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Thèse de Doctorat

Mariam DHAYNI

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Problèmes dans la théorie des semigroupes numériques

JURY

Rapporteurs : **M. Manuel DELGADO**, Professeur, Université de Porto
M. Jorge RAMÍREZ-ALFONSÍN, Professeur, Université de Montpellier
Examineurs : **M. Pedro GARCÍA-SÁNCHEZ**, Professeur, Université de Grenade
M^{me} Monique LEJEUNE-JALABERT, Directeur de recherche au CNRS, Université de Versailles-Saint Quentin
Directeur de thèse : **M. Abdallah ASSI**, Maître de Conférences HDR, Université d'Angers

THÈSE

pour obtenir le grade de

DOCTEUR ÈS MATHÉMATIQUES

présentée à l'Université d'Angers par

Mariam DHAYNI

Problèmes dans la théorie des semigroupes numériques

soutenue le 07 Décembre 2017 devant le jury composé de :

M. Manuel Delgado	Professeur à l'Université de Porto	Rapporteur
M. Jorge Ramírez-Alfonsín	Professeur à l'Université de Montpellier	Rapporteur
M. Pedro García-Sánchez	Professeur à l'Université de Grenade	Examineur
Mme. Monique Lejeune-Jalabert	Directeur de recherche au CNRS	Examineur
M. Abdallah Assi	Maître de Conférences HDR à l'Université d'Angers	Directeur de thèse

préparée au **LAREMA - UMR CNRS 6093**

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Table des matières

Remerciements	iii
Introduction	1
1 Basics and notations	3
2 Wilf's conjecture	9
2.1 Equivalent form of Wilf's conjecture	9
2.2 Technical results	11
2.3 Numerical semigroups with $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$	14
2.4 Numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$	26
2.5 Numerical semigroups with $(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y-x-1) + (y-2) + \lfloor \frac{w_y}{m} \rfloor (x-1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2})\nu \geq m$ and $w_{m-1} - m \geq w_x + w_y$	28
2.6 Numerical semigroups with $m - \nu > \frac{(n-2)(n-3)}{2}$	33
3 Numerical semigroup of the form $\langle m, m+1, \dots, m+l, k(m+l) + r \rangle$	35
3.1 Apéry set of S	36
3.2 Frobenius number of S	38
3.3 Genus of S	43
3.4 Determination of symmetric and pseudo-symmetric numerical semigroups	46
3.4.1 Determination of symmetric numerical semigroup	47
3.4.2 Determination of pseudo-symmetric numerical semigroup	52
3.5 Pseudo-Frobenius Numbers	56
Bibliography	78

Introduction

Let \mathbb{N} denote the set of natural numbers, including 0. A semigroup S is an additive submonoid of $(\mathbb{N}, +)$, that is $0 \in S$ and if $a, b \in S$, then $a + b \in S$. A numerical semigroup S is a submonoid of \mathbb{N} of finite complement, i.e., $\mathbb{N} \setminus S$ is a finite set. It can be shown that a submonoid of \mathbb{N} is a numerical semigroups if and only if the group generated by S in \mathbb{Z} (namely the set of elements $\sum_{i=1}^s \lambda_i a_i, \lambda_i \in \mathbb{Z}, a_i \in S$) is \mathbb{Z} .

There are many invariants associated to a numerical semigroup S . The **Apéry set** of S with respect to an element $a \in S$ is defined as

$$\text{Ap}(S, a) = \{s \in S; s - a \notin S\}.$$

The elements of $\mathbb{N} \setminus S$ are called the **gaps** of S . The largest gap is denoted by

$$f = f(S) = \max(\mathbb{N} \setminus S)$$

and is called the **Frobenius number** of S . The number $f(S) + 1$ is known as the conductor of S and denoted by c or $c(S)$. The number of gaps

$$g = g(S) = |\mathbb{N} \setminus S|$$

is known as **genus** of S . The smallest non zero element $m = m(S)$ of S is called the **multiplicity** of S and the set $\{s \in S; s < f(S)\}$ is denoted by $n(S)$. Every numerical semigroup S is finitely generated, i.e., S is of the form

$$S = \langle g_1, \dots, g_\nu \rangle = \mathbb{N}g_1 + \dots + \mathbb{N}g_\nu$$

for suitable unique coprime integers g_1, \dots, g_ν . The number of minimal set of generators of S is denoted by

$$\nu = \nu(S)$$

and is called the **embedding dimension** of S . An integer $x \in \mathbb{N} \setminus S$ is called a **pseudo-Frobenius number** if $x + S \setminus 0 \subseteq S$. The **type** of the semigroup, denoted by $t(S)$ is the cardinality of set of pseudo-frobenius numbers. We have formulas linking these invariants.

Frobenius in his lectures proposed the problem of giving a formula for the largest integer that is not representable as a linear combination with nonnegative integer coefficients of a given set of positive integers whose greater common divisor is one. He also threw the question of determining how many positive integers do not have such a representation. This problem is known as **Diophantine Frobenius Problem**. Using the terminology of numerical semigroups, the problem is to give a formula, in terms of the elements in a minimal system of generators of a numerical semigroup S , for the greatest integer not in S . This problem, introduced and solved by Sylvester for the case $\nu = 2$ (see [21]), has been widely studied. For $\nu = 3$, in 1962 Brauer and Shockly (see [6]) found a formula for the Frobenius number but their solution was not a polynomial in the generators and it involved magnitudes which could not be expressed by the generators. Later on, more solutions to this case were found by using different methods (for example see [20]). However, all of these methods do not give explicit formula of the Frobenius number in terms of the generators. Generally, it has been proved in [15] that $f(S)$ is not algebraic in the set of generators of S .

In [22] 1978 H. S. Wilf proposed a conjecture suggesting a regularity in the set $\mathbb{N} \setminus S$. It says the following :

$$f(S) + 1 \leq \nu(S)n(S).$$

Although the problem has been considered by several authors (cf. [2], [4], [9], [10], [11], [14], [19], [23]), only special cases have been solved and it remains wide open. In [9], D. Dobbs and G. Matthews proved Wilf's

conjecture for $\nu \leq 3$. In [14], N. Kaplan proved it for $c \leq 2m$ and in [10] S. Eliahou extended Kaplan's work for $c \leq 3m$.

In Chapter 1, we recall some basics about numerical semigroups that will be used through the thesis.

In Chapter 2, we generalize the case covered by A. Sammartano in [19], who showed that Wilf's conjecture holds for $2\nu \geq m$, and $m \leq 8$, based on the idea of counting the elements of S in some intervals of length m . We use different intervals in order to get an equivalent form of Wilf's conjecture and then we prove it in some relevant cases. In particular our calculations cover the case where $2\nu \geq m$, proved by Sammartano in [19]. Here are few more details on the contents of this Chapter. Section 2.1 is devoted to give some notations that will enable us in the same Section to give an equivalent form of Wilf's conjecture. In Section 2.2, we give some technical results needed in the Chapter. Let $\text{Ap}(S, m) = \{0 = w_0 < w_1 < \dots < w_{m-1}\}$. In Section 2.3, first, we show that Wilf's conjecture holds for numerical semigroups that satisfy $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m - 1$ where $c = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. Then, we prove Wilf's conjecture for numerical semigroups with $m - \nu \leq 4$ in order to cover the case where $2\nu \geq m$. We also show that a numerical semigroup with $m - \nu = 5$ verify Wilf's conjecture in order to prove the conjecture for $m = 9$. Finally, we show in this Section, using the previous cases, that Wilf's conjecture holds for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$. In Section 2.4, we prove Wilf's conjecture for numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$ for some $1 < \alpha < m - 1$. In Section 2.5, we show Wilf's conjecture holds for numerical semigroups with $w_{m-1} - m \geq w_x + w_y$ and $(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y-x-1) + (y-2) + \lfloor \frac{w_y}{m} \rfloor (x-1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2})\nu \geq m$. The last Section 2.6 aims to verify the conjecture in the case $m - \nu > \frac{(n-2)(n-3)}{2}$ and also in the case $n \leq 5$.

Exact determination of $\text{Ap}(S, m)$, $f(S)$, $g(S)$ and $PF(S)$ is a difficult problem. When S is generated by an arithmetic sequence $\langle m, m + 1, \dots, m + l \rangle$, Brauer [5] gave a formula for $f(S)$. Roberts [17] extended this result to generators in arithmetic progression (see also [3], [24]). Selmer [20] and Grant [13] generalized this to the case $S = \langle m, hm + d, hm + 2d, \dots, hm + ld \rangle$. In [16], it has been considered the case of semigroups generated by $\{m, m + d, \dots, m + ld, c\}$ (called almost arithmetic semigroups) where it has been given a method to determine $\text{Ap}(S, m)$ and also symmetric almost arithmetic semigroups. In [12], pseudo symmetric almost arithmetic semigroups have been characterized. In Chapter 3, we focus our attention on numerical semigroup consisting of all non-negative integer linear combinations of relatively prime positive integers $m, m + 1, \dots, m + l, k(m + l) + r$ where k, m, l, r are positive integers and $r \leq (k + 1)l + 1$. We give formulas for $\text{Ap}(S, m)$, $f(S)$, $g(S)$ and $PF(S)$. We also determine the symmetric and the pseudo symmetric numerical semigroups of this form. Note that our semigroups $\langle m, m + 1, \dots, m + l, k(m + l) + r \rangle$ are almost arithmetic semigroups. The advantage is that we are able for this class of semigroups to determine all the invariants with simple formulas.

Good references on numerical semigroups are [18] and [1].



Basics and notations

Definition 1.0.1. Let S be a subset of \mathbb{N} . We say that S is a **submonoid** of $(\mathbb{N}, +)$ if the following holds :

- $0 \in S$.
- If $a, b \in S$, then $a + b \in S$.



Remark 1.0.2. All semigroups considered in this thesis are submonoids of $(\mathbb{N}, +)$, hence commutative, that is, $a + b = b + a$ for all $a, b \in S$.



Example 1.0.3. Consider the following examples :

- $\{0\}$ and \mathbb{N} are trivially submonoids of \mathbb{N} .
- Let d be an element of \mathbb{N} , the set $d\mathbb{N} = \{da : a \in \mathbb{N}\}$ is a submonoid of \mathbb{N} .



Definition 1.0.4. Let S be a submonoid of \mathbb{N} . If $\mathbb{N} \setminus S$ is a finite set, then S is said to be a **numerical semigroup**.



We have the following characterization of numerical semigroups :

Proposition 1.0.5. (See Lemma 2.1 in [18]) Let $S \neq \{0\}$, and $S \neq \mathbb{N}$ be a semigroup of \mathbb{N} and let G be the subgroup of \mathbb{Z} generated by S , i.e., $(G = \{\sum_{i=1}^s \lambda_i a_i, s \in \mathbb{N}, \lambda_i \in \mathbb{Z}, a_i \in S\})$. Then, S is a numerical semigroup if and only if $G = \mathbb{Z}$, i.e., $(\gcd(S)=1)$.



Proposition 1.0.6. (See Proposition 2.2 in [18]) Let S be a semigroup of \mathbb{N} . Then, S is isomorphic to a numerical semigroup.



Definition 1.0.7. Let S be a numerical semigroup and let $A \subseteq S$. We say that S is generated by A and we write $S = \langle A \rangle$ if for all $s \in S$, there exist $a_1, \dots, a_r \in A$ and $\lambda_1, \dots, \lambda_r \in \mathbb{N}$ such that $a = \sum_{i=1}^r \lambda_i a_i$. We say that S is finitely generated if $S = \langle A \rangle$ with $A \subseteq S$ and A is a finite set.



Remark 1.0.8. Through this thesis X^* will stand for $X \setminus \{0\}$.



Next, we introduce an important tool associated to a numerical semigroup.

Definition 1.0.9. Let $n \in S^*$. We define the **Apéry set** of S with respect to n , denoted by $\text{Ap}(S, n)$, to be the set

$$\text{Ap}(S, n) = \{s \in S : s - n \notin S\}.$$

Remark 1.0.10. Given a non zero integer n and two integers a and b , we write $a \equiv b \pmod{n}$ to denote that n divides $a - b$. We denote by $b \pmod{n}$ the remainder of the division of b by n .

From Definition 1.0.9, we can easily see the following.

Lemma 1.0.11. Let S be a numerical semigroup and let $n \in S^*$. For all $1 \leq i \leq n$, let $w(i)$ be the smallest element of S such that $w(i) \equiv i \pmod{n}$. We have the following :

$$\text{Ap}(S, n) = \{0, w(1), \dots, w(n-1)\}.$$

Proposition 1.0.12. Let S be a numerical semigroup. Let $n \in S^*$ and let $\text{Ap}(S, n) = \{w_0 < w_1 \dots < w_{n-1}\}$ be the Apéry set of S with respect to n . We have the following :

- $w_0 = w(0) = 0$.
- $|\text{Ap}(S, n)| = n$.

Proposition 1.0.13. (See Lemma 2.6 in [18]) Let S be a numerical semigroup and let $n \in S^*$. For all $s \in S$, there exists a unique $(k, w) \in \mathbb{N} \times \text{Ap}(S, n)$ such that $s = kn + w$.

As a consequence of Proposition 1.0.13, we obtain the following property.

Corollary 1.0.14. (Theorem 2.7 in [18]) Let S be a numerical semigroup. Then, S is finitely generated.

Definition 1.0.15. Let S be a numerical semigroup and let $A \subseteq S^*$. We say that A is a **minimal set of generators** of S if $S = \langle A \rangle$ and for all $x \in A$, x cannot be written as a linear combination with nonnegative integer coefficients of other elements in A .

Corollary 1.0.16. (See Corollary 2.8 in [18]) Let S be a numerical semigroup. Then, S has a minimal set of generators. This set is finite and unique.

Definition 1.0.17. Let S be a numerical semigroup. We define the following invariants :

- The **embedding dimension** of S denoted by $\nu(S)$, or ν for simplicity, is the cardinality of the minimal set of generators of S .
- The **multiplicity** of S denoted by $m(S)$, or m for simplicity, is the smallest non zero element of S .

Lemma 1.0.18. (See Proposition 2.10 in [18]) Let S be a numerical semigroup with multiplicity m and embedding dimension ν . We have $\nu \leq m$.

Let us recall some basic and important invariants of numerical semigroups.

Definition 1.0.19. Let S be a numerical semigroup. We introduce some invariants associated to a numerical semigroup S :

- We define the **Frobenius number** of S , denoted by f or $f(S)$ to be $\max(\mathbb{Z} \setminus S)$.
- We define the **conductor** of S , denoted by c or $c(S)$ to be $f(S) + 1$.

- We define the **set of gaps** of S , denoted by $G(S)$ to be $\mathbb{N} \setminus S$.
- We define the **genus** of S , denoted by $g(S)$ to be the cardinality of $G(S)$.
- We denote by $n(S)$, the cardinality of $\{s \in S : s \leq f(S)\}$.

Remark 1.0.20. Note that $f(S) \geq 1$ for all non trivial numerical semigroups. ■

Lemma 1.0.21. (See in [6], [20]) Let S be a numerical semigroup and let $n \in S$. Then,

- $f(S) = \max(\text{Ap}(S, n)) - n$.
- $g(S) = \frac{1}{n} \sum_{w \in \text{Ap}(S, n)} w - \frac{1}{2}(n - 1)$.

Definition 1.0.22. Let S be a numerical semigroup. We say that $x \in \mathbb{N}$ is a **pseudo-Frobenius number** if $x \notin S$ and $x + s \in S$ for all $s \in S^*$. We denote by $PF(S)$ the set of all pseudo-Frobenius numbers of S . We denote the cardinality of $PF(S)$ by $t(S)$ and we call it the **type** of S . It results from the definition of $f(S)$ that $f(S) \in PF(S)$, and also $f(S) = \max (PF(S))$. ■

Corollary 1.0.23. (See Theorem 20 in [11]) Let S be a numerical semigroup with Frobenius number $f(S)$, type $t(S)$ and $n(S) = |\{s \in S : s < f(S)\}|$. Then, we have

$$f(S) + 1 \leq (t(S) + 1)n(S).$$

Wilf's conjecture : Let the notations be as before. The problem whether $f(S) + 1 \leq n(S)\nu(S)$ is known as Wilf conjecture [22]. For some families of numerical semigroups this conjecture is known to be true, but the general case remains unsolved. ■

Remark 1.0.24. By Corollary 1.0.23, if $t(S) \leq \nu(S) - 1$, then S satisfies Wilf's conjecture. ■

Definition 1.0.25. Let $a, b \in \mathbb{N}$. We define \leq_S as follows : $a \leq_S b$ if and only if $b - a \in S$. ■

Remark 1.0.26. As S is a numerical semigroup, it easily follows that \leq_S is an order relation over S (reflexive, transitive and anti symmetric). ■

Definition 1.0.27. Let S be a numerical semigroup and $n \in S^*$. Let $\text{Ap}(S, n) = \{w_0 = 0 < w_1 < w_2 < \dots < w_{n-1}\}$ be the Apéry set of S with respect to n . Then, define the following sets :

$$\min_{\leq_S}(\text{Ap}(S, n)) = \{w \in \text{Ap}(S, n)^* \text{ such that } w \text{ is minimal with respect to } \leq_S\}.$$

$$\max_{\leq_S}(\text{Ap}(S, n)) = \{w \in \text{Ap}(S, n)^* \text{ such that } w \text{ is maximal with respect to } \leq_S\}.$$

Lemma 1.0.28. (See Lemma 6 in [11]) Let S be a numerical semigroup, $n \in S^*$ and $\text{Ap}(S, n)$ be the Apéry set of S with respect to n . Let $w \in \text{Ap}(S, n)$ and $u \in S$. If there exist $v \in S$ such that $w = u + v$, then $u \in \text{Ap}(S, n)$. ■

Corollary 1.0.29. Let $x \in \text{Ap}(S, n)^*$. We have the following :

- $x \in \min_{\leq_S}(\text{Ap}(S, n))$ if and only if $x \neq w_i + w_j$ for all $w_i, w_j \in \text{Ap}(S, n)^*$.
- $x \in \max_{\leq_S}(\text{Ap}(S, n))$ if and only if $w_i \neq x + w_j$ for all $w_i, w_j \in \text{Ap}(S, n)^*$.

Proof. Let $x \in \text{Ap}(S, n)^*$.

- Let $x \in \min_{\leq_S}(\text{Ap}(S, n))$. Suppose by the way of contradiction that $x = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S, n)^*$. Then, $x = w_i + w_j$ with $w_i \in \text{Ap}(S, n)$ and $w_j \in S$ which implies that $x \notin \min_{\leq_S}(\text{Ap}(S, n))$ and we get a contradiction.

Conversely, suppose that $x \neq w_i + w_j$ for all $w_i, w_j \in \text{Ap}(S, n)^*$. Suppose by the way of contradiction that $x \notin \min_{\leq_S}(\text{Ap}(S, n))$, then there exist $w_i \in \text{Ap}(S, n)$ and $s \in S$ such that $x = w_i + s$. By Lemma 1.0.28, it follows that $s \in \text{Ap}(S, n)$. Thus, $x = w_i + s$ such that $w_i, s \in \text{Ap}(S, n)$ which gives a contradiction.

- Let $x \in \max_{\leq_S}(\text{Ap}(S, n))$. Suppose by the way of contradiction that $w_i = x + w_j$ for some $w_i, w_j \in \text{Ap}(S, n)^*$. Then, $w_i = x + w_j$ with $w_i \in \text{Ap}(S, n)$ and $w_j \in S$ which implies that $x \notin \max_{\leq_S}(\text{Ap}(S, n))$ and we get a contradiction.

Conversely, suppose that $w_i \neq x + w_j$ for all $w_i, w_j \in \text{Ap}(S, n)^*$. Suppose by the way of contradiction that $x \notin \max_{\leq_S}(\text{Ap}(S, n))$, then there exist $w_i \in \text{Ap}(S, m)$ and $s \in S$ such that $w_i = x + s$. By Lemma 1.0.28, it follows that $s \in \text{Ap}(S, n)$. Thus, $w_i = x + s$ such that $w_i, s \in \text{Ap}(S, n)$ which gives a contradiction.

Thus, the proof is complete. ■

Proposition 1.0.30. (See Lemma 3.2 in [7]) Let S be a numerical semigroup with multiplicity m and embedding dimension ν and let $n \in S^*$. Let $\text{Ap}(S, n)$ be the Apéry set of S with respect to n and let $\{g_1 < g_2 < \dots < g_\nu\}$ be the minimal set of generators of S . We have the following :

- $g_1 = m$.
 - $\min_{\leq_S}(\text{Ap}(S, n)) = \{g_2, \dots, g_\nu\}$.
 - $\max_{\leq_S}(\text{Ap}(S, n)) = \{w \in \text{Ap}(S, n) \text{ such that } w - n \text{ is a pseudo-Frobenius number of } S\}$.
-

From Proposition 1.0.30, it follows Corollary 1.0.31.

Corollary 1.0.31. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and $\{g_1 = m, g_2, \dots, g_\nu\}$ the minimal system of generators of S . Let $n \in S^*$ and $\text{Ap}(S, n)$ be the Apéry set of S with respect to n . We have the following :

- $|\min_{\leq_S}(\text{Ap}(S, n))| = \nu - 1$.
 - $|\text{Ap}(S, n)^* \setminus \min_{\leq_S}(\text{Ap}(S, n))| = n - \nu$.
 - $|\max_{\leq_S}(\text{Ap}(S, n))| = t(S)$.
-

We introduce in Definitions 1.0.32 and 1.0.33 special kind of numerical semigroups and give some properties of this kind in Lemma 1.0.34.

Definition 1.0.32. A numerical semigroup is said to **irreducible** if and only if S cannot be expressed as the intersection of two numerical semigroups S_1, S_2 such that $S \subset S_1, S \subset S_2$. ■

Definition 1.0.33. Let S be a numerical semigroup. We have the following :

- S is said to be **symmetric** if and only if S is irreducible and $f(S)$ is odd.
 - S is said to be **pseudo-symmetric** if and only if S is irreducible and $f(S)$ is even.
-

Lemma 1.0.34. (See Corollary 4.5 in [18]) Let S be a numerical semigroup with Frobenius number $f(S)$ and genus $g(S)$. We have the following :

- S is symmetric if and only if $g(S) = \frac{f(S) + 1}{2}$.
- S is pseudo-symmetric if and only if $g(S) = \frac{f(S) + 2}{2}$.

Remark 1.0.35. Consider the following notation that will be used through this thesis :

- We denote by floor $(x) = \lfloor x \rfloor$ the largest integer less than or equal to x .
- We denote by ceil $(x) = \lceil x \rceil$ the smallest integer greater than or equal to x .

Wilf's conjecture

In this chapter, we give an equivalent form of Wilf's conjecture in terms of the elements of the Apéry set of S , embedding dimension and the multiplicity. We also give an affirmative answer to Wilf's conjecture in some cases.

2.1 Equivalent form of Wilf's conjecture

Let the notations be as in the introduction. For the sake of clarity we shall use the notations $\nu, f, n, c...$ for $\nu(S), f(S), n(S), c(S)...$. In this Section, we will introduce some notations and family of numbers that will enable us to give an equivalent form of Wilf's conjecture at the end of this Section.

Notation. Let S be a numerical semigroup with multiplicity m and conductor $c = f + 1$. Denote by

$$q = \lceil \frac{c}{m} \rceil.$$

Thus, $qm \geq c$ and $c = qm - \rho$ with $0 \leq \rho < m$.

Given a non negative integer k , we define the k th interval of length m ,

$$I_k = [km - \rho, (k + 1)m - \rho[= \{km - \rho, km - \rho + 1, \dots, (k + 1)m - \rho - 1\}.$$

We denote by

$$n_k = |S \cap I_k|.$$

For $j \in \{1, \dots, m - 1\}$, we define η_j to be the number of intervals I_k with $n_k = j$.

$$\eta_j = |\{k \in \mathbb{N}; n_k = j\}|.$$

Let $0 \leq k \leq q - 2$. If $s \in S \cap I_k$ then $s + m \in S \cap I_{k+1}$. This implies that $n_k \leq n_{k+1}$. Let for example $S = \langle 4, 6, 13 \rangle$. We have $c(S) = c = 16 = 4 \cdot 4$, hence $I_k = [km, (k + 1)m[$ for all $k \geq 0$. Moreover, $n_0 = 1, n_1 = 2, n_2 = 2, n_3 = 3$, and $n_k = 4$ for all $k \geq 4$. We also have $\eta_1 = 1, \eta_2 = 2, \eta_3 = 1$. ■

Proposition 2.1.1. Under the previous notations, we have :

- i) $1 \leq n_k \leq m - 1$ for all $0 \leq k \leq q - 1$.
- ii) $n_k = m$ for all $k \geq q$.

$$iii) \sum_{k=0}^{q-1} n_k = n(S) = n.$$

$$iv) \sum_{j=1}^{m-1} \eta_j = q.$$

$$v) \sum_{j=1}^{m-1} j\eta_j = \sum_{k=0}^{q-1} n_k = n.$$

Proof.

i) We can easily verify that if S contains m consecutive elements $a, a+1, \dots, a+m-1$, then for all $n \geq a+m$, $n \in S$. Since $(q-1)m - \rho < f < qm - \rho$, then it follows that $n_k \leq m-1$ for all $0 \leq k \leq q-1$. Moreover, $km \in S \cap I_k$ for all $0 \leq k \leq q-1$, thus $n_k \geq 1$.

ii) We have $f = qm - \rho - 1 \in I_{q-1}$. From the definition of the Frobenius number, it follows that $n_k = m$ for all $k \geq q$.

iii) $\sum_{k=0}^{q-1} n_k$ is nothing but the cardinality of $\{s \in S; s < f\}$ which is $n(S)$ by definition.

iv) We have $1 \leq |S \cap I_k| \leq m-1$ if and only if $0 \leq k \leq q-1$. This implies our assertion.

v) The sum $\sum_{j=1}^{m-1} j\eta_j$ is nothing but the cardinality of $|\cup_{k=0}^{q-1} S \cap I_k| = n$. This proves our assertion.

Thus, the proof is complete. ■

Next, we will express η_j in terms of the Apéry set.

Proposition 2.1.2. Let $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}\}$. Under the previous notations, for all $1 \leq j \leq m-1$, we have

$$\eta_j = \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor.$$

Proof. Fix $0 \leq k \leq q-1$ and let $1 \leq j \leq m-1$. We will show that the interval I_k contains exactly j elements of S if and only if $w_{j-1} < (k+1)m - \rho \leq w_j$.

Suppose that I_k contains j elements. Suppose, by contradiction, that $w_{j-1} \geq (k+1)m - \rho$. We have $w_{m-1} > \dots > w_{j-1} \geq (k+1)m - \rho$, thus

$$w_{m-1}, \dots, w_{j-1} \in \cup_{t=k+1}^q I_t.$$

Hence, I_k contains at most $j-1$ elements of S (namely $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_{j-2} + k_{j-2}m$ for some $k_1, \dots, k_{j-2} \in \{0, \dots, k-1\}$). This contradicts the fact that I_k contains exactly j elements of S . Hence, $w_{j-1} < (k+1)m - \rho$.

If $w_j < (k+1)m - \rho$, then $w_0 < \dots < w_j < (k+1)m - \rho$, thus

$$w_0, \dots, w_j \in \cup_{t=0}^k I_t.$$

Then, I_k contains at least $j+1$ elements of S which are: $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_j + k_jm$ for some $k_1, \dots, k_j \in \{0, \dots, k-1\}$, which contradicts the fact that I_k contains exactly j elements of S . Hence, $w_j \geq (k+1)m - \rho$. Consequently, if I_k contains exactly j elements of S , then $w_{j-1} < (k+1)m - \rho \leq w_j$.

Conversely, $w_{j-1} < (k+1)m - \rho$ implies that $w_0 < \dots < w_{j-1} < (k+1)m - \rho$, then

$$w_0, \dots, w_{j-1} \in \cup_{t=0}^k I_t.$$

Hence, I_k contains at least j elements of S which are $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_{j-1} + k_{j-1}m$ for some $k_1, \dots, k_{j-1} \in \{0, \dots, k-1\}$. On the other hand, $w_j \geq (k+1)m - \rho$ implies that $w_{m-1} > \dots > w_j \geq (k+1)m - \rho$, then

$$w_{m-1}, \dots, w_j \in \cup_{t=k+1}^q I_t.$$

Thus, I_k contains at most j elements of S which are : $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_{j-1} + k_{j-1}m$ for some $k_1, \dots, k_{j-1} \in \{0, \dots, k-1\}$. Hence, if $w_{j-1} < (k+1)m - \rho \leq w_j$, then I_k contains exactly j elements of S and this proves our assertion. Consequently,

$$\begin{aligned}
 \eta_j &= |\{k \in \mathbb{N} \text{ such that } |I_k \cap S| = j\}| \\
 &= |\{k \in \mathbb{N} \text{ such that } w_{j-1} < (k+1)m - \rho \leq w_j\}| \\
 &= |\{k \in \mathbb{N} \text{ such that } \frac{w_{j-1} + \rho}{m} < (k+1) \leq \frac{w_j + \rho}{m}\}| \\
 &= |\{k \in \mathbb{N} \text{ such that } \frac{w_{j-1} + \rho}{m} - 1 < k \leq \frac{w_j + \rho}{m} - 1\}| \\
 &= |\{k \in \mathbb{N} \text{ such that } \lfloor \frac{w_{j-1} + \rho}{m} \rfloor \leq k \leq \lfloor \frac{w_j + \rho}{m} \rfloor - 1\}| \\
 &= \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor.
 \end{aligned}$$

Thus, the proof is complete. ■

Proposition 2.1.3 gives an equivalent form of Wilf's conjecture using Propositions 2.1.1 and 2.1.2.

Proposition 2.1.3. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f+1 = qm - \rho$ for some $q \in \mathbb{N}$ and $0 \leq \rho \leq m-1$. Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Then, S satisfies Wilf's conjecture if and only if

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

Proof. By Proposition 2.1.1, we have

$$f+1 \leq n\nu \Leftrightarrow qm - \rho \leq \nu \sum_{k=0}^{q-1} n_k \Leftrightarrow \sum_{k=0}^{q-1} m - \rho \leq \sum_{k=0}^{q-1} n_k \nu \Leftrightarrow \sum_{k=0}^{q-1} (n_k \nu - m) + \rho \geq 0.$$

Equivalently, we obtain

$$\sum_{j=1}^{m-1} \eta_j (j\nu - m) + \rho \geq 0.$$

By applying Proposition 2.1.2, we get

$$\sum_{j=1}^{m-1} \eta_j (j\nu - m) + \rho \geq 0 \Leftrightarrow \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

Thus, the proof is complete. ■

2.2 Technical results

Let S be a numerical semigroup and let the notations be as in Section 2.1. In this Section, we give some technical results will be used through the Chapter.

Remark 2.2.1. Let $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < \dots < w_{m-1}\}$. The following technical remarks will be used through the Chapter :

i) $\lfloor \frac{w_0 + \rho}{m} \rfloor = 0.$

ii) For all $1 \leq i \leq m-1$, we have $\lfloor \frac{w_i + \rho}{m} \rfloor \geq 1.$

- iii) For all $1 \leq i \leq m-1$, we have either $\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor$ or $\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1$.
- iv) If $\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1$, then $\lfloor \frac{w_i + \rho}{m} \rfloor \geq 2$ and $\rho \geq 1$.
- v) For all $0 \leq i < j \leq m-1$, we have $\lfloor \frac{w_i + \rho}{m} \rfloor \leq \lfloor \frac{w_j + \rho}{m} \rfloor$.
- vi) $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor = q$.

Proof.

- i) This is because $w_0 = 0$ and $0 \leq \rho < m$.
- ii) We have $m < w_i$ for all $1 \leq i \leq m-1$. This implies the result since $\rho \geq 0$.
- iii) For all $1 \leq i \leq m-1$, let $w_i = q_i m + r_i$ such that $q_i, r_i \in \mathbb{N}$ and $r_i < m$. We have $\lfloor \frac{w_i}{m} \rfloor = q_i$. Therefore,

$$\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{q_i m + r_i + \rho}{m} \rfloor = \lfloor q_i + \frac{r_i + \rho}{m} \rfloor = q_i + \lfloor \frac{r_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + \lfloor \frac{r_i + \rho}{m} \rfloor.$$

Since $0 \leq \rho, r_i < m$, it follows that $0 \leq \frac{r_i + \rho}{m} < 2$. Consequently, $0 \leq \lfloor \frac{r_i + \rho}{m} \rfloor \leq 1$. Hence,

$$\lfloor \frac{w_i}{m} \rfloor \leq \lfloor \frac{w_i + \rho}{m} \rfloor \leq \lfloor \frac{w_i}{m} \rfloor + 1.$$

Equivalently,

$$\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor \text{ or } \lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1.$$

- iv) Suppose that $\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1$. By using part ii), we get $\lfloor \frac{w_i + \rho}{m} \rfloor \geq 2$. In this case $\rho \geq 1$ (as $\rho \geq 0$).
- v) By definition, we have $w_i < w_j$ for all $0 \leq i < j \leq m-1$. Thus, $\frac{w_i + \rho}{m} < \frac{w_j + \rho}{m}$. Consequently, $\lfloor \frac{w_i + \rho}{m} \rfloor \leq \lfloor \frac{w_j + \rho}{m} \rfloor$.
- vi) By Lemma 1.0.21, we have $f = \max(\text{Ap}(S, m)) - m = w_{m-1} - m$. Hence, $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor = \lfloor \frac{f + m + \rho}{m} \rfloor = \lfloor \frac{qm - \rho - 1 + m + \rho}{m} \rfloor = q$.

Thus, the proof is complete. ■

Let $1 < \alpha < m-1$. Using Remark 2.2.1, we get the following inequalities which will be used later in the Chapter :

$$\begin{aligned} & \sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\ &= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) - \sum_{j=1}^{\alpha} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor (j\nu - m) \\ &= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) - \sum_{j=0}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor ((j+1)\nu - m) \\ &= \sum_{j=1}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) + \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor (\nu - m) - \sum_{j=1}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor ((j+1)\nu - m) \\ &= \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor (\nu - m) - \sum_{j=1}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \\ &= \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor (\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor \nu - \sum_{j=2}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \end{aligned}$$

$$= \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor \nu - \sum_{j=2}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \quad (\text{as } \lfloor \frac{w_0 + \rho}{m} \rfloor = 0).$$

From Remark 2.2.1 (v), we have $\lfloor \frac{w_j + \rho}{m} \rfloor \leq \lfloor \frac{w_\alpha + \rho}{m} \rfloor \forall 2 \leq j \leq \alpha - 1$. Hence,

$$\begin{aligned} & \sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) (j\nu - m) \\ & \geq \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor \nu - \sum_{j=2}^{\alpha-1} \lfloor \frac{w_\alpha + \rho}{m} \rfloor \nu \\ & = \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor \nu - \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha - 2)\nu \\ & = -\lfloor \frac{w_1 + \rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha + \rho}{m} \rfloor (2\nu - m). \end{aligned}$$

Consequently, we have

$$\sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) (j\nu - m) \geq -\lfloor \frac{w_1 + \rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha + \rho}{m} \rfloor (2\nu - m). \quad (2.2.1)$$

Therefore,

$$\begin{aligned} & \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) (j\nu - m) \\ & \geq \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) ((\alpha + 1)\nu - m) \quad (\text{as } j \geq \alpha + 1 \text{ and } \lfloor \frac{w_j + \rho}{m} \rfloor \geq \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) \\ & = ((\alpha + 1)\nu - m) \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) \\ & = ((\alpha + 1)\nu - m) (\sum_{j=\alpha+1}^{m-1} \lfloor \frac{w_j + \rho}{m} \rfloor - \sum_{j=\alpha+1}^{m-1} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) \\ & = ((\alpha + 1)\nu - m) (\sum_{j=\alpha+1}^{m-1} \lfloor \frac{w_j + \rho}{m} \rfloor - \sum_{j=\alpha}^{m-2} \lfloor \frac{w_j + \rho}{m} \rfloor) \\ & = ((\alpha + 1)\nu - m) (\sum_{j=\alpha+1}^{m-2} \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor - \sum_{j=\alpha+1}^{m-2} \lfloor \frac{w_j + \rho}{m} \rfloor) \\ & = (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor) ((\alpha + 1)\nu - m). \end{aligned}$$

Hence, we obtain

$$\sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) (j\nu - m) \geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor) ((\alpha + 1)\nu - m). \quad (2.2.2)$$

The following technical Lemma will be used through the Chapter :

Lemma 2.2.2. Let $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < \dots < w_{m-1}\}$ and suppose that $w_i \geq w_j + w_k$. We have the following :

i) $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k + \rho}{m} \rfloor - 1$.

ii) If $\lfloor \frac{w_i + \rho}{m} \rfloor - \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_k + \rho}{m} \rfloor = -1$, then

$$\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1, \quad \lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \text{ and } \rho \geq 1.$$

In particular,

$$\lfloor \frac{w_j + \rho}{m} \rfloor \geq 2, \lfloor \frac{w_k + \rho}{m} \rfloor \geq 2 \text{ and } \rho \geq 1.$$

Proof.

i) Assume that $w_i \geq w_j + w_k$. Then, $w_i + \rho \geq w_j + w_k + \rho$. Consequently,

$$\frac{w_i + \rho}{m} \geq \frac{w_j + w_k + \rho}{m} \Rightarrow \lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + w_k + \rho}{m} \rfloor.$$

Therefore, we have

$$\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k}{m} \rfloor.$$

By Remark 2.2.1 (*iii*), $\lfloor \frac{w_k}{m} \rfloor \geq \lfloor \frac{w_k + \rho}{m} \rfloor - 1$. Hence,

$$\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k + \rho}{m} \rfloor - 1.$$

ii) Suppose that $w_i \geq w_j + w_k$ and that $\lfloor \frac{w_i + \rho}{m} \rfloor - \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_k + \rho}{m} \rfloor = -1$. Suppose by the way of contradiction that $\lfloor \frac{w_j + \rho}{m} \rfloor \neq \lfloor \frac{w_j}{m} \rfloor + 1$ or $\lfloor \frac{w_k + \rho}{m} \rfloor \neq \lfloor \frac{w_k}{m} \rfloor + 1$ or $\rho < 1$. By Remark 2.2.1 (*iii*) and that $\rho \geq 0$, it follows that

$$\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor \text{ or } \lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor \text{ or } \rho = 0.$$

Since $w_i \geq w_j + w_k$, we have

$$\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + w_k + \rho}{m} \rfloor.$$

Since $\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor$ or $\lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor$ or $\rho = 0$, it follows that

$$\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k + \rho}{m} \rfloor,$$

which contradicts the hypothesis. Hence,

$$\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1, \lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \text{ and } \rho \geq 1.$$

Using Remark 2.2.1 (*ii*), we get that $\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1 \geq 2$, $\lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \geq 2$ and $\rho \geq 1$.

Thus, the proof is complete. ■

2.3 Numerical semigroups with $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$

In this Section, we show that Wilf's conjecture holds for numerical semigroups in the following cases :

1. $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m - 1$.
2. $m - \nu \leq 5$. (Note that the case $m - \nu \leq 3$ results from the fact that Wilf's conjecture holds for $2\nu \geq m$. This case has been proved in [19]), however we shall give a proof in order to cover it through our techniques).

Then, we deduce the conjecture for $m = 9$ and for $(2 + \frac{1}{q})\nu \geq m$. ■

Next, we will show that Wilf's conjecture holds for numerical semigroups with

$$w_{m-1} \geq w_1 + w_\alpha \quad \text{and} \quad (2 + \frac{\alpha-3}{q})\nu \geq m.$$

Theorem 2.3.1. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f+1 = qm - \rho$ for some $q, \rho \in \mathbb{N}$; $0 \leq \rho \leq m-1$. Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m-1$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. We are going to use the equivalent form of Wilf's conjecture given in Proposition 2.1.3. Since $w_{m-1} \geq w_1 + w_\alpha$, by Lemma 2.2.2, it follows that

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_1 + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor - 1.$$

Let $x = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor$. Then, $x \geq -1$ and $\lfloor \frac{w_1 + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - x = q - x$ (by Remark 2.2.1 (vi)). Now, using (2.2.1) and (2.2.2), we have

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &\geq -\lfloor \frac{w_1 + \rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha + \rho}{m} \rfloor (2\nu - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \quad (\text{by (2.2.1) and (2.2.2)}) \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor \left(-\nu + ((\alpha+1)\nu - m) - ((\alpha+1)\nu - m) \right) + \lfloor \frac{w_\alpha + \rho}{m} \rfloor (2\nu - m) \\ &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor (\alpha\nu - m) + \lfloor \frac{w_\alpha + \rho}{m} \rfloor (2\nu - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\ &= (\lfloor \frac{w_1 + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor)(2\nu - m) + \lfloor \frac{w_1 + \rho}{m} \rfloor (\alpha - 2)\nu \\ &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\ &= (q - x)(2\nu - m) + \lfloor \frac{w_1 + \rho}{m} \rfloor (\alpha - 2)\nu + x((\alpha+1)\nu - m) + \rho. \end{aligned}$$

Consequently,

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq (q - x)(2\nu - m) + \lfloor \frac{w_1 + \rho}{m} \rfloor (\alpha - 2)\nu + x((\alpha+1)\nu - m) + \rho. \quad (2.3.1)$$

Since

$$x = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor \geq -1,$$

then we have two cases :

- If $x = -1$, then by Lemma 2.2.2 (ii), we have $\lfloor \frac{w_1 + \rho}{m} \rfloor \geq 2$. From (2.3.1), it follows that

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &\geq (q+1)(2\nu - m) + 2(\alpha - 2)\nu - ((\alpha+1)\nu - m) + \rho \\ &= \nu(2q + 2 + 2\alpha - 4 - \alpha - 1) - qm + \rho \\ &= \nu(2q + \alpha - 3) - qm + \rho \end{aligned}$$

$$\begin{aligned}
 &= q\left(\nu\left(2 + \frac{\alpha - 3}{q}\right) - m\right) + \rho \\
 &\geq 0 \quad (\text{by hypothesis}).
 \end{aligned}$$

- If $x \geq 0$. By Remark 2.2.1 (ii), we have $\lfloor \frac{w_1 + \rho}{m} \rfloor \geq 1$. From (2.3.1), it follows that

$$\begin{aligned}
 &\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 &\geq (q - x)(2\nu - m) + (\alpha - 2)\nu + x((\alpha + 1)\nu - m) + \rho \\
 &= \nu(2q + (\alpha - 2)(x + 1) + x) - qm + \rho \\
 &> \nu(2q + \alpha - 3) - qm + \rho \quad (\text{as } x \geq 0) \\
 &= q\left(\nu\left(2 + \frac{\alpha - 3}{q}\right) - m\right) + \rho \\
 &\geq 0 \quad (\text{by hypothesis}).
 \end{aligned}$$

Using Proposition 2.1.3, we get that S satisfies Wilf's conjecture. Thus, the proof is complete. ■

Example 2.3.2. Consider the following numerical semigroup

$$S = \langle 19, 21, 23, 25, 27, 28 \rangle.$$

Note that $3\nu < m$. We have $w_1 = 21$, $w_{14} = 56$ and $w_{m-1} = 83$ that is $w_{m-1} \geq w_1 + w_{14}$. In addition, $(2 + \frac{\alpha-3}{q})\nu = (2 + \frac{14-3}{4})6 \geq 19 = m$. Thus, the conditions of Theorem 2.3.1 are valid. ■

In the following we shall deduce some cases where Wilf's conjecture holds. We start with the following technical Lemma.

Lemma 2.3.3. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. If $m - \nu > \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ for some $\alpha \in \mathbb{N}^*$, then $w_{m-1} \geq w_1 + w_\alpha$.

Proof. Suppose by contradiction that $w_{m-1} < w_1 + w_\alpha$. Let

$$w \in \text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m)).$$

Thus, $w \leq w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S, m)^*$ this follows from Corollary 1.0.29. Hence, $w \leq w_{m-1} < w_1 + w_\alpha$. Thus, the only possible values for w are included in $\{w_i + w_j; 1 \leq i \leq j \leq \alpha - 1\}$. By Corollary 1.0.31, we have $m - \nu = |\text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m))|$. Therefore, $m - \nu \leq \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$, which is impossible. Hence,

$$w_{m-1} \geq w_1 + w_\alpha.$$

Thus, the proof is complete. ■

Next, we will deduce Wilf's conjecture for numerical Semigroups with

$$m - \nu > \frac{\alpha(\alpha - 1)}{2} \quad \text{and} \quad \left(2 + \frac{\alpha - 3}{q}\right)\nu \geq m.$$

It will be used later to show that the conjecture holds for those with $(2 + \frac{1}{q})\nu \geq m$, and in order also to cover the result in [19] saying that the conjecture is true for $2\nu \geq m$.

Corollary 2.3.4. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f+1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. Suppose that $m - \nu > \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ for some $1 < \alpha < m-1$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. It follows from Lemma 2.3.3 that if $m - \nu > \frac{\alpha(\alpha-1)}{2}$, then $w_{m-1} \geq w_1 + w_\alpha$. Now, use Theorem 2.3.1. Thus, the proof is complete. \blacksquare

As a direct consequence of Theorem 2.3.1, we get the following Corollary.

Corollary 2.3.5. Let S be a numerical semigroup with a given multiplicity m and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. If $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m - 1$ and $m \leq 8 + 4(\frac{\alpha-3}{q})$, then S satisfies Wilf's conjecture.

Proof. By Theorem 2.3.1, we may assume that $(2 + \frac{\alpha-3}{q})\nu < m$. Therefore,

$$\nu < \frac{qm}{2q + \alpha - 3} \leq \frac{8q + \alpha - 12}{2q + \alpha - 3}.$$

Hence, $\nu < 4$. Consequently, S satisfies Wilf's conjecture (see [9]). Thus, the proof is complete. \blacksquare

In the following Lemma, we will show that Wilf's conjecture holds for numerical semigroups with $m - \nu \leq 3$. This will enable us later to prove the conjecture for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$ and cover the result in [19] saying that the conjecture is true for $2\nu \geq m$.

Lemma 2.3.6. Let S be a numerical Semigroup with multiplicity m and embedding dimension ν . If $m - \nu \leq 3$, then S satisfies Wilf's conjecture.

Proof. We may assume that $\nu \geq 4$ ($\nu \leq 3$ is solved [9]). We are going to show that S satisfies Wilf's conjecture by means of Proposition 2.1.3.

Case 1. If $m - \nu = 0$ (S is said to be a numerical semigroup with maximal embedding dimension). Then, $t(S) = m - 1 = \nu - 1$ (Corollary 3.2 [18]). Consequently, S satisfies Wilf's conjecture ([9] Proposition 2.3).

Case 2. If $m - \nu = 1$, then we may assume that $m = \nu + 1 \geq 5$ ($\nu \geq 4$). By taking $\alpha = 1$ in (2.2.2), we get

$$\sum_{j=2}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m). \quad (2.3.2)$$

Hence, we have

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \sum_{j=1}^1 (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \sum_{j=2}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= (\lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_0 + \rho}{m} \rfloor)(\nu - m) + \sum_{j=2}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor(\nu - m) + \sum_{j=2}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \quad (\text{as } \lfloor \frac{w_0 + \rho}{m} \rfloor = 0) \\ &\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(\nu - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + \rho \quad (\text{by (2.3.2)}) \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor(\nu - m + (2\nu - m) - (2\nu - m)) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + \rho \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor(3\nu - 2m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + \rho \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor(m - 3) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(m - 2) + \rho \quad (\text{as } m - \nu = 1). \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq \\ & \lfloor \frac{w_1 + \rho}{m} \rfloor(m-3) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(m-2) + \rho. \end{aligned} \quad (2.3.3)$$

Since $m - \nu = 1 > 0 = \frac{1(0)}{2}$, then by Lemma 2.3.3, it follows that $w_{m-1} \geq w_1 + w_1$. Consequently, by Lemma 2.2.2 (i), we have

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_1 + \rho}{m} \rfloor + \lfloor \frac{w_1 + \rho}{m} \rfloor - 1.$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor = -1$. By Lemma 2.2.2 (ii), we have $\lfloor \frac{w_1 + \rho}{m} \rfloor \geq 2$. From (2.3.3), we obtain

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ & \geq \lfloor \frac{w_1 + \rho}{m} \rfloor(m-3) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(m-2) + \rho \\ & \geq 2(m-3) - (m-2) + \rho \quad (\text{as } \lfloor \frac{w_1 + \rho}{m} \rfloor \geq 2) \\ & \geq 0 \quad (\text{as } m \geq 5). \end{aligned}$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor \geq 0$. From (2.3.3), we get

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ & \geq \lfloor \frac{w_1 + \rho}{m} \rfloor(m-3) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(m-2) + \rho \\ & \geq (m-3) + \rho \\ & \geq 0 \quad (\text{as } m \geq 5). \end{aligned}$$

Using Proposition 2.1.3, we get that S satisfies Wilf's conjecture if $m - \nu = 1$.

Case 3. If $m - \nu \in \{2, 3\}$. We have $m - \nu > 1 = \frac{2(1)}{2}$. If $(2 - \frac{1}{q})\nu \geq m$, then by Corollary 2.3.4 S satisfies Wilf's conjecture. Now, suppose that $(2 - \frac{1}{q})\nu < m$. Since Wilf's conjecture holds for $q \leq 3$ (see [14], [10]), we may assume that $q \geq 4$.

- If $m - \nu = 2$. Then, $(2 - \frac{1}{q})\nu < \nu + 2$. Hence, $\nu < 2(\frac{q}{q-1}) \leq \frac{8}{3}$. By [9], S satisfies Wilf's conjecture.
- If $m - \nu = 3$. Then, $(2 - \frac{1}{q})\nu < \nu + 3$. Hence, $\nu < 3(\frac{q}{q-1}) \leq 4$. By [9], S satisfies Wilf's conjecture.

Thus, Wilf's conjecture holds if $m - \nu \leq 3$. Thus, the proof is complete. ■

The next Corollary covers the result of Sammartano for numerical semigroups with $2\nu \geq m$ ([19]) using Corollary 2.3.4 and Lemma 2.3.6.

Corollary 2.3.7. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . If $2\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. If $m - \nu > 3 = \frac{3(2)}{2}$ and $2\nu \geq m$, then by Corollary 2.3.4 Wilf's conjecture holds. If $m - \nu \leq 3$, by Lemma 2.3.6, S satisfies Wilf's conjecture. Thus, the proof is complete. ■

In the following Corollary we will deduce Wilf's conjecture for numerical semigroups with $m - \nu = 4$. This will enable us later to prove the conjecture for those with $(2 + \frac{1}{q})\nu \geq m$.

Corollary 2.3.8. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . If $m - \nu = 4$, then S satisfies Wilf's conjecture.

Proof. Since Wilf's conjecture holds for $\nu \leq 3$ (see [9]), then we may assume that $\nu \geq 4$. Therefore, $\nu \geq m - \nu$. Consequently, $2\nu \geq m$. Hence, S satisfies Wilf's conjecture. Thus, the proof is complete. ■

The following technical Lemma will be used through the paper.

Lemma 2.3.9. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. If $m - \nu \geq \binom{\alpha}{2} - 1 = \frac{\alpha(\alpha-1)}{2} - 1$ for some $3 \leq \alpha \leq m - 2$, then $w_{m-1} \geq w_1 + w_\alpha$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$.

Proof. Suppose by the way of contradiction that $w_{m-1} < w_1 + w_\alpha$ and $w_{m-1} < w_{\alpha-2} + w_{\alpha-1}$. Let

$$w \in \text{A}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m)).$$

Then, $w \leq w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S, m)^*$ (Corollary 1.0.29). In this case, the only possible values of w are included in $\{w_i + w_j; 1 \leq i \leq \alpha - 1\} \setminus \{w_{\alpha-2} + w_{\alpha-1}, w_{\alpha-1} + w_{\alpha-1}\}$. Consequently, $m - \nu = |\text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m))| \leq \frac{\alpha(\alpha-1)}{2} - 2$. But $\frac{\alpha(\alpha-1)}{2} - 2 < \frac{\alpha(\alpha-1)}{2} - 1$, which contradicts the hypothesis. Hence,

$$w_{m-1} \geq w_1 + w_\alpha$$

or

$$w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}.$$

■

In the next theorem, we will show that Wilf's conjecture holds for numerical semigroups with $m - \nu = 5$.

Theorem 2.3.10. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . If $m - \nu = 5$, then S satisfies Wilf's conjecture.

Proof. Let $m - \nu = 5$. Since Wilf's conjecture holds for $2\nu \geq m$, then we may assume that $2\nu < m$. This implies that $\nu < \frac{m}{2} = \frac{\nu+5}{2}$ i.e., $\nu < 5$. Since the case $\nu \leq 3$ is known [9], then we shall assume that $\nu = 4$. This also implies that

$$m = \nu + 5 = 9.$$

Since $m - \nu = 5 = \frac{4(3)}{2} - 1$, by Lemma 2.3.9, it follows that

$$w_8 \geq w_2 + w_3$$

or

$$w_8 \geq w_1 + w_4.$$

Case 1. If $w_8 \geq w_2 + w_3$. By taking $\alpha = 3$ in (2.2.2) ($m = 9, \nu = 4$), we get

$$\sum_{j=4}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) \geq (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7). \quad (2.3.4)$$

Hence,

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \quad (m = 9) \\ &= \sum_{j=1}^3 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \sum_{j=4}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \end{aligned}$$

$$\begin{aligned}
 &= (\lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_0 + \rho}{9} \rfloor)(-5) + (\lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor)(-1) + (\lfloor \frac{w_3 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor)(3) \\
 &\quad + \sum_{j=4}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\
 &= \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(3) + \sum_{j=4}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \quad (\lfloor \frac{w_0 + \rho}{m} \rfloor = 0) \\
 &\geq \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(3) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho \quad (\text{Using (2.3.4)}).
 \end{aligned}$$

On the other hand, as $\lfloor \frac{w_1 + \rho}{9} \rfloor \leq \lfloor \frac{w_2 + \rho}{9} \rfloor$ and $\lfloor \frac{w_1 + \rho}{9} \rfloor \leq \lfloor \frac{w_3 + \rho}{9} \rfloor$, then

$$4 \lfloor \frac{w_1 + \rho}{9} \rfloor \leq 3 \lfloor \frac{w_2 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor.$$

Consequently, we have

$$\begin{aligned}
 &\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 &\geq \left(\lfloor \frac{w_2 + \rho}{9} \rfloor \left(\left(\frac{-3}{4} \right) 4 \right) + \lfloor \frac{w_3 + \rho}{9} \rfloor \left(\left(\frac{-1}{4} \right) 4 \right) \right) + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(3) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho \\
 &= \lfloor \frac{w_2 + \rho}{9} \rfloor(-7) + \lfloor \frac{w_3 + \rho}{9} \rfloor(2) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho \\
 &= \lfloor \frac{w_3 + \rho}{9} \rfloor(2) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho.
 \end{aligned}$$

Then,

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq \lfloor \frac{w_3 + \rho}{9} \rfloor(2) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho. \tag{2.3.5}$$

Since $w_8 \geq w_2 + w_3$, by Lemma 2.2.2, it follows that $\lfloor \frac{w_8 + \rho}{9} \rfloor \geq \lfloor \frac{w_2 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor - 1$.

- If

$$\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor \geq 0,$$

then (2.3.5) gives

$$\sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \geq 0.$$

- If

$$\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor = -1.$$

By Lemma 2.2.2, we have $\rho \geq 1$. Since for $q \leq 3$ Wilf's conjecture is solved (see [10], [14]), then may assume that $q \geq 4$. Since

$$\lfloor \frac{w_2 + \rho}{9} \rfloor \leq \lfloor \frac{w_3 + \rho}{9} \rfloor \quad \text{and} \quad \lfloor \frac{w_2 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor = \lfloor \frac{w_8 + \rho}{9} \rfloor + 1 = q + 1,$$

in this case, it follows that

$$\lfloor \frac{w_3 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor \geq \lfloor \frac{w_2 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor = q + 1 \geq 5.$$

Hence,

$$\lfloor \frac{w_3 + \rho}{9} \rfloor \geq 3.$$

Now, (2.3.5) gives

$$\sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - m) + \rho \geq 3(2) - 7 + 1 \geq 0.$$

Using Proposition 2.1.3, we get that S satisfies Wilf's conjecture in this case.

Case 2. If $w_8 \geq w_1 + w_4$. We may assume that $w_8 < w_2 + w_3$, since otherwise we are back to case 1. Hence, the possible values of $w \in \text{Ap}(S, 9)^* \setminus \min_{\leq S}(\text{Ap}(S, 9))$ are $\{w_1 + w_j; 1 \leq j \leq 7\} \cup \{w_2 + w_2\}$.

- If $\text{Ap}(S, 9)^* \setminus \min_{\leq S}(\text{Ap}(S, 9)) \subseteq \{w_1 + w_j; 1 \leq j \leq 7\}$. We have

$$5 = m - \nu = |\text{Ap}(S, 9)^* \setminus \min_{\leq S}(\text{Ap}(S, 9))|.$$

Then, there exist five elements in $\text{Ap}(S, 9)^*$ included in $\{w_1 + w_j; 1 \leq j \leq 7\}$. By Corollary 1.0.29, an element x of the Apéry set of S belongs to $\max_{\leq S}(\text{Ap}(S, m))$ if and only if $w_i \neq x + w_j$ for all $w_i, w_j \in \text{Ap}(S, m)^*$, then there exists at least five elements in $\text{Ap}(S, 9)^*$ that are not maximal (five elements from $\{w_1 \dots, w_7\}$), hence,

$$t(S) = |\{\max_{\leq S}(\text{Ap}(S, 9)) - 9\}| \leq 3 = \nu - 1.$$

Consequently, S satisfies Wilf's conjecture (Proposition 2.3 [9]).

- If

$$w_2 + w_2 \in \text{Ap}(S, 9)^* \setminus \min_{\leq S}(\text{Ap}(S, 9)),$$

then

$$w_2 + w_2 \in \text{Ap}(S, 9)$$

namely $w_8 \geq w_2 + w_2$. By Lemma 2.2.2, we have

$$\lfloor \frac{w_8 + \rho}{9} \rfloor \geq 2 \lfloor \frac{w_2 + \rho}{9} \rfloor - 1.$$

In particular,

$$\lfloor \frac{w_2 + \rho}{9} \rfloor \leq \frac{q+1}{2}. \quad (2.3.6)$$

By taking $\alpha = 4$ in (2.2.2) ($m = 9, \nu = 4$), we get

$$\sum_{j=5}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) \geq (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11). \quad (2.3.7)$$

We have

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \quad (m = 9) \\ &= \sum_{j=1}^4 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \sum_{j=5}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\ &= (\lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_0 + \rho}{9} \rfloor)(-5) + (\lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor)(-1) + (\lfloor \frac{w_3 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor)(3) \end{aligned}$$

$$\begin{aligned}
 & +(\lfloor \frac{w_4 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \sum_{j=5}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\
 & = \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_4 + \rho}{9} \rfloor(7) \\
 & \quad + \sum_{j=5}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \quad (\text{as } \lfloor \frac{w_0 + \rho}{m} \rfloor = 0) \\
 & \geq \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_4 + \rho}{9} \rfloor(7) \\
 & \quad + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \quad (\text{by (2.3.7)}) \\
 & \geq \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) + (\frac{q+1}{2})(-4) + \lfloor \frac{w_4 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_4 + \rho}{9} \rfloor(7) + (\lfloor \frac{w_8 + \rho}{9} \rfloor \\
 & \quad - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \quad (\text{by using (2.3.6) and } \lfloor \frac{w_3 + \rho}{9} \rfloor \leq \lfloor \frac{w_4 + \rho}{9} \rfloor) \\
 & = \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) - 2(q+1) + \lfloor \frac{w_4 + \rho}{9} \rfloor(3) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \\
 & = \lfloor \frac{w_1 + \rho}{9} \rfloor(-4 + 11 - 11) - 2(q+1) + \lfloor \frac{w_4 + \rho}{9} \rfloor(3) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \\
 & = \lfloor \frac{w_1 + \rho}{9} \rfloor(7) - 2(q+1) + \lfloor \frac{w_4 + \rho}{9} \rfloor(3) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor)(11) + \rho \\
 & = (\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor)(3) + \lfloor \frac{w_1 + \rho}{9} \rfloor(4) - 2(q+1) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 & \geq (\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor)(3) + \lfloor \frac{w_1 + \rho}{9} \rfloor(4) - 2(q+1) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho.
 \end{aligned} \tag{2.3.8}$$

We have $w_8 \geq w_1 + w_4$, then by Lemma 2.2.2 (i)

$$\lfloor \frac{w_8 + \rho}{9} \rfloor \geq \lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor - 1.$$

- If $\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor \geq 0$. Let $x = \lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor$. Hence, $x \geq 0$ and $\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor = \lfloor \frac{w_8 + \rho}{9} \rfloor - x = q - x$ (Remark 2.2.1 (vi)). Then, (2.3.8) gives

$$\begin{aligned}
 & \sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\
 & \geq (q - x)(3) + 4 - 2(q+1) + 11x + \rho \\
 & = q + 8x + 2 + \rho \geq 0.
 \end{aligned}$$

- If $\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor = -1$. Then, $\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor = \lfloor \frac{w_8 + \rho}{9} \rfloor + 1 = q + 1$ (Remark 2.2.1 (vi)). By Lemma 2.2.2, we have $\lfloor \frac{w_1 + \rho}{9} \rfloor \geq 2$ and $\rho \geq 1$. Since $q \geq 1$ ($S \neq \mathbb{N}$ i.e., $f \geq 1$), then (2.3.8) gives

$$\begin{aligned}
 & \sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\
 & \geq (q+1)(3) + 8 - 2(q+1) - 11 + 1
 \end{aligned}$$

$$= q - 1 \geq 0.$$

Using Proposition 2.1.3, we get that S satisfies Wilf's conjecture in this case.

Thus, Wilf's conjecture holds if $m - \nu = 5$. Thus, the proof is complete. ■

In the next corollary, we will deduce the conjecture for $m = 9$.

Corollary 2.3.11. If S is a numerical Semigroup with multiplicity $m = 9$, then S satisfies Wilf's conjecture.

Proof. By Lemma 2.3.6, Corollary 2.3.8 and Theorem 2.3.10, we may assume that $m - \nu > 5$, hence, $\nu < m - 5 = 4$. By [9], S satisfies Wilf's conjecture. Thus, the proof is complete. ■

The following Lemma will enable us later to show that Wilf's conjecture holds for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$.

Lemma 2.3.12. Let S be a numerical Semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. If $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. Since $m - \nu = 6 \geq \frac{4(3)}{2} - 1$, by Lemma 2.3.9, it follows that

$$w_{m-1} \geq w_1 + w_4$$

or

$$w_{m-1} \geq w_2 + w_3.$$

Case 1. If $w_{m-1} \geq w_1 + w_4$. By hypothesis $(2 + \frac{1}{q})\nu \geq m$ and Theorem 2.3.1 Wilf's conjecture holds in this case.

Case 2. If $w_{m-1} \geq w_2 + w_3$. We may assume that $w_{m-1} < w_1 + w_4$, since otherwise we are back to case i . Hence, $\text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m)) = \{w_1 + w_1, w_1 + w_2, w_1 + w_3, w_2 + w_2, w_2 + w_3, w_3 + w_3\}$ (as $6 = m - \nu = |\text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m))|$). Consequently,

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \sum_{j=1}^3 (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \sum_{j=4}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= (\lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_0 + \rho}{m} \rfloor)(\nu - m) + (\lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + (\lfloor \frac{w_3 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(3\nu - m) \\ & \quad + \sum_{j=4}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_2 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_3 + \rho}{m} \rfloor(3\nu - m) + \sum_{j=4}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\ & \quad + \rho \quad (\text{as } \lfloor \frac{w_0 + \rho}{m} \rfloor = 0) \\ &\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_2 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_3 + \rho}{m} \rfloor(3\nu - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \quad (\text{by (2.3.4)}). \end{aligned}$$

On the other hand, as $\lfloor \frac{w_1 + \rho}{m} \rfloor \leq \lfloor \frac{w_2 + \rho}{m} \rfloor$ and $\lfloor \frac{w_1 + \rho}{m} \rfloor \leq \lfloor \frac{w_3 + \rho}{m} \rfloor$, then

$$2\lfloor \frac{w_1 + \rho}{m} \rfloor \leq \lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor.$$

Consequently, we have

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 & \geq (\lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{-\nu}{2}) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{-\nu}{2})) + \lfloor \frac{w_2 + \rho}{m} \rfloor (-\nu) + \lfloor \frac{w_3 + \rho}{m} \rfloor (3\nu - m) \\
 & \quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
 & = \lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{-3\nu}{2}) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{5\nu}{2} - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
 & = \lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{-3\nu}{2} + (4\nu - m) - (4\nu - m)) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{5\nu}{2} - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
 & = \lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{5\nu}{2} - m) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{5\nu}{2} - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
 & = \lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{3\nu}{2} - 6) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{3\nu}{2} - 6) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(3\nu - 6) + \rho \quad (\text{as } m - \nu = 6).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 & \geq \lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{3\nu}{2} - 6) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{3\nu}{2} - 6) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor + \rho - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(3\nu - 6).
 \end{aligned} \tag{2.3.9}$$

We have $w_{m-1} \geq w_2 + w_3$, by Lemma 2.2.2, it follows that

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor - 1.$$

- If

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor \geq 0,$$

using $\nu \geq 4$ in (2.3.9) ($\nu \leq 3$ is solved [9]), we get

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

- If

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor = -1,$$

then

$$\lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor + 1,$$

that is

$$\lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor = q + 1. \tag{2.3.10}$$

We have $w_3 + w_3 \in \text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m))$ namely $w_3 + w_3 \in \text{Ap}(S, m)$, then $w_{m-1} \geq w_3 + w_3$. By Lemma 2.2.2, we have $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq 2\lfloor \frac{w_3 + \rho}{m} \rfloor - 1$. In particular,

$$\lfloor \frac{w_3 + \rho}{m} \rfloor \leq \frac{q+1}{2}. \tag{2.3.11}$$

Since Wilf's conjecture holds for $q \leq 3$ ([10], [14]), then we may assume that $q \geq 4$. Since $\lfloor \frac{w_2 + \rho}{m} \rfloor \leq \lfloor \frac{w_3 + \rho}{m} \rfloor$, by (2.3.10) and (2.3.11), it follows that $\lfloor \frac{w_2 + \rho}{m} \rfloor = \lfloor \frac{w_3 + \rho}{m} \rfloor = \frac{q+1}{2}$, in particular q is odd, then we have to assume that $q \geq 5$. Now, using (2.3.10), $q \geq 5$ and the hypothesis $(2 + \frac{1}{q})\nu \geq m = \nu + 6$ in (2.3.9), we get

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 & \geq (\lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor)(\frac{3\nu}{2} - 6) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(3\nu - 6) + \rho \\
 & = (q+1)(\frac{3\nu}{2} - 6) - (3\nu - 6) + \rho \\
 & = \nu(\frac{3q}{2} + \frac{3}{2} - 3) - 6q + \rho \\
 & \geq \nu(\frac{3q}{2} - \frac{3}{2}) - q\nu - \nu + \rho \quad (\text{as } 6q \leq q\nu + \nu) \\
 & = \nu(\frac{q}{2} - \frac{5}{2}) + \rho \geq 0 \quad (\text{as } q \geq 5).
 \end{aligned}$$

Using Proposition 2.1.3, we get that S satisfies Wilf's conjecture in this case.

Therefore, Wilf's conjecture holds if $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$. Thus, the proof is complete. ■

Next, we will generalize a result for Sammartano ([19]) and show that Wilf's conjecture holds for numerical semigroups satisfying $(2 + \frac{1}{q})\nu \geq m$, using Lemma 2.3.6, Corollary 2.3.8, Theorem 2.3.10, Lemma 2.3.12 and Corollary 2.3.4.

Theorem 2.3.13. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. If $(2 + \frac{1}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof.

- If $m - \nu \leq 3$, then by Lemma 2.3.6 Wilf's conjecture holds.
- If $m - \nu = 4$, then by Corollary 2.3.8 Wilf's conjecture holds.
- If $m - \nu = 5$, then by Theorem 2.3.10 Wilf's conjecture holds.
- If $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$, then by Lemma 2.3.12 Wilf's conjecture holds.
- If $m - \nu > 6$ and $(2 + \frac{1}{q})\nu \geq m$, then by Corollary 2.3.4 Wilf's conjecture holds.

Thus, the proof is complete. ■

Example 2.3.14. Consider the following numerical semigroup

$$S = \langle 13, 15, 17, 19, 21, 27 \rangle.$$

Note that $2\nu < m$. We have $(2 + \frac{1}{q})\nu = (2 + \frac{1}{4})6 \geq 13 = m$. Thus, the conditions of Theorem 2.3.13 are valid. ■

Corollary 2.3.15. Let S be a numerical semigroup with multiplicity m and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. If $m \leq 8 + \frac{4}{q}$, then S satisfies Wilf's conjecture.

Proof. If $\nu < 4$, then S satisfies Wilf's conjecture (see [9]). Hence, we can suppose that $\nu \geq 4$. Thus,

$$(2 + \frac{1}{q})\nu \geq (2 + \frac{1}{q})4 \geq m.$$

By using Theorem 2.3.13 S satisfies Wilf's conjecture. Thus, the proof is complete. ■

2.4 Numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$

In this Section, we will show that if S is a numerical semigroup such that

$$w_{m-1} \geq w_{\alpha-1} + w_\alpha \quad \text{and} \quad (\frac{\alpha+3}{3})\nu \geq m,$$

then S satisfies Wilf's conjecture.

Theorem 2.4.1. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ for some $1 < \alpha < m - 1$. If $(\frac{\alpha+3}{3})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. We may assume that $\rho \geq \frac{(3-q)\alpha m}{2\alpha+6}$. Indeed, if $0 \leq \rho < \frac{(3-q)\alpha m}{2\alpha+6}$, then $q < 3$ and Wilf's conjecture holds for this case (see [14]). We are going to show that S satisfies Wilf's conjecture by means of Proposition 2.1.3. We have

$$\begin{aligned} & \sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\ &= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) - \sum_{j=1}^{\alpha} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor (j\nu - m) \\ &= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) - \sum_{j=0}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor ((j+1)\nu - m) \\ &= \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor (\nu - m) - \sum_{j=1}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \\ &= \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \nu - \sum_{j=1}^{\alpha-2} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \quad (\text{as } \lfloor \frac{w_0 + \rho}{m} \rfloor = 0) \\ &\geq \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \nu - \sum_{j=1}^{\alpha-2} \frac{1}{2} \left(\lfloor \frac{w_\alpha + \rho}{m} \rfloor + \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \right) \nu \quad (\text{by Remark 2.2.1 (v)}) \\ &= \lfloor \frac{w_\alpha + \rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \nu - \left(\lfloor \frac{w_\alpha + \rho}{m} \rfloor + \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \right) \left(\frac{(\alpha-2)\nu}{2} \right). \end{aligned}$$

Hence,

$$\begin{aligned} & \sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\ & \geq \lfloor \frac{w_\alpha + \rho}{m} \rfloor \left(\frac{(\alpha+2)\nu}{2} - m \right) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \left(\frac{\alpha\nu}{2} \right). \end{aligned} \tag{2.4.1}$$

By (2.2.2), we have

$$\sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha+1)\nu - m).$$

Since $w_{m-1} \geq w_{\alpha-1} + w_\alpha$, by Lemma 2.2.2, it follows that

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor - 1.$$

Let $x = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor$. Then, $x \geq -1$ and $\lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - x = q - x$ (by Remark 2.2.1 vi). Now, using $\rho \geq \frac{(3-q)\alpha m}{2\alpha+6}$ and $(\frac{\alpha+3}{3})\nu \geq m$, we get

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 &= \sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 &\geq \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor ((\frac{\alpha+2}{2})\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor (\frac{\alpha\nu}{2}) \\
 &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \quad (\text{by (2.4.1) and (2.2.2)}) \\
 &= \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor \left(\frac{-\alpha\nu}{2} + (\alpha+1)\nu - m - ((\alpha+1)\nu - m) \right) + \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor ((\frac{\alpha+2}{2})\nu - m) \\
 &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\
 &= (\lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor + \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor)((\frac{\alpha+2}{2})\nu - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\
 &\geq (q-x)((\frac{\alpha+2}{2})\nu - m) + x((\alpha+1)\nu - m) + \rho \\
 &= \nu(q + \frac{q\alpha}{2} + \frac{x\alpha}{2}) - qm + \rho \\
 &\geq \nu(q + \frac{q\alpha}{2} - \frac{\alpha}{2}) - qm + \frac{(3-q)\alpha m}{2\alpha+6} \quad (\text{as } \rho \geq \frac{(3-q)\alpha m}{2\alpha+6}) \\
 &= \nu(q + \frac{q\alpha}{2} - \frac{\alpha}{2}) - m(\frac{q(2\alpha+6) + (q-3)\alpha}{2\alpha+6}) \\
 &= \nu(q + \frac{q\alpha}{2} - \frac{\alpha}{2}) - m(\frac{3q}{\alpha+3} + \frac{3q\alpha}{2(\alpha+3)} - \frac{3\alpha}{2(\alpha+3)}) \\
 &= (q + \frac{q\alpha}{2} - \frac{\alpha}{2})(\frac{3}{\alpha+3})((\frac{\alpha+3}{3})\nu - m) \geq 0 \quad (\text{by hypothesis}).
 \end{aligned}$$

Using Proposition 2.1.3, we get that S satisfies Wilf's conjecture. Thus, the proof is complete. ■

Example 2.4.2. Consider the following numerical semigroup

$$S = \langle 22, 23, 25, 27, 29, 31, 33 \rangle.$$

Note that $3\nu < m$. We have $w_6 = 33$, $w_7 = 46$ and $w_{m-1} = 87$ i.e., $w_{m-1} \geq w_6 + w_7$. Moreover, $(\frac{\alpha+3}{3})\nu = (\frac{7+3}{3})7 \geq 22 = m$, thus the conditions of Theorem 2.4.1 are valid. ■

The following Corollary 2.4.3 is an extension for Corollary 2.3.4 using Theorems 2.3.1 and 2.4.1.

Corollary 2.4.3. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Suppose that $m - \nu \geq \frac{\alpha(\alpha-1)}{2} - 1$ for some $7 \leq \alpha \leq m - 2$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. Since $m - \nu \geq \frac{\alpha(\alpha-1)}{2} - 1$, then by Lemma 2.3.9, we have $w_{m-1} \geq w_1 + w_{\alpha}$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$. Suppose that $w_{m-1} \geq w_1 + w_{\alpha}$. Since $(2 + \frac{\alpha-3}{q})\nu \geq m$, by applying Theorem 2.3.1, S satisfies Wilf's conjecture. Now, suppose that $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$. We may assume that $q \geq 4$ ($q \leq 3$ is solved [14], [10]). Then, for $\alpha \geq 7$, we have $(\frac{\alpha-1+3}{3})\nu \geq (2 + \frac{\alpha-3}{q})\nu$. Consequently, $(\frac{\alpha-1+3}{3})\nu \geq m$. Next, by applying Theorem 2.4.1, S satisfies Wilf's conjecture. Thus, the proof is complete. ■

As a direct consequence of Theorem 2.4.1, we get the following Corollary.

Corollary 2.4.4. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} \geq w_{\alpha-1} + w_{\alpha}$ for some $1 < \alpha < m - 1$. If $m \leq \frac{4(\alpha+3)}{3}$, then S satisfies Wilf's conjecture.

Proof. If $\nu < 4$, then S satisfies Wilf's conjecture (see [9]). Hence, we can suppose that $\nu \geq 4$. Thus, $(\frac{\alpha+3}{3})(\nu) \geq \frac{4(\alpha+3)}{3} \geq m$. By applying Theorem 2.4.1 S satisfies Wilf's conjecture. Thus, the proof is complete. \blacksquare

2.5 Numerical semigroups with $(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y-x-1) + (y-2) + \lfloor \frac{w_y}{m} \rfloor (x-1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2})\nu \geq m$ and $w_{m-1} - m \geq w_x + w_y$

In this Section, we will show that if S is a numerical Semigroup such that

$$w_{m-1} - m \geq w_x + w_y \quad \text{and} \quad (2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y-x-1) + (y-2) + \lfloor \frac{w_y}{m} \rfloor (x-1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2})\nu \geq m,$$

then S satisfies Wilf's conjecture.

Theorem 2.5.1. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} - m \geq w_x + w_y$ for some $0 < x < y < m-1$. If

$$(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y-x-1) + (y-2) + \lfloor \frac{w_y}{m} \rfloor (x-1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2})\nu \geq m,$$

then S satisfies Wilf's conjecture.

Proof. We are going to show that S satisfies Wilf's conjecture by means of Proposition 2.1.3. We have

$$\begin{aligned} & \sum_{j=1}^x (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\ &= \sum_{j=1}^x \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) - \sum_{j=1}^x \lfloor \frac{w_{j-1} + \rho}{m} \rfloor (j\nu - m) \\ &= \sum_{j=1}^x \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) - \sum_{j=0}^{x-1} \lfloor \frac{w_j + \rho}{m} \rfloor ((j+1)\nu - m) \\ &= \sum_{j=1}^{x-1} \lfloor \frac{w_j + \rho}{m} \rfloor (j\nu - m) + \lfloor \frac{w_x + \rho}{m} \rfloor (x\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor (\nu - m) - \sum_{j=1}^{x-1} \lfloor \frac{w_j + \rho}{m} \rfloor ((j+1)\nu - m) \\ &= \lfloor \frac{w_x + \rho}{m} \rfloor (x\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor (\nu - m) - \sum_{j=1}^{x-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \\ &= \lfloor \frac{w_x + \rho}{m} \rfloor (x\nu - m) - \sum_{j=1}^{x-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \quad (\text{as } \lfloor \frac{w_0 + \rho}{m} \rfloor = 0) \\ &\geq \lfloor \frac{w_x + \rho}{m} \rfloor (x\nu - m) - \sum_{j=1}^{x-1} \lfloor \frac{w_x + \rho}{m} \rfloor \nu \quad (\text{by Remark 2.2.1 (v)}) \\ &= \lfloor \frac{w_x + \rho}{m} \rfloor (x\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor (x-1)\nu \\ &= \lfloor \frac{w_x + \rho}{m} \rfloor (\nu - m). \end{aligned}$$

Therefore,

$$\sum_{j=1}^x (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq \lfloor \frac{w_x + \rho}{m} \rfloor (\nu - m). \quad (2.5.1)$$

In addition,

$$\begin{aligned}
 & \sum_{j=x+1}^y (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\
 &= \sum_{j=x+1}^y \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) - \sum_{j=x+1}^y \lfloor \frac{w_{j-1} + \rho}{m} \rfloor(j\nu - m) \\
 &= \sum_{j=x+1}^y \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) - \sum_{j=x}^{y-1} \lfloor \frac{w_j + \rho}{m} \rfloor((j+1)\nu - m) \\
 &= \sum_{j=x+1}^{y-1} \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) + \lfloor \frac{w_y + \rho}{m} \rfloor(y\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor((x+1)\nu - m) - \sum_{j=x+1}^{y-1} \lfloor \frac{w_j + \rho}{m} \rfloor((j+1)\nu - m) \\
 &= \lfloor \frac{w_y + \rho}{m} \rfloor(y\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor((x+1)\nu - m) - \sum_{j=x+1}^{y-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \\
 &\geq \lfloor \frac{w_y + \rho}{m} \rfloor(y\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor((x+1)\nu - m) - \sum_{j=x+1}^{y-1} \lfloor \frac{w_j + \rho}{m} \rfloor \nu \quad (\text{using Remark 2.2.1 (v)}) \\
 &= \lfloor \frac{w_y + \rho}{m} \rfloor(y\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor((x+1)\nu - m) - \lfloor \frac{w_y + \rho}{m} \rfloor(y-x-1)\nu \\
 &= \lfloor \frac{w_y + \rho}{m} \rfloor((x+1)\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor((x+1)\nu - m).
 \end{aligned}$$

Hence,

$$\sum_{j=x+1}^y (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq \lfloor \frac{w_y + \rho}{m} \rfloor((x+1)\nu - m) - \lfloor \frac{w_x + \rho}{m} \rfloor((x+1)\nu - m). \quad (2.5.2)$$

Moreover, we have

$$\begin{aligned}
 & \sum_{j=y+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \\
 &\geq \sum_{j=y+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)((y+1)\nu - m) \quad (\text{using Remark 2.2.1 (v)}) \\
 &= ((y+1)\nu - m) (\sum_{j=y+1}^{m-1} \lfloor \frac{w_j + \rho}{m} \rfloor - \sum_{j=y+1}^{m-1} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) \\
 &= ((y+1)\nu - m) (\sum_{j=y+1}^{m-1} \lfloor \frac{w_j + \rho}{m} \rfloor - \sum_{j=y}^{m-2} \lfloor \frac{w_j + \rho}{m} \rfloor) \\
 &= (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_y + \rho}{m} \rfloor)((y+1)\nu - m).
 \end{aligned}$$

Therefore,

$$\sum_{j=y+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_y + \rho}{m} \rfloor)((y+1)\nu - m). \quad (2.5.3)$$

Consequently,

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho$$

$$\begin{aligned}
 &= \sum_{j=1}^x \left(\left\lfloor \frac{w_j + \rho}{m} \right\rfloor - \left\lfloor \frac{w_{j-1} + \rho}{m} \right\rfloor \right) (j\nu - m) + \sum_{j=x+1}^y \left(\left\lfloor \frac{w_j + \rho}{m} \right\rfloor - \left\lfloor \frac{w_{j-1} + \rho}{m} \right\rfloor \right) (j\nu - m) \\
 &\quad + \sum_{j=y+1}^{m-1} \left(\left\lfloor \frac{w_j + \rho}{m} \right\rfloor - \left\lfloor \frac{w_{j-1} + \rho}{m} \right\rfloor \right) (j\nu - m) + \rho \\
 &\geq \left\lfloor \frac{w_x + \rho}{m} \right\rfloor (\nu - m) + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor ((x+1)\nu - m) - \left\lfloor \frac{w_x + \rho}{m} \right\rfloor ((x+1)\nu - m) \\
 &\quad + \left(\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor - \left\lfloor \frac{w_y + \rho}{m} \right\rfloor \right) ((y+1)\nu - m) + \rho \quad (\text{using (2.5.1), (2.5.2) and (2.5.3)}) \\
 &= \left\lfloor \frac{w_x + \rho}{m} \right\rfloor (-x\nu) + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor ((x+1)\nu - m) \\
 &\quad + \left(\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor - \left\lfloor \frac{w_y + \rho}{m} \right\rfloor \right) ((y+1)\nu - m) + \rho \\
 &= \left\lfloor \frac{w_x + \rho}{m} \right\rfloor \left(-x\nu + ((y+1)\nu - m) - ((y+1)\nu - m) \right) \\
 &\quad + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor ((x+1)\nu - m) + \left(\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor - \left\lfloor \frac{w_y + \rho}{m} \right\rfloor \right) ((y+1)\nu - m) + \rho \\
 &= \left\lfloor \frac{w_x + \rho}{m} \right\rfloor ((y-x+1)\nu - m) + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor ((x+1)\nu - m) \\
 &\quad + \left(\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor - \left\lfloor \frac{w_y + \rho}{m} \right\rfloor - \left\lfloor \frac{w_x + \rho}{m} \right\rfloor \right) ((y+1)\nu - m) + \rho.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 &\sum_{j=1}^{m-1} \left(\left\lfloor \frac{w_j + \rho}{m} \right\rfloor - \left\lfloor \frac{w_{j-1} + \rho}{m} \right\rfloor \right) (j\nu - m) + \rho \\
 &\quad \geq \left\lfloor \frac{w_x + \rho}{m} \right\rfloor ((y-x+1)\nu - m) + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor ((x+1)\nu - m) \\
 &\quad \quad + \left(\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor - \left\lfloor \frac{w_y + \rho}{m} \right\rfloor - \left\lfloor \frac{w_x + \rho}{m} \right\rfloor \right) ((y+1)\nu - m) + \rho.
 \end{aligned} \tag{2.5.4}$$

Since $w_{m-1} - m \geq w_x + w_y$, it follows

$$\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor > \left\lfloor \frac{w_x + w_y + \rho}{m} \right\rfloor. \tag{2.5.5}$$

Consider the following cases :

Case 1. If $\left\lfloor \frac{w_x + \rho}{m} \right\rfloor = \left\lfloor \frac{w_x}{m} \right\rfloor + 1$ and $\left\lfloor \frac{w_y + \rho}{m} \right\rfloor = \left\lfloor \frac{w_y}{m} \right\rfloor + 1$, then (2.5.5) gives

$$\left\lfloor \frac{w_{m-1} + \rho}{m} \right\rfloor \geq \left\lfloor \frac{w_x + \rho}{m} \right\rfloor + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor.$$

Then, from (2.5.4) and the hypothesis, we have

$$\begin{aligned}
 &\sum_{j=1}^{m-1} \left(\left\lfloor \frac{w_j + \rho}{m} \right\rfloor - \left\lfloor \frac{w_{j-1} + \rho}{m} \right\rfloor \right) (j\nu - m) + \rho \\
 &\quad \geq \left\lfloor \frac{w_x + \rho}{m} \right\rfloor ((y-x+1)\nu - m) + \left\lfloor \frac{w_y + \rho}{m} \right\rfloor ((x+1)\nu - m) + \rho \\
 &= \left(\left\lfloor \frac{w_x}{m} \right\rfloor + 1 \right) ((y-x+1)\nu - m) + \left(\left\lfloor \frac{w_y}{m} \right\rfloor + 1 \right) ((x+1)\nu - m) + \rho \\
 &= \left(\left\lfloor \frac{w_x}{m} \right\rfloor + \left\lfloor \frac{w_y}{m} \right\rfloor + 2 \right) \left(\left(2 + \frac{\left\lfloor \frac{w_x}{m} \right\rfloor (y-x-1) + (y-2) + \left\lfloor \frac{w_y}{m} \right\rfloor (x-1)}{\left\lfloor \frac{w_x}{m} \right\rfloor + \left\lfloor \frac{w_y}{m} \right\rfloor + 2} \right) \nu - m \right) + \rho \\
 &\geq 0.
 \end{aligned}$$

By Proposition 2.1.3, we get that Wilf's conjecture holds in this case.

Case 2. If $\lfloor \frac{w_x + \rho}{m} \rfloor = \lfloor \frac{w_x}{m} \rfloor$ and $\lfloor \frac{w_y + \rho}{m} \rfloor = \lfloor \frac{w_y}{m} \rfloor + 1$, then (2.5.5) gives

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor > \lfloor \frac{w_x + \rho}{m} \rfloor + \lfloor \frac{w_y + \rho}{m} \rfloor.$$

Then, from (2.5.4) and the hypothesis, we have

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ & \geq \lfloor \frac{w_x + \rho}{m} \rfloor ((y - x + 1)\nu - m) + \lfloor \frac{w_y + \rho}{m} \rfloor ((x + 1)\nu - m) + ((y + 1)\nu - m) + \rho \\ & = \lfloor \frac{w_x}{m} \rfloor ((y - x + 1)\nu - m) + (\lfloor \frac{w_y}{m} \rfloor + 1)((x + 1)\nu - m) + ((y + 1)\nu - m) + \rho \\ & = \left(\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2 \right) \left(\left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1) + x}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu - m \right) + \rho \\ & > \left(\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2 \right) \left(\left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu - m \right) + \rho \\ & \geq 0. \end{aligned}$$

By Proposition 2.1.3, we get that Wilf's conjecture holds in this case.

Case 3. If $\lfloor \frac{w_x + \rho}{m} \rfloor = \lfloor \frac{w_x}{m} \rfloor + 1$ and $\lfloor \frac{w_y + \rho}{m} \rfloor = \lfloor \frac{w_y}{m} \rfloor$, then (2.5.5) gives

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor > \lfloor \frac{w_x + \rho}{m} \rfloor + \lfloor \frac{w_y + \rho}{m} \rfloor.$$

Then, from (2.5.4) and the hypothesis, we have

$$\begin{aligned} & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ & \geq \lfloor \frac{w_x + \rho}{m} \rfloor ((y - x + 1)\nu - m) + \lfloor \frac{w_y + \rho}{m} \rfloor ((x + 1)\nu - m) + ((y + 1)\nu - m) + \rho \\ & = (\lfloor \frac{w_x}{m} \rfloor + 1)((y - x + 1)\nu - m) + \lfloor \frac{w_y}{m} \rfloor ((x + 1)\nu - m) + ((y + 1)\nu - m) + \rho \\ & = \mathcal{P}_{xy} \left(\left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1) + (y - x)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu - m \right) + \rho \\ & \geq \mathcal{P}_{xy} \left(\left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu - m \right) + \rho \\ & \geq 0, \end{aligned}$$

where

$$\mathcal{P}_{xy} = \left(\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2 \right).$$

By Proposition 2.1.3, we get that Wilf's conjecture holds in this case.

Case 4. If $\lfloor \frac{w_x + \rho}{m} \rfloor = \lfloor \frac{w_x}{m} \rfloor$ and $\lfloor \frac{w_y + \rho}{m} \rfloor = \lfloor \frac{w_y}{m} \rfloor$, then (2.5.5) gives

$$\lfloor \frac{w_{m-1} + \rho}{m} \rfloor > \lfloor \frac{w_x + \rho}{m} \rfloor + \lfloor \frac{w_y + \rho}{m} \rfloor.$$

Then, from (2.5.4) and the hypothesis, we have

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 & \geq \lfloor \frac{w_x + \rho}{m} \rfloor ((y - x + 1)\nu - m) + \lfloor \frac{w_y + \rho}{m} \rfloor ((x + 1)\nu - m) + ((y + 1)\nu - m) + \rho \\
 & = \lfloor \frac{w_x}{m} \rfloor ((y - x + 1)\nu - m) + \lfloor \frac{w_y}{m} \rfloor ((x + 1)\nu - m) + ((y + 1)\nu - m) + \rho.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
 & \geq \left(\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 1 \right) \left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 1) + \lfloor \frac{w_y}{m} \rfloor (x - 1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 1} \right) \nu - m + \rho \\
 & = \left(\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 1 \right) \left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu - m + \rho \\
 & \geq 0.
 \end{aligned}$$

By Proposition 2.1.3, we get that Wilf's conjecture holds in this case.

Thus, Wilf's conjecture holds in all cases. Thus, the proof is complete. ■

As a direct consequence of Theorem 2.5.1, we get the following Corollaries.

Corollary 2.5.2. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} - m \geq w_x + w_y$, $x \geq \alpha + 1$, $y - x \geq \alpha + 1$ for some $\alpha \in \mathbb{N}$. If $(2 + \alpha)\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. We have

$$\begin{aligned}
 \left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu & \geq \left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor \alpha + 2\alpha + \lfloor \frac{w_y}{m} \rfloor \alpha}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu \\
 & \geq (2 + \alpha)\nu \geq m.
 \end{aligned}$$

By Theorem 2.5.1, S satisfies Wilf's conjecture. Thus, the proof is complete. ■

Corollary 2.5.3. Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} - m \geq w_x + w_y$, for some $x \geq \alpha + 1$, $y - x \geq \alpha + 1$ and $\alpha \in \mathbb{N}$. If $m \leq 4(2 + \alpha)$, then S satisfies Wilf's conjecture.

Proof. We may assume that $\nu \geq 4$ ($\nu \leq 3$ is solved [9]), then $(2 + \alpha)\nu \geq (2 + \alpha)4 \geq m$. By applying Corollary 2.5.2, S satisfies Wilf's conjecture. Thus, the proof is complete. ■

Example 2.5.4. Consider the following numerical semigroup

$$S = \langle 19, 21, 23, 25, 27, 28 \rangle.$$

Note that $2\nu < m$. We have $w_4 = 27$, $w_5 = 28$ and $w_{m-1} - m = 64$ i.e., $w_{m-1} - m \geq w_4 + w_5$. Moreover,

$$\left(2 + \frac{\lfloor \frac{w_x}{m} \rfloor (y - x - 1) + (y - 2) + \lfloor \frac{w_y}{m} \rfloor (x - 1)}{\lfloor \frac{w_x}{m} \rfloor + \lfloor \frac{w_y}{m} \rfloor + 2} \right) \nu = \left(2 + \frac{0 + 3 + 3}{1 + 1 + 2} \right) 6 \geq 19 = m.$$

Thus, the conditions of Theorem 2.5.1 are valid. ■

2.6 Numerical semigroups with $m - \nu > \frac{(n-2)(n-3)}{2}$

In this Section, we show that Wilf's conjecture holds for numerical semigroups with $m - \nu > \frac{(n-2)(n-3)}{2}$ and also the conjecture holds for those with $n \leq 5$.

Lemma 2.6.1. Let S be a numerical semigroup with multiplicity m , embedding dimension ν , Frobenius number f and $n = |\{s \in S; s < f\}|$. If $m - \nu > \frac{\alpha(\alpha-1)}{2}$ for some $0 \leq \alpha \leq m - 2$, then $w_\alpha < f$. In particular, $n \geq \alpha + 2$.

Proof. We claim that $w_\alpha < f$. Suppose by the way of contradiction that $w_\alpha > f$ ($w_\alpha \neq f$), and let $w \in \text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m))$. Then, there exists $w_i, w_j \in \text{Ap}(S, m)^*$ such that $w = w_i + w_j$ (Corollary 1.0.29). Suppose that at least one of the two indices, let's say i , is greater than or equal to α . Then, $w = w_i + w_j \geq w_\alpha + m \geq f + 1 + m$. Hence, $w - m \in S$ which contradicts the fact that $w \in \text{Ap}(S, m)$. Consequently, the two indices are necessarily less than or equal to $\alpha - 1$. Since $|\text{Ap}(S, m)^* \setminus \min_{\leq_S}(\text{Ap}(S, m))| = m - \nu$ (Corollary 1.0.31), we deduce that $m - \nu \leq \frac{\alpha(\alpha-1)}{2}$ which is impossible. Consequently, $w_\alpha < f$. Therefore, we get that $\{0, m, w_1, w_2, \dots, w_\alpha\} \subseteq \{s \in S; s < f\}$. Hence, $n \geq \alpha + 2$. ■

Theorem 2.6.2. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and $n = |\{s \in S; s < f\}|$. If $m - \nu > \frac{(n-2)(n-3)}{2}$ with $2 \leq n \leq m$, then S satisfies Wilf's conjecture.

Proof. By Lemma 2.6.1, the condition $m - \nu > \frac{(n-2)(n-3)}{2}$, gives that

$$\{0, m, w_1, w_2, \dots, w_{n-2}\} \subseteq \{s \in S; s < f\}.$$

Therefore, $\{0, m, w_1, w_2, \dots, w_{n-2}\} = \{s \in S; s < f\}$. Hence, $2m > f$. By [14], it follows that S satisfies Wilf's conjecture. ■

In [9], D. Dobbs and G. Matthews proved Wilf's conjecture for $n \leq 4$ in a long technical proof. In [10], S. Eliahou showed Wilf's conjecture has a positive answer for $n \leq 6$. We are going to introduce a simpler proof for $n \leq 5$ using the previous theorem (note that If $n \leq 5$, then we can assume $2 \leq n \leq 5$).

Corollary 2.6.3. Let S be a numerical semigroup with multiplicity m , embedding dimension ν and $n = |\{s \in S; s < f\}|$. If $n \leq 5$, then S satisfies Wilf's conjecture.

Proof. By Theorem 2.3.6 and Theorem 2.3.10, we will assume that $m - \nu > 5$ which is strictly greater than $\frac{\alpha(\alpha-1)}{2}$ for $\alpha \in \{0, 1, 2, 3\}$. Hence, by applying Theorem 2.6.2 for $n = \alpha + 2 \in \{2, 3, 4, 5\}$ Wilf's conjecture holds. ■

Numerical semigroup of the form $\langle m, m + 1, \dots, m + l, k(m + l) + r \rangle$

Throughout this chapter we suppose that S is a numerical semigroup minimally generated by $m, m + 1, \dots, m + l, k(m + l) + r$ with $k, l, m, r \in \mathbb{N}^*$ and $r \leq (k + 1)l + 1$. The aim of this chapter is to determine the Frobenius number $f(S)$ and the genus $g(S)$. Also, it aims to characterize those numerical semigroups which are symmetric (resp. pseudo-symmetric) and to determine the set of pseudo-Frobenius numbers $PF(S)$.

Definition 3.0.1. Let $k, l, m, r \in \mathbb{N}^*$. For every $1 \leq i \leq m - 1$, write, by the euclidean division, $i = \alpha_i(kl + r) + \beta_i l + t_i$ with $0 \leq \beta_i l + t_i < kl + r$ and $0 \leq t_i < l$. In particular

$$\alpha_i = \lfloor \frac{i}{kl + r} \rfloor, \quad \beta_i = \lfloor \frac{i - \alpha_i(kl + r)}{l} \rfloor$$

and

$$t_i = i - \alpha_i(kl + r) - \beta_i l.$$

For the convenience of the statement we will use the following notation :

$$i = \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i$$

where

$$\epsilon_i = \begin{cases} 1 & \text{if } t_i \neq 0, \\ 0 & \text{if } t_i = 0. \end{cases}$$

Clearly $\epsilon_i t_i = t_i$ but we shall use ϵ_i later in the notations. ■

Proposition 3.0.2 and 3.0.3 give some properties that will be used in this Chapter using the notations used in Definition 3.0.1.

Proposition 3.0.2. Let the notations be as in Definition 3.0.1 and suppose that $r \leq (k + 1)l + 1$, we have

$$\beta_i + \epsilon_i \leq 2k + 1.$$

Proof. By using Definition 3.0.1 and $r \leq (k + 1)l + 1$, it follows that

$$\beta_i l + \epsilon_i t_i \leq kl + r - 1 \leq kl + (k + 1)l = (2k + 1)l.$$

Case 1. If $\epsilon_i = 0$. We have $\beta_i l = \beta_i l + \epsilon_i t_i \leq (2k + 1)l$. Consequently, $\beta_i \leq 2k + 1$. Hence, $\beta_i + \epsilon_i \leq 2k + 1$.

Case 2. If $\epsilon_i = 1$. We have $\beta_i l + \epsilon_i t_i \leq (2k + 1)l$ and $t_i \geq 1$ (as $\epsilon_i = 1$). If $\beta_i \geq 2k + 1$, then $\beta_i l + \epsilon_i t_i \geq (2k + 1)l + 1$, which gives a contradiction. Consequently, $\beta_i \leq 2k$. Then, $\beta_i + \epsilon_i \leq 2k + 1$. ■

Proposition 3.0.3. Let the notations be as in Definition 3.0.1 and suppose that $r \leq (k + 1)l + 1$, we may assume that

$$r \leq \min((k + 1)l + 1, m + l - 1).$$

Proof. We claim that we may assume that $r < m + l$. Indeed, if $r \geq m + l$, then there exist $q', r' \in \mathbb{N}$ such that $r = q'(m + l) + r'$ with $r' < m + l$. Let $k' = k + q'$, then $k(m + l) + r = (k + q')(m + l) + r' = k'(m + l) + r'$ with $r' < m + l$ and $r' = r - q'(m + l) \leq (k + 1)l + 1 \leq (k + q' + 1)l + 1 = (k' + 1)l + 1$. Hence, S is a numerical semigroup generated by

$$m, m + 1, \dots, m + l, k'(m + l) + r' \quad \text{with} \quad r' \leq (k' + 1)l + 1.$$

Consequently, we may assume that $r \leq m + l - 1$. By hypothesis, we have $r \leq (k + 1)l + 1$. Hence, we get our assumption. Thus, the proof is complete. ■

3.1 Apéry set of S

Let $\text{Ap}(S, m) = \{0, w(1), \dots, w(m - 1)\}$ be the Apéry set of S with respect to m , where $w(i)$ is the smallest element of S which is congruent to $i \pmod{m}$. The following theorem gives a formula for the Apéry set of S .

Theorem 3.1.1. Let S be a numerical semigroup minimally generated by

$$m, m + 1, \dots, m + l, k(m + l) + r \quad \text{with} \quad r \leq (k + 1)l + 1.$$

For all $1 \leq i \leq m - 1$ where i is written as in Definition 3.0.1, we have :

$$w(i) = m(k\alpha_i + \beta_i + \epsilon_i) + i.$$

Proof. Let $\lambda_i = m(k\alpha_i + \beta_i + \epsilon_i) + i$ where i is written as in Definition 3.0.1. We are going to show that $\lambda_i = w(i)$ for all $1 \leq i \leq m - 1$. To this end, we will show that $\lambda_i \in S$, λ_i is congruent to $i \pmod{m}$ and $\lambda_i - m \notin S$.

• We have $\lambda_i \in S$ and λ_i is congruent to $i \pmod{m}$. It follows from

$$\begin{aligned} \lambda_i &= m(k\alpha_i + \beta_i + \epsilon_i) + i \\ &= m(k\alpha_i + \beta_i + \epsilon_i) + \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i \\ &= \alpha_i(k(m + l) + r) + \beta_i(m + l) + \epsilon_i(m + t_i). \end{aligned} \tag{3.1.1}$$

• We will prove that $\lambda_i - m \notin S$ by the way of contradiction. From (3.1.1), we have $\lambda_i - m = \alpha_i(k(m + l) + r) + \beta_i(m + l) + (\epsilon_i - 1)m + \epsilon_i t_i$. Suppose by the way of contradiction that $\lambda_i - m \in S$. By Definition 1.0.7, there exist $x, x_l, \dots, x_0 \in \mathbb{N}$ such that

$$\lambda_i - m = x(k(m + l) + r) + x_l(m + l) + \dots + x_1(m + 1) + x_0 m.$$

Thus,

$$\begin{aligned} \alpha_i(k(m + l) + r) + \beta_i(m + l) + (\epsilon_i - 1)m + \epsilon_i t_i &= \\ x(k(m + l) + r) + x_l(m + l) + \dots + x_1(m + 1) + x_0 m. \end{aligned}$$

In particular,

$$\begin{aligned} m(k\alpha_i + \beta_i + \epsilon_i - 1) + \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i &= \\ m(kx + x_l + \dots + x_1 + x_0) + x(kl + r) + x_l l + x_{l-1}(l - 1) + \dots + x_1. \end{aligned} \tag{3.1.2}$$

To show that $\lambda_i - m \notin S$, we are going to show first that $x - \alpha_i > 0$, then we will show that $x - \alpha_i \geq 2$ and conclude our assertion from (3.1.2). Since $1 \leq i \leq m - 1$ and $i = \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i$, it follows that

$$\alpha_i(kl + r) + \beta_i l + \epsilon_i t_i \leq m - 1. \tag{3.1.3}$$

We claim that

$$k\alpha_i + \beta_i + \epsilon_i - 1 \geq kx + x_l + x_{l-1} + \dots + x_1 + x_0.$$

Suppose by the way of contradiction that

$$k\alpha_i + \beta_i + \epsilon_i - 1 < kx + x_l + x_{l-1} + \dots + x_1 + x_0.$$

Then, from (3.1.3), we get

$$\begin{aligned} & m(k\alpha_i + \beta_i + \epsilon_i - 1) + \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i \\ & \leq m(kx + x_l + x_{l-1} + \dots + x_1 + x_0 - 1) + m - 1 \\ & = m(kx + x_l + x_{l-1} + \dots + x_1 + x_0) - 1 \\ & \leq m(kx + x_l + \dots + x_1 + x_0) + x(kl + r) + x_l l + x_{l-1}(l - 1) + \dots + x_1 - 1. \end{aligned}$$

This contradicts (3.1.2). Consequently,

$$k\alpha_i + \beta_i + \epsilon_i - 1 \geq kx + x_l + x_{l-1} + \dots + x_1 + x_0. \quad (3.1.4)$$

From (3.1.2) and (3.1.4), it follows that

$$\alpha_i(kl + r) + \beta_i l + \epsilon_i t_i \leq x(kl + r) + x_l l + x_{l-1}(l - 1) + \dots + x_1. \quad (3.1.5)$$

If we multiply (3.1.4) by l , we get

$$\alpha_i kl + \beta_i l \geq (kx + x_l + x_{l-1} + \dots + x_1 + x_0)l + (1 - \epsilon_i)l. \quad (3.1.6)$$

Using (3.1.6) and (3.1.5), it follows that

$$xkl + (x - \alpha_i)r + x_l l + x_{l-1}(l - 1) + \dots + x_1 - \epsilon_i t_i \geq \alpha_i kl + \beta_i l \geq xkl + (x_l + \dots + x_0)l + (1 - \epsilon_i)l.$$

Consequently,

$$(x - \alpha_i)r \geq x_0 l + x_1(l - 1) + \dots + x_{l-1} + (1 - \epsilon_i)l + \epsilon_i t_i.$$

Since $x_0, \dots, x_{l-1}, r \in \mathbb{N}$, $l \in \mathbb{N}^*$ and $\epsilon_i \in \{0, 1\}$, we get that

$$x - \alpha_i > 0. \quad (3.1.7)$$

Next, we aim to show that $x - \alpha_i \geq 2$. From (3.1.2), we have

$$\begin{aligned} m(\beta_i) & = m(k(x - \alpha_i) + 1 - \epsilon_i + x_l + \dots + x_1 + x_0) \\ & \quad + ((x - \alpha_i)(kl + r) - \beta_i l - \epsilon_i t_i) + x_l l + x_{l-1}(l - 1) + \dots + x_1. \end{aligned} \quad (3.1.8)$$

Then, $((x - \alpha_i)(kl + r) - \beta_i l - \epsilon_i t_i) + x_l l + x_{l-1}(l - 1) + \dots + x_1$ is divisible by m . Since $x - \alpha_i > 0$ (by (3.1.7)), $\beta_i l + \epsilon_i t_i < kl + r$ (by definition) and $x_1, \dots, x_l \in \mathbb{N}$, then there exists $p \in \mathbb{N}^*$ such that

$$((x - \alpha_i)(kl + r) - \beta_i l - \epsilon_i t_i) + x_l l + x_{l-1}(l - 1) + \dots + x_1 = pm. \quad (3.1.9)$$

By substituting (3.1.9) in (3.1.8), we get

$$m(\beta_i) = m(k(x - \alpha_i) + 1 - \epsilon_i + x_l + \dots + x_1 + x_0 + p).$$

Consequently,

$$\beta_i = k(x - \alpha_i) + 1 - \epsilon_i + p + x_l + \dots + x_1 + x_0. \quad (3.1.10)$$

From (3.1.9), it follows that

$$\beta_i l = (x - \alpha_i)(kl + r) - \epsilon_i t_i + x_l l + x_{l-1}(l - 1) + \dots + x_1 - pm. \quad (3.1.11)$$

By multiplying (3.1.10) by l and using (3.1.11), we get the following :

$$(x - \alpha_i)r = p(m + l) + (1 - \epsilon_i)l + \epsilon_i t_i + x_{l-1} + \dots + x_1(l - 1) + x_0 l.$$

Since $\epsilon_i \in \{0, 1\}$, $x_{l-1}, \dots, x_0 \in \mathbb{N}$, $p \in \mathbb{N}^*$ and $0 < r < m + l$ (Proposition 3.0.3), we get that

$$x - \alpha_i \geq 2.$$

From (3.1.2), we have

$$\begin{aligned} m(\beta_i + \epsilon_i - 1) + \beta_i l + \epsilon_i t_i = \\ m((x - \alpha_i)k + x_l + \dots + x_1 + x_0) + ((x - \alpha_i)(kl + r) + x_l l + x_{l-1}(l - 1) + \dots + x_1). \end{aligned}$$

Since $x - \alpha_i \geq 2$ and $x_l, \dots, x_0 \in \mathbb{N}$, it follows that

$$m(\beta_i + \epsilon_i - 1) + \beta_i l + \epsilon_i t_i \geq 2km + 2(kl + r). \quad (3.1.12)$$

On the other hand, Since $\beta_i l + \epsilon_i t_i < kl + r$ (by definition) and $\beta_i + \epsilon_i - 1 \leq 2k$ (Proposition 3.0.2), it follows that

$$m(\beta_i + \epsilon_i - 1) + \beta_i l + \epsilon_i t_i < 2km + kl + r. \quad (3.1.13)$$

From (3.1.12) and (3.1.13) we get a contradiction. Consequently, $\lambda_i - m \notin S$.

Hence, $w(i) = \lambda_i = m(k\alpha_i + \beta_i + \epsilon_i) + i$. Thus, the proof is complete. ■

Example 3.1.2. Consider the following numerical semigroup

$$S = \langle 19, 20, 21, 22, 52 \rangle.$$

Note that $m = 19$, $l = 3$, $k = 2$ and $r = 8$. Let $\text{Ap}(S, m) = \{0, w(1), \dots, w(m - 1)\}$ be the Apéry basis of S . By using GAP [8], we obtain

$$\text{Ap}(S, m) = \{0, 20, 21, 22, 42, 43, 44, 64, 65, 66, 86, 87, 88, 108, 52, 72, 73, 74, 94\}$$

and they verify the formula given in Theorem 3.1.1. ■

3.2 Frobenius number of S

Definition 3.2.1. Let the notations be as above. Let

$$q = \lfloor \frac{r-1}{l} \rfloor.$$

Thus, $ql \leq r - 1$ and $r - 1 = ql + t$ with $q, t \in \mathbb{N}$ and $t < l$. Let $\epsilon_t \in \mathbb{N}$ defines as in 3.0.1

$$\epsilon_t = \begin{cases} 1 & \text{if } t \geq 1, \\ 0 & \text{if } t = 0. \end{cases}$$

Proposition 3.2.2 and 3.2.3 give some properties that will be used in this Chapter using the notations used in Definition 3.2.1. ■

Proposition 3.2.2. Under the above notations, we have

$$\beta_i \leq k + q.$$

Proof. By definition, we have

$$\beta_i l + \epsilon_i t_i \leq kl + r - 1 = (k + q)l + t \quad \text{with } t < l.$$

Consequently,

$$\beta_i l + \epsilon_i t_i < (k + q + 1)l.$$

Suppose by the way of contradiction that $\beta_i > k + q$. Hence, $\beta_i l + \epsilon_i t_i \geq (k + q + 1)l$, which is impossible. Consequently, $\beta_i \leq k + q$. Thus, the proof is complete. \blacksquare

Proposition 3.2.3. Under the above notations, we have

$$q + \epsilon_t \leq k + 1.$$

Proof. By definition, we have $r - 1 \leq (k + 1)l$ and $r - 1 = ql + t$ with $q, t \in \mathbb{N}$ and $t < l$. Thus, $ql + t \leq (k + 1)l$.

Case 1. If $\epsilon_t = 0$, then $t = 0$. Hence, $ql \leq (k + 1)l$, which implies that $q \leq k + 1$. Thus, $q + \epsilon_t \leq k + 1$.

Case 2. If $\epsilon_t = 1$, then $t \geq 1$. Hence, $ql + t \leq (k + 1)l$ with $t \geq 1$, it follows that $q \leq k$. Therefore, $q + \epsilon_t \leq k + 1$. Thus, the proof is complete. \blacksquare

Next, we shall focus on the determination of the Frobenius number of S . We shall start with the following proposition that will enable us to determine the Frobenius number and will help us later in determining the Pseudo frobenius number of S .

Proposition 3.2.4. Let S be a numerical semigroup minimally generated by

$$m, m + 1, \dots, m + l, k(m + l) + r \quad \text{with } r \leq (k + 1)l + 1.$$

For all $1 \leq i < j \leq m - 1$ where i and j are written as in Definition 3.0.1, we have :

- If $\alpha_i = \alpha_j - 2$, $\beta_j = \epsilon_j = 0$ and $\beta_i + \epsilon_i = 2k + 1$, then

$$w(i) - w(j) > 0.$$

- If $\alpha_i = \alpha_j - 1$, $\beta_i + \epsilon_i > k + \beta_j + \epsilon_j$ and $\beta_j + \epsilon_j \leq k$, then

$$w(i) - w(j) > 0.$$

- Otherwise,

$$w(i) - w(j) < 0.$$

Proof. Let $1 \leq i < j \leq m - 1$, where

$$i = \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i \quad \text{and} \quad j = \alpha_j(kl + r) + \beta_j l + \epsilon_j t_j$$

be as defined in Definition 3.0.1. By Theorem 3.1.1, we have

$$w(i) = m(k\alpha_i + \beta_i + \epsilon_i) + i \quad \text{and} \quad w(j) = m(k\alpha_j + \beta_j + \epsilon_j) + j.$$

We claim that $\alpha_i \leq \alpha_j$. In fact, since $i < j$, then $\frac{i}{kl+r} < \frac{j}{kl+r}$, which implies that $\lfloor \frac{i}{kl+r} \rfloor \leq \lfloor \frac{j}{kl+r} \rfloor$. Hence, $\alpha_i \leq \alpha_j$.

Case 1. If $\alpha_i = \alpha_j$. We aim to show that $\beta_i \leq \beta_j$. Indeed, suppose by the way of contradiction that $\beta_i > \beta_j$, then $i = \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i = \alpha_j(kl + r) + \beta_i l + \epsilon_i t_i \geq \alpha_j(kl + r) + \beta_i l \geq \alpha_j(kl + r) + (\beta_j + 1)l$. Since $\epsilon_j t_j < l$, we get that $i > \alpha_j(kl + r) + \beta_j l + \epsilon_j t_j = j$ which is a contradiction with $i < j$. Hence, $\beta_i \leq \beta_j$.

- If $\beta_i < \beta_j$. Then,

$$\begin{aligned} w(j) - w(i) &= m(k\alpha_j + \beta_j + \epsilon_j) + j - m(k\alpha_i + \beta_i + \epsilon_i) - i \\ &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &= m((\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i. \end{aligned}$$

Since $\beta_i < \beta_j$, $i < j$ and $\epsilon_i, \epsilon_j \in \{0, 1\}$, it follows that $w(j) - w(i) > 0$.

- If $\beta_i = \beta_j$. We aim to show that $\epsilon_i t_i < \epsilon_j t_j$, in particular $\epsilon_i \leq \epsilon_j$. Suppose by the way of contradiction that $\epsilon_i t_i \geq \epsilon_j t_j$, then

$$i = \alpha_i(kl + r) + \beta_i l + \epsilon_i t_i = \alpha_j(kl + r) + \beta_j l + \epsilon_i t_i \geq \alpha_j(kl + r) + \beta_j l + \epsilon_j t_j = j,$$

which is a contradiction with $i < j$. Hence, $\epsilon_i t_i < \epsilon_j t_j$. As $\epsilon_i, \epsilon_j \in \{0, 1\}$, we get that $\epsilon_i \leq \epsilon_j$. Therefore,

$$\begin{aligned} w(j) - w(i) &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &= m((\epsilon_j - \epsilon_i)) + j - i. \end{aligned}$$

Since $\epsilon_i \leq \epsilon_j$ and $i < j$, we obtain $w(j) - w(i) > 0$.

Consequently, if $i < j$ and $\alpha_i = \alpha_j$, then

$$w(i) - w(j) < 0.$$

Case 2. If $\alpha_i < \alpha_j$.

- If $\alpha_i \leq \alpha_j - 3$. By Proposition 3.0.2, we have $\epsilon_i + \beta_i \leq 2k + 1$. Then,

$$\begin{aligned} w(j) - w(i) &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &\geq m(3k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &\geq m(3k - 2k - 1 + \beta_j + \epsilon_j) + j - i. \end{aligned}$$

Since $k \geq 1$ and $i < j$, it follows that $w(j) - w(i) > 0$. Consequently, if $i < j$ and $\alpha_i \leq \alpha_j - 3$, then

$$w(i) - w(j) < 0.$$

- If $\alpha_i = \alpha_j - 2$.

- If $\beta_j + \epsilon_j > 0$. By Proposition 3.0.2, we have $\epsilon_i + \beta_i \leq 2k + 1$. Then,

$$\begin{aligned} w(j) - w(i) &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &= m(2k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &\geq m(2k + \beta_j + \epsilon_j - 2k - 1) + j - i \\ &\geq m(2k + 1 - 2k - 1) + j - i. \end{aligned}$$

Since $i < j$, we get $w(j) - w(i) > 0$.

- If $\beta_j = \epsilon_j = 0$. Then,

$$\begin{aligned} w(j) - w(i) &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &= m(2k - \beta_i - \epsilon_i) + j - i. \end{aligned}$$

By Proposition 3.0.2, we have $\epsilon_i + \beta_i \leq 2k + 1$.

- If $\beta_i + \epsilon_i \leq 2k$. Since $i < j$, we obtain $w(j) - w(i) > 0$.
- If $\beta_i + \epsilon_i = 2k + 1$. Since $j \leq m - 1$, it follows that $w(j) - w(i) < 0$.

Consequently, if $i < j$ and $\alpha_i = \alpha_j - 2$, then

$$w(i) - w(j) < 0$$

unless in the case where $\beta_j = \epsilon_j = 0$ and $\beta_i + \epsilon_i = 2k + 1$.

- If $\alpha_i = \alpha_j - 1$.
 - If $\beta_i + \epsilon_i \leq k + \beta_j + \epsilon_j$. Then,

$$\begin{aligned} w(j) - w(i) &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &= m(k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &\geq m(k - k) + j - i. \end{aligned}$$

Since $i < j$, we get $w(j) - w(i) > 0$.

- If $\beta_i + \epsilon_i > k + \beta_j + \epsilon_j$. Then,

$$\begin{aligned} w(j) - w(i) &= m((\alpha_j - \alpha_i)k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &= m(k + (\beta_j - \beta_i) + (\epsilon_j - \epsilon_i)) + j - i \\ &\leq m(k - k - 1) + j - i. \end{aligned}$$

Since $j < m$, it follows that $w(j) - w(i) < 0$. Note that if $\beta_i + \epsilon_i > k + \beta_j + \epsilon_j$, then $\beta_j + \epsilon_j \leq k$ it follows from $\beta_i + \epsilon_i \leq 2k + 1$ (Proposition 3.0.2).

Consequently, if $i < j$ and $\alpha_i = \alpha_j - 1$, then

$$w(i) - w(j) < 0$$

unless in the case where $\beta_i + \epsilon_i > k + \beta_j + \epsilon_j$ and $\beta_j + \epsilon_j \leq k$.

In conclusion, if $i < j$, we have

$$w(i) - w(j) > 0,$$

in the case

$$\alpha_i = \alpha_j - 2, \beta_j = \epsilon_j = 0 \text{ and } \beta_i + \epsilon_i = 2k + 1,$$

or

$$\alpha_i = \alpha_j - 1, \beta_i + \epsilon_i > k + \beta_j + \epsilon_j \text{ and } \beta_j + \epsilon_j \leq k,$$

and in the other cases

$$w(i) - w(j) < 0.$$

Thus, the proof is complete. ■

The following theorem gives a formula for the Frobenius $f(S)$.

Theorem 3.2.5. Let S be a numerical semigroup minimally generated by

$$m, m + 1, \dots, m + l, k(m + l) + r \quad \text{with} \quad r \leq (k + 1)l + 1.$$

The Frobenius number $f(S)$ of S is given by :

$$f(S) = \begin{cases} m(k\alpha_{m-1} + q + \epsilon_t - 1) + \alpha_{m-1}(kl + r) - 1 & \text{if } S \text{ satisfies condition (H),} \\ m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) - 1 & \text{otherwise,} \end{cases}$$

where

(H) : $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, with $\alpha_{m-1} \geq 1$, $\beta_{m-1} + \epsilon_{m-1} \leq k$, $q + \epsilon_t > \beta_{m-1} + \epsilon_{m-1}$ and $r > 1$.

Proof. By Lemma 1.0.21, we have $f(S) = \max(\text{Ap}(S, m)) - m$. We are going to show that

$$\max(\text{Ap}(S, m)) = \begin{cases} w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) & \text{if } S \text{ satisfies condition (H)} \\ w(m - 1) & \text{otherwise.} \end{cases}$$

By applying Proposition 3.2.4, we get

$$\begin{aligned} & \max(\text{Ap}(S, m)) = \\ & \max\{w((\alpha_{m-1} - 2)(kl + r) + kl + r - 1), w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1), w(m - 1)\}. \end{aligned}$$

Recall that $r - 1 = ql + t$ with $t < l$, $\epsilon_t = 0$ if $t = 0$ and $\epsilon_t = 1$ if $t \neq 0$. We have

$$\begin{aligned} & w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) - w((\alpha_{m-1} - 2)(kl + r) + kl + r - 1) \\ &= m(k(\alpha_{m-1} - 1) + k + q + \epsilon_t) + (\alpha_{m-1} - 1)(kl + r) + kl + r - 1 \\ & \quad - m(k(\alpha_{m-1} - 2) + k + q + \epsilon_t) - ((\alpha_{m-1} - 2)(kl + r) + kl + r - 1) \\ &= m(k) + kl + r > 0. \end{aligned}$$

Consequently,

$$\max(\text{Ap}(S, m)) = \max\{w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1), w(m - 1)\}.$$

Case 1. If $\alpha_{m-1} = 0$, then $i = (\alpha_{m-1} - 1)(kl + r) + kl + r - 1 < 0$. Hence,

$$\max(\text{Ap}(S, m)) = w(m - 1) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) + m - 1.$$

Case 2. If $\alpha_{m-1} \geq 1$.

- If $r = 1$. Thus, $r - 1 = 0$ and so are q and ϵ_t . Therefore,

$$\begin{aligned} & w(m - 1) - w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ &= w(m - 1) - w((\alpha_{m-1} - 1)(kl + 1) + kl) \\ &= m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) + \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} \\ & \quad - m(k(\alpha_{m-1} - 1) + k) - (\alpha_{m-1} - 1)(kl + 1) - kl \\ &= m(\beta_{m-1} + \epsilon_{m-1}) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 > 0. \end{aligned}$$

Consequently,

$$\max(\text{Ap}(S, m)) = w(m - 1) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) + m - 1.$$

- If $r > 1$ ($r - 1 = ql + t$ where $t < l$, $\epsilon_t = 1$ if $t \geq 1$ and $\epsilon_t = 0$ if $t = 0$). Then,

$$\begin{aligned} & w(m - 1) - w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ &= m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) + \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} \\ & \quad - m(k(\alpha_{m-1} - 1) + k + q + \epsilon_t) - ((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ &= m(\beta_{m-1} + \epsilon_{m-1} - q - \epsilon_t) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1. \end{aligned}$$

- If $q + \epsilon_t > \beta_{m-1} + \epsilon_{m-1}$. We have $\beta_{m-1} + \epsilon_{m-1} \leq k$ in this case this follows from Proposition 3.2.3. Since $\alpha_{m-1} \geq 1$ in this case, we get

$$\beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 \leq \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} = m - 1.$$

Consequently,

$$\begin{aligned} & w(m - 1) - w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ &= m(\beta_{m-1} + \epsilon_{m-1} - q - \epsilon_t) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 \\ &\leq m(-1) + m - 1 \\ &< 0. \end{aligned}$$

Hence,

$$\begin{aligned} \max(\text{Ap}(S, m)) &= w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ &= m(k\alpha_{m-1} + q + \epsilon_t) + \alpha_{m-1}(kl + r) - 1. \end{aligned}$$

- If $q + \epsilon_t \leq \beta_{m-1} + \epsilon_{m-1}$. Then,

$$\begin{aligned} w(m - 1) - w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ = m(\beta_{m-1} + \epsilon_{m-1} - q - \epsilon_t) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 > 0. \end{aligned}$$

Therefore,

$$\max(\text{Ap}(S, m)) = w(m - 1) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) + m - 1.$$

Hence, if S satisfies condition (H), we get

$$\begin{aligned} \max(\text{Ap}(S, m)) &= w((\alpha_{m-1} - 1)(kl + r) + kl + r - 1) \\ &= m(k\alpha_{m-1} + q + \epsilon_t) + \alpha_{m-1}(kl + r) - 1. \end{aligned}$$

Otherwise, we obtain

$$\max(\text{Ap}(S, m)) = w(m - 1) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) + m - 1.$$

By applying Lemma 1.0.21, if S satisfies condition (H), we obtain

$$f(S) = m(k\alpha_{m-1} + q + \epsilon_t - 1) + \alpha_{m-1}(kl + r) - 1,$$

otherwise, we obtain

$$f(S) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) - 1.$$

Thus, the proof is complete. ■

Example 3.2.6. Consider the following numerical semigroups.

- $S = \langle 19, 20, 21, 22, 52 \rangle$. By using GAP [8], we get that $f(S) = 89$. Note that $k = 2, l = 3$ and $r = 8$. In addition, $m - 1 = 18 = 1(14) + 1(3) + 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ with $\alpha_{m-1} = 1, \beta_{m-1} = 1, \epsilon_{m-1} = 1, t_{m-1} = 1$ and $r - 1 = 7 = 2(3) + 1 = ql + t$ with $q = 2, t = 1, \epsilon_t = 1$. We have S verifies condition (H) and verifies the formula given in Theorem 3.2.5 as $89 = 19(2(1) + 2 + 1 - 1) + 1(14) - 1$.
- $S = \langle 11, 12, 13, 14, 36 \rangle$. By using GAP [8], we obtain that $f(S) = 43$. Note that $k = 2, l = 3$ and $r = 8$. In addition, $m - 1 = 10 = 0(14) + 3(3) + 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ with $\alpha_{m-1} = 0, \beta_{m-1} = 3, \epsilon_{m-1} = 1, t_{m-1} = 1$ and $r - 1 = 7 = 2(3) + 1 = ql + t$ with $q = 2, t = 1, \epsilon_t = 1$. We have S does not verify condition (H) as $\alpha_{m-1} = 0$ and verifies the formula given in Theorem 3.2.5 as $43 = 11(2(0) + 3 + 1) - 1$. ■

3.3 Genus of S

The following theorem gives a formula for the genus $g(S)$.

Theorem 3.3.1. Let S be a numerical semigroup minimally generated by $m, m + 1, \dots, m + l, k(m + l) + r$ with $r \leq (k + 1)l + 1$. The genus $g(S)$ of S is given by :

$$\begin{aligned} g(S) &= (k\alpha_{m-1} + \beta_{m-1} + 1) \left(\frac{l(k\alpha_{m-1} + \beta_{m-1})}{2} + \epsilon_{m-1}t_{m-1} \right) \\ &\quad + \frac{k\alpha_{m-1}(\alpha_{m-1} + 1)r}{2} + \frac{(q + 1)(r - 1 + t)\alpha_{m-1}}{2}. \end{aligned}$$

Proof. By using Lemma 1.0.21 and Theorem 3.1.1, we get

$$\begin{aligned} g(S) &= \frac{1}{m} \sum_{w \in \text{Ap}(S, m)} w - \frac{m-1}{2} = \frac{1}{m} \sum_{i=0}^{m-1} (m(k\alpha_i + \beta_i + \epsilon_i) + i) - \frac{m-1}{2} \\ &= \sum_{i=0}^{m-1} (k\alpha_i + \beta_i + \epsilon_i) + \frac{1}{m} \sum_{i=0}^{m-1} i - \frac{m-1}{2} = \sum_{i=0}^{m-1} (k\alpha_i + \beta_i + \epsilon_i). \end{aligned}$$

Now, we are going to divide the set $\{0 \leq i \leq m-1\}$ into subsets and calculate the value of $k\alpha_i + \beta_i + \epsilon_i$ in each subset. We have

$$\begin{aligned} \{0 \leq i \leq m-1\} &= \{j(kl+r) \leq i \leq j(kl+r) + kl + (r-1); 0 \leq j \leq \alpha_{m-1} - 1\} \\ &\cup \{\alpha_{m-1}(kl+r) \leq i \leq \alpha_{m-1} + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &= \cup_{\ell=1}^7 A_\ell, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \{j(kl+r); 0 \leq j \leq \alpha_{m-1} - 1\}, \\ A_2 &= \{j(kl+r) + yl + 1 \leq i \leq j(kl+r) + (y+1)l; 0 \leq j \leq \alpha_{m-1} - 1 \text{ and } 0 \leq y \leq k-1\}, \\ A_3 &= \{j(kl+r) + (k+y)l + 1 \leq i \leq j(kl+r) + (k+y+1)l; 0 \leq j \leq \alpha_{m-1} - 1 \text{ and } 0 \leq y \leq q-1\}, \\ A_4 &= \{j(kl+r) + (k+q)l + 1 \leq i \leq j(kl+r) + (k+q)l + t; 0 \leq j \leq \alpha_{m-1} - 1\}, \\ A_5 &= \{\alpha_{m-1}(kl+r)\}, \\ A_6 &= \{\alpha_{m-1}(kl+r) + yl + 1 \leq i \leq \alpha_{m-1}(kl+r) + (y+1)l; 0 \leq y \leq \beta_{m-1} - 1\}, \\ A_7 &= \{\alpha_{m-1}(kl+r) + \beta_{m-1}l + 1 \leq i \leq \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Next, we will calculate the value of $k\alpha_i + \beta_i + \epsilon_i$ on each subset A_ℓ .

If $i \in A_1$, then $k\alpha_i + \beta_i + \epsilon_i = kj$.

If $i \in A_2$, then $k\alpha_i + \beta_i + \epsilon_i = kj + y + 1$.

If $i \in A_3$, then $k\alpha_i + \beta_i + \epsilon_i = kj + k + y + 1$.

If $i \in A_4$, then $k\alpha_i + \beta_i + \epsilon_i = kj + k + q + 1$.

If $i \in A_5$, then $k\alpha_i + \beta_i + \epsilon_i = k\alpha_{m-1}$.

If $i \in A_6$, then $k\alpha_i + \beta_i + \epsilon_i = k\alpha_{m-1} + y + 1$.

If $i \in A_7$, then $k\alpha_i + \beta_i + \epsilon_i = k\alpha_{m-1} + \beta_{m-1} + 1$.

Therefore,

$$\begin{aligned} g(S) &= \sum_{i=0}^{m-1} (k\alpha_i + \beta_i + \epsilon_i) = \sum_{\ell=1}^7 \sum_{i \in A_\ell} (k\alpha_i + \beta_i + \epsilon_i) \\ &= \sum_{j=0}^{\alpha_{m-1}-1} \left(k\alpha_{j(kl+r)} + \beta_{j(kl+r)} + \epsilon_{j(kl+r)} + \sum_{y=0}^{k-1} \sum_{i=j(kl+r)+yl+1}^{j(kl+r)+(y+1)l} (k\alpha_i + \beta_i + \epsilon_i) \right. \\ &\quad \left. + \sum_{y=0}^{q-1} \sum_{i=j(kl+r)+(k+y)l+1}^{j(kl+r)+(k+y+1)l} (k\alpha_i + \beta_i + \epsilon_i) + \sum_{i=j(kl+r)+(k+q)l+1}^{j(kl+r)+(k+q)l+t} (k\alpha_i + \beta_i + \epsilon_i) \right) \\ &\quad + \left(k\alpha_{\alpha_{m-1}(kl+r)} + \beta_{\alpha_{m-1}(kl+r)} + \epsilon_{\alpha_{m-1}(kl+r)} \right) + \sum_{y=0}^{\beta_{m-1}-1} \sum_{i=\alpha_{m-1}(kl+r)+yl+1}^{\alpha_{m-1}(kl+r)+(y+1)l} (k\alpha_i + \beta_i + \epsilon_i) \\ &\quad + \sum_{i=\alpha_{m-1}(kl+r)+\beta_{m-1}l+1}^{\alpha_{m-1}(kl+r)+\beta_{m-1}l+\epsilon_{m-1}t_{m-1}} (k\alpha_i + \beta_i + \epsilon_i). \end{aligned}$$

Equivalently,

$$\begin{aligned}
 g(S) &= \sum_{j=0}^{\alpha_{m-1}-1} \left(kj + \sum_{y=0}^{k-1} \sum_{i=j(kl+r)+yl+1}^{j(kl+r)+(y+1)l} (kj + y + 1) + \sum_{y=0}^{q-1} \sum_{i=j(kl+r)+yl+1}^{j(kl+r)+(y+1)l} (kj + k + y + 1) \right. \\
 &\quad \left. + \sum_{i=j(kl+r)+(k+q)l+1}^{j(kl+r)+(k+q)l+t} (kj + k + q + 1) \right) + k\alpha_{m-1} + \sum_{y=0}^{\beta_{m-1}-1} \sum_{i=\alpha_{m-1}(kl+r)+yl+1}^{\alpha_{m-1}(kl+r)+(y+1)l} (k\alpha_{m-1} + y + 1) \\
 &\quad + \sum_{i=\alpha_{m-1}(kl+r)+\beta_{m-1}l+1}^{\alpha_{m-1}(kl+r)+\beta_{m-1}l+\epsilon_{m-1}t_{m-1}} (k\alpha_{m-1} + \beta_{m-1} + 1) \\
 &= \sum_{j=0}^{\alpha_{m-1}-1} \left(kj + \sum_{y=0}^{k-1} (kj + y + 1)l + \sum_{y=0}^{q-1} (kj + k + y + 1)l + (kj + k + q + 1)t \right) \\
 &\quad + k\alpha_{m-1} + \sum_{y=0}^{\beta_{m-1}-1} (k\alpha_{m-1} + y + 1)l + (k\alpha_{m-1} + \beta_{m-1} + 1)\epsilon_{m-1}t_{m-1} \\
 &= \sum_{j=0}^{\alpha_{m-1}-1} \left(kj + k(kj + 1)l + \frac{k(k-1)l}{2} + (kj + k + 1)ql + \frac{(q-1)ql}{2} + (kj + k + q + 1)t \right) \\
 &\quad + k\alpha_{m-1} + (k\alpha_{m-1} + 1)\beta_{m-1}l + \frac{\beta_{m-1}(\beta_{m-1} - 1)l}{2} + (k\alpha_{m-1} + \beta_{m-1} + 1)\epsilon_{m-1}t_{m-1} \\
 &= \sum_{j=0}^{\alpha_{m-1}-1} \left(k(kl + 1 + ql + t)j + \frac{(k+1)(kl + 2ql + 2t) + 2qt + q(q-1)l}{2} \right) \\
 &\quad + k\alpha_{m-1} + (k\alpha_{m-1} + 1)\beta_{m-1}l + \frac{\beta_{m-1}(\beta_{m-1} - 1)l}{2} + (k\alpha_{m-1} + \beta_{m-1} + 1)\epsilon_{m-1}t_{m-1} \\
 &= \frac{k(kl + 1 + ql + t)\alpha_{m-1}(\alpha_{m-1} - 1)}{2} + k\alpha_{m-1} + (k\alpha_{m-1} + 1)\beta_{m-1}l \\
 &\quad + \frac{((k+1)(kl + 2ql + 2t) + 2qt + q(q-1)l)\alpha_{m-1}}{2} + \frac{\beta_{m-1}(\beta_{m-1} - 1)l}{2} \\
 &\quad + (k\alpha_{m-1} + \beta_{m-1} + 1)\epsilon_{m-1}t_{m-1} \\
 &= \frac{(k\alpha_{m-1} - k)(kl\alpha_{m-1})}{2} + \frac{k\alpha_{m-1}(\alpha_{m-1} - 1)}{2} + \frac{(ql + t)(k\alpha_{m-1} - k)\alpha_{m-1}}{2} \\
 &\quad + \frac{(k+1)kl\alpha_{m-1}}{2} + \frac{(ql + t)(2k + 2)\alpha_{m-1}}{2} + \frac{(2qt + q(q-1)l)\alpha_{m-1}}{2} \\
 &\quad + k\alpha_{m-1} + \frac{(k\alpha_{m-1} + 1)\beta_{m-1}l}{2} + \frac{\beta_{m-1}(kl\alpha_{m-1})}{2} + \frac{\beta_{m-1}(\beta_{m-1}l)}{2} \\
 &\quad + (k\alpha_{m-1} + \beta_{m-1} + 1)\epsilon_{m-1}t_{m-1} \\
 &= (k\alpha_{m-1} + \beta_{m-1} + 1) \left(\frac{l(k\alpha_{m-1} + \beta_{m-1})}{2} + \epsilon_{m-1}t_{m-1} \right) + \frac{k\alpha_{m-1}(\alpha_{m-1} + 1)}{2} \\
 &\quad + \frac{(ql + t)(k\alpha_{m-1} + k + 2)\alpha_{m-1}}{2} + \frac{(2qt + q(q-1)l)\alpha_{m-1}}{2}.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 g(S) &= (k\alpha_{m-1} + \beta_{m-1} + 1) \left(\frac{l(k\alpha_{m-1} + \beta_{m-1})}{2} + \epsilon_{m-1}t_{m-1} \right) + \frac{k\alpha_{m-1}(\alpha_{m-1} + 1)}{2} \\
 &\quad + \frac{(k + k\alpha_{m-1} + 2)\alpha_{m-1}(r - 1) + (2qt + q(q - 1)l)\alpha_{m-1}}{2} \\
 &= (k\alpha_{m-1} + \beta_{m-1} + 1) \left(\frac{l(k\alpha_{m-1} + \beta_{m-1})}{2} + \epsilon_{m-1}t_{m-1} \right) + \frac{k\alpha_{m-1}(\alpha_{m-1} + 1)r}{2} \\
 &\quad + \frac{(q + 1)(r - 1 + t)\alpha_{m-1}}{2}.
 \end{aligned}$$

Thus, the proof is complete. ■

Example 3.3.2. Consider the examples in Example 3.2.6.

- $S = \langle 19, 20, 21, 22, 52 \rangle$. By using GAP [8], we get that $g(S) = 50$. Note that $k = 2, l = 3$ and $r = 8$. In addition, $m - 1 = 18 = 1(14) + 1(3) + 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ with $\alpha_{m-1} = 1, \beta_{m-1} = 1, \epsilon_{m-1} = 1, t_{m-1} = 1$ and $r - 1 = 7 = 2(3) + 1 = ql + t$ with $q = 2, t = 1$. We have $50 = (2(1) + 1 + 1) \left(\frac{3(2(1)+1)}{2} + 1 \right) + \frac{2(1)(1+1)8}{2} + \frac{(2+1)(8-1+1)(1)}{2}$. Hence, S verifies the formula given in Theorem 3.3.1.
- $S = \langle 11, 12, 13, 14, 36 \rangle$. By using GAP [8], we get that $g(S) = 22$. Note that $k = 2, l = 3$ and $r = 8$. In addition, $m - 1 = 10 = 0(14) + 3(3) + 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ with $\alpha_{m-1} = 0, \beta_{m-1} = 3, \epsilon_{m-1} = 1, t_{m-1} = 1$ and $r - 1 = 7 = 2(3) + 1 = ql + t$ with $q = 2, t = 1$. We have $22 = (2(0) + 3 + 1) \left(\frac{3(2(0)+3)}{2} + 1 \right) + \frac{2(0)(0+1)8}{2} + \frac{(2+1)(8-1+1)(0)}{2}$. Hence, S verifies the formula given in Theorem 3.3.1. ■

3.4 Determination of symmetric and pseudo-symmetric numerical semigroups

Next, we shall focus on the determination of symmetric and pseudo-symmetric numerical semigroups. We shall start with a technical Lemma.

Lemma 3.4.1. Let the notation be as defined in Definition 3.0.1 and in Definition 3.2.1, we have the following :

$$2g(S) - (f(S) + 1) = \begin{cases} \mathcal{F}_1 & \text{if } S \text{ satisfies condition (H),} \\ \mathcal{F}_2 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned}
 \mathcal{F}_1 &= \alpha_{m-1}((1 - \epsilon_t)(kl + r) + kt + (q + 1)(t - 1)) \\
 &\quad + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1 - q + 1 - \epsilon_t) \\
 &\quad + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2 - q + 1 - \epsilon_t) + 1 - \epsilon_t - q,
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{F}_2 &= \alpha_{m-1}(-\epsilon_{m-1}(kl + r) + (k + q - \beta_{m-1})r + k(l - 1) + r + (q + 1)(t - 1)) \\
 &\quad + \beta_{m-1}l(1 - \epsilon_{m-1}) + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + \beta_{m-1} + 2 - \epsilon_{m-1}) - \epsilon_{m-1} - \beta_{m-1}
 \end{aligned}$$

and

$$\text{(H) : } m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}, \text{ with } \alpha_{m-1} \geq 1, \beta_{m-1} + \epsilon_{m-1} \leq k, q + \epsilon_t > \beta_{m-1} + \epsilon_{m-1} \text{ and } r > 1.$$

Proof. By Theorem 3.2.5, we have $f(S) = m(k\alpha_{m-1} + q + \epsilon_t - 1) + \alpha_{m-1}(kl + r) - 1$ if S satisfies condition (H) and $f(S) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) - 1$ otherwise. Now, we use the formulas in Theorem 3.2.5 and Theorem 3.3.1.

Case 1. If $f(S) = m(k\alpha_{m-1} + q + \epsilon_t - 1) + \alpha_{m-1}(kl + r) - 1$. Then,

$$\begin{aligned}
 & 2g(S) - (f(S) + 1) \\
 = & (k\alpha_{m-1} + \beta_{m-1} + 1)(l(k\alpha_{m-1} + \beta_{m-1}) + 2\epsilon_{m-1}t_{m-1}) + k(\alpha_{m-1} + 1)\alpha_{m-1}r \\
 & + (q + 1)(r - 1)\alpha_{m-1} + (q + 1)t\alpha_{m-1} - m(k\alpha_{m-1} + q + \epsilon_t - 1) - \alpha_{m-1}(kl + r) \\
 = & \alpha_{m-1}(k^2l\alpha_{m-1} + kl\beta_{m-1} + 2k\epsilon_{m-1}t_{m-1} + kl\beta_{m-1} + kl + k(\alpha_{m-1} + 1)r \\
 & + (q + 1)(r - 1) + (q + 1)t - km - kl - r) + \beta_{m-1}l(\beta_{m-1} + 1) \\
 & + \epsilon_{m-1}t_{m-1}(2\beta_{m-1} + 2) - m(q + \epsilon_t - 1) \\
 = & \alpha_{m-1}(kql + kt + qr + (q + 1)(t - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1) \\
 & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2) - \alpha_{m-1}(kl + r)(q + \epsilon_t - 1) - \beta_{m-1}l(q + \epsilon_t - 1) \\
 & - \epsilon_{m-1}t_{m-1}(q + \epsilon_t - 1) - (q + \epsilon_t - 1) \\
 = & \alpha_{m-1}((1 - \epsilon_t)(kl + r) + kt + (q + 1)(t - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1 - q + 1 - \epsilon_t) \\
 & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2 - q + 1 - \epsilon_t) + 1 - \epsilon_t - q.
 \end{aligned}$$

Case 2. If $f(S) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) - 1$. Then,

$$\begin{aligned}
 & 2g(S) - (f(S) + 1) \\
 = & (k\alpha_{m-1} + \beta_{m-1} + 1)(l(k\alpha_{m-1} + \beta_{m-1}) + 2\epsilon_{m-1}t_{m-1}) + k(\alpha_{m-1} + 1)\alpha_{m-1}r \\
 & + (q + 1)(r - 1)\alpha_{m-1} + (q + 1)t\alpha_{m-1} - m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) \\
 = & \alpha_{m-1}(k^2l\alpha_{m-1} + kl\beta_{m-1} + 2k\epsilon_{m-1}t_{m-1} + kl\beta_{m-1} + kl + k(\alpha_{m-1} + 1)r \\
 & + (q + 1)(r - 1) + (q + 1)t - km) + \beta_{m-1}l(\beta_{m-1} + 1) \\
 & + \epsilon_{m-1}t_{m-1}(2\beta_{m-1} + 2) - m(\beta_{m-1} + \epsilon_{m-1}).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & 2g(S) - (f(S) + 1) \\
 = & \alpha_{m-1}(kl\beta_{m-1} + kl + kr + (q + 1)(r - 1) + (q + 1)t - k) + \beta_{m-1}l(\beta_{m-1} + 1) \\
 & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2) - \alpha_{m-1}(kl + r)(\beta_{m-1} + \epsilon_{m-1}) \\
 & - \beta_{m-1}l(\beta_{m-1} + \epsilon_{m-1}) - \epsilon_{m-1}t_{m-1}(\beta_{m-1} + \epsilon_{m-1}) - (\beta_{m-1} + \epsilon_{m-1}) \\
 = & \alpha_{m-1}(-\epsilon_{m-1}(kl + r) + (k + q - \beta_{m-1})r + k(l - 1) + r + (q + 1)(t - 1)) \\
 & + \beta_{m-1}l(1 - \epsilon_{m-1}) + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + \beta_{m-1} + 2 - \epsilon_{m-1}) - \epsilon_{m-1} - \beta_{m-1}.
 \end{aligned}$$

Thus, the proof is complete. ■

3.4.1 Determination of symmetric numerical semigroup

The following theorem gives the set of symmetric numerical semigroups.

Theorem 3.4.2. Let S be a numerical semigroup minimally generated by

$$m, m + 1, \dots, m + l, k(m + l) + r \quad \text{with} \quad r \leq (k + 1)l + 1.$$

Then, S is symmetric if and only if it satisfies one of the following :

1. $S = \langle 2k + 3, 2k + 4, k(2k + 4) + k + 2 \rangle$.
2. $S = \langle 2kl + 3, 2kl + 4, \dots, (2k + 1)l + 3, k((2k + 1)l + 3) + kl + 2 \rangle$ with $l \geq 2$.
3. $S = \langle \beta_{m-1} + 1, \beta_{m-1} + 2, k(\beta_{m-1} + 2) + r \rangle$ with $\beta_{m-1} \geq 1$.
4. $S = \langle (\alpha_{m-1} + 1)(k + q + 1), (\alpha_{m-1} + 1)(k + q + 1) + 1, k((\alpha_{m-1} + 1)(k + q + 1) + 1) + q + 1 \rangle$.
5. $S = \langle \beta_{m-1}l + 2, \dots, (\beta_{m-1} + 1)l + 2, k((\beta_{m-1} + 1)l + 2) + ql + t + 1 \rangle$ with $t \geq 1$ and $l \geq 2$.
6. $S = \langle (\alpha_{m-1} + 1)((k + q)l + 2), \dots, (\alpha_{m-1} + 1)((k + q)l + 2) + l, k((\alpha_{m-1} + 1)((k + q)l + 2) + l) + ql + 2 \rangle$ with $l \geq 2$.
7. $S = \langle \beta_{m-1}l + 2, \dots, (\beta_{m-1} + 1)l + 2, k((\beta_{m-1} + 1)l + 2) + ql + 1 \rangle$ with $l \geq 2$.
8. $S = \langle \alpha_{m-1}(kl + 1) + (k - 1)l + 2, \dots, \alpha_{m-1}(kl + 1) + kl + 2, k(\alpha_{m-1}(kl + 1) + kl + 2) + 1 \rangle$ with $l \geq 2$.

Proof. By Lemma 1.0.34, we have

$$S \text{ is symmetric if and only if } 2g(S) - (f(S) + 1) = 0.$$

Case 1. If S satisfies condition (H). By Theorem 3.2.5,

$$f(S) = m(k\alpha_{m-1} + q + \epsilon_t - 1) + \alpha_{m-1}(kl + r) - 1$$

with

$$\alpha_{m-1} > 0, \beta_{m-1} + \epsilon_{m-1} \leq k, q + \epsilon_t > \beta_{m-1} + \epsilon_{m-1} \text{ and } r > 1.$$

By using Lemma (3.4.1), S is symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}((1 - \epsilon_t)(kl + r) + kt + (q + 1)(t - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1 - q + 1 - \epsilon_t) \\ & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2 - q + 1 - \epsilon_t) + 1 - \epsilon_t = q. \end{aligned}$$

- If $\epsilon_t = 0$ ($t = 0$ and $r = ql + 1$ in this case). Then, S is symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}(kl + q(l - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 2 - q) \\ & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 3 - q) + 1 = q. \end{aligned} \tag{3.4.1}$$

Since $l \geq 1$, $\alpha_{m-1} \geq 1$ and $q \leq k + 1$ (Proposition 3.2.3), it follows that $kl + q(l - 1) \geq kl$, $k\alpha_{m-1} + \beta_{m-1} + 2 - q \geq 1$ and $k\alpha_{m-1} + 2\beta_{m-1} + 3 - q \geq 2$. Consequently, (3.4.1) implies that

$$\alpha_{m-1}kl + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}(2) + 1 \leq q. \tag{3.4.2}$$

As $\alpha_{m-1} \geq 1$, $l \geq 1$ and $q \leq k + 1$ (Proposition 3.2.3), then (3.4.2) implies that $l = 1$, $\alpha_{m-1} = 1$, $\beta_{m-1} = 0$, $\epsilon_{m-1}t_{m-1} = 0$ and $q = k + 1$. As $r = ql + 1$ in this case (as $\epsilon_t = t = 0$), we get $r = k + 2$. Therefore, S is symmetric in this case if and only if

$$\alpha_{m-1} = 1, \beta_{m-1} = 0, t_{m-1} = 0, l = 1, r = k + 2. \tag{3.4.3}$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.3) in S , we obtain

$$S = \langle 2k + 3, 2k + 4, k(2k + 4) + k + 2 \rangle.$$

- If $\epsilon_t = 1$ ($t \geq 1$). Then, S is symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}(kt + (q + 1)(t - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1 - q) \\ & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2 - q) = q. \end{aligned} \tag{3.4.4}$$

Since $t \geq 1$, $\alpha_{m-1} \geq 1$ and $q \leq k$ (by Proposition 3.2.3 as $\epsilon_t = 1$), it follows that $kt + (q + 1)(t - 1) \geq kt$, $k\alpha_{m-1} + \beta_{m-1} + 1 - q \geq \beta_{m-1} + 1$ and $k\alpha_{m-1} + 2\beta_{m-1} + 2 - q \geq 2\beta_{m-1} + 2$. Consequently, (3.4.4) implies that

$$\alpha_{m-1}(kt) + \beta_{m-1}l(\beta_{m-1} + 1) + \epsilon_{m-1}t_{m-1}(2\beta_{m-1} + 2) \leq q. \quad (3.4.5)$$

As $\alpha_{m-1} \geq 1$, $t \geq 1$ and $q \leq k$ (by Proposition 3.2.3 as $\epsilon_t = 1$), then (3.4.5) implies that $\alpha_{m-1} = 1$, $t = 1$, $\beta_{m-1} = 0$, $\epsilon_{m-1}t_{m-1} = 0$ and $q = k$. As $r = ql + t + 1$, we get $r = kl + 2$. Since $t \geq 1$, then $l \geq 2$. Therefore, S is symmetric in this case if and only if

$$\alpha_{m-1} = 1, \beta_{m-1} = 0, t_{m-1} = 0, r = kl + 2, l \geq 2. \quad (3.4.6)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.6) in S , we obtain

$$S = \langle 2kl + 3, 2kl + 4, \dots, (2k + 1)l + 3, k((2k + 1)l + 3) + kl + 2 \rangle \text{ with } l \geq 2.$$

Case 2. If S does not satisfy condition (H). By Theorem 3.2.5,

$$f(S) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) - 1$$

with

$$\alpha_{m-1} = 0 \text{ or } \beta_{m-1} + \epsilon_{m-1} > k \text{ or } \beta_{m-1} + \epsilon_{m-1} \geq q + \epsilon_t \text{ or } r = 1. \quad (3.4.7)$$

By using Lemma 3.4.1, S is symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}(-\epsilon_{m-1}(kl + r) + (k + q - \beta_{m-1})r + k(l - 1) + r + (q + 1)(t - 1)) \\ & + \beta_{m-1}l(1 - \epsilon_{m-1}) + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + \beta_{m-1} + 2 - \epsilon_{m-1}) = \beta_{m-1} + \epsilon_{m-1}. \end{aligned}$$

- If $\epsilon_{m-1} = 0$ ($t_{m-1} = 0$). We have $r = ql + t + 1$, then S is symmetric if and only if

$$\alpha_{m-1}((k + q - \beta_{m-1})r + k(l - 1) + q(l - 1) + (q + 2)t) + \beta_{m-1}(l - 1) = 0. \quad (3.4.8)$$

- If $\beta_{m-1} = 0$. Since $\epsilon_{m-1} = 0$ and $m \neq 1$ ($S \neq \mathbb{N}$), it follows that $\alpha_{m-1} \geq 1$. Since $\beta_{m-1} = 0$, $\epsilon_{m-1} = 0$, and $\alpha_{m-1} \geq 1$, by using (3.4.7), we get $r = 1$ in particular $q = 0$ and $\epsilon_t = t = 0$. In this case by using (3.4.8), we obtain S is symmetric if and only if $\alpha_{m-1}(k + k(l - 1)) = 0$. As $\alpha_{m-1} \geq 1$, it follows that $kl = 0$ which is impossible (as $k > 0$ and $l > 0$).
- If $\beta_{m-1} \geq 1$. Since $l \geq 1$ and $\beta_{m-1} \leq k + q$ (Proposition 3.2.2), it follows that $\alpha_{m-1}((k + q - \beta_{m-1})r + k(l - 1) + q(l - 1) + (q + 2)t) \geq 0$ and $\beta_{m-1}(l - 1) \geq 0$. By using (3.4.8), we get

$$\beta_{m-1}(l - 1) = 0 \quad (3.4.9)$$

and

$$\alpha_{m-1}((k + q - \beta_{m-1})r + k(l - 1) + q(l - 1) + (q + 2)t) = 0. \quad (3.4.10)$$

Since $\beta_{m-1} \geq 1$, then (3.4.9) implies that $l = 1$. By substituting $l = 1$ in (3.4.10), we get

$$\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 2)t) = 0. \quad (3.4.11)$$

Now, (3.4.11) implies that

$$\alpha_{m-1} = 0 \quad (3.4.12)$$

or

$$(k + q - \beta_{m-1})r + (q + 2)t = 0. \quad (3.4.13)$$

Since $\beta_{m-1} \leq k + q$ (Proposition 3.2.2), $r > 0$ and $q + 2 > 0$, then (3.4.13) implies that $\beta_{m-1} = k + q$ and $t = 0$. As $r = ql + t + 1$ (with $l = 1$ proved above), then $r = ql + 1 = q + 1$ in this case. Therefore, S is symmetric in this case if and only if

$$\alpha_{m-1} = 0, \beta_{m-1} \geq 1, \epsilon_{m-1} = 0, l = 1 \quad (3.4.14)$$

or

$$\beta_{m-1} = k + q, \epsilon_{m-1} = 0, l = 1, r = q + 1. \quad (3.4.15)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.14) and (3.4.15) in S , we get that S is symmetric if and only if

$$S = \langle \beta_{m-1} + 1, \beta_{m-1} + 2, k(\beta_{m-1} + 2) + r \rangle \text{ with } \beta_{m-1} \geq 1$$

or

$$S = \langle (\alpha_{m-1} + 1)(k + q + 1), (\alpha_{m-1} + 1)(k + q + 1) + 1, \\ k((\alpha_{m-1} + 1)(k + q + 1) + 1) + q + 1 \rangle.$$

- If $\epsilon_{m-1} = 1$ ($t_{m-1} \geq 1$). Then, S is symmetric if and only if

$$\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) + (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 0. \quad (3.4.16)$$

- If $t \geq 1$. Since $\beta_{m-1} \leq k + q$ (Proposition 3.2.2) and $t \geq 1$, it follows that

$$\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) \geq 0.$$

We have $k\alpha_{m-1} + \beta_{m-1} + 1 > 0$ and $t_{m-1} \geq 1$. By using (3.4.16), we get

$$t_{m-1} = 1 \quad (3.4.17)$$

and

$$\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) = 0. \quad (3.4.18)$$

Since $t_{m-1} = 1$, it follows that $l \geq 2$ ($\epsilon_{m-1}t_{m-1} < l$). Now, (3.4.18) gives $\alpha_{m-1} = 0$ or $(k + q - \beta_{m-1})r + (q + 1)(t - 1) = 0$. If $(k + q - \beta_{m-1})r + (q + 1)(t - 1) = 0$, since $\beta_{m-1} \leq k + q$ (Proposition 3.2.2), $r > 0$, $q + 1 > 0$ and $t \geq 1$, it follows that $t = 1$ and $\beta_{m-1} = k + q$. Consequently, S is symmetric if and only if $t_{m-1} = 1$ with $\alpha_{m-1} = 0$, $l \geq 2$ or $t_{m-1} = 1$ with $t = 1$, $\beta_{m-1} = k + q$, $l \geq 2$. Since $r = ql + t + 1$ and $t \geq 1$ in this case, it follows that S is symmetric if and only if

$$\alpha_{m-1} = 0, t_{m-1} = 1, t \geq 1 \text{ with } l \geq 2 \quad (3.4.19)$$

or

$$\beta_{m-1} = k + q, t_{m-1} = 1, r = ql + 2 \text{ with } l \geq 2. \quad (3.4.20)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By using (3.4.19) and (3.4.20), we get S is symmetric if and only if

$$S = \langle \beta_{m-1}l + 2, \dots, (\beta_{m-1} + 1)l + 2, k((\beta_{m-1} + 1)l + 2) + ql + t + 1 \rangle \\ \text{with } t \geq 1 \text{ and } l \geq 2$$

or

$$S = \langle (\alpha_{m-1} + 1)((k + q)l + 2), \dots, (\alpha_{m-1} + 1)((k + q)l + 2) + l, \\ k((\alpha_{m-1} + 1)((k + q)l + 2) + l) + ql + 2 \rangle \text{ with } l \geq 2.$$

- If $t = 0$. We have $r = ql + 1$, then (3.4.16) implies that S is symmetric if and only if

$$\alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) + (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 0. \quad (3.4.21)$$

We have $r = ql + 1$ and $1 \leq \epsilon_{m-1}t_{m-1} \leq l - 1$ in this case. On the other hand, $\beta_{m-1}l + \epsilon_{m-1}t_{m-1} \leq kl + r - 1 = kl + ql$. Hence, $\beta_{m-1} \leq k + q - 1$ (as $\epsilon_{m-1}t_{m-1} \geq 1$). Since $l \geq 1$ and $\beta_{m-1} \leq k + q - 1$ in this case, it follows that $\alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) \geq 0$. We have $t_{m-1} \geq 1$ and $k\alpha_{m-1} + \beta_{m-1} + 1 > 0$. By using (3.4.21), we get

$$t_{m-1} = 1 \quad (3.4.22)$$

and

$$\alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) = 0. \quad (3.4.23)$$

Since $t_{m-1} = 1$, it follows that $l \geq 2$ ($\epsilon_{m-1}t_{m-1} < l$). Now, (3.4.23) gives $\alpha_{m-1} = 0$ or $(k + q - 1 - \beta_{m-1})r + q(l - 1) = 0$. If $(k + q - 1 - \beta_{m-1})r + q(l - 1) = 0$, since $\beta_{m-1} \leq k + q - 1$ in this case (proved above), $r > 0$ and $l \geq 1$, it follows that $\beta_{m-1} = k + q - 1$ and $q(l - 1) = 0$. Since $t_{m-1} = 1$, it follows that $l \geq 2$, in particular $q(l - 1) = 0$ gives $q = 0$. Thus, in the second case, we have $\beta_{m-1} = k + q - 1$ with $q = 0$ ($r = 1$ in this case as $t = 0$). Consequently, S is symmetric if and only if $t_{m-1} = 1$ with $\alpha_{m-1} = 0$, $l \geq 2$ or $t_{m-1} = 1$ with $q = 0$ ($r = 1$), $\beta_{m-1} = k + q - 1 = k - 1$, $l \geq 2$. As $r = ql + 1$ in this case, it follows that S is symmetric if and only if

$$\alpha_{m-1} = 0, t_{m-1} = 1, r = ql + 1 \text{ with } l \geq 2 \quad (3.4.24)$$

or

$$\beta_{m-1} = k - 1, t_{m-1} = 1, r = 1 \text{ with } l \geq 2. \quad (3.4.25)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By using (3.4.24) and (3.4.25), we get S is symmetric if and only if

$$S = \langle \beta_{m-1}l + 2, \dots, (\beta_{m-1} + 1)l + 2, k((\beta_{m-1} + 1)l + 2) + ql + 1 \rangle \\ \text{with } l \geq 2$$

or

$$S = \langle \alpha_{m-1}(kl + 1) + (k - 1)l + 2, \dots, \alpha_{m-1}(kl + 1) + kl + 2, \\ k(\alpha_{m-1}(kl + 1) + kl + 2) + 1 \rangle \text{ with } l \geq 2.$$

Thus, the proof is complete. ■

Example 3.4.3. Consider the following numerical semigroups.

1. $S = \langle 9, 10, 35 \rangle$. By using GAP [8], we get that S is symmetric. Note that $m = 9 = 2k + 3$ with $k = 3$. In addition, $l = 1$ and $r = 5 = k + 2$. Hence, S verifies the formula in Theorem 3.4.2.
2. $S = \langle 15, 16, 17, 18, 44 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 3$ and $m = 15 = 2kl + 3$ with $k = 2$. Moreover, $r = 8 = kl + 2$. Hence, S verifies the formula in Theorem 3.4.2.
3. $S = \langle 8, 9, 48 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 1$. Moreover, $m = 8 = 7(1) + 1$ where $\alpha_{m-1} = 0$, $\beta_{m-1} = 7 \geq 1$ and $t_{m-1} = 0$. In addition, $k = 5$ and $r = 3$. Hence, S verifies the formula in Theorem 3.4.2.
4. $S = \langle 10, 11, 35 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 1$. In addition, $m = 10 = 1(3 + 1 + 1) + 3 + 1 + 1 = (1 + 1)(3 + 1 + 1)$ where $\alpha_{m-1} = 1$, $\beta_{m-1} = 3 + 1 = k + q$ such that $k = 3$ and $q = 1$, $t_{m-1} = 0$ and $r = 2 = q + 1$. Hence, S verifies the formula in Theorem 3.4.2.
5. $S = \langle 18, 19, 20, 21, 22, 54 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 4 \geq 2$ and $m = 18 = 4(4) + 1 + 1$ where $\alpha_{m-1} = 0$, $\beta_{m-1} = 4$ and $t_{m-1} = 1$. In addition $k = 2$ and $r = 10 = 2(4) + 1 + 1 = ql + t + 1$ such that $t = 1 \geq 1$. Hence, S verifies the formula in Theorem 3.4.2.
6. $S = \langle 16, 17, 18, 40 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 2 \geq 2$, $k = 2$, $q = 1$ and $r = 4 = ql + 2$. In addition, $m = 16 = 1((2 + 1)2 + 2) + (2 + 1)2 + 2 = (1 + 1)((2 + 1)2 + 2) = \alpha_{m-1}((k + q)l + 2) + (k + q)l + 2$ where $\alpha_{m-1} = 1$, $\beta_{m-1} = k + q = 3$ and $t_{m-1} = 1$. Hence, S verifies the formula in Theorem 3.4.2.
7. $S = \langle 18, 19, 20, 21, 22, 75 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 4 \geq 2$ and $m = 18 = 4(4) + 1 + 1$ where $\alpha_{m-1} = 0$, $\beta_{m-1} = 4$ and $t_{m-1} = 1$. In addition $k = 3$ and $r = 9 = 2(4) + 1 = ql + 1$. Hence, S verifies the formula in Theorem 3.4.2.
8. $S = \langle 89, 90, 91, 92, 93, 94, 565 \rangle$. By using GAP [8], we get that S is symmetric. Note that $l = 5 \geq 2$, $k = 6$ and $r = 1$. In addition, $m = 2(5(6) + 1) + (6 - 1)(5) + 2$, where $\alpha_{m-1} = 2$, $\beta_{m-1} = 5 = k - 1$ and $t_{m-1} = 1$. Hence, S verifies the formula in Theorem 3.4.2. ■

3.4.2 Determination of pseudo-symmetric numerical semigroup

Now, we shall characterize the set of pseudo-symmetric numerical semigroups.

Theorem 3.4.4. Let S be a numerical semigroup minimally generated by

$$m, m + 1, \dots, m + l, k(m + l) + r \quad \text{with} \quad r \leq (k + 1)l + 1.$$

Then, S is pseudo-symmetric if and only if it satisfies one of the following :

1. $S = \langle 9, 10, 13 \rangle$.
2. $S = \langle 2k + 2, 2k + 3, k(2k + 3) + k + 1 \rangle$.
3. $S = \langle (2k - 1)l + 3, \dots, 2kl + 3, k(2kl + 3) + (k - 1)l + 2 \rangle$ with $l \geq 2$.
4. $S = \langle 2(2l + 2) + 1, \dots, 2(2l + 2) + 1 + l, (2(2l + 2) + 1 + l) + l + 2 \rangle$ with $l \geq 2$.
5. $S = \langle 2k + 1, 2k + 2, k(2k + 2) + 1 \rangle$ with $k \geq 2$.
6. $S = \langle 3, 4, 5 \rangle$.
7. $S = \langle 2kl + 5, \dots, (2k + 1)l + 5, k((2k + 1)l + 5) + 3 \rangle$ with $l \geq 3$.
8. $S = \langle (2k - 2)l + 3, \dots, (2k - 1)l + 3, k((2k - 1)l + 3) + 1 \rangle$ with $l \geq 2$.
9. $S = \langle 4k + 5, 4k + 6, 4k + 7, k(4k + 7) + 3 \rangle$.

Proof. By Lemma 1.0.34, we have

$$S \text{ is pseudo-symmetric if and only if } 2g(S) - (f(S) + 1) = 1.$$

Case 1. If S satisfies condition (H). By Theorem 3.2.5,

$$f(S) = m(k\alpha_{m-1} + q + \epsilon_t - 1) + \alpha_{m-1}(kl + r) - 1$$

with

$$\alpha_{m-1} > 0, \beta_{m-1} + \epsilon_{m-1} \leq k, q + \epsilon_t > \beta_{m-1} + \epsilon_{m-1} \text{ and } r > 1.$$

By using Lemma 3.4.1, S is pseudo-symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}((1 - \epsilon_t)(kl + r) + kt + (q + 1)(t - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1 - q + 1 - \epsilon_t) \\ & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2 - q + 1 - \epsilon_t) + 1 - \epsilon_t = q + 1. \end{aligned}$$

- If $\epsilon_t = 0$ ($t = 0, r = ql + 1$). Since $r = ql + 1$, it follows that S is pseudo-symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}(kl + q(l - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 2 - q) \\ & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 3 - q) = q. \end{aligned} \tag{3.4.26}$$

Since $\alpha_{m-1} \geq 1$ and $q \leq k + 1$ (Proposition 3.2.3), it follows that $k\alpha_{m-1} + \beta_{m-1} + 2 - q \geq \beta_{m-1} + 1$ and $k\alpha_{m-1} + 2\beta_{m-1} + 3 - q \geq 2\beta_{m-1} + 2$. Consequently, (3.4.26) implies that $\alpha_{m-1}(kl + q(l - 1)) + \beta_{m-1}l(\beta_{m-1} + 1) + \epsilon_{m-1}t_{m-1}(2\beta_{m-1} + 2) \leq q$. As $\alpha_{m-1} \geq 1, k \geq 1, l \geq 1$ and $q \leq k + 1$ (Proposition 3.2.3), it follows that

$$l = 1, \beta_{m-1} = 0, \epsilon_{m-1}t_{m-1} = 0, \alpha_{m-1} \in \{1, 2\}. \tag{3.4.27}$$

- If $\alpha_{m-1} = 2$. By substituting (3.4.27) in (3.4.26), it follows that S is pseudo-symmetric if and only if $2k = q$. As $q \leq k + 1$ (Proposition 3.2.3), we get $k = 1$ and $q = 2k = 2$. As $r = ql + 1$ with $l = 1$ in this case, we obtain $r = 3$. Thus, we have

$$\alpha_{m-1} = 2, \beta_{m-1} = 0, t_{m-1} = 0, l = 1, r = 3, k = 1. \tag{3.4.28}$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.28) in S , we get

$$S = \langle 9, 10, 13 \rangle.$$

- If $\alpha_{m-1} = 1$. By substituting (3.4.27) in (3.4.26), it follows that $k = q$. As $r = ql + 1$ with $l = 1$ in this case, we obtain $r = k + 1$. Thus, we have

$$\alpha_{m-1} = 1, \beta_{m-1} = 0, t_{m-1} = 0, l = 1, r = k + 1. \quad (3.4.29)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.29) in S , we get

$$S = \langle 2k + 2, 2k + 3, k(2k + 3) + k + 1 \rangle.$$

- If $\epsilon_t = 1$ ($t \geq 1$). Then, S is pseudo-symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}(kt + (q + 1)(t - 1)) + \beta_{m-1}l(k\alpha_{m-1} + \beta_{m-1} + 1 - q) \\ & + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + 2\beta_{m-1} + 2 - q) = q + 1. \end{aligned} \quad (3.4.30)$$

Since $\alpha_{m-1} \geq 1$ and $q \leq k$ (by Proposition 3.2.3 as $\epsilon_t = 1$), it follows that $k\alpha_{m-1} + \beta_{m-1} + 1 - q \geq \beta_{m-1} + 1$ and $k\alpha_{m-1} + 2\beta_{m-1} + 2 - q \geq 2\beta_{m-1} + 2$. Consequently, (3.4.30) implies that

$$\begin{aligned} & \alpha_{m-1}(kt + (q + 1)(t - 1)) + \beta_{m-1}l(\beta_{m-1} + 1) \\ & + \epsilon_{m-1}t_{m-1}(2\beta_{m-1} + 2) \leq q + 1. \end{aligned} \quad (3.4.31)$$

Since $t \geq 1$, it follows that $l \geq 2$. As $\alpha_{m-1} \geq 1$, $t \geq 1$, $l \geq 1$ and $q \leq k$ (by Proposition 3.2.3 as $\epsilon_t = 1$), then (3.4.31) implies that

$$\alpha_{m-1} \in \{1, 2\}, \beta_{m-1} = 0, \epsilon_{m-1}t_{m-1} = 0, t = 1, l \geq 2. \quad (3.4.32)$$

- If $\alpha_{m-1} = 1$. By substituting (3.4.32) in (3.4.30), it follows that $k = q + 1$. As $r = ql + t + 1$ and $t = 1$, then $r = (k - 1)l + 2$ in this case. Thus, we have

$$\alpha_{m-1} = 1, \beta_{m-1} = 0, t_{m-1} = 0, r = (k - 1)l + 2, l \geq 2. \quad (3.4.33)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.33) in S , we get

$$S = \langle (2k - 1)l + 3, \dots, 2kl + 3, k(2kl + 3) + (k - 1)l + 2 \rangle \text{ with } l \geq 2.$$

- If $\alpha_{m-1} = 2$. By substituting (3.4.32), in (3.4.30), it follows that $2k = q + 1$. As $q \leq k$ (by Proposition 3.2.3 as $\epsilon_t = 1$) in this case, we obtain $k = 1$ and $q = 1$. Since $r = ql + t + 1$ with $t = 1$ and $q = 1$ we get $r = l + 2$. Therefore, we have

$$\alpha_{m-1} = 2, \beta_{m-1} = 0, t_{m-1} = 0, k = 1, r = l + 2, l \geq 2. \quad (3.4.34)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.34) in S , we obtain

$$S = \langle 2(2l + 2) + 1, \dots, 2(2l + 2) + 1 + l, (2(2l + 2) + 1 + l) + l + 2 \rangle \text{ with } l \geq 2.$$

Case 2. If S does not satisfy condition (H). By Theorem 3.2.5,

$$f(S) = m(k\alpha_{m-1} + \beta_{m-1} + \epsilon_{m-1}) - 1$$

with

$$\alpha_{m-1} = 0 \text{ or } \beta_{m-1} + \epsilon_{m-1} > k \text{ or } \beta_{m-1} + \epsilon_{m-1} \geq q + \epsilon_t \text{ or } r = 1. \quad (3.4.35)$$

By using Lemma 3.4.1, S is pseudo-symmetric if and only if

$$\begin{aligned} & \alpha_{m-1}(-\epsilon_{m-1}(kl + r) + (k + q - \beta_{m-1})r + k(l - 1) + r + (q + 1)(t - 1)) \\ & + \beta_{m-1}l(1 - \epsilon_{m-1}) + \epsilon_{m-1}t_{m-1}(k\alpha_{m-1} + \beta_{m-1} + 2 - \epsilon_{m-1}) = \beta_{m-1} + \epsilon_{m-1} + 1. \end{aligned}$$

- If $\epsilon_{m-1} = 0$. We have $r = ql + t + 1$, then S is pseudo-symmetric if and only if

$$\alpha_{m-1}((k + q - \beta_{m-1})r + k(l - 1) + q(l - 1) + (q + 2)t) + \beta_{m-1}(l - 1) = 1. \quad (3.4.36)$$

- If $\beta_{m-1} \geq 1$. Since $\beta_{m-1} \leq k + q$ (Proposition 3.2.2) and $l \geq 1$, it follows that $\alpha_{m-1}((k + q - \beta_{m-1})r + k(l - 1) + q(l - 1) + (q + 2)t) \geq 0$ and $\beta_{m-1}(l - 1) \geq 0$. By using (3.4.36), we get that $l \leq 2$.

- If $l = 2$. From (3.4.36), as $\beta_{m-1} \geq 1$ and $\beta_{m-1} \leq k + q$ (Proposition 3.2.2), it follows that $\alpha_{m-1} = 0$ and $\beta_{m-1} = 1$. In this case as $\epsilon_{m-1} = 0$, we get that $m = 3$. Since $l = 2$, then $\nu = 4 > m$ which is impossible ($\nu \leq m$).
- If $l = 1$. By using (3.4.36), we get S is pseudo-symmetric if and only if $\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 2)t) = 1$. Hence, $\alpha_{m-1} = 1$ and $(k + q - \beta_{m-1})r + (q + 2)t = 1$. Since $\beta_{m-1} \leq k + q$ (Proposition 3.2.2), $r > 0$ and $q + 2 > 1$, we get that $t = 0$, $\beta_{m-1} = k + q - 1$ and $r = 1$ (in particular $q = 0$). Thus, $\beta_{m-1} = k - 1$. We have $\beta_{m-1} \geq 1$, then $k \geq 2$. Thus, we have

$$\alpha_{m-1} = 1, \beta_{m-1} = k - 1, t_{m-1} = 0, r = 1, l = 1 \quad (3.4.37)$$

with $k \geq 2$. We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.37) in S , we obtain

$$S = \langle 2k + 1, 2k + 2, k(2k + 2) + 1 \rangle \text{ with } k \geq 2.$$

- If $\beta_{m-1} = 0$. Since $\epsilon_{m-1} = 0$ and $m \neq 1$ ($S \neq \mathbb{N}$), it follows that $\alpha_{m-1} \geq 1$. As $\alpha_{m-1} \geq 1$, $\beta_{m-1} = 0$ and $\epsilon_{m-1} = 0$, then (3.4.35) implies that $r = 1$ in particular $q = 0$ and $\epsilon_t = t = 0$. Then, (3.4.36) implies that S is pseudo-symmetric if and only if $\alpha_{m-1}(k + k(l - 1)) = 1$. Therefore, $\alpha_{m-1} = 1$ and $kl = 1$. Thus, we have

$$\alpha_{m-1} = 1, \beta_{m-1} = 0, t_{m-1} = 0, k = 1, l = 1, r = 1. \quad (3.4.38)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.38) in S , we get

$$S = \langle 3, 4, 5 \rangle.$$

- If $\epsilon_{m-1} = 1$ ($t_{m-1} \geq 1$). Then, S is pseudo-symmetric if and only if

$$\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) + (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 1. \quad (3.4.39)$$

- If $t \geq 1$. Since $t_{m-1} \geq 1$, $\beta_{m-1} \leq k + q$ (Proposition 3.2.2) and $t \geq 1$, it follows that $\alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) \geq 0$ and $(t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) \geq 0$. From (3.4.39), it follows that

$$\begin{cases} \alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) = 0, \\ (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 1 \end{cases} \quad (3.4.40)$$

or

$$\begin{cases} \alpha_{m-1}((k + q - \beta_{m-1})r + (q + 1)(t - 1)) = 1, \\ (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 0. \end{cases} \quad (3.4.41)$$

From (3.4.40) as $(t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 1$, it follows that $\alpha_{m-1} = 0$, $\beta_{m-1} = 0$ and $t_{m-1} = 2$. Thus, $m = 3$ but $\nu \leq m = 3$, then $l = 1$ for must. On the other hand, $t < l = 1$ which implies that $t = 0$ which is impossible as $t \geq 1$ in this case. Thus, we do not have case (3.4.40). Now, consider (3.4.41). As $k\alpha_{m-1} + \beta_{m-1} + 1 > 0$, then (3.4.41) implies that $t_{m-1} = 1$, $\alpha_{m-1} = 1$ and

$$(k + q - \beta_{m-1})r + (q + 1)(t - 1) = 1. \quad (3.4.42)$$

Since $t \geq 1$, $r > 0$, $q + 1 > 0$ and $\beta_{m-1} \leq k + q$ (Proposition 3.2.2), then (3.4.42) implies that $q = 0$, $t = 2$ and $\beta_{m-1} = k + q = k$ (the case where $\beta_{m-1} = k + q - 1$, $r = 1$ and $t = 1$ is impossible as $r = 1$ implies that $t = 0$ and we get a contradiction). As $r = ql + t + 1$, we get $r = 3$. In addition, as $t = 2$, it follows that $l \geq 3$. Thus, we have

$$\alpha_{m-1} = 1, \beta_{m-1} = k, t_{m-1} = 1, r = 3, l \geq 3. \quad (3.4.43)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By substituting (3.4.43) in S , we obtain

$$S = \langle 2kl + 5, \dots, (2k + 1)l + 5, k((2k + 1)l + 5) + 3 \rangle \text{ with } l \geq 3.$$

- If $t = 0$. We have $r = ql + 1$, from (3.4.39), it follows that S is pseudo-symmetric if and only if

$$\alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) + (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 1. \quad (3.4.44)$$

We have $r = ql + 1$ and $1 \leq \epsilon_{m-1}t_{m-1} \leq l - 1$. On the other hand, $\beta_{m-1}l + \epsilon_{m-1}t_{m-1} \leq kl + r - 1 = kl + ql$. Hence, $\beta_{m-1} \leq k + q - 1$ (as $\epsilon_{m-1}t_{m-1} \geq 1$). Since $l \geq 1$, $t_{m-1} \geq 1$ and $\beta_{m-1} \leq k + q - 1$, it follows that $\alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) \geq 0$ and $(t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) \geq 0$. From (3.4.44), it follows that

$$\begin{cases} \alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) = 0, \\ (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 1, \end{cases} \quad (3.4.45)$$

or

$$\begin{cases} \alpha_{m-1}((k + q - 1 - \beta_{m-1})r + q(l - 1)) = 1, \\ (t_{m-1} - 1)(k\alpha_{m-1} + \beta_{m-1} + 1) = 0. \end{cases} \quad (3.4.46)$$

From (3.4.45), as $k\alpha_{m-1} + \beta_{m-1} + 1 \geq 1$, it follows that $\alpha_{m-1} = \beta_{m-1} = 0$ and $t_{m-1} = 2$. In this case, we have $m = 3$. As $\nu \leq m = 3$, it follows that $l = 1$ which contradicts $t_{m-1} = 2 \leq l - 1$. Thus, we do not have case (3.4.45).

Now, from (3.4.46), as $k\alpha_{m-1} + \beta_{m-1} + 1 \geq 1$, it follows that $t_{m-1} = 1$. Since $t_{m-1} = 1$ ($t_{m-1} \leq l - 1$), we get $l \geq 2$. In addition, (3.4.46) implies that $\alpha_{m-1} = 1$, and

$$(k + q - 1 - \beta_{m-1})r + q(l - 1) = 1. \quad (3.4.47)$$

Since $\beta_{m-1} \leq k + q - 1$ in this case as stated above, $r \geq 1$ and $l \geq 1$, then (3.4.47) implies that

$$r = 1 \ (q = 0), \ \beta_{m-1} = k + q - 2 = k - 2 \quad (3.4.48)$$

or

$$l = 2, \ q = 1, \ \beta_{m-1} = k + q - 1 = k. \quad (3.4.49)$$

Since $r = ql + 1$ in this case, from $\alpha_{m-1} = 1$, $t_{m-1} = 1$, $l \geq 2$, (3.4.48) and (3.4.49), it follows that

$$\alpha_{m-1} = 1, \ \beta_{m-1} = k - 2, \ t_{m-1} = 1, \ r = 1, \ l \geq 2 \quad (3.4.50)$$

or

$$\alpha_{m-1} = 1, \ \beta_{m-1} = k, \ t_{m-1} = 1, \ l = 2, \ r = 3. \quad (3.4.51)$$

We have $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. By (3.4.50) and (3.4.51), we get

$$S = \langle (2k - 2)l + 3, \dots, (2k - 1)l + 3, k((2k - 1)l + 3) + 1 \rangle \text{ with } l \geq 2$$

or

$$S = \langle 4k + 5, 4k + 6, 4k + 7, k(4k + 7) + 3 \rangle.$$

Thus, the proof is complete. ■

Example 3.4.5. Consider the following numerical semigroups.

1. $S = \langle 9, 10, 13 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Moreover, S verifies the formula in Theorem 3.4.4.
2. $S = \langle 32, 33, 511 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 1$, $k = 15$ and $r = 16 = k + 1$. In addition, $m = 32 = 2(15) + 2$. Hence, S verifies the formula in Theorem 3.4.4.
3. $S = \langle 15, 16, 17, 18, 19, 44 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 4$, $k = 2$ and $r = 6 = (k - 1)l + 2$. In addition, $m = 15 = (2(2) - 1)4 + 3 = (2k - 1)l + 3$. Hence, S verifies the formula in Theorem 3.4.4.

4. $S = \langle 17, 18, 19, 20, 25 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 3$, $k = 1$, $r = 5 = l + 2$. Moreover, $m = 17 = 2(2(3) + 2) + 1 = 2(2l + 2) + 1$. Hence, S verifies the formula in Theorem 3.4.4.
5. $S = \langle 9, 10, 41 \rangle$, By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 1$, $r = 1$ and $k = 4 \geq 2$. Moreover, $m = 9 = 2(4) + 1 = 2k + 1$. Hence, S verifies the formula in Theorem 3.4.4.
6. $S = \langle 3, 4, 5 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Moreover, S verifies the formula in Theorem 3.4.4.
7. $S = \langle 45, 46, 47, 48, 49, 50, 203 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 5 \geq 2$, $r = 3$ and $k = 4$. Moreover, $m = 45 = 2(4(5)) + 5 = 2kl + 5$. Hence, S verifies the formula in Theorem 3.4.4.
8. $S = \langle 19, 20, 21, 22, 23, 70 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 4 \geq 2$, $r = 1$ and $k = 3$. Moreover, $m = 19 = (2(3) - 2)4 + 3 = (2k - 2)l + 3$. Hence, S verifies the formula in Theorem 3.4.4.
9. $S = \langle 9, 10, 11, 14 \rangle$. By using GAP [8], we get that S is pseudo-symmetric. Note that $l = 2$, $k = 1$ and $r = 3$. Moreover, $m = 9 = 4(1) + 5 = 4k + 5$. Hence, S verifies the formula in Theorem 3.4.4. ■

3.5 Pseudo-Frobenius Numbers

The aim of this Section, is to determine the set of pseudo-Frobenius Numbers of S . We are going to introduce some Lemmas that will help us in determining the set of pseudo-Frobenius Numbers of S .

Lemma 3.5.1. (see [6]) Let S be a numerical semigroup and $n \in S^*$. Let $\text{Ap}(S, n) = \{w(i); w(i) \equiv i \pmod n, 0 \leq i \leq n - 1\}$ be the Apéry set of S with respect to n and let $PF(S)$ be the set of pseudo-Frobenius numbers of S . Then, $w(x) - n \in PF(S)$ if and only if

$$w(\overline{x+y}) + n \leq w(x) + w(y) \text{ for all } 1 \leq y \leq n - 1 \text{ where } \overline{x+y} = x + y \pmod n.$$

Proof. Let $w(x) - n \in PF(S)$. By definition of $PF(S)$, we have $w(x) - n + S^* \subseteq S$. Then, $w(x) + w(y) - n \in S$ (as $w(y) \in S^*$). On the other hand, both $w(x) + w(y) - n$ and $w(\overline{x+y})$ are congruent to $\overline{x+y} \pmod n$, then by definition of the elements of the Apéry set of S , we get

$$w(\overline{x+y}) \leq w(x) + w(y) - n, \quad \forall 1 \leq y \leq n - 1.$$

Conversely, suppose that $w(\overline{x+y}) + n \leq w(x) + w(y)$, $\forall 1 \leq y \leq n - 1$. By definition of the elements of the Apéry set of S , we have $w(x) - n \notin S$, then it is left to show that $w(x) - n + S^* \subseteq S$ to get that $w(x) - n \in PF(S)$. Indeed, we have $w(x) - n + w(y) \geq w(\overline{x+y})$ for all $1 \leq y \leq n - 1$ and both $w(x) - n + w(y)$ and $w(\overline{x+y})$ are congruent to $\overline{x+y} \pmod n$. Consequently, from the definition of the elements of the Apéry set of S , it follows that $w(x) - n + w(y) \in S$, $\forall 1 \leq y \leq n - 1$ which implies that

$$w(x) - n + S^* \subseteq S.$$

The later follows from the fact that for all $s \in S$, there exists $(k, w) \in \mathbb{N} \times \text{Ap}(S, n)$ such that $s = kn + w$ and that $n \in S$ (Proposition 1.0.13). Hence, our assertion holds. Thus, the proof is complete. ■

By applying Lemma 3.5.1 on our numerical semigroup, we get Proposition 3.5.2. Proposition 3.5.2, mainly equation (3.5.1), will be used later in determining $PF(S)$.

Proposition 3.5.2. Let S be a numerical semigroup minimally generated by $m, m + 1, \dots, m + l, k(m + l) + r$ with $r \leq (k + 1)l + 1$. Then,

$$w(x) - m \in PF(S) \text{ if and only if } \forall 1 \leq y \leq m - 1, \\ m(k\alpha_{\overline{x+y}} + \beta_{\overline{x+y}} + \epsilon_{\overline{x+y}} + 1) + \overline{x+y} \leq m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) + x + y. \quad (3.5.1)$$

Proof. By Lemma 3.5.1, we have $w(x) - m \in PF(S)$ if and only if

$$w(\overline{x+y}) + m \leq w(x) + w(y), \quad \forall 1 \leq y \leq m-1$$

where $\overline{x+y} = x + y \pmod{m}$. By applying Theorem 3.1.1, we get $w(x) - m \in PF(S)$ if and only if

$$m(k\alpha_{\overline{x+y}} + \beta_{\overline{x+y}} + \epsilon_{\overline{x+y}} + 1) + \overline{x+y} \leq m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) + x + y$$

for all $1 \leq y \leq m-1$. Thus, the proof is complete. ■

In Lemma 3.5.3, we give cases where (3.5.1) does not hold. This will allow us to determine some elements that are not in $PF(S)$.

Lemma 3.5.3. Let S be a numerical semigroup minimally generated by $m, \dots, m+l, k(m+l)+r$ with $r \leq (k+1)l+1$. Let $x = \alpha_x(kl+r) + \beta_x l + \epsilon_x t_x$,

$$y_1 = kl+r, \quad y_2 = 1 \quad \text{and} \quad y_3 = l+1 - \epsilon_x t_x.$$

We have the following :

1. Suppose that $x + y_1 \leq m-1$. For all $r \in \mathbb{N}$, x does not satisfy (3.5.1) for y_1 .
2. Suppose that $x + y_2 \leq m-1$ and $\epsilon_x t_x = 0$. If one of the following conditions holds :
 - $r-1 = ql+t$ with $t > 0$;
 - $r-1 = ql$ with $q > 0$ and $\beta_x \neq k+q$;
 - $r=1$ and $\beta_x \neq k$,

then x does not satisfy (3.5.1) for y_2 .

3. Suppose $x + y_3 \leq m-1$. If one of the following conditions holds :
 - $r-1 = ql+t$ with $t > 0$ and $\beta_x \neq k+q$;
 - $r-1 = ql$ with $q > 0$, $\beta_x \neq k+q-1$ and $\beta_x \neq k+q$;
 - $r=1$, $\beta_x \neq k-1$ and $\beta_x \neq k$,

then x does not satisfy (3.5.1) for y_3 .

Proof.

1. We have $m(k\alpha_{y_1} + \beta_{y_1} + \epsilon_{y_1}) = m(k)$. Thus,

$$\begin{aligned} m(k(\alpha_x + \alpha_{y_1}) + \beta_x + \beta_{y_1} + \epsilon_x + \epsilon_{y_1}) + x + y_1 \\ = m(k(\alpha_x + 1) + \beta_x + \epsilon_x) + (\alpha_x + 1)(kl+r) + \beta_x l + \epsilon_x t_x. \end{aligned} \tag{3.5.2}$$

Since $x + y_1 \leq m-1$, it follows that $\overline{x+y_1} = x + y_1$. For all $r \in \mathbb{N}$, we have $x + y_1 = (\alpha_x + 1)(kl+r) + \beta_x l + \epsilon_x t_x$. Hence,

$$\begin{aligned} m(k\alpha_{\overline{x+y_1}} + \beta_{\overline{x+y_1}} + \epsilon_{\overline{x+y_1}} + 1) + \overline{x+y_1} \\ = m(k(\alpha_x + 1) + \beta_x + \epsilon_x + 1) + (\alpha_x + 1)(kl+r) + \beta_x l + \epsilon_x t_x. \end{aligned} \tag{3.5.3}$$

By using (3.5.2) and (3.5.3), it follows that x does not satisfy (3.5.1) for y_1 .

2. If $\epsilon_x = 0$, then $x = \alpha_x(kl+r) + \beta_x l$. We have $m(k\alpha_{y_2} + \beta_{y_2} + \epsilon_{y_2}) = m$. Therefore,

$$\begin{aligned} m(k(\alpha_x + \alpha_{y_2}) + \beta_x + \beta_{y_2} + \epsilon_x + \epsilon_{y_2}) + x + y_2 \\ = m(k\alpha_x + \beta_x + 1) + \alpha_x(kl+r) + \beta_x l + 1. \end{aligned} \tag{3.5.4}$$

Since $x + y_2 \leq m-1$, it follows that $\overline{x+y_2} = x + y_2$. If one of the following conditions holds :

- $r - 1 = ql + t$ with $t > 0$;
- $r - 1 = ql$ with $q > 0$ and $\beta_x \neq k + q$;
- $r = 1$ and $\beta_x \neq k$,

then $x + y_2 = \alpha_x(kl + r) + \beta_x l + 1$. We have

$$\begin{aligned} & m(k\alpha_{x+y_2} + \beta_{x+y_2} + \epsilon_{x+y_2} + 1) + x + y_2 \\ &= m(k\alpha_x + \beta_x + 2) + \alpha_x(kl + r) + \beta_x l + 1. \end{aligned} \quad (3.5.5)$$

By using (3.5.4) and (3.5.5), it follows that x does not satisfy (3.5.1) for y_2 .

3. We have $m(k\alpha_{y_3} + \beta_{y_3} + \epsilon_{y_3}) = m(2)$ if $\epsilon_x = 0$ and $m(k\alpha_{y_3} + \beta_{y_3} + \epsilon_{y_3}) = m$ if $\epsilon_x = 1$. Therefore,

$$\begin{aligned} & m(k(\alpha_x + \alpha_{y_3}) + \beta_x + \beta_{y_3} + \epsilon_x + \epsilon_{y_3}) + x + y_3 \\ &= m(k\alpha_x + \beta_x + 2) + \alpha_x(kl + r) + (\beta_x + 1)l + 1. \end{aligned} \quad (3.5.6)$$

Since $x + y_3 \leq m - 1$, it follows that $\overline{x + y_3} = x + y_3$. If one of the following conditions holds :

- $r - 1 = ql + t$ with $t > 0$ and $\beta_x \neq k + q$;
- $r - 1 = ql$ with $q > 0, \beta_x \neq k + q - 1$ and $\beta_x \neq k + q$;
- $r = 1, \beta_x \neq k - 1$ and $\beta_x \neq k$,

then $x + y_3 = \alpha_x(kl + r) + (\beta_x + 1)l + 1$. We have

$$\begin{aligned} & m(k\alpha_{x+y_3} + \beta_{x+y_3} + \epsilon_{x+y_3} + 1) + x + y_3 \\ &= m(k\alpha_x + \beta_x + 3) + \alpha_x(kl + r) + (\beta_x + 1)l + 1. \end{aligned} \quad (3.5.7)$$

By using (3.5.6) and (3.5.7), it follows that x does not satisfy (3.5.1) for y_3 .

Thus, the proof is complete. ■

Theorem 3.5.4 will determine the elements that do not belong to $PF(S)$.

Theorem 3.5.4. Let S be a numerical semigroup minimally generated by $m, m + 1, \dots, m + l, k(m + l) + r$ with $r \leq (k + 1)l + 1$. For all $1 \leq x \leq m - 1$, write $x = \alpha_x(kl + r) + \beta_x l + \epsilon_x t_x$ as in Definition 3.0.1 and $w(x) = m(k\alpha_x + \beta_x + \epsilon_x) + x$ as in Theorem 3.1.1. We have the following :

Case 1. If $r - 1 = ql + t$ for some $q, t \in \mathbb{N}$ with $t < l$ and $t \neq 0$.

Case 1.1. If $m - 1 = \alpha_{m-1}(kl + r)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$), then

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q)l + 1 \leq x \leq m - 1\}.$$

Case 1.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0, \epsilon_{m-1} = 0$), then

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l\}. \end{aligned}$$

Case 1.3. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ (i.e., $\epsilon_{m-1} = 1$), then

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Case 2. If $r - 1 = ql$ for some $q \in \mathbb{N}^*$.

Case 2.1. If $m - 1 = \alpha_{m-1}(kl + r)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$), then

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 1\}.$$

Case 2.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$), then

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l\}. \end{aligned}$$

Case 2.3. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ (i.e., $\epsilon_{m-1} = 1$), then

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Case 3. If $r = 1$.

Case 3.1. If $m - 1 = \alpha_{m-1}(kl + 1)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$), then

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \leq x \leq m - 1\}.$$

Case 3.2. If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$), then

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l\}. \end{aligned}$$

Case 3.3. If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ (i.e., $\epsilon_{m-1} = 1$), then

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Proof. Case 1. If $r - 1 = ql + t$ for some $q, t \in \mathbb{N}$ with $t < l$ and $t \neq 0$.

Case 1.1. If $m - 1 = \alpha_{m-1}(kl + r)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$). We have

$$(\alpha_{m-1} - 1)(kl + r) + (kl + r) \leq \alpha_{m-1}(kl + r)$$

and

$$((\alpha_{m-1} - 1)(kl + r) + 1) + (kl + r) > \alpha_{m-1}(kl + r).$$

Consequently,

$$x + y_1 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + r).$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + r)$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + 1 \leq x \leq m - 1\}. \quad (3.5.8)$$

Moreover,

$$((\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl + r)$$

and

$$((\alpha_{m-1} - 1)(kl + r) + (k + q)l) + (l + 1) > \alpha_{m-1}(kl + r).$$

Consequently,

$$x + y_3 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 if $x + y_3 \leq m - 1$ and $\beta_x \neq k + q$ (Lemma 3.5.3), by using (3.5.8), we get that if $x \leq (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + l - 1$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q)l \leq x \leq m - 1\}. \quad (3.5.9)$$

In addition, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$, then $\epsilon_x = 0$ and $x + y_2 \leq m - 1$. Since x does not satisfy (3.5.1) for y_2 if $x + y_2 \leq m - 1$ and $\epsilon_x = 0$, we deduce that if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$, then $w(x) - m \notin PF(S)$ (Lemma 3.5.3). By using (3.5.9), we obtain

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q)l + 1 \leq x \leq m - 1\}. \quad (3.5.10)$$

Case 1.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$). We have

$$((\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l) + (kl + r) \leq \alpha_{m-1}(kl + r) + \beta_{m-1}l$$

and

$$((\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l + 1) + (kl + r) > \alpha_{m-1}(kl + r) + \beta_{m-1}l.$$

Consequently,

$$x + y_1 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l.$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$, we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l$ (Lemma 3.5.3), then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\}. \quad (3.5.11)$$

Moreover,

$$(\alpha_{m-1}(kl + r) + (\beta_{m-1} - 2)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl + r) + \beta_{m-1}l$$

and

$$(\alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l) + (l + 1) > \alpha_{m-1}(kl + r) + \beta_{m-1}l.$$

Consequently,

$$x + y_3 \leq m - 1 \text{ iff } x \leq \alpha_{m-1}(kl + r) + (\beta_{m-1} - 2)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 if $x + y_3 \leq m - 1$ and $\beta_x \neq k + q$ (Lemma 3.5.3), by using (3.5.11) we deduce that if $x \leq \alpha_{m-1}(kl + r) + (\beta_{m-1} - 2)l + l - 1$ and $x \neq (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x$ with $(k + q)l + \epsilon_x t_x > \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. In particular,

$$\begin{aligned} PF(S) \subseteq & \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l \leq x \leq m - 1\} \\ & \cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ & \text{with } (k + q)l + \epsilon_x t_x > \beta_{m-1}l\}. \end{aligned} \quad (3.5.12)$$

In addition, if $x = \alpha_{m-1}(kl+r) + (\beta_{m-1} - 1)l$, then $\epsilon_x = 0$ and $x + y_2 \leq m - 1$. Since x does not satisfy (3.5.1) for y_2 if $x + y_2 \leq m - 1$ and $\epsilon_x = 0$, it follows that if $x = \alpha_{m-1}(kl+r) + (\beta_{m-1} - 1)l$, then $w(x) - m \notin PF(S)$ (Lemma 3.5.3). By the same argument we have if $x = (\alpha_{m-1} - 1)(kl+r) + (k+q)l + \epsilon_x t_x$ such that $\epsilon_x = 0$, then $w(x) - m \notin PF(S)$. By using (3.5.12) we deduce that

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl+r) + (k+q)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k+q)l + \epsilon_x t_x > \beta_{m-1}l\}. \end{aligned} \quad (3.5.13)$$

Case 1.3. If $m - 1 = \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, i.e., $(\epsilon_{m-1} = 1)$. We have

$$((\alpha_{m-1} - 1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}) + (kl+r) \leq \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$$

and

$$((\alpha_{m-1} - 1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1) + (kl+r) > \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Consequently,

$$x + y_1 \leq m - 1 \quad \text{iff} \quad x \leq (\alpha_{m-1} - 1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$, we deduce that if $x \leq (\alpha_{m-1} - 1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $w(x) - m \notin PF(S)$ (Lemma 3.5.3). In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 \leq x \leq m - 1\}. \quad (3.5.14)$$

Moreover,

$$(\alpha_{m-1}(kl+r) + (\beta_{m-1} - 1)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$$

and

$$(\alpha_{m-1}(kl+r) + \beta_{m-1}l) + (l + 1) > \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Consequently,

$$x + y_3 \leq m - 1 \quad \text{iff} \quad x \leq \alpha_{m-1}(kl+r) + (\beta_{m-1} - 1)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 if $x + y_3 \leq m - 1$ and $\beta_x \neq k + q$ (Lemma 3.5.3), by using (3.5.14) we deduce that if $x \leq \alpha_{m-1}(kl+r) + (\beta_{m-1} - 1)l + l - 1$ and $x \neq (\alpha_{m-1} - 1)(kl+r) + (k+q)l + \epsilon_x t_x$ with $(k+q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $w(x) - m \notin PF(S)$. In particular,

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+r) + \beta_{m-1}l \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl+r) + (k+q)l + \epsilon_x t_x \\ &\quad \text{with } (k+q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.15)$$

In addition, if $x = \alpha_{m-1}(kl+r) + \beta_{m-1}l$, then $\epsilon_x = 0$ and $x + y_2 \leq m - 1$. Since x does not satisfy (3.5.1) for y_2 if $x + y_2 \leq m - 1$ and $\epsilon_x = 0$, we deduce that if $x = \alpha_{m-1}(kl+r) + \beta_{m-1}l$, then $w(x) - m \notin PF(S)$ (Lemma 3.5.3). By the same argument we have if $x = (\alpha_{m-1} - 1)(kl+r) + (k+q)l + \epsilon_x t_x$ such that $\epsilon_x = 0$, then $w(x) - m \notin PF(S)$. By using (3.5.15), we deduce that

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl+r) + (k+q)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k+q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.16)$$

Case 2. If $r - 1 = ql$ for some $q \in \mathbb{N}^*$.

Case 2.1. If $m - 1 = \alpha_{m-1}(kl+r)$, (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$). We have

$$(\alpha_{m-1} - 1)(kl+r) + (kl+r) \leq \alpha_{m-1}(kl+r)$$

and

$$((\alpha_{m-1} - 1)(kl + r) + 1) + (kl + r) > \alpha_{m-1}(kl + r).$$

Consequently,

$$x + y_1 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + r).$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + r)$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + 1 \leq x \leq m - 1\}. \quad (3.5.17)$$

Moreover,

$$((\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl + r)$$

and

$$((\alpha_{m-1} - 1)(kl + r) + (k + q)l) + (l + 1) > \alpha_{m-1}(kl + r).$$

Consequently,

$$x + y_3 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 in the case $x + y_3 \leq m - 1$, $\beta_x \neq k + q - 1$ and $\beta_x \neq k + q$ (Lemma 3.5.3), by using (3.5.17) we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + r) + (k + q - 2)l + l - 1$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l \leq x \leq m - 1\}. \quad (3.5.18)$$

In addition, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l$, then $\epsilon_x = 0$, $\beta_x \neq k + q$ and $x + y_2 \leq m - 1$. Since x does not satisfy (3.5.1) for y_2 in the case $x + y_2 \leq m - 1$, $\epsilon_x = 0$ and $\beta_x \neq k + q$ (Lemma 3.5.3), we deduce that if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l$, then $w(x) - m \notin PF(S)$. By using (3.5.18), we obtain

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 1\}. \quad (3.5.19)$$

Case 2.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$). We have

$$((\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l) + (kl + r) \leq \alpha_{m-1}(kl + r) + \beta_{m-1}l$$

and

$$((\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l + 1) + (kl + r) > \alpha_{m-1}(kl + r) + \beta_{m-1}l.$$

Consequently,

$$x + y_1 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l.$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\}. \quad (3.5.20)$$

Moreover,

$$(\alpha_{m-1}(kl + r) + (\beta_{m-1} - 2)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl + r) + \beta_{m-1}l$$

and

$$(\alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l) + (l + 1) > \alpha_{m-1}(kl + r) + \beta_{m-1}l.$$

Consequently,

$$x + y_3 \leq m - 1 \text{ iff } x \leq \alpha_{m-1}(kl + r) + (\beta_{m-1} - 2)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 in the case $x + y_3 \leq m - 1$, $\beta_x \neq k + q - 1$ and $\beta_x \neq k + q$ (Lemma 3.5.3), by using (3.5.20) we deduce that if $x \leq \alpha_{m-1}(kl + r) + (\beta_{m-1} - 2)l + l - 1$ such that

$x \neq (\alpha_{m-1}-1)(kl+r) + (k+q-1)l + \epsilon_x t_x$ with $(k+q-1)l + \epsilon_x t_x > \beta_{m-1}l$ and $x \neq (\alpha_{m-1}-1)(kl+r) + (k+q)l$ with $(k+q)l > \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. In particular,

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l \leq x \leq m-1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+r) + (k+q-1)l + \epsilon_x t_x \\ &\quad \text{with } (k+q-1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+r) + (k+q)l \\ &\quad \text{with } (k+q)l > \beta_{m-1}l\}. \end{aligned} \quad (3.5.21)$$

If $x = \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l$, then $\epsilon_x = 0$, $\beta_{m-1}-1 \neq k+q$ (as $\beta_{m-1} \leq k+q$) and $x + y_2 \leq m-1$. Since x does not satisfy (3.5.1) for y_2 in the case $x + y_2 \leq m-1$, $\epsilon_x = 0$ and $\beta_x \neq k+q$ (Lemma 3.5.3), we deduce that if $x = \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l$, then $w(x) - m \notin PF(S)$. By the same argument we have if $x = (\alpha_{m-1}-1)(kl+r) + (k+q-1)l + \epsilon_x t_x$ such that $\epsilon_x = 0$, then $w(x) - m \notin PF(S)$. By using (3.5.21), we get

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + 1 \leq x \leq m-1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+r) + (k+q-1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k+q-1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+r) + (k+q)l \\ &\quad \text{with } (k+q)l > \beta_{m-1}l\}. \end{aligned} \quad (3.5.22)$$

Case 2.3. If $m-1 = \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, i.e., $(\epsilon_{m-1} = 1)$. We have

$$((\alpha_{m-1}-1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}) + (kl+r) \leq \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$$

and

$$((\alpha_{m-1}-1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1) + (kl+r) > \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Consequently,

$$x + y_1 \leq m-1 \quad \text{iff} \quad x \leq (\alpha_{m-1}-1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m-1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1}-1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1}-1)(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 \leq x \leq m-1\}. \quad (3.5.23)$$

Moreover,

$$(\alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + l-1) + (l+1 - (l-1)) \leq \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$$

and

$$(\alpha_{m-1}(kl+r) + \beta_{m-1}l) + (l+1) > \alpha_{m-1}(kl+r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Consequently,

$$x + y_3 \leq m-1 \quad \text{iff} \quad x \leq \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + l-1.$$

Since x does not satisfy (3.5.1) for y_3 in the case $x + y_3 \leq m-1$, $\beta_x \neq k+q-1$ and $\beta_x \neq k+q$ (Lemma 3.5.3), by using (3.5.23) we deduce that if $x \leq \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + l-1$ such that $x \neq (\alpha_{m-1}-1)(kl+r) + (k+q)l$ with $(k+q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ and $x \neq (\alpha_{m-1}-1)(kl+r) + (k+q-1)l + \epsilon_x t_x$ with $(k+q-1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $w(x) - m \notin PF(S)$. In particular,

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+r) + \beta_{m-1}l \leq x \leq m-1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+r) + (k+q)l \\ &\quad \text{with } (k+q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+r) + (k+q-1)l + \epsilon_x t_x \\ &\quad \text{with } (k+q-1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.24)$$

If $x = \alpha_{m-1}(kl + r) + \beta_{m-1}l$, then $\epsilon_x = 0$, $\beta_x = \beta_{m-1} \neq k + q$ (if $\beta_{m-1} = k + q$, then $m - 1 = \alpha_{m-1}(kl + r) + (k + q)l + \epsilon_{m-1}t_{m-1}$ as $\epsilon_{m-1} = 1$ and $r = ql + 1$, we get a contradiction) and $x + y_2 \leq m - 1$. Since x does not satisfy (3.5.1) for y_2 in the case $x + y_2 \leq m - 1$, $\epsilon_x = 0$ and $\beta_x \neq k + q$ (Lemma 3.5.3), we deduce that if $x = \alpha_{m-1}(kl + r) + \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. By the same argument we have if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 0$, then $w(x) - m \notin PF(S)$. By using (3.5.24), we obtain

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.25)$$

Case 3. If $r = 1$.

Case 3.1. If $m - 1 = \alpha_{m-1}(kl + 1)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$). We have

$$(\alpha_{m-1} - 1)(kl + 1) + (kl + 1) \leq \alpha_{m-1}(kl + 1)$$

and

$$((\alpha_{m-1} - 1)(kl + 1) + 1) + (kl + 1) > \alpha_{m-1}(kl + 1).$$

Consequently,

$$x + y_1 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + 1).$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + 1)$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + 1) + 1 \leq x \leq m - 1\}. \quad (3.5.26)$$

Moreover,

$$((\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl + 1)$$

and

$$((\alpha_{m-1} - 1)(kl + 1) + kl) + (l + 1) > \alpha_{m-1}(kl + 1).$$

Consequently,

$$x + y_3 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 in the case $x + y_3 \leq m - 1$, $\beta_x \neq k - 1$ and $\beta_x \neq k$ (Lemma 3.5.3), by using (3.5.26) we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + 1) + (k - 2)l + l - 1$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l \leq x \leq m - 1\}. \quad (3.5.27)$$

If $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l$, then $\epsilon_x = 0$, $\beta_x \neq k$ and $x + y_2 \leq m - 1$. Since x does not satisfy (3.5.1) for y_2 in the case $x + y_2 \leq m - 1$, $\epsilon_x = 0$ and $\beta_x \neq k$ (Lemma 3.5.3), we deduce that if $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l$, then $w(x) - m \notin PF(S)$. Hence,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \leq x \leq m - 1\}. \quad (3.5.28)$$

Case 3.2. If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l$ (i.e., $\epsilon_{m-1} = 0$). We have

$$((\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l) + (kl + 1) \leq \alpha_{m-1}(kl + 1) + \beta_{m-1}l$$

and

$$((\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l + 1) + (kl + 1) > \alpha_{m-1}(kl + 1) + \beta_{m-1}l.$$

Consequently,

$$x + y_1 \leq m - 1 \text{ iff } x \leq (\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l.$$

Since x does not satisfy (3.5.1) for y_1 if $x+y_1 \leq m-1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1}-1)(kl+1) + \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1}-1)(kl+1) + \beta_{m-1}l + 1 \leq x \leq m-1\}. \quad (3.5.29)$$

Moreover,

$$(\alpha_{m-1}(kl+1) + (\beta_{m-1}-2)l + l - 1) + (l+1 - (l-1)) \leq \alpha_{m-1}(kl+1) + \beta_{m-1}l$$

and

$$(\alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l) + (l+1) > \alpha_{m-1}(kl+1) + \beta_{m-1}l.$$

Consequently,

$$x + y_3 \leq m-1 \text{ iff } x \leq \alpha_{m-1}(kl+1) + (\beta_{m-1}-2)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 in the case $x+y_3 \leq m-1$, $\beta_x \neq k-1$ and $\beta_x \neq k$ (Lemma 3.5.3), by using (3.5.29) we deduce that if $x \leq \alpha_{m-1}(kl+1) + (\beta_{m-1}-2)l + l - 1$ such that $x \neq (\alpha_{m-1}-1)(kl+1) + (k-1)l + \epsilon_x t_x$ with $(k-1)l + \epsilon_x t_x > \beta_{m-1}l$ and $x \neq (\alpha_{m-1}-1)(kl+1) + kl$ with $kl > \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. In particular,

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l \leq x \leq m-1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+1) + (k-1)l + \epsilon_x t_x \text{ with } (k-1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+1) + kl \text{ with } kl > \beta_{m-1}l\}. \end{aligned} \quad (3.5.30)$$

If $x = \alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l$, then $\epsilon_x = 0$, $\beta_x = \beta_{m-1} - 1 \neq k$ (as $\beta_{m-1} \leq k+q = k$ in this case as $r = 1$) and $x + y_2 \leq m-1$. Since x does not satisfy (3.5.1) for y_2 in the case $x + y_2 \leq m-1$, $\epsilon_x = 0$ and $\beta_x \neq k$, we deduce that if $x = \alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l$, then $w(x) - m \notin PF(S)$. By the same argument we have if $x = (\alpha_{m-1}-1)(kl+1) + (k-1)l + \epsilon_x t_x$ such that $\epsilon_x = 0$, then $w(x) - m \notin PF(S)$. By using (3.5.30), we get

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l + 1 \leq x \leq m-1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+1) + (k-1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k-1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+1) + kl \text{ with } kl > \beta_{m-1}l\}. \end{aligned} \quad (3.5.31)$$

Now, let $x = (\alpha_{m-1}-1)(kl+1) + (k-1)l + \epsilon_x t_x$. Suppose that $\epsilon_x t_x \geq 2$. If we take $y' = l$, then x does not satisfy (3.5.1) for y' . Indeed, we have $x+y' = \alpha_{m-1}(kl+1) + (\epsilon_x t_x - 1)$ with $\epsilon_x t_x - 1 \geq 1$. We have $x+y' \leq m-1$ which gives $\overline{x+y'} = x+y'$. In addition, $m(k\alpha_{x+y'} + \beta_{x+y'} + \epsilon_{x+y'} + 1) + x+y' = m(k\alpha_{m-1} + 1 + 1) + x+y'$. On the other hand, $m(k(\alpha_x + \alpha_{y'}) + \beta_x + \beta_{y'} + \epsilon_x + \epsilon_{y'}) + x+y' = m(k(\alpha_{m-1}-1) + k-1 + 1 + 1) + x+y' = m(k\alpha_{m-1} + 1) + x+y'$. Consequently, x does not satisfy (3.5.1) for y' if $\epsilon_x t_x \geq 2$. Therefore, if $x = (\alpha_{m-1}-1)(kl+1) + (k-1)l + \epsilon_x t_x$ with $\epsilon_x t_x \geq 2$, then $w(x) - m \notin PF(S)$. By using (3.5.31), we get

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l + 1 \leq x \leq m-1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+1) + (k-1)l + 1 \\ &\quad \text{with } (k-1)l + 1 > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1}-1)(kl+1) + kl \\ &\quad \text{with } kl > \beta_{m-1}l\}. \end{aligned} \quad (3.5.32)$$

Next, let $x = (\alpha_{m-1}-1)(kl+1) + kl$. Suppose that $l \geq 2$. If we take $y'' = 2$, then x does not satisfy (3.5.1) for y'' . Indeed, $x+y'' = \alpha_{m-1}(kl+1) + 1$. We have $x+y'' \leq m-1$ which gives $\overline{x+y''} = x+y''$. In addition, $m(k\alpha_{x+y''} + \beta_{x+y''} + \epsilon_{x+y''} + 1) + x+y'' = m(k\alpha_{m-1} + 1 + 1) + x+y''$. On the other hand, $m(k(\alpha_x + \alpha_{y''}) + \beta_x + \beta_{y''} + \epsilon_x + \epsilon_{y''}) + x+y'' = m(k(\alpha_{m-1}-1) + k+1) + x+y'' = m(k\alpha_{m-1} + 1) + x+y''$.

Hence, x does not satisfy (3.5.1) for y'' if $l \geq 2$. Consequently, if $x = (\alpha_{m-1} - 1)(kl + 1) + kl$ and $l \geq 2$, then $w(x) - m \notin PF(S)$. By using (3.5.32), we get

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l\}. \end{aligned} \quad (3.5.33)$$

Case 3.3 If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, i.e., $(\epsilon_{m-1} = 1)$. We have

$$((\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}) + (kl + 1) \leq \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$$

and

$$((\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1) + (kl + 1) > \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Consequently,

$$x + y_1 \leq m - 1 \quad \text{iff} \quad x \leq (\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Since x does not satisfy (3.5.1) for y_1 if $x + y_1 \leq m - 1$ (Lemma 3.5.3), we deduce that if $x \leq (\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $w(x) - m \notin PF(S)$. In particular,

$$PF(S) \subseteq \{w(x) - m; (\alpha_{m-1} - 1)(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1} + 1 \leq x \leq m - 1\}. \quad (3.5.34)$$

Moreover,

$$(\alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + l - 1) + (l + 1 - (l - 1)) \leq \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$$

and

$$(\alpha_{m-1}(kl + 1) + \beta_{m-1}l) + (l + 1) > \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}.$$

Consequently,

$$x + y_3 \leq m - 1 \quad \text{iff} \quad x \leq \alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + l - 1.$$

Since x does not satisfy (3.5.1) for y_3 in the case $x + y_3 \leq m - 1$, $\beta_x \neq k - 1$ and $\beta_x \neq k$ (Lemma 3.5.3), by using (3.5.34) we deduce that if $x \leq \alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + l - 1$ such that $x \neq (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + \epsilon_x t_x$ with $(k - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ and $x \neq (\alpha_{m-1} - 1)(kl + 1) + kl$ with $kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $w(x) - m \notin PF(S)$. In particular,

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + \epsilon_x t_x \\ &\quad \text{with } (k - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.35)$$

If $x = \alpha_{m-1}(kl + 1) + \beta_{m-1}l$, then $\epsilon_x = 0$, $x + y_2 \leq m - 1$ and $\beta_x = \beta_{m-1} \neq k$ (if $\beta_{m-1} = k$, then $m - 1 = \alpha_{m-1}(kl + 1) + kl + \epsilon_{m-1}t_{m-1}$, as $\epsilon_{m-1} = 1$ and $r = 1$, we get a contradiction). Since x does not satisfy (3.5.1) for y_2 in the case $x + y_2 \leq m - 1$, $\epsilon_x = 0$ and $\beta_x \neq k$ (Lemma 3.5.3), we deduce that if $x = \alpha_{m-1}(kl + 1) + \beta_{m-1}l$, then $w(x) - m \notin PF(S)$. By the same argument we have if $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 0$, then $w(x) - m \notin PF(S)$. By using (3.5.35) we deduce that

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.36)$$

Now, let $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + \epsilon_x t_x$. Suppose that $\epsilon_x t_x \geq 2$. If we take $y' = l$, then x does not satisfy (3.5.1) for y' . Indeed, we have $x + y' = \alpha_{m-1}(kl + 1) + (\epsilon_x t_x - 1)$ with $\epsilon_x t_x - 1 \geq 1$. We have $x + y' \leq m - 1$ which gives $\overline{x + y'} = x + y'$. In addition,

$$m(k\alpha_{x+y'} + \beta_{x+y'} + \epsilon_{x+y'} + 1) + x + y' = m(k\alpha_{m-1} + 1 + 1) + x + y'.$$

On the other hand, $m(k(\alpha_x + \alpha_{y'}) + \beta_x + \beta_{y'} + \epsilon_x + \epsilon_{y'}) + x + y' = m(k(\alpha_{m-1} - 1) + k - 1 + 1 + 1) + x + y' = m(k\alpha_{m-1} + 1) + x + y'$. Consequently, x does not satisfy (3.5.1) for y' if $\epsilon_x t_x \geq 2$. Therefore, if $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + \epsilon_x t_x$ with $\epsilon_x t_x \geq 2$, then $w(x) - m \notin PF(S)$. By using (3.5.36), we get

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.37)$$

Next, let $x = (\alpha_{m-1} - 1)(kl + 1) + kl$. Suppose that $l \geq 2$. If we take $y'' = 2$, then x does not satisfy (3.5.1) for y'' . Indeed, $x + y'' = \alpha_{m-1}(kl + 1) + 1$. We have $x + y'' \leq m - 1$ which gives $\overline{x + y''} = x + y''$. In addition,

$$m(k\alpha_{x+y''} + \beta_{x+y''} + \epsilon_{x+y''} + 1) + x + y'' = m(k\alpha_{m-1} + 1 + 1) + x + y''.$$

On the other hand,

$$\begin{aligned} m(k(\alpha_x + \alpha_{y''}) + \beta_x + \beta_{y''} + \epsilon_x + \epsilon_{y''}) + x + y'' &= m(k(\alpha_{m-1} - 1) + k + 1) + x + y'' \\ &= m(k\alpha_{m-1} + 1) + x + y''. \end{aligned}$$

Hence, x does not satisfy (3.5.1) for y'' . Consequently, if $x = (\alpha_{m-1} - 1)(kl + 1) + kl$ and $l \geq 2$, then $w(x) - m \notin PF(S)$. By using (3.5.37), we get

$$\begin{aligned} PF(S) &\subseteq \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned} \quad (3.5.38)$$

Thus, the proof is complete. ■

Lemmas 3.5.5 and 3.5.6 give cases where (3.5.1) holds. This will allow us to determine later some numbers that belong to $PF(S)$.

Lemma 3.5.5. Let S be a numerical semigroup minimally generated by $m, m+1, \dots, m+l, k(m+l)+r$ with $r \leq (k+1)l+1$. If $m-1 = \alpha_{m-1}(kl+r)$, then $m-1$ satisfies (3.5.1) for all $1 \leq y \leq m-1$.

Proof. Let $1 \leq y \leq m-1$ and write $y = \alpha_y(kl+r) + \beta_y l + \epsilon_y t_y$ as in Definition 3.0.1. Let $x = m-1$. For $1 \leq y \leq m-1$, we have $x + y > m-1$ which gives $x + y = \overline{x + y} + m$, i.e., $m-1 + y = \overline{x + y} + m$. Hence, $\overline{x + y} = y - 1 = \alpha_y(kl+r) + \beta_y l + \epsilon_y t_y - 1$. Therefore,

$$\begin{aligned} m(k\alpha_{\overline{x+y}} + \beta_{\overline{x+y}} + \epsilon_{\overline{x+y}} + 1) + \overline{x+y} &\leq m(k\alpha_y + \beta_y + \epsilon_y + 1) + \overline{x+y} \\ &= m(k\alpha_y + \beta_y + \epsilon_y) + x + y. \end{aligned}$$

On the other hand, $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) + x + y = m(k(\alpha_{m-1} + \alpha_y) + \beta_y + \epsilon_y) + x + y.$$

Therefore,

$$m(k\alpha_{\overline{x+y}} + \beta_{\overline{x+y}} + \epsilon_{\overline{x+y}} + 1) + \overline{x+y} \leq m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) + x + y.$$

Hence, $x = m-1$ satisfies (3.5.1) for all $1 \leq y \leq m-1$. Thus, the proof is complete. ■

Lemma 3.5.6. Let S be a numerical semigroup minimally generated by $m, m + 1, \dots, m + l, k(m + l) + r$ with $r \leq (k + 1)l + 1$. Let $1 \leq x, y \leq m - 1$ such that $x + y > m - 1$ and $x = \alpha_x(kl + r) + \beta_x l + \epsilon_x t_x$ with $\beta_x + \epsilon_x > 0$. Then, x satisfies (3.5.1) for y .

Proof. Suppose by the way of contradiction that x does not satisfy (3.5.1) for y . Consequently,

$$w(\overline{x + y}) + m > w(x) + w(y).$$

We have $x + y > m - 1$ and $1 \leq x, y \leq m - 1$, thus $x + y = \overline{x + y} + m$, $\overline{x + y} < x$ and $\overline{x + y} < y$. Since $w(x) + w(y)$ and $w(\overline{x + y})$ are both elements in S that are congruent to $x + y \pmod{m}$, then from the definition of the element of the Apéry set of S , it follows that $w(x) + w(y) = w(\overline{x + y}) + x_0 m$ for some $x_0 \in \mathbb{N}$. On the other hand, $w(\overline{x + y}) + m > w(x) + w(y)$. Thus, $x_0 = 0$ and

$$w(\overline{x + y}) = w(x) + w(y). \quad (3.5.39)$$

As $x \geq 1$ and $y \geq 1$, it follows that $w(y) > 0$ and $w(x) > 0$. Then, (3.5.39) implies that $w(\overline{x + y}) > w(x)$ with $\overline{x + y} < x$ and $w(\overline{x + y}) > w(y)$ with $\overline{x + y} < y$. By Proposition 3.2.4, we have if $i < j$, then $w(i) > w(j)$ if and only if it satisfies one of the following :

1. $\alpha_i = \alpha_j - 2$, $\beta_j = \epsilon_j = 0$ and $\beta_i + \epsilon_i = 2k + 1$.
2. $\alpha_i = \alpha_j - 1$, $\beta_i + \epsilon_i > k + \beta_j + \epsilon_j$ and $\beta_j + \epsilon_j \leq k$.

In particular, if $i < j$ such that $w(i) > w(j)$, then $\alpha_i \leq \alpha_j - 1$. Write $\overline{x + y} = \alpha_{\overline{x + y}}(kl + r) + \beta_{\overline{x + y}} l + \epsilon_{\overline{x + y}} t_{\overline{x + y}}$ and $y = \alpha_y(kl + r) + \beta_y l + \epsilon_y t_y$. Since $w(\overline{x + y}) > w(x)$ with $\overline{x + y} < x$ and $w(\overline{x + y}) > w(y)$ with $\overline{x + y} < y$, it follows that $\alpha_{\overline{x + y}} \leq \alpha_x - 1$ and $\alpha_{\overline{x + y}} \leq \alpha_y - 1$. By proposition 3.0.2, we have $\beta_{\overline{x + y}} + \epsilon_{\overline{x + y}} \leq 2k + 1$. Hence,

$$\begin{aligned} w(\overline{x + y}) &= m(k\alpha_{\overline{x + y}} + \beta_{\overline{x + y}} + \epsilon_{\overline{x + y}}) + \overline{x + y} \\ &\leq m(k\alpha_{\overline{x + y}} + 2k + 1) + \overline{x + y} \\ &= m(k\alpha_{\overline{x + y}} + 2k) + x + y. \end{aligned} \quad (3.5.40)$$

On the other hand, $w(y) = m(k\alpha_y + \beta_y + \epsilon_y) + y \geq m(k(\alpha_{\overline{x + y}} + 1)) + y$. By using $\beta_x + \epsilon_x > 0$ (hypothesis) and $\alpha_x \geq \alpha_{\overline{x + y}} + 1$, we get $w(x) = m(k\alpha_x + \beta_x + \epsilon_x) + x \geq m(k(\alpha_{\overline{x + y}} + 1) + 1) + x$. Consequently,

$$w(x) + w(y) \geq m(k\alpha_{\overline{x + y}} + 2k + 1) + x + y. \quad (3.5.41)$$

But (3.5.41) and (3.5.40) contradicts (3.5.39). Therefore, x satisfies (3.5.1) for y . ■

Now, we are ready to determine the set of pseudo-Frobenius Numbers of S .

Theorem 3.5.7. Let S be a numerical semigroup minimally generated by $m, m + 1, \dots, m + l, k(m + l) + r$ with $r \leq (k + 1)l + 1$. For all $1 \leq x \leq m - 1$, write $x = \alpha_x(kl + r) + \beta_x l + \epsilon_x t_x$ as in Definition 3.0.1 and $w(x) = m(k\alpha_x + \beta_x + \epsilon_x) + x$ as in Theorem 3.1.1. We have the following :

Case 1. If $r - 1 = ql + t$ for some $q, t \in \mathbb{N}$ with $t < l$ and $t \neq 0$.

Case 1.1. If $m - 1 = \alpha_{m-1}(kl + r)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$), then

$$PF(S) = \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q)l + 1 \leq x \leq m - 1\}.$$

Case 1.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1} l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$), then

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1} l\}. \end{aligned}$$

Case 1.3. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ (i.e., $\epsilon_{m-1} = 1$), then

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Case 2. If $r - 1 = ql$ for some $q \in \mathbb{N}^*$.

Case 2.1. If $m - 1 = \alpha_{m-1}(kl + r)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$), then

$$PF(S) = \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 1\}.$$

Case 2.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$), then

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l\}. \end{aligned}$$

Case 2.3. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ (i.e., $\epsilon_{m-1} = 1$), then

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Case 3. If $r = 1$.

Case 3.1. If $m - 1 = \alpha_{m-1}(kl + 1)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$), then

$$PF(S) = \{w(x) - m; (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \leq x \leq m - 1\}.$$

Case 3.2. If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$), then

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l\}. \end{aligned}$$

Case 3.3. If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ (i.e., $\epsilon_{m-1} = 1$), then

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Proof. Case 1. If $r-1 = ql + t$ for some $q, t \in \mathbb{N}$ with $t < l$ and $t \neq 0$.

Case 1.1. If $m-1 = \alpha_{m-1}(kl+r)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$). We claim that if $(\alpha_{m-1}-1)(kl+r) + (k+q)l + 1 \leq x \leq m-1$, then x satisfies (3.5.1) for all $1 \leq y \leq m-1$. In fact,

- If $x = m-1$, then by using Lemma 3.5.5 x satisfies (3.5.1) for all $1 \leq y \leq m-1$.
- If $(\alpha_{m-1}-1)(kl+r) + (k+q)l + 1 \leq x \leq m-2 = (\alpha_{m-1}-1)(kl+r) + (k+q)l + t$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x+y \leq m-1$ and this implies that $\overline{x+y} = x+y$. In particular

$$(\alpha_{m-1}-1)(kl+r) + (k+q)l + 2 \leq x+y \leq m-1.$$

If $(\alpha_{m-1}-1)(kl+r) + (k+q)l + 2 \leq x+y \leq m-2$, then $m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + q + 1)$ and $m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1})$ if $x+y = m-1 = \alpha_{m-1}(kl+r)$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + q + 1). \quad (3.5.42)$$

Since $x+y \leq m-1$, it follows that $1 \leq y \leq t$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, we have $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + q + 1)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + q + 2). \quad (3.5.43)$$

By using (3.5.43) and (3.5.42), we get that x satisfies (3.5.1) for all y .

Therefore,

$$\{w(x) - m; (\alpha_{m-1}-1)(kl+r) + (k+q)l + 1 \leq x \leq m-1\} \subseteq PF(S). \quad (3.5.44)$$

By using (3.5.10) and (3.5.44), we get

$$PF(S) = \{w(x) - m; (\alpha_{m-1}-1)(kl+r) + (k+q)l + 1 \leq x \leq m-1\}.$$

Case 1.2. If $m-1 = \alpha_{m-1}(kl+r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$). We claim that if $\alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + 1 \leq x \leq m-1$, x satisfies (3.5.1) for all $1 \leq y \leq m-1$. Indeed, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x+y \leq m-1$ and this implies that $\overline{x+y} = x+y$. In particular, $\alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + 2 \leq x+y \leq m-1 = \alpha_{m-1}(kl+r) + \beta_{m-1}l$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_{m-1}). \quad (3.5.45)$$

Since $x+y \leq m-1$, it follows that $1 \leq y \leq l-1$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, we have $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + \beta_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_{m-1} + 1). \quad (3.5.46)$$

By using (3.5.46) and (3.5.45), we get that x satisfies (3.5.1) for all y . Consequently,

$$\{w(x) - m; \alpha_{m-1}(kl+r) + (\beta_{m-1}-1)l + 1 \leq x \leq m-1\} \subseteq PF(S). \quad (3.5.47)$$

Furthermore, if $x = (\alpha_{m-1}-1)(kl+r) + (k+q)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k+q)l + \epsilon_x t_x > \beta_{m-1}l$, then x satisfies (3.5.1) for all $1 \leq y \leq m-1$. In fact, write $y = \alpha_y(kl+r) + \beta_y l + \epsilon_y t_y$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x+y \leq m-1$ and this implies that $\overline{x+y} = x+y$. Since $x+y \leq m-1$ and $(k+q)l + \epsilon_x t_x > \beta_{m-1}l$, it follows that $y = \beta_y l + \epsilon_y t_y$. Thus, $m(k\alpha_y + \beta_y + \epsilon_y) = m(\beta_y + \epsilon_y)$. Since $x = (\alpha_{m-1}-1)(kl+r) + (k+q)l + \epsilon_x t_x$ such that $\epsilon_x = 1$, we get that $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + q + 1)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + q + 1 + \beta_y + \epsilon_y). \quad (3.5.48)$$

We have $1 \leq \epsilon_x t_x \leq t$ (as $\beta_x = k+q$ and $x = (\alpha_{m-1}-1)(kl+r) + (k+q)l + \epsilon_x t_x$). If $1 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x \leq t$, then $x+y = (\alpha_{m-1}-1)(kl+r) + (k+q)l + \beta_y l + \epsilon_y t_y + \epsilon_x t_x$ with $1 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x \leq l-1$ and if $t+1 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x$, then $x+y = \alpha_{m-1}(kl+r) + (\beta_y l + \epsilon_y t_y + \epsilon_x t_x - (t+1))$ with $0 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x - (t+1) \leq \beta_y l + \epsilon_y t_y$ (as $\epsilon_x t_x \leq t$). Since $\beta_y + \epsilon_y \geq 1$ (as $y \geq 1$), it follows that

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + q + \beta_y + \epsilon_y). \quad (3.5.49)$$

By using (3.5.48) and (3.5.49), we get that x satisfies (3.5.1) for $1 \leq y \leq m-1$. Therefore,

$$\begin{aligned} & \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ & \text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l\} \subseteq PF(S). \end{aligned} \quad (3.5.50)$$

By using (3.5.13), (3.5.50) and (3.5.47), we obtain

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ &\text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l\}. \end{aligned}$$

Case 1.3. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, i.e., $(\epsilon_{m-1} = 1)$. We claim that if $\alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. Indeed, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. In particular, $\alpha_{m-1}(kl + r) + \beta_{m-1}l + 2 \leq x + y \leq m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_{m-1} + 1). \quad (3.5.51)$$

Since $x + y \leq m - 1$, it follows that $1 \leq y \leq \epsilon_{m-1}t_{m-1} - 1$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, if $\alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + \beta_{m-1} + 1)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_{m-1} + 2). \quad (3.5.52)$$

By using (3.5.52) and (3.5.51), we get that x satisfies (3.5.1) for all y . Consequently,

$$\{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \subseteq PF(S). \quad (3.5.53)$$

Furthermore, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. In fact, write $y = \alpha_y(kl + r) + \beta_y l + \epsilon_y t_y$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. Since $x + y \leq m - 1$ and $(k + q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, it follows that $y = \beta_y l + \epsilon_y t_y$. Thus, $m(k\alpha_y + \beta_y + \epsilon_y) = m(\beta_y + \epsilon_y)$. Since $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x$ such that $\epsilon_x = 1$, we get that $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + q + 1)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + q + 1 + \beta_y + \epsilon_y). \quad (3.5.54)$$

We have and $1 \leq \epsilon_x t_x \leq t$ (as $\beta_x = k + q$ and $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x$). If $1 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x \leq t$, then $x + y = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \beta_y l + \epsilon_y t_y + \epsilon_x t_x$ with $1 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x \leq l - 1$ and if $t + 1 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x$, then $x + y = \alpha_{m-1}(kl + r) + (\beta_y l + \epsilon_y t_y + \epsilon_x t_y - (t + 1))$ with $0 \leq \beta_y l + \epsilon_y t_y + \epsilon_x t_x - (t + 1) \leq \beta_y l + \epsilon_y t_y$ (as $\epsilon_x t_x \leq t$). Since $\beta_y + \epsilon_y \geq 1$ (as $y \geq 1$), it follows that

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + q + \beta_y + \epsilon_y). \quad (3.5.55)$$

By using (3.5.54) and (3.5.55), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$. Therefore,

$$\begin{aligned} & \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ & \text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \subseteq PF(S). \end{aligned} \quad (3.5.56)$$

By using (3.5.16), (3.5.56) and (3.5.53), we get

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + \epsilon_x t_x \\ &\text{with } \epsilon_x = 1 \text{ and } (k + q)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Case 2. If $r - 1 = ql$ for some $q \in \mathbb{N}^*$.

Case 2.1. If $m - 1 = \alpha_{m-1}(kl + r)$, (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$). We claim that if $(\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 1$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. In fact,

- If $x = m - 1$, then by using Lemma 3.5.5, it follows that x satisfies (3.5.1) for all $1 \leq y \leq m - 1$.
- If $(\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 2 = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$. Indeed, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. In particular,

$$(\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 2 \leq x + y \leq m - 1.$$

If $(\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 2 \leq x + y \leq m - 2$, then $m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + q)$ and $m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1})$ if $x = m - 1$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + q). \quad (3.5.57)$$

Since $x + y \leq m - 1$, it follows that $1 \leq y \leq l$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, if $(\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 2$, then $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + q)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + q + 1). \quad (3.5.58)$$

By using (3.5.58) and (3.5.57), we get that x satisfies (3.5.1) for all y .

Therefore,

$$\{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 1\} \subseteq PF(S). \quad (3.5.59)$$

By using (3.5.19) and (3.5.59), we get

$$PF(S) = \{w(x) - m; (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + 1 \leq x \leq m - 1\}.$$

Case 2.2. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$ (i.e., $\beta_{m-1} > 0$, $\epsilon_{m-1} = 0$). We claim that if $\alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. Indeed, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. In particular, $\alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 2 \leq x + y \leq m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_{m-1}). \quad (3.5.60)$$

Since $x + y \leq m - 1$, it follows that $1 \leq y \leq l - 1$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, if $\alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l$, then $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + \beta_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_{m-1} + 1). \quad (3.5.61)$$

By using (3.5.61) and (3.5.60), we get that x satisfies (3.5.1) for all y . Consequently,

$$\{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \subseteq PF(S). \quad (3.5.62)$$

Furthermore, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l$ or $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$ such that $(k + q)l > \beta_{m-1}l$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. In fact, write $y = \alpha_y(kl + r) + \beta_y l + \epsilon_y t_y$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. Since $x + y \leq m - 1$, it follows that $y = \beta_y l + \epsilon_y t_y$ (as $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l$ or $(k + q)l > \beta_{m-1}l$). Thus, $m(k\alpha_y + \beta_y + \epsilon_y) = m(\beta_y + \epsilon_y)$. Since $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ with $\epsilon_x = 1$ or $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$, we get that $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + q)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + q + \beta_y + \epsilon_y). \quad (3.5.63)$$

- If $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$ such that $(k + q)l > \beta_{m-1}l$. We have

$$x + y = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + 1 + \beta_y l + \epsilon_y t_y - 1 = \alpha_{m-1}(kl + r) + \beta_y l + \epsilon_y t_y - 1$$

with $\beta_y l + \epsilon_y t_y - 1 \geq 0$ as $y = \beta_y l + \epsilon_y t_y \geq 1$. Then,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + \beta_y + \epsilon_y). \quad (3.5.64)$$

By using (3.5.63), (3.5.64) and $q \in \mathbb{N}^*$, we get that x satisfies (3.5.1) for all $1 \leq y \leq m - 1$.

- If $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l$.
 - If $\beta_y \geq 1$. We have

$$\begin{aligned} x + y &= (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x + \beta_y l + \epsilon_y t_y \\ &= \alpha_{m-1}(kl + r) + (\beta_y - 1)l + \epsilon_y t_y + \epsilon_x t_x - 1. \end{aligned}$$

Since $1 \leq \epsilon_x t_x \leq l - 1$ and $\epsilon_y t_y \leq l - 1$, it follows that $0 \leq \epsilon_y t_y + \epsilon_x t_x - 1 \leq 2l - 2$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + \beta_y + \epsilon_y).$$

- If $\beta_y = 0$. Then, $y = \epsilon_y t_y$ with $\epsilon_y = 1$ (as $y \geq 1$). Since $1 \leq \epsilon_x t_x \leq l - 1$ and $1 \leq \epsilon_y t_y \leq l - 1$, it follows that $2 \leq \epsilon_y t_y + \epsilon_x t_x \leq 2l - 2$. If $2 \leq \epsilon_y t_y + \epsilon_x t_x \leq l$, then $x + y = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x + \epsilon_y t_y$ with $\epsilon_x t_x + \epsilon_y t_y \leq l$ and if $\epsilon_y t_y + \epsilon_x t_x \geq l + 1$, then $x + y = \alpha_{m-1}(kl + r) + (\epsilon_x t_x + \epsilon_y t_y - (l + 1))$ with $0 \leq \epsilon_x t_x + \epsilon_y t_y - (l + 1) \leq l - 3$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + q - 1 + 1).$$

Since $q \in \mathbb{N}^*$ and $\beta_y + \epsilon_y \geq 1$ (as $y \geq 1$), it follows that

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + \beta_y + q - 1 + \epsilon_y). \quad (3.5.65)$$

By using (3.5.63) and (3.5.65), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$.

Hence, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l$, or $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$ such that $(k + q)l > \beta_{m-1}l$, then x satisfies (3.5.1) for $1 \leq y \leq m - 1$. Therefore,

$$\begin{aligned} &\{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \text{ with } \epsilon_x = 1 \\ &\text{and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l\} \cup \{w(x) - m; \\ &x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \text{ with } (k + q)l > \beta_{m-1}l\} \subseteq PF(S). \end{aligned} \quad (3.5.66)$$

By using (3.5.22), (3.5.66) and (3.5.62), we get

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l\} \\ &\cup \{m(k\alpha_x + \beta_x + \epsilon_x - 1) + x; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l\}. \end{aligned}$$

Case 2.3. If $m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, i.e., $(\epsilon_{m-1} = 1)$. We claim that if $\alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. Indeed, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. In particular, $\alpha_{m-1}(kl + r) + \beta_{m-1}l + 2 \leq x + y \leq m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_{m-1} + 1). \quad (3.5.67)$$

Since $x + y \leq m - 1$, it follows that $1 \leq y \leq \epsilon_{m-1}t_{m-1} - 1$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, if $\alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1 = \alpha_{m-1}(kl + r) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + \beta_{m-1} + 1)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_{m-1} + 2). \quad (3.5.68)$$

By using (3.5.68) and (3.5.67), we get that x satisfies (3.5.1) for all y . Consequently,

$$\{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \subseteq PF(S). \quad (3.5.69)$$

In addition, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ or $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$ such that $(k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. In fact, write $y = \alpha_y(kl + r) + \beta_y l + \epsilon_y t_y$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. Since $x + y \leq m - 1$, it follows that $y = \beta_y l + \epsilon_y t_y$ (as $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ or $(k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$). Thus, $m(k\alpha_y + \beta_y + \epsilon_y) = m(\beta_y + \epsilon_y)$. Since $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ with $\epsilon_x = 1$ or $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$, we get that $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + q)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + q + \beta_y + \epsilon_y). \quad (3.5.70)$$

- If $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$ such that $(k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. We have $x + y = (\alpha_{m-1} - 1)(kl + r) + (k + q)l + 1 + \beta_y l + \epsilon_y t_y - 1 = \alpha_{m-1}(kl + r) + \beta_y l + \epsilon_y t_y - 1$ with $\beta_y l + \epsilon_y t_y - 1 \geq 0$ as $y = \beta_y l + \epsilon_y t_y \geq 1$. Then,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + \beta_y + \epsilon_y). \quad (3.5.71)$$

By using (3.5.70), (3.5.71) and $q \in \mathbb{N}^*$, we get that x satisfies (3.5.1) for all $1 \leq y \leq m - 1$.

- If $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$.
 - If $\beta_y \geq 1$. We have

$$\begin{aligned} x + y &= (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x + \beta_y l + \epsilon_y t_y \\ &= \alpha_{m-1}(kl + r) + (\beta_y - 1)l + \epsilon_y t_y + \epsilon_x t_x - 1. \end{aligned}$$

Since $1 \leq \epsilon_x t_x \leq l - 1$ and $\epsilon_y t_y \leq l - 1$, it follows that $0 \leq \epsilon_y t_y + \epsilon_x t_x - 1 \leq 2l - 2$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + \beta_y + \epsilon_y).$$

- If $\beta_y = 0$. Then, $y = \epsilon_y t_y$ with $\epsilon_y = 1$ (as $y \geq 1$). Since $1 \leq \epsilon_x t_x \leq l - 1$ and $1 \leq \epsilon_y t_y \leq l - 1$, it follows that $2 \leq \epsilon_y t_y + \epsilon_x t_x \leq 2l - 2$. If $2 \leq \epsilon_y t_y + \epsilon_x t_x \leq l$, then $x + y = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x + \epsilon_y t_y$ with $\epsilon_x t_x + \epsilon_y t_y \leq l$ and if $\epsilon_y t_y + \epsilon_x t_x \geq l + 1$, then $x + y = \alpha_{m-1}(kl + r) + (\epsilon_x t_x + \epsilon_y t_y - (l + 1))$ with $0 \leq \epsilon_x t_x + \epsilon_y t_y - (l + 1) \leq l - 3$. Since $q \in \mathbb{N}^*$, then

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + q - 1 + 1).$$

Since $q \in \mathbb{N}^*$ and $\beta_y + \epsilon_y \geq 1$ (as $y \geq 1$), it follows that

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) \leq m(k\alpha_{m-1} + \beta_y + q - 1 + \epsilon_y). \quad (3.5.72)$$

By using (3.5.70) and (3.5.72), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$.

Hence, if $x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x$ such that $\epsilon_x = 1$ and $(k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, or $x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l$ such that $(k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then x satisfies (3.5.1) for $1 \leq y \leq m - 1$. Therefore,

$$\begin{aligned} &\{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \subseteq PF(S). \end{aligned} \quad (3.5.73)$$

By using (3.5.25), (3.5.73) and (3.5.69), we get

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + r) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q - 1)l + \epsilon_x t_x \\ &\quad \text{with } \epsilon_x = 1 \text{ and } (k + q - 1)l + \epsilon_x t_x > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + r) + (k + q)l \\ &\quad \text{with } (k + q)l > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Case 3. If $r = 1$.

Case 3.1. If $m-1 = \alpha_{m-1}(kl+1)$ (i.e., $\beta_{m-1} = \epsilon_{m-1} = 0$). We claim that if $(\alpha_{m-1}-1)(kl+1) + (k-1)l+1 \leq x \leq m-1$, then x satisfies (3.5.1) for all $1 \leq y \leq m-1$. In fact,

- If $x = m-1$, then by using Lemma 3.5.5 x satisfies (3.5.1) for $1 \leq y \leq m-1$.
- If $(\alpha_{m-1}-1)(kl+1) + (k-1)l+1 \leq x \leq m-2 = (\alpha_{m-1}-1)(kl+1) + kl$. Indeed, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x+y \leq m-1$ and this implies that $\overline{x+y} = x+y$. In particular, $(\alpha_{m-1}-1)(kl+1) + (k-1)l+2 \leq x+y \leq m-1 = \alpha_{m-1}(kl+1)$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1}). \quad (3.5.74)$$

Since $x+y \leq m-1$, it follows that $1 \leq y \leq l$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + 1). \quad (3.5.75)$$

By using (3.5.75) and (3.5.74), we get that x satisfies (3.5.1) for all y .

Consequently,

$$\{w(x) - m; (\alpha_{m-1}-1)(kl+1) + (k-1)l+1 \leq x \leq m-1\} \subseteq PF(S). \quad (3.5.76)$$

By using (3.5.28) and (3.5.76), we get

$$PF(S) = \{w(x) - m; (\alpha_{m-1}-1)(kl+1) + (k-1)l+1 \leq x \leq m-1\}.$$

Case 3.2. If $m-1 = \alpha_{m-1}(kl+1) + \beta_{m-1}l$ (i.e., $\epsilon_{m-1} = 0$). We claim that if $\alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l+1 \leq x \leq m-1$, then x satisfies (3.5.1) for all $1 \leq y \leq m-1$. In fact, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x+y \leq m-1$ and this implies that $\overline{x+y} = x+y$. In particular, $\alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l+2 \leq x+y \leq m-1 = \alpha_{m-1}(kl+1) + \beta_{m-1}l$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_{m-1}). \quad (3.5.77)$$

Since $x+y \leq m-1$, it follows that $1 \leq y \leq l-1$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, if $\alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l+1 \leq x \leq m-1 = \alpha_{m-1}(kl+1) + \beta_{m-1}l$, then $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + \beta_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_{m-1} + 1). \quad (3.5.78)$$

By using (3.5.78) and (3.5.77), we get that x satisfies (3.5.1) for all y . Consequently,

$$\{w(x) - m; \alpha_{m-1}(kl+1) + (\beta_{m-1}-1)l+1 \leq x \leq m-1\} \subseteq PF(S). \quad (3.5.79)$$

Furthermore, if $x = (\alpha_{m-1}-1)(kl+1) + (k-1)l+1$ such that $(k-1)l+1 > \beta_{m-1}l$ or $x = (\alpha_{m-1}-1)(kl+1) + kl$ such that $l = 1$ and $kl > \beta_{m-1}l$, then x satisfies (3.5.1) for all $1 \leq y \leq m-1$. In fact, write $y = \alpha_y(kl+1) + \beta_y l + \epsilon_y t_y$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x+y \leq m-1$ and this implies that $\overline{x+y} = x+y$. Since $x+y \leq m-1$, it follows that $y = \beta_y l + \epsilon_y t_y$ (as $(k-1)l+1 > \beta_{m-1}l$ or $kl > \beta_{m-1}l$). Thus, $m(k\alpha_y + \beta_y + \epsilon_y) = m(\beta_y + \epsilon_y)$. Since $x = (\alpha_{m-1}-1)(kl+1) + (k-1)l+1$ or $x = (\alpha_{m-1}-1)(kl+1) + kl$, we get that $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_y + \epsilon_y). \quad (3.5.80)$$

- If $x = (\alpha_{m-1}-1)(kl+1) + kl$ such that $l = 1$ and $kl > \beta_{m-1}l$. We have $\epsilon_y t_y = 0$ as $l = 1$, thus $y = \beta_y l$ with $\beta_y > 0$ (as $y \geq 1$). In addition, $x+y = (\alpha_{m-1}-1)(kl+1) + kl+1 + \beta_y l - 1 = \alpha_{m-1}(kl+1) + (\beta_y-1)l$ as $l = 1$. Then,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_y - 1). \quad (3.5.81)$$

By using (3.5.80), (3.5.81), we get that x satisfies (3.5.1) for all $1 \leq y \leq m-1$.

- If $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1$ such that $(k - 1)l + 1 > \beta_{m-1}l$.
 - If $\beta_y \geq 1$. We have

$$\begin{aligned} x + y &= (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 + \beta_y l + \epsilon_y t_y \\ &= \alpha_{m-1}(kl + 1) + (\beta_y - 1)l + \epsilon_y t_y. \end{aligned}$$

Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_y - 1 + \epsilon_y). \quad (3.5.82)$$

By using (3.5.80) and (3.5.82), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$.

- If $\beta_y = 0$. Then, $y = \epsilon_y t_y$ with $\epsilon_y = 1$ (as $y \geq 1$). We have $x + y = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 + \epsilon_y t_y$ with $1 + \epsilon_y t_y \leq l$ (as $\epsilon_y t_y < l$). Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1}). \quad (3.5.83)$$

By using (3.5.80) and (3.5.83), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$.

Hence, if $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1$ such that $(k - 1)l + 1 > \beta_{m-1}l$, or $x = (\alpha_{m-1} - 1)(kl + 1) + kl$ such that $l = 1$ and $kl > \beta_{m-1}l$, then x satisfies (3.5.1) for $1 \leq y \leq m - 1$. Thus,

$$\begin{aligned} &\{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l\} \cup \\ &\{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l\} \subseteq PF(S). \end{aligned} \quad (3.5.84)$$

By using (3.5.33), (3.5.79) and (3.5.84), we get

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + 1) + (\beta_{m-1} - 1)l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l\}. \end{aligned}$$

Case 3.3. If $m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, i.e., $(\epsilon_{m-1} = 1)$. We claim that if $\alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. In fact, since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. In particular, $\alpha_{m-1}(kl + 1) + \beta_{m-1}l + 2 \leq x + y \leq m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_{m-1} + 1). \quad (3.5.85)$$

Since $x + y \leq m - 1$, it follows that $1 \leq y \leq \epsilon_{m-1}t_{m-1} - 1$. Then, $m(k\alpha_y + \beta_y + \epsilon_y) = m$. In addition, if $\alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1 = \alpha_{m-1}(kl + 1) + \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1} + \beta_{m-1} + 1)$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_{m-1} + 2). \quad (3.5.86)$$

By using (3.5.86) and (3.5.85), we get that x satisfies (3.5.1) for all y . Consequently,

$$\{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \subseteq PF(S). \quad (3.5.87)$$

Furthermore, if $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1$ such that $(k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ or $x = (\alpha_{m-1} - 1)(kl + 1) + kl$ such that $l = 1$ and $kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. In fact, write $y = \alpha_y(kl + 1) + \beta_y l + \epsilon_y t_y$. Since $\beta_x + \epsilon_x > 0$, then from Lemma 3.5.6 we may assume that $x + y \leq m - 1$ and this implies that $\overline{x + y} = x + y$. Since $x + y \leq m - 1$, it follows that $y = \beta_y l + \epsilon_y t_y$ (as $(k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ or $kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$). Thus, $m(k\alpha_y + \beta_y + \epsilon_y) = m(\beta_y + \epsilon_y)$. Since $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1$ or $x = (\alpha_{m-1} - 1)(kl + 1) + kl$, we get that $m(k\alpha_x + \beta_x + \epsilon_x) = m(k\alpha_{m-1})$. Consequently,

$$m(k(\alpha_x + \alpha_y) + \beta_x + \beta_y + \epsilon_x + \epsilon_y) = m(k\alpha_{m-1} + \beta_y + \epsilon_y). \quad (3.5.88)$$

- If $x = (\alpha_{m-1} - 1)(kl + 1) + kl$ such that $l = 1$ and $kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$. Since $l = 1$, it follows that $\epsilon_y t_y = 0$, thus $y = \beta_y l$ with $\beta_y > 0$ (as $y \geq 1$). In addition, $x + y = (\alpha_{m-1} - 1)(kl + 1) + kl + 1 + \beta_y l - 1 = \alpha_{m-1}(kl + 1) + (\beta_y - 1)l$ as $l = 1$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_y - 1). \quad (3.5.89)$$

By using (3.5.88) and (3.5.89), we get that x satisfies (3.5.1) for all $1 \leq y \leq m - 1$.

- If $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1$ such that $(k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$.
 - If $\beta_y \geq 1$. We have $x + y = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 + \beta_y l + \epsilon_y t_y = \alpha_{m-1}(kl + 1) + (\beta_y - 1)l + \epsilon_y t_y$. Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1} + \beta_y - 1 + \epsilon_y). \quad (3.5.90)$$

By using (3.5.88) and (3.5.90), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$.

- If $\beta_y = 0$. Then, $y = \epsilon_y t_y$ with $\epsilon_y = 1$ (as $y \geq 1$). We have $x + y = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 + \epsilon_y t_y$ with $1 + \epsilon_y t_y \leq l$ (as $\epsilon_y t_y < l$). Hence,

$$m(k\alpha_{x+y} + \beta_{x+y} + \epsilon_{x+y}) = m(k\alpha_{m-1}). \quad (3.5.91)$$

By using (3.5.88) and (3.5.91), we get that x satisfies (3.5.1) for $1 \leq y \leq m - 1$.

Consequently, if $x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1$ such that $(k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$ or $x = (\alpha_{m-1} - 1)(kl + 1) + kl$ such that $l = 1$ and $kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}$, then x satisfies (3.5.1) for all $1 \leq y \leq m - 1$. Thus,

$$\begin{aligned} & \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ & \text{with } (k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \cup \\ & \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ & \text{with } l = 1 \text{ and } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \subseteq PF(S). \end{aligned} \quad (3.5.92)$$

By using (3.5.38), (3.5.87) and (3.5.92), we get

$$\begin{aligned} PF(S) &= \{w(x) - m; \alpha_{m-1}(kl + 1) + \beta_{m-1}l + 1 \leq x \leq m - 1\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + (k - 1)l + 1 \\ &\quad \text{with } (k - 1)l + 1 > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\} \\ &\cup \{w(x) - m; x = (\alpha_{m-1} - 1)(kl + 1) + kl \\ &\quad \text{with } l = 1 \text{ and } kl > \beta_{m-1}l + \epsilon_{m-1}t_{m-1}\}. \end{aligned}$$

Thus, the proof is complete. ■

Example 3.5.8. Consider the following numerical semigroups.

1. $S = \langle 12, 13, 14, 15, 35 \rangle$. By using GAP [8], we get that

$$PF(S) = \{46, 23\} = \{w(10) - 12, w(11) - 12\}.$$

Note that $m = 12$, $k = 2$, $l = 3$ and $r = 5$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 0$, $\epsilon_{m-1}t_{m-1} = 0$, $q = 1$, $t = 1$). S verifies the formula in Theorem 3.5.7.

2. $S = \langle 18, 19, 20, 21, 47 \rangle$. By using GAP [8], we get that

$$PF(S) = \{64, 69, 70, 71\} = \{w(10) - 18, w(15) - 18, w(16) - 18, w(17) - 18\}.$$

Note that $m = 18$, $k = 2$, $l = 3$ and $r = 5$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 2$, $\epsilon_{m-1}t_{m-1} = 0$, $q = 1$, $t = 1$). S verifies the formula in Theorem 3.5.7.

3. $S = \langle 16, 17, 18, 19, 43 \rangle$. By using GAP [8], we get that

$$PF(S) = \{58, 63\} = \{w(10) - 16, w(15) - 16\}.$$

Note that $m = 16$, $k = 2$, $l = 3$ and $r = 5$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 1$, $\epsilon_{m-1}t_{m-1} = 1$, $q = 1$, $t = 1$). S verifies the formula in Theorem 3.5.7.

4. $S = \langle 11, 12, 13, 14, 32 \rangle$. By using GAP [8], we get that

$$PF(S) = \{29, 30, 31, 21\} = \{w(7) - 11, w(8) - 11, w(9) - 11, w(10) - 11\}.$$

Note that $m = 11$, $k = 2$, $l = 3$ and $r = 4$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 0$, $\epsilon_{m-1}t_{m-1} = 0$, $q = 1$, $t = 0$). S verifies the formula in Theorem 3.5.7.

5. $S = \langle 17, 18, 19, 20, 44 \rangle$. By using GAP [8], we get that

$$\begin{aligned} PF(S) &= \{41, 42, 43, 65, 66, 67\} \\ &= \{w(7) - 17, w(8) - 17, w(9) - 17, w(14) - 17, w(15) - 17, w(16) - 17\}. \end{aligned}$$

Note that $m = 17$, $k = 2$, $l = 3$ and $r = 4$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 2$, $\epsilon_{m-1}t_{m-1} = 0$, $q = 1$, $t = 0$). S verifies the formula in Theorem 3.5.7.

6. $S = \langle 15, 16, 17, 18, 40 \rangle$. By using GAP [8], we get that

$$PF(S) = \{37, 38, 39, 59\} = \{w(7) - 15, w(8) - 15, w(9) - 15, w(14) - 15\}.$$

Note that $m = 15$, $k = 2$, $l = 3$ and $r = 4$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 1$, $\epsilon_{m-1}t_{m-1} = 1$, $q = 1$, $t = 0$). S verifies the formula in Theorem 3.5.7.

7. $S = \langle 8, 9, 10, 11, 23 \rangle$. By using GAP [8], we get that

$$PF(S) = \{12, 13, 14, 15\} = \{w(4) - 8, w(5) - 8, w(6) - 8, w(7) - 8\}.$$

Note that $m = 8$, $k = 2$, $l = 3$ and $r = 1$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 0$, $\epsilon_{m-1}t_{m-1} = 0$). S verifies the formula in Theorem 3.5.7.

8. $S = \langle 14, 15, 16, 17, 35 \rangle$. By using GAP [8], we get that

$$PF(S) = \{53, 54, 55\} = \{w(11) - 14, w(12) - 14, w(13) - 14\}.$$

Note that $m = 14$, $k = 2$, $l = 3$ and $r = 1$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 2$, $\epsilon_{m-1}t_{m-1} = 0$). S verifies the formula in Theorem 3.5.7.

9. $S = \langle 12, 13, 14, 15, 31 \rangle$. By using GAP [8], we get that

$$PF(S) = \{47\} = \{w(11) - 12\}.$$

Note that $m = 12$, $k = 2$, $l = 3$ and $r = 1$ ($\alpha_{m-1} = 1$, $\beta_{m-1} = 1$, $\epsilon_{m-1}t_{m-1} = 1$). S verifies the formula in Theorem 3.5.7. ■

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Thèse de Doctorat

Mariam DHAYNI

Problèmes dans la théorie des semigroupes numériques

Problems in numerical semigroups

Résumé

Cette thèse est composée de deux parties. Nous étudions dans la première la conjecture de Wilf pour les semi-groupes numériques et la résolvons dans certains cas. Dans la seconde nous considérons une classe de semi-groupes presque arithmétiques et donnons pour ces semi-groupes des formules explicites pour la base d'Apéry, le nombre de Frobenius, et les nombres de pseudo-Frobenius. Nous caractérisons aussi ceux qui sont symétriques (resp. pseudo-symétriques).

Mots clés

Semigroupes numériques, conjecture de Wilf, semi-groupes presque arithmétiques, nombre de Frobenius, Problème des pièces de monnaie.

Abstract

The thesis is made up of two parts. We study in the first part Wilf's conjecture for numerical semigroups. We give an equivalent form of Wilf's conjecture in terms of the Apéry set, embedding dimension and multiplicity of a numerical semigroup. We also give an affirmative answer for the conjecture in certain cases. In the second part, we consider a class of almost arithmetic numerical semigroups and give for this class of semigroups explicit formulas for the Apéry set, the Frobenius number, the genus and the pseudo-Frobenius numbers. We also characterize the symmetric (resp. pseudo-symmetric) numerical semigroups for this class of numerical semigroups.

Key Words

Numerical semigroups, Wilf's conjecture, almost arithmetic numerical semigroups Frobenius number, money changing problem.