by

$$\Delta_{N_\gamma, \varpi}(\phi, \psi) := \sup_{\substack{(s,t) \in \mathbb{T}_2^+ \\ s \neq t}} \frac{\Delta_{N_\gamma}(\phi_{t,s}, \psi_{t,s})}{\varpi(\omega_{s,t})},$$

where ω, ϖ are defined in Section 5.1.2. The distance $\Delta_{N_\gamma, \varpi}$ may take infinite values.

If d is a distance for which (V, d) is complete, then $(\mathcal{C}(V, V), \Delta_N)$ and $(\mathcal{F}(V), \Delta_{N, \varpi})$ are complete.

Definition 1 (Galaxy). We define the equivalence relation \sim on $\mathcal{F}(V)$ by $\phi \sim \psi$ if and only if there exists a constant C such that

$$d(\phi_{t,s}(a), \psi_{t,s}(a)) \leq CN_\gamma(a)\varpi(\omega_{s,t}), \quad \forall a \in V, \quad \forall (s, t) \in \mathbb{T}_2^+.$$

In other words, $\phi \sim \psi$ if and only if $\Delta_{N_\gamma, \varpi}(\phi, \psi) < +\infty$. Each quotient class of $\mathcal{F}(V)/\sim$ is called a *galaxy*, which contains elements of $\mathcal{F}(V)$ which are at finite distance from each others.

5.1. INTRODUCTION

5.1.3 Summary of the main results

The galaxies partition the space $\mathcal{F}(V)$. Each galaxy may contain two classes of elements on which we focus on this article:

1. The *flows*, that is the families ψ which satisfy

$$\psi_{t,r} = \psi_{t,s} \circ \psi_{s,r} \quad (5.9)$$

for any $(r, s, t) \in \mathbb{T}_3^+$, or equivalently, $\psi^\pi = \psi$ (See (5.7)) for any partition π .

2. The *almost flows* which we see as time-inhomogeneous algorithms. Besides some conditions on the continuity and the growth given in Definition 2 below, an almost flow ϕ is close to a flow with the difference that

$$d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a)) \leq N_\gamma(a) \varpi(\omega_{r,t}), \quad \forall (r, s, t) \in \mathbb{T}_3^+, \quad a \in V,$$

for a suitable function $N_\gamma : V \rightarrow [1, +\infty)$.

Along with an almost flow ϕ comes the notion of *manifold of solutions*, that is a family $\{y_t(a)\}_{t \geq 0, a \in V}$ of paths such that

$$d(y_t(a), \phi_{t,s}(y_s(a))) \leq C \varpi(\omega_{s,t}), \quad \forall (s, t) \in \mathbb{T}_2^+.$$

Each path $y(a)$ is a solution which we called *solution in the sense of Davie* as our definition expands naturally the one introduced by A.M. Davie in [Dav07].

Clearly, a manifold of solutions associated to an almost flow ϕ is also associated to any almost flow in the same galaxy as ϕ . Besides, if a flow ψ exists in the same galaxy as ϕ , then $z_t(a) = \psi_{t,0}(a)$ defines a manifold of solutions. Flows are more constrained object than solutions as (5.9) implies some compatibility conditions, while it is possible to create new paths of solutions from two different ones. As it will be shown in [BL17b], uniqueness of manifold of solutions is strongly related to existence of a flow.

Given an almost flow ϕ , it is natural to study the limit of the net $\{\phi^\pi\}_\pi$ as the mesh of π decreases to 0. Any limit will be a good candidate to be a flow.

Our first main result (Theorem 5.2.2) asserts that if ϕ is an almost flow in a galaxy G , then any iterated map ϕ^π belongs to G whatever the partition π although the map ϕ^π is not necessarily an almost flow. More precisely, a control is given on $\Delta_{N_\gamma, \varpi}(\phi^\pi, \phi)$ which is uniform in the partition π .

An immediate corollary is that any possible limit of $\{\phi^\pi\}_\pi$ also belongs to G as the mesh of the partition decreases to 0.

Our second main result (Theorem 5.3.4) is that when the underlying Banach space V is finite-dimensional for compactness reasons, there exists a least one measurable flow in a galaxy containing an almost flow, even when several manifold of solutions may exists.

Our proof uses a recent result of J.E. Cardona and L. Kapitanski [CK17a] on selection theorems.

Our third result is to give conditions ensuring the existence of at most one flow in a galaxy. It is sufficient for this that a galaxy G contains a Lipschitz flow ψ . In this case, $\{\phi^\pi\}_\pi$ converges to ψ whatever the almost flow ϕ in G . The rate of convergence is also quantified.

Finally, we apply our results to the additive, multiplicative sewing lemmas [FdLPM08] as well as to the algorithms proposed by A.M. Davie in [Dav07] to show existence of measurable flows even without uniqueness. In the sequel [BL17b], we study in details the properties of Lipschitz flows and give some conditions on almost Lipschitz flows to generate them. In addition, we also apply our results to other approximations of RDE, namely the one proposed by P. Friz & N. Victoir [FV10] and the one proposed by I. Bailleul [Bai15a] by using perturbation arguments.

5.1.4 Outline

We show in Section 5.2 that a uniform control of the iterated product of approximation of flows with respect to the subdivision. In Section 5.3, we prove our main result: the existence of a measurable flow under weak conditions of regularity. Then, in Section 5.4 we show the existence and uniqueness of a Lipschitz flow under stronger assumptions. Moreover, we give a rate of the convergence of the iterated product to the flow. In Section 5.5, we show that the convergence of the iterated product of an approximation of flow is preserved under the action of an additive perturbation. Finally, we recover in Section 5.6 the additive [Lyo98], multiplicative [FdLPM08, CL14] and Davie's sewing lemmas [Dav07].

5.2 A uniform control over almost flows

In this section, we define almost flows which serves as approximations. The properties of an almost flow ϕ are weaker than the minimal condition to obtain the convergence of the iterated products ϕ^π as the mesh of the partitions π decreases to 0. However, we prove in Theorem 5.2.2 that we can control ϕ^π uniformly over the partitions π . This justifies our definition.

5.2. A UNIFORM CONTROL OVER ALMOST FLOWS

5.2.1 Definition of almost flows

Definition 2 (Almost flow). An element $\phi \in \mathcal{F}(V)$ is an *almost flow* if for any $(r, s, t) \in \mathbb{T}_3^+$, $a \in V$,

$$\phi_{t,t}(a) = a, \quad (5.10)$$

$$d(\phi_{t,s}(a), a) \leq \delta_T N_\gamma(a), \quad (5.11)$$

$$d(\phi_{t,s}(a), \phi_{t,s}(b)) \leq (1 + \delta_T)d(a, b) + \eta(\omega_{s,t})d(a, b)^\gamma, \quad (5.12)$$

$$d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a)) \leq N_\gamma(a)\varpi(\omega_{r,t}). \quad (5.13)$$

Remark 5.2.1. A family $\phi \in \mathcal{F}(V)$ satisfying condition (5.12) with $\gamma = 1$ is said to be *quasi-contractive*. This notion plays an important role in the fixed point theory [Ber04].

Definition 3 (Iterated almost flow). For a partition π and an almost flow ϕ , we call ϕ^π an *iterated almost flows*, where ϕ^π is the iterated product defined in (5.7).

Definition 4 (A flow). A *flow* ψ is a family of functions $\{\psi_{t,s}\}_{(s,t) \in \mathbb{T}_2^+}$ from V to V (not necessarily continuous in (s, t)) such that $\phi_{t,t}(a) = a$ and $\psi_{t,s} \circ \psi_{s,r}(a) = \psi_{t,r}(a)$ for any $a \in V$ and $(r, s, t) \in \mathbb{T}_3^+$.

5.2.2 A uniform control on iterated almost flows

Before proving our main result in Section 5.3, we prove an important uniform control over ϕ^π .

Theorem 5.2.2. *Let ϕ be an almost flow. Then there exists a time horizon T small enough and a constant $L_T \geq 1$ as well as a constant $K_T \geq 1$ that decreases to 1 as T decreases to 0 such that*

$$d(\phi_{t,s}^\pi(a), \phi_{t,s}(a)) \leq L_T N_\gamma(a)\varpi(\omega_{s,t}), \quad (5.14)$$

$$N_\gamma(\phi_{t,s}^\pi(a)) \leq K_T N_\gamma(a), \quad (5.15)$$

for any $(s, t) \in \mathbb{T}_+^2$, $a \in V$ and any partition π of \mathbb{T} .

Remark 5.2.3. The distance d may be replaced by a pseudo-distance in the statement of Theorem 5.2.2.

The proof of Theorem 5.2.2 is inspired by the one of the Claim in the proof of Lemma 2.4 in [Dav07]. With respect to the one of A.M. Davie, we consider obtaining a uniform control over a family of elements indexed by $(s, t) \in \mathbb{T}_2^+$ which are also parametrized by points in V .

Definition 5 (Successive points / distance between two points). Let π be a partition of $[0, T]$. Two points s and t of π are said to be *at distance k* if there are exactly $k - 1$ points between them in π . We write $d_\pi(t, s) = k$. Points at distance 1 are called *successive points* in π .

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Proof. For $a \in V$, $(r, t) \in \mathbb{T}_2^+$ and a partition π , we set

$$U_{r,t}(a) := d(\phi_{t,r}^\pi(a), \phi_{t,r}(a)).$$

We now restrict ourselves to the case $(r, t) \in \pi_2^+$. To control $U_{r,t}(a)$ in a way that does not depend on π , we use an induction in the distance between r and t .

Our induction hypothesis is that there exist constants $L_T \geq 0$ and $K_T \geq 1$ independent from the partition π such that for any $(r, t) \in \pi_2^+$ at distance at most $m \geq 1$,

$$U_{r,t}(a) \leq L_T N_\gamma(a) \varpi(\omega_{r,t}), \quad (5.16)$$

$$N_\gamma(\phi_{t,s}^\pi(a)) \leq K_T N_\gamma(a), \quad (5.17)$$

with K_T decreases to 1 at T goes to 0.

When $m = 0$, there is only one point r in π . Because $U_{r,r} = 0$ and $N_\gamma(\phi_{r,r}(a)) = N_\gamma(a)$, (5.16) and (5.17) hold.

If r and t are successive points, $\phi_{t,r}^\pi = \phi_{t,r}$ so that $U_{r,t}(a) = 0$. Thus, (5.16) is true for $m = 1$. With (5.11) and since $N_\gamma(a)^\gamma \leq N_\gamma(a)$ as $N_\gamma(a) \geq 1$ by hypothesis,

$$\begin{aligned} N_\gamma(\phi_{t,s}(a)) &\leq N_\gamma(\phi_{t,s}(a)) - N_\gamma(a) + N_\gamma(a) \leq \|N_\gamma\|_\gamma d(\phi_{t,s}(a), a)^\gamma + N_\gamma(a) \\ &\leq \|N_\gamma\|_\gamma \delta_T^\gamma N_\gamma(a)^\gamma + N_\gamma(a) \leq (\|N_\gamma\|_\gamma \delta_T^\gamma + 1) N_\gamma(a). \end{aligned}$$

For $m = 2$, it is also true with $L_T = 1$ as $U_{r,t}(a) = d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a))$ where s is the point in the middle of r and t . This proves (5.16).

In addition, using (5.16),

$$\begin{aligned} N_\gamma(\phi_{t,s}^\pi(a)) &\leq \|N\|_\gamma d(\phi_{t,s}^\pi(a), \phi_{t,s}(a))^\gamma + N_\gamma(\phi_{t,s}(a)) - N_\gamma(a) + N_\gamma(a) \\ &\leq K_T N_\gamma(a) \text{ with } K_T := \|N\|_\gamma \varpi(\omega_{0,T})^\gamma + \|N_\gamma\|_\gamma \delta_T^\gamma + 1. \end{aligned} \quad (5.18)$$

Clearly, K_T decreases to 1 at T decreases to 0. This proves (5.17) whenever (5.16), in particular for $m = 1$.

Assume that that (5.16)-(5.17) when the distance between r and t is smaller than m for some $m \geq 2$.

For $s \in \pi$, $r \leq s \leq t$, with (5.8),

$$\begin{aligned} U_{r,t}(a) &\leq d(\phi_{t,s}^\pi \circ \phi_{s,r}^\pi(a), \phi_{t,s} \circ \phi_{s,r}^\pi(a)) \\ &\quad + d(\phi_{t,s} \circ \phi_{s,r}^\pi(a), \phi_{t,s} \circ \phi_{s,r}(a)) + d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a)). \end{aligned}$$

With (5.12) and (5.13),

$$U_{r,t}(a) \leq U_{s,t}(\phi_{s,r}^\pi(a)) + (1 + \delta_T) U_{r,s}(a) + \eta(\omega_{s,t}) U_{r,s}(a)^\gamma + N_\gamma(a) \varpi(\omega_{r,t}). \quad (5.19)$$

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Using (5.19) on $U_{s,t}(\phi_{s,r}^\pi(a))$ by replacing (r, s, t) by (s, s', t) , where s and s' are two successive points of π leads to

$$U_{s,t}(\phi_{s,r}^\pi(a)) \leq U_{s',t}(\phi_{s',r}^\pi(a)) + N_\gamma(\phi_{s,r}^\pi(a))\varpi(\omega_{s,t})$$

since $\phi_{s',s} \circ \phi_{s,r}^\pi(a) = \phi_{s',r}^\pi(a)$ and $U_{s,s'}(a) = 0$.

With (5.16)-(5.17) and our hypothesis on η , again since $N_\gamma \geq 1$ and s, s' are at distance less than m provided that s is at distance at most 2 from t ,

$$U_{r,t}(a) \leq N_\gamma(a) \left(K_T L_T \varpi(\omega_{s',t}) + (L_T + \delta_T(1 + L_T^\gamma))\varpi(\omega_{r,s}) + (1 + K_T)\varpi(\omega_{r,t}) \right). \quad (5.20)$$

This inequality also holds if $s'=t$ or $r=s$.

We proceed as in [Dav07] to split π in "essentially" two parts. We set

$$s' := \min \left\{ \tau \in \pi \mid \omega_{r,\tau} \geq \frac{\omega_{r,t}}{2} \right\} \text{ and } s := \max \left\{ \tau \in \pi \mid \tau < s' \right\}.$$

Hence, s and s' are successive points with $r \leq s < s' \leq t$ and $\omega_{r,s} \leq \omega_{r,t}/2$. Besides, since ω is super-additive, $\omega_{r,s'} + \omega_{s',t} \leq \omega_{r,t}$. Therefore,

$$\omega_{r,s} \leq \frac{\omega_{r,t}}{2} \text{ and } \omega_{s',t} \leq \frac{\omega_{r,t}}{2}. \quad (5.21)$$

With such a choice, (5.20) becomes with (5.5) and since $L_T \geq 1$,

$$U_{r,t}(a) \leq \left(L_T \frac{K_T + 1 + 2\delta_T}{2} \varkappa + 1 + K_T \right) N_\gamma(a) \varpi(\omega_{r,t}).$$

If T is small enough so that

$$\alpha := \frac{1 + K_T + 2\delta_T}{2} \varkappa < 1 \text{ and } L_T \alpha + 1 + K_T \leq L_T,$$

then one may choose L_T so that $L_T \alpha + 1 + K_T \leq L_T$, that is $L_T \geq \max\{1, (1 + K_T)/(1 - \alpha)\}$. This choice ensures that $U_{r,t}(a) \leq L_T N_\gamma(a) \varpi(\omega_{r,t})$ when r and t are at distance m . Condition (5.17) follows from (5.18) and (5.16).

The choice of T , K_T and L_T does not depend on π . In particular, $d(\phi_{t,s}^\pi(a), \phi_{t,s}(a)) \leq L_T N_\gamma(a) \varpi(\omega_{s,t})$ becomes true for any $(s, t) \in \mathbb{T}_2^+$, is it is sufficient to add the points s and t to π . ■

Corollary 5.2.4. *Let ϕ be an almost flow and π be a partition of \mathbb{T} . Then $\phi^\pi \sim \phi$ (we have not proved that ϕ^π is itself an almost flow).*

Notation 4. For an almost flow ϕ , let us denote by $S_\phi(s, a)$ the set of all the possible limits of the net $\{\phi_{s,\pi}^\pi(a)\}_\pi$ in $(\mathcal{C}([s, T], V), \|\cdot\|_\infty)$ for nested partitions.

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When V is a finite dimensional space with the norm $|\cdot|$, $S_\phi(s, a) \neq \emptyset$. We start with a lemma which will be useful to prove some equi-continuity. We denote $\bar{B}(0, R)$ the closed ball center in 0 of radius $R > 0$.

Lemma 5.2.5. *We assume that V is a finite dimensional vector space and ϕ is an almost flow. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(s, t), (s', t') \in \mathbb{T}_2^+$ with $|t - t'|, |s - s'| < \delta$ and $a \in \bar{B}(0, R)$,*

$$|\phi_{t,s}(a) - \phi_{t',s'}(a)| \leq \epsilon.$$

Proof. In all the proof $(s, t), (s', t') \in \mathbb{T}_2^+$. For any $a \in \bar{B}(0, R)$, $(s, t) \in \mathbb{T}_2^+ \mapsto \phi_{t,s}(a)$ is continuous, so uniformly continuous on the compact \mathbb{T}_2^+ . Let $\epsilon > 0$, there is δ_a such that for all $|t - t'|, |s - s'| < \delta_a$,

$$|\phi_{t,s}(a) - \phi_{t',s'}(a)| \leq \frac{\epsilon}{3}. \quad (5.22)$$

For a $\rho > 0$ with (5.12), for all $b \in B(a, \rho)$,

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq (1 + \delta_T)\rho + \eta(\omega_{0,T})\rho^\gamma,$$

where $B(a, \rho)$ denotes the open ball centred in a of radius ρ . We choose $\rho(\epsilon) > 0$ such that $(1 + \delta_T)\rho(\epsilon) + \eta(\omega_{0,T})\rho(\epsilon)^\gamma \leq \epsilon/3$ to obtain for all $b \in B(a, \rho(\epsilon))$,

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq \frac{\epsilon}{3}. \quad (5.23)$$

We note that $\bigcup_{a \in \bar{B}(0, R)} B(a, \rho(\epsilon))$ is a covering of $\bar{B}(0, R)$ which is a compact of V . There exist an integer N and a finite family of balls $B(a_i, \rho(\epsilon))$ for $i \in \{1, \dots, N\}$ such that $\bar{B}(0, R) \subset \bigcup_{i \in \{1, \dots, N\}} B(a_i, \rho(\epsilon))$.

It follows that for all $b \in \bar{B}(0, R)$ there exists $i \in \{1, \dots, N\}$ such that $b \in B(a_i, \rho(\epsilon))$. From (5.22)-(5.23) there exists $\delta_{a_i} > 0$ such that for all $|t - t'|, |s - s'| < \delta_{a_i}$,

$$|\phi_{t,s}(b) - \phi_{t',s'}(b)| \leq \epsilon.$$

Taking $\delta := \min_{i \in \{1, \dots, N\}} \delta_{a_i}$, we obtain that for all $|t - t'|, |s - s'| < \delta$ and $b \in \bar{B}(0, R)$,

$$|\phi_{t,s}(b) - \phi_{t',s'}(b)| \leq \epsilon,$$

which achieves the proof. ■

Proposition 5.2.6. *Assume that V is finite-dimensional and that ϕ is an almost flow. Then $S_\phi(r, a)$ is not empty for any $(r, a) \in \mathbb{T} \times V$.*

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Proof. Let us show for an almost flow ϕ , $\{\phi_{\cdot,r}^{\pi}(a)\}_{\pi}$ is equi-continuous and bounded. The result is then a direct consequence of the Ascoli-Arzelà theorem.

Let $(\pi_n)_{n \in \mathbb{N}}$ be an increasing sequence of partition of \mathbb{T} such that $|\pi_n| \rightarrow 0$ when $n \rightarrow +\infty$ and $\bigcup_{n \in \mathbb{N}} \pi_n$ dense in \mathbb{T} . Let $R > 0$ and $R' := N_{\gamma}(R)(L_T \varpi(\omega_{0,T}) + \delta_T) + R$. From Theorem 5.2.2, for any $a \in \bar{B}(0, R)$ and $(s, t) \in \mathbb{T}_2^+$,

$$\begin{aligned} |\phi_{t,s}^{\pi_n}(a)| &\leq |\phi_{t,s}^{\pi_n}(a) - \phi_{t,s}(a)| + |\phi_{t,s}(a) - a| + |a| \\ &\leq N_{\gamma}(a)(L_T \varpi(\omega_{0,T}) + \delta_T) + |a| \leq R'. \end{aligned}$$

Let $(r, s, t) \in \mathbb{T}_3^+$ and $s, t \in \bigcap_{n \in \mathbb{N}} \pi_n$, let $\epsilon > 0$ and $\delta > 0$ given by Lemma 5.2.5 for $\epsilon/2$. For $|t - s| < \delta$, let m an integer such that $s, t \in \pi_m$.

We differentiate two cases. If $m \leq n$, then $\pi_m \subset \pi_n$, which implies that $\phi_{t,r}^{\pi_n} = \phi_{t,s}^{\pi_n} \circ \phi_{s,r}^{\pi_n}$. From Theorem 5.2.2 and Lemma 5.2.5, for all $a \in \bar{B}(0, R)$, for all $|t - s| < \delta$

$$|\phi_{t,r}^{\pi_n}(a) - \phi_{s,r}^{\pi_n}(a)| \leq |\phi_{t,s}^{\pi_n} \circ \phi_{s,r}^{\pi_n}(a) - \phi_{s,r}^{\pi_n}(a)| \leq \frac{\epsilon}{2} + L_T N_{\gamma}(\phi_{s,r}^{\pi_n}(a)) \varpi(\omega_{s,t}).$$

Since, $\omega_{s,s} = 0$ and ω is continuous close to diagonal, there exists $\delta' > 0$ such that for all $|t - s| < \delta'$, $L_T N_{\gamma}(\phi_{s,r}^{\pi_n}(a)) \varpi(\omega_{s,t}) < \epsilon/2$. Thus for all $|t - s| < \min(\delta, \delta')$, $|\phi_{t,r}^{\pi_n}(a) - \phi_{s,r}^{\pi_n}(a)| \leq \epsilon$.

In the case $m > n$, let s_-, s_+ be successive points in π_n such that $[s, t] \subset [s_-, s_+]$. Then, $\phi_{t,r}^{\pi_n}(a) = \phi_{t,s_-} \circ \phi_{s_-,r}^{\pi_n}(a)$ and $\phi_{s,r}^{\pi_n}(a) = \phi_{s,s_-} \circ \phi_{s_-,r}^{\pi_n}(a)$. According to Lemma 5.2.5, for $|t - s| < \delta$ with $s, t \in \pi_m$, and all $a \in \bar{B}(0, R)$,

$$|\phi_{t,r}^{\pi_n}(a) - \phi_{s,r}^{\pi_n}(a)| = |\phi_{t,s_-} \circ \phi_{s_-,r}^{\pi_n}(a) - \phi_{s,s_-} \circ \phi_{s_-,r}^{\pi_n}(a)| \leq \epsilon. \quad (5.24)$$

By continuity of $t \mapsto \phi_{t,r}^{\pi_n}(a)$, and density of $\bigcap_{m \in \mathbb{N}} \pi_m$ in \mathbb{T} , we obtain (5.24) for all $(s, t) \in \mathbb{T}_2^+$ with $r \leq s$ and $|t - s| < \delta$. This proves that $\{t \mapsto \phi_{t,r}^{\pi_n}(a)\}_n$ is uniformly equicontinuous for all $a \in V$. We conclude the proof with the Ascoli-Arzelà theorem. ■

5.3 The non-linear sewing lemma

After showing a uniform control of ϕ^{π} with respect to π when ϕ is an almost flow in the previous section, we prove our main result in Theorem 5.3.4. We show that in the finite dimensional case, we can built a flow from ϕ^{π} using a selection principle due to [CK17a].

We restate the notion of solution in the sense introduced by A.M. Davie [Dav07] in the context of a metric space and a general almost flow.

Definition 6 (Solution in the sense of Davie). For an almost flow ϕ , a *solution in the sense of Davie* associated to ϕ is a path $y \in \mathcal{C}(\mathbb{S}, V)$ with $\mathbb{S} = [r, r + T] \subset \mathbb{T}$ if there is a constant $K \geq 0$ such that

$$d(y_t, \phi_{t,s}(y_s)) \leq K N_{\gamma}(y_r) \varpi(\omega_{s,t}), \quad \forall (s, t) \in \mathbb{S}_+^2. \quad (5.25)$$

Remark 5.3.1. Our definition of solution is a natural extension of that of Davie in [Dav07] for a metric space V and a general almost flow ϕ .

Remark 5.3.2. If ϕ is only an almost flow, it is not guarantee that there exists a solution and even less that the uniqueness of the solution holds. When $S_\phi(r, a) \neq \emptyset$, we prove below in Lemma 5.3.8 that a solution in the sense of Davie exists. Even when $S_\phi(r, a) \neq \emptyset$ for any $r \in \mathbb{T}$ and any $a \in V$ does not proves that there exists a choice of families of solutions $\{\psi_{\cdot, r}(a)\}_{r \in \mathbb{T}, a \in V}$ which satisfies the flow property.

Notation 5. We denote $\Omega(r, a)$ the set of continuous path such that $y \in \mathcal{C}(\mathbb{S}, V)$ verifying $y_r = a$. We denote by $G_\phi^K(r, a)$ the set of paths in $\Omega(r, a)$ verifying (5.25) for the constant K .

Definition 7 (Splicing of paths). For $r \leq s$, let us consider $y(r, a) \in \mathcal{C}_a([r, T], V)$ and $z \in \mathcal{C}_b([s, T], V)$ with $b = y_s(r, a)$. Their *splicing* is

$$(y \bowtie_s z)_t(r, a) = \begin{cases} y_t(r, a) & \text{if } t \leq s, \\ z_t(s, y_s(r, a)) & \text{if } t \geq s. \end{cases}$$

We restate here, the definition of a family of abstract integral local funnels (Definition 2 in [CK17a]), which leads to the existence of a measurable flow.

Definition 8. A family $F(r, a) \subset \mathcal{C}(\mathbb{T}, V)$ with $r \in [0, +\infty)$, $a \in V$, will be called a *family of abstract local integral funnels* with terminal times $T(r, a)$ if $F(r, a)$ satisfies

- H0** The map $(r, a) \in [0, +\infty) \times V \mapsto T(r, a)$ is a lower semi-continuous in the sense that if $(r_n, a_n) \rightarrow (r, a)$, then $T(r, a) \leq \liminf_n T(r_n, a_n)$.
- H1** Every set $F(r, a)$ is a non-empty compact in $\mathcal{C}(\mathbb{T}, V)$ and every path $y(r, a) \in F(r, a)$ is a continuous map from $\mathbb{T} = [T_0, T_0 + T(r, a)]$ to V , and $y_r(r, a) = a$.
- H2** For all $r \geq 0$, the map $a \in V \mapsto S(r, a)$ is measurable in the sense that for any closed subset $\mathcal{K} \subset \mathcal{C}([0, 1], V)$, $\{a \in V \mid \tilde{F}(r, a) \cap \mathcal{K} \neq \emptyset\}$ is Borel, where $\tilde{F}(r, a)$ is the set of re-parametrizations of paths of $F(r, a)$ on $[0, 1]$.
- H3** If $y(r, a) \in F(r, a)$ and $\tau < T(r, a)$, then $T(t_0 + \tau, y_{r+\tau}(r, a)) \geq T(r, a) - \tau$ and $t \in [r + \tau, r + \tau + T(t_0 + \tau, y_{r+\tau}(r, a))] \mapsto y_t(r, a) \in F(r + \tau, y_{r+\tau}(r, a))$.
- H4** If $y(r, a) \in F(r, a)$ and $\tau < T(r, a)$, and $z \in F(r + \tau, y_{t+\tau}(r, a))$ then the spliced path (Definition 7) $x := y \bowtie_{r+\tau} z \in F(r, a)$.

Definition 9 (Lipschitz almost flow). A *Lipschitz almost flow* is an almost flow for which (5.12) is satisfied with $\eta = 0$, and $N_\gamma = N_1 = N$ is Lipschitz continuous.

A flow is constructed by assigning to each point of the space of particular choice of a solution in the sense of Davie in a sense which is compatible.

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Hypothesis 1. Let V be a finite dimensional vector space. Let ϕ be a Lipschitz almost flow (Definition 9) with N bounded. We fix a time horizon $T > 0$ such that $\varkappa(1 + \delta_T) < 1$.

Remark 5.3.3. In the case N bounded, we can choose $K_T = 1$ and $L_T = 2/(1 - \varkappa(1 + \delta_T))$ where K_T and L_T are the constants of Theorem 5.2.2.

The main theorem of this paper is the following one.

Theorem 5.3.4 (Non-linear sewing lemma, weak formulation). *Under Hypothesis 1, there exists a family $\psi \in \mathcal{F}(V)$ in the same galaxy as ϕ such that $\psi_{t,s}$ is Borel measurable for any $(s, t) \in \mathbb{T}_+^2$ and ψ satisfies the flow property.*

Remark 5.3.5. Proving such a result with a general Banach space V is false as even existence of solutions to ordinary differential equations may fail [Die50, HJ10].

Remark 5.3.6. To prove Theorem 5.3.4, we show that $(G^{L_T}(r, a))_{r \in [0, +\infty), a \in V}$ is a family of abstract local integral funnels in the sense of Definition 8. Then, we use Theorem 2 of [CK17a].

Lemmas 5.3.7-5.3.12 prove that $G^{L_T}(r, a)$ is a family of local abstract funnels in the sense of the Definition 2 in [CK17a]. Then we apply Theorem 2 in [CK17a] to obtain the following theorem.

Lemma 5.3.7. *Under the Hypothesis 1, the terminal times $T(r, a)$ is independent of the starting time r and the starting point a . We denote $T = T(r, a)$. In particular, H0 holds for $F = G_\phi^{L_T}$.*

Proof. Indeed, \varkappa and δ_T does not depend neither of $a \in V$ and T_0 . ■

We recall that $S_\phi(r, a)$ is defined in Notation 4 Our first result is that when $S_\phi(r, a) \neq \emptyset$, then there exists at least one solution in the sense of Davie in $G_\phi^{L_T}(r, a)$.

Lemma 5.3.8. *Assume that $K \geq K_T L_T$ in Definition 6, where K_T and L_T are constants in Theorem 5.2.2. For any $(r, a) \in \mathbb{T} \times V$, $S_\phi(r, a) \subset G_\phi^K(r, a)$ for an almost flow ϕ (note that $S_\phi(r, a)$ may be empty).*

Proof. If $y \in S_\phi(r, a)$ when $S_\phi(r, a) \neq \emptyset$, then there exists a sequence $\{\pi_k\}_{k \in \mathbb{N}}$ of partitions such that $y_t = \lim \phi_{t,r}^{\pi_k}(a)$ uniformly in $t \in [r, T]$. We note that $y_r = a$. For $k \in \mathbb{N}$ and $s_k \in \pi_k$, with (5.8) and Theorem 5.2.2,

$$\begin{aligned} d(\phi_{t,r}^{\pi_k}(a), \phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a)) &= d(\phi_{t,s_k}^{\pi_k} \circ \phi_{s_k,r}^{\pi_k}(a), \phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a)) \\ &\leq L_T N_\gamma(\phi_{s_k,r}^{\pi_k}(a)) \varpi(\omega_{s_k,t}) \\ &\leq L_T K_T N_\gamma(a) \varpi(\omega_{s_k,t}). \end{aligned} \tag{5.26}$$

And moreover,

$$\begin{aligned} & d(\phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a), \phi_{t,s}(y_s(a))) \\ & \leq d(\phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a), \phi_{t,s_k} \circ y_s(a)) + d(\phi_{t,s_k} \circ y_s(a), \phi_{t,s} \circ y_s(a)) \end{aligned} \quad (5.27)$$

$$\begin{aligned} & \leq (1 + \delta_T) d(\phi_{s_k,r}^{\pi_k}(a), y_s(a)) + \eta(\omega_{0,T}) d(\phi_{s_k,r}^{\pi_k}(a), y_s(a))^\gamma \\ & + d(\phi_{t,s_k} \circ y_s(a), \phi_{t,s} \circ y_s(a)). \end{aligned} \quad (5.28)$$

Choosing $\{s_k\}_{k \in \mathbb{N}}$ so that s_k decreases to s and passing to the limit, we obtain with (5.28) that $\phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a)$ converges uniformly to $\phi_{t,s} \circ y_s(a)$. Thus, when $k \rightarrow +\infty$, (5.26) shows that y is a solution in the sense of Davie. \blacksquare

Lemma 5.3.9. *Under Hypothesis 1, $G_\phi^{LT}(r, a)$ is non-empty compact subset of the path $y \in \mathcal{C}(\mathbb{S}, \mathbb{V})$ such that $y_r = a$ for any $r \in \mathbb{T}$ and $a \in \mathbb{V}$. It shows that H1 holds for $F := G_\phi^{LT}$.*

Proof. It follows directly from Proposition 5.2.6 and Lemma 5.3.8 (with $K_T = 1$ and $K = L_T$) that $G_\phi^K(r, a)$ is not empty. Now, if $\{y^k\}_k$ is a sequence in $G_\phi^K(r, a)$ then $\{y^k\}_k$ is equi-continuous with the same argument as in proof of Proposition 5.2.6. And the subsequence of $\{y^k\}_k$ converges in $G_\phi^K(r, a)$ because of the continuity of $a \in \mathbb{V} \mapsto \phi_{t,s}(a)$ for any $(s, t) \in \mathbb{T}_2^+$. \blacksquare

Let us denote $\tilde{G}_\phi^{LT}(r, a)$, the set of paths in $y \in G_\phi^{LT}(r, a)$ reparametrised on $[0, 1]$ as $t \in [0, 1] \mapsto \tilde{y}_t := y_{r+t(T-r)}$.

Lemma 5.3.10. *Let us assume Hypothesis 1. Let $r \geq 0$, for any closed subset $\mathcal{K} \subset \mathcal{C}([0, 1], \mathbb{V})$, the set $S'(r) := \{a \in \mathbb{V} \mid \tilde{G}_\phi^{LT}(r, a) \cap \mathcal{K} \neq \emptyset\}$ is closed in \mathbb{V} , in particular it is a Borel set in \mathbb{V} . It shows that H2 holds for $F = G_\phi^{LT}$.*

Proof. Let $\{a_k\}_{k \in \mathbb{N}}$ be a convergent sequence of $S'(r)$. For each $k \in \mathbb{N}$, we choose a path $\tilde{y}^k \in \tilde{G}_\phi^{LT}(r, a_k) \cap \mathcal{K}$ (which is not empty by definition). Then, for every $s, t \in [0, 1]$, $s \leq t$,

$$d(\tilde{y}_t^k, \tilde{y}_s^k) \leq d(\tilde{y}_t^k, \phi_{\tilde{t}, \tilde{s}}(\tilde{y}_s^k)) + d(\phi_{\tilde{t}, \tilde{s}}(\tilde{y}_s^k), \tilde{y}_s^k) \leq [L_T \varpi(\omega_{\tilde{s}, \tilde{t}}) + \delta_{\tilde{t}-\tilde{s}}] \|N\|_\infty, \quad (5.29)$$

where $\tilde{t} := r + t(T-r)$ and $\tilde{s} := r + s(T-r)$. Since $\tilde{t} - \tilde{s}$ goes to zero when $t - s \rightarrow 0$, it follows that $\{t \in [0, 1] \mapsto \tilde{y}_t^k\}_{k \in \mathbb{N}}$ is equi-continuous.

The sequence $\{a_k\}_{k \in \mathbb{N}}$ converges, so it is bounded by a constant $A \geq 0$. Applying (5.29) between $s = 0$ and t , we get $|\tilde{y}_t^k| \leq [L_T \varpi(\omega_{0,1}) + \delta_T] \|N\|_\infty + A$, which proves that $t \in [0, 1] \mapsto y^k$ is uniformly bounded.

By Ascoli-Arzelà theorem, there is a convergent subsequence $\{\tilde{y}^{k_i}\}_{i \in \mathbb{N}}$ in $(\mathcal{C}([0, 1], \mathbb{V}), \|\cdot\|_\infty)$ to a path y . This path belongs to \mathcal{K} since \mathcal{K} is closed. Because $\phi_{t,s}$ is continuous, $\tilde{y} \in \tilde{G}_\phi^{LT}(r, a)$. Hence $S'(r)$ is closed and then Borel. \blacksquare

5.3. THE NON-LINEAR SEWING LEMMA

The proof of the next lemma is an immediate consequence of the definition of a solution in the sense of Davie.

Lemma 5.3.11. *If $t \in [r, r + T] \mapsto y(r, a)$ belongs to $G_\phi^{LT}(r, a)$, then for any $r' \geq 0$, its restriction $t \in [r + r', r + r' + T] \mapsto y_t(r, a)$ belongs to $G_\phi^{LT}(r + r', y_{r+r'}(r, a))$. It shows that H3 holds for $F = G_\phi^{LT}$.*

Lemma 5.3.12. *We assume that Hypothesis 1 hold. For $r' \geq 0$, if $y \in G_\phi^{LT}(r, a)$ and $z \in G_\phi^{LT}(r + r', y_{r+r'}(r, a))$, then $y \bowtie_{r+r'} z \in G_\phi^{LT}(r, a)$. It shows that H4 holds for $F := G_\phi^{LT}$*

Proof. Let us write $x := y \bowtie_s z$ where $s := r + r'$ and $U_{\tau,t} := d(x_t, \phi_{t,\tau}(x_\tau))$ for $\tau \leq t$. On the one hand, for any $r \leq \tau \leq s \leq t$ with (5.25), (5.12) and (5.13),

$$\begin{aligned} U_{\tau,t} &\leq d(x_t, \phi_{t,s}(x_s)) + d(\phi_{t,s}(x_s), \phi_{t,s} \circ \phi_{s,\tau}(x_\tau)) + d(\phi_{t,s} \circ \phi_{s,\tau}(x_\tau), \phi_{t,\tau}(x_\tau)) \\ &\leq \|N\|_\infty(2 + \delta_T)L_T\varpi(\omega_{\tau,t}). \end{aligned} \quad (5.30)$$

On another hand, for $s \leq \tau \leq t$ or $\tau \leq t \leq s$ with (5.25)

$$U_{\tau,t} \leq \|N\|_\infty L_T \varpi(\omega_{\tau,t}). \quad (5.31)$$

Thus, combining (5.30) and (5.31), for any $r \leq \tau \leq t \leq T$,

$$U_{\tau,t} \leq \|N\|_\infty(2 + \delta_T)L_T\varpi(\omega_{\tau,t}).$$

Besides, for any $r \leq \tau \leq u \leq t \leq T$ with (5.12) and (5.13),

$$U_{\tau,t} \leq U_{u,t} + (1 + \delta_T)U_{\tau,u} + \|N\|_\infty\varpi(\omega_{\tau,t}). \quad (5.32)$$

Let $\lambda \in (0, 1)$ such that ϖ^λ satisfies (5.5) with $\varkappa_\lambda := 2^{1-\lambda}\varkappa^\lambda < 1$. Let $T(\lambda) > 0$ a real such that $\varkappa_\lambda(1 + \delta_{T(\lambda)}) < 1$. For any, two successive points τ, t of a subdivision π ,

$$U_{\tau,t} \leq D_{L_T}(\pi, \lambda)\varpi^\lambda(\omega_{\tau,t}), \quad (5.33)$$

where $D_{L_T}(\pi, \lambda) := \|N\|_\infty(2 + \delta_T)L_T \sup_{\tau,t \text{ successive points of } \pi} \varpi^{1-\lambda}(\omega_{\tau,t})$.

Let us show by induction over the distance m between points τ and t in $\pi \cap [0, T(\lambda)]^2$ that

$$U_{\tau,t} \leq A_{L_T}(\pi, \lambda)\varpi^\lambda(\omega_{\tau,t}), \quad (5.34)$$

where $A_{L_T}(\pi, \lambda) := \frac{D_{L_T}(\pi, \lambda)(1 + \delta_{T(\lambda)}) + 2\|N\|_\infty\varpi^\lambda(\omega_{0, T(\lambda)})}{1 - \varkappa_\lambda(1 + \delta_{T(\lambda)})}$. When $m = 0$, $U_{\tau,\tau} = 0$ and (5.34) holds. For $m = 1$, τ and t are successive points then (5.34) holds with (5.33). Now, we assume that (5.34) holds for any two points at distance m . Let τ and t be two points at distance $m + 1$ in $\pi \cap [0, T(\lambda)]^2$. Since ω is super-additive, one may choose two successive

points s and s' in π with $\tau \leq s < s' \leq t$ such that $\omega_{\tau,s} \leq \omega_{\tau,t}/2$ and $\omega_{s',t} \leq \omega_{\tau,t}/2$ as in the proof of Theorem 5.2.2. Then, by applying (5.32) between (τ, s, s') and (s, s', t) we obtain,

$$\begin{aligned} U_{\tau,t} &\leq U_{s,t} + (1 + \delta_{T(\lambda)})U_{\tau,s} + \|N\|_{\infty} \varpi^{1-\lambda}(\omega_{0,T(\lambda)}) \varpi^{\lambda}(\omega_{\tau,t}) \\ &\leq U_{s',t} + (1 + \delta_{T(\lambda)})U_{s,s'} + (1 + \delta_{T(\lambda)})U_{\tau,s} + 2\|N\|_{\infty} \varpi^{1-\lambda}(\omega_{0,T(\lambda)}) \varpi^{\lambda}(\omega_{\tau,t}) \\ &\leq [A_{L_T}(\pi, \lambda) \varkappa_{\lambda}(1 + \delta_{T(\lambda)}) + (1 + \delta_{T(\lambda)})D_{L_T}(\pi, \lambda) + 2\|N\|_{\infty} \varpi^{1-\lambda}(\omega_{0,T(\lambda)})] \varpi_{\tau,t}^{\lambda} \\ &\leq A_{L_T}(\pi, \lambda) \varpi_{\tau,t}^{\lambda}, \end{aligned}$$

with our choice of $A_{L_T}(\pi, \lambda)$. This concludes the induction, so (5.34) holds for any $\tau, t \in \pi \cap [0, T(\lambda)]^2$.

Clearly, $D_{L_T}(\pi, \lambda) \rightarrow 0$ when the mesh of π goes to zero. Then, $A_{L_T}(\pi, \lambda) \rightarrow A(\lambda) := 2\|N\|_{\infty} \varpi_{0,T(\lambda)}^{\lambda} / (1 - \varkappa_{\lambda}(1 + \delta_{T(\lambda)}))$ when the mesh of π goes to zero.

By continuity of $(\tau, t) \mapsto U_{\tau,t}$, considering a finer and finer partitions leads to $U_{\tau,t} \leq A(\lambda) \varpi^{\lambda}(\omega_{\tau,t})$ for any $r \leq \tau \leq t \leq T(\lambda)$.

Finally, choosing $T(\lambda)$ so that $T(\lambda)$ increases to T defined in Hypothesis 1 when λ goes to 1, we conclude that for any $r \leq \tau \leq t \leq T$,

$$U_{\tau,t} \leq \|N\|_{\infty} L_T \varpi(\omega_{\tau,t}),$$

where L_T is defined in Hypothesis 1. This proves that $z \in G_{\phi}^{L_T}(r, a)$. ■

Proof of Theorem 5.3.4. Lemma 5.3.7-5.3.12 prove that conditions H0-H4 of Definition 8 hold for $F = G_{\phi}^{L_T}$. This means that $G^{L_T}(r, a)$ is a family of abstract local integral funnels. We apply Theorem 1 in [CK17a]. For any $(r, a) \in \mathbb{T} \times \mathbb{V}$, there exists a measurable map $a \mapsto (t \mapsto \psi_{t,r}(a))$ respect to the Borel subsets of $\mathcal{C}^0(\mathbb{T}, \mathbb{V})$ with the property that $\psi_{r,r}(a) = a$ and $\psi_{t,s} \circ \psi_{s,r}(a) = \psi_{t,r}(a)$, $t \geq r$. ■

5.4 Lipschitz flows

A Lipschitz almost flow which has the flow property is said to be a Lipschitz flow. We recast the definition.

Definition 10 (Lipschitz flow). A flow $\psi \in \mathcal{F}(\mathbb{V})$ is said to be a *Lipschitz flow* is for any $(s, t) \in \mathbb{T}_2^+$, $\psi_{t,s}$ is Lipschitz in space with $\|\psi_{t,s}\|_{\text{Lip}} \leq 1 + \delta_T$.

In this section, we consider galaxies that contain a Lipschitz flow.

We prove that such a Lipschitz flow ψ is the only possible flow in the galaxy (Theorem 5.4.3), and that the iterated almost flow ϕ^{π} of any almost flow ϕ converges to ψ (Theorem 5.4.1). We also characterize the rate of convergence (Theorem 5.4.2).

5.4. LIPSCHITZ FLOWS

Let us choose $\lambda \in (0, 1)$ such that ϖ^λ satisfies the same properties as ϖ up to changing \varkappa to $\varkappa_\lambda := 2^{1-\lambda}\varkappa^\lambda$, provided $\varkappa_\lambda < 1$. This is possible as soon as $\lambda > 1/(1 - \log_2(\varkappa))$ with (5.5).

Clearly, if for $\psi, \chi \in \mathcal{F}(\mathbb{V})$, $\Delta_{N, \varpi}(\psi, \chi) < +\infty$, then

$$\Delta_{N, \varpi^\lambda}(\psi, \chi) \leq \Delta_{N, \varpi}(\psi, \chi) \varpi^{1-\lambda}(\omega_{0, T}) < +\infty, \quad (5.35)$$

where ϖ is defined by (5.5). Hence, the galaxies remain the same when ϖ is changed to ϖ^λ . We define

$$\Theta(\pi) := \sup_{d_\pi(s, s')=1} \varpi^{1-\lambda}(\omega_{s, s'}). \quad (5.36)$$

Theorem 5.4.1. *Let ϕ be an almost flow such that $\|\phi^\pi\|_{\text{Lip}} \leq 1 + \delta_T$ whatever the partition π , we say that ϕ satisfies the uniform Lipschitz (UL) condition. Then there exists a Lipschitz flow $\zeta \in \mathcal{F}(\mathbb{V})$ with $\|\zeta_{s, t}\|_{\text{Lip}} \leq 1 + \delta_T$ such that $\{\phi^\pi\}$ converges to ζ as $|\pi| \rightarrow 0$.*

Theorem 5.4.2. *Let ϕ be an almost flow and ψ be a Lipschitz flow with $\psi \sim \phi$. Then there exists a constant K that depends only on λ , $\Delta_{N, \varpi}(\phi, \psi)$, \varkappa and T (assumed to be small enough) so that*

$$\Delta_{N, \varpi^\lambda}(\psi, \phi^\pi) \leq K\Theta(\pi).$$

In particular, $\{\phi^\pi\}_\pi$ converges to ψ as $|\pi| \rightarrow 0$.

Theorem 5.4.3 (Uniqueness of Lipschitz flows). *If ψ is a Lipschitz flow and χ is a flow (not necessarily Lipschitz a priori) in the same galaxy as ψ , that is $\chi \sim \psi$, then $\chi = \psi$.*

Hypothesis 2. Let us fix a partition π . We consider ψ and χ in $\mathcal{F}(\mathbb{V})$ such that $\psi \sim \chi$ and for any $(r, s, t) \in \pi_+^3$,

$$\|\psi_{t, s}\|_{\text{Lip}} \leq 1 + \delta_T, \quad (5.37)$$

$$N(\chi_{t, s}(a)) \leq (1 + \delta_T)N(a), \quad \forall a \in \mathbb{V}, \quad (5.38)$$

$$\Delta_N(\psi_{t, s} \circ \psi_{s, r}, \psi_{t, r}) \leq \beta_\psi \varpi(\omega_{r, t}) \text{ and } \Delta_N(\chi_{t, s} \circ \chi_{s, r}, \chi_{t, r}) \leq \beta_\chi \varpi(\omega_{r, t}) \quad (5.39)$$

for some constant $\beta_\chi, \beta_\psi \geq 0$.

Remark 5.4.4. In Hypothesis 2, the role of ψ and χ are not exchangeable: ψ is assumed to be Lipschitz, there is no such requirement on χ . The reason of this dissymmetry lies in (5.42).

Remark 5.4.5. If ψ is a Lipschitz almost flow and χ is an almost flow, then (ψ, χ) satisfies Hypothesis 2 for any partition π . The condition (5.38) is a particular case of (5.15).

We choose λ and T so that

$$\frac{1}{1 - \log_2(\varkappa)} < \lambda < 1 \text{ and } 3\delta_T + \delta_T^3 < 2\frac{1 - \varkappa_\lambda}{\varkappa_\lambda}.$$

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We define (recall that $\Theta(\pi)$ is given by (5.36)),

$$\begin{aligned}\rho_T &:= \varpi(\omega_{0,T})^{1-\lambda}, \\ \gamma(\pi) &:= \sup_{d_\pi(s,s')=1} \frac{\Delta_N(\psi, \chi)}{\varpi^\lambda(\omega_{s,s'})} \leq \Delta_{N,\varpi^\lambda}(\psi, \chi)\Theta(\pi), \\ \beta(\pi) &:= (2 + 3\delta_T + \delta_T^2)(\beta_\psi + \beta_\chi)\rho_T + (1 + \delta_T)^2\gamma(\pi) \geq \gamma(\pi), \\ \text{and } L(\pi) &:= \frac{2\beta(\pi)}{2 - \varkappa_\lambda(2 + 3\delta_T + \delta_T^2)} \geq \gamma(\pi).\end{aligned}$$

Here, $\Theta(\pi)$ and thus $\gamma(\pi)$ converge to zero when the mesh of π tends to zero.

Lemma 5.4.6. *Let $\phi, \chi \in \mathcal{F}(V)$ and π be satisfying Hypothesis 2. With the above choice of λ and T , it holds that*

$$d(\phi_{t,r}(a), \chi_{t,r}(a)) \leq L(\pi \cup \{t, r\})N(a)\varpi^\lambda(\omega_{r,t}), \quad \forall (r, t) \in \mathbb{T}_+^2. \quad (5.40)$$

Proof. We set $F_{r,t} := \Delta_N(\psi_{t,r}, \chi_{t,r})$, where Δ_N is defined in Notation 3. From Definition 1, $F_{r,t} \leq \Delta_{N,\varpi}(\psi, \chi)\varpi(\omega_{r,t}) < +\infty$ since $\psi \sim \chi$.

In particular, for $(r, s, t) \in \pi_+^3$, with (5.38) in Hypothesis 2,

$$d(\psi_{t,s} \circ \chi_{s,r}(a), \chi_{t,s} \circ \chi_{s,r}(a)) \leq F_{s,t}N(\chi_{s,r}(a)) \leq (1 + \delta_T)N(a)F_{s,t}. \quad (5.41)$$

For any $(r, s, t) \in \pi_+^3$, the fact that ϕ, χ are almost flow combined with (5.37)-(5.39) and (5.41) imply that for any $a \in V$,

$$\begin{aligned}d(\psi_{t,r}(a), \chi_{t,r}(a)) \\ \leq d(\psi_{t,s} \circ \psi_{s,r}(a), \psi_{t,s} \circ \chi_{s,r}(a)) + d(\psi_{t,s} \circ \chi_{s,r}(a), \chi_{t,s} \circ \chi_{s,r}(a)) + (\beta_\phi + \beta_\chi)N(a)\varpi(\omega_{r,t}) \\ \leq (1 + \delta_T)N(a)F_{r,s} + (1 + \delta_T)N(a)F_{s,t} + (\beta_\chi + \beta_\phi)N(a)\varpi(\omega_{r,t}).\end{aligned} \quad (5.42)$$

Thus, dividing by $N(a)$,

$$F_{r,t} \leq (1 + \delta_T)(F_{r,s} + F_{s,t}) + (\beta_\chi + \beta_\psi)\rho_T\varpi^\lambda(\omega_{r,t}). \quad (5.43)$$

We proceed by induction. Our hypothesis is that

$$F_{r,t} \leq L(\pi)\varpi^\lambda(\omega_{r,t}), \quad \forall (r, t) \in \pi_+^2, \text{ at distance at most } m. \quad (5.44)$$

For $m = 0$, $F_{r,r} = 0$, so (5.16) holds.

When $m = 1$, r and t are successive points. From the very definition of $\gamma(\pi)$,

$$F_{r,t} \leq \gamma(\pi)\varpi^\lambda(\omega_{r,t}). \quad (5.45)$$

The induction hypothesis (5.16) is true for $m = 1$ since $L(\pi) \geq \gamma(\pi)$.

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Assume that the induction hypothesis is true at some level $m \geq 1$. Let $(r, s, t) \in \pi_3^+$ with $r \leq s < t$ and $d_\pi(r, t) = m + 1$. Let s' be such that s and s' are successive points in π (possibly, $s' = t$). Clearly, $d_\pi(r, s) \leq m$ and $d_\pi(s', t) \leq m$. Using (5.43) to decompose $F_{s,t}$ using s' and using (5.45),

$$\begin{aligned} F_{r,t} &\leq (1 + \delta_T)F_{r,s} + (1 + \delta_T)^2 F_{s',t} + (1 + \delta_T)^2 \gamma(\pi) \varpi^\lambda(\omega_{s,s'}) \\ &\quad + (2 + 3\delta_T + \delta_T^2)(\beta_\psi + \beta_\chi) \rho_T \varpi^\lambda(\omega_{r,t}) \\ &\leq (1 + \delta_T)F_{r,s} + (1 + \delta_T)^2 F_{s',t} + \beta(\pi) \varpi^\lambda(\omega_{r,t}) \end{aligned}$$

With the induction hypothesis, since r and s (resp. s' and t) are at distance at most m ,

$$F_{r,t} \leq L(\pi)(1 + \delta_T) \varpi^\lambda(\omega_{r,s}) + L(\pi)(1 + \delta_T)^2 \varpi^\lambda(\omega_{s',t}) + \beta(\pi) \varpi^\lambda(\omega_{r,t}).$$

Choosing s and s' to satisfy (5.21), our choice of $L(\pi)$ and (5.5) imply that

$$F_{r,t} \leq \left(L(\pi) \frac{2 + 3\delta_T + \delta_T^2}{2} \varkappa_\lambda + \beta(\pi) \right) \varpi^\lambda(\omega_{r,t}) \leq L(\pi) \varpi^\lambda(\omega_{r,t}).$$

The induction hypothesis (5.44) is then true at level $m + 1$, and then whatever the distance between the points of the partition.

Finally, (5.40) is obtained by replacing π by $\pi \cup \{r, t\}$. ■

Proof Theorem 5.4.1. Let σ and π be two partitions with $\pi \subset \sigma$. We set $\psi := \phi^\sigma$ and $\chi := \phi^\pi$.

With Theorem 5.2.2,

$$\Delta_{N,\varpi}(\phi^\sigma, \phi^\pi) \leq \Delta_{N,\varpi}(\phi^\sigma, \phi) + \Delta_{N,\varpi}(\phi^\pi, \phi) \leq 2L_T.$$

With (5.35), $\Delta_{N,\varpi^\lambda}(\psi, \chi) \leq 2L_T \rho_T$, so that $\{\Delta_{N,\varpi^\lambda}(\psi, \chi)\}_{\pi,\sigma}$ is bounded.

Again with Theorem 5.2.2, (ψ, χ) satisfies Hypothesis 2 for the subdivision π (up to changing δ_T with $\beta_\psi = \beta_\chi = 0$).

Hence, $L(\pi) = C\gamma(\pi)$ where

$$C := \frac{2(1 + \delta_T)^2}{(2 - \varkappa_\lambda(2 + 3\delta_T + \delta_T^2))}.$$

We may then rewrite (5.40) as

$$d(\phi_{t,r}^\sigma(a), \phi_{t,r}^\pi(a)) \leq C\gamma(\pi \cup \{r, t\})N(a)\varpi^\lambda(\omega_{r,t}). \quad (5.46)$$

Since $\gamma(\pi)$ decreases to 0 as $|\pi|$ decreases to 0 and $|\pi \cup \{r, t\}| \leq |\pi|$, it is easily shown that $\{\phi_{t,s}^\pi\}_\pi$ forms a Cauchy net with respect to the nested partitions. Then, it does converges to a limit $\zeta_{s,t}(a)$. By Theorem 5.4.6 and the continuity of N , $N(\zeta_{s,r}) \leq K_T N(a)$. From the UL condition, $a \mapsto \zeta_{t,s}(a)$ is Lipschitz continuous with $\|\zeta_{t,s}\|_{\text{Lip}} \leq 1 + \delta_T$.

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Moreover ζ does not depend on the subdivision π . Indeed, if $\tilde{\pi}$ is another subdivision, we obtain with (5.46), that $\{\phi^{\tilde{\pi}}\}_{\tilde{\pi}}$ converges to ζ when $|\tilde{\pi}| \rightarrow 0$.

Finally, if $\{\pi_k\}_{k \geq 0}$ is a family of nested partitions, and $(r, s, t) \in \mathbb{T}_3^+$,

$$\phi_{t,r}^{\pi_k \cup \{s\}} = \phi_{t,s}^{\pi_k} \circ \phi_{s,r}^{\pi_k}.$$

Because $|\pi_k \cup \{s\}| \leq |\pi_k|$ and (5.46), $\phi^{\pi_k \cup \{s\}}$ converges to ζ when $k \rightarrow +\infty$. Moreover, for any $a \in V$,

$$\begin{aligned} & d(\zeta_{t,s} \circ \zeta_{s,r}(a), \phi_{t,s}^{\pi_k} \circ \phi_{s,r}^{\pi_k}(a)) \\ & \leq d(\zeta_{t,s} \circ \zeta_{s,r}(a), \phi_{t,s}^{\pi_k} \circ \zeta_{s,r}(a)) + d(\phi_{t,s}^{\pi_k} \circ \zeta_{s,r}(a), \phi_{t,s}^{\pi_k} \circ \phi_{s,r}^{\pi_k}(a)) \\ & \leq C\gamma(\pi_k \cup \{r, t\})N(\zeta_{s,r}(a))\varpi^\lambda(\omega_{r,t}) + (1 + \delta_T)d(\zeta_{s,r}(a), \phi_{s,r}^{\pi_k}(a)) \\ & \leq C\gamma(\pi_k \cup \{r, t\})(1 + K_T)N(a)\varpi(\omega_{r,t}), \end{aligned}$$

because $N(\zeta_{s,r}(a)) \leq K_T N(a)$. So, $\{\phi_{t,s}^{\pi_k} \circ \phi_{s,r}^{\pi_k}\}_{\pi_k}$ converge uniformly to $\zeta_{t,s} \circ \zeta_{s,r}$ when $m \rightarrow +\infty$. Then, the flow property $\zeta_{t,s} \circ \zeta_{s,r} = \zeta_{t,r}$ holds. ■

Proof Theorem 5.4.2. For a partition π , the pair (ψ, ϕ^π) satisfies Hypothesis 2 for the subdivision π with $\beta_\psi = \beta_\chi = 0$. As in the proof of Theorem 5.4.1 (we have assumed for convenience that $\Delta_{N,\varpi}(\phi, \psi) \leq L_T$),

$$\Delta_{N,\varpi^\lambda}(\psi, \phi^\pi) \leq C\gamma(\pi) \leq 2CL_T\rho_T\Theta(\pi)$$

for C given by (5.4). This proves the result. ■

Proof Theorem 5.4.3. For any partition π , ψ and χ satisfy Hypothesis 2 with $\beta_\psi = \beta_\chi = 0$. Thus,

$$\Delta_{N,\varpi^\lambda}(\psi, \chi) \leq C\gamma(\pi)$$

with C given by (5.4). As $\gamma(\pi)$ decreases to 0 when $|\pi|$ decreases to 0, we obtain that $\psi = \chi$. ■

Corollary 5.4.7. *Let ψ and χ be two almost flows with $\psi \sim \chi$ and ψ be Lipschitz. Then for T small enough (in function of some $\lambda < 1$, \varkappa and δ)*

$$\Delta_{N,\varpi}(\psi, \chi) \leq \frac{2(2 + 3\delta_T + \delta_T^2)(\beta_\psi + \beta_\chi)}{2 - \varkappa_\lambda(2 + 3\delta_T + \delta_T^2)}.$$

Proof. With Remark 5.4.5, (ψ, χ) satisfies Hypothesis 2. Letting the mesh of the partition decreasing to 0 as in the proof of Theorem 5.4.3, and then letting λ increasing to 1 leads to the result. ■

5.5. PERTURBATIONS

5.5 Perturbations

In this section, we consider the construction of almost flow by perturbations of existing ones. We assume that V is a Banach space.

Let $\phi \in \mathcal{F}(V)$ be an almost rough path with respect to a function N_γ such that $N_\gamma(a) \geq N_\gamma(0) \geq 1$.

Notation 6. For $\phi \in \mathcal{F}(V)$ when V is a Banach space, we write

$$\phi_{t,s,r}(a) = \phi_{t,s}(\phi_{s,r}(a)) - \phi_{t,r}(a).$$

Definition 11. Let $\epsilon \in \mathcal{F}(V)$ such that for any $(s, t) \in \mathbb{T}_2^+$, $a, b \in V$,

$$\epsilon_{t,t} \equiv 0, \tag{5.47}$$

$$|\epsilon_{t,s}(a)| \leq \lambda N_\gamma(a) \varpi(\omega_{s,t}), \tag{5.48}$$

$$|\epsilon_{t,s}(b) - \epsilon_{t,s}(a)| \leq \eta(\omega_{s,t}) |b - a|^\gamma \tag{5.49}$$

for some $\lambda \geq 0$. We say that ϵ is a *perturbation*.

Proposition 5.5.1. *If $\phi \in \mathcal{F}(V)$ is an almost flow and $\epsilon \in \mathcal{F}(V)$ is a perturbation, then $\psi := \phi + \epsilon$ is an almost flow. Besides, $\psi \sim \chi$.*

Proof. Let $(r, s, t) \in \mathbb{T}_3^+$ and $a, b \in V$. From (5.47), (5.10) is satisfied. With $\delta'_T := \delta_T + \lambda \varpi(\omega_{0,T})$, (5.11) is also true. In addition, with (5.49),

$$|\psi_{t,s}(b) - \psi_{t,s}(a)| \leq (1 + \delta_T) |b - a| + 2\eta(\omega_{s,t}) |b - a|^\gamma.$$

Thus, ψ satisfies (5.12).

To show (5.13), we write

$$\begin{aligned} \phi_{t,s,r}(a) &= \phi_{t,s} \circ \psi_{s,r}(a) + \epsilon_{t,s} \circ \psi_{s,r}(a) - \phi_{t,r}(a) - \epsilon_{t,s}(a) \\ &= \underbrace{\phi_{t,s,r}(a)}_{\text{I}_{r,s,t}} + \underbrace{\phi_{t,s} \circ (\phi_{s,r} + \epsilon_{s,r})(a) - \phi_{t,s} \circ \phi_{s,r}(a)}_{\text{II}_{r,s,t}} \\ &\quad + \underbrace{\epsilon_{t,s} \circ (\phi_{s,r}(a) + \epsilon_{s,r}(a)) - \epsilon_{t,s} \circ \phi_{s,r}(a)}_{\text{III}_{r,s,t}} + \underbrace{\epsilon_{t,s} \circ \phi_{s,r}(a) - \epsilon_{t,s}(a)}_{\text{IV}_{r,s,t}}. \end{aligned} \tag{5.50}$$

We control the first term with (5.13), $|\text{I}_{r,s,t}| \leq N_\gamma(a) \varpi(\omega_{r,t})$. For the second one, we use (5.6), (5.12) and (5.48),

$$\begin{aligned} |\text{II}_{r,s,t}| &\leq (1 + \delta_T) |\epsilon_{s,r}(a)| + \eta(\omega_{s,r}) |\epsilon_{s,r}|^\gamma \\ &\leq (1 + \delta_T) \lambda N_\gamma(a) \varpi(\omega_{r,s}) + \eta(\omega_{s,t}) \lambda^\gamma N_\gamma^\gamma(a) \varpi(\omega_{r,s})^\gamma \\ &\leq [1 + (1 + \lambda^\gamma) \delta_T] N_\gamma(a) \varpi(\omega_{r,t}), \end{aligned}$$

because $N_\gamma(a) \geq 1$ implies that $N_\gamma^\gamma(a) \leq N_\gamma(a)$ for $\gamma \in (0, 1)$.

With (5.48) and (5.47), we obtain for the third term,

$$|\text{III}_{r,s,t}| \leq \eta(\omega_{s,t}) |\epsilon_{s,r}(a)|^\gamma \leq \lambda^\gamma N_\gamma(a)^\gamma \eta(\omega_{s,t}) \varpi(\omega_{r,s})^\gamma \leq \lambda^\gamma N_\gamma(a) \delta_T \varpi(\omega_{s,t}),$$

where the last line comes from (5.6). And for the last term, we use (5.15) and (5.48),

$$\begin{aligned} |\text{IV}_{r,s,t}| &\leq |\epsilon_{t,s} \circ \phi_{s,r}(a)| + |\epsilon_{t,s}(a)| \leq (N_\gamma(\phi_{s,r}) + N_\gamma(a)) \lambda \varpi(\omega_{s,t}) \\ &\leq (K_T + 1) N_\gamma(a) \lambda \varpi(\omega_{s,t}). \end{aligned}$$

Thus, combining estimations for each four terms of (5.50), we obtain (5.13) which proves that ψ is an almost flow.

Besides,

$$|\psi_{t,s}(a) - \phi_{t,s}(a)| = |\epsilon_{t,s}(a)| \leq \lambda N_\gamma(a) \varpi(\omega_{s,t}),$$

which proves that $\psi \sim \phi$ and concludes the proof. \blacksquare

5.6 Applications

On this section, we show that our framework cover former different sewing lemmas.

5.6.1 The additive sewing lemma

The additive sewing lemma is the key to construct integrals the young integral [You36] and the rough integral [Lyo98], [Gub04].

We consider that V is a Banach space with a norm $|\cdot|$. The distance d is $d(a, b) = |b - a|$.

Definition 12 (Almost additive functional). A family $\{\alpha_{s,t}\}_{(s,t) \in \mathbb{T}_2^+}$ is an *almost additive functional* if

$$\alpha_{r,s,t} := \alpha_{r,s} + \alpha_{s,t} - \alpha_{r,t} \text{ satisfies } |\alpha_{r,s,t}| \leq \varpi(\omega_{r,t}), \quad \forall (r, s, t) \in \mathbb{T}_3^+.$$

It is an *additive functional* if $\alpha_{r,s,t} = 0$ for any $(r, s, t) \in \mathbb{T}_3^+$.

Proposition 5.6.1 (The additive sewing lemma [LQ02, FdLPM08]). *If $\{\alpha_{s,t}\}_{(s,t) \in \mathbb{T}_2^+}$ is an almost additive functional with $|\alpha_{s,t}| \leq \delta_T$, there exists an additive additive functional $\{\gamma_{s,t}\}_{(s,t) \in \mathbb{T}_2^+}$ which is unique in the sense that for any constant $C \geq 0$ and any additive functional $\{\beta_{s,t}\}_{(s,t) \in \mathbb{T}_2^+}$, $|\beta_{s,t} - \alpha_{s,t}| \leq C \varpi(\omega_{s,t})$ implies that $\beta = \gamma$.*

Proof. Clearly, $\phi_{t,s}(a) = a + \alpha_{s,t}$ is an almost flow which satisfies the UL condition. Hence the result. \blacksquare

5.6. APPLICATIONS

5.6.2 The multiplicative sewing lemma

Here we recover the results of [FdLPM08] and [CL14]. We consider now that the metric space V has a monoid structure. Hence, for any two elements $a, b \in V$, there exists a product which we write ab which belongs in V . We also assume that there exists a Lipschitz function $N : V \rightarrow [1, +\infty)$ such that

$$d(ac, bc) \leq N(c)d(a, b) \text{ and } d(ca, cb) \leq N(c)d(a, b) \text{ for all } a, b, c \in V.$$

Definition 13. A family $\{\alpha_{s,t}\}_{(s,t) \in \mathbb{T}_2^+}$ is said to be an *almost multiplicative functional* if

$$d(\alpha_{r,s}\alpha_{s,t}, \alpha_{r,t}) \leq \varpi(\omega_{r,t}), \quad \forall (r, s, t) \in \mathbb{T}_+^3.$$

It is a *multiplicative functional* if $\alpha_{r,s}\alpha_{s,t} = \alpha_{r,t}$.

Proposition 5.6.2 (The multiplicative sewing lemma [FdLPM08]). *If $\{\alpha_{s,t}\}_{(s,t) \in \mathbb{T}_+^2}$ is an almost multiplicative functional then there exists a unique multiplicative functional $\{\gamma_{s,t}\}_{(s,t) \in \mathbb{T}_+^2}$ such that any other multiplicative functional $\{\beta_{s,t}\}_{(s,t) \in \mathbb{T}_+^2}$ such that $d(\beta_{s,t}, \alpha_{s,t}) \leq C\varpi(\omega_{s,t})$ for any $(s, t) \in \mathbb{T}_+^2$ satisfies $\beta = \gamma$.*

Proof. For this, it is sufficient to consider $\phi_{t,s}(a) = a\alpha_{s,t}$ which is an almost flow which satisfies the UL condition. ■

Remark 5.6.3. Actually, as for the additive sewing lemma (which is itself a subcase of the multiplicative sewing lemma), we have a stronger statement: No (non-linear) flow satisfies $d(\psi_{t,s}(a), a\alpha_{s,t}) \leq C\varpi(\omega_{s,t})$ except $\{a \mapsto a\beta_{s,t}\}_{(s,t) \in \mathbb{T}_+^2}$ which is constructed as the limit of the products of the $\alpha_{s,t}$ over smaller and smaller intervals.

5.6.3 The multiplicative sewing lemma in a Banach algebra

Consider now that V has a Banach algebra structure with a norm $|\cdot|$ such that $|ab| \leq |a| \times |b|$ and a unit element 1 (the product of two elements is still denoted by ab).

A typical example is the Banach algebra of bounded operators over a Banach space X .

This situation fits in the multiplicative sewing lemma with $d(a, b) = |a - b|$ and $N(a) = |a|$, $a, b \in V$. As seen in [CL14], we have many more properties: continuity, existence of an inverse, Dyson formula, Duhamel principle, ...

In particular, this framework is well suited for considering linear differential equations of type

$$y_{t,s} = a + \int_s^t y_{r,s} d\alpha_r, \quad a \in V, \quad t \geq s \geq 0$$

for an operator valued path $\alpha : \mathbb{T}_2^+ \rightarrow V$. If α is γ -Hölder with $\gamma > 1/2$, then $\phi_{t,s}(a) = a(1 + \alpha_{s,t})$ defines an almost flow which satisfies the condition UL (at the price of imposing some conditions on α , this could be extended to $\gamma < 1/2$).

Defining an “affine flow” $\phi_{t,s}(a) = a(1 + \alpha_{s,t}) + \beta_{s,t}$ where both α and β are γ -Hölder with $\gamma > 1/2$, the associated flow ψ is such that $\psi_{t,s}(a)$ is solution to the perturbed equation

$$\psi_{t,s}(a) = a + \int_s^t \psi_{r,s}(a) d\alpha_r + \beta_{s,t}.$$

This gives an alternative construction to the one of [CL14] where a *backward integral* between β and α was defined in the style of the Duhamel formula. All these results are extended to the rough case $1/3 < \gamma \leq 1/2$.

Example 1 (Lyons extension theorem). With the tensor product \otimes as product and a suitable norm, for any integer k , the tensor algebra

$$\mathbb{T}_k(X) := \mathbb{R} \oplus X \oplus X^{\otimes 2} \oplus \dots \oplus X^{\otimes k}$$

is a Banach algebra. Chen series of iterated integrals (and then rough paths) take their values in some space $\mathbb{T}_k(X)$. The Lyons extension theorem states that any rough path \mathbf{x} of finite p -variation with values in $\mathbb{T}_k(X)$ for some $k \geq [p]$ is uniquely extended to a rough path with values in $\mathbb{T}_\ell(X)$ for any $\ell \geq k$, which leads to the concept of *signature* [Lyo98, LQ02]. This follows $a \mapsto a \otimes \mathbf{x}_{s,t}$ as an almost flow which satisfies the UL condition (see also [FdLPM08] and also [CL14]).

5.6.4 Rough differential equation

Now, we show that our construction is related to the one of A.M. Davie [Dav07]. The main idea is to define an almost flow as the truncated Taylor expansion of the solution of (5.1). The number of term in the Taylor expansion depends directly on the regularity of x . In the Young case one term is needed whereas in the rough case two terms are required.

Here U and V are two Banach spaces, where we use the same notation $|\cdot|$ to denote their norms which can differ. We denote by $\mathcal{L}(U, V)$ the continuous linear maps from U to V . Let f be a map from V to $\mathcal{L}(U, V)$. If f is regular, we denote its Fréchet derivative in $a \in V$, $df(a) \in \mathcal{L}(V, \mathcal{L}(U, V))$.

Moreover, for any $a \in W$ and $(r, s, t) \in \mathbb{T}_3^+$, we set $\phi_{t,s,r}(a) := \phi_{t,s} \circ \phi_{s,r}(a) - \phi_{t,r}(a)$.

Almost flow in the Young case

Let $x : \mathbb{T} \rightarrow U$ be a path of finite p -variation controlled by ω with $1 \leq p < 2$.

We define a family $(\phi_{t,s})_{(s,t) \in \mathbb{T}_2^+}$ in $\mathcal{F}(V)$ such that for all $a \in V$ and $(s, t) \in \mathbb{T}_2^+$,

$$\phi_{t,s}(a) := a + f(a)x_{s,t}, \tag{5.51}$$

where $x_{s,t} := x_t - x_s$.

5.6. APPLICATIONS

Proposition 5.6.4. *Assume that $f \in \mathcal{C}^\gamma(\mathbb{V}, \mathcal{L}(U, V))$, with $1 + \gamma > p$. Then ϕ is an almost flow.*

Proof. We check that assumptions of Definition 2 hold. Let (r, s, t) be in \mathbb{T}_3^+ and let a, b be in \mathbb{V} . First, $\phi_{t,t}(a) = a$ because $x_{t,t} = 0$. Second,

$$|\phi_{t,s}(a) - a| \leq |f(a)| \cdot |x_{s,t}| \leq |f(a)| \cdot \|x\|_p \omega_{s,t}^{1/p},$$

which proves (5.11). Third,

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq |a - b| + |f(a) - f(b)| |x_{s,t}| \leq |a - b| + \|f\|_\gamma \|x\|_p \omega_{s,t}^{1/p} |a - b|^\gamma,$$

which proves (5.12). It remains to prove (5.13). We compute,

$$\phi_{t,s,r}(a) = f(\phi_{s,r}(a))x_{s,t} - f(a)x_{s,t},$$

and we obtain the following estimation

$$\begin{aligned} |\phi_{t,s,r}(a)| &\leq \|f\|_\gamma \|x\|_p \omega_{s,t}^{1/p} |\phi_{s,r}(a) - a|^\gamma \leq \|f\|_\gamma |f(a)|^\gamma \|x\|_p^2 \omega_{r,t}^{2(1+\gamma)/p} \\ &\leq \|f\|_\gamma (1 + |f(a)|) \|x\|_p^2 \omega_{r,t}^{2(1+\gamma)/p}. \end{aligned}$$

Setting $\varpi(\omega_{r,t}) := \omega_{r,t}^{(1+\gamma)/p}$, $\eta(\omega_{s,t}) := \|f\|_\gamma \|x\|_p \omega_{s,t}^{1/p}$ and

$$N_\gamma(a) := (1 + |f(a)|) (\|x\|_p + \|f\|_\gamma \|x\|_p^2),$$

it proves that ϕ is an almost flow.

This concludes the proof. ■

Let ψ be a flow in the same galaxy as the almost flow ϕ . For any $a \in \mathbb{V}$ and any $(r, t) \in \mathbb{T}_+^2$, we set

$$y_t(r, a) := \psi_{t,r}(a) \text{ so that } y_r(r, a) = a.$$

Clearly, $(r, a) \mapsto (t \in [r, T] \mapsto y_t(r, a))$ is a family of continuous paths which satisfies

$$|y_t(r, a) - \phi_{t,r}(y_s(r, a))| \leq CN_\gamma(y_s(r, a)) \omega_{s,t}^{2/p}, \quad \forall (s, t) \in \mathbb{T}_+^2, \quad \forall a \in \mathbb{V}$$

since $y_t(r, a) = \psi_{t,s}(y_s(r, a))$. Besides, $s \in [r, T] \mapsto N_\gamma(y_s(r, a))$ is bounded. Therefore, with our choice of the almost flow ϕ , $\phi_{\cdot,r}(a) = y(\cdot, a)$ is a solution in the sense defined by A.M. Davie [Dav07] for the Young differential equation $z_t = a + \int_r^t f(z_s) dx_s$. Even if several solutions may exist for a given (r, a) , the flow corresponds to a particular choice of a family of solutions which is constructed thanks to a selection principle. This family of solution is stable under splicing (see Definition 7).

Corollary 5.6.5. *We assume that \mathbb{V} is a finite-dimensional vector space and $f \in \mathcal{C}_b^1(\mathbb{V}, \mathcal{L}(U, V))$. Then there exists a flow $\psi \in \mathcal{F}(\mathbb{V})$ in the same galaxy as ϕ such that $\psi_{t,s}$ is Borel measurable for any $(s, t) \in \mathbb{T}_+^2$.*

Remark 5.6.6. Under our condition on f , A.M. Davie shown that several solutions to the Young differential equation $y = a + \int_0^\cdot f(y_s(a)) dx_s$ may exist (Example 1 in [Dav07]). Uniqueness requires f to be $(1 + \gamma)$ -Hölder continuous with $1 + \gamma > p$.

Proof. According to Proposition 5.6.7, ϕ is an almost flow. Here $\gamma = 1$, so ϕ is Lipschitz. Then, we conclude the proof in applying Theorem 5.3.4 to ϕ . \blacksquare

Almost flow in the rough case

When the regularity of x is weaker than in the Young case, we need more terms in the Taylor expansion to obtain an almost flow.

Let $T_2(U) := \mathbb{R} \oplus U \oplus (U \otimes U)$ be the truncated tensor algebra (with addition $+$ and tensor product \otimes). A distance is defined on the subset of elements of $T_2(U)$ on the form $a = 1 + a^1 + a^2$ with $a^i \in U^{\otimes i}$ by $d(a, b) = |a^{-1} \otimes b|$ where $|\cdot|$ is a norm on $T_2(U)$ such that $|a \otimes b| \leq |a| \cdot |b|$ for any $a, b \in U$.

Let $\mathbf{x} = (1, \mathbf{x}^1, \mathbf{x}^2)$ be a rough path with values in $T_2(U)$ of finite p -variation, $2 \leq p < 3$, controlled by ω (See *e.g.*, [LQ02, FH14] for a complete definition).

We define a family $(\phi_{t,s})_{(s,t) \in \mathbb{T}_2^+}$ in $\mathcal{F}(V)$ such that for all $a \in V$ and $(s, t) \in \mathbb{T}_2^+$,

$$\phi_{t,s}(a) := a + f(a)\mathbf{x}_{s,t}^1 + df(a) \cdot f(a)\mathbf{x}_{s,t}^2. \quad (5.52)$$

Proposition 5.6.7. *Assume that $f \in \mathcal{C}_b^{1+\gamma}(V, \mathcal{L}(U, V))$, with $2 + \gamma > p$. Then ϕ is an almost flow.*

Proof. We check that the assumptions of Definition 2 hold. The proofs of (5.10), (5.11) and (5.12) are very similar to the ones in the proof of Proposition 5.6.4. The computation to show (5.13) is a bit more involved.

Indeed, for any $a \in V$, $(r, s, t) \in \mathbb{T}_3^+$,

$$\begin{aligned} \phi_{t,s,r}(a) &= -f(a)\mathbf{x}_{s,t}^1 + f(\phi_{s,r}(a))\mathbf{x}_{s,t}^1 - df(a) \cdot f(a)(\mathbf{x}_{s,t}^2 + \mathbf{x}_{r,s}^1 \otimes \mathbf{x}_{s,t}^1) \\ &\quad + df(\phi_{s,r}) \cdot f(\phi_{s,r}(a))\mathbf{x}_{s,t}^2 \\ &= [f(\phi_{s,t}(a)) - f(a) - df(a) \cdot f(a)\mathbf{x}_{r,s}^1] \otimes \mathbf{x}_{s,t}^1 \\ &\quad + [df(\phi_{s,r}(a)) \cdot f(\phi_{s,r}(a)) - df(a) \cdot f(a)]\mathbf{x}_{s,t}^2 \\ &= \underbrace{f(\phi_{s,t}(a)) - f(a) - df(a) \cdot (\phi_{s,r}(a) - a)}_{\text{I}_{r,s,t}} + \underbrace{df(a) \cdot f(a)\mathbf{x}_{r,s}^2 \otimes \mathbf{x}_{s,t}^1}_{\text{II}_{r,s,t}} \\ &\quad + \underbrace{[df(\phi_{s,t}(a)) \cdot f(\phi_{s,r}(a)) - df(a) \cdot f(a)]\mathbf{x}_{s,t}^2}_{\text{III}_{r,s,t}}. \end{aligned}$$

5.6. APPLICATIONS

For the first term,

$$\begin{aligned} |\mathbf{I}_{r,s,t}| &\leq \|df\|_\gamma \|\mathbf{x}^1\|_p \omega_{s,t}^{1/p} |\phi_{s,r}(a) - a|^{1+\gamma} \\ &\leq \|df\|_\gamma \|\mathbf{x}^1\|_p \omega_{s,t}^{1/p} [\|f\|_\infty \|\mathbf{x}^1\|_p + \|df \cdot f\|_\infty \|\mathbf{x}^2\|_{\frac{p}{2}} \omega_{0,T}^{1/p}]^{1+\gamma} \omega_{r,s}^{(1+\gamma)/p}. \end{aligned}$$

For the two last terms,

$$|\mathbf{II}_{r,s,t}| \leq \|df \cdot f\|_\infty \|\mathbf{x}^1\|_p \|\mathbf{x}^2\|_{\frac{p}{2}} \omega_{r,t}^{3/p} \leq \|df \cdot f\|_\infty \|\mathbf{x}^1\|_p \|\mathbf{x}^2\|_{\frac{p}{2}} \omega_{0,T}^{(1-\gamma)/p} \omega_{r,t}^{(2+\gamma)/p}$$

and

$$|\mathbf{III}_{r,s,t}| \leq \|\mathbf{x}^2\|_{\frac{p}{2}} \omega_{r,t}^{2/p} [\|df\|_\gamma \|f\|_\infty |\phi_{s,r}(a) - a|^\gamma + \|df\|_\infty \|f\|_{\text{Lip}} |\phi_{s,r}(a) - a|] \leq C \omega_{r,t}^{3/p},$$

where C is a constant which depends on f , df , ω , γ , \mathbf{x} . It proves that ϕ is a Lipschitz almost flow.

This concludes the proof. ■

As for the Young case, any flow ψ in the same galaxy as the almost flow ϕ given by (5.52) gives rise to a family of solutions in the sense of A.M. Davie [Dav07] to the Rough Differential Equation $z_t = a + \int_0^t f(z_s) d\mathbf{x}_s$.

Corollary 5.6.8. *We assume that V is a finite-dimensional vector space and $f \in \mathcal{C}_b^2(V, \mathcal{L}(U, V))$. Then there exists a flow $\psi \in \mathcal{F}(V)$ in the same galaxy as ϕ such that $\psi_{t,s}$ is Borel measurable for any $(s, t) \in \mathbb{T}_+^2$.*

Remark 5.6.9. Under our condition on f , A.M. Davie shown that several solutions to the rough differential equation $y = a + \int_0^\cdot f(y_s(a)) d\mathbf{x}_s$ may exist (Example 2 in [Dav07]). Uniqueness requires f to be $(2 + \gamma)$ -Hölder continuous with $2 + \gamma > p$.

Proof. According to Proposition 5.6.7, ϕ is an almost flow. Here $\gamma = 1$, so ϕ is Lipschitz. Then, we conclude the proof in applying Theorem 5.3.4 to ϕ . ■

CHAPTER 5. NON LINEAR SEWING LEMMA I: WEAK FORM

Chapter 6

Non linear sewing lemma II: Lipschitz continuous formulation

This chapter consists of the preprint [BL17b] written in collaboration with Antoine Lejay and submitted for publication.

Abstract

We give an unified framework to solve rough differential equations. Based on flows, our approach unifies the former ones developed by Davie, Friz-Victoir and Bailleul. The main idea is to build a flow from the iterated product of an almost flow which can be viewed as a good approximation of the solution at small time. In this second article, we give some tractable conditions under which the limit flow is Lipschitz continuous and its links with uniqueness of solutions of rough differential equations. We also give perturbation formulas on almost flows which link the former constructions.

Keywords: Rough differential equations; Lipschitz flows; Rough paths

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6.1 Introduction

The rough path theory was introduced to deal with differential equations driven by an irregular deterministic path multidimensional x of the type

$$dy_t = a + \int_s^t f(y_r) dx_r, \quad (6.1)$$

where a is an initial condition and f a regular vector field. Typically, the irregularity of x is measured in α -Hölder ($\alpha \leq 1$) or in p -variation ($p \geq 1$) spaces. Such an equation is called *Rough Differential Equation (RDE)* [Lyo98, FH14].

This theory was very fruitful to study stochastic equations driven by Gaussian process which is not covered by the Itô framework, like the fractional Brownian motion [CQ02, Unt10]. More generally, the rough path framework allows one to separate the probabilistic from the deterministic part in such equation and to overcome some probabilistic conditions such as using adapted or non-anticipative processes.

Recently, the ideas of the rough path theory were extended to stochastic partial differential equations (SPDE) with the works of [Hai14, GIP15] which have led to significant progress in the study of some SPDE. This theory also found applications in machine learning and the recognizing of the Chinese ideograms [Lyo14, CK16].

Since the seminal article [Lyo98] by T. Lyons in 1998, several approaches emerged to solve (6.1). They are based on two main technical arguments: fixed point theorems [Lyo98, Gub04] and flow approximations [FdLPM08, CL14, Dav07, Bai15a, FV10]. In particular, the rough flow theory allows one to extend work about stochastic flows, which has been developed in '80s by Le Jan-Watanabe-Kunita and others, to a non-semimartingale setting [BD15].

The main goal of this article is to give a framework which unifies the approaches by flow and pursue further investigations on their properties and their relations with families of solutions to (6.1).

A *flow* is a family of maps $\{\psi_{t,s}\}$ from a Banach space to itself such that $\psi_{t,s} \circ \psi_{s,r} = \psi_{t,r}$ for any $r \leq s \leq t$. Typically, the map which associates the initial condition a to the solution of (6.1) has a flow property. The existence of a such flow heavily depends on the existence and uniqueness of the solution. However, it was proved in [CK17b, CK17a] and extended to the rough path case in [BL17a] that when non-uniqueness holds, it is possible to build a measurable flow by a selection technique. In this article we are interested by the construction of a Lipschitz flows.

The main idea to build the flow associated is to find a good approximation $\phi_{t,s}$ of $\psi_{t,s}$ when $|t - s|$ is small enough. We iterate this approximation on a subdivision $\pi = \{s \leq t_i \leq \dots \leq t_j \leq t\}$ of $[s, t]$ by setting

$$\phi_{t,s}^\pi := \phi_{t,t_j} \circ \dots \circ \phi_{t_i,s}.$$

6.1. INTRODUCTION

If ϕ^π does converges when the mesh of π goes to zero, ϕ^π , the limit is necessarily a flow. This computation is similar to the ones of numerical schemes as Euler's methods of different order [CMHM78]. Moreover, this idea is found among the Trotter's formulas for bounded or unbounded linear operators which allows to compute the semi-group of the sum of two non-commutative operators only knowing the semi-groups associated to each operator [EN00]. This property can be used to prove the Feynman-Kac formula.

Rather than working with a particular choice for the almost flow ϕ as in [Dav07, FV10], we give here generic conditions on ϕ . We generalize the multiplicative sewing lemma of [FdLPM08] and of [CL14], introduced to solve linear RDE to a non linear situation. In this way, we construct directly some flows. In opposite to the additive and multiplicative sewing lemma, the limit is not necessarily unique. The approximations are assumed to be Lipschitz. This is not the case for the limit.

Our framework is close to the one developed by I. Bailleul in [Bai15a, Bai14b] with a notable difference: in the previous article [BL17a], we have shown that a flow may exist under some weak conditions even if the iterated products ϕ^π are not uniformly Lipschitz continuous. Similarly to I. Bailleul, we have also shown that if ϕ^π is uniformly Lipschitz continuous for any partition π (UL Condition) then it converges to a Lipschitz flow, which is necessarily unique.

The present article gives a sufficient condition, called the *4-points control*, on the almost flow ϕ that ensures the UL Condition. We then study various consequences of this condition: existence of an inverse, unique family of solutions, convergence of the Euler scheme, ... The *4-points control* can be checked on the almost flow, which is then called a *stable almost flow*. This condition is weaker than the one given by I. Bailleul in [Bai15a, Bai14b]: there it should roughly be \mathcal{C}^1 with a Lipschitz continuous spatial derivative while in our case, the spatial derivative may be only Hölder continuous. The question of the existence of a Lipschitz flow without the UL condition, in relation with Stochastic Differential Equations, should be dealt with in a subsequent work.

We also study the relationship between almost flows and family of solutions to (6.1) in the sense of Davie [Dav07] as they are two different objects. In particular, we show that when an almost flow is stable, when the family of solutions to the RDE is unique and Lipschitz continuous. We also relate the distance between two families of solutions with respect to the distance between two almost flows when one is stable. Again, consequences of this result will be drawn in a subsequent work.

We also give several conditions under which perturbations of almost flows, a convenient tool to construct numerical schemes, converge to the same limit flow. These perturbative arguments are the key to unify expansions that are *a priori* of different nature.

Finally, we apply our framework to recover the results of A.M. Davie [Dav07], P. Friz & N. Victoir [FV10, FV08] and I. Bailleul [Bai15a, Bai14b] using various perturbation arguments. Although not done here, our framework could be applied to deal with branched rough paths, that are high-order expansions indexed by trees, which are studied in [CW16] and shown to fit the Bailleul's framework [Bai18].

Outline. After introducing in Section 6.2 the main notations and general definitions, we recall in Section 6.3 the notion of almost flow which is introduced in our previous article [BL17a]. In Section 6.4, we define the 4-point control as well as stable almost flow ϕ . We prove that under these conditions, ϕ^π converges to a Lipschitz flow. In Section 6.5, we give conditions to modify the almost flow ϕ by adding a perturbation ϵ while retains the convergence to a flow. We prove that under suitable conditions, the inverse the approximation ϕ and that ϕ^{-1} is a good approximation of the inverse of the flow. The link between the uniqueness of the solution of (6.1) and the existence of a flow is studied in Section 6.7. In Section 6.8, our formalism links the former approaches based on flow [Dav07, FV10, Bai15a].

6.2 Notations

The following notations and hypotheses will be constantly used throughout all this article.

6.2.1 Controls and remainders

Let us fix $T > 0$, a time horizon. We write $\mathbb{T} := [0, T]$ as well as

$$\begin{aligned} \mathbb{T}_+^2 &:= \{(s, t) \in \mathbb{T}^2 \mid s \leq t\} \text{ and } \mathbb{T}_+^3 := \{(r, s, t) \in \mathbb{T}^3 \mid r \leq s \leq t\}, \\ \mathbb{T}_-^2 &:= \{(s, t) \in \mathbb{T}^2 \mid s \geq t\} \text{ and } \mathbb{T}_-^3 := \{(r, s, t) \in \mathbb{T}^3 \mid r \geq s \geq t\}. \end{aligned}$$

We also set $\mathbb{T}^3 = \mathbb{T}_+^3 \cup \mathbb{T}_-^3$.

A *control* ω is a family from $\mathbb{T}_+^2 := \{0 \leq s \leq t \leq T\}$ to \mathbb{R}_+ which is *super-additive*, that is

$$\omega_{r,s} + \omega_{s,t} \leq \omega_{r,t}, \quad \forall (r, s, t) \in \mathbb{T}_+^3,$$

and continuous close to its diagonal with $\omega_{s,s} = 0$, $s \in \mathbb{T}$. For example $\omega_{s,t} = C|t - s|$ where C is a non-negative constant.

A *remainder* is a continuous, increasing function $\varpi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for some $0 < \varkappa < 1$,

$$2\varpi\left(\frac{\delta}{2}\right) \leq \varkappa\varpi(\delta), \quad \delta > 0. \quad (6.2)$$

A typical example for ϖ is $\varpi(\delta) = \delta^\theta$ for any $\theta > 1$.

Let $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be non-decreasing function with $\lim_{T \rightarrow 0} \delta_T = 0$.

We fix $\gamma \in (0, 1]$. We also consider a continuous, increasing function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\eta(\omega_{s,t})\varpi(\omega_{s,t})^\gamma \leq \delta_T\varpi(\omega_{s,t}), \quad \forall (s, t) \in \mathbb{T}_+^2. \quad (6.3)$$

6.2. NOTATIONS

6.2.2 Functions spaces

We denote by $(V, |\cdot|)$ a Banach spaces. The space of continuous functions from V to V is denoted by $\mathcal{C}(V)$. We set $\|x\|_\infty := \sup_{t \in [0, T]} |x_t|$.

Notation 7. We denote by $\mathcal{F}^+(V)$ the space of families $\{\phi_{t,s}\}_{(s,t) \in \mathbb{T}_+^2}$ with $\phi_{t,s} \in \mathcal{C}(V)$ for each $(s, t) \in \mathbb{T}_+^2$. We also set $\mathcal{F}^-(V)$ the space of families $\{\phi_{s,t}\}_{(s,t) \in \mathbb{T}_+^2}$ with $\phi_{s,t} \in \mathcal{C}(V)$ for each $(s, t) \in \mathbb{T}_+^2$ (note the reversion of the indices).

We now consider a partition $\pi = \{t_0 \leq \dots \leq t_n\}$ of $[0, T]$ with a mesh denoted by $|\pi|$.

Notation 8 (Iterated products). For $\phi \in \mathcal{F}^+(V)$, we write

$$\phi_{t,s}^\pi := \phi_{t,t_j} \circ \phi_{t_j,t_{j-1}} \circ \dots \circ \phi_{t_{i+1},t_i} \circ \phi_{t_i,s},$$

where $[t_i, t_j]$ is the biggest interval of such kind contained in $[s, t]$. We say that $\phi_{t,s}^\pi$ is the *iterated product* of ϕ on a subdivision π . If no such interval exists, then $\phi_{t,s}^\pi = \phi_{t,s}$.

For $\phi \in \mathcal{F}^-(V)$, we define similarly

$$\phi_{s,t}^\pi := \phi_{s,t_1} \circ \phi_{t_1,t_2} \circ \dots \circ \phi_{t_{j-1},t_j} \circ \phi_{t_j,t}.$$

For any partition π , $\phi^\pi \in \mathcal{F}^\pm(V)$ when $\phi \in \mathcal{F}^\pm(V)$. A trivial but important remark is that from the very construction,

$$\phi_{t,s}^\pi = \phi_{t,r}^\pi \circ \phi_{r,s}^\pi \text{ for any } r \in \pi.$$

In particular, $\{\phi_{t,s}^\pi\}_{(s,t) \in \mathbb{T}_\pm^2, s,t \in \pi}$ enjoys a (semi-)flow property (Definition 16). A natural question is then to study the limit of ϕ^π as the mesh of π decreases to 0.

Finally, for any $(r, s, t) \in \mathbb{T}_\pm^3$ we write $\phi_{t,s,r} := \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$.

Notation 9. We extend the norm $|\cdot|$ on $\mathcal{F}^\pm(V)$ by

$$\|\phi\|_\varpi := \sup_{\substack{(s,t) \in \mathbb{T}_\pm^2 \\ s \neq t}} \frac{\|\phi_{t,s}\|_\infty}{\varpi(\omega_{s,t})},$$

where ω, ϖ are defined in Section 6.2. Possibly, $\|\phi\|_\varpi = \infty$. Actually, this norm is mainly used to consider the distance between two elements of $\mathcal{F}^\pm(V)$. With this norm, $(\mathcal{F}^\pm(V), \|\cdot\|_\varpi)$ is a Banach space.

Definition 14. We define the equivalence relation \sim on $\mathcal{F}^\pm(V)$ by $\phi \sim \psi$ if and only if there exists a constant C such that

$$\|\phi_{t,s} - \psi_{t,s}\|_\infty \leq C \varpi(\omega_{s,t}), \quad \forall (s, t) \in \mathbb{T}^2.$$

In other words, $\phi \sim \psi$ if and only if $\|\phi - \psi\|_\varpi < +\infty$. Each quotient class of $\mathcal{F}^\pm(V)/\sim$ is called a *galaxy*, which contains elements of $\mathcal{F}^\pm(V)$ which are at finite distance from each others.

Notation 10 (Lipschitz semi-norm). The Lipschitz semi-norm of a function f from a Banach space $(V, |\cdot|)$ to another Banach space $(W, |\cdot|')$ is

$$\|f\|_{\text{Lip}} := \sup_{\substack{a, b \in V, \\ a \neq b}} \frac{|f(a) - f(b)|'}{|a - b|},$$

whenever this quantity is finite. And if $A \subset V$ is a non-empty subset of V , we say that f is Lipschitz continuous on A when

$$\|f\|_{\text{Lip}, A} := \sup_{\substack{a, b \in A, \\ a \neq b}} \frac{|f(a) - f(b)|'}{|a - b|} < +\infty.$$

Notation 11 (Hölder spaces). For $\gamma \in (0, 1)$ and an integer r , we denote by $\mathcal{C}_b^{r+\gamma}(V)$ the space of bounded continuous functions from V to V with bounded derivatives up to order r and a r order derivative which is γ -Hölder continuous.

6.3 Almost flow and Uniform Lipschitz condition

In this section, we recall some notions and results introduced in [BL17a], which are useful in next sections. As we are working on Banach spaces instead of metric spaces, we have a slightly stronger notion of almost flow than in [BL17a].

We denote by Id the identity map from V to V .

Definition 15 (Almost flow). An element $\phi \in \mathcal{F}^+(V)$ is an *almost flow* if for any $T > 0$ and any $(r, s, t) \in \mathbb{T}_+^3$, $a, b \in V$,

$$\phi_{t,t} = \text{Id}, \tag{6.4}$$

$$\|\phi_{t,s} - \text{Id}\|_\infty \leq \delta_T, \tag{6.5}$$

$$|\phi_{t,s}(b) - \phi_{t,s}(a)| \leq (1 + \delta_T)|b - a| + \eta(\omega_{s,t})|b - a|^\gamma, \tag{6.6}$$

$$\|\phi_{t,s,r}\|_\infty \leq M\varpi(\omega_{r,t}), \tag{6.7}$$

where $M \geq 0$ and $\phi_{t,s,r} := \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$. If we replace $(r, s, t) \in \mathbb{T}_+^3$ by $(r, s, t) \in \mathbb{T}_-^3$, we say that ϕ is a *reverse almost flow*.

Definition 16 (Semi-flow and Flow). A *semi-flow* ψ is a family of functions $(\psi_{t,s})_{(s,t) \in \mathbb{T}_+^2}$ from V to V such that $\phi_{t,t} = \text{Id}$ and

$$\psi_{t,s} \circ \psi_{s,r} = \psi_{t,r} \tag{6.8}$$

for any $a \in V$ and $(r, s, t) \in \mathbb{T}_+^3$. It is a *flow* if each $\phi_{t,s}$ is invertible with an inverse $\phi_{s,t}$ for any $(s, t) \in \mathbb{T}_+^2$ and (6.8) holds for any $(r, s, t) \in \mathbb{T}^3 := \mathbb{T}_+^3 \cup \mathbb{T}_-^3$.

Remark 6.3.1. A flow is invertible and for any $(s, t) \in \mathbb{T}^2$, then $\psi_{t,s}^{-1} = \psi_{s,t}$. Indeed, $\psi_{t,s} \circ \psi_{s,t} = \psi_{t,t} = \text{Id}$ and $\psi_{s,t} \circ \psi_{t,s} = \psi_{s,s} = \text{Id}$.

6.3. ALMOST FLOW AND UNIFORM LIPSCHITZ CONDITION

Theorem 6.3.2 ([BL17a]). *Let ϕ be an almost flow (Definition 15) with $M \geq 0$ and δ_T, κ defined in Section 6.2.1. Then there exists a time horizon T small enough and a constant $L \leq 2M/(1 - (1 + \delta_T)\kappa - \delta_T)$ such that*

$$\|\phi_{t,s}^\pi - \phi_{t,s}\|_\infty \leq L\varpi(\omega_{s,t}) \quad (6.9)$$

for any $(s, t) \in \mathbb{T}_+^2$ and any partition π of \mathbb{T} .

Definition 17 (Condition UL). An almost flow ϕ such that $\|\phi_{s,t}^\pi\|_{\text{Lip}} \leq 1 + \delta_T$ for any $(s, t) \in \mathbb{T}_+^2$ whatever the partition π is said to satisfy the *uniform Lipschitz* (UL) condition.

We give a sufficient condition on an almost flow to get a Lipschitz flow in a galaxy.

Proposition 6.3.3. *Let ϕ be an almost flow which satisfies the condition UL. Then there exists a Lipschitz flow ψ with $\|\psi_{t,s}\|_{\text{Lip}} \leq 1 + \delta_T$ for any $(s, t) \in \mathbb{T}_+^2$ such that $\phi_{t,s}^\pi(a)$ converges to $\psi_{t,s}(a)$ for any $a \in \mathbb{V}$ and any $(s, t) \in \mathbb{T}_+^2$.*

On the other hand, there could be at most one flow in a galaxy if one is Lipschitz.

Proposition 6.3.4. *Assume that there is a Lipschitz flow ψ in a galaxy G . Then ψ is the unique flow in G . Besides, for any almost flow $\phi \sim \psi$, $\phi_{s,t}^\pi(a)$ converges to $\psi_{s,t}(a)$ for any $(s, t) \in \mathbb{T}_+^2$ and $a \in \mathbb{V}$.*

We then complete the results of [BL17a] with the following ones.

Proposition 6.3.5. *Let ϕ be an almost flow which satisfies the condition UL. Then ϕ^π is an almost flow for any partition π .*

The prototypical example for the next result is $\varpi(\delta) = \delta^\theta$ for some $\theta > 1$. In this case, it slightly improves the rate of convergence as $\theta - 1$ with respect to the one given in [BL17a] which is $\theta - 1 - \epsilon$ for any $\epsilon > 0$.

Proposition 6.3.6 (Rate of convergence). *Let ϕ be an almost flow in the same galaxy as a Lipschitz flow ψ with $\|\psi_{t,s}\|_{\text{Lip}} \leq (1 + \delta_T)$ and $\|\psi_{t,s} - \phi_{t,s}\|_\infty \leq K\varpi(\omega_{s,t})$ for any $(s, t) \in \mathbb{T}_+^2$. Let us assume that ϖ is such that for a bounded function μ , $\delta^{-1}\varpi(\delta) \leq \mu(\delta)$ for any $\delta > 0$. Then*

$$\|\psi_{t,s}(a) - \phi_{t,s}^\pi(a)\|_\infty \leq KM(\pi)\omega_{0,T}(1 + \delta_T) \text{ with } M(\pi) := \sup_{\substack{(r,t) \text{ successive} \\ \text{points in } \pi}} \mu(\omega_{r,t}).$$

Proof. The proof follows the one of Theorem 10.30 in [FV10, p. 238]. Let $\{t_i\}_{i=0,\dots,n}$ be the points of $\pi \cup \{s, t\}$ such that $t_0 = s$, $t_n = t$ and (t_i, t_{i+1}) are successive points in $\pi \cup \{s, t\}$. Set $z_k = \psi_{t,t_k}(\phi_{t_k,t}(a))$. Then

$$|\psi_{t,s}(a) - \phi_{t,s}^\pi(a)| = |z_n - z_0| \leq \sum_{i=0}^{n-1} |z_{i+1} - z_i|.$$

Since $\psi_{t,s}$ is Lipschitz and

$$|z_{i+1} - z_i| \leq (1 + \delta_T) |\psi_{t_{k+1}, t_k}(\phi_{t_{k+1}, t_k}(a)) - \psi_{t_{k+1}, t_k}(\phi_{t_{k+1}, t_k}(a))|.$$

The result follows easily. ■

6.4 Stable almost flows

In the previous section, we have recalled some results from [BL17a] which endow the importance of Lipschitz flows. However, the UL condition is not easy to verify. In this section, we give a sufficient condition on an almost flow ϕ to ensure that it satisfies the UL condition and then that its galaxy contains a unique flow which is Lipschitz.

6.4.1 The 4-points control

In this section V, V_1, V_2, V_3 are Banach spaces and we denote by $|\cdot|$ their norms.

Definition 18 (The 4-points control). A function $f : V_1 \rightarrow V_2$ is said to satisfy a *4-points control* if there exists a non-decreasing, continuous function $\hat{f} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a constant $\check{f} \geq 0$ such that

$$\begin{aligned} |f(a) - f(b) - f(c) + f(d)| \\ \leq \hat{f}(|a - b| \vee |c - d|) \times (|a - c| \vee |b - d|) + \check{f}|a - b - c + d| \end{aligned} \quad (6.10)$$

for any $(a, b, c, d) \in V_1$.

Any Lipschitz function f satisfies a 4-points control with $\hat{f} = 2\|f\|_{\text{Lip}}$ and $\check{f} = 0$. However, for our purpose, we need to consider later more restrictive conditions on \hat{f} and \check{f} .

Let us start with a simple example that is found in [Dav07].

Lemma 6.4.1. *Let $f \in \mathcal{C}^{1+\gamma}(V_1, V_2)$, $0 < \gamma \leq 1$, with a bounded derivative. Then f satisfies a 4-points control with*

$$\hat{f}(x) = 2\|\nabla f\|_{\gamma} x^{\gamma}, \quad x \geq 0, \quad \text{and} \quad \check{f} = \|\nabla f\|_{\infty}.$$

Proof. For any $a, b, c, d \in V_1$,

$$\begin{aligned} f(a) - f(b) - f(c) + f(d) \\ = (a - b) \int_0^1 \nabla f(au + (1 - u)b) du - (c - d) \int_0^1 \nabla f(cu + (1 - u)d) du \\ = (a - b) \int_0^1 [\nabla f(au + (1 - u)b) - \nabla f(cu + (1 - u)d)] du \\ + (a - b - c + d) \int_0^1 \nabla f(cu + (1 - u)d) du, \end{aligned}$$

6.4. STABLE ALMOST FLOWS

which yields to

$$|f(a) - f(b) - f(c) + f(d)| \leq |a - b|2\|\nabla f\|_\gamma(|a - b| \vee |c - d|)^\gamma + \|\nabla f\|_\infty|a - b - c + d|.$$

This concludes the proof. ■

Here are a few properties of functions satisfying a 4-points control.

Lemma 6.4.2. *Let f, g satisfying a 4-points control with g Lipschitz continuous.*

- (i) *The function f is locally Lipschitz continuous.*
- (ii) *If $f, g : V_1 \rightarrow V_2$, then for any $\lambda, \mu \in \mathbb{R}$, $\lambda f + \mu g$ satisfies a 4-points control.*
- (iii) *If $f : V_1 \rightarrow V_2$ and $g : V_3 \rightarrow V_1$, then $f \circ g : V_3 \rightarrow V_2$ satisfies a 4-points control.*
- (iv) *If $g : V \rightarrow V$ with $\|g\|_{\text{Lip}} < 1$, then $\text{Id} + g$ is invertible and $k := (\text{Id} + g)^{-1}$ is Lipschitz and satisfies a 4-points control with $\hat{k}(x) = \hat{g}(\|k\|_{\text{Lip}}x)\|k\|_{\text{Lip}}$ for $\|k\|_{\text{Lip}} \leq 1/(1 - \|g\|_{\text{Lip}})$, and $\check{k} = 1/(1 - \check{g})$.*

Proof. For showing (i), we choose $a = d$ and $b = c$ in (6.10) and then,

$$|f(a) - f(b)| \leq \left[\hat{f}(|a - b|) + \check{f} \right] |a - b|,$$

which proves that f is locally Lipschitz continuous.

Moreover, we can choose $\lambda \widehat{f} + \mu \widehat{g} = |\lambda| \hat{f} + |\mu| \hat{g}$ and $\lambda \check{f} + \mu \check{g} = |\lambda| \check{f} + |\mu| \check{g}$ to obtain a 4-points control on $\lambda f + \mu g$. This proves (ii).

To show (iii), we use the fact that g is Lipschitz according to (i). With $h = f \circ g$,

$$\begin{aligned} & |h(a) - h(b) - h(c) + h(d)| \\ & \leq \hat{f}(|g(a) - g(b)| \vee |g(c) - g(d)|) [|g(a) - g(c)| \vee |g(b) - g(d)|] \\ & \quad + \check{f}\hat{g}(|a - b| \vee |c - d|) [|a - c| \vee |b - d|] + \check{f}\check{g}|a - b - c + d| \\ & \leq \hat{f}(\|g\|_{\text{Lip}}|a - b| \vee |c - d|)\|g\|_{\text{Lip}}(|a - c| \vee |b - d|) \\ & \quad + \check{f}\hat{g}(|a - b| \vee |c - d|)[|a - c| \vee |b - d|] + \check{f}\check{g}|a - b - c + d|. \end{aligned}$$

This proves that h satisfies the 4-points control.

It remains to show (iv). From the Lipschitz inverse function theorem [AMR88, p. 124], $\text{Id} + g$ is invertible with an inverse k that satisfies $\|k\|_{\text{Lip}} \leq (1 - \|g\|_{\text{Lip}})^{-1}$. Besides,

$$\begin{aligned} |a - b - c + d + (g(a) - g(b) - g(c) + g(d))| & \geq (1 - \check{g})|a - b - c + d| \\ & \quad - \hat{g}(|a - b| \vee |c - d|) \times [|a - c| \vee |b - d|], \end{aligned}$$

which yields to

$$(1 - \check{g})|k(a) - k(b) - k(c) + k(d)| \leq |a - b - c + d| + \hat{g}(\|k\|_{\text{Lip}}|a - b| \vee |c - d|)\|k\|_{\text{Lip}} [|a - c| \vee |b - d|].$$

Hence, for $x \in \mathbb{R}_+$, $\hat{k} = \hat{g}(\|k\|_{\text{Lip}}x)\|k\|_{\text{Lip}}$ and $\check{k} = (1 - \check{g})^{-1}$. Therefore, k satisfies a 4-points control. ■

The reason for introducing the 4-points control lies in its good behavior with respect to composition. More precisely, if f satisfies a 4-points control while g and h are Lipschitz continuous and bounded,

$$\begin{aligned} \|f \circ g - f \circ h\|_{\text{Lip}} &\leq \hat{f}(\|g - h\|_{\infty})\|g\|_{\text{Lip}} \vee \|h\|_{\text{Lip}} + \check{f}\|g - h\|_{\text{Lip}}, \\ \|f \circ g - f \circ h\|_{\infty} &\leq (\hat{f}(0) + \check{f})\|g - h\|_{\infty}. \end{aligned}$$

6.4.2 Definition of a stable almost flow

Definition 19. A family $\phi \in \mathcal{F}^+(V)$ is said to satisfy a ϖ -compatible 4-points control if there exists a family of functions $(\hat{\phi}_{t,s})_{(s,t) \in \mathbb{T}_+^2}$ and constants $(\check{\phi}_{t,s})_{(s,t) \in \mathbb{T}^2}$ such that for any $[s, t] \subset [0, T]$ the estimation (6.10) holds. Moreover, for any $[s, t] \subset [u, v]$, $(\hat{\phi}_{t,s})_{(s,t) \in \mathbb{T}_+^2}$ is said ϖ -compatible if

$$\begin{aligned} \hat{\phi}_{s,t} &\leq \hat{\phi}_{u,v}, \\ \hat{\phi}_{t,s}(\alpha\varpi(\omega_{s,t})) &\leq \phi^{\otimes}(\alpha)\varpi(\omega_{s,t}), \end{aligned}$$

where $\alpha, \phi^{\otimes}(\alpha)$ are non-negative constants with $\phi^{\otimes}(\alpha)$ which can depend on α .

Definition 20 (Stable almost flow). We say that an almost flow ϕ with $\gamma = 1$ in (6.6) is a *stable almost flow* if

- it satisfies a ϖ -compatible 4-points control with

$$\check{\phi}_{t,s} \leq 1 + \delta_T,$$

- there a constant $C \geq 0$ such that for $(r, s, t) \in \mathbb{T}_+^3$,

$$\|\phi_{t,s,r}\|_{\text{Lip}} \leq C\varpi(\omega_{r,t}), \tag{6.11}$$

where $\phi_{t,s,r} = \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$.

We denote the family of stable almost flow $\mathcal{SA}_{\delta_T, \varpi}(V)$. If we replace assumption $(r, s, t) \in \mathbb{T}_+^3$ by $(r, s, t) \in \mathbb{T}_-^3$, we say that ϕ is a *reverse stable almost flow*.

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6.4.3 Non linear sewing lemma for stable almost flow

The following proposition justifies Definition 20.

Theorem 6.4.3. *If $\phi \in \mathcal{SA}_{\delta_T, \varpi}(\mathbb{V})$ is a stable almost flow then for any partition π ,*

$$\|\phi_{t,s}^\pi - \phi_{t,s}\|_{\text{Lip}} \leq L\varpi(\omega_{s,t}), \forall (s,t) \in \mathbb{T}_+^2, \quad (6.12)$$

where L is a constant that depends on T , $T \mapsto \delta_T$, \varkappa , ω , $\check{\phi}$, ϕ^\otimes and C in (6.11). In particular, the almost flow ϕ satisfies the condition UL , up to changing δ_T .

Remark 6.4.4. When ϕ is a stable almost flow, we assume that $\gamma = 1$ in Definition 15. This implies that (6.6) becomes

$$|\phi_{t,s}(b) - \phi_{t,s}(a)| \leq (1 + \delta_T)|b - a|. \quad (6.13)$$

Proof. Let us choose a partition π . Let $r \in \mathbb{T}$ be fixed and $(s,t) \in \mathbb{T}_+^2$, such that $r \leq s$,

$$U_{s,t}^\pi := \|\phi_{t,r}^\pi - \phi_{t,s} \circ \phi_{s,r}^\pi\|_{\text{Lip}}.$$

For any $0 \leq r \leq s \leq t \leq u \leq T$,

$$U_{s,u}^\pi \leq U_{t,u}^\pi + \|\phi_{u,t} \circ \phi_{t,r}^\pi - \phi_{u,t} \circ \phi_{t,s} \circ \phi_{s,r}^\pi\|_{\text{Lip}} + \|\phi_{u,t} \circ \phi_{t,s} \circ \phi_{s,r}^\pi - \phi_{u,s} \circ \phi_{s,r}^\pi\|_{\text{Lip}}. \quad (6.14)$$

With the 4-points control on $\phi_{u,t}$,

$$\begin{aligned} & \|\phi_{u,t} \circ \phi_{t,r}^\pi - \phi_{u,t} \circ \phi_{t,s} \circ \phi_{s,r}^\pi\|_{\text{Lip}} \\ & \leq \widehat{\phi}_{u,t} (\|\phi_{t,r}^\pi - \phi_{t,s} \circ \phi_{s,r}^\pi\|_\infty) \times (\|\phi_{t,r}^\pi\|_{\text{Lip}} \vee (1 + \delta_T)\|\phi_{s,r}^\pi\|_{\text{Lip}}) + \check{\phi}_{u,t} U_{s,t}^\pi. \end{aligned}$$

According to (6.9) (Theorem 6.3.2) for T small enough,

$$\|\phi_{t,r}^\pi - \phi_{t,s} \circ \phi_{s,r}^\pi\|_\infty \leq C\varpi(\omega_{s,t}).$$

Since $\widehat{\phi}_{u,t}$ is ϖ -compatible and with the control (6.6) with $\gamma = 1$ of the Definition 15 of almost flow,

$$\begin{aligned} & \|\phi_{u,t} \circ \phi_{t,r}^\pi - \phi_{u,t} \circ \phi_{t,s} \circ \phi_{s,r}^\pi\|_{\text{Lip}} \\ & \leq \phi^\otimes(C)\varpi(\omega_{r,t}) (\|\phi_{t,r}^\pi\|_{\text{Lip}} \vee (1 + \delta_T)\|\phi_{s,r}^\pi\|_{\text{Lip}}) + \check{\phi}_{u,t} U_{s,t}^\pi. \end{aligned}$$

For bounding the last term of (6.14), (6.11) yields

$$\|\phi_{u,t} \circ \phi_{t,s} \circ \phi_{s,r}^\pi - \phi_{u,s} \circ \phi_{s,r}^\pi\|_{\text{Lip}} \leq C\varpi(\omega_{s,u})\|\phi_{s,r}^\pi\|_{\text{Lip}}.$$

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Assuming that r, s, t and u belong to π and combining these inequalities and the fact the ϕ is stable (see Definition 20),

$$U_{s,u}^\pi \leq U_{t,u}^\pi + (1 + \delta_T)U_{s,t}^\pi + L^\pi(\phi^{\otimes}(C)(2 + \delta_T) + C)\varpi(\omega_{s,u})$$

where

$$L^\pi := \sup_{\substack{(s,t) \in \mathbb{T}_+^2 \\ s,t \in \pi}} \|\phi_{t,s}^\pi\|_{\text{Lip}}.$$

For two successive points s and $t \geq s$ of π (See Definition 26), $\phi_{t,r}^\pi = \phi_{t,s} \circ \phi_{s,r}^\pi$ so that $U_{s,t}^\pi = 0$.

We assume that T is small enough so that $\varkappa(1 + \delta_T^2 + \delta_T) < 1$. From the Davie Lemma (Lemma 6.A.1 in Appendix),

$$U_{s,t}^\pi \leq L^\pi \alpha_T \varpi(\omega_{s,t}) \text{ with } \alpha_T := (2 + \delta_T) \frac{(\phi^{\otimes}(C)(2 + \delta_T) + C)}{1 - \varkappa(1 + \delta_T^2 + \delta_T)}. \quad (6.15)$$

In particular, for $r = s$, $U_{r,t} = \|\phi_{t,r}^\pi - \phi_{t,r}\|_{\text{Lip}}$.

Let us bound L^π . For this, with (6.15) and (6.6) with $\gamma = 1$,

$$\begin{aligned} L^\pi &\leq \sup_{(s,t) \in \mathbb{T}_+^2} \|\phi_{t,s}^\pi - \phi_{t,s}\|_{\text{Lip}} + \max_{(s,t) \in \mathbb{T}_+^2} \|\phi_{t,s}\|_{\text{Lip}} \\ &\leq L^\pi \alpha_T \varpi(\omega_{0,T}) + 1 + \delta_T. \end{aligned}$$

For T small enough so that $\alpha_T \varpi(\omega_{0,T}) \leq 1/2$, L^π is uniformly bounded. Injecting this control of L^π in (6.15),

$$\|\phi_{t,r}^\pi - \phi_{t,r}\|_{\text{Lip}} \leq K \varpi(\omega_{r,t}), \quad \forall (r, t) \in \mathbb{T}_+^2, \quad r, t \in \pi, \quad (6.16)$$

for some constant K that does not depend on the partition π .

It remains to establish (6.16) for any pair of time $(r, t) \in \mathbb{T}_+^2$.

For this, let t_π (resp. r_π) be the greatest (resp. smallest) point of π below (resp. above) t (resp. r). Then with the definition of ϕ^π (Notation 8),

$$\phi_{t,r}^\pi - \phi_{t,r} = \phi_{t,t_\pi} \circ \phi_{t_\pi,r}^\pi - \phi_{t,t_\pi} \circ \phi_{t_\pi,r} + \phi_{t,t_\pi,r}.$$

With (6.11) and (6.6) of Definition 15 with $\gamma = 1$,

$$\|\phi_{t,r}^\pi - \phi_{t,r}\|_{\text{Lip}} \leq (1 + \delta_T) \|\phi_{t_\pi,r}^\pi - \phi_{t_\pi,r}\|_{\text{Lip}} + C \varpi(\omega_{r,t}). \quad (6.17)$$

Similarly,

$$\phi_{t_\pi,r}^\pi - \phi_{t_\pi,r} = \phi_{t_\pi,r_\pi}^\pi \circ \phi_{r_\pi,r}^\pi - \phi_{t_\pi,r_\pi} \circ \phi_{r_\pi,r} + \phi_{t_\pi,r_\pi,r}.$$

Using (6.9) and (6.16),

$$\begin{aligned} \|\phi_{t_\pi,r}^\pi - \phi_{t_\pi,r}\|_{\text{Lip}} &\leq \|\phi_{t_\pi,r_\pi}^\pi - \phi_{t_\pi,r_\pi}\|_{\text{Lip}} \cdot \|\phi_{r_\pi,r}\|_{\text{Lip}} + C \varpi(\omega_{r,t}) \\ &\leq (1 + \delta_T) K \varpi(\omega_{r,t}) + C \varpi(\omega_{r,t}). \end{aligned} \quad (6.18)$$

Inequality (6.12), which is (6.16) applied for any $(r, t) \in \mathbb{T}_+^2$, stems from (6.16), (6.17) and (6.18). ■

6.5. PERTURBATIONS

Corollary 6.4.5. *If $\phi \in \mathcal{SA}_{\delta_T, \varpi}(\mathbb{V})$ is a stable almost flow then there exists a unique Lipschitz flow ψ in the galaxy containing ϕ . Moreover, there is a constant $L \geq 0$ such that for all $(s, t) \in \mathbb{V}$,*

$$\|\psi_{t,s} - \phi_{t,s}\|_{\text{Lip}} \leq L\varpi(\omega_{s,t}). \quad (6.19)$$

Proof. According to Theorem 6.4.3, ϕ satisfies the UL condition of Proposition 17. Hence, it converges in the sup-norm to a Lipschitz flow $\psi \sim \phi$. According to Proposition 6.3.4, ψ is the only flow in the galaxy of ϕ .

Passing to the limit in (6.12) leads to (6.19). ■

6.5 Perturbations

In [BL17a], we have introduced the notion of perturbation of almost flow. This notion still gives an almost flow.

We recall that η , δ_T and γ are defined in Section 6.2.1.

Definition 21 (Perturbation). *A perturbation is an element $\epsilon \in \mathcal{F}(\mathbb{V})$ such that for any $(s, t) \in \mathbb{T}_+^2$ and $a, b \in \mathbb{V}$,*

$$\epsilon_{t,t} = 0, \quad (6.20)$$

$$\|\epsilon_{t,s}\|_{\infty} \leq C\varpi(\omega_{s,t}), \quad (6.21)$$

$$|\epsilon_{t,s}(b) - \epsilon_{t,s}(a)| \leq \delta_T|b - a| + \eta(\omega_{s,t})|b - a|^\gamma, \quad (6.22)$$

where η is defined by (6.3) and $C \geq 0$ is a constant.

Proposition 6.5.1 ([BL17a, Proposition 1]). *If $\phi \in \mathcal{F}(\mathbb{V})$ is an almost flow and $\epsilon \in \mathcal{F}(\mathbb{V})$ is a perturbation, then $\psi := \phi + \epsilon$ is an almost flow in the same galaxy as ϕ .*

We introduce now the notion of Lipschitz perturbation, which is a perturbation on which a control stronger than (6.22) holds.

Definition 22 (Lipschitz perturbation). *A Lipschitz perturbation is a perturbation $\epsilon \in \mathcal{F}^+(\mathbb{V})$ with satisfies for a constant $C \geq 0$*

$$\|\epsilon_{t,s}\|_{\text{Lip}} \leq C\varpi(\omega_{s,t}), \quad \forall (s, t) \in \mathbb{T}_+^2. \quad (6.23)$$

Stable almost flows remain stable almost flows under Lipschitz perturbations.

Proposition 6.5.2 (Stability of stable almost flow under Lipschitz perturbation). *If $\phi \in \mathcal{SA}_{\delta_T, \varpi}$ is a stable almost flow (see Definition 20) and ϵ is a Lipschitz perturbation, then $\psi := \phi + \epsilon$ is also a stable almost flow.*

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Proof. It is proved in Proposition 6.5.1 that $\phi + \epsilon$ is an almost flow. Here we show that $\phi + \epsilon$ is a stable almost flow.

First, for any $a, b, c, d \in V$,

$$|\epsilon_{t,s}(a) - \epsilon_{t,s}(b) - \epsilon_{t,s}(c) + \epsilon_{t,s}(d)| \leq 2C\varpi(\omega_{s,t})|a - c| \vee |b - d|,$$

so that $\epsilon_{t,s}$ satisfies a ϖ -compatible 4-points control (see Definition 19) with $\widehat{\epsilon}_{t,s} := 2C\varpi(\omega_{s,t})$ and $\check{\epsilon}_{t,s} := 0$. Thus, $\phi + \epsilon$ satisfies a ϖ -compatible 4-points control with $\widehat{\phi + \epsilon} = \widehat{\phi} + 2C\varpi(\omega_{s,t})$ and $\widetilde{\phi_{t,s} + \epsilon_{t,s}} = \widetilde{\phi_{t,s}} \leq 1 + \delta_T$.

It remains to show that for any $(r, s, t) \in \mathbb{T}_+^3$, $\|\psi_{t,s,r}\|_{\text{Lip}} \leq C\varpi(\omega_{r,t})$, with $\psi_{t,s,r} := \psi_{t,s} \circ \psi_{s,r} - \psi_{t,r}$. For any $a \in V$, we write

$$\begin{aligned} \psi_{t,s,r}(a) &= \underbrace{[\phi_{t,s} \circ (\phi_{s,r} + \epsilon_{s,r})(a) - \phi_{t,s} \circ \phi_{s,r}(a)]}_{\text{I}_{r,s,t}(a)} + \underbrace{[\epsilon_{t,s} \circ (\phi_{s,r} + \epsilon_{s,r})(a) - \epsilon_{t,s} \circ \epsilon_{s,r}(a)]}_{\text{II}_{r,s,t}(a)} \\ &\quad + \underbrace{\phi_{t,s,r}(a) + \epsilon_{t,s,r}(a)}_{\text{III}_{r,s,t}(a)}. \end{aligned}$$

On the one hand, using the ϖ -compatible 4-points control of $\phi_{t,s}$, (6.13), (6.21) and (6.23) we write,

$$\begin{aligned} \|\text{I}_{r,s,t}\|_{\text{Lip}} &\leq \widehat{\phi_{t,s}}(\|\epsilon_{s,r}\|_{\varpi})(\|\phi_{s,r}\|_{\text{Lip}} + \|\epsilon_{s,r}\|_{\text{Lip}}) + \widetilde{\phi_{t,s}}\|\epsilon_{s,r}\|_{\text{Lip}} \\ &\leq \widehat{\phi_{t,s}}(C\varpi(\omega_{r,t}))(1 + \delta_T + C\varpi(\omega_{0,T})) + (1 + \delta_T)C\varpi(\omega_{r,t}) \\ &\leq (C'(1 + \delta_T + C\varpi(\omega_{0,T})) + 1 + \delta_T)\varpi(\omega_{r,t}), \end{aligned}$$

where C' is a constant.

On the other hand, with (6.6), (6.23)

$$\begin{aligned} \|\text{II}_{r,s,t}\|_{\text{Lip}} &\leq \|\epsilon_{t,s}\|_{\text{Lip}}(\|\phi_{s,r}\|_{\text{Lip}} + \|\epsilon_{s,r}\|_{\text{Lip}}) + \|\epsilon_{t,s}\|_{\text{Lip}}\|\epsilon_{s,r}\|_{\text{Lip}} \\ &\leq C\varpi(\omega_{r,t})(1 + \delta_T + C\varpi(\omega_{0,T})) + C^2\varpi(\omega_{0,T})\varpi(\omega_{r,t}) \leq K_T\varpi(\omega_{r,t}), \end{aligned}$$

where $K_T \rightarrow 1$ when $T \rightarrow 0$.

Finally, with (6.11) and (6.23),

$$\|\text{III}_{r,s,t}\|_{\text{Lip}} \leq \|\phi_{t,s,r}\|_{\text{Lip}} + \|\epsilon_{t,r}\|_{\text{Lip}} + \|\epsilon_{t,s}\|_{\text{Lip}}\|\epsilon_{s,r}\|_{\text{Lip}} \leq (2C + C^2\varpi(\omega_{0,T}))\varpi(\omega_{r,t}).$$

This concludes the proof. ■

Now, we prove another perturbation formula which is useful in Subsection 6.8.3. We recall the δ_T, η and γ are defined in Section 6.2.1.

6.5. PERTURBATIONS

Proposition 6.5.3. *Let ψ be a flow which may be decomposed as*

$$\psi_{t,s}(a) = \phi_{t,s}(a) + \epsilon_{s,t}(a), \quad a \in V, \quad (s, t) \in \mathbb{T}^2$$

with for any $(s, t) \in \mathbb{T}_+^2$ and $a, b \in V$,

$$\phi_{t,t} = \text{Id}, \quad \epsilon_{t,t} = 0, \quad (6.24)$$

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq (1 + \delta_T)|a - b| + \eta(\omega_{s,t})|a - b|^\gamma, \quad (a, b) \in V, \quad (6.25)$$

$$\|\phi_{t,s} - \text{Id}\|_\infty \leq \delta_T, \quad (6.26)$$

$$\|\epsilon_{s,t}\|_\infty \leq M\varpi(\omega_{s,t}). \quad (6.27)$$

Then ϕ is an almost flow in the same galaxy as ψ . Besides, for any partition π of \mathbb{T} ,

$$\|\phi_{t,s}^\pi - \phi_{t,s}\|_\infty \leq L\varpi(\omega_{s,t}) \quad (6.28)$$

where $L \leq 2[(3 + \delta_T)M + \delta_T M^\gamma] / (1 - (1 + \delta_T)\kappa - \delta_T)$.

Proof. To show that ϕ is an almost flow, it is sufficient to consider (6.24)-(6.27) as well as controlling $\phi_{t,s,r}$. For $(r, s, t) \in \mathbb{T}_+^3$,

$$\psi_{t,s} \circ \psi_{s,r}(a) = \overbrace{\phi_{t,s}(\phi_{s,r}(a) + \epsilon_{s,r}(a)) - \phi_{t,s}(\phi_{s,r}(a))}^{I_{r,s,t}} + \phi_{t,s}(\phi_{s,r}(a)) + \epsilon_{t,s}(\psi_{s,r}(a)).$$

Since ψ is a flow, $\psi_{r,s,t} = 0$ and then

$$\phi_{t,s,r}(a) = I_{r,s,t} + \epsilon_{t,s}(\psi_{s,r}(a)) - \epsilon_{t,r}(a).$$

With (6.25),

$$|I_{r,s,t}| \leq (1 + \delta_T)M\varpi(\omega_{r,s}) + \eta(\omega_{s,t})M^\gamma\varpi(\omega_{r,s})^\gamma.$$

With (6.3),

$$|I_{r,s,t}| \leq A\varpi(\omega_{r,t}) \quad \text{with } A := (1 + \delta_T)M + \delta_T M^\gamma.$$

It follows that $\|\phi_{t,s,r}\|_\infty \leq (2M + A)\varpi(\omega_{r,t})$. This proves that ϕ is an almost flow following Definition 6.3. The control (6.28) follows from Theorem 6.3.2. \blacksquare

Corollary 6.5.4. *Assume that V is a finite-dimensional Banach space. Let $\{\psi^m\}_{m \in \mathbb{N}}$ be a family of flows with decomposition $\psi^m = \phi^m + \epsilon^m$ where (ϕ^m, ϵ^m) satisfy (6.24)-(6.27) uniformly in m . Assume moreover that*

$$\|\phi_{s,t}^m\|_\infty \leq \delta_{t-s}, \quad \forall (s, t) \in \mathbb{T}_+^2. \quad (6.29)$$

Then any possible limit ϕ of $\phi_{t,s}^m$ (at least one exists) satisfies (6.24)-(6.27) as well as (6.29) with the same constants.

Proof. With (6.29), Lemma 1 in [BL17a] can be applied uniformly. As (6.25) is also uniform in m , this proves that for any $R > 0$, $\{\phi^m(a)_{s,t}\}_{(s,t) \in \mathbb{T}_+^2, a \in \overline{B(0,R)}}$ is equi-continuous where $\overline{B(0,R)}$ is the closed ball of center 0 and some radius $R > 0$. The Ascoli-Arzelà shows that at least one limit of ϕ^m exists. Clearly, this limit satisfies the same properties as ϕ^m . \blacksquare

6.6 Inversion of the flow

In this section, we prove that our definition of stable almost (Definition 20) flow is stable respect to inversion.

Proposition 6.6.1. *Let $\phi \in \mathcal{ST}_{\delta_T, \varpi}$ be a stable almost flow and ψ the unique flow in the same galaxy as ϕ (Corollary 6.4.5). We assume that $\chi := \phi - \text{Id}$ satisfies a 4-point control such that*

$$\forall (s, t) \in \mathbb{T}_+^2, \widetilde{\chi}_{t,s} = \widetilde{\phi}_{t,s} - 1, \widehat{\chi}_{t,s} = \widehat{\phi}_{t,s}, \text{ and } \|\chi_{t,s}\|_{\text{Lip}} \leq \delta_T.$$

Then, ϕ is invertible and $(\zeta_{s,t})_{(s,t) \in \mathbb{T}_+^2} := (\phi_{t,s}^{-1})_{(s,t) \in \mathbb{T}_+^2}$ is a stable reverse almost flow which galaxy contains a unique flow which equal to ψ^{-1} .

Proof. According to item (iv) of Lemma 6.4.2 and because $\|\chi_{t,s}\|_{\text{Lip}} \leq \delta_T$ we know that for $T > 0$ such that $\delta_T < 1$, $\phi_{t,s}$ is invertible and that $\phi_{t,s}^{-1}$ satisfies a 4-points control with $\widetilde{\phi_{t,s}^{-1}} = 1/(1 - \widetilde{\chi}_{t,s})$ and $\widehat{\phi_{t,s}^{-1}}(x) = \widehat{\chi}_{t,s}(\|\phi_{t,s}\|_{\text{Lip}}x)\|\phi_{t,s}\|_{\text{Lip}}$ for any $x \in \mathbb{R}_+$. It follows that $\phi_{t,s}^{-1} \leq 1 + \delta'_T$, with $\delta'_T := \delta_T/(1 - \delta_T)$ and that $\phi_{t,s}$ satisfies a ϖ -compatible 4-points control.

Moreover, $\|\phi_{t,s}^{-1}\|_{\text{Lip}} \leq 1/(1 - \|\chi_{t,s}\|_{\text{Lip}})$ and we assume $\|\chi_{t,s}\|_{\text{Lip}} \leq \delta_T$. It follows that $\|\phi_{t,s}^{-1}\|_{\text{Lip}} \leq 1 + \delta'_T$ which proves that (6.6) holds for ϕ^{-1} . In substituting a by $\phi_{t,s}^{-1}(a)$ in (6.5) we show that (6.5) holds for ϕ^{-1} .

To prove that $(\zeta_{s,t})_{(s,t) \in \mathbb{T}_+^2} := (\phi_{t,s}^{-1})_{(s,t) \in \mathbb{T}_+^2}$ is a reverse stable almost flow, it remains to show that the conditions (6.7) and (6.11) hold for any $(r, s, t) \in \mathbb{T}_+^3$. Firstly, we compute with (6.7), since $\phi_{t,s} \circ \phi_{s,r}$ is one-to-one,

$$\begin{aligned} \|\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} \circ \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}\|_{\infty} &= \|\phi_{t,r}^{-1} \circ \phi_{t,r} - \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}\|_{\infty} \\ &\leq (1 + \delta'_T)\|\phi_{t,r} - \phi_{t,s} \circ \phi_{s,r}\|_{\infty} \leq M\varpi(\omega_{r,t}), \end{aligned}$$

which yields with to $\|\zeta_{r,s} \circ \zeta_{s,t} - \zeta_{r,t}\|_{\infty} \leq M\varpi(\omega_{r,t})$.

Secondly, for any $a, b \in \mathbb{V}$ and $(r, s, t) \in \mathbb{T}_+^3$, we set $a' := \phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(a)$, $b' := \phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(b)$, and

$$\begin{aligned} \Phi_{r,s,t} &:= (\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} - \phi_{t,r}^{-1}) \circ \phi_{t,s} \circ \phi_{s,r}(b') - (\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} - \phi_{t,r}^{-1}) \circ \phi_{t,s} \circ \phi_{s,r}(a') \\ &= \phi_{t,r}^{-1} \circ \phi_{t,r}(b') - \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}(b') - \phi_{t,r}^{-1} \circ \phi_{t,r}(a') + \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}(a'). \end{aligned}$$

We know that $\phi_{t,r}^{-1}$ satisfies a ϖ -compatible 4-points control and we use (6.11),

$$\begin{aligned} |\Phi_{r,s,t}| &\leq \widehat{\phi_{t,r}^{-1}}(\|\phi_{t,s,r}\|_{\infty}) [\|\phi_{t,r}\|_{\text{Lip}} \vee \|\phi_{t,s} \circ \phi_{s,r}\|_{\text{Lip}}] |b' - a'| + \widetilde{\phi_{t,r}^{-1}}\|\phi_{t,s,r}\|_{\text{Lip}} |b' - a'| \\ &\leq \phi^{-1, \otimes}(M)\varpi(\omega_{r,t})(1 + \delta_T)^2 |b' - a'| + (1 + \delta_T)C\varpi(\omega_{r,t})|b' - a'|. \end{aligned}$$

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Then substituting a' and b' by $\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(a)$ and $\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(a)$,

$$\begin{aligned} \|\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} - \phi_{t,r}^{-1}\|_{\text{Lip}} &\leq [\phi^{-1, \otimes}(M)\varpi(\omega_{r,t})(1 + \delta_T)^2 + (1 + \delta_T)C\varpi(\omega_{r,t})] \|\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}\|_{\text{Lip}} \\ &\leq [\phi^{-1, \otimes}(M)(1 + \delta_T)^2 + (1 + \delta_T)C] (1 + \delta_T')^2 \varpi(\omega_{r,t}). \end{aligned}$$

Hence ζ is a stable reverse almost flow. According to Corollary 6.4.5, ζ^π converges to a unique Lipschitz flow ζ^∞ in $\mathcal{F}(V)$. But, $\zeta_{s,t}^\pi = (\phi_{t,s}^\pi)^{-1}$, which yields to $\zeta_{s,t}^\pi \circ \phi_{t,s}^\pi = \text{Id}$ and passing to limit $\zeta_{s,t}^\infty \circ \psi_{t,s} = \text{Id}$. This concludes the proof. \blacksquare

6.7 Generalized solution to rough differential equations

Almost flows approximates of flows, similarly to numerical algorithms. In classical analysis, flows are strongly related to solutions of ordinary differential equations (ODE). Rough differential equations (RDE) were solve first using fixed point theorems on paths [Lyo98]. The technical difficulty with this approach is that the solution itself should be a rough path.

Later, A.M. Davie introduced in [Dav07] another notion of solution of RDE which no longer involves solutions as rough paths, but only as paths. We abstract here this approach in order to relate almost flows and paths.

Definition 23 (Generalized solution in the sense of Davie). Let ϕ be an almost flow. Let $a \in V$ and $r \in \mathbb{T}$. A V -valued path $\{y_{r \rightsquigarrow t}\}_{(r,t) \in \mathbb{T}^2}$ is said to be a *solution in the sense of Davie* if $y_{r \rightsquigarrow r} = a$ and there exists a constant C such that

$$|y_{r \rightsquigarrow t} - \phi_{t,s}(y_{r \rightsquigarrow s})| \leq C\varpi(\omega_{s,t}), \quad \forall r \leq s \leq t \leq T. \quad (6.30)$$

Definition 24 (Manifold of solutions). A family $\{y_{r \rightsquigarrow \cdot}(a)\}_{r \in \mathbb{T}, a \in V}$ of solutions satisfying (6.30) and $y_{r \rightsquigarrow r}(a) = a$ is called a *manifold of solutions*. We write $\dot{y} = \phi'(y)$ to denote the whole family of solutions.

Definition 25 (Lipschitz manifold of solutions). If $a \mapsto y_{r \rightsquigarrow \cdot}(a)$ is uniformly Lipschitz continuous from (V, d) to $(\mathcal{C}([r, T], V), \|\cdot\|_\infty)$, then we say that the manifold of solutions is *Lipschitz*.

Remark 6.7.1. When $\phi_{t,s} = \text{Id} + \chi_{t,s}$, then (6.30) may be written $|y_{s,t} - \chi_{t,s}(y_s)| \leq C\varpi(\omega_{s,t})$ with $y_{s,t} := y_{r \rightsquigarrow t} - y_{r \rightsquigarrow s}$. for $(r, s, t) \in \mathbb{T}_2^+$. This is the form used by A.M. Davie in [Dav07].

Flows and manifold of solutions are of closely related. Besides, a manifold of solutions is in relation with a whole galaxy. The proof of the next lemma is immediate so that we skip it.

Lemma 6.7.2. *A flow ψ generates a manifold of solutions to $\dot{y} = \psi'(y)$ through $y_{r \rightsquigarrow t}(a) := \psi_{t,r}(a)$, $(r, t) \in \mathbb{T}_2^+$, $a \in V$. Besides, y is also solution to $\dot{y} = \phi'(y)$ for any almost flow ϕ in the galaxy containing ψ .*

6.7.1 Existence of a flow from a family of solutions

In a first time, we show how to construct a flow from a suitable family of paths.

Proposition 6.7.3. *Consider an almost flow ϕ . Assume that there exists a family $\{y_{0 \rightsquigarrow t}(a)\}_{t \in \mathbb{T}, a \in V}$ of V -valued paths, continuous in time and Lipschitz continuous in space such that such that*

$$y_{0 \rightsquigarrow 0} = \text{Id}, \quad \|y_{0 \rightsquigarrow t} - \phi_{t,s}(y_{0 \rightsquigarrow s})\|_{\infty} \leq C\varpi(\omega_{s,t}), \quad \forall (s,t) \in \mathbb{T}_+^2,$$

$$\sup_{t \in \mathbb{T}} \{\|y_{0 \rightsquigarrow t} - \text{Id}\|_{\text{Lip}} + \|y_{0 \rightsquigarrow t} - \text{Id}\|_{\infty}\} \leq K_T \text{ where } K_T \xrightarrow{T \rightarrow 0} 0.$$

Then the family $\{y_{0 \rightsquigarrow t}(a)\}_{t \in \mathbb{T}, a \in V}$ may be extended to a manifold of solutions to $\dot{y} = \phi'(y)$. Besides, if $\psi_{s,t}(a) := y_{s \rightsquigarrow t}(a)$, then ψ is an invertible, Lipschitz flow in the same galaxy as ϕ .

Proof. The Lipschitz inverse mapping shows that $y_{0 \rightsquigarrow t}$ is invertible with a Lipschitz continuous inverse $y_{0 \rightsquigarrow t}^{-1}$ when $K_T < 1$ ([AMR88] p. 124).

Assuming that T is small enough, we define $\psi_{t,s}(a) := y_{0 \rightsquigarrow t} \circ y_{0 \rightsquigarrow s}^{-1}(a)$ for any $(s,t) \in \mathbb{T}^2$ and $a \in V$. Clearly, ψ is a flow which is invertible. Besides, for any $(s,t) \in \mathbb{T}_+^2$, $\psi_{s,t}$ is Lipschitz continuous since both $y_{0 \rightsquigarrow t}$ and $y_{0 \rightsquigarrow s}^{-1}$ are Lipschitz continuous.

It remains to prove that $\phi \sim \psi$. For $a \in V$, let us set $b := y_{0 \rightsquigarrow s}^{-1}(a)$. Thus,

$$|\psi_{t,s}(a) - \phi_{t,s}(a)| = |\psi_{t,s}(y_{0 \rightsquigarrow s}(b)) - \phi_{t,s}(y_{0 \rightsquigarrow s}(b))|$$

$$\leq |\phi_{t,s}(y_{0 \rightsquigarrow s}(b)) - y_{0 \rightsquigarrow t}(b)| \leq C\varpi(\omega_{s,t}).$$

Thus, $\psi \sim \phi$. ■

6.7.2 Uniqueness and continuity of a solution in the sense of Davie

A stable almost flow ϕ satisfies the condition UL (see Theorem 6.4.3), so that there exists a unique flow ψ in the same galaxy as ϕ . Furthermore, ψ is Lipschitz.

The flow ψ generates a manifold of solutions. We show that there exists only one such manifold with a Lipschitz continuity result. Note that in the following proposition, ζ is not assumed to be stable.

Proposition 6.7.4. *Let ϕ be a stable almost flow and ζ be an almost flow.*

Let y and z be two paths from $[0, T]$ to V such that

$$|y_t - \phi_{t,s}(y_s)| \leq K\varpi(\omega_{s,t}) \text{ and } |z_t - \zeta_{t,s}(z_s)| \leq K\varpi(\omega_{s,t}), \quad \forall (s,t) \in \mathbb{T}_+^2.$$

Let us write $\alpha_{t,s} := \zeta_{t,s} - \phi_{t,s}$ and $\alpha_{t,s,r} := \zeta_{t,s,r} - \phi_{t,s,r}$. Let $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ be such that for any $(r, s, t) \in \mathbb{T}_+^3$,

$$|\alpha_{t,s,r}(z_r)| \leq \epsilon_1\varpi(\omega_{r,t}), \quad \|\alpha_{s,t}\|_{\text{Lip}} \leq \epsilon_2 \text{ and } |\alpha_{s,t}(z_s)| \leq \epsilon_3.$$

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Then there exists a time T small enough and a constant C that depends only on ϕ , K , $T \mapsto \delta_T$, \varkappa and $\sup_{(r,s,t) \in \mathbb{T}^3} \|\phi_{t,s,r}\|_{\text{Lip}}/\varpi(\omega_{r,t})$ such that

$$\begin{aligned} |y_t - z_t| &\leq C(\epsilon_1 + \epsilon_2 + \epsilon_3 + |y_0 - z_0|) \\ \text{and } |y_t - \phi_{t,s}(y_s) - z_t + \zeta_{t,s}(z_s)| &\leq C(\epsilon_1 + \epsilon_2 + \epsilon_3 + |y_0 - z_0|)\varpi(\omega_{s,t}) \end{aligned}$$

for all $(s, t) \in \mathbb{T}^2$.

The following corollary is then immediate by applying $\phi = \zeta$ to Proposition 6.7.4.

Corollary 6.7.5. *If ϕ is a stable almost flow, there exists one and only one manifold of solutions to $\dot{y} = \phi'(y)$. Besides, this manifold of solutions is Lipschitz.*

Remark 6.7.6. As seen in Lemma 6.7.2, the notion of manifold of solution is associated to a galaxy. Hence, a galaxy with a stable almost flow is associated to a unique manifold of solutions (actually, we have not proved that if ϕ is a stable almost flow, then the associated flow is also stable).

Proof of Proposition 6.7.4. We define

$$V_{s,t} = |z_t - \zeta_{t,s}(z_s) - y_t + \phi_{t,s}(y_s)|, \quad \forall (s, t) \in \mathbb{T}^2.$$

Clearly, $V_{s,t} \leq 2K\varpi(\omega_{s,t})$.

For any $(r, s, t) \in \mathbb{T}^3$,

$$\begin{aligned} z_t - \zeta_{t,r}(z_r) - y_t + \phi_{t,r}(y_r) &= z_t - \zeta_{t,s}(z_s) - y_t + \phi_{t,s}(y_s) + \zeta_{t,s}(z_s) - \zeta_{t,s}(\zeta_{s,r}(z_r)) \\ &\quad - \phi_{t,s}(y_s) + \phi_{t,s}(\phi_{s,r}(y_r)) + \zeta_{t,s,r}(z_r) - \phi_{t,s,r}(y_r) \\ &= z_t - \zeta_{t,s}(z_s) - y_t + \phi_{t,s}(y_s) + \alpha_{t,s}(z_s) - \alpha_{t,s}(\zeta_{s,r}(z_r)) \\ &\quad + \phi_{t,s}(z_s) - \phi_{t,s}(\zeta_{s,r}(z_r)) - \phi_{t,s}(y_s) + \phi_{t,s}(\phi_{s,r}(y_r)) \\ &\quad + \alpha_{t,s,r}(z_r) + \phi_{t,s,r}(z_r) - \phi_{t,s,r}(y_r). \end{aligned}$$

Set $L := \sup_{(r,s,t) \in \mathbb{T}^3} \|\phi_{t,s,r}\|_{\text{Lip}}$. With the 4-points control of $\phi_{s,t}$,

$$\begin{aligned} V_{t,r} &\leq V_{t,s} + \check{\phi}_{t,s}V_{r,t} + L|z_r - y_r|\varpi(\omega_{r,t}) \\ &\quad + \hat{\phi}_{t,s}(|z_s - \zeta_{s,r}(z_r)| \vee |y_s - \phi_{s,r}(y_r)|) \cdot (|z_s - y_s| \vee |\zeta_{s,r}(z_r) - \phi_{s,r}(y_r)|) \\ &\quad + |\alpha_{t,s,r}(z_r)| + \|\alpha_{t,s}\|_{\text{Lip}}|z_s - \zeta_{s,r}(z_r)|. \end{aligned}$$

Since $\check{\phi}_{t,s} \leq 1 + \delta_T$ and $\|\phi_{t,s}\|_{\text{Lip}} \leq 1 + \delta_T$,

$$\begin{aligned} V_{t,r} &\leq V_{t,s} + (1 + \delta_T)V_{r,t} + L\|z - y\|_{\infty}\varpi(\omega_{r,t}) \\ &\quad + \hat{\phi}_{t,s}(K\varpi(\omega_{s,r}))((1 + \delta_T)\|z - y\|_{\infty} + \epsilon_3) + (\epsilon_1 + \epsilon_2K)\varpi(\omega_{r,t}). \end{aligned}$$

Since $\widehat{\phi}_{t,s}$ is ϖ -compatible (see Definition 19), $\widehat{\phi}_{t,s}(K\varpi(\omega_{r,s})) \leq \Phi(K)\varpi(\omega_{r,t})$ so that

$$V_{r,t} \leq V_{s,t} + (1 + \delta_T)V_{r,t} + B\varpi(\omega_{r,t}) \quad (6.31)$$

$$\text{with } B := (L + (1 + \delta_T)\Phi(K))\|y - z\|_\infty + \epsilon_1 + \epsilon_2 K + \epsilon_3 \Phi(K). \quad (6.32)$$

Owing to (6.31)-(6.32), from the continuous time version of the Davie's Lemma 6.A.2, for T small enough (depending only on \varkappa and $T \mapsto \delta_T$), for all $(r, t) \in \mathbb{T}^2$,

$$V_{r,t} \leq BC\varpi(\omega_{r,t}) \text{ with } C := \frac{2 + \delta_T}{1 - (\varkappa(1 + \delta_T)^2 + \delta_T)}.$$

For any $t \in [0, T]$, since $\|\phi_{t,0}\|_{\text{Lip}} \leq 1 + \delta_T$,

$$\begin{aligned} |y_t - z_t| &\leq |y_t - z_t - \phi_{t,0}(y_0) + \zeta_{t,0}(z_0)| + |\phi_{t,0}(z_0) - \zeta_{t,0}(z_0)| + |\phi_{t,0}(z_0) - \phi_{t,0}(y_0)| \\ &\leq V_{0,t} + \epsilon_3 + (1 + \delta_T)|y_0 - z_0|. \end{aligned}$$

With the expression of B in (6.32), for any $t \in [0, T]$, we see that there exists constants A and A' that depend only on \varkappa , L , δ_T , K and $\Phi(K)$ such that

$$|y_t - z_t| \leq A\|y - z\|_\infty \varpi(\omega_{0,T}) + A'(\epsilon_1 + \epsilon_2 + \epsilon_3)\varpi(\omega_{0,T}) + \epsilon_3 + (1 + \delta_T)|y_0 - z_0|.$$

Choosing T small enough so that $A\varpi(\omega_{0,T}) \leq 1/2$ implies that

$$\|y - z\|_\infty \leq 2A'(\epsilon_1 + \epsilon_2 + \epsilon_3)\varpi(\omega_{0,T}) + 2\epsilon_3 + 2(1 + \delta_T)|y_0 - z_0|.$$

This concludes the proof. ■

6.8 Application to Rough differential equation

In this section, we show how our framework allows us to link the different flow based approaches. The key is to show that Friz-Victoir's and Bailleul's almost flows are different perturbation of the Davie's almost flow.

We start by giving some notations. We did not recall notions of the rough path theory. The reader can find a clear introduction in [Lej03] and in [FH14].

In this section, the remainder introduced in Section 6.2.1 is of the type

$$\varpi(\delta) = \delta^{(2+\gamma)/p}, \quad \forall \delta > 0.$$

with $\gamma \in (0, 1]$ and a real number $p > 0$ satisfying $2 + \gamma > p$.

6.8. APPLICATION TO ROUGH DIFFERENTIAL EQUATION

6.8.1 Rough path notations

Before showing the link between the different based flow approaches, we set notations of classical objects of the rough path theory.

Given another Banach space $(U, |\cdot|)$ and a real number $p \geq 1$, let us denote by $\mathcal{C}^{p-\omega}(\mathbb{T}, U)$ the space of $1/p$ -Hölder paths controlled by ω , which we equip with the semi-norm

$$\|x\|_p := \sup_{(s,t) \in \mathbb{T}_+^2, s \neq t} \frac{|x_{s,t}|}{\omega_{s,t}^{1/p}},$$

this quantity being bounded by definition.

We define also $\mathcal{C}^{p-\text{var}}([s, t], U)$ the space of bounded p -variation paths from $[s, t]$ to U which we equip with the p -variation semi-norm on $[s, t]$ denoted by $\|x\|_{[s,t],p}$.

Moreover, if $x \in \mathcal{C}^{p-\omega}([0, T], U)$, then $x \in \mathcal{C}^{p-\text{var}}([s, t], U)$ and

$$\|x\|_{[s,t],p} \leq \|x\|_p \omega_{s,t}^{1/p}.$$

We denote by $\lfloor \cdot \rfloor$ the floor function.

For an integer $N \geq \lfloor p \rfloor$, let $\mathcal{T}^{p,N}(U)$ be the space of $1/p$ -Hölder rough path controlled by ω of order N . If $\mathbf{x} \in \mathcal{T}^{p,N}(U)$ we denote by $\mathbf{x}^{(k)}$ the component of \mathbf{x} in $U^{\otimes k}$ with $0 \leq k \leq N$ an integer and $S_j(\mathbf{x}) := \sum_{j=0}^k \mathbf{x}^{(j)}$. Obviously, $\mathbf{x} = S_N(\mathbf{x})$. We denote the homogeneous semi-norm

$$\|\mathbf{x}\|_p := \sup_{k \leq N} \sup_{[s,t] \in \mathbb{T}_+^2, s \neq t} \frac{|\mathbf{x}_{s,t}^{(k)}|}{\omega_{s,t}^{k/p}},$$

which is finite by definition. Moreover we set $\mathcal{T}^p(U) := \mathcal{T}^{p,\lfloor p \rfloor}(U)$.

For $N \geq 0$, we denote $G^N(U)$ the free nilpotent group (Chapter 7 in [FV10]).

Let $\mathcal{G}^p(U) := \mathcal{C}^{p-\omega}([0, T], G^{\lfloor p \rfloor}(U))$ be the set of weak-geometric rough paths of finite $1/p$ -Hölder rough path controlled by ω with values in U .

When $U = \mathbb{R}^\ell$ ($\ell \geq 0$ an integer). For any multi-indices $I := (i_1, \dots, i_k) \in \{1, \dots, \ell\}^k$ we set $|I| := k$ and $e_I := e_{i_1} \otimes \dots \otimes e_{i_k}$ where $\{e_1, \dots, e_\ell\}$ is the canonical basis of \mathbb{R}^ℓ . If $\mathbf{x} \in \mathcal{T}^p(\mathbb{R}^\ell)$. If $\mathbf{x} \in \mathcal{T}^p(\mathbb{R}^\ell)$, then \mathbf{x}^I denote the coordinates of $\mathbf{x}^{(k)}$ in the basis $(e_I)_{|I|=k}$. It follows that $S_k(\mathbf{x}) = \sum_{|I| \leq k} \mathbf{x}^I e_I$. If $x \in \mathcal{C}^{1-\text{var}}(\mathbb{T}, \mathbb{R}^\ell)$ then for any integer $N \geq 0$,

$$S_N(x) = \sum_{|I| \leq N} x^I e_I,$$

where $x_{s,t}^I := \int_{s \leq t_k \leq \dots \leq t_1 \leq t} dx_{t_k}^{i_k} \dots dx_{t_1}^{i_1}$.

6.8.2 The Davie's approach

Let us consider now a p -rough path $\mathbf{x} \in \mathcal{F}^p(U)$ with $2 \leq p < 3$ for a Banach space U . A Rough Differential Equation (RDE) is a solution y taking its values in another Banach space V to

$$y_t = a + \int_s^t f(y_u) d\mathbf{x}_u, \quad \forall (s, t) \in \mathbb{T}_+^2, \quad (6.33)$$

provided that $f : V \rightarrow L(U, V)$ is regular enough.

Existence of solution to (6.33) was proved first by T. Lyons using a Picard fixed point theorem [Lyo98]. In [Dav07], A.M. Davie provided an alternative approach based on Euler-type schemes. Over the approach of T. Lyons, it has the advantage that the solution is sought as a path with values in V and not in the tensor space $T_2(U \oplus V)$.

For $f \in \mathcal{C}_b^1(V, L(U, V))$, we define

$$\phi_{t,s}(a) = a + f(a)\mathbf{x}_{s,t}^{(1)} + df(a) \cdot f(a)\mathbf{x}_{s,t}^{(2)}, \quad (6.34)$$

where $df(a)$ is the differential of f in a . Definition 23 coincides with the one defined by A. M. Davie in [Dav07] for the notion of solution to (6.33).

The following lemma is a generalization in an infinite dimensional setting of Theorems 3.2 and 3.3 in [Dav07] with a bounded function f .

Lemma 6.8.1. *Let $p \in [2, 3)$ and $\gamma > p - 2$.*

(i) *If $f \in \mathcal{C}_b^{1+\gamma}$, then the family ϕ defined by (6.34) is an almost flow.*

(ii) *If f is of class $\mathcal{C}_b^{2+\gamma}$, then ϕ is a stable almost flow.*

Proof. According to Proposition 5 in [BL17a], ϕ is an almost flow as soon as $f \in \mathcal{C}_b^{1+\gamma}$ with $2 + \gamma > p$.

We now assume that $f \in \mathcal{C}_b^{2+\gamma}$. We show that ϕ verifies a ϖ -compatible 4-points control (see Definition 19). For any $a, b, c, d \in V$, $(s, t) \in \mathbb{T}^2$, we compute

$$\begin{aligned} \phi_{t,s}(a) - \phi_{t,s}(b) - \phi_{t,s}(c) + \phi_{t,s}(d) &= a - b - c + d + \underbrace{[f(a) - f(b) - f(c) + f(d)]\mathbf{x}_{s,t}^{(1)}}_{I'_{s,t}} \\ &\quad + \underbrace{[df \cdot f(a) - df \cdot f(b) - df \cdot f(c) + df \cdot f(d)]\mathbf{x}_{s,t}^{(2)}}_{II'_{s,t}}. \end{aligned} \quad (6.35)$$

Then, we apply Lemma 6.4.1 to $f \in \mathcal{C}_b^2$ and to $df \cdot f \in \mathcal{C}_b^{1+\gamma}$ to obtain

$$\begin{aligned} |I'_{s,t}| &\leq \|\mathbf{x}^{(1)}\|_p \omega_{s,t}^{1/p} 4 \|d^2 f\|_\infty (|a - c| \vee |b - d|)^2 |a - b| \\ &\quad + \|\mathbf{x}^{(1)}\|_p \omega_{0,T}^{1/p} \|df\|_\infty |a - b - c + d|, \end{aligned} \quad (6.36)$$

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and

$$|\mathbb{I}'_{s,t}| \leq \|\mathbf{x}^{(2)}\|_{\frac{p}{2}} \omega_{s,t}^{2/p} \left[2 \|d(df \cdot f)\|_{\gamma} (|a-c| \vee |b-d|)^{\gamma} |a-b| + \|d(df \cdot f)\|_{\infty} |a-b-c+d| \right]. \quad (6.37)$$

Combining (6.35), (6.36) and (6.37) we set for all $y \in \mathbb{R}_+$, $\widehat{\phi}_{t,s}(y) := c'_1 \left[\omega_{s,t}^{1/p} y^2 + \omega_{s,t}^{2/p} y^{\gamma} \right]$ where $c'_1 := \|\mathbf{x}^{(1)}\|_{p4} \|d^2 f\|_{\infty} + \|\mathbf{x}^{(2)}\|_{\frac{p}{2}} 2 \|d(df \cdot f)\|_{\gamma}$ and

$$\widetilde{\phi}_{t,s} := 1 + \|\mathbf{x}^{(1)}\|_{p\omega_{0,T}^{1/p}} + \|d(df \cdot f)\|_{\infty} \|\mathbf{x}^{(2)}\|_{\frac{p}{2}} \omega_{0,T}^{2/p}.$$

It follows that $\widetilde{\phi}_{t,s} \leq 1 + \delta_T$ and that $\phi_{t,s}$ is ϖ -compatible. Indeed, for $\alpha \in \mathbb{R}_+$,

$$\begin{aligned} \widehat{\phi}_{t,s} \left(\alpha \omega_{s,t}^{(2+\gamma)/p} \right) &\leq c'_1 (\alpha^2 \vee \alpha^{\gamma}) \left(\omega_{s,t}^{(5+2\gamma)/p} + \omega_{s,t}^{(2+2\gamma+\gamma^2)/p} \right) \\ &\leq c'_1 (\alpha^2 \vee \alpha^{\gamma}) \left(\omega_{0,T}^{(3+\gamma)/p} + \omega_{0,T}^{(\gamma+\gamma^2)/p} \right) \omega_{s,t}^{(2+\gamma)/p} \leq \delta_T \varpi(\omega_{s,t}). \end{aligned}$$

It remains to show that (6.11) holds. As $\phi_{t,s} \in \mathcal{C}^{1+\gamma}$, thus the two semi-norms $\|\phi_{t,s}\|_{\text{Lip}}$ and $\|d\phi_{t,s}\|_{\infty}$ are equivalent. We recall that $\phi_{t,s,r} = \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$. For any $a \in \mathbb{V}$ and $(r, s, t) \in \mathbb{T}_+^3$,

$$\begin{aligned} d\phi_{t,s,r}(a) &= (d\phi_{s,r}(a) df \circ \phi_{s,r}(a) - df(a)) \mathbf{x}_{s,t}^{(1)} - d(df \cdot f)(a) (\mathbf{x}_{s,t}^{(2)} - \mathbf{x}_{r,s}^{(1)} \otimes \mathbf{x}_{s,t}^{(1)}) \\ &\quad + d\phi_{s,r}(a) d(df \cdot f) \circ \phi_{s,r}(a) \mathbf{x}_{s,t}^{(2)} \\ &= \underbrace{(-df(a) + d\phi_{s,r}(a) df \circ \phi_{s,r}(a) - d(df \cdot f)(a) \mathbf{x}_{r,s}^1)}_{\mathbb{I}''_{r,s,t}} \mathbf{x}_{s,t}^{(1)} \\ &\quad - \underbrace{(d(df \cdot f)(a) - d\phi_{s,r}(a) d(df \cdot f) \circ \phi_{s,r}(a)) \mathbf{x}_{s,t}^2}_{\mathbb{II}''_{r,s,t}}. \end{aligned} \quad (6.38)$$

Each term is estimated separately. For the first one,

$$\begin{aligned} |\mathbb{I}''_{r,s,t}| &\leq |df \circ \phi_{s,r}(a) - df(a) - d^2 f(a)(\phi_{s,r}(a) - a)| \|\mathbf{x}_{s,t}^{(1)}\| \\ &\quad + |df(a) df(a) \mathbf{x}_{r,s}^{(2)} \mathbf{x}_{s,t}^1| + |d(df \cdot f)(a) df \circ \phi_{s,r}(a) \mathbf{x}_{r,s}^{(2)} \mathbf{x}_{s,t}^{(1)}| \\ &\quad + |df(a) [df \circ \phi_{s,r}(a) - df(a)] \mathbf{x}_{r,s}^1 \otimes \mathbf{x}_{s,t}^1| \\ &\leq \|d^2 f\|_{\gamma} |\phi_{s,r}(a) - a|^{1+\gamma} \|\mathbf{x}_{s,t}^{(1)}\| + (\|df\|_{\infty}^2 + \|d(df \cdot f) df\|_{\infty}) \|\mathbf{x}_{r,s}^{(2)}\| \|\mathbf{x}_{s,t}^{(1)}\| \\ &\quad + \|df\|_{\infty} \|d^2 f\|_{\infty} |\phi_{s,r}(a) - a| \|\mathbf{x}_{r,s}^1\| \|\mathbf{x}_{s,t}^1\| \\ &\leq c''_1 (\omega_{r,t}^{(2+\gamma)/p} + 2\omega_{r,t}^{3/p}) \leq c''_1 (1 + 2\omega_{0,T}^{(1-\gamma)/p}) \omega_{r,t}^{(2+\gamma)/p}, \end{aligned} \quad (6.39)$$

where c''_1 is a constant which depends on f, \mathbf{x}, γ .

The second one is more simple,

$$\begin{aligned}
 |\mathbb{I}''_{r,s,t}| &\leq |d(df \cdot f) \circ \phi_{s,r}(a) - d(df \cdot f)(a)| |\mathbf{x}_{s,t}^{(2)}| + |d\phi_{s,r}(a) - 1| d(df \cdot f) \circ \phi_{s,r}(a) \mathbf{x}_{s,t}^2 \\
 &\leq \|d(df \cdot f)\|_\infty |\phi_{s,r}(a) - a|^\gamma |\mathbf{x}_{s,t}^{(2)}| + \|d(df \cdot f)\|_\infty |d\phi_{s,r}(a) - 1| |\mathbf{x}_{s,t}^{(2)}| \\
 &\leq c_2'' (\omega_{r,t}^{(2+\gamma)/p} + \omega_{r,t}^{3/p}) \leq c_2'' (1 + \omega_{0,T}^{(1-\gamma)/p}) \omega_{r,t}^{(2+\gamma)/p}.
 \end{aligned} \tag{6.40}$$

Finally, combining (6.38), (6.39) and (6.40), $\|d\phi_{t,s,r}\|_\infty \leq (c_1'' + c_2'')(1 + \omega_{0,T}^{(1-\gamma)/p}) \varpi(\omega_{r,t})$ with $\varpi(\omega_{r,t}) = \omega_{r,t}^{(2+\gamma)/p}$. This proves (6.11) and that ϕ is a stable almost flow. \blacksquare

Combining Lemma 6.8.1, Proposition 6.3.4, Corollaries 6.4.5 and 6.7.5 leads to the following result.

Corollary 6.8.2. *If f is of class $\mathcal{C}_b^{2+\gamma}$, then ϕ^π converges to a unique Lipschitz flow ψ . Moreover, if χ is an almost flow in a galaxy containing ϕ , then, χ^π converges to ψ . Besides, there exists a unique manifold of solutions to $\dot{y} = \phi'(y)$ which is Lipschitz.*

6.8.3 Almost flows constructed from sub-Riemannian geodesics, as in P. Friz and N. Victoir

In [FV10], P. Friz and N. Victoir proposed an approach based on the use of geodesics. The following proposition is one of the fundamental result of their framework.

Now, we assume that $U = \mathbb{R}^\ell$.

Proposition 6.8.3 (Remark 10.10, [FV10, p. 216]). *Let $p \geq 1$ a real number and an integer $N \geq [p]$. For any $\mathbf{x} \in \mathcal{C}^{p-\omega}([0, T], G^N(\mathbb{R}^\ell))$ and any $(s, t) \in \mathbb{T}^2$, there exists a path $x^{s,t} \in \mathcal{C}^{1-\text{var}}(\mathbb{R}^\ell)$ defined on $[s, t]$ such that*

$$S_N(x^{s,t})_{s,t} = \mathbf{x}_{s,t} \text{ and } \|x^{s,t}\|_{[s,t],1} \leq K \|\mathbf{x}\|_p \omega_{s,t}^{1/p} \leq K' \|S_{[p]}(\mathbf{x})\|_p \omega_{s,t}^{1/p}.$$

for some universal constant K (resp. K') that depends only on p (p and N). We say that $x^{s,t}$ is a geodesic path associated to \mathbf{x} .

Remark 6.8.4. If $x \in \mathcal{C}^{1-\text{var}}([s, t], \mathbb{R}^\ell)$, then $\|x\|_{[s,t],1} = \int_s^t |dx_r|$.

For notational convenience, we prefer now to express differential equations with respect to vector fields, that is a family of functions $\vec{f} := (\vec{f}_1, \dots, \vec{f}_\ell)$ that acts on $\mathcal{C}_b^1(V, L(U, V))$. Therefore for $x \in \mathcal{C}^{1-\text{var}}(\mathbb{R}_+, \mathbb{R}^\ell)$, the equation $z_t = a + \int_0^t \vec{f} \text{Id}(z_s) dx_s$ is equivalent to $y_t = a + \int_0^t f(z_s) dx_s$ with $f = \vec{f} \text{Id}$. For a multi-indices $I := (i_1, \dots, i_k) \in \{1, \dots, \ell\}^k$ and $(s, t) \in \mathbb{T}$ we denote $\vec{f}_I := \vec{f}_{i_1} \dots \vec{f}_{i_n}$. By convention, $\vec{f}_\emptyset \text{Id} := \text{Id}$. We employ the Einstein convention of summation.

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Let us fix $n \geq 2$ in $\vec{f}\text{Id} \in \mathcal{C}_b^{\lambda-1}$ for $\lambda \geq n$. Let \mathbf{x} be a rough path with values in $T_n(U)$ and V be a vector field such that $\vec{f}\text{Id} \in \mathcal{C}_b^{n-1}$. Let us define

$$\phi_{t,s}^{(n)}[\mathbf{x}, \vec{f}](a) := \sum_{k=0}^n \sum_{|I|=k} \vec{f}_I \text{Id}(a) \mathbf{x}_{s,t}^I, \quad \forall (s, t) \in \mathbb{T}_+^2. \quad (6.41)$$

The next proposition summarizes various results on RDEs (Theorem 10.26 in [FV10, p. 233], Theorem 10.30 in [FV10, p. 238]). When $\vec{f}\text{Id} \in \mathcal{C}_b^{1+\gamma}$ but not in $\mathcal{C}_b^{2+\gamma}$ with $2 + \gamma > p$, several solutions to the RDE (6.33) may exist (See Example 2 in [Dav07]).

Theorem 6.8.5. *Assume that U and V are finite dimensional Banach spaces. Choose $\lambda > 2$ as well as an integer n with $2 \leq n \leq \lfloor \lambda \rfloor$. Let \mathbf{x} be a p -rough paths with values in $T_n(V)$. Let us assume that $\vec{f}\text{Id} \in \mathcal{C}^{\lambda-1}$ for a vector field $\vec{f}: V \rightarrow L(U, V)$. It holds that*

- (i) *When $\lambda > p$, there exists a flow $\psi[\mathbf{x}, \vec{f}]$ in the same galaxy as $\phi^{(n)}[\mathbf{x}, \vec{f}]$,*
 - (ii) *When $1 + \lambda > p$, then there exists a unique flow as well as a unique Lipschitz manifold of solutions to $\dot{y} = \phi^{(n)'}(y)$ ($\phi^{(n)}$ is defined in (6.41)).*
- In addition, for any partition $\pi = \{t_i\}_{i=0}^k$,*

$$|\psi_{t,s}[\mathbf{x}, \vec{f}](a) - \phi_{t,s}^{(n)\pi}[\mathbf{x}, \vec{f}](a)| \leq C \|\mathbf{x}\|_p \sup_{i=0, \dots, k-1} \omega_{t_i, t_{i+1}}^{\frac{n+\gamma-1}{p}}, \quad (6.42)$$

with $\gamma := \min\{\lambda - n, 1\}$ and a constant C that depends on $\omega_{0,T}$, $\|\mathbf{x}\|_p$ and V .

From the next lemma, we obtain that if $\vec{f}\text{Id} \in \mathcal{C}_b^\lambda$ with $\lambda > p$, then there exists a unique flow associated to $\phi^{(n)}[\mathbf{x}, \vec{f}]$.

Lemma 6.8.6. *For any $n \geq 2$, $\phi^{(n)}[\mathbf{x}, \vec{f}]$ belongs to the same galaxy as $\phi^{(2)}[\mathbf{x}, \vec{f}]$, which the Davie expansion given by (6.34) with $f = \vec{f}\text{Id}$.*

Proof. Let us write for $n \geq 2$,

$$R_{s,t}^{(n)}[\mathbf{x}, \vec{f}] := \phi_{s,t}^{(n)}[\mathbf{x}, \vec{f}] - \phi_{s,t}^{(2)}[\mathbf{x}, \vec{f}] = \sum_{k=3}^n \sum_{|I| \leq k} \vec{f}_I \text{Id}(a) \mathbf{x}_{s,t}^I, \quad \forall (s, t) \in \mathbb{T}_+^2.$$

Clearly, $R^{(n)}[\mathbf{x}, \vec{f}]$ is a perturbation with $\varpi(\delta) = \delta^{3/p}$. Moreover, as $\vec{f}\text{Id} \in \mathcal{C}_b^{\lambda-1}$, $\vec{f}_I \text{Id} \in \mathcal{C}_b^{\lambda-k}$ for any word I with $|I| = k$ so that when $\lambda - n \geq 1$, $R^{(n)}[\mathbf{x}, \vec{f}]$ is a Lipschitz perturbation (Definition 6.5.1). \blacksquare

This is however not sufficient to obtain the rate in (6.42).

For a path $x \in \mathcal{C}^{1-\text{var}}$, we denote for $a \in V$ by $\psi_{\cdot, s}[x, \vec{f}](a)$ the unique solution to

$$\psi_{t,s}[x, \vec{f}](a) = a + \int_s^t \vec{f}\text{Id}(\psi_{r,s}[x, \vec{f}](a)) dx_r, \quad t \geq s. \quad (6.43)$$

The family $\psi[x, \vec{f}]$ satisfies the flow property.

We set for any $(s, t) \in \mathbb{T}_+^2$,

$$\begin{aligned} \phi_{t,s}^{(n)}[x, \vec{f}](a) &:= \phi_{t,s}^{(n)}[S_n(x), \vec{f}](a), \\ \epsilon_{t,s}^{(n)}[x, \vec{f}](a) &= \sum_{|I|=n} \int_{s \leq t_{k-1} \leq \dots \leq t_1 \leq t} \left(\vec{f}_I \text{Id}(\psi_{t_k,s}[x, \vec{f}](a)) - \vec{f}_I \text{Id}(a) \right) dx_{t_k}^I, \end{aligned}$$

where $dx_{t_k}^I = dx_{t_k}^{i_k} \dots dx_{t_1}^{i_1}$. Using iteratively the Newton formula on (6.43),

$$\psi_{t,s}[x, \vec{f}](a) = \phi_{t,s}^{(n)}[x, \vec{f}](a) + \epsilon_{t,s}^{(n)}[x, \vec{f}](a).$$

We denote by $\|f\|_\gamma$ the γ -Hölder semi-norm of a function f . For $\gamma = 1$, this is the Lipschitz semi-norm. Thus, $\gamma := \min\{\lambda - n, 1\}$ is the Hölder indice of $\vec{f}_I \text{Id}$ with $|I| = n$.

From Proposition 10.3 in [FV10, p. 213], for a constant C that depends on λ and on

$$\|V\|_* := \max_{|I| \leq n} \|\vec{f}_I \text{Id}\|_\infty + \max_{|I| \leq n-1} \|\vec{f}_I \text{Id}\|_{\text{Lip}} + \max_{|I|=n} \|\vec{f}_I \text{Id}\|_\gamma,$$

it holds that

$$|\epsilon_{t,s}^{(n)}[x, \vec{f}](a)| \leq C \|x\|_{[s,t],1}^{n+\gamma}. \quad (6.44)$$

Lemma 6.8.7. *Let x be a path of finite 1-variation with $\|x\|_{[s,t],1} \leq A\omega_{s,t}$ for any $(s, t) \in \mathbb{T}_+^2$. Let $x^{s,t}$ be the geodesic path given by Proposition 6.8.3. Assume that there exists a constant K such that $\|x^{s,t}\|_{[s,t],1} \leq K\omega_{s,t}^{1/p}$. Then there exists a time $T > 0$ small enough and a constant D that depend only on ω , K and $\|\vec{f} \text{Id}\|_*$ such that*

$$\|\psi_{t,s}[x^{s,t}, \vec{f}] - \psi_{t,s}[x, \vec{f}]\|_\infty \leq D\omega_{s,t}^{\frac{n+\gamma}{p}}, \quad \forall (s, t) \in \mathbb{T}_+^2.$$

In particular, the choice of T and D does not depend on A .

Proof. Let us assume that $\|x\|_{[s,t],1} \leq A\omega_{s,t}$ and $\|x^{s,t}\|_{[s,t],1} \leq K\omega_{s,t}^{1/p}$.

Let $x^{r,s,t}$ be the concatenation of $x^{r,s}$ and $x^{s,t}$. Then

$$\begin{aligned} \psi_{t,r}[x, \vec{f}](a) - \psi_{t,r}[x^{r,t}, \vec{f}](a) &= \underbrace{\psi_{t,s}[x, \vec{f}](\psi_{s,r}[x, \vec{f}](a)) - \psi_{t,r}[x^{r,s,t}, \vec{f}](a)}_{\text{I}_{r,s,t}} \\ &\quad + \underbrace{\psi_{t,r}[x^{r,s,t}, \vec{f}](a) - \psi_{t,r}[x^{r,t}, \vec{f}](a)}_{\text{II}_{r,s,t}}. \end{aligned}$$

Since $x^{r,s,t}$ is the concatenation between two paths,

$$\psi_{t,r}[x^{r,s,t}, \vec{f}](a) = \psi_{t,s}[x^{s,t}, \vec{f}](\psi_{s,r}[x^{r,s}, \vec{f}](a)).$$

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Thus,

$$\begin{aligned} |I_{r,s,t}| &\leq |\psi_{t,s}[x, \vec{f}](\psi_{s,r}[x, \vec{f}](a)) - \psi_{t,s}[x^{s,t}, \vec{f}](\psi_{s,r}[x, \vec{f}](a))| \\ &\quad + |\psi_{t,s}[x^{s,t}, \vec{f}](\psi_{s,r}[x, \vec{f}](a)) - \psi_{t,s}[x^{s,t}, \vec{f}](\psi_{s,r}[x^{r,s}, \vec{f}](a))|. \end{aligned}$$

Writing $U_{r,t} := \|\psi_{t,r}[x, \vec{f}] - \psi_{t,r}[x^{r,t}, \vec{f}]\|_\infty$, it holds that

$$|I_{s,r,t}| \leq U_{t,s} + \|\psi_{t,s}[x^{s,t}, \vec{f}]\|_{\text{Lip}} U_{r,s}.$$

From (6.43), we derive that for $t \geq s$,

$$\begin{aligned} \|\psi_{t,s}[x^{s,t}, \vec{f}]\|_{\text{Lip}} &\leq 1 + \|\vec{f}\text{Id}\|_{\text{Lip}} \int_s^t \|\psi_{r,s}[x^{s,t}, \vec{f}]\|_{\text{Lip}} dx_r^{s,t} \\ &\leq 1 + \|\vec{f}\text{Id}\|_{\text{Lip}} \int_s^t \|\psi_{r,s}[x^{s,t}, \vec{f}]\|_{\text{Lip}} |\dot{x}_r^{s,t}| dr, \end{aligned} \quad (6.45)$$

where the derivative $\dot{x}^{s,t}$ is almost everywhere defined because $x^{s,t} \in \mathcal{C}^{1-\text{var}}(\mathbb{R}^\ell)$.

Then, using the Grönwall's inequality with (6.45) and Proposition 6.8.3, there is constant C that depends only on K (defined in Proposition 6.8.3), $\|\mathbf{x}\|_p$ and $\|\vec{f}\text{Id}\|_{\text{Lip}}$ such that

$$\|\psi_{t,s}[x^{s,t}, \vec{f}]\|_{\text{Lip}} \leq \exp(C\omega_{0,T}^{1/p}).$$

Besides,

$$S_n(x^{r,s,t})_{r,t} = S_n(x^{r,s})_{r,s} \otimes S_n(x^{s,t})_{s,t} = \mathbf{x}_{r,s} \otimes \mathbf{x}_{s,t} = \mathbf{x}_{r,t} = S_n(x^{r,t})_{r,t}.$$

It follows that $\phi^{(n)}[x^{r,s,t}, \vec{f}] = \phi^{(n)}[x^{r,t}, \vec{f}]$. Thus,

$$\begin{aligned} |II_{r,s,t}| &= |\epsilon_{t,r}^{(n)}[x^{r,s,t}, \vec{f}](a) - \epsilon_{t,r}^{(n)}[x^{r,t}, \vec{f}](a)| \\ &\leq \|\epsilon_{t,r}^{(n)}[x^{r,s,t}, \vec{f}]\|_\infty + \|\epsilon_{t,r}^{(n)}[x^{r,t}, \vec{f}]\|_\infty \leq C' \omega_{s,t}^{\frac{n+\gamma}{p}}, \end{aligned}$$

where $C' \geq 0$ is a new constant and using (6.44) and Proposition 6.8.3 for the last estimation.

Thus,

$$U_{r,t} \leq U_{s,t} + \exp(C\omega_{0,T}^{1/p})U_{r,s} + C' \omega_{s,t}^{\frac{n+\gamma}{p}}.$$

On the other hand, when $\omega_{s,t} \leq 1$,

$$\begin{aligned} U_{s,t} &= \|\psi_{t,s}[x, \vec{f}] - \psi_{t,s}[x^{s,t}, \vec{f}]\|_\infty \\ &\leq \|\psi_{t,s}[x, \vec{f}] - \phi_{t,s}^{(n)}[x, \vec{f}]\|_\infty + \|\psi_{t,s}[x^{s,t}, \vec{f}] - \phi_{t,s}^{(n)}[x^{s,t}, \vec{f}]\|_\infty \\ &\leq C'' \max\{A^{n+\gamma} \omega_{s,t}^{n+\gamma}, K^{\frac{n+\gamma}{p}} \omega_{s,t}^{\frac{n+\gamma}{p}}\} \\ &\leq B \omega_{s,t}^{\frac{n+\gamma}{p}} \quad \text{with } B := C'' \max\{A^{n+\gamma}, K^{\frac{n+\gamma}{p}}\}. \end{aligned}$$

From the continuous time Davie lemma (Lemma 6.A.2 in Appendix), there exists a constant D that does not depend on B (hence on A) such that $U_{s,t} \leq D \omega_{s,t}^{\frac{n+\gamma}{p}}$. \blacksquare

Lemma 6.8.8. *Let $x \in \mathcal{C}^{1-\text{var}}$ be as in Lemma 6.8.8. Then*

$$\|\epsilon_{t,s}^{(n)}[x, \vec{f}]\|_\infty = \|\psi_{t,s}[x, \vec{f}] - \phi_{t,s}^{(n)}[x, \vec{f}]\|_\infty \leq E\omega_{s,t}^{\frac{n+\gamma}{p}},$$

where E depends only on ω , T , and K .

Proof. Since $\phi_{t,s}^{(n)}[x, \vec{f}] = \phi_{t,s}^{(n)}[x^{s,t}, \vec{f}]$,

$$\epsilon_{t,s}^{(n)}[x, \vec{f}] := \psi_{t,s}[x, \vec{f}] - \phi_{t,s}^{(n)}[x, \vec{f}] = \psi_{t,s}[x, \vec{f}] - \psi_{t,s}^{(n)}[x, \vec{f}] + \epsilon_{t,s}^{(n)}[x, \vec{f}].$$

The results is then an immediate consequence of (6.44) and Lemma 6.8.7. \blacksquare

Proof of Theorem 6.8.5. We recall that we assume that U and V are finite dimensional Banach spaces. For any $(a, b) \in V$, any $(s, t) \in \mathbb{T}_+^2$,

$$\begin{aligned} & |\phi_{t,s}^{(n)}[x, \vec{f}](a) - \phi_{t,s}^{(n)}[x, \vec{f}](b)| \\ & \leq \sum_{k=0}^{n-1} \sum_{|I|=k} \|\vec{f}_I \text{Id}\|_{\text{Lip}} |x_{s,t}^I| \cdot |a - b| + \sum_{|I|=n} \|\vec{f}_I \text{Id}\|_\gamma \cdot |x_{s,t}^I| |a - b|^\gamma \\ & \leq \sum_{k=0}^{n-1} \|\vec{f}_k \text{Id}\|_{\text{Lip}} \|x\|_{[s,t],1}^k \cdot |a - b| + \|\vec{f}_n \text{Id}\|_\gamma \|x\|_{[s,t],1}^n |a - b|^\gamma. \end{aligned}$$

It then follows from Proposition 6.5.3 that $\phi^{(n)}[x, \vec{f}]$ is an almost flow with

$$\left\| \phi_{t,s}^{(n)}[x, \vec{f}](\phi_{s,r}^{(n)}[x, \vec{f}](a)) - \phi_{t,r}^{(n)}[x, \vec{f}](a) \right\|_\infty \leq L\omega_{s,t}^{\frac{n+\gamma}{p}}, \forall (r, s, t) \in \mathbb{T}_+^3,$$

for a constant L that depends only on $\|x\|_{[s,t],1}$ and of $\|V\|_*$.

Let $(x^m)_{m \in \mathbb{N}}$ be a sequence of bounded variation paths such that $S_n(x^m) \xrightarrow{m \rightarrow \infty} \mathbf{x}$ uniformly on $[0, T]$ and such that $\sup_{m \in \mathbb{N}} \|S_n(x^m)\|_p \leq c\|\mathbf{x}\|_p$ for a uniform constant c in m . Such a sequence exists according to Remark 10.32 in [FV10]. It consists in concatenating the geodesic approximations given by Proposition 6.8.3. From this, $|S_n(x^m)_{s,t}| \leq K\omega_{s,t}^{1/p}$ with $K = c\|\mathbf{x}\|_p$.

Clearly, $\phi_{t,s}^{(n)}[x^m, \vec{f}](a)$ converges to $\phi_{t,s}^{(n)}[\mathbf{x}, \vec{f}]$ for any $(s, t) \in \mathbb{T}_+^2$ and any $a \in V$. The result follows from Corollary 6.5.4 and Lemma 6.8.8. \blacksquare

6.8.4 Bailleul's approach

Classical control of ODE solutions

For $a \in V$, the solution to the ordinary differential equation

$$y_t(a) = a + \int_0^t \vec{f} \text{Id}(y_s(a)) ds$$

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is a path from $[0, T]$ to V such that

$$\phi(y_t(a)) = \phi(a) + \int_0^t V\phi(y_s(a)) \, ds \quad (6.46)$$

for any $\phi \in \mathcal{C}^1(V, V)$. Assuming enough regularity on both ϕ and \vec{f} , we iterate (6.46) so that

$$\phi(y_t(a)) = \phi(a) + tV\phi(a) + \cdots + \frac{t^k}{k!} V^k \phi(a) + R(V^k \phi, a; t)$$

with $\vec{f}^0 \phi = \phi$, $\vec{f}^{k+1} \phi = \vec{f}(\vec{f}^k \phi)$, $k = 0, 1, \dots$ and

$$R_k(\psi, a; t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} (\psi(y_{t_k}(a)) - \psi(a)) \, dt_k \cdots dt_1.$$

for a function $\psi : V \rightarrow V$.

Lemma 6.8.9. *If $\vec{f}\text{Id}$ is uniformly Lipschitz, then for any $a \in V$ and any $t \geq 0$.*

$$|y_t(a) - a| \leq t |\vec{f}\text{Id}(a)| \exp(\|\vec{f}\text{Id}\|_{\text{Lip}} t). \quad (6.47)$$

Moreover, if $\vec{f}\text{Id}$ satisfies a 4-points control, then for any $a, b \in V$ and any $t \geq 0$,

$$\begin{aligned} \Delta_t(a, b) &:= |y_s(a) - a - y_s(b) + b| \\ &\leq t \widehat{\vec{f}\text{Id}}(\alpha_t(a, b)) \exp\left(\widehat{\vec{f}\text{Id}}(\alpha_t(a, b)) + \vec{f}\text{Id}\right) |a - b| \end{aligned} \quad (6.48)$$

with

$$\alpha_t(a, b) := \sup_{s \in [0, t]} |y_s(a) - a| \vee |y_s(b) - b| \leq t (|\vec{f}\text{Id}(a)| \vee |\vec{f}\text{Id}(b)|) \exp(\|\vec{f}\text{Id}\|_{\text{Lip}} t). \quad (6.49)$$

In particular, if $\vec{f}\text{Id}$ satisfies a 4-points control and is bounded, then $a \mapsto y(a)$ is Lipschitz from V to $(\mathcal{C}([0, T], V), \|\cdot\|_\infty)$.

Proof. Let us write $v := \vec{f}\text{Id} \in \mathcal{C}^1(V, V)$. Since

$$y_t(a) - a = \int_0^t (v(y_s(a)) - v(a)) \, ds + tv(a), \quad (6.50)$$

an immediate application of the Gronwall lemma gives (6.47).

Since v satisfies a 4-points control, for $a, b \in V$, with $\Delta_s(a, b) := |y_s(a) - a - y_s(b) + b|$,

$$\begin{aligned} &|v(y_s(a)) - v(a) - v(y_s(b)) + v(b)| \\ &\leq \widehat{v}(|y_s(a) - a| \vee |y_s(b) - b|) |y_s(a) - y_s(b)| \vee |a - b| + \check{v} \Delta_s(a, b). \end{aligned} \quad (6.51)$$

Besides,

$$|y_s(a) - y_s(b)| \leq |a - b| + \Delta_s(a, b).$$

Injecting (6.51) into (6.50) shows that

$$\Delta_t(a, b) \leq \alpha_t(a, b) t \widehat{v}(\alpha_t(a, b)) |a - b| + (\widehat{v}(\alpha_t(a, b)) + \check{v}) \int_0^t \Delta_s(a, b) \, ds$$

with $\alpha_t(a, b)$ given by (6.49). Again, the Gronwall lemma yields (6.48). ■

Baileul's approach by truncated logarithmic series

Here $U = \mathbb{R}^\ell$ for a dimension $\ell \geq 1$.

Let $V := (\vec{f}_1, \dots, \vec{f}_\ell)$ be a family of vector fields which acts on $C^1(V, V)$ and $\mathbf{x} \in \mathcal{G}^p(\mathbb{R}^\ell)$ be a weak-geometric p -rough path with $2 \leq p < 3$. By definition of the weak geometric rough paths, $\mathbf{x}^{i,j} + \mathbf{x}^{j,i} = \mathbf{x}^i \mathbf{x}^j$ for any $i, j \in \{1, \dots, \ell\}$. We denote by $[\vec{f}_i, \vec{f}_j] := \vec{f}_i \vec{f}_j - \vec{f}_j \vec{f}_i$, the Lie bracket of vector fields \vec{f}_i and \vec{f}_j . The Lie bracket is itself a vector field.

Assuming that V is smooth, we define for any $(s, t) \in \mathbb{T}^2$, $\alpha \in \mathbb{T}$ and $a \in V$, the solution $(\alpha, a) \mapsto y_{s,t}(\alpha, a)$ of the ODE,

$$y_{s,t}(\alpha, a) = a + \int_0^\alpha \vec{f}_i \text{Id}(y_{s,t}(\beta, a)) \mathbf{x}_{s,t}^i d\beta + \frac{1}{2} \int_0^\alpha [\vec{f}_i, \vec{f}_j] \text{Id}(y_{s,t}(\beta, a)) \mathbf{x}_{s,t}^{i,j} d\beta, \quad (6.52)$$

where we omit the summation over all indice $i, j \in \{1, \dots, \ell\}$. We write

$$\chi_{t,s}(a) := y_{s,t}(1, a) = \phi_{s,t} + \epsilon_{t,s}, \quad (6.53)$$

where, by iterating (6.52),

$$\begin{aligned} \phi_{t,s}(a) &:= a + \vec{f}_i \text{Id}(a) \mathbf{x}_{s,t}^i + \frac{1}{2} \vec{f}_i \vec{f}_j \text{Id}(a) \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^j + \frac{1}{2} [\vec{f}_i, \vec{f}_j] \text{Id}(a) \mathbf{x}_{s,t}^{i,j}, \\ \epsilon_{t,s}(a) &:= \int_0^1 \int_0^\beta \vec{f}_i \vec{f}_j (\text{Id}(y_{s,t}(\gamma, a)) - \text{Id}(a)) \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^j d\gamma d\beta \\ &\quad + \frac{1}{2} \int_0^1 \int_0^\beta [\vec{f}_i, \vec{f}_j] (\text{Id}(y_{s,t}(\gamma, a)) - \text{Id}(a)) \mathbf{x}_{s,t}^{i,j} d\gamma d\beta \\ &\quad + \frac{1}{2} \int_0^1 \int_0^\beta \vec{f}_i [\vec{f}_j, \vec{f}_k] \text{Id}(y_{s,t}(\gamma, a)) \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^{j,k} d\gamma d\beta. \end{aligned}$$

With the weak geometric property of $\mathbf{x} : \mathbf{x}_{s,t}^{i,j} + \mathbf{x}_{s,t}^{j,i} = x_{s,t}^i x_{s,t}^j$, we simplify the expression of ϕ such that

$$\begin{aligned} \phi_{t,s}(a) &= a + \vec{f}_i \text{Id}(a) \mathbf{x}_{s,t}^i + \frac{1}{2} \vec{f}_i \vec{f}_j \text{Id}(a) (\mathbf{x}_{s,t}^{i,j} + \mathbf{x}_{s,t}^{j,i}) + \frac{1}{2} [\vec{f}_i, \vec{f}_j] \text{Id}(a) \mathbf{x}_{s,t}^{i,j} \\ &= a + \vec{f}_i \text{Id}(a) \mathbf{x}_{s,t}^i + \vec{f}_i \vec{f}_j \text{Id}(a) \mathbf{x}_{s,t}^{i,j}. \end{aligned}$$

So ϕ corresponds to the Davie's almost flow defined in (6.34).

Proposition 6.8.10. *Assume that $V \in \mathcal{C}_b^{2+\gamma}$ with $2+\gamma > p$. Then, χ defined by (6.53) is an almost flow which generates a Lipschitz manifold of solutions. Moreover χ^π converges to the Davie's flow ψ of the Corollary 6.8.2.*

Proof. We proved in Lemma 6.8.1 that ϕ is a stable almost flow. We shall show that $\epsilon_{t,s}$ is a perturbation in the sense of Definition 21 and then we use Proposition 6.5.1 to

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conclude that χ is an almost flow which is in the galaxy of ϕ . We use Corollary 6.8.2 and Remark 6.7.6 to conclude the proof.

It is straightforward that $\epsilon_{t,t} = 0$. We start by computing an *a priori* estimate of $(\alpha, a) \mapsto y_{s,t}(\alpha, a)$, for any $(s, t) \in \mathbb{T}^2$, $a \in \mathbb{V}$ and $\alpha \in [0, 1]$,

$$\begin{aligned} |y_{s,t}(\alpha, a) - a| &\leq \alpha \|\vec{f}_i \text{Id}\|_\infty \|\mathbf{x}^i\|_p \omega_{s,t}^{1/p} + \frac{\alpha}{2} \|[\vec{f}_i, \vec{f}_j] \text{Id}\|_\infty \|\mathbf{x}^{i,j}\|_{\frac{p}{2}} \omega_{s,t}^{2/p} \\ &\leq \left(\|\vec{f}_i \text{Id}\|_\infty + \|[\vec{f}_i, \vec{f}_j] \text{Id}\|_\infty \omega_{0,T}^{1/p} \right) \|\mathbf{x}\|_p \omega_{s,t}^{1/p} \end{aligned} \quad (6.54)$$

$$\leq C_\infty \|\mathbf{x}\|_p \omega_{s,t}^{1/p}, \quad (6.55)$$

where $C_\infty := \|\vec{f}_i \text{Id}\|_\infty + \|[\vec{f}_i, \vec{f}_j] \text{Id}\|_\infty \omega_{0,T}^{1/p}$. With (6.55), we control the remainder $\epsilon_{t,s}$,

$$\begin{aligned} \|\epsilon_{t,s}\|_\infty &\leq \left[\|\mathbf{x}^i\|_p \|\mathbf{x}^j\|_p \|\vec{f}_i \vec{f}_j \text{Id}\|_{\text{Lip}} + \|\mathbf{x}^{i,j}\|_{2p} \|[\vec{f}_i, \vec{f}_j] \text{Id}\|_{\text{Lip}} \right] C_\infty \|\mathbf{x}\|_p \omega_{s,t}^{3/p} \\ &\quad + \|\vec{f}_i[\vec{f}_j, \vec{f}_k] \text{Id}\|_\infty \|\mathbf{x}^i\|_p \|\mathbf{x}^{j,k}\|_{2p} \omega_{s,t}^{3/p}, \end{aligned}$$

which proves (6.21).

To show the last estimation (6.22), we compute for any $(s, t) \in \mathbb{T}_+^2$ and any $a, b \in \mathbb{V}$,

$$\begin{aligned} \epsilon_{t,s}(b) - \epsilon_{t,s}(a) &= \underbrace{\int_0^1 \int_0^\beta \vec{f}_i \vec{f}_j (\text{Id}(y_{s,t}(\gamma, b)) - \text{Id}(b) - \text{Id}(y_{s,t}(\gamma, a)) + \text{Id}(a)) \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^j \, d\gamma \, d\beta}_{\text{I}} \\ &\quad + \underbrace{\frac{1}{2} \int_0^1 \int_0^\beta [\vec{f}_i, \vec{f}_j] [\text{Id}(y_{s,t}(\gamma, b)) - \text{Id}(b) - \text{Id}(y_{s,t}(\gamma, a)) + \text{Id}(a)] \mathbf{x}_{s,t}^{i,j} \, d\gamma \, d\beta}_{\text{II}} \\ &\quad + \underbrace{\frac{1}{2} \int_0^1 \int_0^\beta \vec{f}_i [\vec{f}_j, \vec{f}_k] [\text{Id}(y_{t,s}(\gamma, b)) - \text{Id}(y_{t,s}(\gamma, a))] \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^{j,k} \, d\gamma \, d\beta}_{\text{III}}. \end{aligned}$$

We assume that $\vec{f} \text{Id} \in \mathcal{C}_b^{2+\gamma}$, so $\vec{f}_i \vec{f}_j \text{Id} \in \mathcal{C}_b^{1+\gamma}$. It follows from Lemma 6.4.1 that $\vec{f}_i \vec{f}_j$ satisfies a 4-points control such that $\widehat{\vec{f}_i \vec{f}_j \text{Id}}(x) = C|x|^\gamma$ where C a positive constant with depends on the γ -Hölder norm of the derivative of $\vec{f}_i \vec{f}_j$. It follows that,

$$\begin{aligned} |\text{I}| &\leq C \sup_{\gamma \in [0,1], a \in \mathbb{V}} |y_{s,t}(\gamma, a) - a|^\gamma \left[\sup_{\gamma \in [0,1]} |y_{s,t}(\gamma, b) - y_{s,t}(\gamma, a)| + |b - a| \right] \|\mathbf{x}^{(1)}\|_p^2 \omega_{s,t}^{2/p} \\ &\quad + \|\vec{f}_i \vec{f}_j\|_{\text{Lip}} \sup_{\gamma \in [0,1]} |y_{s,t}(\gamma, b) - b - y_{s,t}(\gamma, a) + a| \|\mathbf{x}^{(1)}\|_p^2 \omega_{s,t}^{2/p}, \end{aligned}$$

which yields combining with (6.48), (6.49) and (6.55) to

$$|\text{I}| \leq CC_{\infty,T}^\gamma \|\mathbf{x}\|_p^\gamma \omega_{s,t}^{\gamma/p} (1 + C_T) |b - a| \|\mathbf{x}^{(1)}\|_p^2 \omega_{s,t}^{2/p} + \|\vec{f}_i \vec{f}_j\|_{\text{Lip}} C_T |b - a| \|\mathbf{x}^{(1)}\|_p^2 \omega_{s,t}^{2/p},$$

where C_T is a constant which is compute in (6.48). And finally, $|\text{I}| \leq \delta_T |b - a|$ where δ_T is a constant depending on the norms of V , \mathbf{x} which decreases to 0 when $T \rightarrow 0$. Similarly, we obtain the same estimation for II. To estimate III, we note that $\vec{f}_i[\vec{f}_j, \vec{f}_k] \text{Id} \in \mathcal{E}_b^\gamma$. Then with (6.48) it follows that $|\text{III}| \leq C_T'' \omega_{s,t}^{2/p} |b - a|^\gamma$, where C_T'' is another constant which has the same dependencies as C_T' . Thus $|\epsilon(b) - \epsilon(a)| \leq \delta |b - a| + C_T'' \omega_{s,t}^{2/p} |b - a|^\gamma$. This concludes the proof. \blacksquare

Remark 6.8.11. This results can be extend to the case U is a Banach case. It is an advantage compare to the Friz-Victoir's approach of Subsection 6.8.3.

Appendix

6.A The Davie lemma

In this section, we introduce our main tool to control the iterated products (Notation 8) on a partition.

Definition 26 (Successive points). Let π be a partition of $[0, T]$. Two points s and t of π are said to be *at distance k* if there are $k - 1$ points between them in π . Points at distance 1 are then *successive points* in π .

We now state the Davie lemma¹.

Lemma 6.A.1 (The Davie lemma, discrete time version). *Let us consider a family $U := \{U_{s,t}\}_{s,t \in \pi, s \leq t}$ with values in \mathbb{R}_+ satisfying for any $(r, s, t) \in \pi \cap \mathbb{T}^3$, $U_{r,r} = 0$,*

$$\begin{aligned} U_{r,s} &\leq D\varpi(\omega_{r,s}) \text{ when } r \text{ and } s \text{ are successive points,} \\ U_{r,t} &\leq (1 + \alpha_T)U_{r,s} + (1 + \alpha_T)U_{s,t} + B\varpi(\omega_{r,t}), \end{aligned} \quad (6.56)$$

for some constants $D \geq 1$, $B \geq 0$ and $\alpha_T \geq 0$ that decreases to 0 as $T \rightarrow 0$.

Then for all $T > 0$ such that $\varkappa(1 + \alpha_T)^2 < 1$,

$$U_{r,t} \leq A\varpi(\omega_{r,t}), \quad \forall (r, t) \in [0, T] \cap \pi^2, \quad (6.57)$$

$$\text{with } A := \frac{D(1 + \alpha_T)^2 + B(2 + \alpha_T)}{1 - \varkappa(1 + \alpha_T)^2}. \quad (6.58)$$

In particular, A does not depend on the choice of the partition.

¹What we call here the Davie lemma differs from the Davie lemma A and B in [FV10], and also from the one in [Dav07]. However, they all share the same key idea which is due to A.M. Davie.

6.A. THE DAVIE LEMMA

Proof. We perform an induction on the distance m between points in π .

When $m = 0$, $U_{r,r} = 0$ so (6.57) holds.

If $m = 1$, then (6.57) is true since that $A \geq D$.

Let us assume that this is true for any two points at distance m . Fix two points r and t at distance $m + 1$ in π . Hence, there exists two successive points s and s' in π such that

$$\omega_{r,s} \leq \frac{\omega_{r,t}}{2} \text{ and } \omega_{s',t} \leq \frac{\omega_{r,t}}{2},$$

as in the proof of Theorem 5.2.2. Applying (6.56) twice with (r, s, t) and (s, s', t) ,

$$\begin{aligned} U_{r,t} &\leq (1 + \alpha_T)U_{r,s} + (1 + \alpha_T)U_{s,t} + B\varpi(\omega_{r,t}) \\ &\leq (1 + \alpha_T)U_{r,s} + (1 + \alpha_T)^2(U_{s,s'} + U_{s',t}) + (2 + \alpha_T)B\varpi(\omega_{r,t}). \end{aligned}$$

Both $U_{r,s}$ and $U_{s,t}$ satisfy the induction property. With (6.2),

$$\begin{aligned} U_{r,t} &\leq 2A(1 + \alpha_T)^2\varpi\left(\frac{\omega_{r,t}}{2}\right) + (1 + \alpha_T)^2D\varpi(\omega_{s,s'}) + (2 + \alpha_T)B\varpi(\omega_{r,t}) \\ &\leq [A\kappa(1 + \alpha_T)^2 + D(1 + \alpha_T)^2 + (2 + \alpha_T)B]\varpi(\omega_{r,t}). \end{aligned}$$

Our choice of $A \geq D$ in (6.58) ensures the results at level $m + 1$. The control (6.57) is then true whatever the partition. \blacksquare

We could now state a continuous time version of the Davie lemma.

Lemma 6.A.2 (The Davie lemma, continuous time version). *Let us consider a family $U := \{U_{s,t}\}_{s,t \in \mathbb{T}^2}$ with values in \mathbb{R}_+ satisfying for any $(r, s, t) \in \mathbb{T}^3$,*

$$\begin{aligned} U_{r,s} &\leq E\varpi(\omega_{r,s}), \\ U_{r,t} &\leq (1 + \alpha_T)U_{r,s} + (1 + \alpha_T)U_{s,t} + B\varpi(\omega_{r,t}), \end{aligned} \tag{6.59}$$

for some constants $E \geq 1$, $B \geq 0$ and $\alpha_T \geq 0$ that decreases to 0 as $T \rightarrow 0$. Then for any T such that $\kappa(1 + \alpha_T)^2 < 1$,

$$\begin{aligned} U_{r,t} &\leq A\varpi(\omega_{r,t}), \quad \forall (r, t) \in \mathbb{T}^2, \\ \text{with } A &:= B \frac{(2 + \alpha_T)}{1 - \kappa(1 + \alpha_T)^2}. \end{aligned} \tag{6.60}$$

In particular, the choice of A in (6.60) does not depend on the bound E in (6.59).

Proof. The proof is similar as the one of Lemma 7 in [BL17a] from Eq. (31). \blacksquare

CHAPTER 6. NON LINEAR SEWING LEMMA II: LIPSCHITZ CONTINUOUS
FORMULATION

Part III

Inclusions différentielles perturbées

Chapter 7

Introduction aux inclusions différentielles

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7.1 Inclusion différentielle

Étant donné un horizon de temps $T > 0$ et une condition initiale $\xi \in \mathbb{R}^n$, une inclusion différentielle est une équation d'inconnue $y : [0, T] \rightarrow \mathbb{R}^n$ telle que

$$\dot{y}(t) \in F(t, y(t)), \text{ pour presque tout } t \in [0, T], \quad y(0) = \xi, \quad (7.1)$$

où \dot{y} indique la dérivée temporelle de y et F est une application de $\Omega \subset [0, T] \times \mathbb{R}^n$ dans les sous-ensembles de \mathbb{R}^n notés $\mathcal{P}(\mathbb{R}^n)$. Nous dirons que F est multivaluée, puisque chacune de ses images est un ensemble de points. De la même manière que pour les équations différentielles ordinaire (EDO), nous imposerons, pour obtenir une équivalence entre la forme intégrale et différentielle, que y soit absolument continue.

Dans son instructive introduction aux inclusions différentielles [Cel05], A. Cellina explique que l'étude de l'existence de solution à (7.1) débute dans les années trente en tant qu'objet mathématique purement théorique.

A priori, trouver une solution à (7.1) est plus simple que dans le cas d'une EDO

$$\dot{y}(t) = f(t, y(t)), \quad (7.2)$$

où $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ est une fonction. En effet, dans le cas de la première équation (7.1), la dérivée $\dot{y}(t)$ peut appartenir à un ensemble de valeurs alors que pour l'équation (7.2) l'égalité lui impose une valeur particulière. Il semble donc qu'on possède un plus grand degré de liberté pour trouver des solutions à des inclusions différentielles qu'à des EDO.

Cependant, il ne faut pas oublier que l'ensemble $F(t, y(t))$ dépend de lui-même y . Dans le cas d'une EDO où f est continue, si l'on approxime la solution par une suite de fonctions $(y_n)_{n \in \mathbb{N}}$, dès que $y_n(t) \rightarrow y(t)$ alors $f(t, y_n(t)) \rightarrow f(t, y(t))$ et donc l'égalité (7.2) entraîne la convergence de la dérivée $\dot{y}_n(t)$ vers $\dot{y}(t)$. Ce procédé n'est plus automatique pour (7.1) et l'on verra que prouver la convergence de \dot{y}_n constitue la principale difficulté pour montrer l'existence de solutions.

Contrairement aux EDO, l'existence de solution à (7.1) repose non seulement sur des notions de régularité de F mais aussi à des propriétés topologiques de son image (compacité, convexité).

7.2 Notions de régularité des fonctions multivaluées

Nous introduisons dans cette section les notions de continuité des fonctions multivaluées. Nous savons que pour les fonctions définies d'un espace métrique dans un autre, il existe deux définitions équivalentes de la continuité. La première qui est donnée en terme de voisinage et la seconde dite séquentielle. Dans le cas des fonctions multivaluées, nous devons différencier ces deux notions.

Soient (U, d) et (V, d) deux espaces métriques. Nous employons la même notation pour les distances associées à ces espaces métriques bien qu'elles peuvent différer. On note $\overset{\circ}{B}(c, R)$ la boule ouverte d'un espace métrique de rayon $R > 0$ centrée en un point c . Si S est un sous-ensemble non-vide de U , alors $\overset{\circ}{B}(S, R) := \{z \in U \mid d(x, z) < R, x \in S\}$. On suppose ici que F est une fonction multivaluée de U dans les parties non-vides de U .

Définition 7.2.1 (Semi-continuité supérieure). On dit que F est *semi-continue supérieurement* en $x \in U$ si pour tout $\epsilon > 0$ il existe un $\eta > 0$ tel que

$$F(\overset{\circ}{B}(x, \eta)) \subset \overset{\circ}{B}(F(x), \epsilon).$$

Si F vérifie cette propriété pour tout point $x \in U$, on dit que F est semi-continue supérieurement.

Définition 7.2.2 (Semi-continuité inférieure). On dit que F est *semi-continue inférieurement* en $x \in U$ si pour tout $y \in F(x)$ pour tout $\epsilon > 0$, il existe $\eta > 0$ tel que

$$\forall z \in \overset{\circ}{B}(x, \eta), F(z) \cap \overset{\circ}{B}(y, \epsilon) \neq \emptyset.$$

7.2. NOTIONS DE RÉGULARITÉ DES FONCTIONS MULTIVALUÉES

Cela signifie de manière équivalente que pour toute suite $(x_n)_{n \in \mathbb{N}}$ qui converge vers x et $y \in F(x)$, il existe $y_n \in F(x_n)$ tel que $(y_n)_{n \in \mathbb{N}}$ converge vers y . On dit que F est semi-continue inférieurement si elle l'est pour tout $x \in U$.

Remarque 7.2.1. La notion de semi-continuité supérieure correspond à la définition en terme de voisinage de la continuité alors que celle de semi-continu inférieure est une extension de la définition séquentielle.

Définition 7.2.3 (Continuité). Si F est semi-continue supérieurement et inférieurement en $x \in U$, on dit qu'elle est continue en $x \in U$. Si elle vérifie cette propriété pour tout $x \in U$, on dit que F est continue.

Remarque 7.2.2. Si l'on veut être plus général, nous pouvons remplacer les espaces de Banach par des espaces topologiques séparés (espaces de Hausdorff).

Lorsque F à des images compactes, nous pouvons définir la continuité en munissant l'ensemble des parties fermées non-vides de U noté $\bar{\mathcal{P}}(U)$ d'une distance.

Définition 7.2.4 (Distance de Hausdorff). La distance d'Hausdorff est définie pour tout $S_1, S_2 \in \bar{\mathcal{P}}(U)$ par

$$d_h(S_1, S_2) := \max \left(\sup_{s_1 \in S_1} \inf_{s_2 \in S_2} d(s_1, s_2), \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} d(s_1, s_2) \right).$$

Proposition 7.2.3 ([AC12, Corollary 1 p.67]). *Une fonction multivaluée avec des images compactes non-vides est continue si et seulement si elle est continue au sens de la distance d'Hausdorff.*

Donnons maintenant quelques exemples de fonctions multivaluées ayant différentes régularités. Soient

$$F_1(t) := \begin{cases} -1 & \text{si } t < 0 \\ [-1, 1] & \text{si } t = 0 \\ 1 & \text{si } t > 0 \end{cases}, \quad F_2(t) := \begin{cases} 0 & \text{si } t = 0 \\ [-1, 1] & \text{si } t \neq 0 \end{cases}.$$

Pour $t \neq 0$, F_1 et F_2 sont continues. En $t = 0$, F_1 est semi-continu supérieurement puisque pour tout $\epsilon > 0$ et $\eta > 0$ $F_1((-\eta, \eta)) = [-1, 1] \subset (-1 - \epsilon, 1 + \epsilon)$, mais F_2 ne l'est pas car $F_2((-\eta, \eta)) = [-1, 1] \not\subset (-\epsilon, \epsilon)$. Par contre, F_2 est semi-continue inférieurement puisque pour tout $\epsilon > 0$, $y \in F(0) = \{0\}$ et $\eta > 0$ tel que

$$\forall z \in (-\eta, \eta), \quad \{0\} \subset F_2(z) \cap (-\epsilon, \epsilon)$$

mais F_1 ne l'est pas car pour tout $y \in F(0) = [-1, 1]$, et pour $\eta > z > 0$, $F(z) = 1$ et donc $F(z) \cap (-\epsilon, \epsilon) = \emptyset$ si ϵ est trop petit.

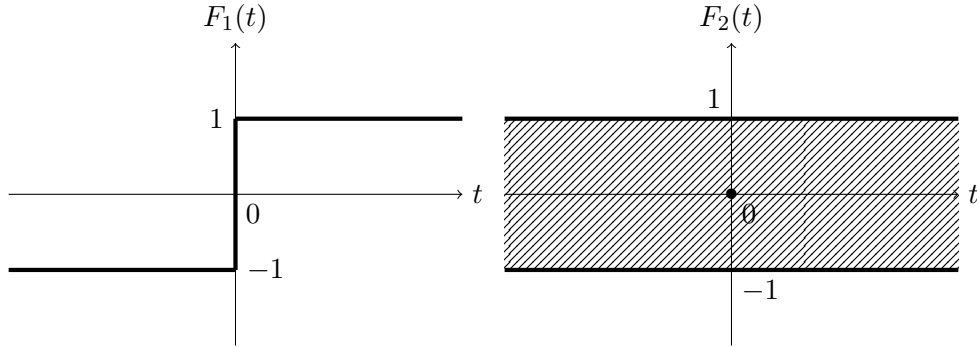


Figure 7.1: À gauche la représentation du graphe de la fonction multivaluée F_1 qui est semi-continue inférieurement. À droite la représentation du graphe de la fonction multivaluée F_2 qui est semi-continue supérieurement.

7.3 Un premier résultat d'existence par sélection continue

Une première stratégie à adopter pour résoudre (7.1) est de chercher une sélection de F , c'est-à-dire une fonction $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ telle que pour tout $t \in [0, T]$, $x \in \mathbb{R}^n$,

$$f(t, x) \in F(t, x).$$

Ainsi si l'on prouve l'existence de solution à l'EDO $\dot{y}(t) = f(t, y(t))$ alors y sera une solution à l'inclusion différentielle (7.2). D'après le théorème de Peano, la régularité requise pour l'existence d'une telle solution est la continuité de la sélection f .

Nous nous demandons donc quelles sont les conditions pour qu'il y ait existence d'une sélection continue.

Théorème 7.3.1 (Michael, [AC12, Theorem 1 p. 82]). *Soient U un espace métrique et V un espace de Banach. Si F est semi-continue inférieurement de U dans les sous-ensembles non-vides compacts convexes de V alors il existe une sélection continue de F .*

Remarque 7.3.2. L'exemple F_1 (Figure 7.2) montre que l'hypothèse de semi-continuité supérieure n'est pas suffisante pour qu'il y ait existence d'une sélection continue.

L'hypothèse de convexité du théorème 7.3.1 n'est pas superflue. Voici un contre-exemple tiré de [AC12]. On va construire une application F_3 continue multivaluée de la boule unité fermée $\bar{B}(0, 1)$ de \mathbb{R}^2 à valeurs de les sous-ensembles compacts du cercle unité C de \mathbb{R}^2 et qui n'a pas de sélection continue. Soit

$$\phi : (\rho, \theta) \in (0, 1] \times [0, 2\pi) \mapsto (\rho \cos(\theta), \rho \sin(\theta)) \in \bar{B}(0, 1) \setminus \{(0, 0)\},$$

alors ϕ est une bijection continue. On définit l'application multivaluée

$$\Phi : (\rho, \theta) \in [0, 1] \times [0, 2\pi) \mapsto [\theta + \pi - \pi(1 - \rho), \theta + \pi + \pi(1 - \rho)] \setminus [0, 2\pi),$$

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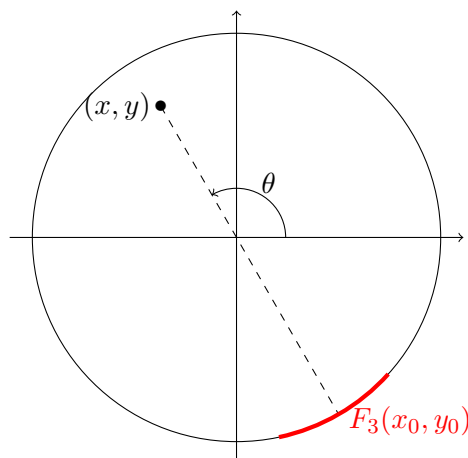


Figure 7.1: Représentation graphique de la fonction multivaluée F_3 .

puis

$$F_3 : (x, y) \in \bar{B}(0, 1) \mapsto \begin{cases} \phi(1, \Phi \circ \phi^{-1}(x, y)), & \text{si } (x, y) \neq (0, 0) \\ C & \text{si } (x, y) = (0, 0) \end{cases} .$$

S'il existe une sélection continue f_3 de F_3 , alors d'après le théorème de Brouwer f_3 a un point fixe $(x_0, y_0) \in C$. Donc $(x_0, y_0) = f_3(x_0, y_0) \in F_3(x_0, y_0)$. Or les images de F_3 sont des arcs de cercle, donc $(x_0, y_0) = (\cos(\theta_0), \sin(\theta_0))$ pour un certain $\theta_0 \in [0, 2\pi)$. Par ailleurs, $F_3(x_0, y_0) = \phi(1, \Phi(1, \theta_0)) = (-\cos(\theta_0), -\sin(\theta_0))$, ce qui entraîne $\cos(\theta_0) = \sin(\theta_0) = 0$. C'est absurde, F_3 n'a ni sélection continue, ni point fixe.

7.4 Résultats d'existence

Nous avons vu dans la section précédente que si l'on cherche à construire une solution de l'inclusion différentielle (7.1) par le biais d'une sélection continue, on se retrouve confronté à deux facteurs limitants : la régularité de F et la convexité de ses valeurs.

On suppose toujours que F est une fonction multivaluée de $[0, T] \times \mathbb{R}^n$ à valeurs dans les compacts de \mathbb{R}^n .

Théorème 7.4.1 ([AC12, Theorem 3 p.98, Corollary 1 p.138]). *Il existe une solution à l'inclusion différentielle (7.1) si F est semi-continue supérieurement à valeurs compactes convexes ou bien si F est continue à valeurs compactes (pas forcément convexes).*

Il existe deux schémas de preuve du théorème 7.4.1: par approximation de la solution et par point fixe.

La difficulté de la première approche est de construire une approximation dont la dérivée converge. Si les images de F sont convexes, la convergence faible de la dérivée suffit pour

conclure. Dans le cas contraire, on peut utiliser un théorème de compacité analogue à celui de Ascoli-Arzelà. Cette stratégie est due à A.F. Filippov [Fil63].

Dans l'approche par point fixe, on considère l'application multivaluée

$$J : f \in B_{\text{Lip}}^\xi \mapsto J(f) := \left\{ g \in B_{\text{Lip}}^\xi \mid \dot{g}(t) \in F(t, f(t)) \text{ presque sûrement sur } [0, T] \right\},$$

où B_{Lip}^ξ est l'ensemble des fonctions M -lipschitziennes (pour un certain $M \geq 0$ fixé) et égalent à ξ en 0. La stratégie est alors de construire une sélection continue ϕ de J qui a d'après le théorème de Brouwer un point fixe correspondant à une solution de (7.1).

7.5 Inclusions différentielles stochastiques

Considérons maintenant une inclusion différentielle de la forme

$$dy(t) \in F(t, y(t))dt + \Sigma_B(y(t))dB_t + \Sigma_H(y(t))dB_t^H, \quad (7.3)$$

avec Σ_B, Σ_H sont des fonctions multivaluées de \mathbb{R}^n dans $\mathcal{P}(\mathbb{R}^{n \times d})$, B (resp. B^H) est un mouvement brownien (resp. un mouvement brownien fractionnaire) à valeurs dans \mathbb{R}^d . Lorsque Σ_B et Σ_H sont nulles, nous avons vu les conditions d'existence de solution.

La première étude du cas stochastique a été faite par G. Da Prato et H. Frankowska dans [DPF94]. Ils se placent dans le cas où Σ_H est nulle et montrent l'existence de solution forte sans hypothèse de convexité sur les valeurs de F et Σ_B . L'existence de solution faible a été montré plus tard par M. Kisielewicz [Kis05] avec F et Σ_B semi-continues inférieurement. A notre connaissance, le cas où Σ_H est non nulle a seulement été traité dans [LV15] avec Σ_H à valeurs convexes. Cependant, d'autres formes d'inclusions différentielles ont été étudiées dans le cadre des trajectoires rugueuses, par exemple le processus de rafle [CMdF17].

7.6 Les objectifs de cette partie

Les objectifs de cette partie sont de donner des conditions d'existence de solution à l'inclusion différentielle perturbée par un bruit rugueux, c'est-à-dire de la forme

$$dy(t) \in F(t, y(t))dt + \sigma(y(t))d\mathbf{X}_t,$$

où $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n \times d}$ est un champ de vecteurs régulier, \mathbf{X} est un trajectoire rugueuse α -Hölder avec $\alpha \in (1/3, 1/2]$. Ici F est une fonction multivaluée d'un sous-ensemble $\Omega \subset [0, T] \times \mathbb{R}^n$ à valeurs dans les sous-ensembles compacts non-vides de \mathbb{R}^n . Nous montrons alors l'existence de solutions sous deux hypothèses différentes

- F est continue;

7.6. LES OBJECTIFS DE CETTE PARTIE

- F est semi-continue inférieurement et bornée sur Ω .

Les preuves de ces résultats reposent sur une adaptation au cadre des espaces Hölder de l'approche par point fixe décrite section 7.4. On remplacera, entre autres, l'espace de travail B_{Lip}^ξ par B_α^ξ où les trajectoires sont supposées non plus Lipschitz mais Hölder.

Le lecteur peut se rapporter à la partie I pour une introduction à la théorie des trajectoires rugueuses.

Les résultats de cette partie se trouvent dans le chapitre 8.

CHAPTER 7. INTRODUCTION AUX INCLUSIONS DIFFÉRENTIELLES

Chapter 8

Differential inclusions perturbed by rough paths

This chapter consists in a joint work with Laure Coutin under finalisation.

Abstract

We study a differential inclusion perturbed by a rough path. We prove that without convexity hypothesis on the multivalued application, there exists solution.

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8.1 Notations

We gather here the main notations of the article.

For a subset $S \subset \mathbb{R}$, we denote by $t \in \mathbb{R} \mapsto \mathbb{1}_S(t)$ the indicator function of S . We set $\|\cdot\|$ the Euclidian norm of \mathbb{R}^d (d a positive integer). Given that $x \in \mathbb{R}^d$, and A a non-empty closed subset of \mathbb{R}^d , we define $d(x, A) = \inf_{y \in A} \|x - y\|$.

If Y is a set, we denote $\mathcal{P}(Y)$ the power set of Y . The space of non-empty, compact subset of \mathbb{R}^d is denoted by $\mathcal{P}_{\text{comp}}(\mathbb{R}^d)$.

Let Ω be an open subset of $\mathbb{R} \times \mathbb{R}^d$ such that $(0, \xi) \in \Omega$ with $\xi \in \mathbb{R}^d$. Let $F : \Omega \rightarrow \mathcal{P}_{\text{comp}}(\mathbb{R}^d)$ be a set-valued map. We denote by $\mathbf{X} := (X, \mathbb{X})$ an α -Hölder rough path (see Definition 3.4.1) with $\alpha \in (1/3, 1/2]$ taking its values in \mathbb{R}^m (m a positive integer).

Let us fix a time horizon $T > 0$. The space of rough paths controlled by X is denoted by $D_X^{2\alpha}([0, T])$ (see Definition 3.4.7). We endow the space $C^0([0, T], \mathbb{R}^d)$ of continuous paths from $[0, T]$ to \mathbb{R}^d with the uniform topology. Let $L^\infty([0, T], \mathbb{R}^d)$ be the space of measurable almost everywhere bounded functions from $[0, T]$ to \mathbb{R}^d .

On $[0, T]$ we denote $\|\cdot\|_{\infty, T}$ the uniform norm, $\|\cdot\|_{\alpha, T}$ the α -Hölder semi-norm, $\|\cdot\|_{\text{Lip}}$ the Lipschitz semi-norm and $\|(\cdot, \cdot)\|_{2\alpha, T}$ the semi-norm of the space of rough paths controlled by X . We denote $L(\mathbb{R}^m, \mathbb{R}^d)$ the set of linear continuous maps from \mathbb{R}^m to \mathbb{R}^d .

Given $M \geq 0$ be a constant, We introduce the following balls :

- $\overset{\circ}{B}$ (resp. B) the open (resp. close) unit ball of \mathbb{R}^d
- $\overset{\circ}{B}_\infty := \{u \in C^0([0, T], \mathbb{R}^d) \mid \|u\|_{\infty, T} \leq 1\}$,
- $B_{\text{Lip}}^\xi := \{u \in C^0([0, T], \mathbb{R}^d) \mid \|u\|_{\text{Lip}} \leq M, u(0) = \xi\}$,
- $B_\alpha^\xi := \{u \in C^0([0, T], \mathbb{R}^d) \mid \|u\|_{\alpha, T} \leq M, u(0) = \xi\}$,
- $B_X^{2\alpha, \xi} := \{(y, y') \in D_X^{2\alpha}([0, T]) \mid \|(y, y')\|_{2\alpha, T} \leq M, y(0) = \xi, y'(0) = \sigma(\xi)\}$, where y' denotes the Gubinelli's derivative (see Definition 3.4.7).

Let $\sigma : \mathbb{R}^d \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$ be a function, which the regularity is defined later.

8.2 Main results

We start by defining a solution to a differential inclusion perturbed by a rough path. Then we outline the two theorems (Theorem 8.2.2 and Theorem 8.2.3) of this article which ensure the existence of a solution for two regularity assumptions on the set-valued map.

Definition 8.2.1 (Solution of a perturbed differential inclusion). We say that the triplet of paths (z, z', x) is a solution on $[0, T]$ of the differential inclusion perturbed by (σ, \mathbf{X}) starting from $\xi \in \mathbb{R}^d$ and denoted by $PDI(F, \sigma, \mathbf{X}, \xi, T)$ if

1. The paths $(z, z') \in D_X^{2\alpha}([0, T])$ and x is an absolute continuous path from $[0, T]$ to \mathbb{R}^d .
2. For almost every $t \in [0, T]$, $x'(t) \in F(t, z(t))$.
3. For every $t \in [0, T]$, $z(t) = \xi + x(t) + \int_0^t \sigma(z(s)) d\mathbf{X}_s$.

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Hypothesis 1. Let $\sigma : \mathbb{R}^d \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$ be C_b^2 , where C_b^2 is the space of bounded functions with the derivatives up to order 2 bounded.

Hypothesis 2. We assume that $F : \Omega \rightarrow \mathcal{P}_{\text{comp}}(\mathbb{R}^d)$ is continuous (see Definition 7.2.3).

Theorem 8.2.2. *Under Hypothesis 1 and 2, there is a time horizon $T > 0$ such that there is a solution to $PDI(F, \sigma, \mathbf{X}, \xi, T)$.*

Hypothesis 3. We assume that $F : \Omega \rightarrow \mathcal{P}_{\text{comp}}(\mathbb{R}^d)$ is lower semicontinuous (see Definition 7.2.1). Besides, there is a time horizon $T > 0$ and a real $M \geq 0$ such that $[0, T] \times \{\xi + TMB\} \subset \Omega$ and $\|F(t, x)\| \leq M$ on $[0, T] \times \{\xi + T^\alpha MB\}$.

Theorem 8.2.3. *Under Hypothesis 1 and 3, there is a time horizon $T > 0$ such that there is a solution to $PDI(F, \sigma, \mathbf{X}, \xi, T)$.*

Remark 8.2.4. The existence of such M and T in Hypothesis 3 is guaranteed when F is continuous.

Remark 8.2.5. The proofs of Theorem 8.2.2 and Theorem 8.2.3 are respectively given in Section 8.5.1 and Section 8.5.2. They are based on the proofs of the non-perturbed case developed in [AC12]. More precisely, we use a fixed point method which is described in Chapter 7.

8.3 Perturbed differential inclusion

Proposition 8.3.1 ensures the existence of a solution to a perturbed differential inclusion (Definition 8.2.1) if there is a particular selection of the set-valued map F . This makes the link between non-perturbed and perturbed differential inclusions and is used in the proofs of Theorem 8.2.2 and Theorem 8.2.2.

Proposition 8.3.1. *We assume Hypothesis 1. If there is a continuous map (for the uniform norm) $\phi : B_\alpha^\xi \rightarrow B_{\text{Lip}}^\xi$ such that for all $u \in B_\alpha^\xi$, $\phi(u)(0) = \xi$ and almost all $t \in [0, T]$, $d(\phi(u)(t))/dt \in F(t, u(t))$. Then, there is an horizon time $T > 0$ such that there is a solution of $PDI(F, \sigma, \mathbf{X}, \xi, T)$.*

Before proving this lemma, we need a compactness result on $B_X^{2\alpha, \xi}$ (Lemma 8.3.2) and classical estimations of the rough integral (Lemma 8.3.4).

Lemma 8.3.2. *The ball $B_X^{2\alpha, \xi}$ is a compact convex subset of*

$$C^0([0, T], \mathbb{R}^d) \times C^0([0, T], \mathbb{R}^{d \times m}).$$

Remark 8.3.3. Lemma 8.3.2 is proved in [FH14] with the β -Hölder norm with $\beta < \alpha$. Here it is done with the uniform norm.

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Proof of Lemma 8.3.2. Let $(y^n, y'^n)_{n \in \mathbb{N}}$ be a sequence in $B_X^{2\alpha, \xi}$. On the one hand, any $s, t \in [0, T]$, $\|y'_{s,t}\| \leq |t - s|^\alpha$ because $\|(y, y')\|_{2\alpha, T} \leq M$. It follows that y^n is uniformly equicontinuous on $[0, T]$. Moreover, $\|y'(t)\| \leq \|\sigma(\xi)\| + MT^\alpha$ for any $t \in [0, T]$. Then, by Arzela-Ascoli theorem there is a subsequence of $y'^{\eta(n)}$ ($\eta : \mathbb{N} \rightarrow \mathbb{N}$ increasing function) which converges uniformly on $[0, T]$ to a path that we denote y' with $\|y'\|_{\alpha, T} \leq M$. On an other hand, for every $s, t \in [0, T]$,

$$\begin{aligned} \|y_{s,t}^{\eta(n)}\| &\leq \|y'^{\eta(n)}(s)\| \|X\|_{\alpha, T} |t - s|^\alpha + \|Ry'^{\eta(n)}\|_{2\alpha, T} |t - s|^{2\alpha} \\ &\leq [(\|\sigma(\xi)\| + MT^\alpha) \|X\|_{\alpha, T} + MT^\alpha] |t - s|^\alpha, \end{aligned} \quad (8.1)$$

which proves the uniform equicontinuity of $y^{\eta(n)}$. In evaluating (8.1) for $s = 0$, we get the uniform bound, $\|y^{\eta(n)}(t)\| \leq \|\xi\| + [(\|\sigma(\xi)\| + MT^\alpha) \|X\|_{\alpha, T} + MT^\alpha] T^\alpha$, for every $t \in [0, T]$. Again by Arzela-Ascoli theorem, there is a subsequence $y^{\theta(n)}$ ($\theta : \mathbb{N} \rightarrow \mathbb{N}$ an increasing function) which converges uniformly to a path that we denote y .

With the relation $y'_{s,t} = y'(s)X_{s,t} + Ry'_{s,t}$ for any $s, t \in [0, T]$, we prove that $Ry^{\theta(n)}$ converges uniformly to Ry on $[0, T]^2$ and $\|Ry\|_{2\alpha, T} \leq M$. Which prove that $(y, y') \in B_X^{2\alpha, \xi}$ and then that $B_X^{2\alpha, \xi}$ is compact. This concludes the proof. \blacksquare

Lemma 8.3.4 ([FH14, Theorem 4.10, Lemma 7.3]). *For $(y, y') \in D_X^{2\alpha}([0, T])$, and $f \in C_b^1$, then $(f(y), f(y)') := (f(y), \dot{f}(y)y') \in D_X^{2\alpha}([0, T])$ and if $f \in C_b^2$*

$$\|(f(y), f(y)')\|_{2\alpha, T} \leq C \|f\|_{C_b^2} \left(\|y'_0\| + \|(y, y')\|_{2\alpha, T} \right), \quad (8.2)$$

where $C \geq 0$ is a constant uniform on $[0, T]$ which decreases to 0 when T goes to 0. Moreover, for any $s, t \in [0, T]$,

$$\left\| \int_s^t y_r d\mathbf{X}_r - y(s)X_{s,t} - y'_s \mathbb{X}_{s,t} \right\| \leq C \|\mathbf{X}\|_{\alpha, T} \|y, y'\|_{2\alpha, T} |t - s|^{3\alpha}, \quad (8.3)$$

where $C \geq 0$ is a constant uniform over $T \in (0, 1]$.

Proof of Proposition 8.3.1. Let $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\beta \subset \mathcal{C}^\alpha$ with $1/3 < \alpha < \beta < 1/2$. We define for $(y, y') \in D_X^{2\alpha}([0, T])$, the map $\Phi : B_X^{2\alpha, \xi} \rightarrow B_X^{2\alpha, \xi}$ such that for every $t \in [0, T]$,

$$\Phi(y, y')(t) := \left(\phi(y)(t) + \int_0^t \sigma(y(s)) d\mathbf{X}_s, \sigma(y(t)) \right).$$

Firstly, we justify the definition in showing that $(z, z') := \Phi(y, y') \in B_X^{2\alpha, \xi}$ for all

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$(y, y') \in B_X^{2\alpha, \xi}$. Indeed, with (8.2) and (8.3) for any $s, t \in [0, T]$,

$$\begin{aligned}
& \|z_{s,t} - z'_s X_{s,t}\| \\
& \leq \|\phi(y)_{s,t}\| + \left\| \int_s^t \sigma(y(r)) d\mathbf{X}_r - \sigma(y(s)) X_{s,t} \right\| \\
& \leq M|t-s| + \|\sigma(y)'\|_{\infty, T} \|\mathbb{X}\|_{\alpha, T} |t-s|^{2\alpha} + C \|\mathbf{X}\|_{\alpha, T} \|\sigma(y), \sigma(y)'\|_{2\alpha, T} |t-s|^{3\alpha} \\
& \leq [MT^{1-2\alpha} + \|\sigma(y)'\|_{\infty, T} \|\mathbb{X}\|_{\alpha, T} + C \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} (\|y'_0\| + \|y, y'\|_{\alpha, T} T^\alpha)] |t-s|^{2\alpha} \\
& \leq [MT^{1-2\alpha} + \|\sigma(y)'\|_{\infty, T} \|\mathbb{X}\|_{\beta, T} T^{\beta-\alpha} + C \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} (\|y'_0\| + \|y, y'\|_{\alpha, T} T^\alpha)] |t-s|^{2\alpha}.
\end{aligned} \tag{8.4}$$

We recall that $(y, y') \in B_X^{2\alpha, \xi}$ implies that $\|(y, y')\|_{2\alpha, T} \leq M$ and $y'(0) = \sigma(\xi)$. It follows from (8.5)

$$\begin{aligned}
& \|z_{s,t} - z'_s X_{s,t}\| \\
& \leq [MT^{1-2\alpha} + \|\sigma(y)'\|_{\infty, T} \|\mathbb{X}\|_{\beta, T} T^{\beta-\alpha} + C \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} (\|\sigma(\xi)\| + M) T^\alpha] |t-s|^{2\alpha} \\
& \leq T^{\beta-\alpha} [M + \|\sigma\|_{C_b^1} \|y\|_{\infty, T} + C \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} (\|\sigma(\xi)\| + M)] |t-s|^{2\alpha},
\end{aligned} \tag{8.6}$$

$$\leq T^{\beta-\alpha} [M + \|\sigma\|_{C_b^1} \|y\|_{\infty, T} + C \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} (\|\sigma(\xi)\| + M)] |t-s|^{2\alpha}, \tag{8.7}$$

because $\|\mathbb{X}\|_{2\alpha, T} \leq T^{\beta-\alpha} \|\mathbb{X}\|_{2\beta, T}$, $T^{\beta-\alpha} \geq T^{1-2\alpha}, T^\alpha$. Since $\|R^y\|_{2\alpha, T} \leq M$,

$$\|y\|_{\infty, T} \leq \|\xi\| + \|\sigma(\xi)\| \|X\|_{\alpha, T} T^\alpha + MT^{2\alpha}. \tag{8.8}$$

Moreover, we obtain with (8.7) and (8.8) that

$$\|R^z\|_{2\alpha, T} \leq T^{\beta-\alpha} [M + 2C(1 + \|\mathbf{X}\|_{\alpha, T})(1 + T^\alpha) \|\sigma\|_{C_b^2} (\|\xi\| + \|\sigma(\xi)\| + M)]. \tag{8.9}$$

Using the fact that $(y, y') \in B_X^{2\alpha, \xi}$,

$$\begin{aligned}
\|z'\|_{\alpha, T} & \leq \|\sigma\|_{C_b^1} \|y\|_{\alpha, T} \\
& \leq \|\sigma\|_{C_b^1} (\|y'\|_{\infty, T} \|X\|_{\alpha, T} + \|R^y\|_{2\alpha, T} T^\alpha) \\
& \leq \|\sigma\|_{C_b^1} (\|y'_0\| + \|y'\|_{\alpha, T} T^\alpha) \|X\|_{\beta, T} T^{\beta-\alpha} + \|R^y\|_{2\alpha, T} T^\alpha \\
& \leq \|\sigma\|_{C_b^1} (\|\sigma(\xi)\| + MT^\alpha) \|X\|_{\beta, T} T^{\beta-\alpha} + MT^\alpha \\
& \leq \|\sigma\|_{C_b^1} (\|\sigma(\xi)\| + MT^\alpha) \|X\|_{\beta, T} + M T^{\beta-\alpha}
\end{aligned} \tag{8.10}$$

With T small enough, we obtain with (8.9) and (8.10) that, $\|z, z'\|_{2\alpha, T} \leq M$, which proves that $\Phi(B_X^{2\alpha, \xi}) \subset B_X^{2\alpha, \xi}$.

Secondly, we show that Φ is continuous for the uniform topology on $B_X^{2\alpha, \xi}$. With an horizon time $T > 0$ small enough, for $(y, y') \in B_X^{2\alpha, \xi}$, $\|y\|_{\alpha, T} \leq (\|\sigma(\xi)\| + M) \|X\|_{\beta, T} + M T^{\beta-\alpha} \leq 1$, which yields to $y \in B_\alpha^\xi$. Let $\psi := \Phi - (\phi, 0)$. On the one hand, with continuity of ϕ , for any $\epsilon > 0$ there is $\delta_1 > 0$ such that for all $(x, x'), (y, y') \in B_X^{2\alpha, \xi}$ and $\|x - y\|_{\infty, T} \leq \delta_1$,

$$\|\phi(x) - \phi(y)\|_{\infty, T} \leq \epsilon/2. \tag{8.11}$$

On the other hand, for all $(x, x'), (y, y') \in B_X^{2\alpha, \xi}$, with (8.2)-(8.3) and choosing $\alpha' < \alpha$,

$$\begin{aligned} & \|\psi(y, y') - \psi(x, x')\|_{\infty, T} \\ & \leq \left\| \int_0^{\cdot} [\sigma(y(s)) - \sigma(x(s))] d\mathbf{X}_s \right\|_{\infty, T} + \|\sigma(y) - \sigma(x)\|_{\infty, T} \end{aligned} \quad (8.12)$$

$$\begin{aligned} & \leq C \|\mathbf{X}\|_{\alpha, T} \|\sigma(y) - \sigma(x), \sigma(y)' - \sigma(x)'\|_{2\alpha', T} T^{3\alpha} + \|\sigma\|_{C_b^1} \|y - x\|_{\infty, T} \\ & \leq CT^{3\alpha} \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} \|y - x, y' - x'\|_{2\alpha', T} + \|\sigma\|_{C_b^1} T^{2\alpha} \|R^y - R^x\|_{2\alpha', T} \\ & \leq (CT^{3\alpha} \|\mathbf{X}\|_{\alpha, T} \|\sigma\|_{C_b^2} + \|\sigma\|_{C_b^1} T^{2\alpha}) \|y - x, y' - x'\|_{2\alpha', T}. \end{aligned} \quad (8.13)$$

Besides, we get the bound

$$\begin{aligned} & \|y - x, y' - x'\|_{2\alpha', T} \\ & \leq \|y' - x'\|_{\alpha', T} + \|R^y - R^x\|_{2\alpha', T} \end{aligned} \quad (8.14)$$

$$\begin{aligned} & \leq 2^{1-\alpha'/\alpha} \left[\|y' - x'\|_{\infty, T}^{1-\alpha'/\alpha} \|y' - x'\|_{\alpha', T}^{\alpha'/\alpha} + \|R^y - R^x\|_{\infty, T}^{1-\alpha'/\alpha} \|R^y - R^x\|_{2\alpha', T}^{\alpha'/\alpha} \right] \\ & \leq 2^{1-\alpha'/\alpha} M^{\alpha'/\alpha} \left[\|y' - x'\|_{\infty, T}^{1-\alpha'/\alpha} + (2\|y - x\|_{\infty, T} + \|y - x\|_{\infty, T} \|X\|_{\alpha, T} T^\alpha)^{1-\alpha'/\alpha} \right]. \end{aligned} \quad (8.15)$$

Then, combining (8.13) and (8.15), we choose $\delta_2 > 0$ such that $\|y - x\|_{\infty, T} + \|y' - x'\|_{\infty, T} < \delta_2$ implies $\|\psi(y, y') - \psi(x, x')\|_{\infty, T} \leq \epsilon/2$.

Thus, with $\delta := \min(\delta_1, \delta_2)$, for all $(x, x'), (y, y') \in B_X^{2\alpha, \xi}$ such that $\|x - y\|_{\infty, T} + \|x' - y'\|_{\infty, T} < \delta$, then $\|\Phi(x, x') - \Phi(y, y')\|_{\infty, T} < \epsilon$. This proves the continuity of Φ for the uniform topology.

Finally, by Lemma 8.3.2 and the continuity of Φ we apply the Schauder theorem. There is a controlled rough path $(z, z') \in B_X^{2\alpha, \xi}$ such that $\Phi(z, z') = (z, z')$.

We conclude the proof in showing that $(z, z', \phi(z))$ is a solution of $PDI(F, \sigma, X, \xi, T)$. ■

8.4 Tools for differential inclusions

We introduce tools as measurable selection and continuous partition segment which are used in the proofs of Theorem 8.2.2 and Theorem 8.2.3. The material of this section is taken from [AC12].

8.4.1 Measurable selection results

Definition 8.4.1 (Selection). Let Y_1, Y_2 be two non-empty sets. We say that $h : Y_1 \rightarrow Y_2$ is a selection of a set-valued map $H : Y_1 \rightarrow \mathcal{P}(Y_2)$ if for all $x \in Y_1$, $h(x) \in H(x)$.

Theorem 8.4.2 (Kuratowski–Ryll–Nardzewski). *Let F from a closed subset I of \mathbb{R} into closed subset of \mathbb{R}^n such that inverse image of open set are Borel. Then, there is a measurable selection $f : I \rightarrow \mathbb{R}^n$ of F .*

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Remark 8.4.3. If F is upper semicontinuous, F has a closed graph and the assumptions of Theorem 8.4.2 hold.

Theorem 8.4.4 ([AC12, Theorem 2, p. 91]). *Let F be a continuous set-valued map from an interval I into the closed subset of \mathbb{R}^d . Let $v : I \rightarrow \mathbb{R}^d$ be a measurable function. Then, there is a measurable selection f of F such that for almost $t \in I$,*

$$\|v(t) - f(t)\| = d(v(t), F(t)).$$

8.4.2 Continuous partition of a segment

Definition 8.4.5 (Continuous partition of unity). Let \mathcal{U} be a compact space. A family $\{\psi_i\}_{i=1}^m$ is called a continuous partition of unity subordinated to the finite open covering $\{\mathcal{U}_i\}_{i=1}^m$ of \mathcal{U} if

- each $\psi_i : \mathcal{U} \rightarrow \mathbb{R}_+$ is continuous
- the support of each ψ_i is closed and contained in \mathcal{U}_i
- for every $x \in \mathcal{U}$, $\sum_{i=1}^m \psi_i(x) = 1$.

Remark 8.4.6. A such continuous partition always exists.

Definition 8.4.7 (Continuous partition of a segment). Let $\{\psi_i\}_{i \in \{1, \dots, m\}}$ be a continuous partition of unity on a compact metric space \mathcal{U} . We set $\tau_0 := 0$ and by induction we define for each $i \in \{1, \dots, m\}$ and every $u \in \mathcal{U}$, $\tau_i(u) := \tau_{i-1}(u) + \psi_i(u)T$. Setting $J_i(u) := [\tau_{i-1}(u), \tau_i(u)]$, we call a *continuous partition of $[0, T]$* corresponding to $\{\psi_i\}_{i \in \{1, \dots, m\}}$ the collection $\{J_i\}_{i \in \{1, \dots, m\}}$.

Remark 8.4.8. By induction $u \mapsto \tau_i(u)$ is continuous from \mathcal{U} to $[0, T]$ for all $i \in \{1, \dots, m\}$.

Lemma 8.4.9 ([AC12, Proposition 2 p.130]). *Let $\{J_i\}_{i \in \{1, \dots, m\}}$ be a continuous partition of $[0, T]$ (Definition 8.4.7). Let v_i belongs of the ball of radius $M \geq 0$ of $L^\infty([0, T], \mathbb{R}^d)$. We define, for all $t \in [0, T]$,*

$$v(u)(t) := \sum_{i=1}^m v_i(t) \mathbf{1}_{J_i(u)}(t).$$

Then

- 1 for all $\epsilon > 0$, there is $\delta > 0$ such that for all $u, w \in \mathcal{U}$ and $\|u - w\|_{\infty, T} \leq \delta$ implies $\text{meas}\{t \in [0, T] \mid \|v(u)(t) - v(w)(t)\| > \epsilon\} < \epsilon$.
- 2 the map $u \in \mathcal{U} \mapsto v(u) \in L^1([0, T], \mathbb{R}^d)$ is continuous where \mathcal{U} and $L^1([0, T], \mathbb{R}^d)$ are respectively endowed with the uniform topology and the L^1 one.

8.5 Proofs of mains results

In this section, we prove Theorem 8.2.2 and Theorem 8.2.3.

8.5.1 If F is continuous

In this section we prove Theorem 8.2.2. For that we combine Proposition 8.3.1 with the following lemma which ensures the existence of a selection.

Proposition 8.5.1. *Under Hypothesis 2, there is an horizon time $T > 0$ and a map $\phi : B_\alpha^\xi \rightarrow B_{\text{Lip}}^\xi$ continuous for the uniform topology such that for all $u \in B_\alpha^\xi$, $\phi(u)(0) = \xi$ and almost all $t \in [0, T]$, $d(\phi(u)(t))/dt \in F(t, u(t))$.*

Remark 8.5.2. Here, the selection ϕ matches with the selection of Proposition 8.3.1.

In Theorem 8.2.2, we assume the continuity of F . Moreover, because F has compact values, the continuity is characterized by the Hausdorff distance.

Definition 8.5.3. Let M, N be two nonempty closed subsets of \mathbb{R}^d . We set $d(y, N) := \inf_{x \in N} d(y, x)$ and $\delta(M, N) := \sup_{y \in M} d(y, N)$. We define the Hausdorff distance d_h as

$$d_h(M, N) := \max(\delta(M, N), \delta(N, M)).$$

We prove the Proposition 8.5.1 by induction. The following lemma is used to initialize this induction.

Lemma 8.5.4. *Under Hypothesis 2, for all $\epsilon > 0$, there is a continuous map $g : B_\alpha^\xi \rightarrow L^1([0, T], \mathbb{R}^d)$ such that for all $u \in B_\alpha^\xi$ and almost every $t \in [0, T]$,*

$$d(g(u)(t), F(t, u(t))) \leq \epsilon.$$

Moreover for all $\eta > 0$, there is $\delta > 0$ such that $\|u - w\|_{\infty, T} \leq \delta$ with $u, w \in B_\alpha^\xi$ implies $\text{meas}\{t \in [0, T] \mid \|g(u)(t) - g(w)(t)\| > 0\} < \eta$.

Proof Lemma 8.5.4. It is the same proof as from [AC12][Lemma 1 p.131]. We work here with the ball of an Hölder space endowed with the uniform norm instead of the ball of Lipschitz space.

Let $\epsilon > 0$. By continuity of F , we know that F is uniformly continuous on the compact $[0, T] \times \{\xi + TMB\}$. Let $\delta > 0$ such that for all $\|x - y\| < \delta$, with $x, y \in \mathbb{R}^d$,

$$d_h(F(t, x), F(t, y)) < \epsilon, \tag{8.16}$$

where d_h is the Hausdorff distance (Definition 8.5.3).

According to Lemma 8.3.2, B_α^ξ is compact for the uniform topology. Let $\{\mathcal{U}_i\}_{i \in \{1, \dots, m\}}$ be an open covering of B_α^ξ such that $\text{diam}(\mathcal{U}_i) < \delta$ for each $i \in \{1, \dots, m\}$. Let $\{\psi_i\}_{i \in \{1, \dots, m\}}$

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be a continuous partition of unity subordinated to $\{\mathcal{U}_i\}_{i \in \{1, \dots, m\}}$. For each $i \in \{1, \dots, m\}$, we choose arbitrary $u_i \in \mathcal{U}_i$. Given that $t \mapsto F(t, u_i(t))$ is upper semicontinuous with closed values, there is a measurable selection v_i of $t \mapsto F(t, u_i(t))$, according to Theorem 8.4.2.

Thus we define, the continuous map $g : B_\alpha^\xi \rightarrow L^1([0, T], \mathbb{R}^d)$ such as in Lemma 8.4.9:

$$g(u)(t) := \sum_{i=1}^m v_i(t) \mathbf{1}_{J_i(u)}(t),$$

where $\{J_i(\cdot)\}_{i \in \{1, \dots, m\}}$ is a continuous partition of $[0, T]$ corresponding to $\{\psi_i\}_{i \in \{1, \dots, m\}}$. According to Lemma 8.4.9, g is continuous from B_α^ξ to $L^1([0, T], \mathbb{R}^d)$ and for every $\eta > 0$ there is $\delta' > 0$ such that $\|u - w\|_{\infty, T} < \delta'$ implies that

$$\text{meas}\{t \in [0, T] \mid \|g(u)(t) - g(w)(t)\| > 0\} < \eta.$$

For each $u \in B_\alpha^\xi$ and $t \in [0, T]$, there is $j \in \{1, \dots, m\}$ such that $t \in J_j(u)$. It follows that $\psi_j(u) > 0$ because $J_j \neq \emptyset$, thus $u \in \mathcal{U}_j$. Then, with (8.16)

$$\begin{aligned} d(g(u)(t), F(t, u(t))) &\leq d(v_j(t), F(t, v_j(t))) + d_h(F(t, v_j(t)), F(t, u(t))) \\ &\leq \epsilon, \end{aligned}$$

which concludes the proof. ■

Proof of Proposition 8.5.1. This proof is based on the proof from [AC12][Theorem 2 p. 132]. However, we work here with the unit ball of an Hölder space instead of a Lipschitz one.

By Lemma 8.5.4, there is a continuous map $g_0 : B_\alpha^\xi \rightarrow L^1([0, T], \mathbb{R}^d)$ such that for all $u, v \in B_\alpha^\xi$ and $t \in [0, T]$

$$d(g_0(u)(t), F(t, u(t))) \leq 1/2$$

and there is a $\delta_0 > 0$ such that $\|u - w\|_{\infty, T} < \delta_0$ implies

$$\text{meas}\{t \in [0, T] \mid \|g_0(u)(t) - g_0(w)(t)\| > 0\} < 1/2.$$

Now, we prove by induction that for all integer $n \geq 0$, there is a continuous map $g_n : B_\alpha^\xi \rightarrow L^1([0, T], \mathbb{R}^d)$ and a constant $\delta_n > 0$ such that for all $u, w \in B_\alpha^\xi$,

- 1) for almost every $t \in [0, T]$, $d(g_n(u)(t), F(t, u(t))) < 1/2^{n+1}$
- 2) if $\|u - w\|_{\infty, T} < \delta_n$ then

$$\text{meas}\{t \in [0, T] \mid \|g_n(u)(t) - g_n(w)(t)\| > 0\} < 1/2^{n+1}$$

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3) if $n \geq 1$, then

$$\text{meas}\{t \in [0, T] \mid \|g_n(u)(t) - g_{n-1}(u)(t)\| > 1/2^n\} < 1/2^n.$$

We assume that g_n and δ_n are defined for an integer $n \geq 0$.

By uniform continuity of F on the compact $[0, T] \times \{\xi + MTB\}$, we choose $\eta > 0$ with $\eta < \delta_n$ such that for all $x, y \in \mathcal{U}_i^{n+1}$ such that $\|x - y\| < \eta$, $d(F(t, x), F(t, y)) < 1/2^{n+2}$ for all $t \in [0, T]$. Let $\{\mathcal{U}_i^{n+1}\}_{i \in \{1, \dots, m\}}$ be an open covering of B_α^ξ such that $\text{diam}(\mathcal{U}_i^{n+1}) < \eta$ for each $i \in \{1, \dots, m\}$ and let $u_i^{n+1} \in \mathcal{U}_i^{n+1}$. Let $\{\psi_i^{n+1}\}_{i \in \{1, \dots, m\}}$ be a continuous partition of unity subordinate to $\{\mathcal{U}_i^{n+1}\}_{i \in \{1, \dots, m\}}$. According to Theorem 8.4.4, there is a measurable selection $v_i^{n+1} : [0, T] \rightarrow \mathbb{R}^d$ of $t \mapsto F(t, u_i^{n+1}(t))$ such that

$$\|g_n(u_i^{n+1})(t) - v_i^{n+1}(t)\| = d(g_n(u_i^{n+1})(t), F(t, u_i^{n+1}(t))). \quad (8.17)$$

We define $g^{n+1} : B_\alpha^\xi \rightarrow L^1([0, T], \mathbb{R}^d)$ such that

$$g_{n+1}(u)(t) := \sum_{i=1}^m v_i^{n+1}(t) \mathbf{1}_{J_i^{n+1}(u)}(t),$$

where $\{J_i^{n+1}\}_{i \in \{1, \dots, m\}}$ the continuous partition of $[0, T]$ associated to $\{\psi_i^{n+1}\}_{i \in \{1, \dots, m\}}$.

We check that g^{n+1} verifies the properties 1), 2), 3).

For every $t \in [0, T]$ and $u \in B_\alpha^\xi$, there is an integer $j \in \{1, \dots, m\}$ such that $t \in J_j^{n+1}(u)$. It follows that $\psi_j^{n+1}(u) > 0$ because $J_j(u) \neq \emptyset$. Then, $u \in \mathcal{U}_j^{n+1}$. Hence,

$$\begin{aligned} d(g^{n+1}(u)(t), F(t, u(t))) &\leq d(v_j^{n+1}(t), F(t, u_j^{n+1}(t))) + d_h(F(t, u_j^{n+1}(t)), F(t, u(t))) \\ &\leq 1/2^{n+2}, \end{aligned}$$

which proves 1).

From Lemma 8.4.9, with $\eta > \delta_n$ small enough, 2) holds.

It remains to show 3). With (8.17)

$$\begin{aligned} \|g_{n+1}(u)(t) - g_n(u)(t)\| &\leq \|v_j^{n+1}(t) - g_n(u_j^{n+1})(t)\| + \|g_n(u_j^{n+1})(t) - g_n(u)(t)\| \\ &= d(g_n(u_j^{n+1})(t), F(t, u_j^{n+1}(t))) + \|g_n(u_j^{n+1})(t) - g_n(u)(t)\|. \end{aligned}$$

By induction hypothesis, $d(g_n(u_j^{n+1})(t), F(t, u_j^{n+1}(t))) < 1/2^{n+1}$. It follows that

$$\text{meas}\{t \in [0, T] \mid \|g_{n+1}(u)(t) - g_n(u)(t)\| > 1/2^{n+1}\} < 1/2^{n+1},$$

because $\text{meas}\{t \in [0, T] \mid \|g_n(u_j^{n+1})(t) - g_n(u)(t)\| > 1/2^{n+1}\} < 1/2^{n+1}$. This concludes the induction.

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From 3) for each $u \in B_\alpha^\xi$, $\{g_n(u)\}_{n \in \mathbb{N}}$ is a Cauchy sequence which converges in measure to a measurable function g . Moreover, a subsequence converges to g pointwise everywhere. By compactness of $F(t, u(t))$, it follows from 1) that for almost every $t \in [0, T]$, $g(u)(t) \in F(t, u(t))$.

We define the map $\phi : B_\alpha^\xi \rightarrow B_{\text{Lip}}^\xi$ such that for all $t \in [0, T]$,

$$\phi(u)(t) := \xi + \int_0^t g(u)(s) ds.$$

It remains to show that ϕ is continuous. Fix $\epsilon > 0$. We choose n such that $(4M + T)/2^{n+1} < \epsilon$ and we set $\delta := \delta_n$. For any $u, w \in B_\alpha^\xi$ such that $\|u - w\|_{\infty, T} < \delta$,

$$\begin{aligned} \int_0^t \|g(u)(s) - g_n(u)(s)\| ds &\leq \sum_{l=n}^{\infty} \int_0^t \|g_{l+1}(u)(s) - g_l(u)(s)\| \\ &\leq \sum_{l=n}^{\infty} 2M/2^{l+1} + 2T/2^{n+1} = (M + T)/2^{n-1}. \end{aligned} \quad (8.18)$$

Hence, with 3) and (8.18),

$$\begin{aligned} \|\phi(u)(t) - \phi(w)(t)\| &\leq \int_0^t \|g(u)(s) - g(w)(s)\| ds \\ &\leq \int_0^t \|g(u)(s) - g_n(u)(s)\| + \int_0^t \|g_n(u)(s) - g_n(w)(s)\| + \int_0^t \|g(w)(s) - g_n(w)(s)\| \\ &\leq 2(M + T)/2^{n-1} + 2M/2^{n+1} \leq (4M + T)/2^{n+1} < \epsilon, \end{aligned}$$

which proves the continuity of ϕ and concludes the proof. ■

Proof Theorem 8.2.2. From hypothesis 2 and Proposition 8.5.5 we know that there is a continuous map $\phi : B_\alpha^\xi \rightarrow \mathbb{R}^d$ such that for all $u \in B_\alpha^\xi$, $\phi(u)(0) = \xi$ and for almost $t \in [0, T]$, $d(\phi(u)(t))/dt \in F(t, u(t))$. Thus adding Hypothesis 1 we show with Proposition 8.3.1 that the map $\Phi : B_X^{2\alpha, \xi} \rightarrow B_X^{2\alpha, \xi}$ such that for every $t \in [0, T]$,

$$\Phi(y, y')(t) := \left(\phi(y)(t) + \int_0^t \sigma(y(s)) d\mathbf{X}_s, \sigma(y(t)) \right)$$

has a fixed point in $B_X^{2\alpha, \xi}$. ■

8.5.2 With lower semi-continuity of F

In this section we prove Theorem 8.2.2. For that we combine Proposition 8.3.1 with the following lemma which ensures the existence of a selection.

Proposition 8.5.5. *Under Hypothesis 3, there exists a map $\phi : B_\alpha^\xi \rightarrow B_{\text{Lip}}^\xi$ continuous for the uniform topology such that for all $u \in B_{\text{Lip}}^\xi$ $\phi(u)(0) = \xi$ and almost all $t \in [0, T]$, $d(\phi(u)(t))/dt \in F(t, u(t))$.*

Unlike the previous section, F is no longer continuous. However, Lemma 8.5.6 allows us to find a finite open covering of B_α^ξ which allows us to approximate $t \mapsto F(t, u(t))$. The proof of this lemma can be found in [AC12, Lemma 1 p. 135].

Lemma 8.5.6 ([AC12, Lemma 1 p. 135]). *Let H be a lower semi-continuous set-valued map from a compact subset K of $[0, T] \times \mathbb{R}^n$ into the subsets of \mathbb{R}^n . For $\epsilon > 0$, we set*

$$\eta^\epsilon(t, x) := \sup \left\{ \eta \mid \bigcap_{(\tau, \epsilon) \in (t, x) + \eta \overset{\circ}{B}} H(\tau, \epsilon) + \epsilon \right\}.$$

Then,

- there is a constant $\eta^\epsilon > 0$ such that for all $(t, x) \in K$, we have $\eta^\epsilon(t, x) > \eta^\epsilon$
- for every continuous u , $(t, u(t))$ in K , there exists a measurable $v : t \mapsto v(t)$ such that $\|(t, x) - (t, u(t))\| < \eta^\epsilon$ implies

$$d(v(t), H(t, x)) \leq \epsilon.$$

We recall the famous Lusin's theorem.

Theorem 8.5.7 (Lusin, [EG15, Theorem 1.14 p. 21]). *Let $f : [0, T] \rightarrow \mathbb{R}$ be a measurable function. For all $\epsilon > 0$, there is a compact $C \in [0, T]$ such that $\text{meas}([0, T] \setminus C) < \epsilon$ and f is continuous on C .*

The following proposition is a basic property of lower semicontinuous set-valued maps. We recall that the domain of a set-valued map H is the set of x such that $H(x) \neq \emptyset$.

Proposition 8.5.8 ([AC12, Proposition 5 p.44]). *Let Y be a metric space, G be a lower semicontinuous set-valued map from Y to subsets of \mathbb{R}^d and $g : Y \rightarrow \mathbb{R}^d$ be a continuous map. Let $x \in Y \mapsto \epsilon(x) \in \mathbb{R}_+$ be a lower semicontinuous map. Then, the map*

$$x \mapsto (g(x) + \epsilon(x) \overset{\circ}{B}) \bigcap G(x),$$

is lower semicontinuous on its domain.

Proof of Proposition 8.5.5. The following proof is an adaptation of [AC12, Theorem 1 p.136]. We apply Lemma 8.5.6, with $H := F$, $\epsilon := 1$. Let $\{\mathcal{U}^i := u^i + \eta_0 \overset{\circ}{B}\}_{i \in \{1, \dots, m_0\}}$ be a finite open covering of the compact B_α^ξ , where η_0 denotes η^ϵ defined in Lemma 8.5.6.

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It is possible because B_α^ξ is compact for the uniform norm. Let v_i be the measurable function associated to u^i such that by Lemma 8.5.6

$$d(v_i(t), F(t, u(t))) < 1,$$

for all $u \in \mathcal{U}^i$. For $u \in B_\alpha^\xi$, we set,

$$g_0(u)(t) := \sum_{i=1}^{m_0} v_i(t) \mathbb{1}_{J_i(u)}(t),$$

where $J_i(u)$ is a continuous partition of $[0, T]$ associated to a continuous partition of unity of $\{\mathcal{U}^i\}_{i \in \{1, \dots, m_0\}}$. Then, for $t \in [0, T]$ and $u \in B_\alpha^\xi$, when $\mathbb{1}_{J_i(u)}(t) = 1$, $u \in \mathcal{U}^i$, it follows that $d(v_i(t), F(t, u(t))) < 1$ and then $d(g_0(u)(t), F(t, u(t))) < 1$. Moreover, with Lemma 8.4.9, $g_0 : u \in B_\alpha^\xi \mapsto g_0(u) \in L^1([0, T], \mathbb{R}^d)$ is continuous and there is $\delta_0 > 0$ such that $\text{meas}\{t \in [0, T] \mid \|g_0(u) - g_0(w)\|_{\infty, T} > 0\} < 1$ when $\|u - w\|_{\infty, T} < \delta$ and $u, w \in B_\alpha^\xi$.

Now, we show by induction that for any integer n there is m_n measurable maps $v_i^n : [0, T] \rightarrow MB$, $i \in \{1, \dots, m_n\}$, $\{J_i^n(\cdot) := [\tau_{i-1}^n(\cdot), \tau_i^n(\cdot)]\}_{i \in \{1, \dots, m_n\}}$ a continuous partition of $[0, T]$, and a continuous map $g_n : B_\alpha^\xi \rightarrow L^1([0, T], \mathbb{R}^d)$ such that for all $t \in [0, T]$,

$$g_n(u)(t) := \sum_{i=1}^{m_n} v_i^n(t) \mathbb{1}_{J_i^n(u)}(t).$$

Moreover there is $\delta_n > 0$ such that

1 for all $u \in B_\alpha^\xi$, $d(g_n(u)(t), F(t, u(t))) < 1/2^n$, for almost every $t \in [0, T]$,

2 for all $u, w \in B_\alpha^\xi$ such that $\|u - w\|_{\infty, T} < \delta_n$,

$$\text{meas}\{t \in [0, T] \mid \|g_n(u)(t) - g_n(w)(t)\| > 0\} < 1/2^n.$$

3 if $n \geq 1$, for $u \in B_\alpha^\xi$,

$$\text{meas}\{t \in [0, T] \mid \|g_n(u)(t) - g_{n-1}(u)(t)\| > 2^{-n+2}\} < 1/2^n.$$

We assume that the above conditions hold for an integer n .

According to Theorem 8.5.7, there is an open set S_{n+1} such that for all $i \in \{1, \dots, m_n\}$ the maps v_i^{n-1} are continuous on $[0, T] \setminus S_{n+1}$ and $\text{meas}(S_{n+1}) < 2^{-(n+1)}$. From Remark 8.4.8 let $\delta > 0$ such that for all $u, v \in B_\alpha^\xi$ and $\|u - w\|_{\infty, T} < \delta$, for every $i \in \{1, \dots, m_n\}$, $|\tau_i^n(u) - \tau_i^n(w)| < 2^{-(n+1)}(2m_n)^{-1}$. There is a finite open cover $\{\mathcal{U}_j := \hat{u}_j + \delta \overset{\circ}{B}_\infty\}_{j \in \{1, \dots, N\}}$ of B_α^ξ .

Then, for any $u \in \mathcal{U}_j$, for

$$t \in [\tau_i^n(\hat{u}_j) + 2^{-(n+1)}(2m_n)^{-1}, \tau_{i+1}^n(\hat{u}_j) - 2^{-(n+1)}(2m_n)^{-1}],$$

$g_n(u)(t) = g_n(\hat{u}_j)(t) = v_i^n(t)$. It follows that for any $u \in \mathcal{U}_j$, $t \in [0, T] \setminus E_j \cup S_{n+1} \mapsto g_n(u)(t)$ is continuous, where

$$E_j := \bigcup_{i=1}^{m_n} (\tau_i^n(\hat{u}_j) - 2^{-(n+1)}(2m_n)^{-1}, \tau_i^n(\hat{u}_j) + 2^{-(n+1)}(2m_n)^{-1}).$$

Indeed, if $t \in [0, T] \setminus E_j \cup S_{n+1}$, for all $i \in \{1, \dots, m_n\}$ and $u \in \mathcal{U}_j$, $t \neq \tau_i^n(u)$.

We define

$$\rho_j^{n+1}(t) := \begin{cases} 2M & \text{if } t \in E_j \cup S_{n+1} \\ 2^{-n} & \text{if } t \in [0, T] \setminus E_j \cup S_{n+1} \end{cases}$$

and $F_j(t, x) := (g_n(\hat{u}_j)(t) + \rho_j^{n+1}(t)\overset{\circ}{B}) \cap F(t, x)$ for every $(t, x) \in \Omega$.

Let us show that F_j is strict, i.e. $F_j(t, x) \neq \emptyset$, for any $(t, x) \in \Omega$ such that

$$\|(t, x) - (t, \hat{u}_j(t))\| \leq \delta.$$

Indeed, if $t \in E_j \cap S_{n+1}$, because $g_n(\hat{u}_j) \in MB$ and $F(t, x) \in B$ according to Hypothesis 3, $F_j(t, x)$ is not empty. Otherwise, if $t \in [0, T] \setminus E_j \cup S_{n+1}$, and $\|x - \hat{u}_j(t)\| < \delta$, let $u \in \mathcal{U}_j$ such that $u(t) = x$. Then, $g_n(u)(t) = g_n(\hat{u}_j)(t)$ for $t \in [0, T] \setminus E_j \cap S_{n+1}$. It follows by induction assumption 1 that $d(x, F(t, x)) = d(g_n(u)(t), F(t, u(t))) < 2^{-n}$. Hence, $F_j(t, x)$ is strict for $t \in [0, T]$ and $\|x - \hat{u}_j(t)\| < \delta$.

The epigraph of $t \mapsto \rho_j^{n+1}(t)$ is closed, then it is lower semi-continuous. It follows with Proposition 8.5.8 that $(t, x) \in [0, T] \times \mathcal{U}_j \mapsto F_j(t, x)$ is lower semi-continuous.

With Lemma 8.5.6, setting $H := F_j$, $\epsilon := 1/2^{n+2}$, let η_j be the constant η^ϵ . Let $\{\mathcal{V}_i := u_j^i + \eta_j \overset{\circ}{B}_\infty\}_{i \in \{1, \dots, N'\}}$ be a finite open covering of $B_\alpha^\xi \cap (\hat{u}_j + \delta \overset{\circ}{B}_\infty)$.

According to Lemma 8.5.6, for every $i \in \{1, \dots, N'\}$ there is a measurable map v_j^i such that $\|(t, x) - (t, u_j^i(t))\| < \eta_j$ implies

$$d(v_j^i(t), F_j(t, x)) \leq 1/2^{n+2} < 1/2^{n+1}. \quad (8.19)$$

Setting $\mathcal{U}_j^i := \mathcal{U}_j \cap \mathcal{V}_i$, $\{\mathcal{U}_j^i\}_{i,j}$ is a finite open covering of B_α^ξ . Let $\{J_j^i\}$ be a continuous partition of $[0, T]$ associated to the open covering $\{\mathcal{U}_j^i\}$.

We define for all $u \in B_\alpha^\xi$ and $t \in [0, T]$,

$$g_{n+1}(u)(t) := \sum_{i=1}^{N'} \sum_{j=1}^N v_j^i(t) \mathbb{1}_{J_j^i(u)}(t).$$

We check that g_{n+1} verifies our induction hypothesis.

8.5. PROOFS OF MAINS RESULTS

Indeed, for $u \in B_\alpha^\xi$ and $t \in [0, T]$, $t \in J_j^i(u)$ and $u \in u_j^i + \eta_j \mathring{B}_\infty$ for some i, j . Hence, with (8.19)

$$d(g_{n+1}(u)(t), F_j(t, u(t))) = d(v_j^i(t), F_j(t, u(t))) < 1/2^{n+1}. \quad (8.20)$$

But $F_j(t, u(t)) \subset F(t, u(t))$, which proves the point 1 of our induction.

The point 2 is a direct consequence of Lemma 8.4.9, with $\delta_n := \delta$, $\epsilon := 2^{-n}$.

To show the point 3, let $u \in B_\alpha^\xi$, we fix $t \in [0, T] \setminus (E_j \cup S_{n+1})$ for some j . We recall that $g_n(u)(t) = g_n(\hat{u}_j)(t)$. By definition $\rho_j^{n+1}(t) = 2^{-n}$, then

$$F_j(t, x) \subset g_n(\hat{u}_j)(t) + 2^{-n} \mathring{B} = g_n(u)(t) + 2^{-n} \mathring{B}.$$

Moreover, $g_{n+1}(u)(t) = v_j^i(t)$ for some i, j , it follows from (8.20) that

$$d(g_{n+1}(u)(t), g_n(u)(t) + 2^{-n} \mathring{B}) \leq d(v_j^i(t), F_j(t, u(t))) \leq 2^{-(n+2)}.$$

Hence, $\|g_{n+1}(u)(t) - g_n(u)(t)\| \leq 2^{-n+1}$ except on $E_j \cup S_{n+1}$ with measure less than 2^{-n} . This proves 3 and concludes the induction.

From 3) for each $u \in B_\alpha^\xi$, $\{g_n(u)\}_{n \in \mathbb{N}}$ is a Cauchy sequence which converges in measure to a measurable function g . Moreover, a subsequence converges to g pointwise everywhere. By compactness of $F(t, u(t))$, it follows from 1) that for almost every $t \in [0, T]$, $g(u)(t) \in F(t, u(t))$.

We define the map $\phi : B_\alpha^\xi \rightarrow B_{\text{Lip}}^\xi$ such that

$$\phi(u)(t) := \xi + \int_0^t g(u)(s) ds.$$

It remains to show that ϕ is continuous. Fix $\epsilon > 0$. We choose n such that $(4M + T)/2^{n+1} < \epsilon$ and we set $\delta := \delta_n$. For any $u, w \in B_\alpha^\xi$ such that $\|u - w\|_{\infty, T} < \delta$,

$$\begin{aligned} \int_0^t \|g(u)(s) - g_n(u)(s)\| ds &\leq \sum_{l=n}^{\infty} \int_0^t \|g_{l+1}(u)(s) - g_l(u)(s)\| \\ &\leq \sum_{l=n}^{\infty} 2M/2^{l+1} + 2T/2^{n+1} = (M + T)/2^{n-1}. \end{aligned} \quad (8.21)$$

Hence, with 3) and (8.21),

$$\begin{aligned} \|\phi(u)(t) - \phi(w)(t)\| &\leq \int_0^t \|g(u)(s) - g(w)(s)\| ds \\ &\leq \int_0^t \|g(u)(s) - g_n(u)(s)\| + \int_0^t \|g_n(u)(s) - g_n(w)(s)\| + \int_0^t \|g(w)(s) - g_n(w)(s)\| \\ &\leq 2(M + T)/2^{n-1} + 2M/2^{n+1} \leq (4M + T)/2^{n+1} < \epsilon, \end{aligned}$$

which proves the continuity of ϕ and concludes the proof. ■

CHAPTER 8. DIFFERENTIAL INCLUSIONS PERTURBED BY ROUGH PATHS

Proof Theorem 8.2.3. From Hypothesis 3 and Proposition 8.5.5 we know that there is a continuous map $\phi : B_\alpha^\xi \rightarrow \mathbb{R}^d$ such that for all $u \in B_\alpha^\xi$, $\phi(u)(0) = \xi$ and for almost $t \in [0, T]$, $d(\phi(u)(t))/dt \in F(t, u(t))$. Thus adding Hypothesis 1 we show with Proposition 8.3.1 that the map $\Phi : B_X^{2\alpha, \xi} \rightarrow B_X^{2\alpha, \xi}$ such that for every $t \in [0, T]$,

$$\Phi(y, y')(t) := \left(\phi(y)(t) + \int_0^t \sigma(y(s)) d\mathbf{X}_s, \sigma(y(t)) \right)$$

has a fixed point in $B_X^{2\alpha, \xi}$. ■

Chapter 9

Perspectives

Nous proposons certaines pistes de recherche en continuité de ce travail de thèse.

9.1 Pistes de recherche sur les flots

9.1.1 Flots stochastiques et flots rugueux

D'après le corollaire 6.8.2 du chapitre 6, nous savons qu'une EDR dirigée par une trajectoire rugueuse α -Hölder avec $\alpha \in (1/3, 1/2]$ a un unique flot de solutions dès que le champ de vecteurs est de classe $C_b^{2+\gamma}$ avec $2 + \gamma > 1/\alpha$. Par ailleurs, ce flot est lipschitzien. Cependant, lorsque la trajectoire rugueuse correspond à la trajectoire du mouvement brownien muni du tenseur du second ordre, nous savons (cf. chapitre 4) qu'il existe un flot stochastique lipschitzien au sens d'Itô dès que le champ de vecteurs est $1 + \gamma$.

Nous souhaiterions montrer que les flots rugueux et stochastiques correspondent dans le cas du mouvement brownien. Lorsque le champ de vecteurs a une régularité supérieure à $2 + \gamma$ les flots stochastiques et rugueux correspondent [Lyo98, LV06, FV10]. Cependant par un argument de régularisation du champ de vecteurs et de stabilité des estimations, il semble possible de réduire cette régularité jusqu'à $1 + \gamma$ pour des flots lipschitzien au sens d'Itô.

9.1.2 Flots génériques

D'après le théorème de Peano, nous savons qu'en dimension finie, une équation différentielle ordinaire (EDO) a toujours au moins une solution lorsque le champ de vecteurs est continu. Ce n'est plus le cas en dimension infinie. J. Dieudonné a montré dans [Die50] que lorsque l'EDO est définie dans un espace de Banach, la continuité du champ de vecteurs ne suffit plus à assurer l'existence d'une solution.

Cependant, F.S. De Blasi et J. Myjak ont prouvé dans [DBM83] qu'un schéma d'Euler associé à une EDO définie dans un espace de Banach converge vers une solution pour «presque» tout point de départ et tout champ de vecteurs.

Précisons cela. On considère un espace de Banach V .

Définition 9.1.1. Une propriété est dite *générique* dans V si elle est vraie sur un sous-ensemble comeagre de V , c'est-à-dire qu'il contient une intersection dénombrable d'ouverts denses de V . On dira aussi que cette propriété est vraie *presque partout*.

Remarque 9.1.1. Cette définition s'étend aux espaces de Baire, donc en particulier aux espaces métriques complets. Cependant, nous nous restreignons ici aux espaces de Banach. On rappelle que dans un espace de Baire une intersection dénombrable d'ouverts denses est dense.

Remarque 9.1.2. Il faut bien différencier l'appellation «presque partout» de cette définition avec celle de la théorie de la mesure.

On note $C_b^0(V, V)$ l'espace des fonctions continues bornées de V dans V . Soient $f \in C_b^0(V, V)$, $a \in V$ et $\pi_n := \{t_j^n\}_{j=1}^n$ une subdivision du segment $[0, 1]$ telle que le pas de la subdivision $|\pi_n|$ tende vers 0 quand n tend vers l'infini. On note $y_a : [0, 1] \rightarrow V$ une solution à l'EDO :

$$\forall t \in [0, 1], \quad \frac{dy(t)}{dt} = f(y(t)), \quad y(0) = a. \quad (9.1)$$

On considère l'approximation $y_a^{\pi_n}$ de la solution y_a par polygone d'Euler-Cauchy, définie pour un entier $n \geq 2$ par

$$y_a^{\pi_n}(t) = a + \int_0^t f \left(\sum_{j=1}^n \mathbb{1}_{[t_j^n, t_{j+1}^n)}(s) y_a^{\pi_n}(t_j^n) \right) ds, \quad (9.2)$$

où $\mathbb{1}_{[t_j^n, t_{j+1}^n)}$ est la fonction indicatrice du segment $[t_j^n, t_{j+1}^n)$.

Theorème 9.1.3 ([DBM83, Theorem 2.2, Theorem 4.2]). *On suppose que V est séparable. Il existe alors un ensemble \mathcal{C} comeagre de $C_b^0(V, V)$ tel que pour $f \in \mathcal{C}$ il y a un ensemble V_f comeagre dans V vérifiant*

- pour tout $a \in V_f$ la suite d'approximations par polygone d'Euler-Cauchy $y_a^{\pi_n}$ définie par (9.2) converge uniformément sur $[0, 1]$ vers une solution y_a à l'EDO (9.1),
- pour toute suite $(a_n)_{n \in \mathbb{N}}$ dans V_f telle que $a_n \rightarrow a$, $y_{a_n} \rightarrow y_a$ uniformément sur $[0, 1]$.

On dit alors que $(a \mapsto y_a(t))_{t \in [0, 1]}$ est un flot générique continu.

Nous souhaiterions étendre ce résultat avec le formalisme des presque flots introduit dans la partie II. *A priori*, il ne semble pas que ce résultat dépende du schéma numérique employé. L'axiomatique des presque flots pourrait suffire.

9.2 Pistes de recherche sur les inclusions

9.2.1 Flots de solutions d'une inclusion différentielle perturbée

Nous avons montré dans le chapitre 5, l'existence d'un flot rugueux mesurable sans hypothèse d'unicité de la solution de l'EDR, c'est-à-dire avec une condition faible de régularité sur le champ de vecteurs. Pour cela nous avons utilisé les résultats de J. E. Cardona et L. Kapitanski [CK17a].

Nous souhaiterions montrer, suite aux résultats du chapitre 8¹, l'existence d'un flot de solutions pour une inclusion différentielle perturbée par une trajectoire rugueuse. Dans [CK17b], les auteurs prouvent qu'il y a un flot de solutions associé à une inclusion différentielle lorsque la fonction multivaluée est semicontinue supérieurement à valeurs compactes, convexes et qu'elle vérifie une condition de croissance particulière.

A priori, les conditions d'existence d'un flot mesurable introduits dans [CK17a] semblent s'appliquer au cadre d'une inclusion différentielle perturbée avec une fonction multivaluée semicontinue supérieurement à valeurs compactes et convexes. En effet, il faut vérifier que l'application qui à la condition initiale associe les solutions soit semicontinue supérieurement. Or d'après [AC12, Theorem 1 p. 104], c'est ce que l'on observe dans le cas non perturbé. Une extension au cas perturbé semble donc raisonnable. Dans un second temps, on peut s'intéresser au cas semicontinu inférieurement et non convexe mais cela semble sortir du cadre développé dans [CK17b, CK17a] et requiert plus d'investigations.

9.2.2 Unicité de la solution d'une inclusion perturbée

Le travail exposé dans le chapitre 8 porte seulement sur l'existence d'une solution à une inclusion différentielle perturbée par une trajectoire rugueuse. Une question naturelle est celle de l'unicité des solutions.

Introduisons la notion de monotonie pour une application multivaluée.

Définition 9.2.1. Soit $F : \mathbb{R}^d \rightarrow \mathcal{P}(\mathbb{R}^d)$ une fonction multivaluée de \mathbb{R}^d dans les parties de \mathbb{R}^d . On dit que F est *monotone* si

$$\forall x_1, x_2 \in \text{Dom}(F), \forall (y_1, y_2) \in F(x_1) \times F(x_2) \langle x_1 - x_2, y_1 - y_2 \rangle \geq 0,$$

où $\text{Dom}(F) := \{x \in \mathbb{R}^d \mid F(x) \neq \emptyset\}$.

Proposition 9.2.1. Soient y, y^* deux solutions à l'inclusion différentielle :

$$\forall t \in [0, 1], \quad \frac{dy(t)}{dt} \in F(y(t)), \quad y(0) = \xi. \quad (9.3)$$

Si $-F$ est monotone alors $y = y^*$ sur $[0, 1]$.

¹théorème 8.2.2 et théorème 8.2.3

Preuve. Pour $t \in [0, 1]$,

$$\begin{aligned} \|y(t) - y^*(t)\|^2 &\leq 2 \int_0^t \langle y(s) - y^*(s), \frac{d}{dt}(y(s) - y^*(s)) \rangle ds \\ &\leq -2 \int_0^t \langle y(s) - y^*(s), F(y(s)) - F(y^*(s)) \rangle ds, \end{aligned}$$

donc comme F est monotone $\|y(t) - y^*(t)\|^2 \leq 0$, ce qui prouve le résultat. ■

Cette preuve s'adapte au cas d'une inclusion différentielle perturbée de manière additive par une trajectoire $X : [0, 1] \rightarrow \mathbb{R}^d$ α -Hölder avec $\alpha \in (0, 1]$. Il serait intéressant de trouver des conditions d'unicité² dans le cas multiplicatif de la forme

$$\forall t \in [0, 1], \quad dy(t) \in F(y(t))dt + \sigma(y(t))dX(t), \quad y(0) = \xi,$$

où σ est une fonction régulière.

9.2.3 Viabilité d'une solution à une inclusion différentielle perturbée

Soient K un sous-ensemble fermé de \mathbb{R}^d et $T > 0$ un horizon de temps. On dit qu'une trajectoire $y : [0, T] \rightarrow \mathbb{R}^d$ est *viaible* dans K lorsque pour tout $t \in [0, T]$, $y(t) \in K$.

On souhaiterait étudier la viabilité d'une solution à une inclusion différentielle perturbée. Plus précisément, on voudrait trouver une relation entre K et la fonction multivaluée F telle que la solution y à

$$\forall t \in [0, T], \quad dy(t) \in F(y(t))dt + \sigma(y(t))dX(t), \quad y(0) = \xi,$$

reste dans K sur $[0, T]$ si $\xi \in K$. Nous dirons alors que la solution est viable.

L'étude de la viabilité des ces solutions est faite dans [AC12] lorsque $\sigma = 0$. Par ailleurs, le cas où F est une fonction à valeurs dans \mathbb{R}^d a été traité par L. Coutin et N. Marie dans [CM17].

²L'existence d'une solution pour $\alpha \in (1/3, 1/2]$ a été montrée dans le chapitre 8.

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Flots rugueux et inclusions différentielles perturbées

Résumé : Cette thèse est composée de trois chapitres indépendants ayant pour thématique commune la théorie des trajectoires rugueuses. Introduite en 1998 par Terry Lyons, cette approche trajectorielle des équations différentielles stochastiques (EDS) permet l'étude d'EDS dirigées par des processus n'ayant pas la propriété de semi-martingale nécessaire à l'application du cadre de l'intégration d'Itô. C'est par exemple le cas du mouvement brownien fractionnaire pour un indice de Hurst différent d'un demi.

Le premier chapitre porte sur les liens entre la théorie des trajectoires rugueuses et celle des structures de régularité qui a été récemment introduite par Martin Hairer pour résoudre une large classe d'équations aux dérivées partielles stochastiques. Nous exposons, avec les outils de cette nouvelle théorie, la définition de l'intégrale rugueuse et de la signature d'une trajectoire irrégulière, ce qui nous mène à la résolution d'équations différentielles rugueuses (EDR).

Dans le second chapitre, nous nous intéressons à la construction de flots d'EDR à partir de leurs approximations en temps petit, appelées presque flots. Nous montrons que sous des conditions faibles de régularité du presque flot, bien que l'unicité des solutions de l'EDR associée ne soit plus assurée, il est possible de sélectionner un flot mesurable. Notre cadre général unifie les précédentes approches par flot dues à I. Bailleul, A. M. Davie, P. Friz et N. Victoir.

Le dernier chapitre s'attache à l'étude d'une inclusion différentielle perturbée par une trajectoire rugueuse, c'est-à-dire d'une EDR dont la dérive est une fonction multivaluée. Nous démontrons, sans hypothèse de convexité et avec différentes conditions de régularité sur la dérive, l'existence de solution.

Abstract: This thesis consists of three independent chapters in the theme of rough path theory. Introduced in 1998 by Terry Lyons, this pathwise approach to stochastic differential equations (SDE) allows one to study SDE driven by processes that do not have the semi-martingale property which is required to apply the framework of the Itô integral. This is for example the case of the fractional Brownian motion for a Hurst index different from one-half.

The first chapter deals with the links between rough path and regularity structure theories. The latter was recently introduced by Martin Hairer to solve a large class of stochastic partial differential equations. With the tools of this new theory, we show how to build the rough integral and the signature of an irregular path, which leads to solve a rough differential equation (RDE).

In the second chapter, we focus on building RDE flows from their approximations at small scale, called almost flows. We show that under weak conditions on regularity of almost flows, although the uniqueness of the associated RDE solutions does not hold, we are able to select a measurable flow. Our general framework unifies the previous approaches by flow due to I. Bailleul, A. M. Davie, P. Friz and N. Victoir.

In the last chapter, we study of a differential inclusion perturbed by a rough path, i.e. a RDE whose drift is a multivalued function. We prove, without convexity hypothesis and several conditions on the regularity of the drift, the existence of a solution.