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Thèse de Doctorat

présentée par

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Pour obtenir le grade de

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Some aspects of the geometry of Lipschitz free spaces

Soutenue le 19 Juin 2018, devant le jury composé de :

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To Olivia

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Introduction

1. Résumé de la thèse

Le principal objectif de cette thèse est d'explorer la structure linéaire des espaces Lipschitz libres. Pour un espace métrique M équipé d'un point distingué noté 0, l'espace Lipschitz libre sur M est défini par

$$\mathcal{F}(M) := \overline{\operatorname{span}}^{\|\cdot\|} \{\delta(x) \,:\, x \in M\} \subset \operatorname{Lip}_0(M)^*.$$

Dans la définition ci-dessus, $\operatorname{Lip}_0(M)$ désigne l'espace de Banach des fonctions Lipschitziennes définies sur M, à valeurs réelles, et satisfaisant $f(0) = 0$. L'espace de Banach $\mathcal{F}(M)$ est en fait un prédual canonique de $\operatorname{Lip}_0(M)$. Le nom "espace Lipschitz libre" provient de l'article fondateur [GK03] que l'on doit à Godefroy et Kalton. Cette terminologie s'explique par le fait que les éléments de l'espace métrique M sont associés à des vecteurs linéairement indépendants dans l'espace libre correspondant $\mathcal{F}(M)$. Cependant, ces "espaces libres" étaient connus et étudiés auparavant par exemple par Weaver dans [Wea99], qui les appelait alors espaces de Arens–Eells. L'article [GK03] contient de profonds résultats qui ont motivé le développement de cette théorie. Ainsi, le travail effectué dans cette thèse s'inscrit dans le programme de recherche lancé par Godefroy et Kalton consistant à déterminer la structure linéaire des espaces libres. Nous allons maintenant détailler le contenu de cette thèse.

Chapitre 1 : Quelques faits généraux à propos des espaces Lipschitz libres.

En premier lieu, nous donnons les propriétés fondamentales des espaces Lipschitz libres. Les résultats présentés en début de chapitre sont standards et bien connus. Un élément central ici est la propriété fondamentale de linéarisation des espaces libres : Toute application Lipschitzienne d'un espace métrique M dans un espace de Banach X se prolonge de manière unique en une application linéaire continue de $\mathcal{F}(M)$ dans X. Pour des raisons pratiques, nous démontrons la plupart des résultats exposés, même lorsqu'ils sont bien connus.

Dans la suite du chapitre, nous nous intéressons à des résultats toujours généraux, mais probablement moins classiques. Il est très facile de voir que l'application $\delta : x \in M \mapsto \delta(x) \in \mathcal{F}(M)$ est une isométrie, et que son image $\delta(M)$ est fermée en norme pour un espace métrique complet M. De plus, nous démontrons le résultat plus général suivant.

Proposition 1.2.1 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique pointé complet. Alors, l'ensemble $\delta(M) \subset \mathcal{F}(M)$ est fermé pour la topologie faible.

On en déduit alors facilement que la topologie faible coïncide avec la topologie de la norme sur $\delta(M)$. Ensuite, nous tournons notre attention sur l'ensemble des molécules. Une molécule est un élément de la forme :

$$m_{xy} := \frac{\delta(x) - \delta(y)}{d(x,y)} \in \mathcal{F}(M), \text{ pour } x \neq y \in M.$$

Nous notons V l'ensemble des molécules. Une propriété importante et très utile de V est qu'il est normant pour $\mathrm{Lip}_0(M)$ et par conséquent $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(V)$. De plus, basé sur le travail de Weaver ([Wea99]), nous démontrons la proposition suivante.

Proposition 1.3.3 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit (M,d) un espace métrique complet. Alors $\overline{V}^w \subset V \cup \{0\}$.

Le résultat ci-dessus aura certaines conséquences sur la structure extrémale de $\mathcal{F}(M)$ que nous aborderons au chapitre 4. Par le biais d'un résultat démontré par Albiac et Kalton dans [AK09], nous en déduisons que l'ensemble V est en fait faiblement séquentiellement fermé. Ce chapitre s'achève par une discussion autour de possibles améliorations.

Chapitre 2 : Dualité de certains espaces Lipschitz libres.

Dans ce chapitre, nous étudions les circonstances sous lesquelles $\mathcal{F}(M)$ est isométrique à un espace dual. Dans ce contexte, l'espace des fonctions petit-Lipschitz apparaît naturellement (avec la convention $\sup \emptyset = 0$) :

$$\mathrm{lip}_0(M) := \Big\{ f \in \mathrm{Lip}_0(M) : \lim_{\varepsilon \to 0} \sup_{0 < d(x,y) < \varepsilon} \frac{|f(x) - f(y)|}{d(x,y)} = 0 \Big\}.$$

En effet, on peut mentionner le travail de Weaver (Théorème 3.3.3 dans [Wea99]) affirmant que $\mathcal{F}(M)$ est isométrique au dual de $\mathrm{lip}_0(M)$ dès que M est compact et $\mathrm{lip}_0(M)$ sépare les points de M uniformément. Nous rappelons que $S \subset \mathrm{Lip}_0(M)$ sépare les points uniformément si il existe $C > 0$ tel que pour tous $x, y \in M$ il existe $f \in S$ vérifiant $f(x) - f(y) = d(x,y)$ et $\|f\|_L \leq C$. Plus généralement, Kalton a démontré sous des hypothèses adaptées que $\mathcal{F}(M)$ est isométrique au dual de $\mathrm{lip}_0(M) \cap \mathcal{C}(M,\tau)$ dès que (M,τ) est compact et $\mathrm{lip}_0(M) \cap \mathcal{C}(M,\tau)$ sépare les points uniformément. En fait, la plupart des préduaux isométriques aux espaces libres considérés dans la littérature sont des sous espaces de $\mathrm{lip}_0(M)$. De plus, ils ont une propriété commune de plus : Ils rendent $\delta(B(0,r))$ préfaiblement fermé dans $\mathcal{F}(M)$ pour tout $r \geq 0$. Nous appelons prédual naturel tout prédual vérifiant cette dernière propriété.

L'un des principaux buts de ce chapitre est d'améliorer légèrement les résultats mentionnés ci-avant. En particulier, nous traitons le cas des espaces métriques non bornés. Pour ces espaces, il s'avère que le sous espace suivant de $\mathrm{lip}_0(M)$ est plus adapté :

$$S_0(M) := \Big\{ f \in \mathrm{lip}_0(M) : \lim_{r \to \infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} = 0 \Big\}.$$

Par exemple, Dalet a prouvé que $\mathcal{F}(M)$ est isométriquement isomorphe au dual de $S_0(M)$ dès que M est un espace propre (i.e. les boules fermées sont compactes) et $S_0(M)$ sépare les points uniformément. Cela nous mène au résultat suivant. Dans la proposition suivante, $\mathcal{C}_b(M,\tau)$ désigne l'ensemble des fonctions τ-continues sur les ensembles bornés de M.

Proposition 2.2.5 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique pointé, séparable et complet. Soit τ une topologie sur M telle que d est τ semi-continue-inférieurement et (M,τ) est τ-propre (i.e les boules fermées pour d sont τ-compactes). On suppose de plus que $X = S_0(M) \cap \mathcal{C}_b(M,\tau)$ sépare les points uniformément. Alors, X est un prédual naturel de $\mathcal{F}(M)$.

Notre démonstration est basée sur le théorème de Petunīn et Plīčko (voir [God87, PP74]). L'avantage de cette démonstration est qu'elle permet de se passer de l'hypothèse de métrisabilité sur τ, présente dans le résultat de Kalton. Par ailleurs, nous démontrons que la Proposition 2.2.5 est le seul moyen de construire un prédual naturel si ce prédual est supposé être un sous espace de $S_0(M)$.

Par la suite, nous nous focalisons sur les espaces métriques uniformément discrets, c'est à dire les espaces métriques tels que $\inf\{d(x,y) : x \neq y \in M\} > 0$. Pour un tel espace métrique M, $\mathcal{F}(M)$ possède un comportement comparable à ℓ_1. Par exemple, Kalton a démontré que $\mathcal{F}(M)$ a la propriété de Schur, la propriété de Radon-Nikodym (RNP) et la propriété d'approximation (AP) ([Kal04, Proposition 4.4]). Un problème ouvert célèbre soulevé dans [Kal04] consiste à déterminer si $\mathcal{F}(M)$ possède la propriété d'approximation bornée (BAP). Dans ce contexte, il est très naturel d'essayer de déterminer si $\mathcal{F}(M)$ est isométriquement un dual. En fait, nous nous concentrons sur le cas des espaces uniformément discrets et bornés pour lesquels nous caractérisons les éventuels prédaux naturels des espaces libres associés.

Corollaire 2.3.4 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit (M,d) un espace métrique pointé, uniformément discret, borné, séparable et complet. Soit X un espace de Banach. Alors les assertions suivantes sont équivalentes :

(i) X est un prédual naturel de $\mathcal{F}(M)$.

(ii) Il existe τ une topologie séparée sur M telle que (M,τ) est compact, d est τ-s.c.i. et $X = \mathrm{Lip}_0(M,d) \cap \mathcal{C}(M,\tau)$ équipé de la norme $\|\cdot\|_L$.

Explorer les espaces libres sur les espaces uniformément discrets nous permet d'isoler des exemples aux propriétés intéressantes. Entre autres, nous présentons un espace métrique M tel que $\mathcal{F}(M)$ est isométrique à un espace dual mais n'admet aucun prédual naturel. Puis, nous mettons en évidence également un espace métrique uniformément discret tel que son espace libre n'est pas isométrique à un espace dual.

Finalement, nous isolons une nouvelle classe d'espaces métriques qui vérifient les hypothèses de la Proposition 2.2.5. Il s'agit des espaces métriques provenant des p-Banach. Plus précisément, nous fixons p dans $]0,1[$ et nous considérons $(X,\|\cdot\|)$ un p-Banach. Maintenant, $d_p(x,y) = \|x-y\|^p$ définit une métrique sur X de telle sorte que l'on peut maintenant étudier $\mathcal{F}(X,d_p)$. En particulier, nous prouvons le corollaire suivant.

Corollaire 2.4.5. *Soit $p \in (0,1)$ et soit $(M_p,d_p) = (X,\|\cdot\|^p)$ où $(X,\|\cdot\|)$ est un p-Banach de dimension finie. Alors, $S_0(M_p)$ est un prédual naturel de $\mathcal{F}(M_p)$.*

Chapitre 3 : Propriétés de Schur et espaces Lipschitz libres.

Dans ce chapitre, nous nous focalisons sur des propriétés de l'espace ℓ_1 telles que la propriété de Schur ou encore des versions quantitatives plus fortes. Kalton a démontré dans [Kal04] que si (M, d) est un espace métrique et que ω est une jauge non triviale (par exemple $\omega(t) = t^p$ avec $p \in]0, 1[$) alors l'espace $\mathcal{F}(M, \omega \circ d)$ a la propriété de Schur. Dans [HLP16], Hájek, Lancien et Pernecká ont démontré qu'il en est de même pour $\mathcal{F}(M)$ où M est un espace métrique propre et dénombrable. Ici, nous donnons une condition suffisante qui unifie ainsi ces deux résultats.

Proposition 3.1.2. *Soit M un espace métrique pointé tel que $\mathrm{lip}_0(M)$ est 1-normant pour $\mathcal{F}(M)$. Alors $\mathcal{F}(M)$ a la propriété de Schur.*

La démonstration est fortement inspirée par celle du résultat de Kalton (Théorème 4.6 dans [Kal04]). Si l'espace métrique est de plus supposé être propre, alors on sait que $S_0(M)$ est isomorphe à un sous espace de c_0 [Dal15c]. Par conséquent, on en déduit alors que $\mathcal{F}(M)$ possède des versions quantitatives de la propriété de Schur.

Proposition 3.2.5. *Soit M un espace métrique pointé et propre tel que $S_0(M)$ sépare les points uniformément. Alors $\mathcal{F}(M)$ a la propriété 1-Schur.*

En ajoutant de plus l'hypothèse que $\mathcal{F}(M)$ a la propriété d'approximation, on est alors capable d'apporter plus d'informations sur la structure de type ℓ_1 de l'espace libre $\mathcal{F}(M)$. Le théorème suivant est basé sur un résultat de Godefroy, Kalton et Li traitant des sous espaces de c_0 ayant la propriété d'approximation.

Théorème 3.3.1. *Soit M un espace métrique pointé et propre tel que $S_0(M)$ sépare les points uniformément et tel que $\mathcal{F}(M)$ a la propriété d'approximation. Alors pour tout $\varepsilon > 0$, il existe $(E_n)_{n=1}^{\infty}$ une suite de sous-espaces de dimensions finies de $\mathcal{F}(M)$ telle que $\mathcal{F}(M)$ est $(1+\varepsilon)$-isomorphe à un sous espace de $\left(\sum_n E_n\right)_{\ell_1}$.*

Pour finir, nous appliquons ces résultats aux espaces métriques provenant des p-Banach.

Chapitre 4 : Structure extrémale des espaces libres.

Dans ce chapitre, nous nous intéressons à la structure extrémale de $\mathcal{F}(M)$. Les deux principales questions de ce domaine sont les suivantes :

a) Si μ est un point extrémal de la boule unité $B_{\mathcal{F}(M)}$, est -ce que μ est nécessairement une molécule ?

b) Si le segment métrique $[x, y] = \{z \in M \,:\, d(x, z) + d(z, y) = d(x, y)\}$ est réduit à $\{x, y\}$, est ce que m_{xy} est un point extrémal de $B_{\mathcal{F}(M)}$?

Weaver a démontré dans [Wea99] que les points extrémaux préservés de $\mathcal{F}(M)$, c'est à dire les points extrémaux de la boule unité $B_{\mathcal{F}(M)^{**}}$ qui appartiennent à $\mathcal{F}(M)$, sont toujours des molécules. Plus récemment, Aliaga et Guirao ont prouvé que les questions a) et b) ont des réponses positives lorsque l'espace métrique est compact. Ils démontrent également une caractérisation métrique des points extrémaux préservés en toute généralité. Nous démontrons cette caractérisation ici par une méthode différente. Davantage de résultats sur ce sujet apparaissent dans [GLPZ17b], où par exemple une caractérisation métrique des points fortement exposés est présentée.

Le but de ce chapitre est de continuer à explorer la structure extrémale des espaces libres et d'apporter des réponses positives aux questions a) et b) dans certains cas particuliers. Après quelques observations rapides, nous démontrons le théorème qui suit.

Théorème 4.1.4 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique pointé. Alors tout point extrémal préservé de $B_{\mathcal{F}(M)}$ est un point de dentabilité, c'est-à-dire, il appartient à des tranches de $B_{\mathcal{F}(M)}$ de diamètre arbitrairement petit.

Nous abordons ensuite le cas des espaces libres admettant un prédual naturel. Notamment, sous des hypothèses on obtient une réponse positive à la question a).

Proposition 4.2.1 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique pointé. Supposons qu'il existe un prédual naturel X de $\mathcal{F}(M)$ qui est un sous espace de $S_0(M)$. Alors, tout point extrémal de $B_{\mathcal{F}(M)}$ est une molécule.

En supposant de plus que l'espace métrique est séparable, on obtient également une réponse positive à la question b). Plus précisément, nous obtenons le corollaire suivant.

Corollary 4.2.2 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique pointé et séparable. Supposons qu'il existe un prédual naturel de $\mathcal{F}(M)$ qui soit un sous espace de $S_0(M)$. Alors $\mu \in B_{\mathcal{F}(M)}$ est un point extrémal si et seulement si c'est un point exposé si et seulement si μ est de la forme m_{xy} avec $x \neq y \in M$ et $[x, y] = \{x, y\}$.

Nous portons alors notre attention une fois de plus sur les espaces métriques uniformément discrets. Entre autre, nous démontrons que si M est de plus supposé borné, alors une molécule m_{xy} est un point extrémal si et seulement si $[x, y] = \{x, y\}$. Nous prouvons également que tout point extrémal de $B_{\mathcal{F}(M)}$ est en fait fortement exposé. Nous terminons en étudiant le cas des espaces métriques propres tels que $S_0(M)$ sépare les points uniformément. Dans ce cas, nous prouvons que $\mathcal{F}(M)$ possède une propriété géométrique, à savoir $\mathcal{F}(M)$ est préfaiblement asymptotiquement uniformément convexe. En particulier, chaque point de la sphère unité possède des voisinages préfaibles (relatifs à $B_{\mathcal{F}(M)}$) de diamètre arbitrairement petit. Comme conséquence directe nous obtenons le résultat suivant.

Corollaire 4.4.2 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique pointé et propre. On suppose que $S_0(M)$ sépare les points uniformément. Alors chaque point extrémal de $B_{\mathcal{F}(M)}$ est un point de dentabilité.

Chapitre 5 : Fonctions Lipschitziennes à valeurs vectorielles.

Nous dirigeons maintenant notre attention sur les fonctions Lipschitziennes à valeurs vectorielles. Par la propriété fondamentale des espaces Lipschitz libres, $\mathrm{Lip}_0(M, X)$ est linéairement isométrique à $\mathcal{L}(\mathcal{F}(M), X)$. Par conséquent, $\mathrm{Lip}_0(M, X)$ est linéairement isométrique à $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$. C'est sûrement la principale motivation à définir l'espace Lipschitz libre à valeur dans un Banach X comme étant : $\mathcal{F}(M, X) := \mathcal{F}(M)\widehat{\otimes}_\pi X$.

Nous débutons cette étude en démontrant des propriétés assez basiques qui peuvent être comparées à celles présentées au Chapitre 1. Par exemple, nous isolons un sous espace $\delta(M, X)$ de $\mathcal{F}(M, X)$ (décrit ci-dessous) pour lequel nous démontrons qu'il est faiblement

fermé dès que M est complet :

$$\delta(M,X) := \{\delta(y)\otimes x \ : \ y\in M,\, x\in X\} \subset \mathcal{F}(M,X).$$

Ensuite, nous étendons certains résultats concernant la dualité des espaces libres. La théorie des produits tensoriels assure que sous certaines hypothèses, si S est un prédual de $\mathcal{F}(M)$ ($\mathcal{F}(M)\equiv S^*$), alors $\mathcal{F}(M,X^*) = \mathcal{F}(M)\widehat{\otimes}_\pi X^* \equiv (S\widehat{\otimes}_\varepsilon X)^*$. Ainsi, le fait que $\mathcal{F}(M,X^*)$ est linéairement isométrique à un dual repose sur le cas scalaire. De plus, nous pouvons également définir une notion plus ou moins légitime de prédual naturel de $\mathcal{F}(M,X^*)$. Nous démontrons que, dans la plupart des cas, le fait que $\mathcal{F}(M,X^*)$ admette un prédual naturel repose également sur le cas scalaire.

Nous cherchons ensuite à représenter un prédual de $\mathcal{F}(M,X^*)$ comme un sous espace de $\mathrm{Lip}_0(M,X^{**})$. Nous commençons par généraliser le résultat de Dalet mentionné au Chapitre 2. Une première méthode pourrait consister à utiliser une nouvelle fois le théorème de Petunīn–Plīčko. Dans le but d'éviter l'hypothèse de séparabilité présente dans ce dernier théorème, nous utilisons un chemin différent. Plus précisément, d'après [GLRZ17], $S_0(M,X)\equiv \mathcal{K}_{w^*,w}(X^*,S_0(M))$ à condition que M est propre (où $\mathcal{K}_{w^*,w}(X^*,S_0(M))$ est l'espace des opérateurs compacts continus de (X^*,w^*) dans $(S_0(M),w)$. Par conséquent, il suffit simplement de montrer que $\mathcal{K}_{w^*,w}(X^*,S_0(M)) \equiv S_0(M)\widehat{\otimes}_\varepsilon X$ afin d'obtenir le résultat suivant.

Théroème 5.2.5 (avec L. García-Lirola et A. Rueda Zoca).
Soit M un espace métrique propre et soit X un espace de Banach. Supposons que $S_0(M)$ sépare les points uniformément. Si $\mathcal{F}(M)$ ou X^* a la propriété d'approximation, alors $S_0(M,X)^*\equiv\mathcal{F}(M,X^*)$

Par le même procédé, nous étendons ensuite le résultat de Kalton mentionné (puis généralisé) au Chapitre 2. La principale tâche consiste alors à caractériser les sous-ensembles relativement compacts de $\mathrm{lip}_0(M)\cap\mathcal{C}(M,\tau)$ où τ est une topologie compacte sur M et d est τ semi-continue-inférieurement. Ceci nous permet, sous certaines hypothèses, d'établir l'identification suivante : $lip_\tau(M,X)\equiv\mathcal{K}_{w^*,w}(X^*,lip_\tau(M))$, où

$$lip_\tau(M,X) := \mathrm{lip}_0(M,X)\cap\{f\colon M\to X : f \text{ is } \tau-\|\cdot\| \text{ continue}\}.$$

Au final, nous en déduisons le théorème suivant :

Theorème 5.2.9 (avec L. García-Lirola et A. Rueda Zoca).
Soit M un espace métrique pointé, séparable, complet, borné et complet. Soit τ une topologie métrisable et compacte sur M telle que $lip_\tau(M)$ est 1-normant. Si $\mathcal{F}(M)$ ou X^* a la propriété d'approximation, alors $lip_\tau(M,X)^*\equiv\mathcal{F}(M,X^*)$.

Nous poursuivons avec l'étude des propriétés de Schur définies au Chapitre 3. L'objectif est de donner des conditions sur M et X forçant $\mathcal{F}(M,X)$ à avoir l'une de ces propriétés. Nous utilisons principalement deux méthodes. La première consiste à analyser la propriété de Dunford–Pettis sur les préduaux éventuels de $\mathcal{F}(M,X^*)$. Ensuite, nous dualisons afin d'en déduire certaines conséquences sur $\mathcal{F}(M,X^*)$. Le seconde méthode utilise des techniques provenant de la théorie des fonctions Lipschitziennes. En fait, nous suivons la démarche utilisée dans le cas des fonctions à valeurs scalaires.

Nous terminons ce chapitre avec quelques considérations sur les fonctions qui atteignent leur norme. Nous considérons deux notions différentes de norme atteinte. L'une

provient de la théorie des opérateurs : $f : M \to X$ atteint sa norme d'opérateur si il existe $\gamma \in S_{\mathcal{F}(M)}$ tel que : $\|\langle f, \gamma\rangle\|_X = \|f\|_L$. L'autre provient de la théorie des fonctions Lipschitziennes : $f : M \to X$ atteint fortement sa norme si il existe $x, y \in M$ tel que : $\|f(x) - f(y)\|_X = \|f\|_L\, d(x, y)$. De manière évidente, une fonction qui atteint fortement sa norme atteint également sa norme d'opérateur. Nous prouvons que ces deux concepts coïncident pour certaines classes d'espaces métriques.

Proposition 5.4.4 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique et X un espace de Banach. On suppose que $\mathcal{F}(M)$ a la propriété de Krein–Milman et que $\operatorname{ext}(B_{\mathcal{F}(M)}) \subseteq V$. Alors une fonction $f \in \operatorname{Lip}_0(M, X)$ qui atteint sa norme sur $\mathcal{F}(M)$ l'atteint également fortement.

A la lumière du théorème de Bishop–Phelps, et également sur la base d'un résultat de Bourgain [Bou77], nous démontrons un résultat sur la densité des fonctions Lipschitziennes qui atteignent leur norme.

Proposition 5.4.5 (avec L. García-Lirola, A. Procházka et A. Rueda Zoca).
Soit M un espace métrique complet et X un espace de Banach. Supposons que $\mathcal{F}(M)$ a la propriété de Radon–Nikodým. Alors l'ensemble des fonctions qui atteignent fortement leur norme est dense dans $\operatorname{Lip}_0(M, X)$.

Annexe A : La conjecture de Demyanov–Ryabova.

Nous appelons polytope tout sous ensemble compact et convexe de $\mathbb{R}^N$ possédant un nombre fini de points extrémaux. Considérons une famille finie $\Re = \{\Omega_1, \ldots, \Omega_\ell\}$ de polytopes de $\mathbb{R}^N$ ainsi qu'une opération transformant la famille initiale $\Re$ en une autre famille $\mathscr{F}(\Re)$ de la même nature. Décrivons brièvement cette opération $\mathscr{F}$: Soit $\operatorname{ext}(\Omega)$ l'ensemble des points extrémaux d'un polytope Ω et soit S la sphère unité de $\mathbb{R}^N$. Pour chaque direction $d \in S$ et chaque polytope $\Omega_i \in \Re$, nous considérons l'ensemble des points extrémaux de Ω_i "actifs" dans la direction d

$$E(\Omega_i, d) := \{x \in \operatorname{ext}(\Omega_i) : \langle x, d\rangle = \max\langle \Omega_i, d\rangle\}.$$

Puis, à chaque $d \in S$ nous associons un polytope $\Omega(d)$ obtenu en prenant l'enveloppe convexe de l'ensemble de tous les points extrémaux actifs dans la direction d (Ω_i parcourant $\Re$). Étant donné que le nombre total de points extrémaux des polytopes de la famille $\Re$ est fini, la famille duale associée $\mathscr{F}(\Re) := \{\Omega(d) : d \in S\}$ contient un nombre fini de polytopes. Cette famille duale est donc bien de la même nature que $\Re$.

En partant alors d'une famille de polytopes $\Re_0$, nous définissons par récurrence une suite $\{\Re_n\}_n$ par applications successives de l'opération $\mathscr{F}$. On est alors en mesure d'énoncer la conjecture de Demyanov et Ryabova : Il existe $n_0 \in \mathbb{N}$ tel que $\Re_{n_0} = \Re_{n_0+2}$.

En dehors de nombreuse expériences numériques, cette conjecture n'est toujours pas prouvée. Le seul résultat concernant cette conjecture est contenu dans [San17]. Dans ce travail, l'auteur prouve la conjecture à condition que tous les points extrémaux $E_{\Re_0}$ de $\Re_0$ soient affinement indépendants.

Dans cette annexe, nous démontrons la conjecture sous des hypothèses assez fortes. Afin d'énoncer le résultat principal, nous devons introduire quelques notations. Nous notons E l'ensemble de tous les points extrémaux des polytopes de la famille considérée,

$R = |E|$ son cardinal et $C := \mathrm{conv}(E)$ son enveloppe convexe. Ensuite nous notons $r(\Omega) := |\Omega \cap E|$ le nombre de points extrémaux d'un polytope $\Omega \in \Re_0$ et nous posons :

$$r_{\min} := \min_{\Omega \in \Re_0} r(\Omega).$$

Théorème A.1.2 (avec A. Daniilidis).
Soit $\Re_0$ une famille de polytope de $\mathbb{R}^N$. Alors, $\Re_1 = \Re_3$ (i.e. une famille réflexive apparaît après seulement une itération) à condition que :

(H1) $\forall x \in E$, $x \notin \mathrm{conv}(E \backslash \{x\})$ (*i.e.* chaque $x \in E$ est un point extrémal de C.).

(H2) $\Re_0$ contient tous les polytopes créés à partir de $r_{\min}$ points de E.

Annexe B : Sur de la géométrie grossière de l'espace de James.

Dans [Kal07], Kalton définit une propriété notée $\mathcal{Q}$ qui sert d'obstruction à la plongeabilité grossière dans les espaces réflexifs. Plus précisément, si un espace de Banach ne possède pas la propriété $\mathcal{Q}$, alors il ne se plonge pas dans les espaces réflexifs. De par sa définition, cette propriété est liée à une famille de graphes notée $(\mathbb{G}_k(\mathbb{N}))_{k\in\mathbb{N}}$. De plus, si un Banach X a la propriété $\mathcal{Q}$, alors les graphes $\mathbb{G}_k(\mathbb{N})$ ne se plongent pas equi-grossièrement dans X. Le but de cette annexe est de démontrer que la réciproque du précédent énoncé est fausse. En effet, Kalton a démontré que l'espace de James $\mathcal{J}$ n'a pas la propriété $\mathcal{Q}$ (Proposition 4.7 dans [Kal07]). Cependant, nous démontrons le théorème suivant.

Théorème B.1.7 (avec G. Lancien et A. Procházka).
La famille $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ ne se plonge pas equi-grossièrement dans l'espace de James $\mathcal{J}$.

Ce dernier résultat isole donc une propriété invariante par équivalence grossière qui est proche mais différente de la propriété $\mathcal{Q}$.

2. Summary of the thesis

The main purpose of this thesis is to explore the linear structure of Lipschitz free spaces. Considering a metric space M equipped with a distinguished point 0, the Lipschitz free space over M is defined by

$$\mathcal{F}(M) := \overline{\text{span}}^{\|\cdot\|} \{\delta(x) \,:\, x \in M\} \subset \text{Lip}_0(M)^*.$$

In the above definition, $\text{Lip}_0(M)$ denotes the Banach space of all real-valued Lipschitz functions defined on M which vanish at 0. The Banach space $\mathcal{F}(M)$ is actually a canonical predual of $\text{Lip}_0(M)$. The name "Lipschitz free space" comes from the seminal paper [GK03] due to Godefroy and Kalton. This terminology is explained by the fact that the elements of a metric space M are associated to linearly independent vectors of the corresponding Lipschitz free space $\mathcal{F}(M)$. Yet, those "free spaces" were known and studied before for instance by Weaver who called them Arens–Eells spaces [Wea99]. The paper [GK03] contains deep results which made popular and motivate the development of this area. So, the work in this thesis takes part in the research program launched by Godefroy and Kalton which consists in determining the linear structure of $\mathcal{F}(M)$. We now describe the content of this thesis.

Chapter 1 : General facts about Lipschitz free spaces.

First and foremost, we introduce the fundamental properties of Lipschitz free spaces. The results presented in the first part of the chapter are standard. Here, the central element is the fundamental linearisation property of Lipschitz free spaces : Every Lipschitz map from a metric space M to a Banach space X extends uniquely to a linear continuous operator from $\mathcal{F}(M)$ to X. We decided to include some proofs for completeness and for convenience of the reader.

The remaining part of the chapter is devoted to some general but less classical results. The map $\delta : x \in M \mapsto \delta(x) \in \mathcal{F}(M)$ is readily seen to be an isometry and we denote $\delta(M)$ the range of this map. Thus, when M is complete, the set $\delta(M)$ is norm closed. Furthermore, we actually prove the following stronger result.

Proposition 1.2.1 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a complete pointed metric space. Then $\delta(M) \subset \mathcal{F}(M)$ is weakly closed.

As an easy consequence, we obtain that the weak topology coincide with the norm topology on $\delta(M)$. Next, we turn to the study of the set of molecules. By a molecule we mean an element of the form :

$$m_{xy} := \frac{\delta(x) - \delta(y)}{d(x,y)} \in \mathcal{F}(M), \text{ for } x \neq y \in M.$$

We denote by V the set of all molecules. A very useful property of V is that it is 1-norming for $\text{Lip}_0(M)$ so that $B_{\mathcal{F}(M)} = \overline{\text{conv}}(V)$. Moreover, based on the work of Weaver ([Wea99]), we show the next proposition.

Proposition 1.3.3 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let (M,d) be a complete pointed metric space. Then $\overline{V}^w \subset V \cup \{0\}$.

The above result will have consequences on the extremal structure of $\mathcal{F}(M)$ which we study later in Chapter 4. Using a result proved by Albiac and Kalton in [AK09], we also deduce that V is actually weakly sequentially closed. We conclude the chapter by discussing some possible improvements.

Chapter 2 : Duality of some Lipschitz free spaces.

In this chapter we study under what circumstances $\mathcal{F}(M)$ is isometric to a dual space. In this context, the space of little Lipschitz functions shows up naturally (with the convention $\sup \emptyset = 0$) :

$$\mathrm{lip}_0(M) := \Big\{ f \in \mathrm{Lip}_0(M) : \lim_{\varepsilon \to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \Big\}.$$

Indeed, we can mention the work of Weaver asserting that $\mathcal{F}(M)$ is isometric to the dual of $\mathrm{lip}_0(M)$ whenever M is compact and $\mathrm{lip}_0(M)$ separates the points of M uniformly (Theorem 3.3.3 in [Wea99]). We recall that $S \subset \mathrm{Lip}_0(M)$ separates points uniformly if there is $C > 0$ such that for every $x, y \in M$ there is $f \in S$ such that $f(x) - f(y) = d(x,y)$ and $\|f\|_L \leq C$. More generally, Kalton proved under suitable assumptions that $\mathcal{F}(M)$ is isometric to the dual of $\mathrm{lip}_0(M) \cap \mathcal{C}(M,\tau)$ whenever (M,τ) is compact and $\mathrm{lip}_0(M) \cap \mathcal{C}(M,\tau)$ separates points uniformly. In fact, most isometric preduals of free spaces considered in the literature are subspaces of the space of little Lipschitz functions. Moreover, they have one more thing in common : they make $\delta(B(0,r))$ weak*-closed in $\mathcal{F}(M)$ for every $r \geq 0$. We call natural predual any isometric predual of $\mathcal{F}(M)$ which satisfies this last property.

One of the main objectives here is to (slightly) improve the results mentioned above. In particular, we deal with unbounded metric spaces. In this case, the following subspace of $\mathrm{lip}_0(M)$ turns out to be very convenient :

$$S_0(M) := \Big\{ f \in \mathrm{lip}_0(M) : \lim_{r \to \infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \Big\}.$$

For instance, Dalet proved that $\mathcal{F}(M)$ is isometrically isomorphic to the dual of $S_0(M)$ whenever M is proper (i.e. closed balls are compact) and $S_0(M)$ separates points uniformly. This leads us to the following main result of the chapter. In the next proposition, $\mathcal{C}_b(M,\tau)$ denotes the set of maps which are τ-continuous on bounded set of M.

Proposition 2.2.5 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a separable complete pointed metric space. Let τ be a topology on M so that d is τ lower-semi-continuous and M is τ-proper (i.e. closed balls for d are τ-compact). Assume that $X = S_0(M) \cap \mathcal{C}_b(M,\tau)$ separates points uniformly. Then, X is a natural predual of $\mathcal{F}(M)$.

Our proof is based on Petunīn–Plīčko theorem (see [God87, PP74]). The benefit is that it avoids the metrisability assumption of the topology τ present in Kalton's result quoted above. Furthermore, we prove that Proposition 2.2.5 is the only way to build a natural predual if the predual is moreover required to be a subspace of the space of $S_0(M)$.

Next, we focus on uniformly discrete metric spaces, that is metric spaces M such that $\inf\{d(x,y) : x \neq y \in M\} > 0$. For such a metric space M, $\mathcal{F}(M)$ is known to have a

strong ℓ_1-behavior. For instance, Kalton proved in [Kal04, Proposition 4.4] that $\mathcal{F}(M)$ has the Schur property, the Radon-Nikodym property (RNP) and the approximation property (AP). A famous open problem raised in [Kal04] consists in determining whether $\mathcal{F}(M)$ enjoys the bounded approximation property (BAP). In this context, it is very natural to try to decide whether $\mathcal{F}(M)$ is isometrically a dual space or not. In fact, we concentrate on uniformly discrete and bounded metric spaces for which we characterise the possible natural preduals of the associated free spaces.

Corollary 2.3.4 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let (M, d) be a uniformly discrete, bounded, separable and complete pointed metric space. Let X be a Banach space. Then it is equivalent :

(i) X is a natural predual of $\mathcal{F}(M)$.

(ii) There is a Hausdorff topology τ on M such that (M, τ) is compact, d is τ-l.s.c. and $X = \mathrm{Lip}_0(M, d) \cap \mathcal{C}(M, \tau)$ equipped with the norm $\|\cdot\|_L$.

Exploring the free spaces over uniformly discrete metric spaces permits us to isolate examples with interesting properties. For instance, there is a metric space M such that $\mathcal{F}(M)$ is isometric to a dual but does not admit any natural predual. We also pin down a uniformly discrete and bounded metric space for which $\mathcal{F}(M)$ does not have any isometric predual.

Finally, we provide a new class of metric spaces which satisfies the assumptions of Proposition 2.2.5, namely the metric spaces originating from p-Banach spaces. More precisely, we fix p in $(0, 1)$ and we consider $(X, \|\cdot\|)$ a p-Banach space. Now $d_p(x, y) = \|x - y\|^p$ defines a metric on X so that we can study $\mathcal{F}(X, d_p)$. In particular, we prove the following corollary.

Corollary 2.4.5. *Let $p \in (0, 1)$ and consider $(M_p, d_p) = (X, \|\cdot\|^p)$ where $(X, \|\cdot\|)$ is a p-Banach space of finite dimension. Then $S_0(M_p)$ is a natural predual of $\mathcal{F}(M_p)$.*

Chapter 3 : Schur properties and Lipschitz free spaces.

In this chapter, we focus on ℓ_1-like properties such as the Schur property or some stronger properties. In [Kal04], Kalton proved that if (M, d) is a metric space and ω is a nontrivial gauge (for instance $\omega(t) = t^p$ with $0 < p < 1$), then the space $\mathcal{F}(M, \omega \circ d)$ has the Schur property. In [HLP16], Hájek, Lancien, and Pernecká proved the same for $\mathcal{F}(M)$ whenever M is proper and countable. Here we give a sufficient condition which unifies the two above mentioned results.

Proposition 3.1.2. *Let M be a pointed metric space such that $\mathrm{lip}_0(M)$ is 1-norming for $\mathcal{F}(M)$. Then the space $\mathcal{F}(M)$ has the Schur property.*

The proof is strongly inspired by Theorem 4.6 in [Kal04]. If the metric space is moreover assumed to be proper, it is known that $S_0(M)$ is isomorphic to a subspace of c_0 [Dal15c]. Thus, we deduce that some quantitative versions of the Schur property are inherited by $\mathcal{F}(M)$.

Proposition 3.2.5. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly. Then $\mathcal{F}(M)$ has the 1-Schur property.*

Adding one more condition, which is $\mathcal{F}(M)$ has the approximation property, we are able to provide more information about the "ℓ_1-structure" of $\mathcal{F}(M)$. The following theorem

is based on a result of Godefroy, Kalton and Li which deals with subspaces of c_0 having the metric approximation property.

Theorem 3.3.1. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly and such that $\mathcal{F}(M)$ has the metric approximation property. Then for any $\varepsilon > 0$, there exists a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $\mathcal{F}(M)$ such that $\mathcal{F}(M)$ is $(1+\varepsilon)$-isomorphic to a subspace of $(\sum_n E_n)_{\ell_1}$.*

Finally, we apply those results to metric spaces originating from p-Banach spaces.

Chapter 4 : Extremal structure of Lipschitz free spaces.

In this chapter we focus on the extremal structure of $\mathcal{F}(M)$. The two main questions in this domain are the following.

a) If μ is an extreme point of the unit ball $B_{\mathcal{F}(M)}$, is μ necessarily a molecule ?

b) If the metric segment $[x,y] = \{z \in M \,:\, d(x,z) + d(z,y) = d(x,y)\}$ is reduced to $\{x,y\}$, is m_{xy} an extreme point of $B_{\mathcal{F}(M)}$?

Weaver proved in [Wea99] that preserved extreme points of $\mathcal{F}(M)$, that is extreme points of the unit ball $B_{\mathcal{F}(M)^{**}}$ which belong to $\mathcal{F}(M)$, are always molecules. More recently Aliaga and Guirao showed that b) have an affirmative answer whenever the metric space is compact. They also give a metric characterisation of preserved extreme points in full generality, which we also prove here by a different argument. More results in the same line appeared in [GLPZ17b], where for instance a metric characterisation of the strongly exposed points is given.

The goal of the present chapter is to continue the effort in exploring the extremal structure of $\mathcal{F}(M)$ and provide affirmative answers to both previous questions a) and b) in some particular cases. After a few easy observations, we begin by proving the following result.

Theorem 4.1.4 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a pointed metric space. Then every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point, that is, it is in slices of $B_{\mathcal{F}(M)}$ of arbitrarily small diameter.

Next, we focus on extreme points in free spaces that admit a natural predual. Notably, under reasonable assumptions we get an affirmative answer to question a).

Proposition 4.2.1 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a pointed metric space. Assume that there is a subspace X of $S_0(M)$ which is a natural predual of $\mathcal{F}(M)$. Then every extreme point of $B_{\mathcal{F}(M)}$ is a molecule.

Assuming moreover the metric space to be separable, we also obtain an affirmative answer to question b). More precisely, we obtain the following corollary.

Corollary 4.2.2 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a separable pointed metric space. Assume that there is a subspace X of $S_0(M)$ which is a natural predual of $\mathcal{F}(M)$. Then a given $\mu \in B_{\mathcal{F}(M)}$ is an extreme point if and only if it is an exposed point if and only if $\mu = m_{xy}$ for some $x \neq y \in M$ with $[x,y] = \{x,y\}$.

Then, once more, we turn our attention to uniformly discrete metric spaces. We show among other things that, when M is uniformly discrete and bounded, a molecule m_{xy}

is an extreme point of $B_{\mathcal{F}(M)}$ if and only if $[x,y] = \{x,y\}$. We also prove that every extreme point of $B_{\mathcal{F}(M)}$ is actually strongly exposed. We finish the section by studying the case of proper metric spaces M such that $S_0(M)$ separates points uniformly. In fact, we prove that $\mathcal{F}(M)$ enjoys a geometrical property as being weak*-asymptotically uniformly convex. In particular, this implies that every point in the unit sphere has relative weak* neighbourhoods of arbitrarily small diameter. As a direct consequence, we obtain the following.

Corollary 4.4.2 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a proper pointed metric space. Assume that $S_0(M)$ 1-separates points uniformly. Then every extreme point of $B_{\mathcal{F}(M)}$ is also a denting point.

Chapter 5 : Vector-valued Lipschitz functions.

We shift now our attention to vector-valued Lipschitz functions. It follows from the fundamental linearisation property of Lipschitz free spaces that $\mathrm{Lip}_0(M,X)$ is linearly isometric to $\mathcal{L}(\mathcal{F}(M),X)$. Consequently $\mathrm{Lip}_0(M,X)$ is linearly isometric to $(\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$. This is the main motivation for defining the X-valued Lipschitz free space over M as being $\mathcal{F}(M,X) := \mathcal{F}(M)\widehat{\otimes}_\pi X$. We start by showing some basic properties comparable to what we proved in Chapter 1. For instance we identify a subset $\delta(M,X)$ of $\mathcal{F}(M,X)$ for which we prove that it is weakly closed whenever M is complete. In fact,

$$\delta(M,X) := \{\delta(y)\otimes x \ : \ y\in M,\, x\in X\} \subset \mathcal{F}(M,X).$$

Afterwards, we extend some duality results presented in Chapter 2. A first observation is that basic theory of tensor products yields $\mathcal{F}(M,X^*) = \mathcal{F}(M)\widehat{\otimes}_\pi X^* \equiv (S\widehat{\otimes}_\varepsilon X)^*$ whenever $\mathcal{F}(M)\equiv S^*$ and under reasonable additional assumptions (where $Y\equiv Z$ means that Y and Z are linearly isometric). So, the fact that $\mathcal{F}(M,X^*)$ is isometric to a dual Banach space actually relies on the scalar case. Moreover, we define a somehow legitimate extension of the notion of natural predual. We show that, in some cases, the fact that $S\widehat{\otimes}_\varepsilon X$ is a natural predual of $\mathcal{F}(M,X^*)$ also relies on the scalar case.

We next try to give a representation of a predual to $\mathcal{F}(M,X^*)$ as a subspace of $\mathrm{Lip}_0(M,X^{**})$. Our first goal is to extend the duality result of Dalet mentioned earlier. A first attempt could be to use Petunīn–Plīčko's theorem. However, to avoid the separability assumption needed in this result, we follow a different path. More precisely, it is proved in [GLRZ17] that $S_0(M,X)\equiv \mathcal{K}_{w^*,w}(X^*,S_0(M))$ whenever M is proper (where $\mathcal{K}_{w^*,w}(X^*,S_0(M))$ is the space of compact operators from X^* to $S_0(M)$ which are weak*-to-weak continuous). Thus, it suffices to prove that $\mathcal{K}_{w^*,w}(X^*,S_0(M))\equiv S_0(M)\widehat{\otimes}_\varepsilon X$ to obtain the next theorem.

Theorem 5.2.5 (with L. García-Lirola and A. Rueda Zoca).
Let M be a proper pointed metric space and let X be a Banach one. Assume that $S_0(M)$ separates points uniformly. If either $\mathcal{F}(M)$ or X^* has the (AP), then the following isometric identification holds : $S_0(M,X)^*\equiv \mathcal{F}(M,X^*)$.

Using the same path, we next extend Kalton's duality result mentioned in Chapter 2. The main task consists in characterising the relatively compact subsets of $\mathrm{lip}_0(M)\cap \mathcal{C}(M,\tau)$ where τ is a topology so that (M,τ) is compact and d is τ lower-semi-continuous.

This allows us to prove under some assumptions that $lip_\tau(M,X) \equiv \mathcal{K}_{w^*,w}(X^*, lip_\tau(M))$, where

$$lip_\tau(M,X) := \mathrm{lip}_0(M,X) \cap \{f\colon M \to X : f \text{ is } \tau - \text{to} - \|\cdot\| \text{ continuous}\}.$$

In the end, we get the next theorem.

Theorem 5.2.9 (with L. García-Lirola and A. Rueda Zoca).
Let M be a separable complete bounded pointed metric space. Suppose that τ is a metrisable topology on M so that (M,τ) is compact and $lip_\tau(M)$ 1-separates points uniformly. If either $\mathcal{F}(M)$ or X^* has the (AP), then $lip_\tau(M,X)^* \equiv \mathcal{F}(M,X^*)$.

Then, we shift our attention to the Schur properties defined in Chapter 3. The main purpose is to give conditions on M and X that force $\mathcal{F}(M,X)$ to have the Schur property. We will use principally two methods. The first one consists in analysing the Dunford–Pettis property on an isometric predual of $\mathcal{F}(M,X^*)$, when it exists, with the help of tensor product theory. Then we dualise to get our desired result about the Schur properties on $\mathcal{F}(M,X^*)$. The second method uses techniques from the theory of Lipschitz maps. In fact, we follow more or less the same pattern as in the real-valued frame.

We will finish the chapter with some considerations of norm attainment of Lipschitz maps. We will actually consider two different notions of norm attainment for a Lipschitz map. One from operator theory : $f : M \to X$ attains its operator norm if there exists $\gamma \in S_{\mathcal{F}(M)}$ such that : $\|\langle f,\gamma\rangle\|_X = \|f\|_L$. The other one from the theory of Lipschitz maps : $f : M \to X$ strongly attains its norm if there are two different points $x,y \in M$ such that : $\|f(x) - f(y)\|_X = \|f\|_L\, d(x,y)$. Obviously, if f strongly attains its norm then f attains its operator norm. We shall prove that both concepts agree for some classes of metric spaces M.

Proposition 5.4.4 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a pointed metric space and X be a Banach space. Assume that $\mathcal{F}(M)$ has the Krein–Milman property and that $\mathrm{ext}(B_{\mathcal{F}(M)}) \subseteq V$. Then every $f \in \mathrm{Lip}_0(M,X)$ which attains its norm on $\mathcal{F}(M)$ also strongly attains it.

In light of the celebrated Bishop–Phelps's theorem and based on a result of Bourgain [Bou77], we also prove a kind of Bishop–Phelps's density result for vector-valued Lipschitz maps.

Proposition 5.4.5 (with L. García-Lirola, A. Procházka and A. Rueda Zoca).
Let M be a complete pointed metric space and X be a Banach space. Assume that $\mathcal{F}(M)$ has the (RNP). Then the set of Lipschitz maps which strongly attain their norm is dense in $\mathrm{Lip}_0(M,X)$.

The thesis is completed with two appendices unrealated to the main subject.

Appendix A : The Demyanov–Ryabova conjecture.

We call polytope any convex compact subset of $\mathbb{R}^N$ with a finite number of extreme points. We consider a finite family $\Re = \{\Omega_1,\dots,\Omega_\ell\}$ of polytopes of $\mathbb{R}^N$ together with an operation which transforms the initial family $\Re$ to a dual family of polytopes that we denote $\mathscr{F}(\Re)$. Let us briefly describe the operation $\mathscr{F}$: let $\mathrm{ext}(\Omega)$ stand for the set

of extreme points of the polytope Ω and let S denote the unit sphere of $\mathbb{R}^N$. For any direction $d \in S$ and polytope $\Omega_i \in \Re$ we consider the set of d-active extreme points of Ω_i

$$E(\Omega_i, d) := \{x \in \mathrm{ext}(\Omega_i) : \langle x, d\rangle = \max\langle \Omega_i, d\rangle\}.$$

We associate to $d \in S$ the polytope $\Omega(d)$ obtained as the convex hull of the set of all d-active extreme points (when Ω_i is taken throughout $\Re$). Since the set of extreme points of all polytopes of the family $\Re$ is finite, the family of polytopes $\mathscr{F}(\Re) := \{\Omega(d) : d \in S\}$ is also finite, hence of the same nature as $\Re$.

Now starting from a given family of polytopes $\Re_0$, we define successively a sequence of families $\{\Re_n\}_n$ by applying repeatedly this duality operation (transformation) $\mathscr{F}$. We are now ready to announce the conjecture of Demyanov and Ryabova : There exists $n_0 \in \mathbb{N}$ we shall have $\Re_{n_0} = \Re_{n_0+2}$.

Besides the recorded numerical evidence, there is still no proof of this conjecture. The only known result in this direction is due to [San17]. In that work, the author establishes the conjecture under the additional assumption that the set $E_{\Re_0}$ of extreme points of the initial family $\Re_0$ is affinely independent.

In this appendix, we prove the conjecture under quite restrictive assumptions. To state the result, we need to fix a few notations. We denote by E the set of extreme points of all polytopes of the family, by $R = |E|$ its cardinality and we set $C := \mathrm{conv}(E)$ its convex hull. Let further $r(\Omega) := |\Omega \cap E|$ denote the number of extreme points of the polytope $\Omega \in \Re_0$ and set

$$r_{\min} := \min_{\Omega \in \Re_0} r(\Omega).$$

Theorem A.1.2 (with A. Daniilidis).
Let $\Re_0$ be a finite family of polytopes in $\mathbb{R}^N$. Then $\Re_1 = \Re_3$ (i.e. a reflexive family occurs after one iteration) provided :

(H1) $\forall x \in E$, $x \notin \mathrm{conv}(E \backslash \{x\})$ (*i.e.* each $x \in E$ is extreme in C.)

(H2) $\Re_0$ contains all $r_{\min}$-polytopes (that is, all polytopes made up of $r_{\min}$ points of E).

Appendix B : On the coarse geometry of the James space.

In [Kal07], Kalton introduced the property $\mathcal{Q}$ which serves as an obstruction to coarse embeddability into reflexive Banach spaces. More precisely, if a Banach space fails the property $\mathcal{Q}$ then it does not coarsely embed into any reflexive Banach space. By its definition, this property has a close relationship with a particular family of graphs that we denote $(\mathbb{G}_k(\mathbb{N}))_{k\in\mathbb{N}}$. Actually, if a Banach space X has property $\mathcal{Q}$, then the graphs $\mathbb{G}_k(\mathbb{N})$ do not equi-coarsely embed into X. The purpose of this appendix is to show that the converse of the previous statement is false. Indeed, Kalton proved (Proposition 4.7 in [Kal07]) that the James space $\mathcal{J}$ fails property $\mathcal{Q}$. However, we are going to show the following.

Theorem B.1.7 (with G. Lancien et A. Procházka).
The family of graphs $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into the James space $\mathcal{J}$.

This last result bring to light a coarse invariant property which is close to but different from the property $\mathcal{Q}$.

3. Notation

We give here some notation that will be used freely throughout the thesis. The notation used later but not mentioned here is either standard or will be given when needed.

- We will denote $|S|$ the cardinal of a finite set S.
- Restriction of maps. Let M, N be two sets and let $f : M \to N$ be any map. If M' is a subset of M, we denote $f\restriction_{M'}$ the map from M' to N which coincides with f on M'.
- A gauge $\omega\colon [0,\infty) \to [0,\infty)$ is an increasing, continuous, and subadditive function which satisfies $\omega(0) = 0$, $\omega(t) \geq t$ for every $0 \leq t \leq 1$. If moreover $\lim_{t\to 0} \omega(t)/t = \infty$, we say that the gauge is nontrivial.

Banach spaces

We will only consider real Banach spaces. Unless otherwise specified, X, Y and Z denote Banach spaces.

- B_X (respectively S_X) denotes the closed unit ball (respectively the unit sphere) of X.
- $\mathcal{C}(T,\tau)$ denotes the space of τ-continuous and real-valued functions defined on a topological space (T,τ).
- $\overline{S}^{\tau}$ denotes the closure of a subset $S \subseteq X$ with respect to a topology τ. When the closure is taken for the norm topology, we shall simply write $\overline{S}$.
- $\operatorname{conv} S$ stands for the convex hull of $S \subseteq X$ and $\overline{\operatorname{conv}}\, S$ for the closed convex hull of S.
- $\mathcal{L}(X,Y)$ is the Banach space of continuous linear operators $T\colon X \to Y$ equipped with its usual operator norm :
$$\|T\| = \sup_{x\in B_X} \|T(x)\|_Y.$$
- $X^* = \mathcal{L}(X,\mathbb{R})$ denotes the topological dual of X.
- $\sigma(X,Y)$ is the topology on X of pointwise convergence on elements of $Y \subseteq X^*$. In particular, we sometimes denote $w^* = \sigma(X^*,X)$ the weak* topology on X^* and $w = \sigma(X,X^*)$ the weak topology on X.
- $\mathcal{K}(X,Y)$ stands for the compact operators from X to Y, that is operators $T \in \mathcal{L}(X,Y)$ such that $\overline{T(B_X)}$ is a compact set in Y for the norm topology. Moreover, $T \in \mathcal{L}(X,Y)$ is called a finite-rank operator if $\dim(T(X)) < \infty$.
- $\mathcal{K}_{\tau_1,\tau_2}(X,Y)$, where τ_1 is a topology on X and τ_2 a topology on Y, is the space of operators $T \in \mathcal{K}(X,Y)$ which are τ_1-to-τ_2 continuous.
- $\mathcal{B}(X \times Y, Z)$ is the Banach space of continuous bilinear operators $B\colon X \times Y \to Z$ equipped with its usual norm :
$$\|B\| = \sup_{x\in B_X, y\in B_Y} \|B(x,y)\|_Z.$$
- ℓ_1-sums of Banach spaces. Let $(X_n)_{n=1}^{\infty}$ be a sequence of Banach spaces. Then the following space
$$\Big(\sum_n X_n\Big)_{\ell_1} := \Big\{x = (x_n)_{n=1}^{\infty} \ :\ \forall n \in \mathbb{N},\ x_n \in X_n;\ \sum_{n=1}^{\infty} \|x_n\|_{X_n} < \infty\Big\}$$
is a Banach space endowed with the following norm $\|x\|_1 := \sum_{n=1}^{\infty} \|x_n\|_{X_n}$.

Metric spaces

Unless otherwise specified, (M, d_M) and (N, d_N) denote pointed metric spaces, that is metric spaces equipped with distinguished points denoted 0_M and 0_N. When there is no ambiguity, we write d instead of d_M or d_N and 0 instead of 0_M or 0_N.

- $B(x, r)$ denotes the closed ball in M centred at $x \in M$ with radius $r \geq 0$.
- $\mathrm{diam}(S)$ denotes the diameter of $S \subset M$, that is :

$$\mathrm{diam}(S) = \sup\{d(x, y) \ : \ x \neq y \in S\}.$$

- A map $L\colon M \to N$ is said to be Lipschitz if there exists $C > 0$ such that for every $x \neq y \in M$, $d_N(L(x), L(y)) \leq C d_M(x, y)$. For such a function, we denote $Lip(L)$ its best Lipschitz constant.
- Proper metric space. A metric space M is said to be proper if each ball $B(x, r)$ is compact.
- τ-Proper metric space. A metric space M is said to be τ-proper (τ being another topology on M) if each ball $B(x, r)$ is τ-compact.
- Given $x, y \in M$, the metric segment between x and y is defined the following way :

$$[x, y] = \{z \in M : d(x, z) + d(z, y) = d(x, y)\}.$$

- Let (M, d) be a metric space and τ another topology on M. We say that d is lower-semi-continuous with respect to τ (in short d is τ-l.s.c.) if the following is satisfied : For every $x_0, y_0 \in M$,

$$\liminf_{x \xrightarrow{\tau} x_0, y \xrightarrow{\tau} y_0} d(x, y) \geq d(x_0, y_0).$$

Classifications

- **Linear isometry.** We say that two Banach spaces X and Y are linearly isometric if there exists a bijective linear map $T : X \to Y$ satisfying $\|T(x)\|_Y = \|x\|_X$ for every $x \in X$. In this case, we write $X \equiv Y$.
- **Linear isomorphism.** We say that two Banach spaces X and Y are linearly isomorphic if there exist $C_1, C_2 > 0$ and a bijective linear map $T : X \to Y$ such that for every $x \in X : C_1\|x\| \leq \|Tx\| \leq C_2\|x\|$. In this case, we write $X \simeq Y$.
- **Lipschitz isomorphism.** We say that two metrics spaces (M, d_M) and (N, d_N) are Lipschitz equivalent (or Lipschitz isomorphic) if there exists a bijective bi-Lipschitz map $L : M \to N$. That is, there exist $C_1, C_2 > 0$ satisfying for every $x, y \in M$: $C_1 d_M(x, y) \leq d_N(L(x), L(y)) \leq C_2 d_M(x, y)$. In this case, we write $X \underset{L}{\sim} Y$.
- **Linear embedding.** We say that a Banach space X linearly embeds into a Banach space Y if X is linearly isomorphic to a subspace of Y.
- **Lipschitz embedding.** We say that a metrics space (M, d_M) Lipschitz embeds into a metric space (N, d_N) if M is Lipschitz equivalent to a subset of N.

Some classical properties of Banach spaces

- **Krein–Milman property.** A Banach space is said to have the Krein–Milman property (in short (KMP)) if every non-empty closed convex bounded subset has an extreme point.
- **Radon–Nikodým property.** We choose to give a geometrical characterisation of the Radon–Nikodým property (in short (RNP)). For more characterisations, we refer the reader to Section VII.6 in [DU77]. A Banach space X has the (RNP) if and only if every bounded subset C of X is dentable, that is, for every $\varepsilon > 0$ there is an open half-space H such that $\operatorname{diam}(C \setminus H) < \varepsilon$ and $C \setminus H \neq \emptyset$.
- **Approximation properties.**

(AP) We say that a Banach space X has the approximation property (AP) if for every $\varepsilon > 0$, for every compact set $K \subset X$, there exists a finite rank operator $T \in \mathcal{B}(X)$ such that $\|Tx - x\| \leq \varepsilon$ for every $x \in K$.

(BAP) Let $\lambda \geq 1$, if in the above definition T can always be chosen so that $\|T\| \leq \lambda$, then we say that X has the λ-bounded approximation property (in short λ-(BAP)).

(MAP) When X has the 1-(BAP) we say that X enjoys the metric approximation property (MAP).

- **Schur property.** We say that X has the Schur property if every weakly null sequence $(x_n)_{n=1}^{\infty}$ in X is also $\|\cdot\|$-convergent to 0.
- **Dunford–Pettis property.** A Banach space X is said to have the Dunford–Pettis property if for any weakly convergent sequences $(x_n)_{n=1}^{\infty} \subset X$ and $(x_n^*)_{n=1}^{\infty} \subset X^*$, converging to x and x^*, the sequence $(x_n^*(x_n))_{n=1}^{\infty}$ converges to $x^*(x)$.
- **Daugavet property.** A Banach space X is said to have the Daugavet property if every rank-one operator $T : X \to X$ satisfies the equality $\|T + Id\| = 1 + \|T\|$. This is known to be equivalent equivalent to the following geometric condition (see [KSSW00, Lemma 2.1]) : For every $x \in S_X$, every slice S of B_X and every $\varepsilon > 0$ there exists another slice T of the unit ball such that $T \subseteq S$ and $\|x + y\| > 2 + \varepsilon$ for every $y \in T$.
- **Finite dimensional decomposition.** A sequence $(X_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of a Banach X is called a finite-dimensional decomposition of X (in short (FDD)) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for every $n \in \mathbb{N}$. Moreover, the (FDD) is called monotone whenever the projections $P_n \colon x = \sum_{i=1}^{\infty} x_n \in X \mapsto \sum_{i=1}^{n} x_n \in \sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$ are such that $\|P_n\| = 1$.

Chapter 1

General facts about Lipschitz free spaces

First and foremost, we are going to give fundamental properties of Lipschitz free spaces (Section 1.1). The name "Lipschitz free space" comes from the seminal paper [GK03] due to Godefroy and Kalton. This terminology is explained by the fact that the elements of a metric space M are associated with linearly independent vectors in the corresponding Lipschitz free space $\mathcal{F}(M)$. The paper [GK03] contains deep results which made popular and motivated the development of this area. Yet, those "free spaces" were known and studied before for instance by Weaver who called them Arens–Eells spaces [Wea99].

Next, we will study the weak closure of some very particular subsets of a Lipschitz free space. More precisely, we will prove in Section 1.2 that $\delta(M)$ (the canonical image of M into $\mathcal{F}(M)$) is weakly closed. Then, we show in Section 1.3 that the set of molecules V is not far from being weakly closed. Indeed, we will actually show that $V \cup \{0\}$ is weakly closed. The Section 1.2 and the Section 1.3 are based on a joint work with Luis García-Lirola, Antonín Procházka and Abraham Rueda Zoca (see [GPPR17]). The results are new but inspired by the work of Weaver (Section 2.4 and Section 2.5 in [Wea99]).

Finally, we will discuss in Section 1.4 some possible extensions of some results obtained in Section 1.2 and Section 1.3. This yields us to a result proved by Albiac and Kalton in [AK09].

1.1 Definition and first properties

For a pointed metric space M and a Banach space X we denote $\mathrm{Lip}_0(M, X)$ the vector space of Lipschitz maps from M to X satisfying $f(0) = 0$. Equipped with the following norm (in fact the best Lipschitz constant of f)

$$\|f\|_L = \sup_{x \neq y \in M} \frac{\|f(x) - f(y)\|_X}{d(x, y)},$$

$\mathrm{Lip}_0(M, X)$ is a Banach space. When the range space is $\mathbb{R}$, we just write $\mathrm{Lip}_0(M)$ instead of $\mathrm{Lip}_0(M, \mathbb{R})$. For any $x \in M$, we denote $\delta(x)$ the evaluation functional defined by $\langle f, \delta(x) \rangle = f(x)$ for every $f \in \mathrm{Lip}_0(M)$. It is readily seen that $\delta(x) \in \mathrm{Lip}_0(M)^*$ with $\|\delta(x)\| = d(x, 0)$.

Definition 1.1.1. The Lipschitz free space over M (also called Arens–Eells space over M) is the following subspace of $\mathrm{Lip}_0(M)^*$:

$$\mathcal{F}(M) := \overline{\mathrm{span}}^{\|\cdot\|}\{\delta(x) \,:\, x \in M\}\,.$$

We say that $\gamma \in \mathcal{F}(M)$ is finitely supported if $\gamma \in \mathrm{span}\,\{\delta(x) \,:\, x \in M\}$. Then, the support of such a γ (denoted $\mathrm{supp}\,\gamma$) is the smallest subset F of M which contains 0 and such that $\gamma \in \mathrm{span}\{\delta(x) \,:\, x \in F\}$.

Since $\|\delta(x) - \delta(y)\| = d(x, y)$ for every $x, y \in M$, the map $\delta\colon x \in M \mapsto \delta(x) \in \mathcal{F}(M)$ is an isometry. The Lipschitz free space $\mathcal{F}(M)$ is characterised by the following property.

Proposition 1.1.2 (**Fundamental linearisation property**). *For every Banach space X, for every $L \in \mathrm{Lip}_0(M, X)$, there exists a continuous linear operator $\overline{L}\colon \mathcal{F}(M) \to X$ with $\|\overline{L}\| = \|L\|_L$ and such that the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, hook, "\delta_M"'] & X \\ \mathcal{F}(M) \arrow[ur, "\overline{L}"'] & \end{tikzcd}$$

Moreover, the linear isometry $\Phi\colon L \in \mathrm{Lip}_0(M, X) \mapsto \overline{L} \in \mathcal{L}(\mathcal{F}(M), X)$ is onto.

Proof. Let us fix a Banach space X.

We start by proving the first part of the proposition. That is, we show that Φ is a linear isometry. Let us fix a Lipschitz map $L \in \mathrm{Lip}_0(M, X)$. Let $\tilde{L}$ be the map defined on $\mathrm{span}\,\{\delta(x) \,:\, x \in M\}$ by $\tilde{L}(\sum_{i=1}^n a_i\delta(x_i)) = \sum_{i=1}^n a_i L(x_i) \in X$. Using the Hahn–Banach theorem we have the following estimate for every $\gamma = \sum_{i=1}^n a_i\delta(x_i)$:

$$\begin{aligned} \|\tilde{L}\gamma\|_X = \Big\|\sum_{i=1}^n a_i L(x_i)\Big\|_X &= \sup\Big\{x^*\Big(\sum_{i=1}^n a_i L(x_i)\Big) \,:\, x^* \in B_{X^*}\Big\} \\ &= \sup\Big\{\sum_{i=1}^n a_i (x^* \circ L)(x_i) \,:\, x^* \in B_{X^*}\Big\} \\ &\leq \sup\Big\{\sum_{i=1}^n a_i f(x_i) \,:\, f \in \|L\|_L B_{\mathrm{Lip}_0(M)}\Big\} \\ &= \|L\|_L\, \|\gamma\|_{\mathcal{F}(M)} \end{aligned}$$

Thus $\|\tilde{L}\| \leq \|L\|_L$. Now we want to prove the reverse inequality. To this end, let us fix $\varepsilon > 0$ and consider $x \neq y \in M$ such that $\|Lx - Ly\|_X \geq (\|L\|_L - \varepsilon)d(x, y)$. Then, let us define $m_{xy} := d(x, y)^{-1}(\delta(x) - \delta(y))$. Clearly $\|m_{xy}\| = 1$ and $\|\tilde{L}m_{xy}\|_X = d(x, y)^{-1}\|Lx - Ly\|_X \geq (\|L\|_L - \varepsilon)$. Since ε was chosen arbitrarily, we actually get $\|\tilde{L}\| \geq \|L\|_L$ and so $\|\tilde{L}\| = \|L\|_L$. To finish, we extend $\tilde{L}$ to $\mathcal{F}(M)$ and we denote $\overline{L}$ this unique continuous extension (which has the same norm).

It remains to show that the linear isometry $\Phi\colon L \in \mathrm{Lip}_0(M, X) \mapsto \overline{L} \in \mathcal{L}(\mathcal{F}(M), X)$ is onto. To this end, consider $T \in \mathcal{L}(\mathcal{F}(M), X)$. Then, define L on M by $Lx = T\delta(x)$ for every $x \in M$. The map L is clearly Lipschitz and satisfies $\Phi L = T$. $\square$

As a direct consequence of the previous proposition (in the case $X = \mathbb{R}$), we obtain that $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$. Moreover, the weak* topology coincides with the topology of pointwise convergence on bounded sets of $\mathrm{Lip}_0(M)$. We also deduce the following variation of the universal property.

Corollary 1.1.3. *Let M and N be two pointed metric spaces. Let $L\colon M \to N$ be a Lipschitz map. Then, there exists a linear bounded operator $\widehat{L}\colon \mathcal{F}(M) \to \mathcal{F}(N)$ with $\|\widehat{L}\| = Lip(L)$ and such that the following diagram commutes*

$$\begin{tikzcd} M_1 \arrow[r, "L"] \arrow[d, "\delta_{M_1}"'] & M_2 \arrow[d, "\delta_{M_2}"] \\ \mathcal{F}(M_1) \arrow[r, "\widehat{L}"'] & \mathcal{F}(M_2) \end{tikzcd}$$

Next, if a metric space N is a subspace of a metric space M, then $\mathcal{F}(N)$ is linearly isometric to a subspace of $\mathcal{F}(M)$. Indeed, denote $Id\colon N \to M$ the identity map. Then the map $\widehat{Id}$ given by Corollary 1.1.3 is the desired isometry. In order to prove this last claim, one needs to use the well known fact that every real-valued Lipschitz function defined on N may be extended to M without increasing its Lipschitz constant. For instance, using the McShane formula : $\tilde{f}(x) = \sup\{f(y) - Lip(f)d(x,y) \,:\, y \in N\}$ (see Theorem 1.5.6 in [Wea99] for instance). Furthermore, we also have the following.

Corollary 1.1.4. *Let M and N be two pointed metric spaces. If N Lipschitz embeds into M, then $\mathcal{F}(N)$ linearly embeds into $F(M)$. Moreover, if N is Lipschitz equivalent to M, then $\mathcal{F}(N)$ is linearly isomorphic to $F(M)$.*

Proof. Let $L\colon N \to M$ be a bi-Lipschitz map. Of course, L is bijective from N into $L(N)$ and the inverse mapping $L^{-1}\colon L(N) \to N$ is Lipschitz. We then consider the bounded operators $\widehat{L}\colon \mathcal{F}(N) \to \mathcal{F}(L(N))$ and $\widehat{L^{-1}}\colon \mathcal{F}(L(N)) \to \mathcal{F}(N)$ given by Corollary 1.1.3. It is easy to see that $\widehat{L} \circ \widehat{L^{-1}} = Id$ and $\widehat{L} \circ \widehat{L^{-1}} = Id$ so that $\widehat{L}$ is an isomorphism from $\mathcal{F}(N)$ to $\mathcal{F}(L(N))$. Since $\mathcal{F}(L(N))$ is isometric to a subspace of $\mathcal{F}(M)$ we get that $\mathcal{F}(N)$ is indeed isomorphic to a subspace of $\mathcal{F}(M)$. The second part of the corollary is clear. □

We now give two classical examples of metric spaces M for which we have a nice representation of the associated Lipschitz free space.

Example 1.1.5.

1. "$M = \mathbb{N}$". The linear operator satisfying $T\colon \delta(n) \in \mathcal{F}(\mathbb{N}) \mapsto \sum_{i=1}^{n} e_i \in \ell_1$ is an onto isometry.
2. "$M = [0,1]$". The linear operator $T\colon \delta(t) \in \mathcal{F}([0,1]) \mapsto \mathbb{1}_{[0,t]} \in L_1([0,1])$ is an onto isometry.

We finish the section with the next remark which justifies that there is no loss of generality in assuming that the metric spaces are complete.

Remark 1.1.6. Let M be a metric space and let $\tilde{M}$ be its completion. Then, the spaces $\mathcal{F}(M)$ and $\mathcal{F}(\tilde{M})$ are linearly isometric. Indeed, $T\colon f \in \mathrm{Lip}_0(\tilde{M}) \mapsto f\restriction_M \in \mathrm{Lip}_0(M)$ is a onto linear isometry which is weak*-to-weak* continuous.

1.2 Weak closure of $\delta(M)$

For a complete metric space M, $\delta(M) = \{\delta(x) : x \in M\} \subset \mathcal{F}(M)$ is obviously norm closed. The next proposition shows that the same result hold for the weak topology.

Proposition 1.2.1. *Let M be a complete pointed metric space. Then $\delta(M) \subset \mathcal{F}(M)$ is weakly closed.*

The proposition could be deduced more or less easily from Proposition 2.1.6 in [Wea99] but we propose a self-contained proof. For the proof, we will need the next observation (essentially already present in [Wea99]). The weak* closures of subsets of $\mathcal{F}(M)$ below are taken in the bidual $\mathcal{F}(M)^{**} \equiv \mathrm{Lip}_0(M)^*$.

Lemma 1.2.2. *Let M be a complete pointed metric space. Let $\mu \in \overline{\delta(M)}^{w^*} \setminus \delta(M)$. Then there exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$ and $q_1, \dots, q_n \in M$ we have that*

$$\mu \in \overline{\delta\left(M \setminus \bigcup_{i=1}^{n} B(q_i, \varepsilon)\right)}^{w^*}.$$

Proof. Indeed, otherwise we could find a sequence $(q_n)_{n=1}^{\infty} \subset M$ such that for every $n \geq 1$: $\mu \in \overline{\delta(B(q_n, 2^{-n}))}^{w^*}$ It follows that $\|\mu - \delta(q_n)\| \leq 2^{-n}$ for every n and thus $(q_n)_{n=1}^{\infty}$ is Cauchy. By completeness of M it follows that $\mu = \lim_n \delta(q_n) \in \delta(M)$. This contradiction proves the claim. □

Proof of Proposition 1.2.1. It is enough to show that if $\mu \in \overline{\delta(M)}^{w^*} \setminus \delta(M)$, then μ is not weak* continuous. Indeed, this yields that $\mu \notin \mathcal{F}(M)$ and so $\overline{\delta(M)}^{w} = \overline{\delta(M)}^{w*} \cap \mathcal{F}(M) = \delta(M)$.

So let $\mu \in \overline{\delta(M)}^{w^*} \setminus \delta(M)$ and let $\varepsilon > 0$ be as in Lemma 1.2.2. Now let U be an open neighbourhood of 0 in $(B_{\mathrm{Lip}_0(M)}, w^*)$. Since the weak* topology and the topology of pointwise convergence coincide on the ball $B_{\mathrm{Lip}_0(M)}$, we may assume that there are $x_1, \dots, x_n \in M$ and $\alpha > 0$ such that $U = \left\{f \in B_{\mathrm{Lip}_0(M)} : |f(x_i)| < \alpha \text{ for } i = 1, \dots n\right\}$. We define $f(x) := \mathrm{dist}(x, \{x_1, \dots, x_n\})$. We clearly have $f \in U$. Moreover since

$$\mu \in \overline{\delta\left(M \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon)\right)}^{w^*},$$

we have that $\mu(f) \geq \varepsilon$. Thus μ is not weak* continuous as U was arbitrary. □

We observe the following curious corollary.

Corollary 1.2.3. *Let M be a complete pointed metric space. If $(\delta(x_\alpha))_\alpha \subset \delta(M)$ is a net weakly converging to some $\mu \in \mathcal{F}(M)$, then there exists $x \in M$ such that $\mu = \delta(x)$ and $(\delta(x_\alpha))_\alpha$ actually converges to $\delta(x)$ in the norm topology. Thus, the weak topology coincide with the norm topology on $\delta(M)$.*

Proof. The fact that $\mu = \delta(x)$ follows from Proposition 1.2.1. For the rest it is enough to pose $f(\cdot) := d(\cdot, x) - d(0, x)$ and use that $d(x_\alpha, x) = \langle \delta(x_\alpha), f\rangle - \langle \delta(x), f\rangle \to 0$. □

Given a complete metric space M and $\mu \in \mathcal{F}(M)\backslash\delta(M)$ there is a weak neighbourhood that separates μ from $\delta(M)$. The next example shows that contrary to what one might expect, such a neighbourhood is not necessarily of the form $\{\gamma \in \mathcal{F}(M) : |\langle f, \gamma - \mu\rangle| < \varepsilon\}$ for some $f \in \mathrm{Lip}_0(M)$ and $\varepsilon > 0$.

Example 1.2.4. Let $M = [0, 1]$ with the usual metric and let μ be the Lebesgue measure on $[0, 1]$. It is well known and can be easily shown using the Riemann sums that $\mu \in \mathcal{F}(M)$. It acts on $\mathrm{Lip}_0([0, 1])$ as follows $\langle \mu, f\rangle = \int_0^1 f(t)dt$. Now the mean value theorem implies that for every $f \in \mathrm{Lip}_0(M)$ there exists $x \in [0, 1]$ such that $\langle \delta(x), f\rangle = \langle \mu, f\rangle$.

1.3 Weak closure of the set of molecules

We shall begin with the following definition.

Definition 1.3.1. By a *molecule* we mean an element of the form :

$$m_{xy} := \frac{\delta(x) - \delta(y)}{d(x, y)} \in \mathcal{F}(M), \text{ for } x \neq y \in M.$$

The set of all molecules will be denoted by V. Note in passing that V is a 1-norming set for $\mathrm{Lip}_0(M)$, that is, for every f in $\mathrm{Lip}_0(M)$,

$$\|f\|_L \leq \sup_{m\in V} |\langle f, m\rangle|.$$

Consequently, $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(V)$.

In view of Proposition 1.2.1, it is natural to wonder if the set V of molecules is weakly closed. Actually, we are going to show that it is almost the case. Indeed, It is known that 0 is in the weak-closure of V whenever M is not bi-Lipschitz embeddable in $\mathbb{R}^N$ (see Lemma 4.2 in [GLRZ17]). The main result of the section (Proposition 1.3.3) shows that 0 is the only point that we can reach taking the weak-closure of V. On the other hand, 0 is never in the sequential closure of V, which we will show in Corollary 1.3.4.

Not surprisingly, we need to study weakly converging nets of molecules. That is the goal of the next lemma which will also be useful later in the proof of Theorem 4.1.4.

Lemma 1.3.2. *Assume $(m_{x_\alpha y_\alpha})$ is a net in V which converges weakly to m_{xy}. Then, $\lim_\alpha d(x_\alpha, x) = 0$ and $\lim_\alpha d(y_\alpha, y) = 0$.*

Proof. Assume that $0 < \varepsilon < \min\{d(x, y), \limsup_\alpha d(x_\alpha, x)\}$. Consider the map f given by $f(t) = (\varepsilon - d(x, t))^+$ and let $g = f - f(0) \in \mathrm{Lip}_0(M)$. Note that $\langle g, m_{xy}\rangle = \frac{\varepsilon}{d(x,y)} > 0$. However,

$$\liminf_\alpha \langle g, m_{x_\alpha y_\alpha}\rangle = \liminf_\alpha \frac{-f(y_\alpha)}{d(x_\alpha, y_\alpha)} \leq 0,$$

a contradiction. Therefore, $\lim_\alpha x_\alpha = x$. Analogously we get that $\lim_\alpha y_\alpha = y$. □

Next, we deal with the weak closure of V. The proof of next proposition is based on [Wea99, Theorem 2.5.3]. So we begin with an explanation of this result. To this end,

consider a complete metric space M and we let $\tilde{M} := \{(x,y) \in M^2 \,:\, x \neq y\}$. We then define the following map :

$$\begin{aligned} \Phi\colon \mathrm{Lip}_0(M) &\to \mathcal{C}_b(\tilde{M}) \\ f &\mapsto \Phi f : (x,y) \in \tilde{M} \mapsto \frac{f(x)-f(y)}{d(x,y)} \end{aligned}$$

(here $\mathcal{C}_b(\tilde{M})$ stands for the continuous and bounded functions on $\tilde{M}$). It is easy to see that Φ is an isometry. Now let us denote $\beta\tilde{M}$ the Stone–Čech compactification of $\tilde{M}$. As usual, we can canonically identify $\mathcal{C}_b(\tilde{M})$ with $\mathcal{C}(\beta\tilde{M})$ so that we now see Φ as a map from $\mathrm{Lip}_0(M)$ to $\mathcal{C}(\beta\tilde{M})$. Thus Φ^* goes from $\mathcal{C}(\beta\tilde{M})^* \equiv \mathcal{M}(\beta\tilde{M})$ to $\mathrm{Lip}_0(M)^*$. According to Weaver, we say that $\mu \in \mathrm{Lip}_0(M)^*$ is normal if $\{\langle \mu, f_i\rangle\}$ converges to $\langle \mu, f\rangle$ whenever (f_i) is a bounded and decreasing (meaning that $f_i \geq f_j$ for $i \leq j$) net in $\mathrm{Lip}_0(M)$ which weak* converges to $f \in \mathrm{Lip}_0(M)$. Clearly normality is implied by weak* continuity. Finally, [Wea99, Theorem 2.5.3] asserts that if $x \in \beta\tilde{M}$ with $\Phi^*\delta(x) \neq 0$, then $\Phi^*\delta(x)$ is normal if and only if $x \in \tilde{M}$. We are now ready to study the weak closure of V.

Proposition 1.3.3. *Let (M,d) be a complete pointed metric space. Then $\overline{V}^w \subset V \cup \{0\}$.*

Proof. Since

$$\overline{V}^w = \overline{V}^{w^*} \cap \mathcal{F}(M) = \{\mu \in \overline{V}^{w^*} \,:\, \mu \text{ is } w^*\text{-continuous}\},$$

it is enough to show that if $\mu \in \overline{V}^{w^*} \backslash (V \cup \{0\})$ then μ is not weak* continuous. So let us fix such a μ. We identify, as we may, $\tilde{M}$ with $\delta(\tilde{M}) \subset \mathcal{M}(\beta\tilde{M})$. We claim that $\delta(\tilde{M})$ is homeomorphic to (V, w^*). Indeed, it is clear that

$$\Phi^*\restriction_{\delta(\tilde{M})} : \delta(x,y) \in \delta(\tilde{M}) \mapsto m_{xy} \in (V, w^*)$$

is continuous and bijective. The fact that the inverse mapping is also continuous follows from Lemma 1.3.2. So the claim is proved. Now $(\overline{V}^{w^*}, w^*)$ is clearly a compactification of V. Thus the universal property of the Stone–Čech compactification provides a unique surjective extension of $\Phi^*\restriction_{\delta(\tilde{M})}$ that goes from $\delta(\beta\tilde{M})$ to $\overline{V}^{w^*}$, which is in fact $\Phi^*\restriction_{\delta(\beta\tilde{M})} : \delta(\beta\tilde{M}) \to \overline{V}^{w^*}$.

Let us now consider $x \in \beta\tilde{M}$ such that $\Phi^*\delta(x) = \mu$. Since $\mu \in \overline{V}^{w^*} \backslash (V \cup \{0\})$, we deduce that $x \notin \tilde{M}$. Thus, according to [Wea99, Theorem 2.5.3], $\Phi^*\delta(x) = \mu$ is not normal and therefore not weak* continuous. This ends the proof. □

From the previous proposition, we deduce a result similar to Corollary 1.2.3.

Corollary 1.3.4. *Let M be a complete pointed metric space. If $(\mu_n)_{n=1}^\infty \subset \mathcal{F}(M)$ is a sequence of molecules ($\mu_n = m_{x_n y_n}$) which converges weakly to some $\mu \in \mathcal{F}(M)$, then there exist $x \neq y \in M$ such that $\mu = m_{xy}$ and $(\mu_n)_{n=1}^\infty$ actually converges in norm to m_{xy}. In particular, a sequence of molecules cannot converge weakly to 0 and so V is weakly sequentially closed.*

Proof. Proposition 1.3.3 shows that $\mu = m_{xy}$ or $\mu = 0$. In the first case the sequence $(\mu_n)_{n=1}^\infty$ actually converges in norm by Lemma 1.3.2.

If $\mu = 0$ then clearly $(\mu_n)_{n=1}^{\infty}$ does not admit any norm convergent subsequence. Therefore it is not totally bounded and so there exist $\varepsilon > 0$ and a subsequence $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that $\|\mu_{n_k} - \mu_{n_l}\| \geq \varepsilon$ for all $k \neq l$.

Now $(\mu_{n_k})_{k=1}^{\infty}$ is a uniformly separated bounded sequence of measures such that the cardinality of their supports is bounded. So the deep Theorem 5.2 in [AK09] shows that $(\mu_{n_k})_{k=1}^{\infty}$ cannot converge weakly to 0 which is a contradiction (see the next section for more details on [AK09, Theorem 5.2]). □

1.4 Perspectives

Let $k \in \mathbb{N}$ and let M be a complete pointed metric space. We consider the following set :

$$FS_k(M) = \{\gamma \in \mathcal{F}(M) \ : \ |\operatorname{supp} \gamma| \leq k\}.$$

In light of the two previous sections, we address the following questions.

Question 1.4.1. Is $FS_k(M)$ norm closed ? Is $FS_k(M)$ weakly closed ?

Note that if $k = 1$ then $FS_1(M) = \mathbb{R} \cdot \delta(M)$. In this case, it is not difficult to show that the answer to both previous questions is yes.

Proposition 1.4.2. *Let M be a complete metric space. Then, $FS_1(M)$ is weakly closed.*

Proof. Let $(\lambda_\alpha \delta(x_\alpha))_\alpha \subset FS_1(M)$ be a net converging to some $\gamma \in \mathcal{F}(M)$. We may assume that $\gamma \neq 0$ because otherwise there is nothing to do. Since $f := d(\cdot, 0)$ belongs to $\mathrm{Lip}_0(M)$, we have $\langle f, \lambda_\alpha \delta(x_\alpha)\rangle \underset{\alpha}{\longrightarrow} \langle f, \gamma\rangle$. Moreover,

$$|\langle f, \lambda_\alpha \delta(x_\alpha)\rangle| = |\lambda_\alpha| d(x_\alpha, 0) = \|\lambda_\alpha \delta(x_\alpha)\|_{\mathcal{F}(M)}.$$

Notice that $\langle f, \gamma\rangle \neq 0$. Indeed, otherwise $(\lambda_\alpha \delta(x_\alpha))_\alpha$ would converge in norm to 0 and this would contradict $\gamma \neq 0$. Thus, there exists α_0 such that whenever $\alpha \geq \alpha_0$, $\lambda_\alpha d(x_\alpha, 0) \neq 0$. Note that we can write for $\alpha \geq \alpha_0$:

$$\lambda_\alpha \delta(x_\alpha) = \lambda_\alpha d(x_\alpha, 0) \frac{\delta(x_\alpha) - \delta(0)}{d(x_\alpha, 0)} = \lambda_\alpha d(x_\alpha, 0) m_{x_\alpha 0}.$$

Consequently, $(m_{x_\alpha 0})_\alpha$ weakly converges to $(\langle f, \gamma\rangle)^{-1}\gamma$. According to Proposition 1.3.3 and the fact that $\gamma \neq 0$, we deduce the existence of $x \in M$ such that $(\langle f, \gamma\rangle)^{-1}\gamma = m_{x0}$. So finally,

$$\gamma = \frac{\langle f, \gamma\rangle}{d(x, 0)} \delta(x) \in FS_1(M).$$

□

In this context, we would like to mention a result of Albiac and Kalton (Theorem 5.2 in [AK09]). Let us state here their result.

Theorem 1.4.3 (Albiac–Kalton). *Let M be a complete pointed metric space. Suppose $(\mu_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{F}(M)$ which satisfies the following properties :*

(i) $(\mu_n)_{n=1}^{\infty}$ is bounded,

(ii) $\|\mu_m - \mu_n\| \geq 1$ *whenever* $m \neq n$,

(iii) there exists $k \in \mathbb{N}$ *such that* $(\mu_n)_{n=1}^{\infty} \subseteq FS_k(M)$.

Then, for any $\varepsilon > 0$ *there exist an infinite subset* $\mathbb{M}$ *of* $\mathbb{N}$ *and* $f \in \mathrm{Lip}_0(M)$ *with* $\|f\|_L \leq 1$ *such that* $\langle f, \mu_n \rangle > \frac{1}{4} - \varepsilon$ *for every* $n \in \mathbb{M}$.

We clearly deduce the following corollary.

Corollary 1.4.4. *Let* M *be a complete pointed metric space. Let* $(\mu_n)_{n=1}^{\infty}$ *be a bounded sequence in* $FS_k(M)$ *which weakly converges to some finitely supported measure* $\mu \in \mathcal{F}(M)$. *Then,* $(\mu_n)_{n=1}^{\infty}$ *converges to* μ *in the norm topology.*

We also would like to mention that the authors develop in [AK09] a theory of "Arens–Eells p-spaces". This is a natural extension of the theory of Lipschitz free spaces in the context of p-metric spaces. In fact, their definition coincides with the one given in Definition 1.1.1 in the case $p = 1$. Exploring this theory, the authors managed to show that there are separable p-Banach spaces $(0 < p < 1)$ which are Lipschitz isomorphic but not linearly isomorphic ([AK09, Theorem 5.4]). Let us recall that it is still not known if such examples exist in the context of separable Banach spaces.

Chapter 2

Duality of some Lipschitz free spaces

Let us consider a separable metric space M. It is clear from Corollary 1.1.4 and Example 1.1.5 that $L_1[0,1]$ linearly embeds into $\mathcal{F}(M)$ whenever $[0,1]$ bi-Lipschitz embeds into M. Since $L_1[0,1]$ does not embed into any separable dual Banach space (see for instance Theorem 6.3.7 in [AK06]), $\mathcal{F}(M)$ is not isomorphic to any dual Banach space provided M contains a line segment.

However, for some metric spaces M it is known that $\mathcal{F}(M)$ is isometric to a dual Banach space. We can mention the work of Weaver (Theorem 3.3.3 in [Wea99]), Kalton (Theorem 6.2 in [Kal04]) and Dalet (Theorem 44 in [Dal15a]). In this chapter, we slightly improve the already known results in this line. We also pin down a distinguished class of preduals, called natural preduals, which turns out to be of particular interest in Chapter 4. In fact, each isometric predual of $\mathcal{F}(M)$ considered in the literature is natural. Nevertheless, non-natural preduals exist.

Then, we focus on the case of uniformly discrete metric spaces (Section 2.3). In fact, we study uniformly discrete and bounded metric spaces for which we characterise the possible natural preduals of the associated free spaces. Finally, we provide a new class of examples for which we may apply the results obtained in Section 2.2, namely the metric spaces originating from p-Banach spaces (Section 2.4).

Part of sections 2.1, 2.2 and 2.3 are based on a joint work with Luis García-Lirola, Antonín Procházka and Abraham Rueda Zoca [GPPR17] while Section 2.4 is based on [Pet17].

2.1 The spaces of little Lipschitz functions

Definition 2.1.1. Let M be a metric space. We define the two following closed subspaces of $\mathrm{Lip}_0(M)$ (with the convention $\sup\emptyset = 0$).

$$\mathrm{lip}_0(M) := \Big\{ f \in \mathrm{Lip}_0(M) : \lim_{\varepsilon\to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \Big\},$$

$$S_0(M) := \Big\{ f \in \mathrm{lip}_0(M) : \lim_{r\to\infty} \sup_{\substack{x \text{ or } y\notin B(0,r)\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \Big\}.$$

Note that the second one differs from the first one when M is unbounded.

The first space $\mathrm{lip}_0(M)$ is called the little Lipschitz space over M. This terminology first appeared in [Wea99] where they were considered on compact metric spaces.

Definition 2.1.2.

- We will say that a subspace $S \subset \mathrm{Lip}_0(M)$ *C-separates points uniformly (shortened C-S.P.U.)* if for every $x, y \in M$ and every $\varepsilon > 0$ there is $f \in S$ such that $f(x) - f(y) = d(x, y)$ and $\|f\|_L < C + \varepsilon$.
- We will say that a subspace $S \subset \mathrm{Lip}_0(M)$ *separates points uniformly* if it C-separates points uniformly for some $C \geq 1$.
- We then recall that a subspace Z of X^*, where X is a Banach space, is *c-norming* (with $c \geq 1$) if for every x in X,

$$\|x\| \leq c \sup_{z^* \in B_Z} |z^*(x)|.$$

It is well known that $\mathrm{lip}_0(M)$ or $S_0(M)$ are c-norming for some $c \geq 1$ if and only if they c-separate points uniformly. This actually follows from the following result of Kalton (see Proposition 3.4 in [Kal04]).

Proposition 2.1.3 (Kalton). *Let S be a subspace of $\mathrm{Lip}_0(M)$ which is also a sublattice with respect to the pointwise order. Then S is c-norming if and only if S c-separates points uniformly.*

Weaver showed in [Wea99] (Theorem 3.3.3) that if M is a compact metric space then $\mathrm{lip}_0(M)$ separates points uniformly if and only if it is an isometric predual of $\mathcal{F}(M)$. Considering $S_0(M)$ instead of $\mathrm{lip}_0(M)$, Dalet showed in [Dal15c] that a similar result holds for proper metric spaces (metric spaces for which each ball is compact). We state here this result for future reference.

Theorem 2.1.4 (Dalet). *Let M be a proper pointed metric space. Then $S_0(M)$ separates points uniformly if and only if it is an isometric predual of $\mathcal{F}(M)$.*

The following proposition provides conditions on M to ensure that $\mathrm{lip}_0(M)$ is 1-norming.

Proposition 2.1.5. *Let M be a pointed metric space. Assume that for every $x \neq y \in M$ and $\varepsilon > 0$, there exist $N \subseteq M$ and a $(1+\varepsilon)$-Lipschitz map $T : M \to N$ such that $\mathrm{lip}_0(N)$ is 1-norming for $\mathcal{F}(N)$, $d(Tx, x) \leq \varepsilon$ and $d(Ty, y) \leq \varepsilon$. Then $\mathrm{lip}_0(M)$ is 1-norming.*

Proof. Let $x \neq y \in M$ and $\varepsilon > 0$. By our assumptions there exist $N \subseteq M$ and a $(1+\varepsilon)$-Lipschitz map $T : M \to N$ such that $\mathrm{lip}_0(N)$ is 1-norming, $d(Tx, x) \leq \varepsilon$ and $d(Ty, y) \leq \varepsilon$. Since $\mathrm{lip}_0(N)$ is 1-norming there exists $f \in \mathrm{lip}_0(N)$ verifying $\|f\|_L \leq 1 + \varepsilon$ and $|f(Tx) - f(Ty)| = d(Tx, Ty)$. Now we define $g = f \circ T$ on M. By composition g is $(1+\varepsilon)^2$-Lipschitz and $g \in \mathrm{lip}_0(M)$. Then a direct computation shows that g does the work.

$$\begin{aligned} |g(x) - g(y)| &= |f(Tx) - f(Ty)| = d(Tx, Ty) \\ &\geq d(x, y) - d(x, Tx) - d(y, Ty) \\ &\geq d(x, y) - 2\varepsilon. \end{aligned}$$

This ends the proof. □

We now give a few examples for which it is known that $\mathrm{lip}_0(M)$ is 1-norming.

Examples 2.1.6.

1. M is a compact countable metric space ([Dal15b, Theorem 2.1]).
2. M is a compact metric space such that there exists a sequence $(\varepsilon_n)_{n=1}^{\infty}$ tending to 0, a real number $\rho < 1/2$ and finite ε_n-separated subsets N_n of M which are $\rho\varepsilon_n$-dense in M ([GO14, Proposition 6]). For instance the middle-third Cantor set.
3. $(M, \omega \circ d)$ where (M, d) is any metric space and ω is a *nontrivial gauge* ([Kal04, Proposition 3.5]). By a *nontrivial gauge* $\omega\colon [0,\infty) \to [0,\infty)$ we mean an increasing, continuous, and subadditive function which satisfies $\omega(0) = 0$, $\omega(t) \geq t$ for every $0 \leq t \leq 1$ and $\lim_{t\to 0} \omega(t)/t = \infty$. For instance, the map defined by $\omega(t) = t^p$ with $0 < p < 1$ is a nontrivial gauge.

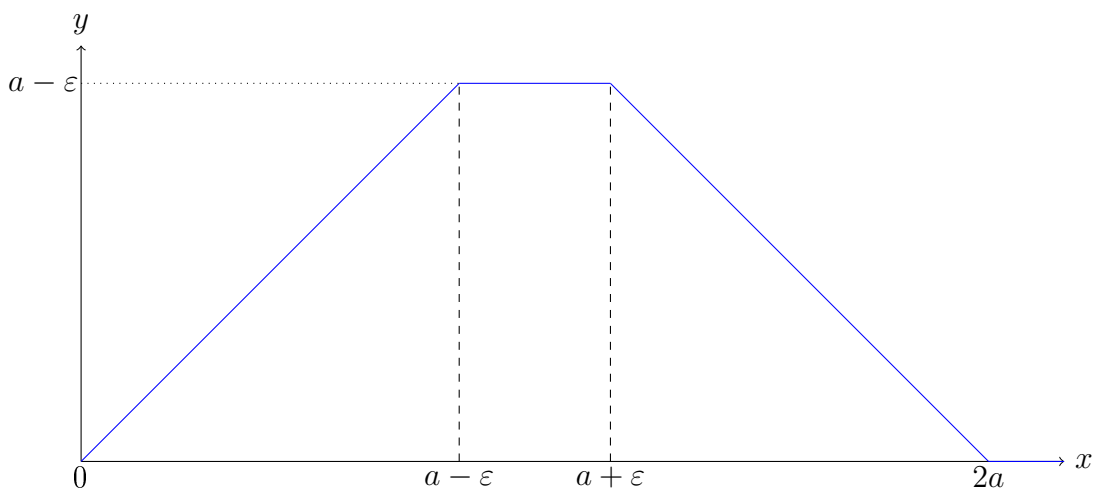
When the metric spaces are considered to be unbounded, it is necessary to work with $S_0(M)$. Thus, we give now examples for which it is known that $S_0(M)$ is 1-norming. Assuming M to be proper and countable, Dalet proved that $S_0(M)$ is 1-norming ([Dal15c, Theorem 2.1]). Moreover, in view of the last example above, we also prove here the following result.

Proposition 2.1.7. *Let (M, d) be a pointed metric space and let ω be a nontrivial gauge. Then, $S_0(M, \omega \circ d)$ is 1-norming.*

Proof. We will adapt the technique used in [Kal04, Proposition 3.5]. To simplify the notation, we denote $d_\omega := \omega \circ d$. We will show that, for every $x \neq y \in M$ and every $\varepsilon > 0$, there exists $f \in S_0(M, d_\omega)$ such that $|f(x) - f(y)| \geq d_\omega(x, y) - \varepsilon$ and $\|f\|_{\mathrm{Lip}_0(M,d_\omega)} \leq 1$.

Fix $\varepsilon > 0$ and let $x \neq y \in M$. We denote $a = d_\omega(x, y)$ and let $\varphi : [0, +\infty[\to [0, +\infty[$ be the following function :

$$\varphi(t) = \begin{cases} t & \text{if } 0 \leq t < a - \varepsilon, \\ a - \varepsilon & \text{if } a - \varepsilon \leq t < a + \varepsilon, \\ -t + 2a & \text{if } a + \varepsilon \leq t < 2a, \\ 0 & \text{if } 2a \leq t. \end{cases}$$



Notice that $\|\varphi\|_L \leq 1$. For every $n \in \mathbb{N}$ we define a new gauge

$$\omega_n(t) = \inf\{\omega(s) + n(t - s)\,;\, 0 \leq s \leq t\}.$$

Note that, for every $t \in [0, +\infty[$, one has that $\omega_n(t) \underset{n\to+\infty}{\longrightarrow} \omega(t)$. Finally, for $n \in \mathbb{N}$, we consider h_n defined on M by $h_n(z) = \varphi(d_{\omega_n}(z, y)) - \varphi(d_{\omega_n}(0, y))$. It is straightforward to check that, for n large enough, $|h_n(x) - h_n(y)| = a - \varepsilon = d_\omega(x, y) - \varepsilon$. Moreover, given z and z' in M, straightforward computations yield the following

$$\begin{aligned} |h_n(z) - h_n(z')| &= |\varphi(d_{\omega_n}(z, y)) - \varphi(d_{\omega_n}(z', y))| \\ &\leq \|\varphi\|_L |d_{\omega_n}(z, y) - d_{\omega_n}(z', y)| \\ &\leq d_{\omega_n}(z, z'). \end{aligned}$$

Furthermore, from the definition of ω_n, it follows

$$d_{\omega_n}(z, z') \leq d_\omega(z, z') \text{ and } d_{\omega_n}(z, z') \leq nd(z, z').$$

Now the first of above inequalities shows that $\|h_n\|_L \leq 1$. The second one proves that $h_n \in \mathrm{Lip}_0(M, d)$, which in fact implies that $h_n \in \mathrm{lip}_0(M, d_\omega)$ (the support of φ is bounded). It remains to prove that $h_n \in S_0(M)$. To this end, fix $\eta > 0$, and pick $r > 2a + d_\omega(0, y)$ such that $\dfrac{a}{r - 2a - d_\omega(0, y)} \leq \eta$. Now let z and z' be in M, and let us discuss by cases :

— If z and z' are not in $\overline{B}(0, r)$, then $|h_n(z) - h_n(z')| = 0 < \eta$.

— Now suppose that $z \notin B(0, r)$ and $z' \in B(0, r)$. Now we can still distinguish two more cases :

  — First assume that $d_\omega(z', y) \geq 2a$. Then $h_n(z) = h_n(z') = 0$ and so $|h_n(z) - h_n(z')| < \eta$ again trivially holds.

  — On the other hand, if $d_\omega(z', y) < 2a$, then $|h_n(z')| \leq a - \varepsilon$ and so

$$\begin{aligned} \frac{|h_n(z) - h_n(z')|}{d_\omega(z, z')} &\leq \frac{a}{d_\omega(z, 0) - d_\omega(z', y) - d_\omega(0, y)} \\ &\leq \frac{a}{r - 2a - d_\omega(0, y)} \leq \eta. \end{aligned}$$

This proves that $h_n \in S_0(M)$ and concludes the proof. □

2.2 Natural preduals

We shall start with the following definition which introduces a new class of isometric preduals of free spaces. Note that in [GPPR17], most results concern bounded metric spaces. Here we treat the case of unbounded metric spaces. Our original definition of natural predual was an isometric predual $\mathcal{F}(M)$ which makes $\delta(M)$ weak* closed in $\mathcal{F}(M)$. However, the previous definition does not fit very well for unbounded metric spaces in contrast to the following one.

Definition 2.2.1. Let M be a pointed metric space. We will say that a Banach space X is a *natural predual* of $\mathcal{F}(M)$ if $X^* \equiv \mathcal{F}(M)$ and $\delta(B(0, r))$ is $\sigma(\mathcal{F}(M), X)$-closed for every $r \geq 0$.

Notice that if $\delta(B(0,r))$ is weak* closed for every $r \geq 0$, then $\delta(B(x,r))$ is weak* closed for every $r \geq 0$ and $x \in M$. Moreover, when M is bounded, it is readily seen that this condition is equivalent to the fact that $\delta(M)$ is $\sigma(\mathcal{F}(M), X)$-closed. Indeed, if $\delta(M)$ is weak* closed then so is $\delta(B(0,r)) = \delta(M) \cap B_{\mathcal{F}(M)}(0,r)$. For the converse it suffices to take r big enough so that $M = B(0,r)$.

Next, it is obvious that if M is a proper metric space then every isometric predual of $\mathcal{F}(M)$ is natural. We will show in Example 2.3.5 and Example 2.3.6 that there are isometric preduals to $\mathcal{F}(M)$ which are not natural. Let us state for future reference an almost obvious characterisation of natural preduals. To this end, we introduce some more notation.

Definition 2.2.2. Let (M,d) be a pointed metric space and let τ be a topology on M.

- We will say that (M,d) is *τ-proper* if for every $x \in M$ and every $r \geq 0$ the closed ball $B(x,r)$ is τ-compact.
- We will denote $\mathcal{C}_b(M,\tau)$ the set of maps which are τ-continuous on bounded set of M.

Proposition 2.2.3. *Let M be a pointed metric space and let X be an isometric predual of $\mathcal{F}(M)$. Then the following are equivalent :*

(i) There is a Hausdorff topology τ on M such that M is τ-proper and X is a subspace of $\mathrm{Lip}_0(M) \cap \mathcal{C}_b(M,\tau)$.

(ii) X is a natural predual of $\mathcal{F}(M)$.

Proof. We start by proving (i)⇒(ii). We may naturally identify M with $\delta(M)$ in such a way that τ is seen as a topology on $\delta(M)$. Let $r \geq 0$. We claim that the weak* topology of $\mathcal{F}(M)$ and the τ-topology coincide on $\delta(B(0,r))$. Indeed, every weak* open set in $\delta(B(0,r))$ is also τ-open since X is made up of τ-continuous functions, so that the weak* topology is weaker than τ on $\delta(B(0,r))$. By compactness of the Hausdorff topology τ, we have that they agree on $\delta(B(0,r))$.

For (ii)⇒(i), we simply define the topology τ on $\delta(M)$ (again identified with M) as being the $w^* = \sigma(\mathcal{F}(M), X)$-topology restricted to $\delta(M)$. □

The natural preduals are quite common. In fact, the known constructions of isometric preduals to $\mathcal{F}(M)$ all produce natural preduals. Indeed, this is the case for Theorem 3.3.3 in [Wea99] because of compactness as well as Theorem 2.1 in [Dal15c] because of properness. We will show that it is also true for Theorem 6.2 in [Kal04]. For convenience, we state here this result.

Theorem 2.2.4 (Kalton). *Let M be a separable complete pointed metric space of finite radius R. Suppose τ is a metrisable topology on M so that (M,τ) is compact and $X = \mathrm{lip}_0(M) \cap \mathcal{C}(M,\tau)$ 1-separates points uniformly. Then, X is a predual of $\mathcal{F}(M)$.*

We will actually generalize Kalton's result to the unbounded setting.

Proposition 2.2.5. *Let M be a separable complete pointed metric space and let τ be a topology on M so that d is τ-l.s.c. and M is τ-proper. Assume that $X = S_0(M) \cap \mathcal{C}_b(M,\tau)$ separates points uniformly. Then, X is a natural predual of $\mathcal{F}(M)$.*

Notice that in Proposition 2.2.5 we assume that d is τ-l.s.c. and that our space X separates points uniformly instead of assuming that X 1-separates the points uniformly (as it is done in Theorem 2.2.4). In fact, we have the following easy observation which restates in a general framework the first step of Kalton's proof.

Lemma 2.2.6. *Let (M,d) be a pointed metric space such that there is a topology τ on M and a subset $X \subset \mathrm{Lip}_0(M) \cap \mathcal{C}(M,\tau)$ which 1-S.P.U. Then $d\colon (M,\tau)^2 \to \mathbb{R}$ is l.s.c.*

Proof. Let (x_α), (y_α) be τ-convergent nets in M with limits x and y, respectively. Given $\varepsilon > 0$, find $f \in X$ such that $f(y) - f(x) \geq d(x,y) - \varepsilon$ and $\|f\|_L = 1$. Then

$$d(x,y) - \varepsilon \leq \lim_\alpha \big(f(y_\alpha) - f(x_\alpha)\big) \leq \liminf_\alpha d(x_\alpha, y_\alpha)$$

and the arbitrariness of ε yields the desired conclusion. □

In what follows we provide a slightly different proof of Proposition 2.2.5 from Kalton's original argument for Theorem 2.2.4. Our proof is based on Petunīn–Plīčko theorem (see [God87, PP74]). Let us state here the assertion of this Theorem for future reference.

Theorem 2.2.7 (Petunīn–Plīčko). *Let $S \subset X^*$ be a closed subspace of the dual of a separable Banach space X. Then, $S^* \equiv X$ if, and only if, the two following conditions are satisfied :*

(i) S is composed of norm-attaining functionals, that is, for every $x \in S$ there exists $x \in S_X$ such that $\langle x^*, x\rangle = \|x^*\|$.

(ii) S separates the points of X, that is, $[\langle x^*, x\rangle = 0,\ \forall x^* \in S] \implies [x = 0]$.

The use of this theorem to produce preduals to free spaces was initiated by Godefroy and has become quite common (see [Dal15a, Dal15b, Dal15c, GLPZ17a] and also our Examples 2.3.5 and 2.3.6). The benefit of this proof is that it avoids the metrisability assumption of the topology τ present in Kalton's original result.

Proof of Proposition 2.2.5. We need to verify the conditions of Petunīn and Plīčko's theorem. It is readily seen that $S_0(M)$ and $C_b(M,\tau)$ are two closed subspaces of $\mathrm{Lip}_0(M)$. Consequently X is closed in $\mathrm{Lip}_0(M)$. Second, X separates the points of $\mathcal{F}(M)$ since it is a lattice and separates the points of M uniformly (see Proposition 2.1.3).

Finally it remains to show that X is made of norm-attaining functionals. To this end, let $f \in S_X$ and take sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ in M such that $\lim_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)} = 1$. Note that $\inf_n d(x_n,y_n) =: \theta > 0$ since $f \in \mathrm{lip}_0(M)$ and that the sequences $(d(x_n,0))_{n=1}^\infty$, $(d(y_n,0))_{n=1}^\infty$ are both bounded since $f \in S_0(M)$. So we may assume that those sequences are contained in $B(0,r)$ for some $r > 0$. By compactness of $(B(0,r),\tau)$, we may consider an accumulation point (x,y,C) of $(x_n,y_n,d(x_n,y_n))_{n=1}^\infty$. Then, since d is τ-l.s.c. we have

$$1 = \frac{f(x)-f(y)}{C} \leq \frac{f(x)-f(y)}{d(x,y)} \leq 1.$$

Thus X is made up of norm-attaining functionals.

To conclude, we get that S is a natural predual by just applying Proposition 2.2.3. □

The next proposition testifies that Proposition 2.2.5 is the only way to build a natural predual if the predual is moreover required to be a subspace of $S_0(M)$.

Proposition 2.2.8. *Let M be a complete separable pointed metric space and let X be a natural predual of $\mathcal{F}(M)$ such that $X \subseteq S_0(M)$. Then there exists a topology τ on M such that M is τ-proper, the metric $d\colon (M,\tau)^2 \to \mathbb{R}$ is l.s.c. and $X = S_0(M) \cap \mathcal{C}_b(M,\tau)$.*

Proof. We define $\tau := \{\delta^{-1}(U) : U \in \sigma(\mathcal{F}(M), X)\}$ and we fix $r \geq 0$. Since the set $\delta(B(0,r))$ is $\sigma(\mathcal{F}(M), X)$-closed and bounded, $(B(0,r),\tau)$ is compact. Remember that $d(x,y) = \|\delta(x) - \delta(y)\|$ and $\|\cdot\|$ is $\sigma(\mathcal{F}(M), X)$-l.s.c., so the metric d is τ-l.s.c.. Since

$$X = \{x^* \in \mathcal{F}(M)^* : x^* \text{ is } \sigma(\mathcal{F}(M), X)\text{-continuous}\}$$

and $X \subset S_0(M)$, we get that $X \subseteq S_0(M) \cap \mathcal{C}(M,\tau) \subset S_0(M) \cap \mathcal{C}_b(M,\tau) =: Y$. This means that $\sigma(\mathcal{F}(M), Y)$ is stronger than $\sigma(\mathcal{F}(M), X)$. On the other hand, Proposition 2.2.5 yields that $Y^* \equiv \mathcal{F}(M)$. Therefore, by compactness, $\sigma(\mathcal{F}(M), X)$ and $\sigma(\mathcal{F}(M), Y)$ coincide on $B_{\mathcal{F}(M)}$. As a consequence of the Banach-Dieudonné theorem (see [FHH+01, Theorem 4.44]), they coincide on $\mathcal{F}(M)$. This means that

$$\begin{aligned} X &= \{x^* \in \mathcal{F}(M)^* : x^* \text{ is } \sigma(\mathcal{F}(M), X)\text{-continuous}\} \\ &= \{x^* \in \mathcal{F}(M)^* : x^* \text{ is } \sigma(\mathcal{F}(M), Y)\text{-continuous}\} = Y. \end{aligned}$$

□

However, one should be aware that not all natural preduals are contained in the space of little Lipschitz functions.

Example 2.2.9. Let $M = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ with the distance coming from the reals. Then it is well known that $\mathcal{F}(M)$ is isometrically isomorphic to ℓ_1. Further, we know (Theorem 2.1 in [Dal15b]) that $\mathrm{lip}_0(M)$ is isometrically a predual. If X is a predual such that $X \subset \mathrm{lip}_0(M)$ then the proof of Proposition 2.2.8 shows that $X = \mathrm{lip}_0(M)$. So any isometric predual of ℓ_1 which is not isometric to $\mathrm{lip}_0(M)$ intersects the complement of $\mathrm{lip}_0(M)$. (Recall that both c_0 and c are isometric preduals of ℓ_1 without being isometric to each other).

2.3 The uniformly discrete case

Now that we have proved that the known constructions of isometric preduals of $\mathcal{F}(M)$ all produce natural preduals, it is quite natural to wonder whether every predual of $F(M)$ (when it exists) is natural. One purpose of the current section is to show that it is actually not the case.

Definition 2.3.1. A metric space M is said to be *uniformly discrete* provided :

$$\inf\{d(x,y) \,:\, x \neq y \in M\} > 0.$$

We would like to motivate the study of Lipschitz free spaces over uniformly discrete metric spaces. So let us consider a uniformly discrete metric space M. Kalton proved in [Kal04, Proposition 4.4] that $\mathcal{F}(M)$ has the Schur property, the Radon-Nikodym property, and the approximation property. Then, Kalton addresses the following question (see the remark after [Kal04, Proposition 4.4]).

Question 2.3.2. Let M be a uniformly discrete metric space. Does $\mathcal{F}(M)$ have the bounded approximation property ?

Both positive and negative answers would have interesting consequences. Indeed, a negative answer would imply that there is an equivalent norm on ℓ_1 which does not have the metric approximation property (which solves an old open problem). A positive answer would imply that every Banach space is *approximable* (see the comments after Problem 1 in [Kal12]). This property may be regarded as a nonlinear version of the (BAP) : We say that a complete metric space M is *approximable* if there is a gauge ω so that for every finite set $E \subset M$ and every $\varepsilon > 0$ we can find a uniformly continuous map $\psi\colon M \to M$ such that $d(x, \psi(x)) < \varepsilon$ for every $x \in E$, $\psi(M)$ is relatively compact and $\omega_\psi \leq \omega$ (where ω_ψ stands for the modulus of continuity of ψ).

In order to study if a Banach space enjoys the metric approximation property, it is convenient to know if it is isometric to a dual. We can mention for instance Grothendieck's theorem (see Theorem 3.3.3) that we will use later. We also would like to mention the following result : If a Banach space have the (RNP), is 1-complemented in its bidual and has the (AP), then it has the (MAP) (see [DU77, Theorem VIII.3.1] or [Rya02, Theorem 5.50]). This result is implicit in Grothendieck's Memoir [Gro55]. It seems that the first explicit formulation of it was published in [Rn75]. Since every dual Banach space is 1-complemented in its bidual, it is relevant to study whether $\mathcal{F}(M)$ is isometric to a dual Banach space for M uniformly discrete. Moreover, we can start with the study of uniformly discrete and bounded metric spaces. Indeed, we have the following related question.

Question 2.3.3. Let M be a uniformly discrete and bounded metric space. Does $\mathcal{F}(M)$ have the metric approximation property ?

A positive answer to Question 2.3.3 would provide a positive answer to Question 2.3.2. Indeed, it follows from [Kal04, Proposition 4.3] that $\mathcal{F}(M)$ always linearly embeds into a complemented subspace of $\left(\sum_{k\in\mathbb{Z}} \mathcal{F}(B(0, 2^k))\right)_{\ell_1}$.

Unfortunately, we managed to find a uniformly discrete and bounded metric space M such that $\mathcal{F}(M)$ is not even 1-complemented in its bidual (but $\mathcal{F}(M)$ has the (MAP), see Example 2.3.7). Nevertheless, looking for such a counterexample helped us to explore the notion of natural predual as we will see with Example 2.3.5 and Example 2.3.6.

We shall begin by characterizing the natural preduals of $\mathcal{F}(M)$. Note that $\mathrm{Lip}_0(M) = \mathrm{lip}_0(M)$ when M is uniformly discrete. This observation and Proposition 2.2.8 yield the following corollary.

Corollary 2.3.4. *Let (M, d) be a uniformly discrete, bounded, separable and complete pointed metric space. Let X be a Banach space. Then the following are equivalent :*

(i) X is a natural predual of $\mathcal{F}(M)$.

(ii) There is a Hausdorff topology τ on M such that (M, τ) is compact, d is τ-l.s.c. and $X = \mathrm{Lip}_0(M, d) \cap \mathcal{C}(M, \tau)$ equipped with the norm $\|\cdot\|_L$.

Proof. (ii)⇒(i) In order to prove this implication, we are going to use a result of Matoušková ([Mat00, Corollary 2.5]). For convenience, we recall here the statement of this result. Let (M, d) be a metric space, τ be a compact topology on M such that d is τ-l.s.c., and

F be a τ-closed set in M. If $g \in C(F,\tau)$ is c-Lipschitz in d, then there exists $f \in C(M,\tau)$ such that $f = g$ on F, $\min_F g \leq f \leq \max_F g$ and f is c-Lipschitz in d.

Given $x, y \in M$, $x \neq y$, define $f\colon \{x,y\} \to \mathbb{R}$ by $f(x) = 0$ and $f(y) = d(x,y)$. By Matoušková's extension theorem, there is $\tilde{f} \in \mathrm{Lip}(M) \cap \mathcal{C}(M,\tau)$ extending f such that $\|\tilde{f}\|_L = 1$. Then, simply consider $g \in \mathrm{Lip}_0(M) \cap \mathcal{C}(M,\tau)$ defined by $g(z) = f(z) - f(0)$. Thus, the hypotheses of Proposition 2.2.5 are satisfied.

The implication (i)⇒(ii) is contained in Proposition 2.2.8. □

We now give some examples in which the preduals of $\mathcal{F}(M)$ have interesting properties. The first one is a uniformly discrete and bounded metric space M such that $\mathcal{F}(M)$ is isometric to a dual Banach space but cannot admit a natural predual. This example comes from [AG17, Example 4.2] where Aliaga and Guirao considered it for a different purpose.

Example 2.3.5. Consider the sequence in c_0 given by $x_0 = 0, x_1 = 2e_1$, and $x_n = e_1 + (1+1/n)e_n$ for $n \geq 2$, where $(e_n)_{n=1}^\infty$ is the canonical basis. Let $M = \{0\} \cup \{x_n : n \in \mathbb{N}\}$. Then

a) $\mathcal{F}(M)$ does not admit any natural predual.

b) The space $X = \{f \in \mathrm{Lip}_0(M) : \lim f(x_n) = f(x_1)/2\}$ satisfies $X^* \equiv \mathcal{F}(M)$.

Our Corollary 2.3.4 guarantees that in order to prove a) it is enough to show that there is no compact topology τ on M such that d is τ-l.s.c. Assume that τ is such a topology. Then the sequence $(x_n)_{n=1}^\infty$ admits a τ-accumulation point $x \in M$. Since d is τ-l.s.c. we get that $x \in B(0,1) \cap B(x_1,1)$. But this is a contradiction as the latter set is clearly empty.

For the proof of b) we will employ the theorem of Petunīn and Plīčhko (see Theorem 2.2.7). The space X is clearly a separable closed subspace of $\mathcal{F}(M)^*$. Further, a simple case check shows that for any $x \neq y \in M$, $y \neq 0$, the function $f(x) = 0, f(y) = d(x,y)$ can be extended as an element of X without increasing the Lipschitz norm. Thus since X is clearly a lattice, Proposition 3.4 of [Kal04] shows that X is separating. Finally, if $f \in X$ and

$$\frac{f(x_{n_k}) - f(x_{m_k})}{d(x_{n_k}, x_{m_k})} \to \|f\|_L$$

then without loss of generality the sequence $(m_k)_{k=1}^\infty$ does not tend to infinity. Passing to a subsequence, we may assume that it is constant, say $m_k = m$ for all $k \in \mathbb{N}$. If $(n_k)_{k=1}^\infty$ does not tend to infinity, then $\frac{f(x_i)-f(x_m)}{d(x_i,x_m)} = \|f\|_L$ for some $i \neq m$. Otherwise, since $f \in X$, we have

$$\frac{f(x_{n_k}) - f(x_m)}{d(x_{n_k}, x_m)} \to \frac{\frac{f(x_1)}{2} - f(m)}{d(x_1, x_m)}.$$

So in this case the norm is attained at $\frac{1}{d(x_1,x_m)}\left(\delta(x_1)/2 - \delta(x_m)\right) \in B_{\mathcal{F}(M)}$. It follows that every $f \in X$ attains its norm. Thus, by Petunīn and Plīčhko's theorem (Theorem 2.2.7), $X^* \equiv \mathcal{F}(M)$.

Next we show that $\mathcal{F}(M)$ can actually have both natural and non-natural preduals.

Example 2.3.6. Let $M = \{0\} \cup \{1,2,3,\ldots\}$ be a graph such that the edges are couples of the form $(0,n)$ with $n \geq 1$. Let d be the shortest path distance on M. Then it is

obvious that $\mathcal{F}(M)$ is isometric to ℓ_1. Moreover $\mathcal{F}(M)$ admits both natural and non-natural preduals. Indeed, an example of a non-natural predual is

$$Y = \{f \in \mathrm{Lip}_0(M) : \lim f(n) = -f(1)\}.$$

An example of a natural predual is $X = \{f \in \mathrm{Lip}_0(M) : \lim f(n) = f(1)\}$ (this is immediate using Corollary 2.3.4).

Our last example shows that there are uniformly discrete bounded metric spaces such that their free space does not admit any isometric predual. As mentioned earlier, such observation is relevant to Question 2.3.2. Nevertheless, for M in this example, $\mathcal{F}(M)$ enjoys the (MAP). In order to prove our assertion, we will need a result about the extremal structure of $\mathcal{F}(M)$ that we will prove later in Chapter 4.

Example 2.3.7. Let $M = \{0\} \cup \{1, 2, 3, \ldots\} \cup \{a, b\}$ with the following distances :

$$\begin{aligned} d(0,n) = d(a,n) = d(b,n) &= 1 + 1/n, \\ d(a,b) = d(0,a) = d(0,b) &= 2, \text{ and} \\ d(n,m) &= 1 \end{aligned}$$

for $n, m \in \{1, 2, 3, \ldots\}$. Then there is no 1-Lipschitz retraction $r\colon \mathcal{F}(M)^{**} \to \mathcal{F}(M)$. In particular $\mathcal{F}(M)$ is not 1-complemented in its bidual and therefore is not isometrically a dual space.

Indeed, let us assume that there is some $r\colon \mathcal{F}(M)^{**} \to \mathcal{F}(M)$ such that $\|r\|_L \leq 1$ and $r(\mu) = \mu$ for all $\mu \in \mathcal{F}(M)$. Let us consider the sets

$$A_n = B_{\mathcal{F}(M)^{**}}\left(0, 1 + \frac{1}{n}\right) \cap B_{\mathcal{F}(M)^{**}}\left(\delta(a), 1 + \frac{1}{n}\right) \cap B_{\mathcal{F}(M)^{**}}\left(\delta(b), 1 + \frac{1}{n}\right).$$

Then $A_{n+1} \subset A_n$ and $\delta(n) \in A_n$ for every $n \in \mathbb{N}$. It follows by weak* compactness that there exists $\varphi \in \bigcap_{n=1}^{\infty} A_n$. Clearly we have $\|\varphi\| = \|\delta(a) - \varphi\| = \|\delta(b) - \varphi\| = 1$. It follows that $\|r(\varphi)\| = \|r(\varphi) - \delta(a)\| = \|r(\varphi) - \delta(b)\| = 1$. But Proposition 4.3.1 implies that $\delta(a)/2$ is an extreme point of $B_{\mathcal{F}(M)}$. This means that $B_{\mathcal{F}(M)}(0,1) \cap B_{\mathcal{F}(M)}(\delta(a), 1) = \{\delta(a)/2\}$ and thus $r(\varphi) = \delta(a)/2$. Similarly for $\delta(b)/2$. Hence $\delta(a)/2 = r(\varphi) = \delta(b)/2$. Contradiction.

Let us now prove that $\mathcal{F}(M)$ has the (MAP). Let $M_n := \{0, a, b, 1, \ldots, n\}$ and define $f_n\colon M \to M_n$ by $f_n(x) = x$ if $x \in M_n$ and $f(x) = n$ otherwise. The function f_n is obviously a retraction from M to M_n. Moreover a simple computation leads to $\|f_n\|_L \leq 1 + 1/n$. Let us denote $\tilde{f}_n\colon \mathcal{F}(M) \to \mathcal{F}(M_n)$ the linearisation of f_n which is in fact a projection of the same norm : $\|\tilde{f}_n\| \leq 1 + 1/n$. Then define $P_n := (1 + 1/n)^{-1}\tilde{f}_n$. Obviously, $\|P_n\| \leq 1$, P_n is of finite rank and $\|P_n\gamma - \gamma\| \to 0$ for every $\gamma \in \mathcal{F}(M)$. Thus $\mathcal{F}(M)$ has the (MAP).

In what follows we are going to develop yet another sufficient condition for an isometric predual to be natural with the goal to show that certain preduals constructed by Weaver in [Wea96] are natural.

Proposition 2.3.8. *Let M be a uniformly discrete, bounded, separable, pointed metric space and let $X \subset \mathrm{Lip}_0(M)$ be a Banach space such that $X^* \equiv \mathcal{F}(M)$. If for every $x \in M \setminus \{0\}$ the indicator function $\mathbb{1}_{\{x\}}$ belongs to X, then X is a natural predual of $\mathcal{F}(M)$. Moreover 0 is the unique accumulation point of $(\delta(M), w^*)$ and X is isomorphic to c_0.*

The proof will be based on the following two general facts.

Lemma 2.3.9. *Let M be a uniformly discrete, bounded, separable, pointed metric space. Then, the sequence $(\delta(x))_{x\in M\setminus\{0\}}$ is a Schauder basis of $\mathcal{F}(M)$ which is equivalent to the unit vector basis of ℓ_1.*

Proof. Let us denote $M\setminus\{0\} := (x_n)_{n=1}^{\infty}$. We are going to show that the map $\delta(x_n)\mapsto e_n$ for $n\geq 1$ defines an isomorphism from $\mathcal{F}(M)$ onto ℓ_1. Indeed, consider $\sum_{i=1}^{n} a_i\delta(x_i)\in\mathcal{F}(M)$. Since M is bounded, there exists $C>0$ such that $\|\sum_{i=1}^{n} a_i\delta(x_i)\|\leq C\sum_{i=1}^{n}|a_i|$. Moreover, since M is uniformly discrete there exists $\theta>0$ such that $d(x,y)>\theta$ for every $x\neq y\in M$. It is easy to check that the map $f\colon M\to\mathbb{R}$ defined by $f(x_i)=\operatorname{sign}(a_i)$ and $f(x)=0$ elsewhere is $2\theta^{-1}$ Lipschitz. Thus, we deduce $\|\sum_{i=1}^{n} a_i\delta(x_i)\|\geq \frac{\theta}{2}\langle f,\sum_{i=1}^{n} a_i\delta(x_i)\rangle\geq\frac{\theta}{2}\sum_{i=1}^{n}|a_i|$. Thus $(\delta(x_n))_{n=1}^{\infty}$ is a Schauder basis for $\mathcal{F}(M)$. □

Lemma 2.3.10. *Let X,Y be Banach spaces such that $X^*\equiv Y$, Y admits a bounded Schauder basis (u_n) and the biorthogonal functionals (u_n^*) belong to X. Then $u_n\to 0$ in the weak* topology.*

Proof. We will show that every subsequence of $(u_n)_{n=1}^{\infty}$ admits a further subsequence that converges to 0 in the weak* topology. So let us consider such subsequence. By weak* compactness and separability, it admits a weak* convergent subsequence, let us call it $(u_n)_{n=1}^{\infty}$ again. So we have $u_n\to u\in X$ in the weak* topology. But this means that for every $m\in\mathbb{N}$ we have

$$0=\lim_{n\to\infty}\langle u_m^*,u_n\rangle=\langle u_m^*,u\rangle.$$

Thus $u=0$. □

Proof of Proposition 2.3.8. Since M is bounded, separable and uniformly discrete, the sequence $(\delta(x))_{x\in M\setminus\{0\}}$ is a Schauder basis which is equivalent to the unit vector basis of ℓ_1. The biorthogonal functionals are exactly the indicator functions $\mathbb{1}_{\{x\}}$ for $x\neq 0$. Applying Lemma 2.3.10 we get that $\delta(M)$ is weak* closed and that 0 is the unique weak* accumulation point of $\delta(M)$. Let τ be the restriction of the weak* topology to M. Now Corollary 2.3.4 yields that $X=\operatorname{Lip}_0(M)\cap\mathcal{C}(M,\tau)$. But, since M is bounded and uniformly discrete, we have that $\operatorname{Lip}_0(M)$ is just all bounded functions that vanish at 0. It follows immediately that $X=c_0(M\setminus\{0\})$. □

Remark 2.3.11. In [Wea96], Weaver proved a duality result for *rigidly locally compact* metric spaces. We recall that a locally compact metric space is said to be rigidly locally compact (see the paragraph before Proposition 3.3 in [Wea96]) if for every $r>1$ and every $x\in M$, the closed ball $B(x,\frac{d(0,x)}{r})$ is compact. The duality result of Weaver in particular implies that for a separable uniformly discrete bounded metric space M which is rigidly compact, the space

$$X=\left\{f\in\operatorname{Lip}_0(M):\frac{f(\cdot)}{d(\cdot,0)}\in C_0(M)\right\}$$

is an isometric predual of $\mathcal{F}(M)$. Here $C_0(M)$ denotes the set of continuous functions which are arbitrarily small out of compact sets. Since it is obvious that the indicator functions $\mathbb{1}_{\{x\}}$ belong to X, Proposition 2.3.8 implies that X is a natural predual of $\mathcal{F}(M)$ and that X is isomorphic to c_0. This shows that in the case of uniformly discrete bounded spaces, Corollary 2.3.4 covers the cases in which Weaver's result ensures the existence of a predual.

Moreover, there is a metric space which satisfies the hypotheses of Corollary 2.3.4 and which is not rigidly locally compact.

Example 2.3.12. Let us consider the metric space $M = \{0,1\} \times \mathbb{N}$ equipped with the following distance : $d((0,n),(1,m)) = 2$ for $n,m \in \mathbb{N}$, and if $n \neq m$ we have $d((0,n),(0,m)) = 1$ and $d((1,n),(1,m)) = 1$. Then M satisfies the assumptions of Corollary 2.3.4. Indeed, declare $(0,1)$ to be the accumulation point of the sequence $\{(0,n)\}$, $(1,1)$ to be the accumulation point of the sequence $\{(1,n)\}$, and then declare all the other points isolated. Now independently of the choice of the distinguished point 0_M, M is not rigidly locally compact. For instance, say that $0_M = (0,n)$. Then for every $r > 1$, the ball $B((1,1), d(0_M,(1,1))/r) = B((1,1),2/r)$ contains all the elements of the form $(1,m)$ with $m \in \mathbb{N}$. Consequently the considered ball is not compact, which proves that M is not rigidly locally compact.

2.4 Metric spaces originating from p-Banach spaces

In this section, we study Lipschitz free spaces over a new family of metric spaces, namely metric spaces originating from p-Banach spaces.

Let X be a real vector space and p in $(0,1)$. We say that a map $N : X \to [0,\infty)$ is p-subadditive if $N(x+y)^p \leq N(x)^p + N(y)^p$ for every x and y in X. Then a homogeneous and p-subadditive map $\|\cdot\| : X \to [0,\infty)$ is called a p-norm if $\|x\| = 0$ if and only if $x = 0$. Moreover the map $(x,y) \in X^2 \mapsto \|x-y\|^p$ defines a metric on X. If X is complete for this metrizable topology, we say that X is a p-Banach space. Note that a p-norm is actually a quasi-norm. That is, there exists a constant $C \geq 1$ such that for every $x,y \in X$: $\|x+y\| \leq C(\|x\| + \|y\|)$. Moreover, an important theorem of Aoki and Rolewicz implies that every quasi-normed space can be renormed to be a p-normed space for some p in $(0,1)$. For background on quasi-Banach spaces and p-Banach spaces we refer the reader to [Kal03, KPR84].

We fix p in $(0,1)$ and we consider a p-Banach space $(X, \|\cdot\|)$. We denote $M_p = (X, d_p)$ the metric space where the metric is the p-norm of X to the power p : $d_p(x,y) = \|x-y\|^p$. Now, M_p being a metric space, we can study its Lipschitz free space.

At this point, we would like to compare our procedure with the one studied by Kalton in [Kal04]. Indeed, Kalton considered metric spaces of the following form : $(M, \omega \circ d)$ where d is a metric on M and ω is a nontrivial gauge (see 3. in Example 2.1.6). In our case, we consider a quasi-norm composed with the nontrivial gauge $\omega(t) = t^p$. Thus, we can expect to have the same kind of results. We will see that we need to use more arguments to overpass the difference between a norm and a p-norm, that is the absence of the triangle inequality for the p-norm. However, techniques that are employed here are inspired by Kalton's ideas in [Kal04].

Moreover, our results are not special cases of results in [Kal04] as it is explained in the following remark.

Remark 2.4.1. In general, we cannot write the distance $\|\cdot\|^p$ originating from a p-norm as the composition of a gauge and another distance.

Let us prove this for instance in $M_p^2 = (\mathbb{R}^2, \|\cdot\|_p^p)$. We argue by contradiction and so we assume that there exists ω a nontrivial gauge and d a distance on $\mathbb{R}^2$ such that $\|x-y\|_p^p = \omega(d(x,y))$ for every x and y in $\mathbb{R}^2$. Now we consider the points $x = (t_x, 0)$,

$y = (0, t_y)$. Straightforward computations show that $\|x-y\|_p^p = |t_x|^p + |t_y|^p = \|x\|_p^p + \|y\|_p^p$. Since d is a distance and ω is a gauge we have

$$\begin{aligned} |t_x|^p + |t_y|^p &= \omega(d(x,y)) \leq \omega(d(x,0) + d(y,0)) \\ &\leq \omega(d(x,0)) + \omega(d(y,0)) = |t_x|^p + |t_y|^p. \end{aligned}$$

Thus $\omega(d(x,0)+d(y,0)) = \omega(d(x,0))+\omega(d(y,0))$. We deduce that ω is additive, and so is such that $\omega(t) = t\omega(1)$. This contradicts the fact that ω is a nontrivial gauge and finally proves our claim. □

As usual, $\|\cdot\|_1$ denotes the ℓ_1-norm on $\mathbb{R}^n$. We also denote $\|\cdot\|_p$ the p-norm on $\mathbb{R}^n$ defined by $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$, for every $x = (x_i)_{i=1}^n \in \mathbb{R}^n$. We begin with a very basic lemma.

Lemma 2.4.2. *Let $p \in (0,1)$ and $n \in \mathbb{N}$. Then we have the following inequalities*

$$\forall x \in \mathbb{R}^n,\ \|x\|_1 \leq \|x\|_p \leq n^{\frac{1-p}{p}} \|x\|_1.$$

Proof. The inequality $\|x\|_1 \leq \|x\|_p$ is obvious and a simple application of Hölder's inequality gives $\|x\|_p \leq n^{\frac{1-p}{p}} \|x\|_1$. □

From now on we write M_p^n for $(\mathbb{R}^n, d_p) = (\mathbb{R}^n, \|\cdot\|_p^p)$. In order to prove our first result about the structure of $\mathcal{F}(M_p^n)$, we need the following technical lemma.

Lemma 2.4.3. *Let $R \in (0,\infty)$, $p \in (0,1)$ and $n \in \mathbb{N}$. Then, there exists a Lipschitz function $\varphi : M_p^n \to M_p^n$ such that φ is the identity map on $\overline{B}(0,R)$, is null on $M_p^n \backslash B(0,2R)$ and φ is n^{2-p}-Lipschitz.*

Proof. Let us define $A = \overline{B}(0,R) \cup (M_p^n \backslash B(0,2R)) \subset M_p^n$ (balls are considered for d_p) and $\phi : (A, d_p) \to M_p^n$ such that ϕ is the identity on $\overline{B}(0,R)$ and is null on $M_p^n \backslash B(0,2R)$. It is easy to check that ϕ is 1-Lipschitz. We now write $\phi = (\phi_1, \cdots, \phi_n)$. Then for every k, $\phi_k : (A, d_p) \to (\mathbb{R}, |\cdot|^p)$ is 1-Lipschitz. Thus $\phi_k : (A, \|\cdot\|_p) \to (\mathbb{R}, |\cdot|)$ is also 1-Lipschitz (with the obvious extension of the notion of Lipschitz maps). Now the right hand side of the inequality in Lemma 2.4.2 implies that $\phi_k : (A, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$ is $n^{\frac{1-p}{p}}$-Lipschitz. So we can extend each ϕ_k without increasing its Lipschitz constant and we denote φ_k those corresponding extensions. Summarizing we have $\varphi_k : (\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$ which is $n^{\frac{1-p}{p}}$ Lipschitz and $\varphi_{k|A} = \phi_k$. Now the left hand side of the inequality in Lemma 2.4.2 implies that $\varphi_k : (\ell_p^n, \|\cdot\|_p) \to (\mathbb{R}, |\cdot|)$ is $n^{\frac{1-p}{p}}$-Lipschitz. So $\varphi_k : M_p^n \to (\mathbb{R}, |\cdot|^p)$ is n^{1-p} Lipschitz. It follows easily that $\varphi = (\varphi_1, \cdots, \varphi_n) : M_p^n \to M_p^n$ is n^{2-p}-Lipschitz and verifies the desired properties.

□

We are now able to prove the following result.

Proposition 2.4.4. *Let $p \in (0,1)$ and $n \in \mathbb{N}$. We still denote $(M_p^n, d_p) = (\mathbb{R}^n, \|\cdot\|_p^p)$. Then, $S_0(M_p^n)$ is a natural predual of $\mathcal{F}(M_p^n)$.*

Proof. In order to prove this result, we will first prove that $S_0(M_p^n)$ is C_n-norming for some $C_n > 0$. And then we will deduce the desired result from Theorem 2.1.4 (and the fact that for a proper metric space M, any predual of $\mathcal{F}(M)$ is natural).

For every $m \in \mathbb{N}$ and $t \geq 0$, we define the following function $\omega_m(t) = \min\{t^p, mt\}$ which is continuous, non-decreasing and subadditive. Note that $\lim\limits_{m\to+\infty} \omega_m(t) = t^p$.

Let $x \neq y \in M_p^n$. Since $(\ell_1^n)^* \equiv \ell_\infty^n$, by the Hahn–Banach theorem there exists $x^* \in \ell_\infty^n$ such that $\|x^*\|_\infty = 1$ and $\langle x^*, x-y\rangle = \|x-y\|_1$. According to Lemma 2.4.2 this gives $\langle x^*, x-y\rangle \geq n^{\frac{p-1}{p}}\|x-y\|_p$. From now on we denote $F := n^{\frac{1-p}{p}}x^*$ and we see F as an element of $(\ell_p^n)^* \equiv \ell_\infty^n$ of norm $\|F\|_{(\ell_p^n)^*} \leq n^{\frac{1-p}{p}}$ which satisfies

$$|F(x) - F(y)| \geq \|x-y\|_p \tag{2.1}$$

Let us consider $R > 2\max(\|x\|_p^p, \|y\|_p^p)$ and $\varphi : M_p^n \to M_p^n$ given by Lemma 2.4.3 (we denote C its Lipschitz constant). Of course, we can see φ as a $C^{\frac{1}{p}}$-Lipschitz function from ℓ_p^n to ℓ_p^n. We then consider f_m defined on M_p^n by

$$f_m(z) = \omega_m(|F(\varphi(z)) - F(y)|) - \omega_m(|F(y)|).$$

Let us prove that those functions f_m belong to $S_0(M_p^n)$ and do the job. For $z \neq z' \in M_p^n$ we compute

$$\begin{aligned} |f_m(z) - f_m(z')| &= |\,\omega_m(|F(\varphi(z)) - F(y)|) - \omega_m(|F(\varphi(z')) - F(y)|)\,| \\ &\leq \omega_m(|F(\varphi(z)) - F(\varphi(z'))|) \\ &= \omega_m(|F(\varphi(z) - \varphi(z'))|). \end{aligned}$$

By its definition $\omega_m(t) \leq t^p$. So we have

$$|f_m(z) - f_m(z')| \leq |F(\varphi(z) - \varphi(z'))|^p \leq n^{1-p} d_p(\varphi(z), \varphi(z')) \leq Cn^{1-p}\, d_p(z, z').$$

Thus, f_m is d_p-Lipschitz with $\|f_m\|_L \leq Cn^{1-p}$. Now since $\omega_m(t) \leq mt$ we get

$$\begin{aligned} |f_m(z) - f_m(z')| &\leq m|F(\varphi(z) - \varphi(z'))| \leq mn^{\frac{1-p}{p}}\|\varphi(z) - \varphi(z')\|_p \leq C^{\frac{1}{p}} mn^{\frac{1-p}{p}}\|z - z'\|_p \\ &\leq (C^{\frac{1}{p}} mn^{\frac{1-p}{p}}\|z - z'\|_p^{1-p})\, d_p(z, z'). \end{aligned}$$

Since $1 - p > 0$, $\|z - z'\|_p^{1-p}$ and thus the Lipschitz constant of f_m can be as small as we want for small distances. This provides the fact that $f_m \in \mathrm{lip}_0(M_p^n)$. It remains to prove that f_m satisfies the flatness condition at infinity to get $f_m \in S_0(M_p^n)$. To this end, fix $\varepsilon > 0$ and pick $k > 2$ such that $\dfrac{2Cn^{1-p}}{(k-2)} \leq \varepsilon$. Now let z and z' be in M, and let us discuss by cases :

(i) If $z \notin \overline{B}(0, kR)$ and $z' \notin \overline{B}(0, 2R)$, then $|f_m(z) - f_m(z')| = 0 < \varepsilon$.

(ii) If $z \notin \overline{B}(0, kR)$ and $z' \in \overline{B}(0, 2R)$, then

$$\begin{aligned} \frac{|f_m(z) - f_m(z')|}{d_p(z, z')} &\leq \frac{|F(\varphi(z'))|^p}{(k-2)R} \\ &\leq \frac{n^{1-p}\|\varphi(z')\|_p^p}{(k-2)R} \\ &\leq \frac{Cn^{1-p}(2R)}{(k-2)R} \leq \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves that $f_m \in S_0(M)$. To finish the first part of the proof just notice now that using the inequality (2.1) and the fact that $\lim_{m\to+\infty} \omega_m(t) = t^p$, we get

$$|f_m(x) - f_m(y)| = \omega_m(|F(x) - F(y)|) \geq \omega_m(\|x-y\|_p) \underset{m\to+\infty}{\longrightarrow} d_p(x,y).$$

Thus $S_0(M_p^n)$ is Cn^{1-p}-norming.

We are now moving to the duality argument. Remark that M_p^n is a proper metric space, so using Theorem 2.1.4, we have that $S_0(M_p^n)^* \equiv \mathcal{F}(M_p^n)$ and $S_0(M_p^n)$ is a natural predual because of properness. □

Of course this last result still holds for every metric space originating from a p-Banach space X_p of finite dimension.

Corollary 2.4.5. *Let $p \in (0,1)$ and consider $(M_p, d_p) = (X, \|\cdot\|^p)$ where $(X, \|\cdot\|)$ is a p-Banach space of finite dimension. Then, $S_0(M_p)$ is a natural predual of $\mathcal{F}(M_p)$.*

Proof. Note that since X_p is of finite dimension, it is isomorphic to ℓ_p^n for some $n \in \mathbb{N}$. Thus there is a bi-Lipschitz map between M_p and M_p^n, let us say $L : M_p \to M_p^n$ is bi-Lipschitz with $C_1 d_{M_p}(x,y) \leq d_{M_p^n}(L(x), L(y)) \leq C_2 d_{M_p}(x,y)$. Now $S_0(M_p)$ is $\frac{C_2}{C_1}$-norming. Indeed pick $x \neq y \in M_p$ and $\varepsilon > 0$. Since $S_0(M_p^n)$ is 1-norming there exists $f \in S_0(M_p^n)$ with Lipschitz constant less than $1+\varepsilon$ such that

$$|f(L(x)) - f(L(y))| = d_{M_p^n}(L(x), L(y)) \geq C_1 d_{M_p}(x,y).$$

Now $f \circ L : M_p \to \mathbb{R}$ is Lipschitz with Lipschitz constant less than $C_2(1+\varepsilon)$. Moreover as the composition of a bi-Lipschitz map with an element of $S_0(M_p^n)$ we know that $f \circ L \in S_0(M_p)$. Thus $S_0(M_p)$ is $\frac{C_2}{C_1}$-norming. Since M_p is proper, it follows again from Theorem 2.1.4 that $\mathcal{F}(M_p)$ is isometric to $S_0(M_p)^*$. □

2.5 Perspectives

An important condition that appears for instance in Proposition 2.2.5 is that a particular subspace of $\mathrm{lip}_0(M)$ (or $\mathrm{lip}_0(M)$ itself) separates points uniformly (or 1-SPU in Theorem 2.2.4 and even later in Proposition 3.1.2). We address the following.

Question 2.5.1. Let M be a complete metric space such that $\mathrm{lip}_0(M)$ separates points uniformly. Does it imply that $\mathrm{lip}_0(M)$ actually 1-separates points uniformly ?

The answer is yes when the metric space is compact. Indeed, it follows from Petunīn–Plīčko's theorem (Theorem 2.2.7) that if M is a compact metric space such that $\mathrm{lip}_0(M)$ separates points uniformly, then $\mathrm{lip}_0(M)^* \equiv \mathcal{F}(M)$ (Theorem 3.3.3 in [Wea99]). Thus, $\mathrm{lip}_0(M)$ is in particular 1-norming and so it 1-separates points uniformly. Of course, the same question makes sense for $S_0(M)$ and the answer is yes when M is proper for the same reason.

Next, in [Kal04] one of the main applications to the duality result of Kalton (Theorem 2.2.4) is the following : Let X^* be a separable dual Banach space and $0 < p < 1$, then $\mathcal{F}(B_{X^*}, \|\cdot\|^p) \equiv \left(\mathrm{lip}_0(B_{X^*}, \|\cdot\|^p) \cap \mathcal{C}(B_{X^*}, \sigma(X^*, X))\right)^*$. In light of our generalisation in the unbounded setting (Proposition 2.2.5), it is quite natural to wonder the following.

Question 2.5.2. Let X^* be a separable Banach space and $0 < p < 1$. Is it true that $\mathcal{F}(X^*, \|\cdot\|^p)$ is isometric to a dual Banach space ? A natural candidate for a predual would be : $X = S_0(X^*, \|\cdot\|^p) \cap \mathcal{C}_b\big(X^*, \sigma(X^*, X)\big)$.

Finally, most of our examples of metric spaces for which $\mathcal{F}(M)$ is isometric to a dual Banach space are either compact, proper, or uniformly discrete. A natural class of metric spaces that contains those three previous examples is the class of locally compact metric spaces. It would be interesting to find, whenever M is locally compact, a new sufficient condition ensuring that $\mathcal{F}(M)$ is a dual Banach space.

Chapter 3

Schur properties and Lipschitz free spaces

In this chapter, we focus on ℓ_1-like properties such as the Schur property (or some stronger properties). In [Kal04], Kalton proved that if (M, d) is a metric space and ω is a nontrivial gauge then $\mathcal{F}(M, \omega \circ d)$ has the Schur property. In [HLP16] Hájek, Lancien, and Pernecká proved that the Lipschitz free space over a proper countable metric space has the Schur property. Here we give a condition on M which ensures that $\mathcal{F}(M)$ has the Schur property, and unifies the two above mentioned results.

Let us briefly describe the content of this chapter. In Section 3.1, generalizing the proof of Theorem 4.6 in [Kal04], we show that $\mathcal{F}(M)$ has the Schur property whenever $\mathrm{lip}_0(M)$ is 1-norming. Assuming moreover that the metric space is proper, we show that some quantitative versions of the Schur property are inherited by $\mathcal{F}(M)$ (Section 3.2). Adding one more condition, which is $\mathcal{F}(M)$ has the approximation property, we are able to provide more information about the "ℓ_1-structure" of $\mathcal{F}(M)$. More precisely, we manage to embed linearly $\mathcal{F}(M)$ into an ℓ_1-sum of its finite dimensional subspaces (Section 3.3).

Finally, in Section 3.4 we focus on metric spaces originating from p-Banach spaces. The aim is to apply some results obtained in the sections described above. This chapter is based on [Pet17].

3.1 The Schur property

Let us recall the definition of the Schur property.

Definition 3.1.1. Let X be a Banach space. We say that X has *the Schur property* if every weakly null sequence $(x_n)_{n=1}^{\infty}$ in X is also $\|\cdot\|$-convergent to 0.

A careful reading of Kalton's proof of [Kal04, Theorem 4.6] reveals that the key ingredient is actually the fact that $\mathrm{lip}_0(M, \omega \circ d)$ is always 1-norming [Kal04, Proposition 3.5]. This leads us to the following result.

Proposition 3.1.2. *Let M be a pointed metric space such that $\mathrm{lip}_0(M)$ is 1-norming for $\mathcal{F}(M)$. Then the space $\mathcal{F}(M)$ has the Schur property.*

Proof. According to Proposition 4.3 in [Kal04], for every $\varepsilon > 0$, $\mathcal{F}(M)$ is $(1+\varepsilon)$-isomorphic to a subspace of $(\sum_{k\in\mathbb{Z}} \mathcal{F}(M_k))_{\ell_1}$ where M_k denotes the ball $\overline{B}(0, 2^k) \subseteq M$ centred at 0

and of radius 2^k. Moreover the Schur property is stable under ℓ_1-sums, under isomorphism and passing to subspaces. So it suffices to prove the result under the assumption that M has finite radius.

Let us consider $(\gamma_n)_{n=1}^{\infty}$ a normalized weakly null sequence in $\mathcal{F}(M)$. We will show that

$$\forall \gamma \in \mathcal{F}(M),\ \liminf_{n\to+\infty} \|\gamma+\gamma_n\| \geq \|\gamma\| + \frac{1}{2}, \tag{3.1}$$

from which it is easy to deduce that for every $\varepsilon > 0$, $(\gamma_n)_{n=1}^{\infty}$ admits a subsequence $(2+\varepsilon)$-equivalent to the ℓ_1-basis (see the end of the proof of Proposition 4.6 in [Kal04]). This contradicts the fact that $(\gamma_n)_{n=1}^{\infty}$ is weakly null.

Fix $\varepsilon > 0$ and $\gamma \in \mathcal{F}(M)$. We can assume that γ is of finite support. Pick $f \in \mathrm{lip}_0(M)$ with $\|f\|_L = 1$ and $\langle f, \gamma\rangle > \|\gamma\| - \varepsilon$. Next pick $\Theta > 0$ so that if $d(x,y) \leq \Theta$ then $|f(x) - f(y)| < \varepsilon d(x,y)$. Choose $\delta < \frac{\varepsilon\Theta}{2(1+\varepsilon)}$. Then by Lemma 4.5 in [Kal04] we have

$$\inf_{|E|<\infty} \sup_n dist(\gamma_n, \mathcal{F}([E]_\delta)) = 0,$$

where $[E]_\delta = \{x \in M : d(x,E) \leq \delta\}$. Thus there exists a finite set $E \subset M$ such that E contains the support of γ and such that for each n we can find $\mu_n \in \mathcal{F}([E]_\delta)$ with $\|\gamma_n - \mu_n\| < \varepsilon$. Remark that $\mathcal{F}(E)$ is a finite dimensional space. Thus :

$$\liminf_{n\to+\infty} dist(\gamma_n, \mathcal{F}(E)) \geq \frac{1}{2}.$$

Then, by the Hahn-Banach theorem, for every n there exists $f_n \in \mathrm{Lip}_0(M)$ verifying $\|f_n\|_L \leq 1+\varepsilon$, $f_n(E) = \{0\}$ and $\liminf_{n\to+\infty}\langle f_n, \gamma_n\rangle \geq \frac{1}{2}$. Now we define $g_n = (f+f_n)_{|[E]_\delta}$, then $g_n \in \mathrm{Lip}_0([E]_\delta)$ and we will show that $\|g_n\|_L < 1+\varepsilon$. We will distinguish two cases to show this last property. First suppose that x and y are such that $d(x,y) \leq \Theta$, then

$$|g_n(x) - g_n(y)| \leq |f(x) - f(y)| + |f_n(x) - f_n(y)| \leq (1+\varepsilon)d(x,y).$$

Second if x and y are such that $d(x,y) > \Theta$, then there exist $u, v \in E$ with $d(x,u) \leq \delta$ and $d(y,v) \leq \delta$, so that

$$\begin{aligned} |g_n(x) - g_n(y)| &\leq |f(x) - f(y)| + |f_n(x)| + |f_n(y)| \\ &= |f(x) - f(y)| + |f_n(x) - f_n(u)| + |f_n(y) - f_n(v)| \\ &\leq d(x,y) + 2(1+\varepsilon)\delta \leq d(x,y) + \varepsilon\Theta \leq (1+\varepsilon)d(x,y). \end{aligned}$$

We extend those functions g_n to M with the same Lipschitz constant and we still denote those extensions g_n for convenience. We now estimate the desired quantities.

$$\|\gamma+\mu_n\| \geq \frac{1}{1+\varepsilon}\langle g_n, \gamma+\mu_n\rangle = \frac{1}{1+\varepsilon}(\langle f,\gamma\rangle + \langle f,\mu_n\rangle + \langle f_n,\gamma\rangle + \langle f_n,\mu_n\rangle),$$

where

(i) $\langle f,\gamma\rangle > \|\gamma\| - \varepsilon$.

(ii) $\limsup_{n\to\infty} |\langle f,\mu_n\rangle| \leq \varepsilon$, since $(\gamma_n)_{n=1}^{\infty}$ is weakly null and $\|\gamma_n - \mu_n\| < \varepsilon$.

(iii) $\langle f_n, \gamma\rangle = 0$, since $\gamma \in \mathcal{F}(E)$.

(iv) $\liminf_{n\to\infty}\langle f_n, \mu_n\rangle \geq \frac{1}{2} - \varepsilon$, since $\liminf_{n\to\infty}\langle f_n, \gamma_n\rangle > \frac{1}{2}$.

Thus,

$$\liminf_{n\to\infty} \|\gamma + \gamma_n\| \quad \geq \quad \frac{1}{1+\varepsilon}(\|\gamma\| + \frac{1}{2} - 3\varepsilon) - \varepsilon.$$

Since ε is arbitrary, this proves (3.1). □

The previous result obviously applies to the metric spaces in Example 2.1.6.

3.2 Quantitative versions of the Schur property

It is well known that the Schur property is equivalent to the following condition : for every $\delta > 0$, every δ-separated sequence $(x_n)_{n=1}^\infty$ in the unit ball of X contains a subsequence that is equivalent to the unit vector basis of ℓ_1. That is there exists $C_1, C_2 > 0$ (which may depend on the sequence considered) and a subsequence $(x_{n_k})_{k=1}^\infty$ such that

$$C_2 \sum_{i=1}^n |a_i| \geq \left\| \sum_{i=1}^n a_i x_{n_i} \right\| \geq C_1 \sum_{i=1}^n |a_i|\,, \text{ for every } (a_i)_{i=1}^n \in \mathbb{R}^n.$$

In this case, we say that the sequence $(x_{n_k})_{k=1}^\infty$ is $\frac{C_2}{C_1}$-equivalent to the unit vector basis of ℓ_1. This equivalence can be easily deduced using Rosenthal's ℓ_1 theorem (see [AK06, Theorem 10.2.1]). Moreover, we may equivalently only consider normalised sequences $(x_n)_{n=1}^\infty \subset S_X$. In this case we necessarily have $C_2 = 1$. This last fact leads us to define the following quantitative version of the Schur property. It seems that the strong Schur property was considered first for subspaces of L_1 by Rosenthal (see [Ros79], and also [BR80]) We also refer to [GKL96] for the 1-strong Schur property.

Definition 3.2.1. Let X be a Banach space. We say that X has *the strong Schur property* if there exists a constant $K > 0$ such that, for every $\delta > 0$, any δ-separated sequence $(x_n)_{n=1}^\infty$ in the unit sphere of X contains a subsequence that is $\frac{K}{\delta}$-equivalent to the unit vector basis of ℓ_1. If in this definition, K can be chosen so that $K = 2+\varepsilon$ for every $\varepsilon > 0$, then we say that X has the 1-*strong Schur property.*

It is clear with the above characterization of the Schur property that the strong Schur property implies the Schur property. It is known that the Schur property is strictly weaker than the strong Schur property (see [Wnu09] or [KS12] for instance). Moreover, Bourgain and Rosenthal have constructed a subspace of L_1 with the 1-strong Schur property that is not isomorphic to a subspace of ℓ_1 [BR80]. We refer the reader to [Kal01, Proposition 2.1] for some equivalent formulations of the strong Schur property. Not surprisingly, ℓ_1 is a space having the 1-strong Schur property. Here are some more examples of spaces enjoying the strong Schur property.

Examples 3.2.2.

1. In [KO89] (Proposition 4.1), Knaust and Odell proved that if X has the property (S) and does not contain any isomorphic copy of ℓ_1, then X^* has the strong Schur property. In particular, ℓ_1 and all its subspaces have the strong Schur property. A Banach space has property (S) if every normalized weakly null sequence contains a subsequence equivalent to the unit vector basis of c_0. This is known to be equivalent to the hereditary Dunford–Pettis property (Proposition 2 in [Cem87])
2. In [GKL96] (Lemma 3.4), Godefroy, Kalton, and Li proved that a subspace of L_1 has the strong Schur property if and only if its unit ball is relatively compact in the topology of convergence in measure.

We now give the second quantitative version of the Schur property which has been introduced more recently by Kalenda and Spurný in [KS12].

Definition 3.2.3. Let X be a Banach space, and let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X. We write $\mathrm{clust}_{X^{**}}(x_n)$ for the set of all weak*-cluster points of $(x_n)_{n=1}^{\infty}$ in X^{**}. Then we define the two following moduli :

$$\begin{aligned} \delta(x_n) &:= \mathrm{diam}\{\mathrm{clust}_{X^{**}}(x_n)\} \\ \mathrm{ca}(x_n) &:= \inf_{n\in\mathbb{N}} \mathrm{diam}\{x_k\,;\, k \geq n\}. \end{aligned}$$

The first modulus measures how far is the sequence from being weakly Cauchy and the second one measures how far is the sequence from being $\|\cdot\|$-Cauchy. We then say that X has *the C-Schur property* if for every bounded sequence $(x_n)_{n=1}^{\infty}$ in X : $\mathrm{ca}(x_n) \leq C\,\delta(x_n)$

In [KS12] the authors proved that the 1-Schur property implies the 1-strong Schur property and that the 1-strong Schur property implies the 5-Schur property. To the best of our knowledge, the question whether the 1-strong Schur property implies the 1-Schur property is open.

In [KS15] it is proved (Theorem 1.1) that if X is a subspace of $c_0(\Gamma)$, then X^* has the 1-Schur property. Moreover, generalizing a proof of Kalton in the compact case (Theorem 6.6 in [Kal04]), Dalet has proved the following lemma (Lemma 3.9 in [Dal15c]).

Lemma 3.2.4 (Dalet). *Let M be a proper pointed metric space. Then, for every $\varepsilon > 0$, the space $S_0(M)$ is $(1+\varepsilon)$-isomorphic to a subspace Z of $c_0(\mathbb{N})$.*

So we easily deduce the following proposition.

Proposition 3.2.5. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly. Then $\mathcal{F}(M)$ has the 1-Schur property.*

Proof. Fix $\varepsilon > 0$. We use Lemma 3.2.4 to find a subspace Z of $c_0(\mathbb{N})$ which is $(1+\varepsilon)$-isomorphic to $S_0(M)$. Now, according to Theorem 1.1 in [KS15], Z^* has the 1-Schur property. Since $S_0(M)$ separates points uniformly, we have that $S_0(M)^* \equiv \mathcal{F}(M)$ (Theorem 2.1.4). Thus $\mathcal{F}(M)$ is $(1+\varepsilon)$-isomorphic to Z^*. Therefore $\mathcal{F}(M)$ has the $(1+\varepsilon)$-Schur property. Since ε is arbitrary, $\mathcal{F}(M)$ has the 1-Schur property. □

3.3 Embeddings into ℓ_1-sums

Before stating the main result of this section we recall a few classical definitions. We say that a Banach space X has the *approximation property* the (AP) if for every $\varepsilon > 0$, for every compact set $K \subset X$, there exists a finite rank operator $T \in \mathcal{B}(X)$ such that $\|Tx - x\| \leq \varepsilon$ for every $x \in K$. Let $\lambda \geq 1$, if in the above definition T can always be chosen so that $\|T\| \leq \lambda$, then we say that X has the λ-*bounded approximation property* (λ-(BAP)). When X has the 1-(BAP) we say that X has the *metric approximation property* (MAP). We can now state and prove our first main result.

Theorem 3.3.1. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly and such that $\mathcal{F}(M)$ has the (MAP). Then for any $\varepsilon > 0$, there exists a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $\mathcal{F}(M)$ such that $\mathcal{F}(M)$ is $(1+\varepsilon)$-isomorphic to a subspace of $(\sum_n E_n)_{\ell_1}$.*

Proof. The proof is based on three results. The first ingredient is the following Lemma (Lemma 3.1 in [GKL96]) :

Lemma 3.3.2 (Godefroy–Kalton–Li). *Let V be a subspace of $c_0(\mathbb{N})$ with the (MAP). Then for any $\varepsilon > 0$, there exists a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of V^* and a weak*-to-weak* continuous linear map $T : V^* \to (\sum_n E_n)_{\ell_1}$ such that for all $x^* \in V^*$*

$$(1-\varepsilon)\|x^*\| \leq \|Tx^*\| \leq (1+\varepsilon)\|x^*\|.$$

The second ingredient is Lemma 3.2.4 which ensures that $S_0(M)$ is $(1+\varepsilon)$-isomorphic to a subspace Z of $c_0(\mathbb{N})$ whenever M is proper. Finally we need the following two results about the (MAP) (see [Gro55] and [DU77, Corollary VIII.3.9, Corollary VIII.4.3]).

Theorem 3.3.3 (Grothendieck). *Let X be a Banach space.*

(G1) If X^* has the (MAP) then X has (MAP).

(G2) If X^* is separable and has the (AP) then X^* has the (MAP).

We are ready to prove Theorem 3.3.1. Let us consider a metric space M satisfying the assumptions of Theorem 3.3.1 and let us take $\varepsilon > 0$ arbitrary. Fix ε' such that $(1+\varepsilon')^3 < 1+\varepsilon$. According to Lemma 3.2.4, there exists a subspace Z of $c_0(\mathbb{N})$ such that $S_0(M)$ is $(1+\varepsilon')$-isomorphic to Z. Then note that Z also has the metric approximation property. Indeed Z^* is $(1+\varepsilon')$-isomorphic to $\mathcal{F}(M)$, so Z^* has the $(1+\varepsilon')$-bounded approximation property. Next, using $(G2)$ of Theorem 3.3.3 we get that Z^* has MAP. Then using $(G1)$ of Theorem 3.3.3 we get that Z also has MAP. Thus we can apply Lemma 3.3.2 to Z so that there exists a sequence $(F_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of Z^* such that Z^* is $(1+\varepsilon')$-isomorphic to a subspace F of $(\sum_n F_n)_{\ell_1}$. Now $\mathcal{F}(M)$ is $(1+\varepsilon')$-isomorphic to Z^* so there exists a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $\mathcal{F}(M)$ such that $(\sum_n E_n)_{\ell_1}$ is $(1+\varepsilon')$-isomorphic to $(\sum_n F_n)_{\ell_1}$. Then there exists a subspace E of $(\sum_n E_n)_{\ell_1}$ which is $(1+\varepsilon')$-isomorphic to F. It is easy to check that $\mathcal{F}(M)$ is $(1+\varepsilon')^3$-isomorphic to E. This completes the proof. □

We now give some examples where Theorem 3.3.1 applies.

Examples 3.3.4. The space $S_0(M)$ separates points uniformly and $\mathcal{F}(M)$ has the (MAP) in any of the following cases.

1. M proper countable metric space (Theorem 2.1 and Theorem 2.6 in [Dal15c]).
2. M proper ultrametric space (Theorem 3.5 and Theorem 3.8 in [Dal15c]).
3. M compact metric space such that there exists a sequence $(\varepsilon_n)_{n=1}^{\infty}$ tending to 0, a real number $\rho < 1/2$ and finite ε_n-separated subsets N_n of M which are $\rho\varepsilon_n$-dense in M (Proposition 6 in [GO14]). For instance the middle-third Cantor set.

Before ending this section, we state a Proposition which says that Theorem 3.3.1 is optimal in some sense. Indeed, in our following example the dimension of the finite dimensional space E_n in Theorem 3.3.1 has to go to infinity when n goes to infinity.

Proposition 3.3.5. *There is a countable compact metric space K, which consists of a convergent sequence and its limit, such that $\mathcal{F}(K)$ fails to have a cotype. In particular, K satisfies the assumptions of Theorem 3.3.1 but does not embed isomorphically into ℓ_1.*

Proof. It is well known that c_0 has no nontrivial cotype. Since c_0 is separable, by Godefroy-Kalton lifting theorem (Theorem 3.1 in [GK03]), there is a subspace of $\mathcal{F}(c_0)$ which is linearly isometric to c_0. Thus $\mathcal{F}(c_0)$ has no nontrivial cotype. So for every $n \geq 1$, there exist $\gamma_1^n, \cdots, \gamma_{k_n}^n \in \mathcal{F}(c_0)$ such that

$$\left(\sum_{i=1}^{k_n} \|\gamma_i^n\|^n\right)^{\frac{1}{n}} > n\left(\mathbb{E}\|\sum_{i=1}^{k_n} \varepsilon_i \gamma_i^n\|^n\right)^{\frac{1}{n}},$$

where $(\varepsilon_i)_{i=1}^{k_n}$ is an independent sequence of Rademacher random variables. Next we approximate each γ_i^n by a finitely supported element $\mu_i^n \in \mathcal{F}(c_0)$ such that

$$\left(\sum_{i=1}^{k_n} \|\mu_i^n\|^n\right)^{\frac{1}{n}} > \frac{n}{2}\left(\mathbb{E}\|\sum_{i=1}^{k_n} \varepsilon_i \mu_i^n\|^n\right)^{\frac{1}{n}}.$$

Then we define $M_n = (\cup_{i=1}^{k_n} \operatorname{supp}(\mu_i^n)) \cup \{0\} \subset c_0$ which is a finite pointed metric space. Since scaling a metric space does not affect the linear isometric structure of the corresponding Lipschitz free space, we may and do assume that the diameter of M_n is less than $\frac{1}{2^{n+1}}$.

Now we construct the desired compact pointed metric space as follows. Let us define the countable set $K := (\cup_{n\geq 2}\{n\} \times M_n) \cup \{e\}$, e being the distinguished point of M. To simplify the notation we write M_n' for $\{n\} \times M_n$. Then we define a metric d on K such that $d(x,e) = \frac{1}{2^n}$ if $x \in M_n'$, $d(x,y) = d_{M_n}(x_n, y_n)$ if $x = (n, x_n), y = (n, y_n) \in M_n'$ and $d(x,y) = d(x,e) + d(y,e)$ if $x \in M_n'$, $y \in M_m'$ with $n \neq m$. Of course, with this metric K is compact since it is a convergent sequence together with its limit. Moreover the fact that $d(x,y) = d(x,e) + d(y,e)$ for $x \in M_n'$, $y \in M_m'$ with $n \neq m$ readily implies that $\mathcal{F}(K) = (\sum \mathcal{F}(M_n \cup \{e\}))_{\ell_1}$ (see Proposition 5.1 in [Kau15] for instance). By construction of M_n, $(\sum \mathcal{F}(M_n))_{\ell_1}$ has no nontrivial cotype. Thus $\mathcal{F}(K)$ also has no cotype. Therefore, $\mathcal{F}(K)$ cannot embed into ℓ_1. But since it is a compact countable metric space, $\operatorname{lip}_0(K)$ separates points uniformly and $\mathcal{F}(K)$ has the (MAP) (Theorem 2.1 and Theorem 2.6 in [Dal15c]). $\square$

3.4 Metric spaces originating from p-Banach spaces

As announced, we apply our results to the case of metric spaces originating from p-Banach spaces. Not surprisingly, the assumptions of Theorem 3.3.1 are satisfied for a metric space originating from a p-Banach space of finite dimension.

Corollary 3.4.1. *Let $p \in (0,1)$. Consider $(M_p, d_p) = (X, \|\cdot\|^p)$ where $(X, \|\cdot\|)$ is a p-Banach space of finite dimension. Then, for any $\varepsilon > 0$ there exists a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $\mathcal{F}(M_p)$ such that $\mathcal{F}(M_p)$ is $(1+\varepsilon)$-isomorphic to a subspace of $(\sum_n E_n)_{\ell_1}$.*

Proof. The aim is to show that all assumptions of Theorem 3.3.1 are satisfied for $\mathcal{F}(M_p)$. According to Corollary 2.4.5, $S_0(M_p)$ is 1-norming. Now it is proved in [LP13, Corollary 2.2] that if M is a doubling metric space (that is there exists $D(M) \geq 1$ such that any ball $B(x,R)$ can be covered by $D(M)$ open balls of radius $R/2$) then $\mathcal{F}(M)$ has the BAP. Since $M_p \subset \mathbb{R}^n$, we get that M_p is doubling. Thus in our case $\mathcal{F}(M_p)$ has the BAP. Since it is a dual space, we get from Theorem 3.3.3 that $\mathcal{F}(M_p)$ actually has the (MAP). Thus all the assumptions of Theorem 3.3.1 are satisfied. □

We now turn to the study of the structure of $\mathcal{F}(M_p)$ with more general assumptions on M_p ($(M_p, d_p) = (X, \|\cdot\|^p)$ where $(X, \|\cdot\|)$ is a p-Banach space). In particular, we now pass to infinite dimensional spaces and the aim is to explore the behavior of $\mathcal{F}(M_p)$ regarding properties such as the (MAP) and the Schur property. To do so, we will assume that X is a p-Banach space which admits a Finite Dimensional Decomposition (shortened in FDD). Especially, a space which admits a Schauder basis such as ℓ_p satisfies this assumption. We start with the study of the Schur property. Using our Proposition 2.1.5 we manage to prove the following result.

Theorem 3.4.2. *Let p in $(0,1)$ and let $(X, \|\cdot\|)$ be a p-Banach space which admits an FDD. Then $\mathcal{F}(X, \|\cdot\|^p)$ has the Schur property.*

Proof. First of all, note that we can assume that X admits a monotone FDD. Indeed, it is classical that we can define an equivalent p-norm $|||\cdot|||$ on X such that the finite dimensional decomposition is monotone for $(X, |||\cdot|||)$ (see Theorem 1.8 in [KPR84] for instance). Now from the fact that $(X, |||\cdot|||)$ and $(X, \|\cdot\|)$ are isomorphic we deduce that $(X, |||\cdot|||^p)$ and $(X, \|\cdot\|^p)$ are Lipschitz equivalent. Thus, using Corollary 1.1.4 we deduce that $\mathcal{F}(X, |||\cdot|||^p)$ and $\mathcal{F}(X, \|\cdot\|^p)$ are isomorphic. Since the Schur property is stable under isomorphism, $\mathcal{F}(X, |||\cdot|||^p)$ has the Schur property if and only if $\mathcal{F}(X, \|\cdot\|^p)$ has the Schur property. So from now on we assume that the FDD is monotone.

The aim is to apply Proposition 2.1.5. We denote again $(M_p, d_p) = (X, \|\cdot\|^p)$. Since X admits a monotone FDD, there exists a sequence $(X_k)_{k=1}^{\infty}$ of finite dimensional subspaces of X such that every $x \in X$ admits a unique representation of the form $x = \sum_{k=1}^{\infty} x_k$ with $x_k \in X_k$. If we denote P_n the projections from X to $\sum_{k=1}^{n} X_k$ defined by $P_n(x) = \sum_{k=1}^{n} x_k$ then $\sup_n \|P_n\| = 1$. Notice that those projections are actually 1-Lipschitz from M_p to $M_{p,n}$ where $M_{p,n} = (\sum_{k=1}^{n} X_k, d_p)$.

Fix $x \neq y \in M_p$ and $\varepsilon > 0$. We can write $x = \sum_{k=1}^{\infty} x_k$, $y = \sum_{k=1}^{\infty} y_k$ with $x_k, y_k \in X_k$ for every k. Now fix $N \in \mathbb{N}$ such that $d_p(x, P_N(x)) < \varepsilon$ and $d_p(y, P_N(y)) < \varepsilon$. Since each X_k is of finite dimension, the space $(\sum_{k=1}^{N} X_k, \|\cdot\|)$ is of finite dimension and thus

by Corollary 2.4.5, $S_0(M_{p,N})$ is 1 norming where $M_{p,N} = (\sum_{k=1}^N X_k, d_p)$. So in particular $\mathrm{lip}_0(M_{p,N})$ is 1-norming. Thus, according to Proposition 2.1.5, $\mathrm{lip}_0(M_p)$ is 1-norming and so $\mathcal{F}(M_p)$ has the Schur property by Proposition 3.1.2. □

We finish here by proving our last result about the (MAP). We keep the same notation as in Theorem 3.4.2.

Proposition 3.4.3. *Let $p \in (0,1)$ and X be a p-Banach space which admits an FDD with decomposition constant K. Then $\mathcal{F}(X, \|\cdot\|^p)$ has the K-(BAP). In particular, if X admits a monotne FDD then $\mathcal{F}(X, \|\cdot\|^p)$ has the (MAP).*

Proof. We still denote $(M_p, d_p) = (X, \|\cdot\|^p)$. Let $\mu_1, ..., \mu_n \in \mathcal{F}(M_p)$ and $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ and $\nu_1, ..., \nu_n \in \mathcal{F}(M_{p,N})$ (where $M_{p,N} = (\sum_{k=1}^N X_k, d_p)$) such that $\|\mu_k - \nu_k\| \leq \frac{\varepsilon}{K}$. We have seen in the proof of Corollary 3.4.1 that $\mathcal{F}(M_{p,N})$ has the (MAP). Thus, there exists $T : \mathcal{F}(M_{p,N}) \to \mathcal{F}(M_{p,N})$ a finite rank operator such that $\|T\| \leq 1$ and $\|T\nu_k - \nu_k\| \leq \varepsilon$ for every k. Since $P_N : M_p \to M_{p,N}$ is a K-Lipschitz retraction, the linearisation $\hat{P_N} : \mathcal{F}(M_p) \to \mathcal{F}(M_{p,N})$ is projection of norm at most K. This leads us to consider the operator $\hat{P}_n \circ T : \mathcal{F}(M_p) \to \mathcal{F}(M_p)$ for which direct computations show that it does the work. Indeed $\hat{P}_n \circ T$ is of finite rank, $\|\hat{P}_n \circ T\| \leq K$ and for every k :

$$\begin{aligned}
\|\hat{P}_n \circ T\mu_k - \mu_k\| &\leq \|\hat{P}_n \circ T\mu_k - \hat{P}_n \circ T\nu_k\| + \|\hat{P}_n \circ T\nu_k - \nu_k\| + \|\mu_k - \nu_k\| \\
&\leq \|\hat{P}_n \circ T\|\|\mu_k - \nu_k\| + \|T\nu_k - \nu_k\| + \varepsilon \\
&\leq 3\varepsilon.
\end{aligned}$$

□

3.5 Perspectives

In Proposition 3.1.2 we stated that if $\mathrm{lip}_0(M)$ is 1-norming then the space $\mathcal{F}(M)$ has the Schur property. It is then natural to try to relax the assumption. For instance, we ask the following question :

Question 3.5.1. Let M be a pointed metric space such that $\mathrm{lip}_0(M)$ is C-norming for some $C > 1$. Does $\mathcal{F}(M)$ have the Schur property ?

A positive answer to Question 2.5.1 ("$\mathrm{lip}_0(M)$ C-norming $\Longrightarrow$ $\mathrm{lip}_0(M)$ 1-norming ?") would obviously imply a positive answer to Question 3.5.1. Furthermore, it would be very nice to obtain a full characterization of free spaces having the Schur property.

Question 3.5.2. Is there a characterization of metric spaces such that their free spaces have the Schur property ?

Now notice that we can deduce Proposition 3.2.5 ("$\mathcal{F}(M)$ has the 1-Schur property") as a direct consequence of Theorem 3.3.1 ("$\mathcal{F}(M)$ embeds into a ℓ_1-sum of finite dimensional spaces") under an additional assumption : $\mathcal{F}(M)$ has the (MAP). However, it seems that most of the examples of proper (in particular compact) metric spaces that we can find in the literature are as follows : Whenever $S_0(M)$ separates points uniformly, $\mathcal{F}(M)$ has the (MAP). So we wonder :

Question 3.5.3. Let M be proper pointed metric space such that $S_0(M)$ separates points uniformly. Then does $\mathcal{F}(M)$ have the (MAP) ? In particular, if M is proper and if ω is a nontrivial gauge, does $\mathcal{F}(M, \omega \circ d)$ have the (MAP) ?

In Section 3.4, we proved (Proposition 3.4.2) that Lipschitz free spaces, over some metric spaces originating from p-Banach spaces, have the Schur property. It is then natural to wonder if we can extend this result to a larger class of metric spaces. Surprisingly, it is really easy to see that we cannot extend this result to every metric space originating from a p-Banach space. Indeed, consider the metric space M originating from $L_p[0,1]$. Then the map $\varphi\colon t \in [0,1] \mapsto \mathbb{1}_{[0,t]} \in M$ is a nonlinear isometry. Therefore there is a linear and isometric embedding of $\mathcal{F}([0,1]) \equiv L_1([0,1])$ into $\mathcal{F}(M)$. Consequently, since the Schur property is stable under passing to subspaces and L_1 does not have it, $\mathcal{F}(M)$ does not have the Schur property. Furthermore, using the same ideas one can show that $\mathcal{F}([0,1]^2)$ linearly embeds into $\mathcal{F}(M)$. Thus, using the fact that $\mathcal{F}([0,1]^2)$ does not embed into L_1 (see [NS07]), we get that $\mathcal{F}(M)$ does not embed into L_1. A major difference between L_p and the p-Banach spaces studied in Section 2.4 is that L_p has trivial dual. In particular, the dual does not separate points of L_p. This suggests the following question.

Question 3.5.4. Consider $(X, \|\cdot\|)$ a p-Banach space whose dual X^* separates points. Then does $\mathcal{F}(X, \|\cdot\|^p)$ have the Schur property ?

Chapter 4

Extremal structure of Lipschitz free spaces

In the present chapter, we focus on the extremal structure of $\mathcal{F}(M)$. For the convenience of the reader, we shall first introduce the different concepts involved in this chapter.

Preliminaries

We start by defining some distinguished families of points (we refer to [Bou83] for background on these concepts). We recall that for a subset C of a Banach space X, a *slice of* C is a set of the following form :

$$S(C, f, \alpha) := \{x \in C : f(x) > \sup f(C) - \alpha\},\ f \in X^* \setminus \{0\}, \alpha > 0.$$

If $C = B_X$, we simply write $S(f, \alpha)$ instead of $S(C, f, \alpha)$.

Definition 4.0.1. Let X be a Banach space and let C be a subset set of X.

- **Extreme point.** A point $x \in C$ is called an *extreme point* of C if $x \notin \operatorname{conv}(C \setminus \{x\})$. The set of all extreme points of C is denoted by $\operatorname{ext}(C)$.
- **Preserved extreme point.** A point $x \in C$ is called a *preserved extreme point* of C if x is an extreme point of the weak*-closure of C in X^{**}. In particular if $C = B_X$, $x \in B_X$ is a preserved extreme point of B_X whenever it is an extreme point of $B_{X^{**}}$.
- **Denting point.** A point $x \in C$ is said to be a *denting point* of C if C admits slices containing x of arbitrarily small diameter. Thus, the slices of C containing x form a neighbourhood basis of x in the relative norm topology
- **Exposed point.** A point $x \in C$ is called an *exposed point* of C if there is $f \in X^*$ such that $f(x) > f(z)$ for every $z \in C \setminus \{x\}$.
  The set of all exposed points of C is denoted by $\exp(C)$.
- **Strongly exposed point.** A point $x \in C$ is called a *strongly exposed point* of C if there exists $f \in X^*$ such that $f(x) = \sup_{z \in C} f(z)$ and for all sequences $(x_n)_{n=1}^{\infty} \subset C$: $\big[\lim_{n\to\infty} f(x_n) = \sup_{z\in C} f(z)\big] \implies \big[\lim_{n\to\infty} \|x_n - x\| = 0\big]$. Equivalently, the family of slices $\{S(C, f, \alpha) : \alpha > 0\}$ is a neighbourhood basis of x in the relative norm topology.
  The set of all strongly exposed points of C is denoted by $\operatorname{strexp}(B_X)$.

It is known that the following chain of implications holds for every Banach space X and every $C \subset X$:

$$\begin{tikzcd}
\text{Strongly exposed} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Exposed} \arrow[dr, Rightarrow] & \\
\text{Denting} \arrow[r, Rightarrow] & \text{Preserved extreme} \arrow[r, Rightarrow] & \text{Extreme}
\end{tikzcd}$$

Next, we are going to give a few well-known properties of those families of distinguished points. We will use them freely during the current chapter. The first two results concern the set of extreme points.

Lemma 4.0.2 (**Choquet**, see Lemma 3.40 in [FHH+01]).

1. *Let C be a weakly compact convex set in a Banach space X. For every $x \in \operatorname{ext}(C)$, the slices of C containing x form a neighbourhood base of x in the relative weak topology.*
2. *Let C be a weak* compact convex set in a dual Banach space X^*. For every $x \in \operatorname{ext}(C)$, the weak* slices of C containing x form a neighbourhood base of x in the relative weak* topology.*

Theorem 4.0.3 (**Milman**, see Theorem 3.41 in [FHH+01]).

1. *Let C be a weakly compact convex set in a Banach space X. If $B \subset C$ is such that $\overline{\operatorname{conv}}(B) = C$, then $\operatorname{ext}(C) \subset \overline{B}^w$.*
2. *Let C be a weak* compact convex set in a dual Banach space X^*. If $B \subset C$ is such that $\overline{\operatorname{conv}}^{w^*}(B) = C$, then $\operatorname{ext}(C) \subset \overline{B}^{w^*}$.*

We will need the following characterisation of preserved extreme points which appeared in [GMZ14].

Proposition 4.0.4 (Proposition 9.1 in [GMZ14]). *Let X be a Banach space, C be a closed bounded convex subset of X and $x \in C$. Then, the following are equivalent :*

(i) *x is a preserved extreme point of C.*

(ii) *The slices of C containing x form a relative neighbourhood basis of x for the weak topology.*

(iii) *For every sequences $(y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ in C such that $\frac{y_n+z_n}{2} \xrightarrow{\|\cdot\|} x$ we have that $y_n, z_n \xrightarrow{w} x$.*

It is easy to check that conditions above are also equivalent to the following :

(iii') For every $\lambda \in (0,1)$ and sequences $(y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ in C such that $\lambda y_n + (1-\lambda) z_n \xrightarrow{\|\cdot\|} x$ we have that $y_n, z_n \xrightarrow{w} x$.

We shall now finish by giving some more notation. Given $x, y \in M$, we recall that $[x, y]$ denotes the *metric segment* between x and y, that is,

$$[x, y] = \{z \in M : d(x, z) + d(z, y) = d(x, y)\}.$$

We will need for every $x, y \in M$, $x \neq y$, the function

$$f_{xy}(t) := \frac{d(x,y)}{2} \frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}. \tag{4.1}$$

The properties collected in the next lemma have been proved in [IKW09]. They turn f_{xy} a useful tool for studying the geometry of $B_{\mathcal{F}(M)}$.

Lemma 4.0.5. *Let $x, y \in M$ with $x \neq y$. We have :*

1. $\dfrac{f_{xy}(u) - f_{xy}(v)}{d(u,v)} \leq \dfrac{d(x,y)}{\max\{d(x,u) + d(u,y), d(x,v) + d(v,y)\}}$ *for all $u \neq v \in M$.*
2. *f_{xy} is Lipschitz and $\|f_{xy}\|_L \leq 1$.*
3. *Let $u \neq v \in M$ and $\varepsilon > 0$ be such that $\dfrac{f_{xy}(u) - f_{xy}(v)}{d(u,v)} > 1 - \varepsilon$. Then,*
$$(1-\varepsilon)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} < d(x,y).$$
4. *If $u \neq v \in M$ and $\dfrac{f_{xy}(u) - f_{xy}(v)}{d(u,v)} = 1$, then $u, v \in [x,y]$.*

The main questions

The study of the extremal structure of $\mathcal{F}(M)$ has probably started in [Wea99], where it is proved that preserved extreme points of the unit ball $B_{\mathcal{F}(M)}$ are always molecules. Recently Aliaga and Guirao pushed further this work in [AG17]. In particular, answering a question of Weaver, they showed in the compact case that m_{xy} is an extreme point if and only it is a preserved extreme point if and only if $[x,y] = \{x,y\}$. They also give a metric characterisation of preserved extreme points in full generality, which we prove also here by a different argument. More results in the same line appeared in [GLPZ17b], where a metric characterisation of strongly exposed points is given. However, the two main questions in this domain remain open :

a) If $\mu \in \operatorname{ext}(B_{\mathcal{F}(M)})$, is μ necessarily of the form $\mu = m_{xy}$ for some $x \neq y \in M$?

b) If the metric segment satisfies $[x,y] = \{x,y\}$, is m_{xy} an extreme point of $B_{\mathcal{F}(M)}$?

The goal of the present chapter is to continue the effort in exploring the extremal structure of $\mathcal{F}(M)$ and provide affirmative answers to both previous questions a) and b) in some particular cases. For instance, we prove that for the following chain of implications

$$\text{strongly exposed} \overset{(1)}{\Longrightarrow} \text{denting} \overset{(2)}{\Longrightarrow} \text{preserved extreme} \overset{(3)}{\Longrightarrow} \text{extreme},$$

the converse of (2) holds true in general but the converse of (1) and (3) are both false. However, some of the previous implications are equivalences in some special classes of metric spaces. The most notable among them is the case when $\mathcal{F}(M)$ admits a natural predual.

The chapter is organised as follows. In Section 4.1 we prove that every preserved extreme point of $B_{\mathcal{F}(M)}$ is also a denting point in full generality and we provide a different proof of the metric characterisation of preserved extreme points given in [AG17].

In Section 4.2, we study the extremal structure of spaces admitting a natural predual. In particular, we show under an additional assumption that the set of extreme points coincides with the set of strongly exposed points. Then in Section 4.3 we focus on the case of uniformly discrete and bounded metric spaces. Under this assumption, question b) has an affirmative answer, the implication (1) admits a converse, and the question a) has also an affirmative answer if moreover $\mathcal{F}(M)$ admits a natural predual. Finally, in Section 4.4 we show that the converse of (3) holds for certain proper metric spaces since the norm of $\mathcal{F}(M)$ turns out to be weak* asymptotically uniformly convex.

This chapter is based on a joint work with Luis García-Lirola, Antonín Procházka and Abraham Rueda Zoca ([GPPR17]).

4.1 General results

The next result of Weaver (see [Wea99, Corollary 2.5.4]) is a starting point for many of our results (and also the main motivation for Question a)). In [Wea99], it follows from Theorem 2.5.3. We include a streamlined proof for reader's convenience which also depends on [Wea99, Theorem 2.5.3]. However, we believe that this approach clearly explains why the preserved extreme points are molecules from one basic principle which is that $\overline{V}^w \subset V \cup \{0\}$.

Corollary 4.1.1. *Let M be a complete pointed metric space and let μ be a preserved extreme point of $B_{\mathcal{F}(M)}$. Then $\mu = m_{xy}$ for some $x \neq y \in M$.*

Proof. Notice that $\overline{\mathrm{conv}}(V) = B_{\mathcal{F}(M)}$ and $\overline{B_{\mathcal{F}(M)}}^{w^*} = B_{\mathrm{Lip}_0(M)^*}$. Thus,

$$\overline{\mathrm{conv}}^{w^*}(V) = B_{\mathrm{Lip}_0(M)^*}.$$

By Milman's theorem $\mathrm{ext}(B_{\mathrm{Lip}_0(M)^*}) \subset \overline{V}^{w^*}$. Finally we get that

$$\mathcal{F}(M) \cap \mathrm{ext}(B_{\mathrm{Lip}_0(M)^*}) \subset \overline{V}^w.$$

So Proposition 1.3.3 yields $\mathcal{F}(M) \cap \mathrm{ext}(B_{\mathrm{Lip}_0(M)^*}) \subset V$. □

As it is emphasized by the previous result, molecules play a significant role in the extremal structure of free spaces. Moreover, we have the next almost obvious necessary condition for a molecule to be an extreme point of $B_{\mathcal{F}(M)}$.

Proposition 4.1.2. *Let M be a pointed metric space. Let $x \neq y \in M$ be such that m_{xy} is an extreme point of $B_{\mathcal{F}(M)}$. Then, $[x, y] = \{x, y\}$.*

Proof. Indeed, assume that there exists $z \in M \setminus \{x, y\}$ such that $z \in [x, y]$. By definition we have $d(x, y) = d(x, z) + d(y, z)$. We thus trivially deduce that m_{xy} cannot be an extreme point from the following equality :

$$\frac{\delta(x) - \delta(y)}{d(x, y)} = \frac{d(x, z)}{d(x, y)} \frac{\delta(x) - \delta(z)}{d(x, z)} + \frac{d(y, z)}{d(x, y)} \frac{\delta(y) - \delta(z)}{d(y, z)}.$$

□

Our next goal is to show that every preserved extreme point of $B_{\mathcal{F}(M)}$ is also a denting point. In order to prove this result, we need the following variation of the Asplund–Bourgain–Namioka superlemma (see Theorem 3.4.1 in [Bou83] for instance).

Lemma 4.1.3. *Let A, B be bounded closed convex subsets of a Banach space X and let $\varepsilon > 0$. Assume that $\operatorname{diam}(A) < \varepsilon$ and that there is $x_0 \in A \setminus B$ which is a preserved extreme point of $\overline{\operatorname{conv}}(A \cup B)$. Then there is a slice of $\overline{\operatorname{conv}}(A \cup B)$ containing x_0 which is of diameter less than ε.*

Proof. For each $r \in [0, 1]$ let

$$C_r = \{x \in X : x = (1 - \lambda)y + \lambda z, y \in A, z \in B, \lambda \in [r, 1]\}.$$

The proof of the superlemma says that there is r so that $\operatorname{diam}(\overline{\operatorname{conv}}(A \cup B) \setminus \overline{C_r}) < \varepsilon$. We will show that $x_0 \notin \overline{C_r}$. Thus, any slice separating x_0 from $\overline{C_r}$ will do the work. To this end, assume that there exist sequences $(y_n)_{n=1}^{\infty} \subset A$, $(z_n)_{n=1}^{\infty} \subset B$ and $\lambda_n \subset [r, 1]$ such that $x_0 = \lim_n (1 - \lambda_n)y_n + \lambda_n z_n$. By extracting a subsequence, we may assume that $(\lambda_n)_{n=1}^{\infty}$ converges to some $\lambda \in [r, 1]$. Note that then $x_0 = \lim_n (1 - \lambda)y_n + \lambda z_n$. Since x_0 is a preserved extreme point, this implies that $(z_n)_{n=1}^{\infty}$ converges weakly to x_0 by Proposition 4.0.4. That is impossible since $x_0 \notin B$ and B is weakly closed as being convex and closed. □

Theorem 4.1.4. *Let M be a pointed metric space. Then every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point.*

Proof. Let μ be a preserved extreme point of $B_{\mathcal{F}(M)}$, which must be an element of V. Denote by $\mathcal{S}$ the set of weak-open slices of $B_{\mathcal{F}(M)}$ containing μ. Consider the order $S_1 \leq S_2$ if $S_2 \subset S_1$ for $S_1, S_2 \in \mathcal{S}$. Using (ii) of Proposition 4.0.4, every finite intersection of elements of $\mathcal{S}$ contains an element of $\mathcal{S}$ and so $(\mathcal{S}, \leq)$ is a directed set. Assume that μ is not a denting point. Then, there is $\varepsilon > 0$ so that $\operatorname{diam}(S) > 2\varepsilon$ for every $S \in \mathcal{S}$.

We distinguish two cases. Assume first that for every slice $S \in \mathcal{S}$ there is μ_S in $(V \cap S) \setminus B(\mu, \varepsilon/4)$. Then (μ_S) is a net in V which converges weakly to μ. By Lemma 1.3.2, it also converges in norm, which is impossible. Thus, there is a slice S of $B_{\mathcal{F}(M)}$ such that $\operatorname{diam}(V \cap S) < \varepsilon/2$. Note that

$$B_{\mathcal{F}(M)} = \overline{\operatorname{conv}}(V) = \overline{\operatorname{conv}}(\overline{\operatorname{conv}}(V \cap S) \cup \overline{\operatorname{conv}}(V \setminus S))$$

and so the hypotheses of Lemma 4.1.3 are satisfied for $A = \overline{\operatorname{conv}}(V \cap S)$, $B = \overline{\operatorname{conv}}(V \setminus S)$, and $\mu \in A \setminus B$ (taking the closed convex hull does not change the diameter). Then there is a slice of $B_{\mathcal{F}(M)}$ containing μ of diameter less than ε, a contradiction. □

In [GLPZ17b, Theorem 3.3], the authors characterized the Lipschitz free spaces having the Daugavet property. More precisely, they proved that $\mathcal{F}(M)$ has the Daugavet property whenever M is a length space. For convenience, we recall the definition of a length space here.

Definition 4.1.5. A metric space M is said to be a length space if, for every pair of points $x, y \in M$, the distance $d(x, y)$ is equal to the infimum of the length of rectifiable curves joining them. If M is complete metric space, this condition is readily seen to be equivalent to the following one : for every $x, y \in M$ and for every $\delta > 0$,

$$B\Big(x, (1 + \delta)\frac{d(x, y)}{2}\Big) \cap B\Big(y, (1 + \delta)\frac{d(x, y)}{2}\Big) \neq \emptyset.$$

From Theorem 4.1.4 and [GLPZ17b, Theorem 3.3], we can show the following result (already proved in [GLPZ17b]).

Corollary 4.1.6. *Let M be a length space. Then $B_{\mathcal{F}(M)}$ does not have any preserved extreme point.*

Proof. If M is a length space then the space $\mathcal{F}(M)$ has the Daugavet property [IKW07]. In particular, every slice of $B_{\mathcal{F}(M)}$ has diameter two. Thus, $B_{\mathcal{F}(M)}$ does not have any denting point. □

In [AG17, Theorem 4.1], Aliaga and Guirao characterized metrically the preserved extreme points of $B_{\mathcal{F}(M)}$. We devote the last part of the section proving this metric characterization by a different argument. In fact, we characterize metrically the denting points which is equivalent to characterize the preserved extreme points (according to Theorem 4.1.4).

Theorem 4.1.7. *Let M be a pointed metric space and $x, y \in M$. The following are equivalent :*

(i) The molecule m_{xy} is a denting point of $B_{\mathcal{F}(M)}$.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that every $z \in M$ satisfies

$$(1-\delta)(d(x,z)+d(z,y)) < d(x,y) \Longrightarrow \min\{d(x,z), d(y,z)\} < \varepsilon.$$

Proof of (i)⇒(ii). In fact, we are going to show that the negation of (ii) implies that m_{xy} is not a preserved extreme point. Since denting points are trivially preserved extreme points, this will show at once that m_{xy} is not denting.

So let us fix $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $z_n \in M$ such that

$$\left(1-\frac{1}{n}\right)(d(x,z_n)+d(z_n,y)) < d(x,y)$$

but $\min\{d(x,z_n), d(y,z_n)\} \geq \varepsilon$. Let μ be a weak* cluster point of $\{z_n\}$ ($\{z_n\}$ is clearly bounded). By the lower-semi-continuity of the norm we have

$$\|\delta(x)-\mu\| + \|\mu-\delta(y)\| = d(x,y).$$

If $\mu \in \{\delta(x), \delta(y)\}$, say $\mu = \delta(x)$, then by Lemma 1.3.2 we get that $z_n \to x$ in (M,d) which is a contradiction.

Thus $\mu \notin \{\delta(x), \delta(y)\}$. Then

$$\frac{\delta(x)-\delta(y)}{\|\delta(x)-\delta(y)\|} = \frac{\|\delta(x)-\mu\|}{\|\delta(x)-\delta(y)\|}\frac{\delta(x)-\mu}{\|\delta(x)-\mu\|} + \frac{\|\mu-\delta(y)\|}{\|\delta(x)-\delta(y)\|}\frac{\mu-\delta(y)}{\|\mu-\delta(y)\|}.$$

Thus μ is a non-trivial convex combination and so it is not preserved extreme which concludes the proof of (i)⇒(ii). □

For the proof of the other implication, we need a couple of lemmata. The first of them shows that the diameter of the slices of the unit ball can be controlled by the diameter of the slices of a subset of the ball that is norming for the dual.

Lemma 4.1.8. *Let X be a Banach space and let $V \subset S_X$ be such that $B_X = \overline{\operatorname{conv}}(V)$. Let $f \in B_{X^*}$ and $0 < \alpha, \varepsilon < 1$. Then*

$$\operatorname{diam}(S(f, \varepsilon\alpha)) \leq 2 \operatorname{diam}(S(f, \alpha) \cap V) + 4\varepsilon.$$

Proof. Fix a point $x_0 \in S(f, \alpha) \cap V$. It suffices to show that $\|x - x_0\| < \operatorname{diam}(S(f, \alpha) \cap V) + 2\varepsilon$ for every $x \in S(f, \varepsilon\alpha) \cap \operatorname{conv}(V)$. To this end, let $x \in B_X$ be such that $f(x) > 1 - \varepsilon\alpha$, and $x = \sum_{i=1}^n \lambda_i x_i$, with $x_i \in V$, $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$ for all $1 \leq i \leq n$. Define

$$G = \{i \in \{1, \ldots, n\} : f(x_i) > 1 - \alpha\}$$

and $B = \{1, \ldots, n\} \setminus G$. We have

$$\begin{aligned} 1 - \varepsilon\alpha < f(x) &= \sum_{i \in G} \lambda_i f(x_i) + \sum_{i \in B} \lambda_i f(x_i) \\ &\leq \sum_{i \in G} \lambda_i + (1 - \alpha) \sum_{i \in B} \lambda_i = 1 - \alpha \sum_{i \in B} \lambda_i, \end{aligned}$$

which yields that $\sum_{i \in B} \lambda_i < \varepsilon$. Now,

$$\|x - x_0\| \leq \sum_{i \in G} \lambda_i \|x_i - x_0\| + \sum_{i \in B} \lambda_i \|x_i - x_0\| \leq \operatorname{diam}(S(f, \alpha) \cap V) + 2\varepsilon.$$

□

We will also need the following technical lemma.

Lemma 4.1.9. *Let $x, y \in M$, $x \neq y$, such that $d(x, y) = 1$. Then, for every $0 < \varepsilon < 1/4$ and $0 < \tau < 1$ there is a function $f \in \operatorname{Lip}_0(M)$ such that $\|f\|_L = 1$, $\langle f, m_{xy} \rangle > 1 - 4\varepsilon\tau$ and satisfying that for every $u, v \in M$, $u \neq v$, if $u, v \in B(x, \varepsilon)$ or $u, v \in B(y, \varepsilon)$, then $\langle f, m_{uv} \rangle \leq 1 - \tau$.*

Proof. Define $f \colon B(x, \varepsilon) \cup B(y, \varepsilon) \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{1}{1+4\varepsilon\tau}(\tau + (1 - \tau)d(y, t)) & \text{if } t \in B(x, \varepsilon), \\ \frac{1}{1+4\varepsilon\tau}(1 - \tau)d(y, t) & \text{if } t \in B(y, \varepsilon). \end{cases}$$

Note that

$$\langle f, m_{xy} \rangle = f(x) - f(y) = \frac{1}{1 + 4\varepsilon\tau} > 1 - 4\varepsilon\tau.$$

Moreover, note that if $u, v \in B(x, \varepsilon)$ or $u, v \in B(y, \varepsilon)$ then $\langle f, m_{uv} \rangle \leq \frac{1-\tau}{1+4\varepsilon\tau} \leq 1 - \tau$, so the last condition in the statement is satisfied. Now we compute the Lipschitz norm of f. It remains to compute $\langle f, m_{uv} \rangle$ with $u \in B(x, \varepsilon)$ and $v \in B(y, \varepsilon)$. In that case we have

$$\begin{aligned} |\langle f, m_{uv} \rangle| &= \frac{|\tau + (1 - \tau)(d(u, y) - d(v, y))|}{(1 + 4\varepsilon\tau)d(u, v)} \leq \frac{\tau + (1 - \tau)d(u, v)}{(1 + 4\varepsilon\tau)d(u, v)} \\ &\leq \frac{1}{1 + 4\varepsilon\tau} \left(\frac{\tau}{1 - 2\varepsilon} + 1 - \tau \right) \leq \frac{\tau(1 + 4\varepsilon) + 1 - \tau}{1 + 4\varepsilon\tau} = 1 \end{aligned}$$

where we are using that $(1 - 2\varepsilon)^{-1} \leq 1 + 4\varepsilon$ since $\varepsilon < 1/4$. This shows that $\|f\|_L \leq 1$. Next, find an extension of f with the same norm. Finally, replace f with the function $t \mapsto f(t) - f(0)$. □

Proof of (ii)⇒(i) of Theorem 4.1.7. Now, assume that (ii) holds. We can assume that $d(x,y)=1$. Fix $0<\varepsilon<1/4$. We will find a slice of $B_{\mathcal{F}(M)}$ containing m_{xy} of diameter smaller than 32ε. Let $\delta>0$ be given by property (ii). Clearly we may assume that $\delta<1$. Let f be the function given by Lemma 4.1.9 with $\tau=\delta/2$. Define

$$h(t):=\frac{f_{xy}(t)+f(t)}{2}.$$

It is clear that $\|h\|_L\leq 1$. Moreover, note that

$$\langle h,m_{xy}\rangle=\frac{\langle f_{xy},m_{xy}\rangle+\langle f,m_{xy}\rangle}{2}>1-2\varepsilon\tau=1-\varepsilon\delta.$$

Take $\alpha=\delta/4$ and consider the slice $S=S(h,\alpha)$. Note that $m_{xy}\in S(h,4\varepsilon\alpha)$. We will show that $\operatorname{diam}(S\cap V)\leq 8\varepsilon$ and as a consequence of Lemma 4.1.8 we will get that $\operatorname{diam}S(h,\alpha)\leq 32\varepsilon$. So let $u,v\in M$ be such that $m_{uv}\in S$. First, note that $\langle f_{xy},m_{uv}\rangle>1-\delta$, since otherwise we would have

$$\langle h,m_{uv}\rangle=\frac{1}{2}(\langle f_{xy},m_{uv}\rangle+\langle f,m_{uv}\rangle)\leq\frac{1}{2}(1-\delta)+\frac{1}{2}=1-\frac{\delta}{2}<1-\alpha.$$

Thus, from the property 3. of the function f_{xy} in Lemma 4.0.5 and the hypothesis (ii) we have that

$$\min\{d(x,u),d(u,y)\}<\varepsilon\quad\text{and}\quad\min\{d(x,v),d(y,v)\}<\varepsilon.$$

On the other hand,

$$1-\alpha<\langle h,m_{uv}\rangle\leq\frac{1}{2}+\frac{1}{2}\langle f,m_{uv}\rangle$$

and so $\langle f,m_{uv}\rangle>1-2\alpha=1-\frac{\delta}{2}=1-\tau$. Thus, we have that u and v do not belong simultaneously to neither $B(x,\varepsilon)$ nor $B(y,\varepsilon)$. If $d(x,v)<\varepsilon$ and $d(y,u)<\varepsilon$, then it is easy to check that $\langle f_{xy},m_{uv}\rangle\leq 0$. So necessarily $d(x,u)<\varepsilon$ and $d(y,v)<\varepsilon$. Now, use the estimate

$$\begin{aligned}\|m_{xy}-m_{uv}\|&=\frac{\|d(u,v)(\delta(x)-\delta(y))-d(x,y)(\delta(u)-\delta(v))\|}{d(x,y)d(u,v)}\\&\leq\frac{\|(\delta(x)-\delta(y))-(\delta(u)-\delta(v))\|}{d(x,y)}+\frac{|d(u,v)-d(x,y)|\|\delta(u)-\delta(v)\|}{d(x,y)d(u,v)}\\&\leq 2\frac{d(x,u)+d(y,v)}{d(x,y)}\leq 4\varepsilon.\end{aligned}$$

Therefore, $\operatorname{diam}(S\cap V)\leq 8\varepsilon$. □

Finally, we would like to point out that there is also a metric characterization of strongly exposed points of $B_{\mathcal{F}(M)}$ which is proved in [GLPZ17b]. We state it here for future reference.

Theorem 4.1.10 (García-Lirola–Procházka–Rueda-Zoca).
Let M be a pointed metric space and let $x\neq y\in M$. Then, the following assertions are equivalent.

(i) $m_{x,y}=\frac{\delta(x)-\delta(y)}{d(x,y)}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$.

(ii) *There is* $f \in \mathrm{Lip}_0(M)$ *peaking at* (x,y)*, that is if* $f(x)-f(y)=d(x,y)$ *and for every open set* U *of* $M^2 \setminus \{(x,x) : x \in M\}$ *containing* (x,y) *and* (y,x)*, there exists* $\delta > 0$ *such that :*

$$\big[(z,t) \notin U\big] \implies \big[|f(z)-f(t)| \leq (1-\delta)d(z,t)\big].$$

(iii) *The pair* (x,y) *does not have the property (Z), that is for every* $\varepsilon > 0$ *there is* $z \in M \setminus \{x,y\}$ *such that* $d(x,z)+d(z,y) \leq d(x,y)+\varepsilon \min\{d(x,z),d(y,z)\}$.

4.2 Extremal structure for spaces with natural preduals

We are going to focus now on extreme points in free spaces that admit a natural predual. Assuming moreover the predual to be a subspace of $S_0(M)$ we get an affirmative answer to one of our main problems. Note that this is an extension of Corollary 3.3.6 in [Wea99], where the same result is obtained under the assumption that M is compact.

Proposition 4.2.1. *Let* M *be a pointed metric space. Assume that there is a subspace* X *of* $S_0(M)$ *which is a natural predual of* $\mathcal{F}(M)$*. Then*

$$\mathrm{ext}(B_{\mathcal{F}(M)}) \subset V.$$

Proof. By the separation theorem we have that $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}^{w^*}(V)$. Thus, according to Milman's theorem (Theorem 4.0.3), we have $\mathrm{ext}(B_{\mathcal{F}(M)}) \subset \overline{V}^{w^*}$. So let us consider $\gamma \in \mathrm{ext}(B_{\mathcal{F}(M)})$. Take a net (m_{x_α,y_α}) in V which weak* converges to γ. Since $X \subset S_0(M)$, we may assume that (x_α) and (y_α) are both contained in $B(0,r)$ for some $r > 0$. Indeed, since $\|\gamma\| = 1$ there is $f \in X$ such that $\langle f,\gamma\rangle > \|\gamma\|/2 = 1/2$. Since $f \in S_0(M)$, there is $r > 0$ such that $|f(z_1)-f(z_2)| \leq \frac{1}{2}d(z_1,z_2)$ whenever z_1 or z_2 does not belong to $B(0,r)$. Since $\langle f, m_{x_\alpha,y_\alpha}\rangle$ tends to $\langle f,\gamma\rangle > \frac{1}{2}$, (x_α) and (y_α) are eventually in $B(0,r)$. By weak* compactness of $\delta(B(0,r))$, we may assume (up to extracting subnets) that $(\delta(x_\alpha))$ and $(\delta(y_\alpha))$ converge to some $\delta(x)$ and $\delta(y)$ respectively.

Next, we claim that we may also assume that $(d(x_\alpha,y_\alpha))$ converges to $C > 0$. Indeed, since M is bounded, we may assume up to extracting a further subnet that $(d(x_\alpha,y_\alpha))$ converges to $C \geq 0$. By the assumption, there is $f \in X$ such that $\langle f,\gamma\rangle > \|\gamma\|/2 = 1/2$. Since $f \in \mathrm{lip}_0(M)$, there exists $\delta > 0$ such that whenever $z_1, z_2 \in M$ satisfy $d(z_1,z_2) \leq \delta$ then we have $|f(z_1)-f(z_2)| \leq \frac{1}{2}d(z_1,z_2)$. Since

$$\lim_\alpha \langle f, m_{x_\alpha,y_\alpha}\rangle = \langle f,\gamma\rangle > \frac{1}{2},$$

there is α_0 such that $\langle f, m_{x_\alpha,y_\alpha}\rangle > 1/2$ for every $\alpha > \alpha_0$. Thus $d(x_\alpha,y_\alpha) > \delta$ for $\alpha > \alpha_0$, which implies that $C \geq \delta > 0$. Summarizing, we have a net (m_{x_α,y_α}) which weak* converges to $\frac{\delta(x)-\delta(y)}{C}$. So, by uniqueness of the limit, $\gamma = \frac{\delta(x)-\delta(y)}{C}$. Since $\gamma \in \mathrm{ext}(B_{\mathcal{F}(M)}) \subset S_{\mathcal{F}(M)}$, we get that $C = d(x,y)$ and so $\gamma = m_{xy}$. □

A weaker version of the following proposition appears in the preprint [AG17] for compact metric spaces. Our approach, which is independent of [AG17], also yields a characterisation of exposed points of $B_{\mathcal{F}(M)}$.

Corollary 4.2.2. *Let* M *be a separable pointed metric space. Assume that there is a subspace* X *of* $S_0(M)$ *which is a natural predual of* $\mathcal{F}(M)$*. Then a given* $\mu \in B_{\mathcal{F}(M)}$ *the following are equivalent :*

(i) $\mu \in \operatorname{ext}(B_{\mathcal{F}(M)})$.

(ii) $\mu \in \exp(B_{\mathcal{F}(M)})$.

(iii) *There are* $x, y \in M$, $x \neq y$, *such that* $[x, y] = \{x, y\}$ *and* $\mu = m_{xy}$.

Proof. *(i)*⇒*(iii)* follows from Proposition 4.2.1. Moreover, *(ii)*⇒*(i)* is clear, so it only remains to show *(iii)*⇒*(ii)*. To this end, let $x, y \in M$, $x \neq y$, be so that $[x, y] = \{x, y\}$. Consider

$$A = \{\mu \in B_{\mathcal{F}(M)} : \langle f_{xy}, \mu \rangle = 1\},$$

where f_{xy} stands for the function defined in (4.1). We will show that $A = \{m_{xy}\}$ and so m_{xy} is exposed by f_{xy} in $B_{\mathcal{F}(M)}$. Let $\mu \in \operatorname{ext}(A)$. Since A is an extremal subset of $B_{\mathcal{F}(M)}$, μ is also an extreme point of $B_{\mathcal{F}(M)}$ and so $\mu \in V \cap A$ (Proposition 4.2.1). Recall that if $\langle f_{xy}, m_{u,v} \rangle = 1$ then $u, v \in [x, y]$, therefore $V \cap A = \{m_{xy}\}$. Thus $\operatorname{ext}(A) \subset \{m_{xy}\}$. Finally note that A is a closed convex subset of $B_{\mathcal{F}(M)}$ and so $A = \overline{\operatorname{conv}}(\operatorname{ext}(A)) = \{m_{xy}\}$ since the space $\mathcal{F}(M)$ has the (RNP) as being a separable dual. □

Another setting in which a complete description of extreme points is possible is revealed below (see Section 4.3 for yet another case).

Proposition 4.2.3. *Let* (M, d) *be a complete bounded pointed metric space for which there is a Hausdorff topology* τ *such that* (M, τ) *is compact and* $d\colon (M, \tau)^2 \to \mathbb{R}$ *is l.s.c.. Let* $0 < p < 1$ *and let* (M, d^p) *be the p-snowflake of* M. *Then given* $\mu \in B_{\mathcal{F}(M)}$ *the following are equivalent :*

(i) $\mu \in \operatorname{ext}(B_{\mathcal{F}(M,d^p)})$.

(ii) $\mu \in \operatorname{strexp}(B_{\mathcal{F}(M,d^p)})$.

(iii) *There are* $x, y \in M$, $x \neq y$, *such that* $\mu = m_{xy}$.

Observe that under the hypotheses above it is not necessarily true that $\mathcal{F}(M)$ is a dual space, but $\mathcal{F}(M, d^p)$ already is. Indeed, similarly as in the proof of Corollary 2.3.4, one can use Matoušková's extension theorem to prove that $\operatorname{Lip}_0(M) \cap C(M, \tau)$ separates points uniformly. Now since $\operatorname{Lip}_0(M) \subset \operatorname{lip}_0(M, d^p)$ when M is bounded, one gets that $\operatorname{lip}_0(M, d^p) \cap C(M, \tau)$ separates points uniformly. Now it suffices to apply Proposition 2.2.5 to obtain that $\operatorname{lip}_0(M, d^p) \cap C(M, \tau)$ is a natural predual of $\mathcal{F}(M, d^p)$.

Proof. $(iii) \implies (ii)$. Let us fix $x \neq y \in M$. Since $0 < p < 1$, it is readily seen that $[x, y] = \{x, y\}$. Moreover it is proved in [Wea99, Proposition 2.4.5] that there is a peaking function at (x, y). Thus m_{xy} is a strongly exposed point ([GLPZ17b, Theorem 4.4]). The implication $(ii) \implies (i)$ is obvious. To finish, the implication $(i) \implies (iii)$ follows directly from Proposition 4.2.1 and the fact that $[x, y] = \{x, y\}$ for every $x \neq y \in M$. □

Next, we will show that the extremal structure of a free space has an impact on its isometric preduals. If a metric space M is countable and satisfies the assumptions of Proposition 4.2.1, then $\operatorname{ext}(B_{\mathcal{F}(M)})$ is also countable. Therefore, any isometric predual of $\mathcal{F}(M)$ is isomorphic to a polyhedral space by a theorem of Fonf [Fon78], and so it is saturated with subspaces isomorphic to c_0. This applies for instance in the following case.

Corollary 4.2.4. *Let* M *be a countable proper pointed metric space. Then any isometric predual of* $\mathcal{F}(M)$ *(in particular* $S_0(M)$*) is isomorphic to a polyhedral space.*

4.3 The uniformly discrete case

We have already witnessed that in the class of uniformly discrete and bounded metric spaces, many results about $\mathcal{F}(M)$ become simpler. Yet another example of this principle is the following main result of this section.

Proposition 4.3.1. *Let (M,d) be a bounded uniformly discrete pointed metric space. Then a molecule m_{xy} is an extreme point of $B_{\mathcal{F}(M)}$ if and only if $[x,y]=\{x,y\}$.*

We will need the following observation, perhaps of independent interest : Since a point $x \in B_X$ is extreme if and only if $x \in \operatorname{ext}(B_Y)$ for every 2-dimensional subspace Y of X, the extreme points of $B_{\mathcal{F}(M)}$ are separably determined. Let us be more precise.

Lemma 4.3.2. *Assume that $\mu_0 \in B_{\mathcal{F}(M)}$ is not an extreme point of $B_{\mathcal{F}(M)}$. Then there is a separable subset $N \subset M$ such that $\mu_0 \in \mathcal{F}(N)$ and $\mu_0 \notin \operatorname{ext}(B_{\mathcal{F}(N)})$.*

Proof. Write $\mu_0 = \frac{1}{2}(\mu_1+\mu_2)$, with $\mu_1,\mu_2 \in B_{\mathcal{F}(M)}$. We can find sequences $(\nu_n^i)_{n=1}^\infty$ of finitely supported measures such that $\mu_i = \lim_{n\to\infty} \nu_n^i$ for $i=0,1,2$. Let $N = \{0\} \cup (\cup_{i,n} \operatorname{supp}\{\nu_n^i\})$. Note that the canonical inclusion $\mathcal{F}(N) \hookrightarrow \mathcal{F}(M)$ is an isometry and $\nu_n^i \in \mathcal{F}(N)$ for each n,i. Since $\mathcal{F}(N)$ is complete, it is a closed subspace of $\mathcal{F}(M)$. Thus $\mu_0,\mu_1,\mu_2 \in \mathcal{F}(N)$ and so $\mu_0 \notin \operatorname{ext}(B_{\mathcal{F}(N)})$. □

Proof of Proposition 4.3.1. Let m_{xy} be a molecule in M such that $[x,y]=\{x,y\}$ and assume that $m_{xy} \notin \operatorname{ext}(B_{\mathcal{F}(M)})$. By Lemma 4.3.2, we may assume that M is countable. Write $M=\{x_n : n \geq 0\}$. Let $(e_n)_{n=1}^\infty$ be the unit vector basis of ℓ_1. The map $\delta(x_n) \mapsto e_n$ for $n \geq 1$ defines an isomorphism from $\mathcal{F}(M)$ onto ℓ_1 (see Lemma 2.3.9). Thus $(\delta(x_n))_{n=1}^\infty$ is a Schauder basis for $\mathcal{F}(M)$.

Assume that $m_{xy} = \frac{1}{2}(\mu+\nu)$ for $\mu,\nu \in B_{\mathcal{F}(M)}$ and write $\mu = \sum_{n=1}^\infty a_n \delta(x_n)$. Fix $n \in \mathbb{N}$ such that $x_n \notin \{x,y\}$. Then, there is $\varepsilon_n > 0$ such that

$$(1-\varepsilon_n)\left(d(x,x_n)+d(x_n,y)\right) \geq d(x,y).$$

Let $g_n = f_{xy} + \varepsilon_n \mathbb{1}_{\{x_n\}}$, which is an element of $\operatorname{Lip}_0(M)$ since M is uniformly discrete. We will show that $\|g_n\|_L \leq 1$. To this end, take $u,v \in M$, $u \neq v$. Since $\|f_{xy}\|_L \leq 1$, it is clear that $|\langle g_n, m_{uv}\rangle| \leq 1$ if $u,v \neq x_n$. Thus we may assume $v = x_n$. Therefore 3. in Lemma 4.0.5 yields that $\langle f_{xy}, m_{uv}\rangle \leq 1-\varepsilon_n$ and so $\langle g_n, m_{uv}\rangle \leq 1$. Exchanging the roles of u and v, we get that $\|g_n\|_L \leq 1$. Moreover, note that

$$1 = \langle g_n, m_{xy}\rangle = \frac{1}{2}(\langle g_n,\mu\rangle + \langle g_n,\nu\rangle) \leq 1$$

and so $\langle g_n,\mu\rangle = 1$. Analogously we show that $\langle f_{xy},\mu\rangle = 1$. Thus $a_n = \langle \mathbb{1}_{\{x_n\}},\mu\rangle = 0$. Therefore $\mu = a\delta(x)+b\delta(y)$ for some $a,b \in \mathbb{R}$. Finally, let $f_1(t) := d(t,x)-d(0,x)$ and $f_2(t) := d(t,y)-d(0,x)$. Then $\|f_i\|_L = 1$ and $\langle f_i, m_{xy}\rangle = 1$, so we also have $\langle f_i,\mu\rangle = 1$ for $i=1,2$. It follows from this that $a=-b=\frac{1}{d(x,y)}$, that is, $\mu = m_{xy}$. This implies that m_{xy} is an extreme point of $B_{\mathcal{F}(M)}$. □

Notice that this result stays true even for those unbounded uniformly discrete spaces such that $\delta(M)$ is a Schauder basis for some enumeration. We do not know though whether it extends to all uniformly discrete spaces. Next, we show that preserved extreme points are automatically strongly exposed for uniformly discrete metric spaces.

Proposition 4.3.3. *Let M be a uniformly discrete pointed metric space. Then every preserved extreme point of $B_{\mathcal{F}(M)}$ is also a strongly exposed point.*

Proof. Let $x, y \in M$ such that m_{xy} is a preserved extreme point of $B_{\mathcal{F}(M)}$. Assume that m_{xy} is not strongly exposed. By Theorem 4.1.10, the pair (x, y) enjoys property (Z). That is, for each $n \in \mathbb{N}$ we can find $z_n \in M \setminus \{x, y\}$ such that

$$d(x, z_n) + d(y, z_n) \leq d(x, y) + \frac{1}{n} \min\{d(x, z_n), d(y, z_n)\}.$$

Thus,

$$(1 - 1/n)(d(x, z_n) + d(y, z_n)) \leq d(x, y)$$

so it follows from condition (ii) in Theorem 4.1.7 that $\min\{d(x, z_n), d(y, z_n)\} \to 0$. Since M is uniformly discrete, this means that a subsequence of $(z_n)_{n=1}^{\infty}$ is eventually equal to either x or y, a contradiction. □

Aliaga and Guirao proved in [AG17] that, in the case of compact metric spaces, every molecule which is an extreme point of $B_{\mathcal{F}(M)}$ is also a preserved extreme point. However, that result is no longer true for general metric spaces, as the following example shows.

Example 4.3.4. Consider the sequence in c_0 given by $x_1 = 2e_1$, and $x_n = e_1 + (1 + 1/n)e_n$ for $n \geq 2$, where $(e_n)_{n=1}^{\infty}$ is the canonical basis. Let $M = \{0\} \cup \{x_n : n \in \mathbb{N}\}$. This metric space is considered in [AG17, Example 4.2], where it is proved that the molecule m_{0x_1} is not a preserved extreme point of $B_{\mathcal{F}(M)}$. Let us note that this fact also follows easily from Theorem 4.1.7. Moreover, by Proposition 4.3.1 we have that $m_{0x_1} \in \operatorname{ext}(B_{\mathcal{F}(M)})$.

On the other hand, if we restrict our attention to uniformly discrete bounded metric spaces satisfying the hypotheses of the duality result, then all the families of distinguished points of $B_{\mathcal{F}(M)}$ that we have considered coincide.

Proposition 4.3.5. *Let (M, d) be a uniformly discrete pointed metric space such that $\mathcal{F}(M)$ admits a natural predual which is a subspace of $S_0(M)$. Then for $\mu \in B_{\mathcal{F}(M)}$ it is equivalent :*

(i) $\mu \in \operatorname{ext}(B_{\mathcal{F}(M)})$.

(ii) $\mu \in \operatorname{strexp}(B_{\mathcal{F}(M)})$.

(iii) There are $x, y \in M$, $x \neq y$, such that $\mu = m_{xy}$ and $[x, y] = \{x, y\}$.

Proof. (i) $\Rightarrow$ (iii) follows from Proposition 4.2.1. Moreover, (ii)$\Rightarrow$(i) is obvious. Now, assume that $\mu = m_{xy}$ with $[x, y] = \{x, y\}$. We will show that the pair (x, y) fails property (Z) and thus μ is a strongly exposed point (see Theorem 4.1.10). Assume, by contradiction, that there is a sequence $(z_n)_{n=1}^{\infty}$ in M such that

$$d(x, z_n) + d(y, z_n) \leq d(x, y) + \frac{1}{n} \min\{d(x, z_n), d(y, z_n)\}.$$

and so

$$(1 - 1/n)(d(x, z_n) + d(y, z_n)) \leq d(x, y).$$

The last inequality trivially implies that the sequence $(z_n)_{n=1}^{\infty}$ is bounded. Let us fix $r > 0$ such that $(z_n)_{n=1}^{\infty} \subset B(0, r)$. The compactness of $\delta(B(0, r))$ with respect to the

weak* topology ensures the existence of a weak* cluster point $\delta(z)$ of $(\delta(z_n))_{n=1}^{\infty}$ (with $z \in (B(0,r))$. Now, by lower-semi-continuity of the distance, we have

$$d(x,z) + d(y,z) \leq \liminf_{n\to\infty}(1-1/n)(d(x,z_n) + d(y,z_n)) \leq d(x,y).$$

Therefore, $z \in [x,y] = \{x,y\}$. Suppose $z = x$. Denote $\theta = \inf\{d(u,v) : u \neq v\} > 0$. The lower-semi-continuity of d yields

$$\begin{aligned}\theta + d(x,y) &\leq \liminf_{n\to\infty}(1-1/n)(\theta + d(y,z_n)) \\ &\leq \liminf_{n\to\infty}(1-1/n)(d(x,z_n) + d(y,z_n)) \leq d(x,y),\end{aligned}$$

which is impossible. The case $z = y$ yields a similar contradiction. Thus the pair (x,y) does not have property (Z). □

4.4 Proper metric spaces

In this section, we focus on the case in which M is a proper metric space and $\mathcal{F}(M)$ is the dual of $S_0(M)$. In this case, all extreme points of $B_{\mathcal{F}(M)}$ are molecules by Corollary 3.3.6 in [Wea99]. We will show that $\mathcal{F}(M)$ satisfies a geometrical property, namely being *weak* asymptotically uniformly convex*, which implies in particular that the norm and the weak* topologies agree in $S_{\mathcal{F}(M)}$ and so every extreme point of the closed ball is also a denting point.

The next definitions are due to V. Milman [Mil71]. Consider a real Banach space X. For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\begin{aligned}\overline{\rho}_X(t,x,Y) &= \sup_{y\in S_Y} \|x+ty\| - 1 && \text{and} && \overline{\delta}_X(t,x,Y) = \inf_{y\in S_Y} \|x+ty\| - 1,\\ \overline{\rho}_X(t,x) &= \inf_{\dim(X/Y)<\infty} \overline{\rho}(t,x,Y) && \text{and} && \overline{\delta}_X(t,x) = \sup_{\dim(X/Y)<\infty} \overline{\delta}(t,x,Y),\\ \overline{\rho}_X(t) &= \sup_{x\in S_X} \overline{\rho}(t,x) && \text{and} && \overline{\delta}_X(t) = \inf_{x\in S_X} \overline{\delta}(t,x).\end{aligned}$$

The norm $\|\cdot\|_X$ is then said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

Moreover, the norm $\|\cdot\|_X$ is called *asymptotically uniformly flat* (in short AUF) if there exists $t_0 > 0$ such that $\overline{\rho}_X(t_0) = 0$. For instance, the space $X = c_0$ is AUF with $\overline{\rho}_{c_0}(t) = 0$ for all $t \in (0,1]$. Moreover, it is proved in [JLPS02, Theorem 2.9] that a separable Banach space has an equivalent AUF norm if and only if it isomorphic to a subspace of c_0. This result was already contained in [GKL00] (see the comments before Theorem 2.4) which was inspired by [KW95].

The norm $\|\cdot\|_X$ is then said to be *asymptotically uniformly convex* (in short AUC) if

$$\forall t > 0,\ \overline{\delta}_X(t) > 0.$$

Similarly, there is in X^* a modulus of weak* asymptotic uniform convexity defined by

$$\overline{\delta}^*_X(t) = \inf_{x^*\in S_{X^*}} \sup_E \inf_{y^*\in S_E} \|x^* + ty^*\| - 1,$$

where E runs through all weak*-closed subspaces of X^* of finite codimension. Then, the norm $\|\cdot\|_{X^*}$ is said to be weak*-*asymptotically uniformly convex* (in short weak*-AUC) if

$$\forall t > 0,\ \overline{\delta}^*_X(t) > 0.$$

We shall start with the following simple lemma.

Lemma 4.4.1. *Let X be a Banach space and assume that for every $\varepsilon > 0$, X is $(1+\varepsilon)$-isomorphic to a subspace of c_0. Then X is AUF.*

Proof. We let $t \in (0,1]$ and we fix $\varepsilon > 0$. By the assumption, there exists $Y \subseteq c_0$ which is $(1+\varepsilon)$-isomorphic to a subspace of X. We consider an isomorphism $T\colon X \to Y$ such that for every $x \in X : \|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|$. Let $x \in S_X$ and $y = Tx$. Since $\overline{\rho}_Y(t) = 0$, there exists a finite codimensional subspace Y_0 of Y verifying $\overline{\rho}_Y(t, \frac{y}{\|y\|}, Y_0) = \sup_{y_0 \in S_{Y_0}} \|\frac{y}{\|y\|} + ty_0\| - 1 \leq \varepsilon$. We then denote $X_0 = T^{-1}Y_0$ which is a finite codimensional subspace of X. For every $x_0 \in S_{X_0}$ we have

$$\begin{aligned}
\|x + tx_0\| - 1 &\leq \|y + tTx_0\| - 1 \\
&\leq \Big\|\frac{y}{\|y\|} + t\frac{Tx_0}{\|Tx_0\|}\Big\| - 1 + \Big\|y - \frac{y}{\|y\|}\Big\| + t\Big\|Tx_0 - \frac{Tx_0}{\|Tx_0\|}\Big\| \\
&\leq \overline{\rho}_Y\Big(t, \frac{y}{\|y\|}, Y_0\Big) + 2\varepsilon \\
&\leq 3\varepsilon.
\end{aligned}$$

Accordingly, for every $x \in S_X : \overline{\rho}_X(t,x) \leq 3\varepsilon$. Thus $\overline{\rho}_X(t) \leq 3\varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, we deduce $\overline{\rho}_X(t) = 0$. □

It is known that the norm $\|\cdot\|_X$ is AUS if and only if the norm $\|\cdot\|_{X^*}$ is weak*-AUC (see [DKLR14] and the references there in). This leads us to the following result.

Proposition 4.4.2. *Let M be a proper pointed metric space. Assume that $S_0(M)$ 1-separates points uniformly. Then the norm of $\mathcal{F}(M)$ is weak*-AUC.*

Proof. According to Lemma 3.2.4, for every $\varepsilon > 0$ the space $S_0(M)$ is $(1+\varepsilon)$-isomorphic to a subspace of c_0. Thus, Lemma 4.4.1 implies that the norm of $S_0(M)$ is AUS and so the norm of $\mathcal{F}(M)$ is weak*-AUC. □

It is easy to see that X^* is weak*-AUC if and only if X^* has the uniform Kadec-Klee property for the weak* topology (in short UKK*). We recall that X^* has UKK*, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that : If $x^* \in B_{X^*}$ satisfies $\|x^*\| > 1 - \delta$, then there exists a weak*-neighbourhoods V of x^* such that $\operatorname{diam}(V \cap B_{X^*}) \leq \varepsilon$. Consequently, every point of the unit sphere has relative weak*-neighbourhoods of arbitrarily small diameter. This fact and Choquet's lemma (Lemma 4.0.2) yield that if $x^* \in \operatorname{ext}(B_{X^*})$ then there are weak*-slices of B_{X^*} containing x^* of arbitrarily small diameter. That is, every extreme point of B_{X^*} is also a denting point.

Corollary 4.4.3. *Let M be a proper pointed metric space. Assume that $S_0(M)$ 1-separates points uniformly. Then every extreme point of $B_{\mathcal{F}(M)}$ is also a denting point.*

At this point, one could be inclined to believe that the denting points and the strongly exposed points of $B_{\mathcal{F}(M)}$ coincide, at least when M is proper (in particular compact). We are going to give an example of a compact metric space for which the inclusion $\mathrm{strexp}(B_{\mathcal{F}(M)}) \subset \mathrm{ext}(B_{\mathcal{F}(M)^{**}}) \cap \mathcal{F}(M)$ is strict.

Example 4.4.4. Let (T, d) be the following set with the shortest path distance

$$[0,1] \times \{0\} \cup \bigcup_{n=2}^{\infty} \left\{1 - \frac{1}{n}\right\} \times \left[0, \frac{1}{n^2}\right].$$

We will consider (Ω, d) as the set

$$\{(0,0),(1,0)\} \cup \left\{\left(1 - \frac{1}{n}, \frac{1}{n^2}\right) : n \geq 2\right\}$$

together with the distance inherited from (T, d). Let us call for simplicity $0 := x_1 := (0,0)$, $x_\infty := (1,0)$ and $x_n := (1 - \frac{1}{n}, \frac{1}{n^2})$ if $n \geq 2$.

Since the couple $(x_\infty, 0)$ has property (Z) (defined in Theorem 4.1.10), the characterisation of strongly exposed points given in [GLPZ17b] yields that $\delta(x_\infty)$ is not a strongly exposed point of $B_{\mathcal{F}(\Omega)}$. Aliaga and Guirao [AG17] have proved that for a compact M, the condition $[x, y] = \{x, y\}$ implies that $\frac{\delta(x)-\delta(y)}{d(x,y)}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$. In particular $\delta(x_\infty)$ is a preserved extreme point of $B_{\mathcal{F}(\Omega)}$.

4.5 Perspectives

We shall begin by recalling the two main questions about the extremal structure of free spaces, that is Question a) and Question b).

Question 4.5.1. Is every extreme point of $B_{\mathcal{F}(M)}$ a molecule ?

Question 4.5.2. Assume that $[x, y] = \{x, y\}$. Is it true that m_{xy} is an extreme point ?

We have seen in Theorem 4.1.7 a metric characterization of preserved extreme points (equivalently denting points) of $B_{\mathcal{F}(M)}$. We have also presented a metric characterization of strongly exposed points of $B_{\mathcal{F}(M)}$ (Theorem 4.1.10). Thus, one may ask the following.

Question 4.5.3. Is there a metric characterization of exposed points of $B_{\mathcal{F}(M)}$?

Moreover, we have proved in full generality that the set of preserved extreme points of $B_{\mathcal{F}(M)}$ coincides with the set of denting points of $B_{\mathcal{F}(M)}$(Theorem 4.1.4). However, there are cases where the set of extreme points of $B_{\mathcal{F}(M)}$ does not coincide with the set of preserved extreme points of $B_{\mathcal{F}(M)}$ (Example 4.3.4). In the same way, there are cases where the set of strongly exposed points of $B_{\mathcal{F}(M)}$ does not coincide with the set of denting points of $B_{\mathcal{F}(M)}$ (Example 4.4.4). Thus, it is quite natural to wonder what happens for the set of exposed points. More precisely, we address the following questions.

Question 4.5.4. Is it true that every exposed point is also a denting point of $B_{\mathcal{F}(M)}$? Conversely, is it true that every denting point is also an exposed point of $B_{\mathcal{F}(M)}$?

Chapter 5

Vector-valued Lipschitz functions

In this chapter, we shift our attention to vector-valued Lipschitz functions. There is a somehow natural definition of vector-valued Lipschitz free spaces which probably first appeared in [BGLPRZ17] : $\mathcal{F}(M,X) := \mathcal{F}(M)\widehat{\otimes}_\pi X$. Since we need some tensor product tools, we shall begin with a reminder about projective and injective tensor products and some related properties.

Our next objective is to extend some known results in the scalar case to the vector-valued one. We are mainly interested in duality results and in results about the Schur properties. In Section 5.2, we first generalize Dalet's duality result (Theorem 2.1.4). Following the same pattern, we also extend Kalton's duality result (Theorem 2.2.4). Then, in Section 5.3 we deal with the Schur properties defined in Chapter 3. The main purpose is to give conditions on M and X that force $\mathcal{F}(M,X)$ to have the Schur property. We will use the theory of tensor product as well as the theory of Lipschitz maps.

We will finish the chapter with a discussion of norm attainment of Lipschitz maps. We will actually consider two different notions of norm attainment for a Lipschitz map. We shall prove that both concepts agree for some classes of metric spaces M. We recall that the celebrated Bishop–Phelps theorem asserts that for every Banach space X the set of functionals in X^* that attain their norm is dense in X^*. We also prove a kind of Bishop–Phelps density result for scalar-valued and vector-valued Lipschitz maps. This chapter is mainly based on a joint work with Luis García-Lirola and Abraham Rueda Zoca ([GLPZ17a]).

5.1 Preliminaries

5.1.1 Tensor products

We start by defining a few classical tools that we will need in the current chapter. Let X, Y and Z be Banach spaces. We denote $\mathcal{B}(X \times Y, Z)$ the space of continuous bilinear operators from $X \times Y$ into Z. If $Z = \mathbb{R}$, we simply write $\mathcal{B}(X \times Y)$ instead of $\mathcal{B}(X \times Y, \mathbb{R})$. We first introduce the projective tensor product of X and Y (we refer the reader to [Rya02] for backgrounds on this concept).

For $x \in X$ and $y \in Y$, define $x \otimes y \in \mathcal{B}(X \times Y)^*$ by :

$$\text{for every } B \in \mathcal{B}(X \times Y),\ \langle x \otimes y, B\rangle = B(x,y).$$

Let us denote $X \otimes Y = \operatorname{span}\{x \otimes y \,:\, x \in X,\, y \in Y\}$. We recall that the norm $\|\cdot\|_{\mathcal{B}(X\times Y)}$ on $\mathcal{B}(X \times Y)$ is defined by $\|B\|_{\mathcal{B}(X\times Y)} = \sup_{x\in B_X, y\in B_Y} |B(x,y)|$. We then denote $\|\cdot\|_\pi$ the dual norm of $\|\cdot\|_{\mathcal{B}(X\times Y)}$. In fact, it is well known (see for instance Proposition VIII. 9. a) in [DU77]) that if $u \in X \otimes Y$ then

$$\|u\|_\pi = \inf\Big\{\sum_{i=1}^n \|x_i\|\|y_i\| \,:\, u = \sum_{i=1}^n x_i \otimes y_i\Big\}.$$

Definition 5.1.1 (Projective tensor product). Let X, Y be Banach spaces. We define the *projective tensor product* of X and Y to be the following space :

$$X\widehat{\otimes}_\pi Y = \overline{\operatorname{span}}^{\|\cdot\|_\pi}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X \times Y)^*.$$

Note that our previous definition of projective tensor products is very similar to the one of Lipschitz free spaces. Indeed, here $x \otimes y$ can be compared with $\delta(x)$ as being evaluation functionals. Moreover, we take the closed linear span of those evaluation functionals $x \otimes y$ in the space where they are considered. More analogy can be made as it is shown by the following fundamental linearisation property (see [Rya02, Theorem 2.9]).

Proposition 5.1.2 (Fundamental linearisation property). *Let X, Y, Z be Banach spaces and consider $B \in \mathcal{B}(X \times Y, Z)$. Then, there exists a unique operator $\bar{B} : X\widehat{\otimes}_\pi Y \to Z$ such that $\|\bar{B}\| = \|B\|$ and such that the following diagram commutes*

$$\begin{tikzcd} X \times Y \arrow[r, "B"] \arrow[d, "i"] & Z \\ X\widehat{\otimes}_\pi Y \arrow[ur, "\bar{B}"'] & \end{tikzcd}$$

where $i(x,y) = x \otimes y$. Thus $\mathcal{B}(X \times Y, Z) \equiv \mathcal{L}(X\widehat{\otimes}_\pi Y, Z)$.

As a direct consequence ($Z = \mathbb{R}$), we obtain $(X\widehat{\otimes}_\pi Y)^* \equiv \mathcal{B}(X \times Y)$. Moreover we have the following well-know identification $\mathcal{B}(X \times Y) \equiv \mathcal{L}(X, Y^*)$. Consequently, we obtain the following corollary.

Corollary 5.1.3. *Let X, Y be Banach spaces. Then $\mathcal{L}(X, Y^*) \equiv (X\widehat{\otimes}_\pi Y)^*$.*

Next, we will also need the injective tensor product of two Banach spaces. In the above definition of projective tensor product we chose to define $x \otimes y$ as an element of $\mathcal{B}(X \times Y)^*$. But we can actually use another point of view. Indeed, we can see $x \otimes y$ as an element of $\mathcal{B}(X^* \times Y^*)$, defined in the following way :

$$\text{For every } (x^*, y^*) \in X^* \times Y^*,\ \langle x \otimes y, (x^*, y^*)\rangle = x^*(x)y^*(y).$$

In this case, we denote $\|\cdot\|_\varepsilon$ the canonical norm on $\mathcal{B}(X^* \times Y^*)$. Thus, if $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ then $\|u\|_\varepsilon = \sup\big\{\big|\sum_{i=1}^n x^*(x_i)y^*(y_i)\big| \,:\, x^* \in B_{X^*}, y^* \in B_{Y^*}\big\}$.

Definition 5.1.4 (Injective tensor product). Let X and Y be Banach spaces. We define the *injective tensor product* of X and Y to be the following space :

$$X\widehat{\otimes}_\varepsilon Y = \overline{\operatorname{span}}^{\|\cdot\|_\varepsilon}\{x \otimes y \,:\, x \in X,\, y \in Y\} \subseteq \mathcal{B}(X^* \times Y^*).$$

We finish this section by giving a famous property of tensor products that will be useful later (see [Rya02, Theorem 5.33]).

Theorem 5.1.5. *Let X and Y be Banach spaces. Assume that X^* or Y^* has the (RNP) and that X^* or Y^* has the (AP). Then, $(X\widehat{\otimes}_\varepsilon Y)^* \equiv X^*\widehat{\otimes}_\pi Y^*$.*

5.1.2 Vector-valued Lipschitz free spaces

Consider a pointed metric space M and a Banach space X. We recall that $\mathrm{Lip}_0(M,X)$ denotes the Banach space of Lipschitz maps from M to X satisfying $f(0)=0$, equipped with the following norm

$$\|f\|_L = \sup_{x\neq y\in M} \frac{\|f(x)-f(y)\|_X}{d(x,y)}.$$

It follows from the fundamental linearisation property of Lipschitz free spaces (Proposition 1.1.2) that $\mathrm{Lip}_0(M,X)\equiv\mathcal{L}(\mathcal{F}(M),X)$. Consequently, according to Corollary 5.1.3, we observe the following isometric identification : $\mathrm{Lip}_0(M,X^*)\equiv(\mathcal{F}(M)\widehat{\otimes}_\pi X)^*$. This is the main motivation for introducing the next definition.

Definition 5.1.6 (**Vector-valued Lipschitz free space**).
Let M be a pointed metric space and let X be a Banach space. We define the X-valued Lipschitz free space over M to be : $\mathcal{F}(M,X):=\mathcal{F}(M)\widehat{\otimes}_\pi X$.

Recall that $\delta(M)$ is weakly closed in $\mathcal{F}(M)$ provided M is complete (Proposition 1.2.1). Accordingly, we wonder if there is a similar result in the vector-valued setting. For this purpose, we need to identify a set that corresponds to $\delta(M)$ in the vector-valued case. It seems to us that a legitimate set to look at could be the following :

$$\delta(M,X):=\{\delta(y)\otimes x\ :\ y\in M,\, x\in X\}\subset\mathcal{F}(M,X).$$

Notice that this does not exactly correspond to $\delta(M)$ in the case $X=\mathbb{R}$ since we have $\delta(M,\mathbb{R})=\mathbb{R}\cdot\delta(M)$. However, we also proved in Proposition 1.4.2 that $\mathbb{R}\cdot\delta(M)$ is weakly closed when M is complete. So, for M a complete pointed metric space and X a Banach space, we wonder whether $\delta(M,X)$ is $\sigma(\mathcal{F}(M,X),\mathrm{Lip}_0(M,X))$-closed. The next proposition shows that it is actually the case.

Proposition 5.1.7. *Let M be a complete pointed metric space and X be a Banach space such that $\mathcal{F}(M)$ or X have the approximation property. Then $\delta(M,X)$ is weakly closed in $\mathcal{F}(M,X)$.*

For the proof, we will need to the following lemma.

Lemma 5.1.8. *Let X,Y be two Banach spaces such that X or Y have the approximation property. Then the set of elementary tensors $\mathcal{T}=\{x\otimes y\ :\ x\in X,\, y\in Y\}$ is weakly closed in $X\widehat{\otimes}_\pi Y$.*

Proof. We are going to prove that if $T\in X\widehat{\otimes}_\pi Y$, then $T\in\mathcal{T}$ if and only if for every linearly independent families $\{x_1^*,x_2^*\}\subset X^*$ and $\{y_1^*,y_2^*\}\subset Y^*$ we have :

$$\begin{vmatrix}\langle T,x_1^*\otimes y_1^*\rangle & \langle T,x_1^*\otimes y_2^*\rangle\\ \langle T,x_2^*\otimes y_1^*\rangle & \langle T,x_2^*\otimes y_2^*\rangle\end{vmatrix}=0.$$

The lemma directly follows from this claim since one can write $\mathcal{T}$ as the intersection of weakly closed sets (given by the inverse image of continuous functions). So let us prove the claim. If $T\in\mathcal{T}$, then $T=x\otimes y$ for some $x\in X$ and $y\in Y$. And then simple computations yield the desired estimate.

Now assume that $T \notin \mathcal{T}$. It is known that every $T \in X\widehat{\otimes}_\pi Y$ can be written $T = \sum_{n=1}^\infty x_n \otimes y_n$ with $\sum_{n=1}^\infty \|x_n\|\|y_n\| \leq 2\|T\|$ (see [Rya02, Proposition 2.8]). Moreover, $\Phi : X\widehat{\otimes}_\pi Y \to \mathcal{L}(X^*, Y)$ defined by $\Phi(\sum_{n=1}^\infty x_n \otimes y_n)(x^*) = \sum_{n=1}^\infty x^*(x_n)y_n$ is a bounded linear operator. Since X or Y have the (AP), Φ is moreover injective [Rya02, Proposition 4.6]. Consequently, $\Phi(T)$ is an operator of rank greater than 2 in $\mathcal{L}(X^*, Y)$. Thus, there exists a linearly independent family $\{x_1^*, x_2^*\} \subset X^*$ such that $\Phi(T)(x_1^*) \neq 0$, $\Phi(T)(x_2^*) \neq 0$ and $\{\Phi(T)(x_1^*), \Phi(T)(x_2^*)\} \subset Y$ is a linearly independent family. To finish the proof, simply pick a linearly independent family $\{y_1^*, y_2^*\} \subset Y^*$ satisfying :

$$\begin{array}{ll} \langle \Phi(T)(x_1^*), y_1^* \rangle \neq 0 & \langle \Phi(T)(x_1^*), y_2^* \rangle = 0 \\ \langle \Phi(T)(x_2^*), y_1^* \rangle = 0 & \langle \Phi(T)(x_2^*), y_2^* \rangle \neq 0. \end{array}$$

□

Proof of Proposition 5.1.7. In what follows, $\mathcal{T}$ now denotes the elementary tensors of $\mathcal{F}(M)\widehat{\otimes}_\pi X$. That is, $\mathcal{T} = \{\gamma \otimes x \,:\, \gamma \in \mathcal{F}(M),\, x \in X\}$. Consider a net $(\delta(m_\alpha) \otimes x_\alpha)_\alpha \subset \delta(M, X)$ which weakly converges to some $\gamma \otimes x \in \mathcal{T}$ ($\mathcal{T}$ is weakly closed). We may assume that $x \neq 0$ otherwise there is nothing to do. Pick $x^* \in X^*$ such that $x^*(x) \neq 0$. Then, for every $f \in \mathrm{Lip}_0(M)$ we have that $f(m_\alpha)x^*(x_\alpha) \to f(\gamma)x^*(x)$. So the net $\left(\frac{x^*(x_\alpha)}{x^*(x)}\delta(m_\alpha)\right)_\alpha \subset \mathbb{R} \cdot \delta(M)$ weakly converges to γ. Since $\mathbb{R} \cdot \delta(M)$ is weakly closed, there is $\lambda \in \mathbb{R}$ and $m \in M$ such that $\gamma = \lambda\delta(m)$. Consequently $\gamma \otimes x = \delta(m) \otimes \lambda x \in \delta(M, X)$. This finishes the proof. □

Taking into account Section 2.2, one may wonder if there exists a reasonable extension of the notion of natural predual in the vector-valued setting. Considering Proposition 5.1.7, a reasonable definition of natural predual in the vector-valued case may be the following.

Definition 5.1.9. Let M be a pointed metric space and X be a Banach space with $\dim(X) \geq 2$. We say that a Banach Y is a natural predual of $\mathcal{F}(M, X^*)$ if $Y^* \equiv \mathcal{F}(M, X^*)$ and

$$\delta(B(0,r), X^*) = \{\delta(m) \otimes x^* \,:\, m \in B(0,r),\, x^* \in X^*\} \subset \mathcal{F}(M, X^*)$$

is $\sigma(\mathcal{F}(M, X^*), Y)$-closed for every $r \geq 0$.

Notice again that $\delta(B(0,r), \mathbb{R}) = \mathbb{R} \cdot \delta(B(0,r))$. So, the above definition is not exactly the equivalent of Definition 2.2.1. To avoid problem of consistence we assume $\dim(X) \geq 2$. Let us start with the following observation.

Lemma 5.1.10. *Let M be a separable pointed metric space. Suppose that $S \subset \mathrm{lip}_0(M)$ is a natural predual of $\mathcal{F}(M)$ (in the sense of Definition 2.2.1). Then, for every $r \geq 0$, $\mathbb{R} \cdot \delta(B(0,r))$ is weak* closed in $\mathcal{F}(M)$.*

Proof. Let us fix $r \geq 0$. Let $(\lambda_n\delta(x_n))_{n=1}^\infty \subset \mathbb{R} \cdot \delta(B(0,r))$ be a sequence weak* converging to some $\gamma \in \mathcal{F}(M)$. We may assume that $\gamma \neq 0$ because otherwise there is nothing to do. Since a weak* convergent sequence is bounded and by weak* lower-semi-continuity of the norm we may assume that there exists $C > 0$ such that for every n :

$$0 < \frac{\|\gamma\|}{2} \leq |\lambda_n|\|\delta(x_n)\| = |\lambda_n| d(x_n, 0) \leq C.$$

Thus, $d(x_n, 0) \neq 0$ and $\lambda_n \neq 0$ for every n. Up to extracting a further subsequence, we may assume that the sequence $(\lambda_n d(x_n, 0))_{n=1}^\infty$ converges to some $\ell \neq 0$. Since $(x_n)_{n=1}^\infty \subset$

$B(0,r)$, we also assume that $(d(x_n,0))_{n=1}^{\infty}$ converges to some d. We will distinguish two cases.

If $d \neq 0$, then $(\lambda_n)_{n=1}^{\infty}$ converges to $\lambda := \dfrac{\ell}{d}$ and so $(\delta(x_n))_{n=1}^{\infty}$ weak* converges to $\dfrac{\gamma}{\lambda}$. By assumption, $\delta(B(0,r))$ is weak* closed in $\mathcal{F}(M)$, so there exists $x \in M$ such that $\gamma = \lambda\delta(x)$.

If $d = 0$, then $(\delta(x_n))_{n=1}^{\infty}$ converges to 0 in the norm topology (and $(\lambda_n)_{n=1}^{\infty}$ tends to infinity). Note that we may write :

$$\lambda_n\delta(x_n) = \lambda_n d(x_n,0)\frac{\delta(x_n)-\delta(0)}{d(x_n,0)} = \lambda_n d(x_n,0)m_{x_n0}.$$

Since $S \subset \mathrm{lip}_0(M)$, the sequence $(m_{x_n0})_{n=1}^{\infty}$ weak* converges to 0. Remember that the sequence $(\lambda_n d(x_n,0))_{n=1}^{\infty}$ converges to $\ell \neq 0$. Consequently $(\lambda_n\delta(x_n))_{n=1}^{\infty}$ weak* converges to 0 and so $\gamma = 0$, which is a contradiction. □

Assume now that there exists a subspace S of $\mathrm{Lip}_0(M)$ such that $S^* \equiv \mathcal{F}(M)$. Note that Theorem 5.1.5 yields the following :

$$\mathcal{F}(M,X^*) = \mathcal{F}(M)\widehat{\otimes}_\pi X^* \equiv (S\widehat{\otimes}_\varepsilon X)^*$$

whenever either $\mathcal{F}(M)$ or X^* has the (AP) and either $\mathcal{F}(M)$ or X^* has the Radon-Nikodým property the (RNP). One may wonder if there are conditions which ensure that $S\widehat{\otimes}_\varepsilon X$ is a natural predual. The next result asserts that the above problem sometimes relies on the scalar case.

Proposition 5.1.11. *Let M be a separable pointed metric space, $S \subset \mathrm{lip}_0(M)$ be a natural predual of $\mathcal{F}(M)$ (in the sense of Definition 2.2.1) and X be a Banach space (with $\dim(X) \geq 2$). Assume moreover that either $\mathcal{F}(M)$ or X^* has the (AP) and either $\mathcal{F}(M)$ or X^* has the (RNP). Then, $S\widehat{\otimes}_\varepsilon X$ is a natural predual of $\mathcal{F}(M,X^*)$.*

Proof. It follows from Theorem 5.1.5 that $(S\widehat{\otimes}_\varepsilon X)^* \equiv \mathcal{F}(M,X^*)$. To show that $S\widehat{\otimes}_\varepsilon X$ is a natural predual, we essentially follow the proof of Proposition 5.1.7.

First of all, we show that $\mathcal{T} := \{\gamma \otimes x^* : \gamma \in \mathcal{F}(M),\, x \in X^*\}$ is weak* closed in $\mathcal{F}(M,X^*)$. Indeed, it is not hard to show that if $T \in \mathcal{F}(M,X^*)$, then $T \in \mathcal{T}$ if and only if for every linearly independent families $\{f_1,f_2\} \subset S$ and $\{x_1,x_2\} \subset X$ we have :

$$\begin{vmatrix} \langle T, f_1\otimes x_1\rangle & \langle T, f_1\otimes x_2\rangle \\ \langle T, f_2\otimes x_1\rangle & \langle T, f_2\otimes x_2\rangle \end{vmatrix} = 0.$$

Accordingly, $\mathcal{T}$ is weak* closed. Now we fix $r > 0$. Let us consider a net $(\delta(m_\alpha)\otimes x_\alpha^*)_\alpha \subset \delta(B(0,r),X^*)$ which weak* converges to some $\gamma\otimes x^* \in \mathcal{T}$. We may assume that $x^* \neq 0$ otherwise there is nothing to do. Consider $x \in X$ such that $x^*(x) \neq 0$. Then, for every $f \in S$ we have that $f(m_\alpha)x^*(x_\alpha) \to f(\gamma)x^*(x)$. So the net $\left(\frac{x^*(x_\alpha)}{x^*(x)}\delta(m_\alpha)\right)_\alpha \subset \mathbb{R}\cdot\delta(M)$ weak* converges to γ. Since $\mathbb{R}\cdot\delta(M)$ is weak* closed (Lemma 5.1.10), there is $\lambda \in \mathbb{R}$ and $m \in M$ such that $\gamma = \lambda\delta(m)$. □

5.2 Duality results

The problem that we explore in this section is whether we can give a representation of a predual of $\mathcal{F}(M,X)$ as a subspace of $\mathrm{Lip}_0(M,X^*)$. For this purpose, we introduce the vector-valued version of Definition 2.1.1.

Definition 5.2.1. Let M be a pointed metric space and X be a Banach space. We define the two following closed subspaces of $\mathrm{Lip}_0(M,X)$ (with the convention $\sup\emptyset = 0$).

$$\mathrm{lip}_0(M,X) := \Big\{ f \in \mathrm{Lip}_0(M,X) : \lim_{\varepsilon\to 0} \sup_{0<d(x,y)<\varepsilon} \frac{\|f(x)-f(y)\|_X}{d(x,y)} = 0 \Big\},$$

$$S_0(M,X) := \Big\{ f \in \mathrm{lip}_0(M,X) : \lim_{r\to\infty} \sup_{\substack{x \text{ or } y\notin B(0,r)\\ x\neq y}} \frac{\|f(x)-f(y)\|_X}{d(x,y)} = 0 \Big\}.$$

Our first goal is to extend Theorem 2.1.4 asserting that $S_0(M)$ is an isometric predual of $\mathcal{F}(M)$ whenever M is proper and $S_0(M)$ separates points uniformly. As we saw in Chapter 2, a very useful tool in this context is Petunīn–Plīčko's theorem (Theorem 2.2.7). In fact, we obtained our first desired vector-valued duality result using this theorem. However, we had to assume that X^* is separable because of Petunīn–Plīčko's assumptions. So this was not completely satisfactory. Here we present an alternative way to obtain the vector-valued version of Theorem 2.1.4 which does not need any assumption on separability.

In [GLRZ17, Theorem 5.2], it is proved that $S_0(M,X) \equiv \mathcal{K}_{w^*,w}(X^*, S_0(M))$ whenever M is proper (where $\mathcal{K}_{w^*,w}(X^*, S_0(M))$ is the space of compact operators from X^* to $S_0(M)$ which are weak*-to-weak continuous.). Consequently, in order to prove that $S_0(M,X)^* \equiv \mathcal{F}(M,X^*) = \mathcal{F}(M)\widehat{\otimes}_\pi X^*$ under natural assumptions on M, we shall begin by analysing when the equality $\mathcal{K}_{w^*,w}(X^*,Y) \equiv X\widehat{\otimes}_\varepsilon Y$ holds. In order to do that, we shall need to introduce two results.

Lemma 5.2.2. *Let X, Y be Banach spaces, then $T \in \mathcal{K}_{w^*,w}(X^*,Y) \mapsto T^* \in \mathcal{K}_{w^*,w}(Y^*,X)$ defines an onto isometry.*

Proof. Let $T \in \mathcal{K}(X^*,Y) \cap L_{w^*,w}(X^*,Y)$. Then $T^* \in \mathcal{K}(Y^*,X^{**})$. Moreover, given $y^* \in Y^*$, we have that $T^*(y^*) = y^* \circ T : X^* \to \mathbb{R}$ is weak* continuous and thus $T^*(y^*) \in X$. Therefore $T^* \in \mathcal{K}(Y^*,X)$. Since T^* is $\sigma(Y^*,Y) - \sigma(X^{**},X^*)$-continuous, we get $T^* \in L_{w^*,w}(Y^*,X)$. Conversely, if $R \in \mathcal{K}(Y^*,X) \cap L_{w^*,w}(Y^*,X)$ then $R^* \in \mathcal{K}(X^*,Y) \cap L_{w^*,w}(X^*,Y)$ and $R^{**} = R$. □

Next proposition is well-known (see Remark 1.2 in [RS82]), although we have not found any proof in the literature. We include it here for the sake of completeness.

Proposition 5.2.3. *Let X and Y be Banach spaces and assume that either X or Y has the (AP). Then $\mathcal{K}_{w^*,w}(X^*,Y) \equiv X\widehat{\otimes}_\varepsilon Y$.*

Proof. By the above lemma we may assume that Y has the (AP). Clearly the inclusion $\supseteq$ holds, so let us prove the reverse one. To this end pick $T : X^* \longrightarrow Y$ a compact operator which is weak*-to-weak continuous. We will approximate T in norm by a finite-rank operator following word by word the proof of [Rya02, Proposition 4.12]. As Y has

the (AP) we can find $R : Y \longrightarrow Y$ a finite-rank operator such that $\|x - R(x)\| < \varepsilon$ for every $x \in T(B_{X^*})$, and define $S := R \circ T$. S is clearly a finite-rank operator such that $\|S - T\| < \varepsilon$. As S is a finite rank operator, then $S = \sum_{i=1}^n x_i^{**} \otimes y_i$ for suitable $n \in \mathbb{N}, x_i^{**} \in X^{**}$ and $y_i \in Y$. Moreover S is weak*-to-weak continuous. Indeed, the fact that S is weak*-to-weak continuous means that, for every $y^* \in Y^*$, one has

$$y^* \circ S = \sum_{i=1}^n y^*(y_i) x_i^{**} : X^* \longrightarrow \mathbb{R}$$

is a weak* continuous functional, so $\sum_{i=1}^n y^*(y_i) x_i^{**} \in X$ for each $y^* \in Y^*$. Note that an easy argument of bilinearity allows us to assume that $\{y_1, \ldots, y_n\}$ are linearly independent. Now, a straightforward application of Hahn–Banach theorem yields that, for every $i \in \{1, \ldots, n\}$, there exists $y_i^* \in Y^*$ such that $y_j^*(y_i) = \delta_{ij}$. Therefore, for every $j \in \{1, \ldots, n\}$, one has

$$X \ni y_j^* \circ S = \sum_{i=1}^n y_j^*(y_i) x_i^{**} = \sum_{i=1}^n \delta_{ij} x_i^{**} = x_j^{**}.$$

Consequently we get that $S \in X \otimes Y$. Summarising, we have proved that each element of $\mathcal{K}_{w^*,w}(X^*, Y)$ can be approximated in norm by an element of $X \otimes Y$, so

$$\mathcal{K}_{w^*,w}(X^*, Y) \equiv X \widehat{\otimes}_\varepsilon Y.$$

□

As a consequence of Proposition 5.2.3 and [GLRZ17, Theorem 5.2] we get the following.

Corollary 5.2.4. *Let M be a proper pointed metric space. If either $S_0(M)$ or X has the (AP), then $S_0(M, X) \equiv S_0(M) \widehat{\otimes}_\varepsilon X$.*

The above corollary as well as the basic theory of tensor product spaces give us the key to proving our first duality result in the vector-valued setting.

Theorem 5.2.5. *Let M be a proper pointed metric space and let X be a Banach space. Assume that $S_0(M)$ separates points uniformly. If either $\mathcal{F}(M)$ or X^* has the (AP), then*

$$S_0(M, X)^* \equiv \mathcal{F}(M, X^*).$$

Proof. As $S_0(M)$ separates points uniformly, $S_0(M)^* \equiv \mathcal{F}(M)$ (Theorem 2.1.4). Thus $S_0(M)$ is an Asplund space. Consequently, we get from the above corollary and from Theorem 5.1.5 that

$$S_0(M, X)^* \equiv (S_0(M) \widehat{\otimes}_\varepsilon X)^* \equiv \mathcal{F}(M) \widehat{\otimes}_\pi X^* = \mathcal{F}(M, X^*).$$

□

Now we will exhibit some examples of metric and Banach spaces in which Theorem 5.2.5 applies.

Corollary 5.2.6. *Let M be a proper pointed metric space and X be a Banach space. Then $S_0(M, X)^* \equiv \mathcal{F}(M, X^*)$ whenever M and X satisfy one of the following assumptions :*

1. *M is countable.*
2. *$(M, \omega \circ d)$ is a metric space where ω is a non trivial gauge, and either $\mathcal{F}(M)$ or X^* has the (AP).*
3. *M is the middle third Cantor set.*

Proof. If M satisfies (1), then $S_0(M)$ separates points uniformly and $\mathcal{F}(M)$ has the approximation property [Dal15c]. Thus Theorem 5.2.5 applies. Moreover, if M satisfies (2) then Proposition 2.1.7 does the work. Finally, [Wea99, Proposition 3.2.2] yields (3). □

Throughout the rest of the section we will consider a bounded metric space (M, d) and a topology τ on M such that (M, τ) is compact and d is τ lower-semi-continuous. We will consider

$$lip_\tau(M) = \mathrm{lip}_0(M) \cap \mathcal{C}(M, \tau),$$

the space of little-Lipschitz functions which are τ-continuous on M. Since M is bounded, $lip_\tau(M)$ is a closed subspace of $\mathrm{lip}_0(M)$ and thus it is a Banach space. Moreover, Kalton proved in [Kal04, Theorem 6.2] that $lip_\tau(M)^* \equiv \mathcal{F}(M)$ whenever M is separable, complete, and $lip_\tau(M)$ 1-separates points uniformly. Recall that this condition holds if, and only if, $lip_\tau(M)$ is 1-norming for $\mathcal{F}(M)$ (see Proposition 2.1.3).

Now we can wonder whether there is a natural extension of this result to the vector-valued case. We will prove, following ideas similar to the ones of [GLRZ17, Section 5], that under suitable assumptions the space

$$lip_\tau(M, X) := \mathrm{lip}_0(M, X) \cap \{f \colon M \to X : f \text{ is } \tau - \text{to} - \|\cdot\| \text{ continuous}\}$$

is a predual of $\mathcal{F}(M, X^*)$. For this, we shall begin by characterising relative compactness in $lip_\tau(M)$.

Lemma 5.2.7. *Let (M, d) be a pointed metric space of radius R and τ a topology on M such that (M, τ) is compact and d is τ lower-semi-continuous. Let $\mathcal{F}$ be a subset of $lip_\tau(M)$. Then $\mathcal{F}$ is relatively compact in $lip_\tau(M)$ if, and only if, the following three conditions hold :*

1. *$\mathcal{F}$ is bounded.*
2. *$\mathcal{F}$ satisfies the following uniform little-Lipschitz condition : for every $\varepsilon > 0$ there exists a positive $\delta > 0$ such that*

$$\sup_{0<d(x,y)<\delta} \frac{|f(x) - f(y)|}{d(x, y)} < \varepsilon$$

   for every $f \in \mathcal{F}$.
3. *$\mathcal{F}$ is equicontinuous in $\mathcal{C}(M, \tau)$, i.e. for every $x \in M$ and every $\varepsilon > 0$ there exists U a τ-neighbourhood of x such that $y \in U$ implies $\sup_{f \in \mathcal{F}} |f(x) - f(y)| < \varepsilon$.*

Proof. In [Kal04, Theorem 6.2] it is proved that $lip_\tau(M)$ is isometrically isomorphic to a subspace of a space of continuous functions on a compact set. Indeed, let $K := \{(x, y, t) \in (M, \tau) \times (M, \tau) \times [0, 2R] : d(x, y) \leq t\}$. Then K is compact by τ lower-semi-continuity of d. Moreover, the map $\Phi \colon lip_\tau(M) \to \mathcal{C}(K)$ defined by

$$\Phi(f)(x, y, t) := \begin{cases} \frac{f(x)-f(y)}{t} & t \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

is a linear isometry. Therefore, we have that $\mathcal{F}$ is relatively compact if, and only if, $\Phi(\mathcal{F})$ is relatively compact. By Ascoli-Arzelà theorem we get that $\mathcal{F}$ is relatively compact if, and only if, $\Phi(\mathcal{F})$ is bounded and equicontinuous in $\mathcal{C}(K)$. We will first assume that conditions (1), (2) and (3) hold. It is clear that $\Phi(\mathcal{F})$ is bounded, so let us prove the equicontinuity of $\Phi(\mathcal{F})$. To this end pick $(x, y, t) \in K$. Now we have two possibilities :

(i) If $t \neq 0$ we can find a positive number $\eta < t$ such that $t' \in]t-\eta, t+\eta[$ implies $\left|\frac{1}{t} - \frac{1}{t'}\right| < \frac{\varepsilon}{4R\alpha}$, where $\alpha = \sup_{f\in\mathcal{F}} \|f\|$. Now, as x and y are two points of M and $\mathcal{F}$ satisfies condition (3), we conclude the existence of U a τ-neighbourhood of x and V a τ-neighbourhood of y in M verifying $x' \in U, y' \in V$ implies $|f(x)-f(x')|+|f(y)-f(y')| < \frac{\varepsilon t}{2}$ for every $f \in \mathcal{F}$. Now, given $(x', y', t') \in (U \times V \times]t-\eta, t+\eta[) \cap K$, one has

$$|\Phi f(x,y,t) - \Phi f(x',y',t')| = \left|\frac{f(x)-f(y)}{t} - \frac{f(x')-f(y')}{t'}\right| \leq$$

$$\left|\frac{1}{t} - \frac{1}{t'}\right| |f(x')-f(y')| + \frac{1}{t}|f(x)-f(x')+f(y)-f(y')|$$

$$\leq \frac{\varepsilon}{4R\alpha}\|f\|d(x',y') + \frac{\varepsilon t}{2t} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $f \in \mathcal{F}$, which proves equicontinuity of $\Phi(f)$ at (x, y, t).

(ii) If $t = 0$ then $x = y$. Pick an arbitrary $\varepsilon > 0$. By (2) we get a positive δ such that $0 < d(x, y) < \delta$ implies $\frac{|f(x)-f(y)|}{d(x,y)} < \varepsilon$ for every $f \in \mathcal{F}$. Now, given $(x', y', t) \in (M \times M \times [0, \delta[) \cap K$ we have $d(x', y') \leq t < \delta$ and so, given $f \in \mathcal{F}$, it follows

$$|\Phi f(x',y',t)| \leq \frac{|f(x')-f(y')|}{t} < \varepsilon\frac{d(x',y')}{t} \leq \varepsilon,$$

which proves equicontinuity at $(x, x, 0)$.

Both previous cases prove that $\Phi(\mathcal{F})$ is equicontinuous whenever conditions (1), (2) and (3) are satisfied.

Conversely, assume that $\Phi(\mathcal{F})$ is equicontinuous in $\mathcal{C}(K)$. It is clear that $\mathcal{F}$ is bounded, so let us prove that conditions (2) and (3) are satisfied. We shall begin by proving (3), for which we fix $x \in M$ and $\varepsilon > 0$. Given $t \in [0, 2R]$, by equicontinuity of $\Phi(\mathcal{F})$ at the point (x, x, t), we can find U_t a τ-neighbourhood of x and $\eta_t > 0$ such that $x' \in U_t$ and $t' \in (t-\eta_t, t+\eta_t)$ implies $|\Phi f(x, x', t')| < \frac{\varepsilon}{2R}$ for every $f \in \mathcal{F}$. Then $[0, 2R] \subset \bigcup_t (t-\eta_t, t+\eta_t)$ and thus there exist $t_1, \ldots, t_n$ such that $[0, 2R] \subset \bigcup_{i=1}^n (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Now take $U = \bigcap_{i=1}^n U_{t_i}$. We will show that U is the desired τ-neighbourhood of x. Pick $x' \in U$. Then there exists t_i such that $d(x, x') \in (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Since $x' \in U_{t_i}$ we get

$$|\Phi f(x, x', d(x, x'))| = \left|\frac{f(x)-f(x')}{d(x,x')}\right| < \frac{\varepsilon}{2R}$$

and thus $|f(x) - f(x')| < \varepsilon$ for every $x' \in U$ and $f \in \mathcal{F}$. This proves that $\mathcal{F}$ is equicontinuous at every $x \in M$.

Finally, let us prove condition (2). To this end pick a positive ε. For every $x \in M$ we have, from equicontinuity of $\Phi(\mathcal{F})$ at $(x, x, 0)$, the existence of U_x a τ- open neighbourhood of x in M and a positive $\delta_x > 0$ such that $x', y' \in U_x$ and $0 < t < \delta_x$ implies $|\Phi f(x', y', t)| < \varepsilon$ for every $f \in \mathcal{F}$.

As $M \times M = \bigcup_{x \in M} U_x \times U_x$, we get by compactness the existence of $x_1, \ldots, x_n \in M$ such that $M \times M \subseteq \bigcup_{i=1}^{n} U_{x_i} \times U_{x_i}$. Pick $\delta := \min_{1 \leq i \leq n} \delta_{x_i}$. Now, if $x, y \in M$ verify that $0 < d(x,y) < \delta$ then there exists $i \in \{1, \ldots, n\}$ such that $x, y \in U_{x_i}$. As $d(x,y) < \delta \leq \delta_{x_i}$ we get

$$\frac{|f(x) - f(y)|}{d(x,y)} = |\Phi f(x, y, d(x,y))| < \varepsilon$$

for every $f \in \mathcal{F}$, which proves (2) and finishes the proof. □

The previous lemma allows us to identity $lip_\tau(M, X)$ as a space of compact operators from X^* to $lip_\tau(M)$.

Theorem 5.2.8. *Let M be a pointed metric space and let τ be a topology on M such that (M, τ) is compact and d is τ lower-semi-continuous. Then,*

$$lip_\tau(M, X) \equiv \mathcal{K}_{w^*,w}(X^*, lip_\tau(M)).$$

Moreover, if either $lip_\tau(M)$ or X has the (AP), then $lip_\tau(M, X) \equiv lip_\tau(M) \widehat{\otimes}_\varepsilon X$.

Proof. It is shown in [JVSVV14] that $f \mapsto f^t$, where $f^t(x^*) = x^* \circ f$, defines an isometry from $\mathrm{Lip}_0(M, X)$ onto $L_{w^*,w^*}(X^*, \mathrm{Lip}_0(M)))$. Let f be in $lip_\tau(M, X)$ and let us prove that $f^t \in \mathcal{K}_{w^*,w}(X^*, lip_\tau(M))$. Notice that $x^* \circ f$ is τ-continuous for every $x^* \in X^*$. Moreover, for every $x \neq y \in M$ and every $x^* \in X^*$, we have

$$\frac{|x^* \circ f(x) - x^* \circ f(y)|}{d(x,y)} \leq \|x^*\| \frac{\|f(x) - f(y)\|}{d(x,y)} \tag{5.1}$$

thus $x^* \circ f \in \mathrm{lip}_0(M)$. Therefore $f^t(X^*) \subset lip_\tau(M)$. We claim that $f^t(B_{X^*})$ is relatively compact in $lip_\tau(M)$. In order to show that, we need to check the conditions in Lemma 5.2.7. First, it is clear that $f^t(B_{X^*})$ is bounded. Moreover, it follows from (5.1) that the functions in $f^t(B_{X^*})$ satisfy the uniform little-Lipschitz condition. Finally, $f^t(B_{X^*})$ is equicontinuous in the sense of Lemma 5.2.7. Indeed, given $x \in M$ and $\varepsilon > 0$, there exists a τ-neighbourhood U of x such that $\|f(x) - f(y)\| < \varepsilon$ whenever $y \in U$. That is,

$$\sup_{x^* \in B_{X^*}} |x^* \circ f(x) - x^* \circ f(y)| < \varepsilon$$

whenever $y \in U$, as we wanted. Now, Lemma 5.2.7 implies that $f^t(B_{X^*})$ is a relatively compact subset of $lip_\tau(M)$ and thus $f^t \in \mathcal{K}(X^*, lip_\tau(M)) \cap L_{w^*,w^*}(X^*, \mathrm{Lip}_0(M))$. Finally, the set $\overline{f^t(B_{X^*})}$ is norm-compact and thus every coarser Hausdorff topology agrees on it with the norm topology. In particular, the weak topology of $lip_\tau(M)$ agrees on $f^t(B_{X^*})$ with the inherited weak* topology of $\mathrm{Lip}_0(M)$. Thus $f^t|_{B_{X^*}} : B_{X^*} \to lip_\tau(M)$ is weak*-to-weak continuous. By [Kim13, Proposition 3.1] we have that $f^t \in \mathcal{K}_{w^*,w}(X^*, lip_\tau(M))$.

It only remains to prove that the isometry is onto. So take $T \in \mathcal{K}_{w^*,w}(X^*, lip_\tau(M))$. We claim that T is weak*-to-weak continuous from X^* to $\mathrm{Lip}_0(M)$. Indeed, assume that (x^*_α) is a net in X^* weak* convergent to some $x^* \in X^*$. Since every $\gamma \in \mathcal{F}(M)$ is also an element in $lip_\tau(M)^*$, we get that $\langle \gamma, Tx^*_\alpha \rangle$ converges to $\langle \gamma, Tx^* \rangle$. Thus, $T \in L_{w^*,w^*}(X^*, \mathrm{Lip}_0(M)))$. By the isometry described above, there exists $f \in \mathrm{Lip}_0(M, X)$ such that $T = f^t$. Let us

prove that f actually belongs to $lip_\tau(M,X)$. As $f^t(B_{X^*})$ is relatively compact, then by Lemma 5.2.7 we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{0<d(x,y)<\delta} \frac{|x^* \circ f(x) - x^* \circ f(y)|}{d(x,y)} < \varepsilon$$

for each $x^* \in B_{X^*}$. By taking supremum with $x^* \in B_{X^*}$ we get that

$$\sup_{0<d(x,y)<\delta} \frac{\|f(x) - f(y)\|}{d(x,y)} \leq \varepsilon,$$

so $f \in \mathrm{lip}_0(M,X)$. We will prove, to finish the proof, that f is τ-to-$\|\cdot\|$ continuous. To this end pick $y \in M$ and $\varepsilon > 0$. By equicontinuity of $f^t(B_{X^*})$ we can find U a τ-neighbourhood of y such that $|x^* \circ f(y') - x^* \circ f(y)| < \varepsilon$ for every $x^* \in B_{X^*}$ and $y' \in U$. Now,

$$\|f(y') - f(y)\| = \sup_{x^* \in B_{X^*}} |x^*(f(y') - f(y))| \leq \varepsilon$$

for every $y' \in U$. Consequently, f is τ-to-$\|\cdot\|$ continuous. So $f \in lip_\tau(M,X)$, as desired.

Finally, if either $lip_\tau(M)$ or X has the approximation property, then Proposition 5.2.3 yields the equality $\mathcal{K}_{w^*,w}(X^*, lip_\tau(M)) \equiv lip_\tau(M)\widehat{\otimes}_\varepsilon X$. □

Now we get our second duality result for vector-valued Lipschitz free Banach spaces, which extends Theorem 2.2.4.

Theorem 5.2.9. *Let M be a separable, complete, and bounded pointed metric space. Let τ be a compat and metrisable topology on M so that $lip_\tau(M)$ 1-separates points uniformly. If either $\mathcal{F}(M)$ or X^* has the (AP), then $lip_\tau(M,X)^* \equiv \mathcal{F}(M,X^*)$.*

Proof. By [Kal04, Theorem 6.2] we have that $lip_\tau(M)$ is a predual of $\mathcal{F}(M)$. Consequently, $\mathcal{F}(M)$ has the (RNP). Therefore, we get from Theorem 5.2.8 and from Theorem 5.1.5 that

$$lip_\tau(M,X)^* \equiv (lip_\tau(M)\widehat{\otimes}_\varepsilon X)^* \equiv \mathcal{F}(M)\widehat{\otimes}_\pi X^* = \mathcal{F}(M,X^*),$$

which finishes the proof. □

Last result applies to the following particular case (see Proposition 6.3 in [Kal04]). Given two Banach spaces X, Y, and ω a non trivial gauge, we will denote $lip_{\omega,*}(B_{X^*},Y) := lip_{w^*}\big((B_{X^*}, \omega \circ \|\cdot\|), Y\big)$.

Corollary 5.2.10. *Let X and Y be Banach spaces, and let ω be a non trivial gauge. Assume that X^* is separable and that either $\mathcal{F}(B_{X^*}, \omega \circ \|\cdot\|)$ or Y^* has the (AP). Then, $lip_{\omega,*}(B_{X^*},Y) \equiv lip_{\omega,*}(B_{X^*})\widehat{\otimes}_\varepsilon Y$ and $lip_{\omega,*}(B_{X^*},Y)^* \equiv \mathcal{F}\big((B_{X^*}, \omega \circ \|\cdot\|), Y^*\big)$.*

5.3 Schur properties in the vector-valued case

In consideration of Chapter 3, the main purpose of the section is to find sufficient conditions on M and X which imply that $\mathcal{F}(M,X)$ has the Schur property (or a stronger version of it). Bearing in mind the definition $\mathcal{F}(M,X) = \mathcal{F}(M)\widehat{\otimes}_\pi X$, one may think that this kind of result relies on tensor product theory and thus on the scalar case ($X = \mathbb{R}$).

Nevertheless, to the best of our knowledge, it is an open problem how projective tensor product preserves the Schur property [GG01, Remark 6].

In fact, we will consider two different points of view. On the one hand, we will work on the "predual version" of the Schur property, namely the *Dunford–Pettis property*. We use tensor product theory at many times in our proofs. On the other hand, we will follow a different pattern which does not rely on tensor product theory. In fact, we use techniques inspired by the scalar case. Therefore, we get examples of Banach spaces having Schur property whose projective tensor product still has the Schur property.

We start with the following reminder.

Definition 5.3.1. A Banach space X is said to have the *Dunford–Pettis property* whenever every weakly compact operator from X into a Banach space Y is *completely continuous*, i.e. carries weakly compact sets into norm compact sets.

An equivalent definition is that for any weakly convergent sequences $(x_n)_{n=1}^{\infty} \subset X$ and $(x_n^*)_{n=1}^{\infty} \subset X^*$, converging (weakly) to x and x^* respectively, the sequence $(x_n^*(x_n))_{n=1}^{\infty}$ converges to $x^*(x)$ (see Theorem 5.4.4 in [AK06]).

It is known that a dual Banach space X^* has the Schur property if, and only if, X has the Dunford–Pettis property and does not contain any isomorphic copy of ℓ_1 [GJL95, Theorem 5.2]. So, in order to analyse the Schur property in $\mathcal{F}(M, X^*)$, it can be useful analysing the Dunford–Pettis property in the predual in case such a predual exists. For this, in the proper case, we can go much further.

Theorem 5.3.2. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly. Then $S_0(M)$ does not contain any isomorphic copy of ℓ_1 and has the hereditary Dunford–Pettis property, i.e. every closed subspace of $S_0(M)$ has the Dunford–Pettis property.*

Proof. By Lemma 3.2.4 we get that $S_0(M)$ is $(1+\varepsilon)$ isometric to a subspace of c_0, which is known to have the hereditary Dunford–Pettis (see e.g. [Cem87]). Consequently, $S_0(M)$ has the hereditary Dunford–Pettis property. Obviously, previous condition also implies that $S_0(M)$ does not contain any isomorphic copy of ℓ_1. □

The above theorem not only applies to the scalar valued version of $S_0(M)$ but also in the vector-valued one. Indeed, we get the following result.

Theorem 5.3.3. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly. Assume that X is a Banach space having the hereditary Dunford–Pettis property and which does not contain any isomorphic copy of ℓ_1. If either X or $S_0(M)$ has the (AP), then $S_0(M, X)$ does not contain any isomorphic copy of ℓ_1 and has the hereditary Dunford–Pettis property.*

Proof. As $S_0(M)\widehat{\otimes}_\varepsilon X \equiv S_0(M, X)$ holds because of Theorem 5.2.5, it does not contain any isomorphic copy of ℓ_1 [Ros07, Corollary 4]. Moreover, as $S_0(M)$ is isomorphic to a subspace of c_0, then $S_0(M, X) \equiv S_0(M)\widehat{\otimes}_\varepsilon X$ is isomorphic to a subspace of $c_0\widehat{\otimes}_\varepsilon X \equiv c_0(X)$. As $c_0(X)$ has the hereditary Dunford–Pettis property whenever X has the hereditary Dunford–Pettis property [KO89, Theorem 3.1] we get that $S_0(M, X)$ has the hereditary Dunford–Pettis property. □

As it is known that a dual Banach space X^* has the strong Schur property (Definition 3.2.1) whenever X does not contain any isomorphic copy of ℓ_1 and has the hereditary Dunford–Pettis property (see Example 3.2.2), we get the following corollary.

Corollary 5.3.4. *Let M be a proper pointed metric space such that $S_0(M)$ separates points uniformly. Assume that X is a Banach space with the hereditary Dunford–Pettis property and that X does not contain any isomorphic copy of ℓ_1. If either X^* or $\mathcal{F}(M)$ has the (AP), then $\mathcal{F}(M, X^*)$ has the strong Schur property.*

Above corollary should be compared with Proposition 3.2.5 in the real case. As announced earlier, we now turn to methods from the scalar case. To this aim, we will analyse the uniformly discrete case, for which Kalton proved in [Kal04] that the scalar valued Lipschitz free Banach space has the Schur property. Here we extend this result to a vector-valued setting.

Proposition 5.3.5. *Let (M, d) be a uniformly discrete pointed metric space and let X be a Banach space with the Schur property. Then $\mathcal{F}(M, X)$ has the Schur property.*

Proof. For this purpose we will need Kalton's decomposition (see Lemma 4.2 in [Kal04]). That is, there exist a universal constant $C > 0$ and a sequence of operators $T_k \colon \mathcal{F}(M) \to \mathcal{F}(M_k)$, where $k \in \mathbb{Z}$ and M_k denotes the closed ball $\overline{B}(0, 2^k)$, satisfying

$$\gamma = \sum_{k\in\mathbb{Z}} T_k\gamma \text{ unconditionally and } \sum_{k\in\mathbb{Z}} \|T_k\gamma\| \leq C\|\gamma\|$$

for every $\gamma \in \mathcal{F}(M)$. Now, using this decomposition, we can consider $S \colon \mathcal{F}(M) \to (\sum \mathcal{F}(M_k))_{\ell_1}$ defined by $S\gamma = (T_k\gamma)_{k=1}^{\infty}$. So S defines an isomorphism between $\mathcal{F}(M)$ and a closed subspace of $(\sum \mathcal{F}(M_k))_{\ell_1}$.

We will show that the image of $\mathcal{F}(M)$ is complemented in $(\sum \mathcal{F}(M_k))_{\ell_1}$. To achieve this we define $P : (\sum \mathcal{F}(M_k))_{\ell_1} \to S(\mathcal{F}(M))$ by $P((\gamma_k)_k) = (T_k\gamma)_{k=1}^{\infty}$, where $\gamma = \sum_k \gamma_k$. Then P is a well defined projection. Indeed, if $(\gamma_k)_k \in (\sum \mathcal{F}(M_k))_{\ell_1}$ then $P(P((\gamma_k)_k)) = P((T_k\gamma)_{k=1}^{\infty})$. Now, if we define $\gamma := \sum_{k\in\mathbb{Z}} T_k\gamma$, it follows that $P((T_k\gamma)_{k=1}^{\infty}) = (T_k\gamma)_{k=1}^{\infty}$, which proves that $P \circ P = P$. Notice that P is continuous since, given $(\gamma_k) \in (\sum \mathcal{F}(M_k))_{\ell_1}$, if we define $\gamma := \sum_{k\in\mathbb{Z}} \gamma_k$, we have the following chain of inequalities

$$\|P((\gamma_k))\| = \left\| \sum_{k\in\mathbb{Z}} T_k\gamma \right\| \leq \sum_{k\in\mathbb{Z}} \|T_k\gamma\| \leq C\|\gamma\| = C \left\| \sum_{k\in\mathbb{Z}} \gamma_k \right\| \leq C \sum_{k\in\mathbb{Z}} \|\gamma_k\|.$$

Thus $\mathcal{F}(M)\widehat{\otimes}_\pi X$ is isomorphic to a subspace of $(\sum \mathcal{F}(M_k))_{\ell_1}\widehat{\otimes}_\pi X$ [Rya02, Proposition 2.4]. It is not difficult to prove that $(\sum \mathcal{F}(M_k))_{\ell_1}\widehat{\otimes}_\pi X$ is isometrically isomorphic to $(\sum \mathcal{F}(M_k)\widehat{\otimes}_\pi X)_{\ell_1}$. Consequently, we have that $\mathcal{F}(M, X)$ is isomorphic to a subspace of $(\sum \mathcal{F}(M_k, X))_{\ell_1}$.

In order to finish the proof, we will prove that $\mathcal{F}(M_k, X)$ has the Schur property for every k, which will be enough since the Schur property is stable under ℓ_1 sums [Tan98] and by passing to subspaces. To do that, we will show that $\mathcal{F}(M_k, X)$ is isomorphic to $\ell_1(M_k, X)$ (the space of all absolutely summable families in X indexed by M_k), which enjoys the Schur property since X has it. Consider F a finite set and $\gamma = \sum_{i\in F} \delta_{m_i} \otimes x_i \in \mathcal{F}(M_k, X)$. Using the triangle inequality we have

$$\|\gamma\| \leq \sum_{i\in F} \|\delta_{m_i}\| \|x_i\| \leq 2^k \sum_{i\in F} \|x_i\|.$$

Moreover, for each $i \in F$, pick $x_i^* \in S_{X^*}$ such that $x_i^*(x_i) = \|x_i\|$ and define $f\colon M_k \to X^*$ by the equation

$$f(m) := \begin{cases} x_i^* & \text{if } m = m_i \text{ for some } i \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Since $2^{-k}\|f\|_\infty \leq \|f\|_L \leq 2\theta^{-1}\|f\|_\infty$, we get that $\|f\|_L \leq 2\theta^{-1}$. Thus,

$$\|\gamma\| \geq \langle \frac{\theta}{2} f, \gamma \rangle = \frac{\theta}{2} \sum_{i \in F} \|x_i\|.$$

This proves that the linear operator $T : \mathcal{F}(M_k, X) \to \ell_1(M_k, X)$ defined by

$$T\Big(\sum_{i \in F} \delta_{m_i} \otimes x_i\Big) = (z_m)_{m \in M_k},$$

where $z_{m_i} = x_i$ and $z_m = 0$ otherwise, is an isomorphism. □

Remark 5.3.6. Since $\mathcal{F}(M_k, X^*)$ is isomorphic to $\ell_1(X^*)$, we get that $\mathcal{F}(M, X^*)$ has the strong Schur property whenever X^* has it in the above proposition. Indeed, this follows from the two next propositions.

Proposition 5.3.7. *Let $(X_k)_{k=1}^N$ be a finite family of Banach spaces. Assume that each X_k has the strong Schur property with the same constant K in the definition of this property. Then $X = (\sum_{k=1}^N X_k)_{\ell_1}$ has the strong Schur property with constant $K+\varepsilon$ for every $\varepsilon > 0$.*

Proof. Fix $\delta > 0$ and let $(x_n)_{n=1}^\infty$ be a δ-separated sequence in the unit ball of X. We denote $(x'_n)_n \prec (x_n)_{n=1}^\infty$ to mean that $(x'_n)_{n=1}^\infty$ is a subsequence of $(x_n)_{n=1}^\infty$. We consider the following quantity :

$$\Delta = \sup\left\{\sum_{k=1}^N \delta_k : \exists (x'_n)_{n=1}^\infty \prec (x_n)_{n=1}^\infty, \forall k = 1 \cdots N, (x'_n(k))_{n=1}^\infty \text{ is } \delta_k\text{-separated}\right\}$$

We will show that $\Delta \geq \delta$. Let $\varepsilon > 0$ arbitrary, and assume that $\Delta < \delta - \varepsilon$. Then there exist $(x'_n)_{n=1}^\infty$ a subsequence of $(x_n)_{n=1}^\infty$ and $(\delta_1, \cdots, \delta_N) \in [0,2]^N$ such that, for every k, $(x'_n(k))_{n=1}^\infty$ is δ_k-separated and $\sum_{k=1}^N \delta_k > \Delta - \frac{\varepsilon}{N}$. Ramsey theorem provides a subsequence $(x''_n)_{n=1}^\infty$ of $(x'_n)_{n=1}^\infty$ such that one of the following conditions hold :

1. $\|x''_n(1) - x''_m(1)\| > \delta_1 + \frac{\varepsilon}{N}$ for every $n \neq m$.
2. $\|x''_n(1) - x''_m(1)\| \leq \delta_1 + \frac{\varepsilon}{N}$ for every $n \neq m$.

Let us see that the first case (1) does not hold. Otherwise $(x''_n(1))_{n=1}^\infty$ is $(\delta_1 + \frac{\varepsilon}{N})$-separated and so $\Delta \geq \sum_{k=1}^N \delta_k + \frac{\varepsilon}{N} > \Delta$, which is impossible. Therefore the second case holds. By iterating this argument we can assume that, for every $n \neq m$ and every k, $\|x_n(k) - x_m(k)\| \leq \delta_k + \frac{\varepsilon}{N}$. Thus $\sum_{k=1}^N \|x_n(k) - x_m(k)\| \leq \sum_{k=1}^N \delta_k + \varepsilon < \delta$, which contradicts the δ-separation of the original sequence. Consequently $\Delta \geq \delta$.

We now consider $(x'_n)_{n=1}^\infty$ a subsequence of $(x_n)_{n=1}^\infty$ such that, for every k, $(x'_n(k))_{n=1}^\infty$ is δ_k-separated and $\sum_{k=1}^N \delta_k \geq \delta - \varepsilon$. Since each X_k has the strong Schur property, for every k with $\delta_k > 0$, there exists a subsequence of $(x'_n(k))_{n=1}^\infty$, still denoted by $(x'_n(k))_{n=1}^\infty$ for convenience, such that $(x'_n(k))_{n=1}^\infty$ is (K/δ_k)-equivalent to the ℓ_1-basis. Next, by a

diagonal argument, there exists $(x''_n)_{n=1}^\infty$ a subsequence of $(x'_n)_{n=1}^\infty$ such that, for every k with $\delta_k > 0$, $(x''_n(k))_{n=1}^\infty$ is (K/δ_k)-equivalent to the ℓ_1-basis. Then a simple computation shows that, for every $(a_i)_{i=1}^m \in \mathbb{R}^m$, it follows

$$\begin{aligned}\left\|\sum_{i=1}^m a_i x''_i\right\| &= \sum_{k=1}^N \left\|\sum_{i=1}^m a_i x''_i(k)\right\| \\ &\geq \sum_{k=1}^N \frac{\delta_k}{K}\sum_{i=1}^m |a_i| \\ &\geq \frac{\delta-\varepsilon}{K}\sum_{i=1}^m |a_i|.\end{aligned}$$

This proves that $(x''_n)_{n=1}^\infty$ is $\frac{K}{\delta-\varepsilon}$-equivalent to the ℓ_1-basis. □

Now extend the previous result to infinite ℓ_1-sums. To achieve this we need to assume that the spaces X_k are dual ones.

Proposition 5.3.8. *Let $(X_k)_{k\in\mathbb{N}}$ be a family of Banach spaces. Assume that each X_k^* has the strong Schur property with the same constant K in the definition of this property. We consider $X = (\sum_{k\in\mathbb{N}} X_k)_{c_0}$ and its dual space $X^* = (\sum_{k\in\mathbb{N}} X_k^*)_{\ell_1}$. Then X^* has the strong Schur property with constant $\max\{2K+\varepsilon, 4+\varepsilon\}$ for every $\varepsilon > 0$.*

Proof. For $N \in \mathbb{N}$, we denote $P_N : X^* \to (\sum_{k=1}^N X_k^*)_{\ell_1}$ the norm-one projection on the N first coordinates. Fix $\delta > 0$ and let $(x_n)_{n=1}^\infty$ be a δ-separated sequence in the unit ball of X^*. Fix also $\varepsilon > 0$. Now two cases may occur.

First case. There exist $N \in \mathbb{N}$ such that there is $(x'_n)_{n=1}^\infty$ a subsequence of $(x_n)_{n=1}^\infty$ satisfying $d(x'_n, \sum_{k=1}^N X_k^*) \leq \delta/4$. Then a straightforward computation using the triangle inequality shows that $(P_N(x'_n))_{n=1}^\infty$ is $(\delta/2)$-separated. Thus, according to Proposition 5.3.7, $(P_N(x'_n))_{n=1}^\infty$ admits a subsequence $(\frac{2K+\varepsilon}{\delta})$-equivalent to the ℓ_1-basis. For convenience we still denote the same way the subsequence considered. Now consider $(a_i)_{i=1}^m \in \mathbb{R}^m$, and let us estimate the following norm

$$\left\|\sum_{i=1}^m a_i x'_i\right\| \geq \left\|\sum_{i=1}^m a_i P_N(x'_i)\right\| \geq \frac{\delta}{2K+\varepsilon}\sum_{i=1}^m |a_i|.$$

This ends the first case.

Second case. For every $N \in \mathbb{N}$ and every subsequence $(x'_n)_{n=1}^\infty$, there exist n such that $d(x'_n, \sum_{k=1}^N X_k^*) > \frac{\delta}{4}$. Passing to a subsequence and using [Pet17, Lemma 2.13] we can assume that $(x_n)_{n=1}^\infty$ is weak* convergent to 0 and that $\|x_n\| \geq \frac{\delta}{2}$ for every n. We will construct by induction a subsequence with the desired property. To achieve this, fix $(\varepsilon_i)_{i=1}^\infty$ a sequence of positives real numbers smaller than $\frac{\delta}{4}$ such that $\prod_{i=1}^{+\infty}(1-\varepsilon_i) \geq 1-\varepsilon$ and take $C := 4\sum_{k=1}^{+\infty}\frac{\varepsilon_k}{\delta} < \varepsilon$. We begin with the construction of a sequence in X very close to $(x_n)_{n=1}^\infty$ which is equivalent to the ℓ_1-basis, and after this we will deduce what we want from the principle of small perturbations (see for example [AK06, Theorem 1.3.9]). More precisely we will construct a sequence $(P_{K_i}(x_{n_i}))_{i=1}^\infty$ which is $\frac{4}{\delta(1-\varepsilon)}$-equivalent to the ℓ_1-basis and such that $\|P_{K_i}(x_{n_i}) - x_{n_i}\| \leq \varepsilon_i$.

First of all, we set $n_1 = 1$ and $N_1 \in \mathbb{N}$ such that $\|P_{N_1}x_{n_1}\| \geq \|x_{n_1}\| - \varepsilon_1$.

Construction of $n_2 > n_1$. Since P_{N_1} is weak* continuous, $(P_{N_1}(x_n))_{n=1}^{\infty}$ is weak* null. We apply [AK06, Lemma 1.5.1], so there exists $m > n_1$ such that for all $n \geq m$ and for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2$,

$$\|\lambda_1 P_{N_1}(x_{n_1}) + \lambda_2 P_{N_1}(x_n)\| \geq (1-\varepsilon_2)\|\lambda_1 P_{N_1}(x_{n_1})\|.$$

Now, using the assumption of the second case, there exist $n_2 \geq m$ such that $\|x_{n_2} - P_{K_1}(x_{n_2})\| > \frac{\delta}{4}$. We then pick $N_2 > N_1$ such that $\|P_{N_2}(x_{n_2}) - P_{N_1}(x_{n_2})\| > \frac{\delta}{4}$ and $\|P_{N_2}(x_{n_2}) - x_{n_2}\| < \varepsilon_2$. Next the following inequalities hold :

$$\begin{aligned}
\|\lambda_1 P_{N_1}(x_{n_1}) + \lambda_2 P_{N_2}(x_{n_2})\| &= \|\lambda_1 P_{N_1}(x_{n_1}) + \lambda_2 P_{N_1}(x_{n_2})\| \\
&+ \|\lambda_2 [P_{N_2} - P_{N_1}](x_{n_2})\| \\
&> (1-\varepsilon_2)\|\lambda_1 P_{N_1}(x_{n_1})\| + |\lambda_2|\frac{\delta}{4} \\
&> (1-\varepsilon_1)(1-\varepsilon_2)\frac{\delta}{4}|\lambda_1| + |\lambda_2|\frac{\delta}{4} \\
&> (1-\varepsilon_1)(1-\varepsilon_2)\frac{\delta}{4}(|\lambda_1| + |\lambda_2|).
\end{aligned}$$

We continue this construction by induction to get a sequence $(P_{N_i}(x_{n_i}))_{i=1}^{\infty}$ which is $\frac{4}{\delta(1-\varepsilon)}$-equivalent to the ℓ_1-basis and verifying $\|P_{N_i}(x_{n_i}) - x_{n_i}\| \leq \varepsilon_i$. By choice we have $C < \varepsilon$. Thus we can apply the principle of small perturbations which gives the following inequalities

$$\begin{aligned}
\left\|\sum_{i=1}^{m} a_i x_{n_i}\right\| &\geq \frac{1-C}{1+C}\left\|\sum_{i=1}^{m} a_i P_{N_i}(x_{n_i})\right\| \\
&\geq \frac{1-\varepsilon}{1+\varepsilon}(1-\varepsilon)\frac{\delta}{4}\sum_{i=1}^{m}|a_i| \\
&\geq \frac{\delta}{4}\frac{(1-\varepsilon)^2}{1+\varepsilon}\sum_{i=1}^{m}|a_i|,
\end{aligned}$$

for every $(a_i)_{i=1}^{m} \in \mathbb{R}^m$. This ends the second case and finishes the proof. □

5.4 Norm attainment

The isometric identification $\mathrm{Lip}_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$ yields two natural definitions of norm attainment for $f \in \mathrm{Lip}_0(M, X)$. On the one hand we can see f as a linear operator denoted $\overline{f} \colon \mathcal{F}(M) \to X$ in Proposition 1.1.2. We can consider the classical definition of norm attainment : $\overline{f}$ attains its norm if there exists $\gamma \in S_{\mathcal{F}(M)}$ such that :

$$\|\langle f, \gamma\rangle\|_X = \|\overline{f}\|_{\mathcal{L}(\mathcal{F}(M),X)} = \|f\|_L.$$

By an abuse of notation, we will use the same notation f for referring to either the bounded operator from $\mathcal{F}(M)$ to X or the Lipschitz map from M to X. We denote $NA(\mathcal{F}(M), X)$ the set of continuous operators which attain their norm.

On the other hand, considering $f \in \mathrm{Lip}_0(M,X)$, we say that f *strongly attains its norm* if there are two different points $x, y \in M$ such that :

$$\|f(x) - f(y)\| = \|f\|_L \, d(x,y).$$

Obviously, if $f \in \mathrm{Lip}_0(M,X)$ strongly attains its norm, say on $(x,y) \in M^2$, then the associated operator $f \in \mathcal{L}(\mathcal{F}(M), X)$ attains its operator norm on the molecule m_{xy}. We will mean by $Lip_{SNA}(M,X)$ the class of all functions in $\mathrm{Lip}_0(M,X)$ which strongly attain their norm.

A natural question here is wondering when both concepts of norm-attainment agree and, connected with this, wondering about the density of the class of Lipschitz functions which strongly attain their norm in $\mathrm{Lip}_0(M,X)$. Nice results recently appeared in this line. On the one hand, negative results can be found in [KMS16], where it is proved that $Lip_{SNA}(X,\mathbb{R})$ is never dense in $\mathrm{Lip}_0(X)$ when X is a Banach space [KMS16, Theorem 2.3]. On the other hand, positive results in this line appear in [God15], where it is proved that if M is a compact metric space such that $\mathrm{Lip}_0(M)$ separates points uniformly and if E is finite dimensional, then $Lip_{SNA}(M,E)$ is norm-dense in $\mathrm{Lip}_0(M,E)$.

Let us recall that a Banach space is said to have the *Krein–Milman property* the (KMP) if every non-empty closed convex bounded subset has an extreme point. It is well known that the (RNP) implies the (KMP), although the converse is still an open question. We shall begin with a result for the case $X = \mathbb{R}$.

Proposition 5.4.1. *Let (M,d) be a pointed metric space such that $\mathcal{F}(M)$ has the (KMP) and such that $\mathrm{ext}(B_{\mathcal{F}(M)}) \subseteq V$. Then every $f \in \mathrm{Lip}_0(M)$ which attains its norm on $\mathcal{F}(M)$ also strongly attains it. In other words, the following equality holds :*

$$NA(\mathcal{F}(M), \mathbb{R}) = \mathrm{Lip}_{SNA}(M, \mathbb{R}).$$

Therefore, $\overline{\mathrm{Lip}_{SNA}(M,\mathbb{R})}^{\|\cdot\|} = \mathrm{Lip}_0(M)$.

Proof. Notice that the inclusion $\mathrm{Lip}_{SNA}(M,\mathbb{R}) \subseteq NA(\mathcal{F}(M),\mathbb{R})$ always holds. Thus we just have to prove the reverse one. Let f be a function in $\mathrm{Lip}_0(M)$ which attains its norm on $B_{\mathcal{F}(M)}$. Since $\mathcal{F}(M)$ has the (KMP), f also attains its norm at an extreme point. Indeed, the set

$$C = \{\mu \in B_{\mathcal{F}(M)} : \langle f, \mu \rangle = 1\}$$

is a non-empty closed convex bounded subset of $\mathcal{F}(M)$, so there is $\mu \in \mathrm{ext}(C)$. Since C is a face, μ is also an extreme point of $B_{\mathcal{F}(M)}$. Since $\mathrm{ext}(B_{\mathcal{F}(M)}) \subseteq V$, f attains its norm on a molecule m_{xy} with $x \neq y$.

The last part follows from the Bishop–Phelps theorem (see [FHH+01, Theorem 3.54]). □

As a consequence of Proposition 4.2.1, we get the following.

Corollary 5.4.2. *Let M be a separable pointed metric space such that $\mathcal{F}(M)$ admits a natural predual $X \subset S_0(M)$. Then*

$$NA(\mathcal{F}(M),\mathbb{R}) = \mathrm{Lip}_{SNA}(M,\mathbb{R}) \text{ and } \overline{\mathrm{Lip}_{SNA}(M,\mathbb{R})}^{\|\cdot\|} = \mathrm{Lip}_0(M).$$

We give some examples where the previous corollary applies.

Example 5.4.3.

1. M a compact metric space such that $\mathrm{lip}_0(M)$ separates points uniformly (note that this result was first proved by Godefroy using M-ideal theory, see [God15]). For instance M being compact and countable [Dal15a], being the middle third Cantor set [Wea99], or being any compact metric space where the distance is composed with a nontrivial gauge [Kal04].
2. M a proper metric space such that $S_0(M)$ separates points uniformly.
3. M a uniformly discrete metric space satisfying the assumptions of Proposition 4.3.5.
4. The metric spaces of Propostion 4.2.3. In particular, $(B_{X^*}, \|\cdot\|^p)$ the unit ball of a separable dual Banach space where the distance is the norm to the power $p \in (0,1)$ (see Proposition 6.3 in [Kal04]).

We turn to the study of vector-valued Lipschitz functions. The next result trivially extends the first part of Proposition 5.4.1 to vector-valued Lipschitz maps.

Proposition 5.4.4. *Let (M,d) be a pointed metric space and X be a Banach space. Assume that $\mathcal{F}(M)$ has the (KMP) and that $\mathrm{ext}(B_{\mathcal{F}(M)}) \subseteq V$. Then every $f \in \mathrm{Lip}_0(M,X)$ which attains its norm on $\mathcal{F}(M)$ also strongly attains it. Thus the following equality holds : $NA(\mathcal{F}(M),X) = Lip_{SNA}(M,X)$.*

Proof. Assume that $\gamma \in \mathcal{F}(M)$ is such that $\|\gamma\| \leq 1$ and $\|f(\gamma)\| = \|f\|_L$. Then, by Hahn-Banach theorem, there exists $x^* \in S_{X^*}$ verifying $\langle x^*, f(\gamma)\rangle = \|f(\gamma)\|$. But $x^* \circ f : M \to \mathbb{R}$ is a real-valued Lipschitz function which attains its operator norm on γ. Thus, Proposition 5.4.1 gives the conclusion. □

Since Bishop-Phelps theorem fails in the vector-valued case, we cannot deduce the same density result as in Proposition 5.4.1. To do so, we need a different argument. Actually, a result of Bourgain will do the trick. We first state this result. We say that an operator $T\colon X \to Y$ is strongly exposing if there exists $x \in S_X$ such that for every sequence $(x_n)_{n=1}^\infty \subset B_X$ such that $\lim_n \|Tx_n\|_Y = \|T\|$, there is a subsequence $(x_{n_k})_{k=1}^\infty$ which converges to either x or $-x$. Clearly every strongly exposing operator is norm attaining. Bourgain proved that if X has the (RNP) then for every Banach space Y the set of strongly exposing operators from X to Y is dense in $\mathcal{L}(X,Y)$ (see [Bou77, Theorem 5]). This leads us to the following result.

Proposition 5.4.5. *Let M be a complete pointed metric space and X be a Banach space. Assume that $\mathcal{F}(M)$ has the (RNP). Then $\mathrm{Lip}_{SNA}(M,X)$ is norm dense in $\mathrm{Lip}_0(M,X)$.*

Proof. By Bourgain's theorem quoted above, it suffices to show that every strongly exposing operator $T \in \mathcal{L}(\mathcal{F}(M),X)$ attains its norm at a molecule, and so $T \circ \delta \in \mathrm{Lip}_{SNA}(M,X)$. Let $T\colon \mathcal{F}(M) \to X$ and $\mu \in \mathcal{F}(M)$ witnessing the definition of strongly exposing operator (defined above). Take a sequence $(x_n^*)_{n=1}^\infty \subset S_{X^*}$ such that

$$\|T^* x_n^*\|_{X^*} > \|T\| - 1/n,$$

for every $n \in \mathbb{N}$. Since V is 1-norming, there is a sequence $(m_{x_n,y_n})_{n=1}^{\infty} \subset V$ such that $\langle T^* x_n^*, m_{x_n,y_n} \rangle \geq \|T\| - 1/n$ for every n. Note that :

$$\langle T^* x_n^*, m_{x_n,y_n} \rangle = \langle x_n^*, T m_{x_n,y_n} \rangle \leq \|T m_{x_n,y_n}\|_X.$$

So, $\lim_n \|T m_{x_n,y_n}\|_X = \|T\|$. Thus there is a subsequence $(m_{x_n,y_n})_{n=1}^{\infty}$ which converges to either μ or $-\mu$. Since V is norm-closed (being weakly sequentially closed, see Corollary 1.3.4) we get that $\mu \in V$ as desired. □

5.5 Perspectives

It is known that the injective tensor product preserves the Schur property (see [BR09, Proposition 4.1]). However, as we mentioned earlier it is an open problem to know how the projective tensor product preserves the Schur property ([GG01, Remark 6]). Thus, related to our Section 5.3 we may try to explore this open problem when one factor of the projective tensor product is a Lipschitz free space.

Question 5.5.1. Let M be a metric space and let X be a Banach one. If both $\mathcal{F}(M)$ and X have the Schur property, can we deduce that $\mathcal{F}(M, X)$ has the Schur property ?

Note that an affirmative answer holds for $X = \ell_1(I)$, for any arbitrary set I, since

$$\mathcal{F}(M, \ell_1(I)) = \mathcal{F}(M) \widehat{\otimes}_\pi \ell_1(I) \equiv \ell_1(I, \mathcal{F}(M))$$

has the Schur property if, and only if, $\mathcal{F}(M)$ has the Schur property [Tan98, Proposition, Section 2].

In the context of norm attainment, we assumed at many times that $\mathcal{F}(M)$ enjoys the (RNP) (sometimes only the (KMP)). Nevertheless, there is still no good description of metric spaces such that $\mathcal{F}(M)$ has the (RNP). Of course, if M contains a line segment, say $[0, 1]$, then $\mathcal{F}(M)$ fails the (RNP). Indeed, in that case $L_1[0, 1]$ linearly embeds into $\mathcal{F}(M)$ but $L_1[0, 1]$ fails the (RNP) and this property is stable under passing to subspaces. Surprisingly, it seems that every example of metric space M works as follows. Either $\mathcal{F}(M)$ has both the Schur property and the (RNP), or $\mathcal{F}(M)$ has none of these properties. Yet, we proved in Proposition 3.1.2 that $\mathcal{F}(M)$ has the Schur property provided $\mathrm{lip}_0(M)$ is 1-norming. So we may wonder :

Question 5.5.2. Let M be a pointed metric space such that $\mathrm{lip}_0(M)$ is 1-norming. Is it true that $\mathcal{F}(M)$ has the (RNP) ?

In fact, we may also wonder :

Question 5.5.3. Let M be a pointed metric space. Is it true that $\mathcal{F}(M)$ has the Schur property if and only if $\mathcal{F}(M)$ has the (RNP) ?

Most likely, the answer to the previous question is negative. To the best of our knowledge, the known proofs of "$\mathcal{F}(M)$ has the (RNP)" use the fact that $\mathcal{F}(M)$ is isomorphic to a subspace of a separable dual Banach space. However, disproving a conjecture of Uhl, Bourgain and Delbaen constructed in [BD80] a Banach space X having the (RNP) which cannot be embedded isomorphically into a separable dual Banach space. So we address the following question.

Question 5.5.4. Is there a pointed metric space M such that $\mathcal{F}(M)$ have the (RNP) but $\mathcal{F}(M)$ does not embed isomorphically into a separable dual Banach space ?

Appendix A

The Demyanov–Ryabova conjecture

The subject of this appendix is completely independent of the main topic of the thesis. This joint work with Aris Daniilidis (see [DP18]) has been done during a two-month visit (October 2016 and April 2017) to the Center of Mathematical Modeling in Santiago de Chile. The author is deeply grateful to Aris and to the CMM for hospitality and excellent working conditions during his visit.

A.1 Introduction

A.1.1 The conjecture

We call *polytope* any convex compact subset of $\mathbb{R}^N$ with a finite number of extreme points. Throughout this work we consider a finite family $\Re = \{\Omega_1, \dots, \Omega_\ell\}$ of polytopes of $\mathbb{R}^N$ together with an operation which transforms the initial family $\Re$ to a dual family of polytopes that we denote $\mathscr{F}(\Re)$. (Motivation and origin of this operation will be given at the end of the introduction).

Let us now describe the operation $\mathscr{F}$: let $\mathrm{ext}(\Omega)$ stand for the set of extreme points of the polytope Ω and let S denote the unit sphere of $\mathbb{R}^N$. Then given a family $\Re$ as before, for any direction $d \in S$ and polytope $\Omega_i \in \Re$ ($i \in \{1, \dots, \ell\}$) we consider the set of d-active extreme points of Ω_i

$$E(\Omega_i, d) := \{x \in \mathrm{ext}(\Omega_i) : \langle x, d\rangle = \max\langle \Omega_i, d\rangle\}.$$

We associate to $d \in S$ the polytope

$$\Omega(d) := \mathrm{conv}\Big(\bigcup_{\Omega_i \in \Re} E(\Omega_i, d)\Big), \tag{A.1}$$

that is, the polytope obtained as convex hull of the set of all d-active extreme points (when Ω_i is taken throughout $\Re$). Since the set of extreme points of all polytopes of the family $\Re$

$$E_\Re = \bigcup_{\Omega_i \in \Re} \mathrm{ext}(\Omega_i) \tag{A.2}$$

is finite, the family of polytopes

$$\mathscr{F}(\Re) := \{\Omega(d) : d \in S\} \tag{A.3}$$

is also finite, hence of the same nature as $\mathfrak{R}$. We call $\mathscr{F}(\mathfrak{R})$ the dual family of $\mathfrak{R}$.

Now starting from a given family of polytopes $\mathfrak{R}_0$, we define successively a sequence of families $\{\mathfrak{R}_n\}_n$ by applying repeatedly this duality operation (transformation) $\mathscr{F}$, that is, setting $\mathfrak{R}_{n+1} := \mathscr{F}(\mathfrak{R}_n)$, for all $n \in \mathbb{N}$. Since the transformation $\mathscr{F}$ cannot create new extreme points, the sequence

$$E_{\mathfrak{R}_n} = \bigcup_{\Omega \in \mathfrak{R}_n} \operatorname{ext}(\Omega) \quad \text{(extreme points of polytopes in } \mathfrak{R}_n) \qquad n \in \mathbb{N}$$

is nested (decreasing) and eventually becomes stable, equal to a finite set E. By a standard combinatorial argument, we now deduce that for some $k \geq 1$ and $n_0 \geq 0$ we necessarily get $\mathfrak{R}_n = \mathfrak{R}_{n+k}$ (and $E_{\mathfrak{R}_n} = E$), for all $n \geq n_0$. Therefore, a k-cycle $(\mathfrak{R}_{n_0}, \mathfrak{R}_{n_0+1}, \cdots, \mathfrak{R}_{n_0+k-1})$ is always formed. We are now ready to announce the conjecture of Demyanov and Ryabova :

— **Conjecture** (Demyanov–Ryabova, [DR11]). Let $\mathfrak{R}_0$ be a finite family of polytopes in $\mathbb{R}^N$. Then for some $n_0 \in \mathbb{N}$ we shall have $\mathfrak{R}_{n_0} = \mathfrak{R}_{n_0+2}$.

In other words, after some threshold n_0 the sequence

$$\mathfrak{R}_0, \quad \mathfrak{R}_1 = \mathscr{F}(\mathfrak{R}_0), \cdots, \quad \mathfrak{R}_{n+1} = \mathscr{F}(\mathfrak{R}_n), \cdots$$

stabilizes to either a 1-cycle (self-dual family $\mathfrak{R}_n = \mathscr{F}(\mathfrak{R}_n) = \mathfrak{R}_{n+1}$) or to a 2-cycle (reflexive family $\mathfrak{R}_n = \mathscr{F}(\mathscr{F}(\mathfrak{R}_n)) = \mathfrak{R}_{n+2}$) for $n \geq n_0$. In [DR11], the authors carried out generic numerical experiments over two hundred families of polytopes, where only 1-cycles or 2-cycles eventually arise.

However, during the preparation of this thesis, Vera Roshchina communicated to us a family of polytopes $\mathfrak{R}_0$ in dimension 2 such that $\mathfrak{R}_5 = \mathfrak{R}_1$ but $\mathfrak{R}_3 \neq \mathfrak{R}_1$. So, for this particular family, a 4-cycle arise but no 2-cycle eventually arise. Consequently, the conjecture is false in full generality.

Nevertheless, the are special configurations under which the conjecture is true. The only known positive result in this direction is due to [San17]. In that work, the author establishes the conjecture under the additional assumption that the set $E_{\mathfrak{R}_0}$ of extreme points of the initial family $\mathfrak{R}_0$ is affinely independent. In this appendix, we prove the conjecture under different assumptions.

Before we state and prove our main result, let us mention that in 1–dimension the conjecture is trivially true.

Proposition A.1.1 (The conjecture is true in 1–dim)**.** *Let $\mathfrak{R}_0$ be a finite family of closed bounded intervals of $\mathbb{R}$. Then $\mathfrak{R}_1 = \mathfrak{R}_3$.*

Proof. Let us denote $\{I_1, \ldots, I_\ell\}$ the elements of $\mathfrak{R}_0$ with $I_j = [a_j, b_j]$, $j \in \{1, \ldots, \ell\}$. Since the unit sphere $S = S_{\mathbb{R}} = \{1, -1\}$ consists of only two directions, the construction of the dual family $\mathfrak{R}_1 = \mathscr{F}(\mathfrak{R}_0)$ is very simple. To this end, we set $a_- := \min_{i \in \{1..\ell\}} a_i$, $a_+ := \max_{i \in \{1..\ell\}} a_i$, $b_- := \min_{i \in \{1..\ell\}} b_i$, $b_+ := \max_{i \in \{1..\ell\}} b_i$. This leads to the family

$$\mathfrak{R}_1 = \{\Omega_1(-1), \Omega_1(1)\} = \{[a_-, a_+], [b_-, b_+]\}.$$

The construction of $\mathfrak{R}_2 = \mathscr{F}(\mathfrak{R}_1)$ is even simpler, since we only have two intervals (polytopes) to consider. We actually have

$$\mathfrak{R}_2 = \{\Omega_2(-1), \Omega_2(1)\} = \{[a_-, b_-], [a_+, b_+]\}.$$

It now suffices to compute $\Re_3$ and obtain directly that $\Re_1 = \Re_3$. (Notice that if it happens $a_+ = b_-$ then we actually get a 1-cycle : $\Re_1 = \Re_2$.) $\square$

The extreme simplicity of the problem in dimension 1 is due to the fact that the family that arises after any new iteration has at most 2 elements (corresponding to the directions 1 and -1 of the unit sphere $S_{\mathbb{R}}$). The problem gets much more complicated though in higher dimensions, where no prior efficient control on the cardinality of the iterated families can be obtained (apart from an absolute combinatorial bound on the number of all possible polytopes that can be obtained by convexifying subsets of the prescribed set of extreme points E). We shall now treat this general case.

Let $\Re_0$ be a finite family of polytopes in $\mathbb{R}^N$ ($N \geq 2$). We denote by $E := E_{\Re_0}$ the set of extreme points of all polytopes of the family, see (A.2), by $R = |E|$ its cardinality and we set

$$C := \mathrm{conv}(E)$$

its convex hull. Notice that every polytope Ω of the family $\Re_0$ (or of any family $\Re_n$ obtained after n-iterations, for every $n \in \mathbb{N}$), is contained in C. Let further

$$r(\Omega) := |\Omega \cap E|$$

denote the number of extreme points of the polytope $\Omega \in \Re_0$ and set

$$r_{\min} := \min_{\Omega \in \Re_0} r(\Omega). \tag{A.4}$$

We now state the main result of this chapter.

Theorem A.1.2 (Main result)**.** *Let $\Re_0$ be a finite family of polytopes in $\mathbb{R}^N$ and $r_{\min} \in \{1, \ldots, R\}$ as in (A.4). Then $\Re_1 = \Re_3$ (i.e. a reflexive family occurs after one iteration) provided :*

(H1) $\forall\, x \in E,\ x \notin \mathrm{conv}(E \backslash \{x\})$ *(i.e. each $x \in E$ is extreme in C.)*

(H2) $\Re_0$ contains all $r_{\min}$-polytopes (that is, all polytopes made up of $r_{\min}$ points of E).

Remark A.1.3. (i) Assumption (H_1) easily yields that the set of extreme points remains stable from the very beginning, that is,

$$E_{\Re_n} = E_{\Re_0} = E, \qquad \text{for all } n \in \mathbb{N}.$$

Indeed pick $x \in E$ and $\mathrm{e}_x \in S$ which exposes x in C. Let $\Omega \in \Re_0$ be such that $x \in \mathrm{ext}(\Omega)$ (there is clearly at least one such a polytope in $\Re_0$). Then e_x exposes x in Ω, that is $x \in E(\Omega, \mathrm{e}_x)$. It follows readily that $x \in \Omega(\mathrm{e}_x) \subset E_{\Re_1}$ (see the definition in (A.1)) and by a simple induction, $x \in E_{\Re_n}$, for every $n \geq 1$.

(ii) Assumption (H_2) will be weakened in the sequel.

A.1.2 Origin of the conjecture

The initial motivation which eventually led to the formulation of the above conjecture stems from the problem of stable representation of positively homogeneous polyhedral functions as a finite minima of sublinear ones, or its geometric counterpart, the representation of a closed polyhedral cone as a finite union of closed convex polyhedral cones. Let us recall that a function $f : \mathbb{R}^N \to \mathbb{R}$ is called positively homogeneous provided $f(\lambda x) = \lambda f(x)$ for every $x \in \mathbb{R}^N$ and $\lambda > 0$. It is called sublinear (respectively, superlinear) if it is positively homogeneous and convex (respectively, concave).

Following [Psh80], a sublinear function $\overline{g} : \mathbb{R}^N \to \mathbb{R}$ is called an upper convex approximation of f if $\overline{g}$ majorises f on $\mathbb{R}^N$, that is, $\overline{g}(x) \geq f(x)$, for every $x \in \mathbb{R}^N$. In the same way, a superlinear function $\underline{g} : \mathbb{R}^N \to \mathbb{R}$ is called a lower concave approximation of f if $\underline{g}$ minorises f on $\mathbb{R}^N$, that is, $\underline{g}(x) \leq f(x)$ for all $x \in \mathbb{R}^N$. Then we say that a set of sublinear functions E^* is an upper exhaustive family for f if the following equality holds for every $x \in \mathbb{R}^N$:

$$f(x) = \inf_{\overline{g} \in E^*} \overline{g}(x). \tag{A.5}$$

Similarly, we say that a set of superlinear functions E_* is a lower exhaustive family for f if the following equality holds for every $x \in \mathbb{R}^N$:

$$f(x) = \sup_{\underline{g} \in E_*} \overline{g}(x). \tag{A.6}$$

In [DR00] the authors established the existence of an upper exhaustive family of upper convex approximations (respectively lower exhaustive family of lower concave approximations) when f is upper semi-continuous on $\mathbb{R}^N$ (respectively lower-semi-continuous). In particular, if f is continuous, the existence of both such families is guaranteed.

It is well known (see [HUL01, Phe93] *e.g.*) that a function $\overline{g} : \mathbb{R}^N \to \mathbb{R}$ is sublinear if and only if $\overline{g}(x) = \max_{h \in \partial \overline{g}(0)} \langle x, h \rangle$. Using this fact we are able to restate (A.5) in the following way :

$$f(x) = \inf_{\overline{g} \in E^*} \overline{g}(x) = \inf_{\overline{g} \in E^*} \max_{h \in \partial \overline{g}(0)} \langle x, h \rangle = \inf_{\overline{\Omega} \in \overline{\mathfrak{R}}} \max_{h \in \overline{\Omega}} \langle x, h \rangle,$$

where $\overline{\mathfrak{R}} = \{\partial \overline{g}(0) : \overline{g} \in E^*\}$ is the family of subdifferentials of the sublinear functions $\overline{g}$ that represent f and $\overline{\Omega} = \partial \overline{g}(0)$. In a similar way, considering superlinear functions $\underline{g}$ (lower concave approximations of f) and denoting by $\underline{\mathfrak{R}} = \{-\partial(-\underline{g})(0) : \underline{g} \in E_*\}$ the family of superdifferentials $\underline{\Omega} = -\partial(-\underline{g})(0)$, we can restate (A.6) as follows :

$$f(x) = \sup_{\underline{\Omega} \in \underline{\mathfrak{R}}} \min_{h \in \underline{\Omega}} \langle x, h \rangle.$$

In case of a polyhedral function f the exhaustive families E^* and E_* can be taken to be finite, with elements being polyhedral functions ($\overline{g}$ and $\underline{g}$ respectively). In this case, the corresponding families $\overline{\mathfrak{R}}$ and $\underline{\mathfrak{R}}$ —called upper (respectively lower) exhausters— are made up of finite polytopes. In [DR11], the authors presented a procedure —that they called *converter*— which permits to define from a given lower exhauster $\underline{\mathfrak{R}}$ an upper exhauster $\overline{\mathfrak{R}} = \mathscr{F}(\underline{\mathfrak{R}})$ and vice-versa (this is actually the same procedure and coincides with the

described operator $\mathscr{F}$ in the beginning of the introduction). A lower (respectively, an upper) exhauster $\underline{\mathfrak{R}}$ (respectively, $\overline{\mathfrak{R}}$) is called *stable* or *reflexive*, if

$$\underline{\mathfrak{R}} = \mathscr{F}\left(\mathscr{F}(\underline{\mathfrak{R}})\right) \qquad \text{(respectively, } \overline{\mathfrak{R}} = \mathscr{F}\left(\mathscr{F}(\overline{\mathfrak{R}})\right)\text{).}$$

An equivalent way to formulate the Demyanov–Ryabova conjecture is to assert that starting with any finite (upper or lower) exhaustive family of polyhedral functions, we eventually end up to a stable one.

A.2 Preliminary results

Notation.
$\mathfrak{R}_0$ is a finite set of polytopes in $\mathbb{R}^N$ with $N \geq 2$.
S denotes the unit sphere of $\mathbb{R}^N$.
$\mathrm{ext}(\Omega)$ is the set of extreme points of a given polytope $\Omega \in \mathfrak{R}_0$.
$E = \bigcup_{\Omega \in \mathfrak{R}_0} \mathrm{ext}(\Omega)$ is the set of extreme points of all polytopes in $\mathfrak{R}_0$.
$R := |E|$
$C := \mathrm{conv}(E)$
We assume throughout this chapter that E satisfies the assumption (H1) of Theorem A.1.2. For the proof of Theorem A.1.2, we shall need the two following notions.

Definition A.2.1.

— **(d-compatible enumeration)** An enumeration $\{x_i\}_{i=1}^R$ of E is called **d-compatible** with respect to a direction $d \in S$, provided

$$\langle x_1, d\rangle \leq \langle x_2, d\rangle \leq \cdots \leq \langle x_R, d\rangle. \tag{A.7}$$

Notice that a d-compatible enumeration is not necessarily unique : indeed, whenever $\langle x_i, d\rangle = \langle x_j, d\rangle$, for $1 \leq i < j \leq R$ the elements x_i and x_j can be interchanged in the above enumeration.

— **(strict p-location)** A direction $d \in S$ is said to **locate strictly** an element $\bar{x} \in E$ at the p-position (where $p \in \{1, \ldots, R\}$), if there exists a d-compatible enumeration $\{x_i\}_{i=1}^R$ of E for which $x_p = \bar{x}$ and

$$\ldots \leq \langle x_{p-1}, d\rangle < \langle x_p, d\rangle < \langle x_{p+1}, d\rangle \leq \ldots$$

In case $p = 1$ (resp. $p = R$) the left strict inequality $\langle x_{p-1}, d\rangle < \langle x_p, d\rangle$ (resp. the right strict inequality $\langle x_p, d\rangle < \langle x_{p+1}, d\rangle$) is vacuous. Notice further that since C is a polytope, assumption (H_1) yields that for every $\bar{x} \in E$ the normal cone

$$N_C(\bar{x}) = \{d \in \mathbb{R}^N : \ \langle d, y - \bar{x}\rangle \leq 0, \ \forall y \in C\}$$

of C at $\bar{x}$ has nonempty interior (see [Roc97, HUL01] *e.g.*), and every $d \in \ S \cap \mathrm{int}\, N_C(\bar{x})$ strictly locates $\bar{x}$ in the R-position, under any d-compatible enumeration $\{x_i\}_{i=1}^R$ of E.

— **(selection)** A map $x \in E \mapsto \mathrm{e}_x \in S$ is called a selection if

$$\forall x \in E, \ \mathrm{e}_x \in S \cap \mathrm{int}\, N_C(x).$$

Thus, for every $x \in E$, e_x is a direction that strictly exposes x.

We now begin a series of "reordering results". The main goal is the following. Given a d-compatible enumeration of E which locates an element x at some position, say i, we construct a direction $d' \in S$ and a d'-compatible enumeration of E which locates strictly x to a possibly different position $p \geq i$. To construct such a d', the general idea is to do small perturbations on d using other well-chosen directions. These perturbations need to be quantified and adequately controlled. We start with the following simple lemma.

Lemma A.2.2 (Uniform control). *Let $d \in S$ and fix $x \in E \mapsto \mathrm{e}_x \in S \cap \operatorname{int} N_C(x)$ a selection. Then, there exist constants $M > 0$ and $m > 0$ such that, for every $x \in E$, the map $D_x : \mathbb{R} \to \mathbb{R}^N$ defined for every $t \in \mathbb{R}$ by $D_x(t) = d + t\mathrm{e}_x$ satisfies the following properties :*

1. *D_x is continuous and $D_x(0) = d$.*
2. *For $t > 0$ (respectively $t < 0$) large enough in absolute value, any $(D_x(t)/\|D_x(t)\|)$-compatible enumeration $(x_i)_{i=1}^R$ of E strictly locates x at the R-position (resp. at the 1-position). That is, for every $y \in E$, $y \neq x$: $\langle x, D_x(t)\rangle > \langle y, D_x(t)\rangle$ (resp. $\langle x, D_x(t)\rangle < \langle y, D_x(t)\rangle$).*
3. *For every y_1, $y_2 \in E$: $|\langle y_1 - y_2, D_x(t) - D_x(0)\rangle| \leq M|t|$.*
4. *For every $y \in E$, $y \neq x$: $|\langle x - y, D_x(t) - D_x(0)\rangle| \geq m|t|$.*

Proof. The first assertion is obvious. The second assertion is a simple consequence of the fact that $\mathrm{e}_x \in \operatorname{int} N_C(x)$ exposes x.

Now let us prove 3. We define $M = \max\{\|y_1 - y_2\| \,:\, y_1,\, y_2 \in E\} > 0$. Then, for every y_1, $y_2 \in E$,

$$|\langle y_1 - y_2, D_x(t) - D_x(0)\rangle| = |\langle y_1 - y_2, t\mathrm{e}_x\rangle| \leq |t|\,\|y_1 - y_2\|\,\|\mathrm{e}_x\| \leq M|t|.$$

In the same way we prove 4. Define $m = \min\{|\langle x - y, \mathrm{e}_x\rangle| \,:\, x,\, y \in E,\, x \neq y\} > 0$. Then for every x, $y \in E$ with $y \neq x$,

$$|\langle x - y, D_x(t) - D_x(0)\rangle| = |t|\,|\langle x - y, \mathrm{e}_x\rangle| \geq m|t|.$$

□

Remark A.2.3. Note that, whenever the selection $x \in E \mapsto \mathrm{e}_x \in S \cap \operatorname{int} N_C(x)$ is fixed, the constants m and M in the previous lemma hold for every function D_x (and do not depend neither on d, nor on x).

The next lemma will play a key role in the sequel.

Lemma A.2.4 (Strict location in the very next position). *Let $\{x_j\}_{j=1}^R$ be a d-compatible enumeration of E such that for some $1 \leq i \leq R-1$ we have :*

$$\ldots \leq \langle x_{i-1}, d\rangle < \langle x_i, d\rangle \leq \langle x_{i+1}, d\rangle \leq \ldots$$

Then there exist a direction $d' \in S$ and a d'-compatible enumeration $\{y_j\}_{j=1}^R$ satisfying

$$\{x_1, \cdots, x_{i-1}\} \subset \{y_1 \cdots, y_i\}$$

and locating strictly x_i at the $i+1$-position, that is,

$$\begin{cases} y_{i+1} = x_i \\ \ldots \leq \langle y_i, d'\rangle < \langle y_{i+1}, d'\rangle < \langle y_{i+2}, d'\rangle \leq \ldots \end{cases}$$

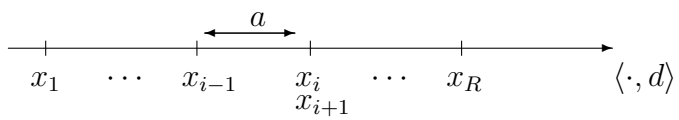
Proof. Throughout the proof, we fix $x \in E \mapsto \mathrm{e}_x \in S \cap \operatorname{int} N_C(x)$ a selection and $m, M > 0$ the universal constants given in Lemma A.2.2 (*c.f.* Remark A.2.3).

Case 1 : x_i is not strictly located in the i-position, that is the d-compatible enumeration $\{x_j\}_{j=1}^R$ verifies

$$\ldots \leq \langle x_{i-1}, d\rangle < \langle x_i, d\rangle = \langle x_{i+1}, d\rangle \leq \ldots$$

An additional difficulty here is that they may exist more than one $y \in E$ such that $\langle x_i, d\rangle = \langle y, d\rangle$ (that is x_{i+1} may not be the unique point with this property). So let $k \in \{i-1, \ldots, R\}$ be the maximum index such that $\langle x_i, d\rangle = \langle x_k, d\rangle$. Our strategy would be to do a small perturbation on d with a good control in order to put x_i at the i-position strictly. Of course this creates a new direction d' together with a new ordering of elements in E through d'. Then, we consider an element y which is right after x_i in the d'-ordering. Again, we do a small perturbation of d' with a good control in order to reverse the order of x_i and y. The key point is the uniform control of the employed perturbations ensuring that the element x_i reaches the $i+1$-position and not a further position.

Let us write $a = \langle x_i - x_{i-1}, d\rangle > 0$, $c = M/m$ and let $\varepsilon > 0$ such that $a - 2c\varepsilon > a/2 > 0$. Let us summarize our notations with the following picture



Step 1 : We locate x_i strictly in the i-position but in a controlled way. Consider the map $D_{x_i}(t) = d + t\mathrm{e}_{x_i}$ defined in Lemma A.2.2, and then define the function

$$\Phi : t \in \mathbb{R} \mapsto \min_{j \in \{i+1, \ldots, R\}} \langle x_j - x_i, D_{x_i}(t)\rangle.$$

The map Φ is continuous, satisfies $\Phi(0) = 0$ and $\lim_{t \to -\infty} \Phi(t) = +\infty$. Thus, by the intermediate value theorem, there exists $t_0 < 0$ such that $\Phi(t_0) = \varepsilon$. That is

$$\min_{j \in \{i+1, \ldots, R\}} \langle x_j, D_{x_i}(t_0)\rangle = \langle x_i, D_{x_i}(t_0)\rangle + \varepsilon.$$

Taking $\varepsilon > 0$ small enough we ensure that if $y \in (x_i)_{j=i+1}^R$ is such that $\langle y - x_i, D_{x_i}(t_0)\rangle = \varepsilon$, then $y \in \{x_{i+1}, \ldots, x_k\}$. Pick such a $y \in (x_i)_{j=i+1}^k$. Thanks to the assertion 4 of Lemma A.2.2, we have

$$\varepsilon = |\langle y - x_i, D_{x_i}(t_0) - D_{x_i}(0)\rangle| \geq m|t_0|.$$

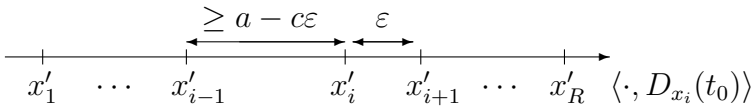
Thus $|t_0| \leq \varepsilon/m$. Next, thanks to the assertion 3 of Lemma A.2.2, for every j in $\{1, \ldots, i-1\}$ we have :

$$|\langle x_i - x_j, D_{x_i}(t_0) - D_{x_i}(0)\rangle| \leq M|t_0| \leq c\varepsilon.$$

This implies that

$$\langle x_i - x_j, D_{x_i}(t_0)\rangle \geq \langle x_i - x_j, D_{x_i}(0)\rangle - c\varepsilon \geq a - c\varepsilon. \tag{A.8}$$

Therefore we obtain a $(D_{x_i}(t_0)/\|D_{x_i}(t_0)\|)$-compatible enumeration $(x'_i)_{i=1}^R$ satisfying $x_i = x'_i$, $\{x_1, \ldots, x_{i-1}\} = \{x'_1, \ldots, x'_{i-1}\}$ and $y = x'_{i+1} \in \{x_{i+1}, \ldots, x_k\}$. We resume the situation in the following picture :



Step 2 : We define a new direction $\tilde{d}$ together with a $\tilde{d}$-enumeration which locates x_i at the $(i+1)$-position and such that there is only one element y in E, $y \neq x_i$, verifying $\langle x_i, \tilde{d}\rangle = \langle y, \tilde{d}\rangle$. Consider $D_{x'_{i+1}}(t) = D_{x_i}(t_0) + t\mathrm{e}_{x'_{i+1}}$. Reasoning as before, by the intermediate value theorem, there exists $t_1 < 0$ such that $\langle x'_{i+1} - x'_i, D_{x'_{i+1}}(t_1)\rangle = 0$. Thanks to the assertion 4 of Lemma A.2.2, we have

$$\varepsilon = |\langle x'_{i+1} - x'_i, D_{x'_{i+1}}(t_1) - D_{x'_{i+1}}(0)\rangle| \geq m|t_1|.$$

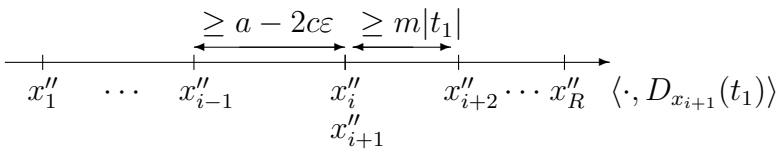
Thus $|t_1| \leq \varepsilon/m$. Next, thanks to the assertion 3 of Lemma A.2.2, evoking A.8 under the new enumeration $\{x'_i\}_{i=1}^R$, for every $j \in \{1, \ldots, i-1\}$ we deduce :

$$\langle x'_i - x'_j, D_{x'_{i+1}}(t_1)\rangle \geq \langle x'_i - x'_j, D_{x'_{i+1}}(0)\rangle - M|t_1| \geq (a - c\varepsilon) - c\varepsilon = a - 2c\varepsilon.$$

Note that we also have $\langle x'_j - x'_{i+1}, D_{x'_{i+1}}(t_1)\rangle \geq m|t_1|$ for $j \geq i+2$. Therefore, denoting $\tilde{d} := D_{x'_{i+1}}(t_1)/\|D_{x'_{i+1}}(t_1)\|$, we may fix $(x''_i)_{i=1}^R$ a $\tilde{d}$-compatible enumeration satisfying

$$\{x''_1, \ldots, x''_{i-1}\} = \{x_1, \ldots, x_{i-1}\},\ x''_i = x'_{i+1},\ x''_{i+1} = x'_i = x_i.$$

This leads us to the following configuration :



Step 3 : Conclusion. To complete the proof. It suffices to evoke a continuity argument and take $t_2 \in (-\infty, t_1)$ such that :

$$\begin{aligned}
&\langle x''_i, D_{x'_{i+1}}(t_2)\rangle > \langle x''_j, D_{x'_{i+1}}(t_2)\rangle,\ \forall\, j \in \{1, \ldots, i-1\}\\
&\langle x''_{i+1}, D_{x'_{i+1}}(t_2)\rangle > \langle x''_i, D_{x'_{i+1}}(t_2)\rangle\\
&\langle x''_\ell, D_{x'_{i+1}}(t_2)\rangle > \langle x''_{i+1}, D_{x'_{i+1}}(t_2)\rangle,\ \forall\, \ell \in \{i+2, \ldots, R\}.
\end{aligned}$$

Setting $d' = D_{x'_{i+1}}(t_2)/\|D_{x'_{i+1}}(t_2)\|$, we deduce the existence of a d'-compatible enumeration $\{y_j\}_{j=1}^R$ satisfying the desired conditions. That is $\{y_1 \cdots, y_{i-1}\} = \{x''_1, \cdots, x''_{i-1}\} = \{x_1, \cdots, x_{i-1}\}$, $y_i = x''_i$, $y_{i+1} = x''_{i+1} = x_i$ and

$$\ldots \leq \langle y_i, d'\rangle < \langle y_{i+1}, d'\rangle < \langle y_{i+2}, d'\rangle \leq \ldots$$

This finishes the first part of the proof.

Case 2 : x_i is strictly located in the i-position, that is $\langle x_i, d\rangle < \langle x_{i+1}, d\rangle$. We prove that this case reduces to the first case. Indeed, consider $D_{x_i} : t \mapsto d + t\mathrm{e}_{x_i}$ the map given by Lemma A.2.2. Applying again the intermediate value theorem we deduce the existence of $t_0 > 0$ such that

$$\langle x_i, D_{x_i}(t_0)\rangle = \min_{j \in \{i+1, \ldots, R\}} \langle x_j, D_{x_i}(t_0)\rangle.$$

Thus, replacing d by $\tilde{d} := D_{x_i}(t_0)/\|D_{x_i}(t_0)\|$ we obtain a $\tilde{d}$-compatible enumeration $(y_j)_{i=1}^R$ of E verifying $\{x_1, \ldots, x_{i-1}\} \subset \{y_1, \ldots, y_{i-1}\}$, $y_i = x_i$ and $\langle y_i, d'\rangle = \langle y_{i+1}, d'\rangle$. Therefore we rejoined the first case. □

Remark A.2.5. It might seem strange, at a first sight, to get back to the first case, since the first step of the latter was precisely to apply a perturbation that strictly locates x_i in the i-position. However, as we pointed out in the proof, this is done in a precise quantified way.

The following corollary is an easy consequence of the previous lemma and will be recalled in several occasion in the proof of Theorem A.1.2.

Corollary A.2.6 (Reordering lemma)**.** *Let $\{x_i\}_{i=1}^R$ be a d-compatible enumeration of E and assume that $\langle x_i, d\rangle < \langle x_p, d\rangle$ for $1 \leq i < p \leq R$. Then there exist a direction $d' \in S$ and a d'-compatible enumeration $\{y_j\}_{j=1}^R$ satisfying $\{x_1, \cdots, x_{i-1}\} \subseteq \{y_1, \cdots, y_{p-1}\}$ and strictly locating x_i at the p-position, that is,*

$$\begin{cases} y_p = x_i \\ \ldots \leq \langle y_{p-1}, d'\rangle < \langle y_p, d'\rangle < \langle y_{p+1}, d'\rangle \leq \ldots \end{cases}$$

Proof. First note that if $\langle x_{i-1}, d\rangle < \langle x_i, d\rangle$, then the result follows from Lemma A.2.4 applied successively $p - i$ times. So let us assume that $\langle x_{i-1}, d\rangle = \langle x_i, d\rangle$. Fix $\mathrm{e}_{x_i} \in \mathrm{int}N_C(x_i)$ and consider the map $D_{x_i} : t \in \mathbb{R} \mapsto d + t\mathrm{e}_{x_i}$ given by Lemma A.2.2. Recall that D_{x_i} is continuous with $D_{x_i}(0) = d$. Since $\langle x_i, d\rangle < \langle x_p, d\rangle$, there exists $t_0 > 0$ such that $\langle x_i, D_{x_i}(t_0)\rangle < \langle x_p, D_{x_i}(t_0)\rangle$ and x_i is strictly located at some position, say k, in every $(D_{x_i}(t_0)/\|D_{x_i}(t_0)\|)$-compatible enumeration. Thus we set $\tilde{d} := D_{x_i}(t_0)/\|D_{x_i}(t_0)\|$ and we fix $(x'_i)_{i=1}^R$ a $\tilde{d}$-compatible enumeration. Of course we have $\{x_1, \cdots, x_{i-1}\} \subseteq \{x'_1, \cdots, x'_{k-1}\}$ and x_i is strictly located at the k-position in this $\tilde{d}$-compatible enumeration. Now the result follows from Lemma A.2.4 applied $p - k$ times. □

A.3 Proof of the main result

Extra notation. We keep the notation introduced at the beginning of Section A.2. For the needs of the proof, we introduce some extra notation.
Given $E_1 \subset E$ we shall often use the abbreviate notation $[E_1] = \mathrm{conv}(E_1)$. Under this notation we trivially have $C = [E]$.
Starting from a finite family of polytopes $\mathcal{R}_0$, we recall that $\mathcal{R}_n = \mathscr{F}^n(\mathcal{R}_0)$ $(n \geq 1)$ where $\mathscr{F}^n$ means applying the operator $\mathscr{F}$ defined in (A.3) n times. For $d \in S$ and $n \geq 1$ we denote

$$\Omega_n(d) = [\bigcup_{P \in \Re_{n-1}} E(P, d)],$$

where $E(P, d) = \{x \in \mathrm{ext}(P) : \langle x, d\rangle = \max\langle P, d\rangle\}$. Under this notation,

$$\mathcal{R}_n = \mathscr{F}(\mathcal{R}_{n-1}) = \{\Omega_n(d) : d \in S\}. \tag{A.9}$$

We recall that a subset F of a polytope Ω is called a face of Ω if there exists a direction $d \in S$ such that

$$F = \{x \in \Omega \,:\, \langle x, d\rangle = \min_{z \in \Omega}\langle z, d\rangle\}. \tag{A.10}$$

In this case we denote the face by $F(\Omega, d)$. Notice that for any $d \in S$ it holds :

$$F(C, d) = [E(C, -d)].$$

We are ready to proceed to the proof of Theorem A.1.2.

Proof of Theorem A.1.2. In view of Proposition A.1.1 we may assume $N \geq 2$. Let us first treat the case $r_{\min} = 1$, that is, the case where the initial family $\Re_0$ contains all singletons. In this case, pick any $x \in E$ and $d \in S$. Since $\Omega_x = [x] \in \Re_0$, we deduce that $E(\Omega_x, d) = \{x\}$ and consequently, $x \in \Omega_1(d)$. It follows that $\Omega_1(d) = [E] = C$ for all $d \in S$, that is, $\Re_1 = \{C\}$. Consequently, the family $\Re_2$ consists of all faces of C, that is,

$$\Re_2 = \{[E(C, d)] : d \in S\} = \{F(C, d) : d \in S\}.$$

In particular, for $\bar{x} \in E$ and $d \in \operatorname{int} N_C(\bar{x})$ (direction that exposes $\bar{x}$ in C) we get $F(C, -d) = [\bar{x}]$, therefore $\Re_2$ contains all singletons and $\Re_3 = \{C\} = \Re_1$.
Let us now treat the case $r_{\min} = R$. In this case $\Re_0 = \{C\}$ and we deduce, as before, that $\Re_1$ is the family of all faces of C and $\Re_2 = \{C\} = \Re_0$.

It remains to treat the case $r_{\min} \notin \{1, R\}$ which is what we assume in the sequel. In this case, we show that $\Re_1 = \Re_3$ which in view of (A.9) yields $\mathscr{F}(\Re_0) = \mathscr{F}(\Re_2))$, *i.e.*

$$\Omega_1(d) = \Omega_3(d) \quad \text{for every } d \in S. \tag{A.11}$$

To establish (A.11) we shall proceed in three steps (Subsections A.3.1–A.3.3), characterizing respectively, the polytopes belonging to the families $\Re_1$, $\Re_2$ and respectively $\Re_3$.

A.3.1 Characterization of polytopes in $\Re_1$.

In this step, by means of geometric conditions on C we characterize membership of a given polytope to the family $\Re_1$. We start with the biggest possible polytope, namely C.

Proposition A.3.1. *Assume $\Re_0$ satisfies (H_1), (H_2). Then the following are equivalent :*
(i) $C = \Omega_1(d_0)$, for some $d_0 \in S$ (that is, $C \in \Re_1 = \mathscr{F}(\Re_0)$) ;
(ii) $|F(C, d_0) \cap E| \geq r_{\min}$ (that is, C has a face containing at least $r_{\min}$ points).

Proof. [(ii)$\Longrightarrow$(i)] Let us first assume that for $d_0 \in S$ assertion (ii) holds and let us prove that

$$\Omega_1(d_0) = [\bigcup_{\Omega \in \Re_0} E(\Omega, d_0)] = C. \tag{A.12}$$

It suffices to prove that for each $\bar{x} \in E$ there exists a polytope $\Omega \in \Re_0$ such that $\bar{x} \in E(\Omega, d_0)$. Since $F(C, d_0)$ contains at least $r_{\min} - 1$ extreme points different than $\bar{x}$, by assumption (H_2) the family $\Re_0$ contains the polytope Ω obtained by convexification of $\bar{x}$ and the aforementioned $r_{\min} - 1$ points. Recalling (A.10) we deduce

$$E(\Omega, d_0) = \begin{cases} \{\bar{x}\}, & \text{if } \bar{x} \notin F(C, d_0) \\ \Omega \cap E, & \text{if } \bar{x} \in F(C, d_0). \end{cases}$$

In all cases $\bar{x} \in E(\Omega, d_0) \subset \Omega_1(d_0)$, which shows that (A.12) holds true.

[(i)$\Longrightarrow$(ii)] Let us now assume that $C = \Omega_1(d_0)$, for some $d_0 \in S$, and let $\{x_i\}_{i=1}^R$ be a d_0-compatible enumeration. Let $k = \max\{i : \langle x_i, d_0\rangle = \langle x_1, d_0\rangle\}$ so that

$$F(C, d_0) = [x_1, \ldots, x_k].$$

Assume towards a contradiction, that $k < r_{\min}$, and fix $i_0 \in \{1, \ldots, k\}$. Then (in view of the definition of $r_{\min}$, see (A.4)) any polytope $\Omega \in \Re_0$ that contains x_{i_0} should necessarily contain some element x_j with $j > k$. In particular, $\langle x_{i_0}, d_0\rangle < \langle x_j, d_0\rangle$, hence $x_{i_0} \notin E(\Omega, d_0)$. Thus $x_{i_0} \notin \Omega_1(d_0)$, contradicting (i). $\square$

Let us now characterize membership of smaller polytopes to $\Re_1$.

Proposition A.3.2. *Assume $\Re_0$ satisfies (H_1), (H_2). Let $x_1, x_2, \ldots, x_k$ be distinct points in E with $1 \le k < r_{\min}$. The following are equivalent :*

(i) $C_k := [E \setminus \{x_1, \ldots, x_k\}] = \Omega_1(d_k)$, for some $d_k \in S$ (that is, $[E \setminus \{x_1, \ldots, x_k\}] \in \Re_1 = \mathscr{F}(\Re_0)$) ;

(ii) There exists a d_k-compatible enumeration $\{x'_i\}_{i=1}^R$ of E such that

$$\langle x'_1, d_k\rangle \le \ldots \le \langle x'_k, d_k\rangle < \langle x'_{k+1}, d_k\rangle = \cdots = \langle x'_{r_{\min}}, d_k\rangle \le \ldots \le \langle x_R, d_k\rangle, \tag{A.13}$$

and

$$\{x_1, \ldots, x_k\} = \{x'_1, \ldots, x'_k\}. \tag{A.14}$$

Proof. [(ii)$\Longrightarrow$(i)] The proof is very similar to the previous one. Let us first assume that (ii) holds for any $1 \le k < r_{\min}$ and distinct points $x_1, x_2, \ldots, x_k \in E$. We shall prove

$$\bigcup_{\Omega \in \Re_0} E(\Omega, d_k) = E \setminus \{x_1, \ldots, x_k\},$$

which obviously yields $\Omega_1(d_k) = C_k$. Pick any $i \in \{1, \ldots, k\}$. Then by (A.14) there exists $i_0 \in \{1, \ldots, k\}$ with $x_i = x'_{i_0}$. Let $\Omega \in \Re_0$ be such that $x_i \in \Omega$. Then since $|\Omega| \ge r_{\min} > k$, Ω should contain some $x'_j \in E$ with $\langle x_i, d_k\rangle < \langle x'_j, d_k\rangle$ (see (A.13)). Thus $x_i \notin E(\Omega, d_k)$. This shows that

$$\bigcup_{\Omega \in \Re_0} E(\Omega, d_k) \subset E \setminus \{x_1, \ldots, x_k\}.$$

Let now $\bar{x} \in E \setminus \{x_1, \ldots, x_k\}$. Then $\langle \bar{x}, d_k\rangle \ge \langle x'_{r_{\min}}, d_k\rangle := \alpha$ and by assumption, there exist at least $r_{\min} - 1$ extreme points with values less or equal to α, forming, together with $\bar{x}$ an $r_{\min}$-polytope $\Omega \in \Re_0$ for which $\bar{x} \in E(\Omega, d_k)$. This shows that

$$C_k := [E \setminus \{x_1, \ldots, x_k\}] = \Omega_1(d_k) \in \Re_1,$$

that is (i) holds.

[(i)$\Longrightarrow$(ii)]. Assume now that for some $d_k \in S$ we have $[E \setminus \{x_1, \ldots, x_k\}] = \Omega_1(d_k)$, consider a d_k-compatible enumeration $\{x'_i\}_{i=1}^R$ of E, set $\alpha := \langle x'_{r_{\min}}, d_k\rangle$ and let $i_1 \in \{1, \ldots, r_{\min}\}$ (respectively, $i_2 \in \{r_{\min}, \ldots, R\}$) be the minimum (respectively, maximum) integer i such that $\langle x'_i, d_k\rangle = \alpha$. If $i_1 = 1$, then in view of (A.10) the face $F(C, d_k)$ contains $i_2 \ge r_{\min}$ extreme points $\{x'_1, \ldots, x'_{i_2}\}$. Then, according to Proposition A.3.1, $\Omega_1(d_k) = C$

which is a contradiction. It follows that $i_1 > 1$. Then the d_k-compatible enumeration satisfies

$$\langle x'_1, d_k\rangle \leq \ldots \leq \langle x'_{i_1-1}, d_k\rangle < \langle x'_{i_1}, d_k\rangle = \cdots = \langle x'_{r_{\min}}, d_k\rangle \cdots = \langle x'_{i_2}, d_k\rangle < \ldots \leq \langle x_R, d_k\rangle.$$

Applying [(ii)$\Longrightarrow$(i)] for $k = i_1 - 1 \in \{1, \ldots, r_{\min} - 1\}$, we get $C_k := [E \setminus \{x'_1, \ldots, x'_{i_1-1}\}]$, whence $i_1 - 1 = k$ and $\{x'_1, \ldots, x'_{i_1-1}\} = \{x_1, \ldots, x_k\}$. The proof is complete.

□

Let us complete this part with the following result.

Proposition A.3.3. *Assume $\Re_0$ satisfies (H_1), (H_2). Then $\Re_1$ does not contain any polytope of the form $[E \setminus \{x_1, \ldots, x_k\}]$ where $x_1, \ldots, x_k \in E$ are distinct and $k \geq r_{\min}$.*

Proof. This fact is obvious since $\Re_0$ contains all possible $r_{\min}$-polytopes. In particular, there exists a polytope Ω entirely contained in $[x_1, \ldots, x_k]$, and consequently for every $d \in S$ it holds

$$E(\Omega, d) \cap [x_1, \ldots, x_k] \neq \emptyset.$$

The proof is complete. □

To resume the above results, we have established that a polytope Ω belongs to the family $\Re_1$ if and only if there is a d_k-compatible enumeration $\{x'_i\}_{i=1}^R$ of E such that

$$\Omega = [E \setminus \{x'_1, \ldots, x'_k\}] \qquad (0 \leq k < r_{\min})$$

with the obvious abuse of notation : $k = 0 \Longrightarrow \{x'_1, \ldots, x'_k\} = \emptyset$.

A.3.2 Characterization of polytopes in $\Re_2$.

In this step, we shall describe the elements of the family

$$\Re_2 = \mathscr{F}(\Re_1) = \{\Omega_2(d) : d \in S\}$$

where as usual,

$$\Omega_2(d) = [\bigcup_{\Omega \in \Re_1} E(\Omega, d)].$$

Let us proceed to a complete description of the above elements. To this end, let us fix a direction $d_0 \in S$. By the previous step (Subsection A.3.1), there exists a d_0-compatible enumeration $\{x'_i\}_{i=1}^R$ of E and $k \in \{0, \ldots, r_{\min} - 1\}$ such that

$$\Omega_1(d_0) = [\bigcup_{\Omega \in \Re_0} E(\Omega, d_0)] = [E \setminus \{x'_1, \ldots, x'_k\}] \in \Re_1. \tag{A.15}$$

Proposition A.3.4. *Let $\{x'_i\}_{i=1}^R$ denote the above d_0-compatible enumeration of E for which (A.15) holds. Then*

$$\Omega_2(-d_0) := [\bigcup_{\Omega \in \Re_1} E(\Omega, -d_0)] = [x'_1, \ldots, x'_\ell] \in \Re_2,$$

where

$$\ell = \max\{i : \langle x'_i, d_0\rangle = \langle x'_{r_{\min}}, d_0\rangle\} \quad (\in \{r_{\min}, \ldots, R\}). \tag{A.16}$$

Proof. Let us first assume $k \geq 1$. According to Proposition A.3.2, we have

$$\langle x'_k, d_0\rangle < \langle x'_{k+1}, d_0\rangle = \langle x'_{r_{\min}}, d_0\rangle = \langle x'_\ell, d_0\rangle = a.$$

Since $\Omega_1(d_0) \in \Re_1$ the above yields

$$E(\Omega_1(d_0), -d_0) = \{x'_{k+1}, \ldots, x'_\ell\},$$

therefore

$$\{x'_{k+1}, \ldots, x'_\ell\} \subset \Omega_2(-d_0).$$

Let further $m \in \{1, \ldots, k\}$ be such that

$$F(C, d_0) = [x'_1, \ldots, x'_m].$$

It follows easily that

$$\{x'_1, \ldots, x'_m\} = E(C, -d_0) \subset \Omega_2(-d_0).$$

Finally, let $i \in \{m+1, \ldots, k\}$ and let us show that $x'_i \in \Omega_2(-d_0)$. To this end, we need to exhibit a direction $d' \in S$ such that the polytope $\Omega_1(d') \in \Re_1$ contains x'_i but does not contain any x'_j for $1 \leq j < i$. (In such a case we would get $x'_i \in E(\Omega_1(d'), -d_0) \subset \Omega_2(-d_0)$ and we are done.) Indeed, let d' be given by Corollary A.2.6 for $p = r_{\min}$. Then there exists a d'-compatible enumeration $\{y_i\}_{i=1}^R$ of E locating strictly x'_i in the $p = r_{\min}$ position (*i.e.* $y_{r_{\min}} = x'_i$) and $\{x'_1, \cdots, x'_{i-1}\} \subseteq \{y_1, \cdots, y_{r_{\min}-1}\}$. Applying Proposition A.3.2 [(ii) $\Longrightarrow$ (i)] for d' we deduce

$$\Omega_1(d') = [E \diagdown \{y_1, \ldots, y_{r_{\min}-1}\}] \in \Re_1$$

and consequently

$$x'_i \in \Omega_1(d') \quad \text{and} \quad \{x'_1, \cdots, x'_{i-1}\} \cap \Omega_1(d') = \emptyset.$$

This proves that $\{x'_1, \ldots, x'_\ell\} \subset \Omega_2(-d_0)$. It remains to show that if $j > \ell$ then $x'_j \notin \Omega_2(-d_0)$. Indeed, since $\ell \geq r_{\min}$, it follows from Proposition A.3.3 that any polytope of $\Re_1$ should contain at least one of the elements $\{x'_i : 1 \leq i \leq \ell\}$. Therefore $x'_j \notin E(\Omega_1, -d_0)$ for all $\Omega_1 \in \Re_1$. It follows that $\Omega_2(-d_0) = [x_1, \ldots, x_\ell]$, as asserted.

Let us now assume $k = 0$, that is, $\Omega_1(d_0) = C$. Then according to Proposition A.3.1 the face $F(C, d_0)$ contains at least $r_{\min}$ points of E. In view of (A.16) we deduce that

$$[x'_1, \ldots, x'_\ell] = F(C, d_0) = E(C, -d_0) \subset \Omega_2(-d_0).$$

Using the same argument as before, we get that $x'_j \notin \Omega_2(-d_0)$ whenever $j \geq \ell+1$. Indeed, according to Proposition A.3.3, since $l \geq r_{min}$ any polytope of $\Re_1$ should contain at least one of the elements $\{x'_i : 1 \leq i \leq \ell\}$. Thus for any polytope Ω_1 in $\mathcal{R}_1$ containing x_j we have $x_j \notin E(\Omega_1, -d_0)$. The proof is complete. $\square$

Since Proposition A.3.4 can be applied to all directions $d \in S$ we eventually recover a full description of polytopes in $\Re_2$.

A.3.3 Construction of $\Re_3$ and conclusion.

In this part we prove the following assertion : For every $d \in S$, we have $\Omega_1(d) = \Omega_3(d)$. This last statement trivially implies that $\Re_1 = \Re_3$ and finishes the proof of the theorem. Let us proceed to the proof of the assertion. Fix any direction $d_0 \in S$. According to Subsection A.3.1, we can fix a d_0-compatible enumeration $(x'_i)_{i=1}^R$ such that

$$\Omega_1(d_0) = [E \setminus \{x'_1, \dots, x'_k\}] \in \Re_1,$$

where $k \in \{0, \dots, r_{\min}\}$ (under the convention that $\{x'_1, \dots, x'_k\} = \emptyset$ for $k = 0$). Then, according to Proposition A.3.2,

$$\Omega_2(-d_0) = [x'_1, \dots, x'_\ell] \in \Re_2,$$

where $\ell \geq r_{\min}$ being defined in (A.16). Thus, we are in the following configuration :

$$\dots \leq \langle x'_k, d_0\rangle < \langle x'_{k+1}, d_0\rangle = \dots = \langle x'_\ell, d_0\rangle < \langle x'_{\ell+1}, d_0\rangle \leq \dots$$

The above readily yields that

$$E(\Omega_2(-d_0), d_0) = \{x'_{k+1}, \cdots, x'_\ell\} \subset \Omega_3(d_0).$$

Let $m \in \{\ell + 1, \dots, R\}$ be such that

$$[x'_m, \dots, x'_R] = F(C, -d_0) = E(C, d_0).$$

It follows that

$$\{x'_{k+1}, \cdots, x'_\ell\} \cup \{x'_m, \cdots, x'_R\} \subset \Omega_3(d_0).$$

Let us prove that $x'_j \in \Omega_3(d_0)$ for all $j \in \{\ell + 1, \dots, m - 1\}$. Notice that x'_j is located in the $(R - j)$-position in the inverse $(-d)$-compatible enumeration. Applying Corollary A.2.6 we obtain a direction $(-d')$ that pushed forward x_j to the $(R - r_{\min})$-position, locating it there strictly. So we obtain a d'-compatible enumeration $\{y_j\}_{j=1}^R$ such that

$$\begin{cases} \langle y_R, -d'\rangle \leq \cdots \leq \langle y_{r_{\min}}, -d'\rangle < \langle y_{r_{\min}-1}, -d'\rangle < \cdots \leq \langle y_1, -d'\rangle \\ y_{r_{\min}} = x'_j \\ \{x'_{j+1}, \dots, x'_R\} \subseteq \{y_{r_{\min}+1}, \dots, y_R\} \end{cases}$$

Writing the above assertion in reverse order yields

$$\langle y_1, d'\rangle \leq \cdots < \langle y_{r_{\min}-1}, d'\rangle < \langle y_{r_{\min}}, d'\rangle < \cdots \leq \langle y_R, d'\rangle.$$

It follows by Proposition A.3.2 [(ii) $\Longrightarrow$ (i)] that

$$\Omega_1(d') = [E \setminus \{y_1, \dots, y_{r_{\min}-1}\}] \in \Re_1$$

and consequently, $y_{r_{\min}} = x'_j \in E(\Omega_1(d'), d_0) \subset \Omega_3(d_0)$.

It remains to prove that $x'_j \notin \Omega_3(d_0)$ whenever $j \in \{1, \dots, k\}$. Indeed, if this were not the case, then there would exist a polytope $\Omega \in \Re_2$ such that $x'_j \in E(\Omega, d_0)$ and consequently the polytope Ω cannot contain any other element $x \in E$ with $\langle x, d_0\rangle > \langle x'_j, d_0\rangle$. In particular $\{x'_{k+1}, \dots, x'_R\} \cap \Omega = \emptyset$. Thus such a polytope could contain at most k points of E with $k < r_{min}$, which is impossible according to Proposition A.3.4 (every polytope of $\Re_2$ contains at least r_{min} points of E). It follows that

$$\Omega_3(d_0) = [E \setminus \{x'_1, \dots, x'_k\}] = \Omega_1(d_0),$$

which proves the assertion and the theorem. □

A.3.4 Weakening assumption (H2)

A careful inspection of the previous proof reveals that some $r_{\min}$-polytopes do not intervene in the construction of the family $\Re_1 = \mathscr{F}(\Re_0)$ and consequently assumption (H_2) can be relaxed as follows (we leave the details to the reader) :

(H_2') The family $\Re_0$ contains all $r_{\min}$-polytopes of the form $[x_1, \ldots, x_{r_{\min}}]$ for which there exists a direction $d \in S$ and a d-compatible enumeration $\{x_i'\}_{i=1}^{R}$ such that

$$\{x_1, \ldots, x_{r_{\min}}\} = \{x_1', \ldots, x_{r_{\min}}'\} \quad \text{and} \quad \langle x_{r_{\min}}', d\rangle < \langle x_{r_{\min}+1}', d\rangle.$$

Appendix B

On the coarse geometry of the James space

The purpose of this appendix is to explore the coarse geometry of the James space. This is a joint work with Gilles Lancien and Antonín Procházka.

B.1 Introduction

In [Kal07], Kalton introduced a property of metric spaces that he named property $\mathcal{Q}$. In particular, it served as an obstruction to coarse embededdability into reflexive Banach spaces. This property is related to the behavior of Lipschitz maps defined on a particular family of graphs that we shall denote $(\mathbb{G}_k(\mathbb{N}))_{k\in\mathbb{N}}$. We will recall the precise definitions of the $\mathbb{G}_k(\mathbb{N})$'s and of property $\mathcal{Q}$ in Section B.1.2. Let us just say, vaguely speaking for the moment, that a Banach space X has property $\mathcal{Q}$ if for every Lipschitz map f from $G_k(\mathbb{N})$ to X, there exists a full subgraph $G_k(\mathbb{M})$ of $G_k(\mathbb{N})$, with $\mathbb{M}$ infinite subset of $\mathbb{N}$, on which f satisfies a strong concentration phenomenon. It is then easy to see that if a Banach space X has property $\mathcal{Q}$, then the family of graphs $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into X (see the definition in Section B.1.1). One of the main results in [Kal07] is that any reflexive Banach space has property $\mathcal{Q}$. It then readily follows that a reflexive Banach cannot contain a coarse copy of all separable metric spaces, or equivalently does not contain a coarse copy of the Banach space c_0. In fact, with a sophistication of this argument, Kalton proved an even stronger result in [Kal07] : if a separable Banach space contains a coarse copy of c_0, then there is an integer n such that the dual of order n of X is non separable. It was therefore natural to extend the study of property Q outside the range of reflexive spaces. Then Kalton was able to exhibit non reflexive but quasi-reflexive spaces with property $\mathcal{Q}$. However, he proved that the most famous example of a quasi-reflexive space, namely the James space $\mathcal{J}$, as well as its dual $\mathcal{J}^*$, fail property $\mathcal{Q}$. The main purpose of this paper is to show that, although $\mathcal{J}$ does not obey the concentration phenomenon described by property $\mathcal{Q}$, the family of graphs $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into $\mathcal{J}$ (Theorem B.2.1). This provides a coarse invariant, namely "not containing equi-coarsely the $G_k(\mathbb{N})$'s", that is very close to but different from property $\mathcal{Q}$. This could allow to find obstructions to coarse embeddability between seemingly close Banach spaces.
Before proceeding with the proof, let us give a few definitions and fix the notation.

B.1.1 Coarse embeddings

Let M, N be two metric spaces and $f\colon M \to N$ be a map. We define the compression modulus ρ_f and the expansion modulus ω_f as follows :

$$\rho_f(t) = \inf\{d_N(f(x), f(y)) \,:\, d_M(x,y) \geq t\},$$
$$\omega_f(t) = \sup\{d_N(f(x), f(y)) \,:\, d_M(x,y) \leq t\}.$$

We adopt the convention $\sup(\emptyset) = 0$ and $\inf(\emptyset) = \infty$.
Note that for every $x, y \in M$,

$$\rho_f(d_M(x,y)) \leq d_N(f(x), f(y)) \leq \omega_f(d_M(x,y)).$$

Then, we say that f is a *coarse embedding* if $\omega_f(t) < \infty$ for every $t \in (0, +\infty)$ and $\lim_{t\to\infty} \rho_f(t) = \infty$.
Next, let $(M_i)_{i\in I}$ be a family of metric spaces. We say that the family $(M_i)_{i\in I}$ *equi-coarsely embeds* into a metric space N if there exist two maps ρ, $\omega\colon [0, +\infty) \to [0, +\infty)$ and maps $f_i\colon M_i \to N$ for $i \in I$ such that :

(i) $\lim_{t\to\infty} \rho(t) = \infty$,

(ii) $\omega(t) < \infty$ for every $t \in (0, +\infty)$,

(iii) $\rho(t) \leq \rho_{f_i}(t)$ and $\omega_{f_i}(t) \leq \omega(t)$ for every $i \in I$ and $t \in (0, \infty)$.

B.1.2 Property Q and Kalton's graphs

For an infinite subset $\mathbb{M}$ of $\mathbb{N}$, we denote $G_k(\mathbb{M})$ the set of all subsets of $\mathbb{M}$ of size k. We shall need a classical version of Ramsey's theorem that we state it here for future reference (see [Gow03, Corollary 1.2]).

Theorem B.1.1 (Ramsey)**.** *Let (K, d) be a compact metric space, $k \in \mathbb{N}$ and $f\colon G_k(\mathbb{N}) \to K$. Then for every $\varepsilon > 0$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that $d(f(\overline{n}), f(\overline{m})) < \varepsilon$ for every $\overline{n}, \overline{m} \in G_k(\mathbb{M})$.*

Let us now recall the definition of two key notions for our paper : Kalton's interlaced graphs and Kalton's property $\mathcal{Q}$. Let $\mathbb{M}$ be a infinite subset of $\mathbb{N}$. We will always write an element $\overline{n}$ of $G_k(\mathbb{M})$ as follows : $\overline{n} = (n_1, \ldots, n_k)$ with $n_1 < \ldots < n_k$. Next we equip $G_k(\mathbb{M})$ with the graph metric d satisfying $d(\overline{n}, \overline{m}) = 1$ whenever $\overline{n} \neq \overline{m}$ and

$$n_1 \leq m_1 \leq n_2 \ldots \leq n_k \leq m_k \text{ or } m_1 \leq n_1 \leq m_2 \ldots \leq m_k \leq n_k.$$

In particular, if $\overline{n}$ and $\overline{m}$ are disjoint we have $d(\overline{n}, \overline{m}) = k_0 \leq k$ if the following is satisfied : in the increasing enumeration of $\overline{n} \cup \overline{m}$ all blocks of consecutive n_i's or consecutive m_i's are of size at most k_0 and there is at least one of these blocks which is of size k_0.

Remark B.1.2. Let X be a Banach space and let $f\colon G_k(\mathbb{N}) \to X$ be a map with finite expansion modulus ω_f. Since d is a graph distance on $G_k(\mathbb{N})$, f is actually $\omega_f(1)$-Lipschitz.

In [Kal07] the property $\mathcal{Q}$ is defined in the setting of metric spaces. For homogeneity reasons, its definition can be simplified for Banach spaces. Let us recall it here.

Definition B.1.3. Let X be a Banach space. We say that X has *property* $\mathcal{Q}$ if there exists $C \geq 1$ such that for every $k \in \mathbb{N}$ and every Lipschitz map $f : G_k(\mathbb{N}) \to X$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that :

$$\forall\, \overline{n}, \overline{m} \in G_k(\mathbb{M}),\ \|f(\overline{n}) - f(\overline{m})\| \leq C\omega_f(1).$$

The following proposition should be clear from the definitions. We shall however include its short proof.

Proposition B.1.4. *Let X be a Banach space. If X has property $\mathcal{Q}$, then the family of graphs $(\mathbb{G}_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into X.*

Proof. Let $C \geq 1$ be given by the definition of property $\mathcal{Q}$. Aiming for a contradiction, assume that the family $(\mathbb{G}_k(\mathbb{N}))_{k\in\mathbb{N}}$ equi-coarsely embeds into X. That is, there are maps $f_k \colon \mathbb{G}_k(\mathbb{N}) \to X$, there are two functions $\rho, \omega \colon [0, +\infty) \to [0, +\infty)$ such that $\lim_{t\to\infty} \rho(t) = \infty$ and

$$\forall k \in \mathbb{N}\ \ \forall t > 0\ \ \rho(t) \leq \rho_{f_k}(t) \text{ and } \omega_{f_k}(t) \leq \omega(t) < \infty.$$

Thus, for every $k \in \mathbb{N}$, there exists an infinite subset $\mathbb{M}_k$ of $\mathbb{N}$ such that $\operatorname{diam}(f(G_k(\mathbb{M}_k))) \leq C\omega(1)$. Since $\operatorname{diam}(G_k(\mathbb{M}_k)) = k$, this implies that for all $k \in \mathbb{N}$, $\rho(k) \leq C\omega(1)$. This contradicts the fact that $\lim\limits_{t\to\infty} \rho(t) = \infty$. □

Let $(s_n)_{n=1}^{\infty}$ denote the summing basis of c_0. That is $s_n = \sum_{i=1}^{n} e_i$, where $(e_i)_{i=1}^{\infty}$ is the canonical basis of c_0. It is easily checked that for any $k \in \mathbb{N}$, the map $f_k : G_k(\mathbb{N}) \to c_0$ defined by $f_k(\overline{n}) = \sum_{i=1}^{k} s_{n_i}$ is a bi-Lipschitz embedding. On the other hand, Kalton proved in [Kal07] that any reflexive Banach space has property $\mathcal{Q}$. As an immediate consequence he could answer an important question by deducing that c_0 does not coarsely embed into any reflexive Banach space. In fact, as we already mentioned, he even showed with additional arguments, that if c_0 coarsely embeds into a separable Banach space X, then one of the iterated duals of X has to be non separable. Inspecting further this property $\mathcal{Q}$ he exhibited non reflexive quasi-reflexive spaces with the property $\mathcal{Q}$ but showed that $\mathcal{J}$ and $\mathcal{J}^*$ fail property $\mathcal{Q}$.

B.1.3 The James space

We now recall the definition and some basic properties of the James space $\mathcal{J}$. We refer the reader to [AK06](Section 3.4) for more details. The James space $\mathcal{J}$ is the real Banach space of all sequences $x = (x(n))_{n\in\mathbb{N}}$ of real numbers with finite square variation and verifying $\lim_{n\to\infty} x(n) = 0$. The space $\mathcal{J}$ is endowed with the following norm

$$\|x\|_{\mathcal{J}} = \sup\Big\{\Big(\sum_{i=1}^{k-1}(x(p_{i+1}) - x(p_i))^2\Big)^{1/2} \ :\ 1 \leq p_1 < p_2 < \ldots < p_k\Big\}.$$

The standard unit vector basis $(e_n)_{n=1}^{\infty}$ ($e_n(i) = 1$ if $i = n$ and $e_n(i) = 0$ otherwise) is a monotone shrinking basis for $\mathcal{J}$. We denote as usual P_n the basis projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$ and $\operatorname{supp} x = \{i \in \mathbb{N} \ :\ x(i) \neq 0\}$ for $x \in \mathcal{J}$. For $x, y \in \mathcal{J}$, we denote :

(N1) $x \prec y$ whenever $\max\operatorname{supp} x < \min\operatorname{supp} y$,

(N2) $x \ll y$ whenever $\max\operatorname{supp} x + 1 < \min\operatorname{supp} y$.

Furthermore, the summing basis $(s_n)_{n=1}^{\infty}$ ($s_n(i) = 1$ if $i \leq n$ and $s_n(i) = 0$ otherwise) is a monotone and boundedly complete basis for $\mathcal{J}$. Thus, $\mathcal{J}$ is naturally isometric to a dual Banach space $\mathcal{J} = X^*$ with X being the closed linear span of the biorthogonal functionals $(e_n - e_{n+1})_{n=1}^{\infty}$ associated with $(s_n)_{n=1}^{\infty}$ in $\mathcal{J}^*$. Thus, a bounded sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{J}$ converges to 0 in the $\omega^* = \sigma(\mathcal{J}, X)$ topology if and only if $\lim_{n\to\infty} \big(x_n(i) - x_n(j)\big) = 0$ for every $i \neq j \in \mathbb{N}$. Consequently, we can state the following.

Lemma B.1.5. *For every weak*-null sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{J}$, there exists C in $\mathbb{R}$ and a subsequence $(x'_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that*

$$\forall i \in \mathbb{N}, \ \lim_{n\to\infty} x'_n(i) = C.$$

Proof. Since the sequence $(x_n)_{n=1}^{\infty}$ is weak*-null, it is bounded in $\mathcal{J}$. So there is C in $\mathbb{R}$ and a subsequence $(x'_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that the sequence $((x'_n)(1))_{n=1}^{\infty}$ converges to C. Then the conclusion follows from our description of weak*-null sequences in $\mathcal{J}$. □

We will need two more basic properties of the norm of $\mathcal{J}$. We state them now and will use them freely in the next section.

Lemma B.1.6. *Let $x_1, \ldots, x_n$ in $\mathcal{J}$.*
(P1) If $x_1 \prec x_2 \prec \ldots \prec x_n \in \mathcal{J}$, then

$$\Big\| \sum_{i=1}^{n} x_i \Big\|_{\mathcal{J}}^2 \leq 5 \sum_{i=1}^{n} \|x_i\|_{\mathcal{J}}^2.$$

(P2) If $x_1 \ll x_2 \ll \ldots \ll x_n \in \mathcal{J}$, then

$$\sum_{i=1}^{n} \|x_i\|_{\mathcal{J}}^2 \leq \Big\| \sum_{i=1}^{n} x_i \Big\|_{\mathcal{J}}^2.$$

Proof. See for instance Lemma 2.2 in [Net16] for a proof of property (P1).
Property (P2) is elementary and based on the fact that, thanks to the "holes" between the supports of the x_i's, we can find a single sequence of integers $(p_j)_j$ which maximizes the quadratic variation of each of the x_i's. □

B.2 Kalton's graphs do not embed into the James space

In this section we state and prove our main result which is the following.

Theorem B.2.1. *The family of graphs $(G_k(\mathbb{N}))_{k\in\mathbb{N}}$ does not equi-coarsely embed into $\mathcal{J}$.*

Therefore the converse of Proposition B.1.4 does not hold. Although this proof may seem a bit technical, the general idea is rather simple. Aiming for a contradiction, we assume that there are $f_k \colon G_k(\mathbb{N}) \to \mathcal{J}$ which form a family of equi-coarse embeddings of the graphs $G_k(\mathbb{N})$ into $\mathcal{J}$. After a few extractions we manage to decompose each f_k on a subgraph $G_k(\mathbb{M})$ into the sum of two maps that we call g_k and h_k. The first family (g_k) can be simply decomposed with the help of the summing basis of $\mathcal{J}$, whereas the second family (h_k) can be described using vectors with well separated supports and therefore

adding in an ℓ_2 way as indicated by Lemma B.1.6. Then we manage to study them almost separately and to show that those two families of maps cannot be equi-coarse embeddings. Combining the reasons why it is so, we will obtain that (f_k) itself cannot be a family of equi-coarse embeddings.

We now state precisely and prove our decomposition result. Although the graph metrics considered in [BLS17] were different, it can somehow be seen as an adaptation of Proposition 4.1 in [BLS17].

Proposition B.2.2. *Let $f\colon G_k(\mathbb{N}) \to \mathcal{J}$ be a Lipschitz map. Then, for every $\varepsilon > 0$ there exist $\lambda\colon \mathbb{N} \to \mathbb{N}$ increasing and $y \in \mathcal{J}$ satisfying the following :*
for every $\overline{n} = (n_1, \ldots, n_k) \in G_k(\mathbb{N})$ there exist $y_1(n_1), y_2(n_1, n_2), \ldots, y_k(\overline{n}) \in \mathcal{J}$ of finite supports and for all $i \in \{1, \ldots, k\}$ there exists a map $u(., i) : \{n \in \mathbb{N},\ n \geq i\} \mapsto \mathbb{N}$ such that :

1. $\|f(\lambda(n_1), \ldots, \lambda(n_k)) - (y + y_1(n_1) + \ldots + y_k(\overline{n}))\| \leq \varepsilon$.
2. $y_i(n_1, \ldots, n_i) = C_i s_{u(n_i,i)} + v_i(n_1, \ldots, n_i)$,
   with $C_i \in \mathbb{R}$, $s_{u(n_i,i)} \prec v_i(n_1, \ldots, n_i)$ *and*
   - *(i)* $\forall i \leq k\ \forall j \in \mathbb{N},\ u(j+1, i) \geq u(j, i) + 3$.
   - *(ii)* *For every* $(n_1, \ldots, n_i)$ *in* $G_i(\mathbb{N})$ *:*
     $\operatorname{supp} v_i(n_1, \ldots, n_i) \subset \{u(n_i, i) + 1, \ldots, u(n_i + 1, i) - 2\}$.
   - *(iii)* $u(j_1, i_1) + 2 \leq u(j_2, i_2)$,
     whenever $j_1 \leq j_2$, $i_1 \leq i_2$ *and* $(j_1, i_1) \neq (j_2, i_2)$.
3. $\|C_i s_{u(n_i,i)}\| \leq Lip(f) + \varepsilon$ *and* $\|v_i(n_1, \ldots, n_i)\| \leq 2Lip(f)$.

Proof. We shall prove this statement by induction on k. Let us write $\mathbb{M}_0 \in \mathcal{P}_\infty(\mathbb{N})$ to mean that $\mathbb{M}_0$ is an infinite subset of $\mathbb{N}$. In the entire proof, we use the weak* topology described in Section B.1.3. We recall that we denote P_n the basis projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$.

So assume first that $k = 1$ and fix $\varepsilon > 0$. Since f is Lipschitz, $f(G_1(\mathbb{N}))$ is bounded in $\mathcal{J}$. Thus, using weak* compactness and Lemma B.1.5 there exist $\mathbb{M}_1 \in \mathcal{P}_\infty(\mathbb{N})$ such that $(f(n))_{n\in\mathbb{M}_1}$ converges to some y in the weak* topology and $C_1 \in \mathbb{R}$ such that for any $N \in \mathbb{N}$:

$$\lim_{n\in\mathbb{M}_1} \|P_N(y - f(n)) - C_1 s_N\| = 0.$$

Without any loss of generality, we may assume that $\mathbb{M}_1 = \mathbb{N}$. By a gliding hump argument, using the above property and the fact that $(e_n)_{n=1}^\infty$ is a basis of $\mathcal{J}$, we manage to construct $\lambda\colon \mathbb{N} \to \mathbb{N}$, $u(\cdot, 1)\colon \mathbb{N} \to \mathbb{N}$ increasing such that $u(1, 1) = 1$, and such that for every $j \in \mathbb{N}$:

- $u(j+1, 1) \geq u(j, 1) + 3$
- $\|P_{u(j,1)}\big(y - f(\lambda(j))\big) - C_1 s_{u(j,1)}\| \leq \frac{\varepsilon}{2}$.
- $\|\big(f(\lambda(j)) - y\big) - P_{u(j+1,1)-2}(f(\lambda(j)) - y)\| \leq \frac{\varepsilon}{2}$,

Next, for $j \in \mathbb{N}$, we define :

$$\begin{aligned} v_1(j) &= \big(P_{u(j+1,1)-2} - P_{u(j,1)}\big)\big(f(\lambda(j)) - y\big), \\ y_1(j) &= C_1 s_{u(j,1)} + v_1(j). \end{aligned}$$

Note that, by weak* lower-semi-continuity of the norm, we have that $\|f(\lambda(j)) - y\| \leq Lip(f)$. Then, since $(e_n)_{n=1}^{\infty}$ is a monotone basis of $\mathcal{J}$, we deduce :

$$\forall j \in \mathbb{N} \quad \|C_1 s_{u(j,1)}\| \leq \|P_{u(j,1)}\big(y - f(\lambda(j))\big)\| + \frac{\varepsilon}{2} \leq Lip(f) + \frac{\varepsilon}{2}.$$

Using again the monotonicity of $(e_n)_{n=1}^{\infty}$ we obtain that $\|v_1(j)\| \leq 2Lip(f)$ for all j in $\mathbb{N}$. To conclude the case $k = 1$, a direct application of the triangle inequality yields the desired estimate :

$$\|f(\lambda(j)) - (y + y_1(j))\| \leq \varepsilon.$$

Assume now that the statement holds for some $k \in \mathbb{N}$. Let us consider a Lipschitz map $f\colon G_{k+1}(\mathbb{N}) \to \mathcal{J}$ and fix $\varepsilon > 0$. Since f is Lipschitz and $G_{k+1}(\mathbb{N})$ has diameter equal to $k+1$, $f(G_{k+1}(\mathbb{N}))$ is bounded in $\mathcal{J}$. Thus, by weak* compactness, Lemma B.1.5 and a diagonal argument we can find $\mathbb{M}_1 \in \mathcal{P}_\infty(\mathbb{N})$ such that :

— For every $\overline{n} \in G_k(\mathbb{M}_1)$, $(f(\overline{n}, n_{k+1}))_{n_{k+1} \in \mathbb{M}_1}$ weak* converges to some $g(\overline{n}) \in \mathcal{J}$.

— For every $\overline{n} \in G_k(\mathbb{M}_1)$, there exists $C_{\overline{n}} \in \mathbb{R}$ such that :

$$\forall N \in \mathbb{N} \quad \lim_{n_{k+1} \in \mathbb{M}_1} \|P_N(f(\overline{n}, n_{k+1}) - g(\overline{n})) - C_{\overline{n}} s_N\| = 0.$$

Using Ramsey's theorem (see Theorem B.1.1), we can find $\mathbb{M}_2 \in \mathcal{P}_\infty(\mathbb{M}_1)$ and $C_{k+1} \in \mathbb{R}$ (which does not depend on $\overline{n}$), such that :

$$\forall N \in \mathbb{N} \quad \forall \overline{n} \in G_k(\mathbb{M}_2), \ \limsup_{n_{k+1} \in \mathbb{M}_2} \|P_N(f(\overline{n}, n_{k+1}) - g(\overline{n})) - C_{k+1} s_N\| < \frac{\varepsilon}{4}.$$

Then, by weak* lower-semi-continuity of the norm it is readily seen that $g\colon G_k(\mathbb{M}_2) \to \mathcal{J}$ is Lipschitz with $Lip(g) \leq Lip(f)$. Thus, we may apply our induction hypothesis to g in order to find $\varphi\colon \mathbb{M}_2 \to \mathbb{M}_2$ increasing and $y \in \mathcal{J}$ such that, for every $\overline{n} \in G_k(\mathbb{M}_2)$, there exist elements $y_1(n_1), y_2(n_1, n_2), \ldots, y_k(\overline{n}) \in \mathcal{J}$ of finite supports, constants $C_1, \ldots, C_k$ in $\mathbb{R}$ and maps $u(., i)$ for $i \leq k$ which satisfy all the conditions given by our proposition for $\frac{\varepsilon}{2}$.

In order to simplify the notation, let us assume, as we may, that $\mathbb{M}_2 = \mathbb{N}$ and φ is the identity on $\mathbb{N}$. We are going to construct $\lambda\colon \mathbb{N} \to \mathbb{N}$ increasing, maps $u(\cdot, k+1)\colon \{k+1, k+2, \ldots\} \to \mathbb{N}$, and the desired elements $y_{k+1}(n_1, \ldots, n_{k+1})$ by induction. Let us start the construction by setting $\lambda(j) = j$ for $j \leq k$ and $u(k+1, k+1) = u(k+1, k) + 2$.

Let $k' \geq k$. Assume that $\lambda(j)$ has been defined for every $j \leq k'$ as well as $u(j, k+1)$ for every $j \leq k'+1$. We also assume that the required elements $y_{k+1}(n_1, \ldots, n_{k+1})$ with $n_{k+1} \leq k'$ have been defined. Note that this last assumption is void for $k' = k$, but as it will soon be clear this will not affect our induction. We continue the construction with the aim of defining $\lambda(k'+1)$, $u(k'+2, k+1)$ and $y_{k+1}(n_1, \ldots, n_{k+1})$ for any $n_1 < \ldots < n_k$ in $\{\lambda(1), \ldots, \lambda(k')\}$ and $n_{k+1} = \lambda(k'+1)$. Let us denote :

$$S := \big\{(n_1, \ldots, n_k) \in G_k(\mathbb{N}) \ : \ \{n_1, \ldots, n_k\} \subset \{\lambda(1), \ldots, \lambda(k')\}\big\}.$$

Since S is finite, we can pick $N \in \mathbb{N} > \lambda(k')$ such that :

$$\forall \overline{n} \in S, \ \|P_{u(k'+1,k+1)}\big(f(\overline{n}, N) - g(\overline{n})\big) - C_{k+1} s_{u(k'+1,k+1)}\| \leq \frac{\varepsilon}{4}.$$

We then define $\lambda(k'+1) = N$. Next, there exists an integer that we denote $u(k'+2, k+1)$ which satisfies :

$$u(k'+2,k+1) \geq \max\big(u(k'+1,k+1)+3, u(k'+2,k)+2\big)$$
$$\forall \overline{n} \in S \quad \|\big(I - P_{u(k'+2,k+1)-2}\big)\big(f\big(\overline{n}, \lambda(k'+1)\big) - g(\overline{n})\big)\| \leq \frac{\varepsilon}{4}.$$

Now we define the following elements for every $\overline{n} \in S$:

$$\begin{aligned} v_{k+1}(\overline{n}, k'+1) &= \big(P_{u(k'+2,k+1)-2} - P_{u(k'+1,k+1)}\big)\big(f\big(\overline{n}, \lambda(k'+1)\big) - g(\overline{n})\big), \\ y_{k+1}(\overline{n}, k'+1) &= C_{k+1} s_{u(k'+1,k+1)} + v_{k+1}(\overline{n}, k'+1). \end{aligned}$$

By weak* lower-semi-continuity of the norm and the nature of our graph metric, we have :

$$\forall \overline{n} \in S, \ \|f\big(\overline{n}, \lambda(k'+1)\big) - g(\overline{n})\| \leq Lip(f).$$

Thus,

$$\|C_{k+1} s_{u(k'+1,k+1)}\| \leq \|P_{u(k'+1,k+1)}\big(f\big(\overline{n}, \lambda(k'+1)\big) - g(\overline{n})\big)\| + \frac{\varepsilon}{4} \leq Lip(f) + \frac{\varepsilon}{4}.$$

We also have $\|v_{k+1}(\overline{n}, k'+1)\| \leq 2\mathrm{Lip}_0(f)$. Gathering all these estimates, we obtain that for every $\overline{n} \in S$:

$$\begin{aligned} & \|f(\overline{n}, \lambda(k'+1)) - (y + y_1(n_1) + \cdots + y_{k+1}(\overline{n}, k'+1))\| \\ \leq\ & \|f(\overline{n}, \lambda(k'+1)) - g(\overline{n}) - y_{k+1}(\overline{n}, k'+1))\| \\ & \qquad + \|g(\overline{n}) - (y + y_1(n_1) + \ldots + y_k(\overline{n}))\| \\ \leq\ & \|\big(I - P_{u(k'+2,k+1)-2}\big)\big(f\big(\overline{n}, \lambda(k'+1)\big) - g(\overline{n})\big)\| \\ & + \|P_{u(k'+2,k+1)-2}\big(f(\overline{n}, \lambda(k'+1)) - g(\overline{n})\big) - y_{k+1}(\overline{n}, k'+1)\| + \frac{\varepsilon}{2} \\ \leq\ & \frac{3\varepsilon}{4} + \|P_{u(k'+1,k+1)}\big(f(\overline{n}, \lambda(k'+1)) - g(\overline{n})\big) - C_{k+1} s_{u(k'+1,k+1)}\| \\ \leq\ & \varepsilon. \end{aligned}$$

□

We now show that the "summing basis part" of the above decomposition cannot provide equi-coarse embeddings of the graphs $G_k(\mathbb{M})$ into $\mathcal{J}$. We actually need a slightly stronger statement (see below). It is also worth mentioning that this is the part of our proof that gives an obstruction to equi-coarse embeddability despite the absence of a concentration phenomenon as described by property $\mathcal{Q}$.
First, we need a definition.
For $k \in \mathbb{N}$, we say that a map $g : G_k(\mathbb{N}) \to \mathcal{J}$ is of *type (S)* if

$$\forall \overline{n} \in G_k(\mathbb{N}), \quad g(\overline{n}) = \sum_{i=1}^{k} C_i s_{u(n_i,i)},$$

where $(C_i)_{i=1}^k \subset \mathbb{R}$, $u(\cdot, i)\colon \mathbb{N} \to \mathbb{N}$ (for $i \in \{1, \ldots, k\}$) is increasing and $u(n,i) + 2 \leq u(m,j)$ whenever $i \leq j$, $n \leq m$ and $(n,i) \neq (m,j)$.
We can now state our lemma.

Lemma B.2.3. *For every $N \in \mathbb{N}$ and for every $C > 0$, there exists an integer $k(N, C) \in \mathbb{N}$ such that for every $k \geq k(N, C)$ and for every C-Lipschitz map $g : G_k(\mathbb{N}) \to \mathcal{J}$ of type (S) we have that $\rho_g(N) < 1$.*
In particular, a family $(g_k)_{k\in\mathbb{N}}$ of maps such that for all k, $g_k : G_k(\mathbb{N}) \to \mathcal{J}$ is of type (S) cannot be a family of equi-coarse embeddings.

Proof. Let us fix $N \in \mathbb{N}$. Pick $\varepsilon > 0$ such that $2N^3\varepsilon^2 < 1$ and define

$$k(N, C) = \max\big(N + 1, \big(\frac{C^2 + 1}{\varepsilon^2} + 1\big)N\big).$$

Consider now $k \geq k(N, C)$ and a C-Lipschitz map $g : G_k(\mathbb{N}) \to \mathcal{J}$ of type (S). Keeping the notation from our definition of type (S) maps, we begin with an easy observation. Let $\overline{n}, \overline{m} \in \mathbb{G}_k(\mathbb{N})$ be such that $n_1 < m_1 < n_2 < \ldots < n_k < m_k$. It is clear that $s_{u(n_i,i)} - s_{u(m_i,i)} \ll s_{u(n_{i+1},i+1)} - s_{u(m_{i+1},i+1)}$. Using Lemma B.1.6 we get the following estimate :

$$\begin{aligned}
\|g(\overline{n}) - g(\overline{m})\|^2 &= \Big\| \sum_{i=1}^{k} C_i(s_{u(n_i,i)} - s_{u(m_i,i)}) \Big\|^2 \\
&\geq \sum_{i=1}^{k} |C_i|^2 \|s_{u(n_i,i)} - s_{u(m_i,i)}\|^2 \\
&= 2\sum_{i=1}^{k} |C_i|^2.
\end{aligned}$$

Since $d(\overline{n}, \overline{m}) = 1$, we deduce that $2\sum_{i=1}^{k} |C_i|^2 \leq C^2$.
We claim now that there is $j \in \{1, \ldots, k - N\}$ such that for every $i \in \{j + 1, \ldots, j + N\}$, $|C_i^k| \leq \varepsilon$. Indeed, otherwise the cardinality of $\{i,\ |C_i^k| > \varepsilon\}$ would be at least $\frac{k}{N} - 1 > \frac{C^2}{\varepsilon^2}$, a contradiction.
Let us now consider $\overline{n}, \overline{m} \in \mathbb{G}_k(\mathbb{N})$ satisfying :

$$\begin{aligned}
&n_1 = m_1 < n_2 = m_2 < \ldots < n_j = m_j \\
&n_{j+1} < n_{j+2} < \ldots < n_{j+N} < m_{j+1} < \ldots < m_{j+N} < n_{j+N+1} \\
&n_{j+N+1} = m_{j+N+1} < \ldots < n_k = m_k.
\end{aligned}$$

It is clear that $d(\overline{n}, \overline{m}) = N$. Our next aim is to estimate the norm of $x = g(\overline{n}) - g(\overline{m}) \in \mathcal{J}$. Let us collect the values $x(i)$ for $i \in \mathbb{N}$. Note first that $\operatorname{supp} x \subseteq [u(n_{j+1}, j + 1) + 1, \ldots, u(m_{j+N}, j + N)]$. Moreover, the value $x(i)$ possibly changes at most $2N$ times and the possible values are of the form $x(i) = -\big(\sum_{i=l_1}^{l_2} C_i\big)$ with $j + 1 \leq l_1 \leq l_2 \leq j + N$ or $x(i) = 0$. Thus, by definition of the norm of $\mathcal{J}$ we may pick an increasing sequence of integers $(p_i)_{i=1}^{l}$ such that

$$\begin{aligned}
&(p_i)_{i=1}^{l} \subset [u(n_{j+1}, j + 1), \ldots, u(m_{j+N}, j + N) + 1] \\
&\|x\|_{\mathcal{J}}^2 = \sum_{i=1}^{l} (x(p_{i+1}) - x(p_i))^2.
\end{aligned}$$

In the above expression, each of the terms $(x(p_{i+1})-x(p_i))^2$ is at most $(N\varepsilon)^2$ and at most $2N$ of them are non zero. So we obtain that

$$\|g(\overline{n})-g(\overline{m})\|^2 \leq 2N(N\varepsilon)^2 < 1.$$

□

We are now ready to conclude the proof of our main result.

Proof of Theorem B.2.1. Assume that $f_k\colon G_k(\mathbb{N}) \to \mathcal{J}$ for $k \in \mathbb{N}$ is a family of equi-coarse embeddings. So there are two maps ρ, $\omega\colon [0,+\infty) \to [0,+\infty)$ such that $\lim_{t\to\infty} \rho(t) = \infty$, and for all $k \in \mathbb{N}$ and all $t > 0$:

$$\rho(t) \leq \rho_{f_k}(t) \quad \text{and} \quad \omega_{f_k}(t) \leq \omega(t) < \infty.$$

We choose $k \in \mathbb{N}$ large enough and denote $f = f_k$. The choice of k will be made precise later. Using Proposition B.2.2 with $\varepsilon = 1$ and after re-indexing, we may assume that for every $\overline{n} \in G_k(\mathbb{N})$:

$$\|f(\overline{n}) - (y + y_1(n_1) + \ldots + y_k(\overline{n}))\| \leq 1,$$

the elements $y, y_1(n_1), \ldots, y_k(\overline{n})$ satisfying the properties (2) and (3) of Proposition B.2.2 (with the same notation). Let us now define the following maps :

$$\begin{aligned} g\colon \overline{n} \in G_k(\mathbb{N}) &\mapsto \sum_{i=1}^{k} C_i s_{u(n_i,i)} \in \mathcal{J} \\ h\colon \overline{n} \in G_k(\mathbb{N}) &\mapsto \sum_{i=1}^{k} v_i(n_1,\ldots,n_i) \in \mathcal{J}. \\ \varphi(\overline{n}) &= g(\overline{n}) + h(\overline{n}). \end{aligned}$$

Thus, for every $\overline{n} \neq \overline{m} \in G_k(\mathbb{N})$ we have :

$$\|\varphi(\overline{n}) - \varphi(\overline{m})\| \leq \|f(\overline{n}) - f(\overline{m})\| + 2.$$

Next, notice that the values $x(j)$ of an element of the form $x = g(\overline{n}) - g(\overline{m})$ possibly change only between $u(n_i,i)$ and $u(n_i,i)+1$, or between $u(m_i,i)$ and $u(m_i,i)+1$, for some $i \leq k$. Thus, we may choose a finite sequence of integers $(p_j)_{j=1}^{l}$ so that :

(i) $\|g_k(\overline{n}) - g_k(\overline{m})\|^2 = \sum_{j=1}^{l-1}(x(p_{i+1}) - x(p_i))^2$.

(ii) $(p_j)_{j=1}^{l-1} \subset A = \big\{u(n_i,i) \,:\, i \in \{1,\ldots,k\}\big\} \cup \big\{u_k(m_i,i) \,:\, i \in \{1,\ldots,k\}\big\}$,

(iii) $p_l > \max\left(\operatorname{supp} v(\overline{n}) \cup \operatorname{supp} v(\overline{m})\right)$,

Let us denote $y = \varphi(\overline{n}) - \varphi(\overline{m})$. Remark that it follows from property (2-ii) in Proposition B.2.2 that for any $j \in A$, $x(j) = y(j)$. Therefore, for all $\overline{n}, \overline{m} \in G_k(\mathbb{N})$:

$$\|g(\overline{n}) - g(\overline{m})\| \leq \|\varphi(\overline{n}) - \varphi(\overline{m})\| \leq \omega(d(\overline{n},\overline{m})) + 2.$$

Note, for further use, that $\mathrm{Lip}_0(g) \leq \omega(1) + 2$.
As a consequence, we also have that for every $\overline{n}, \overline{m} \in G_k(\mathbb{N})$:

$$\|h(\overline{n}) - h(\overline{m})\| \leq \|\varphi(\overline{n}) - \varphi(\overline{m})\| + \|g(\overline{n}) - g(\overline{m})\| \leq 2\,\omega(d(\overline{n},\overline{m})) + 4.$$

We now claim that

$$\forall \overline{n} \in G_k(2\mathbb{N}),\ \|h(\overline{n})\| \leq 3(2\omega(1)+4).$$

Indeed, let $\overline{n} \in G_k(2\mathbb{N})$ and let $\overline{m} \in G_k(\mathbb{N})$ be such that $m_i = n_i + 1$ for every $i \leq k$. We have $d(\overline{n}, \overline{m}) = 1$. Moreover, properties (2-ii) and (2-iii) of Proposition B.2.2 insure that $v_i(n_1, \ldots, n_i) \ll v_i(m_1, \ldots, m_i)$ and $v_i(m_1, \ldots, m_i) \ll v_{i+1}(n_1, \ldots, n_{i+1})$. Therefore, by property (P2) of Lemma B.1.6 :

$$\sum_{i=1}^{k} \|v_i(n_1, \ldots, n_i)\|^2 + \|v_i(m_1, \ldots, m_i)\|^2 \leq \|h(\overline{n}) - h(\overline{m})\|^2 \leq (2\omega(1)+4)^2.$$

Using property (P1) of Lemma B.1.6 we also get :

$$\|h(\overline{n})\|^2 \leq 5 \sum_{i=1}^{k} \|v_i(n_1, \ldots, n_i)\|^2 \leq 5(2\omega(1)+4)^2.$$

This proves our claim, which implies that

$$\forall \overline{n}, \overline{m} \in G_k(2\mathbb{N}),\ \|h(\overline{n}) - h(\overline{m})\| \leq 6(2\omega(1)+4)$$

Finally, since $\lim_{t\to\infty} \rho(t) = \infty$, we can pick $N \in \mathbb{N}$ such that

$$\rho(N) > 3 + 6(2\omega(1)+4).$$

Since $\mathrm{Lip}_0(g) \leq \omega(1)+2$, if our initial choice of k was made so that $k = k(N, \omega(1)+2)$ given by Lemma B.2.3, then there exist $\overline{n}, \overline{m} \in G_k(2\mathbb{N})$ so that $d(\overline{n}, \overline{m}) = N$ and $\|g(\overline{n}) - g(\overline{m})\| < 1$, which in view of the previous estimates implies that

$$\|f(\overline{n}) - f(\overline{m})\| \leq \|g(\overline{n}) - g(\overline{m})\| + \|h(\overline{n}) - h(\overline{m})\| + 2 \leq 3 + 6(2\omega(1)+4).$$

This is a contradiction. □

Bibliographie

[AG17] R. Aliaga and A. Guirao. On the preserved extremal structure of Lipschitz-free spaces. arXiv :1705.09579, 2017. 37, 57, 60, 63, 66, 69

[AK06] F. Albiac and N. J. Kalton. *Topics in Banach space theory*, volume 233 of *Graduate Texts in Mathematics*. Springer, New York, 2006. 29, 47, 82, 85, 86, 109

[AK09] F. Albiac and N. J. Kalton. Lipschitz structure of quasi-Banach spaces. *Israel J. Math.*, 170 :317–335, 2009. 4, 12, 21, 27, 28

[BD80] J. Bourgain and F. Delbaen. A class of special $\mathcal{L}_\infty$ spaces. *Acta Math.*, 145(3-4) :155–176, 1980. 89

[BGLPRZ17] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca. Octahedrality in Lipschitz free Banach spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 2017. To appear. 71

[BLS17] F. Baudier, G. Lancien, and T. Schlumprecht. Not every infinite dimensional Banach space coarsely contains Hilbert space. *ArXiv e-prints*, May 2017. 111

[Bou77] J. Bourgain. On dentability and the Bishop-Phelps property. *Israel J. Math.*, 28(4) :265–271, 1977. 9, 16, 88

[Bou83] R. D. Bourgin. *Geometric aspects of convex sets with the Radon-Nikodým property*, volume 993 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983. 55, 59

[BR80] J. Bourgain and H. P. Rosenthal. Martingales valued in certain subspaces of L^1. *Israel J. Math.*, 37(1-2) :54–75, 1980. 47

[BR09] G. Botelho and P. Rueda. The Schur property on projective and injective tensor products. *Proc. Amer. Math. Soc.*, 137(1) :219–225, 2009. 89

[Cem87] P. Cembranos. The hereditary Dunford-Pettis property on $C(K, E)$. *Illinois J. Math.*, 31(3) :365–373, 1987. 48, 82

[Dal15a] A. Dalet. *Étude des espaces Lipschitz-libres*. PhD thesis, Université de Franche-Comté, 2015. 29, 34, 88

[Dal15b] A. Dalet. Free spaces over countable compact metric spaces. *Proc. Amer. Math. Soc.*, 143(8) :3537–3546, 2015. 31, 34, 35

[Dal15c] A. Dalet. Free spaces over some proper metric spaces. *Mediterr. J. Math.*, 12(3) :973–986, 2015. 6, 13, 30, 31, 33, 34, 48, 50, 78

[DKLR14] S. J. Dilworth, D. Kutzarova, G. Lancien, and N. L. Randrianarivony. Asymptotic geometry of Banach spaces and uniform quotient maps. *Proc. Amer. Math. Soc.*, 142(8) :2747–2762, 2014. 68

[DP18] A. Daniilidis and C. Petitjean. A partial answer to the demyanov-ryabova conjecture. *Set-Valued and Variational Analysis*, 26(1) :143–157, Mar 2018. 91

[DR00] V. F. Demyanov and A. Rubinov, editors. *Quasidifferentiability and related topics*, volume 43 of *Nonconvex Optimization and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000. 94

[DR11] V. F. Demyanov and J. A. Ryabova. Exhausters, coexhausters and converters in nonsmooth analysis. *Discrete Contin. Dyn. Syst.*, 31(4) :1273–1292, 2011. 92, 94

[DU77] J. Diestel and J. J. Uhl, Jr. *Vector measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15. 20, 36, 49, 72

[FHH+01] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, and V. Zizler. *Functional analysis and infinite-dimensional geometry*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8. Springer-Verlag, New York, 2001. 35, 56, 87

[Fon78] V. P. Fonf. Massiveness of the set of extremal points of the unit sphere of some conjugate Banach spaces. *Ukrain. Mat. Zh.*, 30(6) :846–849, 863, 1978. 64

[GG01] M. González and J. Gutiérrez. The Dunford-Pettis property on tensor products. *Math. Proc. Cambridge Philos. Soc.*, 131(1) :185–192, 2001. 82, 89

[GJL95] J. M. Gutiérrez, J. A. Jaramillo, and J. G. Llavona. Polynomials and geometry of Banach spaces. *Extracta Math.*, 10(2) :79–114, 1995. 82

[GK03] G. Godefroy and N. J. Kalton. Lipschitz-free Banach spaces. *Studia Math.*, 159(1) :121–141, 2003. 3, 11, 21, 50

[GKL96] G. Godefroy, N. J. Kalton, and D. Li. On subspaces of L^1 which embed into l_1. *J. Reine Angew. Math.*, 471 :43–75, 1996. 47, 48, 49

[GKL00] G. Godefroy, N. J. Kalton, and G. Lancien. Subspaces of $c_0(\mathbf{N})$ and Lipschitz isomorphisms. *Geom. Funct. Anal.*, 10(4) :798–820, 2000. 67

[GLPZ17a] L. García-Lirola, C. Petitjean, and A. Rueda Zoca. On the structure of spaces of vector-valued Lipschitz functions. *Studia Math.*, 239(3) :249–271, 2017. 34, 71

[GLPZ17b] L. García-Lirola, A. Procházca, and A. Rueda Zoca. A characterisation of the Daugavet property in spaces of Lipschitz functions. arXiv :1705.05145, 2017. 6, 14, 57, 59, 60, 62, 64, 69

[GLRZ17] L. García-Lirola and A. Rueda Zoca. Unconditional almost squareness and applications to spaces of Lipschitz functions. *J. Math. Anal. Appl.*, 451(1) :117–131, 2017. 8, 15, 25, 76, 77, 78

[GMZ14] A. J. Guirao, V. Montesinos, and V. Zizler. On preserved and unpreserved extreme points. In *Descriptive topology and functional analysis*, volume 80 of *Springer Proc. Math. Stat.*, pages 163–193. Springer, Cham, 2014. 56

[GO14] G. Godefroy and N. Ozawa. Free Banach spaces and the approximation properties. *Proc. Amer. Math. Soc.*, 142(5) :1681–1687, 2014. 31, 50

[God87] G. Godefroy. Boundaries of a convex set and interpolation sets. *Math. Ann.*, 277(2) :173–184, 1987. 5, 12, 34

[God15] G. Godefroy. A survey on Lipschitz-free Banach spaces. *Comment. Math.*, 55(2) :89–118, 2015. 87, 88

[Gow03] W. T. Gowers. Ramsey methods in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1071–1097. North-Holland, Amsterdam, 2003. 108

[GPPR17] L. García-Lirola, C. Petitjean, A. Procházka, and A. Rueda Zoca. Extremal structure and Duality of Lipschitz free spaces. *ArXiv e-prints*, July 2017. 21, 29, 32, 58

[Gro55] A. Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, No. 16 :140, 1955. 36, 49

[HLP16] P. Hájek, G. Lancien, and E. Pernecká. Approximation and Schur properties for Lipschitz free spaces over compact metric spaces. *Bull. Belg. Math. Soc. Simon Stevin*, 23(1) :63–72, 2016. 6, 13, 45

[HUL01] J.B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis.* Grundlehren Text Editions. Springer-Verlag, Berlin, 2001. Abridged version of ıt Convex analysis and minimization algorithms. I [Springer, Berlin, 1993 ; MR1261420 (95m :90001)] and ıt II [ibid. ; MR1295240 (95m :90002)]. 94, 95

[IKW07] Y. Ivakhno, V. Kadets, and D. Werner. The Daugavet property for spaces of Lipschitz functions. *Math. Scand.*, 101(2) :261–279, 2007. 60

[IKW09] Y. Ivakhno, V. Kadets, and D. Werner. Corrigendum to : The Daugavet property for spaces of Lipschitz functions. *Math. Scand.*, 104(2) :319, 2009. 57

[JLPS02] W. B. Johnson, J. Lindenstrauss, D. Preiss, and G. Schechtman. Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces. *Proc. London Math. Soc. (3)*, 84(3) :711–746, 2002. 67

[JVSVV14] A. Jiménez-Vargas, J. M. Sepulcre, and M. Villegas-Vallecillos. Lipschitz compact operators. *J. Math. Anal. Appl.*, 415(2) :889–901, 2014. 80

[Kal01] N. J. Kalton. On subspaces of c_0 and extension of operators into $C(K)$-spaces. *Q. J. Math.*, 52(3) :313–328, 2001. 47

[Kal03] N. J. Kalton. Quasi-Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1099–1130. North-Holland, Amsterdam, 2003. 40

[Kal04] N. J. Kalton. Spaces of Lipschitz and Hölder functions and their applications. *Collect. Math.*, 55(2) :171–217, 2004. 5, 6, 13, 29, 30, 31, 33, 35, 36, 37, 40, 43, 45, 46, 48, 78, 81, 83, 88

[Kal07] N. J. Kalton. Coarse and uniform embeddings into reflexive spaces. *Q. J. Math.*, 58(3) :393–414, 2007. 10, 17, 107, 108, 109

[Kal12] N. J. Kalton. The uniform structure of Banach spaces. *Math. Ann.*, 354(4) :1247–1288, 2012. 36

[Kau15] P. L. Kaufmann. Products of Lipschitz-free spaces and applications. *Studia Math.*, 226(3) :213–227, 2015. 50

[Kim13] J. M. Kim. On spaces of weak* to weak continuous compact operators. *Bull. Korean Math. Soc.*, 50(1) :161–173, 2013. 80

[KMS16] V. Kadets, M. Martín, and M. Soloviova. Norm-attaining Lipschitz functionals. *Banach J. Math. Anal.*, 10(3) :621–637, 2016. 87

[KO89] H. Knaust and E. Odell. On c_0 sequences in Banach spaces. *Israel J. Math.*, 67(2) :153–169, 1989. 48, 82

[KPR84] N. J. Kalton, N. T. Peck, and J. W. Roberts. *An F-space sampler*, volume 89 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1984. 40, 51

[KS12] O. F. K. Kalenda and J. Spurný. On a difference between quantitative weak sequential completeness and the quantitative Schur property. *Proc. Amer. Math. Soc.*, 140(10) :3435–3444, 2012. 47, 48

[KS15] O. F. K. Kalenda and J. Spurný. On quantitative Schur and Dunford-Pettis properties. *Bull. Aust. Math. Soc.*, 91(3) :471–486, 2015. 48

[KSSW00] V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner. Banach spaces with the Daugavet property. *Trans. Amer. Math. Soc.*, 352(2) :855–873, 2000. 20

[KW95] N. J. Kalton and D. Werner. Property (M), M-ideals, and almost isometric structure of Banach spaces. *J. Reine Angew. Math.*, 461 :137–178, 1995. 67

[LP13] G. Lancien and E. Pernecká. Approximation properties and Schauder decompositions in Lipschitz-free spaces. *J. Funct. Anal.*, 264(10) :2323–2334, 2013. 51

[Mat00] E. Matoušková. Extensions of continuous and Lipschitz functions. *Canad. Math. Bull.*, 43(2) :208–217, 2000. 36

[Mil71] V. D. Milman. Geometric theory of Banach spaces. II. Geometry of the unit ball. *Uspehi Mat. Nauk*, 26(6(162)) :73–149, 1971. 67

[Net16] F. Netillard. Coarse Lipschitz embeddings of James spaces. *ArXiv e-prints*, June 2016. 110

[NS07] A. Naor and G. Schechtman. Planar earthmover is not in L_1. *SIAM J. Comput.*, 37(3) :804–826 (electronic), 2007. 53

[Pet17] C. Petitjean. Lipschitz-free spaces and Schur properties. *J. Math. Anal. Appl.*, 453(2) :894–907, 2017. 29, 45, 85

[Phe93] R. R. Phelps. *Convex functions, monotone operators and differentiability*, volume 1364 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 1993. 94

[PP74] Ju. Ī. Petunīn and A. N. Plīčko. Some properties of the set of functionals that attain a supremum on the unit sphere. *Ukrain. Mat. Ž.*, 26 :102–106, 143, 1974. 5, 12, 34

[Psh80] B. N. Pshenichnyi. *Convex Analysis and Extremal Problems*. Nauka, Moscou, 1980. 94

[Rn75] O. I. Reĭ nov. RN type operators in Banach spaces. *Dokl. Akad. Nauk SSSR*, 220 :528–531, 1975. 36

[Roc97] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks. 95

[Ros79] H. P. Rosenthal. Sous espaces de L^1. *Cours de troisième cycle, Université Paris VI (unpublished)*, 1979. 47

[Ros07] H. P. Rosenthal. Some new characterizations of Banach spaces containing ℓ^1. *ArXiv e-prints*, October 2007. 82

[RS82] W. M. Ruess and C. P. Stegall. Extreme points in duals of operator spaces. *Math. Ann.*, 261(4) :535–546, 1982. 76

[Rya02] R. A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002. 36, 71, 72, 74, 76, 83

[San17] T. Sang. On the conjecture by Demyanov-Ryabova in converting finite exhausters. *J. Optim. Theory Appl.*, 174(3) :712–727, 2017. 9, 17, 92

[Tan98] B. Tanbay. Direct sums and the Schur property. *Turkish J. Math.*, 22(3) :349–354, 1998. 83, 89

[Wea96] N. Weaver. Duality for locally compact Lipschitz spaces. *Rocky Mountain J. Math.*, 26(1) :337–353, 1996. 38, 39

[Wea99] N. Weaver. *Lipschitz algebras*. World Scientific Publishing Co., Inc., River Edge, NJ, 1999. 3, 4, 6, 11, 12, 14, 21, 23, 24, 25, 26, 29, 30, 33, 43, 57, 58, 63, 64, 67, 78, 88

[Wnu09] W. Wnuk. The strong Schur property in Banach lattices. *Positivity*, 13(2) :435–441, 2009. 47

Quelques aspects de la géométrie des espaces Lipschitz libres

En premier lieu, nous donnons les propriétés fondamentales des espaces Lipschitz libres. Puis, nous démontrons que l'image canonique d'un espace métrique M est faiblement fermée dans l'espace libre associé $\mathcal{F}(M)$. Nous prouvons un résultat similaire pour l'ensemble des molécules.
Dans le second chapitre, nous étudions les conditions sous lesquelles $\mathcal{F}(M)$ est isométriquement un dual. En particulier, nous généralisons un résultat de Kalton sur ce sujet. Par la suite, nous nous focalisons sur les espaces métriques uniformément discrets et sur les espaces métriques provenant des p-Banach.
Au chapitre suivant, nous explorons le comportement de type ℓ_1 des espaces libres. Entre autres, nous démontrons que $\mathcal{F}(M)$ a la propriété de Schur dès que l'espace des fonctions petit-Lipschitz est 1-normant pour $\mathcal{F}(M)$. Sous des hypothèses supplémentaires, nous parvenons à plonger $\mathcal{F}(M)$ dans une somme ℓ_1 d'espaces de dimension finie.
Dans le quatrième chapitre, nous nous intéressons à la structure extrémale de $\mathcal{F}(M)$. Notamment, nous montrons que tout point extrémal préservé de la boule unité d'un espace libre est un point de dentabilité. Si $\mathcal{F}(M)$ admet un prédual, nous obtenons une description précise de sa structure extrémale.
Le cinquième chapitre s'intéresse aux fonctions Lipschitziennes à valeurs vectorielles. Nous généralisons certains résultats obtenus dans les trois premiers chapitres. Nous obtenons également un résultat sur la densité des fonctions Lipschitziennes qui atteignent leur norme.

Mots-clefs : Espace Lipschitz libre ; Fonction petit-Lipschitz ; Dualité ; Propriété de Schur ; Structure extrémale ; Fonction Lipschitzienne à valeurs vectorielles.

Some aspects of the geometry of Lipschitz free spaces

First and foremost, we give the fundamental properties of Lipschitz free spaces. Then, we prove that the canonical image of a metric space M is weakly closed in the associated free space $\mathcal{F}(M)$. We prove a similar result for the set of molecules.
In the second chapter, we study the circumstances in which $\mathcal{F}(M)$ is isometric to a dual space. In particular, we generalize a result due to Kalton on this topic. Subsequently, we focus on uniformly discrete metric spaces and on metric spaces originating from p-Banach spaces.
In the next chapter, we focus on ℓ_1-like properties. Among other things, we prove that $\mathcal{F}(M)$ has the Schur property provided the space of little Lipschitz functions is 1-norming for $\mathcal{F}(M)$. Under additional assumptions, we manage to embed $\mathcal{F}(M)$ into an ℓ_1-sum of finite dimensional spaces.
In the fourth chapter, we study the extremal structure of $\mathcal{F}(M)$. In particular, we show that any preserved extreme point in the unit ball of a free space is a denting point. Moreover, if $\mathcal{F}(M)$ admits a predual, we obtain a precise description of its extremal structure.
The fifth chapter deals with vector-valued Lipschitz functions. We generalize some results obtained in the first three chapters. We finish with some considerations of norm attainment. For instance, we obtain a density result for vector-valued Lipschitz maps which attain their norm.

Key words : Lipschitz free space ; Little Lipschitz function ; Duality ; Schur property ; Extremal structure ; Vector-valued Lipschitz map.

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