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## Modal proof theory through a focused telescope

Sonia Marin

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# Modal proof theory through a focused telescope

Thèse de doctorat de l'Université Paris-Saclay  
préparée à l'École Polytechnique

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et de la Communication (ED-STIC)  
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**Mme. Sonia Marin**

## Composition du Jury :

|   |                       |
|---|-----------------------|
| Mme. Agata Ciabattoni<br>Professor, TU Wien                                 | Examinatrice          |
| M. Gilles Dowek<br>Directeur de Recherche, ENS de Cachan                    | Examinateur           |
| M. Andreas Herzig<br>Directeur de Recherche, IRIT, Université Paul Sabatier | Rapporteur            |
| M. Alexander Kurz<br>Professor, University of Leicester                     | Rapporteur            |
| M. Dale Miller<br>Directeur de Recherche, Inria Saclay                      | Co-directeur de thèse |
| M. Michel Parigot<br>Chargé de Recherche, IRIF, Université Paris Diderot    | Examinateur           |
| Mme. Elaine Pimentel<br>Associate Professor, UFRN                           | Examinatrice          |
| M. Lutz Straßburger<br>Chargé de Recherche, Inria Saclay                    | Directeur de thèse    |



# Résumé

Dans cette thèse, nous observons les preuves en logique modale à travers un “téléscope” focalisé. Plus précisément, nous utilisons le concept de *règles d'inférence synthétiques* obtenues à partir d'un système de preuve *focalisé* de deux manières; d'un côté du “téléscope”, la focalisation nous permet d'analyser les mécanismes internes des règles d'inférence; de l'autre côté, elle nous permet de nous en abstraire et de nous intéresser à un comportement plus global. Cette thèse s'inscrit donc dans la récente ligne de travaux qui valorisent le concept de focalisation, non plus seulement comme un outil améliorant l'efficacité de la recherche de preuves, mais comme une véritable notion de théorie de la preuve permettant de définir des systèmes déductifs et de démontrer leurs propriétés méta-théoriques de manière élégante. Elle est organisée en trois parties, précédées d'un chapitre d'introduction.

Dans la première partie, nous passons en revue les systèmes de preuves existants pour la logique modale en nous concentrant autour du calcul des séquents et ses extensions. Nous mettons en valeur les problématiques qui animent le domaine de la théorie de la démonstration de la logique modale, notamment en déconstruisant la distinction usuelle entre systèmes étiquetés et non-étiquetés. Nous présentons ces questions et concepts en parallèle dans les chapitres 2 et 3 pour la logique modale classique et intuitionniste, respectivement. Nous détaillons en particulier les calculs des séquents emboîtés et des séquents emboîtés indexés; pour ce dernier nous démontrons un nouveau résultat de complétude via élimination des coupures.

La deuxième partie introduit d'abord la notion de focalisation et de règles d'inférence synthétiques dans le chapitre 4, puis présente deux contributions dans les chapitres 5 et 6. Premièrement, nous démontrons comment émuler le calcul étiqueté pour la logique modale intuitionniste de Simpson à l'aide du calcul focalisé pour la logique du premier ordre de Liang et Miller, étendant ainsi les travaux de Miller et Volpe au cas intuitionniste. Chaque règle d'inférence du calcul étiqueté est simulé par un *bipôle* (enchaînement d'une règle synthétique positive puis négative) en logique du premier ordre, et réciproquement, tout bipôle d'une dérivation de la traduction en langage du premier ordre d'une formule modale correspond à une unique règle d'inférence du calcul étiqueté. Comme dans le cas classique, cette correspondance peut également être étendue aux logiques modales définie par des axiomes géométriques. Deuxièmement, nous proposons un encodage similaire mais pour le calcul des séquents ordinaire (non-étiqueté) à l'aide d'une structure intermédiaire basée sur une version focalisée du calcul des séquents étiquetés.

La troisième partie peut être résumée également à deux contributions, les preuves de complétude de deux calculs de séquents emboîtés pour la logique modale, à la fois classique et intuitionniste. Nous définissons d'abord une version focalisée du calcul des séquents emboîtés, puis un système basé uniquement sur des règles d'inférence synthétiques; celles-ci ne retiennent que les transitions entre étapes positives et négatives en rendant invisibles la plupart des règles du calcul focalisé. Cela offre une présentation du système claire et élégante et simplifie grandement les preuves d'élimination des coupures et de complétude.



# Table of Contents

|  |            |
|--|------------|
| <b>Résumé</b>  | <b>i</b>   |
| <b>Table of Contents</b>   | <b>iii</b> |
| <b>List of Figures</b>   | <b>v</b>   |
| <b>1 Introduction</b>  | <b>1</b>   |
| 1.1 Proof theory . . . . .                                       | 1          |
| 1.2 Modal logic and its proof theory . . . . .                   | 3          |
| 1.3 Focusing and synthetic rules . . . . .                       | 4          |
| 1.4 The use of a focused telescope . . . . .                     | 5          |
| <br>   |            |
| <b>I « Il est possible à tout métaphysicien d’être danseur »</b> | <b>11</b>  |
| <b>2 Classical modal logic</b>                                   | <b>13</b>  |
| 2.1 Hilbert axiomatisations . . . . .                            | 13         |
| 2.2 Sequent and hypersequent calculi . . . . .                   | 14         |
| 2.3 Possible-world semantics . . . . .                           | 18         |
| 2.4 Labelled deduction . . . . .                                 | 19         |
| 2.5 Unlabelled deduction . . . . .                               | 22         |
| 2.6 Bridges . . . . .  | 27         |
| <b>3 Intuitionistic modal logic</b>                              | <b>35</b>  |
| 3.1 Hilbert axiomatisations . . . . .                            | 35         |
| 3.2 Sequent and hypersequent calculi . . . . .                   | 36         |
| 3.3 Possible-world semantics . . . . .                           | 37         |
| 3.4 Labelled deduction. . . . .                                  | 39         |
| 3.5 Unlabelled deduction . . . . .                               | 41         |
| 3.6 Bridges . . . . .  | 44         |
| <br>   |            |
| <b>II « Unfolding »</b>  | <b>53</b>  |
| <b>4 Focusing for first-order logic</b>                          | <b>55</b>  |
| 4.1 The ABC of focusing . . . . .                                | 55         |
| 4.2 From focused to synthetic reasoning . . . . .                | 59         |
| <b>5 Unfolding labelled deduction via first-order logic</b>      | <b>61</b>  |
| 5.1 From labelled modal to focused first-order . . . . .         | 61         |
| 5.2 From focused first-order to labelled modal . . . . .         | 65         |
| 5.3 Extensions . . . . .   | 65         |
| <b>6 Unfolding unlabelled deduction via labelling</b>            | <b>67</b>  |

|            |   |            |
|------------|---|------------|
| 6.1        | A focused labelled framework . . . . .            | 67         |
| 6.2        | Emulation of sequent proof systems . . . . .      | 69         |
| <b>III</b> | <b>« Folding »</b>                                | <b>79</b>  |
| <b>7</b>   | <b>Folding nested sequents: focused systems</b>   | <b>81</b>  |
| 7.1        | Focused nested sequents . . . . .                 | 81         |
| 7.2        | Focused meta-theory . . . . .                     | 85         |
| <b>8</b>   | <b>Folding nested sequents: synthetic systems</b> | <b>95</b>  |
| 8.1        | Synthetic nested sequents . . . . .               | 95         |
| 8.2        | Synthetic meta-theory . . . . .                   | 98         |
| <b>9</b>   | <b>Conclusion</b>                                 | <b>105</b> |
|            | <b>Bibliography</b>                               | <b>109</b> |

# List of Figures

|     |  |    |
|-----|--|----|
| 2.1 | The classical S5 cube . . . . .                              | 14 |
| 2.2 | Sequent system and extensions for k, 4, 5, d and t . . . . . | 15 |
| 2.3 | System hS4 and extensions for 5 and 2 . . . . .              | 17 |
| 2.4 | System labK . . . . .  | 21 |
| 2.5 | System nK and extensions for t, b, d, 4, and 5 . . . . .     | 23 |
| 2.6 | Nested structural rules for t, b, d, 4, and 5 . . . . .      | 26 |
| 2.7 | System inK . . . . .   | 29 |
| 3.1 | System labIK . . . . .                                       | 40 |
| 3.2 | System nIK and extensions for t, b, 4, and 5 . . . . .       | 42 |
| 3.3 | System inIK . . . . .  | 45 |
| 4.1 | Focused system LKF . . . . .                                 | 56 |
| 4.2 | Focused system LJF . . . . .                                 | 58 |
| 6.1 | Framework LMF <sub>*</sub> . . . . .                         | 68 |
| 7.1 | System fnK and extensions for t, b, d, 4, and 5 . . . . .    | 82 |
| 7.2 | System fnIK and extensions for t, b, 4, and 5 . . . . .      | 84 |
| 7.3 | System wnK and extensions for t, b, d, 4, and 5 . . . . .    | 86 |
| 8.1 | System snK and extensions for t, b, d, 4, and 5 . . . . .    | 96 |
| 8.2 | System snIK and extensions for t, b, 4, and 5 . . . . .      | 98 |



# 1

## Introduction

When one looks through a telescope from one side, one can see remote objects bigger and analyse them; but when one looks through the other side of the lens, the closer objects will suddenly look distant. In this work, it is the *proof theory of modal logic* that we want to observe through a *focused telescope*. Because proof theory for modal logic has proven quite tricky throughout the second half of the 20th century, and even though we might say that it is well handled practically now, it is still a wonderful playground for new proof-theoretical ideas. So let us first introduce these concepts: (i) proof theory; (ii) modal logic; (iii) focusing; and (iv) our telescope.

### 1.1 Proof theory

Mathematics is made of theorems and their proofs. Sometimes a theorem may have several proofs and we want to understand what is different and what is common in them, and maybe how to transform one into the other in some way. Proofs of different theorems may also share common patterns, for example induction or reduction to the absurd, and we want to understand in which contexts those particular forms of reasoning apply and how. However, in order to communicate proofs, mathematicians usually use natural language written in plain text (with some symbols). So the urge to answer these questions calls for the formalisation of proofs (yes, every single word of them) in a purely mathematical language, and for a whole discipline to study their properties: *proof theory*. Frege pioneered this discipline suggesting that proofs be considered themselves as objects of mathematical study in 1879 in his *Begriffsschrift* [50]. Hilbert followed Frege's ideas and proposed the definition of a deductive system to formalise reasoning [64].

Proof theory is now established as one of the four pillars of mathematical logic alongside model theory, set theory, and recursion theory. In designing various formalisms to view proofs as formal mathematical objects, we may understand their properties via formal mathematical methods. One direct application would be to develop computer programs that can prove theorems automatically (or almost), or check mathematical proofs (automatic theorem proving/checking). Another application could be to extract from a proof an algorithm; for example if the theorem states the existence of an object, this object could also be effectively constructed; or to extract from failed proof-search a counter-example to that invalid theorem (proof mining). Some more abstract consequences are that we can try to understand which axioms are required to prove which theorems (reverse mathematics), as well as compare different proof methods and in particular the sizes of the proofs they output (proof complexity).

Looking back to our first sentence, perhaps a better picture for anyone who has done a little bit of mathematics would be: Mathematics is made of lots of lemmas, a few theorems and their proofs. Indeed, the way it goes is that mathematicians rarely prove a theorem from scratch, they build up its proof by proving intermediate statements that are called

lemmas. In a deductive system *à la Hilbert and Frege*, such a reasoning step would be the main *inference rule* combining axioms. That is, to prove a theorem  $T$ , it is possible to use a given lemma  $L$  if one can prove  $L$  on the one hand and if one can infer  $T$  from  $L$  on the other hand. This step of reasoning, which is of course not restricted to mathematics but widespread in daily reasoning too, is called *Modus Ponens*. Obviously, the hard part is usually to find the appropriate lemma  $L$  to use amongst the infinity of mathematical statements.

Actually, a lot of the beauty of mathematics resides in lemmas, as they can appeal to some very clever tricks such as using some topology to solve a numerical equation or appealing to complex numbers to integrate a real function. However, for the applications we mentioned, shifting to a different theory or a different language than that of the theorem one is trying to prove can be a nightmare, and we often prefer a proof without lemmas, even if it is likely to be much less clever and cluttered with a lot of repetitions. (Though not all mathematical proofs can be made lemma-free.)

A lemma-free proof is usually said to be *analytic*. The designation comes from Bolzano, who published in 1817 a second proof of his intermediate value theorem that he calls a *Purely analytic proof of the theorem that between any two values which give results of opposite sign, there lies at least one real root of the equation* [11]. His first proof of the intermediate value theorem was indeed using geometrical intuitions of lines crossing each other at a given point. The second proof, however, was completely free of geometrical or algebraic methods, but used only concepts from analysis (which is why he called it analytic) and therefore stayed fully in the language and in the theory of the theorem itself.

The term was then later reused by Gentzen who, in his *Untersuchungen* in 1934 [56, 57], introduced fundamental proof-theoretic methods to prove results in mathematical logic. In particular, he developed the *sequent calculus*, an alternative representation of proofs promoting inference rules over axioms. The *Hauptsatz* then states that any proof in the purely logical sequent calculus can be transformed into an analytic normal form. From this follows the *subformula property*, that every formula appearing in the premiss(es) of an inference rule is a subformula of its conclusion, which ensures that a proof is internal to the language of what it proves. To prove the *Hauptsatz* one needs to show that a rule called the *cut*, reformulating *Modus Ponens* in the sequent calculus, is redundant in the system by an inductive elimination procedure transforming a given proof into another one with cuts of smaller induction measure.

Another crucial contribution of Gentzen's work was the parallel treatment of classical and intuitionistic logics. *Intuitionistic logic* was conceived (starting in 1907 with Brouwer [14]) as a formalisation of constructive reasoning, rejecting in particular the principle of *Excluded Middle*, that a statement has to be either true or false. In classical logic, this principle is accepted, while in intuitionistic logic, on the contrary, there might be an uncertainty whether a proposition holds or not. Gentzen observed that, in the sequent calculus, classical and intuitionistic reasoning could be formally dissociated, as the intuitionistic calculus can be obtained as a simple syntactical restriction of the classical one. If one considers a sequent as a list of formulas that are annotated as either *input* or *output*, i.e. hypothesis or conclusion, then a classical sequent can have zero to many outputs, while an intuitionistic sequent must have exactly one output, and hence the inference rules of the system must preserve the uniqueness of the output formula. This trick actually has widespread applications in the constructive foundations of mathematics as well as in computer science. For instance, the connection between proofs in intuitionistic logic and computational behaviours stimulated the development of type theories for programming languages [65].

Finally, one goal of *structural* proof theory is to extend Gentzen's results to other mathematical theories, usually given in terms of axioms which, if added as such to the sequent calculus, break the cut-elimination proof [58]. It is usually a tedious task for proof theorists to find a cut-free deductive system in a sequent(-like) calculus corresponding to a given axiomatisation. The best solution would be to automate this process, and to find general

criteria for determining when a set of axioms can be transformed into an equivalent set of inference rules such that cut-elimination is preserved. This has been achieved for instance for *geometric* theories in classical first-order logic [104] and for some *algebraic* theories in sub-structural and intermediate logics [30]. In the next section we consider more specifically these questions of extending sequent systems in the context of *modal logics*.

## 1.2 Modal logic and its proof theory

The study of modal logic, going back to Aristotle, comes from the desire to analyse certain philosophical arguments, and thus qualify finely the truth of a proposition: for example a proposition may be false now but true later, or on the contrary true and necessarily so, and so on. What is now called modal logic describes the behaviour of the abstract *modalities*  $\Box$  and  $\Diamond$ , but covers a wide range of ‘real’ modalities in linguistic expressions: time, necessity, possibility, obligation, knowledge, belief...

The most well-studied modal logics are based on classical reasoning (which is why we will call them *classical modal logics*, and not following the convention of [27]). The interest in intuitionistic versions of modal logics has come much later and from two different sources: on the one hand, logicians had a theoretical interest in obtaining intuitionistically relevant versions of modal logics, and on the other hand, certain applications in computer science naturally gave rise to some modal logics with a constructive flavour. For example, as we have said that intuitionistic logic allows us to capture some aspects of computation via type theories, the idea to add modal operators on top of type theories to capture further aspects of computation seems quite natural [98, 110, 8]. As we shall see, the purely logical and the type-theoretical approaches do not give rise to the same logics.

Classical, as well as intuitionistic, modal logics used to be studied as systems of axioms that describe the conditions regulating each modal operator depending on the context of application. Structural proof theory on the other hand was considered a difficult topic for modal logics, as ordinary sequents did not provide a satisfactory answer, that is, a way to systematically build *analytic* proof systems translating modal axioms into inference rules. Proof systems based on Gentzen’s (ordinary) sequents that exist for specific modal logics cannot be easily extended to encompass more axioms. On the other hand, some combinations of axioms have so far escaped all efforts towards a corresponding sequent system at all. In general, each proposal was built independently and in an *ad hoc* fashion. One reason to explain this phenomenon is that there was no uniform semantical framework to guide the authors attempting these constructions.

Finally in the 1960s, a semantics was given by Kripke [69] (among others) in terms of *relational structures* for a large subset of the classical modal logics previously studied, now called the *normal* modal logics. It allows one to see modal logic as a language for graphs, which also uncovered new applications to describe phenomena in computer science, as graphs are very common structures to model concepts in the domain.

The consequence of this new perspective in terms of proof theory was the possibility to build *semantic tableaux* proof procedures for a wide range of modal logics. The formalism of tableaux was the most studied for proofs in modal logics, and probably still is the most developed and used (see [122] for a variety of tools, most of which use tableaux). However, whereas tableaux are arguably well-suited for automated deduction and decision procedures, that is, to find proofs, sequent-like systems are usually more appropriate when trying to distinguish proof shapes and to study proof transformations, in particular cut-elimination.

In the last two decades, some extensions of ordinary sequents were successfully proposed to handle modalities. Two approaches are commonly distinguished: systems that incorporate explicitly relational semantics in the formalism itself, called *labelled deductive systems* (e.g. semantic tableaux [45] and labelled sequents [101]), and systems that use syntactical devices to recover the modal language that we will regroup under the counter-relief term *unlabelled deductive systems* (e.g. display calculi [68], hypersequents [6], nested sequents [15]).

So proof theorists interested in modal logic now have a wide range of different proof formalisms to choose from, each of them presenting its own features and drawbacks. Several results concerning correspondences and connections between the different formalisms are also known [47, 61, 75]. In the most general way, we would like to understand which modal logics can be given a proof system in which formalism (and if not, why). The different formalisms that exist for studying the proof theory of modal logic should not be seen as a *doxa*; they rather bring different angles, and complementary tools in order to explore the subject. The work of this thesis is an illustration of the variety of points of view since it uses labelled as well as unlabelled systems to analyse modal proofs. In the next section, we introduce the main tools we will use to organise proofs in these systems.

### 1.3 Focusing and synthetic rules

One major application of proof systems is proof search: given a mathematical statement, can we construct a proof of it from a set of other statements, which are accepted as valid hypotheses, in order to consider it a valid theorem? One way proof search can be conducted in a sequent calculus is *backward proof search*. The idea is to consider all the inference rules that could be applied in order to obtain the desired conclusion, collect all the premisses that would be obtained from applying these rules, and recursively apply this process to these premisses until either we have reached one of our valid hypotheses or no more rules can be applied.

The search of a proof is necessarily a non-deterministic process as it typically requires some choices to be made in a certain order to terminate. That is, even when starting from a valid conclusion, not every possible succession of backward inference steps will lead to a correct proof and sometimes a deadlock may be reached which forces the search to backtrack. However, one can make a simple observation: not every step in the proof search procedure does introduce non-determinism. For some inference rules, called *invertible*, each of the premisses can actually be deduced from the conclusion, so for a valid conclusion, these rules cannot bring the proof search to a deadlock. Other rules, however, require some choice or some interaction with different elements in the sequent, so if we make the wrong choice or we are not in the appropriate context, we might not be able to find the correct proof and thus need to backtrack to that critical step.

*Focusing* is a general technique that was designed to improve proof search by reducing the number of non-deterministic choices [95, 3], and indeed it led to concrete optimisations of standard proof search procedures [26, 90]. The essential idea of focusing is to identify and merge the non-deterministic choices in a proof: the normal form of a focused proof is given by an alternation of *inversion* phases, where invertible rules are applied (bottom-up) eagerly, and *focus* phases, where applications of the other rules are confined and controlled. It is a rather strict way to build a proof, but it remains *complete*, which means that every provable statement actually has a focused proof.

The first proof one could think of for such completeness result would be by permutations of rules, that is, one would need to show that invertible rules can be permuted down and the other rules can be regrouped as needed in a focus phase. This is a quite long and tedious proof [3]. Another one is by defining a subsystem of the original system, following the invertibility discipline of connectives, that is constrained in a way that it can only produce focused proofs; and then prove that this forms a complete subset of all the proofs, i.e. that this subsystem is equivalent to the original one. One proof of the equivalence pioneered by Laurent [74] goes in two steps: first the simulation of all the rules from the original system by the rules in the new “focused” subsystem, and then, as this simulation makes intensive use of the cut rule, show that this rule is admissible in the focused system, namely that eliminating the cut rule from a focused proof still preserves the alternation of inversion and focusing phases. So in the general case, the focused subsystem is provably complete. In some particular cases, the subsystem is even canonical; which means that it produces a unique

representative element of each family of proofs that are equivalent up to permutations, in the same way as proof nets or expansion trees can do it for respectively MALL or classical first-order logic [24, 21].

The theory of focusing was originally developed for the sequent calculus for linear logic, but it has since been extended to a variety of logics [83, 117] and proof systems [20, 13]. This generality suggests that the ability to transform a proof system into a focused form is a good indication of its syntactic quality, in a similar manner to how cut-admissibility shows that a proof system is syntactically well-designed. So focusing is not only an efficient proof search strategy, it has also strong representational benefits. In particular, it gives us a systematic notion of grouping together inference rules into meaningful entities: we can just ignore the internal structure of the phases and the details of the focused logical rules and only remember the sequents at a transition between two phases to see each phase as one rigid *synthetic rule* [139, 24].

We want to illustrate in this thesis two properties of such synthetic rules, in the context of modal logic. On the one hand, a focused proof system can be used to provide *adequate* encodings of other non-focused systems. We show in particular how to use focusing mechanisms to build a general framework emulating existing systems for modal logics. The adequacy of the emulation amounts to the fact that each inference step in the unfocused system corresponds to a specific chain of synthetic rules. On the other hand, designing a proof system using synthetic rules provides a *simplification* of both the proof system and its associated meta-theory. In particular, the proof of completeness of focusing becomes more manageable in a synthetic formulation. Such a simplification has been observed for ordinary intuitionistic logic by Zeilberger [139] but when we add more features to a logic (and in our case modalities), the effect of this simplification becomes more noticeable.

## 1.4 The use of a focused telescope

In this thesis, we are going to look at modal proofs through a focused ‘telescope’. More precisely, the way we use the concept of synthetic rules obtained in a focused system is twofold; from one side of the telescope, we see details of the internal mechanisms of inference rules; from the other side, we abstract from them and observe more global behaviour.

### Unfolding: analysing the machinery behind a proof system

Considerable progress has been made in extending Gentzen’s purely logical results to more expressive mathematical theories, as already mentioned [100, 38, 30, 33, 77, 76]. We now need to look back and see if we can understand why the proposed systems work and how the different methods used to obtain them relate to each other.

Through focusing, we dissect the *operational semantics* of the inference rules that are induced by these methods, i.e. the way they operate on some sequents to produce new ones. This gives us a better understanding of why they work, how they can be extended, and as a by-product how to verify proofs they form through a *universal* proof checker. With Miller and Volpe, following their first successful attempt in [96], we wanted to understand this line of work in a uniform and systematic manner from the point of view of focusing. This is an ongoing effort and in this thesis we present some of our first results. Their interest lies in understanding the machinery behind the general construction of inference rules for axiomatic theories in a structured way.

This research line is part of the ERC project *ProofCert* [94] led by Miller, which is concerned with providing a uniform format for proofs across many types of proof systems. The end goal would be that theorem provers come equipped with the semantics of the kind of proofs they issue, and then a universal proof checker could take as input not only the proof to be checked, but a *proof certificate* containing such semantical information alongside. The basis chosen for this checker is a focused sequent system for classical or intuitionistic logic

which can be augmented with suitable proof certificates and some devices able to extract and organise the information it contains [29].

We were interested in how to use these mechanisms to emulate proof systems for modal logics. We present first an encoding of unfocused labelled sequent calculi for modal logics into the focused sequent calculus for first-order logic, and then an encoding of unfocused sequent calculus for modal logic into focused labelled sequent calculus for modal logic. This second encoding is an illustrative example of a more general framework [86] that acts as an interface between any modal system and focused first-order logic.

By analysing the case of ordinary sequents, we observed that a modal rule usually corresponds to the application of a distinctive chain of synthetic rules in our focused framework and vice versa. The (focused labelled) framework, when properly instantiated, can emulate modal proof systems with good precision. Thus, implementations of that framework can be used to provide a proof checker for the emulated proof systems [84], contributing to the mechanisation of proof checking.

### **Folding: simplifying a proof system and its meta-theory**

When one adds features to a proof system, one generally expects that the meta-theory will become more complicated. Indeed, at the start, a sequent calculus for classical propositional logic can have just as many logical rules as connectives and two additional structural rules of identity and cut (not exactly Gentzen's original system, but the one refined successively by Ketonen and Kleene does). Eliminating cuts from this system is relatively straightforward: there is a single cut-rule and a simple lexicographic induction. Extending the system with the modal operators  $\Box$  and  $\Diamond$  and modal axioms adds new forms of cuts and further complications to the measure, or even requires us to generalise the very notion of a sequent, as already mentioned, and so the entire cut-elimination procedure needs to be re-examined. Finally, moving to a two-sided sequent calculus, which is essential for intuitionistic versions of these calculi, doubles the number of inference rules, and hence doubles the number of cases to address in the cut-elimination argument.

However, it appears that re-designing a proof system to enforce focused normal forms can reduce its syntactic and meta-theoretic complexity. This is what we observed in the case study of focused and synthetic proof systems for modal logics based on nested sequents in a joint work with Chaudhuri and Straßburger [22, 23]. We started by studying a standard family of classical modal logics between the basic logic  $K$  and the well-known  $S5$ ; we could already observe the effects of the simplification both on the system design and on the meta-theoretical proofs. Then we applied the method to intuitionistic modal logics, as it contains all the complications mentioned above, and the simplification is even more striking. We were surprised to discover that: (i) the input/output annotations of intuitionistic sequents turn out to be redundant, as they can always be uniquely inferred in the focused system; and (ii) the synthetic version of the intuitionistic system has fewer rules than the non-focused version, and the same number of structural rules as the classical system, which we did not expect would be the case.

The soundness of focused and synthetic systems is usually straightforward. Our contribution is a proof of completeness of the focused systems with respect to the non-focused systems by showing that the rules of the non-focused systems can be simulated by derivations in the focused systems using cuts, and then proving a cut-elimination result internal to the focused nested calculus, and similarly for the completeness of the synthetic systems.

### **Outline**

This thesis is structured as follows:

**Chapter 2** introduces classical modal logic, its syntax, its relational semantics, and different proof formalisms, separated into labelled systems and unlabelled systems. We mention many different forms that both classes can take, but we concentrate our attention

on variations of the sequent framework, from ordinary sequents to labelled sequents, via hypersequents, nested sequents, and indexed nested sequents.

**Chapter 3** parallels the structure of Chapter 2 to introduce intuitionistic modal logic, its syntax, its bi-relational semantics, and its proof theory, whether labelled or unlabelled. We highlight the particular challenges that the addition of the intuitionistic dimension to modal logic has brought, compared to the classical case.

**Chapter 4** reviews formally the focused proof systems for classical and intuitionistic first-order logic [83] and how to build synthetic rules from them.

**Chapter 5** presents the encoding of the labelled sequent system for intuitionistic modal logic into focused first-order intuitionistic logic. This is an adaptation of the work of [96], which only considered classical modal logic, to the intuitionistic case.

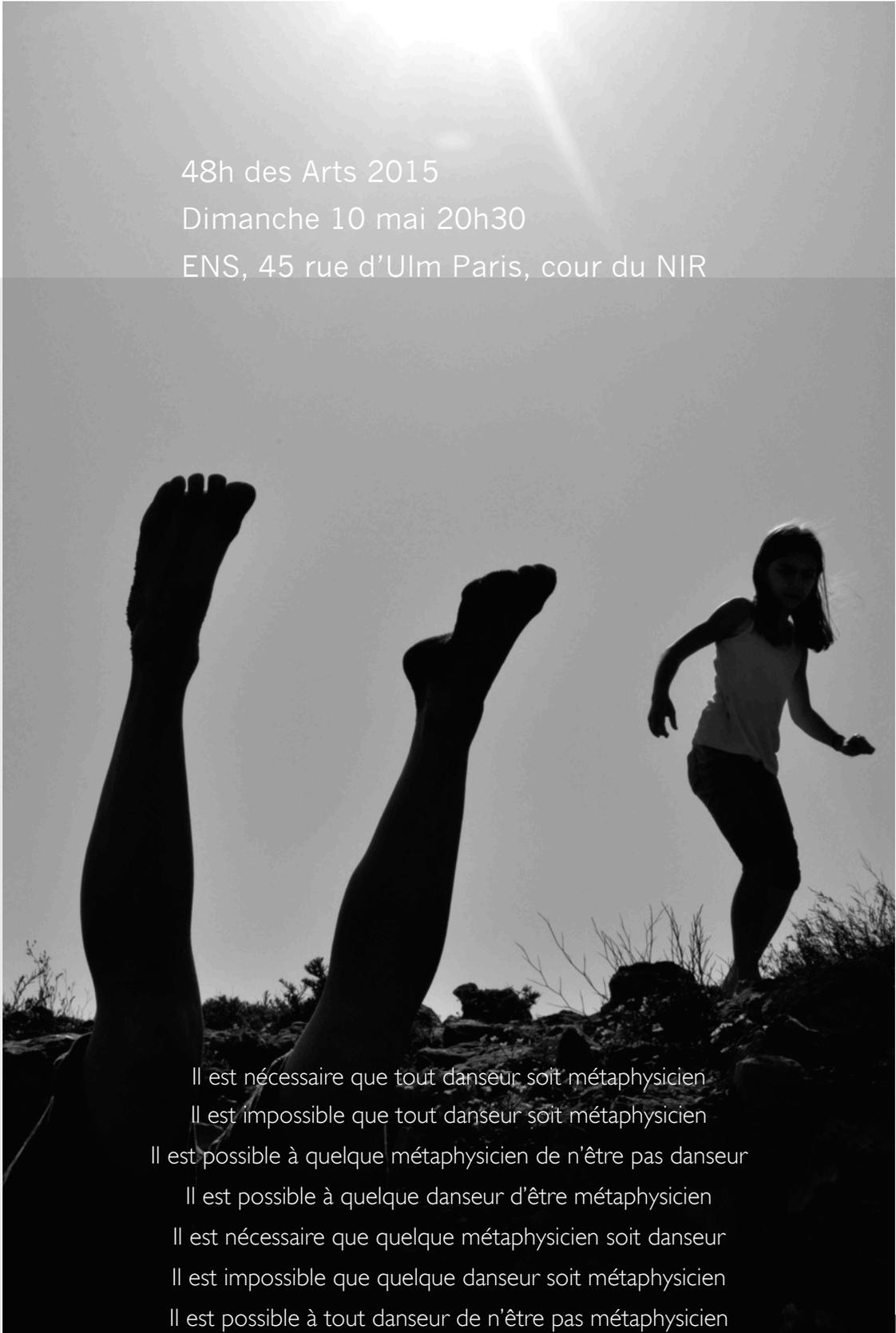
**Chapter 6** presents the encoding of ordinary sequent systems for classical modal logic into a focused labelled framework for classical modal logic [86].

**Chapter 7** presents focused nested sequent systems for classical and intuitionistic modal logics, the main results being their completeness, obtained via a cut-elimination argument.

**Chapter 8** presents synthetic nested sequent systems for classical and intuitionistic modal logics, which allows us to give an alternative and simpler proof of completeness for the focused systems. These two chapters are based on [22] for the classical systems and on [23] for the intuitionistic systems.

**Chapter 9** concludes this thesis with some perspectives and ideas for future work.





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Dimanche 10 mai 20h30

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Il est nécessaire que tout danseur soit métaphysicien  
Il est impossible que tout danseur soit métaphysicien  
Il est possible à quelque métaphysicien de n'être pas danseur  
Il est possible à quelque danseur d'être métaphysicien  
Il est nécessaire que quelque métaphysicien soit danseur  
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Il est possible à tout danseur de n'être pas métaphysicien

Il est possible à tout métaphysicien d'être danseur

Création de danse à propos de logique modale par le club danse impro !

Direction Marguerite Papazoglou



# I « Il est possible à tout métaphysicien d'être danseur »

We present the *Milky Way* of proof systems for modal logics. In Chapter 1, we set foot on the questions and the methods we will explore. In Chapter 2 and 3, we introduce in parallel the universes of classical and intuitionistic modal proof theory respectively. *It is possible not to be a metaphysicist to understand the many approaches that have been developed in order to capture modal logics through proof theoretic methods. However, we will see that it is certainly necessary to be a dancer to cross the bridges between the labelled and unlabelled hemispheres.*



## Classical modal logic

In this chapter we introduce the concepts and notations that will be used in this thesis for *classical* modal logic, the modal extension of classical propositional logic. (Note, however, that the term *classical modal logic* is sometimes used by other authors, following ??, to denote a weaker modal logic that does not satisfy necessitation and distributivity.) For a more detailed presentation we refer to the surveys by Fitting [46], Negri [102], and Wansing [136], on which this chapter is partly based. We start by introducing Hilbert-style axiomatisations in Section 2.1 and Gentzen-style sequent systems in Section 2.2 for standard modal logics, and then by giving their relational semantics in Section 2.3. Building on the semantic interpretation of the logics, we can introduce the labelled approach to the proof theory of classical modal logic in Section 2.4 and by contrast the unlabelled approach in Section 2.5. Finally in Section 2.6, we describe some ways to build bridges between the two approaches, and in particular the formalism of *indexed nested sequents* that we studied in [88].

### 2.1 Hilbert axiomatisations

Modal logicians were at first interested in producing specialised axiomatic systems to describe intuitively modal-flavoured behaviours like strict implication, belief or knowledge [81]. The way we usually see classical modal logic nowadays is, as proposed first in [79], as an extension of classical propositional logic.

The language of classical modal logic is obtained from the one of classical propositional logic by adding the modal connectives  $\Box$  and  $\Diamond$ , standing for example for *necessity* and *possibility*. Starting with a set of atomic propositions denoted  $a$  and their duals  $\bar{a}$ , modal formulas are constructed from the following grammar:

$$A ::= a \mid \bar{a} \mid A \wedge A \mid \top \mid A \vee A \mid \perp \mid \Box A \mid \Diamond A$$

In a classical setting, we always assume that formulas are in negation normal form, that is, negation is restricted to atoms. When we write  $\neg A$  in this case, we mean the result of computing the de Morgan dual of connectives and atomic propositions within  $A$ , i.e.  $\neg\neg A \equiv A$ ,  $\neg(A \wedge B) \equiv \neg A \vee \neg B$  and  $\neg\Box A \equiv \Diamond\neg A$ , where  $\equiv$  denotes syntactic equality. Implication can be defined from this set of connectives by  $A \supset B := \neg A \vee B$ .  $\top$  and  $\perp$  are the usual *units* of the binary connectives  $\wedge$  and  $\vee$  respectively.

The classical modal logic  $K$  is then obtained from classical propositional logic by adding to any standard formulation of the latter, such as its Hilbert axiomatisation,

- the *necessitation* rule: if  $A$  is a theorem of  $K$  then  $\Box A$  is too; and
- the axiom of *distributivity*, commonly written  $k := \Box(A \supset B) \supset (\Box A \supset \Box B)$ .

It is important to mention that what we are calling an *axiom* here is in fact, as is standard in Hilbert-style axiomatisations, an axiom *schema*, in order to avoid explicit mention of



$$\begin{array}{cccccc}
\text{id}^\circ \frac{}{\Gamma, \bar{a}, a} & \wedge^\circ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} & \top^\circ \frac{}{\Gamma, \top} & \vee^\circ \frac{\Gamma, A, B}{\Gamma, A \vee B} & \text{w}^\circ \frac{\Gamma}{\Gamma, A} & \text{c}^\circ \frac{\Gamma, A, A}{\Gamma, A} \\
\hline
\Box_k^\circ \frac{\Gamma, A}{\Diamond \Gamma, \Box A} & \Box_{k4}^\circ \frac{\Diamond \Gamma, \Gamma, A}{\Diamond \Gamma, \Box A} & \Box_{k45}^\circ \frac{\Diamond \Gamma_1, \Gamma_1, A, \Box \Gamma_2}{\Diamond \Gamma_1, \Box A, \Box \Gamma_2} & \Diamond_d^\circ \frac{\Gamma}{\Diamond \Gamma} & \Diamond_t^\circ \frac{\Gamma, A}{\Gamma, \Diamond A} & 
\end{array}$$

Figure 2.2: Sequent system and extensions for k, 4, 5, d and t

Gentzen's formalism of sequent calculus [56, 57], as it represents a more intuitive way of searching for a proof.

In the classical setting, we will define a *sequent*  $\Gamma := A_1, \dots, A_n$  as a multiset of formulas, as usual omitting set-brackets and simply using the comma to denote the multiset union. It is interpreted as the formula  $\text{fm}^\circ(\Gamma) := A_1 \vee \dots \vee A_n$ . A *sequent inference rule* is an expression

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_m}{\Gamma}$$

for  $n \geq 0$  such that the bottom sequent  $\Gamma$ , the *conclusion*, can be deduced from the top one(s)  $\Gamma_1, \dots, \Gamma_m$ , the *premiss(es)* (or none if  $m = 0$ ), in the logic at hand.

A *derivation*, denoted  $\mathcal{D}$ , is then constructed according to these rules; it will have the structure of a tree, where each edge is a sequent and each internal node is a rule. A derivation is a *proof* of the sequent at the root, if each leaf is a rule with no premisses. The *height* of a derivation  $\mathcal{D}$ , denoted by  $\text{ht}(\mathcal{D})$ , is the height of  $\mathcal{D}$  when seen as a tree, *i.e.*, the length of the longest path in the tree from its root to one of its leaves.

Designing a sound and complete sequent system for a given logic means defining the right set of sequent rules such that each theorem of the logic at hand has a proof in the sequent system (*completeness*) and that any proof that can be built in the sequent system derives a theorem of the logic (*soundness*).

We say that a rule  $r$  is *admissible* for a system  $S$  if, whenever its premisses are provable in  $S$  there is a proof of its conclusion in  $S$ . We say it is *derivable* if there exist a derivation in  $S$  from its premisses to its conclusion, possibly using premisses multiple times. As mentioned in the introduction, one important rule to show admissible in a deductive system is the *cut-rule*, which can be written as:

$$\boxed{\text{cut} \frac{\Gamma, C \quad \Gamma, \neg C}{\Gamma}}$$

for some formula  $C$  and its de Morgan dual  $\neg C$ . The *rank* of an instance of cut is the depth of the formula  $C$  introduced by the cut (read bottom-up). The *depth* of a formula  $A$ , denoted by  $\text{dp}(A)$ , being inductively given as:

$$\begin{array}{ll}
\text{dp}(a) = \text{dp}(\bar{a}) = \text{dp}(\perp) = \text{dp}(\top) = 1 & \text{dp}(A_1 \wedge A_2) = \max(\text{dp}(A_1), \text{dp}(A_2)) + 1 \\
\text{dp}(\Diamond A) = \text{dp}(\Box A) = \text{dp}(A) + 1 & \text{dp}(A_1 \vee A_2) = \max(\text{dp}(A_1), \text{dp}(A_2)) + 1
\end{array}$$

We also write  $\text{cut}_r$  to denote an instance of cut with rank at most  $r$ . The *cut-rank* of a derivation  $\mathcal{D}$ , denoted by  $\text{rk}(\mathcal{D})$ , is the maximal rank of a cut in  $\mathcal{D}$ .

Proving that this rule is admissible in a given proof system somewhat ensures that it is well-designed for backward proof-search. Such a result was first proven by Gentzen as the *Hauptsatz* [56] for classical and intuitionistic (non modal) logic. The standard proof proceeds by a double-induction on the cut-rank and on the height of a derivation.

The first sequent systems for modal logics seem to have been defined for logics  $K$ ,  $T$ ,  $S4$  and  $S5$  by Ohnishi and Matsumoto in [105]. However, as they announced in a follow-up note [106], the cut-rule is not admissible in their proposed system for  $S5$ . Since then, cut-free sequent systems have been proposed for other extensions of  $K$  with the axiom  $d$  [59], with

the axiom 4 [120], and with the axioms 4 and 5 [125], but to our knowledge no cut-free system based on ordinary sequents has been found for the modal logic S5 (or for K5, D5 and KB5).

Figure 2.2 sums up the rules of these systems. We use here a common abuse of notation for sets of modal formulas:  $\Box\Gamma := \{\Box A \mid A \in \Gamma\}$  and  $\Diamond\Gamma := \{\Diamond A \mid A \in \Gamma\}$ . The systems oK, oK4, and oK45 are obtained by adding to the rules in the first line of Figure 2.2 the rule  $\Box_k^\circ$ ,  $\Box_{k4}^\circ$ , or  $\Box_{k45}^\circ$  respectively. This is enough for getting sound and complete systems for the logics K, K4, and K45. They can then be extended modularly by adding either the rule  $\Diamond_d^\circ$  or the rule  $\Diamond_t^\circ$  to get combinations of these logics with axiom d or t respectively.

**2.2.1 Theorem** ([105, 59, 120, 125]) *Let  $X = \emptyset$  or  $\{d\}$  or  $\{t\}$ . For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $K + X$  or  $K4 + X$  or  $K45 + X$ .
2.  $A$  is provable in the sequent calculus  $\text{oK} + \Diamond_X^\circ + \text{cut}$  or  $\text{oK4} + \Diamond_X^\circ + \text{cut}$  or  $\text{oK45} + \Diamond_X^\circ + \text{cut}$ .
3.  $A$  is provable in the sequent calculus  $\text{oK} + \Diamond_X^\circ$  or  $\text{oK4} + \Diamond_X^\circ$  or  $\text{oK45} + \Diamond_X^\circ$  respectively.

This theorem can be proved for any of these systems in three steps. The proof follows the standard pattern found for example in [133]. First,  $2 \rightarrow 3$  is the admissibility of the cut-rule in the sequent calculus, that is usually obtained, as already mentioned, by induction on derivation cut-rank and height. Then,  $3 \rightarrow 1$  is the soundness of the sequent calculus with respect to the Hilbert system, for which it is enough to show for every sequent rule that if the premisses are provable in the Hilbert system, so is the conclusion. Finally,  $1 \rightarrow 2$  is the completeness of the sequent calculus (with cut) with respect to the Hilbert system, for which it suffices to show that the axioms of the Hilbert system are provable and the rules of the Hilbert system admissible in the sequent system. We do not prove these statements (and all the other similar results of this chapter) formally, but we give one detailed proof of cut admissibility in Section 2.6 to illustrate how the others would proceed.

A list of criticisms is usually addressed to these sequent systems, namely that they (or at least some of them) lack (i) locality; (ii) modularity; (iii) generality; and (iv) analyticity.

(i) These systems do not have local rules for modalities and do not separate rules for a connective and its dual. The modal rules do not only introduce one connective in the conclusion, some of them introduce at the same time  $\Box$  and  $\Diamond$ , and some of them involve restriction on the context of application, e.g. require a context to be exclusively composed by modal formulas.

(ii) These systems are not fully modular. A modular system would be one where each rule can be freely mixed with others as required by the logic to be defined, and not one where the rule corresponding to the axiom k varies depending on whether other axioms like 4 or 5 are also present.

(iii) These systems are not general enough. Already in the S5 cube, the logics K5, D5, KB5 and S5 could not be given a complete sequent system so far, and that is without considering logics beyond the cube, in particular those axiomatised by the Scott-Lemmon axioms, defined in the previous section.

(iv) We could obtain a system for S5 by adding to the classical sequent calculus the rules  $\Box_{k45}^\circ$  and  $\Diamond_t^\circ$ , but the resulting system would not be analytic. Indeed, the formula  $\bar{a} \vee \Box\Diamond a$  which is a theorem of S5, is provable in  $\text{oK} + \Box_{k45}^\circ + \Diamond_t^\circ + \text{cut}$  as follows

$$\frac{\frac{\frac{\text{id}^\circ \frac{}{\bar{a}, a}}{\bar{a}, \Box\Diamond a, a}}{\bar{a}, \Box\Diamond a, a}}{\bar{a}, \Box\Diamond a, \Diamond a}}{\bar{a}, \Box\Diamond a} \quad \frac{\frac{\frac{\text{id}^\circ \frac{}{\bar{a}, a}}{\bar{a}, \Diamond a}}{\bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a}}{\bar{a}, \bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a}$$

$$\begin{array}{cccc}
\text{id}^h \frac{}{\mathcal{H} \mid \Gamma, \bar{a}, a} & \wedge^h \frac{\mathcal{H} \mid \Gamma, A \quad \Gamma, B}{\mathcal{H} \mid \Gamma, A \wedge B} & \top^h \frac{}{\mathcal{H} \mid \Gamma, \top} & \vee^h \frac{\mathcal{H} \mid \Gamma, A, B}{\mathcal{H} \mid \Gamma, A \vee B} \\
\Box_{k4}^h \frac{\mathcal{H} \mid \Diamond \Gamma, \Gamma, A}{\mathcal{H} \mid \Diamond \Gamma, \Box A} & \Diamond_t^h \frac{\mathcal{H} \mid \Gamma, A}{\mathcal{H} \mid \Gamma, \Diamond A} & w^h \frac{\mathcal{H} \mid \Gamma}{\mathcal{H} \mid \Gamma, A} & c^h \frac{\mathcal{H} \mid \Gamma, A, A}{\mathcal{H} \mid \Gamma, A} \\
& ew^h \frac{\mathcal{H}}{\mathcal{H} \mid \Gamma} & ec^h \frac{\mathcal{H} \mid \Gamma \mid \Gamma}{\mathcal{H} \mid \Gamma} & \\
\cdots & & & \\
\Box_5^h \frac{\mathcal{H} \mid \Gamma_1, \Diamond \Gamma_2, \Box \Gamma_3}{\mathcal{H} \mid \Gamma_1 \mid \Diamond \Gamma_2, \Box \Gamma_3} & & \Box_2^h \frac{\mathcal{H} \mid \Diamond \Gamma_1, \Diamond \Gamma_2}{\mathcal{H} \mid \Diamond \Gamma_1 \mid \Diamond \Gamma_2} & 
\end{array}$$

Figure 2.3: System hS4 and extensions for 5 and 2

but the cut-rule is impossible to remove without breaking provability. Indeed, the only rules that can be applied in a cut-free derivation ending in  $\bar{a}, \Box \Diamond a$  are  $w$  and  $c$  possibly followed by  $\Box_{k45}^o$ . Hence, such a derivation can only contain sequents of the form:  $\bar{a}^m, \Box \Diamond a^n$  or  $\Diamond a^k, a^\ell, \Box \Diamond a^j$ , for some arbitrary  $j, k, \ell, m, n \geq 0$ , which do not include an identity sequent, so the derivation cannot be closed [77].

Many solutions for the analyticity issue of S5 in the context of ordinary sequents have been explored, but they usually involve breaking some other properties of proof-theory orthodoxy like the *subformula property* or the locality constraint [97, 121, 12]. One alternative treatment has been via the formalism of hypersequents. It was apparently independently developed by Mints [97], Pottinger [113], and Avron [6].

A *hypersequent*  $\mathcal{H} := \Gamma_1 \mid \dots \mid \Gamma_n$  is a finite collection of sequents separated by vertical bars, and each  $\Gamma_i$  for  $1 \leq i \leq n$  is called a *hypersequent component*. In the context of modal logic, it can be interpreted as the modal logic formula  $\text{fm}^h(\mathcal{H}) := \Box \text{fm}^o(\Gamma_1) \vee \dots \vee \Box \text{fm}^o(\Gamma_n)$ . Intuitively it allows us to consider several streams of proofs in parallel that can communicate through *structural rules*; structural in the sense that they interact only with a structural element, here the vertical bar.

On Figure 2.3 we present, in the first part, rules of a basic hypersequent system hS4 corresponding to the logic S4, the extension of K with axioms  $t$  and 4. Note that the first two lines are almost identical to the sequent rules for S4 on Figure 2.2, namely the rules for the propositional connectives, the rules  $\Box_{k4}^o$  and  $\Diamond_t^o$ , and the (internal) structural rules  $c$  and  $w$ . The only difference is the presence of a hypersequent context in every rule, but the latter does not play a significant role in these rules. Two structural rules need to be added to handle the management of this context, the external contraction  $ec$  and external weakening  $ew$ .

Finally, in the second part of Figure 2.3, the rules  $\Box_5^h$  [6] and  $\Box_2^h$  [71] can be added to the basic system to obtain a sound and complete system for the logics S5, the extension of S4 by the axiom 5, and S4.2, the extension of S4 by the axiom 2.

**2.2.2 Theorem** (Avron [6], Kurokawa [71]) *Let  $x$  be the axiom 5 or 2. For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of S5 if  $x = 5$  or S4.2 if  $x = 2$ .
2.  $A$  is provable in hS4 +  $\Box_x^h$  + cut where cut is the rule
3.  $A$  is provable in the hypersequent calculus hS4 +  $\Box_x^h$ .

$$\boxed{\text{cut} \frac{\mathcal{H} \mid \Gamma, C \quad \mathcal{H} \mid \Gamma, \neg C}{\mathcal{H} \mid \Gamma}}$$

These sequent and hypersequent systems were all developed *ad hoc* for a specific logic in a specific context, very probably by “trial and error”. However, we can observe patterns in the way each system relates to its corresponding logic. Lellmann and Pattinson [77, 76], inspired by a successful line of work by Ciabattoni *et al.* on intermediate and substructural logics [30, 33], have explored this idea fully and explained a way to generalise these patterns

in order to understand which modal logics can be given a sequent system based on sequents *with context restrictions*, and which ones cannot and require hypersequents *with context restrictions*. They show in particular that **S5** falls into this second category; it does not prove that **S5** cannot be given a sequent system at all, but that its system would have to be quite special.

There is a noticeable difference between rules in the sequent and in the hypersequent calculi. In the sequent calculi, some modal operators are introduced in the conclusion of the rules, similarly to what we are used to for propositional connectives. In the hypersequent rules corresponding to **5** and **2** on the other hand, the modal operators are only shifted from one hypersequent component to another one, being already present in the premisses, and there is furthermore an interaction with the structural element in the context. This point will be relevant in the other proof systems for modal logic that we will introduce in Section 2.4 and 2.5.

## 2.3 Possible-world semantics

Probably the most crucial result of the 20th century for modal logic has been the definition of *possible-worlds semantics*. It is commonly attributed to Kripke (and even alternatively called Kripke semantics), as he gave its first systematic treatment [69], though we know now that this concept was “in the air” around that time (see the history in [35]). The idea is to use graphs, relational structures, to give a meaning to modal formulas, or the other way around, to use the modal language to describe properties of graphs. A standard reference for this section is for instance [10].

We start with a graph: a *frame*  $\mathcal{F}$  is a pair  $\langle W, R \rangle$  of a non-empty set  $W$  of *possible worlds* and a binary relation  $R \subseteq W \times W$ , called the *accessibility relation*. Then we add a mechanism to evaluate formulas: a *model*  $\mathfrak{M}$  is a frame together with a *valuation* function  $V: W \rightarrow 2^A$ , which assigns to each world  $w$  a subset of propositional variables that are “true” in  $w$ . The *truth* of a modal formula at a world  $w$  in a relational structure is the smallest relation  $\Vdash$  satisfying:

$$\begin{aligned} w \Vdash a & \quad \text{iff} \quad a \in V(w) \\ w \Vdash \bar{a} & \quad \text{iff} \quad a \notin V(w) \\ w \Vdash A \wedge B & \quad \text{iff} \quad w \Vdash A \text{ and } w \Vdash B \\ w \Vdash A \vee B & \quad \text{iff} \quad w \Vdash A \text{ or } w \Vdash B \\ w \Vdash \Box A & \quad \text{iff} \quad \text{for all } v \in W \text{ such that } (w, v) \in R \text{ one has } v \Vdash A \\ w \Vdash \Diamond A & \quad \text{iff} \quad \text{there exists } v \in W \text{ such that } (w, v) \in R \text{ and } v \Vdash A \end{aligned}$$

Notice the correspondence between  $\Box$  and universal quantification  $\forall$ , as well as  $\Diamond$  and existential quantification  $\exists$ . This allows us to identify classical modal logic as a fragment of first-order classical logic.

We say that a formula  $A$  is *satisfied in a model*  $\mathfrak{M} = \langle W, R, V \rangle$ , denoted by  $\mathfrak{M} \models A$ , if for every  $w \in W$ ,  $w \Vdash A$ . We say that a formula  $A$  is *valid in a frame*  $\mathcal{F} = \langle W, R \rangle$ , denoted by  $\mathcal{F} \models A$ , if for every valuation  $V$ ,  $\langle W, R, V \rangle \models A$ .

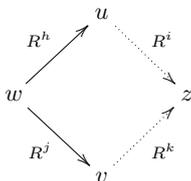
The syntax presented in Section 2.1 and this possible-world semantics are then linked by the fact that the logic **K** is sound and complete with respect to these general frames.

**2.3.1 Theorem** (Kripke [70]) *A formula  $A$  is derivable in the Hilbert system **K** if and only if  $A$  is valid in every frame.*

Furthermore, the power of this construction is that this link is not restricted to **K**: some classes of modal formulas correspond to specific properties of frames. An alternative way, then, to obtain modal logics stronger than **K** is by restricting the class of frames we want to consider, by imposing some constraints on the accessibility relation. For example, for a given 4-tuple of natural numbers  $\langle h, i, j, k \rangle$ , it is possible to recover the logic axiomatised

by the Scott-Lemmon axiom  $\mathfrak{g}_{hijk}$  by considering the formulas that are valid in all frames satisfying the following confluence condition:

$$\text{for all } w, u, v \in W \text{ with } wR^h u \text{ and } wR^j v \text{ there is a } z \in W \text{ s.t. } uR^i z \text{ and } vR^k z. \quad (2.3)$$



where we define as usual  $R^n$  to be  $R$  composed  $n$  times with itself, that is,  $R^0$  is the diagonal of  $W$  and  $R^{n+1} = \{(w, v) \mid \exists u. (w, u) \in R \text{ and } (u, v) \in R^n\}$ .

The Hilbert system  $\mathbf{K} + \mathbf{G}$  actually completely axiomatises classical models of this type.

**2.3.2 Theorem** (Lemmon and Scott [80]) *Let  $\mathbb{G} \subseteq \mathbb{N}^4$  and let  $\mathbf{G}$  be the corresponding set of Scott-Lemmon axioms. A formula is provable in  $\mathbf{K} + \mathbf{G}$  if and only if it is valid in all frames satisfying (2.3) for each  $\langle h, i, j, k \rangle \in \mathbb{G}$ .*

The following table recapitulates the modal axioms constituent of the S5-cube, and in parallel their corresponding first-order counterpart and the frame property it entails.

| Axiom  | Frame property | First-order correspondence  |
|--|----------------|---|
| <b>t</b> : $A \supset \Diamond A$                    | Reflexivity    | $\forall x. xRx$  |
| <b>b</b> : $A \supset \Box \Diamond A$               | Symmetry       | $\forall x, y. xRy \supset xRy$   |
| <b>d</b> : $\Box A \supset \Diamond A$               | Seriality      | $\forall x \exists y. xRy$  |
| <b>4</b> : $\Diamond \Diamond A \supset \Diamond A$  | Transitivity   | $\forall x, y, z. (xRy \wedge yRz) \supset xRz$                         |
| <b>5</b> : $\Diamond A \supset \Box \Diamond A$      | Euclideaness   | $\forall x, y, z. (xRy \wedge xRz) \supset yRz$                         |
| <b>2</b> : $\Diamond \Box A \supset \Box \Diamond A$ | Confluence     | $\forall x, y, z. (xRy \wedge xRz) \supset \exists u. (yRu \wedge zRu)$ |

The possible-worlds semantics has rejuvenated the field of modal logics since graphs are ubiquitous in modelling, and possessing a language to describe their properties is highly useful. More surprisingly perhaps, this idea not only allowed the model theory of modal logic to grow, but also gave a new perspective on its proof theory as it offered another view on the way proof systems could work, along semantical lines.

## 2.4 Labelled deduction

Once possible-world semantics was established as a solid base to define modal logics, the idea of incorporating these notions into the proof theory of modal logics emerged. Fitch seems to have been the first one to formalise it, directly including symbols representing worlds into the language of his proofs in *natural deduction* [43].

The concept was popularised when Fitting described a tableau proof system for the logics of the S5-cube called *prefixed tableaux* where each formula is prefixed by a label, referring to a world in a relational structure [44]. Modal operators can only be introduced while changing the prefix of the formula, which corresponds to the fact that the semantics of  $\Box A$  and  $\Diamond A$  uses the semantics given to  $A$  in every/some accessible worlds. Fitting also remarked that the correspondence between modalities of propositional modal logic and quantifiers of first-order logic observed at the semantical level, namely between  $\Box$  and  $\forall$ , and between  $\Diamond$  and  $\exists$ , is also noticeable in the similarity of the proof rules, and even in the meta-theory. His systems were later refined and given a modular treatment by Massacci [89].

Labelled deduction has been more generally proposed by Gabbay in the 80's as a unifying framework throughout proof theory in order to provide proof systems for a wide range of logics [52]. For modal logics it can also take the form of labelled natural deduction

and labelled sequent systems as used, for example, by Simpson [127], Viganò [134], and Negri [101]. These formalisms make explicit use not only of labels, but also of *relational atoms*. We will continue with our choice of the sequent presentation; we present the calculus of Negri [101] for classical modal logic in this section and come back to Simpson’s system for intuitionistic modal logic [127] in the next chapter.

## Geometric theories

The approach here is not to obtain a proof system for a logic by converting modal axioms into inference rules as before, but to consider the logic as defined by a given class of frames. The most general frame properties which have been considered in this setting are based on *geometric theories*. They generalise, in particular, the Scott-Lemmon frame properties that we introduced in (2.3).

Interest in the study of geometric theories arises from different areas of logic and mathematics, such as topology and category theory. The proof-theoretical analysis of systems for first-order classical logic extended by such theories was investigated in detail by Negri and von Plato [104]. Here we recall some basic definitions and results, adapted mainly from this reference.

A *geometric formula* is a first-order formula of the form:

$$\forall \mathbf{z}(A \supset B)$$

where  $A$  and  $B$  contain neither  $\supset$  nor  $\forall$ , and  $\mathbf{z}$  is a vector of variables  $z_1, \dots, z_n$ .

It has been observed (e.g. [127] and for a detailed proof see [130]) that geometric implications can be expressed in the following canonical form.

**2.4.1 Proposition** *For each geometric formula, there exists an equivalent formulation as a conjunction of geometric clauses of the form:*

$$\forall \mathbf{z}((a_1 \wedge \dots \wedge a_n) \supset (\exists \mathbf{x}_1 A_1 \vee \dots \vee \exists \mathbf{x}_m A_m))$$

where each  $a_i$  for  $1 \leq i \leq n$  is an atomic formula, each  $A_j$  for  $1 \leq j \leq m$  is a conjunction of atomic formulas  $b_{j1}, \dots, b_{jk_j}$ , and none of the variables in the vectors  $\mathbf{x}_j$  are free in any  $a_i$ .

If one tries to add axioms from a Hilbert-style system as hypotheses of a sequent system it will in general break the cut-admissibility property [58]. However, for certain classes of axioms, in particular the geometric ones just defined, it is possible to overcome this issue by converting the axioms into inference rules. The idea is to incorporate implicitly some of the logical content of the axioms into the ‘meta-language’ of premisses and conclusions of a sequent rule. A general rule scheme has been proposed for geometric axioms [100], and we will present its modal version.

## Labelled sequents

Labelled sequents are formed from by *labelled formulas* of the form  $x : A$  and *relational or equality atoms* of the form  $xRy$  or  $x = y$  respectively, where  $x, y$  range over a set of variables (called *labels*) and  $A$  is a modal formula. A (one-sided) labelled sequent is then of the form  $\mathcal{G} \Rightarrow \mathcal{R}$  where  $\mathcal{G}$  denotes a set of relational or equality atoms, and  $\mathcal{R}$  a multiset of labelled formulas. A simple proof system (shown on Figure 2.4) for classical modal logic  $\mathbf{K}$  can be obtained in this formalism.

This can be extended to a wide class of modal logics by applying the general ideas described in the previous section. In the previous paragraph, we remained purposefully vague about the context of application as these rule schemes can occur in different contexts and different calculi, with a little bit of adaptation. The context that interests us here is when it is applied to modal logics, more precisely when modal logics are defined in terms

|   |
|---|
| $\text{id}^{\text{lab}} \frac{}{\mathcal{G} \Rightarrow \mathcal{R}, x : a, x : \bar{a}} \quad \top^{\text{lab}} \frac{}{\mathcal{G} \Rightarrow \mathcal{R}, x : \top}$  |
| $\wedge^{\text{lab}} \frac{\mathcal{G} \Rightarrow \mathcal{R}, x : A \quad \mathcal{G} \Rightarrow \mathcal{R}, x : B}{\mathcal{G} \Rightarrow \mathcal{R}, x : A \wedge B} \quad \vee^{\text{lab}} \frac{\mathcal{G} \Rightarrow \mathcal{R}, x : A, x : B}{\mathcal{G} \Rightarrow \mathcal{R}, x : A \vee B}$       |
| $\square^{\text{lab}} \frac{\mathcal{G}, xRy \Rightarrow \mathcal{R}, y : A}{\mathcal{G} \Rightarrow \mathcal{R}, x : \square A} \quad y \text{ is fresh} \quad \diamond^{\text{lab}} \frac{\mathcal{G}, xRy \Rightarrow \mathcal{R}, x : \diamond A, y : A}{\mathcal{G}, xRy \Rightarrow \mathcal{R}, x : \diamond A}$ |

Figure 2.4: System labK

of the frames they characterise. Let us consider a modal logic that characterises a class of frames satisfying a set of geometric properties. Note that in this case the only atomic propositions we can refer to are relational or equality atoms. Therefore a *geometric frame property* will be of the form:

$$\forall \mathbf{u} \forall \mathbf{v} ((a_1 \wedge \dots \wedge a_n) \supset (\exists \mathbf{x}_1 A_1 \vee \dots \vee \exists \mathbf{x}_m A_m)) \quad (2.4)$$

where each  $a_i$  for  $1 \leq i \leq n$  is a relational or an equality atom,  $u_i R v_i$  or  $u_i = v_i$ , for  $1 \leq j \leq m$  each  $A_j = b_{j1} \wedge \dots \wedge b_{jk_j}$  is also a conjunction of relational or equality atoms, and the variables in  $\mathbf{x}_j$  do not appear among  $\mathbf{u}$  or  $\mathbf{v}$ .

The corresponding *geometric rule scheme* is obtained from the generic shape of [100] as:

$$\boxtimes_{\text{grs}}^{\text{lab}} \frac{A_1(\mathbf{y}_1/\mathbf{x}_1), a_1, \dots, a_n, \mathcal{G} \Rightarrow \mathcal{R} \quad \dots \quad A_m(\mathbf{y}_m/\mathbf{x}_m), a_1, \dots, a_n, \mathcal{G} \Rightarrow \mathcal{R}}{a_1, \dots, a_n, \mathcal{G} \Rightarrow \mathcal{R}} \quad (2.5)$$

where  $\mathcal{G}$  is a set of relational or equality atoms,  $\mathcal{R}$  is a multiset of labelled modal formulas, each  $a_i$  for  $1 \leq i \leq n$  is a relational or an equality atom, each  $A_j$  for  $1 \leq j \leq m$  denotes the set of relational or equality atoms present in  $A_j$ , and  $\mathbf{y}_1, \dots, \mathbf{y}_m$  are vectors of fresh eigenvariables, i.e. they do not occur in the conclusion.

**2.4.2 Theorem** (Negri [101]) *Let  $\mathcal{C}$  be a set of geometric frame properties as in (2.4) and  $\boxtimes_{\mathcal{C}}^{\text{lab}}$  be the corresponding set of rules following schema (2.5). For any formula  $A$ , the following are equivalent.*

1.  $A$  is provable in  $\text{labK} + \boxtimes_{\mathcal{C}}^{\text{lab}} + \text{cut}$  where cut is
2.  $A$  is provable in  $\text{labK} + \boxtimes_{\mathcal{C}}^{\text{lab}}$
3.  $A$  is valid in all frames satisfying the properties in  $\mathcal{C}$ .

|   |
|---|
| $\text{cut} \frac{\mathcal{G} \Rightarrow \mathcal{R}, z : C \quad \mathcal{G} \Rightarrow \mathcal{R}, z : \neg C}{\mathcal{G} \Rightarrow \mathcal{R}}$ |
|---|

As examples, we sum up in this table the rules that are obtained via this process for the most common axioms we reviewed in previous sections.

| Axiom  | Geometric rule scheme  |
|--|--|
| t: $A \supset \diamond A$                          | $\boxtimes_{\text{t}}^{\text{lab}} \frac{\mathcal{G}, xRx \Rightarrow \mathcal{R}}{\mathcal{G} \Rightarrow \mathcal{R}} \quad x \text{ appears in } \mathcal{R}$                                 |
| b: $A \supset \square \diamond A$                  | $\boxtimes_{\text{b}}^{\text{lab}} \frac{\mathcal{G}, xRy, yRx \Rightarrow \mathcal{R}}{\mathcal{G}, xRy \Rightarrow \mathcal{R}}$   |
| d: $\square A \supset \diamond A$                  | $\boxtimes_{\text{d}}^{\text{lab}} \frac{\mathcal{G}, xRy \Rightarrow \mathcal{R}}{\mathcal{G} \Rightarrow \mathcal{R}} \quad x \text{ appears in } \mathcal{R} \text{ but } y \text{ is fresh}$ |
| 4: $\diamond \diamond A \supset \diamond A$        | $\boxtimes_4^{\text{lab}} \frac{\mathcal{G}, xRy, yRz, xRz \Rightarrow \mathcal{R}}{\mathcal{G}, xRy, yRz \Rightarrow \mathcal{R}}$  |
| 5: $\diamond A \supset \square \diamond A$         | $\boxtimes_5^{\text{lab}} \frac{\mathcal{G}, xRy, xRz, yRz \Rightarrow \mathcal{R}}{\mathcal{G}, xRy, xRz \Rightarrow \mathcal{R}}$  |
| 2: $\diamond \square A \supset \square \diamond A$ | $\boxtimes_2^{\text{lab}} \frac{\mathcal{G}, xRy, xRz, yRu, zRu \Rightarrow \mathcal{R}}{\mathcal{G}, xRy, xRz \Rightarrow \mathcal{R}} \quad u \text{ is fresh}$                                |

Let us re-examine the points that we faltered upon in the case of sequents and hypersequents: (i) locality; (ii) modularity; (iii) generality; and (iv) analyticity. The rules introducing modalities in the labelled sequent calculus are local; they separate the rule for  $\Box$  from the rule for  $\Diamond$  and they do not assume any restriction on the form of the context. The labelled sequent calculus allows for a modular treatment of modal logics defined by (geometric) frame properties, namely for any set of geometric formulas, one can add the corresponding set of rules to a basic system and obtain a complete system for the given logic. Furthermore, it is the most general system we know of as a lot of the most common modal logics axiomatise frames satisfying geometric axioms, including the ones defined by Scott-Lemmon axioms. Finally, it is possible to prove a general cut-elimination that applies for the basic system and all its extensions by geometric rules.

## 2.5 Unlabelled deduction

We have already presented instances of unlabelled deduction systems for modal logics in the form of sequent and hypersequent calculi. We have also seen their drawbacks, and how labelled systems offer a smart solution, bringing back expressivity and modularity. However, some proof-theorists are not satisfied with the idea of labels in proofs that would be seen as ‘semantical pollution’ because some ingredients of a labelled formalism resemble model-theoretic objects [116]. They advocate then for ‘label-free’ alternatives that would have the same qualities as labelled systems, but with no labelling, arguably more in line with original sequents or hypersequents.

*Display calculus* is a proof formalism which does not use labelling and provides a systematic way to obtain structural rules for every Scott-Lemmon (SL) axiom via Kracht’s algorithm in [68], unfortunately only in the context of *tense logic*, an extension of modal logic with an adjoint for each connective. To our knowledge, the modal case, i.e. how to get display rules corresponding to the purely modal SL axioms, is still open (and perhaps impossible).

A generalisation of tableaux for modal logics without labels was proposed in [17]. Instead of using labelling, Castilho *et al* propose to write tableaux rules as graphical rewrite rules. Moreover, whereas prefixed tableaux are usually built in a tree structure, these label-free tableaux use rooted directed acyclic graphs as their basis of construction. This allowed them to express many more modal logics than Fitting’s [45] or Massacci’s [89] systems.

### Nested sequents

Another way to tackle the proof theory of modal logics without labelling is offered by the formalism of *nested sequents*. Nested sequents are a generalisation of sequents from a multiset of formulas to a *tree* of multisets of formulas. They were first defined by Kashima [66] for tense logic and by Cerrato [18] for modal logic (under the name *tree-sequents*), and then independently rediscovered by Poggiolesi [112] (who called them *tree-hypersequents*) and Brünnler [15].

In nested sequent notation, brackets are used to indicate the parent-child relation in the tree, and can be interpreted as the modal  $\Box$  (similarly to how the comma is interpreted as  $\vee$ ). In other words, nested sequents are defined according to the following grammar:

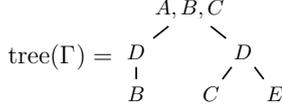
$$\Gamma ::= \emptyset \mid A, \Gamma \mid \Gamma_1, [\Gamma_2] \quad (2.6)$$

For any nested sequent  $\Gamma$ , a *corresponding formula*, written  $\text{fm}^n(\Gamma)$ , gives the meaning of  $\Gamma$  as a modal logic formula. It is defined as:  $\text{fm}^n(\emptyset) := \perp$ ,  $\text{fm}^n(A, \Gamma) := A \vee \text{fm}^n(\Gamma)$ , and  $\text{fm}^n(\Gamma_1, [\Gamma_2]) := \text{fm}^n(\Gamma_1) \vee \Box \text{fm}^n(\Gamma_2)$ . Note that the sequent  $[\emptyset]$  is different from  $\emptyset$ .

**2.5.1 Example** The nested sequent  $\Gamma = A, B, C, [D, [B]], [D, [C], [E]]$  is the term used to describe the following tree:

$$\begin{array}{c}
\text{id}^n \frac{}{\Gamma\{a, \bar{a}\}} \quad \wedge^n \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \top^n \frac{}{\Gamma\{\top\}} \quad \vee^n \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \Box^n \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond_k^n \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \\
\text{-----} \\
\Diamond_t^n \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \quad \Diamond_b^n \frac{\Gamma_1\{[\Gamma_2, \Diamond A], A\}}{\Gamma_1\{[\Gamma_2, \Diamond A]\}} \quad \Diamond_d^n \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} \quad \Diamond_4^n \frac{\Gamma_1\{\Diamond A, [\Diamond A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond_5^n \frac{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\Diamond A\}}{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\emptyset\}}
\end{array}$$

Figure 2.5: System nK and extensions for t, b, d, 4, and 5



It can be interpreted as the modal formula:  $\text{fm}^n(\Gamma) = A \vee B \vee C \vee \Box(D \vee \Box B) \vee \Box(D \vee \Box C \vee \Box E)$ .

The inference rules for nested sequents operate on sub-trees of such sequents. To identify the sub-trees, we use the notions of *context* and *substitution*. An  $n$ -ary *context*  $\Gamma\{^1\} \cdots \{^n\}$  is like a nested sequent, but contains  $n$  pairwise distinct numbered *holes*  $\{\}$  in place of formulas (and never inside a formula). We can assume by default that holes are independently numbered in left-to-right order and so, unless there is any ambiguity, we will omit the hole index for better readability. Note that a 0-ary context is the same as a sequent.

Given such a context  $\Gamma\{\} \cdots \{\}$  and  $n$  nested sequents as *arguments*  $\Gamma_1, \dots, \Gamma_n$ , we write  $\Gamma\{\Gamma_1\} \cdots \{\Gamma_n\}$  to stand for the *substitution* nested sequent where for  $1 \leq i \leq n$  the  $i$ -th hole in the context has been replaced by  $\Gamma_i$ , with the understanding that if  $\Gamma_i$  is empty then the hole is simply removed.

The *depth* of  $\Gamma\{\}$ , written  $\text{dp}(\Gamma\{\})$ , is given inductively by:  $\text{dp}(\{\}) = 0$ ,  $\text{dp}(\Gamma_1, \Gamma_2\{\}) = \text{dp}(\Gamma_2\{\})$ , and  $\text{dp}([\Gamma\{\}]) = \text{dp}(\Gamma\{\}) + 1$ .

**2.5.2 Example** Let  $\Gamma\{\}\{\} = A, [B, \{\}, [\{\}], C]$ . For the sequents  $\Gamma_1 = D$  and  $\Gamma_2 = A, [C]$ , we get:  $\Gamma\{\Gamma_1\}\{\Gamma_2\} = A, [B, D, [A, [C]], C]$  and  $\Gamma\{\emptyset\}\{\Gamma_2\} = A, [B, [A, [C]], C]$ . We also have that  $\text{dp}(\Gamma\{\}\{\emptyset\}) = 1$  and  $\text{dp}(\Gamma\{\emptyset\}) = 2$ .

The basic modal logic K is captured using nested sequents as the cut-free proof system nK shown in the first line of Figure 2.5. The rules  $\text{id}^n$ ,  $\wedge^n$ ,  $\top^n$ , and  $\vee^n$  are basically identical to the corresponding ones in the ordinary sequent system on Figure 2.2. They do not interact with the bracketing structure, but they can occur at any node in the context  $\Gamma\{\}$ . The rules  $\Box^n$  and  $\Diamond^n$  can also occur at any depth in  $\Gamma\{\}$ ; however, they justify the need of generalising sequents to nested sequents as  $\Box^n$  transforms a bracketed formula into a  $\Box$ -formula and  $\Diamond^n$  shifts a formula from the inside of a bracket to the outside by prefixing it with a  $\Diamond$ . The two modal rules implement the same idea as the ones in the labelled calculus on Figure 2.4: the parent-child relation in the bracketing tree of a nested sequent replaces here the relational notation  $xRy$ .

**2.5.3 Example** In nK there are two possible derivations of the sequent  $\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [b]$  (which is equivalent to axiom k):

$$\begin{array}{c}
\text{id}^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [a, \bar{a}, b]} \quad \text{id}^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [\bar{b}, \bar{a}, b]} \\
\wedge^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [a \wedge \bar{b}, \bar{a}, b]} \quad \Diamond_k^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [a, b]} \quad \text{id}^n \frac{}{\Diamond(a \wedge b), \Diamond \bar{a}, [\bar{b}, b]} \\
\Diamond_k^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [a, b]} \quad \wedge^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [a \wedge \bar{b}, b]} \\
\Diamond_k^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [\bar{a}, b]} \quad \Diamond_k^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [a \wedge \bar{b}, b]} \\
\Diamond_k^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [b]} \quad \Diamond_k^n \frac{}{\Diamond(a \wedge \bar{b}), \Diamond \bar{a}, [b]}
\end{array}$$

We now want to extend this system modularly into cut-free systems for axiomatic extensions of K. In their tableaux systems, Castilho *et al* [17] distinguish two classes of axioms that correspond to two kinds of rules: propagation rules and structural rules. The first one allows them to rewrite the content of one of the nodes in the graph and more precisely

corresponds to a step of the form “if in some node of such a given pattern there is some formula  $A$ , then propagate the formula  $B$  (which can be  $A$  or another one)”; while the other one allows them to directly change the structure of the graph itself, which implements a step of the form “if there is such a given pattern then add some new node(s) and edge(s)”. We will investigate these two types of rules in the formalism of nested sequents.

### Propagation rules.

Brünnler and Poggiolesi obtained nested sequents systems for all the logics in the S5-cube, without giving a systematic algorithm to transform axioms into rules. Following the ideas of Goré, Postniece and Tiu in [60] but adapting the presentation slightly, we present an algorithm to define nested sequent systems for the logic  $K$  extended with any subset of the Scott-Lemmon path axioms, defined in (2.2). Note that the authors worked in the context of tense logic so they also had adjoint modalities in their language. Here, we keep our language purely modal and use the alphabet  $\diamond$  and  $\blacklozenge$ , with no intended logical meaning.

For any Scott-Lemmon (SL) path axiom  $p := \diamond^h \Box A \supset \diamond^j A$ , we define its *reflections* as the two following pairs of a sequence (denoted  $\cdot$  if empty) and an element of  $\{\diamond, \blacklozenge\}$ :

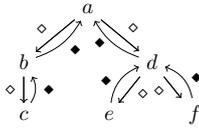
$$p^\diamond := (\blacklozenge^h \diamond^j, \diamond) \qquad p^\blacklozenge := (\diamond^h \blacklozenge^j, \blacklozenge)$$

where  $\blacklozenge^h$  denotes  $h$  occurrences of  $\blacklozenge$  and  $\diamond^h$  denotes  $h$  occurrences of  $\diamond$ .

Let  $s = (\langle s_1 \rangle \cdots \langle s_m \rangle, \langle s_0 \rangle)$  and  $t = (\langle t_1 \rangle \cdots \langle t_n \rangle, \langle t_0 \rangle)$  be pairs of a sequence and one item of  $\{\diamond, \blacklozenge\}$ , i.e. for  $0 \leq i \leq m$  and for  $0 \leq k \leq n$ ,  $\langle s_i \rangle, \langle t_k \rangle \in \{\diamond, \blacklozenge\}$ .  $s$  is *composable* with  $t$  at position  $k$  if  $\langle s_0 \rangle = \langle t_k \rangle$ . Then, the composition of  $s$  with  $t$  at position  $k$  is the pair:  $s \triangleright^k t := (\langle t_1 \rangle \cdots \langle t_{k-1} \rangle \langle s_1 \rangle \cdots \langle s_m \rangle \langle t_{k+1} \rangle \cdots \langle t_n \rangle, \langle t_0 \rangle)$ . Finally, for a set of SL path axioms  $P$ , the *completion* of  $P$  is the smallest set that (i) contains the two reflections  $p^\diamond$  and  $p^\blacklozenge$  for any  $p \in P$ , as well as the two identity pairs  $(\diamond, \diamond)$  and  $(\blacklozenge, \blacklozenge)$ , and (ii) is closed under composition (at any position).

Let  $\Gamma$  be a nested sequent. The *propagation graph* of  $\Gamma$  is a directed graph defined on the set of nodes of  $\text{tree}(\Gamma)$  such that for any nodes  $n_1$  and  $n_2$ , if  $n_2$  is a child of  $n_1$  in  $\text{tree}(\Gamma)$ , then in the propagation graph there is an edge  $(n_1, n_2)$  labelled  $\diamond$  and there is an edge  $(n_2, n_1)$  labelled  $\blacklozenge$ . If  $\pi$  is a path  $n_1 \langle l_1 \rangle n_2 \langle l_2 \rangle \dots \langle l_{k-1} \rangle n_k$  in the propagation graph of  $\Gamma$  where each edge-label  $\langle l_i \rangle \in \{\diamond, \blacklozenge\}$  for  $1 \leq i \leq k-1$ , then we denote by  $\langle \pi \rangle$  the sequence of labels  $\langle l_1 \rangle \langle l_2 \rangle \dots \langle l_{k-1} \rangle$  along that path.

**2.5.4 Example** Consider again the nested sequent  $\Gamma = A, B, C, [D, [B]], [D, [C], [E]]$ , its propagation graph is:



The sequence  $\langle \pi \rangle = \blacklozenge \blacklozenge \diamond \diamond$  corresponds to a path going for example from the node  $e$  to the node  $f$ .

For a set  $P$  of SL path axioms, the set  $\diamond_\pi^n$  of *propagation rules* for  $P$  contains any rule

$$\diamond_\pi^n \frac{\Gamma \{^u \diamond A\} \{^v A\}}{\Gamma \{^u \diamond A\} \{^v \emptyset\}}$$

if there is a path  $\pi$  from  $u$  to  $v$  in the propagation graph of  $\Gamma \{^u \emptyset\} \{^v \emptyset\}$  such that  $(\langle \pi \rangle, \diamond)$  appears in the completion of  $P$ . Note that since the identity pairs are always present in the completion of  $P$ , the rule  $\diamond_\pi^n$  necessarily appears in the set of propagation rules.

**2.5.5 Theorem** (Goré, Postniece and Tiu [60]) *Let  $P$  be a set of Scott-Lemmon path axioms and  $\diamond_\pi^n$  be the corresponding set of propagation rules. For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $K + P$ .
2.  $A$  is provable in the nested sequent calculus  $nK + \diamond_p^n$

This is not an ideal definition of a deductive system because, even if the set of nodes in  $\text{tree}(\Gamma)$  and the set  $P$  are finite, (i) the completion of  $P$  can be infinite, and (ii) there might be an infinite number of (looping) paths between any two nodes  $u$  and  $v$  in the propagation graph of  $\Gamma$ . However, Goré, Postniece and Tiu proved, using automaton theoretic intuitions, that the decision problem whether a propagation rule can be applied to two nodes of a nested sequents actually is decidable. Their idea is to see path axioms as a context-free grammar and the propagation graph of a nested sequent as a finite-state automaton, and then to reduce their decision problem to emptiness of the intersection of a context-free grammar and the regular language associated to an automaton, that we know to be a decidable problem.

They moreover show that some specific axioms can be described by simple regular expression which allow them to redefine the set of propagation rules for these axioms and in particular to recover some already known rules as previously defined by Brünnler or Poggiolesi. (We omit the rule  $\diamond_k^n$  in the set of propagation rules of t and b.)

| Axiom                                       | Completion  | Set of propagation rules  |
|---|---|---|
| t: $A \supset \diamond A$                   | $(\cdot, \diamond), (\cdot, \blacklozenge), (\diamond, \diamond), (\blacklozenge, \blacklozenge)$   | $\diamond_t^n \frac{\Gamma\{\diamond A, A\}}{\Gamma\{\diamond A\}}$   |
| b: $A \supset \square \diamond A$           | $(\diamond, \blacklozenge), (\blacklozenge, \diamond), (\diamond, \diamond), (\blacklozenge, \blacklozenge)$  | $\diamond_b^n \frac{\Gamma_1\{\Gamma_2, \diamond A\}, A\}}{\Gamma_1\{\Gamma_2, \diamond A\}}$   |
| 4: $\diamond \diamond A \supset \diamond A$ | $(\diamond^n, \diamond), (\blacklozenge^n, \blacklozenge)$<br>for $n \in \mathbb{N}, n \geq 1$  | $\diamond_{4+k}^n \frac{\Gamma\{\diamond A, [\Gamma_1[\dots[\Gamma_n, A]\dots]]\}}{\Gamma\{\diamond A, [\Gamma_1, [\dots[\Gamma_n]\dots]]\}}$<br>for $n \in \mathbb{N}, n \geq 1$ |
| 5: $\diamond A \supset \square \diamond A$  | $(\blacklozenge\langle l_1 \rangle \dots \langle l_n \rangle \diamond, \langle l_0 \rangle), (\diamond, \diamond), (\blacklozenge, \blacklozenge)$<br>$n \in \mathbb{N}$ and $\langle l_i \rangle \in \{\diamond, \blacklozenge\}, 0 \leq i \leq n$ | $\diamond_{5+k}^n \frac{\Gamma_1\{\{\diamond A, \Gamma_2\}\}\{[A, \Gamma_3]\}}{\Gamma_1\{\{\diamond A, \Gamma_2\}\}\{[\Gamma_3]\}}$   |

Actually the rules introduced by Brünnler and Poggiolesi allow the propagation of  $\diamond$  without introducing it. It does not affect t and b, but it gives a simpler notation of the rules for 4 and 5 (though a trickier proof of cut-elimination).

$$\diamond_4^n \frac{\Gamma_1\{\diamond A, \{\diamond A, \Gamma_2\}\}}{\Gamma_1\{\diamond A, [\Gamma_2]\}} \quad \diamond_5^n \frac{\Gamma_1\{\{\diamond A, \Gamma_2\}\}\{\diamond A\}}{\Gamma_1\{\{\diamond A, \Gamma_2\}\}\{\emptyset\}}$$

Furthermore, Brünnler and Poggiolesi define a rule corresponding to the axiom d, which is not an SL path axiom, as follows:

$$\diamond_d^n \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}} \quad (2.7)$$

This allows them to give cut-free complete systems for  $nK$  extended to any logic of the S5-cube. A nice property of the logics in the S5 cube, already observed in [17], is that even though completion sets might become complex when combining axioms, in the case of these fifteen logics we can consider for each of them a specific axiomatisation for which we only need to combine the rules corresponding to each chosen axiom in order to obtain a complete nested sequent system for the whole logic. We say that this set of axioms is *axiomatically complete*: if a logic in the S5 cube has a unique axiomatisation, then it is necessarily axiomatically complete; otherwise, for KB5 and S5 that can be axiomatised by several distinct set of axioms,  $\{b, 4, 5\}$  and  $\{t, 4, 5\}$  respectively are axiomatically complete sets to define them.

$$\boxed{\boxtimes_t^n \frac{\Gamma_1\{\Gamma_2\}}{\Gamma_1\{\Gamma_2\}} \quad \boxtimes_b^n \frac{\Gamma_1\{\Gamma_2, [\Gamma_3]\}}{\Gamma_1\{\Gamma_2, \Gamma_3\}} \quad \boxtimes_d^n \frac{\Gamma\{\emptyset\}}{\Gamma\{\emptyset\}} \quad \boxtimes_4^n \frac{\Gamma_1\{\Gamma_2, [\Gamma_3]\}}{\Gamma_1\{[\Gamma_2], \Gamma_3\}} \quad \boxtimes_5^n \frac{\Gamma_1\{\Gamma_2, [\Gamma_3]\}\{\emptyset\}}{\Gamma_1\{\Gamma_2\}\{\Gamma_3\}}}$$

Figure 2.6: Nested structural rules for t, b, d, 4, and 5

**2.5.6 Theorem** (Brünnler [15]) *Let  $X \subseteq \{t, b, d, 4, 5\}$  be axiomatically complete. For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $K + X$ .
2.  $A$  is provable in  $nK + \diamond_X^n + \text{cut}$  where  $\text{cut}$  is the rule
3.  $A$  is provable in  $nK + \diamond_X^n$ .

$$\boxed{\text{cut} \frac{\Gamma\{C\} \quad \Gamma\{\neg C\}}{\Gamma\{\emptyset\}}}$$

The proof follows the chain  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ . The fact that  $X$  is axiomatically complete is used in the cut-elimination proof  $2 \rightarrow 3$  only.

### Structural rules.

Another cut-free completeness result can be formulated which allows us to drop the requirement that  $X$  be axiomatically complete. This means that not only every logic of the S5 cube, but even every single axiomatisation of them can be given a corresponding nested sequent system in this way. The idea, described in details in [87], is to add another form of rules to the mix, namely *structural rules*, one for each axiom in  $\{t, b, d, 4, 5\}$  as shown on Figure 2.6. (These very rules were already defined by Brünnler and Poggiolesi as they are needed in the course of the cut-elimination proof, but get eliminated at the end.) They are called structural as they do not introduce any new connective, but only modify the structure of the nested sequent (bracketing) tree.

**2.5.7 Theorem** (Marin and Straßburger [87]) *Let  $X$  be any subset of  $\{t, b, d, 4, 5\}$ . For a formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $K + X$ .
2.  $A$  is provable in  $nK + \diamond_X^n + \boxtimes_X^n$ .

One idea that was suggested by Brünnler and Straßburger in [16] was that perhaps having only the structural rules corresponding to each axiom would be enough to give cut-free systems for the S5 cube. Unfortunately, their argument had a flaw: it is true for logics of the front face of the cube, that is, axiomatised with a subset of  $\{t, b, d\}$ , but not for logics using axioms 4 or 5. For example, the formula  $\diamond \Box q \vee \Box(\diamond \bar{p} \vee \diamond \diamond p)$  is provable in  $K + 4$ , but not in  $nK + \boxtimes_4^n$ . Details can be found in [87].

However, a direction that has not yet been explored to our knowledge is to consider structural rules defined using the algorithm described before for building propagation rules. For a set  $P$  of SL path axioms, the set  $\boxtimes_P^n$  of *structural rules* for  $P$  would contain any rule

$$\boxtimes_\pi^n \frac{\Gamma_1\{^u\Gamma_2\}\{^v\emptyset\}}{\Gamma_1\{^u\emptyset\}\{^v\Gamma_2\}} \quad (2.8)$$

if there is a path  $\pi$  from  $u$  to  $v$  in the propagation graph of  $\Gamma\{^u\emptyset\}\{^v\emptyset\}$  such that  $(\langle \pi \rangle, \diamond)$  appears in the completion of  $P$ . This would give us the same structural rules for t and b, but different rules for 4 and 5, which is advantageous compared to  $\boxtimes_4^n$  and  $\boxtimes_5^n$ .

**2.5.8 Open question** Is the system  $nK + \boxtimes_P^n$  sound and complete for the logic  $K + P$ ? If not, is there still some utility in this set of rules, e.g. in alleviating the need for strong completion similarly to Theorem 2.5.7?

In any case, we conjecture that these structural rules should play a role in the cut-elimination procedure similar to the structural rules for  $\{t, b, d, 4, 5\}$  in the proof of Theorem 2.5.6. The reduction step for the rules in  $\diamond_p^n$  should proceed as follows: a cut between  $\diamond A$  and  $\square \neg A$  is replaced by a cut on the same formula but on proofs of smaller height and a cut on the smaller formulas  $A$  and  $\neg A$  (modulo weakening).

$$\diamond_p^n \frac{\text{cut} \frac{\frac{\Gamma\{^u \diamond A\}\{^v A\}}{\Gamma\{^u \diamond A\}\{^v \emptyset\}} \quad \frac{\Gamma\{^u \neg A\}\{^v \emptyset\}}{\Gamma\{^u \square \neg A\}\{^v \emptyset\}}}{\Gamma\{^u \emptyset\}\{^v \emptyset\}}}{\Gamma\{^u \emptyset\}\{^v \emptyset\}} \rightsquigarrow \text{cut} \frac{\frac{\Gamma\{^u \diamond A\}\{^v A\}}{\Gamma\{^u \emptyset\}\{^v A\}} \quad \frac{\Gamma\{^u \square \neg A\}\{^v A\}}{\Gamma\{^u \emptyset\}\{^v \neg A\}}}{\Gamma\{^u \emptyset\}\{^v \emptyset\}}}{\Gamma\{^u \emptyset\}\{^v \emptyset\}}$$

## 2.6 Bridges

Following the proliferation of the extensions of sequents mentioned so far, proof-theorists have worked to understand how these different formalisms could relate. For example, hypersequents can be understood in a formalism that restricts the nested sequent tree to a single branch called *linear nested* sequents [75]; they can also be translated into the display calculus [135]. Display calculus, on the other hand, can be embedded in labelled sequents [118]. Similarly, prefixed tableaux turned out to be the same as nested sequents [47] which can be embedded in labelled sequents as a subclass called *labelled tree sequents* [61]. Of course each formalism has its strengths and weaknesses, and the choice between one or the other is usually led by the application that one has in mind. Nevertheless, it seems from this quick summary that labelled sequents are general enough to encompass most existing formalisms, and we will develop this aspect in detail in Part II.

However, as we already mentioned, labelled systems have been accused of incorporating semantics into the syntax, *viz.* model theory into proof theory. Although we do not agree with this statement on philosophical grounds, as clearly semantical intuitions are always at hand when building a logical proof system even if it is less obvious in the case of propositional classical logic, we do share the concerns that Restall underlines in [118].

(i) There is a *multiplicity* issue, i.e. the same labelled sequent can be written in infinitely many ways by swapping labels. Restall recalls the straightforward solution of having a *canonical* way of labelling a sequent, for example both  $xRy \Rightarrow x : \diamond A, y : A$  and  $uRv \Rightarrow u : \diamond A, v : A$  would be, for example, written as  $x_0Rx_1 \Rightarrow x_0 : \diamond A, x_1 : A$  following the order of introduction of labels.

(ii) There is a more fundamental issue with the *subformula property*. This property applies to an inference rule if it satisfies that any formula appearing in its premiss(es) is a subformula of its conclusion. Of course, the introduction rule for  $\square$  in the labelled system shown on Figure 2.4 does not validate this property since its premiss contains the relational atom  $xRy$  that is not present in any formula in the conclusion. To handle this, Restall suggests that we return to a more geometrical representation of a labelled sequent, reminiscent for example of tree-proofs [18] or rdag-tableaux [17], by forgetting the labelling and only keeping the graphical links between nodes. In our opinion, this is not an ideal solution as it is quite inconvenient to write already on paper, and hides the fact that some sort of encoding would still be needed in order to formalise this representation properly.

Recently, Fitting has introduced the formalism of *indexed nested sequents*. They are an extension of nested sequents from a tree structure to a graph structure. The embedding of nested sequents into labelled sequents can be adapted to embed indexed nested sequents into labelled sequents as the subclass of *labelled tree sequents with equality* [115].

Therefore they are merely a new way of representing a labelled sequent, but apparently a powerful one as they seem to escape the two criticisms addressed to labelled deductive systems reviewed above. The multiplicity issue is indeed handled by using a canonical enumeration of indices and then having an explicit structural rule of *index substitution* on the one hand. On the other hand, indexed nested sequents provide a convenient notation for the graph structure of labelled sequents, not an abstract geometrical one, which indeed

helps to recover the subformula property in full and allows us to combine easily ideas from labelled and unlabelled deductive systems.

### Indexed nested sequents.

An *indexed nested sequent*, as defined in [48], is a nested sequent where each sequent node (either the root or any interior node) carries an *index* taken from a countable set, so we write an indexed sequent by extending (2.6) in the following way

$$\Gamma ::= \emptyset \mid A, \Gamma \mid \Gamma_1, [^x \Gamma_2]$$

where  $\Gamma$  is now an indexed sequent, but the index of the root is not explicitly shown (e.g., we can assume that it is  $x_0$ ).

Intuitively, once indexed, nested sequents are no longer trees; by identifying nodes carrying the same index we obtain any kind of *rooted* directed graphs. (A *rooted* graph is a graph where one node is distinguished as the root and every node is reachable from it, i.e., the whole graph can be obtained as the minimal upward closure of this root by the edge relation.) Indeed the structure of a rooted directed graph is equivalent to that of a tree where certain nodes are identified, as used in [115] to represent the structure of indexed nested sequents.

The definition of contexts is also extended to indexed nested sequents. We still write  $\Gamma\{^{x_1} \} \dots \{^{x_n} \}$  for an  $n$ -ary context but, in this case, the indices  $x_k$  ( $1 \leq k \leq n$ ) attributed to the hole are not only abstract pointers, they correspond to the index of the sequent node that contains this hole, in the order of their appearance in the sequent. This means that some holes might carry the same index even though they appear in different places of the context. We might still omit the index on context-braces if this information is clear or not relevant.

**2.6.1 Example**  $A, [^{x_1} B, [^{x_2} C, \{ \}]], [^{x_3} D, [^{x_1} \{ \}, A]], [^{x_2} D, \{ \}]$  is a ternary context that we can write as  $\Gamma\{^{x_2} \}\{^{x_1} \}\{^{x_2} \}$  as the first and the third holes appear in brackets indexed by  $x_2$  and the second one in a bracket indexed by  $x_3$ .

If we substitute the sequents  $\Gamma_1 = D, [^{x_4} E]$ ;  $\Gamma_2 = F$ ; and  $\Gamma_3 = [^{x_5} G]$  into the holes, we get:

$$\Gamma\{^{x_2} \Gamma_1\}\{^{x_1} \Gamma_2\}\{^{x_2} \Gamma_3\} = A, [^{x_1} B, [^{x_2} C, D, [^{x_4} E]]], [^{x_3} D, [^{x_1} F, A]], [^{x_2} D, [^{x_5} G]]$$

For an indexed nested sequent  $\Gamma$ , we write  $\text{graph}(\Gamma)$  to denote the underlying graph of  $\Gamma$  similarly to the underlying tree of a nested sequent. It consists of  $I_\Gamma$ , the set of indices occurring in  $\Gamma$ , and  $R_\Gamma$  the accessibility relation induced by  $\Gamma$ , that is, the binary relation  $R_\Gamma \subseteq I_\Gamma \times I_\Gamma$  defined as:  $(w, v) \in R_\Gamma$  iff  $\Gamma = \Gamma_1\{^w [^v \Gamma_2]\}$  for some context  $\Gamma_1\{ \}$  and some indexed nested sequent  $\Gamma_2$ , i.e.  $v$  is the index of a child of  $w$ .

**2.6.2 Example** If we consider the sequent that we obtained above:

$$\Gamma = A, [^{x_1} B, [^{x_2} C, D, [^{x_4} E]]], [^{x_3} D, [^{x_1} F, A]], [^{x_2} D, [^{x_5} G]],$$

$$I_\Gamma = \{x_0, x_1, x_2, x_3, x_4, x_5\} \text{ with } x_0 \text{ being the index of the root, and}$$

$$R_\Gamma = \{(x_0, x_1), (x_0, x_2), (x_0, x_3), (x_1, x_2), (x_2, x_4), (x_2, x_5), (x_3, x_1)\}.$$

In Figure 2.7, the classical system that we call *inK* is an adaptation of the system described by Fitting in [48] to our notations and to the one-sided setting. It can also be seen as Br unnler's system [15] extended with indices.

What is different from the pure nested sequent system is the addition of the two structural rules *tp* and *bc*, called *teleportation* and *bracket-copy* respectively, which are variants of the formula-contraction *FC* and the sequent-contraction *SC* of [48]. Note that we need two versions of *bc* to take care of every possible context where the rule may be applied. They are needed to adapt the system to indexed sequents, namely to maintain the intended semantics by allowing two brackets with the same index to be identified. Another peculiarity is that in the rules for  $\Box$  we demand that the index of the new bracket in the premiss does not occur in the conclusion.

$$\begin{array}{c}
\text{id}^{\text{in}} \frac{}{\Gamma\{a, \bar{a}\}} \quad \vee^{\text{in}} \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \wedge^{\text{in}} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \top^{\text{in}} \frac{}{\Gamma\{\top\}} \\
\quad \quad \quad \diamond^{\text{in}} \frac{\Gamma_1\{\diamond A, [^u A, \Gamma_2]\}}{\Gamma_1\{\diamond A, [^u \Gamma_2]\}} \quad \square^{\text{in}} \frac{\Gamma\{[^v A]\}}{\Gamma\{\square A\}} \quad v \text{ is fresh} \\
\cdots \\
\text{tp}^{\text{in}} \frac{\Gamma\{^w \emptyset\}\{^w A\}}{\Gamma\{^w A\}\{^w \emptyset\}} \quad \text{bc}_1^{\text{in}} \frac{\Gamma_1\{^w [^u \Gamma_2]\}\{[^u \emptyset]\}}{\Gamma_1\{^w [^u \Gamma_2]\}\{^w \emptyset\}} \quad \text{bc}_2^{\text{in}} \frac{\Gamma_1\{^w [^u \Gamma_2\{^w [^u \emptyset]\}]\}}{\Gamma_1\{^w [^u \Gamma_2\{^w \emptyset\}]\}}
\end{array}$$

Figure 2.7: System inK

Fitting describes in [48] a way to obtain structural rules using indexed nested sequents for any subset of the Scott-Lemmon axioms, defined in (2.1). For a 4-tuple of natural numbers  $\langle h, i, j, k \rangle$  with  $i + k \neq 0$ , the rule  $\boxtimes_{\mathfrak{G}_{hijk}}^{\text{in}}$  is defined as:

$$\boxtimes_{\mathfrak{G}_{hijk}}^{\text{in}} \frac{\Gamma\{^{u_0} [^{u_1} \Delta_1, \dots [^{u_h} \Delta_k, [^{v_1} \dots [^{v_i} \dots]] \dots], [^{w_1} \Sigma_1, \dots [^{w_j} \Sigma_m, [^{x_1} \dots [^{x_k} \dots]] \dots]] \}}{\Gamma\{^{u_0} [^{u_1} \Delta_1, \dots [^{u_h} \Delta_h] \dots], [^{w_1} \Sigma_1, \dots [^{w_j} \Sigma_j] \dots]\}} \quad (2.9)$$

It must satisfy that  $v_1 \dots v_i$  and  $x_1 \dots x_k$  are fresh indices which are pairwise different, except for the *confluence condition*: we always have  $v_i = x_k$ . When one or more elements of the tuple  $\langle h, i, j, k \rangle$  are equal to 0, there are special cases:

- if  $h = 0$  (or  $j = 0$ ) then  $u_1$  to  $u_h$  (resp.  $w_1$  to  $w_j$ ) all collapse to  $u_0$ .
- if  $i = 0$  then  $w_1$  to  $w_i$  all collapse to  $u_h$ , and similarly, if  $k = 0$  then  $x_1$  to  $x_k$  all collapse to  $v_j$ . In particular, if  $h = 0$  and  $i = 0$ , we must have  $x_k = u_0$ , and similarly, if  $j = 0$  and  $k = 0$ , we demand that  $v_i = u_0$ .

The case where  $i = 0$  and  $k = 0$  was not handled by Fitting in [48]; we give the following corresponding rule:

$$\boxtimes_{\mathfrak{G}_{h0j0}}^{\text{in}} \frac{\sigma\Gamma\{^{u_0} [^{u_1} \Delta_1, \dots [^{\sigma(u_h)} \Delta_h], \dots], [^{w_1} \Sigma_1, \dots [^{\sigma(w_j)} \Sigma_j], \dots]\}}{\Gamma\{^{u_0} [^{u_1} \Delta_1, \dots [^{u_h} \Delta_h], \dots], [^{w_1} \Sigma_1, \dots [^{w_j} \Sigma_j], \dots]\}} \quad (2.10)$$

In that case, not only do we identify  $u_h$  and  $w_j$ , but it is also necessary to apply a substitution  $\sigma: I_\Gamma \rightarrow I_\Gamma$  to the indices in the context  $\Gamma\{^{u_0}\}$ , giving the new context  $\sigma\Gamma\{^{u_0}\}$ , such that  $\sigma(u_h) = \sigma(w_j)$  in the whole sequent, but  $\sigma(y) = y$  for any other  $y \in I_\Gamma$ . For a given set of SL axioms  $\mathfrak{G}$ , we write  $\boxtimes_{\mathfrak{G}}^{\text{in}}$  to be the set of rules corresponding to  $\mathfrak{G}$  according to these definitions. As usual, we write  $\text{inK} + \boxtimes_{\mathfrak{G}}^{\text{in}}$  for the system obtained from inK by adding the rules in  $\boxtimes_{\mathfrak{G}}^{\text{in}}$ .

System  $\text{inK} + \boxtimes_{\mathfrak{G}}^{\text{in}}$  is sound and complete wrt. the logic  $\mathbf{K} + \mathfrak{G}$ .

**2.6.3 Theorem** (Fitting [48], Marin and Straßburger [88]) *Let  $\mathfrak{G}$  be a set of SL axioms and let  $\boxtimes_{\mathfrak{G}}^{\text{in}}$  be the corresponding set of rules. For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $\mathbf{K} + \mathfrak{G}$ .
2.  $A$  is provable in  $\text{inK} + \boxtimes_{\mathfrak{G}}^{\text{in}}$ .

Soundness is proven by Fitting (for  $i + k \neq 0$ ) wrt. relational frames, and completeness via a translation to another equivalent formalism he calls *set-prefixed tableaux* for which in turn he gives a semantic completeness proof [48]. In the sequel, we give an alternative proof of completeness via a cut-elimination procedure (that covers also the case  $i + k = 0$ ).

## Cut-elimination

We present a cut-elimination proof for the indexed nested sequent system defined above. Of course, the admissibility of cut, the fact that the system is complete without the cut-rule, is implied by Theorem 2.6.3, the proof of which appeals to semantical tools. A syntactic proof of cut-admissibility has also been given in [115] via the embedding into labelled tree sequents with equality which are known to satisfy cut-admissibility. However, proof-theorists prefer to have an explicit syntactical procedure to eliminate cut from proofs, that works internally to the system at hand. The proof we present turns out to be nevertheless the same in structure as the one for labelled sequents with the difference that we use an explicit index substitution rule. In any case, this (more general) argument can serve as a model for the proofs of the various cut-admissibility results only stated throughout this chapter.

**2.6.4 Theorem** (Marin and Straßburger [88]) *Let  $\mathbf{G}$  be a set of Scott-Lemmon axioms. For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $\mathbf{K} + \mathbf{G}$ .

2.  $A$  is provable in  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}} + \text{cut}$  where **cut** is the rule

$$\text{cut} \frac{\Gamma\{C\} \quad \Gamma\{-C\}}{\Gamma\{\emptyset\}}$$

3.  $A$  is provable in  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}}$ .

This result will follow from Lemma 2.6.5 and Lemma 2.6.8. The proof relies on a standard nested induction, the main one being on the number of cuts of maximal rank, and the nested one on the multiset of the ranks of a given cut and the height of the derivation above a cut.

To facilitate the overall argument, we consider a variant of system  $\text{inK}$ , that we call *system  $\dot{\text{inK}}$* , that is obtained from  $\text{inK}$  by removing the teleportation rule **tp** (but keeping the **bc**-rules), and by replacing the rules  $\text{id}^{\text{in}}$  and  $\diamond^{\text{in}}$  by

$$\dot{\text{id}} \frac{}{\Gamma\{^u a\}\{^u \bar{a}\}} \quad \text{and} \quad \dot{\diamond} \frac{\Gamma_1\{^u \diamond A\}\{^u [A, \Gamma_2]\}}{\Gamma_1\{^u \diamond A\}\{^u [\Gamma_2]\}}$$

respectively. The reason behind this is that  $\text{inK}$  and  $\dot{\text{inK}}$  are equivalent (with and without **cut**), but the **tp**-rule is admissible in the new system, and it is helpful that we do not need to consider it in the cut-elimination argument. Therefore we will actually prove the following.

**2.6.5 Lemma** *Let  $\mathbf{G}$  be a set of SL axioms and let  $\boxtimes_{\mathbf{G}}^{\text{in}}$  be the corresponding set of rules. If a sequent  $\Gamma$  is derivable in  $\dot{\text{inK}} + \boxtimes_{\mathbf{G}}^{\text{in}} + \text{cut}$ , then it is also derivable in  $\dot{\text{inK}} + \boxtimes_{\mathbf{G}}^{\text{in}}$ .*

The idea is to show that if there exists a proof  $\mathcal{D}$  of  $\Gamma$  in  $\dot{\text{inK}} + \boxtimes_{\mathbf{G}}^{\text{in}} + \text{cut}$  then  $\Gamma$  is also derivable in  $\dot{\text{inK}} + \boxtimes_{\mathbf{G}}^{\text{in}}$ , by an induction on the cut rank of  $\mathcal{D}$ . The induction step also uses an induction on the number of occurrences of cut with the maximal rank as well as the main Reduction Lemma 2.6.9 to eliminate the topmost occurrence in the proof.

To complete this proof we will need some additional structural rules called *weakening*, *contraction*, *necessitation*, and *index substitution* respectively:

$$\text{w} \frac{\Gamma_1\{\emptyset\}}{\Gamma_1\{\Gamma_2\}} \quad \text{c} \frac{\Gamma_1\{\Gamma_2, \Gamma_2\}}{\Gamma_1\{\Gamma_2\}} \quad \text{nec} \frac{\Gamma}{[\Gamma]} \quad \text{isub} \frac{\Gamma}{\sigma\Gamma}$$

The rules for weakening and contraction are similar to the standard sequent ones except that they can apply deeply inside a context and to a whole nested structure. The rules **nec** and **isub** on the other hand cannot be applied deep inside a context; they always work on the whole sequent. In **isub**, the sequent  $\sigma\Gamma$  is obtained from  $\Gamma$  by applying the substitution  $\sigma: I_{\Gamma} \rightarrow I_{\Gamma}$  on the indices occurring in  $\Gamma$ , here  $\sigma$  can be an arbitrary renaming. We will show that these structural rules are admissible in the proof system.

For a given system  $\mathbf{S}$ , a rule  $r \notin \mathbf{S}$  with  $n$  premisses ( $n \geq 1$ ) is *height* (or *cut-rank*) *preserving admissible* in  $\mathbf{S}$ , if for each proof  $\mathcal{D}_1, \dots, \mathcal{D}_n$  of its premisses in  $\mathbf{S}$ , there is a proof

$\mathcal{D}'$  of its conclusion in  $\mathcal{S}$ , such that  $\text{ht}(\mathcal{D}') \leq \text{ht}(\mathcal{D}_i)$  (resp.  $\text{rk}(\mathcal{D}') \leq \text{rk}(\mathcal{D}_i)$ ) for  $1 \leq i \leq n$ . Similarly, a rule  $r$  is *height* (or *cut-rank*) *preserving invertible* in a system  $\mathcal{S}$ , if for every derivation of the conclusion of  $r$  there are derivations for each of its premisses with at most the same height (resp. at most the same cut-rank).

**2.6.6 Lemma** *Let  $\mathcal{G}$  be a set of SL axioms and let  $\boxtimes_{\mathcal{G}}^{\text{in}}$  be the corresponding set of rules.*

1. *The rules  $\text{nec}$ ,  $\text{w}$ ,  $\text{isub}$  and  $\text{c}$  are cut-rank and height preserving admissible for  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$ .*
2. *All rules of  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$  (except for  $\text{id}$ ) are cut-rank and height-preserving invertible.*

*Proof.* This proof is by induction on the height of the derivation above an instance of the rule, analogously to that for the pure nested sequent systems in [15]. For  $\text{bc}$  and  $\boxtimes_{\text{gklmn}}^{\text{in}}$ , note that their inverses are just application of the weakening rule.  $\odot$

**2.6.7 Lemma** *Let  $\mathcal{G}$  be a set of SL axioms and let  $\boxtimes_{\mathcal{G}}^{\text{in}}$  be the corresponding set of rules. The rule  $\text{tp}$  is admissible for  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$  (and for  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}} + \text{cut}$ ).*

*Proof.* The proof uses an induction on the number of instances of  $\text{tp}$  in a proof, eliminating topmost instances first, by an induction on the height of the proof above it and a case analysis of the rule  $r$  applied just before  $\text{tp}$ . The only non-trivial case is when  $r = \square^{\text{in}}$ :

$$\square^{\text{in}} \frac{\text{tp} \frac{\Gamma\{^u [^v A]\}\{^u\}}{\Gamma\{^u \square A\}\{^u\}}}{\Gamma\{^u\}\{^u \square A\}} \rightsquigarrow \frac{\text{w} \frac{\Gamma\{^u [^v A]\}\{^u\}}{\Gamma\{^u [^v A]\}\{^u [^v]\}}}{\text{tp} \frac{\Gamma\{^u [^v]\}\{^u [^v A]\}}{\Gamma\{^u\}\{^u [^v A]\}}} \text{bc} \frac{\Gamma\{^u\}\{^u [^v A]\}}{\Gamma\{^u\}\{^u \square A\}} \square^{\text{in}}$$

after this transformation, we use the admissibility of weakening (Lemma 2.6.6) and the induction hypothesis to conclude.  $\odot$

**2.6.8 Lemma** *Let  $\mathcal{G}$  be a set of SL axioms and let  $\boxtimes_{\mathcal{G}}^{\text{in}}$  be the corresponding set of rules. A sequent  $\Gamma$  is provable in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$  (or in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}} + \text{cut}$ ) if and only if it is provable in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$  (resp. in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}} + \text{cut}$ ).*

*Proof.* Given a proof of  $\Gamma$  in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$ , we can observe that the rules  $\text{id}^{\text{in}}$  and  $\diamond^{\text{in}}$  are just special cases of the rules  $\text{id}$  and  $\diamond$ , respectively. Thus, we obtain a proof of  $\Gamma$  in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$  from admissibility of  $\text{tp}$  (Lemma 2.6.7).

Conversely, if we have a proof of  $\Gamma$  in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$ , we can obtain a proof of  $\Gamma$  in  $\text{inK} + \boxtimes_{\mathcal{G}}^{\text{in}}$  by replacing every instance of  $\text{id}$  and  $\diamond$  by the following derivations:

$$\text{tp} \frac{\text{id} \frac{\Gamma\{^u \emptyset\}\{^u a, \bar{a}\}}{\Gamma\{^u a\}\{^u \bar{a}\}}}{\Gamma\{^u \emptyset\}\{^u a, \bar{a}\}} \text{ and } \diamond \frac{\text{tp} \frac{\Gamma_1\{^u \diamond A\}\{^u [A, \Gamma_2]\}}{\Gamma_1\{^u \emptyset\}\{^u \diamond A, [A, \Gamma_2]\}}}{\Gamma_1\{^u \emptyset\}\{^u \diamond A, [\Gamma_2]\}} \text{tp} \frac{\Gamma_1\{^u \diamond A\}\{^u [\Gamma_2]\}}{\Gamma_1\{^u \diamond A\}\{^u [\Gamma_2]\}}$$

respectively. The same proof goes for the system with  $\text{cut}$ .  $\odot$

Finally we can prove the main lemma of this section which corresponds to the induction step of the cut-elimination proof.

**2.6.9 Lemma** *If there is a proof  $\mathcal{D}$  of shape*

$$\text{cut}_{r+1} \frac{\mathcal{D}_1 \parallel \Gamma\{A\} \quad \mathcal{D}_2 \parallel \Gamma\{\neg A\}}{\Gamma\{\emptyset\}}$$

in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}} + \text{cut}$  such that  $\text{rk}(\mathcal{D}_1) \leq r$  and  $\text{rk}(\mathcal{D}_2) \leq r$ , then there is proof  $\mathcal{D}'$  of  $\Gamma\{\emptyset\}$  in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}} + \text{cut}$  such that  $\text{rk}(\mathcal{D}') \leq r$ .

*Proof.* We proceed by induction on  $\text{ht}(\mathcal{D}_1) + \text{ht}(\mathcal{D}_2)$ , making a case analysis on the bottom-most rules in  $\mathcal{D}_1$  and  $\mathcal{D}_2$ .

1. If  $\mathcal{D}_1$  is just  $\text{id}$ , there are two sub-cases:

- If the cut-formula  $A$  is one of the atoms in the identity, then

$$\text{cut}_1 \frac{\text{id} \frac{\Gamma\{^u a\}\{^u \bar{a}\} \quad \mathcal{D}_2 \parallel}{\Gamma\{^u \emptyset\}\{^u \bar{a}\}}}{\Gamma\{^u \emptyset\}\{^u \bar{a}\}} \rightsquigarrow \text{c} \frac{\text{tp} \frac{\mathcal{D}_2 \parallel \quad \Gamma\{^u \bar{a}\}\{^u \bar{a}\}}{\Gamma\{^u \emptyset\}\{^u \bar{a}, \bar{a}\}}}{\Gamma\{^u \emptyset\}\{^u \bar{a}\}}$$

where we apply the admissibility of  $\text{tp}$  (Lemma 2.6.7) and  $\text{c}$  (Lemma 2.6.6).

- If the cut-formula  $A$  is not one of the atoms in the identity then we can apply the  $\text{id}$ -rule directly to  $\Gamma\{^u \emptyset\}\{^u \bar{a}\}$ .

2. If the bottommost rule  $r$  of  $\mathcal{D}_1$  is  $\text{bc}$  or  $\boxtimes_{\text{gnjk}}^{\text{in}}$  then we have

$$\text{cut}_{r+1} \frac{r \frac{\mathcal{D}'_1 \parallel \quad \mathcal{D}_2 \parallel}{\Gamma\{A\} \quad \Gamma\{\neg A\}}}{\Gamma\{\emptyset\}} \rightsquigarrow \text{cut}_{r+1} \frac{r \frac{\mathcal{D}'_1 \parallel \quad \mathcal{D}_2 \parallel}{\Gamma\{A\} \quad \Gamma\{\neg A\}}}{\Gamma\{\emptyset\}} \quad \text{w} + \text{isub} \frac{\mathcal{D}_2 \parallel \quad \Gamma\{\neg A\}}{\Gamma\{\neg A\}}$$

and we proceed by induction hypothesis and height-preserving admissibility of weakening, and, in the special case of  $\boxtimes_{\text{ghoj0}}^{\text{in}}$ , of index substitution (Lemma 2.6.6).

3. If the bottommost rule  $r$  of  $\mathcal{D}_1$  is one of  $\wedge$ ,  $\vee$ ,  $\square$ , or  $\diamond$ , such that the principal formula of  $r$  is not the cut-formula  $A$ , then we proceed as in the previous case; we apply the height-preserving invertibility (Lemma 2.6.6) of the rules  $\wedge$ ,  $\vee$ ,  $\square$ , or  $\diamond$  (and apply it twice in the case of the  $\wedge$ -rule as illustrated below), denoting the admissible inverse of  $r$  by  $r^{-1}$ , and proceed by induction hypothesis.

$$\text{cut}_{r+1} \frac{\wedge \frac{\mathcal{D}'_1 \parallel \quad \mathcal{D}'_2 \parallel}{\Gamma\{A\}\{C\} \quad \Gamma\{B\}\{C\}} \quad \mathcal{D}_2 \parallel}{\Gamma\{A \wedge B\}\{C\} \quad \Gamma\{A \wedge B\}\{-C\}} \rightsquigarrow \text{cut}_{r+1} \frac{\wedge \frac{\mathcal{D}'_1 \parallel \quad \mathcal{D}'_2 \parallel}{\Gamma\{A\}\{C\} \quad \Gamma\{B\}\{C\}} \quad \wedge^{-1} \frac{\mathcal{D}_2 \parallel \quad \Gamma\{A \wedge B\}\{-C\}}{\Gamma\{A\}\{-C\}}}{\Gamma\{A\}\{\emptyset\} \quad \Gamma\{B\}\{\emptyset\}} \quad \wedge^{-1} \frac{\mathcal{D}_2 \parallel \quad \Gamma\{A \wedge B\}\{-C\}}{\Gamma\{A \wedge B\}\{-C\}}$$

Cases 1-3 are repeated for the bottommost rule of  $\mathcal{D}_2$ . Let us now consider the key cases:

4. If the bottommost rules  $r_1$  of  $\mathcal{D}_1$  and  $r_2$  of  $\mathcal{D}_2$  are among  $\wedge$ ,  $\vee$ ,  $\square$ , or  $\diamond$ , and for both the cut-formula is principal, we have the following cases:

- $A = B \vee C$ :

$$\text{cut}_{r+1} \frac{\vee \frac{\mathcal{D}'_1 \parallel \quad \mathcal{D}'_2 \parallel \quad \mathcal{D}'_3 \parallel}{\Gamma\{B, C\} \quad \Gamma\{\neg B\} \quad \Gamma\{\neg C\}} \quad \wedge \frac{\mathcal{D}_2 \parallel \quad \Gamma\{\neg B\} \quad \Gamma\{\neg C\}}{\Gamma\{\neg B \wedge \neg C\}}}{\Gamma\{B \vee C\} \quad \Gamma\{\neg B \wedge \neg C\}} \rightsquigarrow \text{cut}_r \frac{\text{cut}_r \frac{\mathcal{D}'_1 \parallel \quad \mathcal{D}'_2 \parallel}{\Gamma\{B, C\} \quad \Gamma\{\neg B, C\}} \quad \text{w} \frac{\mathcal{D}_2 \parallel \quad \Gamma\{\neg B\}}{\Gamma\{\neg B, C\}}}{\Gamma\{C\}} \quad \text{cut}_r \frac{\mathcal{D}'_3 \parallel}{\Gamma\{\neg C\}}}{\Gamma\{\emptyset\}}$$

where we apply the height-preserving admissibility of weakening (Lemma 2.6.6).

- $A = B \wedge C$ : Likewise.
- $A = \diamond B$ :

$$\begin{array}{c}
\frac{\frac{\frac{\mathcal{D}'_1 \parallel \Gamma_1\{^w \diamond B\}\{^w [^u B, \Gamma_2]\}}{\Gamma_1\{^w \diamond B\}\{^w [^u \Gamma_2]\}} \quad \frac{\frac{\mathcal{D}'_2 \parallel \Gamma_1\{^w [^v \neg B]\}\{^w [^u \Gamma_2]\}}{\Gamma_1\{^w \square \neg B\}\{^w [^u \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u B, \Gamma_2]\}} \quad \frac{\frac{\frac{\mathcal{D}'_2 \parallel \Gamma_1\{^w [^v \neg B]\}\{^w [^u \Gamma_2]\}}{\Gamma_1\{^w \square \neg B\}\{^w [^u B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \neg B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \Gamma_2]\}} \rightsquigarrow \\
\frac{\frac{\frac{\mathcal{D}'_1 \parallel \Gamma_1\{^w \diamond B\}\{^w [^u B, \Gamma_2]\}}{\Gamma_1\{^w \emptyset\}\{^w [^u B, \Gamma_2]\}} \quad \frac{\frac{\frac{\mathcal{D}'_2 \parallel \Gamma_1\{^w [^v \neg B]\}\{^w [^u \Gamma_2]\}}{\Gamma_1\{^w \square \neg B\}\{^w [^u B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u B, \Gamma_2]\}} \quad \frac{\frac{\frac{\mathcal{D}'_2 \parallel \Gamma_1\{^w [^v \neg B]\}\{^w [^u \Gamma_2]\}}{\Gamma_1\{^w \square \neg B\}\{^w [^u B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \neg B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \neg B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \neg B, \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \Gamma_2]\}}}{\Gamma_1\{^w \emptyset\}\{^w [^u \Gamma_2]\}}
\end{array}$$

where on the left branch we use height-preserving admissibility of weakening and proceed by induction hypothesis, and on the right branch we use admissibility of the rules *isub* and *tp* (Lemmas 2.6.6 and 2.6.7).

- $A = \square B$ : Likewise. ☺



## Intuitionistic modal logic

In this chapter we introduce the concepts and notations that will be used throughout this thesis for *intuitionistic* modal logic. For a more detailed presentation we refer to the Ph.D. thesis of Simpson [127] which still is the most comprehensive survey of everything intuitionistic modal logic related. We proceed in the same order as in Chapter 2 even though it does not fully reflect the chronological order in the case of intuitionistic modal logic. We review Hilbert-style axiomatisations and Gentzen-style sequent systems for intuitionistic modal logics in Section 3.1 and Section 3.2, then present the appropriate *bi*-relational semantics in Section 3.3. We then introduce the labelled and unlabelled deductive systems that exist for intuitionistic modal logic, in Section 3.4 and Section 3.5. Finally in Section 3.6, we explain why bridges are even more difficult to build in the intuitionistic setting, and we close on a presentation of our work on the intuitionistic variant of indexed nested sequents [88].

### 3.1 Hilbert axiomatisations

In the intuitionistic case, we work with a different set of connectives. Starting with a set of atomic propositions still denoted  $a$ , formulas are constructed from the following grammar:

$$A ::= a \mid A \wedge A \mid \top \mid A \vee A \mid \perp \mid A \supset A \mid \Box A \mid \Diamond A$$

When we write  $\neg A$ , we mean  $A \supset \perp$ .

Obtaining the intuitionistic variant of  $\mathbf{K}$  is more involved than the classical variant. Lacking De Morgan duality, there are several variants of  $\mathbf{k}$  that are classically but not intuitionistically equivalent. Five axioms have been considered as primitives in the literature. An intuitionistic variant of the modal logic  $\mathbf{K}$  can then be obtained from ordinary intuitionistic propositional logic IPL by

- adding the *necessitation rule*:  $\Box A$  is a theorem if  $A$  is a theorem; and
- adding a subset of the following five variants of the  $\mathbf{k}$  axiom.

$$\begin{array}{lll} \mathbf{k}_1: \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathbf{k}_3: \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) & \mathbf{k}_5: \Diamond \perp \supset \perp \\ \mathbf{k}_2: \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathbf{k}_4: (\Diamond A \supset \Box B) \supset \Box(A \supset B) & \end{array} \quad (3.1)$$

The idea is that intuitionistic propositional logic does not allow the principle of *Excluded Middle*, so the modalities  $\Box$  and  $\Diamond$  are not de Morgan duals any more, but one can choose to design the axiomatisation in order to relate them in different ways. The most basic intuitionistic modal system one can think of would be to consider only the  $\Box$  modality as regulated by the  $\mathbf{k}$  axiom (or as called here,  $\mathbf{k}_1$ ), which gives the system  $\text{IPL} + \text{nec} + \mathbf{k}_1$ . However this would give strictly no information on the behaviour of the  $\Diamond$  modality.

It seems that Fitch [42] was the first one to propose a way to treat  $\Diamond$  in an intuitionistic system by considering the system  $\text{IPL} + \text{nec} + \mathbf{k}_1 + \mathbf{k}_2$ , which is now sometimes called  $\mathbf{CK}$  for

*constructive modal logic*. Wijekesera [137] also considered the axiom  $k_5$ , which states that  $\diamond$  distributes over 0-ary disjunctions, but did not assume that it would always distribute over binary disjunctions; the system he proposed was therefore  $\text{IPL} + \text{nec} + k_1 + k_2 + k_5$ . These systems are well-designed for some applications, for example to analyse some type systems [7] or to reason about states of a machine under partial information [138]. However, they cannot satisfy the logicians as the addition of the *Excluded Middle* principle to it does not yield classical modal logic  $\text{K}$ , that is, it is not possible to retrieve the De Morgan duality of  $\square$  and  $\diamond$  in this case.

The axiomatisation that is now generally accepted as *intuitionistic modal logic* denoted by  $\text{IK}$  was given by Plotkin and Stirling [111] and is equivalent to the one proposed by Fischer-Servi [124], and by Ewald [39] in the case of intuitionistic tense logic. It is taken to be  $\text{IPL} + \text{nec} + k_1 + k_2 + k_3 + k_4 + k_5$ . It then was investigated in detail in [127], in which strong arguments are given in favour of this axiomatic definition: it allows for adapting to intuitionistic logic the standard embedding of modal logic into first-order logic, and also provides an extension of the standard Kripke semantics for classical modal logic to the intuitionistic case (see Section 3.3). Although  $\text{CK}$  and  $\text{IK}$  are the most studied constructive versions of modal logic  $\text{K}$ , we would advocate that it might still be interesting to consider the intermediate axiomatisations between  $\text{CK}$  and  $\text{IK}$  in the future.

We can consider extensions of those basic intuitionistic modal logics with axioms for the modalities as we did in the classical case, for example by the class of Scott-Lemmon axioms defined in (2.1). However, over an intuitionistic basis, and if  $h \neq j$  or  $i \neq k$ , the axiom  $\diamond^h \square^j a \supset \square^j \diamond^k a$  is no longer equivalent to the axiom  $\diamond^j \square^k a \supset \square^h \diamond^i a$ . So one can consider extensions with either one of these two axioms [111], or with both in conjunction in an attempt to recover the intuitionistic version of the corresponding classical Scott-Lemmon logic [127]. In particular, we will study the intuitionistic variants of the logics in the  $\text{S5}$  cube, for which we always mean the conjunction of the two versions of each axioms, that is, in an intuitionistic context:

$$\begin{aligned}
\mathbf{t} &:= (a \supset \diamond a) \wedge (\square a \supset a) \\
\mathbf{b} &:= (a \supset \square \diamond a) \wedge (\diamond \square a \supset a) \\
\mathbf{d} &:= \square a \supset \diamond a \\
\mathbf{4} &:= (\diamond \diamond a \supset \diamond a) \wedge (\square a \supset \square \square a) \\
\mathbf{5} &:= (\diamond a \supset \square \diamond a) \wedge (\diamond \square a \supset \square a) \\
\mathbf{2} &:= \diamond \square a \supset \square \diamond a
\end{aligned}$$

Extensions of other versions of constructive modal logics and for instance of the logic  $\text{CK}$  are however, to our knowledge, quite less understood. In particular, we lack an understanding of the power of such extensions as it has been observed for example that the axiom  $\mathbf{b}$  already entails axioms  $k_3$  and  $k_5$  [4].

## 3.2 Sequent and hypersequent calculi

Following Gentzen [56], a first attempt at a sequent calculus for the intuitionistic modal logic  $\text{IK}$  would be to consider a two-sided version of the classical sequent calculus for  $\text{K}$  with the restriction that only one formula can occur on the right-hand side. A intuitionistic sequent is a multiset of formulas with one formula occurrence specifically distinguished, it is written  $\Lambda \Rightarrow C$ , where  $\Lambda$  is a multiset of formulas and  $\Rightarrow$  is the delimiter. This would give the following rules:

$$\begin{array}{c}
\frac{\Lambda, A \Rightarrow B}{\square \Lambda, \diamond A \Rightarrow \diamond B} \diamond_k^o \quad \frac{\Lambda \Rightarrow A}{\square \Lambda \Rightarrow \square A} \square_k^o
\end{array}$$

as proposed for example in [34]. However, adding these rules to a sequent calculus for intuitionistic propositional logic does not result in a complete calculus for intuitionistic modal logic IK, but rather for the logic called CK. Indeed, the axioms  $k_3$ ,  $k_4$  and  $k_5$  are not derivable via these rules.

Although this is seen as a strong drawback for logicians, this calculus and extensions built in a similar fashion from the rules in Figure 2.2 have been exploited in modal type theories (in particular the extension with the rules corresponding to the axioms t and 4, known as CS4). These correspond to the calculi studied proof-theoretically in [114, 9, 62, 92]; which originated from the analysis of logics comparing divergent processes in AI [129], reasoning about hardware circuits [40], or defining a programming language with explicit binding time constructors [109].

Another attempt was proposed by Amati and Pirri in [2]. They expose a uniform method to give tableau and (multi-conclusion) sequent systems for all the intuitionistic versions of the logics in the S5 cube. For example, they generalise the rules  $\Box_k^\circ$  and  $\Diamond_k^\circ$  above, as follows:

$$\Diamond_k^\circ \frac{\Lambda, A \Rightarrow \Pi}{\Box \Lambda, \Diamond A \Rightarrow \Box \Pi} \quad \Box_k^\circ \frac{\{\Gamma_1, \Gamma_2, \Diamond \Lambda_i \Rightarrow \Box \Pi_i\}_{1 \leq i \leq p} \quad \{\Gamma_2, \Lambda_i \Rightarrow \Pi_i\}_{p+1 \leq i \leq n}}{\Gamma_1, \Box \Gamma_2 \Rightarrow \Box A}$$

if there exists a derivation  $\mathcal{D}$  containing only structural and logical rules: 
$$\begin{array}{c} \{\Gamma_2, \Lambda_i \Rightarrow \Pi_i\}_{1 \leq i \leq n} \\ \mathcal{D} \parallel \\ \Gamma_2 \Rightarrow A \end{array}$$

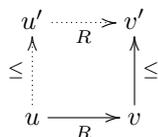
As far as hypersequents go, in [54] Galmiche and Salhi provide a proof system for intuitionistic modal logic IS5 based on what they call *multi-conclusion sequents*, that can be seen as the intuitionistic version of hypersequents. This is the only hypersequent calculus of which we are aware for intuitionistic versions of modal logics.

Investigating further hypersequents calculi for modal logics in a constructive setting would be an interesting avenue of future research. The formalism has been employed successfully to treat intermediate logics [30], and the situation we are facing with intuitionistic modal logic is similar to the intermediate case. Indeed, we can see the various logics between the  $\Box$ -only fragment IPL + k1 + nec and full intuitionistic IK as intermediate logics, as suggested by de Medeiros in [37] who studied in detail IPL + k1 + k2 + k3 + k5. Furthermore, in [54], Galmiche and Salhi actually also provide a proof system for an intermediate logic between IS5 and S5 called IM5 defined in [49], and the links between intuitionistic modal logics and intermediate predicate logics have already started to be explored by Ono and Suzuki in [107, 108, 132], so we hope to continue in this direction in the future.

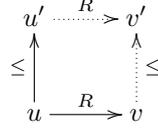
### 3.3 Possible-world semantics

The Kripke semantics for IK was first defined by Fischer-Servi in [124]. It combines the Kripke semantics for intuitionistic propositional logic and the one for classical modal logic, using two distinct relations on the set of worlds. Indeed, a *bi-relational frame*  $\mathcal{F}$  is a triple  $\langle W, \leq, R \rangle$  of a non-empty set of worlds  $W$  with two binary relations:  $R \subseteq W \times W$  and  $\leq$  a pre-order on  $W$  (*i.e.* a reflexive and transitive relation) satisfying the conditions:

(F1) For all worlds  $u, v, v'$ , if  $uRv$  and  $v \leq v'$ , there exists a  $u'$  such that  $u \leq u'$  and  $u'Rv'$ :



(F2) For all worlds  $u', u, v$ , if  $u \leq u'$  and  $uRv$ , there exists a  $v'$  such that  $u'Rv'$  and  $v \leq v'$ :



A *bi-relational model*  $\mathfrak{M}$  is a quadruple  $\langle W, \leq, R, V \rangle$  with  $\langle W, \leq, R \rangle$  a frame and  $V$  a monotone *valuation* function  $V: W \rightarrow 2^A$  which is a function that maps each world  $w$  to the subset of propositional atoms that are true in  $w$ , subject to:

$$w \leq w' \Rightarrow V(w) \subseteq V(w')$$

As in the classical case, we write  $w \Vdash a$  if  $a \in V(w)$  and we extend this relation to all formulas by induction, following the rules for both intuitionistic and modal Kripke models:

$$\begin{aligned}
 w &\not\Vdash \perp \\
 w \Vdash A \wedge B &\text{ iff } w \Vdash A \text{ and } w \Vdash B \\
 w \Vdash A \vee B &\text{ iff } w \Vdash A \text{ or } w \Vdash B \\
 w \Vdash A \supset B &\text{ iff for all } w' \text{ with } w \leq w', \text{ if } w' \Vdash A \text{ then also } w' \Vdash B \\
 w \Vdash \Box A &\text{ iff for all } w' \text{ and } u \text{ with } w \leq w' \text{ and } w'Ru, \text{ we have } u \Vdash A \\
 w \Vdash \Diamond A &\text{ iff there is a } u \in W \text{ such that } wRu \text{ and } u \Vdash A
 \end{aligned}$$

We write  $w \not\Vdash A$  if it is not the case that  $w \Vdash A$ , but contrarily to the classical case, we do not have  $w \Vdash \neg A$  iff  $w \not\Vdash A$  (since  $\neg A$  is defined as  $A \supset \perp$ ).

From the monotonicity of the valuation function  $V$ , we get a monotonicity property for the relation  $\Vdash$ .

**3.3.1 Proposition** (Monotonicity) *For any formula  $A$  and for  $w, w' \in W$ , if  $w \leq w'$  and  $w \Vdash A$ , then  $w' \Vdash A$ .*

We say that a formula  $A$  is *satisfied in a model*  $\mathcal{M} = \langle W, R, \leq, V \rangle$ , if for all  $w \in W$  we have  $w \Vdash A$ . A formula  $A$  is *valid in a frame*  $\langle W, R, \leq \rangle$ , if for all valuations  $V$ ,  $A$  is valid in  $\langle W, R, \leq, V \rangle$ .

**3.3.2 Theorem** (Fischer-Servi [124], Plotkin and Stirling [111]) *A formula  $A$  is derivable in the Hilbert system  $\mathbb{IK}$  if and only if  $A$  is valid in every bi-relational frame.*

Now, we would like to consider special restrictions on these bi-relational frames, similarly to what we did in the classical case. For example, we would be interested in having a correspondence between the logic  $\mathbb{IK}$  extended with some variants of SL axioms and bi-relational frames satisfying the corresponding condition defined in (2.3) similarly to Theorem 2.3.2.

In the intuitionistic case, the correspondence theory is much more tedious, and a lot of questions are still open. For the general form of intuitionistic SL logic, which would be  $\mathbb{IK}$  extended with a set of axioms of the form:

$$(\Diamond^h \Box^i a \supset \Box^j \Diamond^k a) \wedge (\Diamond^j \Box^k a \supset \Box^h \Diamond^i a)$$

it is not clear that the class of frames satisfying condition (2.3) is even complete (see [127] for a detailed analysis). However, Theorems 6.2.1 and 8.1.4 of [127] entail a completeness result for bi-relational models with respect to the restricted family of *intuitionistic SL path axioms*.

**3.3.3 Theorem** (Fischer-Servi [124], Simpson [127]) *Let  $\mathbf{P}$  be a set of axiom schemes of the form:*

$$(\diamond^h \Box a \supset \Box^j a) \wedge (\diamond^j a \supset \Box^h \diamond a)$$

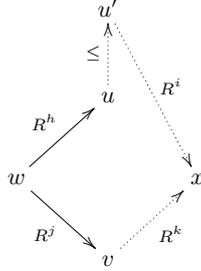
*A is a theorem of  $\mathbf{IK} + \mathbf{P}$  if and only if A is satisfied in every bi-relational model  $\langle W, R, \leq, V \rangle$  that satisfies:*

$$\text{for all } w, u, v \in W \text{ if } wR^h u \text{ and } wR^j v \text{ then } uRv.$$

The completeness of  $\mathbf{IK}$ ,  $\mathbf{IKT}$ ,  $\mathbf{IKTB}$ ,  $\mathbf{IS4}$  and  $\mathbf{IS5}$  with respect to their corresponding bi-relational models already appeared in [124]. However this completeness result is not a correspondence result: the axioms do not characterise the considered class of frames. Plotkin and Stirling give proper correspondence results, but they must distinguish the two dual parts of each axiom and appeal to both relations of the frames,  $R$  and  $\leq$ .

**3.3.4 Theorem** (Plotkin and Stirling [111]) *An intuitionistic modal frame  $\langle W, R, \leq \rangle$  validates  $\mathfrak{g}_{\text{hijk}}$  if and only if the frame satisfies:*

*if  $wR^h u$  and  $wR^j v$  then there exists  $u'$  and  $x$  such that  $u \leq u'$ ,  $u'R^i x$ , and  $vR^k x$*



The semantics of other versions of intuitionistic modal logics, for example of  $\mathbf{CK}$  or  $\mathbf{CS4}$ , has also been studied [137, 41, 1, 91, 67]. However we are not aware of any general correspondence or even completeness results for extensions of the basic logic with wider families of axioms or frames properties. We do not give more details here as we will focus on  $\mathbf{IK}$  and its extensions in this thesis.

### 3.4 Labelled deduction.

Echoing to the definition of bi-relational structures, the straightforward extension of labelled deduction to the intuitionistic setting would be to use two sorts of relational atoms, one for the modal relation  $R$  and another one for the intuitionistic relation  $\leq$ . This is the approach developed by Maffezioli, Naibo and Negri in [85]. To our knowledge this has not yet been investigated much further, but could be a fruitful perspective; it would be interesting for example to integrate Plotkin and Stirling's completeness result (Theorem 3.3.4) into this framework, or to combine the advances on labelled systems based on neighbourhood semantics for non-normal modal logics with the semantics for  $\mathbf{CK}$  proposed in [67].

However, another approach was taken by Simpson [127] even earlier than Maffezioli *et al*'s work. Simpson followed the lines of Gentzen in a labelled context, namely, he developed a labelled natural deduction framework for modal logics and then converted it into sequent systems with the consequent restriction to one formula on the right-hand side of each sequent. This worked as well in the labelled setting as in the ordinary sequent case; we will present Simpson's sequent system  $\mathbf{labIK}$  and its extensions in the rest of this section.

Intuitionistic labelled sequents are written  $\mathcal{G}, \mathcal{L} \Rightarrow z : C$  for some multiset of labelled formulas  $\mathcal{L}$ , some formula  $C$ , some label  $z$  and a set of relational atoms  $\mathcal{G}$ . The rules of  $\mathbf{labIK}$  (Figure 3.1) are very similar to the one of  $\mathbf{labK}$  (Figure 2.4) with the standard restrictions for intuitionistic version of the right-rules  $\vee_{R1}^{\text{lab}}$  and  $\vee_{R2}^{\text{lab}}$  and the left-rule  $\supset_{L}^{\text{lab}}$ . It is possible to

$$\begin{array}{c}
\text{id}^{\text{lab}} \frac{}{\mathcal{G}, \mathcal{L}, x : a \Rightarrow x : a} \quad \perp_{\text{L}}^{\text{lab}} \frac{}{\mathcal{G}, \mathcal{L}, x : \perp \Rightarrow z : A} \quad \top_{\text{R}}^{\text{lab}} \frac{}{\mathcal{G}, \mathcal{L} \Rightarrow x : \top} \\
\wedge_{\text{L}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \wedge B, x : A, x : B \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : A \wedge B \Rightarrow z : C} \quad \wedge_{\text{R}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow x : A \quad \mathcal{L} \Rightarrow x : B}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \wedge B} \\
\vee_{\text{L}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \vee B, x : A \Rightarrow z : C \quad \mathcal{G}, \mathcal{L}, x : A \vee B, x : B \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : A \vee B \Rightarrow z : C} \\
\vee_{\text{R1}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow x : A}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \vee B} \quad \vee_{\text{R2}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow x : B}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \vee B} \\
\supset_{\text{L}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \supset B \Rightarrow x : A \quad \mathcal{G}, \mathcal{L}, x : A \supset B, x : B \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : A \supset B \Rightarrow z : C} \quad \supset_{\text{R}}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \Rightarrow x : B}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \supset B} \\
\Box_{\text{L}}^{\text{lab}} \frac{\mathcal{G}, xRy, \mathcal{L}, x : \Box A, y : A \Rightarrow z : B}{\mathcal{G}, xRy, \mathcal{L}, x : \Box A \Rightarrow z : B} \quad \Box_{\text{R}}^{\text{lab}} \frac{\mathcal{G}, xRy, \mathcal{L} \Rightarrow y : A}{\mathcal{G}, \mathcal{L} \Rightarrow x : \Box A} \quad y \text{ is fresh} \\
\Diamond_{\text{L}}^{\text{lab}} \frac{\mathcal{G}, xRy, \mathcal{L}, x : \Diamond A, y : A \Rightarrow z : B}{\mathcal{G}, \mathcal{L}, x : \Diamond A \Rightarrow z : B} \quad y \text{ is fresh} \quad \Diamond_{\text{R}}^{\text{lab}} \frac{\mathcal{G}, xRy, \mathcal{L} \Rightarrow y : A}{\mathcal{G}, xRy, \mathcal{L} \Rightarrow x : \Diamond A}
\end{array}$$

Figure 3.1: System labIK

extend this basic calculus with the *geometric rule scheme* in the same way as we presented in Section 2.4, adapted to the intuitionistic setting. Namely, for every geometric frame property  $\forall \mathbf{u} \forall \mathbf{v} ((a_1 \wedge \dots \wedge a_n) \supset (\exists \mathbf{x}_1 A_1 \vee \dots \vee \exists \mathbf{x}_m A_m))$  we can construct the schematic rule:

$$\Box_{\text{grs}}^{\text{lab}} \frac{\mathcal{A}_1(\mathbf{y}_1/\mathbf{x}_1), a_1, \dots, a_n, \mathcal{G}, \mathcal{L} \Rightarrow z : C \quad \dots \quad \mathcal{A}_m(\mathbf{y}_m/\mathbf{x}_m), a_1, \dots, a_n, \mathcal{G}, \mathcal{L} \Rightarrow z : C}{a_1, \dots, a_n, \mathcal{G}, \mathcal{L} \Rightarrow z : C} \quad (3.2)$$

where again  $\mathcal{G}$  is a set of relational or equality atoms,  $\mathcal{L}$  is a multiset of labelled modal formulas,  $z : C$  is a given labelled formula, each  $a_i$  for  $1 \leq i \leq n$  is a relational or an equality atom, each  $\mathcal{A}_j$  for  $1 \leq j \leq m$  denotes the set of relational or equality atoms present in  $A_j$ , and  $\mathbf{y}_1, \dots, \mathbf{y}_m$  are vectors of fresh eigenvariables, i.e. do not occur in the conclusion.

Simpson also considers a particular subclass of geometric formulas: a *Horn clause* is a first-order formula of the form:

$$\forall \mathbf{z} ((a_1 \wedge \dots \wedge a_n) \supset b)$$

where  $b$  and all the  $a_i$  for  $1 \leq i \leq n$  are atomic formulas, so in our case relational or equality atoms. When the geometric frame condition actually is a Horn clause, Simpson observes that the structural rule can be replaced by the following two propagation rules:

$$\Box_{\text{L}^*}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, y : A \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : \Box A \Rightarrow z : C} \quad xRy \in \mathcal{G}^* \quad \Diamond_{\text{R}^*}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow y : A}{\mathcal{G}, \mathcal{L} \Rightarrow x : \Diamond A} \quad xRy \in \mathcal{G}^* \quad (3.3)$$

where  $\mathcal{G}^*$  is the closure of the relational set  $\mathcal{G}$  with respect to the considered Horn clause.

Completeness of the system labIK extended with the general geometric rule scheme is non-trivial. However, in the restricted case of *intuitionistic SL paths axioms*, we can observe a correspondence with the Hilbert axiomatisation of both the system extended with structural rules and the one with propagation rules. Indeed, SL paths axioms correspond to frame properties which are Horn clauses, as we have seen in Theorem 3.3.3.

**3.4.1 Theorem** (Simpson [127]) *Let  $\mathcal{P}$  be a set of intuitionistic Scott-Lemmon path axioms. Let  $\Box_{\mathcal{P}}^{\text{lab}}$  be the corresponding set of structural geometric rules (3.2) and  $\Box_{\mathcal{P}}^{\text{lab}} + \Diamond_{\mathcal{P}}^{\text{lab}}$  be the corresponding set of labelled propagation rules (3.3). For any formula  $A$ , the following are equivalent.*



$$\begin{array}{c}
\text{id}^n \frac{}{\Lambda\{a, a^\circ\}} \quad \wedge_R^n \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}} \quad \top_R^n \frac{}{\Lambda\{\top^\circ\}} \quad \wedge_L^n \frac{\Pi\{A, B\}}{\Pi\{A \wedge B\}} \quad \top_L^n \frac{\Pi\{\emptyset\}}{\Pi\{\top\}} \\
\vee_{R1}^n \frac{\Lambda\{A^\circ\}}{\Lambda\{A \vee B^\circ\}} \quad \vee_{R2}^n \frac{\Lambda\{B^\circ\}}{\Lambda\{A \vee B^\circ\}} \quad \vee_L^n \frac{\Pi\{A\} \quad \Pi\{B\}}{\Pi\{A \vee B\}} \quad \perp_L^n \frac{}{\Pi\{\perp\}} \\
\supset_R^n \frac{\Lambda\{A, B^\circ\}}{\Lambda\{A \supset B^\circ\}} \quad \supset_L^n \frac{\Pi^*\{A \supset B, A^\circ\} \quad \Pi\{B\}}{\Pi\{A \supset B\}} \\
\diamond_{Rk}^n \frac{\Lambda_1\{[\Lambda_2, A^\circ]\}}{\Lambda_1\{[\Lambda_2, \diamond A^\circ]\}} \quad \diamond_L^n \frac{\Pi\{[A]\}}{\Pi\{\diamond A\}} \quad \square_R^n \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\square A^\circ\}} \quad \square_{Lk}^n \frac{\Delta_1\{\square A, [A, \Delta_2]\}}{\Delta_1\{\square A, [\Delta_2]\}} \\
\dots \\
\diamond_{Rt}^n \frac{\Lambda\{A^\circ\}}{\Lambda\{\diamond A^\circ\}} \quad \diamond_{Rb}^n \frac{\Lambda_1\{[\Lambda_2, A^\circ]\}}{\Lambda_1\{[\Lambda_2, \diamond A^\circ]\}} \quad \diamond_{R4}^n \frac{\Lambda_1\{[\Lambda_2, \diamond A^\circ]\}}{\Lambda_1\{[\Lambda_2, \diamond A^\circ]\}} \quad \diamond_{R5}^n \frac{\Lambda_1\{[\Lambda_2]\}\{\diamond A^\circ\}}{\Lambda_1\{[\Lambda_2, \diamond A^\circ]\}\{\emptyset\}} \\
\square_{Lt}^n \frac{\Pi\{\square A, A\}}{\Pi\{\square A\}} \quad \square_{Lb}^n \frac{\Delta_1\{\{\square A, \Delta_2\}, A\}}{\Delta_1\{\{\square A, \Delta_2\}\}} \quad \square_{L4}^n \frac{\Delta_1\{\square A, [\square A, \Delta_2]\}}{\Delta_1\{\square A, [\Delta_2]\}} \quad \square_{L5}^n \frac{\Delta_1\{\{\Delta_2, \square A\}\}\{\square A\}}{\Delta_1\{\{\Delta_2, \square A\}\}\{\emptyset\}}
\end{array}$$

Figure 3.2: System nIK and extensions for t, b, 4, and 5

Similarly to the classical case, an  $n$ -ary context  $\Delta\{^1\} \dots \{^n\}$  is like a sequent but contains  $n$  pairwise distinct numbered *holes*  $\{ \}$  wherever a formula may otherwise occur. A *full context* is a context of the form  $\Pi\{^1\} \dots \{^n\}$ , where there is an output formula in  $\Pi\{^1\emptyset\} \dots \{^n\emptyset\}$ . Thus, all the arguments passed to this context must be input sequents. On the other hand, an *input context* is of the form  $\Lambda\{^1\} \dots \{^n\}$  and contains only input formulas, so when it is used to build a sequent exactly one of its arguments will itself be a full sequent.

Given a context that contains no output formulas, *e.g.*, of the form  $\Lambda\{ \}$ , it is possible to replace the hole with a full sequent  $\Pi$ , in which case the substitution  $\Lambda\{\Pi\}$  is also a full sequent. If the context contains an output formula, however, then this formula must be removed before such a substitution is syntactically well-formed. We write  $\Delta^*\{^1\} \dots \{^n\}$  for the result of deleting any output formula from an  $n$ -holed context  $\Delta\{^1\} \dots \{^n\}$ .

**3.5.2 Example** Consider the input context  $\Lambda\{ \} = [[B, C], \{ \}]$ ;  $C$ ; the full context  $\Pi_1\{ \} = C, [\{ \}, [B, C^\circ]]$ ; and the full sequent  $\Pi_2 = A, [B^\circ]$ . Then,  $\Lambda\{\emptyset\} = [[B, C], C$  is an input sequent; while  $\Lambda\{\Pi_2\} = C, [[B, C], A, [B^\circ]]$  and  $\Pi_1\{\Lambda\{\emptyset\}\} = C, [[[B, C], C, [B, C^\circ]]$  are full sequents.  $\Pi_1\{\Pi_2\}$  on the other hand is not well-formed because it would contain both  $C^\circ$  and  $B^\circ$ , but  $\Pi_1^*\{\Pi_2\} = C, [[B], A, [B^\circ]]$  is a full sequent.

In Figure 3.2 we display the rules of system nIK and some extensions from [131]. The rules are similar to the classical ones (Figure 2.5) except that they present a right and a left version in the intuitionistic case, which apply respectively to an output or an input formula respectively.

**3.5.3 Example** We give as an example the proof of  $k_4 : (\diamond a \supset \square b) \supset \square(a \supset b)$  in nIK. Observe how  $b^\circ$  is deleted from the first premiss of  $\supset_L^n$ .

$$\begin{array}{c}
\begin{array}{c}
\text{id}^n \frac{}{\diamond a \supset \square b, [a^\circ, a]} \quad \text{id}^n \frac{}{\square b, [a, b, b^\circ]} \\
\diamond_R^n \frac{}{\diamond a \supset \square b, \diamond a^\circ, [a]} \quad \square_L^n \frac{}{\square b, [a, b^\circ]} \\
\supset_L^n \frac{}{\diamond a \supset \square b, [a, b^\circ]}
\end{array} \\
\supset_R^n \frac{}{\diamond a \supset \square b, [a \supset b^\circ]} \\
\square_R^n \frac{}{\diamond a \supset \square b, \square(a \supset b)^\circ} \\
\supset_R^n \frac{}{(\diamond a \supset \square b) \supset \square(a \supset b)^\circ}
\end{array}$$

The basic system  $\mathfrak{nIK}$  in the first part of Figure 3.2 can be extended modularly by the left and right rules  $\square_{Lx}^n + \diamond_{Rx}^n$  corresponding to each axiom  $x$  among  $\{t, b, 4, 5\}$  in the second part.

**3.5.4 Theorem** (Straßburger [131]) *Let  $X$  be an axiomatically complete subset of  $\{t, b, 4, 5\}$ . For any formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $\mathfrak{IK} + X$ .

2.  $A$  is provable in  $\mathfrak{nIK} + \square_{LX}^n + \diamond_{RX}^n + \text{cut}$  where  $\text{cut}$  is the rule

$$\text{cut} \frac{\Pi^*\{C^\circ\} \quad \Pi\{C\}}{\Pi\{\emptyset\}}$$

3.  $A$  is provable in  $\mathfrak{nIK} + \square_{LX}^n + \diamond_{RX}^n$ .

Note the particular form of the cut-rule; in the left premiss the output formula of  $\Pi$  has to be pruned in order to maintain the output uniqueness when  $A^\circ$  is added to the sequent. In the proof of this theorem, as for Theorem 2.5.6, axiomatic completeness is only needed for cut-elimination and cut-free completeness, not for soundness and completeness with cut.

**3.5.5 Conjecture** *We would expect that this result can be extended to any logic obtained by adding to  $\mathfrak{IK}$  a set  $P$  of intuitionistic path SL axioms such as  $p: (\diamond^h \square a \supset \square^j a) \wedge (\diamond^j a \supset \square^h \diamond a)$ , by using rules (for each  $p \in P$ ) derived from the classical case described in Section 2.5 of the form:*

$$\square_{Lp}^n \frac{\Pi\{\square A\}\{^v A\}}{\Pi\{\square A\}\{\emptyset\}} \quad \diamond_{Rp}^n \frac{\Lambda\{\emptyset\}\{^v A^\circ\}}{\Lambda\{\emptyset\}\{^v A^\circ\}}$$

if there is a path  $\pi$  from  $u$  to  $v$  in the propagation graph of  $\Delta\{\emptyset\}\{\emptyset\}$  such that  $(\langle \pi \rangle, \diamond)$  appears in the completion of  $P$ . Indeed, this would correspond to Theorem 3.4.1 in the formalism of nested sequents.

Note that, as we already mentioned,  $d$  is not a path axiom. In [131] Straßburger claimed to have built rules corresponding to axiom  $d$  from the classical rule in (2.7), in a similar fashion, which gives:

$$\square_{Ld}^n \frac{\Pi\{\square A, [A]\}}{\Pi\{\square A\}} \quad \diamond_{Rd}^n \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\diamond A^\circ\}}$$

These rules, once added to the system  $\mathfrak{nIK}$ , give a sound and complete system (for the logic  $\mathfrak{ID} = \mathfrak{IK} + d$ ), but the combination of these rules with the ones of Figure 3.2 do not necessarily give a sound and complete system for the respective extensions of  $\mathfrak{ID}$ . For example, the system  $\mathfrak{nIK} + \square_{Ld}^n + \diamond_{Rd}^n + \square_{L4}^n + \diamond_{R4}^n$  is not cut-free complete for the logic  $\mathfrak{IK} + d + 4$ . As an illustration, we can see that  $\mathfrak{IK} + 4 + d \vdash (\diamond \diamond a \supset \perp) \supset \diamond(a \supset b)$ , but  $\mathfrak{nIK} + \square_{Ld}^n + \diamond_{Rd}^n + \square_{L4}^n + \diamond_{R4}^n \not\vdash (\diamond \diamond a \supset \perp) \supset \diamond(a \supset b)$ . This formula is only derivable with the aid of a cut-rule:

$$\frac{\begin{array}{c} \text{id}^n \frac{}{[[a^\circ, a]]} \\ \diamond_{Rk}^n \frac{}{\diamond \diamond a^\circ, [[a]]} \\ \supset_L^n \frac{}{\diamond \diamond a \supset \perp, [[a, b^\circ]]} \\ \supset_R^n \frac{}{\diamond \diamond a \supset \perp, [[a \supset b^\circ]]} \\ \diamond_{Rd}^n \frac{}{\diamond \diamond a \supset \perp, [\diamond(a \supset b)^\circ]} \\ \square_R^n \frac{}{\diamond \diamond a \supset \perp, \square \diamond(a \supset b)^\circ} \\ \text{cut} \end{array}}{\supset_R^n \frac{}{(\diamond \diamond a \supset \perp) \supset \diamond(a \supset b)^\circ}} \quad \frac{\begin{array}{c} \text{id}^n \frac{}{\diamond \diamond a \supset \perp, [[a, a^\circ]]} \quad \text{id}^n \frac{}{\diamond \diamond a \supset \perp, [[b, a, b^\circ]]} \\ \supset_L^n \frac{}{\diamond \diamond a \supset \perp, [[a \supset b, a, b^\circ]]} \\ \supset_R^n \frac{}{\diamond \diamond a \supset \perp, [[a \supset b, a \supset b^\circ]]} \\ \diamond_{Rk}^n \frac{}{\diamond \diamond a \supset \perp, [[a \supset b], \diamond(a \supset b)^\circ]} \\ \diamond_L^n \frac{}{\diamond \diamond a \supset \perp, [\diamond(a \supset b), \diamond(a \supset b)^\circ]} \\ \diamond_{R4}^n \frac{}{\diamond \diamond a \supset \perp, [\diamond(a \supset b)], \diamond(a \supset b)^\circ} \\ \square_{Ld}^n \frac{}{\diamond \diamond a \supset \perp, \square \diamond(a \supset b), \diamond(a \supset b)^\circ} \end{array}}{\supset_R^n \frac{}{(\diamond \diamond a \supset \perp) \supset \diamond(a \supset b)^\circ}}$$

If we follow the procedure proposed in [131] to eliminate this cut, we will end up with a proof that contains an occurrence of the structural rule  $\boxtimes_d^n$  which cannot be eliminated:

$$\begin{array}{c}
\text{id}^n \frac{}{[[a, a^\circ]]} \\
\frac{\diamond_{\text{Rk}}^n \frac{}{\diamond \diamond a^\circ, [[a]]} \quad \perp_{\text{L}}^n \frac{}{\perp, [[a, b^\circ]]}}{\supset_{\text{L}}^n} \\
\frac{\supset_{\text{R}}^n \frac{}{\diamond \diamond a \supset \perp, [[a, b^\circ]]}}{\diamond_{\text{Rd}}^n} \\
\frac{\diamond_{\text{Rd}}^n \frac{}{\diamond \diamond a \supset \perp, [\diamond(a \supset b)^\circ]}}{\diamond_{\text{R4}}^n} \\
\frac{\diamond_{\text{R4}}^n \frac{}{\diamond \diamond a \supset \perp, \diamond(a \supset b)^\circ, [ ]}}{\boxtimes_d^n} \\
\frac{\boxtimes_d^n \frac{}{\diamond \diamond a \supset \perp, \diamond(a \supset b)^\circ}}{\supset_{\text{R}}^n} \\
\frac{}{(\diamond \diamond a \supset \perp) \supset \diamond(a \supset b)^\circ}
\end{array}$$

It therefore seems that, contrarily to the classical case, the axiom **d** cannot be handled via a propagation rule in the intuitionistic case (which is consistent with Theorem 3.4.1).

Finally, in [87] we also worked towards eliminating the axiomatic completeness condition from the assumptions of this theorem. It is possible to do so by using exactly the same structural rules as in the classical case (presented on Figure 2.6). Because here structural rules are needed anyway, it is not a problem to handle **d** in this case. As before, this result should extend to any set **P** containing intuitionistic SL path axioms and the axiom **d** using the structural rule that we defined in (2.8).

**3.5.6 Theorem** (Marin and Straßburger [87]) *Let  $X$  be any subset of  $\{\mathbf{t}, \mathbf{b}, \mathbf{d}, \mathbf{4}, \mathbf{5}\}$ . For a formula  $A$ , the following are equivalent.*

1.  $A$  is a theorem of  $\text{IK} + X$ .
2.  $A$  is provable in  $\text{nIK} + \square_{\text{L}X}^n + \diamond_{\text{R}X}^n + \boxtimes_X^n$ .

A theory of nested sequent systems for extensions of constructive modal logic **CK** has been conducted in [4]. We do not present it here as we will not be using this logic in the rest of our exposition, but it would be interesting to understand how the two approaches relate to each other and to the different approaches using labelled sequents.

## 3.6 Bridges

We do not know of many attempts to build bridges between the different approaches at hand for intuitionistic modal logics. Some students tried to start the groundwork in their Master's theses such as Murawska [99] or Hein [63], and probably others that we have not heard of. However, because in the intuitionistic case the different formalisms give rise to distinct logics (including axioms  $k_3$ ,  $k_4$ , and  $k_5$ , or not), the task of relating the various systems is subtler than in the classical case.

We have already argued in Section 2.6 for the formalism of *indexed nested sequent* as a potential candidate to close some technical debates on proof formalisms for modal logics. In this section we will review our results on extending this formalism to the intuitionistic setting [88].

We define intuitionistic indexed nested sequents according to a similar grammar as intuitionistic nested sequents:

$$\Lambda ::= \emptyset \mid A, \Lambda \mid [{}^y \Lambda_1], \Lambda_2 \qquad \Pi ::= \Lambda, A^\circ \mid \Lambda, [{}^x \Pi] \qquad \Delta ::= \Lambda \mid \Pi$$

The intuitionistic indexed nested system  $\text{inIK}$  is shown in Figure 3.3. Observe that the structural rules in the bottom line are identical to system  $\text{inK}$ . It is also the case that each system can be extended with the rules presented in (2.9) and (2.10). This system can be seen, similarly to the classical case, as a compact way to rewrite the labelled system  $\text{labIK}$ , but unlike the classical case, its relation to the Hilbert axiomatisation and its semantical interpretation is not straightforward.

$$\begin{array}{c}
\text{id}^{\text{in}} \frac{}{\Lambda\{a, a^\circ\}} \quad \wedge_{\text{R}}^{\text{in}} \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}} \quad \top_{\text{R}}^{\text{in}} \frac{}{\Lambda\{\top^\circ\}} \quad \wedge_{\text{L}}^{\text{in}} \frac{\Pi\{A, B\}}{\Pi\{A \wedge B\}} \quad \top_{\text{L}}^{\text{in}} \frac{}{\Pi\{\top\}} \\
\vee_{\text{R1}}^{\text{in}} \frac{\Lambda\{A^\circ\}}{\Lambda\{A \vee B^\circ\}} \quad \vee_{\text{R2}}^{\text{in}} \frac{\Lambda\{B^\circ\}}{\Lambda\{A \vee B^\circ\}} \quad \vee_{\text{L}}^{\text{in}} \frac{\Pi\{A\} \quad \Pi\{B\}}{\Pi\{A \vee B\}} \quad \perp_{\text{L}}^{\text{in}} \frac{}{\Pi\{\perp\}} \\
\supset_{\text{R}}^{\text{in}} \frac{\Lambda\{A, B^\circ\}}{\Lambda\{A \supset B^\circ\}} \quad \supset_{\text{L}}^{\text{in}} \frac{\Pi^*\{A \supset B, A^\circ\} \quad \Pi\{B\}}{\Pi\{A \supset B\}} \\
\Box_{\text{L}}^{\text{in}} \frac{\Delta_1\{\Box A, [{}^w A, \Delta_2]\}}{\Delta_1\{\Box A, [{}^w \Delta_2]\}} \quad \Box_{\text{R}}^{\text{in}} \frac{\Lambda\{[{}^v A^\circ]\}}{\Lambda\{\Box A^\circ\}} \quad \Diamond_{\text{L}}^{\text{in}} \frac{\Pi\{[{}^v A]\}}{\Pi\{\Diamond A\}} \quad \Diamond_{\text{R}}^{\text{in}} \frac{\Lambda_1\{[{}^w A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond A^\circ, [{}^w \Lambda][2]\}} \\
\cdots \\
\text{tp}_1^{\text{in}} \frac{\Pi\{[{}^w \emptyset]\}\{[{}^w A]\}}{\Pi\{[{}^w A]\}\{[{}^w \emptyset]\}} \quad \text{tp}_2^{\text{in}} \frac{\Lambda\{[{}^w \emptyset]\}\{[{}^w A^\circ]\}}{\Lambda\{[{}^w A^\circ]\}\{[{}^w \emptyset]\}} \quad \text{bc}_1^{\text{in}} \frac{\Delta_1\{[{}^w [{}^u \Delta_2]\}\{[{}^u \emptyset]\}}{\Delta_1\{[{}^w [{}^u \Delta_2]\}\{[{}^w \emptyset]\}} \quad \text{bc}_2^{\text{in}} \frac{\Delta_1\{[{}^w [{}^u \Delta_2]\}\{[{}^w [{}^u \emptyset]\}\}}{\Delta_1\{[{}^w [{}^u \Delta_2]\}\{[{}^w \emptyset]\}}
\end{array}$$

Figure 3.3: System inK

### Completeness

Completeness of  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$  with respect to  $\text{IK} + \mathbb{G}$  is proved in a standard manner. As a first step, the cut-elimination proof conducted in  $\text{inK}$  can be reproduced in a similar fashion in the intuitionistic systems.

**3.6.1 Theorem** (Marin and Straßburger [88]) *Let  $\mathbb{G}$  be a set of intuitionistic Scott-Lemmon axioms. For any formula  $A$ , the following are equivalent.*

1.  $A$  is provable in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}} + \text{cut}$  where  $\text{cut}$  is the rule
2.  $A$  is provable in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$

$$\text{cut} \frac{\Delta^*\{A^\circ\} \quad \Delta\{A\}}{\Delta\{\emptyset\}}$$

*Proof sketch.* The proof works similarly to the one of Theorem 2.6.4. We need to transform the system in a similar fashion as we did with  $\text{inK}$ , removing the  $\text{tp}$ -rule and changing the rules  $\text{id}$ ,  $\Box_{\text{L}}^{\text{in}}$ ,  $\Diamond_{\text{R}}^{\text{in}}$  for respectively

$$\ddot{\text{id}} \frac{}{\Delta\{[{}^u a^\circ]\}\{[{}^u a]\}} \quad \ddot{\Box}_{\text{L}}^{\text{in}} \frac{\Delta_1\{[{}^u \Box A]\}\{[A, \Delta_2]\}}{\Delta_1\{[{}^u \Box A]\}\{[{}^u \Delta_2]\}} \quad \ddot{\Diamond}_{\text{R}}^{\text{in}} \frac{\Delta_1\{[{}^u \emptyset]\}\{[{}^u [A^\circ, \Delta_2]\}}{\Delta_1\{[{}^u \Diamond A^\circ]\}\{[\Delta_2]\}}$$

Then, we can easily extend Lemma 2.6.6 and 2.6.7 to the intuitionistic setting. And so, we can prove a reduction lemma like Lemma 2.6.9 for the intuitionistic system. The proof is almost identical, except that the reduction cases now occur between the left and the right rule for each connective, and that there are some additional cases for  $\supset$ . Finally, we can prove a result similar to Lemma 2.6.8 in the intuitionistic setting to complete the proof.  $\odot$

We deduce then that every theorem of  $\text{IK} + \mathbb{G}$  is a theorem of  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$  using the cut-elimination result. Note that, in the classical case, a similar proof can be conducted, and it provides an alternative to Fitting's proof of cut-free completeness [48].

**3.6.2 Theorem** (Marin and Straßburger [88]) *Let  $\mathbb{G}$  be a set of intuitionistic SL axioms and  $\boxtimes_{\mathbb{G}}^{\text{in}}$  be the corresponding set of rules. If a formula  $A$  is provable in the Hilbert system  $\text{IK} + \mathbb{G}$ , then the sequent  $A^\circ$  is provable in the indexed nested sequent system  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$ .*

*Proof.* The axioms of intuitionistic propositional logic as well as the axioms  $k_1$ - $k_5$  can be derived in  $\text{inK}$ , in the same way as in the usual nested sequent system presented in [131]. The *necessitation* rule of the Hilbert system can be simulated by the structural rule  $\text{nec}$ , which is admissible in  $\text{inK}$  (by a similar result as Lemma 2.6.6), and *Modus Ponens* can

be simulated by the cut-rule, which is also admissible (Theorem 3.6.1). Thus, it remains to show that any  $\mathbf{g}_{klmn}$  axiom can be derived, using the corresponding rule  $\boxtimes_{\mathbf{g}_{klmn}}^{\text{in}}$  :

$$\begin{array}{c}
\text{id} \frac{}{\frac{[u_1 \dots [u_k \Box^l a, [v_1 \Box^{l-1} a, \dots [v_{l-1} \Box a, [v_l]] \dots]] \dots], [w_1 \dots [w_m [x_1 \dots [x_{n-1} [x_n a, a^\circ]] \dots]] \dots]}{[u_1 \dots [u_k \Box^l a, [v_1 \Box^{l-1} a, \dots [v_{l-1} \Box a, [v_l]] \dots]] \dots], [w_1 \dots [w_m [x_1 \dots [x_{n-1} [x_n a^\circ]] \dots]] \dots]} \\
\text{tp} \frac{}{[u_1 \dots [u_k \Box^l a, [v_1 \Box^{l-1} a, \dots [v_{l-1} \Box a, [v_l]] \dots]] \dots], [w_1 \dots [w_m [x_1 \dots [x_{n-1} [x_n a^\circ]] \dots]] \dots]} \\
\frac{l \cdot \Box_{\mathbb{L}}^{\text{in}}, n \cdot \Diamond_{\mathbb{R}}^{\text{in}}}{\boxtimes_{\mathbf{g}_{klmn}}^{\text{in}}} \frac{[u_1 \dots [u_k \Box^l a, [v_1 \dots [v_l]] \dots], [w_1 \dots [w_m \Diamond^n a^\circ, [x_1 \dots [x_n]] \dots]] \dots]}{[u_1 \dots [u_k \Box^l a] \dots], [w_1 \dots [w_m \Diamond^n a^\circ] \dots]} \\
\frac{k \cdot \Diamond_{\mathbb{L}}^{\text{in}}, m \cdot \Box_{\mathbb{R}}^{\text{in}}}{\supset_{\mathbb{R}}^{\text{in}}} \frac{[u_1 \dots [u_k \Box^l a] \dots], [w_1 \dots [w_m \Diamond^n a^\circ] \dots]}{\Diamond^k \Box^l a \supset \Box^m \Diamond^n a^\circ} \\
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\end{array}$$

However, there are examples of theorems of  $\text{inIK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  that are not theorems of  $\text{IK} + \mathbf{G}$ , that is, the indexed nested sequent system is not sound with respect to the Hilbert axiomatisation using what we gave above as the intuitionistic alternative to Scott-Lemmon axioms. There is already a simple counter-example when  $\mathbf{G}$  consists of the sole axiom  $\mathbf{g}_{1111} : \Diamond \Box A \supset \Box \Diamond A$ . The formula

$$F = (\Diamond(\Box(a \vee b) \wedge \Diamond a) \wedge \Diamond(\Box(a \vee b) \wedge \Diamond b)) \supset \Diamond(\Diamond a \wedge \Diamond b) \quad (3.4)$$

is derivable in  $\text{inIK} + \boxtimes_{\mathbf{g}_{1111}}^{\text{in}}$ , but is not a theorem of  $\text{IK} + \boxtimes_{\mathbf{g}_{1111}}^{\text{in}}$  (as mentioned in [127]).

Thus, for some set of SL axioms  $\mathbf{G}$ , the logic given by the Hilbert axiomatisation  $\text{IK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  and the one given by the indexed nested sequent system  $\text{inIK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  actually differ in the intuitionistic case.

## Soundness

In the classical case, the indexed nested sequent system is not only equivalent to the Hilbert axiomatisation using Scott-Lemmon axioms, it is actually sound and complete wrt. the corresponding Kripke semantics. In this section, we investigate the behaviour of system  $\text{inIK}$  and its extensions, regarding the Kripke semantics for intuitionistic modal logics.

For proving the soundness of our system, we must also define what is the *validity* of a sequent in a model. For this, we adapt here the method of Fitting [48] to the intuitionistic setting. The first step is to put intuitionistic indexed nested sequent in correspondence with intuitionistic models.

Let  $\Delta$  be an indexed nested sequent and let  $\mathcal{M} = \langle W, R, \leq, V \rangle$  be an intuitionistic Kripke model. A *homomorphism*  $h: \Delta \rightarrow \mathcal{M}$  is a mapping  $h: I_\Delta \rightarrow W$ , such that  $R_\Delta(w, v)$  implies  $R(h(w), h(v))$  for all  $w, v \in I_\Delta$ . A pre-order relation between homomorphisms can be obtained from the pre-order in an intuitionistic model: For  $h, h': \Delta \rightarrow \mathcal{M}$  two homomorphisms, we write  $h \leq h'$  if  $h(w) \leq h'(w)$  in  $\mathcal{M}$  for all  $w \in I_\Delta$ .

The notion of validity can then be defined by induction on the sub-sequents of a given sequent. However, the correspondence between indices in a sequent and worlds in a model brings us to consider the particular class of exhaustive sub-sequents. Let  $\Delta$  and  $\Delta_w$  be indexed nested sequents, and  $w \in I_\Delta$ . We say that  $\langle \Delta_w, w \rangle$  is an *exhaustive subsequent* of  $\Delta$  if either  $\Delta_w = \Delta$  and  $w = 0$ , or  $\Delta = \Delta_0 \{ [^w \Delta_w] \}$  for some context  $\Delta_0 \{ \}$ . Note that for a given index  $w$  of  $\Delta$ , there might be more than one  $\Delta_w$  such that  $\langle \Delta_w, w \rangle$  is an exhaustive subsequent of  $\Delta$ , simply because  $w$  can occur more than once in  $\Delta$ . For this reason we will write  $\dot{w}$  to denote a particular occurrence of  $w$  in  $\Delta$  and  $\Delta|_{\dot{w}}$  for the subsequent of  $\Delta$  rooted at the node  $\dot{w}$ . Then  $\langle \Delta|_{\dot{w}}, w \rangle$  stands for a uniquely defined exhaustive subsequent of  $\Delta$ .

Let  $h: \Delta \rightarrow \mathcal{M}$  be a homomorphism from a sequent  $\Delta$  to a model  $\mathcal{M}$ . Let  $w \in I_\Delta$  and let  $\langle \Delta_w, w \rangle$  be an exhaustive subsequent of  $\Delta$ . Then  $\Delta_w$  has one of the following forms:

- (i) either  $\Delta_w = B_1, \dots, B_l, [^{v_1} \Lambda_1], \dots, [^{v_n} \Lambda_n]$ . Then we define  $\langle h, w \rangle \Vdash_i \Delta_w$  as  $h(w) \Vdash B_i$  for some  $i \leq l$  or  $\langle h, v_j \rangle \Vdash_i \Lambda_j$  for some  $j \leq n$ ;

- (ii) or  $\Delta_w = B_1, \dots, B_l, [^{v_1}\Lambda_1], \dots, [^{v_n}\Lambda_n], A^\circ$ . Then we define  $\langle h, w \rangle \Vdash_i \Delta_w$  as either  $h(w) \not\models B_i$  for some  $i \leq l$  or  $\langle h, v_j \rangle \Vdash_i \Lambda_j$  for some  $j \leq n$  or  $h(w) \Vdash A$ ;
- (iii) or  $\Delta_w = B_1, \dots, B_l, [^{v_1}\Lambda_1], \dots, [^{v_n}\Lambda_n], [^u\Pi]$ . Then we define  $\langle h, w \rangle \Vdash_i \Delta_w$  as either  $h(w) \not\models B_i$  for some  $i \leq l$  or  $\langle h, v_j \rangle \Vdash_i \Lambda_j$  for some  $j \leq n$  or for all homomorphisms  $h' \geq h$ , we have that  $\langle h', u \rangle \Vdash_i \Pi$ .

If, for all  $h' \geq h$ , we have  $\langle h', w \rangle \Vdash_i \Delta_w$ , then we say that  $\langle \Delta_w, w \rangle$  is *satisfied in  $\mathcal{M}$  under  $h$* . Then, a sequent  $\Delta$  is *satisfied* in a model  $\mathcal{M}$ , if  $\langle \Delta, 0 \rangle$  is satisfied in  $\mathcal{M}$  under every  $h: \Delta \rightarrow \mathcal{M}$ .

Informally, an indexed nested sequent is satisfied in a modal  $\mathcal{M}$  if it contains somewhere in the sequent tree an output formula that is satisfied in  $\mathfrak{M}$  or an input formula that is not. The following lemma formalises this idea.

**3.6.3 Lemma** *Let  $\Delta$  be an indexed nested sequent. Let  $\langle \Delta_v, v \rangle$  be an exhaustive subsequent of  $\Delta$ . Suppose  $\Delta_v = \Delta_0 \{^w A\}$  for some context  $\Delta_0 \{^w \}$  and some formula  $A$ . Let  $\mathcal{M}$  be a bi-relational model and  $h: \Delta \rightarrow \mathcal{M}$  a homomorphism.*

- If  $A = A^\circ$  and  $h(w) \Vdash A$ , then  $\langle h, v \rangle \Vdash_i \Delta_v$ .
- If  $A = A$  and  $h(w) \not\models A$ , then  $\langle h, v \rangle \Vdash_i \Delta_v$ .

*Proof sketch.* By a straightforward induction on the height of the tree rooted at the considered occurrence of  $v$ . The base case occurs when  $A^\circ$  (or  $A$ ) is at the root of that tree.  $\odot$

We now make explicit the class of model that we are going to consider in order to interpret system  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$ . We need to appeal to the notion of graph-consistency introduced by Simpson [127]. A bi-relational model  $\mathcal{M}$  is called *graph-consistent* if for any sequent  $\Gamma$ , given any homomorphism  $h: \Gamma \mapsto \mathcal{M}$ , any  $w \in \Gamma$ , and any  $w' \geq h(w)$ , there exists  $h' \geq h$  such that  $h'(w) = w'$ .

**3.6.4 Theorem** (Marin and Straßburger [88]) *Let  $\mathbb{G}$  be a set of intuitionistic SL axioms and  $\boxtimes_{\mathbb{G}}^{\text{in}}$  be the corresponding set of rules. If a sequent  $\Delta$  is provable in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$  then it is satisfied in every graph-consistent bi-relational model  $\mathcal{M} = \langle W, R, \leq, V \rangle$  that obeys condition (2.3), i.e. for each  $\langle k, l, m, n \rangle$  with  $\mathbf{g}_{klmn} \in \mathbb{G}$ :*

*for all  $w, u, v \in W$  with  $wR^k u$  and  $wR^m v$  there is a  $z \in W$  such that  $uR^l z$  and  $vR^n z$ .*

The proof proceeds by induction on the height of the derivation, i.e., the length of the longest path in the tree from its root to one of its leaves, and crucially relies on Lemma 3.6.5, which states that each rule of  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$  is sound when interpreted in these graph-consistent models that obey condition (2.3).

**3.6.5 Lemma**  $\Delta_1$  *Let  $\mathbb{G}$  be a set of intuitionistic SL axioms and  $\boxtimes_{\mathbb{G}}^{\text{in}}$  be the corresponding set of rules. Let  $r \frac{\Delta_1 \dots \Delta_n}{\Delta}$  be an instance of an inference rule in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$  for  $n = 0, 1, 2$ .*

*If all of  $\Delta_1, \dots, \Delta_n$  are satisfied in every graph-consistent model that satisfies (2.3), then so is  $\Delta$ .*

*Proof.* First, assume that  $r$  is  $\boxtimes_{\mathbf{g}_{klmn}}^n \frac{\Phi}{\Psi}$ , for some  $k, l, m, n > 0$ .

By way of contradiction, suppose that  $\Phi$  is satisfied in every graph-consistent model that satisfies (2.3) and that there exist a model  $\mathcal{M} = \langle W, R, \leq, V \rangle$  that satisfies (2.3), and a homomorphism  $h: \Psi \rightarrow \mathcal{M}$  such that  $\langle \Psi, 0 \rangle$  is not satisfied in  $\mathcal{M}$  under  $h$ . Recall that  $\Psi$  has the form:

$$\Psi = \Gamma \{^{u_0} [^{u_1} \Delta_1, \dots [^{u_k} \Delta_k] \dots], [^{w_1} \Sigma_1, \dots [^{w_m} \Sigma_m] \dots]\}$$

Therefore, there exist  $u_0, u_k, w_m$  in  $W$  such that  $u_0 = h(u_0)$ ,  $u_k = h(u_k)$ ,  $w_m = h(w_m)$ , and  $u_0 R^k u_k$ , and  $u_0 R^m w_m$ . Hence, as  $\mathcal{M}$  is in particular satisfies (2.3), for  $\langle k, l, m, n \rangle$ , there exists  $y \in W$  with  $u_k R^l y$  and  $w_m R^n y$ . Namely, there are worlds  $v_1, \dots, v_l, x_1, \dots, x_n$  in  $W$  such that  $u_k R v_1 \dots v_{l-1} R v_l$ ,  $w_m R x_1 \dots x_{n-1} R x_n$ , and  $v_l = y = x_n$ .

By noting that

$$\Phi = \Gamma \left\{ u_0 \left[ u_1 \Delta_1, \dots \left[ u_k \Delta_k, \left[ v_1 \dots \left[ v_l \dots \right] \dots \right] \dots \right], \left[ w_1 \Sigma_1, \dots \left[ w_m \Sigma_m, \left[ x_1 \dots \left[ x_n \dots \right] \dots \right] \dots \right] \right\}$$

we can define a homomorphism  $h': \Phi \rightarrow \mathcal{M}$  with  $h'(z) = h(z)$  for all  $z \in I_\Psi$ ,  $h'(v_i) = v_i$  for  $1 \leq i \leq l$  and  $h'(x_j) = x_j$  for  $1 \leq j \leq n$ .

We are now going to show that for every  $h: \Psi \rightarrow \mathcal{M}$ , and every occurrence  $\dot{z}$  of an index  $z \in I_\Psi$ , we have  $\langle h, z \rangle \Vdash_i \Psi|_{\dot{z}}$  iff  $\langle h', z \rangle \Vdash_i \Phi|_{\dot{z}}$ . We proceed by induction on the height of the tree rooted at  $\dot{z}$ . There are four cases:

1. The node of  $\dot{z}$  is a leaf node of  $\Psi$ , and  $z \neq u_k$  and  $z \neq w_m$ . Then we have  $\Psi|_{\dot{z}} = \Phi|_{\dot{z}}$  and the claim holds trivially.
2. The node of  $\dot{z}$  is an inner node of  $\Psi$ , and  $z \neq u_k$  and  $z \neq w_m$ . By the induction hypothesis, for every  $t \in I_\Psi$  with  $z R_\Psi t$ , every occurrence  $\dot{t}$  of  $t$  in  $\Psi|_{\dot{z}}$ , and every  $h: \Psi \rightarrow \mathcal{M}$ ,  $\langle h, t \rangle \Vdash_i \Psi|_{\dot{t}}$  iff  $\langle h', t \rangle \Vdash_i \Phi|_{\dot{t}}$ . The statement follows then by unravelling the definition of the intuitionistic satisfiedity.
3.  $z = u_k$ . For any occurrence  $\dot{z}$  in the context  $\Gamma \{ z_0 \}$ , the proof is similar to one of the previous cases. Otherwise, we know that  $\Psi|_{\dot{z}} = \Delta_k$  and  $\Phi|_{\dot{z}} = \Delta_k, \left[ v_1 \dots \left[ v_l \dots \right] \dots \right]$ . Furthermore, for all  $i \leq l$  and  $h'' \geq h$  we have  $\langle h'', v_i \rangle \not\Vdash_i \left[ v_{i+1} \dots \left[ v_l \dots \right] \dots \right]$ , and therefore  $\langle h, z \rangle \Vdash_i \Psi|_{\dot{z}}$  iff  $\langle h', z \rangle \Vdash_i \Phi|_{\dot{z}}$ .
4.  $v = w_m$ . This case is similar to the previous one.

Since we assumed that  $\langle \Psi, 0 \rangle$  is not satisfied in  $\mathcal{M}$  under  $h$ , we can conclude that  $\langle \Phi, 0 \rangle$  is not satisfied in  $\mathcal{M}$  under  $h'$ , contradicting the satisfiedity of  $\Phi$ .

The proofs for **bc**, **tp**, and the other cases of  $\boxtimes_{\mathbf{gklmn}}^n$  when one of the parameters is 0, are similar. For the logical rules, we will consider in details the case for  $\square_{\mathbf{R}}^{\text{in}}$ , the others are similar or simpler (the cases for  $\supset_{\mathbf{R}}^{\text{in}}$  also makes use of the graph-consistency property).

Suppose that  $\Phi = \Gamma \{ w \left[ v A^\circ \right] \}$  is satisfied in every graph-consistent model that satisfies (2.3). For  $\Psi = \Gamma \{ w \square A^\circ \}$ , suppose that there exists a graph-consistent model  $\mathcal{M} = \langle W, R, \leq, V \rangle$  obeying (2.3), and a homomorphism  $h: \Psi \mapsto \mathcal{M}$  such that  $\langle \Psi, 0 \rangle$  is not satisfied in  $\mathcal{M}$  under  $h$ . Therefore, there exists  $h' \geq h$  such that  $\langle h', 0 \rangle \not\Vdash_i \Psi$ , in particular by Lemma 3.6.3,  $h'(w) \not\Vdash \square A$ . So there exist  $w$  and  $v$  such that  $w R v$ ,  $h'(w) \leq w$  and  $v \not\Vdash A$ . As  $\mathcal{M}$  is graph-consistent, there exists  $h''$  such that  $w = h''(w)$ . Thus, we can extend  $h''$  by setting  $h''(v) = v$  to obtain a homomorphism  $h'': \Phi \mapsto \mathcal{M}$ , indeed  $\Phi$  and  $\Psi$  have the same set of indices related by the same underlying structure, but for the fresh index  $v$  that does not appear in  $\Psi$ . Finally, as  $h''(v) \not\Vdash A$ , we have by Lemma 3.6.3 that  $\langle \Phi, 0 \rangle$  is not satisfied in  $\mathcal{M}$  under  $h''$ , which contradicts the assumption of satisfiedity of  $\Phi$ .  $\odot$

In the classical case, for a given set of SL axioms  $\mathbf{G}$ , the logic given by  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  corresponds exactly to the logic axiomatised by the extension of the Hilbert system  $\mathbf{K}$  with the corresponding Scott-Lemmon axioms  $\mathbf{G}$ . We could construct the proof as follows. First, every theorem of  $\mathbf{K} + \mathbf{G}$  is a theorem of  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  (as a corollary of cut-elimination in  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}}$ ). Then, every theorem of  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  is satisfied in any classical model that satisfies (2.3). And in the end, the cornerstone that would allow one to conclude that  $\text{inK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  is indeed sound and complete wrt.  $\mathbf{K} + \mathbf{G}$ , is Theorem 2.3.2 that states that the Hilbert system  $\mathbf{K} + \mathbf{G}$  actually completely axiomatises classical models that obey condition (2.3).

In the intuitionistic case, we do have Theorem 3.6.2 giving that every theorem of  $\text{IK} + \mathbf{G}$  is a theorem of  $\text{inIK} + \boxtimes_{\mathbf{G}}^{\text{in}}$ , and Theorem 3.6.4 giving that every theorem of  $\text{inIK} + \boxtimes_{\mathbf{G}}^{\text{in}}$  is

satisfied in graph-consistent models that obey (2.3), but there is no proper equivalent of Theorem 2.3.2 to “link” the two theorems into a actual soundness and completeness result for  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$ . As we have seen, the first inclusion is strict, since the formula in (3.4) is provable in  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$ , but not in  $\text{IK} + \text{G}$ . However, the strictness of the second inclusion is open.

**3.6.6 Open question** Is there a certain set  $\mathbb{G} \subseteq \mathbb{N}^4$  such that there exists a formula that is satisfied in every directed graph-consistent model that obeys (2.3), but that is not a theorem of  $\text{inK} + \boxtimes_{\mathbb{G}}^{\text{in}}$ ?

To conclude, we can say that the accurate definition of intuitionistic modal logics might actually come from structural proof-theoretical studies rather than Hilbert axiomatisations or semantical considerations. For Simpson [127] there are two different (but equivalent) ways to define intuitionistic modal logics, either the natural deduction systems he proposes, or the extension of the standard translation for intuitionistic modal logics into first-order intuitionistic logic. Equivalence between the natural deduction systems and the Hilbert axiomatisations, or direct interpretation of the natural deduction systems in intuitionistic (bi-relational) structures are just side-results. He therefore sees their failure for the majority of logics not as a problem, but rather as another justification of the validity of the proof-theoretic approach.



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## II « Unfolding »

We will observe through our focused telescope some of the systems presented in the previous part in order to get closer to their core composition. The course of this part will allow us to unfold inference rules in unlabelled calculus by stripping away layers of abstraction. Taking an unlabelled inference step, we can understand its behaviour in terms of a synthetic rule in the world of labelled sequents, and so any labelled rule can itself be interpreted as a synthetic rule in the world of focused first-order logic. In Chapter 4, we will introduce the base system for first-order logic and see how it can build synthetic rules. In Chapter 5, we will unfold labelled rules into this system for first-order logic. And finally, in Chapter 6, we will unfold rules from unlabelled calculi into the labelled framework.



## Focusing for first-order logic

To analyse the proof theory of modal logic, it is useful to have some tools that are a bit stronger which can be applied not just to propositional logic, but to the wider first-order world. *First-order logic* is an extension of propositional logic (whether classical or intuitionistic) to include *predicates* in the language. Those predicates can apply to some *terms* that are built from term variables, constants and functions. Two binding operators are added to the language in order to introduce the possibility to quantify over these terms. That is, if  $A$  is a given first-order formula, so are  $\exists x.A$  and  $\forall x.A$ , where the variable  $x$  that could appear freely in formula  $A$  is now bound. The intended semantics is that there *exists* a term  $t$  such that the formula  $(x/t)A$ , obtained by substituting  $t$  for the variable  $x$  everywhere in  $A$ , holds and respectively that *for all* term  $t$ ,  $(x/t)A$  holds.

The standard sequent systems introduced by Gentzen for (classical or intuitionistic) first-order logic are respectively called LK and LJ, extending the propositional sequent calculi with rules for the quantifiers [56]. We do not give the standard unfocused version of these rules, but we start by recalling their focused version in Section 4.1. For this thesis, we give a different presentation of the focused systems LKF and LJF than the original in [83]; in particular, we will be using an explicitly polarised syntax which considerably alleviates the notations. We end this chapter by explaining in Section 4.2 how to build *macro* synthetic rules from *micro* focused rules [139].

### 4.1 The ABC of focusing

#### Classical system

The way we will present the notion of focusing relies heavily on a certain *polarisation* of connectives. That is, we classify formulas into *negative* formulas whose rules are invertible, and *positive* formulas, whose rules require either some form of choice or some interaction between the main formula and the rest of the sequent. For instance, existential formulas are positive, as in the sequent they tend to remain as side formulas until it is possible to instantiate them with the adequate term; but universal formulas never need to be preserved and so can be decomposed eagerly as the rule introducing the universal quantifier is invertible, and as such they are classified as negative.

For simplicity we use an explicitly *polarised syntax* [73] consisting of the classes of positive and negative formulas, with a pair of *shift* connectives which allows for formulas to embed the classes into each other. Outside of the quantifiers which, as we saw, have fixed polarity, the rest of the connectives have ambiguous polarities, in the sense that it is possible to design inference rules for them that give them a positive or a negative interpretation. In the classical case, we will introduce polarised versions of  $\vee$ ,  $\wedge$ ,  $\perp$ , and  $\top$ , that have the same truth tables as the original unpolarised ones, but different inference rules in a sequent system. We can also play around with the polarity of the atomic predicates: we can say

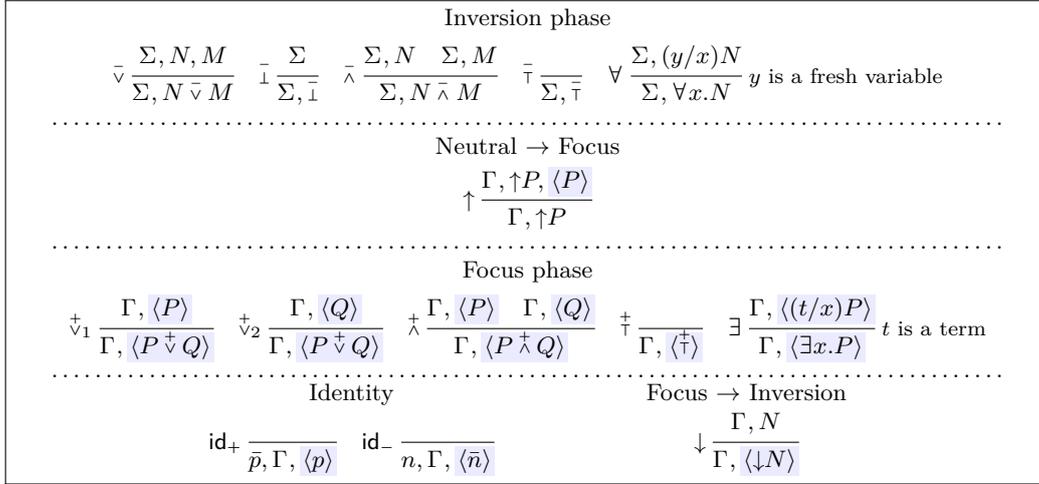


Figure 4.1: Focused system LKF

that they are all positive, or all negative, or a mix of both, as long as each atom is assigned exactly one polarity. In the classical case, the dual of a negative atom is positive, and vice-versa. The different polarisations do not change provability but can radically influence the structure of proofs (distinguishing for example what is known as backward vs. forward chaining of implications [25]). The polarity of a formula is given by the polarity of its top-level connective according to the following grammar:

$$\begin{aligned}
 P &::= p \mid \bar{n} \mid P \wedge P \mid \dagger \mid P \downarrow P \mid \dagger \mid \exists x.P \mid \downarrow N & L &::= p \mid \bar{n} \mid \downarrow N \\
 N &::= \bar{p} \mid n \mid N \bar{\vee} N \mid \bar{\perp} \mid N \bar{\wedge} N \mid \bar{\top} \mid \forall x.N \mid \uparrow P & R &::= \bar{p} \mid n \mid \uparrow P
 \end{aligned}$$

We use the notation  $p$  and  $n$  to vary over positive and negative atomic predicates respectively, but note that they can contain free first-order variables. As we are in a classical setting, each column in the grammar above defines a pair of de Morgan duals; note that the negation of a positive formula is a negative formula, and vice versa. Every polarised formula  $P$  or  $N$  is related to an unpolarised formula by a *depolarisation* map  $\partial(\cdot)$  which simply erases the shifts  $\uparrow$  and  $\downarrow$ , collapses  $\wedge$  and  $\bar{\wedge}$  into  $\wedge$ , as well as  $\downarrow$  and  $\bar{\vee}$  into  $\vee$ .

A formula is said to be *neutral* if its main connective is one of the shifts, or if it is an atom or a negated atom. More precisely, we can distinguish negative neutral formulas, denoted  $R$  above, and positive neutral formulas, denoted  $L$  above.

A polarised sequent is the same as in the non-focused setting, with the difference that all formulas are polarised. The class of *invertible* sequents, written  $\Sigma$ , is a subclass of polarised sequents consisting only of negative formulas. An invertible sequent is *neutral* if it is built from only negative neutral formulas, according to the grammar:

$$\Gamma ::= \emptyset \mid \Gamma, n \mid \Gamma, \bar{p} \mid \Gamma, \uparrow P$$

A *focused* sequent is of the form  $\Gamma, \langle P \rangle$ , that is, it is a neutral sequent  $\Gamma$  with an additional highlighted positive formula  $P$  which is called its *focus*.

As we have said in Chapter 1, during backward proof search, it is possible to apply all invertible rules at any time, and therefore in particular as first rules. What Andreoli [3] observed is a parallel behaviour of non-invertible rules, namely that it is also fine to apply them in sequences, to a formula and all its sub-formulas, as long as their main connectives are of the same non-invertible kind. A *focused* proof is morally a proof where the *decision* to apply a non-invertible rule to a neutral formula must be explicitly taken, and this decision commits the proof to retaining focus on its sub-formulas until there is a polarity change.

This drastically reduces the search space, and the main theorem of focusing is that this strategy is complete, i.e., every theorem that has a proof in LK has a focused proof.

The easiest way to prove this result is to design, from LK the standard system for first-order classical logic, a sort of subsystem, called LKF, that formally implements this idea, i.e. it is constrained in order to build only proofs of the focused form. Then, showing that this system is complete, that is, that every provable formula in LK also has a proof in this restricted system LKF, is equivalent to show that any provable formula has a proof that is focused.

**4.1.1 Theorem** (Liang and Miller [83]) *The system LKF is sound and complete with respect to LK, i.e. a negative neutral formula  $R$  is provable in LKF if and only if  $\partial(R)$  is provable in LK.*

We are going to explain how the focused system LKF, presented in Figure 4.1, can only build focused proofs. A proof in LKF begins—reading from conclusion upwards—with a neutral end-sequent, to which only the rule  $\uparrow$  may be applied (sometimes called in the literature the *decide rule*). In each case one shifted positive formula is selected in the conclusion and a copy of it (the rule implicitly contains an instance of contraction) is put under focus in the premiss, at which point the proof enters the *focus* phase, where non-invertible rules can be applied to the formula in focus until the focus becomes neutral again (but positive neutral this time). At this point, the proof either finishes with *id* if the focus is atomic, or it enters the *inversion* phase using the rule  $\downarrow$  (sometimes called in the literature the *release rule*) that drops the focus on a shifted formula. In the inversion phase, negative formulas are decomposed, in an arbitrary order, using invertible rules, until eventually the sequent becomes neutral again.

Note however, that our system does not include an equivalent of the rule called *store*. In fact, it is unnecessary at the theoretical level and is mainly used in the literature for implementation purposes. Removing the *store* rule makes the *decision* and *release* rules of the system correspond exactly to the introduction rules for the two shift connectives  $\downarrow$  and  $\uparrow$ , respectively, that inject each polarised class into the other. This simplification is completely independent of the classical or intuitionistic flavour of the logic.

## Intuitionistic system

In the previous chapter, input and output formulas were differentiated in an intuitionistic sequent using some sort of structural marking like  $\circ$ -annotations or a sequent delimiter  $\Rightarrow$ , but without any particular restrictions on which kinds of formulas may receive which annotations or be placed on which side of the arrow. It turns out that the polarity of connectives actually make them naturally belong to one or the other annotation/side. The use of the polarised syntax will therefore make the input/output annotations unnecessary.

In the intuitionistic case, the *polarised* syntax separates the classes of *positive* formulas, whose *right*-rules are non-invertible and *negative* formulas, whose *left*-rules are non-invertible. The classification is almost canonically determined; in particular, contrarily to the classical case,  $\vee$  and  $\perp$  can only be positive and  $\supset$  is negative. The exceptions are the atomic predicates which will be separated again into the positive ones  $p$  and the negative ones denoted  $n$ , as well as the  $\wedge$  and  $\top$  connectives which we divide into their polarised versions  $\overset{\dagger}{\wedge}$  and  $\bar{\wedge}$ , and  $\overset{\dagger}{\top}$  and  $\bar{\top}$ . Formulas are therefore divided into the positive and negative classes according to the following grammar:

$$\begin{aligned} P &::= L \mid P \overset{\dagger}{\wedge} P \mid \overset{\dagger}{\top} \mid P \vee P \mid \perp \mid \exists x.P & L &::= p \mid \downarrow N \\ N &::= R \mid N \bar{\wedge} N \mid \bar{\top} \mid P \supset N \mid \forall x.N & R &::= n \mid \uparrow P \end{aligned}$$

As before, we write  $L$  for particular positive formulas that we call *left-neutral formulas*, and  $R$  for particular negative formulas that we call *right-neutral formulas*. They can be atoms or built from the polarity *shifts*  $\downarrow$  and  $\uparrow$ , which are used to move between the two polarised

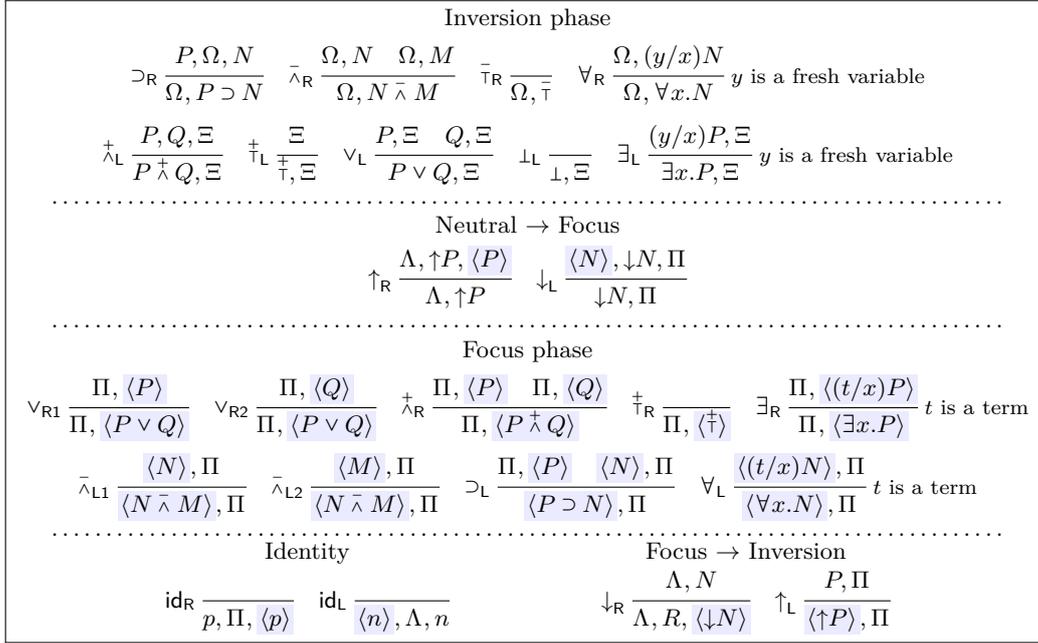


Figure 4.2: Focused system LJF

classes. We can define a *depolarisation* map in the intuitionistic case too, to remove the decorations.

Intuitionistically meaningful polarised sequents are exactly those sequents with a single negative formula interpreted as the output. We can define *invertible* sequents as a specific type of polarised sequents. We denote by  $\Omega$  left invertible sequents, which are composed only of positive formulas, and by  $\Xi$  full invertible sequents, which contain additionally one single instance of a negative formula. We write  $\Theta$  when the distinction is unnecessary or impossible.

$$\Omega ::= \emptyset \mid P, \Omega \qquad \Xi ::= \Omega, N \qquad \Theta ::= \Omega \mid \Xi$$

Then, *neutral* sequents are specific types of invertible sequents. A neutral left sequent  $\Lambda$  contains only left-neutral formulas, and if one instance of a right-neutral formula is added, it becomes a neutral (full) sequent  $\Pi$ . When we do not need to distinguish between left and full sequents, we write  $\Delta$ .

$$\Lambda ::= \emptyset \mid \downarrow N, \Lambda \mid p, \Lambda \qquad \Pi ::= \Lambda, \uparrow P \mid \Lambda, n \qquad \Delta ::= \Lambda \mid \Pi$$

*Focused* sequents are neutral sequents with an occurrence of  $\langle P \rangle$  or  $\langle N \rangle$ . We can relate these notations to the usual concepts in the literature, and see a sequent  $\Pi, \langle P \rangle$  with a positive focus as focused “on the right”, and a sequent  $\langle N \rangle, \Pi$  with a negative focus as focused “on the left”.

A focused intuitionistic proof follows a similar discipline to a focused classical proof. Starting from a neutral end-sequent, one neutral formula is chosen as a focus by one of  $\uparrow_R$  or  $\downarrow_L$ , then non-invertible rules can be applied to it and to its sub-formulas during a *focus phase* until the focus becomes neutral again. At this point either the proof ends with an identity, or the focus is released by one of  $\downarrow_R$  or  $\uparrow_L$ . The proof then enters an *inversion phase* to decompose any negative formula until the whole sequent is back to neutral. Note that, because the  $\downarrow_R$  rule introduces a negative formula to the premiss sequent, any negative formula appearing in the conclusion must be deleted in the premiss. However, this is the only place where a negative formula has to be removed; the polarised syntax and focused system allow us otherwise to keep the output formula until the end of the focused phase.

This restricted way of building proofs is again sound and complete with the standard proof system LJ for first-order intuitionistic logic.

**4.1.2 Theorem** (Liang and Miller [83]) *The system LJF is sound and complete with respect to LJ, i.e. a right-neutral formula  $R$  is provable in LJF if and only if  $\Rightarrow \partial(R)$  is provable in LJ.*

## 4.2 From focused to synthetic reasoning

Once we have imposed the focused structure on proofs, we gain a new perspective on proofs and may distinguish between what could be called *micro rules*, the ordinary sequent calculus rules, and *macro rules*, that contain an entire phase of a focused proof, and as such can be seen as rules introducing a *synthetic connective*, called *synthetic rules* [36]. Indeed, we can observe what a *focus phase* would actually look like. Bottom-up, we start with a neutral sequent, and focus on a positive formula, which gets decomposed into sub-formulas until we reach sub-formulas that are again neutral. This piece of a derivation should look as follows in the classical case:

$$\begin{array}{c}
 \left\{ \Gamma, \uparrow P, \langle \downarrow N_i \rangle \right\}_{1 \leq i \leq k} \quad \left\{ \text{id}_+ \frac{\bar{p}_i \in \Gamma}{\Gamma, \uparrow P, \langle p_i \rangle} \right\}_{1 \leq i \leq k_p} \quad \left\{ \text{id}_- \frac{n_i \in \Gamma}{\Gamma, \uparrow P, \langle \bar{n}_i \rangle} \right\}_{1 \leq i \leq k_n} \quad \left\{ \frac{\dagger}{\Gamma, \uparrow P, \langle \dagger \rangle} \right\} \\
 \hline
 \text{pos} \\
 \hline
 \frac{\Gamma, \uparrow P, \langle P \rangle}{\Gamma, \uparrow P}
 \end{array}$$

where the braces enclose multi-sets of premisses, the index  $i$  ranges over their arbitrary size. The leftmost set of premisses is open and ends on a neutral sequent containing an occurrence of a neutral focus, and the last three sets are closed premisses, either by an identity rule  $\text{id}_+$  or  $\text{id}_-$ , or by a rule introducing  $\dagger$ .

A synthetic rule corresponding to this whole phase would only record the open premisses, but the closed premisses constrain the shape of the context as some atoms have to be present to be able to close the branches:

$$\text{pos} \frac{\left\{ \Gamma, \bar{p}_1, \dots, \bar{p}_{k_p}, n_1, \dots, n_{k_n}, \uparrow P, \langle \downarrow N_i \rangle \right\}_{1 \leq i \leq k}}{\Gamma, \bar{p}_1, \dots, \bar{p}_{k_p}, n_1, \dots, n_{k_n}, \uparrow P}$$

Note however that the  $\dagger$  rule introduces a choice in the sub-formulas of  $P$  that are retained as premisses. For example, if  $P = (\downarrow N_1 \uparrow \downarrow N_2) \dagger (\downarrow N_3 \uparrow \downarrow N_4)$ , we can extract the two following distinct rules for the synthetic connective  $(\cdot \uparrow \cdot) \dagger (\cdot \uparrow \cdot)$ :

$$(\cdot \uparrow \cdot) \dagger (\cdot \uparrow \cdot)_1 \frac{\frac{\langle \downarrow N_1 \rangle}{\uparrow (\downarrow N_1 \uparrow \downarrow N_2)} \quad \frac{\langle \downarrow N_2 \rangle}{\dagger (\downarrow N_3 \uparrow \downarrow N_4)}}{\uparrow (\downarrow N_1 \uparrow \downarrow N_2) \dagger (\downarrow N_3 \uparrow \downarrow N_4)} \quad (\cdot \uparrow \cdot) \dagger (\cdot \uparrow \cdot)_2 \frac{\frac{\langle \downarrow N_3 \rangle}{\uparrow (\downarrow N_1 \uparrow \downarrow N_2)} \quad \frac{\langle \downarrow N_4 \rangle}{\dagger (\downarrow N_3 \uparrow \downarrow N_4)}}{\uparrow (\downarrow N_1 \uparrow \downarrow N_2) \dagger (\downarrow N_3 \uparrow \downarrow N_4)}$$

Similarly we can observe how an inversion phase looks like in general. Starting, still bottom-up, with a neutral sequent that contains a focus on a shifted negative formula, the focus is released and the negative formula is decomposed into sub-formulas that are again neutral. This piece of a derivation should look as follows in the classical case:

$$\begin{array}{c}
 \left\{ \Gamma, \uparrow P_{i1}, \dots, \uparrow P_{in_i} \right\}_{1 \leq i \leq j} \quad \left\{ \frac{\bar{\tau}}{\Gamma, \Sigma_i, \bar{\tau}} \right\} \\
 \hline
 \text{neg} \\
 \hline
 \frac{\Gamma, N}{\Gamma, \langle \downarrow N \rangle}
 \end{array}$$

where again the brackets indicate sets of premisses, the first one being the open premisses, the second one the premisses that are closed by a  $\bar{\tau}$  rule. Here there is no choice involved, all the positive sub-formulas of  $N$  are present in the premisses. Furthermore, the context is not constrained by the presence or not of atomic formulas. This gives the following abstract rule for a synthetic negative connective:

$$\text{neg} \frac{\{\Gamma, \uparrow P_{i1}, \dots, \uparrow P_{in_i}\}_{1 \leq i \leq j}}{\Gamma, \langle \downarrow N \rangle}$$

For example if  $N = (\uparrow P_1 \bar{\wedge} \uparrow P_2) \bar{\vee} (\uparrow P_3 \bar{\wedge} \uparrow P_4)$ , there can be only one synthetic rule for the synthetic connective  $(\cdot \bar{\wedge} \cdot) \bar{\vee} (\cdot \bar{\wedge} \cdot)$ :

$$(\cdot \bar{\wedge} \cdot) \bar{\vee} (\cdot \bar{\wedge} \cdot) \frac{\uparrow P_1, \uparrow P_3 \quad \uparrow P_1, \uparrow P_4 \quad \uparrow P_2, \uparrow P_3 \quad \uparrow P_2, \uparrow P_4}{\downarrow (\uparrow P_1 \bar{\wedge} \uparrow P_2) \bar{\vee} (\uparrow P_3 \bar{\wedge} \uparrow P_4)}$$

Now, we can moreover see that it would be possible to plug in one of the **neg** synthetic rule on top of each premiss of the synthetic rule **pos**. Following Andreoli [3], such a pair of a focus phase and an inversion phase is called a *bipole*, with the bottom phase corresponding to the application of a **pos** synthetic rule, and the top phase to the adequate set of **neg** synthetic rules. For example starting with the formula  $P = (\uparrow P_1 \bar{\wedge} \uparrow P_2) \bar{\vee} (\downarrow \uparrow P_3 \bar{\wedge} \downarrow (\uparrow P_4 \bar{\vee} \uparrow P_5))$ , we can obtain one of the two following bipoles:

$$\text{pos} \frac{\text{neg} \frac{\uparrow P_0, \uparrow P_1 \quad \uparrow P_2}{\langle \downarrow ((\uparrow P_0 \bar{\vee} \uparrow P_1) \bar{\wedge} \uparrow P_2) \rangle}}{\downarrow ((\uparrow P_0 \bar{\vee} \uparrow P_1) \bar{\wedge} \uparrow P_2) \bar{\vee} (\downarrow \uparrow P_3 \bar{\wedge} \downarrow (\uparrow P_4 \bar{\vee} \uparrow P_5))}} \quad \text{pos} \frac{\text{neg} \frac{\uparrow P_3}{\langle \downarrow \uparrow P_3 \rangle} \quad \text{neg} \frac{\uparrow P_4, \uparrow P_5}{\langle \downarrow (\uparrow P_4 \bar{\wedge} \uparrow P_5) \rangle}}{\downarrow (\uparrow P_1 \bar{\wedge} \uparrow P_2) \bar{\vee} (\downarrow \uparrow P_3 \bar{\wedge} \downarrow (\uparrow P_4 \bar{\vee} \uparrow P_5))}}$$

A bipole is therefore a derivation  $\mathcal{D}$  such that, its root is a neutral sequent that is the conclusion of a  $\uparrow$  rule application; its leaves are either closed by an identity rule, a  $\dagger$  or a  $\bar{\tau}$  rule, or open on neutral sequents with no focus; and all the non-invertible rule applications occur below any invertible rule application.

Zeilberger [139] has proposed a way to describe formally synthetic rules and has designed a *synthetic* proof system (for intuitionistic propositional logic) that can only build proofs from these macro rules. This might certainly not be ideal in terms of complexity. The number of rules in the positive case and the number of premisses in the negative case would in fact grow exponentially as a function of the number of  $\wedge/\vee$  alternations in a given synthetic connective (as we could see on our examples). However, the applications we are going to explore in the rest of this thesis take place at a more abstract proof-theoretical level: we will use the notion of synthetic rules in two different ways.

In the rest of this part, we consider the rules of the modal proof systems introduced in Part I (namely labelled sequent and ordinary sequent rules), and *unfold* their behaviour as synthetic rules. This means that we can see some standard rules of modal proof systems as macro-rules over a more fine grained system (namely LKF or LJF) and this informs us about some of the internal mechanics of these rules. In Part III, on the other hand, we will take the rules from modal proof systems of Part I (namely nested sequent rules) and, considering them as micro rules this time, *fold* them together in bunch in order to extend the systems of [139] to modal logics. This allows for a simplification of the design of the nested sequent systems, as well as of the study of their meta-theoretical properties.

## Unfolding labelled deduction via first-order logic

In [96], Miller and Volpe show how to emulate the classical labelled sequent system  $\text{labK}$  [101] using the focused calculus for first-order classical logic LKF [83]. They propose an encoding of classical modal formulas into polarised first-order formulas that allows them to obtain a precise correspondence between each rule of  $\text{labK}$ , along with any of its extension by geometric axioms, and a bipole in LKF.

In this chapter, we extend the results of [96] to the intuitionistic case; that is, we show that we can emulate the intuitionistic labelled calculus  $\text{labIK}$  of [127] via the focused calculus for first-order intuitionistic logic LJF [83]. We extend the polarisation proposed in [96] to the intuitionistic syntax and the two-sided calculus. Our notations differ as we are using an explicitly polarised syntax, but the proofs are basically identical, sometimes even simpler thanks to this choice. As in the classical case, the most interesting cases are the ones for modalities, the only point where the use of first-order logic is important. We furthermore consider extensions of intuitionistic modal logic  $\text{IK}$  with geometric frame conditions, and show that in the intuitionistic case as well they can be polarised such as to correspond to a bipole.

### 5.1 From labelled modal to focused first-order

The idea of *unfolding* the labelled system for modal logic into the focused system for first-order logic is to see each rule of  $\text{labK}$  or  $\text{labIK}$  as a *macro* rule composed of *micro* focused rules in LKF or LJF respectively. It relies heavily on the fact that modal logic can actually be expressed as a fragment of first-order logic via what is called as the *standard translation* (see e.g., [10]). The key point is the way the modalities are interpreted, following the semantics presented in Section 2.3,  $\Box$  corresponding to a universal quantification  $\forall$  and  $\Diamond$  to an existential  $\exists$ .

We recall the result of [96] for the classical case. It makes use of polarised version of the standard translation to interpret every classical labelled sequent  $S := \mathcal{G} \Rightarrow \mathcal{R}$  as a LKF sequent, and then to transform each proof in  $\text{labK}$  as a proof in LKF.

**5.1.1 Theorem** (Miller and Volpe [96]) *There exists a proof  $\mathcal{D}$  in the labelled system  $\text{labK}$  of a sequent  $S$  if and only if there is a proof  $\mathcal{D}'$  in LKF of the polarised translation of  $S$  with an exact correspondence of each rule application in  $\mathcal{D}$  and each bipole in  $\mathcal{D}'$ .*

We are now going to adapt this result and its proof to the intuitionistic case. We first present a translation from the language of  $\text{labIK}$  into the language of LJF, which is based on the standard translation for modal formulas, but polarised for our purposes. Given a world  $x$ , we define respectively a *left translation*  $[\cdot]_x$  and a *right translation*  $[\cdot]_x$  from modal formulas into polarised neutral first-order formulas as follows, where for each propositional atom  $a$ , we pick a positive predicate  $p_a(x)$  to correspond to  $a$  and  $r(x, y)$  is a positive

predicate that is only used to correspond to relational atoms:

$$\begin{array}{ll}
[a]_x = p_a(x) & [a]_x = \uparrow p_a(x) \\
[\top]_x = \downarrow \uparrow^\dagger & [\top]_x = \uparrow^\dagger \\
[\perp]_x = \downarrow \uparrow \perp & [\perp]_x = \uparrow \perp \\
[\Box A]_x = \downarrow \forall y. (r(x, y) \supset \uparrow [A]_y) & [\Box A]_x = \uparrow \downarrow \forall y. (r(x, y) \supset [A]_y) \\
[A \wedge B]_x = \downarrow \uparrow ([A]_x \wedge [B]_x) & [A \wedge B]_x = \uparrow (\downarrow [A]_x \wedge \downarrow [B]_x) \\
[\Diamond A]_x = \downarrow \uparrow \exists y. (r(x, y) \wedge [A]_y) & [\Diamond A]_x = \uparrow \exists y. (r(x, y) \wedge \downarrow [A]_y) \\
[A \vee B]_x = \downarrow \uparrow ([A]_x \vee [B]_x) & [A \vee B]_x = \uparrow (\downarrow [A]_x \vee \downarrow [B]_x) \\
[A \supset B]_x = \downarrow (\downarrow [A]_x \supset \uparrow [B]_x) & [A \supset B]_x = \uparrow \downarrow ([A]_x \supset [B]_x)
\end{array}$$

We extend this translation from formulas to labelled sequents. As seen in Section 3.4, an intuitionistic labelled sequent is of the shape  $\mathcal{G}, \mathcal{L} \Rightarrow z : B$ , where  $\mathcal{G}$  is a set of relational atoms e.g.  $xRy$  for some labels  $x$  and  $y$ ,  $\mathcal{L}$  is a multiset of labelled formulas  $x : A$  with  $x$  a label and  $A$  a modal formula, and finally  $z : B$  is a distinguished labelled formula, separated from the others by the  $\Rightarrow$  symbol to give an intuitionistic reading to the sequent. We write:

$$\begin{array}{l}
[\mathcal{G}] \text{ for } \{r(x, y) \mid xRy \in \mathcal{G}\} \\
[\mathcal{L}] \text{ for } \{[A]_x \mid x : A \in \mathcal{L}\}
\end{array}$$

Then, we can translate the labelled sequent  $S := \mathcal{G}, \mathcal{L} \Rightarrow z : B$  into the LJF neutral sequent  $[\mathcal{G}], [\mathcal{L}], [B]_z$  that we denote  $[S]$ .

The first direction of the encoding, from labIK to LJF, can then be formulated as the following theorem.

**5.1.2 Theorem** *Let  $\mathcal{D}$  be a proof in the labelled system labIK of a sequent  $S$ . There exists a proof  $\mathcal{D}'$  in LJF of the polarised neutral sequent  $[S]$  such that each rule application in  $\mathcal{D}$  corresponds to a bipole in  $\mathcal{D}'$ .*

*Proof.* The idea of the proof is to transform the labIK derivation bottom-up, by swapping each rule  $r \frac{\{S_i\}_{i \leq n}}{S}$  of labIK (with  $n \leq 2$ ) for a derivation  $\frac{\{[S_i]\}_{i \leq n}}{\parallel} [S]$  which is a bipole in LJF.

At each step, we get LJF sequents as leaves that are the translations of the premisses of the labIK rule, so we can continue up to the identity leaves of the proof.

For each rule of labIK, the corresponding bipole in LJF is obtained as follows:

- For  $r = \text{id}^{\text{lab}} \frac{}{\mathcal{G}, \mathcal{L}, x : a \Rightarrow x : a}$

$$\frac{\text{id}_R \frac{}{[\mathcal{G}], [\mathcal{L}], p_a(x), [a]_x, \langle p_a(x) \rangle}}{\uparrow_R \frac{}{[\mathcal{G}], [\mathcal{L}], [a]_x, [a]_x}}$$

- For  $r = \perp_L^{\text{lab}} \frac{}{\mathcal{G}, \mathcal{L}, x : \perp \Rightarrow z : A}$

$$\frac{\perp_L \frac{}{[\mathcal{G}], [\mathcal{L}], [\perp]_x, \perp, [A]_z}}{\uparrow_L \frac{}{[\mathcal{G}], [\mathcal{L}], [\perp]_x, \langle \uparrow \perp \rangle, [A]_z}}{\downarrow_L \frac{}{[\mathcal{G}], [\mathcal{L}], [\perp]_x, [A]_z}}$$

- For  $r = \uparrow_R^{\text{lab}} \frac{}{\mathcal{G}, \mathcal{L} \Rightarrow x : \top}$

$$\uparrow \frac{\uparrow \frac{[\mathcal{G}], [\mathcal{L}], [\top]_x, \langle \uparrow \rangle}{[\mathcal{G}], [\mathcal{L}], [\top]_x}}{[\mathcal{G}], [\mathcal{L}], [\top]_x}}$$

- For  $r = \wedge_L^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \wedge B, x : A, x : B \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : A \wedge B \Rightarrow z : C}$

$$\downarrow_L \frac{\uparrow_L \frac{\uparrow \frac{\uparrow \frac{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, [A]_x, [B]_x, [C]_z}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, [A]_x \uparrow [B]_x, [C]_z}}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, \langle \uparrow([A]_x \uparrow [B]_x) \rangle, [C]_z}}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, [C]_z}}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, [C]_z}}$$

- For  $r = \wedge_R^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow x : A \quad \mathcal{G}, \mathcal{L} \Rightarrow x : B}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \wedge B}$

$$\uparrow_R \frac{\downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [A]_x}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, \langle \downarrow [A]_x \rangle} \quad \downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [B]_x}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, \langle \downarrow [B]_x \rangle}}{\uparrow \frac{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x, \langle \downarrow [A]_x \uparrow \downarrow [B]_x \rangle}{[\mathcal{G}], [\mathcal{L}], [A \wedge B]_x}}$$

- For  $r = \vee_L^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \vee B, x : A \Rightarrow z : C \quad \mathcal{G}, \mathcal{L}, x : A \vee B, x : B \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : A \vee B \Rightarrow z : C}$

$$\downarrow_L \frac{\uparrow_L \frac{\uparrow \frac{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, [A]_x, [C]_z}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, [A]_x \vee [B]_x, [C]_z}}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \uparrow([A]_x \vee [B]_x) \rangle, [C]_z}}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, [C]_z}}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, [C]_z}}$$

- For  $r = \vee_{R1}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow x : A}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \vee B}$

$$\uparrow_R \frac{\downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [A]_x}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \downarrow [A]_x \rangle} \quad \downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [B]_x}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \downarrow [B]_x \rangle}}{\uparrow \frac{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \downarrow [A]_x \vee \downarrow [B]_x \rangle}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x}}$$

- For  $r = \vee_{R2}^{\text{lab}} \frac{\mathcal{G}, \mathcal{L} \Rightarrow x : B}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \vee B}$

$$\uparrow_R \frac{\downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [B]_x}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \downarrow [B]_x \rangle} \quad \downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [A]_x}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \downarrow [A]_x \rangle}}{\uparrow \frac{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x, \langle \downarrow [A]_x \vee \downarrow [B]_x \rangle}{[\mathcal{G}], [\mathcal{L}], [A \vee B]_x}}$$

- For  $r = \supset_L^{\text{lab}} \frac{\mathcal{G}, \mathcal{L}, x : A \supset B \Rightarrow x : A \quad \mathcal{G}, \mathcal{L}, x : A \supset B, x : B \Rightarrow z : C}{\mathcal{G}, \mathcal{L}, x : A \supset B \Rightarrow z : C}$

$$\downarrow_L \frac{\downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, [A]_x}{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, \langle \downarrow [A]_x \rangle, [C]_z} \quad \uparrow_L \frac{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, [B]_x, [C]_z}{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, \langle \uparrow [B]_x \rangle, [C]_z}}{\downarrow \frac{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, \langle \downarrow [A]_x \supset \uparrow [B]_x \rangle, [C]_z}{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, [C]_z}}$$

- For  $r = \supset_R^{\text{lab}}$   $\frac{\mathcal{G}, \mathcal{L}, x : A \Rightarrow x : B}{\mathcal{G}, \mathcal{L} \Rightarrow x : A \supset B}$

$$\frac{\supset_R \frac{[\mathcal{G}], [\mathcal{L}], [A]_x, [B]_x}{[\mathcal{G}], [\mathcal{L}], [A]_x \supset [B]_x}}{\downarrow_R \frac{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x, \langle \downarrow ([A]_x \supset [B]_x) \rangle}{\uparrow_R \frac{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x}{[\mathcal{G}], [\mathcal{L}], [A \supset B]_x}}}}$$

- For  $r = \square_L^{\text{lab}}$   $\frac{\mathcal{G}, xRy, \mathcal{L}, x : \square A, y : A \Rightarrow z : B}{\mathcal{G}, xRy, \mathcal{L}, x : \square A \Rightarrow z : B}$

$$\frac{\text{id}_L \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, \langle r(x, y) \rangle, [B]_z}{\downarrow_L \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, \langle r(x, y) \rangle, [B]_z}{\uparrow_L \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, [A]_y, [B]_z}{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, \langle \uparrow [A]_y \rangle, [B]_z}}}}}{\forall_L \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, \langle r(x, y) \supset \uparrow [A]_y \rangle, [B]_z}{\downarrow_L \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, \langle \forall y. (r(x, y) \supset \uparrow [A]_y) \rangle, [B]_z}{[\mathcal{G}], r(x, y), [\mathcal{L}], [\square A]_x, [B]_z}}}}}}$$

- For  $r = \square_R^{\text{lab}}$   $\frac{\mathcal{G}, xRy, \mathcal{L} \Rightarrow y : A}{\mathcal{G}, \mathcal{L} \Rightarrow x : \square A}$   $y$  is fresh

$$\frac{\supset_R \frac{[\mathcal{G}], [\mathcal{L}], r(x, y), [A]_y}{[\mathcal{G}], [\mathcal{L}], r(x, y) \supset [A]_y}}{\forall_R \frac{[\mathcal{G}], [\mathcal{L}], \forall y. (r(x, y) \supset [A]_y)}{[\mathcal{G}], [\mathcal{L}], [\square A]_x, \langle \downarrow \forall y. (r(x, y) \supset [A]_y) \rangle}} \frac{y \text{ is fresh}}{\uparrow_R \frac{[\mathcal{G}], [\mathcal{L}], [\square A]_x}{[\mathcal{G}], [\mathcal{L}], [\square A]_x}}}}$$

- For  $r = \diamond_L^{\text{lab}}$   $\frac{\mathcal{G}, xRy, \mathcal{L}, x : \diamond A, y : A \Rightarrow z : B}{\mathcal{G}, \mathcal{L}, x : \diamond A \Rightarrow z : B}$   $y$  is fresh

$$\frac{\uparrow_L \frac{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, r(x, y), [A]_y, [B]_z}{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, r(x, y) \wedge [A]_y, [B]_z}}{\exists_L \frac{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, \exists y. (r(x, y) \wedge [A]_y), [B]_z}{\uparrow_L \frac{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, \langle \uparrow \exists y. (r(x, y) \wedge [A]_y) \rangle, [B]_z}{\downarrow_L \frac{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, [B]_z}{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, [B]_z}}}}}} \frac{y \text{ is fresh}}{\downarrow_L \frac{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, [B]_z}{[\mathcal{G}], [\mathcal{L}], [\diamond A]_x, [B]_z}}}}$$

- For  $r = \diamond_R^{\text{lab}}$   $\frac{\mathcal{G}, xRy, \mathcal{L} \Rightarrow y : A}{\mathcal{G}, xRy, \mathcal{L} \Rightarrow x : \diamond A}$

$$\frac{\uparrow_R \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle r(x, y) \rangle}{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle \downarrow [A]_y \rangle}}{\exists_R \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle r(x, y) \wedge \downarrow [A]_y \rangle}{\uparrow_R \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle \exists y. (r(x, y) \wedge \downarrow [A]_y) \rangle}{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x}}}} \frac{\downarrow_R \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [A]_y}{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle \downarrow [A]_y \rangle}}{\downarrow_R \frac{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle \downarrow [A]_y \rangle}{[\mathcal{G}], r(x, y), [\mathcal{L}], [\diamond A]_x, \langle \downarrow [A]_y \rangle}}}}$$

⊙

So we see here how each single rule from **labIK** unfolds into a derivation of LJF. Such a derivation however is always a bipole, which suggests to inspect the adequacy to this unfolding. In the next section, we show that indeed any bipole proof of a first-order polarised formula which is the translation of a labelled modal formula actually corresponds to the application of a rule of **labIK**. That is, the encoding of **labIK** into LJ is adequate at the level of proofs.

## 5.2 From focused first-order to labelled modal

The second side of the encoding, from LJF to labIK can then be stated as follows.

**5.2.1 Theorem** *Let  $\mathcal{D}'$  be a proof of an LJF sequent that is the translation  $\lceil S \rceil$  of some intuitionistic labelled sequent  $S$ . There exists a proof  $\mathcal{D}$  of  $S$  in labIK such that each bipole in  $\mathcal{D}'$  corresponds to a single rule application in  $\mathcal{D}$ .*

We transform the LJF proof into a labIK proof by mapping each bipole that composes it to a corresponding rule in labIK. The proof  $\mathcal{D}'$  has to start, looking bottom-up, by a decision on some formula which is either  $\lceil C \rceil_x$  or  $\lceil C \rceil_x$  for some given  $x$  and  $C$ . Depending on which formula is chosen as focus, the proof will then be forced to unfold according to one of the bipoles we described in the proof of Theorem 5.1.2.

For instance, if the decision is on a formula of the form  $\lceil \Diamond A \rceil_x$ , then the bottom of  $\mathcal{D}'$  will have to look like:

$$\frac{\frac{\frac{\frac{\frac{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, r(x, y), \lceil A \rceil_y, \lceil B \rceil_z}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, r(x, y) \wedge \lceil A \rceil_y, \lceil B \rceil_z}}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \exists y. (r(x, y) \wedge \lceil A \rceil_y), \lceil B \rceil_z} \exists_L \text{ } y \text{ is fresh}}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \langle \exists y. (r(x, y) \wedge \lceil A \rceil_y) \rangle, \lceil B \rceil_z} \uparrow_L}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \lceil B \rceil_z} \downarrow_L$$

which can be replaced by the rule  $\diamond_L^{\text{lab}} \frac{xRy, \mathcal{G}, \mathcal{L}, x : \Diamond A, y : A \Rightarrow z : B}{\mathcal{G}, \mathcal{L}, x : \Diamond A \Rightarrow z : B}$  where  $y$  is fresh.

Similarly, if the decision is on a formula of the form  $\lceil \Diamond A \rceil_x$ , then the bottom of  $\mathcal{D}'$  will have to look like:

$$\frac{\frac{\frac{\frac{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \langle r(x, y) \rangle}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \langle \downarrow \lceil A \rceil_y \rangle} \downarrow_R}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \langle r(x, y) \wedge \downarrow \lceil A \rceil_y \rangle} \exists_R}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x, \langle \exists y. (r(x, y) \wedge \downarrow \lceil A \rceil_y) \rangle} \uparrow_R}{\lceil \mathcal{G} \rceil, \lceil \mathcal{L} \rceil, \lceil \Diamond A \rceil_x} \uparrow_R$$

However, for this derivation to be a valid bipole, the left premiss has to be closed by an application of  $\text{id}_R$  as the sequent contains a focus on the positive atom  $r(x, y)$ . Hence, as we started with  $\mathcal{D}'$  which is indeed a valid proof, we know that  $r(x, y)$  has to appear in  $\lceil \mathcal{G} \rceil$ .

Therefore, this bipole can be matched to the rule  $\diamond_R^{\text{lab}} \frac{xRy, \mathcal{G}, \mathcal{L} \Rightarrow y : A}{xRy, \mathcal{G}, \mathcal{L} \Rightarrow x : \Diamond A}$ .

There is one such case for each possible modal formula  $B$ , just by reverting the unfolding we described in the proof of Theorem 5.1.2, which ensures that the encoding is adequate.

## 5.3 Extensions

In this section, we want to investigate the same question of encoding labelled systems for modal logic into focused systems for first-order logic, but for modal logics beyond IK, namely the ones that are defined by geometric frame conditions. We recall from Section 3.4 that a *geometric frame property* is of the following form:

$$\forall \mathbf{u} \forall \mathbf{v} ((a_1 \wedge \dots \wedge a_n) \supset (\exists \mathbf{x}_1 (b_{11} \wedge \dots \wedge b_{1k_1}) \vee \dots \vee \exists \mathbf{x}_m (b_{m1} \wedge \dots \wedge b_{mk_m})))$$

where each  $a_i$  for  $1 \leq i \leq n$  and each  $b_{jl}$  for  $1 \leq j \leq m$ ,  $1 \leq l \leq k_j$ , is a relational or an equality atom,  $u_i R v_i$  or  $u_i = v_i$ , and the variables in  $\mathbf{x}_j$  do not appear among  $\mathbf{u}$  or  $\mathbf{v}$ ; the corresponding geometric rule scheme is:

$$\boxtimes_{\text{grs}}^{\text{lab}} \frac{A_1(y_1/x_1), a_1, \dots, a_n, \mathcal{G}, \mathcal{L} \Rightarrow z : C \quad \dots \quad A_n(y_n/x_n), a_1, \dots, a_n, \mathcal{G}, \mathcal{L} \Rightarrow z : C}{a_1, \dots, a_n, \mathcal{G}, \mathcal{L} \Rightarrow z : C}$$

where  $\mathcal{G}$  is a set of relational or equality atoms,  $\mathcal{L}$  is a multiset of labelled modal formulas, each  $a_i$  for  $1 \leq i \leq n$  is a relational or an equality atom, each  $\mathcal{A}_j$  for  $1 \leq j \leq m$  denotes the set of relational or equality atoms  $\{b_{j1}, \dots, b_{jk_j}\}$ , and the eigenvariables  $y_1, \dots, y_m$  are fresh, i.e. do not occur in the conclusion.

We propose the following polarisation for connectives and atoms of such a geometric frame property:

$$\downarrow \forall \mathbf{u} \forall \mathbf{v} ((a_1 \dagger \dots \dagger a_n) \supset (\exists \mathbf{x}_1 (b_{11} \dagger \dots \dagger b_{1k_1}) \vee \dots \vee \exists \mathbf{x}_m (b_{m1} \dagger \dots \dagger b_{mk_m}))) \quad (5.1)$$

with all the relational and equality atoms  $a_i$  and  $b_{jl}$  considered positive.

Via this polarisation, we show that each application of a rule  $\boxtimes_{\text{grs}}^{\text{lab}}$  corresponds to a single bipole in LJF. This fact ensures that the statements of Theorems 5.1.2 and 5.2.1 hold also for any geometric extension of IK with a set of geometric frame properties and its corresponding system  $\text{labIK} + \boxtimes_{\text{grs}}^{\text{lab}}$ .

**5.3.1 Theorem** *Let  $\mathcal{C}$  be a set of neutral polarised geometric frame conditions and  $\boxtimes_{\mathcal{C}}^{\text{lab}}$  be the corresponding set of geometric rule schemes as in (3.2). There exists a proof in  $\text{labIK} + \boxtimes_{\mathcal{C}}^{\text{lab}}$  of the sequent  $\mathcal{G}, \mathcal{L} \Rightarrow z : A$  if and only if there exists a proof in LJF of the sequent  $\mathcal{C}, [\mathcal{G}], [\mathcal{L}], [A]_z$ , with a strict correspondence between each rule of the proof in  $\text{labIK} + \boxtimes_{\mathcal{C}}^{\text{lab}}$  and each bipole in the proof in LJF.*

Consider one element of the set  $\mathcal{C}$ , for instance  $\downarrow \forall y. ((a_1 \dagger \dots \dagger a_n) \supset \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}))$ . When this formula is chosen as the focus, it is *processed* along a single bipole and gives rise to a macro rule that matches exactly the application of the corresponding geometric rule scheme.

$$\begin{array}{c} \left\{ \text{id}_R \frac{a_i \in [\mathcal{G}]}{\mathcal{C}, \langle a_i \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z} \right\}_{1 \leq i \leq n} \quad \vee_L \frac{\left\{ \begin{array}{l} \dagger_{\mathcal{L}} \frac{\mathcal{C}, (z_j/x_j)b_{j1}, \dots, (z_j/x_j)b_{jk_j}, [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, (z_j/x_j)(b_{j1} \dagger \dots \dagger b_{jk_j}), [\mathcal{G}], [\mathcal{L}], [A]_z} \\ \exists_L \frac{\mathcal{C}, \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}), [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}), [\mathcal{G}], [\mathcal{L}], [A]_z} \end{array} \right\}_{1 \leq j \leq m}}{\mathcal{C}, \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}), [\mathcal{G}], [\mathcal{L}], [A]_z} \\ \dagger_R \frac{\mathcal{C}, \langle a_1 \dagger \dots \dagger a_n \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, \langle a_1 \dagger \dots \dagger a_n \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z} \quad \uparrow_L \frac{\mathcal{C}, \langle \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, \langle \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z} \\ \supset_L \frac{\mathcal{C}, \langle (a_1 \dagger \dots \dagger a_n) \supset \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j}) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, \langle \forall y. ((a_1 \dagger \dots \dagger a_n) \supset \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j})) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z} \\ \forall_L \frac{\mathcal{C}, \langle \forall y. ((a_1 \dagger \dots \dagger a_n) \supset \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j})) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, \langle \forall y. ((a_1 \dagger \dots \dagger a_n) \supset \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j})) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z} \\ \downarrow_L \frac{\mathcal{C}, \langle \forall y. ((a_1 \dagger \dots \dagger a_n) \supset \uparrow \forall_{j=1}^m \exists x_j. (b_{j1} \dagger \dots \dagger b_{jk_j})) \rangle, [\mathcal{G}], [\mathcal{L}], [A]_z}{\mathcal{C}, [\mathcal{G}], [\mathcal{L}], [A]_z}$$

The results in this chapter can be seen as part of the *ProofCert* project [93, 94]. One of the products of this project is a general proof checker [28], based on LKF for its classical version and on LJF for its intuitionistic version. Its aim is to check formal proofs that are obtained from proof search tools based in many different formalisms. To achieve this, the basic focused systems have to be augmented with a *proof certificate* [29], which describes the proof evidence to be checked and the operational semantics of the proof formalism that was used to output it, which consists in a translation of the output language into the language of focused first-order logic and the notions of *expert* and *clerk*. Experts appear in the focus phase to extract information from the certificate and conduct the proof checking, while clerks appear in the inversion phase to perform simple procedures on the sequents without interacting with the certificate. Volpe and Libal have started implementing such certificates for some classical modal proof systems [84]; the results in this chapter allow  $\text{labIK}$  derivations to be described as first-order focused derivations, and therefore could be the basis for an extension of their work to the intuitionistic case in order to check proofs from the intuitionistic labelled calculus  $\text{labIK}$  within the augmented LJF.

## Unfolding unlabelled deduction via labelling

In this chapter we present our joint work with Miller and Volpe [86]; it is another follow-up on their paper [96]. With a little bit of magic, we can extend the results from the previous chapter to unfold unlabelled systems as well. Indeed, we saw that rules in the labelled systems, in particular geometric rule schemes, can be seen as synthetic inference rules that correspond to bipoles. So we unfold unlabelled systems into a focused labelled framework that can be used as a “springboard” to reach the base system LKF. In Section 6.1 we present a framework that will be general enough to emulate various unlabelled proof systems. In Section 6.2 we illustrate the way to use the framework on the example of sequent systems.

### 6.1 A focused labelled framework

In this section, we present the general framework  $\text{LMF}_*$ . It can be seen as a focused version of the labelled system  $\text{labK}$  [101] presented in Section 2.4, further augmented with some elements that help control the structure of proofs and, in particular, emulate the behaviour of existing modal proof systems.

In [96], Miller and Volpe derive a focused labelled system LMF for classical modal logics from the focused system for classical first-order logic. We can formulate it using an explicit polarised syntax; namely, in the classical setting, *polarised* modal formulas are build from the following grammar:

$$\begin{aligned} P &::= p \mid \downarrow N \mid P \uparrow P \mid \dagger \mid P \downarrow P \mid \ddagger \mid \diamond P \\ N &::= \bar{p} \mid \uparrow P \mid N \downarrow N \mid \bar{\dagger} \mid N \wedge N \mid \bar{\ddagger} \mid \square N \end{aligned}$$

Observe that  $\diamond$  is a positive connective like the existential quantifier  $\exists$  and  $\square$  is a negative connective like the universal quantifier  $\forall$ . We assume that all atoms are polarised positively. The *depolarisation* map is straightforwardly extended to modal formulas by  $\partial(\square N) = \square\partial(N)$  and  $\partial(\diamond P) = \diamond\partial(P)$ .

$\text{LMF}_*$  on Figure 6.1 is a reformulation of LMF with some additional features that do not affect the soundness of the system but allow us to constrain proofs to some specific shapes if we want.

1. When rule applications do not interfere with each other, they can, in fact, be applied in parallel. This justifies what is called *multi-focusing*, a variant of focusing where one can focus on several positive formulas at the same time [24, 21]. In this way, we can group several positive rules within a single phase.
2. In the focus phase, we use a generalisation of labels; each formula, instead of being labelled by a single label like in  $\text{labK}$ , is labelled by a sequence of labels that we write  $x\sigma$  (with  $x$  a label and  $\sigma$  a possibly empty sequence of labels), more similar to the prefixed tableaux approach [45]. We say that  $x$  is the *present* of the formula and  $\sigma$  is

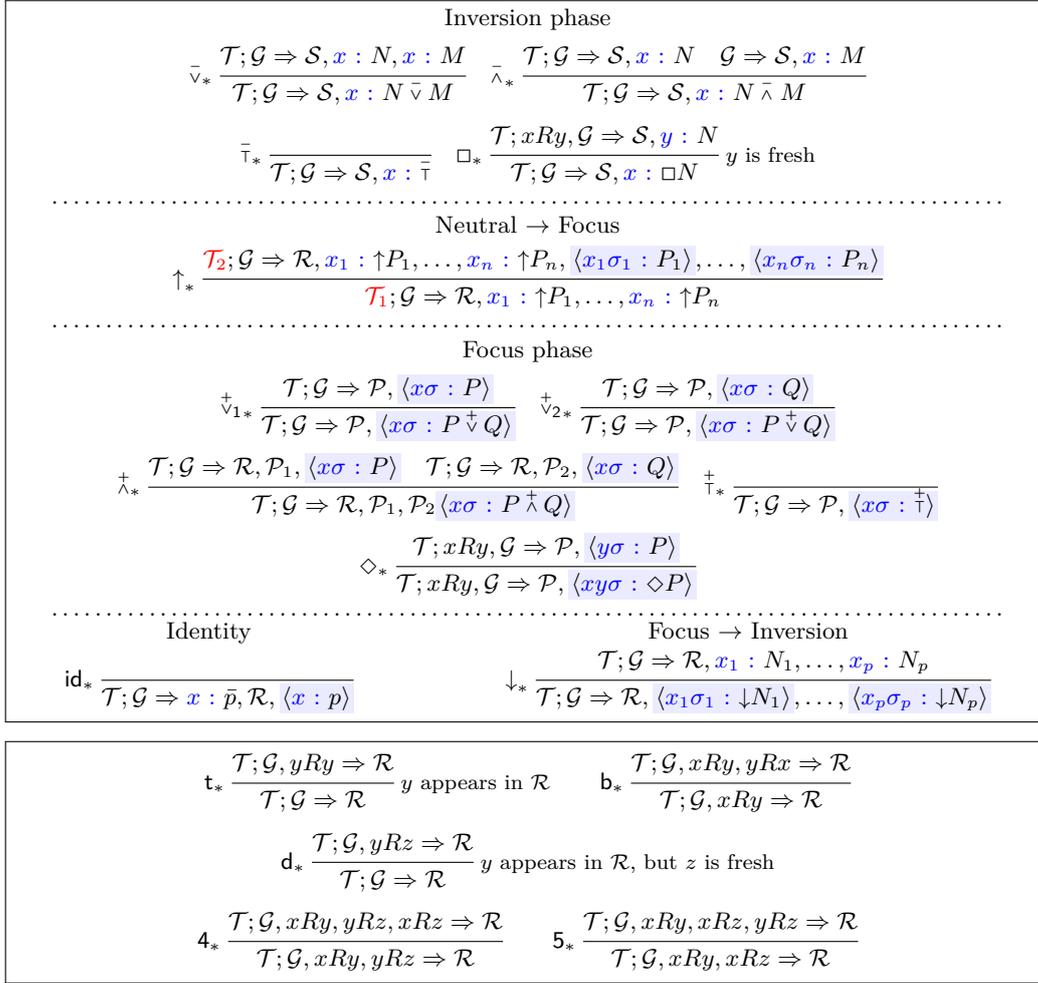


Figure 6.1: Framework LMF<sub>\*</sub>

its *future*. We note that for the emulation of the calculi presented in this chapter, a future consisting of a singleton is enough, but we want to keep this framework general to allow for capturing also other behaviours.

3. In each sequent, we have to keep track of what we call the *time context* denoted  $\mathcal{T}$ . In this chapter we will consider this context to be a pair  $(x, \mathcal{F})$  of a label  $x$ , the *present* of the sequent, and a set of labels  $\mathcal{F}$ , the *forbidden futures*, but it could however be generalised for further applications.
4. When applying  $\uparrow_*$ , if  $\mathcal{T}_1 = (x, \mathcal{F})$  for some  $\mathcal{F}$ , we can either (i) (multi)focus on non- $\diamond$ -formulas labelled with  $x$ , that is, the foci are of the form  $x\sigma : P$ , where  $P$  is not a  $\diamond$ -formula; or (ii) “move” to a  $y$  reachable from  $x$  (by (multi)focusing on  $\diamond$ -formulas) if  $y$  is not in the set  $\mathcal{F}$  of *forbidden futures* for  $x$ , that is, the foci are of the form  $zy\sigma : \diamond P$ , while  $xRy$  and  $zRy \in \mathcal{G}$ , but  $y \notin \mathcal{F}$ .
5. The time changes from  $\mathcal{T}_1$  to  $\mathcal{T}_2$  at an application of  $\uparrow_*$  (red in Figure 6.1) when a decision is taken to focus on some formulas. In this general formulation, the set  $\mathcal{T}_2$  that we get in the premiss of the rule can be defined in an arbitrary way; a specific way of defining it will be proposed in next section, but again it can be designed in order to obtain particular behaviours in proofs.

An  $\text{LMF}_*$  *sequent* has the form  $\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{S}$  where  $\mathcal{S}$  is a multiset of labelled negative formulas,  $\mathcal{G}$  is a set of relational atoms, and  $\mathcal{T}$  is the *time* of the sequent. As the framework is focused, we can consider the restricted class of *neutral* sequents of the shape  $\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{R}$  where  $\mathcal{R}$  is composed of neutral formulas, namely:

$$\mathcal{R} ::= \emptyset \mid \mathcal{R}, x : \uparrow P \mid \mathcal{R}, x : \bar{p}$$

A *focused* sequent contains furthermore (potentially several) occurrences of *foci*, so is of the shape  $\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{P}$  with:

$$\mathcal{P} ::= \mathcal{R} \mid \mathcal{P}, \langle x\sigma : P \rangle$$

The rules of the framework are mostly simple adaptations of the ones in  $\text{LMF}$ , which are focused versions of the rules of  $\text{labK}$ . The important differences are that: in  $\uparrow_*$ , we assign a future  $\sigma$  to the formulas  $x : A$  in the context, so that we actually focus on  $x\sigma : A$ ; when we apply the rule  $\diamond_*$  with respect to a formula  $xy\sigma : \diamond A$ , it is predetermined that we get  $y\sigma : A$  in the premiss; and finally since futures of formulas are only relevant during the focus phase, applications of  $\downarrow_*$  remove all such futures.

The parameter  $X$  is a subset of  $\{\mathbf{t}, \mathbf{b}, \mathbf{d}, 4, 5\}$ , specifying which modal logic we are considering. The system  $\text{LMF}_*^\emptyset$  is a system for the logic  $\mathbf{K}$  and is obtained by including only the first table of rules of Figure 6.1. Any other system  $\text{LMF}_*^X$  is obtained by adding to  $\text{LMF}_*^\emptyset$  the set of relational rules  $x_*$ , for each  $x \in X$ , in the second table of Figure 6.1.

**6.1.1 Proposition** *Let  $X$  be a subset of  $\{\mathbf{t}, \mathbf{b}, \mathbf{d}, 4, 5\}$  and  $N$  be a neutral modal formula. There is a proof of  $N$  in  $\text{LMF}_*^X$  if and only if  $\partial(N)$  is a theorem of  $\mathbf{K} + X$ .*

*Proof (Sketch).* The system  $\text{LMF}_*$  is a multi-focused version of the system  $\text{LMF}$  presented in [96], augmented with some devices for controlling the application of rules. Soundness follows from the fact that such devices can only introduce restrictions to the application of rules and multi-focusing can be simulated in  $\text{LMF}$  by several rule applications. Completeness is also a direct consequence of that of  $\text{LMF}$ , since in the liberal version presented here all new devices (including multi-focusing) can just be ignored, or used in a trivial way, such that each proof in the previous system is also a valid proof in  $\text{LMF}_*$ .  $\odot$

The idea behind this framework is that we can obtain different concrete proof systems by properly specifying the behaviour of the new devices introduced in  $\text{LMF}_*$ . These will be defined by specialising the rule  $\uparrow_*$ , i.e., in particular, by tuning the following parameters: the class of formulas on which multi-focusing can be applied in the conclusion, the future  $\sigma$  of foci, and the set  $\mathcal{T}_2$  in the premiss.

## 6.2 Emulation of sequent proof systems

In this section we illustrate the usage of  $\text{LMF}_*$  by specialising its various parameters to emulate the modal proof systems based on ordinary sequents, presented in Figure 2.2.

Intuitively, one can see modal inference rules such as  $\Box_k^\circ$  in the ordinary sequent calculus (reading from conclusion to premiss) as moving from one world to another reachable world in a suitable Kripke structure. Such a change of world becomes apparent when we consider the corresponding deduction steps in a labelled system, as in this case, modal rules will explicitly change the label of the formula under consideration. In order to properly mimic the behaviour of the ordinary sequent rules in the labelled system, we need to be able to force all the formulas involved in one rule to move to the same new world. This is where having labelled formulas with a *sequence* of labels comes in handy, since the rule  $\diamond_*$  exactly passes formulas to the future world indicated in the label.

In a labelled sequent system like  $\text{labK}$ , when constructing a proof tree, it is possible to switch freely from one label to another in the deduction process. On the contrary, in ordinary

sequent calculus rules, only formulas having a modal operator as the main connective can be “promoted” to a different world. This amounts to considering a single world at a time, in such a way that when moving to a new one, formulas standing at previously encountered worlds are no longer accessible. So the set  $\mathcal{T}$  that keeps track of the *time* of the sequent, specifies which world is currently under consideration and which worlds are enabled, with the intended meaning that we can decide on a formula only if its label is available, and we cannot return back to the future (that is, to the forbidden ones).

A particularly challenging aspect of the emulation of modal rules such as the  $\Box_k^o$  rule is that many introductions of connectives must be performed at once, which means that a single inference rule can correspond to more than one bipole in our emulation. We capture part of the parallel applications of inference rules using *multi-focusing*. This allows us to capture a modal rule via two synthetic inference rules: one of that performs the  $\Box$ -introduction (which corresponds to creating a new reachable world) and one that performs the  $\Diamond$ -introduction (which correspond to moving all the  $\Diamond$ -formulas to this newly created world).

We also specialise the rule  $\uparrow_*$  in order to emulate the behaviour of the ordinary sequent system  $\text{oK}$  and its extensions. In addition to the general conditions we gave earlier, we have that:

- if there exists  $y$  s.t.  $xRy \in \mathcal{G}$  then we can focus on a multiset of formulas of the form  $zy : \Diamond P$  s.t.  $zRy \in \mathcal{G}$ ,  $z \in \mathcal{F}$ , and if  $x \neq y$  then  $\mathcal{T}_2 = (y, \mathcal{F} \cup \{x\}) \neq \mathcal{T}_1$ , but if  $x = y$  then  $\mathcal{T}_2 = (x, \mathcal{F}) = \mathcal{T}_1$ ;

$$\uparrow_o \frac{\mathcal{T}_2; xRy, z_1Ry, \dots, z_nRy, \mathcal{G} \Rightarrow \mathcal{R}, z_1 : \uparrow \Diamond P_1, \dots, z_n : \uparrow \Diamond P_n, \langle z_1y : \Diamond P_1 \rangle, \dots, \langle z_ny : \Diamond P_n \rangle}{(x, \mathcal{F}); xRy, z_1Ry, \dots, z_nRy, \mathcal{G} \Rightarrow \mathcal{R}, z_1 : \uparrow \Diamond P_1, \dots, z_n : \uparrow \Diamond P_n}$$

- otherwise, we can choose a single focus, such that  $P$  is not a  $\Diamond$ -formula:

$$\uparrow_o \frac{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : \uparrow P, \langle x : P \rangle}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : \uparrow P}$$

Concretely, the specialisation with respect to the general framework consists in: restricting the use of multi-focusing to  $\Diamond$ -formulas, forcing such  $\Diamond$ -formulas to be labelled with the same future, and when moving to a new label, adding the current label to the set of forbidden futures. The structure of the proofs obtained by using these restrictions can be described as a sequence of ‘blocks’, each of which is related to a specific world/label. For each block, we first apply a number of  $\wedge$ ,  $\vee$ , and  $\Box$ -introductions in the current world (and some relational rules, if we are considering extensions of  $\text{K}$ ) and then move to a new world by means of a  $\Diamond$ -phase. The mechanism that we use, in  $\uparrow_o$ , for updating the time  $\mathcal{T}$  of the sequent ensures that we never go back to an already encountered world. We call  $\text{LMF}_o$  the system obtained from  $\text{LMF}_*$  by replacing the rule  $\uparrow_*$  with the rule  $\uparrow_o$ .

As we have seen in the previous chapter, the key to embed an unpolarised system into a focused system is the way we translate and polarise the formulas of the former in order to generate bipoles in the latter. So, we present a polarisation that allows us to enforce the behaviour of the ordinary sequent inference rules in our focused labelled framework:

$$\begin{aligned} [a]_o &:= \uparrow p_a & [\bar{a}]_o &:= \bar{p}_a \\ [A \wedge B]_o &:= \uparrow \downarrow ([A]_o \bar{\wedge} [B]_o) & [A \vee B]_o &:= \uparrow \downarrow ([A]_o \bar{\vee} [B]_o) \\ [\Box A]_o &:= \uparrow \downarrow \Box [A]_o & [\Diamond A]_o &:= \uparrow \Diamond \downarrow [A]_o \end{aligned}$$

where we associate a positively polarised atom  $p_a$  to any unpolarised atom  $a$ .

## Modal logics without the 4 axiom

In an  $\text{LMF}_*$  proof, we can encounter neutral sequents that do not correspond adequately to any sequent of a proof in ordinary sequent calculi for modal logic. We will thus base our correspondence results on an interpretation that takes this fact into account.

Let  $X = \emptyset$  or  $\{d\}$  or  $\{t\}$ . Formally, we define the interpretation  $I_{K+X}(\cdot)$  of neutral labelled sequents as multisets of modal formulas as follows:

$$I_{K+X}((x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}) = \{\partial(N) \mid x : N \in \mathcal{R}\} \cup \{\Box \partial(N) \mid y : N \in \mathcal{R}, xRy \in \mathcal{G}^*, y \notin \mathcal{F}\}$$

where  $\partial()$  is the depolarisation map defined in Section 4.1 and  $\mathcal{G}^*$  denotes the closure of  $\mathcal{G}$  taking into account the property of reflexivity or seriality that might be indicated by  $X$ .

**6.2.1 Theorem** (Marin, Miller and Volpe [86]) *Let  $X = \emptyset$  or  $\{d\}$  or  $\{t\}$ . Let  $\mathcal{D}$  be an  $\mathfrak{oK} + \diamond_X^\circ$  proof of a formula  $A$ . There exists an  $\text{LMF}_\circ^X$  proof  $\mathcal{D}'$  of  $(x, \emptyset); \emptyset \Rightarrow x : [A]_\circ$  for some label  $x$  such that each rule application in  $\mathcal{D}$  corresponds to one or two bipoles in  $\mathcal{D}'$ .*

This theorem is proved in a similar fashion to Theorem 5.1.2. For simplicity, we assume that in  $\mathcal{D}$  the rule  $c$  is only applied to a given formula immediately below a rule that introduces an occurrence of such a formula. An  $\text{LMF}_\circ$  derivation tends to keep information that is lost in the corresponding  $\mathfrak{oK} + \diamond_X^\circ$  derivation. We define a notion of *sequent extension* that will help compare the two systems. A neutral sequent  $\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{R}_1$  extends  $\Gamma$  if there exists  $\mathcal{R}_2 \supseteq \mathcal{R}_1$  such that  $I_{K+X}(\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{R}_2) = \Gamma$  with  $\mathcal{R}_1$  and  $\mathcal{R}_2$  containing only formulas of the form  $x : [A]_\circ$ . We proceed bottom-up by starting from the root of  $\mathcal{D}$  and build  $\mathcal{D}'$  by repeatedly applying the next lemma. At each step, we get as leaves sequents that are extensions of the ones in  $\mathcal{D}$ , so that the lemma can be applied again until we reach the initial sequent rules.

**6.2.2 Lemma** *Let  $X = \emptyset$  or  $\{d\}$  or  $\{t\}$ . Let  $r \frac{\{\Gamma_i\}_{i \leq n}}{\Gamma}$  be an application of a non-structural rule in  $\mathfrak{oK} + \diamond_X^\circ$  (with  $n \leq 2$ ). For any neutral sequent  $\mathcal{T}_1; \mathcal{G}_1 \Rightarrow \mathcal{R}$  that extends  $\Gamma$ , there exists a derivation  $\frac{\mathcal{T}_2; \mathcal{G}_2 \Rightarrow \mathcal{R}_i}{\mathcal{T}_1; \mathcal{G}_1 \Rightarrow \mathcal{R}}$  in  $\text{LMF}_\circ^X$ , such that each  $\mathcal{T}_2; \mathcal{G}_2 \Rightarrow \mathcal{R}_i$  extends  $\Gamma_i$ .*

*Proof.* The proof proceeds by considering all the non-structural rules of  $\mathfrak{oK} + \diamond_X^\circ$ . The cases for the axiom and the introduction of classical connectives are similar to the proof of Theorem 5.1.2.

Let us consider an application of the rule  $\Box_k^\circ \frac{\Gamma, A}{\diamond \Gamma, \Box A}$ .

Take a neutral sequent  $(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\diamond \Gamma, \Box A$ . It follows that  $x : [\diamond \Gamma]_\circ \subseteq \mathcal{R}$  and that either (a)  $x : [\Box A]_\circ \in \mathcal{R}$ ; or (b)  $y : [A]_\circ \in \mathcal{R}$  and  $xRy \in \mathcal{G}^*$ . Then the  $\text{LMF}_\circ$  derivation corresponding to this rule application consists in the following steps (reading the derivation bottom-up):

1. if we are in case (a), decide on  $x : [\Box A]_\circ$ , which has the consequence of adding  $xRy$  to the left-hand side:

$$\begin{array}{c} \Box_* \frac{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow x : [\diamond \Gamma]_\circ, y : [A]_\circ}{(x, \mathcal{F}); \mathcal{G} \Rightarrow x : [\diamond \Gamma]_\circ, x : [\Box A]_\circ} \\ \downarrow_* \frac{(x, \mathcal{F}); \mathcal{G} \Rightarrow x : [\diamond \Gamma]_\circ, \langle x : \downarrow [\Box A]_\circ \rangle}{(x, \mathcal{F}); \mathcal{G} \Rightarrow x : [\diamond \Gamma]_\circ, x : [\Box A]_\circ} \\ \uparrow_\circ \end{array}$$

or if we are in case (b), apply a series of relational rules (second table of Figure 6.1) in order to make  $xRy$  appear in the relational context too;

2. then, multifocus on  $x : [\diamond \Gamma]_\circ$  choosing  $y$  as the future:

$$\begin{array}{c} \downarrow_* \frac{(y, \{x\} \cup \mathcal{F}); xRy, \mathcal{G} \Rightarrow y : [\Gamma]_\circ, y : [A]_\circ}{(y, \{x\} \cup \mathcal{F}); xRy, \mathcal{G} \Rightarrow \langle y : \downarrow [\Gamma]_\circ \rangle, y : [A]_\circ} \\ \diamond_* \frac{(y, \{x\} \cup \mathcal{F}); xRy, \mathcal{G} \Rightarrow \langle xy : \diamond \downarrow [\Gamma]_\circ \rangle, y : [A]_\circ}{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow x : [\diamond \Gamma]_\circ, y : [A]_\circ} \\ \uparrow_\circ \end{array}$$

Let us consider an application of the rule  $\diamond_t^{\circ} \frac{\Gamma, A}{\Gamma, \diamond A}$

Take a neutral sequent  $(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\Gamma, \diamond A$ . We have that  $x : [\diamond A]_{\circ} \in \mathcal{R}$ . The  $\text{LMF}_{\circ}$  derivation corresponding to this rule application consists in the following bipole:

$$\begin{array}{c} \downarrow_* \frac{(x, \mathcal{F}); xRx, \mathcal{G} \Rightarrow \mathcal{R}, x : [A]_{\circ}}{(x, \mathcal{F}); xRx, \mathcal{G} \Rightarrow \mathcal{R}, \langle x : \downarrow [A]_{\circ} \rangle} \\ \diamond_*^* \frac{(x, \mathcal{F}); xRx, \mathcal{G} \Rightarrow \mathcal{R}, \langle xx : \diamond \downarrow [A]_{\circ} \rangle}{(x, \mathcal{F}); xRx, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond A]_{\circ}} \\ \uparrow_{\circ} \frac{(x, \mathcal{F}); xRx, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond A]_{\circ}}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond A]_{\circ}} \\ \text{t}_*^* \end{array}$$

Let us consider an application of the rule  $\diamond_d^{\circ} \frac{\Gamma}{\diamond \Gamma}$

Take a neutral sequent  $\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\diamond \Gamma$ , which means that  $x : [\diamond \Gamma]_{\circ} \subseteq \mathcal{R}$ . The  $\text{LMF}_{\circ}$  derivation corresponding to this rule application consists in the following bipole:

$$\begin{array}{c} \downarrow_* \frac{(y, \{x\} \cup \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, y : [\Gamma]_{\circ}}{(y, \{x\} \cup \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle y : \downarrow [\Gamma]_{\circ} \rangle} \\ \diamond_*^* \frac{(y, \{x\} \cup \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle xy : \diamond \downarrow [\Gamma]_{\circ} \rangle}{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond \Gamma]_{\circ}} \\ \uparrow_{\circ} \frac{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond \Gamma]_{\circ}}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}} \text{ y is fresh} \\ \text{d}_*^* \end{array} \quad \odot$$

## Modal logics with the 4 axiom

In the case of logics whose frames enjoy transitivity, our interpretation will also have to consider that in the rule  $\square_{k4}^{\circ}$  and  $\square_{k45}^{\circ}$ ,  $\diamond$ -formulas stay in the sequent when going from conclusion to premiss, and such a behaviour can only be captured in  $\text{LMF}_{\circ}$  by applying more than one step. The interpretation of neutral sequents has to be modified in this context, so we define for  $X \subseteq \{d, t, 5\}$ :

$$\begin{aligned} I_{K4+X}((x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}) = & \{A \mid x : [A]_{\circ} \in \mathcal{R}\} \cup \{\square B \mid y : [B]_{\circ} \in \mathcal{R}, xRy \in \mathcal{G}^*, y \notin \mathcal{F}\} \\ & \cup \{\diamond C \mid z : [\diamond C]_{\circ} \in \mathcal{R}, zRx \in \mathcal{G}^*, z \in \mathcal{F}\} \end{aligned}$$

where  $\mathcal{G}^*$  denotes the closure of  $\mathcal{G}$  taking into account the property of euclideaness, reflexivity or seriality that might be indicated by  $X$ .

**6.2.3 Theorem** (Marin, Miller and Volpe [86]) *Let  $X = \emptyset$  or  $\{d\}$  or  $\{t\}$ . Let  $\mathcal{D}$  be a proof of a formula  $A$  in  $\text{oK4} + \diamond_X^{\circ}$  or  $\text{oK45} + \diamond_X^{\circ}$ . There exists a proof  $\mathcal{D}'$  in  $\text{LMF}_{\circ}^{4+X}$  or in  $\text{LMF}_{\circ}^{4+5+X}$  respectively of  $(x, \emptyset); \emptyset \Rightarrow x : [A]_{\circ}$  for some label  $x$ .*

This theorem is proved in exactly the same way as Theorem 6.2.1. We just have to adapt the notion of sequent extension to the new interpretation, namely a neutral sequent  $\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{R}_1$  extends  $\Gamma$  if there exists  $\mathcal{R}_2 \supseteq \mathcal{R}_1$  such that  $I_{K4+X}(\mathcal{T}; \mathcal{G} \Rightarrow \mathcal{R}_2) = \Gamma$  with  $\mathcal{R}_1$  and  $\mathcal{R}_2$  containing only formulas of the form  $x : [A]_{\circ}$ . Then, we need to adapt Lemma 6.2.2 to this context with 4.

**6.2.4 Lemma** *Let  $X = \emptyset$  or  $\{d\}$  or  $\{t\}$ . Let  $r \frac{\{\Gamma_i\}_{i \leq n}}{\Gamma}$  be an application of a non-structural rule in  $\text{oK4} + \diamond_X^{\circ}$  or in  $\text{oK45} + \diamond_X^{\circ}$  (with  $n \leq 2$ ). For any neutral sequent  $\mathcal{T}_1; \mathcal{G}_1 \Rightarrow \mathcal{R}$  that extends  $\Gamma$ , there exists a derivation  $\{\mathcal{T}_2; \mathcal{G}_2 \Rightarrow \mathcal{R}_i\}_{i \leq n}$  in  $\text{LMF}_{\circ}^{4+X}$  or in  $\text{LMF}_{\circ}^{4+5+X}$ , such that each  $\mathcal{T}_2; \mathcal{G}_2 \Rightarrow \mathcal{R}_i$  extends  $\Gamma_i$ .  $\mathcal{T}_1; \mathcal{G}_1 \Rightarrow \mathcal{R}$*

*Proof.* The only cases that are affected by the presence of 4 are the modal rules.

Let us consider an application of the rule  $\square_{k4}^{\circ} \frac{\diamond \Gamma, \Gamma, A}{\diamond \Gamma, \diamond A}$ .

Take  $(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\diamond\Gamma, \square A$ . As in the previous proof, we can have either (a)  $x : [\square A]_o \in \mathcal{R}$ ; or (b)  $y : [A]_o \in \mathcal{R}$  and  $xRy \in \mathcal{G}^*$ . Moreover, here for each  $B \in \Gamma$ , either (c)  $x : [\diamond B]_o \in \mathcal{R}$ ; or (d)  $z : [\diamond B]_o \in \mathcal{R}$  and  $zRx \in \mathcal{G}^*$  for some  $z$ . The  $\text{LMF}_o$  derivation corresponding to this rule application consists in the following bipoles:

1. if we are in case (a), decide on  $x : [\square A]_o$ , which adds  $xRy$  to the left-hand side:

$$\begin{array}{c} \frac{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, y : [A]_o}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : \square[A]_o} \\ \downarrow_* \\ \frac{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, \langle x : \downarrow \square[A]_o \rangle}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : [\square A]_o} \\ \uparrow_o \end{array}$$

and if we are in case (b), apply some relational rules leading to a sequent containing  $xRy$ .

2. for the  $B_i \in \Gamma$  such that case (d) holds, apply some relational rules that lead to a sequent containing the corresponding  $z_iRx$ ; then apply  $4_*$  to each  $z_iRx$  and  $xRy$  such that  $z_iRy$  is added to the relational set; finally multifocus on all the  $w : [\diamond B]_o$  such that  $wRy$  is in the relation set and  $B \in \Gamma$ , choosing  $y$  as the future:

$$\begin{array}{c} \downarrow_* \frac{y, \{x\} \cup \mathcal{F}; z_iRx, xRy, z_iRy, \mathcal{G} \Rightarrow \mathcal{R}, y : [\Gamma_0]_o, \{y : [B_i]_o\}}{(y, \{x\} \cup \mathcal{F}); z_iRx, xRy, z_iRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle y : \downarrow [\Gamma_0]_o \rangle, \{ \langle y : \downarrow [B_i]_o \rangle \}} \\ \diamond_* \frac{(y, \{x\} \cup \mathcal{F}); z_iRx, xRy, z_iRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle x : \diamond \downarrow [\Gamma_0]_o \rangle, \{ \langle z_i : \diamond \downarrow [B_i]_o \rangle \}}{(x, \mathcal{F}); z_iRx, xRy, z_iRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond \Gamma_0]_o, \{z_i : [\diamond B_i]_o\}} \\ \uparrow_o \\ 4_* \frac{(x, \mathcal{F}); z_iRx, xRy, z_iRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond \Gamma_0]_o, \{z_i : [\diamond B_i]_o\}}{(x, \mathcal{F}); z_iRx, xRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\diamond \Gamma_0]_o, \{z_i : [\diamond B_i]_o\}} \end{array}$$

Let us consider an application of the rule  $\square_{k45}^o \frac{\diamond\Gamma_1, \Gamma_1, \square\Gamma_2, A}{\diamond\Gamma_1, \square\Gamma_2, \square A}$ .

Take a sequent  $\mathcal{T}, \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\diamond\Gamma_1, \square\Gamma_2, \square A$ . We focus on the treatment of the formulas in  $\Gamma_2$ , which is the difference with respect to the case for  $\square_{k4}^o$ . For  $B \in \Gamma_2$ , either (a)  $x : [\square B]_o \in \mathcal{R}$ ; or (b)  $y : [B]_o \in \mathcal{R}$  and  $xRy \in \mathcal{G}^*$ . For the formulas for which case (a) holds, the following derivation needs to be inserted between step 1 and step 2 of  $\square_{k4}^o$ :

$$\begin{array}{c} 5_* \frac{(x, \mathcal{F}); xRy, xRu, yRu, \mathcal{G} \Rightarrow \mathcal{R}, u : [B]_o}{(x, \mathcal{F}); xRy, xRu, \mathcal{G} \Rightarrow \mathcal{R}, u : [B]_o} \\ \square_* \frac{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, x : \square[B]_o}{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle x : \downarrow \square[B]_o \rangle} \\ \downarrow_* \\ \uparrow_o \frac{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\square B]_o}{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}, x : [\square B]_o} \end{array}$$

otherwise, in case (b) holds, then a series of relational rules (including  $5_*$ ) to obtain  $xRy$  in the relational set, suffices.

Let us consider an application of the rule  $\diamond_t^o \frac{\Gamma, A}{\Gamma, \diamond A}$ .

Take  $(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\Gamma, \diamond A$ . We have that either (a)  $x : [\diamond A]_o \in \mathcal{R}$ ; or (b)  $z : [\diamond A]_o \in \mathcal{R}$  and  $zRx \in \mathcal{G}^*$ . Case (a) is identical to the previous proof for  $\diamond_t^o$  without  $4$  so we focus on case (b). After possible applications of relational rules leading to a sequent whose relational set contains  $zRx$ , the  $\text{LMF}_o$  derivation corresponding to this rule application consists in the following bipole:

$$\begin{array}{c} \downarrow_* \frac{(x, \mathcal{F}); zRx, \mathcal{G} \Rightarrow \mathcal{R}, x : [A]_o}{(x, \mathcal{F}); zRx, \mathcal{G} \Rightarrow \mathcal{R}, \langle x : \downarrow [A]_o \rangle} \\ \diamond_* \frac{(x, \mathcal{F}); zRx, \mathcal{G} \Rightarrow \mathcal{R}, \langle zx : \diamond \downarrow [A]_o \rangle}{(x, \mathcal{F}); zRx, \mathcal{G} \Rightarrow \mathcal{R}, z : [\diamond A]_o} \\ \uparrow_o \end{array}$$

Let us consider an application of the rule  $\diamond_d^o \frac{\Gamma}{\diamond \Gamma}$ .

Take  $(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}$  that extends  $\diamond\Gamma$ . For each  $B \in \Gamma$ , one of the following two cases holds: either (a)  $x : \lceil \diamond B \rceil_{\circ} \in \mathcal{R}$  or (b)  $z : \lceil \diamond B \rceil_{\circ} \in \mathcal{R}$  and  $zRx \in \mathcal{G}^*$ . The  $\text{LMF}_{\circ}$  derivation corresponding to this application consists in the following bipole:

1. apply the rule  $\mathbf{d}_*$  in order to add  $xRy$  to  $\mathcal{G}$  for some fresh  $y$ ;

$$\mathbf{d}_* \frac{(x, \mathcal{F}); xRy, \mathcal{G} \Rightarrow \mathcal{R}}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}} \quad y \text{ is fresh}$$

2. for those  $B \in \Gamma$  such that case (b) holds, apply a series of relational rules leading to a sequent whose relational set contains  $zRx$  and then the rule  $\mathbf{4}_*$  to  $zRx$  and  $xRy$ , which adds  $zRy$  to the relation set;

$$\mathbf{4}_* \frac{(x, \mathcal{F}); zRx, xRy, zRy, \mathcal{G} \Rightarrow \mathcal{R}}{(x, \mathcal{F}); zRx, xRy, \mathcal{G} \Rightarrow \mathcal{R}}$$

3. finally multifocus on all the  $w : \lceil \diamond B \rceil_{\circ}$  such that  $wRy$  is in the relation set and  $B \in \Gamma$ , choosing  $y$  as the future.

$$\begin{array}{c} \downarrow_* \\ \diamond_* \\ \uparrow_{\circ} \end{array} \frac{\frac{(x, \mathcal{F}); wRy, \mathcal{G} \Rightarrow \mathcal{R}, y : \lceil B \rceil_{\circ}}{(x, \mathcal{F}); wRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle y : \downarrow \lceil B \rceil_{\circ} \rangle}}{(x, \mathcal{F}); wRy, \mathcal{G} \Rightarrow \mathcal{R}, \langle wy : \diamond \downarrow \lceil B \rceil_{\circ} \rangle}}{(x, \mathcal{F}); wRy, \mathcal{G} \Rightarrow \mathcal{R}, w : \lceil \diamond B \rceil_{\circ}} \quad \odot$$

The second direction of the encoding can be stated as follows (encompassing both cases when  $\mathbf{4}$  is or is not part of the considered logic).

**6.2.5 Theorem** (Marin, Miller and Volpe [86]) *Let  $\mathsf{Y} = \{\mathbf{k}\}$  or  $\{\mathbf{k}, \mathbf{4}\}$  or  $\{\mathbf{k}, \mathbf{4}, \mathbf{5}\}$ . Let  $\mathsf{X} = \emptyset$  or  $\{\mathbf{d}\}$  or  $\{\mathbf{t}\}$ . Let  $\mathcal{D}'$  be a proof in  $\text{LMF}_{\circ}^{\mathsf{X}+\mathsf{Y}}$  of a sequent  $(x, \emptyset); \emptyset \Rightarrow x : \lceil A \rceil_{\circ}$  for some  $x$ . There exists a proof  $\mathcal{D}$  of  $A$  in the ordinary sequent calculus built from  $\square_{\mathsf{Y}}^{\circ} + \diamond_{\mathsf{X}}^{\circ}$ , such that each bipole in  $\mathcal{D}'$  corresponds to one rule application in  $\Pi$ , plus possible applications of the contraction rule  $\mathbf{c}$ .*

*Proof (Sketch).* This proof is similar to the one of Theorem 5.2.1. We proceed bottom-up to transform  $\mathcal{D}'$ , bipole by bipole, into  $\mathcal{D}$ . At the root of each bipole, one or several formulas are chosen as foci, and we can distinguish cases according to the main connective of the formula(s) on which we decide.

The cases of classical connectives are straightforward. The case of a formula with  $\square$  as the main connective is also simple as the interpretation of the conclusion and the premiss of the following bipole under  $I_{\mathbf{k}+\mathsf{X}}(\cdot)$  or  $I_{\mathbf{k}+\mathbf{4}+\mathsf{X}}(\cdot)$  are identical:

$$\begin{array}{c} \square_* \\ \downarrow_* \\ \uparrow_{\circ} \end{array} \frac{\frac{(x, \mathcal{F}); \mathcal{G}, xRy \Rightarrow \mathcal{R}, y : \lceil A \rceil_{\circ}}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : \square \lceil A \rceil_{\circ}}}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, \langle x : \downarrow \square \lceil A \rceil_{\circ} \rangle}}{(x, \mathcal{F}); \mathcal{G} \Rightarrow \mathcal{R}, x : \lceil \square A \rceil_{\circ}}$$

Relational rules  $\mathbf{x}_*$  for  $\mathbf{x} \in \{\mathbf{t}, \mathbf{b}, \mathbf{d}, \mathbf{4}, \mathbf{5}\}$  do not change interpretation of the sequent from conclusion to premiss either. Finally, if we consider an application of  $\uparrow_{\circ}$  on a multiset of  $\diamond$ -formulas, one can see, by inspecting the cases arising from the first condition in the definition of  $\uparrow_{\circ}$ , that the bipole corresponds, bottom-up, to an application of  $\mathbf{c}$  followed by  $\square_{\mathbf{k}}^{\circ}$ ,  $\square_{\mathbf{k}+\mathbf{4}}^{\circ}$ ,  $\square_{\mathbf{k}+\mathbf{4}+\mathbf{5}}^{\circ}$ ,  $\diamond_{\mathbf{t}}^{\circ}$  or  $\diamond_{\mathbf{d}}^{\circ}$ , depending on the logic considered and the label chosen as the next one.  $\odot$

The results of this chapter seem to require heavy notations, even though our the explicit polarised syntax already considerably lightened the presentation. We would like, in the future, to simplify the results and maybe come up with a more elegant framework, but for the moment this one is functional, and every one of its devices is necessary. We will illustrate on an example one of them, namely the importance of the concept of forbidden futures.

If we did not have a way to prevent the proof search to go back to forbidden futures, we could for example build in  $\text{LMF}_\circ^{t+4+5}$  a cut-free proof of  $\emptyset \Rightarrow x : \bar{p}_a, x : \uparrow\downarrow\Box\uparrow\Diamond\downarrow\uparrow p_a$ :

$$\begin{array}{c}
\text{id}_* \frac{}{xRy, xRx, yRx \Rightarrow x : \bar{p}_a, \langle x : p_a \rangle} \\
\uparrow_\circ \frac{}{xRy, xRx, yRx \Rightarrow x : \bar{p}_a, x : \uparrow p_a} \\
\downarrow_* \frac{}{xRy, xRx, yRx \Rightarrow x : \bar{p}_a, \langle x : \downarrow\uparrow p_a \rangle} \\
\Diamond_* \frac{}{xRy, xRx, yRx \Rightarrow x : \bar{p}_a, \langle yx : \Diamond\downarrow\uparrow p_a \rangle} \\
\uparrow_\circ \frac{}{xRy, xRx, yRx \Rightarrow x : \bar{p}_a, y : \uparrow\Diamond\downarrow\uparrow p_a} \quad (*) \\
\text{t}_*, 5_* \frac{}{xRy \Rightarrow x : \bar{p}_a, y : \uparrow\Diamond\downarrow\uparrow p_a} \\
\Box_* \frac{}{\emptyset \Rightarrow x : \bar{p}_a, x : \Box\uparrow\Diamond\downarrow\uparrow p_a} \\
\downarrow_* \frac{}{\emptyset \Rightarrow x : \bar{p}_a, \langle x : \downarrow\Box\uparrow\Diamond\downarrow\uparrow p_a \rangle} \\
\uparrow_\circ \frac{}{\emptyset \Rightarrow x : \bar{p}_a, x : \uparrow\downarrow\Box\uparrow\Diamond\downarrow\uparrow p_a}
\end{array}$$

Yet, this sequent is interpreted in our framework as  $\bar{a}, \Box\Diamond a$ , and we have seen in Part I that it does not have a cut-free proof in  $\text{oK45} + \Diamond_\circ$ . However, once we do consider the time of each sequent, the rule  $\uparrow_\circ$  cannot be applied to  $y : \uparrow\Diamond\downarrow\uparrow p_a$  at  $(*)$ . Indeed, the time  $(x, \emptyset)$  in the conclusion would need to be updated to  $(y, \{x\})$  in the premiss, which prevents the focus to be  $\langle yx : \Diamond\downarrow\uparrow p_a \rangle$ , since  $x$  would then be a forbidden future.

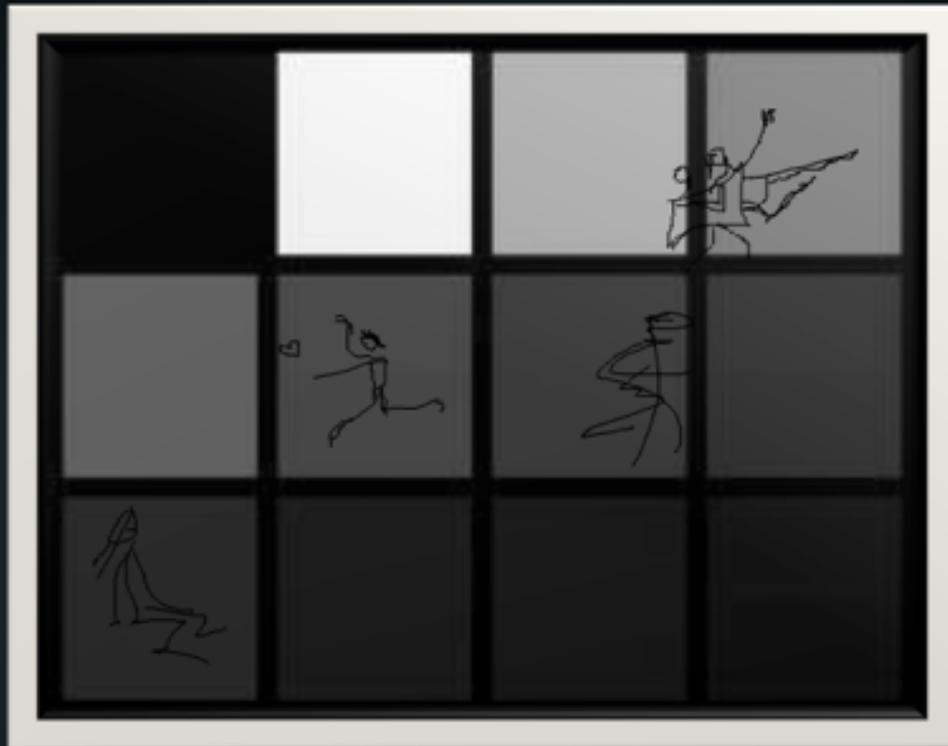
As a result, the proof system  $\text{LMF}_\circ$  can emulate modal sequent systems with adequate precision: individual modal inference rules correspond to certain distinctive chains of bipoles in  $\text{LMF}_\circ$  and *vice versa*. The case of ordinary sequents is interesting, because such calculi have a natural complexity-optimal proof search procedure. Using the additional gadgets of our framework that specify which world we are currently working on and which worlds are no longer reachable, we are able to reproduce the mechanism that constrains proof search in such calculi. We remark that a similar result is obtained by Lellmann and Pimentel in [78], by using a different technique (but also based on the idea of *focusing*). it might be good to compare in more detail.)

As for the previous chapter, this work is part of the *ProofCert* project [93, 94]. Implementations of the  $\text{LMF}_*$  framework can therefore provide a proof checker for the modal sequent systems considered, given checkers for first-order logic (such as those described in [29]) that do not have any special knowledge of modal operators and Kripke frames. The framework is also general enough to capture modal proof systems defined in other formalisms, such as prefixed tableaux [45] and nested sequents [15]; Libal and Volpe have already started implementing these extensions [84]. We are currently working on extending the application of the framework to hypersequents [6].



L'atelier d'improvisation /danse du vendredi présente

# FOLDING



*Organisation tridimensionnelle spécifique à la séquence peptidique des protéines dont les désordres sont responsables de maladies neurodégénératives telles que celle de la vache folle.*

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### III « Folding »

We will take our same telescope, but look through the opposite side of the lense. Instead of seeing existing rules unfold as synthetic rules in another more basic system, we will actually consider folding those existing rules themselves into synthetic entities. Our main technical contribution is the proof of completeness of a focused nested sequent calculus by means of an internal cut-elimination proof. More precisely two such proofs; the second one, in the synthetic system, is considerably shorter and simpler, using a more standard induction measure, than the first one in the non-synthetic system. In Chapter 7 we introduce the focused version of nested sequents systems for both classical and intuitionistic modal logics and give a first proof of cut-free completeness. In Chapter 8 we present the simplified synthetic version of these systems, whose cut-elimination result gives us a second proof of completeness for the focused systems. In Chapter 9 we present some concluding remarks and perspectives.



## Folding nested sequents: focused systems

In this chapter, we present focused nested sequent systems. After the design of the systems, the main contributions are: (i) a proof of cut-elimination for the focused nested calculus, given in terms of a traditional rewriting procedure to eliminate cuts; and (ii) a proof of completeness of the focused system with respect to the non-focused system by showing that the focused system admits the rules of the non-focused system. It generalises similar proofs of cut-elimination and focusing completeness for (non-nested) sequent calculi [74, 83].

### 7.1 Focused nested sequents

We build a focused variant of the nested sequent systems introduced in Section 2.5 and 3.5 for all modal logics of the classical S5 cube and a large subset of the intuitionistic S5 cube.

#### Classical system

In the classical setting, *polarised* formulas are built from the following grammar:

$$\begin{aligned} P &::= p \mid \bar{n} \mid \downarrow N \mid P \overset{\dagger}{\wedge} P \mid \ddagger \mid P \overset{\nabla}{\vee} P \mid \bar{\imath} \mid \diamond P & L &::= p \mid \bar{n} \mid \downarrow N \\ N &::= \bar{p} \mid n \mid \uparrow P \mid N \bar{\vee} N \mid \bar{\imath} \mid N \bar{\wedge} N \mid \bar{\tau} \mid \square N & R &::= \bar{p} \mid n \mid \uparrow P \end{aligned}$$

We recall that a formula is said to be *neutral* in the classical case if it is a shifted formula, an atom, or a negated atom, which can be a positive neutral formula  $L$  or a negative neutral formula  $R$ .

A *polarised* nested sequent is the same as a nested sequent in the non-focused setting, with the difference that all formulas are polarised. An *invertible* nested sequent, written  $\Sigma$ , is a polarised nested sequent that contains only negative formulas, according to the following grammar:

$$\Sigma ::= \emptyset \mid \Sigma, N \mid \Sigma_1, [\Sigma_2]$$

The *meaning* of a polarised nested sequent  $\Sigma$ , written  $\text{fm}^n(\Sigma)$ , is a corresponding negative formula obeying:  $\text{fm}^n(\emptyset) \equiv \bar{\imath}$ ;  $\text{fm}^n(\Sigma, N) \equiv \text{fm}^n(\Sigma) \bar{\vee} N$ ;  $\text{fm}^n(\Sigma_1, [\Sigma_2]) \equiv \text{fm}^n(\Sigma_1) \bar{\vee} \square \text{fm}^n(\Sigma_2)$ ;

In building the focused proof system, we will largely confine ourselves to *neutral* nested sequents, written  $\Gamma$ , which are invertible nested sequents only built from neutral formulas and bracketed neutral nested sequents:

$$\Gamma ::= \emptyset \mid \Gamma, \uparrow P \mid \Gamma, \neg p \mid \Gamma, n \mid \Gamma_1, [\Gamma_2]$$

A context  $\Sigma\{\}\dots\{\}$  is *invertible* if  $\Sigma\{\emptyset\}\dots\{\emptyset\}$  is invertible. Likewise, a context  $\Gamma\{\}\dots\{\}$  is *neutral* if  $\Gamma\{\emptyset\}\dots\{\emptyset\}$  is neutral.

A *focused sequent* is like a neutral sequent but contains an additional single occurrence of  $\langle P \rangle$  wherever a negative formula may otherwise occur, called its *focus*. We depict such

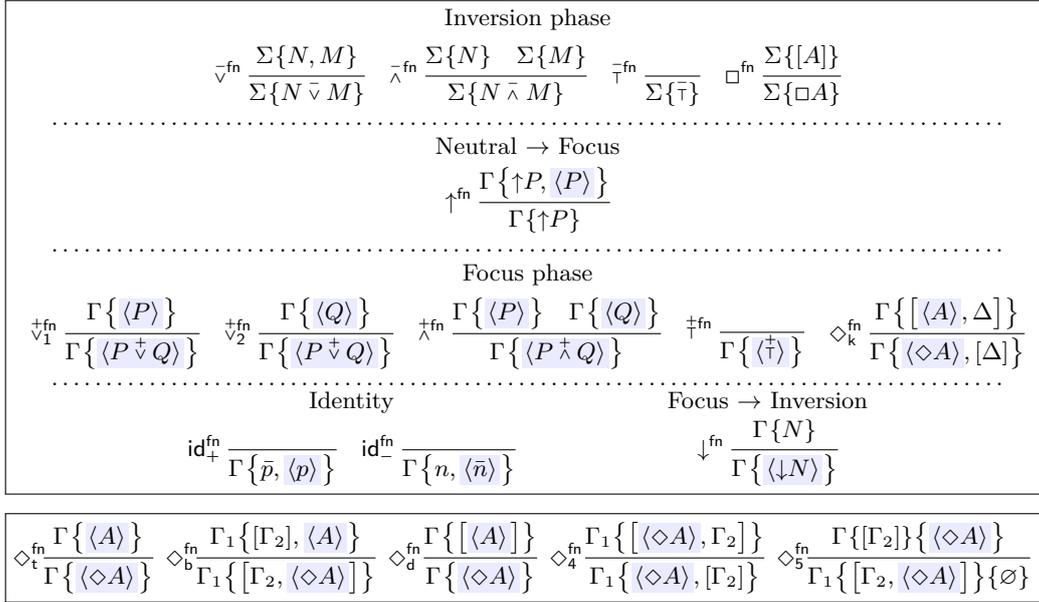


Figure 7.1: System fnK and extensions for t, b, d, 4, and 5

sequents as  $\Gamma\{\langle P \rangle\}$  where  $\Gamma\{\}$  is a neutral context and  $P$  is a positive formula. The meaning of a focused sequent is written by extending  $\text{fm}^n(\cdot)$ , so that  $\text{fm}^n(\Gamma\{\langle P \rangle\}) \equiv \text{fm}^n(\Gamma\{\uparrow P\})$ .

The *classical focused nested system* fnK for the logic K, and its extensions with rules  $\diamond_x^{\text{fn}}$  for any axiom  $x$  in  $\{t, b, d, 4, 5\}$ , is shown in Figure 7.1. Observe that the rules for negative formulas are exactly the same as in the unfocused system (Figure 2.5), while the rules for positive formulas can only be applied if the principal formula is in focus. Similarly, the modal rules in the focused system, i.e. the rule  $\diamond_k^{\text{fn}}$  and the rules  $\diamond_x^{\text{fn}}$  shown on the last line of the figure, are similar to the ones in the non-focused system. However, they can only be applied if the focus is on the principal  $\diamond$ -formula. Note also that they do not internalise contraction; this is not needed because of the explicit contraction appearing in the rule  $\uparrow^{\text{fn}}$ .

**7.1.1 Example** As examples of derivations in fnK we present the focused versions of the nK derivations we gave in Example 2.5.3.

$\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [n]$  is one possible polarisation of  $\diamond(a \wedge \bar{b}), \diamond \bar{a}, [b]$  as a neutral sequent. Once this polarisation is fixed, it constrains the shape of the proof as follows:

$$\begin{array}{c}
\text{id}^{\text{fn}} \frac{}{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\langle p \rangle, \bar{p}, n]} \quad \text{id}^{\text{fn}} \frac{}{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\langle \bar{n} \rangle, \bar{p}, n]} \\
\bar{\wedge}^{\text{fn}} \frac{}{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\langle p \rangle, \bar{p}, n]} \\
\begin{array}{c}
\diamond_k^{\text{fn}} \frac{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\langle p \dagger \bar{n} \rangle, \bar{p}, n]}{\uparrow \diamond(p \dagger \bar{n}), \langle \diamond(p \dagger \bar{n}) \rangle, \uparrow \diamond \downarrow \bar{p}, [\bar{p}, n]} \\
\uparrow^{\text{fn}} \frac{}{\uparrow \diamond(p \dagger \bar{n}), \langle \diamond(p \dagger \bar{n}) \rangle, \uparrow \diamond \downarrow \bar{p}, [\bar{p}, n]} \\
\downarrow^{\text{fn}} \frac{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\bar{p}, n]}{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\langle \downarrow \bar{p} \rangle, n]} \\
\diamond_k^{\text{fn}} \frac{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [\langle \downarrow \bar{p} \rangle, n]}{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, \langle \diamond \downarrow \bar{p} \rangle, [n]} \\
\uparrow^{\text{fn}} \frac{}{\uparrow \diamond(p \dagger \bar{n}), \uparrow \diamond \downarrow \bar{p}, [n]}
\end{array}
\end{array}$$

$\uparrow \diamond \downarrow (n \bar{\wedge} \uparrow p), \uparrow \diamond \bar{n}, [\bar{p}]$  is a second possible polarisation of  $\diamond(a \wedge \bar{b}), \diamond \bar{a}, [b]$  as a neutral



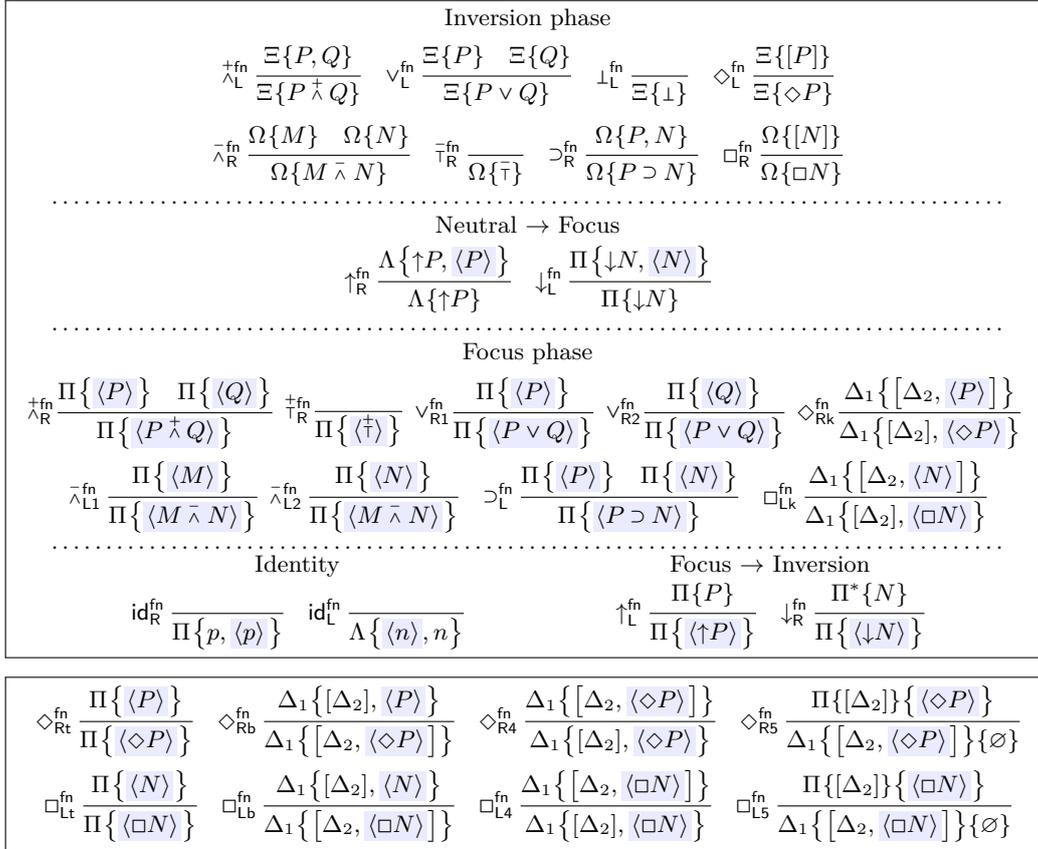


Figure 7.2: System fnIK and extensions for t, b, 4, and 5

The *meaning* of a polarised sequent  $\Theta$ , written  $\text{fm}^n(\Theta)$ , is a polarised formula obeying:  $\text{fm}^n(\emptyset) \equiv \dagger$ ;  $\text{fm}^n(P, \Omega) \equiv P \uparrow \text{fm}^n(\Omega)$ ;  $\text{fm}^n([\Omega_1], \Omega_2) \equiv \diamond \text{fm}^n(\Omega_1) \uparrow \text{fm}^n(\Omega_2)$ ;  $\text{fm}^n(\Omega, N) \equiv \text{fm}^n(\Omega) \supset N$ ; and  $\text{fm}^n(\Omega, [\Xi]) \equiv \text{fm}^n(\Omega) \supset \square \text{fm}^n(\Xi)$

We write  $\Sigma^*\{\dots\}$  for the context formed by deleting any negative formula from  $\Sigma\{\dots\}$ . We reuse the right-deletion notation from Section 3.5 in the polarised case since the concepts are similar, replacing “output formula” with “negative (right-neutral) formula”.

A *focused sequent* is a neutral sequent with an occurrence of a *focus*  $\langle P \rangle$  or  $\langle N \rangle$  in the place where a positive formula may otherwise occur, that is,  $\Pi\{\langle P \rangle\}$  or  $\Pi\{\langle N \rangle\}$  where  $\Pi\{\dots\}$  is a neutral context. The notion of corresponding formula is extended with  $\text{fm}^n(\Pi\{\langle P \rangle\}) = \text{fm}^n(\Pi^*\{\uparrow P\})$  and  $\text{fm}^n(\Pi\{\langle N \rangle\}) = \text{fm}^n(\Pi\{\downarrow N\})$ .

The inference rules of the intuitionistic focused nested sequent systems are given in Figure 7.2. They are the polarised and focused versions of the nIK rules; note however, that similarly to LJF, the left-rule for  $\supset$  does not need to make use of the right-deletion operation in the first premiss. The only place where a formula needs to be deleted from the context is in the rule  $\downarrow_{\text{L}}^{\text{fn}}$ , as in the premiss a new negative formula needs to appear. Like with fnK above, for any  $X \subseteq \{t, b, 4, 5\}$ , we can extend the system fnIK, consisting of the rules in the upper section of the figure, with the rules  $\square_{\text{Lx}}^{\text{f}}$  and  $\diamond_{\text{Rx}}^{\text{f}}$  (for each  $x \in X$ ) in the lower section of the figure.

**7.1.3 Theorem** (Chaudhuri, Marin, and Straßburger [23]) *Let  $X \subseteq \{t, b, 4, 5\}$  be axiomatically complete. For any polarised right-neutral formula  $R$ , the following are equivalent.*

1.  $\partial(R)$  is a theorem of  $\text{IK} + X$ .
2.  $R$  is provable in  $\text{fnIK} + \square_{\text{Lx}}^{\text{f}} + \diamond_{\text{Rx}}^{\text{f}}$ .

**7.1.4 Example** As an example, we give a focused version of the derivation in Example 3.5.3 below. Namely, the derivation in  $\text{fnK}$  of  $R = \uparrow\downarrow(\downarrow(\diamond p \supset \square n) \supset \square(p \supset n))$ , which is a right-neutral polarised form of  $k_4$  with a minimal number of shifts. Contrary to the unfocused derivation, when  $\supset_L^{\text{fn}}$  is applied,  $n$  can remain in the left branch all the way up to the application of  $\text{id}_R^{\text{fn}}$ . Observe also that the instance of  $\downarrow_L^{\text{fn}}$  cannot be applied any lower in the derivation, since its conclusion would not then be neutral.

$$\begin{array}{c}
\begin{array}{c}
\text{id}_R^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), [\langle p \rangle, p, n]} \\
\downarrow_{\text{Rk}}^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle \diamond p \rangle, [p, n]} \\
\supset_L^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), [\langle p \rangle, p, n]}
\end{array}
\quad
\begin{array}{c}
\text{id}_L^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), [\langle n \rangle, p, n]} \\
\downarrow_{\text{Lk}}^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle \square n \rangle, [p, n]}
\end{array} \\
\downarrow_L^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle \diamond p \supset \square n \rangle, [p, n]} \\
\supset_R^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), [p, n]} \\
\downarrow_{\text{R}}^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), [p \supset n]} \\
\supset_R^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n), \square(p \supset n)} \\
\downarrow_{\text{R}}^{\text{fn}} \frac{}{\downarrow(\diamond p \supset \square n) \supset \square(p \supset n)} \\
\uparrow_{\text{R}}^{\text{fn}} \frac{}{R, \langle \downarrow(\downarrow(\diamond p \supset \square n) \supset \square(p \supset n)) \rangle} \\
R
\end{array}$$

## 7.2 Focused meta-theory

We will present the meta-theory of the system  $\text{fnK}$ , namely its soundness and completeness wrt. the unfocused system  $\text{nK}$ , via a cut-elimination result, a technique pioneered by Laurent for linear logic [74]. Parallel results are available for system  $\text{fnIK}$  wrt.  $\text{nIK}$ , and obtained by almost the same proofs, with twice as many cases as the system contains twice as many rules.

### Soundness

**7.2.1 Theorem** *Let  $X \subseteq \{\text{t}, \text{b}, \text{d}, 4, 5\}$ . Let  $R$  be a negative neutral formula. If  $R$  is provable in  $\text{fnK} + \diamond_X^{\text{fn}}$ , then  $\partial(R)$  is provable in  $\text{nK} + \diamond_X^{\text{n}}$ .*

*Proof.* By forgetting the polarity information, every  $\text{fnK} + \diamond_X^{\text{fn}}$  proof of  $N$  is transformed into a  $\text{nK} + \diamond_X^{\text{n}}$  proof of  $\partial(N)$ . The proof makes use of the following two inference rules

$$\text{w} \frac{\Gamma_1\{\emptyset\}}{\Gamma_1\{\Gamma_2\}} \quad \text{and} \quad \text{c} \frac{\Gamma_1\{\Gamma_2, \Gamma_2\}}{\Gamma_1\{\Gamma_2\}}$$

defined on polarised sequents, and the fact that they are admissible for  $\text{fnK} + \diamond_X^{\text{fn}}$  by induction, similarly to the unfocused case [15].

Any polarised sequent  $\Sigma$  can be transformed into an unpolarised sequent  $\partial(\Sigma)$  with the same meaning by replacing every formula  $N$  in  $\Sigma$  by  $\partial(N)$ , and similarly for contexts  $\Sigma\{\}$ . Moreover we can define  $\partial(\Sigma\{\langle P \rangle\}) = \partial(\Sigma)\{\partial(P)\}$ . It follows that every rule in  $\text{fnK} + \diamond_X^{\text{fn}}$  either becomes trivial or can be simulated by a derivation consisting of an instance of a rule in  $\text{nK} + \diamond_X^{\text{n}}$  and an instance of  $\text{w}$ , except for  $\uparrow^{\text{n}}$  which becomes an instance of  $\text{c}$ . Thus, a proof of  $A$  in  $\text{fnK} + \diamond_X^{\text{fn}}$  is transformed into a proof of  $\partial(A)$  in  $\text{nK} + \diamond_X^{\text{n}} + \text{w} + \text{c}$ . The theorem finally follows from admissibility of  $\text{w}$  and  $\text{c}$  for  $\text{nK} + \diamond_X^{\text{n}}$ .  $\odot$

### Completeness

We want to show completeness of the focused systems  $\text{fnK} + \diamond_X^{\text{fn}}$  with respect to  $\text{nK} + \diamond_X^{\text{n}}$ , that is, every formula which can be proved within the unfocused system, also has a focused proof.

**7.2.2 Theorem** *Let  $X \subseteq \{\text{t}, \text{b}, \text{d}, 4, 5\}$  be axiomatically complete. Let  $R$  be a negative neutral formula. If  $\partial(R)$  is provable in  $\text{nK} + \diamond_X^{\text{n}}$ , then  $R$  is provable in  $\text{fnK} + \diamond_X^{\text{fn}}$ .*

|   |   |   |  |   |
|---|---|---|--|---|
| Invertible rules  |   |   |  |   |
| $\bar{\vee}^{\text{wn}} \frac{\Sigma\{N, M\}}{\Sigma\{N \bar{\vee} M\}}$                                | $\bar{\perp}^{\text{wn}} \frac{\Sigma\{\emptyset\}}{\Sigma\{\bar{\perp}\}}$                             | $\bar{\wedge}^{\text{wn}} \frac{\Sigma\{N\} \quad \Sigma\{M\}}{\Sigma\{N \bar{\wedge} M\}}$   | $\bar{\top}^{\text{wn}} \frac{}{\Sigma\{\bar{\top}\}}$ | $\bar{\square}^{\text{wn}} \frac{\Sigma\{[A]\}}{\Sigma\{\bar{\square}A\}}$  |
| Inversion $\rightarrow$ Focus   |   |   |  |   |
| $\uparrow^{\text{wn}} \frac{\Sigma\{\uparrow P, \langle P \rangle\}}{\Sigma\{\uparrow P\}}$             |   |   |  |   |
| Focused rules   |   |   |  |   |
| $\downarrow_1^{\text{wn}} \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle P \downarrow Q \rangle\}}$ | $\downarrow_2^{\text{wn}} \frac{\Sigma\{\langle Q \rangle\}}{\Sigma\{\langle P \downarrow Q \rangle\}}$ | $\uparrow^{\text{wn}} \frac{\Sigma\{\langle P \rangle\} \quad \Sigma\{\langle Q \rangle\}}{\Sigma\{\langle P \uparrow Q \rangle\}}$ | $\dagger^{\text{wn}} \frac{}{\Sigma\{\dagger\}}$       | $\diamond_k^{\text{wn}} \frac{\Sigma\{[\langle A \rangle, \Delta]\}}{\Sigma\{\langle \diamond A \rangle, [\Delta]\}}$ |
| Identity  |   | Focus $\rightarrow$ Inversion   |  |   |
| $\text{id}_+^{\text{wn}} \frac{}{\Sigma\{\bar{p}, \langle p \rangle\}}$                                 | $\text{id}_-^{\text{wn}} \frac{}{\Sigma\{n, \langle \bar{n} \rangle\}}$                                 | $\downarrow^{\text{wn}} \frac{\Sigma\{N\}}{\Sigma\{\langle \downarrow N \rangle\}}$   |  |   |

|   |   |   |  |   |
|---|---|---|--|---|
| $\diamond_t^{\text{wn}} \frac{\Sigma\{\langle A \rangle\}}{\Sigma\{\langle \diamond A \rangle\}}$ | $\diamond_b^{\text{wn}} \frac{\Sigma_1\{[\Sigma_2], \langle A \rangle\}}{\Sigma_1\{[\Sigma_2], \langle \diamond A \rangle\}}$ | $\diamond_d^{\text{wn}} \frac{\Sigma\{[\langle A \rangle]\}}{\Sigma\{\langle \diamond A \rangle\}}$ | $\diamond_4^{\text{wn}} \frac{\Sigma_1\{[\langle \diamond A \rangle, \Sigma_2]\}}{\Sigma_1\{\langle \diamond A \rangle, [\Sigma_2]\}}$ | $\diamond_5^{\text{wn}} \frac{\Sigma_1\{[\Sigma_2]\}\{\langle \diamond A \rangle\}}{\Sigma_1\{[\Sigma_2], \langle \diamond A \rangle\}\{\emptyset\}}$ |
|---|---|---|--|---|

Figure 7.3: System  $\text{wnK}$  and extensions for  $\text{t}$ ,  $\text{b}$ ,  $\text{d}$ ,  $4$ , and  $5$

The proof that we present in this section requires an intermediate step. In the next chapter, we will present an alternative proof which is much simpler and shorter, based on synthetic rules.

We consider the system called  $\text{wnK}$  for *weakly focused* which is a variant  $\text{fnK}$  with a weaker focusing discipline: the rule  $\uparrow^{\text{fn}}$  can be applied in a context that contains also negative formulas, and the invertible rules can occur anywhere, even in a focus phase. This system is given in Figure 7.3; it relies on *weakly focused sequents* which are invertible sequents with an occurrence of a focus, i.e. of the form  $\Sigma\{\langle P \rangle\}$ . Furthermore, we abuse the notation  $\Sigma$  to encompass sequents that can contain a (unique) occurrence of a focus  $\langle P \rangle$ ; indeed, the invertible rules can occur even when the context contains a focus elsewhere, but not the focused rules. We can show directly that any proof within  $\text{wnK}$  can be transformed into one in  $\text{fnK}$  and vice-versa.

**7.2.3 Lemma** *Let  $X \subseteq \{\text{t}, \text{b}, \text{d}, 4, 5\}$ . A negative neutral formula  $R$  is provable in  $\text{fnK} + \diamond_X^{\text{fn}}$  if and only if it is provable in  $\text{wnK} + \diamond_X^{\text{wn}}$ .*

*Proof.* A derivation in  $\text{fnK} + \diamond_X^{\text{fn}}$  is by definition also a derivation in  $\text{wnK} + \diamond_X^{\text{wn}}$ . Conversely, to convert a derivation in  $\text{wnK} + \diamond_X^{\text{wn}}$  into one in  $\text{fnK} + \diamond_X^{\text{fn}}$ , we first have to replace all instances of  $\text{id}^{\text{wn}}$  with a sequence of applications of  $\{\bar{\vee}^{\text{fn}}, \bar{\wedge}^{\text{fn}}, \bar{\square}^{\text{fn}}\}$  followed (reading from conclusion upwards) by  $\text{id}^{\text{fn}}$ , to ensure that the conclusion of the  $\text{id}^{\text{fn}}$  rule is neutral. Then, any negative rules from  $\{\bar{\vee}^{\text{fn}}, \bar{\wedge}^{\text{fn}}, \bar{\square}^{\text{fn}}\}$ , in particular the ones that we just added, can be permuted down by straightforward rule permutations, such that  $\uparrow^{\text{fn}}$  is only applied to neutral sequents and therefore no negative rule occurs within a focus phase.  $\odot$

We are now going to simulate  $\text{nK} + \diamond_X^{\text{n}}$  within  $\text{wnK} + \diamond_X^{\text{wn}}$ . The simulation crucially relies on the use of a cut-rule of the form

$$\text{cut} \frac{\Sigma\{\uparrow P\} \quad \Sigma\{\neg P\}}{\Sigma\{\emptyset\}}$$

which is admissible in  $\text{wnK} + \diamond_X^{\text{wn}}$ , as we will show in the next section. But first, we need another technical property of  $\text{wnK}$ , called *identity reduction*.

The *depth* of a polarised formula  $P$  or  $N$ , denoted by  $\text{dp}(P)$  or  $\text{dp}(N)$ , is computed in the same inductive way as for a non-polarised formula:  $\text{dp}(p) = \text{dp}(n) = 1$ ;  $\text{dp}(\uparrow P) = \text{dp}(\diamond P) = \text{dp}(P) + 1$ ;  $\text{dp}(\downarrow N) = \text{dp}(\square N) = \text{dp}(N) + 1$ ;  $\text{dp}(P \uparrow Q) = \text{dp}(P \downarrow Q) = \max(\text{dp}(P), \text{dp}(Q)) + 1$ ; and  $\text{dp}(M \wedge N) = \text{dp}(N \bar{\vee} M) = \max(\text{dp}(M), \text{dp}(N)) + 1$ .

**7.2.4 Lemma** *The following generalised identity rule is derivable in  $\text{wnK}$ :*

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle P \rangle, \neg P \}}$$

*Proof.* We proceed by induction on the depth of the formula  $P$ . The base case is when  $P$  is an atom or a negated atom, where we replace the instance of  $\text{gid}^{\text{wn}}$  by an actual  $\text{id}^{\text{wn}}$ :

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle p \rangle, \bar{p} \}} \rightsquigarrow \text{id}_+^{\text{wn}} \frac{}{\Sigma \{ \langle p \rangle, \bar{p} \}}$$

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle \bar{n} \rangle, n \}} \rightsquigarrow \text{id}_-^{\text{wn}} \frac{}{\Sigma \{ \langle \bar{n} \rangle, n \}}$$

Otherwise, we replace  $\text{gid}^{\text{wn}}$  by a derivation which contains instances of  $\text{gid}^{\text{wn}}$  applied to formulas of smaller depth:

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle P_1 \uparrow P_2 \rangle, \neg P_1 \bar{\vee} \neg P_2 \}} \rightsquigarrow \frac{\frac{\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle P_1 \rangle, \neg P_1, \neg P_2 \}}}{\bar{\vee}^{\text{wn}}} \quad \frac{\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle P_2 \rangle, \neg P_1, \neg P_2 \}}}{\bar{\vee}^{\text{wn}}}}{\uparrow^{\text{wn}}} \frac{}{\Sigma \{ \langle P_1 \uparrow P_2 \rangle, \neg P_1 \bar{\vee} \neg P_2 \}}$$

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle P_1 \downarrow P_2 \rangle, \neg P_1 \bar{\wedge} \neg P_2 \}} \rightsquigarrow \frac{\frac{\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle P_1 \rangle, \neg P_1 \}}}{\downarrow^{\text{wn}}} \quad \frac{}{\Sigma \{ \langle P_1 \downarrow P_2 \rangle, \neg P_1 \}}}{\bar{\wedge}^{\text{wn}}} \frac{}{\Sigma \{ \langle P_1 \downarrow P_2 \rangle, \neg P_1 \bar{\wedge} \neg P_2 \}}$$

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle \diamond Q \rangle, \square \neg Q \}} \rightsquigarrow \frac{\frac{\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle Q \rangle, \neg Q \}}}{\diamond_k^{\text{wn}}} \quad \frac{}{\Sigma \{ \langle \diamond Q \rangle, [\neg Q] \}}}{\square^{\text{wn}}} \frac{}{\Sigma \{ \langle \diamond Q \rangle, \square \neg Q \}}$$

$$\text{gid}^{\text{wn}} \frac{}{\Sigma \{ \langle \downarrow N \rangle, \uparrow \neg N \}} \rightsquigarrow \frac{\frac{\text{gid}^{\text{wn}} \frac{}{\Sigma \{ N, \neg N, \langle \neg N \rangle \}}}{\uparrow^{\text{wn}}} \quad \frac{}{\Sigma \{ N, \uparrow \neg N \}}}{\downarrow^{\text{wn}}} \frac{}{\Sigma \{ \langle \downarrow N \rangle, \uparrow \neg N \}}$$

Note that we have to apply some invertible rules even within a focus phase, so it is important that we reason in  $\text{wnK}$  and not in  $\text{fnK}$ .  $\odot$

**7.2.5 Lemma** *Let  $X \subseteq \{t, b, d, 4, 5\}$ . Let  $N$  be a negative formula. If  $\partial(N)$  is provable in  $\text{nK} + \diamond_X^n$ , then  $N$  is provable in  $\text{wnK} + \diamond_X^{\text{wn}} + \text{cut}$ .*

*Proof.* First, any unpolarised  $\text{nK}$  sequent can be transformed into an invertible polarised sequent with the same meaning. The connectives are turned into their polarised variant, that is, in particular, a polarity is arbitrarily chosen for every atom, every  $\vee$ , and every  $\wedge$ ; then shifts are added as needed to produce well-formed polarised formulas. If the obtained formula is positive, an extra shift  $\downarrow$  needs to be prenexed.

Then, we show that each rule in  $\text{nK} + \diamond_X^n$  is derivable in  $\text{wnK} + \diamond_X^{\text{wn}} + \text{cut}$ , and the lemma follows by induction on the height of the proof of  $A$ : replacing in the proof the bottom

most rule  $r$  by an equivalent in  $\text{wnK} + \diamond_X^{\text{wn}}$ , appealing to the admissibility of weakening  $w$  in  $\text{nK}$  [15] and of the generalised identity  $\text{gid}^{\text{wn}}$  in  $\text{wnK}$  (Lemma 7.2.4), and continuing for the rest of the proof which is of smaller height.

$$\begin{array}{l}
\text{id}^n \frac{}{\Gamma\{a, \bar{a}\}} \text{ becomes } \frac{\text{id}_+^{\text{wn}} \overline{\Sigma\{\uparrow p, \langle p \rangle, \bar{p}\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{\uparrow p, \bar{p}\}}} \text{ or } \frac{\text{id}_-^{\text{wn}} \overline{\Sigma\{n, \uparrow \bar{n}, \langle \bar{n} \rangle\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{n, \uparrow \bar{n}\}}} \\
\vee^n \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \text{ becomes } \frac{\vee^{\text{wn}} \overline{\Sigma\{N, M\}}}{\vee^{\text{wn}} \overline{\Sigma\{N \vee M\}}} \text{ or} \\
\text{cut} \frac{\frac{w \overline{\Sigma\{\uparrow P, \uparrow Q\}}}{\text{cut} \overline{\Sigma\{\uparrow(P \downarrow Q), \uparrow P, \uparrow Q\}}} \quad \frac{\frac{\text{gid}^{\text{wn}} \overline{\Sigma\{\uparrow(P \downarrow Q), \langle Q \rangle, \uparrow P, \neg Q\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{\uparrow(P \downarrow Q), \langle P \downarrow Q \rangle, \uparrow P, \neg Q\}}} \quad \frac{\text{gid}^{\text{wn}} \overline{\Sigma\{\uparrow(P \downarrow Q), \langle P \rangle, \neg P\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{\uparrow(P \downarrow Q), \langle P \downarrow Q \rangle, \neg P\}}}}{\text{cut} \overline{\Sigma\{\uparrow(P \downarrow Q), \uparrow P\}}} \quad \frac{\text{gid}^{\text{wn}} \overline{\Sigma\{\uparrow(P \downarrow Q), \neg P\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{\uparrow(P \downarrow Q), \neg P\}}}}{\text{cut} \overline{\Sigma\{\uparrow(P \downarrow Q)\}}} \\
\wedge^n \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \text{ becomes } \frac{\wedge^{\text{wn}} \overline{\Sigma\{N\} \quad \Sigma\{M\}}}{\wedge^{\text{wn}} \overline{\Sigma\{N \wedge M\}}} \text{ or} \\
\text{cut} \frac{\frac{w \overline{\Sigma\{\uparrow P\}}}{\text{cut} \overline{\Sigma\{\uparrow P, \uparrow(P \bar{\wedge} Q)\}}} \quad \frac{\frac{w \overline{\Sigma\{\uparrow Q\}}}{\text{cut} \overline{\Sigma\{\neg P, \uparrow(P \bar{\wedge} Q), \uparrow Q\}}} \quad \frac{\frac{\text{gid}^{\text{wn}} \overline{\Sigma\{\neg P, \neg Q, \uparrow(P \bar{\wedge} Q), \langle P \rangle\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{\neg P, \neg Q, \uparrow(P \bar{\wedge} Q), \langle P \bar{\wedge} Q \rangle\}}} \quad \vdots}{\uparrow^{\text{wn}} \overline{\Sigma\{\neg P, \uparrow(P \bar{\wedge} Q), \neg Q\}}}}{\text{cut} \overline{\Sigma\{\uparrow(P \bar{\wedge} Q)\}}} \quad \frac{\text{gid}^{\text{wn}} \overline{\Sigma\{\neg P, \uparrow(P \bar{\wedge} Q)\}}}{\uparrow^{\text{wn}} \overline{\Sigma\{\neg P, \uparrow(P \bar{\wedge} Q)\}}}
\end{array}$$

(where the omitted third premiss derivation is the similar branch for  $\langle Q \rangle$ )

$$\begin{array}{l}
\Box^n \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \text{ becomes } \frac{\Box^{\text{wn}} \overline{\Sigma\{[N]\}}}{\Box^{\text{wn}} \overline{\Sigma\{\Box N\}}} \\
\Diamond_k^n \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \text{ becomes } \text{cut} \frac{\frac{\text{gid}^{\text{wn}} \overline{\Sigma_1\{\uparrow \Diamond P, [\langle P \rangle, \neg P, \Sigma_2]\}}}{\Diamond_k^{\text{wn}} \overline{\Sigma_1\{\uparrow \Diamond P, \langle \Diamond P \rangle, [\neg P, \Sigma_2]\}}} \quad \frac{\uparrow^{\text{wn}} \overline{\Sigma_2\{\uparrow \Diamond P, [\neg P, \Sigma_2]\}}}{\Sigma_1\{\uparrow \Diamond P, [\Sigma_2]\}}}{\Sigma_1\{\uparrow \Diamond P, [\Sigma_2]\}} \\
\Diamond_t^n \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \text{ becomes } \text{cut} \frac{\frac{\text{gid}^{\text{wn}} \overline{\Sigma\{\uparrow \Diamond P, \langle P \rangle, \neg P\}}}{\Diamond_t^{\text{wn}} \overline{\Sigma\{\uparrow \Diamond P, \langle \Diamond P \rangle, \neg P\}}} \quad \frac{\uparrow^{\text{wn}} \overline{\Sigma\{\uparrow \Diamond P, \neg P\}}}{\Sigma\{\uparrow \Diamond P, \uparrow P\}}}{\Sigma\{\uparrow \Diamond P\}} \\
\Diamond_b^n \frac{\Gamma_1\{[\Gamma_2, \Diamond A], A\}}{\Gamma_1\{[\Gamma_2, \Diamond A]\}} \text{ becomes } \text{cut} \frac{\frac{\text{gid}^{\text{wn}} \overline{\Sigma_1\{[\Sigma_2, \uparrow \Diamond P], \langle P \rangle, \neg P\}}}{\Diamond_b^{\text{wn}} \overline{\Sigma_1\{[\Sigma_2, \uparrow \Diamond P], \langle \Diamond P \rangle, \neg P\}}} \quad \frac{\uparrow^{\text{wn}} \overline{\Sigma_1\{[\Sigma_2, \uparrow \Diamond P], \neg P\}}}{\Sigma_1\{[\Sigma_2, \uparrow \Diamond P]\}}}{\Sigma_1\{[\Sigma_2, \uparrow \Diamond P]\}}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\text{gid}^{\text{wn}} \\
\overline{\overline{\Sigma\{\uparrow\Diamond P, [\langle P \rangle, \neg P], \uparrow\Diamond\downarrow\neg P\}}} \\
\Diamond_k^{\text{wn}} \\
\overline{\Sigma\{\uparrow\Diamond P, \langle\Diamond P\rangle, [\neg P], \uparrow\Diamond\downarrow\neg P\}} \\
\uparrow^{\text{wn}} \\
\overline{\Sigma\{\uparrow\Diamond P, [\neg P], \uparrow\Diamond\downarrow\neg P\}} \\
\downarrow^{\text{wn}} \\
\overline{\Sigma\{\uparrow\Diamond P, [\langle\downarrow\neg P\rangle], \uparrow\Diamond\downarrow\neg P\}} \\
\Diamond_d^{\text{wn}} \\
\overline{\Sigma\{\uparrow\Diamond P, \langle\Diamond\downarrow\neg P\rangle, \uparrow\Diamond\downarrow\neg P\}} \\
\uparrow^{\text{wn}} \\
\overline{\Sigma\{\uparrow\Diamond P, \uparrow\Diamond\downarrow\neg P\}} \\
\text{cut} \\
\overline{\Sigma\{\uparrow\Diamond P\}}
\end{array} \\
\Diamond_d^n \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} \text{ becomes}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\text{gid}^{\text{wn}} \\
\overline{\overline{\Sigma_1\{\uparrow\Diamond P, [\langle\Diamond P\rangle, \square\neg P, \Sigma_2\}}} \\
\Diamond_4^{\text{wn}} \\
\overline{\Sigma_1\{\uparrow\Diamond P, \langle\Diamond P\rangle, [\square\neg P, \Sigma_2\}}} \\
\uparrow^{\text{wn}} \\
\overline{\Sigma_1\{\uparrow\Diamond P, [\square\neg P, \Sigma_2\}}} \\
\text{cut} \\
\overline{\Sigma_1\{\uparrow\Diamond P, [\Sigma_2\}}}
\end{array} \\
\Diamond_4^n \frac{\Gamma_1\{\Diamond A, [\Diamond A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \text{ becomes}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\text{gid}^{\text{wn}} \\
\overline{\overline{\Sigma_1\{[\Sigma_2, \uparrow\Diamond P]\}\{\langle\Diamond P\rangle, \square\neg P\}}} \\
\Diamond_5^{\text{wn}} \\
\overline{\Sigma_1\{[\Sigma_2, \uparrow\Diamond P, \langle\Diamond P\rangle]\}\{\square\neg P\}}} \\
\uparrow^{\text{wn}} \\
\overline{\Sigma_1\{[\Sigma_2, \uparrow\Diamond P]\}\{\square\neg P\}}} \\
\text{cut} \\
\overline{\Sigma_1\{[\Sigma_2, \uparrow\Diamond P]\}\{\emptyset\}}
\end{array} \\
\Diamond_5^n \frac{\Gamma_1\{\Diamond A, \Gamma_2\}\{\Diamond A\}}{\Gamma_1\{\Diamond A, \Gamma_2\}\{\emptyset\}} \text{ becomes}
\end{array} \quad \textcircled{\ast}$$

At this point, we have almost completed the proof of completeness. Indeed, suppose that we have a proof of  $\partial(R)$  in  $\text{nK} + \Diamond_X^n$  for some negative neutral formula  $R$ . In particular,  $R$  is negative, so by Lemma 7.2.5, there is a proof of  $R$  in  $\text{wnK} + \Diamond_X^{\text{wn}} + \text{cut}$ . And we know by Lemma 7.2.3 that, if we manage to get a proof of  $R$  in  $\text{wnK} + \Diamond_X^{\text{wn}}$ , we can transform it into a proof in  $\text{fnK} + \Diamond_X^{\text{fn}}$ , which is what we are after. So the last step we are missing is a way to eliminate this cut.

## Cut Elimination

In this section we will show that the cut-rule we have been using is admissible for  $\text{wnK} + \Diamond_X^{\text{wn}}$ . However, as is common in inductive proofs, we have to slightly generalise the statement in order to prove it.

As before, we extend our notation of polarised (invertible or neutral) sequents to contain an occurrence of a focus  $\langle P \rangle$  (at most one). For a sequent  $\Sigma$ , we write  $\Sigma^\diamond$  to delete its focus if there is one, i.e. if  $\Sigma = \Sigma_1\{\langle P \rangle\}$  for some polarised context  $\Sigma_1\{\}$ , then  $\Sigma^\diamond = \Sigma_1\{\emptyset\}$ ; otherwise  $\Sigma^\diamond = \Sigma$ . This definition extends straightforwardly to contexts  $\Sigma\{\}$ .

The cut-rules that we will show admissible in  $\text{wnK} + \Diamond_X^{\text{wn}}$  can then be written as follows:

$$\text{cut}_1 \frac{\Sigma\{\uparrow P\} \quad \Sigma^\diamond\{\neg P\}}{\Sigma\{\emptyset\}} \qquad \text{cut}_2 \frac{\Sigma\{\langle P \rangle\} \quad \Sigma\{\neg P\}}{\Sigma\{\emptyset\}}$$

The *rank* of an instance of one of the cut rules is the depth of its cut formula  $P$ .

The theorem we want to prove is the following. However, the actual elimination procedure will happen in the weaker system  $\text{wnK}$ .

**7.2.6 Theorem** *Let  $X \subseteq \{t, b, d, 4, 5\}$  be axiomatically complete. If a sequent  $\Sigma$  is provable in  $\text{fnK} + \Diamond_X^{\text{fn}} + \{\text{cut}_1, \text{cut}_2\}$ , then it is also provable in  $\text{fnK} + X^{\text{fn}}$ .*

In order to show the admissibility of the cut rules, it will be useful, as we have already seen in Section 2.5, to appeal to a collection of other admissible and invertible rules, in particular:

$$\text{w} \frac{\Sigma\{\emptyset\}}{\Sigma\{N\}} \quad \text{c}_+ \frac{\Sigma\{\bar{p}, \bar{p}\}}{\Sigma\{\bar{p}\}} \quad \text{c}_- \frac{\Sigma\{n, n\}}{\Sigma\{n\}} \quad \boxtimes_k^n \frac{\Sigma_1\{[\Sigma_2], [\Sigma_3]\}}{\Sigma_1\{[\Sigma_2, \Sigma_3]\}}$$

Note that here we use contraction only on negative atoms because that is all that is needed in the cut-elimination proof below. One can indeed show that the general contraction rule on arbitrary sequents (and not just formulas) is admissible, but this requires a more involved argument for focused sequents.

**7.2.7 Lemma** *Let  $X \subseteq \{t, b, d, 4, 5\}$ . The rules  $w$ ,  $c_+$ ,  $c_-$ , and  $\boxtimes_k^n$  are height-preserving admissible for  $\text{wnK} + \diamond_X^{\text{wn}}$ , and the rules  $\square^{\text{wn}}$ ,  $\bar{\wedge}^{\text{wn}}$ , and  $\bar{\vee}^{\text{wn}}$  are invertible for  $\text{wnK} + \diamond_X^{\text{wn}}$ . Moreover, if  $X$  is axiomatically complete, for every  $x \in X$ , the rule  $\boxtimes_x^n$  (shown in Figure 2.6) is admissible in  $\text{wnK} + \diamond_X^{\text{wn}}$ .*

*Proof.* The first part is by induction on the height of the derivation, similarly to Lemma 2.6.6 and the original proof [15]. The second part is less straightforward, but also by induction on the height of the derivation, following [15]. We carry out a case analysis on the last rule  $r$  which appears above  $\boxtimes_x^n$  in the derivation. If  $r = \text{id}$  then the occurrence of  $\boxtimes_x^n$  can just be removed, while in the other cases, we use the induction hypothesis to conclude. We show here the critical cases that require the axiomatic completeness condition, translated into our setting.

- For  $\boxtimes_x^n = \boxtimes_t^n$  and  $r = \diamond_5^{\text{wn}}$ : the only non-trivial cases appear when the focus is situated at the root of the sequent. We are then in one of the following cases:

$$\begin{array}{ccc} \diamond_5^{\text{wn}} \frac{[\Sigma_1], \Sigma_2 \{ \langle \diamond P \rangle \}}{\boxtimes_t^n \frac{[\Sigma_1, \langle \diamond P \rangle], \Sigma_2 \{ \emptyset \}}{\Sigma_1, \langle \diamond P \rangle, \Sigma_2 \{ \emptyset \}}} & \rightsquigarrow & \boxtimes_t^n \frac{[\Sigma_1], \Sigma_2 \{ \langle \diamond P \rangle \}}{\Sigma_1, \Sigma_2 \{ \langle \diamond P \rangle \}} \\ & & \parallel \diamond_4^{\text{wn}} \\ & & \Sigma_1, \langle \diamond P \rangle, \Sigma_2 \{ \emptyset \} \\ \\ \diamond_5^{\text{wn}} \frac{\Sigma_1, [\Sigma_2 \{ \langle \diamond P \rangle \}]}{\boxtimes_t^n \frac{\Sigma_1, [\langle \diamond P \rangle, \Sigma_2 \{ \emptyset \}]}{\Sigma_1, \langle \diamond P \rangle, \Sigma_2 \{ \emptyset \}}} & \rightsquigarrow & \boxtimes_t^n \frac{\Sigma_1, [\Sigma_2 \{ \langle \diamond P \rangle \}]}{\Sigma_1, \Sigma_2 \{ \langle \diamond P \rangle \}} \\ & & \parallel \diamond_4^{\text{wn}} \\ & & \Sigma_1, \langle \diamond P \rangle, \Sigma_2 \{ \emptyset \} \end{array}$$

By axiomatic completeness,  $\diamond_4^{\text{wn}}$  is in  $\diamond_X^{\text{wn}}$  when  $\{t, 5\} \subseteq X$ .

- For  $\boxtimes_x^n = \boxtimes_b^n$  and  $r = \diamond_4^{\text{wn}}$ :

$$\diamond_4^{\text{wn}} \frac{\Sigma_1 \{ [\Sigma_2, [\langle \diamond P \rangle, \Sigma_3]] \}}{\boxtimes_b^n \frac{\Sigma_1 \{ [\Sigma_2, \langle \diamond P \rangle, [\Sigma_3]] \}}{\Sigma_1 \{ [\Sigma_2, \langle \diamond P \rangle], \Sigma_3 \}}} \rightsquigarrow \boxtimes_b^n \frac{\Sigma_1 \{ [\Sigma_2, [\langle \diamond P \rangle, \Sigma_3]] \}}{\Sigma_1 \{ [\Sigma_2, \langle \diamond P \rangle, \Sigma_3 \}} \diamond_5^{\text{wn}} \frac{\Sigma_1 \{ [\Sigma_2, \langle \diamond P \rangle], \Sigma_3 \}}{\Sigma_1 \{ [\Sigma_2, \langle \diamond P \rangle], \Sigma_3 \}}$$

By axiomatic completeness,  $\diamond_5^{\text{wn}}$  is in  $\diamond_X^{\text{wn}}$  when  $\{b, 4\} \subseteq X$ .

- For  $\boxtimes_x^n = \boxtimes_b^n$  and  $r = \diamond_5^{\text{wn}}$ : similarly, the non-trivial case appears when the focus is at the root level of the sequent.

$$\diamond_5^{\text{wn}} \frac{[\Sigma_1, [\Sigma_2]], \Sigma_3 \{ \langle \diamond P \rangle \}}{\boxtimes_b^n \frac{[\Sigma_1, [\langle \diamond P \rangle, \Sigma_2]], \Sigma_3 \{ \emptyset \}}{\langle \diamond P \rangle, [\Sigma_1], \Sigma_2, \Sigma_3 \{ \emptyset \}}} \rightsquigarrow \boxtimes_b^n \frac{[\Sigma_1, [\Sigma_2]], \Sigma_3 \{ \langle \diamond P \rangle \}}{[\Sigma_1], \Sigma_2, \Sigma_3 \{ \langle \diamond P \rangle \}} \parallel \diamond_4^{\text{wn}} \frac{[\Sigma_1], \Sigma_2, \langle \diamond P \rangle, \Sigma_3 \{ \emptyset \}}{[\Sigma_1], \Sigma_2, \langle \diamond P \rangle, \Sigma_3 \{ \emptyset \}}$$

By axiomatic completeness,  $\diamond_4^{\text{wn}}$  is in  $\diamond_X^{\text{wn}}$  when  $\{b, 5\} \subseteq X$ .

- For  $\boxtimes_x^n = \boxtimes_4^n$  and  $r = \diamond_b^{\text{wn}}$ :

$$\diamond_b^{\text{wn}} \frac{\Sigma_1 \{ \langle P \rangle, [\Sigma_2], [\Sigma_3] \}}{\boxtimes_4^n \frac{\Sigma_1 \{ [\Sigma_2, \langle \diamond P \rangle], [\Sigma_3] \}}{\Sigma_1 \{ [[\Sigma_2, \langle \diamond P \rangle], \Sigma_3] \}}} \rightsquigarrow \diamond_b^{\text{wn}} \frac{\Sigma_1 \{ \langle P \rangle, [\Sigma_2], [\Sigma_3] \}}{\Sigma_1 \{ \langle P \rangle, [[\Sigma_2], \Sigma_3] \}} \diamond_5^{\text{wn}} \frac{\Sigma_1 \{ [[\Sigma_2, \langle \diamond P \rangle], \Sigma_3] \}}{\Sigma_1 \{ [[\Sigma_2, \langle \diamond P \rangle], \Sigma_3] \}}$$

By axiomatic completeness,  $\diamond_5^{\text{wn}}$  is in  $\diamond_X^{\text{wn}}$  when  $\{b, 4\} \subseteq X$ .

These cases illustrate why axiomatic completeness is required, and also show that the height of the proof can increase for the admissibility of the  $\boxtimes_X^n$  rules.  $\odot$

We are now ready to prove the admissibility of cuts. Specifically, we show both  $\text{cut}_1$  and  $\text{cut}_2$  simultaneously admissible. Note that this collection of cuts is just sufficiently large to make the standard cut admissible.

**7.2.8 Lemma** *Let  $X \subseteq \{\mathbf{t}, \mathbf{b}, \mathbf{d}, 4, 5\}$  be axiomatically complete.*

$\mathcal{D}_1 \parallel \quad \mathcal{D}_2 \parallel$   
 $\Sigma_1 \quad \Sigma_2$   
 $\Sigma$

For every derivation  $\text{cut}_i \frac{\Sigma_1 \quad \Sigma_2}{\Sigma}$  where  $i \in \{1, 2\}$  and  $\mathcal{D}_1$  and  $\mathcal{D}_2$  in  $\text{wnK} + \diamond_X^{\text{wn}}$  are cut-free, there is a cut-free derivation of  $\Sigma$  in  $\text{wnK} + \diamond_X^{\text{wn}}$ .

*Proof.* Let  $\mathcal{D}_1$  always stand for the derivation with the positive cut formula. We proceed by lexicographic induction: the induction hypothesis may be applied whenever (i) the rank of the cut decreases, or (ii) the rank stays the same and a  $\text{cut}_1$  is replaced by a  $\text{cut}_2$ , or (iii) the rank stays the same and the height of  $\mathcal{D}_1$  decreases. The height of  $\mathcal{D}_2$  does not matter for the induction, which is an important difference with the proof of Lemma 2.6.9. The proof is then given in terms of a terminating rewrite sequence, written with  $\rightsquigarrow$ , that reduces the topmost instances of  $\text{cut}_i$ .

We start with the cases of  $\text{cut}_1$ , which are mostly commutative cases. There are two main possibilities to consider: First, three cases where the last rule  $r$  in  $\mathcal{D}_1$  applies to a negative formula in the context;

- The first case is the ordinary commutative case, where  $r$  is a negative rule.

$$\text{cut}_1 \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma_1 \{\uparrow P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma^\emptyset \{-P\}}}{\Sigma \{\emptyset\}} \rightsquigarrow \text{cut}_1 \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma_1 \{\uparrow P\}} \quad \frac{\frac{\mathcal{D}_2 \parallel}{\Sigma^\emptyset \{-P\}}}{r^{-1} \frac{\Sigma^\emptyset \{-P\}}{\Sigma_1^\emptyset \{-P\}}}}{r \frac{\Sigma_1 \{\emptyset\}}{\Sigma \{\emptyset\}}}$$

Here  $r$  is invertible (Lemma 7.2.7) and we denote again by  $r^{-1}$  its admissible inverse. We can appeal to the inductive hypothesis because the height is reduced. If  $r$  is a binary rule the situation is similar:

$$\text{cut}_1 \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma \{N\} \{\uparrow P\}} \quad \frac{\mathcal{D}''_1 \parallel}{\Sigma \{M\} \{\uparrow P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma^\emptyset \{N \bar{\wedge} M\} \{-P\}}}{\bar{\wedge}^{\text{wn}} \frac{\Sigma \{N \bar{\wedge} M\} \{\uparrow P\}}{\Sigma \{N \bar{\wedge} M\} \{\emptyset\}}} \rightsquigarrow \text{cut}_1 \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma \{N\} \{\uparrow P\}} \quad \frac{(\bar{\wedge}^{\text{wn}})^{-1} \frac{\Sigma^\emptyset \{N \bar{\wedge} M\} \{-P\}}{\Sigma^\emptyset \{N\} \{-P\}}}{\bar{\wedge}^{\text{wn}} \frac{\Sigma \{N\} \{\emptyset\}}{\Sigma \{N \bar{\wedge} M\} \{\emptyset\}}} \quad \frac{\mathcal{D}''_1 \parallel}{\Sigma \{M\} \{\uparrow P\}} \quad \frac{(\bar{\wedge}^{\text{wn}})^{-1} \frac{\Sigma^\emptyset \{N \bar{\wedge} M\} \{-P\}}{\Sigma^\emptyset \{M\} \{-P\}}}{\Sigma \{M\} \{\emptyset\}}}{\Sigma \{N \bar{\wedge} M\} \{\emptyset\}}$$

- The second case is similar; it occurs when a decision on a positive formula is taken in the context.

$$\text{cut}_1 \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma \{\uparrow Q, \langle Q \rangle\} \{\uparrow P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma^\emptyset \{\uparrow Q\} \{-P\}}}{\Sigma \{\uparrow Q\} \{\emptyset\}} \rightsquigarrow \text{cut}_1 \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma \{\uparrow Q, \langle Q \rangle, \uparrow P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma^\emptyset \{Q\} \{\uparrow \neg P\}}}{\uparrow^{\text{wn}} \frac{\Sigma \{\uparrow Q, \langle Q \rangle\} \{\emptyset\}}{\Sigma \{\uparrow Q\} \{\emptyset\}}}$$

We can appeal to the inductive hypothesis because the height is reduced.

- The last case occurs when the decision is on the cut formula. (Here,  $\Sigma$  cannot contain a focus.)

$$\frac{\frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma\{\uparrow P, \langle P \rangle\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{-P\}}}{\text{cut}_1 \frac{\Sigma\{\uparrow P\}}{\Sigma\{\emptyset\}}} \quad \sim \quad \frac{\frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma\{\uparrow P, \langle P \rangle\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{-P\}}}{\text{cut}_1 \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\emptyset\}}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{-P\}}}{\text{cut}_2 \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\emptyset\}}}$$

We first appeal to the inductive hypothesis for the upper cut, which is possible because the height is reduced. Then we appeal to the inductive hypothesis for the lower cut, which is allowed because a  $\text{cut}_1$  is replaced by a  $\text{cut}_2$ .

And second, the following three cases, where the last rule  $r$  in  $\mathcal{D}_1$  operates on the formula with the focus and can simply be permuted under the cut:

- If  $r$  is an identity:

$$\frac{\frac{\text{id}_+^{\text{wn}} \frac{\Sigma\{\bar{p}, \langle p \rangle\}\{P\}}{\Sigma\{\bar{p}, \langle p \rangle\}\{\emptyset\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\bar{p}\}\{-P\}}}{\text{cut}_1} \quad \sim \quad \frac{\text{id}_+^{\text{wn}} \frac{\Sigma\{\bar{p}, \langle p \rangle\}\{P\}}{\Sigma\{\bar{p}, \langle p \rangle\}\{\emptyset\}}}{\text{id}_+^{\text{wn}} \frac{\Sigma\{\bar{p}, \langle p \rangle\}\{P\}}{\Sigma\{\bar{p}, \langle p \rangle\}\{\emptyset\}}}$$

$$\frac{\frac{\text{id}_-^{\text{wn}} \frac{\Sigma\{\langle \bar{n} \rangle, n\}\{P\}}{\Sigma\{\langle a \rangle\}\{\emptyset\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{n\}\{-P\}}}{\text{cut}_1} \quad \sim \quad \frac{\text{id}_-^{\text{wn}} \frac{\Sigma\{\langle \bar{n} \rangle, n\}\{P\}}{\Sigma\{\langle \bar{n} \rangle, n\}\{\emptyset\}}}{\text{id}_-^{\text{wn}} \frac{\Sigma\{\langle \bar{n} \rangle, n\}\{P\}}{\Sigma\{\langle \bar{n} \rangle, n\}\{\emptyset\}}}$$

- If the rule keeps the focus we proceed as follows:

$$\frac{\frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma_1\{\langle Q_1 \rangle\}\{P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}\{-P\}}}{r \frac{\Sigma\{\langle Q \rangle\}\{P\}}{\Sigma\{\langle Q \rangle\}\{\emptyset\}}} \quad \sim \quad \frac{\frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma_1\{\langle Q_1 \rangle\}\{P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}\{-P\}}}{\text{cut}_1 \frac{\Sigma_1\{\langle Q_1 \rangle\}\{P\}}{\Sigma_1\{\langle Q_1 \rangle\}\{\emptyset\}}} \quad \frac{\Sigma\{\langle Q \rangle\}\{P\}}{r \frac{\Sigma\{\langle Q \rangle\}\{P\}}{\Sigma\{\langle Q \rangle\}\{\emptyset\}}}$$

and similarly if  $r$  is a binary rule (with two copies of  $\mathcal{D}_2$ ). We can appeal to the inductive hypothesis because the height is reduced.

- If the focus is released, we proceed in a similar fashion and use the admissibility of  $w$  (Lemma 7.2.7):

$$\frac{\frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma\{N\}\{P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}\{-P\}}}{\downarrow^{\text{wn}} \frac{\Sigma\{\langle \downarrow N \rangle\}\{P\}}{\Sigma\{\langle \downarrow N \rangle\}\{\emptyset\}}} \quad \sim \quad \frac{\frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma\{N\}\{P\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}\{-P\}}}{\text{cut}_1 \frac{\Sigma\{N\}\{P\}}{\Sigma\{N\}\{\emptyset\}}} \quad \frac{\Sigma\{\emptyset\}\{-P\}}{w \frac{\Sigma\{N\}\{-P\}}{\Sigma\{N\}\{-P\}}}}{\downarrow^{\text{wn}} \frac{\Sigma\{N\}\{P\}}{\Sigma\{\langle \downarrow N \rangle\}\{\emptyset\}}}$$

It is important to note that the formula we are weakening is negative which can only occur as we are reasoning in the weakly focused  $\text{wnK}$  and not in the strongly focused  $\text{fnK}$ .

Finally, we consider the cases for  $\text{cut}_2$ , which correspond to the key cases (because the bottom-most rule in  $\mathcal{D}_1$  must work on the cut formula which has the focus):

- In the axiom case we use admissibility of contraction (Lemma 7.2.7):

$$\frac{\frac{\text{id}_+^{\text{wn}} \frac{\Sigma\{\bar{p}, \langle p \rangle\}}{\Sigma\{\bar{p}\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\bar{p}, \bar{p}\}}}{\text{cut}_2} \quad \sim \quad \frac{\mathcal{D}_2 \parallel}{c_+ \frac{\Sigma\{\bar{p}, \bar{p}\}}{\Sigma\{\bar{p}\}}} \quad \frac{\text{id}_-^{\text{wn}} \frac{\Sigma\{n, \langle \bar{n} \rangle\}}{\Sigma\{n\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{n, n\}}}{\text{cut}_2} \quad \sim \quad \frac{\mathcal{D}_2 \parallel}{c_- \frac{\Sigma\{n, n\}}{\Sigma\{n\}}}$$

- If we release the focus we have:

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \downarrow^{\text{wn}} \frac{\Sigma\{N\}}{\Sigma\{\langle \downarrow N \rangle\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}} \rightsquigarrow \text{cut}_1 \frac{\mathcal{D}_2 \parallel \quad \mathcal{D}'_1 \parallel}{\Sigma\{\uparrow \neg N\} \quad \Sigma\{N\}} \Sigma\{\emptyset\}$$

We can appeal to the inductive hypothesis because the cut-rank is reduced. Note that we do not need to appeal to the height of  $\mathcal{D}_2$ .

- The cases of the binary connectives are standard and use the admissibility of  $w$  and invertibility of the negative rules (Lemma 7.2.7):

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \quad \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\} \quad \Sigma\{\langle Q \rangle\}}{\Sigma\{\langle P \uparrow Q \rangle\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}} \frac{\Sigma\{\neg P \bar{\vee} \neg Q\}}{\Sigma\{\emptyset\}} \rightsquigarrow \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle P \rangle\}} \end{array} \quad \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle Q \rangle\}}{\Sigma\{\langle Q \rangle\}} \end{array} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\neg P \bar{\vee} \neg Q\}}}{\Sigma\{\neg P \bar{\vee} \neg Q\}}}{\Sigma\{\emptyset\}} \frac{\Sigma\{\neg P \bar{\vee} \neg Q\}}{\Sigma\{\emptyset\}}$$

and

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle P \downarrow Q \rangle\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}} \frac{\Sigma\{\neg P \bar{\wedge} \neg Q\}}{\Sigma\{\emptyset\}} \rightsquigarrow \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle P \rangle\}} \end{array} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\neg P \bar{\wedge} \neg Q\}}}{\Sigma\{\neg P \bar{\wedge} \neg Q\}}}{\Sigma\{\emptyset\}} \frac{\Sigma\{\neg P \bar{\wedge} \neg Q\}}{\Sigma\{\emptyset\}}$$

and similarly for  $\downarrow_2^{\text{wn}}$ . We can appeal to the inductive hypothesis because the cut-rank is reduced.

- If the cut formula is a  $\diamond$ -formula, there are six cases:

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma_1\{\langle P \rangle, [\Sigma_2]\}}{\Sigma_1\{\langle \diamond P \rangle, [\Sigma_2]\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma_1\{[\Sigma_2]\}} \frac{\Sigma_1\{\square \neg P, [\Sigma_2]\}}{\Sigma_1\{[\Sigma_2]\}} \rightsquigarrow \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma_1\{\langle P \rangle, [\Sigma_2]\}}{\Sigma_1\{\langle P \rangle, [\Sigma_2]\}} \end{array} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma_1\{\square \neg P, [\Sigma_2]\}}}{\Sigma_1\{[\Sigma_2]\}} \frac{\Sigma_1\{\square \neg P, [\Sigma_2]\}}{\Sigma_1\{[\Sigma_2]\}}$$

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle \diamond P \rangle\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}} \frac{\Sigma\{\square \neg P\}}{\Sigma\{\emptyset\}} \rightsquigarrow \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle P \rangle\}} \end{array} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\square \neg P\}}}{\Sigma\{\emptyset\}} \frac{\Sigma\{\square \neg P\}}{\Sigma\{\emptyset\}}$$

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma_1\{\langle P \rangle, [\Sigma_2]\}}{\Sigma_1\{\langle \diamond P \rangle, [\Sigma_2]\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma_1\{[\Sigma_2]\}} \frac{\Sigma_1\{\square \neg P, [\Sigma_2]\}}{\Sigma_1\{[\Sigma_2]\}} \rightsquigarrow \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma_1\{\langle P \rangle, [\Sigma_2]\}}{\Sigma_1\{\langle P \rangle, [\Sigma_2]\}} \end{array} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma_1\{\square \neg P, [\Sigma_2]\}}}{\Sigma_1\{[\Sigma_2]\}} \frac{\Sigma_1\{\square \neg P, [\Sigma_2]\}}{\Sigma_1\{[\Sigma_2]\}}$$

$$\text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle \diamond P \rangle\}} \end{array} \quad \mathcal{D}_2 \parallel}{\Sigma\{\emptyset\}} \frac{\Sigma\{\square \neg P\}}{\Sigma\{\emptyset\}} \rightsquigarrow \text{cut}_2 \frac{\begin{array}{c} \mathcal{D}'_1 \parallel \\ \frac{\Sigma\{\langle P \rangle\}}{\Sigma\{\langle P \rangle\}} \end{array} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{\square \neg P\}}}{\Sigma\{\emptyset\}} \frac{\Sigma\{\square \neg P\}}{\Sigma\{\emptyset\}}$$

$$\begin{array}{ccc}
\frac{\frac{\mathcal{D}'_1 \parallel}{\text{cut}_2^{\diamond_4^{\text{wn}}} \frac{\Sigma_1\{[\langle \diamond P \rangle, \Sigma_2]\}}{\Sigma_1\{\langle \diamond P \rangle, [\Sigma_2]\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma_1\{\square \neg P, [\Sigma_2]\}}}{\Sigma_1\{[\Sigma_2]\}}}{\Sigma_1\{[\Sigma_2]\}} & \rightsquigarrow & \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma_1\{[\langle \diamond P \rangle, \Sigma_2]\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma_1\{[\Sigma_2]\}}}{\text{cut}_2} \quad \frac{(\square^{\text{wn}})^{-1} \frac{\Sigma_1\{\square \neg P, [\Sigma_2]\}}{\Sigma_1\{[\neg P], [\Sigma_2]\}} \quad \frac{\boxtimes_4^n \frac{\Sigma_1\{[\neg P], [\Sigma_2]\}}{\Sigma_1\{[\neg P], [\Sigma_2]\}}}{\square^{\text{wn}} \Sigma_1\{[\square \neg P, \Sigma_2]\}}}{\Sigma_1\{[\Sigma_2]\}}
\\
\frac{\frac{\mathcal{D}'_1 \parallel}{\text{cut}_2^{\diamond_5^{\text{wn}}} \frac{\Sigma\{[\Sigma_2]\}\{\langle \diamond P \rangle\}}{\Sigma\{[\Sigma_2, \langle \diamond P \rangle]\}\{\emptyset\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma\{[\Sigma_2, \square \neg P]\}\{\emptyset\}}}{\Sigma_1\{[\Sigma_2]\}\{\emptyset\}} & \rightsquigarrow & \frac{\frac{\mathcal{D}'_1 \parallel}{\Sigma_1\{[\Sigma_2]\}\{\langle \diamond P \rangle\}} \quad \frac{\mathcal{D}_2 \parallel}{\Sigma_1\{[\Sigma_2, \square \neg P]\}\{\emptyset\}}}{\text{cut}_2} \quad \frac{(\square^{\text{wn}})^{-1} \frac{\Sigma_1\{[\Sigma_2, \square \neg P]\}\{\emptyset\}}{\Sigma_1\{[\Sigma_2, [\neg P]]\}\{\emptyset\}} \quad \frac{\boxtimes_5^n \frac{\Sigma_1\{[\Sigma_2, [\neg P]]\}\{\emptyset\}}{\Sigma_1\{[\Sigma_2]\}\{[\neg P]\}}}{\square^{\text{wn}} \Sigma_1\{[\Sigma_2]\}\{\square \neg P\}}}{\Sigma_1\{[\Sigma_2]\}\{\emptyset\}}
\end{array}$$

In all six cases we use the admissibility of the structural rules, which is where we need the axiomatic completeness condition, and the invertibility of the  $\square$  rule (Lemma 7.2.7). In the first four cases we can apply the inductive hypothesis because the cut-rank is reduced. In the last two cases we can proceed by appealing to the inductive hypothesis because of a smaller height.  $\odot$

By applying this lemma to all cut instances in the derivation, starting with a topmost one, we can transform a proof in  $\text{wnK} + \diamond_X^{\text{wn}} + \{\text{cut}_1, \text{cut}_2\}$  into a cut-free proof in  $\text{wnK} + \diamond_X^{\text{wn}}$ , from which we get a cut-free proof in  $\text{fnK} + \diamond_X^{\text{fn}}$  using Lemma 7.2.3. This finalises the proof of the cut-elimination theorem (Theorem 7.2.6), and therefore of the completeness theorem (Theorem 7.2.2).

Although we were ultimately interested in strongly focused proofs, the cut-elimination theorem (Theorem 7.2.6) was proved in the weakly focused system and made essential use of the admissibility of weakening by arbitrary formulas, including negative ones, and of the possibility of applying invertible rules even within a focus phase. Of course, thanks to Lemma 7.2.3, we also have a cut-elimination proof for  $\text{fnK} + \diamond_X^{\text{fn}}$ , but this is not entirely satisfactory: it is not an internal proof, i.e., a sequence of cut reductions for that stays inside this system.

One possible response to this issue might be to try to redo the cut-elimination using just  $\text{fnK} + \diamond_X^{\text{fn}}$ , but this quickly gets rather complicated because we no longer have access to the weakening rules in the case where the weakened structure contains negative formulas. Indeed, similar attempts to re-prove this meta-theory (for the sequent calculus) usually solve this problem by adding a larger collection of cuts leading to a lengthy and intricate argument [26, 82, 126]. To avoid this complexity, it is better to consider the focused proof system in a *synthetic* form where the logical inference rules for the various connectives are composed as much as possible, as we shall see in the next chapter.

## Folding nested sequents: synthetic systems

In this chapter, we present synthetic nested sequent systems. After the design of the systems, the main contributions are: (i) a proof of cut-elimination for the synthetic nested calculus, which this time is fully internal, that is, the reductions steps stay inside the system itself; and (ii) a proof of completeness of the focused system with respect to the non-focused system by completeness of the synthetic system. These proofs might seem a little redundant once spelt out, but they witness a quite different type of reasoning than in the previous focused system. Synthetic rules are a compact notation of the focused rules that somehow gives us access to a slightly more global behaviour while still being able to reason meta-theoretically by local transformations on proofs.

### 8.1 Synthetic nested sequents

In this section we are going to introduce synthetic systems for the same logics as in the previous chapter: for the whole S5 cube in the classical case, and for a subset of it in the intuitionistic case. As we have seen in Chapter 4, since the boundary rules between phases (the rules for the shift operators,  $\downarrow$  and  $\uparrow$ ) in a focused proof are limited to conclusions that are either neutral or focused, we can see any focused derivation as progressing in large *synthetic* steps where the rest of the rules in between the shifts rules are ignored. Therefore, we can design a system that basically only contains the shift rules [139, 19].

The basis of the design for these synthetic systems is to generalise the subformula relation into a *substructure relation*, written  $\in$ , that determines, for a given focus, which formulas would be present as the beginning and the end of a focused phase that would start by deciding on this formula. Since only neutral formulas occur at the boundaries, these substructures would consist of neutral formulas.

#### Classical system

We recall that a *neutral sequent* is a polarised sequent consisting only of neutral formulas and therefore built according to the grammar:

$$\Gamma ::= \emptyset \mid \Gamma, n \mid \Gamma, \bar{p} \mid \Gamma, \uparrow P \mid \Gamma_1, [\Gamma_2]$$

For any negative formula, there is a collection of corresponding neutral nested sequents that represents one of the possible branches taken in a sequence of negative rules applied to the formula. The nested sequent  $\Gamma$  *matches* the negative formula  $N$ , written  $\Gamma \in N$ , if it is derivable from the rules in the first line of Figure 8.1.

We introduce a new type of sequents called *matching sequents*, that we can see as the complement of neutral sequents, and that are built according to the grammar:

$$\Lambda ::= \emptyset \mid p, \Lambda \mid \bar{n}, \Lambda \mid \downarrow N, \Lambda \mid \Lambda_1, [\Lambda_2]$$

|   |                                |  |   |   |  |
|---|--------------------------------|--|---|---|--|
| Substructure matching   |                                |  |   |   |  |
| $\frac{}{n \in n}$  | $\frac{}{\bar{p} \in \bar{p}}$ | $\frac{}{\uparrow P \in \uparrow P}$                                       | $\frac{\Gamma_1 \in M \quad \Gamma_2 \in N}{\Gamma_1, \Gamma_2 \in M \bar{\vee} N}$ | $\frac{}{\emptyset \in \bar{\perp}}$  | $\frac{\Gamma \in M}{\Gamma \in M \bar{\wedge} N}$   |
| $\frac{}{p \in p}$  | $\frac{}{\bar{n} \in \bar{n}}$ | $\frac{}{\downarrow N \in \downarrow N}$                                   | $\frac{\Lambda \in P}{\Lambda \in P \bar{\vee} Q}$                                  | $\frac{\Lambda \in Q}{\Lambda \in P \bar{\vee} Q}$  | $\frac{\Lambda_1 \in P \quad \Lambda_2 \in Q}{\Lambda_1, \Lambda_2 \in P \bar{\wedge} Q}$  |
| Neutral $\rightarrow$ Focus   |                                |  |   |   |  |
| $\frac{\uparrow^{\text{sn}} \Lambda \in P \quad \Gamma \{ \uparrow P, \langle \Lambda \rangle \}}{\Gamma \{ \uparrow P \}}$   |                                |  |   |   |  |
| Focus phase   |                                |  |   |   |  |
| $\text{spl}^{\text{sn}} \frac{\Gamma \{ \langle \Lambda_1 \rangle \} \quad \Gamma \{ \langle \Lambda_2 \rangle \}}{\Gamma \{ \langle \Lambda_1, \Lambda_2 \rangle \}} \quad \# \Lambda_1 > 0, \# \Lambda_2 > 0$   |                                |  | $\text{fin}^{\text{sn}} \frac{}{\Gamma \{ \langle \emptyset \rangle \}}$            |   | $\diamond_k^{\text{sn}} \frac{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \}}{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \}}$ |
| Identity  |                                |  | Focus $\rightarrow$ Neutral   |   |  |
| $\text{id}_+^{\text{sn}} \frac{}{\Gamma \{ \bar{p}, \langle p \rangle \}}$  |                                | $\text{id}_-^{\text{sn}} \frac{}{\Gamma \{ n, \langle \bar{n} \rangle \}}$ |   | $\downarrow^{\text{sn}} \frac{\{ \Gamma_1 \{ \Gamma_2 \} : \Gamma_2 \in N \}}{\Gamma_1 \{ \langle \downarrow N \rangle \}}$ |  |
| $\diamond_t^{\text{sn}} \frac{\Gamma \{ \langle \Lambda \rangle \}}{\Gamma \{ \langle \Lambda \rangle \}} \quad \diamond_b^{\text{sn}} \frac{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \}}{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \}} \quad \diamond_d^{\text{sn}} \frac{\Gamma \{ \langle \Lambda \rangle \}}{\Gamma \{ \langle \Lambda \rangle \}} \quad \diamond_4^{\text{sn}} \frac{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \}}{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \}} \quad \diamond_5^{\text{sn}} \frac{\Gamma_1 \{ \langle \Gamma_2 \rangle \{ \langle \Lambda \rangle \} \}}{\Gamma_1 \{ \langle \Gamma_2, \langle \Lambda \rangle \rangle \} \{ \emptyset \}}$ |                                |  |   |   |  |

Figure 8.1: System snK and extensions for t, b, d, 4, and 5

For any positive formula, there is a collection of corresponding matching nested sequents that represents one of the possible choices if a sequence of positive rules are applied to the formula. The nested sequent  $\Lambda$  *matches* the positive formula  $P$ , written  $\Lambda \in P$ , if it is derivable from the rules in the second line of Figure 8.1.

Finally, a *contextualising sequent* is of the form  $\Gamma \{ \langle \Lambda \rangle \}$  where  $\Gamma \{ \}$  is a neutral sequent context and  $\Lambda$  is a matching sequent. The meaning is again extended to obey  $\text{fm}^n(\Gamma \{ \langle \Lambda \rangle \}) \equiv \text{fm}^n(\Gamma \{ \uparrow \text{fm}^n(\Lambda) \})$ .

The synthetic system is built using neutral and contextualising sequents. The rules of  $\text{snK} + \diamond_X^{\text{sn}}$  (for any  $X \subseteq \{t, b, d, 4, 5\}$ ) are shown in Figure 8.1. As before, we define  $\text{snK} + \diamond_X^{\text{sn}}$  to be  $\text{snK}$  extended with  $\diamond_x^{\text{sn}}$  for every  $x \in X$ . The rule  $\uparrow^{\text{sn}}$  is similar to the rule  $\uparrow^{\text{fn}}$  from  $\text{fnK}$ , except that, instead of focusing on the formula  $N$  itself, it selects one of its substructures for *contextualisation*. The contextualisation rules consist of the rules  $\{ \text{spl}^{\text{sn}}, \text{fin}^{\text{sn}}, \diamond_k^{\text{sn}}, \diamond_t^{\text{sn}}, \diamond_b^{\text{sn}}, \diamond_4^{\text{sn}}, \diamond_d^{\text{sn}}, \diamond_5^{\text{sn}} \}$  that divide up or move the focus among the premisses of the rule. The rule  $\text{spl}^{\text{sn}}$  has a side condition that neither of the foci in the premisses is empty, otherwise it could be applied infinitely often; the  $\text{fin}^{\text{sn}}$  rule handles the case of an empty focus instead. The modal rules require the focus in the conclusion to be bracketed.

Once the focus has been reduced to a single formula by the other rules, it must be neutral again, namely either atomic or a shifted formula. If it is an atom, we apply one of the  $\text{id}^{\text{sn}}$  rules. If it is a shifted formula, we use  $\downarrow^{\text{sn}}$  which produces one premiss per substructure of the principal formula.

The duality between the synthetic rules  $\downarrow^{\text{sn}}$  and  $\uparrow^{\text{sn}}$  can be seen as meta-quantification over substructures: the positive rule  $\uparrow^{\text{sn}}$  quantifies *existentially* over the substructures of  $P$  and pick one such  $\Lambda$  as a focus in the unique premiss, while the negative rule quantifies *universally*, and so the rule actually consists of one premiss for each way in which to prove  $\Gamma \in N$ . For example, if  $N$  is  $\bar{a} \bar{\wedge} \square(\bar{b} \bar{\vee} \uparrow P)$ , we know that  $\bar{a} \in N$  and  $[\bar{b}, P] \in N$ , so the instance

of  $\downarrow^{\text{sn}}$  in this case is:

$$\downarrow^{\text{sn}} \frac{\Gamma\{\bar{a}\} \quad \Gamma\{\bar{b}, P\}}{\Gamma\{\langle \downarrow(\bar{a} \bar{\wedge} \square(\bar{b} \bar{\vee} \uparrow P)) \rangle\}}$$

It is instructive to compare  $\text{snK} + \diamond_{\mathbb{X}}^{\text{sn}}$  with  $\text{fnK} + \diamond_{\mathbb{X}}^{\text{fn}}$  for a given  $\mathbb{X} \subseteq \{\text{t}, \text{b}, \text{d}, 4, 5\}$ . In the latter system, the focus  $\langle \diamond P \rangle$  is used to drive the modal rules  $\{\diamond_{\text{k}}^{\text{fn}}, \diamond_{\text{t}}^{\text{fn}}, \diamond_{\text{b}}^{\text{fn}}, \diamond_{\text{d}}^{\text{fn}}, \diamond_4^{\text{fn}}, \diamond_5^{\text{fn}}\}$ . Such modal rules can be applied only a fixed number of times before  $\langle \diamond P \rangle$  needs to be reduced to  $\langle P \rangle$ , and some logical or identity rules need to be used; this is necessary to finish the proof since foci can never be weakened. Thus, the analysis of  $P$  is forced to be interleaved with the modal rules for  $\diamond P$ , as shown by the alternation of  $\diamond_{\text{k}}^{\text{fn}}$  and  $\downarrow^{\text{fn}}$  in the derivation on the left below. In  $\text{snK} + \diamond_{\mathbb{X}}^{\text{sn}}$ , in contrast, the  $\uparrow^{\text{sn}}$  rule itself performs the analysis of  $P$  up front to produce a synthetic substructure; the modal rules  $\{\diamond_{\text{k}}^{\text{sn}}, \diamond_{\text{t}}^{\text{sn}}, \diamond_{\text{b}}^{\text{sn}}, \diamond_{\text{d}}^{\text{sn}}, \diamond_4^{\text{sn}}, \diamond_5^{\text{sn}}\}$  then work entirely at the level of focused substructures, as in the derivation on the right below.

$$\begin{array}{c} \text{id} \frac{}{\diamond(p \downarrow \diamond p), [\langle \langle p \rangle, \bar{p} \rangle]} \\ \diamond_{\text{k}}^{\text{fn}} \frac{}{\diamond(p \downarrow \diamond p), [\langle \langle \diamond p \rangle, \bar{p} \rangle]} \\ \downarrow^{\text{fn}} \frac{}{\diamond(p \downarrow \diamond p), [\langle \langle p \downarrow \diamond p \rangle, \bar{p} \rangle]} \\ \diamond_{\text{k}}^{\text{fn}} \frac{}{\uparrow \diamond(p \downarrow \diamond p), \langle \langle \diamond(p \downarrow \diamond p) \rangle \rangle, [\bar{p}]} \\ \uparrow^{\text{fn}} \frac{}{\uparrow \diamond(p \downarrow \diamond p), [\bar{p}]} \end{array} \quad \begin{array}{c} \text{id} \frac{}{\diamond(p \downarrow \diamond p), [\langle \langle \bar{p} \rangle, \bar{p} \rangle]} \\ \diamond_{\text{k}}^{\text{sn}} \frac{}{\diamond(p \downarrow \diamond p), [\langle \langle \bar{p} \rangle \rangle, \bar{p}]} \\ \diamond_{\text{k}}^{\text{sn}} \frac{}{\diamond(p \downarrow \diamond p), \langle \langle \langle \bar{p} \rangle \rangle \rangle, [\bar{p}]} \\ \uparrow^{\text{sn}} \frac{[\bar{p}] \in \diamond(p \downarrow \diamond p)}{\diamond(p \downarrow \diamond p), [\bar{p}]} \end{array}$$

Thus, the modal rules of  $\text{snK} + \diamond_{\mathbb{X}}^{\text{sn}}$  are some sort of hybrids between structural rules and logical propagation rules.

**8.1.1 Theorem** (Chaudhuri, Marin, and Straßburger [22]) *Let  $\mathbb{X} \subseteq \{\text{t}, \text{b}, \text{d}, 4, 5\}$  be axiomatically complete. Let  $R$  be a negative neutral formula, the following are equivalent.*

1.  $\partial(R)$  is a theorem of  $\text{K} + \mathbb{X}$ .
2.  $R$  is provable in  $\text{snK} + \diamond_{\mathbb{X}}^{\text{sn}}$ .

### Intuitionistic system

As before, we recall the definition of *neutral input sequents* (written  $\Lambda$ ) and *neutral full sequents* (written  $\Pi$ ):

$$\Lambda ::= \emptyset \mid L, \Lambda \mid [\Lambda_1], \Lambda_2 \quad \Pi ::= \Lambda, R \mid \Lambda, [\Pi] \quad \Delta ::= \Lambda \mid \Pi$$

In the intuitionistic case, a *contextualising sequent* is then a neutral full sequent that contains an occurrence of a *focus* that can be either an input or a full neutral sequent  $\Delta$ , i.e.  $\Pi\{\langle \Delta \rangle\}$  where  $\Pi\{\}$  is a neutral context. The *meaning* of a focused sequent is written using  $\text{fm}^n(\cdot)$  and obeys:  $\text{fm}^n(\Pi\{\langle \Lambda \rangle\}) = \text{fm}^n(\Pi^*\{\uparrow \text{fm}^n(\Lambda)\})$  and  $\text{fm}^n(\Pi_1\{\langle \Pi_2 \rangle\}) = \text{fm}^n(\Pi_1\{\downarrow \text{fm}^n(\Pi_2)\})$ .

The synthetic system  $\text{snIK} + \diamond_{\mathbb{X}}^{\text{sn}}$  is then built in a similar manner as  $\text{snK}$ , as shown in Figure 8.2. Remarkably, the intuitionistic synthetic system has far fewer rules than the focused one  $\text{fnIK} + \mathbb{X}^{\text{fn}}$ . In particular, there is exactly one modal rule for every modal axiom, unlike  $\text{fnIK} + \mathbb{X}^{\text{fn}}$  that needed both left  $\square_{\text{L}}^{\text{fn}}$  and right  $\diamond_{\text{R}}^{\text{fn}}$  versions. Nevertheless, this system will be sound and complete with respect to both  $\text{nlK} + \diamond_{\mathbb{X}}^{\text{fn}}$  and  $\text{fnIK} + \mathbb{X}^{\text{fn}}$ , and therefore to  $\text{IK} + \mathbb{X}$  as well. Note that as in the rule  $\uparrow_{\text{R}}^{\text{fn}}$  in the focused system, we need to remove the right-neutral formula from the surrounding context in the  $\uparrow_{\text{R}}^{\text{sn}}$  rule.

**8.1.2 Theorem** (Chaudhuri, Marin, and Straßburger [23]) *Let  $\mathbb{X} \subseteq \{\text{t}, \text{b}, 4, 5\}$  be axiomatically complete. Let  $R$  be a right-neutral formula, the following are equivalent.*

1.  $\partial(R)$  is a theorem of  $\text{IK} + \mathbb{X}$ .

|   |  |  |  |
|---|--|--|--|
| Substructure matching   |  |  |  |
| $\frac{}{n \in n}$  | $\frac{}{\uparrow P \in \uparrow P}$   | $\frac{\Lambda \in P \quad \Pi \in N}{\Lambda, \Pi \in P \supset N}$   | $\frac{\Pi \in M}{\Pi \in M \bar{\wedge} N}$   |
|   |  | $\frac{\Pi \in N}{\Pi \in M \bar{\wedge} N}$   | $\frac{\Pi \in N}{[\Pi] \in \square N}$  |
| $\frac{}{p \in p}$  | $\frac{}{\downarrow N \in \downarrow N}$   | $\frac{\Lambda \in P}{\Lambda \in P \vee Q}$   | $\frac{\Lambda \in Q}{\Lambda \in P \vee Q}$   |
|   |  | $\frac{\Lambda_1 \in P \quad \Lambda_2 \in Q}{\Lambda_1, \Lambda_2 \in P \bar{\wedge} Q}$  | $\frac{}{\emptyset \in \dagger}$   |
|   |  | $\frac{\Lambda \in P}{[\Lambda] \in \diamond P}$   |  |
| Neutral $\rightarrow$ Focus   |  |  |  |
| $\frac{\uparrow_R^{\text{sn}} \Lambda_2 \in P \quad \Lambda_1 \{\uparrow P, \langle \Lambda_2 \rangle\}}{\Lambda_1 \{\uparrow P\}}$           |  | $\frac{\downarrow_L^{\text{sn}} \Pi_2 \in N \quad \Pi_1 \{\downarrow N, \langle \Pi_2 \rangle\}}{\Pi_1 \{\downarrow N\}}$  |  |
| Focus phase   |  |  |  |
| $\text{spl}^{\text{sn}} \frac{\Pi \{\langle \Delta \rangle\} \quad \Pi \{\langle \Lambda \rangle\}}{\Pi \{\langle \Delta, \Lambda \rangle\}}$ | $\# \Delta > 0, \# \Lambda > 0$  | $\text{fin}^{\text{sn}} \frac{}{\Pi \{\langle \emptyset \rangle\}}$  | $\diamond_k^{\text{sn}} \frac{\Delta_1 \{\langle \Delta_2, \langle \Delta_3 \rangle \rangle\}}{\Delta_1 \{\langle \Delta_2 \rangle, \langle \Delta_3 \rangle \}}$                  |
| Identity  |  | Focus $\rightarrow$ Neutral  |  |
| $\text{id}_R^{\text{sn}} \frac{}{\Pi \{p, \langle p \rangle\}}$   | $\text{id}_L^{\text{sn}} \frac{}{\Lambda \{\langle n \rangle, n\}}$  | $\downarrow_R^{\text{sn}} \frac{\{\Pi_1^* \{\Pi_2\} : \Pi_2 \in N\}}{\Pi_1 \{\langle \downarrow N \rangle\}}$  | $\uparrow_L^{\text{sn}} \frac{\{\Pi \{\Lambda\} : \Lambda \in P\}}{\Pi \{\langle \uparrow P \rangle\}}$  |
| $\diamond_t^{\text{sn}} \frac{\Delta_1 \{\langle \Delta_2 \rangle\}}{\Delta_1 \{\langle \Delta_2 \rangle\}}$                                  | $\diamond_b^{\text{sn}} \frac{\Delta_1 \{\langle \Delta_2 \rangle, \langle \Delta_3 \rangle\}}{\Delta_1 \{\langle \Delta_2 \rangle, \langle \Delta_3 \rangle\}}$ | $\diamond_4^{\text{sn}} \frac{\Delta_1 \{\langle \Delta_2 \rangle, \langle \Delta_3 \rangle\}}{\Delta_1 \{\langle \Delta_2 \rangle, \langle \Delta_3 \rangle\}}$ | $\diamond_5^{\text{sn}} \frac{\Gamma \{\langle \Delta_2 \rangle, \langle \Delta_3 \rangle\} \{\emptyset\}}{\Delta_1 \{\langle \Delta_2 \rangle \} \{ \langle \Delta_3 \rangle \}}$ |

Figure 8.2: System snIK and extensions for t, b, 4, and 5

2.  $R$  is provable in  $\text{snIK} + \diamond_X^{\text{sn}}$ .

**8.1.3 Example** As an example we give the synthetic version of the derivation in Example 3.5.3:

$$\begin{array}{c}
\text{id}^{\text{sn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle [p], p, n \rangle} \quad \text{id}^{\text{sn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle [n], p, n \rangle} \\
\downarrow_k^{\text{sn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle [p] \rangle, [p], n} \quad \downarrow_k^{\text{sn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle [n] \rangle, [p], n} \\
\text{spl}^{\text{sn}} \frac{}{\downarrow(\diamond p \supset \square n), \langle [p], [n] \rangle, [p], n} \\
\downarrow_L^{\text{sn}} \frac{}{\downarrow(\diamond p \supset \square n), [p], n} \\
\downarrow_R^{\text{sn}} \frac{}{R, \langle \downarrow(\downarrow(\diamond p \supset \square n) \supset \square(p \supset n)) \rangle} \\
\uparrow_R^{\text{sn}} \frac{}{R}
\end{array}$$

together with the corresponding substructures derivations for applying the rules  $\uparrow_R^{\text{sn}}$  and  $\downarrow_L^{\text{sn}}$ :

$$\begin{array}{cc}
\frac{p \in p \quad n \in n}{p, n \in p \supset n} & \frac{p \in p \quad n \in n}{[p] \in \diamond p \quad [n] \in \square n} \\
\frac{\downarrow(\diamond p \supset \square n) \in \downarrow(\diamond p \supset \square n) \quad [p], n \in \square(p \supset n)}{\downarrow(\diamond p \supset \square n), [p], n \in \downarrow(\diamond p \supset \square n) \supset \square(p \supset n)} & \frac{[p] \in \diamond p \quad [n] \in \square n}{[p], [n] \in \diamond p \supset \square n}
\end{array}$$

## 8.2 Synthetic meta-theory

In this section, we study the meta-theory, soundness, cut-elimination, and completeness, of these synthetic systems. Contrary to the last chapter, the synthetic meta-theory is as simple in the intuitionistic setting as in the classical one. As evidence of this phenomenon, we will make the opposite choice from last chapter and present all the results for the intuitionistic version. (Of course they are translatable to the classical system with minor usual adjustments.) Besides, the completeness of the synthetic systems gives an alternative, refined proof of completeness of the focused systems, not requiring us to appeal to the weakly focused systems.

## Soundness

In this section we show that the synthetic system is sound wrt. the unfocused system.

**8.2.1 Theorem** *Let  $X \subseteq \{t, b, 4, 5\}$ . Let  $R$  be a right-neutral formula. If  $R$  is provable in  $\text{snIK} + \diamond_X^{\text{sn}}$ , then  $\partial(R)^\circ$  is provable in  $\text{nIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$ .*

This theorem is easier to prove in two steps, by showing first that if  $R$  is provable in  $\text{snIK} + \diamond_X^{\text{sn}}$  then it is provable in  $\text{fnIK} + \diamond_X^{\text{fn}}$  (below), and then by stating a parallel result to Theorem 7.2.1 for the intuitionistic systems, namely that if  $R$  is provable in  $\text{fnIK} + \diamond_X^{\text{fn}}$ , then it is also provable in  $\text{nIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$ .

**8.2.2 Lemma** *Let  $Y \subseteq \{t, b, 4, 5\}$  and let  $\Pi$  be a neutral full sequent. If  $\Pi$  is provable in  $\text{snIK} + \diamond_X^{\text{sn}}$ , then it is also provable in  $\text{fnIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$ .*

*Proof (Sketch).* The essential idea is to interpret any  $\text{snIK}$  focused sequent  $\Pi\{\langle\Delta\rangle\}$  as the  $\text{fnIK}$  focused sequent  $\Pi\{\langle\text{fm}^n(\Delta)\rangle\}$ . The rules of  $\text{snIK}$  can be simulated by  $\text{fnIK}$  because  $\Delta \in \text{fm}^n(\Delta)$ . Examples 7.1.4 and 8.1.3 illustrates this interpretation. The whole proof then works by induction on the given  $\text{snIK} + \diamond_X^{\text{sn}}$  derivation. The decision  $\uparrow_{\text{R}}^{\text{sn}}$  and  $\downarrow_{\text{L}}^{\text{sn}}$  rules are simulated by repeating the derivation of the substructure in the  $\text{fnIK}$  sequent rather than as a side premiss, in particular we use a matching derivation to determine how to choose between the two  $\vee$  rules. The  $\downarrow_{\text{L}}^{\text{sn}}$  and  $\uparrow_{\text{R}}^{\text{sn}}$  rules are easily simulated since the active rules of  $\text{fnIK}$  are precisely matched by the  $\in$  inferences. The  $\diamond_X^{\text{sn}}$  rules are simulated by  $\diamond_{\text{RX}}^{\text{fn}}$  or  $\square_{\text{LX}}^{\text{fn}}$  rules, depending on whether the focus contains a negative formula or not. Finally,  $\text{spl}^{\text{sn}}$  and  $\text{fin}^{\text{sn}}$  are simulated by  $\uparrow_{\text{R}}^{\text{fn}}$  and  $\dagger_{\text{R}}^{\text{fn}}$  respectively.  $\odot$

## Cut-elimination

In this section, we will show that the cut rule is admissible in  $\text{snIK} + \diamond_X^{\text{sn}}$ . To formulate the cut rule with a minimum of redundancy, we will need to slightly extend the *pruning* operations that we have already seen. As before, we allow the notation for neutral sequents to stand for sequents that contain at most one occurrence of a focus  $\langle\Delta\rangle$ . For such a sequent  $\Delta$ , we write  $\Delta^\emptyset$  to prune its focus if there is one (i.e., if  $\Delta = \Delta_1\{\langle\Delta_2\rangle\}$  for some  $\Delta_1\{\}$ , then  $\Delta^\emptyset = \Delta_1\{\emptyset\}$ ; otherwise  $\Delta^\emptyset = \Delta$ ). As before, this definition extends to contexts  $\Delta\{\}$ . For a context  $\Delta_1\{\}$ , we now write  $\Delta_1^*\{\Delta_2\}$  to mean  $\Delta_1\{\Delta_2\}$  if  $\Delta_2$  is an input sequent, and  $\Delta_1^*\{\Delta_2\}$  if  $\Delta_2$  is a full sequent.

The synthetic cut rule for  $\text{snIK} + \diamond_X^{\text{sn}}$  can then be written concisely as follows:

$$\text{cut} \frac{\Pi^*\{\Delta\} \quad \Pi^\emptyset\{\langle\Delta\rangle\}}{\Pi\{\emptyset\}}$$

The *rank* of this cut is defined as the multiset of the depths of the formulas in  $\Delta$ .

Before we can show that  $\text{cut}$  is admissible, we need to show the admissibility of the usual structural rules though in a synthetic setting. Note that the weakening rule  $w$  can only be applied to input contexts, and that we only need the instance of the contraction rule  $c$  applied to positive atoms.

**8.2.3 Lemma** *Let  $Y \subseteq \{t, b, 4, 5\}$ . The rules  $w$ ,  $c$ , and  $\boxtimes_k^n$  restricted to neutral and extended sequents (as appropriate) are height-preserving admissible in  $\text{snIK} + \diamond_X^{\text{sn}}$ , and if  $Y$  is axiomatically complete, then for every  $x \in Y$ , the rule  $\boxtimes_x^n$  (Figure 2.6) is admissible in  $\text{snIK} + \diamond_X^{\text{sn}}$ .*

This can be proved exactly in the same way as Lemma 7.2.7, i.e. by induction on the height of the derivation, only translating each derivation into the synthetic setting. Note that we do not allow, e.g., weakening  $\Delta_1\{\langle\Delta_2\rangle\}\{\emptyset\}$  to  $\Delta_1\{\langle\Delta_2\rangle\}\{N\}$  top-down; the latter is, in fact, not even a well-formed focused sequent.

**8.2.4 Lemma** *Let  $\mathsf{Y} \subseteq \{\mathsf{t}, \mathsf{b}, 4, 5\}$  be axiomatically complete. Given a proof of the form:*

$$\text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{\Delta\} \quad \mathcal{D}_2 \parallel \Pi^\emptyset \{\langle \Delta \rangle\}}{\Pi \{\emptyset\}}$$

where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are in  $\text{snIK} + \Phi_{\mathsf{X}}^{\text{sn}}$ , there is a proof of  $\Pi \{\emptyset\}$  in  $\text{snIK} + \Phi_{\mathsf{X}}^{\text{sn}}$ .

*Proof.* By lexicographic induction on the tuple  $\langle \text{rk}(\Delta), \text{ht}(\mathcal{D}_2), \text{ht}(\mathcal{D}_1) \rangle$ , splitting cases on the last rule instances in  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . Note that the last rule in  $\mathcal{D}_2$  always applies to the focus  $\langle \Delta \rangle$ . We will rewrite the derivation, written using  $\rightsquigarrow$ , by moving the instance of cut to a position of strictly lower measure or eliminating it entirely.

- First, let us consider the cases where  $\mathcal{D}_2$  ends with a structural rule:

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{\Delta, \Lambda\} \quad \text{spl}^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Pi^\emptyset \{\langle \Delta \rangle\} \quad \mathcal{D}''_2 \parallel \Pi^\emptyset \{\langle \Lambda \rangle\}}{\Pi^\emptyset \{\langle \Delta, \Lambda \rangle\}}}{\Pi \{\emptyset\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{\Delta, \Lambda\} \quad \text{w} \frac{\mathcal{D}'_2 \parallel \Pi^\emptyset \{\langle \Delta \rangle\}}{\Pi^\emptyset \{\langle \Delta \rangle, \Lambda \rangle\}} \quad \mathcal{D}''_2 \parallel \Pi^\emptyset \{\langle \Lambda \rangle\}}{\text{cut} \frac{\Pi \{\Lambda\}}{\Pi \{\emptyset\}}} \end{array}$$

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Delta_1^* \{[\Delta_2^*], [\Delta_3]\} \quad \Phi_k^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}{\Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}}{\Delta_1 \{[\Delta_2]\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Delta_1^* \{[\Delta_2^*], [\Delta_3]\} \quad \mathcal{D}'_2 \parallel \Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}{\text{cut} \frac{\Delta_1^* \{[\Delta_2^*], [\Delta_3]\}}{\Delta_1 \{[\Delta_2]\}}} \end{array}$$

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{[\Delta]\} \quad \Phi_d^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Pi^\emptyset \{\langle [\Delta] \rangle\}}{\Pi^\emptyset \{\langle [\Delta] \rangle\}}}{\Pi \{\emptyset\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{[\Delta]\} \quad \mathcal{D}'_2 \parallel \Pi^\emptyset \{\langle [\Delta] \rangle\}}{\text{cut} \frac{\Pi \{\emptyset\}}{\Pi \{\emptyset\}}} \end{array}$$

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{[\Delta]\} \quad \Phi_t^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Pi^\emptyset \{\langle [\Delta] \rangle\}}{\Pi^\emptyset \{\langle [\Delta] \rangle\}}}{\Pi \{\emptyset\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{[\Delta]\} \quad \mathcal{D}'_2 \parallel \Pi^\emptyset \{\langle [\Delta] \rangle\}}{\Pi \{\emptyset\}} \end{array}$$

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Delta_1^* \{[\Delta_2^*], [\Delta_3]\} \quad \Phi_b^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}{\Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}}{\Delta_1 \{[\Delta_2]\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Delta_1^* \{[\Delta_2^*], [\Delta_3]\} \quad \mathcal{D}'_2 \parallel \Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}{\text{cut} \frac{\Delta_1^* \{[\Delta_2^*], [\Delta_3]\}}{\Delta_1 \{[\Delta_2]\}}} \end{array}$$

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Delta_1^* \{[\Delta_2^*], [\Delta_3]\} \quad \Phi_4^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}{\Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}}{\Delta_1 \{[\Delta_2]\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Delta_1^* \{[\Delta_2^*], [\Delta_3]\} \quad \mathcal{D}'_2 \parallel \Delta_1^\emptyset \{[\Delta_2^\emptyset], \langle [\Delta_3] \rangle\}}{\text{cut} \frac{\Delta_1^* \{[\Delta_2^*], [\Delta_3]\}}{\Delta_1 \{[\Delta_2]\}}} \end{array}$$

$$\begin{array}{c} \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{[\Delta]\} \{\emptyset\} \quad \Phi_5^{\text{sn}} \frac{\mathcal{D}'_2 \parallel \Pi^\emptyset \{\emptyset\} \{\langle [\Delta] \rangle\}}{\Pi^\emptyset \{\emptyset\} \{\langle [\Delta] \rangle\}}}{\Pi \{\emptyset\} \{\emptyset\}} \rightsquigarrow \text{cut} \frac{\mathcal{D}_1 \parallel \Pi^* \{[\Delta]\} \{\emptyset\} \quad \mathcal{D}'_2 \parallel \Pi^\emptyset \{\emptyset\} \{\langle [\Delta] \rangle\}}{\text{cut} \frac{\Pi^* \{\emptyset\} \{\emptyset\}}{\Pi \{\emptyset\} \{\emptyset\}}} \end{array}$$

In the first case we can apply the induction hypothesis (twice) because  $\text{rk}(\Delta), \text{rk}(\Lambda) < \text{rk}(\Delta, \Lambda)$  and in the other cases because  $\text{ht}(\mathcal{D}'_2) < \text{ht}(\mathcal{D}_2)$  as the rank stays the

same. We abuse the pruning notation in the cases for  $\Phi_k^{\text{sn}}$ ,  $\Phi_b^{\text{sn}}$  and  $\Phi_4^{\text{sn}}$  by writing  $\Delta_1^* \{[\Delta_2^*, \{\}]\}$  or  $\Delta_1^* \{[\Delta_2^*, \{\}]\}$  to denote the pruned context  $\Delta^* \{\}$  when  $\Delta \{\} = \Delta_1 \{[\Delta_2], \{\}\}$  or  $\Delta \{\} = \Delta_1 \{[\Delta_2, \{\}]\}$  respectively. Note the use of Lemma 8.2.3.

- If the last rule in  $\mathcal{D}_2$  is an axiom, we have one of the following three cases:

$$\begin{array}{c} \mathcal{D}_1 \parallel \\ \text{cut} \frac{\Pi \{\emptyset\} \quad \text{fin}^{\text{sn}} \frac{}{\Pi^\emptyset \{ \langle \emptyset \rangle \}}}{\Pi \{\emptyset\}} \quad \rightsquigarrow \quad \mathcal{D}_1 \parallel \\ \Pi \{\emptyset\} \end{array} \quad \begin{array}{c} \mathcal{D}_1 \parallel \\ \text{cut} \frac{\Lambda \{n\} \quad \text{id}^{\text{sn}} \frac{}{\Lambda^\emptyset \{n, \langle n \rangle\}}}{\Lambda \{n\}} \quad \rightsquigarrow \quad \mathcal{D}_1 \parallel \\ \Lambda \{n\} \end{array}$$

$$\begin{array}{c} \mathcal{D}_1 \parallel \\ \text{cut} \frac{\Pi \{p, p\} \quad \text{id}^{\text{sn}} \frac{}{\Pi^\emptyset \{p, \langle p \rangle\}}}{\Pi \{p\}} \quad \rightsquigarrow \quad \text{c} \frac{\mathcal{D}_1 \parallel \\ \Pi \{p, p\}}{\Pi \{p\}} \end{array}$$

For the third case we use the admissibility of atomic contraction.

- Finally, the last rule in  $\mathcal{D}_2$  can be  $\downarrow_{\text{L}}^{\text{sn}}$  or  $\uparrow_{\text{R}}^{\text{sn}}$ , and if at the same time the last rule in  $\mathcal{D}_1$  is the corresponding  $\downarrow_{\text{R}}^{\text{sn}}$  or  $\uparrow_{\text{L}}^{\text{sn}}$  on the cut formula (the cut sequent has to be a singleton in that case), we have one of the two principal cases:

$$\begin{array}{c} \mathcal{D}'_1 \parallel \\ \downarrow_{\text{L}}^{\text{sn}} \frac{\Pi_2 \in N \quad \Pi_1 \{ \downarrow N, \langle \Pi_2 \rangle \}}{\text{cut} \frac{\Pi_1 \{ \downarrow N \}}{\Pi_1 \{\emptyset\}}} \quad \downarrow_{\text{R}}^{\text{sn}} \frac{\left\{ \begin{array}{c} \mathcal{D}_{\Pi_2} \parallel \\ \Pi_1^* \{ \Pi_2 \} \end{array} \right\}_{\Pi_2 \in N}}{\Pi_1 \{ \langle \downarrow N \rangle \}} \quad \rightsquigarrow \quad \begin{array}{c} \mathcal{D}'_1 \parallel \quad \mathcal{D}_2 \parallel \\ \text{cut} \frac{\Pi_1 \{ \downarrow N, \langle \Pi_2 \rangle \} \quad \Pi_1 \{ \langle \downarrow N \rangle \}}{\text{cut} \frac{\Pi_1 \{ \langle \Pi_2 \rangle \}}{\Pi_1 \{\emptyset\}}} \quad \mathcal{D}_{\Pi_2} \parallel \\ \Pi_1^* \{ \Pi_2 \} \end{array} \end{array}$$

$$\begin{array}{c} \mathcal{D}'_1 \parallel \\ \uparrow_{\text{R}}^{\text{sn}} \frac{\Lambda \in P \quad \Pi^* \{ \uparrow P, \langle \Lambda \rangle \}}{\text{cut} \frac{\Pi^* \{ \uparrow P \}}{\Pi \{ f \}}} \quad \uparrow_{\text{L}}^{\text{sn}} \frac{\left\{ \begin{array}{c} \mathcal{D}_\Lambda \parallel \\ \Pi \{ \Lambda \} \end{array} \right\}_{\Lambda \in P}}{\Pi \{ \langle \uparrow P \rangle \}} \quad \rightsquigarrow \quad \begin{array}{c} \mathcal{D}'_1 \parallel \quad \mathcal{D}_2 \parallel \\ \text{cut} \frac{\Pi^* \{ \uparrow P, \langle \Lambda \rangle \} \quad \Pi \{ \langle \uparrow P \rangle \}}{\text{cut} \frac{\Pi \{ \langle \Lambda \rangle \}}{\Pi \{\emptyset\}}} \quad \mathcal{D}_\Lambda \parallel \\ \Pi \{ \Lambda \} \end{array} \end{array}$$

In both cases we have to apply the induction hypothesis twice: first to the upper cut because  $\text{ht}(\mathcal{D}'_1) < \text{ht}(\mathcal{D}_1)$ , and then to the lower cut because  $\text{rk}(\Pi_2) < \text{rk}(\downarrow N)$  and  $\text{rk}(\Lambda) < \text{rk}(\uparrow P)$ . After the reduction step the focus is not in the same branch any more, so the branch considered to be  $\mathcal{D}_1$  or  $\mathcal{D}_2$  may change, but since the rank has decreased strictly this proceeds along the same inductive argument.

- Of course, when the last rule in  $\mathcal{D}_2$  is  $\downarrow_{\text{L}}^{\text{sn}}$  or  $\uparrow_{\text{R}}^{\text{sn}}$ , the last rule in  $\mathcal{D}_1$  does not need to be the corresponding  $\downarrow_{\text{R}}^{\text{sn}}$  or  $\uparrow_{\text{L}}^{\text{sn}}$  rule. In that case we have a commutative case, that is, the last rule in  $\mathcal{D}_1$  is permuted under the cut:

$$\begin{array}{c} \mathcal{D}'_1 \parallel \\ \text{r} \frac{\Pi_1^* \{ \Delta \}}{\Pi^* \{ \Delta \}} \quad \mathcal{D}_2 \parallel \\ \text{cut} \frac{\Pi^* \{ \Delta \} \quad \Pi^\emptyset \{ \langle \Delta \rangle \}}{\Pi \{\emptyset\}} \quad \rightsquigarrow \quad \begin{array}{c} \mathcal{D}'_1 \parallel \quad \mathcal{D}_2 \parallel \\ \text{cut} \frac{\Pi_1^* \{ \Delta \} \quad \Pi_1^\emptyset \{ \langle \Delta \rangle \}}{\text{r} \frac{\Pi_1 \{\emptyset\}}{\Pi \{\emptyset\}}} \end{array} \end{array}$$

The situation above applies if  $\text{r}$  is  $\Phi_k^{\text{sn}}$  or any of the  $\Phi_x^{\text{sn}}$  rules, because then there is a focus in  $\Pi \{\}$  which is moved by  $\text{r}$ , and we have  $\Pi_1^\emptyset \{\} = \Pi^\emptyset \{\}$ . It also applies if  $\text{r}$  is one of  $\uparrow_{\text{L}}^{\text{sn}}$  or  $\downarrow_{\text{R}}^{\text{sn}}$  because then  $\Pi \{\}$  contains no focus and therefore  $\Pi_1^\emptyset \{\} = \Pi^\emptyset \{\}$ . If the last rule in  $\mathcal{D}_1$  is  $\text{spl}^{\text{sn}}$  the situation is similar, and if it is one of  $\text{id}^{\text{sn}}$  or  $\text{fin}^{\text{sn}}$ , then the cut disappears trivially. Note that the last rule in  $\mathcal{D}_1$  is not applying to  $\Delta$  (which is a singleton) because otherwise it would be a principal case.

The only non-trivial commutative cases are when the focus in  $\Pi\{\}$  is released by the last rule in  $\mathcal{D}_1$  which can be either a  $\uparrow_L^{\text{sn}}$  or a  $\downarrow_R^{\text{sn}}$ . In the  $\uparrow_L^{\text{sn}}$ -case, we can reduce as follows:

$$\frac{\uparrow_L^{\text{sn}} \left\{ \frac{\mathcal{D}_\Lambda \parallel}{\Pi^*\{\Lambda\}\{\Delta\}} \right\}_{\Lambda \in P} \quad \frac{\mathcal{D}_2 \parallel}{\Pi\{\emptyset\}\{\langle \Delta \rangle\}}}{\text{cut} \frac{\Pi^*\{\langle \uparrow P \rangle\}\{\Delta\}}{\Pi\{\langle \uparrow P \rangle\}\{\emptyset\}}} \rightsquigarrow \uparrow_L^{\text{sn}} \left\{ \frac{\mathcal{D}_\Lambda \parallel \quad \frac{\mathcal{D}_2 \parallel}{\Pi\{\emptyset\}\{\langle \Delta \rangle\}}}{\text{cut} \frac{\Pi^*\{\Lambda\}\{\Delta\} \quad \frac{\Pi\{\emptyset\}\{\langle \Delta \rangle\}}{\Pi\{\Lambda\}\{\emptyset\}}}{\Pi\{\langle \uparrow P \rangle\}\{\emptyset\}}} \right\}_{\Lambda \in P}$$

and we only need height-preserving admissibility of weakening in order to apply the induction hypothesis, using  $\text{ht}(\mathcal{D}_\Lambda) < \text{ht}(\mathcal{D}_1)$ .

In the  $\downarrow_R^{\text{sn}}$ -case we need to distinguish whether  $\Delta$  is of the form  $\uparrow P$  or  $\downarrow N$ . In the first case the cut disappears:

$$\frac{\downarrow_R^{\text{sn}} \left\{ \frac{\mathcal{D}_{\Pi_2} \parallel}{\Pi_1^*\{\Pi_2\}\{\emptyset\}} \right\}_{\Pi_2 \in N} \quad \frac{\mathcal{D}_2 \parallel}{\Pi_1\{\emptyset\}\{\langle \uparrow P \rangle\}}}{\text{cut} \frac{\Pi_1^*\{\langle \downarrow N \rangle\}\{\uparrow P\}}{\Pi_1\{\langle \downarrow N \rangle\}\{\emptyset\}}} \rightsquigarrow \downarrow_R^{\text{sn}} \frac{\left\{ \frac{\mathcal{D}_{\Pi_2} \parallel}{\Pi_1^*\{\Pi_2\}\{\emptyset\}} \right\}_{\Pi_2 \in N}}{\Pi_1\{\langle \downarrow N \rangle\}\{\emptyset\}}$$

and in the second we again use height-preserving admissibility of weakening in order to apply the induction hypothesis, as  $\text{ht}(\mathcal{D}_{\Pi_2}) < \text{ht}(\mathcal{D}_1)$ :

$$\frac{\downarrow_R^{\text{sn}} \left\{ \frac{\mathcal{D}_{\Pi_2} \parallel}{\Pi_1^*\{\Pi_2\}\{\downarrow M\}} \right\}_{\Pi_2 \in N} \quad \frac{\downarrow_R^{\text{sn}} \left\{ \frac{\mathcal{D}_{\Pi_3} \parallel}{\Pi_1^*\{\emptyset\}\{\Pi_3\}} \right\}_{\Pi_3 \in M}}{\text{cut} \frac{\Pi_1\{\langle \downarrow N \rangle\}\{\downarrow M\}}{\Pi_1\{\langle \downarrow N \rangle\}\{\emptyset\}}} \rightsquigarrow \downarrow_R^{\text{sn}} \left\{ \frac{\mathcal{D}_{\Pi_2} \parallel \quad \frac{\downarrow_R^{\text{sn}} \left\{ \frac{\mathcal{D}_{\Pi_3} \parallel}{\Pi_1^*\{\emptyset\}\{\Pi_3\}} \right\}_{\Pi_3 \in M}}{\text{w} \frac{\Pi_1^*\{\emptyset\}\{\Pi_3\}}{\Pi_1^*\{\Pi_2\}\{\downarrow M\}}}_{\Pi_1^*\{\Pi_2\}\{\downarrow M\}}}{\text{cut} \frac{\Pi_1^*\{\Pi_2\}\{\downarrow M\}}{\Pi_1^*\{\Pi_2\}\{\emptyset\}}} \right\}_{\Pi_2 \in N} \quad \textcircled{\ast}$$

**8.2.5 Theorem** *Let  $Y \subseteq \{t, b, 4, 5\}$  be axiomatically complete. If a sequent  $\Pi$  is provable in  $\text{snIK} + \diamond_X^{\text{sn}} + \text{cut}$ , then it is also provable in  $\text{snIK} + \diamond_X^{\text{sn}}$ .*

This is as Theorem 7.2.6 proved by induction on the number of cuts in the proof, by repeatedly applying the previous lemma, always starting with a topmost cut.

## Completeness

In this section we will show that  $\text{snIK} + \diamond_X^{\text{sn}}$  extended with the cut rule can simulate  $\text{nIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$  derivations under a certain interpretation of the annotations. Thus, the cut-elimination theorem will allow us to conclude that  $\text{nIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$  rules under that interpretation are admissible in  $\text{snIK} + \diamond_X^{\text{sn}}$ , i.e.,  $\text{snIK} + \diamond_X^{\text{sn}}$  is complete with respect to  $\text{nIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$ .

**8.2.6 Theorem** *Let  $Y \subseteq \{t, b, 4, 5\}$  be axiomatically complete. For any right-neutral formula  $R$ , if  $\partial(R)^\circ$  is provable in  $\text{nIK} + \square_{\text{LX}}^n + \diamond_{\text{RX}}^n$ , then  $R$  is provable in  $\text{snIK} + \diamond_X^{\text{sn}}$ .*

First, we can obtain a similarly synthetic version of identity reduction (Lemma 7.2.4).

**8.2.7 Lemma** *The following rule is derivable in snIK.*

$$\text{gid}^{\text{sn}} \frac{}{\Delta_1 \{ \langle \Delta_2 \rangle, \Delta_2 \}}$$

*Proof.* By induction on the structure of the focus. We present it in the form of an expansionary rewrite.

$$\begin{aligned} \text{gid}^{\text{sn}} \frac{}{\Pi \{ \langle \emptyset \rangle \}} &\rightsquigarrow \text{fin}^{\text{sn}} \frac{}{\Pi \{ \langle \emptyset \rangle \}} \\ \text{gid}^{\text{sn}} \frac{}{\Lambda \{ \langle n \rangle, n \}} &\rightsquigarrow \text{id}_L^{\text{sn}} \frac{}{\Lambda \{ \langle n \rangle, n \}} \\ \text{gid}^{\text{sn}} \frac{}{\Pi \{ \langle p \rangle, p \}} &\rightsquigarrow \text{id}_R^{\text{sn}} \frac{}{\Pi \{ \langle p \rangle, p \}} \\ \text{id}^{\text{sn}} \frac{}{\Lambda_1 \{ \langle \uparrow P \rangle, \uparrow P \}} &\rightsquigarrow \uparrow_L^{\text{sn}} \frac{\left\{ \frac{\uparrow_R^{\text{sn}} \Lambda_2 \in \uparrow P \quad \text{gid}^{\text{sn}} \frac{}{\Lambda_1 \{ \Lambda_2, \uparrow P, \langle \Lambda_2 \rangle \}}}{\Lambda_1 \{ \Lambda_2, \uparrow P \}} \right\}_{\Lambda_2 \in \uparrow P}}{\Lambda_1 \{ \langle \uparrow P \rangle, \uparrow P \}} \\ \text{id}^{\text{sn}} \frac{}{\Pi_1 \{ \langle \downarrow N \rangle, \downarrow N \}} &\rightsquigarrow \downarrow_R^{\text{sn}} \frac{\left\{ \frac{\downarrow_R^{\text{sn}} \Pi_2 \in \downarrow N \quad \text{gid}^{\text{sn}} \frac{}{\Pi_1^* \{ \Pi_2, \downarrow N, \langle \Pi_2 \rangle \}}}{\Pi_1^* \{ \Pi_2, \downarrow N \}} \right\}_{\Pi_2 \in \downarrow N}}{\Pi_1 \{ \langle \downarrow N \rangle, \downarrow N \}} \\ \text{gid}^{\text{sn}} \frac{}{\Delta_1 \{ \langle \Delta_2, \Lambda \rangle, \Delta_2, \Lambda \}} &\rightsquigarrow \text{spl}^{\text{sn}} \frac{\text{gid}^{\text{sn}} \frac{}{\Delta_1 \{ \langle \Delta_2 \rangle, \Delta_2, \Lambda \}} \quad \text{gid}^{\text{sn}} \frac{}{\Delta_1 \{ \langle \Lambda \rangle, \Delta_2, \Lambda \}}}{\Delta_1 \{ \langle \Delta_2, \Lambda \rangle, \Delta_2, \Lambda \}} \\ \text{gid}^{\text{sn}} \frac{}{\Delta_1 \{ \langle [\Delta_2] \rangle, [\Delta_2] \}} &\rightsquigarrow \text{id}_k^{\text{sn}} \frac{\text{gid}^{\text{sn}} \frac{}{\Delta_1 \{ \langle \Delta_2 \rangle, \Delta_2 \}}}{\Delta_1 \{ \langle [\Delta_2] \rangle, [\Delta_2] \}} \quad \textcircled{\ast} \end{aligned}$$

This allows us to simulate derivations from nIK sequents in the synthetic system.

**8.2.8 Lemma** *Let  $Y \subseteq \{t, b, 4, 5\}$ . Let  $R$  be a right-neutral formula. If  $\partial(R)^\circ$  is provable in  $\text{nIK} + \square_{LX}^n + \diamond_{RX}^n$ , then  $R$  is provable in  $\text{snIK} + \text{id}_X^{\text{sn}} + \text{cut}$ .*

*Proof.* First, any nIK sequent can be transformed into a neutral polarised sequent with the same meaning. The connectives are turned into their polarised variant and in particular, a polarity is arbitrarily chosen for every atom, every  $\top$ , and every  $\wedge$ ; then shifts are added as needed to produce well-formed polarised formulas. Once the formulas are polarised, one can obtain neutrality, and remove the  $^\circ$ -annotation, by adding extra shifts in front of each formula in the sequent as follows: if  $P$  is a positive formula,  $P \mapsto \downarrow \uparrow P$  and  $P^\circ \mapsto \uparrow P$ , and if  $N$  is a negative formula,  $N \mapsto \downarrow N$  and  $N^\circ \mapsto \uparrow \downarrow N$ . Each rule of  $\text{nIK} + \square_{LX}^n + \diamond_{RX}^n$  can therefore be considered as a rule between neutral polarised sequents. As such, it can be shown to be derivable in  $\text{snIK} + \text{id}_X^{\text{sn}}$ . We show the cases for the rules  $\text{id}$ ,  $\wedge_L^n$  and  $\diamond_{Rd}^n$ . The other cases are similar.

$$\text{id} \frac{}{\Lambda \{ a, a^\circ \}} \quad \text{becomes} \quad \uparrow_L^{\text{sn}} \frac{\text{id}^{\text{sn}} \frac{}{\Lambda \{ \downarrow \uparrow p, \langle \uparrow p \rangle, \uparrow p \}}}{\Lambda \{ \downarrow \uparrow p, \uparrow p \}} \quad \text{or} \quad \downarrow_R^{\text{sn}} \frac{\text{id}^{\text{sn}} \frac{}{\Lambda \{ \downarrow n, \uparrow \downarrow n, \langle \downarrow n \rangle \}}}{\Lambda \{ \downarrow n, \uparrow \downarrow n \}}$$



## Conclusion

We have presented various contributions to the proof theory of classical and intuitionistic modal logics. In this chapter, we sum up the content of each part of this thesis and uncover some directions for future investigations.

In **Part I**, we established an inventory of various systems that have been proposed to handle modal logics proof theoretically so far, whether in labelled or unlabelled formalisms. It led to our joint work with Straßburger on nested sequents [87] and on indexed nested sequents [88] for both classical and intuitionistic modal logics.

We observed that the distinction between labelled and unlabelled deduction is rather superficial and even cosmetic. Indeed, labelling, bracketing, etc. are merely alternative representations of the underlying structure that could even be made explicit graphically (as in some tableau formalisms for instance). It is the case that some formalisms can be naturally interpreted in the language of modal formulas, but it is not the choice of representation that is the cause. A formula is, by definition, a tree, so if the underlying structure of the lines of a proof system is restricted to a tree, it can be translated as a formula; if it is a more generic graph in general it cannot.

Maybe a better distinction between the formalisms is their use of structural or propagation rules, rather than their labelled/unlabelled nature. In any case, we can say that it is still crucial to study the different formalisms at hand as they give us different insights on the proof shapes.

- We do not yet fully understand which modal logics can be given which types of proof system. The ideal result would be in the form of a necessary and sufficient condition for a modal logic to be a system with propagation/structural rules, similarly to Lellmann's one in the case of sequents and hypersequents *with context-restriction* [77, 76].

We would like to understand why the axiom  $d : \Box A \supset \Diamond A$ , which is not a path axiom, stands as an exception by having a seemingly simple proof theory. On the other hand, pseudo-transitive modal logics, obtained by extending  $K$  with axioms of the form  $\Diamond A \supset \Diamond^n A$  which are also not path axioms, seem to be definitely beyond the scope of propagation rules; having a terminating proof search procedure for them would be very interesting since their decidability is an open problem.

Stewart and Stouppa [128] had conjectured that the frontier between modal axioms that could and could not be given propagation rules was what they called *3/4 SL axioms*, namely the class of axioms of the form:  $\mathbf{g}_{hij0} := \Diamond^h \Box^i a \supset \Box^j a$  which is situated in between SL paths axioms and general SL axioms. We did not confirm or reject this conjecture, but it seems that we now might have the right tools to re-examine it.

- There are also possible directions to stretch further the limits of our current knowledge: (i) study proof systems for modal logics that are not characterisable by first-order frame properties, for example extensions of  $K$  with axiom  $\Box \Diamond A \supset \Diamond \Box A$ ; (ii) study

proof systems for non-normal modal logics, i.e. logics that do not assume the  $k$ -axiom, as it is clear that all systems we presented are structurally forced to derive it (there is some progress in this direction [51, 78, 103]); *(iii)* study proof systems for extensions of constructive and intuitionistic modal logics with one-sided SL axioms, as we already mentioned in Chapter 3, and investigate their relationships with intermediate logic.

- Two applications of cut-free sequent calculi that can be seen as complementary are in the fields of *justification logic* and *type theory*. In justification logic, the goal is to give an explicit provability interpretation to a modal formula  $\Box A$  as *there exists a proof of A*. With Kuznets and Straßburger, we worked on an extension of justification logic to the constructive setting [72] in order to give a justification logic equivalent to the  $\Diamond$  modality as well. For this first step we only considered the case of constructive modal logic CK as it can easily be given a sequent calculus, but we would like to extend our results to IK, perhaps by using a nested sequent system instead.

On the other hand, we have said that modal type theories usually also restrict their expressivity to the CK fragment. It would be interesting to expand their reach to IK too, by proposing a term language for nested sequents; to this end there is work in progress by Chaudhuri, based on the natural deduction system in [53].

However, both these topics underline an important question: does  $\Diamond$  make any sense at all in such applications of intuitionistic modal logic? There might be a fundamental reason why justification logic and type theory have a weak treatment of  $\Diamond$  or do not consider it at all, namely  $\Box$  as a negative connective is more well-suited for constructive theories.

In **Part II**, we started by presenting the concepts of focused and synthetic proofs, which have been influential in many respects in proof theory, from the initial concrete application to proof search and logic programming, to the more structural issues of proof representation. With Miller and Volpe we are investigating the various techniques in the proof-theory literature to transform axiomatic theories to inference rules in the sequent calculus. We are seeking to gain a uniform understanding of what seems quite different techniques as simply different ways to build synthetic macro rules from micro focused rules. This has practical applications as a continuation of Miller’s *ProofCert* project which aims at designing a universal format to describe the operational semantics of any piece of a proof built in any proof formalism.

We presented in Chapter 5 the first results of this investigation, namely a study of the rules from labelled sequent systems for modal logics extended with geometric frame properties, which was conducted by Miller and Volpe [96] in the classical setting and that we extended to the intuitionistic setting in this thesis. In this case, we indeed have shown that each rule can be expressed as a bipole, i.e. a pair of a positive and a negative synthetic rules, in LKF or in LJF respectively, under an appropriate translation of formulas.

Once labelled sequent calculi are embeddable in the basic focused systems, we suggested using this embedding as a ‘plug-in’ to encode further systems. In Chapter 6, we presented a framework that extends LKF with labels in order to be suitable for modal logics. We illustrated how it can emulate unlabelled systems for modal logics with the example of certain sequent calculi.

One could object that, since any of the systems we described could provisionally be emulated by a labelled framework, we have a contradiction with our previous point that there is a lot of value in the variety of proof systems for modal logics. We would like to anticipate this criticism and answer it by hijacking a quote of Dana Scott [123]: « There is no weight to the claim that the original system must therefore be replaced by the new one. What is essential is to single out important concepts and to investigate their properties.»

- We presented technical details for the first steps of this line of work, namely for the labelled sequent systems, but we expect that all similar methods to build inference

rules from axiomatic theories can be seen as synthetic rules in a focused system, as for example the ones for sequents and hypersequents in [30, 33], for display calculi in [32] and for systems of rules in [103]. Studying this last example, in particular, could give an alternative view on the relationship between hypersequents and the restriction called two-level *system of rules* in [31].

- One direction we have already started to investigate is the emulation of other unlabelled proof systems in our framework  $\text{LMF}_*$ . For nested sequents, it seems quite simple: it is necessary to have a set of labels present, but it needs to be restricted to a tree-structure at all times (which is possible following [61]). Hypersequents, in contrast, are a sort of hybrid between ordinary sequents and nested sequents, and less straightforward to encode. One key-point is that the formula interpretation of a hypersequent is not uniformly given over the different logics, contrary to nested sequents.
- Another route that needs to be explored is, of course, the emulation of intuitionistic systems in an intuitionistic version of the framework. However, as the labelled sequent system  $\text{labIK}$  implements extensions of  $\text{IK}$  while ordinary sequent systems implement extensions of  $\text{CK}$ , it will probably be much more difficult to find the right framework design in the intuitionistic case.

In **Part III**, we presented our joint work with Chaudhuri and Straßburger on focused and synthetic nested sequent systems for both classical [22] and intuitionistic [23] modal logics. It highlights another use of the synthetic formulation of rules; it allows us to give an elegant presentation of the nested sequent systems and to simplify the proofs of their meta-theoretical properties, namely the admissibility of the cut-rule and the completeness of focusing.

- A first extension of this work would be to define focused and synthetic variants of the propagation rules for path axioms discussed in Section 2.5. It would probably not address the quest of Goré *et al* [60] for an efficient proof search strategy in this system as termination would remain an issue, but it would still uncover some interesting features of this system. In general, expanding this work towards other proof formalisms would be worth considering; for labelled sequent calculi it seems straightforward, but for hypersequents it would be more involved for similar reasons as evoked above.
- Another extension would be relaxing the restriction that there can be at most one focus in a  $\text{fnK}$  proof, which would take us from ordinary focusing to multi-focusing. Certain well chosen multi-focusing systems can yield syntactically canonical representatives of equivalence classes of sequent proofs [24, 21]. To our knowledge, there are only a few results in the field of canonical proofs for modal logics, e.g. in the form of proof-nets [119], so this constitutes an exciting fresh research program.
- In this thesis, we crucially interpret  $\diamond$  as positive and  $\square$  as negative, which differs from the polarities that would be assigned to these connectives if they were interpreted in terms of  $?$  and  $!$ , respectively, from linear logic [109]. With Chaudhuri and Straßburger, we wondered to which extent this choice was canonical and if this difference came from the fact that we were considering nested sequents instead of sequents. So we tried to transform the focused proof system for classical modal logic defined in Chapter 7 into a proof system for linear logic with a positive  $?$  and a negative  $!$ .

We found a counter-example that nullified this program, which makes us believe now that, at least from a polarisation point of view, exponentials do not behave like modalities. The problem actually does not seem to come from the depth of the nested sequent formalism, but from the interaction between the exponentials and the multiplicative connectives of linear logic. We would be interested in understanding more precisely which forces are at play, and if the polarities of the exponentials can be twisted if we consider other fragments of linear logic for example.

To conclude, we would say that exploring the applications of the focusing tool-kit to the proof theory of modal logics gave us, in general, a better understanding of the internal machinery of the variety of proof systems that existed in the literature and of the mechanisms at hand when considering their different properties. This thesis has been a study of modal proofs, but moreover we have conducted this study under some of the most recent developments in structural proof theory; as such, this work could be a useful template for other logics and other proof systems in the future.

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**Titre :** Voir la théorie de la démonstration de la logique modale à travers un télescope focalisé

**Mots clés :** théorie de la démonstration, focalisation, logique modale

**Résumé :** Dans cette thèse, nous utilisons de deux manières le concept de *règles d'inférence synthétiques focalisées*; d'un côté du "télescope", la focalisation nous permet d'analyser les mécanismes internes des règles d'inférence; de l'autre côté, elle nous permet de nous intéresser à leur comportement plus global.

Dans la première partie, nous passons en revue les systèmes de preuves existants pour la logique modale en nous concentrant autour du calcul des séquents et ses extensions. Nous mettons en valeur les problématiques qui animent le domaine de la théorie de la démonstration de la logique modale, notamment en déconstruisant la distinction usuelle entre systèmes étiquetés et non-étiquetés. Nous présentons ces questions et concepts en parallèle dans les chapitres 2 et 3 pour la logique modale *classique* et *intuitionniste* respectivement. Nous détaillons en particulier le calcul des *séquents emboîtés indexés* de Fitting pour lequel nous avons démontré un nouveau résultat de complétude via élimination des coupures.

La deuxième partie rappelle d'abord les notions de focalisation et de règles d'inférence synthétiques dans le

chapitre 4, puis présente deux contributions dans les chapitres 5 et 6. Premièrement, nous démontrons comment émuler le calcul étiqueté pour la logique modale intuitionniste de Simpson à l'aide du calcul focalisé pour la logique du premier ordre de Liang et Miller, étendant ainsi les travaux de Miller et Volpe au cas intuitionniste. Deuxièmement, nous proposons un encodage similaire mais pour le calcul des séquents ordinaire (non-étiqueté) à l'aide d'une structure intermédiaire basée sur une version focalisée du calcul des séquents étiquetés de Negri. La troisième partie rapporte deux autres de nos contributions: les preuves de complétude de deux calculs de séquents emboîtés pour la logique modale (classique et intuitionniste), d'abord une version focalisée dans le chapitre 7, puis un système basé uniquement sur des règles d'inférence synthétiques dans le chapitre 8. Ces règles ne retiennent que les transitions principales du raisonnement et rendent invisibles la plupart des règles du calcul focalisé, ce qui rend la présentation du système claire et élégante et simplifie grandement les preuves d'élimination des coupures et de complétude.

**Title :** Modal proof theory through a focused telescope

**Keywords :** proof theory, focusing, modal logic

**Abstract :** In this thesis, we use in two ways the concept of *synthetic inference rules* that can be obtained from a *focused* proof system; from one side of the "telescope", focusing allows us to analyse the internal machinery of inference rules; on the other side, it allows us to consider more global behaviours.

In the first part, we review existing proof systems for modal logic, concentrating our efforts around the sequent calculus and its extensions. We underline the issues that drive the modal proof theory community, such as the usual distinction between labelled and unlabelled systems that we aim at deconstructing. We present these questions and concepts in parallel for *classical* and *intuitionistic* modal logic in chapter 2 and 3 respectively. We in particular go through Fitting's *indexed nested sequents*, for which we demonstrated a new completeness result via cut-elimination.

The second part recalls first the notion of focusing and of synthetic inference rules in chapter 4, then presents two

of our contributions in chapter 5 and 6. Firstly, we show how to emulate Simpson's labelled sequent calculus for intuitionistic modal logic with Liang and Miller's focused sequent calculus for first-order logic, therefore extending the result of Miller and Volpe. Secondly, we propose a similar encoding though for ordinary (unlabelled) sequent calculus via an intermediate focused framework based on Negri's labelled sequent calculus.

The third part reports on two other contributions, namely the completeness proofs of two nested sequent calculi for both classical and intuitionistic modal logic, first a focused version in chapter 7, and then a system merely based on synthetic inference rules in chapter 8. These rules only retain the transitions between big steps of reasoning forgetting most of the focused rules, which renders the system presentation clear and elegant while also simplifying the cut-elimination and completeness proofs.

