



# XVA Analysis, Risk Measures and Applications to Centrally Cleared Trading

Yannick Armenti

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# THÈSE DE DOCTORAT

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Chambres de compensation : analyse XVA, mesures de risque et  
applications

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# Introduction

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La crise des subprimes de 2007 aux Etats-Unis a occasionné un bouleversement dans le fonctionnement des marchés financiers. Cette crise a souligné l'importance de réguler l'ensemble des marchés de gré à gré pour maintenir un équilibre financier mondial. En effet, bien que les dérivés OTC ne soient pas à l'origine de cette crise, leur utilisation massive a permis cette dernière de se propager à travers les différents acteurs.

En parallèle, les membres des chambres de compensation ont été moins impactés. C'est pour cette raison qu'en Septembre 2009 le G20 a conclu que l'ensemble des dérivés OTC devaient, à l'avenir, être traités à travers les chambres de compensation afin de limiter l'ensemble des risques (opérationnel, de contrepartie, de liquidité, et évidemment systémique). C'est à travers la loi Dodd-Frank (Dodd-Frank Wall Street Reform and Consumer Protection Act) aux Etats-Unis et du texte EMIR (European Market Infrastructure Regulation) en Europe que ces réglementations virent le jour à partir de 2010.

Les chambres de compensation (CCPs) sont des institutions financières qui, lors d'une transaction, se placent entre l'acheteur et le vendeur. Elles deviennent la contrepartie "acheteur" pour chaque vendeur et "vendeur" pour chaque acheteur.

Aujourd'hui, plusieurs problèmes se posent. Les CCPs diminuent le risque de contrepartie en devenant l'intermédiaire de chaque acteur (appelé membre ou *clearing member*) lors d'une transaction. Mais le réseau devenant totalement centralisé, le risque le devient également. S'il est mal géré, une bulle systémique peut alors se créer. Pour empêcher la formation de cette dernière, la CCP impose à chacun de ses membres un niveau de collatéral suffisant pour couvrir l'intégralité des pertes de leur défaut potentiel ainsi qu'un fonds de garantie mutualisé entre les membres.

Cette thèse traite de diverses problématiques ayant trait à la gestion du collatéral dans le contexte du trading centralisé au travers des chambres de compensation : coûts comparés pour une banque de trader de manière bilatérale ou au travers d'une chambre de compensation et méthodologies de calcul des marges initiales et du fonds de garantie.

Un travail préliminaire consiste en une réflexion sur les notions de coûts de capital et coût de financement pour une banque, en les replaçant dans un cadre Black-Scholes élémentaire où le payoff d'un call standard tient lieu d'exposition au défaut d'une contrepartie. Cependant, on suppose que la banque ne couvre qu'imparfaitement ce call et doit faire face à un coût de financement supérieur au taux sans risque, d'où des corrections de pricing de type FVA (*funding valuation adjustment*) et KVA (*capital valuation adjustment*) par rapport au prix Black-Scholes. Ces corrections sont obtenues comme solutions d'EDP qui, dans le cas où le spread de crédit de la banque et l'erreur de couverture tendent vers zéro, convergent vers zéro. Ce cadre simplifié nous permet également d'étudier le risque de modèle, un enjeu important s'agissant des coûts XVA, que nous appréhendons dans des formalismes de modèle à volatilité incertaine puis de transport martingale optimal.

Nous nous intéressons ensuite aux coûts auxquels une banque doit faire face lorsqu'elle négocie à travers une CCP. À cette fin, nous transposons au trading centralisé le cadre d'analyse XVA du trading bilatéral. Sur la base d'une analyse dynamique des flux financiers échangés, le coût

total pour un membre de trader au travers d'une CCP est ainsi décomposé en une CVA correspondant au coût pour le membre de renflouer sa contribution au fonds de garantie en cas de pertes consécutives du fait de défauts d'autres membres, une MVA correspondant au coût de financement de sa marge initiale et une KVA correspondant au coût du capital mis à risque par le membre sous la forme de sa contribution au fonds de garantie. Ces différents coûts peuvent alors être comparés avec les coûts analogues dans le cadre du trading bilatéral, ce que nous illustrons par une étude numérique.

L'analyse de coûts précédente est menée sur la base des spécifications réglementaires actuelles concernant le calcul des marges initiales et des contributions au fonds de garantie des membres. Nous remettons alors en cause ces hypothèses, nous intéressant à des alternatives dans lesquelles les membres auraient recours pour leur marge initiale à une tierce partie, qui posterait le collatéral à la place du membre en échange d'une rémunération. Nous considérons également un mode de calcul du fonds de garantie et de son allocation qui prendraient en compte le risque de la chambre au sens des fluctuations de son profit-and-loss (P&L) sur l'année suivante, tel qu'il résulte de la combinaison des risques de marché mais aussi des risques de défaut des membres (par opposition aux modes de calculs actuels de type Cover 2, purement basés sur le risque de marché des membres).

Enfin, nous proposons l'application de méthodologies de type mesures de risque multivariées pour le calcul des marges et/ou du fonds de garantie des membres. Nous introduisons une notion de mesures de risque systémiques au sens où elles présentent une sensibilité non seulement aux risques marginaux des composantes d'un système financier (par exemple, mais non nécessairement, les positions des membres d'une CCP), mais aussi à leur dépendance. Plus spécifiquement, nous étendons à un cadre multivarié la notion de *shortfall risk*. Dans une suite empirique nous mettons en œuvre de telles mesures de risque sur la base de données réelles de chambre de compensation fournies par LCH.

En appendice, nous reprenons certains des textes réglementaires inscrits dans EMIR, relatifs aux CCPs. Le lecteur pourra ainsi se référer à ces articles concernant les calculs de marges initiales, de fonds de garantie, de calibration des paramètres ou encore des procédures de back-tests et stress-tests.

Dans la suite de cette introduction, nous présentons les principaux résultats exposés dans chaque chapitre.

## 0.1 Les équations de Black–Scholes en marchés incomplets

Le Chapitre 1 de cette thèse consiste à introduire les notions de coût de financement (FVA) et coût du capital économique (KVA) pour une banque dans le modèle de Black–Scholes. Ce cadre, bien que simpliste, nous permet d'appréhender les problèmes relatifs aux coûts additionnels des banques.

Dans les marchés incomplets, les équations de Black–Scholes ne suffisent plus pour valoriser des portefeuilles composés de dérivés européens. Le modèle doit alors être complété par la valorisation des imperfections. Nous nous intéressons à un portefeuille composé de calls, tenant lieu d'exposition au défaut (pour la suite de nos travaux). Afin d'introduire l'imperfection du marché, nous supposons que la banque ne se couvre que partiellement.

Ce faisant, la valorisation du portefeuille fait intervenir son prix Black–Scholes ainsi que deux coûts additionnels : un coût de financement et un coût de capital économique. Le premier est un coût semi-linéaire dû au financement du portefeuille au-delà du taux sans risque, tandis que le second correspond à la rémunération des actionnaires (à un certain taux de dividende  $h$ ) du risque résiduel (couverture imparfaite).

On se place sur un espace probabilisé  $(\Omega, \mathbb{G}, \mathbb{Q})$  où  $\mathbb{Q}$  désigne une probabilité risque neutre.

On considère un portefeuille composé de  $n$  différents calls de caractéristiques  $(\omega_i, T_i, K_i)$  où  $T_i$  (resp.  $K_i$ ) représente la maturité (resp. le strike) de l'option  $i$  sur un même sous-jacent  $S$  et  $\omega_i$  la position de la banque sur l'option  $i$ . Du fait des imperfections de couverture, un montant de capital économique  $EC = EC_t(\varrho)$  doit être réservé par la banque et calculé comme une mesure de risque conditionnelle de la perte potentielle  $\varrho$ . En suivant les arguments de [Albanese and Crépey \(2017, Section 5.3\)](#), le coût du capital économique (KVA) pour rémunérer les actionnaires à un taux constant  $h$  doit être formulé de la manière suivante :

$$KVA_t(\varrho) = h \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T e^{-\int_t^s (r_u + h) du} EC_s(\varrho) ds \right) \quad (0.1.1)$$

où  $\mathbb{E}_t^{\mathbb{Q}}$  désigne l'espérance conditionnelle sous  $\mathbb{Q}$  sachant  $\mathcal{G}_t$  et le processus  $r$  représente le taux d'intérêt sans risque.

On suppose également que la banque peut investir au taux  $r$  mais qu'elle emprunte au taux  $(r + \lambda) > r$ .

On définit par  $\Theta$  la valeur de marché du portefeuille de la banque incluant la FVA et nous explicitons la dynamique du processus de perte  $\varrho$  :

$$\begin{aligned} d\varrho_t = & -d\Theta_t - \sum_{i=1}^n \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt) \\ & + \left( \lambda_t (\Theta_t - EC_t(\varrho))^+ + r_t \Theta_t \right) dt + \eta_t d\mathcal{M}_t \end{aligned} \quad (0.1.2)$$

où  $\eta$  représente le vecteur prévisible de positions sur les instruments de couverture  $\mathcal{M}$ , sachant que la couverture peut être imparfaite. On découple cette équation de celle de la KVA qui est calculé dans un second temps. En supposant l'hypothèse d'absence d'opportunité d'arbitrage,  $\varrho$  est une  $\mathbb{Q}$ -martingale locale et le processus de prix  $\Theta$  est solution de l'EDSR suivante pour  $t \in [0, T]$  (et en notant  $\beta_t = e^{-\int_0^t r_s ds}$ ) :

$$\begin{aligned} \Theta_t = & \underbrace{\mathbb{E}_t^{\mathbb{Q}} \left( \sum_{t < T_i} \beta_t^{-1} \beta_{T_i} \omega_i (S_{T_i} - K_i)^+ \right)}_{\Theta_t^0} \\ & - \underbrace{\mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \beta_t^{-1} \beta_s \lambda_s (\Theta_s - EC_s(\varrho))^+ ds \right)}_{FVA_t} \end{aligned} \quad (0.1.3)$$

On se place alors dans le modèle Black–Scholes dans lequel le sous-jacent  $S$  possède un drift constant  $r$ , un taux de dividendes constant  $q$  et une volatilité constante  $\sigma$ . Notons que dans ce cadre  $d\mathcal{M}_t = dS_t - (r - q)S_t dt = S_t \sigma dW_t$ . Nous spécifions également un cadre markovien pour le capital économique modélisé par :

$$EC_t(\varrho) = f \times \sqrt{\frac{d\langle \varrho \rangle_t}{dt}} = f \sigma S_t |\partial_S u(t, S_t) - \eta(t, S_t)| \quad (0.1.4)$$

dans le cas où  $\Theta_t := u(t, S_t)$ ,  $\eta := \eta(t, S_t)$  et  $f$  est un coefficient multiplicatif représentant un certain niveau de quantile. En spécifiant ensuite  $\eta(t, S_t) := (1 - \alpha) \partial_S u(t, S_t)$ , avec  $\alpha \in [0, 1]$  un paramètre d'erreur sur la couverture, nous obtenons que la solution  $u$  est définie par une suite de fonctions  $(u_i)_{1 \leq i \leq n}$  sur chaque intervalle  $[T_{i-1}, T_i] \times \mathbb{R}_*^+$ . Ces dernières sont elles-mêmes uniques solutions des EDP en cascades suivantes, pour  $i$  décroissant de  $n$  vers 1 (en spécifiant que  $u_{n+1} = 0$  et  $T_0 = 0$ ) :

$$\begin{cases} u_i(T_i, S) = u_{i+1}(T_i, S) + \omega_i (S - K_i)^+ & \text{sur } \mathbb{R}_*^+ \\ \partial_t u_i + \mathcal{A}_S^{bs} u_i - \lambda(u_i - \alpha f \sigma S |\partial_S u_i|)^+ - r u_i = 0 & \text{sur } [T_{i-1}, T_i[ \times \mathbb{R}_*^+ \end{cases} \quad (0.1.5)$$

avec  $\mathcal{A}_S^{bs} = (r - q)S\partial_S + \frac{\sigma^2}{2}S^2\partial_S^2$  le générateur Black-Scholes risque neutre. Les processus de FVA  $:= \Theta^0 - \Theta$  et de KVA sont alors définis par les équations suivantes :

$$\begin{aligned} \text{FVA}_t &= \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T e^{-r(s-t)} \lambda(s, S_s) (u(s, S_s) - \alpha f \sigma S_s |\partial_S u(s, S_s)|)^+ ds \right) \\ &= v(t, S_t) = u_{bs}(t, S_t) - u(t, S_t) \end{aligned} \quad (0.1.6)$$

$$\text{KVA}_t = h \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T e^{-(r+h)(s-t)} \alpha f \sigma S_s |\partial_S u(s, S_s)| ds \right) = w(t, S_t) \quad (0.1.7)$$

où  $u_{bs}$  est la valeur Black-Scholes du portefeuille. Les fonctions  $v$  et  $w$  sont définies comme solutions de :

$$\begin{cases} v(T, S) = w(T, S) = 0 \text{ sur } \mathbb{R}_*^+ \text{ et sur } [0, T[ \times \mathbb{R}_*^+ : \\ \partial_t v + \mathcal{A}_S^{bs} v + \lambda(u_{bs} - v - \alpha f \sigma S |\Delta_{bs} - \partial_S v|)^+ - rv = 0 \\ \partial_t w + \mathcal{A}_S^{bs} w + h \alpha f \sigma S |\Delta_{bs} - \partial_S v| - (r + h)w = 0 \end{cases} \quad (0.1.8)$$

avec  $\Delta_{bs} = \partial_S u_{bs}$ .

Dans un second temps, nous présentons les résultats obtenus en introduisant le modèle à volatilité incertaine de [Avellaneda, Levy, and Parás \(1995\)](#). On suppose alors que  $d\mathcal{M}_t = S_t \sigma_t dW_t$  avec  $\sigma_t \in [\underline{\sigma}, \bar{\sigma}]$  pour tout  $t$ . On montre dès-lors que  $u$  définie initialement par (0.1.5) devient la solution des EDP suivantes :

$$\begin{cases} u_i(T_i, S) = u_{i+1}(T_i, S) + \omega_i(S - K_i)^+ \text{ sur } \mathbb{R}_*^+ \text{ et sur } [T_{i-1}, T_i[ \times \mathbb{R}_*^+ : \\ \partial_t u_i + \inf_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} [\mathcal{A}_S^{bs} u_i - \lambda(u_i - \alpha f \sigma S |\partial_S u_i|)^+] - ru_i = 0 \end{cases} \quad (0.1.9)$$

tandis que la KVA définie par  $w$  devient la solution de :

$$\begin{cases} w(T, S) = 0 \text{ sur } \mathbb{R}_*^+ \text{ et sur } [T_{i-1}, T_i[ \times \mathbb{R}_*^+ : \\ \partial_t w + \sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} [\mathcal{A}_S^{bs} w + \alpha h f \sigma S |\partial_S u|] - (r + h)w = 0 \end{cases} \quad (0.1.10)$$

Cependant, puisque les calls sont des produits liquides, leurs prix  $\mathbb{E}^{\mathbb{Q}}[\beta_{T_i}(S_{T_i} - K)^+]$  ne doivent pas être pris en compte dans le risque de modèle, mais calibrés au marché. Au-delà du modèle de volatilité incertaine, nous souhaitons ajouter les contraintes de distribution terminale sur notre sous-jacent. On considère la KVA (supposant que  $\lambda = 0$ ) correspondant à un unique call de paramètres  $(T, K)$ . On utilise alors les résultats de [Tan and Touzi \(2013\)](#) sur les problèmes de transport martingale optimal : on cherche à maximiser  $\text{KVA}_0$  dans le modèle à volatilité incertaine sous contrainte de distribution terminale  $\mu_T$ . On définit alors la KVA à la date 0 comme :

$$\text{KVA}_0 := h \sup_{\mathbb{Q} \in \mathcal{Q}(\mu_0, \mu_T)} \mathbb{E}^{\mathbb{Q}} \left( \int_0^T e^{-(r+h)t} f \alpha \sigma_t S_t |\partial_S u(t, S_t)| dt \right) \quad (0.1.11)$$

avec  $\mathcal{Q}$  l'ensemble des probabilités associées aux processus de volatilité incertaine,  $\mathcal{Q}(\mu_0) = \{\mathbb{Q} \in \mathcal{Q} : \mathbb{Q} \circ S_0^{-1} = \mu_0\}$  et  $\mathcal{Q}(\mu_0, \mu_T) = \{\mathbb{Q} \in \mathcal{Q}(\mu_0) : \mathbb{Q} \circ S_T^{-1} = \mu_T\}$ .

Les résultats numériques montrent qu'à moins que la couverture soit très bonne, la KVA domine la FVA. Ce résultat est logique sachant que le capital économique n'impacte qu'indirectement la FVA alors qu'elle est le cœur même de la KVA. Par ailleurs, ils montrent que dans le cadre du modèle à volatilité incertaine, la FVA ne change que peu alors que  $u$  et la KVA sont énormément impactées par cette incertitude. Enfin, dans le cadre de transport martingale optimal, le risque de modèle de la KVA n'est pas grandement impacté par la contrainte de distribution terminale. Cela est dû à l'utilisation de  $u$  définie par (0.1.9) faute de théorie pour l'évaluation de  $u$  à des dates futures dans le modèle à volatilité incertaine contraint par la distribution terminale de  $S_T$ .

## 0.2 Analyse XVA en trading centralisé

Dans le chapitre 2, nous nous intéressons à la valorisation des portefeuilles des membres de chambres de compensation. Nous transposons le cadre XVA du monde bilatéral au monde centralisé. Comparativement aux premiers travaux dans ce domaine initiés par [Arnsdorf \(2012\)](#) et [Ghamami \(2015\)](#), notre “CCVA” (*Central Clearing Valuation Adjustment*) prend en compte non seulement la CVA correspondant aux pertes que les membres subissent suite aux défauts des autres membres via la chambre, mais aussi une MVA correspondant au coût de financement des marges ainsi que la KVA, coût du capital implicitement requis à travers leur contribution au fonds de garantie.

Dans l’ensemble de ce chapitre, nous supposons que la CCP est composée de  $n + 1$  différents membres, dénotés par  $i \in \{0, \dots, n\}$ .

On se place sur un espace probabilisé filtré  $(\Omega, \mathbb{G}, \mathbb{Q})$  avec  $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$ , tel que tous les processus définis sont  $\mathbb{G}$  adaptés et que tous les temps aléatoires utilisés sont des  $\mathbb{G}$ -temps d’arrêts. On rappelle que  $\mathbb{E}_t^{\mathbb{Q}}$  désigne l’espérance conditionnelle sous  $\mathbb{Q}$  sachant  $\mathcal{G}_t$ . Enfin, on rappelle que le processus de taux sans risque est noté  $r$  et on définit le facteur d’actualisation  $\beta$  défini par  $\beta_t = e^{-\int_0^t r_s ds}$ .

Pour chaque membre  $i$ , on dénote par  $P_t^i$  la valeur de marché de son portefeuille ignorant le risque de contrepartie et le coût de financement, i.e.

$$\beta_t P_t^i = \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^{\bar{T}} \beta_s dD_s^i \right), \quad t \in [0, \bar{T}] \quad (0.2.1)$$

où  $D$  est le processus de dividendes promis, et  $\bar{T}$  est la maturité finale du portefeuille de la CCP.

Le but principal d’une chambre de compensation étant de diminuer le risque systémique, du collatéral est demandé à chaque membre afin de couvrir son défaut potentiel. En effet, lors d’un défaut, la CCP se substitue au membre défaillant en récupérant son portefeuille et le collatéral correspondant durant la période de liquidation (typiquement 5 jours). Durant cet intervalle, le risque de marché du portefeuille est porté par la chambre avant que le portefeuille ne soit liquidé auprès des membres survivants.

La cascade (*waterfall*) de collatéral est la suivante.

En premier lieu, la CCP exige que les membres échangent la marge de variation, notée VM, correspondant à la valeur de marché du portefeuille. On a donc  $VM_{lh}^i = P_{lh-}^i$ , avec  $h$  l’intervalle de temps entre deux appels de marge et  $l \in \mathbb{N}$ . Ce premier niveau de collatéral permet de réduire le risque du portefeuille accumulé sur l’intervalle  $[0, lh]$ .

Lors d’un défaut à la date  $\tau$ , le risque de marché sur la période de liquidation  $[\tau, \tau + \delta]$  n’est pas couvert par ce premier niveau de collatéral. C’est pourquoi la chambre exige un second niveau de collatéral : la marge initiale. Notée IM, elle est définie comme étant une mesure de risque sur la perte potentielle du portefeuille  $L_{t,t+\delta}$  durant la période de liquidation. On a donc  $IM_{lh}^i = \rho_{lh}(L_{lh,lh+\delta})$ , avec  $\rho$  une mesure de risque univariée évaluée à la date  $lh$ .

La réglementation européenne en place (EMIR) demande également un troisième niveau de collatéral, appelé fonds de défaillance ou fonds de garantie. Celui-ci doit :

“Permettre au moins aux contreparties centrales de résister, dans des conditions de marché extrêmes mais plausibles, soit à la défaillance du membre compensateur vis-à-vis duquel elles présentent la plus forte exposition, soit à la défaillance du deuxième et du troisième membres compensateurs vis-à-vis desquels elles présentent les plus fortes expositions, si la somme de ces expositions est supérieure”.

La contribution au fonds de garantie de chaque membre, notée DFC, est donc basée sur une mesure de risque multivariée  $\varrho$  sur l’ensemble des portefeuilles des membres. La fréquence de mise à jour  $T$  de son niveau est plus faible que pour les marges de variations et initiales (typiquement

tous les mois). De plus, contrairement aux autres niveaux de collatéral, la contribution au fonds de garantie est mutualisée : elle peut être utilisée en dernier recours pour couvrir les pertes liées aux défauts des autres membres. Elle doit par conséquent être considérée comme du capital à risque pour les membres. On note alors  $\text{DFC}_{lT}^i = \varrho_{lT}^i((L_{lT+\delta}^1 - \text{IM}_{lT}^1, \dots, L_{lT+\delta}^n - \text{IM}_{lT}^n))$ ,  $l \in \mathbb{N}$ .

Afin d'inclure le risque systémique dans notre modélisation, nous autorisons les membres à faire défaut simultanément. Dès lors, nous montrons que la “brèche”  $B$  subie par la CCP à chaque date de liquidation  $\tau_Z + \delta$  vaut :

$$B_{\tau_Z+\delta} = \sum_{i \in Z} (1 - R_i) \underbrace{\left( P_{\tau_i+\delta}^i + \int_{[\tau_i, \tau_i+\delta]} e^{\int_s^{\tau_i+\delta} r_u du} dD_s^i - \mathcal{C}_{\tau_i}^i \right)^+}_{\varepsilon_i} \quad (0.2.2)$$

où  $\tau_Z$  désigne la date de défaut joint des membres  $i \in Z$ , la somme étant sur l'ensemble  $Z \subseteq \{0, \dots, n\}$  tel que  $i \in Z$  si et seulement si  $\tau_i = \tau_Z$ , en notant  $\hat{\tau}_i$  la dernière date d'appel de marge avant le défaut du membre  $i$ , en supposant qu'un recouvrement au taux  $R_i$  puisse être attendu par la chambre pour tous les membres  $i \in \{0, \dots, n\}$  et enfin en notant  $\mathcal{C} = \text{VM} + \text{IM} + \text{DFC}$  l'ensemble du collatéral posté par le membre.

À chaque date de liquidation, si la brèche  $B$  est positive, la perte au-delà du collatéral du membre est d'abord absorbée par le “skin-in-the-game”  $E$ . Ce collatéral, fourni cette fois par la CCP, est typiquement mis à jour une fois par an. Ainsi, à chaque date  $\tau_Z + \delta$ , le skin-in-the game  $E$  varie de :

$$\Delta E_{\tau_Z+\delta} = -(B_{\tau_Z+\delta} \wedge E_{\tau_Z+\delta-}) \quad (0.2.3)$$

Dès-lors, la perte résiduelle  $\epsilon$  non couverte par ce dernier niveau de collatéral est absorbée par l'ensemble des membres survivants et allouée via une clé de répartition, typiquement basée sur leur contribution au fonds de garantie :

$$\epsilon_{\tau_Z+\delta}^i = (B_{\tau_Z+\delta} - E_{\tau_Z+\delta-})^+ \frac{J_{\tau_Z+\delta}^i \text{DFC}_{\tau_Z+\delta}^i}{\sum_{0 \leq j \leq n} J_{\tau_Z+\delta}^j \text{DFC}_{\tau_Z+\delta}^j} \quad (0.2.4)$$

avec  $J^i = \mathbb{1}_{[0, \tau_i)}$  le processus d'indicatrice de survie.

Dès-lors, plaçons nous du point de vue du membre 0. Par souci de simplifications de lecture, nous retirons l'indice 0 de nos notations.

Consistants avec notre méthodologie de non-arbitrabilité via la mesure risque neutre  $\mathbb{Q}$ , nous supposons que le vecteur de gains unitaires  $\mathcal{M}$  sur les actifs de couverture est une  $(\mathbb{G}, \mathbb{Q})$ -martingale. Nous supposons également que le membre réalise une stratégie de couverture  $(-\zeta)$  de son portefeuille, i.e. un vecteur de processus prévisibles correspondant aux positions prises sur les actifs de couverture.

Notons que l'ensemble du collatéral requis pour traiter via la CCP doit être financé. Nous assumons que la marge de variation  $\text{VM}_t = P_{t-}^{\zeta}$  consiste en du cash ré-hypothécable rémunéré au taux sans risque, tandis que les marges initiales sont considérées être des actifs liquides évoluant au taux sans risque. De plus, nous faisons l'hypothèse que les marges initiales et les contributions au fonds de garantie sont sujettes à des frais de la CCP au taux  $c$  (aux alentours de 30 bps). Nous supposons enfin que le membre peuvent investir à un taux  $(r + \lambda)$  et emprunter à un taux  $(r + \bar{\lambda})$ .

En écrivant la dynamique du P&L du portefeuille du membre (en supposant qu'un taux de recouvrement  $\bar{R}$  est attendu par le membre envers son financeur), on déduit que la valeur du

portefeuille  $\Pi$  vérifie l'EDSR suivante :

$$\begin{aligned} \Pi_{\bar{\tau}^\delta} &= \mathbb{1}_{\tau < \bar{T}} \left[ -\mathbb{1}_{\varepsilon=0} \left( P_{\tau+\delta} + \int_{[\tau, \tau+\delta]} e^{\int_s^{\tau+\delta} r_u du} dD_s \right) - \mathbb{1}_{\varepsilon>0} (\mathcal{C}_\tau + R\varepsilon) \right] \\ \text{et, pour } t \leq \bar{\tau}^\delta, \\ d\Pi_t &= r_t \Pi_t dt + \mathbb{1}_{\tau < \bar{T}} (1 - \bar{R}) (\Pi_{t-} + \mathcal{C}_t^*)^+ dJ_t \\ &\quad + J_t \left( dD_t + \sum_{Z \subseteq N} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt) + g_t(\Pi_t) dt \right) + d\nu_t, \end{aligned} \quad (0.2.5)$$

avec  $\nu$  une martingale locale, en utilisant la notation  $\bar{\tau}^\delta = \mathbb{1}_{\tau < \bar{T}} \tau^\delta + \mathbb{1}_{\tau \geq \bar{T}} \bar{T}$  et en notant, pour tout  $\pi \in \mathbb{R}$ ,

$$g_t(\pi) = c_t (\mathcal{C}_t - P_{t-}) + \bar{\lambda}_t (\pi + \mathcal{C}_t^*)^+ - \lambda_t (\pi + \mathcal{C}_t^*)^- \quad (0.2.6)$$

où  $\mathcal{C}_t^* = \text{VM}_t + \text{IM}_t$ . On définit alors la CCVA comme le processus  $\Theta$  défini par

$$\Theta := - \left( P + \int_{[\tau, \cdot]} e^{\int_s^{\tau} r_u du} dD_s + \Pi \right) \quad (0.2.7)$$

et on montre alors que :

$$\begin{aligned} \beta_t \Theta_t &= \mathbb{E}_t^{\mathbb{Q}} \left[ \sum_{t < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} - \mathbb{1}_{\tau < \bar{T}} \left( \beta_{\tau^\delta} \xi + \beta_\tau (1 - \bar{R}) (P_{\tau-} - \mathcal{C}_\tau + \Theta_{\tau-})^- J_t \right) \right. \\ &\quad \left. + \int_t^{\bar{\tau}} \beta_s (g_s(-P_s - \Theta_s)) ds \right], t \in [0, \bar{\tau}^\delta] \end{aligned} \quad (0.2.8)$$

Par ailleurs, au-delà de  $\Theta$ , on se doit d'ajouter la KVA correspondant au coût du capital requis par le membre afin de rémunérer ses actionnaires à un certain taux  $k$ . À ce titre, le capital mis à disposition par le membre est  $K = \text{DFC} + K^{cm}$  où  $K^{cm}$  représente le capital réglementaire exigé par la réglementation. La KVA, ajustement à prendre en compte au-delà de  $\Theta$  est alors :

$$\text{KVA}_t = k \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^{\bar{\tau}} e^{-\int_t^s (r_u + k) du} K_s ds \right), t \in [0, \bar{\tau}] \quad (0.2.9)$$

La CCVA définie comme étant  $\Theta + \text{KVA}$  peut alors être comparée à la valeur d'ajustement du portefeuille du membre en trading bilatéral. Cette comparaison est réalisée avec des simulations numériques où les quantités sont calculées par méthodes de Monte-Carlo sur des portefeuilles constitués de swaps.

Nous concluons que les différences de coûts entre trading bilatéral et centralisé sont principalement dûs au “netting” des positions en trading centralisé. En effet, la CCP se trouvant au centre du système dans lequel évoluent les membres, le risque qu'elle porte concerne uniquement les positions agrégées des membres contrairement à la somme des risques contrepartie-par-contrepartie en trading bilatéral.

### 0.3 Optimisation de marges pour le trading centralisé

L'objectif du chapitre 3 est de présenter de nouvelles méthodologies concernant, dans un premier temps, le calcul du fonds de garantie et son allocation ; puis de l'emprunt de la marge initiale pour les membres.

En effet, la taille du fonds de garantie est principalement basée sur la méthodologie Cover 2 présentée ci-avant. Cette méthode est basée sur des considérations de mouvements de marché stressé. À l'instar du trading bilatéral en marché incomplet, nous introduisons le capital économique (EC) de la CCP (initialement introduit par [Ghamami \(2015\)](#) et [Albanese \(2015\)](#)) et proposons d'utiliser cette valeur pour le fonds de garantie. Dès lors, nous pouvons également définir le coût du capital (KVA) de la CCP.

Nous nous plaçons dans le même espace probabilisé filtré introduit précédemment et utilisons les mêmes notations que la section précédente. Néanmoins, contrairement au chapitre 2, nous supposons que le collatéral est capitalisé entre le défaut du membre et la liquidation de son portefeuille. Ainsi la perte du portefeuille du  $i^{\text{ème}}$  membre est :

$$L_{\tau_i+\delta}^i = \left( P_{\tau_i+\delta}^i + \int_{[\tau_i, \tau_i+\delta]} e^{\int_s^{\tau_i+\delta} r_u du} dD_s^i - \beta_{\tau_i+\delta}^{-1} \beta_{\tau_i} (\text{VM}_{\tau_i}^i + \text{IM}_{\tau_i}^i + \text{DFC}_{\tau_i}^i) \right)^+ \quad (0.3.1)$$

Dans l'optique du calcul du fonds de garantie, notons  $L_{\tau_i+\delta}^{i, \text{DFC}=0}$  la valeur de la perte du membre  $i$  sans sa contribution au fonds de garantie. Nous supposons que la CCP ne peut faire défaut dans notre modèle. C'est pourquoi nous supposons qu'elle peut emprunter et prêter au taux OIS  $r$ , réduisant ainsi les XVAs de la CCP à sa CVA :

$$\text{CVA}_t^{\text{ccp}} = \mathbb{E}_t^{\mathbb{Q}} \left( \sum_{t < \tau_i + \delta < T} \beta_t^{-1} \beta_{\tau_i + \delta} L_{\tau_i + \delta}^{i, \text{DFC}=0} \right) \quad (0.3.2)$$

Le processus de pertes correspondant s'écrit :

$$\begin{cases} L_0^{\text{ccp}} &= z^{\text{ccp}}, \quad \text{la perte initiale de la CCP} \\ \beta_t dL_t^{\text{ccp}} &= \beta_t (d\text{CVA}_t^{\text{ccp}} - r_t \text{CVA}_t^{\text{ccp}} dt) \\ &+ \sum_i \left( \beta_{\tau_i + \delta} L_{\tau_i + \delta}^{i, \text{DFC}=0} \right) \delta_{\tau_i + \delta}(dt) \end{cases} \quad (0.3.3)$$

Nous définissons ensuite le capital économique (EC) permettant de couvrir les pertes exceptionnelles de la CCP comme l'*expected shortfall* des pertes sur l'année à venir :

$$\text{EC}_t = \mathbb{ES}_t^{\mathbb{Q}} \left( \int_t^{t+1} \beta_s^{-1} \beta_s dL_s^{\text{ccp}} \right) =: \text{DF}_t \quad (0.3.4)$$

où  $\mathbb{ES}_t$  représente l'*expected shortfall* conditionnelle. La KVA de la CCP est donnée par :

$$\text{KVA}_t^{\text{ccp}} = h \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^{\bar{T}} e^{-\int_t^s (r_u + h) du} \text{DF}_s ds \right) \quad (0.3.5)$$

Concernant l'allocation du fonds de garantie à travers les membres, nous pouvons également comparer l'allocation "classique" utilisant comme clé de répartition la marge initiale

$$\mu^i = \frac{\text{IM}^i}{\sum_j \text{IM}^j} \quad (0.3.6)$$

à une allocation de type "X-incrémentale" au sens suivant (cf. [Albanese \(2015\)](#)) :

$$\mu^i = \frac{\Delta_i \text{EC}^{(-i)}}{\sum_j \Delta_j \text{EC}^{(-j)}}, \quad \text{ou} \quad \mu^i = \frac{\Delta_i \text{KVA}^{(-i)}}{\sum_j \Delta_j \text{KVA}^{(-j)}} \quad (0.3.7)$$

où  $X^{(-i)}$  représente la valeur  $X$  sans le  $i^{\text{ème}}$  membre et en notant  $\Delta_i X^{(-i)} := X - X^{(-i)}$ .

Dans un second temps, nous proposons de comparer le coût de financement de la marge initiale, déjà présenté dans le chapitre 2 avec le schéma de financement suggéré par Albanese (2015), selon lequel le collatéral de chaque membre est posté par une tierce partie en échange d'un certain coût. En notant  $\lambda = \gamma(1 - R)$  le spread CDS du membre 0 avec  $\gamma$  son processus d'intensité de défaut, sa  $MVA_0^{ub}$  (*ub* pour *unsecured borrowing*), i.e. le coût de financement de sa marge initiale à la date 0, est :

$$MVA_0^{ub} = \mathbb{E}^{\mathbb{Q}} \left( \int_0^{\bar{\tau}} \beta_s \lambda_s \text{IM}_s \, ds \right). \quad (0.3.8)$$

Supposons qu'il existe une institution financière sans risque de défaut, appelée *prêteur spécialisé*. En contrepartie d'une certaine rémunération, ce prêteur se substitue au membre vis-à-vis de la CCP concernant les appels de marge initiale. De plus, en cas de défaut du membre, on suppose que ce prêteur récupère la partie de la marge initiale non utilisée pour l'absorption des pertes. L'exposition de ce prêteur spécialisé vis-à-vis du membre est donc :

$$(G_{\tau+\delta}^+ \wedge \beta_{\tau+\delta}^{-1} \beta_{\tau} \text{IM}_{\tau}) \quad (0.3.9)$$

où  $G_t$  représente la différence du portefeuille entre le défaut et sa date de liquidation :

$$G_t = P_t + \int_{[t, t+\delta]} e^{\int_s^{t+\delta} r_u \, du} \, dD_s^i - \beta_t^{-1} \beta_{t-\delta} \text{VM}_{t-\delta} \quad (0.3.10)$$

Le coût de financement de la marge initiale dans ce nouveau schéma de financement à la date 0 est  $MVA_0^{sl}$  (*sl* pour *specialist lender*) :

$$\begin{aligned} MVA_0^{sl} &= \mathbb{E}^{\mathbb{Q}} [\beta_{\tau+\delta} \mathbf{1}_{\tau < \bar{T}} ((G_{\tau+\delta}^+ \wedge \beta_{\tau+\delta}^{-1} \beta_{\tau} \text{IM}_{\tau}))] \\ &= \mathbb{E}^{\mathbb{Q}} \left( \int_0^{\bar{\tau}} \beta_s \gamma_s \xi_s \, ds \right) \end{aligned} \quad (0.3.11)$$

avec  $\xi$  un processus prévisible tel que  $\mathbb{E}_{\tau-}^{\mathbb{Q}} (\beta_{\tau+\delta} G_{\tau+\delta}^+ \wedge \beta_{\tau} \text{IM}_{\tau}) = \beta_{\tau} \xi_{\tau}$ .

Par identification avec le schéma classique, ce nouveau schéma de financement correspond à un coût implicite  $\bar{\lambda} = \frac{1}{1-R} \frac{\xi}{\text{IM}} \lambda$ . La marge initiale étant calculée sur une mesure de risque de  $G$ , avec la supposition de taux de recouvrement  $R$  compris entre 20% et 40%, on a généralement  $\bar{\lambda} \ll \lambda$  et ensuite  $MVA_0^{sl}$  significativement inférieur à  $MVA_0^{ub}$ .

En dernier lieu, ces deux optimisations sont illustrées numériquement sur les mêmes portefeuilles du chapitre précédent après avoir dérivé l'ensemble des formules (semi-fermées) pour calculer les quantités nécessaires.

## 0.4 Mesures de risques multivariées et allocation de fonds de défaut

Dans le chapitre 4, nous étendons la définition de *shortfall risk* introduite par Föllmer and Schied (2002), ou *optimized certainty equivalent* présentée par Ben-Tal and Teboulle (2007), au cadre multidimensionnel. Cette nouvelle mesure évalue les risques de composants dépendants et peut être utilisée pour l'optimisation de portefeuille, ou, dans le cadre de cette thèse, à la valorisation du fonds de garantie de la CCP.

Nous nous plaçons sur  $\mathbb{R}^d$ , et notons  $\leq$  son ordre partiel. Ainsi,  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \mathbf{x} \leq \mathbf{y}$  si et seulement si pour tout  $k$  tel que  $1 \leq k \leq d$  on a  $x_k \leq y_k$ . Par ailleurs, on se place dans un espace probabilisé  $(\Omega, \mathcal{A}, \mathbb{P})$  et notons  $L^0(\mathbb{R}^d)$  l'espace des vecteurs aléatoires de dimension  $d$ ,  $\mathcal{A}$ -mesurables. Nous représentons un vecteur de pertes par la notation  $\mathbf{X} \in L^0(\mathbb{R}^d)$ .

Dans un premier temps, nous définissons des fonctions appelées *fonctions de perte*. Une fonction  $\ell : \mathbb{R}^d \rightarrow (-\infty, \infty]$  est une fonction de perte si elle est croissante, convexe, semi-continue

inférieurement avec  $\inf \ell < 0$ , et enfin sur-linéaire. Ces fonctions permettent l'aggrégation des pertes des différents composants.

Pour des raisons d'intégrabilité, nous restreignons notre analyse au sous-espace des vecteurs aléatoires suivant :

$$M^\theta := \{ \mathbf{X} \in L^0 \text{ tel que } \mathbb{E}(\theta(\lambda \mathbf{X})) < \infty \text{ pour tout } \lambda \in \mathbb{R}^+ \} \quad (0.4.1)$$

avec la notation  $\theta(\mathbf{x}) := \ell(|\mathbf{x}|)$ ,  $\mathbf{x} \in \mathbb{R}^d$ .

Nous définissons qu'une position  $(-\mathbf{X})$  est acceptable si elle vérifie  $\mathbb{E}(\ell(\mathbf{X})) \leq 0$ . L'ensemble des *allocations monétaires acceptables* est défini par

$$A(\mathbf{X}) := \{ \mathbf{m} \in \mathbb{R}^d : \mathbb{E}(\ell(\mathbf{X} - \mathbf{m})) \leq 0 \} \quad (0.4.2)$$

Nous définissons le *multivariate shortfall risk*  $R(\mathbf{X})$ , pour  $\mathbf{X} \in M^\theta$ , par :

$$R(\mathbf{X}) = \inf \left\{ \sum_k m_k : \mathbf{m} \in A(\mathbf{X}) \right\} = \inf \left\{ \sum_k m_k : \mathbb{E}(\ell(\mathbf{X} - \mathbf{m})) \leq 0 \right\}. \quad (0.4.3)$$

Nous démontrons que cette mesure de risque est une mesure de risque monotone, convexe, invariante par translation, continue et sous-différentiable. De plus, si  $\ell$  est positivement homogène,  $R$  l'est également. Enfin,  $R$  admet une représentation duale de la forme

$$R(\mathbf{X}) = \max_{\mathbb{Q} \in \mathcal{Q}^{\theta^*}} \{ \mathbb{E}_{\mathbb{Q}}(\mathbf{X}) - \alpha(\mathbb{Q}) \} \quad (0.4.4)$$

où  $\mathcal{Q}^{\theta^*}$  est défini par

$$\mathcal{Q}^{\theta^*} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} := (Z_1, \dots, Z_d) : \mathbf{Z} \in L^{\theta^*}, \mathbf{Z} \geq \mathbf{0} \text{ tel que } \mathbb{E}[Z_k] = 1 \text{ pour tout } k \right\} \quad (0.4.5)$$

avec  $L^{\theta^*}$  le dual de  $M^\theta$  tandis que la fonction de pénalité  $\alpha$  est définie par

$$\alpha(\mathbb{Q}) = \inf_{\lambda > 0} \mathbb{E} \left[ \lambda \ell^* \left( \frac{d\mathbb{Q}}{\lambda d\mathbb{P}} \right) \right], \quad \mathbb{Q} \in \mathcal{Q}^{\theta^*}. \quad (0.4.6)$$

Au-delà de son niveau global, l'allocation de cette mesure de risque entre les différentes composantes du système est un point essentiel. Nous étudions les questions d'existence d'une allocation, de son unicité et sa sensibilité par rapport à la dépendance entre les pertes  $X_i$ ,  $1 \leq i \leq d$ . Nous montrons que si une fonction de perte est invariante par permutation de ces coordonnées, alors l'allocation optimale est uniquement caractérisée par les conditions du premier ordre suivantes :

$$\mathbf{1} \in \lambda^* \mathbb{E}(\nabla \ell(\mathbf{X} - \mathbf{m}^*)) \quad \text{et} \quad \mathbb{E}(\ell(\mathbf{X} - \mathbf{m}^*)) = 0 \quad (0.4.7)$$

où  $\lambda^*$  est un multiplicateur de Lagrange. Nous montrons également que l'allocation est invariante par translation et positivement homogène si  $\ell$  l'est.

Dans une dernière optique, nous nous intéressons à la sensibilité de notre mesure de risque par rapport à un choc extérieur. Nous définissons la contribution marginale en risque de  $\mathbf{Y} \in M^\theta$  par rapport à  $\mathbf{X} \in M^\theta$  par la quantité

$$R(\mathbf{X}; \mathbf{Y}) := \limsup_{t \searrow 0} \frac{R(\mathbf{X} + t\mathbf{Y}) - R(\mathbf{X})}{t} \quad (0.4.8)$$

Si  $\ell$  est invariante par permutation de ces coordonnées alors nous démontrons que

$$R(\mathbf{X}; \mathbf{Y}) = \min_{\mathbf{m} \in B(\mathbf{X})} \max_{\lambda \in C(\mathbf{X})} \lambda \mathbb{E}[\nabla \ell(\mathbf{X} - \mathbf{m}) \cdot \mathbf{Y}] \quad (0.4.9)$$

où  $B(\mathbf{X}) \times C(\mathbf{X})$  est l'ensemble des points selles de la fonction  $(\mathbf{m}, \lambda, \mathbf{X}) \mapsto L(\mathbf{m}, \lambda, \mathbf{X}) = \sum_k m_k + \lambda \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})]$ .

Nous réalisons ensuite une étude comparant différents schémas numériques de calculs de notre mesure de risque et d'allocation lorsque  $\mathbf{X}$  est un vecteur gaussien de dimension variable. Nous analysons les temps de calcul de  $R(\mathbf{X})$  et  $RA(\mathbf{X})$  lorsque les espérances sont calculées par méthodes de Fourier, Monte-Carlo, et lorsque nous interpolons les fonctions de perte par la méthode de Chebychev.

Nous terminons enfin notre analyse sur des données réelles fournies par la chambre de compensation LCH. Nous comparons l'allocation du fonds de garantie proportionnellement aux marges initiales avec l'allocation proportionnelle aux  $\mathbf{m}^*$  (en ajoutant la contrainte des allocations marginales positives).



# The sustainable Black-Scholes equations

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This chapter is based on [Armenti, Crépey, and Zhou \(2016\)](#).

## 1.1 Introduction

In incomplete markets, a basic Black–Scholes perspective has to be complemented by the valuation of market imperfections. Otherwise this results in Black–Scholes Ponzi schemes, such as the ones at the core of the last global financial crisis, where always more derivatives need to be issued for remunerating the capital attracted by the already opened positions. In this chapter we consider the sustainable Black–Scholes equations that arise for a portfolio of options if one adds to their trade additive Black–Scholes price, on top of a nonlinear funding cost, the cost of remunerating at a hurdle rate the residual risk left by imperfect hedging. We assess the impact of model uncertainty in this setup.

Section 1.2 revisits the pricing of a book of options accounting for cost of capital and cost of funding, which are material in incomplete markets. Section 1.3 specializes the pricing equations to a Markovian Black–Scholes setup. Section 1.4 assesses the impact of model risk in an UVM (uncertain volatility model) setup. Section 1.5 refines the model risk add-ons by accounting for calibrability constraints.

We consider a portfolio of options made of  $\omega_i$  vanilla call options of maturity  $T_i$  and strike  $K_i$  on a stock  $S$ , with  $0 < T_1 < \dots < T_n = T$ . Note that, if a corporate holds a bank payable, it typically has an appetite to close it, receive cash, and restructure the hedge otherwise with a par contract (the bank would agree to close the deal as a market maker, charging fees for the new trade). Because of this natural selection, a bank is mostly in the receivables (i.e. “ $\omega_i \geq 0$ ”) in its derivative business with corporates.

We write  $x^\pm = \max(\pm x, 0)$ .

## 1.2 Cost of Capital and Cost of Funding

### 1.2.1 Cost of Capital

Let  $r_t$  denote a risk-free OIS short term interest rate and  $\beta_t = e^{-\int_0^t r_s ds}$  be the corresponding risk-neutral discount factor.

In presence of hedging imperfections resulting in a nonvanishing loss (and profit) process  $\varrho$  of the bank, a conditional risk measure  $\text{EC} = \text{EC}_t(\varrho)$  must be dynamically computed and reserved by the bank as economic capital.

It is established in [Albanese and Crépey \(2017, Section 5.3\)](#) that the capital valuation adjustment (KVA) needed by the bank in order to remunerate its shareholders for their capital at risk at some average hurdle rate  $h$  (e.g. 10%) at any point in time in the future is :

$$\text{KVA} = \text{KVA}_t(\varrho) = h \mathbb{E}_t \left( \int_t^T e^{-\int_t^s (r_u + h_u) du} \text{EC}_s(\varrho) ds \right) \quad (1.2.1)$$

where  $\mathbb{E}_t$  stands for the conditional expectation with respect to some probability measure  $\mathbb{Q}$  and model filtration.

In principle, the probability measure used in capital and cost of capital calculations should be the historical probability measure. But, in the present context of optimization of a portfolio of derivatives, the historical probability measure is hard to estimate in a relevant way, especially for long maturities. As a consequence, we do all our price and risk computations under a risk-neutral measure  $\mathbb{Q}$  calibrated to the market (or a family of pricing measures, in the context of model uncertainty later below), assuming no arbitrage.

### 1.2.2 Cost of Funding

We assume that the bank can invest at the risk-free rate  $r$  but can only obtain unsecured funding at a shifted rate  $r + \lambda > r$ . This entails funding costs over OIS and a related funding valuation adjustment (FVA) for the bank. Given our focus on capital and funding in this chapter, we ignore counterparty risk for simplicity, so that  $\lambda$  is interpreted as a pure funding liquidity basis.

In order to exclude arbitrages in the primary market of hedging instruments, we assume that the vector gain process  $\mathcal{M}$  of unit positions held in the hedging assets is a risk-neutral martingale. The bank “marks to the model” its derivative portfolio, assumed bought from the client at time 0, by means of an FVA-deducted value process  $\Theta$ . The bank may also set up a (possibly imperfect) hedge  $(-\eta)$  in the hedging assets, for some predictable row-vector process  $\eta$  of the same dimension as  $\mathcal{M}$ . We assume that the depreciation of  $\Theta$ , the funding expenditures and the loss  $\eta d\mathcal{M}$  on the hedge, minus the option payoffs as they mature, are instantaneously realized into the loss(-and-profit) process  $\varrho$  of the bank. In particular, at any time  $t$ , the amount on the funding account of the bank is  $\Theta_t$ . Moreover, we assume that the economic capital can be used by the trader for her funding purposes provided she pays to the shareholders the OIS rate on EC that they would make otherwise by depositing it (assuming it all cash for simplicity).

Note that the value process  $\Theta$  of the trade already includes the FVA as a deduction, but ignores the KVA, which is considered as a risk adjustment computed in a second step (in other words, we assume that the trader’s account and the KVA account are kept separate from each other). Rephrasing in mathematical terms the above description, the loss equation of the trader is written, for  $t \in (0, T]$ , as (starting from  $\varrho_0 = y$ , the accrued loss of the portfolio) :

$$\begin{aligned}
d\varrho_t = & \underbrace{- \sum_i \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt)}_{\text{Call payoffs}} \\
& + \underbrace{r_t \text{EC}_t(\varrho) dt}_{\text{Payment of internal lending of the EC funding source at OIS rate}} \\
& + \underbrace{\left( (r_t + \lambda_t) (\Theta_t - \text{EC}_t(\varrho))^+ - r_t (\Theta_t - \text{EC}_t(\varrho))^- \right) dt}_{\text{Portfolio funding costs / benefits}} \\
& + \underbrace{(-d\Theta_t)}_{\text{Depreciation of } \Theta} + \underbrace{\eta_t d\mathcal{M}_t}_{\text{Loss on the hedge}} \\
= & -d\Theta_t - \sum_i \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt) + \left( \lambda_t (\Theta_t - \text{EC}_t(\varrho))^+ + r_t \Theta_t \right) dt + \eta_t d\mathcal{M}_t
\end{aligned} \tag{1.2.2}$$

Hence, a no-arbitrage condition that the loss process  $\varrho$  of the bank should follow a risk-neutral martingale (assuming integrability) and the terminal condition  $\Theta_T = 0$  lead to the

following FVA-deducted risk-neutral valuation BSDE :

$$\Theta_t = \underbrace{\mathbb{E}_t \left[ \sum_{t < T_i} \beta_t^{-1} \beta_{T_i} \omega_i (S_{T_i} - K_i)^+ \right]}_{\Theta_t^0} - \underbrace{\mathbb{E}_t \left[ \int_t^T \beta_t^{-1} \beta_s \lambda_s (\Theta_s - \text{EC}_s(\varrho))^+ ds \right]}_{\text{FVA}_t}, \quad t \in [0, T] \quad (1.2.3)$$

(since we consider a portfolio of options with several maturities, we treat option payoffs as cash-flows at their maturity times rather than a terminal condition in the equations, in particular  $\Theta_T = 0$ ).

The funding source provided by economic capital creates a feedback loop from EC into FVA, which makes the FVA smaller.

Note that, in the usual case of a risk measure EC only affected by the time fluctuations of  $\varrho$ , the equations (1.2.3) and in turn (1.2.1) are independent of the accrued loss  $y$ , which eventually does not affect  $\Theta$  nor the KVA.

If  $\lambda = 0$ , then, whatever the hedge  $\eta$ ,  $\Theta$  reduces to  $\Theta^0$ , which corresponds to the usual trade additive (linear) no-arbitrage pricing formula for a portfolio of options, with zero FVA, but with a KVA given by (1.2.1), depending on the hedge  $\eta$ .

If  $\lambda \neq 0$ , and if there exists a replicating hedge  $\eta$  such that the corresponding  $\varrho$  is constant in (1.2.2), i.e.  $\eta d\mathcal{M}$  coincides with the martingale part of  $\Theta$ , then the resulting  $\varrho$ , EC and KVA vanish (since we assumed  $\text{EC}(0) = 0$ ), and the ensuing FVA-deducted value process is given the following process  $\Theta^*$  :

$$\Theta_t^* = \mathbb{E}_t \left[ \sum_{t < T_i} \beta_t^{-1} \beta_{T_i} \omega_i (S_{T_i} - K_i)^+ - \int_t^T \beta_t^{-1} \beta_s \lambda_s (\Theta_s^*)^+ ds \right], \quad t \in [0, T] \quad (1.2.4)$$

This is a monotone driver BSDE, admitting as such a unique square integrable solution  $\Theta^*$  (see e.g. Kruse and Popier (2016, Section 4)), provided  $\lambda$  is bounded from below and  $\Theta^0$  is square integrable.

**Example 1.2.1 (Single option positions)** If  $n = 1$  and  $\omega_1 = 1$  (one long call position), then, by application of the comparison theorem for BSDEs with a monotonic generator (see Kruse and Popier (2016, Section 4)), we have  $\Theta^* \geq 0$ , hence

$$\Theta_t^* = \mathbb{E}_t \left[ \tilde{\beta}_t^{-1} \tilde{\beta}_{T_1} (S_{T_1} - K_1)^+ \right] \quad (1.2.5)$$

where  $\tilde{\beta}_t = e^{-\int_0^t (r_s + \lambda_s) ds}$ . With respect to  $\Theta^0$ , the value  $\Theta^*$  corresponds to an FVA rebate on the buying price by the bank (since we assumed a positive liquidity basis  $\lambda$ ).

If  $n = \omega_1 = -1$  (one short call position), then we deduce likewise that  $\Theta^* \leq 0$ , hence  $\Theta^* = \Theta^0$ .

But, apart from the above special cases where  $\lambda = 0$  or  $\eta = \eta^*$ , the BSDE (1.2.3) for  $\Theta$  is nonstandard due to the term  $\text{EC} = \text{EC}_t(\varrho)$  in the FVA.

### 1.3 Markovian Black–Scholes Setup

In this section we assume a constant risk-free rate  $r$  and a stock price  $S$  following a geometric Brownian motion with volatility  $\sigma$  and constant dividend yield  $q$ . The risk-neutral martingale  $\mathcal{M}$  is then taken as the gain process of a continuously rolled unit position on the stock  $S$ , assumed funded at the risk-free rate via a repo market, i.e.  $d\mathcal{M}_t = dS_t - (r - q)S_t dt$ . We denote by  $\mathcal{A}_S^{bs} = (r - q)S\partial_S + \frac{1}{2}\sigma^2 S^2 \partial_{S^2}^2$  the corresponding risk-neutral Black–Scholes generator.

Doing our modeling exercise in the context of the Black-Scholes model, where perfect replication, hence no KVA, is possible, may seem rather artificial. However, doing all the computations in a stylized Black-Scholes setup with a single risk factor  $S$  yields useful practical insights. In addition, this conveys the message that, in real-life incomplete markets, a basic Black-Scholes perspective has to be complemented by the valuation of market imperfections, otherwise this unavoidably results in Black-Scholes Ponzi schemes, such as the ones that have been involved in the global financial crisis, where always more derivatives are issued to remunerate the capital required by the already opened positions (if priced and risk-managed in a basic Black-Scholes way ignoring the cost of capital).

In the Black-Scholes setup and assuming a stylized Markovian specification

$$\text{EC}_t(\varrho) = f \sqrt{\frac{d\langle \varrho \rangle_t}{dt}} \quad (1.3.1)$$

(the stylized Value-at-Risk which is proportional to the instantaneous volatility of the loss process  $\varrho$  modulo a suitable “quantile level”  $f$ ) as well as  $\lambda_t = \lambda(t, S_t)$ ,  $\eta_t = \eta(t, S_t)$ , then the above FVA and KVA equations can be reduced to the “sustainable Black-Scholes PDEs” (1.3.7), as follows (resulting in an FVA- and KVA-deducted price that would be sustainable for the bank even in the limit case of a portfolio held on a run-off basis, with no new trades ever entered in the future).

First, observe that given a tentative FVA-deducted price process of the form  $\Theta_t = u(t, S_t)$  for some to-be-determined function  $u = u(t, S)$ , we have, assuming (1.3.1) :

$$\sqrt{\frac{d\langle \varrho \rangle_t}{dt}} = \sigma S_t |\partial_S u(t, S_t) - \eta(t, S_t)| \quad (1.3.2)$$

Accordingly, let the function  $u$  be defined by  $u_i(t, S)$  on each strip  $(T_{i-1}, T_i] \times (0, \infty)$ , where  $(u_i)_{1 \leq i \leq n}$  is the unique sequence of viscosity solutions, which can then shown to be classical solutions, to the following PDE cascade, for  $i$  decreasing from  $n$  to 1 (closing the system by setting  $u_{n+1} = 0$  and  $T_0 = 0$ ) :

$$\begin{cases} u_i(T_i, S) = u_{i+1}(T_i, S) + \omega_i(S - K_i)^+ \text{ on } (0, \infty) \\ \partial_t u_i + \mathcal{A}_S^{bs} u_i - \lambda(u_i - f\sigma S |\partial_S u_i - \eta|)^+ - ru_i = 0 \text{ on } [T_{i-1}, T_i] \times (0, \infty) \end{cases} \quad (1.3.3)$$

Itô's calculus shows that the process  $(\Theta_t)_t = (u(t, S_t))_t$  solves the Markovian, monotonic driver (assuming  $\lambda$  bounded from below) BSDE

$$\begin{aligned} u(t, S_t) = \mathbb{E}_t \left[ \sum_{t < T_i} \beta_t^{-1} \beta_{T_i} \omega_i(S_{T_i} - K_i)^+ \right. \\ \left. - \int_t^T \beta_t^{-1} \beta_s \lambda_s (u(s, S_s) - f\sigma S_s |\partial_S u(s, S_s) - \eta(s, S_s)|)^+ ds \right], t \in [0, T] \end{aligned} \quad (1.3.4)$$

which in view of (1.3.1)-(1.3.2) is precisely (1.2.3).

The ensuing FVA =  $\Theta^{(0)} - \Theta$  and KVA processes are given as (cf. (1.2.3) and (1.2.1)) :

$$\begin{aligned} \text{FVA}_t(\varrho) &= \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \lambda_s \left( u(s, S_s) - f \sqrt{\frac{d\langle \varrho \rangle_s}{ds}} \right)^+ ds \right] \\ \text{KVA}_t(\varrho) &= h \mathbb{E}_t \left[ \int_t^T e^{-(r+h)(s-t)} f \sqrt{\frac{d\langle \varrho \rangle_s}{ds}} ds \right] \end{aligned} \quad (1.3.5)$$

where  $\sqrt{\frac{d\langle \varrho \rangle_t}{dt}}$  is given by (1.3.2). We set  $\eta = (1 - \alpha) \partial_S u$ , where  $\alpha \in [0, 100\%]$  is the mis-hedge parameter (noting that, for  $\alpha = 0$ , the BSDE (1.3.4) reduces to the replication BSDE (1.2.4)),

then the latter reduces to  $\alpha\sigma S_t |\partial_S u(t, S_t)|$  and we have

$$\begin{aligned} \text{FVA}_t(\varrho) &= \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \lambda_s (u(s, S_s) - \alpha f \sigma S_s |\partial_S u(s, S_s)|)^+ ds \right] \\ &= v(t, S_t) = u_{bs}(t, S_t) - u(t, S_t), \\ \text{KVA}_t(\varrho) &= h \mathbb{E}_t \left[ \int_t^T e^{-(r+h)(s-t)} \alpha f \sigma S_s |\partial_S u(s, S_s)| ds \right] \\ &= w(t, S_t) \end{aligned} \tag{1.3.6}$$

where  $u_{bs}$  is the trade additive Black–Scholes portfolio value and where the FVA and KVA pricing functions  $v$  and  $w$  satisfy

$$\begin{cases} v(T, S) = w(T, S) = 0 \text{ on } (0, \infty) \\ \partial_t v + \mathcal{A}_S^{bs} v + \lambda (u_{bs} - v - \alpha f \sigma S |\Delta_{bs} - \partial_S v|)^+ - rv = 0 \text{ on } [0, T) \times (0, \infty) \\ \partial_t w + \mathcal{A}_S^{bs} w + \alpha h f \sigma S |\Delta_{bs} - \partial_S v| - (r + h)w = 0 \text{ on } [0, T) \times (0, \infty) \end{cases} \tag{1.3.7}$$

in which  $\Delta_{bs} = \partial_S u_{bs}$ .

These “sustainable Black–Scholes PDEs” (1.3.7) allow computing an FVA and KVA deducted price

$$u - w = u_{bs} - v - w \tag{1.3.8}$$

that would be sustainable for the bank even in the limit case of a portfolio held on a run-off basis, with no new trades ever entered in the future.

## 1.4 With Volatility Uncertainty

An important and topical issue, referred to by the regulation as AVA (additional valuation adjustment), is the magnifying impact of model risk on the different XVA metrics.

In this section, we assess model risk from the angle of [Avellaneda, Levy, and Parás \(1995\)](#)’s uncertain volatility model (UVM). Namely, we only assume bounds  $\underline{\sigma}$  and  $\bar{\sigma}$  but we do not assume any specific dynamic on the stock volatility process  $\sigma$ . Therefore, there is a model uncertainty about it. That is, we only consider  $d\mathcal{M}_t := \sigma_t S_t dW_t = dS_t - (r - q)S_t dt$ , where  $\sigma_t \in [\underline{\sigma}, \bar{\sigma}]$  for every  $t$ .

We call  $\mathcal{C}$  the space of continuous paths on  $\mathbb{R}^+$ ,  $C$  the canonical process on the space  $\mathcal{C}$ ,  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  the canonical filtration generated by  $C$  and  $\mathcal{Q}$  the set of  $\mathbb{F}$  local martingale probability measures for  $C$ . We recall from [Soner, Touzi, and Zhang \(2012\)](#) and [Soner, Touzi, Zhang, et al. \(2013\)](#) that, for any probability measure  $\mathbb{Q} \in \mathcal{Q}$ , the process  $C$  satisfies  $dC_t = a_t^{1/2} dW_t^{\mathbb{Q}}$ , for some  $\mathbb{Q}$  Brownian motion  $W^{\mathbb{Q}}$ , where  $a_t$  is the Lebesgue density of the aggregated quadratic variation of  $C$ . In the following, we restrict attention to the probability measures  $\mathbb{Q}$  such that  $a_t^{1/2} \in [\underline{\sigma}, \bar{\sigma}]$  holds  $dt \times \mathbb{Q}$  almost surely, still denoting by  $\mathcal{Q}$  the (restricted) set of measures, and we model  $d\mathcal{M}_t = dS_t - (r - q)S_t dt$  as  $S_t dC_t$ .

Under each  $\mathbb{Q}$ , similarly to (1.2.2), the loss equation of the trader is written, for  $t \in (0, T]$ , as :

$$\begin{aligned} d\varrho_t^{\mathbb{Q}} &= -d\Theta_t^{\mathbb{Q}} - \sum_i \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt) \\ &\quad + \left( \lambda_t (\Theta_t^{\mathbb{Q}} - \text{EC}_t^{\mathbb{Q}}(\varrho^{\mathbb{Q}}))^+ + r_t \Theta_t^{\mathbb{Q}} \right) dt + \eta_t d\mathcal{M}_t \end{aligned} \tag{1.4.1}$$

where  $\text{EC}^{\mathbb{Q}}$  is some conditional risk measure under  $\mathbb{Q}$ . The ensuing equation for the  $\mathbb{Q}$  FVA-deducted value  $\Theta^{\mathbb{Q}}$  appears as

$$\Theta_t^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}} \left[ \sum_{t < T_i} \beta_t^{-1} \beta_{T_i} \omega_i (S_{T_i} - K_i)^+ - \int_t^T \beta_t^{-1} \beta_s \lambda_s \left( \Theta_s^{\mathbb{Q}} - \text{EC}_s^{\mathbb{Q}}(\varrho^{\mathbb{Q}}) \right)^+ ds \right], t \in [0, T] \quad (1.4.2)$$

For all  $\mathbb{Q} \in \mathcal{Q}$ , the trader should value the derivative portfolio  $\Theta_0^{\mathbb{Q}}$  at time 0 (or  $\Theta_t^{\mathbb{Q}}$  at time  $t$ ). However, due to the model uncertainty, the trader values it  $\Theta_0 = \inf_{\mathbb{Q} \in \mathcal{Q}} \Theta_0^{\mathbb{Q}}$  (or at time  $t$ ,  $\Theta_t = \text{ess inf}_{\mathbb{Q} \in \mathcal{Q}} \Theta_t^{\mathbb{Q}}$ ), which is a non-arbitrage price.

At time  $t$ ,  $\text{EC}_t^{\mathbb{Q}}(\varrho^{\mathbb{Q}})$  may depend on the whole future of the process  $(\varrho_s^{\mathbb{Q}})$ ,  $s \geq t$ . This makes (1.4.2) a so-called anticipated BSDE under  $\mathbb{Q}$  (ABSDE in the sense of Peng, Yang, et al. (2009)), with generator  $\lambda_t(\Theta_t^{\mathbb{Q}} - \text{EC}_t^{\mathbb{Q}}(\varrho^{\mathbb{Q}}))^+$ , where  $\Theta^{\mathbb{Q}}$  corresponds to the “Y-component” and  $(d\varrho_s^{\mathbb{Q}} - \eta_s S_s d\mathcal{B}_s)$  to the “Z-component” of the solution. However, in the Markovian setting of Section 1.3,  $\text{EC}_t^{\mathbb{Q}}(\varrho^{\mathbb{Q}})$  only depends on  $(\varrho_t^{\mathbb{Q}})$  at time  $t$ , so that the ABSDE (1.4.2) reduces to a BSDE.

In order to take model risk into consideration (i.e. the impact of several  $\mathbb{Q}$ ), we need the notion of second order BSDE. Wellposedness results regarding second order anticipated BSDEs are not yet available in the literature. Hence, we only give heuristic formulations in this regard. Namely, by analogy with the second order BSDEs theory introduced by Soner, Touzi, and Zhang (2012) we should have the following representation, where  $\mathbb{F}_+ = (\mathcal{F}_t^+)_{0 \leq t \leq T}$  the right limit of  $\mathbb{F}$ , i.e.  $\mathcal{F}_t^+ = \cap_{s > t} \mathcal{F}_s$  for all  $t \in [0, T)$  and  $\mathcal{F}_T^+ = \mathcal{F}_T$  :

There exists a process  $\varrho$  such that, for each  $\mathbb{Q} \in \mathcal{Q}$ ,  $\varrho$  is a  $\mathbb{Q}$ -local martingale and it  $\mathbb{Q} - a.s.$  holds that

$$\begin{aligned} d\varrho_t &= -d\Theta_t - \sum_i \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt) \\ &\quad + \left( \lambda_t(\Theta_t - \text{EC}_t^{\mathbb{Q}}(\varrho))^+ + r_t \Theta_t \right) dt \\ &\quad + \eta_t S_t d\mathcal{B}_t + dA_t^{\mathbb{Q}} \end{aligned} \quad (1.4.3)$$

where  $\text{EC}^{\mathbb{Q}}$  is some conditional risk measure and the family  $A^{\mathbb{Q}}$  of non-decreasing processes satisfies the minimality condition

$$A_t^{\mathbb{Q}} = \text{ess inf}_{\mathbb{Q}' \in \mathcal{Q}(t, \mathbb{Q}, \mathbb{F}_+)} \mathbb{E}^{\mathbb{Q}'} \left[ A_T^{\mathbb{Q}'} | \mathcal{F}_t^{Q+} \right], 0 \leq t \leq T, \mathbb{Q} - a.s., \forall \mathbb{Q} \in \mathcal{Q} \quad (1.4.4)$$

where  $\mathcal{Q}(t, \mathbb{Q}, \mathbb{F}_+) := \left\{ \mathbb{Q}' \in \mathcal{Q} \text{ s.t. } \mathbb{Q}' = \mathbb{Q} \text{ on } \mathcal{F}_t^+ \right\}$ .

The corresponding equation for the FVA-deducted value  $\Theta$  would appear as

$$\begin{aligned} \Theta_t &= \text{ess inf}_{\mathbb{Q}' \in \mathcal{Q}(t, \mathbb{Q}, \mathbb{F}_+)} \mathbb{E}_t^{\mathbb{Q}'} \left[ \sum_{t < T_i} \beta_t^{-1} \beta_{T_i} \omega_i (S_{T_i} - K_i)^+ \right. \\ &\quad \left. - \int_t^T \beta_t^{-1} \beta_s \lambda_s \left( \Theta_s - \text{EC}_s^{\mathbb{Q}'}(\varrho) \right)^+ ds \right], t \in [0, T], \mathbb{Q} - a.s. \end{aligned} \quad (1.4.5)$$

## Equations in the Markovian Setting

By contrast, in the Markovian setting of Section 1.3 with VaR-like specification of economic capital, we can make rigorous statements. According to the second order BSDE theory introduced in Soner, Touzi, and Zhang (2012), the PDE (1.3.3) becomes :

$$\begin{cases} u_i(T_i, S) = u_{i+1}(T_i, S) + \omega_i (S - K_i)^+ \text{ on } (0, \infty) \\ \partial_t u_i + \inf_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} \left[ \mathcal{A}_S^{bs} u_i - \lambda(u_i - f\sigma S |\partial_S u_i - \eta|)^+ \right] - ru_i = 0 \text{ on } [T_{i-1}, T_i) \times (0, \infty) \end{cases} \quad (1.4.6)$$

Let  $u$  be defined by  $u_i(t, S)$  on each strip  $(T_{i-1}, T_i] \times (0, \infty)$ . The FVA can be defined as  $\Theta^{\lambda=0} - \Theta$  and the ensuing KVA process is given as (cf. (1.2.3) and (1.2.1)) :

$$\text{KVA}_t(\varrho) = h \operatorname{ess\,sup}_{\mathbb{Q}' \in \mathcal{Q}(t, \mathbb{Q}, \mathbb{F}_+)} \mathbb{E}_t^{\mathbb{Q}'} \left[ \int_t^T e^{-(r+h)(s-t)} f \sqrt{\frac{d\langle \varrho \rangle_s}{ds}} ds \right], t \in [0, T], \mathbb{Q} - a.s. \quad (1.4.7)$$

where  $\sqrt{\frac{d\langle \varrho \rangle_t}{dt}} = a_t^{1/2} S_t |\partial_S u(t, S_t) - \eta(t, S_t)|$ . In the case where  $\eta = (1 - \alpha) \partial_S u$ , we obtain

$$\text{KVA}_t(\varrho) = w(t, S_t)$$

where

$$\begin{cases} w(T, S) = 0 \text{ on } (0, \infty) \\ \partial_t w + \sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} [\mathcal{A}_S^{bs} w + \alpha h f \sigma S |\partial_S u|] - (r + h)w = 0 \text{ on } [0, T) \times (0, \infty) \end{cases} \quad (1.4.8)$$

in which (cf. (1.4.6)) :

$$\begin{cases} u_i(T_i, S) = u_{i+1}(T_i, S) + \omega_i(S - K_i)^+ \text{ on } (0, \infty) \\ \partial_t u_i + \inf_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} [\mathcal{A}_S^{bs} u_i - \lambda(u_i - \alpha f \sigma S |\partial_S u_i|)^+] - r u_i = 0 \text{ on } [T_{i-1}, T_i) \times (0, \infty) \end{cases} \quad (1.4.9)$$

## 1.5 Optimal Transportation Approach

Since vanilla call options are liquidly traded, their time 0 price components

$$\mathbb{E} [\beta_{T_i}(S_{T_i} - K_i)^+]$$

should not be seen as subject to model risk, but calibrated to the market. Hence, we need to refine our preliminary UVM assessment of model risk in order to account for these calibration constraints. For simplicity we consider a single call option  $(T, K)$  and we set  $\lambda = 0$ , focusing on KVA in this section. Hence, the system (1.4.6) reduces to a single PDE with  $\lambda = 0$ , with solution denoted by  $u$ .

Tan and Touzi (2013) consider the optimal transportation problem consisting of minimizing a cost among all continuous semimartingales given initial and terminal distributions. They show an extension of the Kantorovitch duality to this context and suggest a finite-difference scheme combined with the gradient projection algorithm to approximate the dual value. Their results can be applied to our setup as follows.

Let  $\mu_0 = \delta_{S_0}$  denote the Dirac measure on the initial value of  $S_0$  and let  $\mu_T$  denote the marginal distribution of  $S_T$ , inferred by calibration to the market prices of all European call options with maturity  $T$  (assuming quotations available for all strikes). Let

$$\begin{aligned} \mathcal{Q}(\mu_0) &= \{ \mathbb{Q} \in \mathcal{Q} : \mathbb{Q} \circ S_0^{-1} = \mu_0 \} \text{ and} \\ \mathcal{Q}(\mu_0, \mu_T) &= \{ \mathbb{Q} \in \mathcal{Q}(\mu_0) : \mathbb{Q} \circ S_T^{-1} = \mu_T \} \end{aligned}$$

the last not being empty in our setting given the arguments of Tan and Touzi (2013, Remark 2.3).

The time-0 KVA with model uncertainty and terminal marginal constraint is defined as follows :

$$\text{KVA}_0(\varrho) = h \sup_{\mathbb{Q} \in \mathcal{Q}(\mu_0, \mu_T)} \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T e^{-(r+h)s} f \sqrt{\frac{d\langle \varrho \rangle_s}{ds}} ds \right], \quad (1.5.1)$$

where  $\varrho$  represents the portfolio loss in this setting, that is, the loss and profit of the bank in a world with uncertain volatility subject to the law of  $S_T$ . However, it is not clear how to

extrapolate the theory of [Tan and Touzi \(2013\)](#) to valuation at future time points when only the unconditional law of  $S_T$  is known. Hence for the sake of tractability we conservatively assume that  $\varrho$  in (1.5.1) is the UVM one and we only apply the constraint to the outer expectation in (1.5.1) (as opposed to the conditional expectations that are hidden in  $\varrho$ ).

With this understanding of (1.5.1), given any measure  $\nu$ , we define

$$\nu(\phi) = \int_{\mathbb{R}} \phi(x) \nu(dx)$$

on the set  $C_b(\mathbb{R}^d)$  of all bounded continuous functions  $\phi$  on  $\mathbb{R}^d$ . We can readily check that [Tan and Touzi \(2013, Assumptions 3.1-3.3\)](#) are satisfied. Hence, by an application of their main duality result, we can rewrite the  $KVA_0$  as

$$KVA_0(\varrho) = \inf_{\phi \in C_b(\mathbb{R}^d)} \left\{ \mu_0(\Phi_0) - e^{-(r+h)T} \mu_T(\phi) \right\} \quad (1.5.2)$$

where the “pseudo-payoff function”  $\phi$  corresponds to a Lagrangian for the constrained optimization problem (1.5.1) and where

$$\Phi_0(x) = \sup_{\mathbb{Q} \in \mathcal{Q}(\delta_x)} \mathbb{E}^{\mathbb{Q}} \left[ e^{-(r+h)T} \phi(S_T) + \int_0^T e^{-(r+h)s} hf \sqrt{\frac{d\langle \varrho \rangle_s}{ds}} ds \right] \quad (1.5.3)$$

Hence, the KVA in an optimal transportation (OT) setting can be represented as an infimum of KVAs in modified UVM setting.

### 1.5.1 Equations in the Markovian Setting

In the Markovian setting of Section 1.3, we consider the probability measures  $\mathbb{Q}$  on the canonical space  $(\Omega, \mathcal{F}_T)$ , under which the canonical process  $C$  is a local martingale on  $[t, T]$ . Define  $\mathcal{Q}_t$  as the collection of all such martingale probability measures  $\mathbb{Q}$  such that  $a_s^{1/2} \in [\underline{\sigma}, \bar{\sigma}]$   $d\mathbb{Q} \times ds$  -a.e. on  $\Omega \times [t, T]$ . Denote  $\mathcal{Q}_{t,x} := \{\mathbb{Q} \in \mathcal{Q}_t \text{ s.t. } \mathbb{Q}(S_s = x, 0 \leq s \leq t) = 1\}$ . For any  $\phi \in C_b(\mathbb{R}^d)$ , let

$$\Phi(t, x) = \sup_{\mathbb{Q} \in \mathcal{Q}_{t,x}} \mathbb{E}^{\mathbb{Q}} \left[ e^{-(r+h)(T-t)} \phi(S_T) + \int_t^T e^{-(r+h)(s-t)} hf \sqrt{\frac{d\langle \varrho \rangle_s}{ds}} ds \right] \quad (1.5.4)$$

where  $\sqrt{\frac{d\langle \varrho \rangle_t}{dt}} = a_t^{1/2} S_t |\partial_S u(t, S_t) - \eta(t, S_t)|$ , in which  $u$  is the solution to (1.4.6) with  $\lambda = 0$ .

Then, in the case where  $\eta = (1 - \alpha) \partial_S u$ ,  $\Phi$  is a viscosity solution to the dynamic programming equation

$$\begin{cases} \Phi(T, S) = \phi(S) \text{ on } (0, \infty) \\ \partial_t \Phi + \sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} [\mathcal{A}_S^{bs} \Phi + \alpha hf \sigma S |\partial_S u|] - (r + h) \Phi = 0 \text{ on } [0, T) \times (0, \infty) \end{cases} \quad (1.5.5)$$

In view of (1.5.2), in the present OT setup,  $KVA_0$  is obtained as the minimum of

$$\Phi(0, S_0) - e^{-(r+h)T} \int_{\mathbb{R}} \phi(x) \mu_T(dx) \quad (1.5.6)$$

over  $\phi \in C_b(\mathbb{R}^d)$ . This minimization is numerically achieved by the Nelder-Mead simplex algorithm.

As a sanity check, observe that, if  $\mu_T$  is the log-normal probability density function and  $\underline{\sigma} = \bar{\sigma} = \sigma$  then (1.5.6) is exactly the time 0 KVA of Section 1.3, independent of  $\phi$ .

## 1.6 Numerical Results

Figure 1.1 shows the results obtained by solving the related PDEs (and minimizing (1.5.6) in the OT setup) without model uncertainty as of Section 1.3 (top panel), with UVM uncertainty as of Section 1.4 (middle panel) and with OT uncertainty as of Section 1.5 (bottom panel), for a level of the mis-hedge parameter  $\alpha$  increasing from 0 to 100%.

The main observation from the top panel is that, unless the hedge is very good (of the order of 25% of mis-hedge or less), the KVA dominates the FVA, and becomes about ten times greater than the FVA in the absence of hedge ( $\alpha = 1$ ). This is logical given that economic capital (EC) has only an indirect reduction effect on the FVA, whereas it directly sizes the KVA.

Going to the middle panel, the FVA changes little, but both  $u$  and the KVA (unless the hedge is almost perfect) are tremendously impacted by the uncertainty on the volatility. Regarding the KVA, this is in line with the fact that it is the cost of a risk measure, which nonlinearly amplifies the impact of perturbations to its input data.

In reality the time 0 price of a vanilla option such as the one considered in our numerics is given by the market, so there is no model risk on it, but only on the KVA. This is what is reflected by the OT bottom panel. The model risk on the KVA component however is essentially the same as in the UVM case, because it is conservatively assessed by using the UVM  $u$  in (1.5.5), fault of a developed theory of valuation at future time points under uncertain volatility subject to the unconditional law of  $S_T$ .

XVA desks, KVA in particular, are the first consulted desks in all major trades today. Our results in a toy model where all the quantities of interest can be computed exactly (modulo the numerical error on the PDE solutions) emphasize that, accounting for model risk, the relative importance of the KVA should become even larger. Moreover one can easily imagine how to transpose these results to the setup of Albanese and Crépey (2017) where each option payoff  $(S_{T_i} - K_i)^+$  is replaced by the CVA exposure of the bank to the default at time of its counterparty  $i$ , at the (random) time  $T_i$ , with corresponding position of the bank  $\omega_i S_{T_i}$  and margins received by the bank  $\omega_i K_i$ . However in this case a relevant risk measure really needs to be computed at a one-year horizon (as opposed to instantaneous in (1.3.1)), in order to leave time to credit events to develop. This points out to developments of a slightly different nature, which would be interesting to develop.

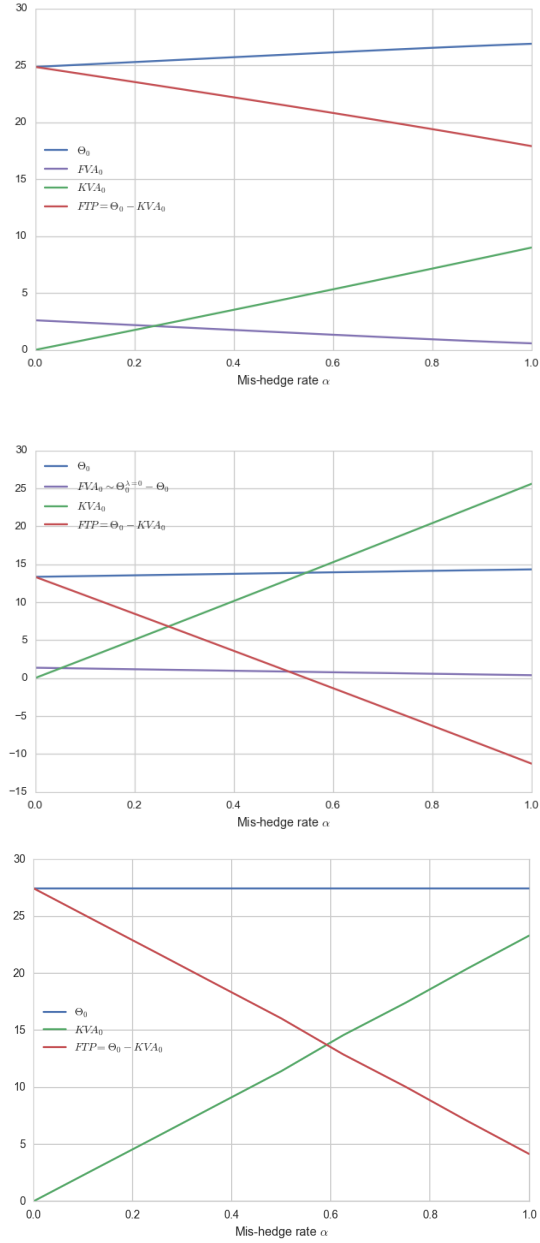


FIGURE 1.1 – XVAs and FTP as a function of the mis-hedge parameter  $\alpha$ . *Top* : Without model uncertainty. *Middle* : With UVM uncertainty ( $\underline{\sigma} = 15\%$ ,  $\bar{\sigma} = 60\%$ ). *Bottom* : With OT uncertainty ( $\underline{\sigma} = 15\%$ ,  $\bar{\sigma} = 60\%$ ,  $\sigma = 30\%$ ).

# Central Clearing Valuation Adjustment

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This chapter is based on [Armenti and Crépey \(2017a\)](#).

## 2.1 Introduction

To cope with counterparty risk, the current trend in regulation is to push dealers to clear their trades via CCPs, i.e. central counterparties (also known as clearing houses). Progressively, central clearing is even becoming mandatory for vanilla products. Centrally cleared trading mitigates counterparty risk through an extensive netting of all transactions. Moreover, on top of the variation and initial margins that are used in the context of bilateral transactions, a CCP imposes its members to mutualize losses through an additional layer of protection, called the default or guarantee fund, which is pooled between the clearing members.

In this paper we develop the vision of a clearing house effectively eliminating counterparty risk (we do not incorporate the default of the clearing house in our setup), but at a certain cost for the members that we analyze. For this purpose, we develop an XVA (costs) analysis of centrally cleared trading, parallel to the one that has been developed in the last years for bilateral transactions.

### 2.1.1 Review of the CCP Literature

[Duffie \(2010\)](#) and [Cont, Santos, and Moussa \(2013\)](#) dwell upon the danger of creating “too big to fail” financial institutions, including, potentially, clearing houses.

Collateralization, whether in the context of centrally cleared trading or of bilateral trading under “standard CSA” (credit support annex), which is the emerging bilateral trading alternative to centrally cleared trading, requires a huge amount of cash or liquid assets. This puts a high pressure on liquidity, an issue addressed in [Aitken and Singh \(2009\)](#), [Singh \(2010\)](#), [Levels and Capel \(2012\)](#) and [Duffie, Scheicher, and Vuillemeys \(2015\)](#). Relying on metrics à la [Eisenberg and Noe \(2001\)](#), [Amini, Filipović, and Minca \(2015\)](#) assess the systemic risk and incentivization properties of a CCP design where, in order to spare the clearing members from liquidation costs, in situations of financial distress, the clearing members could temporarily withdraw from their default fund contributions to post variation margin.

[Avellaneda and Cont \(2013\)](#) consider the optimal liquidation of the portfolio of a defaulted member by the clearing house.

Clearing is typically organized by asset classes, so that service closure of the CCP on one asset class does not harm its activity on other markets—and also because otherwise, in case of the default of a member, holders of less liquid assets (e.g. CDS contracts) are penalized with respect to holders of more liquid assets (e.g. interest rate swaps). As a consequence, the multilateral netting benefit of CCPs comes at the expense of a loss of bilateral netting across asset classes (see [Duffie and Zhu \(2011\)](#)). [Cont and Kokholm \(2014\)](#) claim that the former effect typically dominates the latter.

But Ghamami and Glasserman (2017) show that, accounting for bilateral cross-asset netting, the higher regulatory capital and margin requirements adopted for bilateral contracts do not necessarily create the intended cost incentive in favor of central clearing.

Cont, Mondescu, and Yu (2011) and Pallavicini and Brigo (2013) analyze the pricing implications of the differences between the margining procedures involved in bilateral and centrally cleared transactions.

Until recently, the cost analysis of CCPs, our focus in this paper, was only considered in an old business finance literature reviewed in Knott and Mills (2002), notably Fenn and Kupiec (1993). In the last years, new papers have appeared in this direction. Under stylized assumptions, Arnsdorf (2012) derives an explicit approximation to a CCVA (using the terminology of the present paper), including effects such as wrong way risk (meant as procyclicality of the margins), credit dependence between members and left tailed distributions of their P&Ls. Ghamami (2015) proposes a static one-period model where a CCVA can be priced by Monte-Carlo. Brigo and Pallavicini (2014) extend the bilateral counterparty risk dynamic setup of their previous papers to centrally cleared trading. However, they ignore the default fund and the credit risk dependence issues that are inherent to the position of a clearing member.

### 2.1.2 Contributions and Outline

This paper develops an XVA (costs) analysis of centrally cleared trading, parallel to the one that has been developed in the last years for bilateral transactions (see e.g. Crépey, Bielecki, and Brigo (2014, Parts II and III) or Brigo, Morini, and Pallavicini (2013)). A dynamic framework incorporates the sequence of cash flows involved in the waterfall of resources of the clearing house. As compared with Arnsdorf (2012) and Ghamami (2015), our CCVA accounts not only for the central clearing analog of the CVA, which is the cost for a member of its losses on the default fund in case of other members' defaults, but also for the cost of funding its margins (MVA) and for the cost of the capital (KVA) that is implicitly required from members through their default fund contributions (and for completeness and reference we also compute a DVA term).

The framework of this paper can be used by a clearing house to find the right balance between initial margins and default fund in order to minimize the CCVA (subject to the regulatory constraints), hence optimize its costs to the members for a given level of resilience. A clearing house can also use it to analyze the benefit for a dealer to trade centrally as a member rather than on a bilateral basis, or to help its members manage their CCVA (regarding the question for instance of how much of these costs they could consider passing to their clients).

The paper is organized as follows. Section 2.2 presents our clearing house setup. The waterfall of resources of the CCP is described in Section 2.3. The CCVA analysis is conducted in Section 2.4. Section 2.5 introduces the common shock model that is used for the default times of the members of the clearing house. Section 2.6 provides an executive summary of the centrally cleared XVA analysis of this paper and recalls for comparison purposes the bilateral CSA XVA methodology of Crépey and Song (2016). Section 2.7 designs an experimental framework used in the numerics of Section 2.8. Section 2.9 concludes. Regulatory formulas are recalled in Section 2.10.1. **Proofs of all lemmas are deferred to Section 2.10.4.**

### 2.1.3 Basic Notation and Terminology

$\int_a^b = \int_{(a,b]}$ ;  $x^\pm = \max(\pm x, 0)$ ;  $\delta_a$  represents a Dirac measure at a point  $a$ ;  $\lambda$  denotes the Lebesgue measure on  $\mathbb{R}_+$ . Unless otherwise stated, a filtration satisfies the usual conditions; a price process is a *special semi-martingale* in a càdlàg version; all inequalities between random quantities are meant almost surely or almost everywhere, as suitable; all the cash flows are assumed to be integrable whenever required; by “martingale” we mean local martingale unless

otherwise stated, but true martingale is assumed whenever necessary. This means that we only derive local martingale properties. Usually in applications one needs true martingales, but this is not a real issue in our case, where even square integrability follows from additional assumptions postulated when dealing with BSDEs, which are our main pricing tool in this paper.

## 2.2 Clearing house Setup

We model a service of a clearing house dedicated to trading between its members, labeled by  $i \in N = \{0, \dots, n\}$ .

### 2.2.1 From Bilateral to Centrally Cleared Trading

In a centrally cleared setup, the clearing house interposes itself in all transactions, becoming “the buyer to every seller and the seller to every buyer”. All the transactions between the clearing house and a given member are netted together. See Figure 2.1 for an example, where the circled numbers in the left (respectively right) diagram show the gross positions of  $n = 3$  counterparties in a CSA setup (respectively their net positions with the CCP after the introduction of the latter in the middle).

In addition to interfacing all trades, the clearing house asks for several layers of guarantee to be posted by the members against counterparty risk, including a default fund that is pooled between the clearing members.

The benefits of centrally cleared trading are multilateral netting benefit and mutualization of risk. The drawbacks are an increase of systemic risk, where “too big to fail” CCPs might be created, liquidity risk, due to the margin requirements, and a loss of bilateral netting across asset classes (cf. [Duffie \(2010\)](#) and [Cont, Santos, and Moussa \(2013\)](#)).

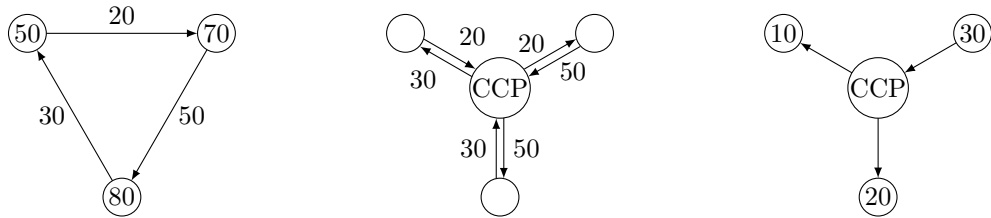


FIGURE 2.1 – From bilateral to centrally cleared trading.

### 2.2.2 Liquidation Procedure

The mandate of a CCP is to liquidate over a few days the portfolio of a defaulted member. During the liquidation period, the CCP bears the risk of the portfolio. The trades with a defaulted member are typically reallocated by means of auctions among the surviving members and/or by a gradual liquidation of its assets in the market.

For ease of analysis in this paper, we assume the existence of a risk-free “buffer” that is used by the clearing house for replacing defaulted members in their transactions with others at the end of a liquidation period of length  $\delta$  (the defaulted transactions already involving the buffer as one counterparty are simply terminated). We assume that during the liquidation period, the promised contractual cash flows and the hedge of a defaulted member are taken over by the CCP.

### 2.2.3 Pricing Framework

Let  $(\Omega, \mathbb{G}, \mathbb{Q})$  represent a stochastic pricing basis, with  $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$ , such that all our processes are  $\mathbb{G}$  adapted and all the random times of interest are  $\mathbb{G}$  stopping times. Expectation under  $\mathbb{Q}$  and  $(\mathcal{G}_t, \mathbb{Q})$  conditional expectation are denoted by  $\mathbb{E}$  and  $\mathbb{E}_t$ . We denote by  $r$  a  $\mathbb{G}$  progressive OIS rate process and by  $\beta_t = e^{-\int_0^t r_s ds}$  the corresponding discount factor. An OIS (overnight index swap) rate is together the best market proxy for a risk-free rate and the reference rate for the remuneration of the collateral.

For each member  $i$ , we denote by  $D_t^i$  the process of the cumulative contractual cash flows of its portfolio with the CCP (“promised dividend” process ignoring counterparty and funding risk), assumed of finite variation. We denote by  $P_t^i$  the mark-to-market of its portfolio ignoring counterparty and funding risk, i.e.

$$\beta_t P_t^i = \mathbb{E}_t \left( \int_t^{\bar{T}} \beta_s dD_s^i \right), \quad t \in [0, \bar{T}] \quad (2.2.1)$$

where  $\bar{T}$  is the final maturity of the CCP service portfolio, assumed held on a run-off basis (as is standard in any pricing or risk model). All cash flows and values are considered from the point of view of the clearing house, e.g.  $P_t^i = 1$  means that the member  $i$  is short of a mark-to-market value equal to one (disregarding margins) toward the clearing house at time  $t$ . Since all trades are between the members, we have  $\sum_{i \in N} P^i = 0$ .

## 2.3 Margin Waterfall Analysis

The mark-to-market pricing formula (2.2.1) ignores the counterparty risk of the member  $i$ , with default time  $\tau_i$  and survival indicator process  $J^i = \mathbb{1}_{[0, \tau_i)}$ . As a first counterparty risk mitigation tool, the members are required to exchange variation margins that track the mark-to-market of their portfolios. A clearing house can call for variation margins at every time of a margin grid of step  $h$ , e.g. twice a day.

However, various frictions and delays, notably the liquidation period  $\delta$ , imply gap risk, which is the risk of a gap between the variation margin and the debt of a defaulted member at the time of liquidation of its portfolio. This is a special concern for certain classes of assets, such as credit derivatives, that may have quite unpredictable cash flows (see Crépey and Song (2016)).

This is why another layer of collateralization, called initial margins, is maintained in centrally cleared transactions as well as in bilateral transactions under standard CSA (the emerging bilateral trading alternative to centrally cleared trading). Initial margins are also dynamically updated, based on some risk measure of the variation-margined P&L of each member computed over the time horizon  $\delta' = \delta + h$  of the so called margin period of risk (maximal time  $h$  elapsed since the last margin call before the default plus liquidation period  $\delta$  between default and liquidation).

Gap risk is magnified by wrong-way risk, which is the risk of adverse dependence between the positions and the credit risks of the members. One may also face credit contagion effects between members (wrong-way and contagion risk are of special concern regarding credit derivatives). Clearing houses deal with such extreme and systemic risk through a default fund mutualized between the clearing members. The default fund contribution of each member is primarily intended to reimburse the losses triggered by its own default, but, if rendered necessary by exhaustion of the previous layers of the waterfall, it can also be used for reimbursing the losses due to the defaults of other members.

### 2.3.1 Margins

Let  $lh$ , with  $l \geq 0$ , represent the times of the variation and initial margin calls, and let  $lT$ , with  $T$  a multiple of  $h$  (e.g.  $h = \text{one day}$  and  $T = \text{one month}$ ), represent the times of update of the default fund contributions.

Consistent with our sign convention that all cash flows and values are seen from the perspective of the clearing house, we count a margin positively when it is posted by a member and we define the variation margin  $VM^i$ , initial margin  $IM^i$  and default fund contribution  $DFC^i$  of the member  $i$  as the piecewise constant process reset at the respective grid times following, respectively (while the member  $i$  is alive) :

$$VM_{lh}^i = P_{lh-}^i, \quad IM_{lh}^i = \rho_{lh}^i, \quad DFC_{lT}^i = \varrho_{lT}^i \quad (2.3.1)$$

where  $\rho^i$  and  $\varrho^i$  refer to suitable risk measures as explained below. Note that (2.3.1) defines the level of reset of the respective cumulative amounts. Starting from  $VM_0^i = P_{0-}^i$ ,  $IM_0^i = \rho_0^i$  and  $DFC_0^i = \varrho_0^i$ , the corresponding updates at grid times are  $(P_{lh-}^i - P_{(lh-h)-}^i)$ ,  $(\rho_{lh}^i - \rho_{lh-h}^i)$  and  $(\varrho_{lT}^i - \varrho_{lT-T}^i)$ .

**Remark 2.3.1** In practice, the variation margin only tracks the mark-to-market of the portfolio up to some thresholds, or free credit lines of the members, and up to minimal transfer amounts devoted to avoiding useless updates. These features, which can be important in the case of bilateral transactions, are omitted here as negligible in the case of centrally cleared transactions.

Let

$$L_{t,t+\delta'}^i = P_{t+\delta'}^i + \int_{[t,t+\delta']} e^{\int_s^{t+\delta'} r_u du} dD_s^i - P_{t-}^i. \quad (2.3.2)$$

represents the loss process of the CCP between  $t$  and  $t + \delta'$ . In particular, at margin call times  $t = lh$ , we have, in view of the specification of the variation margin by the first identity in (2.3.1) :

$$L_{lh,lh+\delta'}^i = P_{lh+\delta'}^i + \int_{[lh,lh+\delta']} e^{\int_s^{lh+\delta'} r_u du} dD_s^i - VM_{lh}^i \quad (2.3.3)$$

which is the variation-margined loss-and-profit of the member  $i$  at the time horizon  $\delta' = \delta + h$  of the margin period of risk (cumulative loss-and-profit also accounting for all the contractual cash flows capitalized at the risk-free rate during the margin period of risk  $[t, t + \delta']$ ). The risk measure used for fixing the initial margins is a univariate risk measure computed at the level of each member individually, which we write as

$$\rho_{lh}^i = \rho(L_{lh,lh+\delta'}^i) \quad (2.3.4)$$

where  $\rho$  can be value at risk, expected shortfall, etc. . The dependence between the portfolios of the members is only reflected in the initial margins through the structural constraint that  $\sum_{i \in N} P^i = 0$ .

**Remark 2.3.2** Historically, for computing initial margins, CCPs have been mostly using the SPAN methodology, introduced by the Chicago Mercantile Exchange in the 80s. This methodology is based, for each member, on the consideration of the most unfavorable among sixteen reference scenarios (see Kupiec and White (1996)). Nowadays, value at risk methodologies tend to become the standard.

Unless defaults happen, margins do not imply any transfer of ownership and can be seen in this sense as a loan by the posting member. By contrast, default fund contributions can be consumed in case of other members' defaults, hence they should really be viewed as capital put at the disposal of the CCP by the clearing members. The "Cover two" EMIR rule prescribes to size

the default fund as, at least, the maximum of its largest exposure and of the sum of its second and third largest exposures to the clearing members (see Section 2.10.2). This is only a regulatory minimum and sometimes more conservative rules are used, such as a default fund set as the sum of the two largest exposures. It is then allocated between the clearing members by some rule, e.g. proportionally to their initial margins. At a more theoretical level, the mutualization rationale of the default fund calls for the use of multivariate risk measures, which we write in an abstract fashion as

$$\varrho_{lT}^i = \varrho_i \left( \left( L_{lT, lT+\delta'}^j - \text{IM}_{lT}^j \right)_{j; J_{lT}^j=1} \right) \quad (2.3.5)$$

(or an analog formula involving not only the  $L_{lT, lT+\delta'}^j$ , but also intermediary  $L_{\cdot, \cdot+\delta'}^j$  between  $(l-1)T$  and  $lT$  to refrain members from temporarily closing their positions right before  $lT$  in order to avoid to contribute to the default fund).

Regarding the distributions that are used for members loss-and-profits in all these risk measure computations, since the crisis, the focus has shifted from the cores of the distributions, dominated by volatility effects, to their queues, dominated by scenarios of crisis and default events. For the determination of the initial margins, Gaussian VaR models are generally banned since the crisis and CCPs typically focus on either Pareto laws or on historical VaR. Stressed scenarios and parameters are used for the determination of the default fund.

Note that margin schemes as above, even, in the case of the default fund contributions, possibly based on multivariate risk measures (cf. (2.3.5)), only account for asset dependence between the portfolios of the members, ignoring credit risk and contagion effects between members. This is in line with the mandate of a clearing house to mitigate (i.e. put a cap on) its exposure to the members by means of the margins, in case a default would happen, where a default is viewed as a totally unpredictable event. On top of the margins, add-ons are sometimes required from members with particularly high credit or concentration risk.

We refer the reader to Ghamami (2015), Lopez, Harris, Hurlin, and Pérignon (2017), Menkveld (2014) or Chapter 4 for alternative margin schemes and default fund specifications. Good margining schemes should guarantee the required level of resilience to the clearing house at a bearable cost for the members. Two points of concern are procyclicality, in particular with haircuts that increase with the distress of a member, and liquidity, given the generalization of central clearing and collateralization. Capponi and Cheng (2016) construct a model which endogenizes collateral, making it part of an optimization problem where the CCP maximizes profit by controlling collateral and fee levels.

### 2.3.2 Breaches

The default time of the member  $i$  is modeled as a stopping time  $\tau_i$  with an intensity process  $\gamma^i$ . In particular, any event  $\{\tau_i = t\}$ , for a fixed time  $t$ , has zero probability and can be ignored in the analysis. For every time  $t \geq 0$ , let

$$\bar{t} = t \wedge \bar{T}, \quad t^\delta = t + \delta, \quad \bar{t}^\delta = \mathbb{1}_{t < \bar{T}} t^\delta + \mathbb{1}_{t \geq \bar{T}} \bar{T} \quad (2.3.6)$$

and let  $\hat{t}$  denote the greatest margin call time  $lh \leq t$ . We denote by

$$\mathcal{C}^i = \text{VM}^i + \text{IM}^i + \text{DFC}^i \quad (2.3.7)$$

the overall collateral process of the member  $i$ . We assume that collateral posted is remunerated OIS and that the CCP substitutes itself to a defaulted member during its liquidation period, including regarding these collateral OIS remuneration cash flows. In our model, collateral earns OIS but collateral OIS earnings are transferred as a remuneration to the posting member, they do not stay in the collateral accounts. Hence, the amount of available collateral for the liquidation of a defaulted member does not accrue at the OIS rate but stays constant during the liquidation

period. As a consequence, we have  $C_t^i = \bar{C}_t^i$  for  $t \leq \tau_i$  and the process  $C$  is stopped at time  $\hat{\tau}_i$ . For each member  $i$ , we write

$$\begin{aligned}\Delta_t^i &= \int_{[\tau_i, t]} e^{\int_s^t r_u du} dD_s^i, \quad Q_t^i = P_t^i + \Delta_t^i, \quad \varepsilon_i = \left(Q_{\tau_i^\delta}^i - \bar{C}_{\tau_i}^i\right)^+, \\ \chi_i &= -\mathbb{1}_{\varepsilon_i=0} Q_{\tau_i^\delta}^i - \mathbb{1}_{\varepsilon_i>0} \left(\bar{C}_{\tau_i}^i + R_i \varepsilon_i\right), \\ \xi_i &= Q_{\tau_i^\delta}^i + \chi_i = \mathbb{1}_{\varepsilon_i>0} \left(Q_{\tau_i^\delta}^i - \bar{C}_{\tau_i}^i - R_i \varepsilon_i\right) = (1 - R_i) \varepsilon_i\end{aligned}\tag{2.3.8}$$

where  $\Delta_t^i$  represents the cumulative contractual dividends capitalized at the risk-free rate that fail to be paid by member  $i$  from time  $\tau_i$  onwards. These dividends are promised but unpaid due to the default of the member  $i$  at  $\tau_i$ . Hence, they also belong to the exposure of the CCP to the default of the member  $i$ . More precisely, as will be understood in more detail from the proof of Lemma 2.3.1,  $\chi_i$  corresponds to a terminal cash flow closing the position of the defaulted member  $i$ , paid by the CCP to the estate of the defaulted member at time  $\tau_i^\delta$ ;  $\varepsilon_i$  corresponds to the raw exposure of the CCP to the default of the member  $i$ ;  $\xi_i$  is the exposure accounting for an assumed recovery rate  $R_i$  of the member  $i$ .

In fact, in the context of centrally cleared trading, by liquidation of a defaulted member, we simply mean the liquidation of its CCP portfolio, as opposed to the legal liquidation, by a mandatory liquidator, that can take several years (the New York branch of Lehman was legally liquidated in December 2013, more than five years after Lehman's default). In particular, there is typically no recovery to expect on a defaulted member, i.e.  $R_i = 0$ . Moreover, in our context, we suppose that losses are defined as pure “market losses” only. The reason why we introduce recovery coefficients is for the discussion regarding DVA and DVA2 in Section 2.4 and for comparison with the bilateral trading setup of Section 2.6.

Note that we do not exclude joint defaults in our setup. In fact, joint defaults, which can be viewed as a form of “instantaneous contagion”, is the way we will introduce credit dependence between members in Section 2.5. For  $Z \subseteq N = \{0, \dots, n\}$ , we denote by  $\tau_Z \in \mathbb{R}^+ \cup \{\infty\}$  the time of joint default of names in the subset  $Z$  and only in  $Z$ . At this stage we consider all the costs from the perspective of the CCP and the community of the surviving members altogether. The allocation of these costs between the CCP and the surviving members will be considered in Section 2.3.3. We call realized breach of a (possibly joint) default event the residual loss to the CCP after all the collateral of the defaulted member(s) has been consumed.

**Lemma 2.3.1** *At each liquidation time  $\tau_Z^\delta = \tau_Z + \delta$  with  $\tau_Z < \bar{T}$ , the realized breach of the CCP is given by*

$$B_{\tau_Z^\delta} = \sum_{i \in Z} \xi_i \tag{2.3.9}$$

### 2.3.3 Equity and Default Fund Replenishment Principle

We proceed with the description of the next layers of the waterfall of resources of the clearing house, namely the equity and the default fund.

If the default of a member entails a positive breach, then the first payer (although to a typically quite limited extent) is the clearing house itself (before the surviving members), via its equity  $E$ .

**Remark 2.3.3** The regulation (e.g. EMIR) does not necessarily require that the CCP be the first payer in case of a realized breach. However, CCPs typically take the equity tranche of this risk, as a good management incentive. See for example [Capponi, Cheng, and Sethuraman \(2017\)](#) where the authors provide an economic explanation for this management incentive.

Specifically, at times  $lY$ ,  $l \geq 0$ , where  $Y$  is a multiple of  $T$  (e.g. one year whereas  $T$  is one month), the equity process  $E$  is reset by the clearing house at some target level  $E_{lY}^*$ , the “skin-in-the-game” of the clearing house for the time period  $[lY, (l+1)Y]$ . In the meantime, the equity is used as first resource for covering the realized breaches, i.e., at each  $t = \tau_Z^\delta$  with  $\tau_Z < \bar{T}$ , we have

$$\Delta E_t = -(B_t \wedge E_{t-}) \quad (2.3.10)$$

The part of the realized breach left uncovered by the equity,  $(B_t - E_{t-})^+$ , is covered by the surviving members through the default fund, which they refill instantaneously by the following rule, at each  $t = \tau_Z^\delta$  with  $\tau_Z < \bar{T}$  (see Figure 2.2) :

$$\epsilon_t^i = (B_t - E_{t-})^+ \frac{J_t^i \text{DFC}_t^i}{\sum_{j \in N} J_t^j \text{DFC}_t^j} \quad (2.3.11)$$

proportionally to their current default fund contributions  $\text{DFC}_t^i$  (or other keys of repartition such as their initial margins, the notionals of their positions, or for example the multivariate shortfall risk allocation presented in Chapter 4).

In sum, the margins and the default fund contributions  $\text{VM}_{lh}^i, \text{IM}_{lh}^i$  and  $\text{DFC}_{lT}^i$  are reset at their respective grid times by the surviving members according to (2.3.1); the equity is reset at the times  $lY$  by the clearing house and is used for covering the first levels of realized breaches at liquidation times according to (2.3.10); the losses in case of realized breaches above the residual equity are covered at liquidation times by the surviving members according to (2.3.11) (see Figure 2.2).

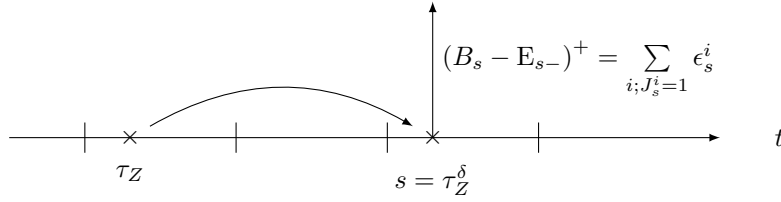


FIGURE 2.2 – Margin cash flows : resets at margin call grid times and refill of the default fund at liquidation times.

**Remark 2.3.4** The total size of the default fund is  $\sum_{j \in N} J^j \text{DFC}^j$ , a quantity also referred to as the funded default fund. The unfunded default fund refers to the additional amounts members may have to pay via the above default fund replenishment principle in case of defaults of other members.

More precisely,

$$u_{lT}^i = \left( \sum_{lT-T < \tau_Z^\delta < lT} \epsilon_{\tau_Z^\delta}^i - \text{DFC}_{lT-T}^i \right)^+ \quad (2.3.12)$$

represents the unfunded default fund contribution of the member  $i$  for the period  $(lT - T, lT)$ .

The service closure, i.e. the closure of the activity of the clearing house on a given market or service, is usually specified in terms of events such as the unfunded default fund  $\sum_{j \in N} J_{lT}^j u_{lT}^j$  reaching a cap given as, e.g.,  $2 \sum_{j \in N} J_{lT-T}^j \text{DFC}_{lT-T}^j$ , i.e. twice the funded default fund. Given the high levels of initial margins that are used in practice, this is a very extreme tail event. Moreover, in case of service closure, the risk of a member is bounded above by the sum between its margins, three times its default fund contribution (assuming the above specification of service closure) and the cost of the liquidation of the service for this member. This cost is itself bounded

by the notional of the member position, which would only be the actual cost in a scenario where all the assets of the CCP would jump to zero, also a very unlikely situation. In conclusion, the service closure event does not really matter regarding our present purpose of the XVA cost analysis of CCP membership. The default of the CCP as a whole (i.e. the closure of all its services) is an even more unlikely event, especially because a central bank would hardly allow it to occur in view of its systemic consequences. Hence we may and do ignore the service closure and the default of the clearing house in the context of this paper. See [Armakola and Laurent \(2015\)](#) about CCP resilience and see [Duffie \(2014\)](#) about alternative approaches to the design of insolvency and failure resolution regimes for CCPs.

## 2.4 Central Clearing Valuation Adjustment

We refer to the (generic) member 0 as “the member” henceforth, the other members being collectively referred to as “the clearing house”. For notational simplicity, we remove the index 0 referring to the reference member.

We call value of the CCP portfolio of the member its value inclusive of counterparty and funding risk (as opposed to the mark-to-market of the portfolio).

We assume that the member enters its portfolio at time 0, against an upfront payment of a certain amount  $\Pi_0$ , where the semi-martingale  $\Pi$  is a tentative value process of the CCP portfolio of the member.

We also assume that profit-and-losses are marked to the model value process  $\Pi$  and realized in continuous time.

In this section, we derive a representation of the (no arbitrage) value  $\Pi$  of the CCP portfolio of a member as the difference (cf. the remark 2.4.2 below) between the mark-to-market of the portfolio and a correction  $\Theta$ . We call  $\Theta$  the central clearing valuation adjustment (CCVA).

The KVA-inclusive CCVA is obtained in a second step by adding to  $\Theta$  a capital valuation adjustment (KVA) meant as the cost that it would require for remunerating the member at some hurdle rate for its CCP capital at risk (including its default fund contribution).

### 2.4.1 DVA and DVA2 Issues

From the perspective of the member, the effective time horizon of interest is  $\bar{\tau}^\delta$  (cf. (2.3.6)). The position of the member is closed at  $\tau^\delta$  (if  $\tau < \bar{T}$ ), with a terminal cash flow from the member’s perspective given, in view of (2.3.8) and of the analysis developed in the proof of Lemma 2.3.1 (for  $i = 0$  here), by

$$\chi = -\mathbb{1}_{\varepsilon=0}Q_{\tau^\delta} - \mathbb{1}_{\varepsilon>0}(\mathcal{C}_\tau + R\varepsilon) \quad (2.4.1)$$

In particular, if  $\varepsilon > 0$ , i.e.  $Q_{\tau^\delta} > \mathcal{C}_\tau$ , then the member receives

$$\begin{aligned} -\mathcal{C}_\tau - R\varepsilon &= -\mathcal{C}_\tau - R(Q_{\tau^\delta} - \mathcal{C}_\tau) \\ &= (-Q_{\tau^\delta}) + (1 - R)(Q_{\tau^\delta} - \mathcal{C}_\tau) \end{aligned}$$

However, for this amount to benefit to the member’s shareholders, it needs to be hedged so that they can monetize it before  $\tau$  (otherwise it is only a profit to the member’s bondholders). But, in order to hedge this amount, the member would basically need to sell credit protection on itself, which is barely possible in practice. Consequently, from an entry (i.e. transaction) price perspective, the member should ignore such a windfall benefit at own default and the ensuing debt valuation adjustment (DVA).

This means formally setting  $R = 1$ , which results in  $\chi = -Q_{\tau^\delta}$  in (2.4.1) and  $\xi = 0$  later in (2.4.9).

Then  $R$  becomes disconnected from what the clearing house would actually recover (if anything) from the member in case it defaults, but this is immaterial for analyzing the costs of this

member itself, it only matters for the others. In sum, it is possible and convenient to analyze the no DVA case for the reference member just by formally setting  $R = 1$ .

If, however, some DVA is accounted for (i.e. if  $R < 1$ ), then one may want to reckon likewise a funding benefit of the member at its own default, a windfall benefit called DVA2 in the terminology of [Hull and White \(2012\)](#), corresponding to an additional cash flow to the member of the form

$$(1 - \bar{R}) \left( \Pi_{\tau-} + \mathcal{C}_{\tau}^{\star} \right)^+ \quad (2.4.2)$$

at time  $\tau$  (if  $< \bar{T}$ ).

Here  $\mathcal{C}^{\star} = \text{VM} + \text{IM}$  and  $\bar{R}$  is a recovery rate of the member to its funder, so that the amount  $\left( \Pi_{\tau-} + \mathcal{C}_{\tau}^{\star} \right)^+$  in (2.4.2) represents the funding debt of the member at its default (having assumed profit-and-losses marked-to-model and realized in real time, see the proof of Lemma 2.4.1 below for more detail).

The funder of the member corresponds to a third party, possibly composed in practice of several entities or devices and assumed default-free for simplicity, playing the role of lender/borrower of last resort after exhaustion of the internal sources of funding provided to the member through its collateral and its hedge.

More generally, even if one considers that the “true” recovery rate of the member is simply zero, playing with formal recovery coefficients  $R$  and  $\bar{R}$  somewhere between 0 and 1 allows reaching any desired level of interpolation between the entry price point of view  $R = \bar{R} = 1$  and the reference exit price point of view  $R = \bar{R} = 0$ . On the DVA and DVA2 issues, see [Hull and White \(2012\)](#), [Burgard and Kjaer \(2012\)](#), [Albanese and Andersen \(2015\)](#), [Albanese, Andersen, and Iabichino \(2015\)](#), [Andersen, Duffie, and Song \(2017\)](#) and [Albanese and Crépey \(2017\)](#).

### 2.4.2 Gain Process

The member can hedge its collateralized portfolio and needs to fund its whole position (portfolio, margins and hedge). Regarding hedging, we restrict ourselves to the situation of a fully securely funded hedge, entirely implemented by means of swaps, short sales and repurchase agreements (all traded outside the clearing house, given our assumption of a constant CCP portfolio of the member), at no upfront payment. As explained in [Crépey, Bielecki, and Brigo \(2014, Section 4.2.1 page 87\)](#)<sup>1</sup>, this assumption encompasses the vast majority of hedges that are used in practice.

Consistent with arbitrage requirements and our terminology of a risk-neutral measure  $\mathbb{Q}$ , we assume that the vector-valued gain process  $\mathcal{M}$  of unit positions in the hedging assets is a  $\mathbb{Q}$  martingale (see [Crépey, Bielecki, and Brigo \(2014, Remark 4.4.2 pages 96-97\)](#)<sup>2</sup> or [Bielecki and Rutkowski \(2015, Proposition 3.3\)](#)). We assume that the member sets up a related hedge  $(-\zeta)$ , i.e. a predictable row-vector process with components yielding the (negative of) positions in the hedging assets. The “short” negative notation in front of  $\zeta$  is used for consistency with the idea, just to fix the mindset, that the portfolio is “bought” by the member, which therefore “sells” the corresponding hedge.

Regarding funding, we assume that variation margins  $\text{VM}_t = P_{t-}^{\star}$  consist of cash re-hypothecable and remunerated at OIS rates, while initial margins consist of segregated liquid assets accruing at OIS rates. Initial margins and default fund contributions are supposed to be subject to CCP fees  $c_t$ , e.g. 30 basis points. We postulate that the member can invest excess-cash at a rate  $(r_t + \lambda_t)$  and obtain unsecured funding at a rate  $(r_t + \bar{\lambda}_t)$ .

Let  $e$  denote the gain process (or profit-and-loss, hedging error, etc.) of the member’s position, held by the member itself before  $\bar{\tau}$  and then, if  $\tau < \bar{T}$ , by the clearing house (as liquidator of the member’s position) on  $[\bar{\tau}, \bar{\tau}^{\delta}]$ .

1. Or [Crépey \(2015, Part I, Section 2.1\)](#) in journal version.

2. Or [Crépey \(2015, Part I, Remark 4.1\)](#) in journal version.

**Lemma 2.4.1** We have  $e_0 = 0$  and, for  $0 < t \leq \bar{\tau}^\delta$ ,

$$\begin{aligned} de_t = & d\Pi_t - r_t \Pi_t dt - J_t \left( dD_t + \sum_{Z \subseteq N} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt) + g_t(\Pi_t) dt \right) \\ & - \mathbb{1}_{\tau < \bar{T}} (1 - \bar{R}) \left( \Pi_{t-} + \mathcal{C}_t^* \right)^+ dJ_t - \zeta_t d\mathcal{M}_t, \end{aligned} \quad (2.4.3)$$

where, for any  $\pi \in \mathbb{R}$ ,

$$g_t(\pi) = c_t \left( \mathcal{C}_t - P_{t-} \right) + \bar{\lambda}_t (\pi + \mathcal{C}_t^*)^+ - \lambda_t (\pi + \mathcal{C}_t^*)^- \quad (2.4.4)$$

**Remark 2.4.1** The self-financing equation (2.4.3) holds for any funding coefficient  $g_t = g_t(\pi)$  there, not necessarily given by (2.4.4), as soon as  $(r_t \Pi_t + g_t(\Pi_t)) dt$  represents the  $dt$ -funding cost of the member (whilst the member is alive, and net of the funding cost of its hedge that is already comprised in the local martingale  $\zeta_t d\mathcal{M}_t$ ).

### 2.4.3 Pricing BSDE

**Definition 2.4.1** We call  $\Pi$  a (no arbitrage) value process for the member's portfolio if  $\Pi_{\bar{\tau}^\delta} = \mathbb{1}_{\tau < \bar{T}} \chi$  and the ensuing gain process  $e$  (cf. (2.4.3)) is a risk-neutral local martingale.

**Proposition 2.4.1** A semi-martingale  $\Pi$  is a value process for the member's portfolio if and only if it satisfies the following valuation BSDE on  $[0, \bar{\tau}^\delta]$  :

$$\begin{aligned} \Pi_{\bar{\tau}^\delta} = & \mathbb{1}_{\tau < \bar{T}} \chi \text{ and, for } t \leq \bar{\tau}^\delta, \\ d\Pi_t = & r_t \Pi_t dt + \mathbb{1}_{\tau < \bar{T}} (1 - \bar{R}) (\Pi_{t-} + \mathcal{C}_t^*)^+ dJ_t \\ & + J_t \left( dD_t + \sum_{Z \subseteq N} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt) + g_t(\Pi_t) dt \right) + d\nu_t, \end{aligned} \quad (2.4.5)$$

for some local martingale  $\nu$ .

**Proof:** In view of (2.4.3), (2.4.5) is equivalent to  $de_t = d\nu_t - \zeta_t d\mathcal{M}_t$ . Since  $\zeta_t d\mathcal{M}_t$  defines a local martingale, therefore  $e$  and  $\nu$  are jointly local martingales or not, which establishes the proposition. ■

Note that, assuming  $\nu$  a true martingale, equivalently to the differential formulation (2.4.5), we can write (absorbing the  $r_t \Pi_t dt$  term from (2.4.5) into the risk-neutral discount factor  $\beta$  in (2.4.6)) :

$$\begin{aligned} \beta_t \Pi_t = & \mathbb{E}_t \left[ \mathbb{1}_{\tau < \bar{T}} \left( \beta_{\tau^\delta} \chi + \beta_\tau (1 - \bar{R}) \left( \Pi_{\tau-} + \mathcal{C}_\tau^* \right)^+ J_t \right) \right. \\ & \left. - \sum_{t < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} - \int_t^{\bar{\tau}} \beta_s J_s (dD_s + g_s(\Pi_s) ds) \right], \quad 0 \leq t \leq \bar{\tau}^\delta \end{aligned} \quad (2.4.6)$$

### 2.4.4 CCVA Representation

In this section we define the central counterparty valuation adjustment (CCVA) and derive the corresponding BSDE.

**Definition 2.4.2** Given a value  $\Pi$  for the member, the corresponding CCVA is the process defined on  $[0, \bar{\tau}^\delta]$  as  $\Theta = -(Q + \Pi)$ .

**Remark 2.4.2** Recall from (2.3.8) that  $Q = P + \Delta$ , with all values viewed from the perspective of the clearing house. Consistent with the usual definition of a valuation adjustment (see Brigo, Morini, and Pallavicini (2013) or Crépey, Bielecki, and Brigo (2014)), we have  $\Theta = (-Q) - \Pi$ , where  $(-Q)$  corresponds to the perspective of the member.

Let

$$\bar{\xi}_t = \mathbb{E}_t (\beta_t^{-1} \beta_{\tau^\delta} \xi) \quad (2.4.7)$$

where  $\xi = (1 - R) (Q_{\tau^\delta} - \mathcal{C}_{\tau^\delta}^+)^+$  as before (cf. (2.3.8)). Let  $\hat{\xi}$  be a  $\mathbb{G}$  predictable process, which exists by Corollary 3.23 2) in He and Yan (1992), such that

$$\hat{\xi}_\tau = \mathbb{E}_{\tau-} (\beta_\tau^{-1} \beta_{\tau^\delta} \xi) = \mathbb{E}_{\tau-} (\bar{\xi}_\tau) \quad (2.4.8)$$

In particular, in the no-DVA case with  $R = 1$ , then  $\xi = \bar{\xi} = \hat{\xi} = 0$ .

**Proposition 2.4.2** *Let there be given semi-martingales  $\Pi$  and  $\Theta$  such that  $\Theta = -(Q + \Pi)$  on  $[0, \bar{\tau}^\delta]$ . The process  $\Pi$  is a value process for the member's portfolio if and only if the process  $\Theta$  satisfies the following BSDE :*

$$\begin{aligned} \beta_t \Theta_t = \mathbb{E}_t \left[ \sum_{t < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} - \mathbb{1}_{\tau < \bar{T}} \left( \beta_{\tau^\delta} \xi + \beta_\tau (1 - \bar{R}) (P_{\tau-} - \mathcal{C}_{\tau^\delta}^+ + \Theta_{\tau-})^- J_t \right) \right. \\ \left. + \int_t^{\bar{\tau}} \beta_s (g_s(-P_s - \Theta_s)) ds \right], t \in [0, \bar{\tau}^\delta] \end{aligned} \quad (2.4.9)$$

**Proof:** Assuming  $\Theta$  defined as  $-(Q + \Pi)$  for some value process  $\Pi$  on  $[0, \bar{\tau}^\delta]$ , then the terminal condition  $\Theta_{\bar{\tau}^\delta} = -\mathbb{1}_{\tau < \bar{T}} \xi$  that is implicit in (2.4.9) results from (2.3.8) and the terminal condition for  $\Pi$  in (2.4.5). Moreover, we have, for  $t \in [0, \bar{\tau}^\delta]$ ,

$$-\beta_t \Theta_t = \beta_t Q_t + \beta_t \Pi_t = \beta_t P_t + \int_0^t \beta_s dD_s + \left( \beta_t \Pi_t - \int_0^t \beta_s J_s dD_s \right) \quad (2.4.10)$$

hence

$$\begin{aligned} -\beta_t \Theta_t - \int_0^t \beta_s J_s \left( \sum_{Z \subseteq N} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(ds) + g_s(-P_s - \Theta_s) ds \right) \\ - \mathbb{1}_{\tau < \bar{T}} \int_0^t (1 - \bar{R}) (-P_{s-} - \Theta_{s-} + \mathcal{C}_s^*)^+ dJ_s \\ = \left( \beta_t P_t + \int_0^t \beta_s dD_s \right) + \int_0^t \beta_s d\nu_s \end{aligned}$$

by the pricing BSDE (2.4.5) satisfied by  $\Pi$ . In view also of (2.2.1) (used for  $i = 0$  here), this is a (local) martingale, hence it coincides with the conditional expectation of its terminal condition (assuming it is a true martingale), which establishes (2.4.9). The converse implication is proven similarly.  $\blacksquare$

**Remark 2.4.3** As an alternative argument equivalent to the above, one can substitute the right-hand side in (2.4.6) for  $\beta_t \Pi_t$  in (2.4.10), which, after an application of the tower rule, yields (2.4.9). One can proceed similarly to show (2.4.6) if (2.4.9) is assumed.

Let, for  $\vartheta \in \mathbb{R}$ ,

$$\begin{aligned}\widehat{f}_t(\vartheta) &= g_t(-P_t - \vartheta) - \gamma_t \widehat{\xi}_t - (1 - \bar{R})\gamma_t(P_t - \mathcal{C}_t^* + \vartheta)^- \\ &= \underbrace{-\gamma_t \widehat{\xi}_t}_{dva_t} + \underbrace{\left(c_t(\mathcal{C}_t - P_{t-}) + \tilde{\lambda}_t(P_t - \mathcal{C}_t^* + \vartheta)^- - \lambda_t(P_t - \mathcal{C}_t^* + \vartheta)^+\right)}_{fva_t(\vartheta)}\end{aligned}\quad (2.4.11)$$

by definition (2.4.4) of  $g$ , where  $\tilde{\lambda} = \bar{\lambda} - (1 - \bar{R})\gamma$  (recall  $\gamma = \gamma^0$  is the assumed intensity of  $\tau$ ). From the perspective of the member, the two terms in the decomposition (2.4.11) of the coefficient  $\widehat{f}_t(\vartheta)$  can respectively be interpreted as a beneficial debt valuation adjustment coefficient ( $dva_t$  that can be ignored by setting  $R = 1$ ) and a funding valuation adjustment coefficient ( $fva_t(\vartheta)$  in which the DVA2 component can be ignored by setting  $\bar{R} = 1$ ).

**Proposition 2.4.3** *The “full CCVA BSDE” (2.4.9) for a semi-martingale  $\Theta$  on  $[0, \bar{\tau}^\delta]$  is equivalent to the following “reduced CCVA BSDE” for a semi-martingale  $\widehat{\Theta}$  on  $[0, \bar{\tau}]$  :*

$$\beta_t \widehat{\Theta}_t = \mathbb{E}_t \left[ \sum_{t < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} + \int_t^{\bar{\tau}} \beta_s \widehat{f}_s(\widehat{\Theta}_s) ds \right], \quad t \in [0, \bar{\tau}] \quad (2.4.12)$$

equivalent in the sense that if  $\Theta$  solves (2.4.9), then  $\widehat{\Theta} = J\Theta$  solves (2.4.12), whilst if  $\widehat{\Theta}$  solves (2.4.12), then  $\Theta = J\widehat{\Theta} - (1 - J)\mathbb{1}_{\tau < \bar{T}} \bar{\xi}$  solves (2.4.9).

**Proof:** The full CCVA BSDE (2.4.9) is obviously equivalent to  $\Theta = -\mathbb{1}_{\tau < \bar{T}} \bar{\xi}$  on  $[\bar{\tau}, \bar{\tau}^\delta]$  and

$$\begin{aligned}\beta_t \Theta_t &= \mathbb{E}_t \left[ \sum_{t < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} - \mathbb{1}_{\tau < \bar{T}} \beta_\tau \left( \bar{\xi}_\tau + (1 - \bar{R}) \left( P_{\tau-} - \mathcal{C}_\tau^* + \Theta_{\tau-} \right)^- \right) \right. \\ &\quad \left. + \int_t^{\bar{\tau}} \beta_s g_s(-P_s - \Theta_s) ds \right]\end{aligned}$$

on  $[0, \bar{\tau})$ , which is in turn equivalent to

$$\begin{aligned}\Theta &= -\mathbb{1}_{\tau < \bar{T}} \bar{\xi} \text{ on } [\bar{\tau}, \bar{\tau}^\delta] \text{ and, on } [0, \bar{\tau}), \\ \beta_t \Theta_t &= \mathbb{E}_t \left[ \sum_{t < \tau_Z^\delta \leq \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} + \int_t^{\bar{\tau}} \beta_s \widehat{f}_s(\Theta_s) ds \right]\end{aligned}\quad (2.4.13)$$

because on  $[0, \bar{\tau})$  :

$$\begin{aligned}\mathbb{E}_t \left[ \mathbb{1}_{\tau < \bar{T}} \beta_\tau \left( \bar{\xi}_\tau + (1 - \bar{R}) \left( P_{\tau-} - \mathcal{C}_\tau^* + \Theta_{\tau-} \right)^- \right) \right] \\ &= \mathbb{E}_t \left[ \mathbb{1}_{t < \tau < \bar{T}} \beta_\tau \left( \widehat{\xi}_\tau + (1 - \bar{R}) \left( P_{\tau-} - \mathcal{C}_\tau^* + \Theta_{\tau-} \right)^- \right) \right] \\ &= \mathbb{E}_t \left[ - \int_t^{\bar{T}} \beta_s \left( \widehat{\xi}_s + (1 - \bar{R}) \left( P_{s-} - \mathcal{C}_s^* + \Theta_{s-} \right)^- \right) dJ_s \right] \\ &= \mathbb{E}_t \left[ \int_t^{\bar{T}} \beta_s \gamma_s \left( \widehat{\xi}_s + (1 - \bar{R}) \left( P_{s-} - \mathcal{C}_s^* + \Theta_{s-} \right)^- \right) ds \right]\end{aligned}$$

Here the last identity holds by consideration of the (local, assumed true) martingale

$$\beta_t \left( \widehat{\xi}_t + (1 - \bar{R}) \left( P_{t-} - \mathcal{C}_{t-}^* + \Theta_{t-} \right)^- \right) \left( dJ_t + \gamma_t dt \right)$$

One readily checks that if  $\Theta$  solves (2.4.13), then  $\hat{\Theta} = J\Theta$  solves (2.4.12), whilst if  $\hat{\Theta}$  solves (2.4.12), then  $\Theta = J\hat{\Theta} - (1 - J)\mathbb{1}_{\tau < \bar{T}}\tilde{\xi}$  solves (2.4.13). ■

Note that, provided  $r$  and  $\tilde{\lambda}$  are bounded from below, the reduced BSDE coefficient  $\hat{f}_t(\vartheta)$  in (2.4.11) satisfies the monotonicity assumption

$$\left(\hat{f}_t(\vartheta) - \hat{f}_t(\vartheta')\right)(\vartheta - \vartheta') \leq C \times (\vartheta - \vartheta')^2$$

for some constant  $C$ . Then, under mild integrability conditions, the reduced CCVA BSDE (2.4.12) is well-posed in the space of square integrable solutions (see e.g. Kruse and Popier (2016, Sect. 5)). By virtue of Proposition 2.4.3, so is in turn the full CCVA BSDE (2.4.9).

**Remark 2.4.4** In the terminology of Crépey and Nguyen (2016), (2.4.12) is the “partially reduced” CCVA BSDE (cf. also Lemma 2.3 in Crépey and Song (2015)), while the “fully reduced” BSDE (simply called “reduced” in Crépey and Song (2016)) is the BSDE on the time interval  $[0, T]$  obtained from (2.4.12) by projection on a smaller filtration (the market or reference filtration myopic to the defaults of the two parties). In this paper we only work with the partially reduced BSDE in order to avoid the enlargement of filtration technicalities.

### 2.4.5 Cost of Capital

The capital at risk of the member is composed of its default fund contribution  $\text{DFC}_t$ , which represents implicit capital at risk, and of its regulatory CCP capital  $K_t^{cm}$  as of (2.10.3). Along the lines of Albanese and Crépey (2017), we define the capital valuation adjustment (KVA) of the member as the cost of remunerating its capital at risk  $K_t = \text{DFC}_t + K_t^{cm}$  at some hurdle rate  $k$  throughout the whole life of the portfolio (or until the member defaults). Such a KVA is given by the following formula (cf. Albanese and Crépey (2017)) :

$$\text{KVA}_t = k \mathbb{E}_t \int_t^{\bar{\tau}} e^{-\int_t^s (r_u + k) du} K_s ds, \quad t \in [0, \bar{\tau}] \quad (2.4.14)$$

The KVA-inclusive CCVA is then defined as the sum between our previous CCVA  $\Theta$  and this KVA.

## 2.5 Common Shock Model of Default Times

We use a dynamic Marshall-Olkin (DMO) copula model of the default times  $\tau_i$  (see Crépey, Bielecki, and Brigo (2014, Chapt. 8–10)<sup>3</sup> and Crépey and Song (2016)). As demonstrated in Crépey, Bielecki, and Brigo (2014, Sect. 8.4)<sup>4</sup>, such a model can be efficiently calibrated to marginal and portfolio credit data, e.g. CDS and CDO data (or proxies) on the members. The joint defaults feature of the DMO model is also interesting in regard of the EMIR “cover two” default fund sizing rule (cf. Section 2.10.2).

Let there be given a family  $\mathcal{Y}$  of “shocks”, i.e. subsets  $Y$  of members, typically the singletons  $\{0\}, \{1\}, \dots, \{n\}$  and a small number of “common shocks” representing simultaneous defaults. For  $Y \in \mathcal{Y}$ , we define

$$\eta_Y = \inf \left\{ t > 0; \int_0^t \gamma_Y(s) ds > \mathcal{E}_Y \right\}, \quad J^Y = \mathbb{1}_{[0, \eta_Y)}$$

for a shock intensity function  $\gamma_Y(t)$  and an independent standard exponential random variable  $\mathcal{E}_Y$ . We then set

$$\tau_i = \min_{\{Y \in \mathcal{Y}; i \in Y\}} \eta_Y, \quad i \in N$$

3. Or Bielecki et al. (2014b, 2014a) for the journal versions.

4. Or Bielecki, Cousin, Crépey, and Herbertsson (2014a, Part II) in journal version.

**Example 2.5.1** Figure 2.3 shows one possible default path in a common shock model with  $n = 5$  and

$$\mathcal{Y} = \left\{ \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{3, 4\}, \{1, 2, 3\}, \{0, 1\} \right\}$$

The inner ovals show which shocks happen and cause the observed defaults at successive default times. First, the default of name 1 occurs as the consequence of the shock  $\{1\}$ . Second, names 3 and 4 default simultaneously as a consequence of the shock  $\{3, 4\}$ . Third, the shock  $\{1, 2, 3\}$  triggers the default of name 2 alone (as name 1 and 3 have already defaulted). Fourth, the default of name 0 alone occurs as the consequence of shock  $\{0, 1\}$ .

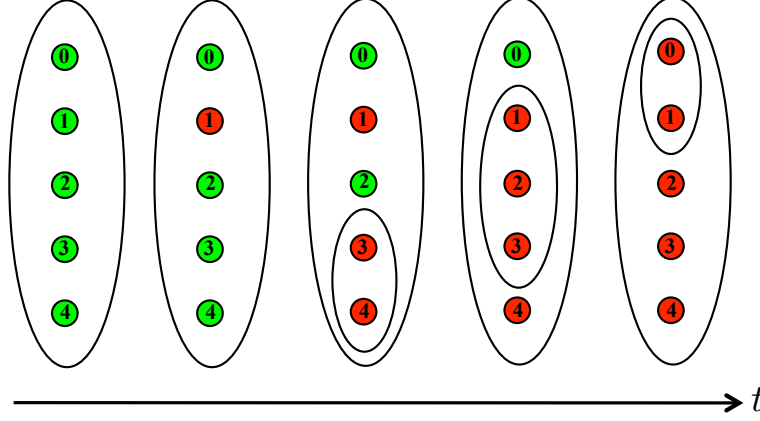


FIGURE 2.3 – One possible default path in a model with  $n = 4$  and  $\mathcal{Y} = \left\{ \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{3, 4\}, \{1, 2, 3\}, \{0, 1\} \right\}$ .

Again, in the case of the reference member (labeled 0), we omit the superscript 0 in the notation. In particular, we have  $J = \mathbb{1}_{[0, \tau)} = \prod_{Y \in \mathcal{Y}} J^Y$ , where  $\mathcal{Y}_\bullet = \{Y \in \mathcal{Y}; 0 \in Y\}$ , hence the intensity  $\gamma$  of  $\tau$  is given as

$$\gamma = J_- \gamma_\bullet, \text{ where } \gamma_\bullet = \sum_{Y \in \mathcal{Y}_\bullet} \gamma_Y \quad (2.5.1)$$

We assume that all the market risk factors are gathered in a vector process  $\mathbf{X}$  without jump at  $\tau$  and that the processes  $\mathbf{X}$  and  $X = (\mathbf{X}, \mathbf{J})$ , where  $\mathbf{J} = (J^Y)_{Y \in \mathcal{Y}}$ , are Markov in the full model filtration  $\mathbb{G}$  given as the filtration of  $\mathbf{X}$  progressively enlarged by the random times  $\eta_Y, Y \in \mathcal{Y}$  (in Section 2.7-2.8,  $\mathbf{X}$  is simply a Black–Scholes stock  $S$ , augmented by additional factors in order to cope with the potential path dependence of dividends and collateral). Setting  $\hat{\Delta}_t = \int_0^t e^{\int_s^t r_u du} dD_s$  so that  $\beta_t \Delta_t = \beta_t \hat{\Delta}_t - \beta_\tau \hat{\Delta}_{\tau-}$  for  $t \geq \tau$ , we assume, consistent with the interpretation of each respective quantity, that

$$\begin{aligned} \epsilon_t &= \epsilon(t, X_t) \text{ for } t = \tau_Z^\delta, Z \subseteq N \\ P_t &= P(t, \mathbf{X}_t), \hat{\Delta}_t = \hat{\Delta}(t, \mathbf{X}_t), C_t = C(t, X_t), \quad t \in [0, \bar{\tau}] \end{aligned}$$

(having augmented  $\mathbf{X}$  by  $\hat{\Delta}$  and/or  $C$  if need be), for continuous functions  $\epsilon(t, x)$ ,  $P(t, \mathbf{x})$ ,  $\hat{\Delta}(t, \mathbf{x})$  and  $C(t, x)$ . In particular, it holds that

$$\Delta_\tau = \hat{\Delta}_\tau - \hat{\Delta}_{\tau-} = \hat{\Delta}(\tau, \mathbf{X}_\tau) - \hat{\Delta}(\tau, \mathbf{X}_{\tau-}) = 0$$

by continuity of  $\mathbf{X}$  at  $\tau$  (as opposed to  $\Delta_\tau \neq 0$  in the gap risk model of Crépey and Song (2016)).

**Lemma 2.5.1** *We have*

$$dva_t = dva(t, X_t) = -J_t \bar{\xi}(t, X_t) \gamma_\bullet, \quad \mathbb{Q} \times \lambda \text{ a.e.}$$

for a function  $\bar{\xi}(t, x)$  such that  $\bar{\xi}_\tau = \bar{\xi}(\tau, X_{\tau-})$ .

## 2.6 XVA Engines

In this section, we summarize in algorithmic terms the central clearing XVA methodology of this paper, as well as a bilateral trading XVA methodology recalled for comparison purposes from [Crépey and Song \(2016\)](#). In both cases we use the common shock model of Section 2.5 for modeling the default times involved.

### 2.6.1 CCVA Engine

In spite of the nonlinearity inherent to the funding component of the CCVA, standard Monte Carlo loops can be used for estimating a linearized first order CCVA obtained replacing  $fva_s(\Theta_s)$  by  $fva_s(0)$  in (2.4.11), i.e.  $\hat{f}_s(\hat{\Theta}_s)$  by  $\hat{f}_s(0)$  in (2.4.12). A nonlinear correction can be estimated based on the Monte Carlo expansion of [Fujii and Takahashi \(2012a,2012b\)](#) (further studied in [Gobet and Pagliarani \(2015\)](#)) in vanilla cases, with explicit formulas for  $P_t$ , or by the branching particles scheme of [Henry-Labordère \(2012\)](#) in more exotic situations. In the bilateral trading setup of [Crépey and Song \(2016\)](#) (see also [Crépey and Nguyen \(2016\)](#)), the nonlinear correction is consistently found less than 5% to 10% of the linear part. Hence, in this paper, we just use the linear part. We obtain by first order linear approximation in the reduced CCVA BSDE (2.4.12) :

$$\begin{aligned} \Theta_0 = \hat{\Theta}_0 &\approx \mathbb{E} \left[ \sum_{0 < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} + \int_0^{\bar{\tau}} \beta_s \hat{f}_s(0) ds \right] \\ &= \underbrace{\mathbb{E} \sum_{0 < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta}}_{\text{CVA}} + \underbrace{\mathbb{E} \int_0^{\bar{\tau}} \beta_s dva_s ds}_{\text{DVA}} \\ &\quad + \underbrace{\mathbb{E} \int_0^{\bar{\tau}} \beta_s \left( \tilde{\lambda}_s (C_s^* - P_s)^+ - \lambda_s (C_s^* - P_s)^- \right) ds}_{\text{MVA}} \\ &\quad + \underbrace{\mathbb{E} \int_0^{\bar{\tau}} \beta_s c_s (C_s - P_{s-}) ds}_{\text{MLA}} \end{aligned} \tag{2.6.1}$$

where  $\beta_t = e^{-\int_0^t r_s ds}$ ,  $\tilde{\lambda} = \bar{\lambda} - (1 - \bar{R})\gamma_\bullet$ ,  $C^* = \text{VM} + \text{IM}$  and, for each  $t = \tau_Z^\delta < \bar{\tau}$ ,

$$\epsilon_t = (B_t - E_{t-})^+ \frac{\text{DFC}_t}{\sum_{j \in N} J_t^j \text{DFC}_t^j} \text{ in which } B_t = \sum_{i \in Z} (P_t^i + \Delta_t^i - C_t^i)^+$$

with, for each member  $i$ ,  $C^i = \text{VM}^i + \text{IM}^i + \text{DFC}^i$  (cf. (2.3.11) and (2.3.7)-(2.3.9)). In addition,  $dva = -\gamma \hat{\xi}$ , where  $\hat{\xi}$  is a predictable process such that  $\hat{\xi}_\tau = \mathbb{E}_{\tau-} (\beta_\tau^{-1} \beta_{\tau^\delta} \xi)$  (cf. (2.4.8)), with  $\xi = (1 - R)(P_{\tau^\delta} + \Delta_{\tau^\delta} - C_\tau)^+$ .

The  $\epsilon$  terms in (2.6.1) give rise to a CVA paid by the member through its contributions to the refill of realized breaches. The terms dubbed MVA and MLA in (2.6.1), where

$$\mathcal{C}_s^* - P_s = P_{s-} + \text{IM}_s - P_s \approx \text{IM}_s$$

and

$$\mathcal{C}_s - P_{s-} = \text{IM}_s + \text{DFC}_s$$

are interpreted as a margin valuation adjustment (cost to the member of funding its initial margins, essentially) and a margin liquidity adjustment (cost to the member of the CCP margin fees). The positive (respectively negative) terms in (2.6.1) can be considered as deal adverse (respectively deal friendly) as they increase (respectively decrease) the CCVA  $\Theta$ . The DVA and the DVA2 can be ignored in  $\Theta$  by setting  $R = 1$  and  $\bar{R} = 1$ , respectively.

For numerical purposes, we use the following randomized version of (2.6.1) :

$$\mathbb{E} \left[ \sum_{0 < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} + \mathbb{1}_{\zeta < \bar{\tau}} \frac{e^{\mu\zeta}}{\mu} \beta_\zeta \hat{f}_\zeta(0) \right] \quad (2.6.2)$$

where  $\zeta$  denotes an independent exponential time of parameter  $\mu$ . Moreover, to deal with the  $dva_\zeta$  term in  $\hat{f}_\zeta(0)$ , we use the following result.

**Lemma 2.6.1** *For any predictable process  $h$  and independent atomless random variable  $\zeta$ , we have :*

$$\mathbb{E} [\mathbb{1}_{\zeta < \bar{\tau}} h_\zeta \beta_\zeta dva(\zeta, X_\zeta)] = -\mathbb{E} \left[ \mathbb{1}_{\zeta < \bar{\tau}} h_\zeta \beta_{\zeta+\delta} (1-R) \gamma_\bullet(\zeta) (Q_{\zeta^\delta} - \mathcal{C}_\zeta^*)^+ \right] \quad (2.6.3)$$

Plugging  $h_\zeta = \frac{e^{\mu\zeta}}{\mu}$  in (2.6.3) to deal with the  $dva_\zeta$  term in  $\hat{f}_\zeta(0)$ , (2.6.2) is rewritten as

$$\begin{aligned} \hat{\Theta}_0 \approx \mathbb{E} \left\{ \sum_{0 < \tau_Z^\delta < \bar{\tau}} \beta_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} + \mathbb{1}_{\zeta < \bar{\tau}} \frac{e^{\mu\zeta}}{\mu} \times \right. \\ \left. \left[ -\beta_{\zeta^\delta} \gamma_\bullet(\zeta) (1-R) (Q_{\zeta^\delta} - \mathcal{C}_\zeta)^+ \right. \right. \\ \left. \left. + \beta_\zeta \left( \tilde{\lambda}_\zeta (\mathcal{C}_\zeta^* - P_\zeta)^+ - \lambda_\zeta (\mathcal{C}_\zeta^* - P_\zeta)^- \right) \right] \right\} \quad (2.6.4) \end{aligned}$$

The KVA-inclusive CCVA is then defined as the sum between (2.6.4) and a KVA as of (2.4.14), valued at time  $t = 0$  by simulation and randomization of the time integral there.

## 2.6.2 BVA Engine

Here we provide an executive summary of a bilateral CSA trading setup recalled for comparison purposes from Crépey and Song (2016) (cf. also Brigo and Pallavicini (2014) or Bichuch, Capponi, and Sturm (2017) for related bilateral counterparty risk analyses with asymmetric funding costs).

**Remark 2.6.1** In Crépey and Song (2016), the cash flows are viewed from the perspective of the bank, which will be taken as the reference member here, whereas we view them in this paper from the perspective of the clearing house, i.e. opposite to the one of the member. Hence, the sign conventions are opposite, i.e.  $P, \Delta, Q$ , etc. in this paper correspond to their opposites in Crépey and Song (2016), which is why we see  $\cdot^\mp$  here whenever we have  $\cdot^\pm$  there.

In the context of bilateral trading between a bank, taken as the reference member labeled by 0 in the previous CCP setup, and a counterparty taken as another member  $i \neq 0$ , let VM denote the variation margin, where  $VM \geq 0$  (resp.  $\leq 0$ ) means collateral posted by the bank and received by the counterparty (resp. posted by the counterparty and received by the bank). Let  $I^b \geq 0$  and  $I^c \leq 0$  represent the initial margin posted by the bank and the negative of the initial margin posted by the counterparty. Hence,

$$C^b = VM + I^b \quad \text{and} \quad C^c = VM + I^c \quad (2.6.5)$$

are the total collateral guarantee for the counterparty and the negative of the total collateral guarantee for the bank. Assuming the variation margins re-hypothecable and the initial margins segregated (as typically so in practice), the collateral funded by the bank is  $C = VM + I^b$ . For consistency with our CCP setup,  $VM_t$  will be taken as  $P_{t-}$ . So, in the spirit of a standard CSA, we are considering full collateralization, and even over-collateralization through the initial margins. We assume that VM and  $I^b$  are remunerated at the OIS rate  $r$ . Following [Crépey and Song \(2016\)](#), at time 0, the difference  $\Theta_0$  between the mark-to-market of the portfolio and its value inclusive of counterparty and funding risk (both from the perspective of the bank, cf. the remark 2.4.2), difference dubbed BVA for bilateral valuation adjustment, can be linearized as follows :

$$\begin{aligned} \Theta_0 = \bar{\Theta}_0 &\approx \mathbb{E} \left[ \int_0^{\bar{\tau}} \beta_s \bar{f}_s(0) ds \right] \\ &= \underbrace{\mathbb{E} \int_0^{\bar{\tau}} \beta_s cdva_s ds}_{\text{CDVA}} \\ &\quad + \underbrace{\mathbb{E} \int_0^{\bar{\tau}} \beta_s \left( \tilde{\lambda}_s (C_s - P_s)^+ - \lambda_s (C_s - P_s)^- \right) ds}_{\text{MVA}} \end{aligned} \quad (2.6.6)$$

Here :

- $P$  means the mark-to-market of the position of the member with the counterparty (viewed from the perspective of the latter),
- the meaning of  $\beta$ ,  $\tilde{\lambda}$  and  $\lambda$  is as in the CCP setup,
- $\tau = \tau_b \wedge \tau_c$  is the first-to-default time of the bank and the counterparty,
- $cdva = \gamma \hat{\xi}$ , where  $\hat{\xi}$  is a predictable process such that  $\hat{\xi}_\tau = \mathbb{E}_{\tau-} (\beta_\tau^{-1} \beta_{\tau^\delta} \xi)$ , with

$$\xi = \mathbb{1}_{\tau_c \leq \tau_b^\delta} (1 - R_c) (P_{\tau^\delta} + \Delta_{\tau^\delta} - C_\tau^c)^- - \mathbb{1}_{\tau_b \leq \tau_c^\delta} (1 - R_b) (P_{\tau^\delta} + \Delta_{\tau^\delta} - C_\tau^b)^+$$

in which the recovery rates  $R_c$  of the counterparty to the bank and  $R_b$  of the bank to the counterparty are usually taken in a bilateral trading setup as 40%.

For numerical purposes, we use the following randomized version of (2.6.6) :

$$\mathbb{E} \left[ \mathbb{1}_{\zeta < \bar{\tau}} \frac{e^{\mu\zeta}}{\mu} \beta_\zeta \bar{f}_\zeta(0) \right] \quad (2.6.7)$$

where  $\zeta$  denotes an independent exponential time of parameter  $\mu$ . The  $cdva_\zeta$  term in  $\bar{f}_\zeta(0)$  is treated by the following bilateral analog of Lemma 2.6.1. We write  $\mathcal{Y}_b = \{Y \in \mathcal{Y}; 0 \in Y\}$ ,  $\mathcal{Y}_c = \{Y \in \mathcal{Y}; i \in Y\}$  and we recall that  $X = (\mathbf{X}, \mathbf{J})$  denotes the market risk and common shock factor process introduced in Section 2.5, assumed without jump at  $\tau$ . Similar to Lemma 2.5.1, it holds that  $cdva_t = cdva(t, X_t)$ . In addition (see Lemma 8.2 and its proof in [Crépey and Song \(2016, hal version 2\)](#), in a slightly more general setup where  $\mathbf{X}$  may jump at  $\tau$ ) :

**Lemma 2.6.2** *For any predictable process  $h$  and independent atomless random variable  $\zeta$ , we have :*

$$\begin{aligned} & \mathbb{E} [\mathbb{1}_{\zeta < \bar{\tau}} h_{\zeta} \beta_{\zeta} c d v a(\zeta, X_{\zeta})] \\ &= \mathbb{E} \left[ \mathbb{1}_{\zeta < \bar{\tau}} h_{\zeta} \beta_{\zeta} \times \left( \left( \sum_{Y \in \mathcal{Y}_c} \gamma_Y(\zeta) + \mathbb{1}_{\tau_c \leq \zeta^\delta} \sum_{Y \in \mathcal{Y}_b \setminus \mathcal{Y}_c} \gamma_Y(\zeta) \right) (1 - R_c) (Q_{\zeta^\delta} - C_{\zeta}^-) \right. \right. \\ & \quad \left. \left. - \left( \sum_{Y \in \mathcal{Y}_b} \gamma_Y(\zeta) + \mathbb{1}_{\tau_b \leq \zeta^\delta} \sum_{Y \in \mathcal{Y}_c \setminus \mathcal{Y}_b} \gamma_Y(\zeta) \right) (1 - R_b) (Q_{\zeta^\delta} - C_{\zeta}^b)^+ \right) \right] \end{aligned} \quad (2.6.8)$$

Plugging  $h_{\zeta} = \frac{e^{\mu\zeta}}{\mu}$  in (2.6.8) to deal with the  $c d v a_{\zeta}$  term in  $\bar{f}_{\zeta}(0)$ , (2.6.7) is rewritten as (compare (2.6.4)) :

$$\begin{aligned} \bar{\Theta}_0 \approx \mathbb{E} \left\{ \mathbb{1}_{\zeta < \bar{\tau}} \frac{e^{\mu\zeta}}{\mu} \beta_{\zeta} \times \right. \\ \left( \left( \sum_{Y \in \mathcal{Y}_c} \gamma_Y(\zeta) + \mathbb{1}_{\tau_c \leq \zeta^\delta} \sum_{Y \in \mathcal{Y}_b \setminus \mathcal{Y}_c} \gamma_Y(\zeta) \right) (1 - R_c) (Q_{\zeta^\delta} - C_{\zeta}^-) \right. \\ \left. - \left( \sum_{Y \in \mathcal{Y}_b} \gamma_Y(\zeta) + \mathbb{1}_{\tau_b \leq \zeta^\delta} \sum_{Y \in \mathcal{Y}_c \setminus \mathcal{Y}_b} \gamma_Y(\zeta) \right) (1 - R_b) (Q_{\zeta^\delta} - C_{\zeta}^b)^+ \right) \\ \left. + \beta_{\zeta} \left( \tilde{\lambda}_{\zeta} (P_{\zeta} - C_{\zeta})^- - \lambda_{\zeta} (P_{\zeta} - C_{\zeta})^+ \right) \right\} \end{aligned} \quad (2.6.9)$$

Such adjustments are then computed counterparty by counterparty and added over  $i = 1, \dots, n$  to obtain the BVA of the bank.

**Remark 2.6.2** In practice, netting sets typically merge into a unique funding set, meaning that one should solve for a single MVA at the level of the whole portfolio of the bank. However, in the present frictionless variation-margining case (cf. the remark 2.3.1),

$$\begin{aligned} C_{\zeta} - P_{\zeta} &= P_{\zeta}^- + \text{IM}_{\zeta} - P_{\zeta} \\ &\approx \text{IM}_{\zeta} \geq 0 \end{aligned}$$

holds counterparty by counterparty, so that a unique funding set or funding by netting sets makes a negligible difference in practice.

Similar as in the CCP setup, the KVA-inclusive BVA is obtained by adding to (2.6.9) a KVA in the sense of the formula (2.4.14) (valued at  $t = 0$ ), except that  $K$  is now the bilateral regulatory capital given by the formulas of Section 2.10.3.

## 2.7 Experimental Framework

In this section we design an experimental framework that is used for the XVA comparative numerical analysis of Section 2.8.

### 2.7.1 Driving Asset

Given an interest rate process  $S$ , we consider a stylized swap of strike  $\bar{S}$  with cash flows  $h_l(\bar{S} - S_{T_{l-1}})$  at increasing times  $T_l$ ,  $l = 1, \dots, d$ , where  $h_l = T_l - T_{l-1}$ . We suppose a stylized

Black–Scholes dynamics with risk-neutral drift  $\kappa$  and volatility  $\sigma$  for the interest rate process  $S$ . Denoting by  $T_l$  the smallest  $T_l > t$ , the mark-to-market of the swap for the party receiving the above cash flows is given, for  $T_0 = 0 \leq t \leq T_d = \bar{T}$ , by  $P_t = \beta_t^{-1} \beta_{T_l} h_l (\bar{S} - S_{T_{l-1}}) + P_t^*$ , where

$$P_t^* = \beta_t^{-1} \bar{S} \sum_{l=l_t+1}^d \beta_{T_l} h_l - \beta_t^{-1} S_t \sum_{l=l_t+1}^d \beta_{T_l} h_l e^{\kappa(T_{l-1}-t)} = P_*(t, S_t) \quad (2.7.1)$$

We choose the notional  $Nom$  of the swap and its strike  $\bar{S}$  in such a way that each leg of the swap has a mark-to-market equal to one at time 0.

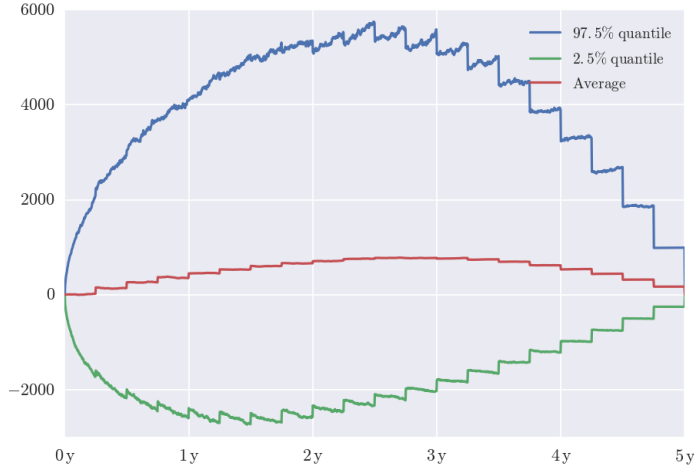


FIGURE 2.4 – Mark-to-market process of the swap viewed from the point of view of a party receiving floating and paying fix in the swap (party with a long unit position in the swap) expressed in bps. The mean and quantiles as a function of time are computed by Monte Carlo simulation of the process  $(-P_t)$  based on the formula (2.7.1) for  $P_*$ , used along  $m = 10^4$  simulated trajectories of  $S$ .

The following numerical values are used :

$$r = 2\%, \quad S_0 = 100, \quad \kappa = 12\%, \quad \sigma = 20 \quad h_l = 3 \text{ months}, \quad \bar{T} = 5 \text{ years}$$

resulting in the mark-to-market process displayed in Figure 2.4 from the point of view of a party receiving floating and paying fix, which we call a long unit position in the swap. Figure 2.4 exhibits the typical profile of an interest rate swap in an increasing term structure of interest rates, where expectations of increasing rates make the swap in the money on average (i.e. the average curve is in the positive in Figure 2.4). This yields to the product the XVA flavor that would be absent in a flat interest rates environment where the mark-to-market process of the swap would be zero and not give rise to any adjustments. The present Black–Scholes setup and values of the parameters for the process  $S$  allow us to obtain this stylized pattern without having to introduce a full flesh interest rate model, which would add useless complexity with respect to our goal in this paper.

## 2.7.2 Structure of the Clearing house

We consider a clearing house with  $(n + 1)$  members chosen among the 125 names of the CDX index as of 17 December 2007, a particular day toward the beginning of the global financial crisis. The default times of the 125 names are modeled by a common shock model with piecewise

constant intensities  $\gamma_Y$  constant on the time intervals  $[0, 3]$  and  $[3, 5]$  years, calibrated to the corresponding 3 and 5 year CDS and 5 year CDO data. With five nested common shocks  $Y$  on top of an idiosyncratic shock  $Y = \{i\}$  for each of the 125 names, a nearly perfect calibration can be achieved, as developed in Crépey, Bielecki, and Brigo (2014, Sect. 8.4.3)<sup>5</sup> We consider a subset of nine representative members of the index, with increasing CDS spreads shown in the first row of Table 2.1.

$\Sigma_i$	45	52	56	61	73	108	176	367	1053
$\alpha_i$	(0.46)	0.09	0.23	(0.05)	0.34	(0.04)	0.69	(0.44)	(0.36)

TABLE 2.1 – (*Top*) Average 3 and 5 year CDS spreads  $\Sigma_i$ , in basis points (bp), for a representative subset of nine members of the CDX index as of 17 December 2007. (*Bottom*) Coefficients  $\alpha_i$  summing up to 0 used for determining the swap positions of the nine members.

The coefficients  $\alpha_i$  in the second row, where parentheses mean negative numbers, will be used in a way explained below for determining the positions in the swap of the nine members in the simulations. These coefficients were obtained as the difference between a vector of nine uniform numbers and its cyclic shift, so that  $\sum_{i \in N} \alpha_i = 0$ .

### 2.7.3 Member Portfolios

We represent in an antisymmetric matrix form

$$\varpi = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & \dots & n \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix} & \begin{pmatrix} 0 & \varpi_{0,1} & \varpi_{0,2} & \varpi_{0,3} & \dots & \varpi_{0,n} \\ \cdot & 0 & \varpi_{1,2} & \varpi_{1,3} & \dots & \varpi_{1,n} \\ \cdot & \cdot & 0 & \varpi_{2,3} & \dots & \varpi_{2,n} \\ \cdot & \cdot & \cdot & 0 & \dots & \varpi_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \dots & 0 \end{pmatrix} \end{matrix}$$

where each “ $\cdot$ ” represents the negative of the symmetric entry in the matrix, the positions of each member  $i$  with respect to each member  $j$  (or short positions of  $j$  with respect to  $i$ ) in the swap. Note that the data of the CCVA BSDE related to the member 0, or of the linearized time-0 CCVA formula (2.6.4), only depend on the matrix  $\varpi$  through the sums of each of its rows, corresponding to the vector of the short positions of the different clearing members against the CCP. By contrast, the data of the BVA BSDE related to the member 0, or of the linearized time-0 BVA formula (2.6.9), only depend on the matrix  $\varpi$  through its first row (vector of the short positions of the different counterparties  $i = 1, \dots, n$  against the reference member 0). Hence, we can forget about the detail of the above matrix, focusing on the  $\omega_i^{csa} := \varpi_{0,i}$  and  $\omega_i^{ccp} := \sum_{l \neq i} \varpi_{l,i}$ ,  $i \neq 0$ , for comparing two trading setups :

- A CSA setup as of Section 2.6.2, where each member  $i \neq 0$  trades a short  $\omega_i^{csa} \in \mathbb{R}$  position in the swap with the member 0, whichever other trades members  $i \neq 0$  may have between each others.
  - For instance, but non necessarily, each member  $i \neq 0$  has a short  $\omega_i^{csa} \in \mathbb{R}$  position with the member 0 and there are no other trades between members (at least after netting at the level of each pair of members), which corresponds to the situation where only the first row and column are nonzero in the matrix  $\varpi$ .

5. Or Bielecki, Cousin, Crépey, and Herbertsson (2014a, Part II) in journal version.

- In any case, the netted long position of the member 0 is  $\sum_{i \neq 0} \omega_i^{csa}$ . However, netting does not apply across different counterparties in the CSA setup. We call compression factor  $\nu_0$  the gross position of the reference member 0, i.e. the number  $\nu_0 = \sum_{i \neq 0} |\omega_i^{csa}|$  of trades the member 0 is engaged into in the CSA setup.
- A CCP setup as of Section 2.6.1, where each member  $i \neq 0$  trades a short  $\omega_i^{ccp} \in \mathbb{R}$  position in the swap through the CCP ( $\omega_i^{ccp} \leq 0$  effectively means a long position of member  $i$ ), whichever way this position may be distributed among other members.
  - For instance, but non necessarily, each member  $i \neq 0$  has a short  $\omega_i^{ccp} \in \mathbb{R}$  position with the member 0 and there are no other trades between members, which again corresponds to the situation where only the first row and column are nonzero in  $\varpi$ .
  - In any case, since members trade between themselves, the member has a  $\sum_{i \neq 0} \omega_i^{ccp}$  position in the driving asset after netting through the CCP, instead of a non netted position of size  $\nu_0$  before clearing through the CCP.

Moreover, in order to obtain diverse while comparable setups, we will alternately consider as reference member 0 each of the nine members in Table 2.1, for positions in the driving asset determined by the coefficients  $\alpha_i$  (summing up to zero) in the second row of Table 2.1 through the following rule :  $\omega_i = -\frac{\alpha_i}{\alpha_0}$ ,  $i \neq 1$  (where  $\omega = \omega^{csa}$  or  $\omega^{ccp}$ , as suitable). Since the coefficients  $\alpha_i$  add up to 0, this specification ensures  $\sum_{i \neq 0} \omega_i = 1$ , i.e. a netted position of the member 0 (whoever it is), always equal to 1 in the CCP setup. We also define  $\omega_0 = -\frac{\alpha_0}{\alpha_0} = -1$ , consistent with the member 0 being long a +1, i.e. short a -1, net position in the swap in the CCP setup (in the CSA setup this value of  $\omega_0$  is purely conventional).

Note that

$$\nu_0 = \sum_{i \neq 0} |\omega_i| = \sum_{i \neq 0} \frac{|\alpha_i|}{|\alpha_0|} = \frac{\sum_{i \in N} |\alpha_i|}{|\alpha_0|} - 1$$

so the smaller  $|\alpha_0|$ , the larger the compression factor  $\nu_0$  (gross position of the reference member when trading bilaterally, whereas its net, centrally cleared position is equal to one).

**Example 2.7.1** Table 2.2 shows the resulting values of the  $\omega_i$  of the different members  $i \neq 0$  when the name with CDS spread 61 bp (name with the second smallest  $|\alpha_i|$  in Table 2.1, with corresponding entries emphasized in bold in Table 2.2) is taken as reference member 0 (prototype of a name with a large gross position). Hence, the  $\omega_i$  in Table 2.2 are proportional to the  $\alpha_i$  in Table 2.1, modulo a scaling factor so that the  $\omega_i$  of this particular name (then labeled as 0) is -1. In this case  $\nu_0 = \sum_{i \neq 0} |\omega_i| = 53.00$ .

$\Sigma$	45	52	56	<b>61</b>	73	108	176	367	1053
$\omega$	(9.20)	1.80	4.60	<b>(1.00)</b>	6.80	(0.80)	13.80	(8.80)	(7.20)

TABLE 2.2 – Positions  $\omega_i$  in the swap of the nine members with CDS spreads  $\Sigma_i$ , in the respective  $\omega_i = \omega_i^{csa}$  or  $\omega_i^{ccp}$  meaning, when the reference member 0 is the name with CDS spread 61 bp and the second smallest  $|\alpha_i|$  in Table 2.1.

**Example 2.7.2** Table 2.3 is the analog of Table 2.2 when the member with spread 367 bp (name with the second largest credit spread in Table 2.1, with corresponding entries emphasized in bold in Table 2.2) is taken as reference member 0 (prototype of a risky name). In this case  $\nu_0 = \sum_{i \neq 0} |\omega_i| = 5.14$ .

$\Sigma_i$	45	52	56	61	73	108	176	<b>367</b>	1053
$\omega_i$	(1.05)	0.20	0.52	(0.11)	0.77	(0.09)	1.57	<b>(1.00)</b>	(0.82)

TABLE 2.3 – Analog of Table 2.2 when the reference member 0 is the name with CDS spread 367 bp (name with the second largest credit spread  $\Sigma_i$ ) in Table 2.1.

### 2.7.4 Margins

**CCP setup** The initial margin  $\text{IM}^i$  posted by each member  $i \in N$  is set through (2.3.4), using as risk measure  $\rho$  the “risk-neutral” value at risk (that is, computed under  $\mathbb{Q}$ ) of some level  $a$  “close to 1”. Since the pricing function  $P_\star$  in (2.7.1) is decreasing in  $S$ , therefore  $\text{IM}^i$  can be proxied, at each simulated time  $\zeta$  in (2.6.4) or (2.6.9), by

$$\text{IM}_\zeta^i = \text{Nom} \times |\omega_i| \times \begin{cases} P_\star(\zeta, S_\zeta) - P_\star(\zeta, S_\zeta e^{\sigma\sqrt{\delta'}\Phi^{-1}(a) + (\kappa - \frac{\sigma^2}{2})\delta'}), & \omega_i \geq 0 \\ P_\star(\zeta, S_\zeta e^{\sigma\sqrt{\delta'}\Phi^{-1}(1-a) + (\kappa - \frac{\sigma^2}{2})\delta'}) - P_\star(\zeta, S_\zeta), & \omega_i \leq 0 \end{cases} \quad (2.7.2)$$

where  $\Phi$  is the standard normal cdf and where we recall that  $\delta' = \delta + h$  is the margin period of risk.

For instance, the reference member 0, with  $\omega_0^{\text{ccp}} = -1$ , is long one unit in the swap with mark-to-market profile shown in Figure 2.4, hence the exposure of the CCP to member 0 is the opposite profile. Accordingly (recalling that Figure 2.4 shows  $(-P_t)$ ), the CCP asks initial margins to the member 0 based on  $P_\star(\zeta, S_\zeta e^{\sigma\sqrt{\delta'}\Phi^{-1}(1-a) + (\mu - \frac{\sigma^2}{2})\delta'}) - P_\star(\zeta, S_\zeta)$ , consistent with the second line in (2.7.2) in case  $\omega_0 \leq 0$ .

Consistently with a “cover two” EMIR rule (see Section 2.10.2), the default fund contributions are set as the sum of the two largest exposures of the clearing members (exposures in the sense of their EADs as explained in Section 2.7.5), allocated between them proportionally to their initial margins.

**CSA setup** The initial margin  $-I^c \geq 0$  required by the member 0 from the member  $i \neq 0$  (cf. (2.6.5)) is given by the right-hand side formula in (2.7.2) valued at some quantile level  $a$  (possibly different from the one used in the CCP setup).

For instance, if  $\omega_i^{\text{csa}} = +2$ , meaning that the member 0 has a “double Figure 2.4 exposure” with regard to counterparty  $i$ , then the member 0 asks the counterparty  $i$  to post initial margins based on  $P_\star(\zeta, S_\zeta) - P_\star(\zeta, S_\zeta e^{\sigma\sqrt{\delta'}\Phi^{-1}(a) + (\kappa - \frac{\sigma^2}{2})\delta'})$  (recall again that Figure 2.4 shows  $(-P_t)$ ), consistent with the use of the first branch in (2.7.2) in the case where  $\omega_i^{\text{csa}} \geq 0$  (for  $i \neq 0$ ).

Symmetrically, the formula for the initial margin  $I^b \geq 0$  required by the member  $i$  from the member 0 reads

$$I_\zeta^b = -\text{Nom} \times \omega_i \times \begin{cases} P_\star(\zeta, S_\zeta) - P_\star(\zeta, S_\zeta e^{\sigma\sqrt{\delta'}\Phi^{-1}(a) + (\kappa - \frac{\sigma^2}{2})\delta'}), & \omega_i \leq 0 \\ P_\star(\zeta, S_\zeta) - P_\star(\zeta, S_\zeta e^{\sigma\sqrt{\delta'}\Phi^{-1}(1-a) + (\kappa - \frac{\sigma^2}{2})\delta'}), & \omega_i \geq 0 \end{cases}$$

### 2.7.5 Exposure-at-defaults

The prime motivation for the Black–Scholes model used for  $S$  and for our risk-neutral value-at-risk for the IMs is that these give rise to an explicit formula for the exposure-at-defaults

(EAD), which are the basic primitive of all the regulatory capital formulas. This avoids the computational burden of nested Monte Carlo simulations (see the introductory paragraph to Section 2.10.1). We also use EADs as a proxy of the exposures of the members in the context of our EMIR “cover two” default fund computations (cf. Section 2.10.2).

In fact, for any grid time  $v = t + \epsilon p$  involved in EAD computations (cf. (2.10.2), (2.3.2) and (2.7.1), with  $\epsilon$  taken as one month in the numerics), we have in our model :

$$\begin{aligned} & \mathbb{E}_t \left[ \left( P_{v+\delta'} + \int_{[v, v+\delta']} e^{\int_s^{v+\delta'} r_u du} dD_s - P_{v-} - \text{IM}_v \right)^+ \right] \\ &= \mathbb{E}_t \left[ \left( P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v) - \text{VaR}_t(P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v)) \right)^+ \right] \\ &= \mathbb{E}_t \mathbb{E}_v \left[ \left( P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v) - \text{VaR}_t(P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v)) \right)^+ \right] \end{aligned}$$

where  $\text{VaR}$  represents the risk-neutral value-at-risk of level  $a$ . Denoting by  $\mathbb{E}\mathbb{S}$  the corresponding expected shortfall, the conditional version of the identity  $\mathbb{E}(X \mathbb{1}_{X \geq \text{VaR}(X)}) = (1-a)\mathbb{E}\mathbb{S}(X)$  yields

$$\begin{aligned} & \mathbb{E}_v \left[ \left( P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v) - \text{VaR}_t(P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v)) \right)^+ \right] \\ &= (1-a) \left( \mathbb{E}\mathbb{S}_v(P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v)) - \text{VaR}_v(P_\star(v + \delta', S_{v+\delta'}) - P_\star(v, S_v)) \right) \\ &= (1-a) \left( e^{\sigma\sqrt{\delta'}\Phi^{-1}(a)} - e^{\sigma\sqrt{\delta'}\frac{\phi(\Phi^{-1}(a))}{1-a}} \right) \beta_{v+\delta'}^{-1} e^{-\kappa v - \frac{\sigma^2}{2}\delta'} S_v \sum_{l=l_{v+\delta'}}^d \beta_{T_l} h_l e^{\kappa T_{l-1}} \end{aligned}$$

where  $\Phi$  and  $\phi$  are the standard normal cdf and density. Hence,

$$\mathbb{E}_t \left[ \left( P_{v+\delta'} + \int_{[v, v+\delta']} e^{\int_s^{v+\delta'} r_u du} dD_s - P_{v-} - \text{IM}_v \right)^+ \right] = f_v^{a, \delta'} \times (1-a) e^{-\kappa t} S_t \quad (2.7.3)$$

where

$$f_v^{a, \delta'} = \left( e^{\sigma\sqrt{\delta'}\Phi^{-1}(a)} - e^{\sigma\sqrt{\delta'}\frac{\phi(\Phi^{-1}(a))}{1-a}} \right) \beta_{v+\delta'}^{-1} e^{-\frac{\sigma^2}{2}\delta'} \sum_{l=l_{v+\delta'}}^d \beta_{T_l} h_l e^{\kappa T_{l-1}} \quad (2.7.4)$$

Likewise, we have

$$\mathbb{E}_t \left[ \left( P_{v+\delta'} + \int_{[v, v+\delta']} e^{\int_s^{v+\delta'} r_u du} dD_s - P_{v-} - \text{IM}_v \right)^- \right] = g_v^{a, \delta'} \times (1-a) e^{-\kappa t} S_t \quad (2.7.5)$$

where

$$g_v^{a, \delta'} = - \left( e^{-\sigma\sqrt{\delta'}\Phi^{-1}(a)} - e^{-\sigma\sqrt{\delta'}\frac{\phi(\Phi^{-1}(a))}{1-a}} \right) \beta_{v+\delta'}^{-1} e^{-\frac{\sigma^2}{2}\delta'} \sum_{l=l_{v+\delta'}}^d \beta_{T_l} h_l e^{\kappa T_{l-1}} \quad (2.7.6)$$

Based on (2.7.3) through (2.7.4), explicit formulas for the EADs follow. Figure 2.5 shows the time-0 EADs of the nine CCP members for their positions in the swap corresponding to the choice of the name of examples 2.7.1 or 2.7.2 as reference member.

## 2.7.6 XVA Data

The following numerical values are used in the sequel :

$$\bar{R} = 1, \quad \bar{\lambda} = \frac{1}{2}\Sigma_0, \quad \lambda = 0, \quad k = 10\%, \quad h = 1 \text{ day}, \quad \mu = \frac{2}{T}, \quad m = 10^4 \quad (2.7.7)$$

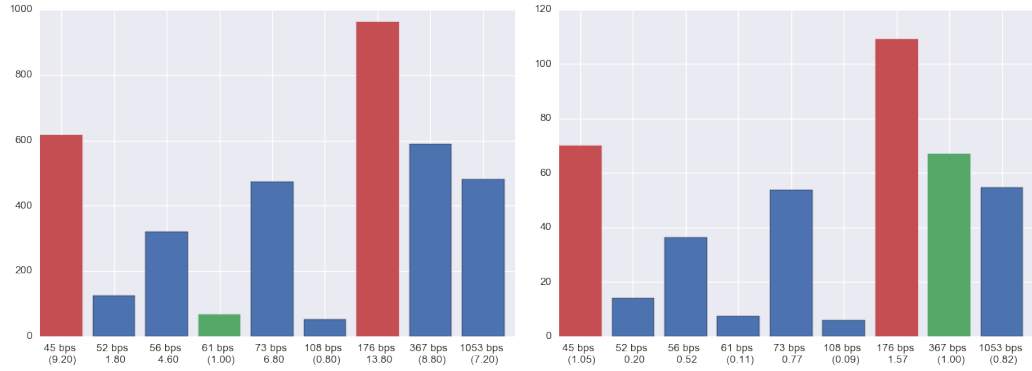


FIGURE 2.5 – Time-0 EADs in basis points (IM quantile level  $a = 70\%$ , liquidation period  $\delta = 5$  days). The two largest EADs, in red, size the default fund. The reference member EAD is in green. The corresponding positions  $\omega_i$  of the member are displayed at the bottom. *Left* : Reference member with  $\Sigma_0 = 61$  bps and  $\nu_0 = 53.00$ . *Right* : Reference member with  $\Sigma_0 = 367$  bps and  $\nu_0 = 5.14$ .

where  $m$  is the number of simulations used for estimating the expectations in (2.6.4) and (2.6.9). The level of 10% used for  $k$  is consistent with reference orders of magnitude for a hurdle rate.

Moreover, in a CCP setup, unless otherwise stated, we set

$$\begin{aligned} R &= 0, \quad \delta = 5 \text{ days}, \quad a = 70\%, \\ T &= 1 \text{ month}, \quad Y = 1 \text{ year}, \\ E^* &= 25\%K^{ccp}, \quad c = 30 \text{ bps} \end{aligned} \tag{2.7.8}$$

where  $K^{ccp}$  is defined in (2.10.4). The low quantile level used to set the initial margins is meant to compensate the excessive simplicity of the Black–Scholes setup without wrong-way risk used for  $S$  (it also leads to moderate standard errors with a relatively small number  $m = 10^4$  of simulations). Margin fees of  $c = 30$  bp are consistent with current CCP practices. These margin fees are distinct from the commission fees, not included in our setup, that a CCP is also charging to its members. In practice, commission fees are of the order of a few basis points of the size of the positions, i.e. a few basis points in the case of a unit position in our swap with each leg equal to one at time 0.

In a CSA setup, alternatively to (2.7.8), unless otherwise stated, we set

$$R_b = R_c = 40\%, \quad \delta = 15 \text{ days}, \quad a = 80\%$$

The value  $a = 80\%$  used in the bilateral case is higher than the value  $a = 70\%$  used in the CCP setup, where the protection offered by the default fund allows requiring less initial margins.

## 2.8 Numerical Results

All our XVA numbers are stated in basis points (recall that both legs of the swap are worth one at time 0). For comparability purposes, common random inputs are used in all our Monte Carlo estimates, i.e. we use the same sampled trajectories of  $S$  and sampled sets of default times  $\tau_i$  in all cases, it is only the way these  $m = 10^4$  random input sets are used which changes. The computation times are proportional to the number of members  $n$  and model trajectories  $m$ , e.g. about 5 minutes on a standard laptop to compute a full set of XVAs in Table 2.4 (four or five XVA components and their sum), where  $n = 8$  and  $m = 10^4$ , using pre-simulated values for

all the random inputs. Negative (e.g. DVA) numbers are displayed in parentheses. Regarding the aggregated XVA numbers in the tables, i.e. BVA in the CSA setup, CCVA in the CCP setup and TVA sometimes used as a common acronym for covering both cases, they are all KVA-inclusive, but they do not include the corresponding DVA numbers, which are only showed for reference. In other words, all the displayed TVA numbers correspond to entry price TVAs. The CCP MLA number are consistently found one order of magnitude smaller than the other XVA numbers, which is a sanity check that the CCP margin fees do not drive the comparison between the CCP and the CSA setup.

Note that, for simplicity, we are comparing a situation where all the trading is centrally cleared with a situation where all the trading is bilateral. In practice, vanilla products (hedges) tend to be cleared and exotics tend to be bilaterally traded. Therefore, in a more realistic setup, the multilateral netting benefit that CCPs provide is balanced by a loss of bilateral netting across asset classes (see [Duffie and Zhu \(2011\)](#) and [Cont, Santos, and Moussa \(2013\)](#)). To correct this bias, we will also show bilateral XVA figures scaled by the compression factor  $\nu_0$  of the reference name.

### 2.8.1 Multilateral Netting Benefit

Table 2.4 shows the XVA numbers obtained by considering alternately each of the nine members in Table 2.1 as reference member, using the  $\alpha_i$  coefficients for setting the positions of the members in each case as explained in Section 2.7.3 (cf. the examples 2.7.1 and 2.7.2). The different cases in Table 2.4 are ordered by increasing values of the compression factor  $\nu_0$ , i.e. by decreasing  $|\alpha_0|$ . We can see from Table 2.4 that the MVA and the KVA are the main contributors in the respective CSA and CCP setup. Moreover, the CSA XVA numbers vary roughly proportionally to the compression factor  $\nu_0$ , whereas the CCP XVA numbers are essentially not impacted by  $\nu_0$ . This illustrates the multilateral netting benefit provided by the CCP, especially for members with a large compression factor  $\nu_0$ .

$\nu_0$	2.91	4.87	5.14	6.50	6.94	10.74	29	53	66.50
$\alpha_0$	0.69	(0.46)	(0.44)	(0.36)	0.34	0.23	0.09	(0.05)	(0.04)
$\Sigma_0$	176	45	367	1053	73	56	52	61	108
CVA	11.07	25.06	19.34	14.06	28.37	42.69	111.38	238.22	299.37
DVA	(8.76)	(4.49)	(30.85)	(90.10)	(8.08)	(13.59)	(28.77)	(52.70)	(111.33)
MVA	30.38	13.63	110.50	339.69	31.41	39.34	98.46	204.72	449.68
KVA	11.17	21.16	19.40	21.14	29.26	46.28	122.20	221.63	275.87
BVA	52.62	59.85	149.24	374.90	89.04	128.31	332.05	664.57	1024.92
CVA	7.88	11.33	6.54	3.57	10.85	11.73	11.91	11.60	9.23
DVA	(2.57)	(0.69)	(5.43)	(13.03)	(1.07)	(0.89)	(0.81)	(0.90)	(1.57)
MVA	5.19	1.39	10.33	24.24	2.22	1.76	1.61	1.86	3.23
MLA	1.17	1.22	1.09	0.89	1.22	1.22	1.22	1.22	1.20
KVA	10.79	11.59	10.00	7.97	11.44	11.52	11.54	11.58	11.21
CCVA	25.03	25.54	27.95	36.67	25.73	26.23	26.27	26.26	24.87

TABLE 2.4 – XVA numbers obtained by considering alternately each of the nine members in Table 2.1 as reference member 0, using the  $\alpha_i$  for setting the positions of the members in each case as explained in Section 2.7.3. (*Up*) Credit spread  $\Sigma_0$ , coefficient  $\alpha_0$  and compression factor  $\nu_0$  of the reference member in each case (ordered by increasing  $\nu_0$ , i.e. decreasing  $|\alpha_0|$ ). (*Middle*) CSA XVA numbers. (*Bottom*) CCP XVA numbers.

Table 2.5 shows the percentage standard errors corresponding to the Monte Carlo estimates of Table 2.4. As we can see from the table, the standard errors are typically no more than a few percents in relative terms. Standard errors of Monte Carlo estimates are no longer shown in the sequel.

$\nu_0$	2.91	4.87	5.14	6.50	6.94	10.74	29	53	66.50
$\alpha_0$	0.69	(0.46)	(0.44)	(0.36)	0.34	0.23	0.09	(0.05)	(0.04)
$\Sigma_0$	176	45	367	1053	73	56	52	61	108
CVA	3.40	2.87	3.40	4.97	3.22	3.22	3.22	2.90	2.89
DVA	5.66	10.38	4.08	2.58	8.92	9.21	9.49	9.28	7.05
MVA	0.79	0.78	0.75	0.96	0.77	0.64	0.63	0.84	0.80
KVA	0.58	0.54	0.64	0.81	0.54	0.54	0.54	0.54	0.55
CVA	2.55	2.93	3.13	4.49	2.69	2.71	2.70	2.91	2.66
DVA	3.11	3.02	3.05	3.42	3.15	2.92	2.94	3.27	3.21
MVA	0.86	0.78	0.77	0.96	0.91	0.67	0.69	0.95	0.93
MLA	0.65	0.60	0.71	0.88	0.61	0.61	0.60	0.60	0.62
KVA	0.58	0.58	0.65	0.84	0.57	0.59	0.59	0.59	0.58

TABLE 2.5 – Percentage standard errors corresponding to the Monte Carlo estimates of Table 2.4.

### 2.8.2 Impact of the Credit Spread of the Reference Member

The CCP multilateral netting benefit dominates the comparison between our CSA and CCP XVA numbers. However, in our stylized setup, we cannot see the netting benefit across assets of bilateral trading. In order to compensate for this bias and obtain comparative results net of the first order CCP multilateral netting benefit, Table 2.6 shows the same results as Table 2.4, but with all the CSA XVA numbers scaled by the corresponding compression factor  $\nu_0$  (we will present in this way all the CSA XVA results in the sequel) and ordered by increasing credit spread  $\Sigma_0$  of the reference name, instead of increasing  $\nu_0$  in Table 2.4.

From Table 2.6 we can see that, if we get rid of the CCP multilateral netting benefit through this scaling, then the CSA and CCP XVA numbers become of a similar order of magnitude. The aggregated TVA numbers even become in favor of the CSA setup, except for the reference name with the largest (actually huge) credit spread of 1053 bps. These results can be put in perspective with the ones in Ghamami and Glasserman (2017) (see Section 2.1.1).

Regarding the comparison between the nine different cases within the CCP setup, as also within the CSA setup after scaling by the compression factor, Table 2.6 shows that the main explanatory factor of the results is the credit spread of the reference member, risky members being heavily penalized in terms of MVA, especially in the CSA setup. In both cases, the dominant patterns are a logarithmic decrease of the CVA numbers and a linear increase of the |DVA| and MVA numbers with respect to the credit spread of the reference name.

### 2.8.3 Impact of the Liquidation Period

Focusing on the reference members of the examples 2.7.1 and 2.7.2, respectively dubbed “safe member” and “risky member” henceforth (with respective credit spread of  $\Sigma_0 = 61$  and 367 bp), Table 2.7 shows the impact of changing the length  $\delta$  of the liquidation period from 5 days to 15 days in the CSA setup and vice versa in the CCP setup. The CSA 15 day and CCP 5 day numbers in Table 2.7 are simply retrieved from Table 2.6, for comparison purposes with the additional CSA 5 day and CCP 15 day numbers. The results are consistent with a  $\sqrt{\delta}$  pattern in line with the distributional properties of the Black–Scholes model used for  $S$ .

### 2.8.4 Margin Optimization

Table 2.8 shows the impact of using higher quantile levels  $a$  for the initial margins, which were only 80% and 70% in the respective CSA and CCP setup so far (with the motivation exposed in Section 2.7.6). The left column in each of the two main panels, retrieved from Table 2.6, corresponds to our base case where  $a = 70\%$  and  $a = 80\%$ . When higher values are used for the quantile levels, i.e. going from left to right in each panel, we observe the same qualitative

$\nu_0$	4.87	29	10.74	53	6.94	66.5	2.91	5.14	6.5
$\alpha_0$	(0.46)	0.09	0.23	(0.05)	0.34	(0.04)	0.69	(0.44)	(0.36)
$\Sigma_0$	45	52	56	61	73	108	176	367	1053
CVA / $\nu_0$	5.15	3.84	3.97	4.49	4.09	4.50	3.80	3.76	2.16
DVA / $\nu_0$	(0.92)	(0.99)	(1.27)	(0.99)	(1.16)	(1.67)	(3.01)	(6.00)	(13.86)
MVA / $\nu_0$	2.80	3.40	3.66	3.86	4.53	6.76	10.44	21.50	52.26
KVA / $\nu_0$	4.34	4.21	4.31	4.18	4.22	4.15	3.84	3.77	3.25
BVA / $\nu_0$	12.29	11.45	11.95	12.54	12.83	15.41	18.08	29.03	57.68
CVA	11.33	11.91	11.73	11.60	10.85	9.23	7.88	6.54	3.57
DVA	(0.69)	(0.81)	(0.89)	(0.90)	(1.07)	(1.57)	(2.57)	(5.43)	(13.03)
MVA	1.39	1.61	1.76	1.86	2.22	3.23	5.19	10.33	24.24
MLA	1.22	1.22	1.22	1.22	1.22	1.20	1.17	1.09	0.89
KVA	11.59	11.54	11.52	11.58	11.44	11.21	10.79	10.00	7.97
CCVA	25.54	26.27	26.23	26.26	25.73	24.87	25.03	27.95	36.67

TABLE 2.6 – XVA numbers obtained by considering alternately each of the nine members in Table 2.1 as reference member 0, using the  $\alpha_i$  for setting the positions of the members in each case as explained in Section 2.7.3. (*Up*) Credit spread  $\Sigma_0$ , coefficient  $\alpha_0$  and compression factor  $\nu_0$  of the reference member in each case (ordered by increasing  $\Sigma_0$ ). (*Middle*) CSA XVA numbers scaled by the compression factors  $\nu_0$ . (*Bottom*) CCP XVA numbers.

Member	61 bps, $\nu_0 = 53.00$		367 bps, $\nu_0 = 5.14$	
$\delta$	5 d	15 d	5 d	15 d
CVA / $\nu_0$	2.17	4.49	1.82	3.76
DVA / $\nu_0$	(0.50)	(0.99)	(2.90)	(6.00)
MVA / $\nu_0$	2.34	3.86	13.14	21.50
KVA / $\nu_0$	2.43	4.18	2.18	3.77
BVA / $\nu_0$	6.94	12.54	17.15	29.03
CVA	11.60	19.54	6.54	10.78
DVA	(0.90)	(1.41)	(5.43)	(8.91)
MVA	1.86	3.40	10.33	18.96
MLA	1.22	2.25	1.09	2.00
KVA	11.58	21.60	10.00	18.59
CCVA	26.26	46.79	27.95	50.34

TABLE 2.7 – Impact of the liquidation period. (*Left*) Safe reference member. (*Right*) Risky reference member. (*Top*) CSA XVA numbers scaled by  $\nu_0$ . (*Bottom*) CCP XVA numbers.

patterns as before in terms of the comparison between the CSA and the CCP setup. Considering now the impact of higher quantile levels inside each CSA or CCP setup, we can see a shift from CVA(/DVA) and KVA into MVA.

Ultimately, for very high quantiles, the CVA(/DVA) and KVA would reach zero whereas the MVA would keep increasing, since excessive margins become useless and a pure cost to the system, in the CSA as in the CCP setup.

Figure 2.6 illustrates this further by showing the aggregated TVA numbers and the relative weight of their CVA, FVA and KVA contributions when the quantile level  $a$  used for setting the IM goes from 55% to 100%, where FVA means MVA in the CSA setup (left graphs) and MVA + MLA in the CCP setup (right graphs). In each of the four cases considered in the upper panels (left CSA vs. right CCP curve and blue safe vs. green risky reference member curve), the numerical values of the TVA exhibit a convex dependence with respect to  $a$  (although, mathematically speaking, this depends on the values of the numerical parameters that are used, see for instance the CVA curve in the left graph of Figure 2.7, which shows a more detailed XVA decomposition of the safe reference member CCVA curve in the upper right graph of Figure 2.6). In the case of the risky reference member in the CSA setup, the level of initial margins is too high already with a 55% quantile level : The risky reference member (green) BVA curve in the upper left graph of Figure 2.6 keeps increasing when  $a$  increases from 55% to 100%. In each of the other three cases, the TVA has a minimum at some value  $a < 1$ . For both reference names, the optimal quantile level is larger in the CCP than in the CSA setup. This is because, in the CCP setup, the member is happy to post more initial margins, which “cost” her  $\bar{\lambda} = \frac{1}{2}\Sigma_0$ , in order to reduce her default fund contribution, which “costs” her a greater  $k = 10\%$  (cf. (2.7.7)). In each of the four considered cases, the FVA becomes preponderant and even hegemonic (as it tends to infinity) when  $a$  goes to 100%.

Capponi and Cheng (2016) construct a model which endogenizes collateral, making it part of an optimization problem where the CCP maximizes profit by controlling collateral and fee levels. They conclude that the collateral level should decrease with funding costs, on top of increasing with market volatility. The above numerical results are quite in line with such statements.

Member	$\Sigma_0 = 61 \text{ bp}, \nu_0 = 53.00$			$\Sigma_0 = 367 \text{ bp}, \nu_0 = 5.14$		
	80%	90%	99%	80%	90%	99%
CVA / $\nu_0$	4.49	2.64	0.74	3.76	2.23	0.62
DVA / $\nu_0$	(0.99)	(0.56)	(0.15)	(6.00)	(3.51)	(1.02)
MVA / $\nu_0$	3.86	5.87	10.66	21.50	32.99	60.18
KVA / $\nu_0$	4.18	1.78	0.13	3.77	1.61	0.12
BVA / $\nu_0$	12.54	10.29	11.53	29.03	36.83	60.92
$a$	70%	80%	95%	70%	80%	95%
CVA	11.60	9.15	4.64	6.54	5.17	2.62
DVA	(0.90)	(0.66)	(0.22)	(5.43)	(4.02)	(1.43)
MVA	1.86	2.83	5.32	10.33	15.71	29.53
MLA	1.22	1.54	2.56	1.09	1.38	2.31
KVA	11.58	6.55	1.19	10.00	5.66	1.03
CCVA	26.26	20.07	13.72	27.95	27.91	35.49

TABLE 2.8 – Impact of the level of the quantile level  $a$  that is used for setting the initial margins. (Left) Safe reference member. (Right) Risky reference member. (Top) CSA setup with all XVA numbers scaled by  $\nu_0$ . (Bottom) CCP setup.

### 2.8.5 Impact of the Number of Members

Another interesting question is what happens when we vary the number of members of the CCP. Obviously, more members means more mutualization of risk. However, the main effects in a CCP are already visible with nine members as above : with more members, things would mainly happen as in the projection of the system onto the ten (or so) greatest members anyway.

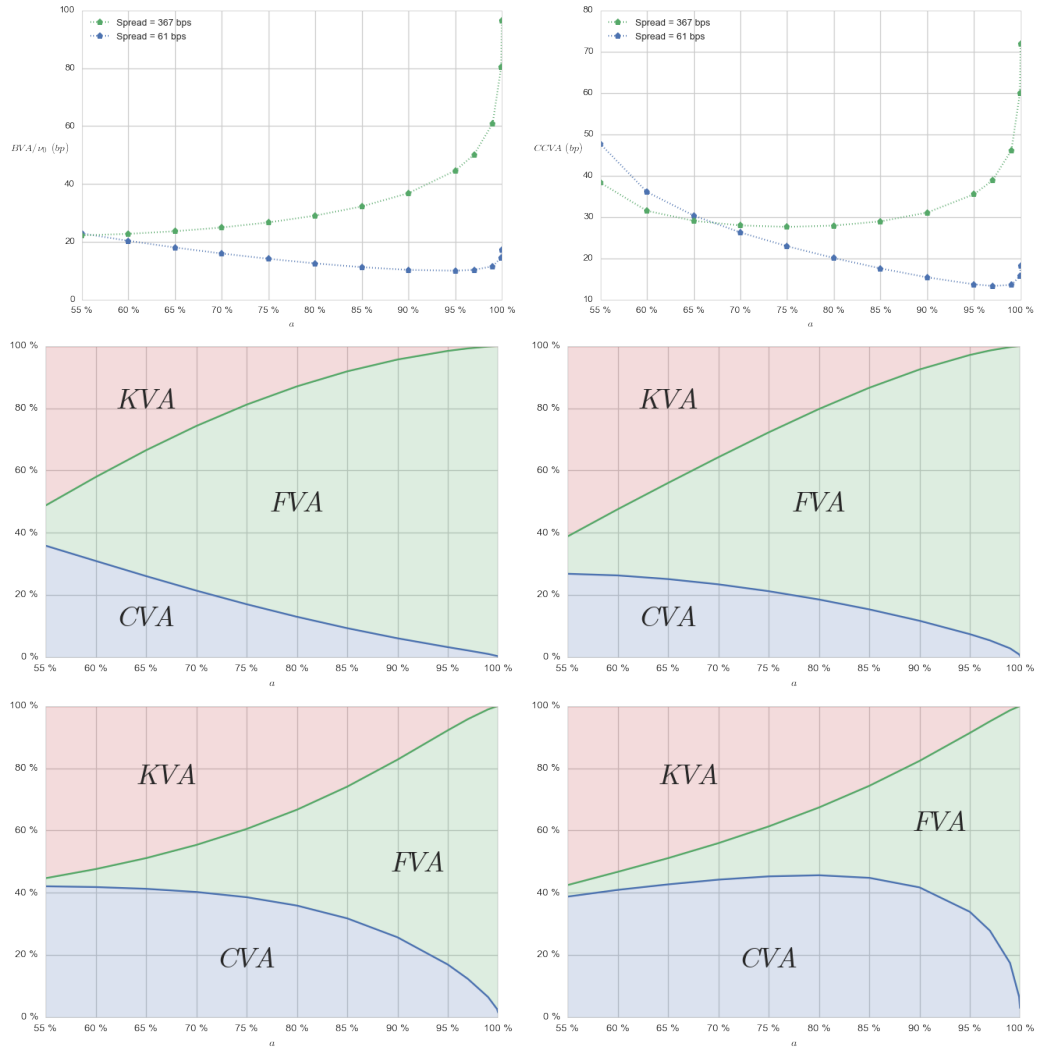


FIGURE 2.6 – Varying the initial margins quantile level  $\alpha$ . *Left* : CSA setup. *Right* : CCP setup. *Top* :  $BVA/\nu_0$  vs. CCVA. *Bottom* : XVA relative contributions in the case of the safe reference member. *Middle* : XVA relative contributions in the case of the risky reference member.

Figure 2.7 illustrates that, if there are now not enough members, a regulatory “cover two” default fund specification sized to the two largest exposures of the clearing members may result in very heavy default fund contributions and KVA for the small members in the common situation of heterogeneous members’ exposure.

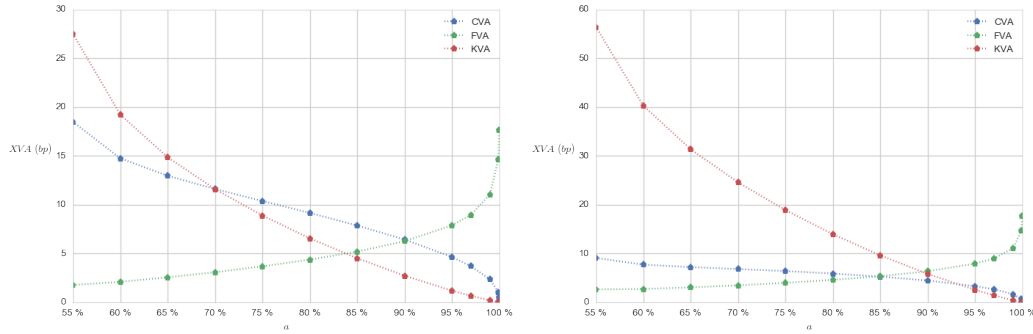


FIGURE 2.7 – CCP XVA results for the reference member with  $\Sigma_0 = 61$  bps and  $\nu_0 = 53.00$ . *Left* : Results in our previous CCP with nine members. *Right* : Results in a CCP restricted to three members : the reference member and two other members. The reference member, with  $\omega_i = -1$  by definition, corresponds to the member with time-0 EAD displayed in green in the left panel of Figure 2.5. The two other members are the members of the original CCP with the greatest time-0 EADs, i.e. the members with the time-0 EADs displayed in red in the left panel of Figure 2.5. Moreover, we modified the positions of these two members as  $\omega_i = -9$  and 10, instead of  $-9.2$  and  $13.8$  in the left panel of Figure 2.5, for being in line with the clearing condition  $\sum_{i \in N} P^i = 0$ .

## 2.9 Conclusions

In this chapter we study the cost of the clearance framework for a member of a clearing house. The overall cost, dubbed CCVA for central clearing valuation adjustment, is decomposed into CVA, MVA and KVA components. The CVA is the cost for a member of its losses on the default fund due to the defaults of other members, the MVA is the cost of funding initial margins and the KVA is the cost of its capital at risk, including its default fund contribution in the CCP setup. The numerical experiments show the multilateral netting benefit of central clearing. Multilateral netting has actually been, together with transparency and mutualization, one of the main motivation for the incentivisation of CCPs by regulators. But this multilateral netting comes at the expense of a loss of netting across asset classes. If we compensate the first order multilateral netting effect by a suitable scaling factor accounting for the loss of netting across asset classes, then the bilateral and centrally cleared XVA numbers become comparable. The second more explanatory factor of the numerical results is the credit risk of the members and the ensuing MVA, especially in the CSA setup where even more initial margins are required.

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## 2.10 Appendix

### 2.10.1 Regulatory Capital and Default Fund Formulas

A primitive of all the regulatory capital formulas are the so-called exposure-at-defaults given, for  $i \in N = 0, 1, \dots, n$  and  $t \in [0, \bar{T}]$ , as

$$EAD_t^i = 1.4 \epsilon \sum_{\epsilon p < 1 \wedge (T-t)} EEE_t^i(t_p), \quad (2.10.1)$$

where (see [Basel Committee et al. \(2005b\)](#), formulas (1)-(2)-(3) pages 26-27)) :

- the factor 1.4 is a wrong-way risk multiplier,
- $\epsilon$  is a time-integration step (e.g. one month),
- $t_p = t + \epsilon p$ ,
- the effective expected exposures  $EEE_t^i(t_p)$  are defined through the following iteration :  
 $EEE_t^i(t_{-1}) = 0$  and, for  $p \geq 0$ ,

$$EEE_t^i(t_p) = \max \left( EEE_t^i(t_{p-1}), \mathbb{E}_t \left[ \left( L_{t_p, t_p + \delta'}^i - \text{IM}_{t_p}^i \right)^+ \right] \right) \quad (2.10.2)$$

where  $L_{t_p, t_p + \delta'}^i$  has been defined in (2.3.2).

In our case, we also use EADs as a proxy of the exposure of the CCP to the members in the context of EMIR “cover two” default fund computations (see Section 2.10.2). For our default fund and KVA computations, such EADs must then be computed at any randomization time  $t = \zeta$  used in (2.6.4) or for simulating the time integral in (2.4.14). Unless an explicit formula is available for the conditional expectations in the right-hand side of (2.10.2), such EAD exposures can only be done by means of nested Monte Carlo simulations.

Note that in both our centrally cleared and bilateral trading setups, we neglect capital for market risk in the paper, as if the reference member (or bank) was perfectly hedged in terms of market risk. Otherwise one more capital term is required for market risk.

### 2.10.2 CCP Setup

Under centrally cleared trading, the “cover two” EMIR rule prescribes to size the default fund as, at least, the maximum of the greatest and of the sum of the second and third greatest exposures “under extreme but plausible market conditions” (see [European Parliament \(2012b\)](#), article 42, paragraph 3, page 37)). This total amount is then allocated between the clearing members according to some repartition key, e.g. proportional to their initial margins.

As explained in the paper, default fund contributions are “implicit capital” that the clearing members put at the disposal of the CCP. In addition, to cover their residual risk beyond the guarantee provided by the different margin layers of the CCP, the regulatory capital  $K = K^{cm}$  of a generic reference member is defined, following [BCBS \(2012\)](#), page 11), as :

$$K^{cm} = \max \left( K^{ccp} \times \frac{\text{DFC}}{E + \sum_{i \in N} J^i \text{DFC}^i}, 8\% \times 2\% \times \text{DFC} \right), \quad (2.10.3)$$

where DFC is the default fund contribution of the reference member and where

$$K^{ccp} = RW \times Cap_{Ratio} \times \sum_{i \in N} J^i EAD^i \quad (2.10.4)$$

with  $RW = 20\%$  and  $Cap_{Ratio} = 8\%$ .

**Remark 2.10.1** Accordingly, [Ghamami \(2015\)](#) argues that the CCP regulatory capital  $K^{cm}$  of a member should rather be based on its expected future unfunded default fund contributions (see the remark 2.3.4), which represent the losses of the member beyond the level already funded via its default fund contribution.

### 2.10.3 CSA Setup

In the bilateral setup, the capital at risk  $K$  of the bank reduces to its regulatory capital (there is no bilateral trading analog of the default fund), which comprises a first contribution for counterparty default losses and a second one for the volatility of the CVA (the market risk of the bank being supposed to be hedged out). Since we focus on the reference member 0 with  $n$  counterparties  $i \in N^* = \{1, 2, \dots, n\}$ , the capital formulas below all need to be summed over  $i \in N^*$ .

**$K^{ccr}$**  The Basel II regulatory capital specified for *counterparty risk* is defined as

$$K^{ccr} = CapRatio \sum_{i \in N^*} RWA^i$$

where

$$RWA^i = 12.5 \times w_i \times 1.4 \times EAD^i$$

Here  $CapRatio \geq 8\%$  (which is the value that we use in the numerics) is a chosen capital ratio that the bank must hold. The capital weight  $w_i$  is given by the internal ratings-based formula

$$w_i = (1 - R_i) \left( \Phi \left( \frac{\Phi^{-1}(DP_i)}{\sqrt{1 - corr_i}} + \sqrt{\frac{corr_i}{1 - corr_i}} \Phi^{-1}(0.999) \right) - DP_i \right) \frac{1 + (\hat{T}^i - 2.5)b(DP_i)}{1 - 1.5b(DP_i)}$$

(see [Basel Committee et al. \(2005a, page 7\)](#)), where :

- $R_i$  is the recovery rate of the counterparty  $i$ ,
- $\Phi$  is the standard normal cdf,
- $DP_i$  is the one year default probability of the counterparty  $i$ , historical in principle, proxied in our numerics by the risk-neutral default probability extracted from the corresponding CDS spread,
- $corr_i$  is the asset-counterparty  $i$  correlation in the sense of

$$corr_i = 0.12 \frac{1 - e^{-50DP_i}}{1 - e^{-50}} + 0.24 \frac{1 - (1 - e^{-50DP_i})}{1 - e^{-50}}$$

- $\hat{T}^i$  is the effective time to maturity of the netting set  $i$ , i.e. the time to maturity of the swap in our numerical case study where a single derivative is considered,
- $b(p) = [0.11852 - 0.05478 \ln(p)]^2$

**$K^{cva}$**  The standardized CVA risk capital charge in [Basel Committee et al. \(2010, §104\)](#) reads as

$$K^{cva} = 2.33\sqrt{Y} \left[ \left( 0.5 \sum_{i \in N^*} w_i \hat{T}^i E \tilde{A} D^i \right)^2 + 0.75 \sum_{i \in N^*} \left( w_i \hat{T}^i E \tilde{A} D^i \right)^2 \right]^{\frac{1}{2}}$$

which we approximate as in [Green, Kenyon, and Dennis \(2014\)](#) by

$$\frac{2.33}{2} \sqrt{Y} \sum_{i \in N^*} w_i \hat{T}^i E \tilde{A} D^i$$

where :

- $Y$  is the one year risk horizon, i.e.  $Y = 1$ ,
- $\hat{T}^i$  is defined above,

- $E\tilde{A}D^i = \frac{1 - e^{0.05T^i}}{0.05T^i} EAD^i$ ,
- $w_i$  is a weight based on the external rating extracted from the one year default probability  $DP_i$  as of the following table, where the left part comes from Moody's and the right part is taken from [Basel Committee et al. \(2010, §104\)](#) :

Default Prob	Rating	Weight
0.00%	AAA	0.7%
0.02%	AA	0.7%
0.06%	A	0.8%
0.17%	BBB	1.0%
1.06%	BB	2.0%
3.71%	B	3.0%
12.81%	CCC	10.0%

#### 2.10.4 Proofs of Auxiliary Results

**Proof of Lemma 2.3.1** Under our stylized model of the liquidation procedure, during the liquidation period  $[\tau_Z, \tau_Z^\delta]$ , where  $\tau_Z = \tau_i$  if and only if  $i \in Z$ , the clearing house substitutes itself to the defaulting members, taking care of all their dividend cash flows, which represent a cumulative cost of  $\sum_{i \in Z} \Delta_{\tau_i^\delta}^i$  (including a funding cost at the risk-free rate comprised in the  $\Delta_{\tau_i^\delta}^i$ ). At the liquidation time  $\tau_Z^\delta$ , the clearing house substitutes the buffer to itself as counterparties in all the concerned contracts (or simply puts an end to the contracts that were already with the buffer), which represents a supplementary cost  $\sum_{i \in Z} P_{\tau_i^\delta}^i$ . In addition, for any  $i \in Z$  :

- If  $\varepsilon_i = 0$ , meaning that  $Q_{\tau_i^\delta}^i \leq \mathcal{C}_{\tau_i}^i$ , then either  $Q_{\tau_i^\delta}^i \leq 0$  and an amount  $(-Q_{\tau_i^\delta}^i)$  is paid by the clearing house to the member  $i$  (who keeps ownership of all its collateral), or  $Q_{\tau_i^\delta}^i \geq 0$  and the ownership of an amount  $Q_{\tau_i^\delta}^i$  of collateral is transferred to the clearing house. In both cases, the clearing house gets  $Q_{\tau_i^\delta}^i$  ;
- Else, i.e. if  $\varepsilon_i > 0$ , meaning that the overall collateral  $\mathcal{C}^i$  of a member  $i \in Z$  does not cover the totality of its debt to the clearing house, then, at time  $\tau_i^\delta$ , the ownership of  $\mathcal{C}^i$  is transferred in totality to the clearing house. If  $R_i > 0$  then the clearing house also gets a recovery  $R_i \varepsilon_i$ .

In conclusion, the realized breach of the CCP is the sum over  $i \in Z$  of the

$$\begin{aligned}
& P_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i - \mathbb{1}_{\varepsilon_i > 0} \left( \mathcal{C}_{\tau_i}^i + R_i \varepsilon_i \right) - \mathbb{1}_{\varepsilon_i = 0} Q_{\tau_i^\delta}^i \\
&= Q_{\tau_i^\delta}^i - \mathbb{1}_{\varepsilon_i = 0} Q_{\tau_i^\delta}^i - \mathbb{1}_{\varepsilon_i > 0} \left( \mathcal{C}_{\tau_i}^i + R_i \varepsilon_i \right) \\
&= \mathbb{1}_{\varepsilon_i > 0} \left( Q_{\tau_i^\delta}^i - \mathcal{C}_{\tau_i}^i - R_i \varepsilon_i \right) \\
&= (1 - R_i) \varepsilon_i \\
&= \xi_i
\end{aligned}$$

which is the desired result.

**Proof of Lemma 2.4.1** To formulate in mathematical terms the above-described margining, hedging and funding policy of the member, we introduce three funding assets  $B^0$ ,  $B^f$  and  $\bar{B}^f$

evolving on  $[0, \bar{\tau}^\delta]$  as

$$\begin{aligned} dB_t^0 &= r_t B_t^0 dt \\ dB_t^f &= (r_t + \lambda_t) B_t^f dt \\ d\bar{B}_t^f &= (r_t + \bar{\lambda}_t) \bar{B}_t^f dt + (1 - \bar{R}) \bar{B}_{t-}^f dJ_t \end{aligned} \quad (2.10.5)$$

These represent the risk-free OIS deposit asset and the assets used by the bank for its respective investing and unsecured funding purposes. Under our continuous-time mark-to-model and realization assumption on profit-and-losses, the amount on the funding accounts of the bank is

$$-\Pi_t = -(\Pi_t + \mathcal{C}_t^*) + \mathcal{C}_t^*$$

where  $\mathcal{C}^* = \text{VM} + \text{IM}$  is the amount of margins that need to be funded by the member (its default fund contribution is assumed to be taken on its uninvested equity, hence does not need to be funded), so that the terms in the parenthesis represent the amount invested or borrowed unsecured (depending on its sign) by the bank, and where we recall that collateral is remunerated OIS by the receiving party. Defining

$$\eta_t^f = \frac{(\Pi_t + \mathcal{C}_t^*)^-}{B_t^f}, \quad \bar{\eta}_t^f = -\frac{(\Pi_t + \mathcal{C}_t^*)^+}{\bar{B}_t^f}, \quad \eta_t^0 = \frac{\mathcal{C}_t^*}{B_t^0}, \quad \bar{\eta}_t^0 = -\frac{(\Pi_t + \mathcal{C}_t^*)}{B_t^0} \quad (2.10.6)$$

we can write

$$-\Pi_t = J_t \eta_t^f B_t^f + J_t \bar{\eta}_t^f \bar{B}_t^f + \eta_t^0 B_t^0 + (1 - J_t) \bar{\eta}_t^0 B_t^0 \quad (2.10.7)$$

where, by self-financing condition,

$$\begin{aligned} d \left( J_t \eta_t^f B_t^f + J_t \bar{\eta}_t^f \bar{B}_t^f + \eta_t^0 B_t^0 + (1 - J_t) \bar{\eta}_t^0 B_t^0 \right) \\ = J_t \eta_t^f dB_t^f + J_t \bar{\eta}_t^f d\bar{B}_t^f + \eta_t^0 dB_t^0 + (1 - J_t) \bar{\eta}_t^0 dB_t^0 \end{aligned} \quad (2.10.8)$$

A left-limit in time is required in  $J_{t-} \bar{\eta}_{t-}^f$  because  $\bar{B}_t^f$  in (2.10.5) jumps at time  $\tau$ , so that the process  $\bar{\eta}^f$ , which is defined through (2.10.6), is not predictable.

In view of (2.10.7)-(2.10.8) and of the additional cash flows that affect the member (contractual cash flows, margin fees, realized breaches refills and hedging cash flows), the gain process  $e$  associated with the member's valuation-and-hedge policy  $(\Pi, \zeta)$  satisfies the following forward SDE :  $e_0 = 0$  and, for  $0 < t \leq \bar{\tau}^\delta$ ,

$$\begin{aligned} de_t = & \underbrace{\frac{d\Pi_t}{\text{Gain on the portfolio}}}_{\text{Gain on the portfolio}} - \underbrace{\frac{J_t dD_t}{\text{Contractual dividends}}}_{\text{Contractual dividends}} - \underbrace{\frac{J_t c_t (\mathcal{C}_t - P_{t-}^\zeta) dt}{\text{Margin fees}}}_{\text{Margin fees}} \\ & - \underbrace{\frac{J_t \sum_{Z \subseteq N} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt)}{\text{Refill of realized breaches}}}_{\text{Refill of realized breaches}} - \underbrace{\frac{\zeta_t d\mathcal{M}_t}{\text{Loss on the hedge}}}_{\text{Loss on the hedge}} \\ & + J_t \eta_t^f dB_t^f + J_t \bar{\eta}_{t-}^f d\bar{B}_t^f + \eta_t^0 dB_t^0 + (1 - J_t) \bar{\eta}_t^0 dB_t^0 \end{aligned}$$

Substituting (2.10.5) into the above yields

$$\begin{aligned} de_t = & d\Pi_t - r_t \Pi_t dt - \zeta_t d\mathcal{M}_t - \mathbb{1}_{\tau < \bar{T}} (1 - \bar{R}) \left( \Pi_t + \mathcal{C}_t^* \right)^+ dJ_t \\ & - J_t \left( dD_t + \sum_{Z \subseteq N} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt) \right) \\ & + \left( c_t (\mathcal{C}_t - P_{t-}^\zeta) + \bar{\lambda}_t (\Pi_t + \mathcal{C}_t^*)^+ - \lambda_t (\Pi_t + \mathcal{C}_t^*)^- \right) dt \end{aligned}$$

which is (2.4.3), by definition (2.4.4) of  $g$ .

**Proof of Lemma 2.5.1** Since  $\xi = (1 - R) (Q_{\tau^\delta} - \mathcal{C}_\tau^+)$  (cf. (2.3.8)), where

$$\begin{aligned} \mathcal{C}_\tau^+ &= \mathcal{C}_{\tau-} = \mathcal{C}(\tau, X_{\tau-}) \text{ and} \\ Q_{\tau^\delta} &= P_{\tau^\delta} + \Delta_{\tau^\delta} \\ &= P(\tau^\delta, \mathbf{X}_{\tau^\delta}) + \widehat{\Delta}(\tau^\delta, \mathbf{X}_{\tau^\delta}) - e^{\int_\tau^{\tau^\delta} r(u, \mathbf{X}_u) du} \widehat{\Delta}(\tau, \mathbf{X}_\tau) \end{aligned}$$

we have by definition (2.4.7) of  $\bar{\xi}$  :

$$\begin{aligned} \bar{\xi}_\tau &= (1 - R) \mathbb{E}_\tau \left[ e^{-\int_\tau^{\tau^\delta} r(u, \mathbf{X}_u) du} \times \right. \\ &\quad \left. \left( P(\tau^\delta, \mathbf{X}_{\tau^\delta}) + \widehat{\Delta}(\tau^\delta, \mathbf{X}_{\tau^\delta}) - e^{\int_\tau^{\tau^\delta} r(u, \mathbf{X}_u) du} \widehat{\Delta}(\tau, \mathbf{X}_\tau) - \mathcal{C}(\tau, X_{\tau-}) \right)^+ \right] \end{aligned} \quad (2.10.9)$$

Therefore, the Markov property of  $\mathbf{X}$  and the continuity of  $\mathbf{X}$  at time  $\tau$  imply that  $\bar{\xi}_\tau$  can be represented in functional form as  $\bar{\xi}(\tau, X_{\tau-})$ . Hence (cf. Crépey and Song (2016, Lemma 5.1)), it holds that

$$\gamma_t \widehat{\xi}_t = \gamma_t \bar{\xi}(t, X_t), \quad \mathbb{Q} \times \boldsymbol{\lambda} - a.e.,$$

where (2.5.1) yields  $\gamma = J_- \gamma_\bullet$ . This gives the result since  $dva = -\gamma \widehat{\xi}$ .

**Proof of Lemma 2.6.1** We denote by  $\mathcal{T}_\delta$  the transition function of the homogeneous Markov process  $(t, \mathbf{X}_t, \beta_t)$  over the time horizon  $\delta$ , i.e.

$$\begin{aligned} \mathcal{T}_\delta : (\varphi, (t, \mathbf{x}, b)) &\rightarrow \mathcal{T}_\delta[\varphi](t, \mathbf{x}, b) = \mathbb{E}[\varphi(t^\delta, \mathbf{X}_{t^\delta}, \beta_{t^\delta}) \mid \mathbf{X}_t = \mathbf{x}, \beta_t = b] \\ &= \mathbb{E}_t[\varphi(t^\delta, \mathbf{X}_{t^\delta}, \beta_{t^\delta})] \end{aligned}$$

Recalling (2.10.9) and using the fact that  $\mathbf{X}$  does not jump at time  $\tau$ , we have

$$\begin{aligned} \bar{\xi}_\tau &= \mathcal{T}_\delta \left[ \xi_\star \left( \cdot, \cdot, \cdot, \beta_\tau, \mathcal{C}_{\tau-}^*, \widehat{\Delta}_{\tau-} \right) \right] (\tau, \mathbf{X}_\tau, \beta_\tau) \\ &= \mathcal{T}_\delta \left[ \xi_\star \left( \cdot, \cdot, \cdot, \beta_\tau, \mathcal{C}_{\tau-}^*, \widehat{\Delta}_{\tau-} \right) \right] (\tau, \mathbf{X}_{\tau-}, \beta_\tau) \end{aligned} \quad (2.10.10)$$

where we set

$$\xi_\star(t, \mathbf{x}, b, \beta_\tau, \mathcal{C}_{\tau-}^*, \widehat{\Delta}_{\tau-}) = (1 - R) \beta_\tau^{-1} b \left( P(t, \mathbf{x}) + \widehat{\Delta}(t, \mathbf{x}) - \beta_\tau b^{-1} \widehat{\Delta}_{\tau-} - \mathcal{C}_{\tau-}^* \right)^+ \quad (2.10.11)$$

in which  $\beta_\tau$ ,  $\mathcal{C}_{\tau-}^*$  and  $\widehat{\Delta}_{\tau-}$  are considered as  $\mathcal{G}_{\tau-}$  measurable parameters. In view of (2.10.10), we have (cf. Crépey and Song (2016, Lemma 5.1))

$$\begin{aligned} -dva_t &= \gamma_t \widehat{\xi}_t \\ &= J_{t-} \gamma_t \mathcal{T}_\delta \left[ \xi_\star \left( \cdot, \cdot, \cdot, \beta_t, \mathcal{C}_t^*, \widehat{\Delta}_{t-} \right) \right] (t, \mathbf{X}_{t-}, \beta_t) \quad \mathbb{Q} \times \boldsymbol{\lambda} \text{ a.e.} \end{aligned} \quad (2.10.12)$$

As a consequence, given an independent random variable  $\zeta$  with density  $p$ , we can write, using (2.10.12), the definition of  $\mathcal{T}_\delta$  and (2.5.1) to pass to the second, third and fourth line, respectively :

$$\begin{aligned}
-\mathbb{E}[h_\zeta \mathbf{1}_{\zeta \leq \tau} \beta_\zeta dva(\zeta, X_\zeta)] &= -\int_0^T \mathbb{E}[h_t \beta_t \mathbf{1}_{t < \tau} dva(t, X_t)] p(t) dt \\
&= \int_0^T \mathbb{E}\left[h_t \beta_t \mathbf{1}_{t \leq \tau} \gamma_t \mathcal{T}_\delta\left[\xi_\star\left(\cdot, \cdot, \cdot, \beta_t, \mathcal{C}_t^\star, \widehat{\Delta}_t\right)\right](t, \mathbf{X}_t, \beta_t)\right] p(t) dt \\
&= \int_0^T \mathbb{E}\left[h_t \beta_t \mathbf{1}_{t \leq \tau} \gamma_t \mathbb{E}\left[\xi_\star\left(t^\delta, \mathbf{X}_{t^\delta}, \beta_{t^\delta}, \beta_t, \mathcal{C}_t^\star, \widehat{\Delta}_t\right)\right]\right] p(t) dt \\
&= \int_0^T \mathbb{E}\left[h_t \beta_t \mathbf{1}_{t \leq \tau} \gamma_\bullet(t) \xi_\star\left(t^\delta, \mathbf{X}_{t^\delta}, \beta_{t^\delta}, \beta_t, \mathcal{C}_t^\star, \widehat{\Delta}_t\right)\right] p(t) dt \\
&= \mathbb{E}\left[\mathbf{1}_{\zeta \leq T} h_\zeta \beta_\zeta \mathbf{1}_{\zeta \leq \tau} \gamma_\bullet(\zeta) \xi_\star\left(\zeta^\delta, \mathbf{X}_{\zeta^\delta}, \beta_{\zeta^\delta}, \beta_\zeta, \mathcal{C}_\zeta^\star, \widehat{\Delta}_\zeta\right)\right]
\end{aligned}$$



# XVA Metrics for CCP Optimization

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This chapter is based on [Armenti and Crépey \(2017b\)](#).

## 3.1 Introduction

In the aftermath of the financial crisis, the banking regulators undertook a number of initiatives to cope with counterparty risk. One major evolution is the generalization of central counterparties (CCPs), also known as clearing houses. A clearing house serves as an intermediary during the completion of the transactions between its clearing members. It organizes the collateralization of their transactions and takes care of the liquidation of the CCP portfolio of defaulted members. Non-members can have access to the services of a CCP through external accounts by the clearing members.

In order to mitigate counterparty risk, the CCP asks its clearing members to meet several collateralization requirements. Apart from the variation and initial margin (VM and IM) that are also required in bilateral trading (as gradually implemented since September 2016, regarding the IM), the clearing members contribute to a mutualized default fund (DF) set against extreme and systemic risk. See [Khwaja \(2016\)](#) for a review of margin and default schemes used by different CCPs on different asset classes.

In the light of the literature, pros and cons of CCPs can be summarized as follows :

**Counterparty credit risk and systemic risk :** Counterparty risk of the CCP itself low and default contagion effects between members reduced, but concentration risk if a major CCP were to default, with 30 major CCPs today and only a few prominent ones. CCPs also pose joint membership and feedback liquidity issues. On these and related issues see [Capponi, Cheng, and Rajan \(2015\)](#), [Glasserman, Moallemi, and Yuan \(2015\)](#) and [Barker, Dickinson, Lipton, and Virmani \(2016\)](#).

**Netting :** Multilateral netting benefit versus loss of bilateral netting across asset classes. See [Duffie and Zhu \(2011\)](#), [Cont and Kokholm \(2014\)](#), [Armenti and Crépey \(2017a\)](#) and [Ghamami and Glasserman \(2017\)](#).

**Transparency :** Portfolio wide information of the CCP and easier access to the data for the regulator, versus opacity of the default fund for the clearing members and joint membership issues again. On related (and other) CCP issues, see [Gregory \(2014\)](#).

**Efficiency :** Default resolution cheaper. Bilateral trading means an arbitrary network of transactions. An orderly default procedure cannot be done manually ; it requires an IT network, whether it is CCP, blockchain, SIMM reconciliation appliance, or whatever. However, the way CCPs are designed today entails two major inefficiencies for the clearing members, one related to the fact that default fund contributions are capital at risk not remunerated at a hurdle rate and another one related to the cost of borrowing their IM. See [Albanese \(2015\)](#) and [Ghamami \(2015\)](#).

## Contents of this chapter

The margins and the default fund mitigate counterparty risk, but they generate substantial costs. Chapter 2 studies the cost of the clearance framework for a member of a CCP, under standard regulatory assumptions on its default fund contribution and assuming unsecurely funded initial margin. Following up on the last item in the above list, the present work challenges these assumptions in two directions.

First, we confront the current default fund Cover 2 EMIR sizing rule with a broader risk based approach, advocated in [Ghamami \(2015\)](#) and [Albanese \(2015\)](#), relying on a suitable notion of economic capital ( $EC^{ccp}$ ) of a CCP. Regarding the allocation of the default fund between the clearing members, we compare a classical IM based allocation with the one based on member incremental  $EC^{ccp}$ .

Second, we assess the efficiency of an initial margin funding scheme, suggested in [Albanese \(2015\)](#), whereby a third party provides the IM in exchange of some service fee, as opposed to the standard procedure where clearing members unsecurely borrow their IM.

Note that such ideas, which may look rather orthogonal to current market practice, are actually not complete aliens to the industry. A default fund approach in the direction of the one of this work (calibrated to the Cover 2 EMIR regulatory prescription as we also suggest could be done in Section 3.3.1 is actually used by the Swiss CCP SIX X-Clear Ltd : see <https://www.six-securities-services.com/dam/downloads/clearing/clearing-notice/2017/clr-170420-clearing-notice-margin-en.pdf>. Likewise, the specialist lending business already exists at the early stages : in practice specialist lenders are private equity funds. See also [Albanese, Brigo, and Oertel \(2013\)](#) for similar ideas regarding VM. However, such funding schemes are much more difficult to implement for VM because VM is far larger and more volatile than IM.

The chapter is outlined as follows. Section 3.2 applies the XVA principles of [Albanese, Cae-nazzo, and Crépey \(2017b\)](#) to the assessment of the cost of the clearance framework for a clearing member of a CCP. The critical cost centers are the cost of funding their initial margin (MVA) and the cost of the capital (KVA) that they have to put at risk as their default fund contribution. Section 3.3 studies ways of compressing the related market inefficiencies. Section 3.4 presents a CCP toy model, where the above is illustrated numerically in Section 3.5, based on the analytics of Section 3.6.1.

## 3.2 Clearing Member XVA Analysis

In this section, we apply the XVA principles of [Albanese and Crépey \(2017\)](#) to a bank trading as a member of a clearing house with  $n$  other clearing members.

A clearing house eliminates the direct impact of the defaults of other clearing members on the surviving ones (as detailed in Remark 3.2.2, the default of the clearing house itself is essentially irrelevant to XVA analysis). But this comes at a certain cost for the clearing members. In this section we analyse the cost of the clearance framework on a reference clearing member bank. For other XVA frameworks, see, for instance :

- [Brigo and Pallavicini \(2014\)](#), [Bichuch, Capponi, and Sturm \(2017\)](#) (without KVA) ;
- Or, with a KVA meant as an additional contra-asset like the CVA and the FVA (as opposed to a risk premium in our case) : [Green \(2015\)](#), [Green, Kenyon, and Dennis \(2014\)](#), or [Elouerkhaoui \(2016\)](#).

The first reference in each bullet point includes a treatment of CCPs. However, in the first case, the default fund is ignored (the network of the clearing members is not introduced explicitly), whereas, in the second case, the emphasis is on regulatory capital, instead of economic capital in this work.

In practice banks tend to “clear the delta” of their (corporate) client derivative portfolio with CCPs. The CCP portfolio of the bank thus provides a fully collateralized, back-to-back hedge to

its client portfolio. This comes at the cost of the clearance framework for the bank, which then passes these costs to its clients.

Moreover, in reality, a bank is involved with many different clients, in centrally cleared vanilla trades and bilateral exotic ones, and CCPs (or CCP services), which are typically siloed by asset classes (see Figures 3.1 and 3.2). However, it is enough to understand the extreme cases of purely bilateral XVA analysis and of the XVA analysis of a bank performing all its trading as member of a single CCP. More complex situations (cf. e.g. Sherif (2017)) can then be tackled by combination of these two extreme cases. The XVA analysis of bilateral portfolios is detailed in Albanese, Caenazzo, and Crépey (2017a). In this chapter we focus on the “orthogonal” case of a bank clearing its delta with a single CCP.

Note that, assuming the client portfolio of the bank entirely cleared, it is not necessary to introduce the latter explicitly : it is enough to know that the contractually promised cash flows between the bank and the CCP are exactly compensated by mirroring cash flows between the bank and its clients.

### 3.2.1 Cash Flows

We consider a pricing stochastic basis  $(\Omega, \mathbb{G}, \mathbb{Q})$ , with model filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$  and risk-neutral pricing measure  $\mathbb{Q}$ , such that all the processes of interest are  $\mathbb{G}$  adapted and all the random times of interest are  $\mathbb{G}$  stopping times. The corresponding expectation and conditional expectation are denoted by  $\mathbb{E}$  and  $\mathbb{E}_t$ . We also introduce the value at risk and expected shortfall of level  $a$ ,  $\text{VaR}^a$  and  $\text{ES}^a$ , and their conditional versions  $\text{VaR}_t^a$  and  $\text{ES}_t^a$ .

We denote by  $r$  a  $\mathbb{G}$  progressive OIS (overnight indexed swap) rate process, which is together the best market proxy for a risk-free rate and the reference rate for the remuneration of the collateral. We write  $\beta = e^{-\int_0^\cdot r_s ds}$  for the corresponding risk-neutral discount factor.

By mark-to-market of a derivative portfolio, we mean the trade additive risk-neutral conditional expectation of its future discounted promised cash flows, ignoring counterparty risk and its capital and funding implications, i.e. without any XVAs. We consider a CCP with  $(n + 1)$  risky members, labelled by  $i = 0, 1, 2, \dots, n$ . We denote by :

- $T$  : an upper bound on the maturity of all claims in the CCP portfolio, also accounting for the time  $\delta > 0$  (assumed constant for simplicity) of liquidating the position between the bank and any of its counterparties in case of default ;
- $\bar{t} = t \wedge T$ ,  $t^\delta = t + \delta$ , for every  $t \geq 0$  ;
- $\tau_i$  : The default time of the member  $i$ , with non-default indicator process  $J^i = \mathbb{1}_{[0, \tau_i)}$  ;
- $D_t^i$  : The cumulative contractual cash flow process of the CCP portfolio of the member  $i$ , cash flows being counted positively when they flow from the clearing member to the CCP ;
- $\text{MtM}_t^i = \mathbb{E}_t \left[ \int_t^T \beta_t^{-1} \beta_s dD_s^i \right]$  : The mark-to-market of the CCP portfolio of the member  $i$  ;
- $\Delta_t^i = \int_{[t-\delta, t]} \beta_t^{-1} \beta_s dD_s^i$  : The cumulative contractual cash flows of the member  $i$ , accrued at the OIS rate, over a past period of length  $\delta$  ;
- $\text{VM}_t^i, \text{IM}_t^i \geq 0, \text{DFC}_t^i \geq 0$  : VM, IM, and DFC posted by the member  $i$  at time  $t$ .

We do not exclude simultaneous defaults, but we suppose that all the default times are positive and endowed with an intensity (in particular, defaults at any constant or  $\mathbb{G}$  predictable time have zero probability). Regarding the liquidation procedures, for ease of analysis, we assume the existence of a risk-free (hence, non IM or DFC posting) “buffer” replacing defaulted members in their transactions with the surviving members, after a liquidation period of length  $\delta$ . In the interim, the positions of the defaulted members are carried by the clearing house. Accounting

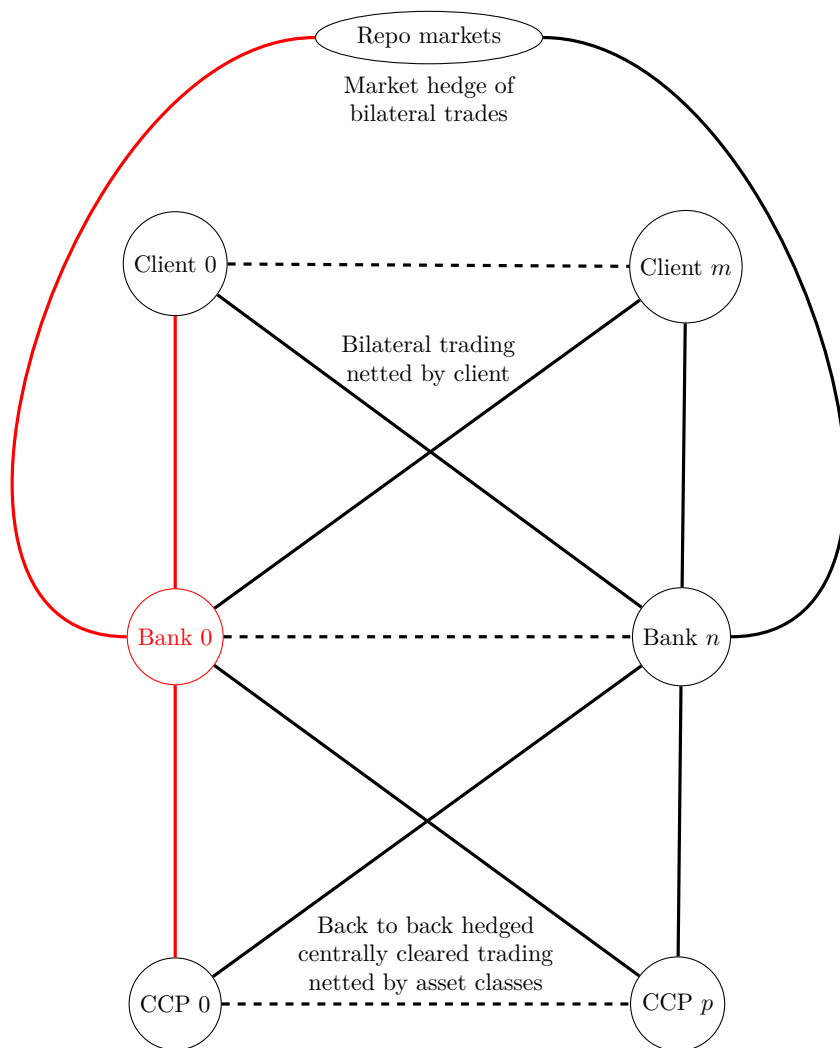


FIGURE 3.1 – Financial network of clients, banks, and CCPs. Solid edges represent cash flows between the related entities. Bilateral trades correspond to the upper part of the picture (banks and above) and centrally cleared trades to the lower part (banks and below). We assume that centrally cleared client trades are back-to-back hedged for the banks, in terms of market risk, by offsetting CCP trades, whereas bilateral client trades are hedged by banks through repo markets. Figure 3.2 provides a focus on the red part of the graph with more detail.

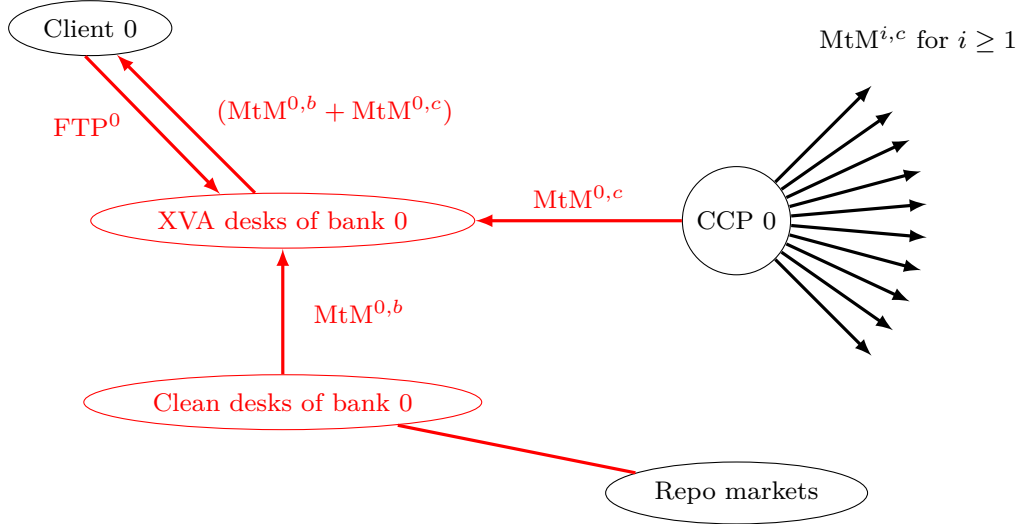


FIGURE 3.2 – Zoom on a reference bank, labelled by 0, focusing on its transactions with client 0 and CCP 0, corresponding to the red part in Figure 3.1. The XVA desks of the bank filter out counterparty risk and its capital and funding implications from client trades, so that the other (“clean”) trading desks of the bank can focus on the market risk of their business lines, as if there was no counterparty risk. This is at least the picture for bilateral transactions (see Albanese, Caenazzo, and Crépey (2017b, Section 2.2)), labelled by  $\cdot^b$  in the picture. In the case of centrally cleared transactions, labelled by  $\cdot^c$ , the task of the bank reduces to its interaction with the CCP through its XVA desks, whereas the CCP itself provides fully collateralized deals to its clearing members. The arrows represent the direction of deal entry payments between the bank, the client and the CCP, under the convention that the reference clearing member bank 0 “buys” assets from its clients, at an FTP (all-inclusive XVA rebate) deducted price with respect to their “clean valuation” ignoring counterparty risk. Clean valuation is denoted by  $MtM^{0,b}$  (for bilateral trades) or  $MtM^{0,c}$  (for centrally cleared cleared). Repo traded hedges of its bilateral transactions are assumed entered at no upfront payment by the bank. The trading of the CCP 0 with the other clearing members is suggested by the arrows  $MtM^{i,c}$ , for  $i = 1, \dots, n$ . The trading of the CCP clears, i.e.  $\sum_{i=0}^n MtM^{i,c} = 0$ .

for the OIS accrued value  $\Delta_{\tau_i^\delta}^i$  of the cash flows contractually due by the member  $i$  to the other clearing members from time  $\tau_i$  onward (cash flows unpaid due to the default of the member  $i$  at  $\tau_i$ ), the loss triggered by the liquidation of the member  $i$  at time  $\tau_i^\delta$  is

$$\left( \text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i - \beta_{\tau_i^\delta}^{-1} \beta_{\tau_i} (\text{VM}_{\tau_i}^i + \text{IM}_{\tau_i}^i + \text{DFC}_{\tau_i}^i) \right)^+ \quad (3.2.1)$$

(assuming that margin and DFC accounts accrue at rate  $r$ ). Note that, contrary to the bilateral case, there is no recovery stemming from the liquidation of the CCP portfolio of a defaulted member.

**Remark 3.2.1** The expression (3.2.1) for the loss of the CCP given the default of the member  $i$  is a stylized formulation ignoring the possibility of CCP close-out losses in relation to illiquidity of OTC markets in the aftermath of a major default in addition to the “gap risk of slippage” over the length  $\delta$  of the liquidation period. Accordingly, we will use for the embedded  $\text{IM}^i$  a basic specification in the form of a value-at-risk of the  $\delta = \text{one week}$  increment of  $\text{MtM}^i$  ( $\text{VM}_{\tau_i}^i$  being taken as  $\text{MtM}_{\tau_i}^i$ ). In reality, OTC markets are sensitive to the potential illiquidity experienced by a CCP in macro-hedging and then auctioning a potentially large and illiquid OTC portfolio in the aftermath of a major default. CCPs are of course aware of this and account for it by means of suitable initial margin liquidity add-ons. All in one, at the conceptual level of this work, it is simpler to stay with the basic formulation (3.2.1), playing with the quantile levels that are used for setting the IM (and also the DFC as we will see) in order to emulate more or less conservative CCP setups.

**Remark 3.2.2** The actual size of the default fund is sometimes referred to as the funded default fund, as opposed to the unfunded default fund, which refers to the additional amounts members may have to pay via the above default fund replenishment principle in case of defaults of other members. The service closure, i.e. the closure of the activity of the clearing house on a given market or service, is usually specified in terms of a cap on the unfunded default fund, such as the unfunded default fund reaching twice the initial level of the funded default fund. Given the high levels of initial margins that are used in practice, this is a very extreme tail event. Moreover, in case of service closure, the risk of a member is bounded by the sum between its margins, three times its default fund contribution (assuming the above specification of service closure) and the cost of the liquidation of the service for this member. This cost is itself bounded by the notional of the member position, which would only be the actual cost in a scenario where all the assets of the CCP would jump to zero, also a very unlikely situation. In conclusion, including the service closure event through a finite cap on member refill would only negligibly affect the XVA amounts. The default of the CCP as a whole (i.e. the closure of all its services) is an even more unlikely event, especially because a central bank would hardly allow it to occur in view of its systemic consequences. Hence we may and do ignore the service closure and the default of the clearing house in the context of XVA analysis.

In the sequel the bank corresponds to the reference member 0. For notational simplicity we remove any index 0 referring to it and we write  $\bar{\tau} = \tau \wedge T = \tau_0 \wedge T$ . The CCP is simply an interface between the clearing members. Hence the overall CCP portfolio clears, i.e.

$$\text{MtM} = \text{MtM}^0 = - \sum_{i \neq 0} \text{MtM}^i \quad (3.2.2)$$

and we assume likewise

$$\text{VM} = \text{VM}^0 = - \sum_{i \neq 0} \text{VM}^i$$

Recall that we do not exclude simultaneous defaults. For any  $Z \subseteq \{1, 2, \dots, n\}$ , let  $\tau_Z$  denote the time when members in  $Z$  and only in  $Z$  default (or  $+\infty$  if this never happens). At each

$t = \tau_Z^\delta < \bar{\tau}$ , the loss of the bank, assumed instantaneously realized as refill to its default fund contribution, is (also accounting for the unwinding of the corresponding client trades)

$$\epsilon_{\tau_Z^\delta} = w_{\tau_Z^\delta} \sum_{i \in Z} \left( \text{MtM}_{\tau_Z^\delta}^i + \Delta_{\tau_Z^\delta}^i - \beta_{\tau_Z^\delta}^{-1} \beta_{\tau_Z} (\text{VM}_{\tau_Z} + \text{IM}_{\tau_Z}^i + \text{DFC}_{\tau_Z}^i) \right)^+ \quad (3.2.3)$$

for some refill allocation key  $w_t$ . A typical specification is proportional to the default fund contributions of the surviving members, i.e.

$$w_t = \frac{\text{DFC}_t}{\text{DFC}_t + \sum_{i \neq 0} J_t^i \text{DFC}_t^i} \quad (3.2.4)$$

Note that (3.2.3) conservatively ignore the impact of netting in the context of the joint liquidation of several defaulted members (and we ignore the equity or “skin-in-the-game” of the CCP, which is typically small and therefore negligible from a loss-absorbing point of view).

We assume that the bank can invest cash in excess of its funding requirements at the OIS rate  $r$ , borrow collateral to post as VM at its unsecured funding spread  $\lambda$  over  $r$ , and borrow collateral to post as IM at a possibly blended spread (see Section 3.3.2)  $\bar{\lambda}$ .

### 3.2.2 Contra-Assets Valuation

Contra-assets are the liability triggered by the derivative portfolio of the bank, given in particular the impossibility for the bank to hedge its own jump-to-default exposure (see Albanese, Caenazzo, and Crépey (2017b, Section 5.1)). As contra-assets are marked to the model, their value process, denoted by CA, is part of the trading loss(-and-profit) of the bank. Moreover, before resorting to unsecured borrowing for raising collateral, the bank can first use the amount CA charged to the client of the deals and deposited on the so-called reserve capital account of the bank.

We denote by  $\delta_t$  the Dirac measure at time  $t$ .

**Lemma 3.2.1** *In the case of a centrally cleared portfolio of trades between a reference clearing member bank and  $n$  other clearing members  $i = 1, \dots, n$ , given a putative CA process, the trading loss (and profit) process  $L$  of the bank satisfies the following forward SDE :*

$$\begin{aligned} L_0 &= z \text{ (the initial trading loss of the bank) and, for } t \in (0, \bar{\tau}], \\ dL_t &= d\text{CA}_t - r_t \text{CA}_t dt + J_t \sum_Z \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt) \\ &\quad + \left( \lambda_t (\text{VM}_t - \text{MtM}_t - \text{CA}_t)^+ + \bar{\lambda}_t \text{IM}_t \right) dt \end{aligned} \quad (3.2.5)$$

(and  $L$  is constant from time  $\bar{\tau}$  onward).

**Proof:** Collecting all the trading cash flows of the reference clearing member bank, we obtain :

$$\begin{aligned}
L_0 &= z \text{ and, for } t \in (0, \bar{\tau}], \\
dL_t &= \underbrace{J_t \sum_Z \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt)}_{\text{Counterparty default losses of the bank}} \\
&+ \underbrace{\left( (r_t + \lambda_t) (VM_t - MtM_t - CA_t)^+ - r_t (VM_t - MtM_t - CA_t)^- \right) dt}_{\text{Bank costs/benefits of funding the VM posted on its CCP portfolio, net of MtM received as VM on its client portfolio and of the reserve capital amount } CA_t \text{ also available as a funding source for the bank}} \\
&+ \underbrace{(r_t + \bar{\lambda}_t) IM_t dt}_{\text{Bank IM funding costs}} \\
&- \underbrace{r_t (VM_t - MtM_t + IM_t) dt}_{\text{Posted VM is remunerated OIS by the receiving party and IM accrues at the OIS rate}} \\
&- \underbrace{r_t CA_t dt}_{\text{Risk-free funding of the bank position taken over by the CCP during the bank liquidation period}} \\
&- \underbrace{(-dCA_t)}_{\text{Depreciation of the liability CA of the bank}}
\end{aligned}$$

which gives (3.2.5) ■

In the spirit of a bank shareholder no-arbitrage condition, we assume that the trading loss process  $L$  must be a risk-neutral local martingale. Moreover, assuming all the assets of the bank wiped out at time  $\tau$  (cf. Albanese, Caenazzo, and Crépey (2017b, Section 8)), the  $CA$  process satisfies the terminal condition  $CA_{\bar{\tau}} = 0$ . Therefore :

**Proposition 3.2.1 (i)** *The contra-asset value process  $CA$  of the bank satisfies the following backward SDE on  $[0, \bar{\tau}]$  :*

$$\begin{aligned}
CA_{\bar{\tau}} &= 0 \text{ and, for } t \in (0, \bar{\tau}], \\
dCA_t &= -J_t \sum_Z w_{\tau_Z^\delta} \epsilon_{\tau_Z^\delta} \delta_{\tau_Z^\delta}(dt) + r_t CA_t dt + dL_t \\
&\quad - \left( \lambda_t (VM_t - MtM_t - CA_t)^+ + \bar{\lambda}_t IM_t \right) dt
\end{aligned} \tag{3.2.6}$$

for some risk-neutral local martingale  $L$  corresponding to the trading loss process of the bank.

(ii) Assuming integrability, it holds that

$$\begin{aligned}
 \text{CA}_t = & \mathbb{E}_t \left[ \underbrace{\sum_{t < \tau_Z^\delta < \bar{\tau}} \beta_t^{-1} \beta_{\tau_Z^\delta} w_{\tau_Z^\delta} \sum_{i \in Z} \left( \text{MtM}_{\tau_Z^\delta}^i + \Delta_{\tau_Z^\delta}^i - \beta_{\tau_Z^\delta}^{-1} \beta_{\tau_Z} \left( \text{VM}_{\tau_Z^\delta}^i + \text{IM}_{\tau_Z^\delta}^i + \text{DFC}_{\tau_Z^\delta}^i \right) \right)^+}_{\text{CVA}_t \text{ (credit valuation adjustment)}} \right] \\
 & + \underbrace{\mathbb{E}_t \left[ \int_t^{\bar{\tau}} \beta_t^{-1} \beta_s \lambda_s (\text{VM}_s - \text{MtM}_s - \text{CA}_s)^+ ds \right]}_{\text{FVA}_t \text{ (funding valuation adjustment)}} \\
 & + \underbrace{\mathbb{E}_t \left[ \int_t^{\bar{\tau}} \beta_t^{-1} \beta_s \bar{\lambda}_s \text{IM}_s ds \right]}_{\text{MVA}_t \text{ (margin valuation adjustment)}}, \quad 0 \leq t \leq \tau^\delta
 \end{aligned} \tag{3.2.7}$$

**Proof:** Accounting for the risk-neutral martingale condition on  $L$  and the terminal condition  $\text{CA}_{\bar{\tau}} = 0$ , the SDE (3.2.5) in Lemma 3.2.1 implies (i), hence (assuming integrability) (ii). ■

**Remark 3.2.3** The initial (actually unknown) condition  $L_0 = z$  in (3.2.5) is immaterial.

### 3.2.3 Capital Valuation Adjustment

On top of no arbitrage in the sense of risk-neutral CA valuation, bank shareholders need to be remunerated at some hurdle rate  $h$  for their capital at risk. As default fund contributions are loss-absorbing and survivor-pay (beyond the level of losses covered by the margins and the DFC of the defaulted members), they are capital at risk of the clearing members. In fact, the capital at risk of a bank operating as clearing member of a CCP takes the form of its default fund contribution.

**Remark 3.2.4** Regulatory capital is also required from the bank for dealing with potential losses beyond its margin and default fund contribution. But, given the regulatory incentivization of central clearing, such regulatory capital is negligible in practice (see Chapter 2 for numerical illustration).

As a result, in a centrally cleared trading setup, the KVA formula (65) in Albanese, Caenazzo, and Crépey (2017b), corresponding to a remuneration of bank shareholder capital at risk at a constant hurdle rate  $h$ , needs be amended as

$$\text{KVA}_t = h \mathbb{E}_t \left( \int_t^{\bar{\tau}} e^{-\int_t^s (r_u + h) du} \text{DFC}_s ds \right) \quad t \in [0, \bar{\tau}] \tag{3.2.8}$$

(assuming all the assets of the bank wiped out at time  $\tau$ , see Albanese, Caenazzo, and Crépey (2017b, Section 8)). The formula (3.2.8) can be seen as a continuous-time analogous to the risk margin formula in the Solvency II eurozone insurance regulation (itself adapted from Swiss Solvency Test (2017)), where  $h$  is set as 6%.

This perspective opens the door to an organization of a clearance framework, whereby a CCP could remunerate the clearing members for their default fund contributions. This would make the clearing members less reluctant to put capital at risk in the default fund. In fact, if it was remunerated at some hurdle rate, the default fund of a CCP could even become attractive and be open to external investors (if that could be done without prejudice to the other key role of the default fund, which is to give the clearing members incentive to bid in the auctions setup by the CCP to liquidate the CCP portfolios of defaulted members).

Capital and cost of capital calculations are supposed to be performed under the historical probability measure  $\mathbb{P}$ . But  $\mathbb{P}$  is hardly estimable for the purpose of cost of capital calculations, which involve projections over decades in the future. As a consequence, we do all our price and risk computations under a risk-neutral measure  $\mathbb{Q}$  calibrated to the market. In other words, we work under the modelling assumption that  $\mathbb{P} = \mathbb{Q}$ , leaving the residual uncertainty about  $\mathbb{P}$  to model risk.

**Remark 3.2.5** As soon as quantitative methodologies are used regarding the default fund and/or initial margins, an important topic is the related model risk, which is of course high as soon as risk measures (hence the tail of the distribution of the P&L of the member banks) are involved. This topic is left for future research.

### 3.2.4 Funds Transfer Price

In the context of XVA computations, derivative portfolios are typically modelled on a run-off basis, i.e. assuming that no new trades will enter the portfolio in the future. Otherwise the bank could be led into snowball Ponzi schemes, where always more deals are entered for the sole purpose of funding previously entered ones. Moreover the trade-flow of a price-maker bank does not have a stationarity property that could allow the bank forecasting future trades.

Of course in reality a bank deals with incremental portfolios, where trades are added or removed as time goes on. Accordingly, incremental XVAs are computed at every new trade, as the differences between the portfolio XVAs with and without the new trade, the portfolio being assumed held on a run-off basis in both cases.

The incremental all-inclusive XVA of a new deal, called funds transfer price (FTP), corresponds for the bank to the “fabrication cost” of the deal, computed on an incremental run-off basis given the endowment (legacy portfolio) of the bank. Summing up the above, in case of a new deal through the CCP, the FTP of the reference clearing member bank is given by (cf. (3.2.7) and (3.2.8)) :

$$\text{FTP} = \Delta\text{CA} + \Delta\text{KVA} = \Delta\text{CVA} + \Delta\text{FVA} + \Delta\text{MVA} + \Delta\text{KVA} \quad (3.2.9)$$

computed on an incremental run-off basis relatively to the portfolios with and without the new deal.

Given the high level of collateralization that applies in the context of centrally cleared trading, the credit valuation adjustment (CVA) of a clearing member, i.e. its expected loss due to other members’ defaults, is typically quite small. Moreover, for daily (or even more frequent) re-margining on the derivative portfolio, the variation margins of a clearing member on its derivative portfolio and on its back-to-back hedge tend to match each other. Hence the funding variation adjustment (FVA) of a member (cf. (3.2.7)) is also quite small and much smaller than its MVA. As a consequence, in a centrally cleared setup, the prominent XVA numbers of a clearing member are its MVA and its KVA.

## 3.3 Default Fund Contributions and Initial Margin Funding Schemes

As of today :

- Posted IM is typically borrowed unsecured by the bank, resulting in  $\bar{\lambda} = \lambda$  in (3.2.5) ;
- The default fund of a (European) CCP is sized according to the EMIR Cover 2 rule, i.e. enough to cover the joint default of the two clearing members with the greatest CCP exposures ;
- The typical allocation of the total amount between the clearing members is proportional to their initial margins or to a suitable notion of losses over IM.

Proportionality to initial margins makes the default fund contributions in the same direction as the initial margins, which is felt as unfair by the clearing members. Proportionality to losses over IM makes the evolution of the default fund contributions completely unpredictable to them, which they like even less. In the sequel, we use the IM proportional rule as a benchmark.

In any case, both the size and the allocation of the default fund are purely based on market risk, irrespective of the credit risk of the clearing members. The latter is only accounted for marginally and in a second step, by means of specific add-ons to the IM of the riskiest members (cf. Remark 3.2.1).

However, whatever the prevailing regulation and market practice in terms of capital and funding policies, for XVA computations that entail projections of these over decades, an economical specification is more appropriate than the ad-hoc and ever-changing regulatory specifications supposed to approximate it. Two important considerations in this regard are the specification of the default fund and of the funding policy for initial margins.

### 3.3.1 Economic Capital Based Default Fund

As explained in Section 3.2.3, through their default fund contributions, the clearing members provide capital at risk to the CCP (ignoring the skin-in-the-game of the CCP, which is negligible from a loss-absorbing point of view). The economical capital and KVA methodology of [Albanese, Caenazzo, and Crépey \(2017b\)](#) can be used for designing an economically sound and sustainable specification of the default fund and of its allocation between the clearing members. Beyond the theoretical interest and message to the regulator, this approach can yield valuable specifications, even under the current regulatory environment, for the default fund and its allocation that intervene as data in the CA equation (3.2.6) and KVA formula (3.2.8). In this perspective, an economical specification can also be calibrated at time 0 to the actual regulatory capital amounts of the bank.

In view of the losses (3.2.3) summed over all members, we define an aggregated loss process of a CCP that would be in charge of dealing with member counterparty default losses through a  $CVA^{ccp}$  account (earning OIS) and capital at risk at the aggregated CCP level as (cf. (3.2.5))

$$\begin{aligned} L_0^{ccp} &= z^{ccp} \text{ (the initial loss of the CCP) and, for } t \in (0, T], \\ \beta_t dL_t^{ccp} &= \sum_i \left( \beta_{\tau_i^\delta} \left( \text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i \right) - \beta_{\tau_i} \left( \text{VM}_{\tau_i}^i + \text{IM}_{\tau_i}^i \right) \right)^+ \delta_{\tau_i^\delta}(dt) \\ &\quad + \beta_t (dCVA_t^{ccp} - r_t CVA_t^{ccp}) dt \end{aligned} \quad (3.3.1)$$

(and  $L$  constant from time  $T$  onward), where the CVA of the CCP is given as

$$CVA_t^{ccp} = \mathbb{E}_t \left[ \sum_{t < \tau_i^\delta < T} \beta_t^{-1} \left( \beta_{\tau_i^\delta} \left( \text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i \right) - \beta_{\tau_i} \left( \text{VM}_{\tau_i}^i + \text{IM}_{\tau_i}^i \right) \right)^+ \right], 0 \leq t \leq T \quad (3.3.2)$$

The ensuing economic capital process of the CCP

$$\text{EC}_t^{ccp} = \mathbb{E}_t^{a,df} \left( \int_t^{t+1} \beta_t^{-1} \beta_s dL_s^{ccp} \right) \quad (3.3.3)$$

where, in view of (3.3.1),

$$\begin{aligned} \int_t^{t+1} \beta_s dL_s^{ccp} &= \sum_{t < \tau_i^\delta \leq t+1} \left( \beta_{\tau_i^\delta} \left( \text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i \right) - \beta_{\tau_i} \left( \text{VM}_{\tau_i}^i + \text{IM}_{\tau_i}^i \right) \right)^+ \\ &\quad - (\beta_t CVA_t^{ccp} - \beta_{t+1} CVA_{t+1}^{ccp}) \end{aligned} \quad (3.3.4)$$

yields the size of an overall risk based default fund at the confidence quantile level  $a_{df}$ . The current regulatory Cover 2 EMIR rule purely relies on market risk. By contrast, the sizing rule (3.3.3) reflects a broader notion of risk of the CCP, in the form of a risk measure of its one-year ahead loss-and-profit if there was no default fund, as it results from the combination of the credit risk of the clearing members and of the market risk of their portfolios.

The KVA of the CCP estimates how much it would cost the CCP to remunerate all clearing members at some hurdle rate  $h$  for their capital at risk in the default fund, namely, for  $t \leq T$  (cf. (3.2.8)) :

$$\text{KVA}_t^{\text{ccp}} = h\mathbb{E}_t \left( \int_t^T e^{-(r+h)s} \text{DF}_s \, ds \right) \quad (3.3.5)$$

A member incremental  $\text{EC}^{\text{ccp}}$  or  $\text{KVA}^{\text{ccp}}$  allocation of the default fund between the  $(n+1)$  clearing members could be used as an alternative to the usual IM proportional allocation.

### 3.3.2 Specialist Lending of Initial Margin

Let  $\lambda = \gamma(1-R)$  denote the instantaneous CDS spread of the bank, where  $\gamma$  is its risk-neutral default intensity and  $R$  its recovery rate as implicit in its CDS spread quotations.

The time-0 margin valuation adjustment (MVA) of the bank when its IM is funded through unsecured borrowing is given by (cf. 3.2.7)

$$\text{MVA}_0^{ub} = \mathbb{E} \left( \int_0^{\bar{\tau}} \beta_s \lambda_s \text{IM}_s \, ds \right) \quad (3.3.6)$$

However, instead of assuming its IM borrowed by the bank on an unsecured basis, we can consider an alternative scheme whereby IM is provided by a liquidity supplier, dubbed “specialist lender”, lending IM in exchange of some fee. Under the terms of a legal agreement concluded between the CCP and the specialist lender, in case of default of the bank, the specialist lender would receive back from the CCP the portion of IM unused to cover losses. Hence, as opposed to unsecured borrowing, where, in case of default of the bank, IM unused to cover losses just increases the recovery rate of the bank creditors, by contrast, with specialist lending of initial margin, IM unused to cover losses stays with the shareholders of the specialist lender. As a result, specialist lending compresses the MVA wealth transfer from bank shareholders to creditors triggered by the derivative portfolio of the bank (see Albanese, Caenazzo, and Crépey (2017b, Section 5.1)).

More precisely, assuming as standard that IM is subordinated to own DFC, i.e. that the first levels of losses are absorbed by IM, the exposure of the specialist lender to the default of the bank is

$$(G_{\tau^\delta}^+ \wedge \beta_{\tau^\delta}^{-1} \beta_\tau \text{IM}_\tau)$$

for a time- $t$  gap  $G_t$  given as

$$G_t = \text{MtM}_t + \Delta_t - \beta_t^{-1} \beta_{t-\delta} \text{VM}_{t-\delta} \quad (3.3.7)$$

The time-0 MVA of the bank under such a third party arrangement follows as

$$\text{MVA}_0^{sl} = \mathbb{E} [\beta_{\tau^\delta} \mathbf{1}_{\tau < T} (G_{\tau^\delta}^+ \wedge \beta_{\tau^\delta}^{-1} \beta_\tau \text{IM}_\tau)] = \mathbb{E} \left( \int_0^{\bar{\tau}} \beta_s \gamma_s \xi_s \, ds \right) \quad (3.3.8)$$

where  $\xi$  is a  $\mathbb{G}$  predictable process, which exists by Corollary 3.23 2) in He and Yan (1992), such that  $\mathbb{E}_{\tau-} (\beta_{\tau^\delta} G_{\tau^\delta}^+ \wedge \beta_\tau \text{IM}_\tau) = \beta_\tau \xi_\tau$ . The process  $\gamma \xi \, ds$  corresponds to the fees to be paid by the

bank to the specialist lender. By identification with the generic expression  $\bar{\lambda}_s \text{IM}_s$  in (3.2.5), the formula (3.3.8) corresponds to a blended IM funding spread

$$\bar{\lambda} = \frac{\gamma \xi}{\text{IM}} = \frac{1}{(1-R)} \frac{\xi}{\text{IM}} \lambda$$

Under a common specification where  $\beta_s \text{IM}_s$  is set as a high quantile (value-at-risk) of  $\beta_{s\delta} G_{s\delta}$  (cf. (3.4.3) below, assuming there for simplicity continuous-time variation margining  $\text{VM}_t = \text{MtM}_t$  until time  $\tau$  in (3.3.7)), for a bank with an unsecured recovery rate  $R$  commonly estimated in a range between 20% and 40%, the blending factor

$$\frac{1}{(1-R)} \frac{\xi}{\text{IM}}$$

is typically significantly less than one. Hence  $\bar{\lambda}$  is significantly less than  $\lambda$  and  $\text{MVA}_0^{sl}$  significantly less than  $\text{MVA}_0^{ub}$ .

**Remark 3.3.1** The initial margin determined by a CCP relies on historical data which is unconditional in the sense that very few days reflect market conditions in the aftermath of a bank default. Since initial margin is only ever required in default scenarios, the confidence levels used for computation may be misleading and losses above initial margin may easily be understated. This is why the regulation imposes very high quantile levels  $a_{im}$  in the value-at-risk used for setting the IM, e.g.  $a_{im} \geq 99\%$  under EMIR requirements. Moreover, it is not uncommon that CCPs use even higher quantile levels, e.g. 99.7% at LCH SA. In addition, most CCPs charge, on top of the corresponding value-at-risk, various IM add-ons meant to account for liquidity, credit risk, etc. (cf. Remark 3.2.1). In conclusion, even if it is theoretically possible to consider situations in which bespoke initial margin funding may actually be more expensive than traditional unsecured funding (contradicting (3.3.2)), such a scenario is quite unlikely to occur in practice.

Note that such an IM funding policy is not a violation of *pari passu* rules. It just compresses the MVA wealth transfer from bank shareholders to creditors triggered by the derivative portfolio of the bank. Subordinating own DFC to IM would result in less IM consumption upon defaults, hence even more efficient specialist lender IM funding schemes.

The remaining of the chapter is a case study of our approach in the CCP toy model of Armenti and Crépey (2017a, Section 7). The actual number of members in CCP services varies from four or five in starting services to several hundreds on certain asset classes. However, most CCP services are driven by no more than a dozen of major players, with typically two or three prominent ones (see e.g. Armenti, Crépey, Drapeau, and Papantoleon (2016, Sections 6.1 and C)). Hence we want to consider a family of members, not necessarily large, but well diversified in terms of market and credit risk, which are the two main features of interest for the points we want to illustrate in this chapter. The resulting XVA numbers should be considered not so much in absolute terms than in terms of comparison between the clearing members and of sensitivities with respect to the market and credit risks of the latter.

Note that running a similar exercise on a real CCP dataset would necessitate to implement the totality of the CCP pricers and to dynamically compute the economic capital of a real CCP at all the nodes of a computational tree with final maturity  $T$  of all claims in the CCP portfolio. Such an implementation effort would be out of reach in the context of an academic paper.

## 3.4 CCP Toy Model

In this section we briefly recap the CCP setup of Section 2.7, to which we refer the reader for more details. In particular,  $\text{CVA}^{ccp}$  is analytic in this model (see Section 3.6.1), which avoids

the numerical burden of nested Monte Carlo that is required otherwise for simulating the loss and profit processes involved in capital computations.

### 3.4.1 Market Model

As common asset driving all our clearing member portfolios, we consider a stylized swap with strike rate  $\bar{S}$  and maturity  $T$  on an underlying interest rate process  $S$ . At discrete time points  $T_l$  such that  $0 < T_1 < T_2 < \dots < T_d = T$ , the swap pays an amount  $h_l (S_{T_{l-1}} - \bar{S})$ , where  $h_l = T_l - T_{l-1}$ . The underlying rate process  $S$  is supposed to follow a standard Black-Scholes dynamics with risk-neutral drift  $\kappa$  and volatility  $\sigma$ , so that the process  $\hat{S}_t = e^{-\kappa t} S_t$  is a Black martingale with volatility  $\sigma$ . For  $t \in [T_0 = 0, T_d = T]$ , we denote by  $l$  the index such that  $T_{l-1} \leq t < T_l$ . The following numerical parameters are used :

$$r = 2\%, S_0 = 100, \kappa = 12\%, \sigma = 20\%, h_l = 3 \text{ months}, T = 5 \text{ years}.$$

The nominal (Nom) of the swap is set so that each leg has a time-0 mark-to-market of one (i.e.  $10^4$  basis points). Figure 3.3 shows the resulting mark-to-market (process  $\text{MtM}^*$  in (3.6.1) below) viewed from the perspective of a party long one unit position, i.e. receiving floating, in the swap.

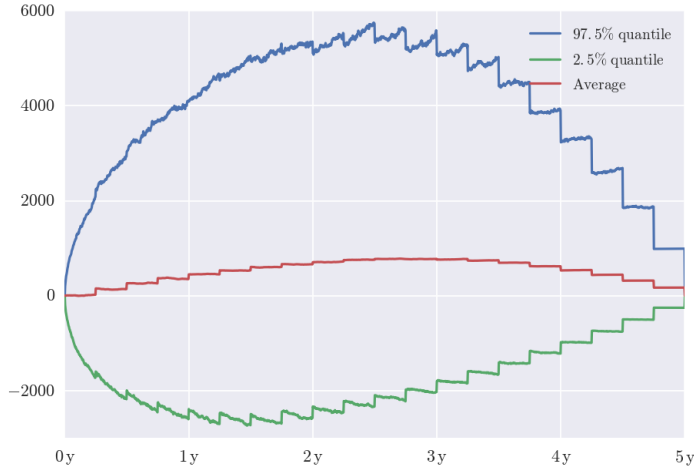


FIGURE 3.3 – Mean and 2.5% and 97.5% quantiles, in basis points as a function of time, of the process  $\text{MtM}^*$  in (3.6.1), calculated by Monte Carlo simulation of 5000 Euler paths of the process  $S$ .

### 3.4.2 Credit Model

For the default times  $\tau_i$  of the clearing members, we use the “common shock” or dynamic Marshall-Olkin copula (DMO) model of Crépey, Bielecki, and Brigo (2014, Chapters 8–10) and Crépey and Song (2016) (see also Elouerkhaoui (2007) and Elouerkhaoui (2017)). In this model defaults can happen simultaneously with positive probabilities. The model is made dynamic, as required for XVA computations, by the introduction of the filtration of the indicator processes of the  $\tau_i$ .

First we define shocks as pre-specified subsets of the clearing members, i.e. the singletons  $\{0\}, \{1\}, \{2\}, \dots, \{n\}$ , for single defaults, and a small number of groups representing members susceptible to default simultaneously.

**Example 3.4.1** A shock  $\{1, 2, 4, 5\}$  represents the event that all the (non-defaulted names among the) members 1, 2, 4, and 5 default at that time.

As demonstrated numerically in Crépey, Bielecki, and Brigo (2014, Section 8.4), a few common shocks are typically enough to ensure a good calibration of the model to market data regarding the credit risk of the clearing members and their default dependence (or to expert views about these).

Given a family  $\mathcal{Y}$  of shocks, the times  $\eta_Y$  of the shocks  $Y \in \mathcal{Y}$  are modelled as independent time-inhomogeneous exponential random variables with intensity functions  $\gamma_Y$ . For each clearing member  $i = 0, \dots, n$ , we then set

$$\tau_i = \min_{\{Y \in \mathcal{Y}; i \in Y\}} \eta_Y \quad (3.4.1)$$

(we recall that the default time  $\tau$  of the reference clearing member bank corresponds to  $\tau_0$ ). The specification (3.4.1) means that the default time of the member  $i$  is the first time of a shock  $Y$  that contains  $i$ . As a consequence, the (pre-default) intensity  $\gamma_i$  of  $\tau_i$  is the constant

$$\gamma_i = \sum_{\{Y \in \mathcal{Y}; i \in Y\}} \gamma_Y$$

with associated CDS spread  $\lambda_i = (1 - R_i) \gamma_i$ , where  $R_i = 40\%$  is taken as recovery rate implicit in CDS spread market quotations.

**Example 3.4.2** Consider a family of shocks

$$\mathcal{Y} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{2, 3\}, \{0, 1, 2, 4, 5\}\}$$

(with  $n = 5$ ). The following illustrates a possible default path in the model.

$t = 0.9 :$	$\{3\}$	0	1	2	③	4	5	$\tau_3 = 0.9$
$t = 1.4 :$	$\{5\}$	0	1	2	3	4	⑤	$\tau_5 = 1.4$
$t = 2.6 :$	$\{1, 3\}$	0	①	2	3	4	5	$\tau_1 = 2.6$
$t = 5.5 :$	$\{0, 1, 2, 4, 5\}$	④	1	②	3	④	5	$\tau_0 = \tau_2 = \tau_4 = 5.5$

At time  $t = 0.9$ , shock  $\{3\}$  occurs. This is the first time that a shock involving member 3 appears, hence the default time of member 3 is 0.9. At  $t = 1.4$ , member 5 defaults as the consequence of the shock  $\{5\}$ . At time 2.6, the shock  $\{1, 3\}$  triggers the default of member 1 alone as member 3 has already defaulted. Finally, only members 0, 2 and 4 default simultaneously at  $t = 5.5$  since members 1, 3 and 5 have already defaulted before.

In the sequel we consider a CCP with  $n + 1 = 9$  members, chosen among the 125 names of the CDX index on 17 December 2007, in the turn of the sub-prime crisis. The default times of the 125 names of the index are modelled by a DMO model with 5 common shocks, with<sup>1</sup> shock intensities  $\gamma_Y$  calibrated to the CDS and CDO market data of that day (see Crépey, Bielecki, and Brigo (2014, Sect. 8.4.3)). Table 3.1 shows the (market) credit spread  $\Sigma_i$  and the (fictitious) swap position  $\omega_i$  of each of our nine clearing members. Hence

$$\text{MtM}^i = (-\omega_i) \times \text{MtM}^* \quad (3.4.2)$$

(recalling that the CCP member portfolio mark-to-market processes  $\text{MtM}^i$  are considered from the point of view of the CCP). We write  $\text{Nom}_i = \text{Nom} \times |\omega_i|$ .

---

1. Piecewise-constant 0–3y and 3y–5y.

$\omega_i$	9.20	(1.80)	(4.60)	1.00	(6.80)	0.80	(13.80)	8.80	7.20
$\Sigma_i$	45	52	56	61	73	108	176	367	1053

TABLE 3.1 – (*Top*) Swap position  $\omega_i$  of each member, where parentheses mean negative numbers (i.e. short positions). (*Bottom*) Average 3 and 5 year CDS spread  $\Sigma_i$  of each member at time 0 (17 December 2017), in basis points.

### 3.4.3 Margin Schemes

We assume that the margins and default fund contribution of each clearing member are updated in continuous time<sup>2</sup> while the member has not defaulted and are then stopped at its default time, until the liquidation of its portfolio occurs after a period of length  $\delta =$  one week. Hence we set

$$\begin{aligned} \text{VM}_t^i &= \text{MtM}_t^i \text{ and} \\ \beta_t \text{IM}_t^i &= \mathbb{V}a_{\mathbb{R}_t^{a_{im}}}(\beta_{t\delta}(\text{MtM}_{t\delta}^i + \Delta_{t\delta}^i) - \beta_t \text{MtM}_t^i) \end{aligned} \quad (3.4.3)$$

for  $t \leq \tau_i \wedge T$  and some IM quantile level  $a_{im}$ .

## 3.5 Numerical Results

In the CCP toy model, we have semi-explicit formulas for all the quantities that we need (see Section 3.6.1), except for (a term structure of) the economical capital process of the CCP, which is obtained by Monte Carlo simulation.

### 3.5.1 Economic Capital of the CCP

In this section we consider a default fund that would be set, for fact or in the context of XVA computations, as the economic capital of the CCP, in the sense of a conditional expected shortfall of its one-year ahead loss (and profit) as per (3.3.3). However, for numerical tractability, we use  $\mathbb{ES}_0^{a_{df}}$  instead of  $\mathbb{ES}_t^{a_{df}}$  in (3.3.3). In other terms we compute a default fund term structure as opposed to a whole process. The ensuing KVA of the CCP follows by numerical time integration based on (3.3.5). Instead, computing a full-flesh conditional expected shortfall process would require nested Monte Carlo simulation (and even doubly nested Monte Carlo in more complex models where  $\text{CVA}^{ccp}$  is not known analytically), at not much difference modulo some second order convexity adjustment (see Abbas-Turki, Crépey, and Diallo (2017)).

We use  $m = 10^5$  simulated paths of  $S$  and default scenarios. All the reported numbers are in basis points (bps). We recall that the nominal “Nom” of the swap was fixed so that each leg equals  $1 = 10^4$  bps at time 0. Unless stated otherwise we use  $a_{im} = 85\%$  and  $a_{df} = 99\%$ . The solid blue curves in Figure 3.4 show the resulting default fund term structures for  $a_{df} = 85\%, 95.5\%$  and  $99\%$  (top to bottom). The respective dotted red and dashed green curves represent the analogous results using value at risk instead of expected shortfall in (3.3.3), respectively ignoring the CVA terms (the second line) in (3.3.4).

The broadly decreasing feature of all curves reflects the run-off feature of the modelling setup. The comparison between the solid blue and the dotted red curves shows that for too low DF quantile levels  $a_{df}$ , the corresponding value-at-risk misses the right tail of the distribution of the losses : the 85% value at risk curve in the upper panel is visually indistinguishable from 0, so that the corresponding expected shortfall reduces to an expectation of the positive part of the losses. The comparison between the solid blue and the dashed green curves in Figure 3.4 reveals that when the DF quantile level  $a_{df}$  increases, the impact of the CVA terms in (3.3.3) decreases.

2. Instead of daily and monthly under typical market practice.

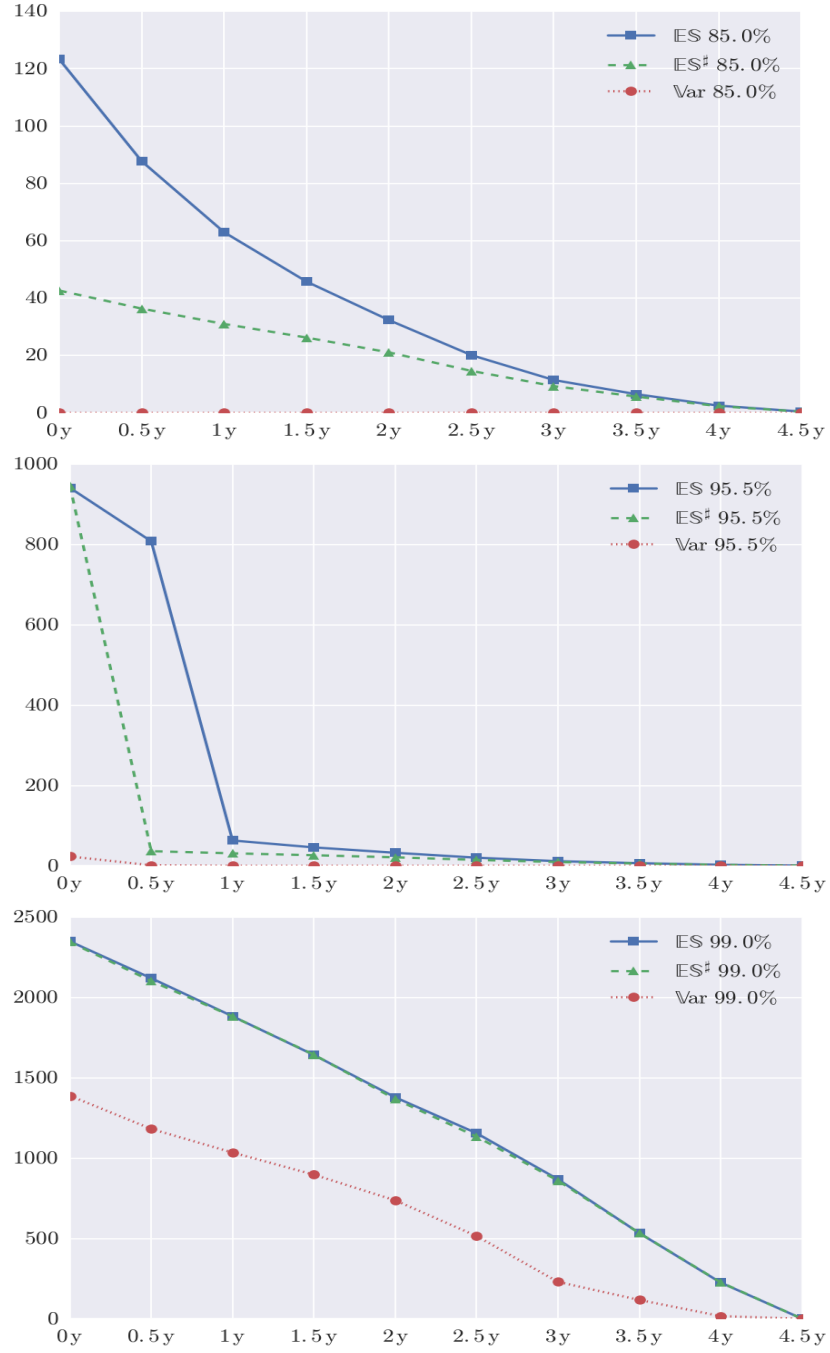


FIGURE 3.4 – *Solid blue curves* : Economic capital based default fund of the CCP, as a function of time, for  $a_{df} = 85\%, 95.5\%$  and  $99\%$  ( $a_{im} = 85\%$ ). *Dotted red curves* : Analogous results using value at risk instead of expected shortfall in (3.3.3). *Dashed green curves* : Analogous results ignoring the CVA terms (the second line) in (3.3.4).

It shows that the right tail of the distribution of the losses is driven by the counterparty default losses rather than by the volatile swings of  $CVA^{ccp}$ . This could be expected given the intensity model that we use for the default times.<sup>3</sup> Extreme swings of  $CVA^{ccp}$  could only arise in more structural “distance to default” credit models,<sup>4</sup> where defaults are announced by volatile swings of CDS spreads.

This analysis is confirmed by Figure 3.5, which shows, for each of the (overlapping) time intervals  $(0, 1), (0.5, 1.5), \dots, (4.5, 5.5)$ , the proportion of defaults per simulated path (upper panel) and the expectation and standard deviation of the corresponding losses (bottom panel). For instance, a proportion of 30% in the upper panel means that, over the  $10^5$  simulated paths,  $30\% \times 10^5 = 3 \times 10^4$  defaults happened on the corresponding time interval. The run-off feature of the setup (see after (3.2.9)) means that the clearing member portfolios purely amortize as time passes, but it also implies that defaulted clearing members are not replaced by new ones in the CCP. Hence, as time passes, there are less and less defaults on average (the mean and standard deviation of the losses take much more time to amortize, as the bottom panel of Figure 3.5 illustrates). Since the right tail of the losses is driven by the defaults, the EC based default fund exhibits the decreasing term structure shown by the solid blue curves in Figure 3.4.

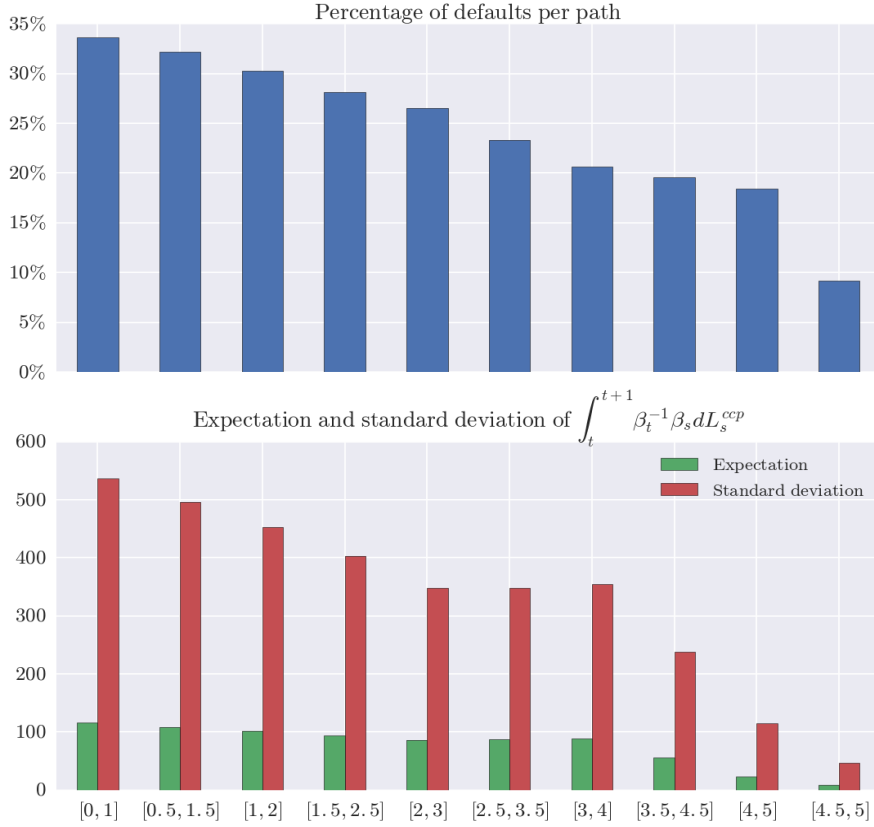


FIGURE 3.5 – *Top* : Proportion of defaults per simulated path. *Bottom* : Expectation and standard deviation of the losses.

Figure 3.6 represents, as a function of the IM quantile level  $a_{im}$ , the time-0 DF quantile level  $a_{df}$  calibrated to the objective of a total default fund equal to 10% (solid blue curve), 15%

3. Even if embedding credit dependence between the clearing members through the joint defaults.

4. But then the challenge is the default dependence modelling.

(dashed green curve) or 30% (dotted red curve) of the total IM of all the clearing members—a range of values commonly encountered in the case of a CCP clearing interest rate derivatives. With  $m = 10^5$  scenarios as we take, the  $a_{df}$  quantile level corresponding to a default fund equal to 50% or more of the total IM of the CCP, an order of magnitude not uncommon in the case of a CCP clearing CDS contracts, would be visually indistinguishable from 100% on the whole range of values used for  $a_{im}$  in Figure 3.6.

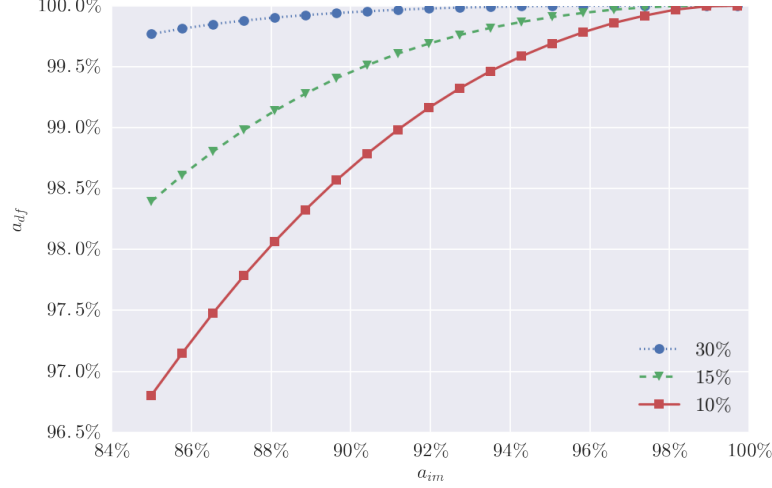


FIGURE 3.6 – Time-0 DF quantile level  $a_{df}$  resulting in a default fund equal to 10% (solid blue curve), 15% (dashed green curve) or 30% (dotted red curve) of the total IM of the CCP, plotted as a function of the IM quantile level  $a_{im}$  of the clearing members.

Figure 3.7 shows the  $KVA^{ccp}$  term structures corresponding to Formula (3.3.5) for a default fund sized by the EC (solid blue) curves of Figure 3.4, for a hurdle rate  $h = 10\%$ .

### 3.5.2 Default Fund Contributions

Let  $EC^{ccp(-j)}$  denote the economic capital of the CCP deprived from its  $j^{th}$  member, i.e. with the  $j^{th}$  member replaced by the risk-free “buffer” in all its CCP transactions. Namely, at time  $t$  (cf. (3.3.3)-(3.3.4)) :

$$EC_t^{ccp(-j)} = \mathbb{E}_t^{a_{df}} \left( \sum_{t < \tau_i^{\delta} \leq t+1, i \neq j} \left( \beta_{\tau_i^{\delta}} \left( MtM_{\tau_i^{\delta}}^i + \Delta_{\tau_i^{\delta}}^i \right) - \beta_{\tau_i} \left( MtM_{\tau_i}^i + IM_{\tau_i}^i \right) \right)^+ \right. \\ \left. - \left( \beta_t CVA_t^{ccp(-j)} - \beta_{t+1} CVA_{t+1}^{ccp(-j)} \right) \right)$$

where  $CVA_t^{ccp(-j)}$  corresponds to the CVA of the CCP (cf. (3.3.2)) deprived from its  $j^{th}$  member.

In the line of Section 3.5.1, we can consider an allocation of the default fund between the clearing members proportional to the incremental change in  $EC^{ccp}$  attributable to each of them. Namely, as long as all the clearing members are non-default (at time 0, in particular), we have

$$\mu_t^{ec,i} = \frac{\Delta_i EC_t^{ccp}}{\sum_j \Delta_j EC_t^{ccp}}, \text{ where } \Delta_j EC_t^{ccp} = EC_t^{ccp} - EC_t^{ccp(-j)}$$

A variant would be to allocate the default fund proportionally to the member incremental  $KVA^{ccp}$ . Let  $KVA_t^{ccp(-j)} = h \mathbb{E}_t \left( \int_t^T e^{-(r+h)s} EC_s^{ccp(-j)} ds \right)$  denote the value of the KVA of the

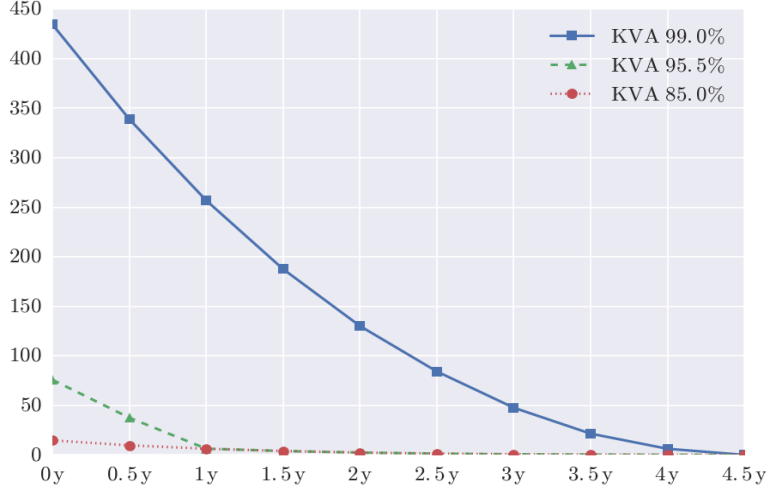


FIGURE 3.7 –  $KVA^{ccp}$  term structures corresponding to the  $EC^{ccp}$  (solid blue) curves of Figure 3.4 ( $h = 10\%$ ).

CCP deprived from its  $j^{th}$  member. The corresponding allocation is written as

$$\mu_t^{kva,i} = \frac{\Delta_i KVA_t^{ccp}}{\sum_j \Delta_j KVA_t^{ccp}}, \text{ where } \Delta_j KVA_t^{ccp} = KVA_t^{ccp} - KVA_t^{ccp(-j)}$$

Figure 3.8 shows the time-0 default fund allocations based on member initial margin, member incremental  $EC^{ccp}$  and member incremental  $KVA^{ccp}$ , respectively represented by blue, red and green bars. In the upper panel the clearing members in the  $x$ -axis are ordered by increasing position  $|\omega_i|$ , whereas in the lower panel they are ordered by increasing credit spread  $\Sigma_i$  (cf. Table 3.1). In the present setup where all portfolios are driven by a single Black–Scholes underlying, the initial margins, hence the blue bars in Figure 3.8, are simply proportional to the size  $|\omega_i|$  (or nominal  $Nom_i$ ) of the member positions. By contrast, the member incremental  $EC^{ccp}$  or  $KVA^{ccp}$  allocations (green and red bars) also take the credit risk of the members into account.

Figure 3.9 shows the term structures of the  $EC^{ccp}$  and  $KVA^{ccp}$  based allocation weights for each of the clearing members. We clearly see the impact of market but also credit risk on these term structures. At the beginning of the time period (and in particular at time 0), where defaults are, on average, still to come, with probabilities reflected by the time-0 credit spreads of the clearing members, the impact of credit risk is even predominant on the allocation weights.

### 3.5.3 Funding Strategies for Initial Margins

Figure 3.10 shows the time-0 MVAs of the nine clearing members for unsecuredly borrowed (top) vs. specialist lender (bottom) initial margin funding policies, for  $a_{im} = 70\%$  (blue),  $80\%$  (green),  $90\%$  (red) and  $97.5\%$  (purple). For each of the clearing members, its specialist lender MVA appears several times cheaper than its unsecured borrowing MVA (note the different scale of the  $y$ -axis between the top and the lower panel in Figure 3.10).

As explained in Section 3.2.4, in a centrally cleared setup with daily re-margining, the most important XVA numbers of a clearing member are its MVA and its KVA. Figure 3.11 compares the MVA and the KVA of each of the nine clearing members in our case study, under alternative specifications : unsecuredly borrowed vs. specialist lender initial margin regarding the MVA, member incremental  $EC^{ccp}$  vs. member incremental  $KVA^{ccp}$  allocation of an  $EC^{ccp}$  based default fund regarding the KVA. The credit risk of the clearing members appears to be a more important

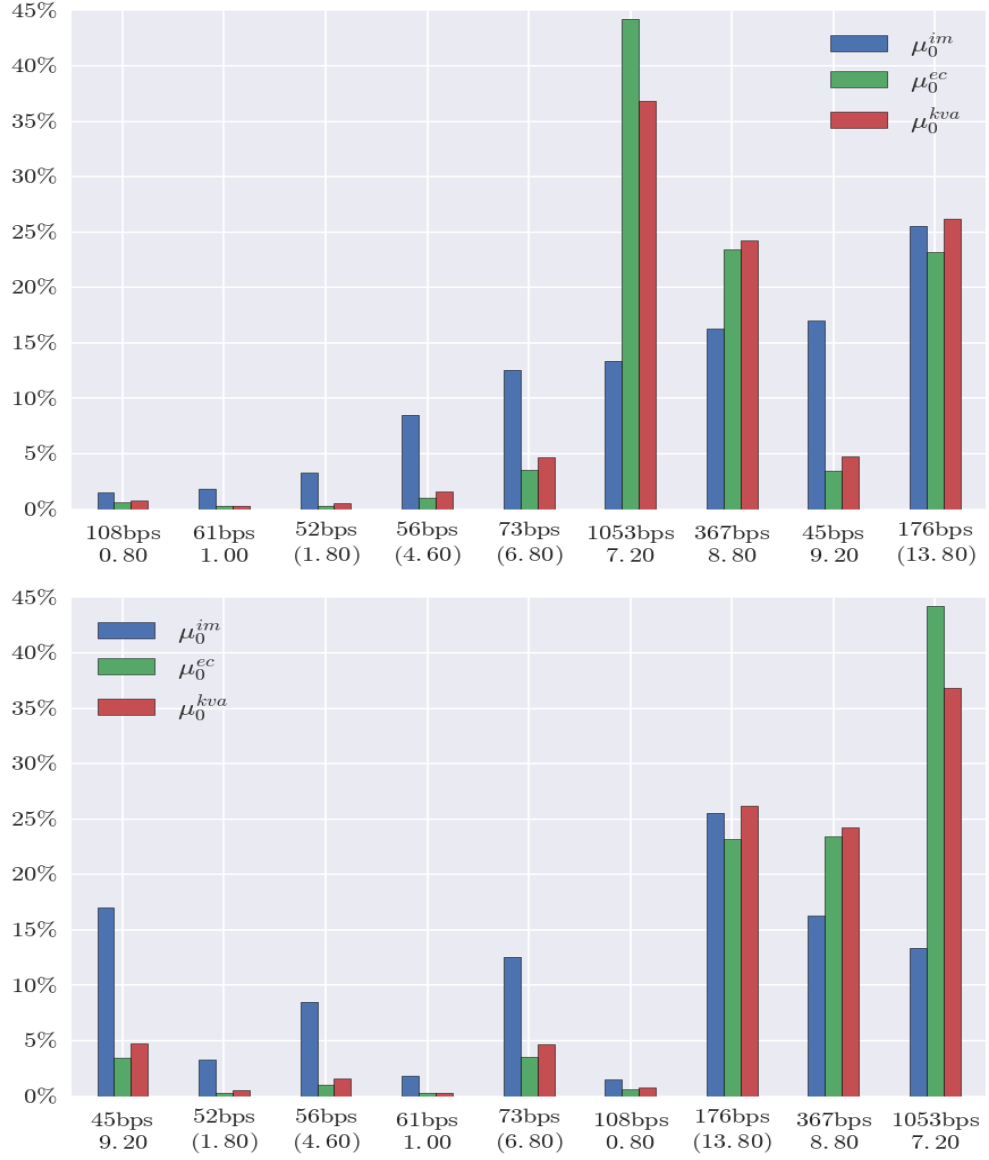


FIGURE 3.8 – Time-0 default fund allocation based on member initial margin, member incremental  $EC^{cp}$  and member incremental  $KVA^{cp}$ . *Top* : Members ordered by increasing position  $|\omega_i|$ . *Bottom* : Members ordered by increasing credit spread  $\Sigma_i$ .

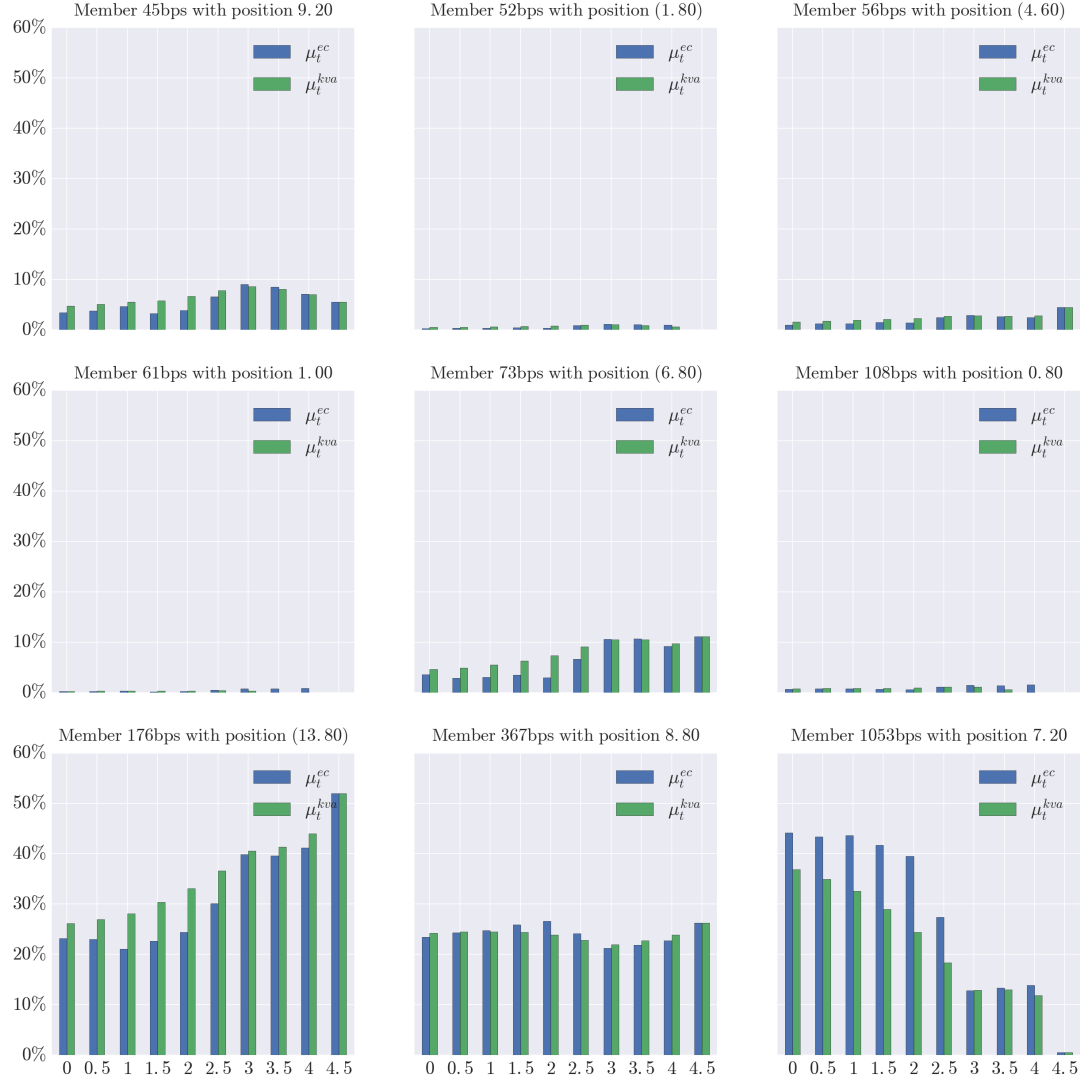


FIGURE 3.9 – Default fund allocation weights term structures based on member incremental  $EC^{cp}$  (in blue) or  $KVA^{cp}$  (in green) for each member, ordered from left to right and top to bottom per increasing credit spread, as a function of time  $t = 0, \dots, 4.5$ .

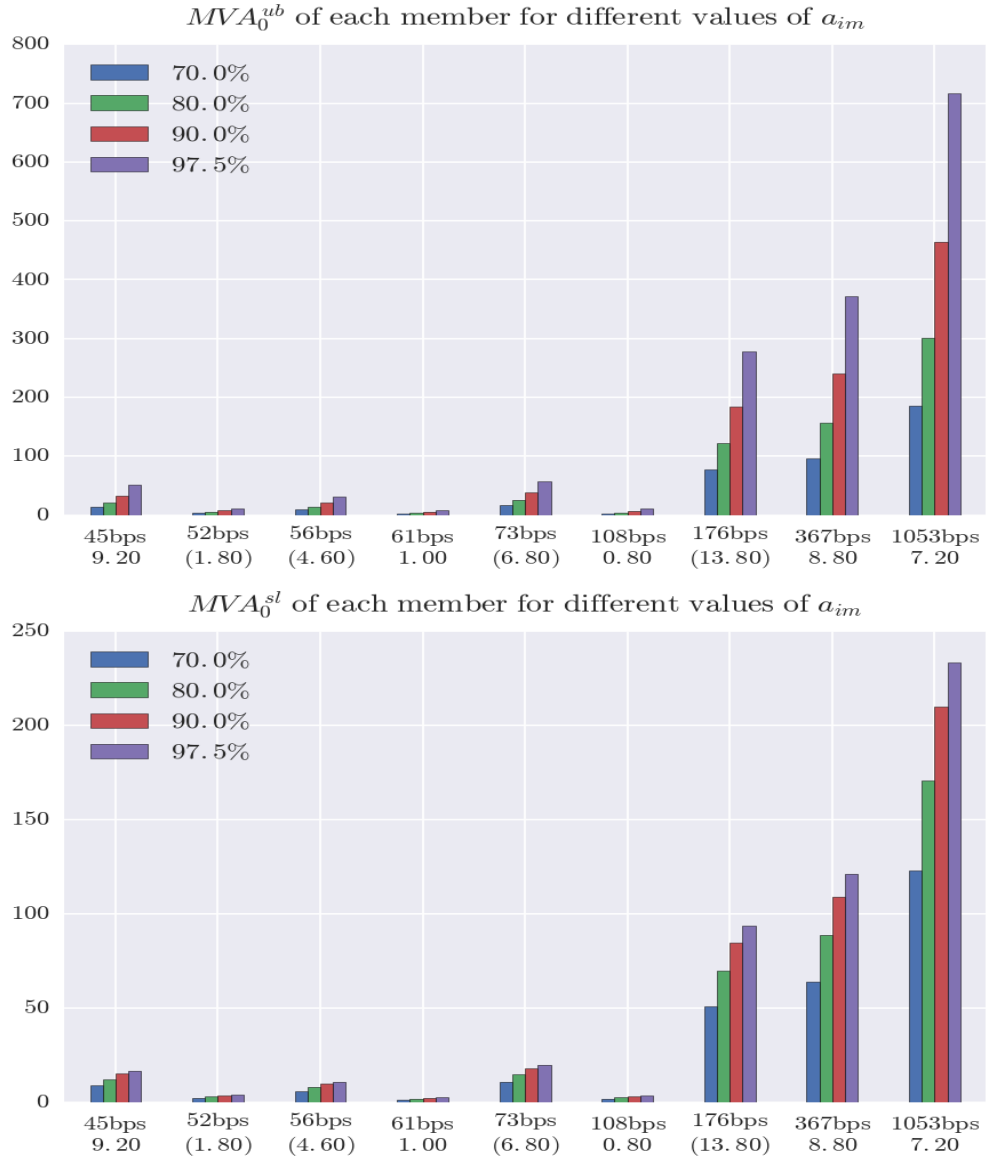


FIGURE 3.10 – MVAs of the nine clearing members for unsecurely borrowed (top) vs. specialist lender (bottom) initial margin funding policies, for  $a_{im} = 70\%$  (blue),  $80\%$  (green),  $90\%$  (red) and  $97.5\%$  (purple).

driver of their MVA and KVA than their market risk : the bars of each given colour are roughly ordered in the bottom panel, where they are ranked by increasing credit spread of the clearing members. By contrast, no particular ordering emerges in the upper panel, where they are ranked by increasing position of the clearing members.

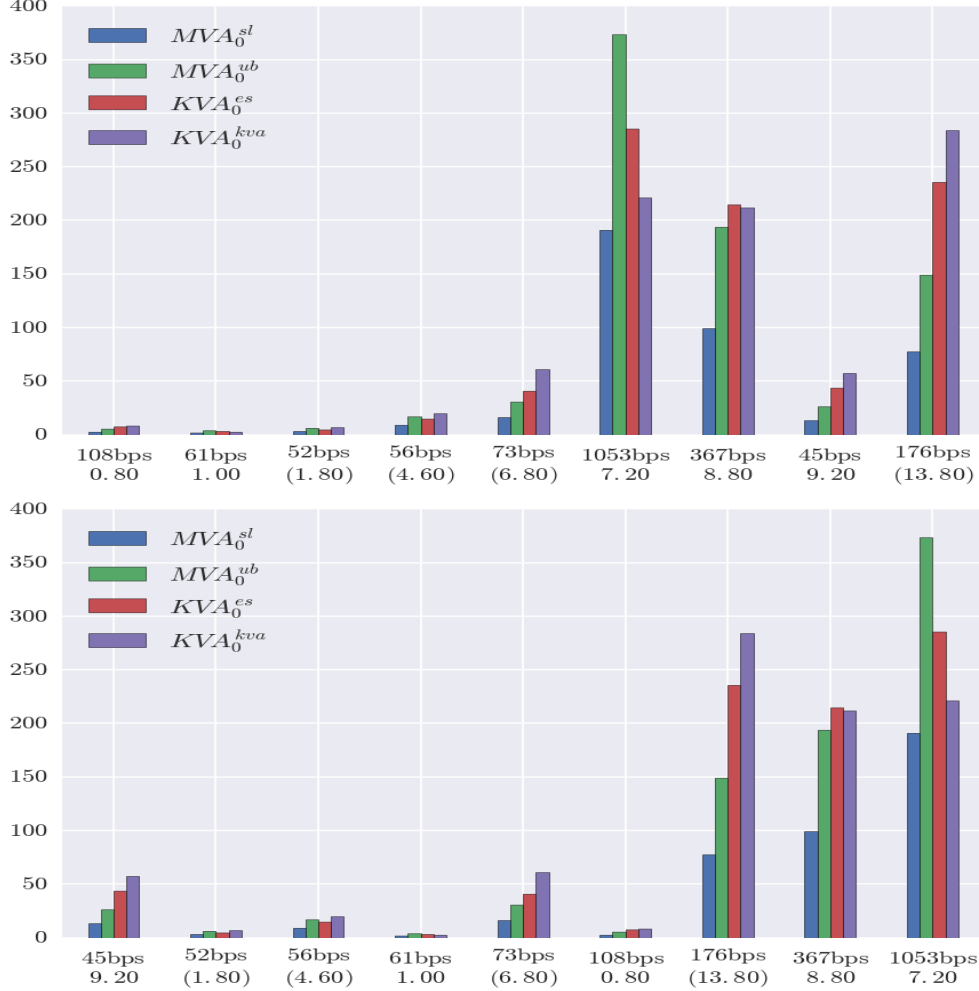


FIGURE 3.11 – MVA and KVA for each of the clearing members ordered along the  $x$  axis by increasing position (top) or increasing credit spread (bottom).

## 3.6 Appendix

### 3.6.1 Analytics in the CCP Toy Model

We denote by  $\Phi$  and  $\phi$  the standard normal cumulative distribution and density functions.

### Mark-to-Market and Initial Margin

The mark-to-market of a long position in the swap of Section 3.4.1 is given by

$$\begin{aligned} \text{MtM}_t &= \text{Nom} \times \mathbb{E}_t \left( \beta_t^{-1} \beta_{T_{l_t}} h_{l_t} (S_{T_{l_t-1}} - \bar{S}) + \sum_{l=l_t+1}^d \beta_t^{-1} \beta_{T_l} h_l (S_{T_{l-1}} - \bar{S}) \right) \\ &= \text{Nom} \times \left( \beta_t^{-1} \beta_{T_{l_t}} h_{l_t} (S_{T_{l_t-1}} - \bar{S}) + \beta_t^{-1} \sum_{l=l_t+1}^d \beta_{T_l} h_l (e^{\kappa T_{l-1}} \hat{S}_t - \bar{S}) \right) \end{aligned} \quad (3.6.1)$$

by the martingale property of the process  $\hat{S}$ . By (3.6.1) and (3.4.2)

$$\beta_{t^\delta} (\text{MtM}_{t^\delta}^i + \Delta_{t^\delta}^i) - \beta_t \text{MtM}_t^i = \text{Nom} \times \omega_i \times f(t) \times (\hat{S}_t - \hat{S}_{t^\delta}) \quad (3.6.2)$$

with  $f(t) = \sum_{l=l_t+1}^d \beta_{T_l} h_l e^{\kappa T_{l-1}}$ .

**Remark 3.6.1** At least (3.6.2) holds whenever there is no coupon date between  $t$  and  $t^\delta$  (cf. Andersen, Pykhtin, and Sokol (2017)). Otherwise, i.e. whenever  $l_{t^\delta} = l_t + 1$ , the term  $\beta_{T_{l_t}} h_{l_t} (S_{T_{l_t-1}} - \bar{S})$  in (3.6.1) induces a small and centered difference

$$\text{Nom} \times \omega_i \times h_{l_{t^\delta}} \beta_{T_{l_{t^\delta}}} (e^{\kappa T_{l_t}} \hat{S}_t - S_{T_{l_t}}) \quad (3.6.3)$$

between the left hand side and the right hand side in (3.6.2). As  $\delta \approx$  a few days, a coupon between  $t$  and  $t^\delta$  is the exception rather than the rule. Moreover the resulting error (3.6.3) is not only “rare”, but also small and centered. As all XVA numbers are time and space averages over future scenarios, we can and do neglect this feature in the chapter.

**Lemma 3.6.1** For  $t \leq \tau_i \wedge T$ , we have  $\beta_t \text{IM}_t^i = \text{Nom}_i \times B_i(t) \times \hat{S}_t$  where

$$B_i(t) = f(t) \times \begin{cases} e^{\sigma \sqrt{\delta} \Phi^{-1}(a_{im}) - \frac{\sigma^2}{2} \delta} - 1, & \omega_i \leq 0 \\ 1 - e^{\sigma \sqrt{\delta} \Phi^{-1}(1-a_{im}) - \frac{\sigma^2}{2} \delta}, & \omega_i > 0 \end{cases}$$

**Proof:** This follows from (3.4.3) and (3.6.2) in view of the Black model used for  $\hat{S}$ . ■

### 3.6.2 CVA of the CCP

**Lemma 3.6.2** We have, for  $s \leq T$  :

$$\mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s (\text{MtM}_s^i + \text{IM}_s^i) \right)^+ \right] = \text{Nom}_i \times A_i(s) \times \hat{S}_s$$

where

$$A_i(t) = (1 - a_{im}) \times f(t) \times e^{-\frac{\sigma^2 \delta}{2}} \begin{cases} e^{\sigma \sqrt{\delta} \frac{\Phi^{-1}(\frac{a_{im}}{1-a_{im}})}{1-a_{im}}} - e^{\sigma \sqrt{\delta} \Phi^{-1}(a_{im})}, & \omega_i \leq 0 \\ e^{\sigma \sqrt{\delta} \Phi^{-1}(1-a_{im})} - e^{-\sigma \sqrt{\delta} \frac{\Phi^{-1}(\frac{a_{im}}{1-a_{im}})}{1-a_{im}}}, & \omega_i > 0 \end{cases}$$

**Proof:** In view of (3.4.3) and (3.6.2), the conditional version of the identity

$$\mathbb{E}[X \mathbb{1}_{X \geq \text{VaR}^a(X)}] = (1 - a) \mathbb{E} S^a(X)$$

yields

$$\begin{aligned} & \mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s (\text{MtM}_s^i + \text{IM}_s^i) \right)^+ \right] \\ &= \text{Nom} \times (1 - a_{im}) \times f(s) \left[ \mathbb{E} \mathbb{S}_s^{aim} \left( \omega_i \left( \hat{S}_t - \hat{S}_{t^\delta} \right) \right) - \mathbb{V} a \mathbb{R}_s^{aim} \left( \omega_i \left( \hat{S}_t - \hat{S}_{t^\delta} \right) \right) \right] \end{aligned}$$

The result follows in view of the Black model used for  $\hat{S}$ . ■

**Proposition 3.6.1** *We have, for  $s \leq T$  :*

$$\begin{aligned} \beta_t \text{CVA}_t^{ccp} &= \sum_i \text{Nom}_i \times \\ & \left( \mathbb{1}_{t < \tau_i} \hat{S}_t \int_t^T A_i(s) \gamma_i(s) e^{-\int_t^s \gamma_i(u) du} ds + \mathbb{1}_{\tau_i < t < \tau_i^\delta} E_i(\tau_i, \hat{S}_{\tau_i}, t, \hat{S}_t) \right) \end{aligned}$$

where, setting  $y_\pm^i = \frac{\ln(\hat{S}_t / \hat{S}_{\tau_i})}{\sigma \sqrt{\tau_i^\delta - t}} \pm \frac{1}{2} \sigma \sqrt{\tau_i^\delta - t}$ ,

$$E_i(\tau_i, \hat{S}_{\tau_i}, t, \hat{S}_t) = f(\tau_i) \times \begin{cases} \hat{S}_t \Phi(y_+^i) - \hat{S}_{\tau_i} \Phi(y_-^i), & \omega_i \leq 0 \\ \hat{S}_{\tau_i} \Phi(-y_-^i) - \hat{S}_t \Phi(-y_+^i), & \omega_i > 0 \end{cases}$$

**Proof:** We have

$$\begin{aligned} & \beta_t \text{CVA}_t^{ccp} \\ &= \sum_i \mathbb{1}_{t < \tau_i^\delta} \mathbb{E}_t \left[ \left( \beta_{\tau_i^\delta} (\text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i) - \beta_{\tau_i} (\text{MtM}_{\tau_i}^i + \text{IM}_{\tau_i}^i) \right)^+ \right] \\ &= \sum_i \mathbb{1}_{t < \tau_i} \mathbb{E}_t \left[ \mathbb{E}_{\tau_i-} \left[ \left( \beta_{\tau_i^\delta} (\text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i) - \beta_{\tau_i} (\text{MtM}_{\tau_i}^i + \text{IM}_{\tau_i}^i) \right)^+ \right] \right] \\ & \quad + \sum_i \mathbb{1}_{\tau_i < t < \tau_i^\delta} \mathbb{E}_t \left[ \left( \beta_{\tau_i^\delta} (\text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i) - \beta_{\tau_i} (\text{MtM}_{\tau_i}^i + \text{IM}_{\tau_i}^i) \right)^+ \right] \\ &= \sum_i \mathbb{1}_{t < \tau_i} \mathbb{E}_t \int_t^T \mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s (\text{MtM}_s^i + \text{IM}_s^i) \right)^+ \right] \gamma_i(s) e^{-\int_t^s \gamma_i(u) du} ds \\ & \quad + \text{Nom} \sum_i \mathbb{1}_{\tau_i < t < \tau_i^\delta} f(\tau_i) \mathbb{E}_t \left[ \left( \omega_i \left( \hat{S}_{\tau_i} - \hat{S}_{\tau_i^\delta} \right) \right)^+ \right] \end{aligned} \tag{3.6.4}$$

by virtue of (3.6.2) and of the conditional distribution properties of the DMO model exposed in Crépey, Bielecki, and Brigo (2014, Section 8.2.1). We conclude the proof by an application of Lemma 3.6.2 to the first line in (3.6.4) and of the Black formula to the second line. ■

### 3.6.3 Unsecured Borrowing vs. Specialist Lender MVAs

In the setup of our case study, the generic expressions (3.3.6) and (3.3.8) for the unsecured borrowing vs. specialist lender MVAs can be computed by deterministic time integration based on the following formulas.

**Proposition 3.6.2** *The unsecured borrowing MVA of member  $i$  is given, at time 0, by*

$$\text{MVA}_0^{ub,i} = \text{Nom}_i S_0 \int_0^T B_i(s) \lambda_i(s) e^{-\int_0^s \gamma_i(u) du} ds$$

**Proof:** By virtue of (3.3.6) and of the distributional properties of the DMO model, we have

$$\text{MVA}_0^{ub,i} = \mathbb{E} \int_0^{T \wedge \tau_i} \beta_s \lambda_i(s) \text{IM}_s^i \, ds = \mathbb{E} \int_0^T \beta_s \lambda_i(s) e^{-\int_0^s \gamma_i(u) \, du} \text{IM}_s^i \, ds$$

Hence the result follows from Lemma 3.6.1.  $\blacksquare$

**Lemma 3.6.3** *We have, for  $s \leq \tau_i \wedge T$ ,*

$$\mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s \text{MtM}_s^i \right)^+ \right] = \text{Nom}_i C(s) \widehat{S}_s \quad (3.6.5)$$

where

$$C(s) = f(s) \left[ \Phi \left( \frac{\sigma \sqrt{\delta}}{2} \right) - \Phi \left( -\frac{\sigma \sqrt{\delta}}{2} \right) \right] \quad (3.6.6)$$

**Proof:** In view of (3.6.2), it comes :

$$\left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s \text{MtM}_s^i \right)^+ = \text{Nom} \times f(s) \left( \omega_i(\widehat{S}_s - \widehat{S}_{s^\delta}) \right)^+$$

Hence the result follows from the Black formula.  $\blacksquare$

**Proposition 3.6.3** *The specialist lender MVA of member  $i$  is given, at time 0, by*

$$\text{MVA}_0^{sl,i} = \text{Nom}_i S_0 \int_0^T \left( C(s) - A_i(s) \right) \gamma_i(s) e^{-\int_0^s \gamma_i(u) \, du} \, ds$$

**Proof:** Let, for  $s \leq \tau_i \wedge T$ ,

$$\begin{aligned} \xi_s^i &= \mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s \text{MtM}_s^i \right)^+ \wedge \beta_s \text{IM}_s^i \right] \\ &= \mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s \text{MtM}_s^i \right)^+ \right] \\ &\quad - \mathbb{E}_s \left[ \left( \beta_{s^\delta} (\text{MtM}_{s^\delta}^i + \Delta_{s^\delta}^i) - \beta_s (\text{MtM}_s^i + \text{IM}_s^i) \right)^+ \right] \\ &= \text{Nom}_i \widehat{S}_s (C(s) - A_i(s)) \end{aligned}$$

by Lemmas 3.6.3 and 3.6.2. Note this is a predictable process. Hence (cf. (3.3.8))

$$\begin{aligned} \text{MVA}_0^{sl,i} &= \mathbb{E} \left[ \mathbf{1}_{\tau_i < T} \left( \left( \beta_{\tau_i^\delta} (\text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i) - \beta_{\tau_i} \text{MtM}_{\tau_i}^i \right)^+ \wedge \beta_{\tau_i} \text{IM}_{\tau_i}^i \right) \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\tau_i < T} \mathbb{E}_{\tau_i} \left( \left( \beta_{\tau_i^\delta} (\text{MtM}_{\tau_i^\delta}^i + \Delta_{\tau_i^\delta}^i) - \beta_{\tau_i} \text{MtM}_{\tau_i}^i \right)^+ \wedge \beta_{\tau_i} \text{IM}_{\tau_i}^i \right) \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\tau_i < T} \xi_{\tau_i}^i \right] = \mathbb{E} \left[ \int_0^T \gamma_i(s) e^{-\int_0^s \gamma_i(u) \, du} \xi_s^i \, ds \right] \end{aligned}$$

where the conditional distribution properties of the DMO model were used in the last equality (see Crépey, Bielecki, and Brigo (2014, Section 8.2.1)).  $\blacksquare$



# Multivariate Shortfall Risk Allocation

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This chapter is based on [Armenti, Crepey, Drapeau, and Papapantoleon \(2016\)](#).

## 4.1 Introduction

The ongoing concern about systemic risk since the onset of the global financial crisis has prompted intensive research on the design and properties of multivariate risk measures. In this paper, we study the risk assessment for financial systems with interconnected risky components, focusing on two major aspects, namely :

- The quantification of a monetary risk measure corresponding to an overall reserve of liquidity such that the whole system can overcome unexpected stress or default scenarios ;
- The allocation of this overall amount between the different risk components in a way that reflects the systemic risk of each one.

Our goal is fourfold. First, we introduce a theoretically sound and numerically tractable class of systemic risk measures. Second, we study the impact of the intrinsic dependence on the risk allocation and its sensitivity. Third, we address the computational aspects and challenges of systemic risk allocation. Finally, we present empirical results, based on real data provided by LCH S.A., on the risk allocation of the default fund of a CCP.

**Review of the Literature :** Monetary risk measures have been the subject of intensive research since the seminal paper [Artzner, Delbaen, Eber, and Heath \(1999\)](#), which was further extended by [Föllmer and Schied \(2002\)](#) and [Frittelli and Gianin \(2002\)](#), among others. The corresponding risk measures, including conditional value-at-risk in [Artzner, Delbaen, Eber, and Heath \(1999\)](#), shortfall risk measures in [Föllmer and Schied \(2002\)](#) or optimized certainty equivalents by [Ben-Tal and Teboulle \(2007\)](#), can be applied in a multivariate framework that models the dependence of several financial risk components. Multivariate market data-based risk measures include the marginal expected shortfall of [Acharya, Pedersen, Philippon, and Richardson \(2017\)](#), law invariant convex risk measures for portfolio vectors of [Rüschendorf \(2006\)](#), the systemic risk measure of [Acharya, Engle, and Richardson \(2012\)](#) and [Brownlees and Engle \(2012\)](#), the delta conditional value-at-risk of [Adrian and Brunnermeier \(2016\)](#) or the contagion index of [Cont, Santos, and Moussa \(2013\)](#). In parallel, theoretical economical and mathematical considerations have led to multivalued and set-valued risk measures, in static or even dynamic setup ; see for instance [Molchanov and Cascos \(2016\)](#), [Hamel, Heyde, and Rudloff \(2011\)](#) and [Jouini, Meddeb, and Touzi \(2004\)](#).

Recently, the risk management of financial institutions raised concerns about the allocation of the overall risk among the different components of a financial system. A bank, for instance, for real time monitoring purposes, wants to channel to each trading desk a cost reflecting its responsibility in the overall capital requirement of the bank. A central clearing counterparty — CCP for short, also known as a clearing house — is interested in quantifying the size of the so-called default fund and allocating it in a meaningful way among the different clearing members, see [Cont \(2015\)](#), [Armenti and Crépey \(2017a\)](#) or [Ghamami and Glasserman \(2017\)](#).

On a macroeconomic level, regulators are considering to require from financial institutions an amount of capital reflecting their systemic relevance. The aforementioned approaches can only address the allocation problem indirectly, through the sensitivity of the risk measure with respect to the different risk components. For instance, the so-called *Euler rule* allocates the total amount of risk according to the marginal impact of each risk factor. However, a practical limitation of the Euler rule is that it is based on Gâteaux derivatives which in general is difficult to compute beyond simple cases. Also the Euler rule considers the marginal risk of one element with respect to the full system rather than the marginal risk with respect to each individual component. In addition, the Euler risk allocation does not add up to the total risk, unless the univariate risk measure that is used in the first place is sub-additive, see [Tasche and Resti \(2008\)](#). In other words, the Euler rule does not automatically fulfill the so-called *full allocation* property. The work by [Brunnermeier and Cheridito \(2014\)](#) addresses systematically the question of allocation of systemic risk with regard to certain economic properties :

- Full allocation : the sum of the components of the risk allocation is equal to the overall risk measure ;
- Riskless allocation : if a risk factor is riskless, the corresponding component of the risk allocation is equal to it ;
- Causal responsibility : any system component bears the entire additional costs of any additional risk that it takes.

More specifically, [Brunnermeier and Cheridito \(2014\)](#) propose a framework where an overall capital requirement is first determined by utility indifference principles and then allocated according to a rule such that the above three properties are fulfilled, at least at a first order level of approximation. In fact, as far as dependence is concerned, whether the last two properties should hold is debatable. One may argue that each component in the system is not only responsible for its own risk taking but also for its relative exposure to other components. This is also what comes out from the present study, see Section 4.4.3. In a general framework, [Kromer, Overbeck, and Zilch \(2016\)](#) characterized systemic risk out of axioms allowing for a decomposition between and aggregation function and a univariate risk measure. In the spirit of this aggregation function, in two recent papers, [Feinstein, Rudloff, and Weber \(2017\)](#) and [Biagini, Fouque, Frittelli, and Meyer-Brandis \(2015\)](#) proposed a general approach similar in spirit to ours. We make precise thereafter and later in the paper the relationship to these references and in which sense our approach differs.

**Contribution and Outline of the Paper :** Our approach addresses simultaneously the design of an overall risk measure regarding a financial system of interconnected components and the allocation of this risk measure among the different risk components ; the emphasis lies on the allocation and its sensitivities. In contrast to [Brunnermeier and Cheridito \(2014\)](#) or [Chen, Iyengar, and Moallemi \(2013\)](#), we *first allocate* the monetary risk among the different risk components and *then aggregate* and minimize the risk allocations in order to obtain the overall capital requirement. As previously mentioned, [Kromer, Overbeck, and Zilch \(2016\)](#), [Feinstein, Rudloff, and Weber \(2017\)](#) and [Biagini, Fouque, Frittelli, and Meyer-Brandis \(2015\)](#) develop approaches in a similar spirit, covering allocation first followed by aggregation, in general frameworks with different aggregation procedures. They focus on the resulting risk measure, conducting systematic studies of their properties in terms of set valued functions, diversification and monotonicity, among others. The multivariate shortfall risk measure of this paper can be viewed as a special case of their definition, in a way made precise in Remark 4.2.2. Sharing with these references the “allocate first, then aggregate” perspective, our approach is restricted to a systemic extension of shortfall risk measures, see [Föllmer and Schied \(2002\)](#), based on multivariate loss functions. However, in contrast to the aforementioned references, we focus on the resulting risk allocation in terms of existence, uniqueness, sensitivities and numerical applications. In our framework, the *systemic risk* is the risk that stems specifically from the intrinsic dependence structure of

an interconnected system of risk components. In this perspective, the risk allocation and its properties provide a “cartography” of the systemic risk, see Section 4.5 on the numerical aspects of risk allocation and the empirical study in Section 4.6 on real data for an illustration thereof. It turns out that special care has to be given to the specifications of the loss function in order to stress the systemic risk. In Biagini, Fouque, Frittelli, and Meyer-Brandis (2015), by allowing random allocations, the impact of the interdependence structure can be observed in the future. Such random allocations may be interesting in view of a posterior management of defaults. By contrast, our deterministic allocation is sensitive to the dependence of the system already at the moment of the quantification, see Section 4.4 and see *a contrario* Proposition 4.3.1. We study the sensitivity of the risk allocation with respect to external shocks as well as internal dependence structure. We show in particular that a causal responsibility can be derived in marginal terms, see Proposition 4.4.1. In addition, we discuss computational aspects of risk allocation and finally, we provide an empirical study on the risk allocation of a default fund of a CCP based on real data provided by LCH S.A.

The univariate shortfall risk measure as a law invariant risk measure holds additional properties as an operator on probability distributions. Indeed, as shown by Weber (2006) and Krättschmer, Schied, and Zähle (2014), it has some continuity properties with respect to the  $\psi$ -weak topology on distributions. It has been furthermore characterised in Weber (2006) as the only law invariant convex risk measure on the level of distributions and therefore the unique one having elicibility properties, a wishful statistical property, see Osband (1985) or Bellini and Bignozzi (2015). Extensions of these results, such as elicibility characterization in multi-dimensional case as proposed by Ziegel (2016) and Fissler, Ziegel, et al. (2016), as well as the axiomatic characterization along the lines of Weber (2006), are highly non trivial and therefore let for further study. A set-valued multivariate shortfall risk measure has been introduced by Ararat, Hamel, and Rudloff (2017). However, allocation is not the focus of their work and the loss function that they then consider is decoupled in the sense of (C2), which from our viewpoint is too restrictive in view of Proposition 4.3.1.

The paper is organized as follows : Section 4.2 introduces the class of systemic loss functions, acceptance sets and risk measures that we use in the paper. Section 4.3 establishes the existence and uniqueness of a risk allocation. Section 4.4 focuses on sensitivities with respect to external shocks, dependence structure, nature of the loss function as well as the properties of full allocation, causal responsibility and riskless allocation mentioned beforehand. Section 4.5 discusses the computational aspects and challenges of risk allocation. Section 4.6, applies our approach to the concrete allocation of the default fund of a CCP. Appendices 4.7.1 and 4.7.2 gather classical facts from convex optimization and results on multivariate Orlicz spaces. Appendix 4.7.3 provides additional insight on the data of the empirical study.

#### 4.1.1 Basic Notation

Let  $x_k$  denote the generic coordinate of a vector  $\mathbf{x} \in \mathbb{R}^d$ , and  $e_k$  the  $k$ -th unit vector. By  $\geq$  we denote the lattice order on  $\mathbb{R}^d$ , that is,  $\mathbf{x} \geq \mathbf{y}$  if and only if  $x_k \geq y_k$  for every  $1 \leq k \leq d$ . We denote by  $\|\cdot\|$  the Euclidean norm and by  $\pm, \wedge, \vee, |\cdot|$  the lattice operations on  $\mathbb{R}^d$ . For  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ , we write  $\mathbf{x} > \mathbf{y}$  for  $x_k > y_k$  component-wise,  $\mathbf{x} \cdot \mathbf{y} = \sum_k x_k y_k$ ,  $\mathbf{x}\mathbf{y} = (x_1 y_1, \dots, x_d y_d)$  and  $\mathbf{x}/\mathbf{y} = (x_1/y_1, \dots, x_d/y_d)$ . We denote by  $f^*(\mathbf{y}) = \sup_{\mathbf{x}} \{\mathbf{x} \cdot \mathbf{y} - f(\mathbf{x})\}$  the convex conjugate of a function  $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ , and for  $C \subseteq \mathbb{R}^d$ , we denote by  $\delta(\cdot|C)$  the indicator function of  $C$  defined as  $\delta(\mathbf{x}, C) = 0$  for  $\mathbf{x}$  in  $C$  and  $\infty$  otherwise.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and denote by  $L^0(\mathbb{R}^d)$  the space of  $\mathcal{F}$ -measurable  $d$ -variate random variables on this space identified in the  $\mathbb{P}$ -almost sure sense. The space  $L^0(\mathbb{R}^d)$  inherits the lattice structure of  $\mathbb{R}^d$ , hence we can use the above notation in a  $\mathbb{P}$ -almost sure sense. For instance, for  $\mathbf{X}$  and  $\mathbf{Y}$  in  $L^0(\mathbb{R}^d)$ , we say that  $\mathbf{X} \geq \mathbf{Y}$  or  $\mathbf{X} > \mathbf{Y}$  if  $\mathbb{P}[\mathbf{X} \geq \mathbf{Y}] = 1$  or  $\mathbb{P}[\mathbf{X} > \mathbf{Y}] = 1$ , respectively. Since we mainly deal with multivariate functions or random variables, to simplify notation we drop the reference to  $\mathbb{R}^d$  in  $L^0(\mathbb{R}^d)$ , writing simply  $L^0$  unless

necessary.

## 4.2 Multivariate Shortfall Risk

Let  $\mathbf{X} = (X_1, \dots, X_d) \in L^0$  be a random vector of financial losses, that is, negative values of  $X_k$  represent actual profits. We want to determine an overall monetary measure  $R(\mathbf{X})$  of the risk of  $\mathbf{X}$  as well as a sound risk allocation  $RA_k(\mathbf{X}), k = 1, \dots, d$ , of  $R(\mathbf{X})$  among the  $d$  risk components. We consider a flexible class of risk measures defined by means of loss functions and sets of acceptable monetary allocations. This class allows us to discuss in detail the properties of the resulting risk allocation as an indicator of systemic risk. Inspired by the shortfall risk measure introduced in Föllmer and Schied (2002) in the univariate case, we start with a loss function  $\ell$  defined on  $\mathbb{R}^d$ , used to measure the expected loss  $\mathbb{E}[\ell(\mathbf{X})]$  of the financial loss vector  $\mathbf{X}$ .

**Definition 4.2.1** A function  $\ell : \mathbb{R}^d \rightarrow (-\infty, \infty]$  is called a *loss function* if

- (A1)  $\ell$  is increasing, that is,  $\ell(\mathbf{x}) \geq \ell(\mathbf{y})$  if  $\mathbf{x} \geq \mathbf{y}$ ;
- (A2)  $\ell$  is convex, lower semi-continuous with  $\inf \ell < 0$ ;
- (A3)  $\ell(\mathbf{x}) \geq \sum_k x_k - c$  for some constant  $c$ .

A loss function  $\ell$  is *permutation invariant* if  $\ell(\mathbf{x}) = \ell(\pi(\mathbf{x}))$  for every permutation  $\pi$  of its components.

A risk neutral assessment of the losses corresponds to  $\mathbb{E}[\sum X_k] = \sum \mathbb{E}[X_k]$ . Thus, (A3) expresses a form of risk aversion, whereby the loss function puts more weight on high losses than a risk neutral evaluation. As for (A1) and (A2), they express the respective normative facts about risk that “the more losses, the riskier” and “diversification should not increase risk”; see Drapeau and Kupper (2013) for related discussions.

**Remark 4.2.1** The choice of the terminology “loss function” stems from Föllmer and Schied (2002) for which this paper is a multivariate extension. Our notion of a loss function coincides with the one of “aggregation function” in Feinstein, Rudloff, and Weber (2017) or Biagini, Fouque, Frittelli, and Meyer-Brandis (2015), in the sense that it aggregates several loss profiles into a univariate random variable for which it can be decided whether or not it is acceptable, see Remark 4.2.2. Due to the obvious extension from the shortfall risk measure, throughout this paper we stick to the terminology “loss function”.

As for the permutation invariance, the considered risk components are often of the same type — banks, members of a clearing house or trading desks within a trading floor. In that case, the loss function should not discriminate a particular component against another.

**Example 4.2.1** Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a one-dimensional loss function, that is, a function satisfying conditions (A1), (A2) and (A3) in one dimension, such as for instance

$$h(x) = \beta x^+ - \alpha x^-, \quad 0 < \alpha < 1 < \beta, \quad h(x) = x + \frac{(x^+)^2}{2} \quad \text{or} \quad h(x) = e^x - 1$$

Using these as building blocks, we obtain the following classes of multivariate loss functions,<sup>1</sup> which will be used for illustrative purposes in the discussion of systemic risk, see Sections 4.3 and 4.4.

- (C1)  $\ell(\mathbf{x}) = h(\sum_k x_k)$ ;
- (C2)  $\ell(\mathbf{x}) = \sum_k h(x_k)$ ;
- (C3)  $\ell(\mathbf{x}) = \alpha h(\sum_k x_k) + (1 - \alpha) \sum_k h(x_k)$  for every  $0 \leq \alpha \leq 1$ .

---

1. A simple check shows that the following examples satisfy condition (A1), (A2) and (A3) in  $d$ -dimensions.

Note that each of these loss functions are permutation invariant.

For integrability reasons we consider loss vectors in the following multivariate Orlicz heart :<sup>2</sup>

$$M^\theta = \{ \mathbf{X} \in L^0 : \mathbb{E}[\theta(\lambda \mathbf{X})] < \infty \text{ for all } \lambda \in \mathbb{R}_+ \},$$

where  $\theta(\mathbf{x}) = \ell(|\mathbf{x}|)$ ,  $\mathbf{x} \in \mathbb{R}^d$ ; see Appendix 4.7.2.

**Definition 4.2.2** A monetary allocation  $\mathbf{m} \in \mathbb{R}^d$  is *acceptable* for  $\mathbf{X}$  if

$$\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0$$

We denote by

$$A(\mathbf{X}) := \{ \mathbf{m} \in \mathbb{R}^d : \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0 \} \quad (4.2.1)$$

the corresponding set of *acceptable monetary allocations*.

**Example 4.2.2** In a centrally cleared trading setup, each clearing member  $k$  is required to post a default fund contribution  $m_k$  in order to make the risk of the clearing house acceptable with respect to a risk measure accounting for extreme and systemic risk. The default fund is a pooled resource of the clearing house, in the sense that the default fund contribution of a given member can be used by the clearing house not only in case the liquidation of this member requires it, but also in case the liquidation of another member requires it. For the determination of the default fund contributions, the methodology of this paper can be applied to the vector  $\mathbf{X}$  defined as the vector of stressed losses-and-profits over initial margins of the clearing members. According to the findings of Section 4.3 and 4.4, a “systemic” loss function such as (A3) with  $\alpha > 0$  would be consistent with the purpose of a default fund. Note however that our setup applied to clearing houses takes the view of a closed system, so an internal assessment. In principle we ignore additional systemic risk such as a competition between clearing houses with common membership, or the external risk to which these members may be subject to, as addressed for instance in Glasserman, Moallemi, and Yuan (2015). However, our method could also assess such a systemic risk by taking  $\mathbf{X}$  as the overall vector of positions of each member in each clearing house.

The next proposition gathers the main properties of the sets of acceptable monetary allocations. The convexity property in (i) means that a diversification between two acceptable monetary allocations remains acceptable. If a monetary allocation is acceptable, then any greater amount of money should also be acceptable, which is the monotonicity property in (i). As for (ii), it says that, if the losses  $\mathbf{X}$  are less than  $\mathbf{Y}$  almost surely, then any monetary allocation that is acceptable for  $\mathbf{Y}$  is also for  $\mathbf{X}$ . Next, (iii) means that a convex combination of acceptable allocations in two markets is still acceptable in the diversified market. In particular, the acceptability concept pushes towards greater diversification among the different risk components. From the viewpoint of a clearing house for instance, a diversified position of its members is preferable to a concentrated one and therefore may enforce default fund allocations that incite its members towards this goal. Also, from a trading floor supervision, an overall diversified position of the traders is preferable, an incentive which is a current practice, see example 4.5.2. Finally, (iv) means that acceptable positions translate with cash in the sense of scalar monetary risk measures à la Artzner, Delbaen, Eber, and Heath (1999), Föllmer and Schied (2002) or Frittelli and Gianin (2002). As an immediate consequence of these properties,  $\mathbf{X} \mapsto A(\mathbf{X})$  defines a monetary set-valued risk measure in the sense of Hamel, Heyde, and Rudloff (2011), that is, a set-valued map  $A$  from  $M^\theta$  into the set of monotone, closed and convex subsets of  $\mathbb{R}^d$ .

2. Orlicz spaces are natural spaces in this context. The theory of Orlicz spaces has been used for long in the theory of risk measures, see Delbaen (2002), Biagini, Frittelli, et al. (2008), Cheridito and Li (2009) and Biagini and Frittelli (2009).

**Proposition 4.2.1** For  $\mathbf{X}, \mathbf{Y}$  in  $M^\theta$ , it holds :

- (i)  $A(\mathbf{X})$  is convex, monotone and closed;
- (ii)  $A(\mathbf{X}) \supseteq A(\mathbf{Y})$  whenever  $\mathbf{X} \leq \mathbf{Y}$ ;
- (iii)  $A(\alpha\mathbf{X} + (1 - \alpha)\mathbf{Y}) \supseteq \alpha A(\mathbf{X}) + (1 - \alpha)A(\mathbf{Y})$ , for any  $\alpha \in (0, 1)$ ;
- (iv)  $A(\mathbf{X} + \mathbf{m}) = A(\mathbf{X}) + \mathbf{m}$ , for any  $\mathbf{m} \in \mathbb{R}^d$ ;
- (v)  $\emptyset \neq A(\mathbf{X}) \neq \mathbb{R}^d$ .

If furthermore

- (vi)  $\ell$  is positive homogeneous, then  $A(\lambda\mathbf{X}) = \lambda A(\mathbf{X})$  for every  $\lambda > 0$ ;
- (vii)  $\ell$  is permutation invariant, then  $A(\pi(\mathbf{X})) = \pi(A(\mathbf{X}))$  for every permutation  $\pi$ ;

**Proof:** Since  $\ell$  is convex, increasing and lower semi-continuous, it follows that  $(\mathbf{m}, \mathbf{X}) \mapsto \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})]$  is convex and lower semi-continuous, decreasing in  $\mathbf{m}$  and increasing in  $\mathbf{X}$ . This implies the properties (i) through (iii) by Definition 4.2.2 of  $A(\mathbf{X})$ . Regarding (iv), a change of variables yields

$$\begin{aligned} A(\mathbf{X} + \mathbf{m}) &= \{\mathbf{n} \in \mathbb{R}^d : \mathbb{E}[\ell(\mathbf{X} + \mathbf{m} - \mathbf{n})] \leq 0\} \\ &= \{\mathbf{n} + \mathbf{m} \in \mathbb{R}^d : \mathbb{E}[\ell(\mathbf{X} - \mathbf{n})] \leq 0\} \\ &= A(\mathbf{X}) + \mathbf{m}. \end{aligned}$$

As for (v), on the one hand,  $\ell(\mathbf{X} - \mathbf{m}) \searrow \ell(-\infty) < 0$  as  $\mathbf{m} \rightarrow +\infty$  component-wise. Since  $\mathbf{X} \in M^\theta$  it follows that  $\ell(\mathbf{X}) \in L^1$ , thus monotone convergence yields  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \searrow \ell(-\infty) < 0$  and in turns the existence of  $\mathbf{m} \in \mathbb{R}^d$  such that  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0$ , showing that  $A(\mathbf{X}) \neq \emptyset$ . On the other hand,  $\ell$  being increasing and such that  $\ell(\mathbf{x}) \geq \sum_k x_k - c$ , it implies that  $\ell(\mathbf{X} - \mathbf{m}) \geq \sum_k X_k - \sum_k m_k - c \nearrow +\infty$  as  $\mathbf{m} \rightarrow -\infty$ , component-wise. Hence, monotone convergence yields  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \nearrow +\infty > 0$ , therefore there exists  $\mathbf{m} \in \mathbb{R}^d$  such that  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] > 0$ , that is,  $\mathbf{m} \notin A(\mathbf{X})$ . As for (vi), if  $\ell$  is positive homogeneous, for any  $\lambda > 0$  it holds  $\mathbb{E}[\ell(\lambda\mathbf{X} - \mathbf{m})] = \lambda \mathbb{E}[\ell(\mathbf{X} - \frac{1}{\lambda}\mathbf{m})]$ . Hence  $\mathbf{m}$  is in  $A(\lambda\mathbf{X})$  if and only if  $\frac{1}{\lambda}\mathbf{m}$  is in  $A(\mathbf{X})$  if and only if  $\mathbf{m}$  is in  $\lambda A(\mathbf{X})$ . Finally, if  $\ell$  is permutation invariant, for any permutation  $\pi$  it holds  $\mathbb{E}[\ell(\pi(\mathbf{X}) - \mathbf{m})] = \mathbb{E}[\ell(\pi(\mathbf{X} - \pi^{-1}(\mathbf{m})))] = \mathbb{E}[\ell(\mathbf{X} - \pi^{-1}(\mathbf{m}))]$ . Hence  $\mathbf{m}$  is in  $A(\pi(\mathbf{X}))$  if and only if  $\pi^{-1}(\mathbf{m})$  is in  $A(\mathbf{X})$ , if and only if  $\mathbf{m}$  is in  $\pi(A(\mathbf{X}))$  showing (vii). ■

Figure 4.1 shows sets of acceptable monetary allocations for a bivariate normal distribution with varying correlation coefficient. The location and shape of these sets change with the correlation : the higher the correlation, the more costly the acceptable monetary allocations, as expected in terms of systemic risk. As discussed in Sections 4.3 and 4.4, this feature is not always immediate and depends on the specification of the loss function.

Given an acceptable monetary allocation  $\mathbf{m} \in A(\mathbf{X})$ , its aggregated liquidity cost is  $\sum_k m_k$ . The smaller the cost, the better, which motivates the following definition.

**Definition 4.2.3** The *multivariate shortfall risk* of  $\mathbf{X} \in M^\theta$  is

$$\begin{aligned} R(\mathbf{X}) &:= \inf \left\{ \sum_k m_k : \mathbf{m} \in A(\mathbf{X}) \right\} \\ &= \inf \left\{ \sum_k m_k : \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0 \right\} \end{aligned} \tag{4.2.2}$$

**Example 4.2.3** Following up on the central clearing house Example 4.2.2, any acceptable allocation  $\mathbf{m} \in A(\mathbf{X})$  yields a corresponding value for the default fund. Clearing houses are in competition with each other, hence they are looking for the cheapest acceptable allocation to require from their members.

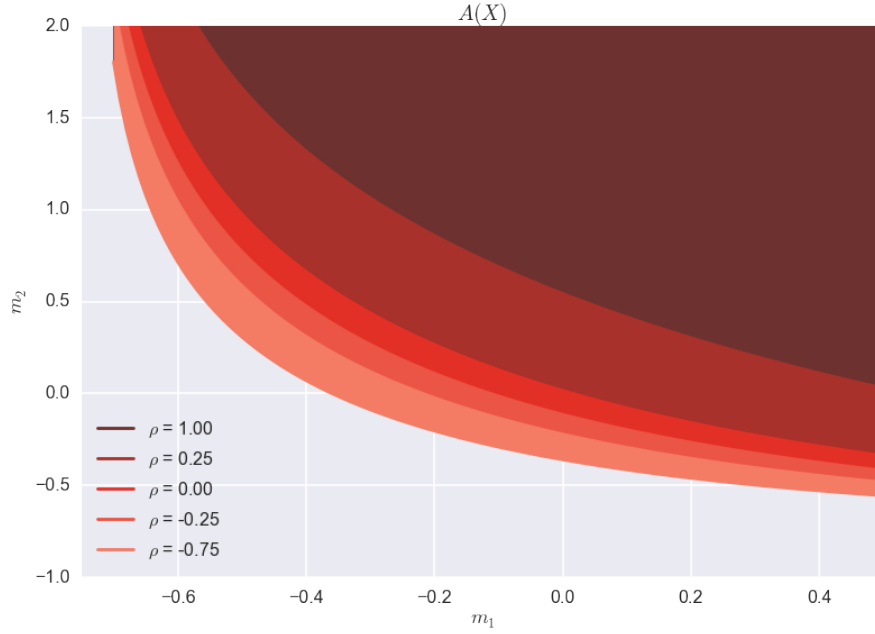


FIGURE 4.1 – Acceptance sets  $A(\mathbf{X})$  corresponding to the case study of Section 4.3.3 for different correlations.

**Remark 4.2.2** When  $d = 1$ , the above definition corresponds exactly to the shortfall risk measure in Föllmer and Schied (2002), of which this paper is a multivariate extension.

The set valued risk measure  $\mathbf{X} \mapsto A(\mathbf{X})$  introduced in (4.2.1) can be seen as an example of the set valued systemic risk measures presented in Feinstein, Rudloff, and Weber (2017), which in their notation translates as follows

$$A(\mathbf{X}) = R(\mathbf{Y}, \mathbf{k}) = \{\mathbf{m} \in \mathbb{R}^d : Y_{\mathbf{k}+\mathbf{m}} \in \mathcal{A}\}$$

where the aggregation is given by  $Y_{\mathbf{k}+\mathbf{m}} = \Lambda(\mathbf{X} - \mathbf{k} - \mathbf{m})$  for  $\Lambda(\mathbf{x}) = \ell(\mathbf{x})$  and the acceptance set is  $\mathcal{A} := \{X : \mathbb{E}[X] \leq 0\}$ . Their setting considers more general random fields  $Y_{\mathbf{k}}$  associated with capital allocations denoted by  $\mathbf{k}$  accommodating for instance the modelling of financial networks, among others. The case we consider can be embedded into Feinstein, Rudloff, and Weber (2017, Case (ii), Page 5). Even if set valued risk measure is not the primary focus of Biagini, Fouque, Frittelli, and Meyer-Brandis (2015), it is included in the definition of the acceptance family which, in their notation, is given as follows

$$\mathcal{A}^{\mathbf{m}} = \mathcal{A}^{\mathbf{Y}} = \{\mathbf{X} : \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0\}, \quad \mathbf{Y} \in \mathcal{C}$$

where  $\mathcal{C} = \mathbb{R}^d$  and  $\mathbf{Y} = \mathbb{R}^d$ . The resulting systemic risk measure can also be translated in their notation and denomination in terms of an aggregating function  $\Lambda(\mathbf{x}) = \ell(\mathbf{x})$ , acceptance set  $\mathcal{A} = \{X : \mathbb{E}[X] \leq 0\}$  and a measure of risk  $\pi(\mathbf{m}) = \sum_k m_k$ , resulting into

$$R(\mathbf{X}) = \inf \{\pi(\mathbf{m}) : \Lambda(\mathbf{X} - \mathbf{m}) \in \mathcal{A}\}$$

Therefore the case we consider can be embedded into the class presented in Biagini, Fouque, Frittelli, and Meyer-Brandis (2015, Section 1.3).

Our next result, which uses the concepts and notation of Appendix 4.7.2, shows that all the classical properties of the shortfall risk measure, including its dual representation, can be

extended to the multivariate case. We denote by

$$\mathcal{Q}^{\theta^*} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} := (Z_1, \dots, Z_d) : \mathbf{Z} \in L^{\theta^*}, Z_k \geq 0 \text{ and } \mathbb{E}[Z_k] = 1 \text{ for every } k \right\}$$

the set of  $d$ -dimensional measure densities in  $L^{\theta^*}$ , dual space of  $M^\theta$  according to 4.7.2. For the sake of simplicity, we use the notation  $\mathbb{E}_{\mathbb{Q}}[\mathbf{X}] := \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} \cdot \mathbf{X}]$  for  $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{Q}^{\theta^*}$  and  $\mathbf{X} \in M^\theta$ .

**Theorem 4.2.1** The function

$$R(\mathbf{X}) = \inf \left\{ \sum_k m_k : \mathbf{m} \in A(\mathbf{X}) \right\}, \quad \mathbf{X} \in M^\theta,$$

is real valued, convex, monotone and translation invariant.<sup>3</sup> In particular, it is continuous and sub-differentiable. If  $\ell$  is positive homogeneous, then so is  $R$ . Moreover, it admits the dual representation

$$R(\mathbf{X}) = \max_{\mathbb{Q} \in \mathcal{Q}^{\theta^*}} \left\{ \mathbb{E}_{\mathbb{Q}}[\mathbf{X}] - \alpha(\mathbb{Q}) \right\}, \quad \mathbf{X} \in M^\theta \quad (4.2.3)$$

where the penalty function is given by

$$\alpha(\mathbb{Q}) = \inf_{\lambda > 0} \mathbb{E} \left[ \lambda \ell^* \left( \frac{1}{\lambda} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right], \quad \mathbb{Q} \in \mathcal{Q}^{\theta^*} \quad (4.2.4)$$

**Remark 4.2.3** This robust representation can also be inferred from the general results of Farakas, Koch-Medina, and Munari (2015). However, for the sake of completeness and since the multivariate shortfall risk measure is closely related to a multidimensional version of the optimized certainty equivalent, we give a self contained proof tailored to our context.

The argumentation follows the original one by Föllmer and Schied (2002), which however cannot be directly applied on the product space  $\Omega \times \{1, \dots, d\}$  since the optimization is done here according to multidimensional allocations  $\mathbf{m} \in \mathbb{R}^d$  rather than one dimensional allocations  $m \in \mathbb{R}$ . Moreover, in the course of our derivation of the dual representation we extend to the multidimensional setting the following relationship between the optimized certainty equivalent and the shortfall risk provided in Ben-Tal and Teboulle (2007, Section 5.2)

$$R(X) = \inf_{m \in \mathbb{R}} \{m : \mathbb{E}[\ell(X - m)] \leq 0\} = \sup_{\lambda > 0} S(\lambda, X),$$

where

$$\begin{aligned} S(\lambda, X) &:= \inf_{m \in \mathbb{R}} \{m + \lambda \mathbb{E}[\ell(X - m)]\} \\ &= \sup_{\mathbb{Q} \ll P} \left\{ \mathbb{E}_{\mathbb{Q}}[X] - \mathbb{E} \left[ \lambda \ell^* \left( \frac{1}{\lambda} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \right\} \end{aligned}$$

is the optimized certainty equivalent of  $X$ .<sup>4</sup>

**Proof:** By Proposition 4.2.1 (v), we have  $A(\mathbf{X}) \neq \emptyset$  and in turn  $R(\mathbf{X}) < \infty$ . If  $R(\mathbf{X}) = -\infty$  for some  $\mathbf{X} \in M^\theta$ , then there exists a sequence  $(\mathbf{m}^n) \subseteq A(\mathbf{X})$  such that  $\sum_k m_k^n \rightarrow -\infty$ , in contradiction with  $0 \geq \mathbb{E}[\ell(\mathbf{X} - \mathbf{m}^n)] \geq \mathbb{E}[\sum_k X_k] - \sum_k m_k^n - c$ . Hence,  $R(\mathbf{X}) > -\infty$ . Monotonicity, convexity and translation invariance readily follow from Proposition 4.2.1 (ii), (iii) and (iv), respectively. In particular,  $R$  is a convex, real-valued and increasing functional on the Banach lattice  $M^\theta$ . Hence, by Cheridito and Li (2009, Theorem 4.1),  $R$  is continuous

3. In the sense that  $R(\mathbf{X} + \mathbf{m}) = R(\mathbf{X}) + \sum_k m_k$ .

4. Here  $\ell$  is a one dimensional loss function and  $X$  a one dimensional random variable.

and sub-differentiable. Therefore, the results recalled in Appendix 4.7.2 and the Fenchel-Moreau theorem imply

$$\begin{aligned} R(\mathbf{X}) &= \sup_{\mathbf{Y} \in L^{\theta^*}} \left\{ \mathbb{E}[\mathbf{X} \cdot \mathbf{Y}] - R^*(\mathbf{Y}) \right\} \\ &= \max_{\mathbf{Y} \in L^{\theta^*}} \left\{ \mathbb{E}[\mathbf{X} \cdot \mathbf{Y}] - R^*(\mathbf{Y}) \right\} \end{aligned} \quad (4.2.5)$$

where

$$R^*(\mathbf{Y}) = \sup_{\mathbf{X} \in M^\theta} \left\{ \mathbb{E}[\mathbf{X} \cdot \mathbf{Y}] - R(\mathbf{X}) \right\}, \quad \mathbf{Y} \in L^{\theta^*}$$

By the bipolar theorem, for  $\mathbf{Y} \not\geq \mathbf{0}$ , there exists  $\mathbf{K} \in M^\theta$ ,  $\mathbf{K} \geq \mathbf{0}$  with  $\mathbb{E}[\mathbf{Y} \cdot \mathbf{K}] < -\varepsilon < 0$  for some  $\varepsilon > 0$ . By monotonicity of  $R$ , it follows that  $R(-\lambda \mathbf{K}) \leq R(\mathbf{0}) < \infty$  for every  $\lambda > 0$ . Hence

$$\begin{aligned} R^*(\mathbf{Y}) &= \sup_{\mathbf{X} \in M^\theta} \left\{ \mathbb{E}[\mathbf{Y} \cdot \mathbf{X}] - R(\mathbf{X}) \right\} \\ &\geq \sup_{\lambda > 0} \left\{ -\lambda \mathbb{E}[\mathbf{Y} \cdot \mathbf{K}] - R(-\lambda \mathbf{K}) \right\} \\ &\geq \sup_{\lambda} \lambda \varepsilon - R(\mathbf{0}) = +\infty \end{aligned}$$

Furthermore, by translation invariance, setting  $\mathbf{X} = (0, \dots, r, \dots, 0)$  for  $r \in \mathbb{R}$  at the  $k$ -th component, it follows that

$$R^*(\mathbf{Y}) \geq r \mathbb{E}[Y_k] - R(\mathbf{0}) - r = r (\mathbb{E}[Y_k] - 1) - R(\mathbf{0})$$

where the right hand side can be made arbitrarily large whenever  $\mathbb{E}[Y_k] \neq 1$ . It shows that the supremum and maximum in (4.2.5) can be restricted to the set of those  $\mathbf{Y} \in L^{\theta^*}$  such that  $Y_k \geq 0$  and  $\mathbb{E}[Y_k] = 1$  for every  $k$ . Hence, it can be identified to  $\mathcal{Q}^{\theta^*}$ . In order to obtain a more explicit expression of the penalty function  $\alpha(\mathbb{Q}) := R^*\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = R^*(\mathbf{Y})$ , we set

$$\begin{aligned} L(m, \lambda, \mathbf{X}) &= \sum_k m_k + \lambda \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \\ S(\lambda, \mathbf{X}) &= \inf_{\mathbf{m} \in \mathbb{R}^d} L(m, \lambda, \mathbf{X}) \\ &= \inf_{\mathbf{m} \in \mathbb{R}^d} \left\{ \sum_k m_k + \lambda \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \right\} \end{aligned}$$

The functional  $\mathbf{X} \mapsto S(\lambda, \mathbf{X})$  is a multivariate version of the so called optimized certainty equivalent, see Ben-Tal and Teboulle (2007). Clearly,

$$R(\mathbf{X}) = \inf_{\mathbf{m} \in \mathbb{R}^d} \sup_{\lambda > 0} L(m, \lambda, \mathbf{X}) \geq \sup_{\lambda > 0} \inf_{\mathbf{m} \in \mathbb{R}^d} L(m, \lambda, \mathbf{X}) = \sup_{\lambda > 0} S(\lambda, \mathbf{X})$$

Since  $A(\mathbf{X})$  is non-empty and monotone, there exists  $\mathbf{m} \in \text{Int}(A(\mathbf{X}))$  and so the Slater condition is fulfilled. As a consequence of Rockafellar (1970, Theorem 28.2), there is no duality gap. Namely,  $R(\mathbf{X}) = \sup_{\lambda > 0} S(\lambda, \mathbf{X})$ . Via the first part of the proof, an easy multivariate adaptation of Ben-Tal and Teboulle (2007, Section 4) and Drapeau, Kupper, and Papapantoleon (2014, Section 2) yields

$$S(\lambda, \mathbf{X}) = \sup_{\mathbb{Q} \in \mathcal{Q}^{\theta^*}} \left\{ \mathbb{E}_{\mathbb{Q}}[\mathbf{X}] - \mathbb{E} \left[ (\ell_\lambda)^* \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \right\}$$

where  $\ell_\lambda(\mathbf{m}) = \lambda \ell(\mathbf{m})$ , hence  $\ell_\lambda^*(\mathbf{m}^*) = \lambda \ell^*(\frac{1}{\lambda} \mathbf{m}^*)$ . Combining this with  $R(\mathbf{X}) = \sup_{\lambda > 0} S(\lambda, \mathbf{X})$ , the dual representation (4.2.4) follows. ■

**Example 4.2.4** We consider the following two positive homogeneous loss functions that will be used later in the empirical study :

$$\ell_1(\mathbf{x}) = \beta \sum_k x_k^+ - \alpha \sum_k x_k^- \quad (4.2.6)$$

$$\ell_2(\mathbf{x}) = \beta \sum_k x_k^+ - \alpha \sum_k x_k^- + \beta \sum_{k < j} (x_k + x_j)^+ - \alpha \sum_{k < j} (x_k + x_j)^- \quad (4.2.7)$$

for  $0 < \alpha < 1 < \beta$ . A simple computation yields that  $\ell_i^* = \delta(\cdot | C_i)$  where <sup>5</sup>

$$C_1 = \{x : \alpha \leq x_k \leq \beta \text{ for all } k\}$$

$$C_2 = \left\{ x = \sum_{1 \leq j \leq d} x_{0j} e_k + \sum_{1 \leq k < j \leq d} x_{kj} (e_k + e_j) : \alpha \leq x_{kj} \leq \beta \text{ for all } 0 \leq k < j \leq d \right\}$$

Note that  $[\alpha, \beta] = C_1 \subseteq C_2 \subseteq [\alpha, d\beta]$  where  $\alpha$  and  $\beta$  are identified with their vector of equal components. Furthermore,  $d\beta$  is an extreme point of  $C_2$ . It follows in particular that  $R_1 \leq R_2$ . By positive homogeneity,  $\alpha_i^*$  only takes values 0 or  $\infty$ . It follows that  $\alpha_i^*(\mathbb{Q}) = 0$  if and only if there exists  $\lambda > 0$  such that  $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \lambda C_i$  almost surely. Since  $\mathbf{1}$  has to be in  $\lambda C_i$ , for this to happen, we can constrain  $\frac{1}{\beta} \leq \lambda \leq \frac{1}{\alpha}$  in the case of  $C_1$  and  $\frac{1}{d\beta} \leq \lambda \leq \frac{1}{\alpha}$  in the case of  $C_2$ . Thus

$$R_1(\mathbf{X}) = \sup \left\{ \mathbb{E}_{\mathbb{Q}}[\mathbf{X}] : \frac{d\mathbb{Q}}{d\mathbb{P}} \in \lambda C_1 \text{ for some } \frac{1}{\beta} \leq \lambda \leq \frac{1}{\alpha} \right\}$$

$$R_2(\mathbf{X}) = \sup \left\{ \mathbb{E}_{\mathbb{Q}}[\mathbf{X}] : \frac{d\mathbb{Q}}{d\mathbb{P}} \in \lambda C_2 \text{ for some } \frac{1}{d\beta} \leq \lambda \leq \frac{1}{\alpha} \right\}$$

### 4.3 Risk Allocation

We have established in Theorem 4.2.1 that the infimum over all allocations  $\mathbf{m} \in \mathbb{R}^d$  used for defining  $R(\mathbf{X})$  is real valued and has the desired properties of a risk measure. Beyond the question of the overall liquidity reserve, the allocation of this amount between the different risk components is key for systemic risk purposes. We therefore address in this section the following questions :

- The existence of a risk allocation ;
- The uniqueness of a risk allocation ;
- The impact of the interdependence structure,

The first question is important in some applications such as the default fund contribution of each member of a clearing house or the allocation of the capital among the different business lines of a bank. As for the second question, non-uniqueness can become an issue when this allocation is a regulatory cost for the different members or desks. If no additional clear rule is provided, the members would then face arbitrariness as for their contributions for the same overall risk. As for the last question, systemic risk should reflect the level of dependence of the system. For instance, highly correlated losses, while having the same marginal risk, should result into a higher systemic risk and different optimal allocations.

**Definition 4.3.1** A *risk allocation* is an acceptable monetary allocation  $\mathbf{m} \in A(\mathbf{X})$  such that  $R(\mathbf{X}) = \sum_k m_k$ . When a risk allocation is uniquely determined, we denote it by  $RA(\mathbf{X})$ .

**Remark 4.3.1** By definition, if a risk allocation exists, then the full allocation property automatically holds ; see also Section 4.4.3.

5. In particular, since  $1 \in C_i$  for  $i = 1, 2$ , it follows that  $\ell_i$  satisfies condition (A3) of a loss function.

In contrast to the univariate case, where the unique risk allocation is given by  $m = R(X)$ , existence and uniqueness are no longer straightforward in the multivariate case. The following example shows that existence may fail.

**Example 4.3.1** Consider the loss function

$$\ell(x, y) = \begin{cases} x + y + \frac{(x+y)^+}{1-y} - 1 & \text{if } y < 1 \\ \infty & \text{otherwise} \end{cases}$$

It follows that

$$A(\mathbf{0}) = \left\{ \mathbf{m} \in \mathbb{R}^2 : m_2 > -1 \text{ and } 1 \geq -m_1 - m_2 + \frac{(-m_1 - m_2)^+}{1 + m_2} \right\}$$

Computations yield

$$R(\mathbf{0}) = \inf_{m_2 > -1} \left\{ m_2 - \frac{m_2^2 + 3m_2 + 1}{m_2 + 2} \right\} = -1$$

However, the infimum is not attained.

Note that the loss function used in Example 4.3.1 is not permutation invariant. Our next result introduces conditions towards the existence and uniqueness of a risk allocation. We denote by  $Z = \{\mathbf{u} \in \mathbb{R}^d : \sum_k u_k = 0\}$  the set of *zero-sum allocations*.

**Theorem 4.3.1** If  $\ell$  is a permutation invariant loss function, then, for every  $\mathbf{X} \in M^\theta$ , risk allocations  $m^*$  exist. They are characterized by the first order conditions

$$\mathbf{1} \in \lambda^* \mathbb{E} [\nabla \ell(\mathbf{X} - \mathbf{m}^*)] \quad \text{and} \quad \mathbb{E} [\ell(\mathbf{X} - \mathbf{m}^*)] = 0, \quad (4.3.1)$$

where  $\lambda^*$  is a Lagrange multiplier. In particular, when  $\ell$  has no zero-sum direction of recession<sup>6</sup> except  $\mathbf{0}$ , the set of the solutions  $(\mathbf{m}^*, \lambda^*)$  to the first order conditions (4.3.1) is bounded.

If  $\ell(\mathbf{x} + \cdot)$  is strictly convex along zero-sums allocations for every  $\mathbf{x}$  with  $\ell(\mathbf{x}) \geq 0$ , then the risk allocation is unique.

**Proof:** Let  $\mathbf{m}$  in  $A(\mathbf{X})$ , according to Theorem 4.7.1, it holds

$$\begin{aligned} 0^+ A(\mathbf{X}) &= \{\mathbf{u} \in \mathbb{R}^d : \mathbb{E} [\ell(\mathbf{X} - \mathbf{m} - r\mathbf{u})] \leq 0, \text{ for all } r > 0\} \\ &= \left\{ \mathbf{u} \in \mathbb{R}^d : \sup_{r>0} \mathbb{E} \left[ \frac{\ell(\mathbf{X} - \mathbf{m} - r\mathbf{u}) - \ell(\mathbf{X} - \mathbf{m})}{r} \right] \leq 0 \right\} \\ &= \left\{ \mathbf{u} \in \mathbb{R}^d : \mathbb{E} \left[ \sup_{r>0} \frac{\ell(\mathbf{X} - \mathbf{m} - r\mathbf{u}) - \ell(\mathbf{X} - \mathbf{m})}{r} \right] \leq 0 \right\} \\ &= -0^+ \ell \end{aligned}$$

Further, we define  $f(\mathbf{m}) = \sum_k m_k + \delta(\mathbf{m} | A(\mathbf{X}))$ . It follows that  $f$  is increasing, convex, lower semi-continuous, proper and such that  $R(\mathbf{X}) = \inf f$ . Let  $B = \{\mathbf{m} : f(\mathbf{m}) \leq \gamma\}$  be non-empty for some  $\gamma$  large enough and  $\mathbf{b} \in B$ . By Theorem 4.7.1 and the definition,  $\mathbf{u} \in 0^+ B = 0^+ f$  if and only if

$$R(\mathbf{X}) \leq \sum_k b_k + r \sum_k u_k \leq \gamma \quad \text{and} \quad \mathbf{b} + r\mathbf{u} \in A(\mathbf{X}) \text{ for all } r > 0$$

However,  $-\infty < R(\mathbf{X}) \leq \gamma < \infty$  showing that  $0^+ f = Z \cap 0^+ A(\mathbf{X}) = -Z \cap 0^+ \ell$ . By Rockafellar (1970, Theorem 27.1 (b)), the existence of a risk allocation follows from  $f$  being constant along its

6. We refer the reader to Appendix 4.7.1 regarding the notions and properties of recession cones and functions.

directions of recession  $0^+f$ , which according to Theorem 4.7.1, is equivalent to  $\mathbf{u} \in 0^+f$  implies  $(-\mathbf{u}) \in 0^+f$ . However, since  $\ell$  is permutation invariant it follows that  $0^+\ell = -0^+\ell$  and therefore  $\mathbf{u} \in 0^+f$  implies that  $-\mathbf{u} \in 0^+f$ . Thus the existence of a risk allocation.<sup>7</sup> In particular, if  $0^+\ell = \mathbf{0}$ , then by Rockafellar (1970, Theorem 27.1, (d)), the set of risk allocations is non-empty and bounded. Furthermore, since  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] < 0$  for some  $\mathbf{m}$  large enough, the Slater condition for the convex optimization problem  $R(\mathbf{X}) = \inf_{\mathbf{m}} f(\mathbf{m})$  is fulfilled. Hence, according to Rockafellar (1970, Theorems 28.1, 28.2 and 28.3), optimal solutions  $\mathbf{m}^*$  are characterized by (4.3.1).

Finally, let  $\mathbf{m} \neq \mathbf{n}$  be two risk allocations. It follows that  $\alpha\mathbf{m} + (1 - \alpha)\mathbf{n}$  is a risk allocation as well for every  $\alpha \in [0, 1]$ . Furthermore,  $(\mathbf{m} - \mathbf{n})$  is a zero sum allocation. By convexity, it follows that  $0 = \mathbb{E}[\ell(\mathbf{X} - \alpha\mathbf{m} - (1 - \alpha)\mathbf{n})] \leq \alpha\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] + (1 - \alpha)\mathbb{E}[\ell(\mathbf{X} - \mathbf{n})] = 0$  for every  $0 \leq \alpha \leq 1$ , which shows that  $\alpha\ell(\mathbf{X} - \mathbf{m}) + (1 - \alpha)\ell(\mathbf{X} - \mathbf{n}) = \ell(\mathbf{X} - \alpha\mathbf{m} - (1 - \alpha)\mathbf{n})$   $\mathbb{P}$ -almost surely for every  $0 \leq \alpha \leq 1$ . By assumption,  $\ell(\mathbf{x} + \cdot)$  is strictly convex on  $Z$  for every  $\mathbf{x}$  such that  $\ell(\mathbf{x}) \geq 0$ . From  $\mathbf{m} - \mathbf{n} \in Z$ , it holds that  $\mathbf{X} - \alpha\mathbf{m} + (1 - \alpha)\mathbf{n} + Z$  entails the segment  $[\mathbf{X} - \mathbf{m}, \mathbf{X} - \mathbf{n}]$ . From  $\alpha\ell(\mathbf{X} - \mathbf{m}) + (1 - \alpha)\ell(\mathbf{X} - \mathbf{n}) = \ell(\mathbf{X} - \alpha\mathbf{m} - (1 - \alpha)\mathbf{n})$ ,  $\mathbf{z} \mapsto \ell(\mathbf{X} - \alpha\mathbf{m} - (1 - \alpha)\mathbf{n} + \mathbf{z})$  is almost surely constant on this segment and therefore not strictly convex. Hence  $\mathbb{P}[\ell(\mathbf{X} - \alpha\mathbf{m} - (1 - \alpha)\mathbf{n}) < 0] = 1$  for every  $0 \leq \alpha \leq 1$ , showing in particular that  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] < 0$ , a contradiction. ■

**Corollary 4.3.1** *Let  $\ell$  be a permutation invariant loss function, such that  $\ell(\mathbf{x} + \cdot)$  is strictly convex along zero-sum allocations for every  $\mathbf{x}$  with  $\ell(\mathbf{x}) \geq 0$ . It holds*

$$RA(\mathbf{X} + \mathbf{r}) = RA(\mathbf{X}) + \mathbf{r}, \quad \text{for every } \mathbf{X} \in M^\theta \text{ and } \mathbf{r} \in \mathbb{R}^d.$$

*If  $\ell$  is additionally positive homogeneous, it holds*

$$RA(\lambda\mathbf{X}) = \lambda RA(\mathbf{X}), \quad \text{for every } \mathbf{X} \in M^\theta \text{ and } \lambda > 0$$

**Proof:** From Theorem 4.3.1, the assumptions on  $\ell$  ensure the existence and uniqueness of a risk allocation uniquely characterized, together with the Lagrange multiplier, by the first order conditions. Let  $\mathbf{m} = RA(\mathbf{X} + \mathbf{r})$ , for which there exists a unique  $\lambda$  such that  $\lambda\mathbb{E}[\nabla\ell(\mathbf{X} + \mathbf{r} - \mathbf{m})] = \mathbf{1}$  and  $\mathbb{E}[\ell(\mathbf{X} + \mathbf{r} - \mathbf{m})] = 0$ . Hence,  $\mathbf{n} = \mathbf{m} - \mathbf{r}$  and  $\lambda$  satisfy the first order conditions  $\lambda\mathbb{E}[\nabla\ell(\mathbf{X} - \mathbf{n})] = \mathbf{1}$  and  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{n})] = 0$ , which by uniqueness shows that  $\mathbf{n} = RA(\mathbf{X}) = \mathbf{m} - \mathbf{r} = RA(\mathbf{X} + \mathbf{r}) - \mathbf{r}$ . As for the second assertion, it follows from  $A(\lambda\mathbf{X}) = \lambda A(\mathbf{X})$  for every  $\lambda > 0$  according to Proposition 4.2.1. ■

**Remark 4.3.2** In general, the positivity of the risk allocation is not required. However, if positivity or any other convex constraint is imposed, for instance by regulators, it can easily be embedded in our setup. In case of positivity, this would modify the definition of  $R(\mathbf{X})$  into

$$R(\mathbf{X}) = \inf \left\{ \sum_k m_k : \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0 \text{ and } m_k \geq 0 \text{ for every } k \right\},$$

with accordingly modified first order conditions.

As already mentioned, the following example illustrates the importance of the uniqueness.

**Example 4.3.2** Any loss function of class (C1), that is,  $\ell(\mathbf{x}) = h(\sum_k x_k)$ , is permutation invariant. Thus, a risk allocation  $\mathbf{m}^* \in A(\mathbf{X})$  exists by means of Theorem 4.3.1. However, for any zero-sum allocation  $\mathbf{u}$ , we have  $R(\mathbf{X}) = \sum_k (m_k^* + u_k) = \sum_k m_k^*$  and  $\mathbb{E}[h(\sum_k X_k - (m_k^* + u_k))] =$

7. Note that this computation shows that the condition  $Z \cap 0^+\ell = -Z \cap 0^+\ell$  is sufficient to get the existence of a risk allocation.

$\mathbb{E}[h(\sum X_k - m_k^*)] \leq 0$ , so that  $\mathbf{m}^* + \mathbf{u}$  is another risk allocation. In terms of regulatory costs, this is a problematic situation. Indeed, consider two banks and require from them 110M€ and 500M€, respectively, as capital allocation. In such a case, one could equally well require 610M€, from the first bank and nothing from the second. Such arbitrariness is unlikely to be accepted in that case.

Example 4.3.2 shows that loss functions of the class (C1) lack the uniqueness of a risk allocation. By contrast, for loss functions of class (C2), that is,  $\ell(\mathbf{x}) = \sum_k h(x_k)$ , the following proposition shows that, while there exists a unique risk allocation under very mild conditions, the risk allocation only depends on the marginal distributions of the loss vector  $\mathbf{X} = (X_1, \dots, X_d)$ . In other words, the risk measure and the risk allocation do not reflect the dependence structure of the system.

**Proposition 4.3.1** *Let  $\ell(\mathbf{x}) := \sum_k h(x_k)$  for some strictly convex univariate loss function  $h : \mathbb{R} \rightarrow \mathbb{R}$ . For every  $\mathbf{X} \in M^\theta$ , there exists a unique optimal risk allocation  $RA(\mathbf{X})$  and we have  $RA(\mathbf{X}) = RA(\mathbf{Y})$ , for every  $\mathbf{Y} \in M^\theta$  such that  $Y_k$  has the same distribution as  $X_k$  for  $k = 1, \dots, d$ .*

**Proof:** The loss function is permutation invariant and strictly convex. According to Theorem 4.3.1, there exists a unique risk allocation for every  $\mathbf{X} \in M^\theta$ . The first order conditions (4.3.1) are written as

$$1 \in \lambda \mathbb{E}[\partial h(X_k - m_k)], \text{ for } k = 1, \dots, d, \quad \text{and} \quad \sum_k \mathbb{E}[h(X_k - m_k)] = 0$$

which only depend on the marginal distributions of  $\mathbf{X}$ . ■

Following Rüschendorf (2004), we can characterise in terms of supermodular, directionally convex and upper orthant stochastic ordering the risk of positive dependence in terms of  $\ell$ . For a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  we define

$$\Delta_{k,\mathbf{y}} f(\mathbf{x}) = f(x_0, \dots, x_k + y_k, \dots, x_d) - f(\mathbf{x}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, k \in \{1, \dots, d\}$$

We say that a continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is

- super-modular, if  $\Delta_{k,\mathbf{y}} \Delta_{l,\mathbf{y}} f(\mathbf{x}) \geq 0$  for every  $1 \leq k < l \leq d$ ;
- directionally convex, if  $\Delta_{k,\mathbf{y}} \Delta_{l,\mathbf{y}} f(\mathbf{x}) \geq 0$  for every  $1 \leq k \leq l \leq d$ ;
- $\Delta$ -monotone, if  $\Delta_{i_1,\mathbf{y}} \dots \Delta_{i_n,\mathbf{y}} f(\mathbf{x}) \geq 0$  for every  $\{i_1, \dots, i_n\} \subseteq \{1, \dots, d\}$ ;

for every  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^d$  with  $\mathbf{y} \geq \mathbf{0}$ . We denote by  $\succ^{sm}$ ,  $\succ^{dc}$  and  $\succ^{uo}$  the integral orders given by the respective class of functions. We refer to Rüschendorf (2004) for a discussion of these orders in terms of dependence risk. Note that  $X \succ^{uo} Y$  if and only if  $\mathbb{P}[X \geq \mathbf{x}] \geq \mathbb{P}[Y \geq \mathbf{x}]$  for every  $\mathbf{x} \in \mathbb{R}^d$ .

**Proposition 4.3.2** *The shortfall risk measure  $R$  is monotone with respect with  $\succ^{sm}$ ,  $\succ^{dc}$  or  $\succ^{uo}$  whenever  $\ell$  is super-modular, directionally convex, or  $\Delta$ -monotone, respectively.*

**Proof:** The assertion follows immediately from the fact that if  $\ell$  is one of super-modular, directionally convex, or  $\Delta$ -monotone function, so is  $\ell(\cdot - \mathbf{m})$  for every  $\mathbf{m}$ . Therefore if  $\mathbf{X} \succ^x \mathbf{Y}$  with  $x$  either  $sm$ ,  $dc$ , or  $uo$  according to  $\ell$ , it follows that  $\mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \geq \mathbb{E}[\ell(\mathbf{Y} - \mathbf{m})]$  showing that  $A(\mathbf{Y}) \subseteq A(\mathbf{X})$ . ■

**Remark 4.3.3** Any loss function of the form (C1), (C2) and (C3) are directionally convex and therefore super-modular. They are  $\Delta$ -monotone if  $d = 2$ . As for the specific loss functions used in this paper in several places for illustration

$$\sum_k \frac{(x_k^+)^2}{2} + \alpha \sum_{k < j} x_k^+ x_j^+ - 1 \quad \text{and} \quad \sum_k x_k^+ + \alpha \sum_{k < j} (x_j + x_j)^+ - 1,$$

they are both directionally convex and  $\Delta$ -monotone. However, if  $\alpha = 0$  they are degenerated in terms of these monotonicity since  $\Delta_{k,y}\Delta_{j,y}\ell(\mathbf{x}) = 0$  for every  $k \neq j$ . As soon as  $\alpha > 0$ , these loss functions are strictly monotone on  $\mathbb{R}_+^d$ .

**Remark 4.3.4** A loss function can be chosen in view of an a-priori list of wished properties in terms of risk measurement and allocation as the Proposition above mentioned. However, loss functions may also arise in systemic risk problems as an intrinsic property of the system as presented by [Eisenberg and Noe \(2001\)](#) or recently by [Weber and Weske \(2017\)](#).

**Example 4.3.3** The following simple example shows the impact of the dependence in a simple case for a loss function<sup>8</sup>

$$\ell(x_1, x_2) = \frac{1}{1+\alpha} \left[ \frac{1}{2}e^{2x_1} + \frac{1}{2}e^{2x_2} + \alpha e^{x_1}e^{x_2} \right] - 1, \quad (4.3.2)$$

that is  $\Delta$ -monotone and bivariate normal vector  $\mathbf{X} = (X_1, X_2) \sim \mathcal{N}(0, \Sigma)$  with  $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ .

Solving the first order conditions yield

$$RA_i(\mathbf{X}) = \sigma_i^2 + \frac{1}{2}SRC(\rho, \sigma_1, \sigma_2, \alpha) \quad R(\mathbf{X}) = \sigma_1^2 + \sigma_2^2 + SRC(\rho, \sigma_1, \sigma_2, \alpha),$$

showing that the risk allocations are disentangled into the respective individual contributions  $\sigma_i^2$ ,  $i = 1, 2$ , and a *systemic risk contribution*

$$SRC = \ln \left( 1 + \alpha e^{\rho\sigma_1\sigma_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)} \right) \quad (4.3.3)$$

which depends on the correlation parameter  $\rho$  and on the systemic weight  $\alpha$  of the loss function. Figure 4.2 shows the value of this systemic risk contribution as a function of  $\rho$  and  $\sigma_1$ . Computing

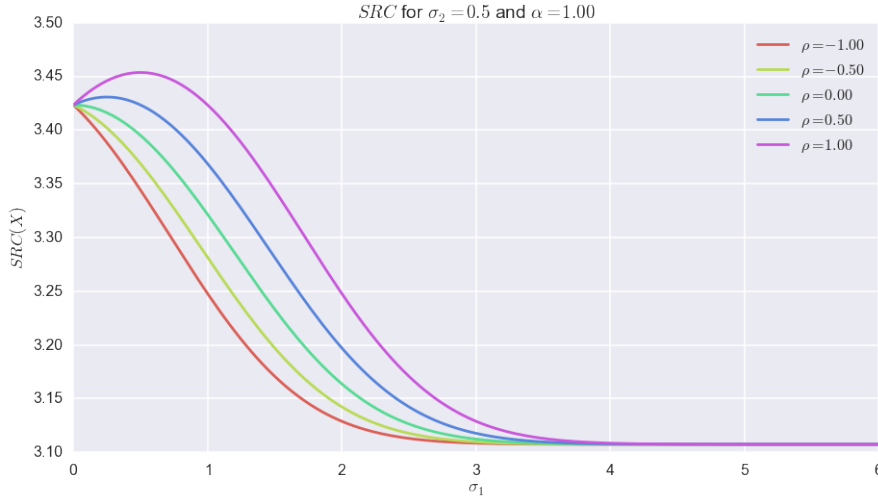


FIGURE 4.2 – SRC (4.3.3) as a function of  $\sigma_1$  for different values of the correlation  $\rho$  in the case where  $\alpha = 1$ .

the partial derivatives with respect to  $\sigma_i$  and  $\rho$  yields

$$\begin{aligned} \frac{\partial SRC}{\partial \sigma_1} &= \frac{\alpha(\rho\sigma_2 - \sigma_1)}{2} \frac{e^{\rho\sigma_1\sigma_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)}}{1 + \alpha e^{\rho\sigma_1\sigma_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)}}, \\ \frac{\partial SRC}{\partial \rho} &= \frac{\alpha\sigma_1\sigma_2}{2} \frac{e^{\rho\sigma_1\sigma_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)}}{1 + \alpha e^{\rho\sigma_1\sigma_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)}}. \end{aligned}$$

8. A simple check shows that it is indeed a loss function satisfying (A1), (A2) and (A3).

showing that the systemic risk contribution is

- increasing with respect to the correlation  $\rho$ ;
- decreasing with respect to  $\sigma_1$  if the correlation is negative;
- increasing up to  $\rho\sigma_2$  and then decreasing with respect to  $\sigma_1$  if the correlation is positive as the individual risk of  $X_1$  dominates the risk of the system.

## 4.4 Systemic Sensitivity of Shortfall Risk and its Allocation

The previous results emphasize the importance of using a loss function that adequately captures the systemic risk inherent to the system. This motivates the study of the sensitivity of shortfall risk and its allocation so as to identify the systemic features of a loss function.

**Definition 4.4.1** The *marginal risk contribution* of  $\mathbf{Y} \in M^\theta$  to  $\mathbf{X} \in M^\theta$  is defined as the sensitivity of the risk of  $\mathbf{X}$  with respect to the impact of  $\mathbf{Y}$ , that is

$$R(\mathbf{X}; \mathbf{Y}) := \limsup_{t \searrow 0} \frac{R(\mathbf{X} + t\mathbf{Y}) - R(\mathbf{X})}{t}$$

In the case where  $R(\mathbf{X} + t\mathbf{Y})$  admits a unique risk allocation  $RA(\mathbf{X} + t\mathbf{Y})$  for every  $t$ , the *risk allocation marginals* of the risk of  $\mathbf{X}$  with respect to the impact of  $\mathbf{Y}$  are given by

$$RA_k(\mathbf{X}; \mathbf{Y}) = \limsup_{t \searrow 0} \frac{RA_k(\mathbf{X} + t\mathbf{Y}) - RA_k(\mathbf{X})}{t}, \quad k = 1, \dots, d$$

Theorem 4.2.1 and its proof show that the determination of the risk measure  $R(\mathbf{X})$  reduces to the saddle point problem

$$R(\mathbf{X}) = \min_{\mathbf{m}} \max_{\lambda > 0} L(\mathbf{m}, \lambda, \mathbf{X}) = \max_{\lambda > 0} \min_{\mathbf{m}} L(\mathbf{m}, \lambda, \mathbf{X})$$

Using Rockafellar (1970), the “argminmax” set of saddle points  $(\mathbf{m}^*, \lambda^*)$  is a product set that we denote by  $B(\mathbf{X}) \times C(\mathbf{X})$ .

**Theorem 4.4.1** Assuming that  $\ell$  is permutation invariant, then

$$R(\mathbf{X}; \mathbf{Y}) = \min_{\mathbf{m} \in B(\mathbf{X})} \max_{\lambda \in C(\mathbf{X})} \lambda \mathbb{E} [\nabla \ell(\mathbf{X} - \mathbf{m}) \cdot \mathbf{Y}]$$

Supposing further that  $\ell$  is twice differentiable and that  $(\mathbf{m}, \lambda) \in B(\mathbf{X}) \times C(\mathbf{X})$  is such that

$$M = \begin{bmatrix} \lambda \mathbb{E} [\nabla^2 \ell(\mathbf{X} - \mathbf{m})] & -\frac{1}{\lambda} \mathbf{1} \\ 1 & 0 \end{bmatrix}$$

is non-singular, then

- there exists  $t_0 > 0$  such that  $B(\mathbf{X} + t\mathbf{Y}) \times C(\mathbf{X} + t\mathbf{Y})$  is a singleton, for every  $0 \leq t \leq t_0$ ;
- the corresponding unique saddle point  $(\mathbf{m}_t, \lambda_t) = (RA(\mathbf{X} + t\mathbf{Y}), \lambda_t)$  is differentiable as a function of  $t$  and we have

$$\begin{bmatrix} RA(\mathbf{X}; \mathbf{Y}) \\ \lambda(\mathbf{X}; \mathbf{Y}) \end{bmatrix} = M^{-1} V,$$

where  $\lambda(\mathbf{X}; \mathbf{Y}) = \limsup_{t \searrow 0} \frac{\lambda_t - \lambda_0}{t}$  and

$$V = \begin{bmatrix} \lambda \mathbb{E} [\nabla^2 \ell(\mathbf{X} - \mathbf{m}) \mathbf{Y}] \\ R(\mathbf{X}; \mathbf{Y}) \end{bmatrix}$$

**Proof:** Let  $L(\mathbf{m}, \lambda, t) = \sum_k m_k + \lambda \mathbb{E}[\ell(\mathbf{X} + t\mathbf{Y} - \mathbf{m})]$ . Theorem 4.2.1 yields

$$R(\mathbf{X} + t\mathbf{Y}) = \min_{\mathbf{m}} \max_{\lambda} L(\mathbf{m}, \lambda, t) = \max_{\lambda} \min_{\mathbf{m}} L(\mathbf{m}, \lambda, t) = L(\mathbf{m}_t, \lambda_t, t),$$

for every selection  $(\mathbf{m}_t, \lambda_t) \in B(\mathbf{X} + t\mathbf{Y}) \times C(t + t\mathbf{Y})$ . Regarding the first assertion of the theorem, since  $\ell$  has no zero-sum direction of recession other than  $\mathbf{0}$ , it follows from Theorem 4.3.1 that  $B(\mathbf{X}) \times C(\mathbf{X})$  is non empty and bounded. Hence, the assumptions of Golshtein's Theorem on the perturbation of saddle values, see Rockafellar and Wets (2009, Theorem 11.52), are satisfied and the first assertion follows. As for the second assertion, the assumptions of Fiacco and McCormick (1990, Theorem 6, pp. 34–45) are fulfilled. The Jacobian of the vector

$$\begin{bmatrix} \nabla_{\mathbf{m}} L(\mathbf{m}, \lambda, 0) \\ \lambda \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \end{bmatrix}$$

that is used to specify the first order conditions is given by the matrix  $M$ . Hence, the second assertion follows from Fiacco and McCormick (1990, Theorem 6, pp. 34–35). ■

Theorem 4.4.1 allows to explicitly derive the impact of an independent exogenous shock as stated in the following proposition.

**Proposition 4.4.1** *Under the assumptions of Theorem 4.4.1 ensuring the uniqueness of a saddle point, suppose that  $\mathbf{Y}$  is independent of  $\mathbf{X}$ . Then*

$$R(\mathbf{X}; \mathbf{Y}) = \sum_k \mathbb{E}[Y_k] \quad \text{and} \quad RA(\mathbf{X}; \mathbf{Y}) = \mathbb{E}[\mathbf{Y}]$$

**Proof:** Since  $\mathbf{Y}$  is independent of  $\mathbf{X}$ , denoting by  $\mathbf{m} = RA(\mathbf{X}; \mathbf{Y})$ , it follows from the first order conditions that

$$R(\mathbf{X}; \mathbf{Y}) = \lambda \mathbb{E}[\nabla \ell(\mathbf{X} - \mathbf{m}) \cdot \mathbf{Y}] = \lambda \mathbb{E}[\nabla \ell(\mathbf{X} - \mathbf{m})] \cdot \mathbb{E}[\mathbf{Y}] = \mathbf{1} \cdot \mathbb{E}[\mathbf{Y}] = \sum_k \mathbb{E}[Y_k]$$

Furthermore, we have

$$M = \begin{bmatrix} \lambda A & -B \\ C & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \lambda \mathbb{E}[\nabla^2 \ell(\mathbf{X} - \mathbf{m}) \mathbf{Y}] \\ R(\mathbf{X}; \mathbf{Y}) \end{bmatrix} = \begin{bmatrix} \lambda A \mathbb{E}[\mathbf{Y}] \\ C \mathbb{E}[\mathbf{Y}] \end{bmatrix}$$

where  $A = \mathbb{E}[\nabla^2 \ell(\mathbf{X} - \mathbf{m})]$ ,  $B = [\frac{1}{\lambda} \quad \dots \quad \frac{1}{\lambda}]^\top$ , and  $C = [1 \quad \dots \quad 1]$ . Using the classical formula of block matrix inversion, we obtain

$$\begin{aligned} RA(\mathbf{X}; \mathbf{Y}) &= \left[ \frac{A^{-1}}{\lambda} - \frac{A^{-1}BCA^{-1}}{\lambda CA^{-1}B} \quad \frac{A^{-1}B}{CA^{-1}B} \right] \begin{bmatrix} \lambda A \mathbb{E}[\mathbf{Y}] \\ C \mathbb{E}[\mathbf{Y}] \end{bmatrix} \\ &= \mathbb{E}[\mathbf{Y}] - \frac{A^{-1}BC \mathbb{E}[\mathbf{Y}]}{CA^{-1}B} + \frac{A^{-1}BC \mathbb{E}[\mathbf{Y}]}{CA^{-1}B} = \mathbb{E}[\mathbf{Y}] \end{aligned}$$

■

According to the discussion about causal responsibility in Section 4.4.3, it follows that each member is marginally paying for the additional risk it takes provided this one is independent of the system. In particular, if the risk factor  $k$  is affected by a shock  $Y_k$  independent of the system, it follows that  $R(\mathbf{X}; \mathbf{Y}) = \mathbb{E}[Y_k] = RA_k(\mathbf{X}; \mathbf{Y})$ , showing that the member  $k$  pays for the full risks it takes.

#### 4.4.1 Impact of an Exogenous Shock

The following Section illustrates the case when the exogenous shock may depend on  $\mathbf{X}$ . We consider a bivariate situation where  $\mathbf{X} = (X_1, X_2)$ , and exogenous factor  $\mathbf{Y} = (Y_1, 0)$  impacting only the first component. We consider the loss function

$$\ell(x_1, x_2) = \frac{(x_1^+)^2 + (x_2^+)^2}{2} + \alpha x_1^+ x_2^+ - 1, \quad 0 \leq \alpha \leq 1$$

which gives rise to a unique risk allocation by virtue of Theorem 4.3.1. Note that  $\ell$  is  $\Delta$ -monotone, and strictly  $\Delta$ -monotone on  $\mathbb{R}_+^2$  if  $\alpha > 0$ . For ease of notation, we assume that  $X_1 \sim X_2$ , which, since  $\ell$  is permutation invariant, implies that  $m = RA_1(\mathbf{X}) = RA_2(\mathbf{X})$ . Let  $p := \mathbb{P}[X_1 \geq m] = \mathbb{P}[X_2 \geq m]$  and  $r = \mathbb{P}[X_1 \geq m; X_2 \geq m]$ . According to Theorem 4.4.1, and the first order condition (4.3.1), we have

$$R(\mathbf{X}; \mathbf{Y}) = \frac{\mathbb{E}[Y_1(X_1 - m_1)^+] + \alpha p \mathbb{E}[Y_1(X_2 - m_2)^+ | X_1 \geq m_1]}{\mathbb{E}[(X_1 - m_1)^+] + \alpha p \mathbb{E}[(X_2 - m_2)^+ | X_2 \geq m_2]}$$

As for the allocation of this marginal risk contribution, in the notation of Theorem 4.4.1, we have :

$$M = \begin{bmatrix} \lambda p & \lambda \alpha r & \frac{-1}{\lambda} \\ \lambda \alpha r & \lambda p & \frac{-1}{\lambda} \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \lambda p \mathbb{E}[Y_1 | X_1 \geq m_1] \\ \lambda \alpha r \mathbb{E}[Y_1 | X_1 \geq m_1; X_2 \geq m_2] \\ R(\mathbf{X}; \mathbf{Y}) \end{bmatrix}$$

which by inverting  $M$  yields

$$RA_1(\mathbf{X}; \mathbf{Y}) = \frac{R(\mathbf{X}; \mathbf{Y})}{2} + \frac{1}{2} \frac{\mathbb{E}[Y_1 \mathbb{1}_{\{X_1 \geq m_1\}}] - \alpha \mathbb{E}[Y_1 \mathbb{1}_{\{X_1 \geq m_1; X_2 \geq m_2\}}]}{p - \alpha r}$$

$$RA_2(\mathbf{X}; \mathbf{Y}) = \frac{R(\mathbf{X}; \mathbf{Y})}{2} - \frac{1}{2} \frac{\mathbb{E}[Y_1 \mathbb{1}_{\{X_1 \geq m_1\}}] - \alpha \mathbb{E}[Y_1 \mathbb{1}_{\{X_1 \geq m_1; X_2 \geq m_2\}}]}{p - \alpha r}$$

Beyond the fact that according to Proposition 4.4.1, if  $\mathbf{Y}$  is independent of  $\mathbf{X}$  then  $R(\mathbf{X}; \mathbf{Y}) = RA_1(\mathbf{X}; \mathbf{Y})$  and  $RA_2(\mathbf{X}; \mathbf{Y}) = 0$ , observe in general that :

- The two risk components marginally share first equally the additional cost of the exogenous impact in terms of  $\frac{1}{2}R(\mathbf{X}; \mathbf{Y})$  each.
- The asymmetry of the shock that concerns only  $X_1$  is reflected in the correction with respect to the second term which is added to the first one and subtracted to the second. Furthermore,  $\mathbb{1}_{\{X_1 \geq m_1\}} \geq \alpha \mathbb{1}_{\{X_1 \geq m_1; X_2 \geq m_2\}}$  for every  $0 \leq \alpha \leq 1$ . It implies that the additional risk taken by the first risk factor is always positively proportional to  $Y_1$  while the second one is negatively proportional to  $Y_1$ .
- If  $\alpha = 0$ , then the marginal change impact the risk factors according to  $\pm \frac{\mathbb{E}[Y_1] - \mathbb{E}[Y_1 | X_1 \geq m]}{2}$ .
- If  $\alpha = 1$  and  $X_1$  and  $X_2$  are strongly anti-correlated, then  $\mathbb{1}_{\{X_1 \geq m; X_2 \geq m\}}$  is likely very small and therefore the effect is similar to the case where  $\alpha = 0$ . On the other hand, if  $X_1$  and  $X_2$  are strongly correlated, then  $\mathbb{1}_{\{X_1 \geq m\}} \approx \mathbb{1}_{\{X_1 \geq m; X_2 \geq m\}}$  and in that case  $RA_1(\mathbf{X}; \mathbf{Y}) \approx RA_2(\mathbf{X}; \mathbf{Y}) \approx \frac{1}{2}R(\mathbf{X}; \mathbf{Y})$  showing that the full dependence with  $\alpha = 1$  yields an equal share of the marginal risk changes.

#### 4.4.2 Sensitivity to Dependence

Following the previous section where the loss function depends on  $\alpha$  that impacts the risk allocation with respect to the degree of dependence between risk factors, we apply the techniques of Theorem 4.4.1 to study the sensitivity with respect to  $\alpha$ . To this end we consider a loss function of the following form

$$\ell(\mathbf{x}) = \sum_k g(x_k) + \alpha h(\mathbf{x}),$$

where  $g$  is a one dimensional loss function and  $h$  a multidimensional function such that  $\ell$  is a loss function for all  $\alpha \geq 0$  close enough to 0.<sup>9</sup> For instance a loss function of the class **(C3)**. We also suppose that  $g$  is twice differentiable. Using the same strategy as in the proof of the Theorem 4.4.1, we can provide the marginal risk contribution and allocation as a function of  $\alpha$  around 0, stressing the dependence part of the loss function. Computations yield

$$\partial_\alpha R(\mathbf{X}) = \lambda \mathbb{E}[h(\mathbf{X} - \mathbf{m})] \quad \text{and} \quad \partial_\alpha \begin{bmatrix} R(\mathbf{X}) \\ \lambda' \end{bmatrix} = M^{-1} \begin{bmatrix} \lambda \mathbb{E}[\nabla h(\mathbf{X} - \mathbf{m})] \\ \partial_\alpha R(\mathbf{X}) \end{bmatrix}$$

where  $M$  is given by  $M = \begin{bmatrix} \lambda A & -B \\ C & 0 \end{bmatrix}$  and  $A = \text{diag}(g''(X_k - m_k))$  and  $B$  and  $C$  as in the proof of Proposition 4.4.1. In the case where

$$\ell(\mathbf{x}) = \frac{1}{2} \sum_{k=1}^3 (x_k^+)^2 + \alpha \sum_{1 \leq k < j \leq 3} x_k^+ x_j^+ - 1$$

and  $\mathbf{X} = (X_1, X_2, X_3)$  with  $X_1 \sim X_2 \sim X_3$ ,  $(X_1, X_2) \sim (X_2, X_1)$  and  $X_3$  independent of  $(X_1, X_2)$ , it follows that  $\mathbf{m} = RA_k(\mathbf{X})$  for every  $k = 1, 2, 3$ . Defining  $Z = (X_1 - m)^+ \sim (X_2 - m)^+ \sim (X_3 - m)^+$ , computations yields

$$\partial_\alpha R(\mathbf{X}) = \mathbb{E}[Z] \left( 2 + \frac{\mathbb{E}[(X_1 - m)^+(X_2 - m)^+]}{\mathbb{E}[Z]^2} \right)$$

Hence, with increasing correlation between  $X_1$  and  $X_2$  the marginal risk increases. As for the impact on the risk allocation, since  $\mathbb{E}[(X_1 - m)^+ | X_2 \geq m] = \mathbb{E}[(X_2 - m)^+ | X_1 \geq m]$  it simplifies to

$$\begin{aligned} \partial_\alpha RA_{1 \text{ or } 2}(\mathbf{X}) &= \frac{\mathbb{E}[Z]}{3} \left( 1 + \frac{\mathbb{E}[(X_2 - m)^+ | X_1 \geq m]}{\mathbb{E}[Z]} + \frac{\mathbb{E}[(X_1 - m)^+(X_2 - m)^+]}{\mathbb{E}[Z]^2} \right) \\ \partial_\alpha RA_3(\mathbf{X}) &= \frac{\mathbb{E}[Z]}{3} \left( 4 - 2 \frac{\mathbb{E}[(X_2 - m)^+ | X_1 \geq m]}{\mathbb{E}[Z]} + \frac{\mathbb{E}[(X_1 - m)^+(X_2 - m)^+]}{\mathbb{E}[Z]^2} \right) \end{aligned}$$

Due to the asymmetric dependence of the system :

- One the one hand, if  $X_1$  and  $X_2$  are highly anti-correlated, then

$$\partial_\alpha RA_{1 \text{ or } 2}(\mathbf{X}) \approx \frac{\mathbb{E}[Z]}{3} \quad \text{and} \quad \partial_\alpha RA_3(\mathbf{X}) \approx 4 \frac{\mathbb{E}[Z]}{3}$$

The systemic risk factor is advantaging those who are anti-correlated, with respect to the others.

- On the other hand, if  $X_1$  and  $X_2$  are highly correlated, then for  $p = \mathbb{P}[X_1 \geq m]$ ,

$$\partial_\alpha RA_{1 \text{ or } 2}(\mathbf{X}) \approx \frac{\mathbb{E}[Z]}{3} \left( \frac{p+1}{p} + \frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \right)$$

while

$$\partial_\alpha RA_3(\mathbf{X}) \approx \frac{\mathbb{E}[Z]}{3} \left( 2 \frac{p-1}{p} + \frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \right)$$

Since  $p \leq 1$ , the systemic risk factor penalizes those who are highly correlated and reduces the costs for the one who is independent with respect to the previous case.

Figure 4.3 illustrate this fact for different correlation values in the case of a 3-variate normal distribution

$$\mathbf{X} \sim \mathcal{N} \left( 0, \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

9. For instance when  $h$  is positive.

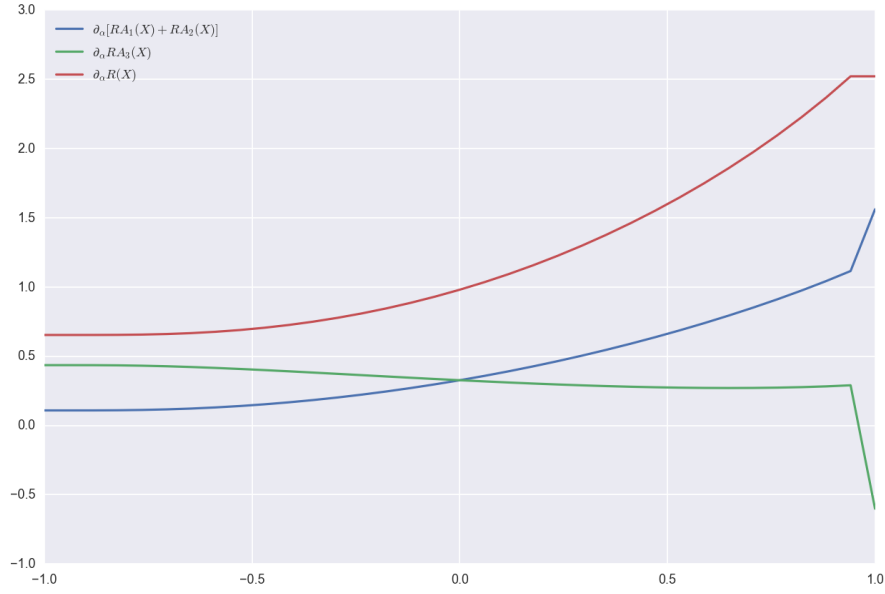


FIGURE 4.3 – Systemic factor marginal change of the risk allocation and total risk for different correlations  $\rho$ .

#### 4.4.3 Riskless Allocation, Causal Responsibility and Additivity

We conclude this section regarding risk allocation and its sensitivity by a discussion of their properties in light of the following economic features of risk allocations introduced in Brunnermeier and Cheridito (2014).

**(FA) Full Allocation :**  $\sum_k RA_k(\mathbf{X}) = R(\mathbf{X})$ ;

**(RA) Riskless Allocation :**  $RA_k(\mathbf{X}) = X_k$  if  $X_k$  is deterministic;

**(CR) Causal Responsibility :**  $R(\mathbf{X} + \Delta X_k) - R(\mathbf{X}) = RA_k(\mathbf{X} + \Delta X_k) - RA_k(\mathbf{X})$ , where  $\Delta X_k$  is a loss increment of the  $k$ -th risk component;

As mentioned before, per design, shortfall risk allocations always satisfy the full allocation property **(FA)**. As visible from the above case studies, riskless allocation **(RA)** and causal responsibility **(CR)** are not satisfied in general. In fact, from a systemic risk point of view, we think that **(RA)** and **(CR)** are not desirable properties. Indeed, both imply that risk taking, or non-taking, should only impact the concerned risk component. However, the risk components are interdependent and any move in one of them bears consequences to the rest of the system. The search for an optimal allocation is a non-cooperative game between the different system components, each of them respectively looking for its own minimal risk allocation while impacting the others by doing so. In other words, everyone is responsible for its own risk but also for its relative exposure with respect to the others. The sensitivity analysis of this section however shows that external shocks are primarily born by the risk component that is hit at least in a first order. In the case where this shock is independent of the system, by Proposition 4.4.1 it is then a full causal responsibility. Otherwise, a correction appears and a fraction of the shock is offloaded to the other risk components according to their relative exposure to the concerned component and dependence with the shock.

## 4.5 Computational Aspects of Risk Allocation

In this section we present computational results based on the loss function<sup>10</sup>

$$\ell(\mathbf{x}) = \sum_{k=1}^d x_k + \frac{1}{2} \sum_{k=1}^d (x_k^+)^2 + \alpha \sum_{1 \leq j < k \leq d} x_j^+ x_k^+ - 1, \quad (4.5.1)$$

for  $\alpha = 0$  or  $1$ . In that case, the constrained problem (4.2.2) becomes :

$$R(\mathbf{X}) := \inf \left\{ \sum_k m_k : \sum_{k=1}^d \mathbb{E}[X_k - m_k] + \frac{1}{2} \sum_{k=1}^d \mathbb{E}[(X_k - m_k)^+]^2 + \alpha \sum_{1 \leq j < k \leq d} \mathbb{E}[(X_j - m_j)^+ (X_k - m_k)^+] \leq 1 \right\} \quad (4.5.2)$$

According to Theorem 4.3.1, the risk allocation is determined by the first order conditions (4.3.1), which read in this case :

$$\begin{cases} \lambda \mathbb{E} \left[ (X_k - m_k)^+ + \alpha \sum_{j=1, j \neq k}^d (X_j - m_j)^+ \mathbb{1}_{\{X_k \geq m_k\}} \right] = 1 - \lambda, & \text{for } k = 1, \dots, d; \\ \mathbb{E} \left[ \sum_{k=1}^d (X_k - m_k) + \frac{1}{2} ((X_k - m_k)^+)^2 + \alpha \sum_{1 \leq j < k \leq d} (X_k - m_k)^+ (X_j - m_j)^+ \right] = 1 \end{cases} \quad (4.5.3)$$

We use Gaussian distributions with mean vector  $\mu$  and variance-covariance matrix  $\Sigma$  for the loss vector  $\mathbf{X}$ . In the bi- and tri-variate cases the variance-covariance matrix is parameterized by a single correlation factor  $\rho$  and the variances  $\sigma_k^2$  of  $X_k$  for all  $k$ . In other words,  $\Sigma_{ij} = \rho \sigma_i \sigma_j$  for  $i \neq j$ . We write CT for computational time. The implementation was done on standard desktop computers in the Python programming language. To solve the constrained problem (4.2.2), we use the root finding scheme Sequential Least Squares Programming (SLSQP) algorithm, in combination with Monte Carlo, Fourier or Chebychev interpolation schemes, briefly described below, for the computation of the expectations in (4.5.3).

**Fourier methods** Assuming that the moment generating functions of the considered distributions are available, Fourier methods allow us to compute the different expectations in (4.5.3), based on methods presented, among others, in Eberlein, Glau, and Papapantoleon (2010) and Drapeau, Kupper, and Papapantoleon (2014). The main advantage of this method is that it is theoretically possible to compute the value of the integrals at any level of precision, while the basic computational time is roughly doubled for every additional digit of accuracy. However, as seen in the subsequent computations this method suffers from the large number of double integrals to be computed, for which the computational time can become prohibitively long.

**Monte Carlo Methods** We can also use Monte Carlo simulations for the estimation of the many integrals in (4.5.3). An important observation here is that we can generate and store all realizations in advance, and then use them for the estimation of the functions for different  $\mathbf{m}$  in every step of the root-finding procedure. The main advantage of Monte Carlo relative to Fourier methods is that a wider variety of models can be considered ; think, for example, of models with copulas or of random variables with Pareto type distributions as considered in the empirical study in Section 4.6. The main disadvantage is the slow statistical convergence of the scheme, yet, in our context, it is fast enough. In addition, the time to generate the samples once and for all, independently of the value of  $\mathbf{m}$ , as well as to compute the Monte Carlo averages is very fast.

10. A direct check shows that this function satisfies (A1), (A2) and (A3).

**Chebyshev interpolation** A numerical scheme well-suited to approximate the large numbers of functions in the context of optimization routines is the Chebyshev interpolation method. This method, recently applied to option pricing by [Gaß, Glau, Mahlstedt, and Mair \(2015\)](#), can be summarized as follows : Suppose you want to evaluate quickly a function  $F(\mathbf{m})$ , of one or several variables, for a large number of  $\mathbf{m}$ 's. The first step of the Chebyshev method is to evaluate the function  $F(\mathbf{m})$  on a given set of nodes  $\mathbf{m}^i$ ,  $1 \leq i \leq N$ . These evaluations can be computed by Fourier or Monte Carlo schemes, are independent of each other and can thus be realized in parallel. The next step, in order to compute  $F(\mathbf{m})$  for an  $\mathbf{m}$  outside the nodes  $\mathbf{m}^i$ , is to perform a polynomial interpolation of the  $F(\mathbf{m}^i)$ 's using the Chebyshev coefficients. In other words, the Chebyshev method provides a polynomial approximation  $\hat{F}(\mathbf{m})$  of  $F(\mathbf{m})$ .

**Discussion :** Whether it is advantageous to use the Chebyshev interpolation or not, is a matter of two competing factors that affect the computational time : On the one hand, the number of iterations  $I(d)$  needed to find the root of the system and, on the other hand, the size of the grid  $N^2$  used in the Chebyshev interpolation. Our findings reveals that the Monte Carlo schemes are better than the Fourier schemes in the range of our accuracy requirements, since they require the least amount of work during each step of the root-finding procedure or for the pre-processing computations in the Chebyshev method. Only when the dimension is low, less than three or  $\alpha = 0$  can the Fourier methods be faster. Next, the choice between Chebyshev or not is a matter of comparison between  $I(d)$  and  $N^2$ . In high dimensions, when  $I(d)$  dominates  $N^2$ , with  $I(d)$  being in principle of order  $d$  and  $N$  usually between 10 and 20, then the Chebyshev method is less costly. Furthermore, the Chebyshev method can intensively benefit from parallel computing as the pre-processing step is not sequential.

**Remark 4.5.1** The numerical methods outlined above can be further improved by considering variance reduction techniques for the Monte Carlo simulations. Sparse grids and analogous numerical techniques can be developed to reduce the computational work for the Fourier and Chebyshev schemes. Another avenue to be explored is the application of stochastic approximation schemes, instead of deterministic root-finding methods, for the computation of multivariate risk measures. In the one dimensional case, a stochastic gradient algorithm has been proposed for the computation of shortfall risk measure by [Dunkel and Weber \(2010\)](#) or [Hu and Dali \(2016\)](#). With respect to deterministic optimization or root finding schemes, stochastic gradient algorithm present the advantage of being incremental, less sensitive to the dimension, and offer a flexible framework that can be conveniently combined with other features such as importance sampling (see [Glasserman \(2013\)](#), [Asmussen and Glynn \(2007\)](#) and [Dunkel and Weber \(2010\)](#)), model uncertainty, or the quest of, not only the risk measure itself, but also its sensitivities to model parameters. This is all left for future research.

#### 4.5.1 Bivariate case

We suppose that  $d = 2$  and consider a bivariate Gaussian distribution with zero mean,  $\sigma_1 = \sigma_2 = 1$  and correlation

$$\rho \in \{-0.9, -0.5, -0.2, 0, 0.2, 0.5, 0.9\}$$

When setting  $\alpha = 0$ , that is without systemic risk weight, the result  $\mathbf{m}^*$  does not depend on the correlation value. Since  $\sigma_1 = \sigma_2 = 1$  the allocation is symmetric and we find  $m_1^* \approx -0.173$ . Explicit formulas for the involved expectations are available in this case and this yields of course the fastest computation. Fourier methods are quite fast (CT  $\approx 3 \times$  explicit formula) as we only need to compute 1-dimensional integrals. In order to get a high approximation in the Chebyshev approximation, one must use 20 nodes for each integral. Since the number of iterations in the optimizations is low, the Chebyshev method coupled with Fourier transforms is slower than Fourier without it. Finally, Monte Carlo is about 20 to 40 times slower than Fourier, becoming the

slowest method in that case. When setting  $\alpha = 1$ , the values of the risk allocation are increasing with respect to  $\rho$ , as expected, see Table 4.1. The Monte Carlo method becomes the fastest one. Indeed, we now need to compute bi-variate integrals in (4.5.3). Even if Fourier methods are fast, from 30 seconds to almost 3 minutes, they are still approximately 10 to 50 slower than Monte Carlo. Moreover, using even as little as 10 nodes in the Chebychev interpolation, which is not very accurate, increases the total computational time because of the number of 2-dimensional integrals to compute in the preprocessing step.

$\rho$	Fourier		Fourier + Chebychev 10 nodes		Monte Carlo 2 Mio	
	$m_1^*$	CT	$m_1^*$	CT	$m_1^*$	CT
-0.9	-0.167	61520 ms	-0.150	45 m 18 s	-0.167	3257 ms
-0.5	-0.143	37100 ms	-0.132	30 m 27 s	-0.143	3357 ms
-0.2	-0.120	45200 ms	-0.113	25 m 21 s	-0.120	3414 ms
0	-0.103	51800 ms	-0.098	24 m 52 s	-0.103	3302 ms
0.2	-0.085	75700 ms	-0.082	27 m 55 s	-0.085	3417 ms
0.5	-0.057	158000 ms	-0.055	32 m 10 s	-0.056	3250 ms
0.9	-0.013	88900 ms	-0.012	55 m 04 s	-0.012	3387 ms

TABLE 4.1 – Bivariate case with systemic weight, that is, for  $\alpha = 1$ .

#### 4.5.2 Trivariate Case

In this section, we illustrate the systemic contribution of the loss function with three risk components and study the impact of the interdependence of two components with respect to the third one. We start with a Gaussian vector with the variance-covariance matrix

$$\Sigma = \begin{bmatrix} 0.5 & 0.5\rho & 0 \\ 0.5\rho & 0.5 & 0 \\ 0 & 0 & 0.6 \end{bmatrix},$$

for different correlations  $\rho \in \{-0.9, -0.5, -0.2, 0, 0.2, 0.5, 0.9\}$ . Here the third risk component has a higher marginal risk than the first two so that, in the absence of systemic weight, it should contribute most to the overall risk. When  $\alpha = 0$ , this is indeed the case. The result is independent of the correlation and is typically overall lower, charging the risk component with the highest variance more –  $m_3^* \approx -0.12$  – than the other two –  $m_1^* = m_2^* \approx -0.166$ . However, with systemic risk weight, the contribution of the first two overcomes the third one for high correlation, as emphasised in red in Table 4.2. These results illustrate that the systemic risk weights correct the risk allocation as the correlation between the first two risk components increases. The Monte Carlo scheme in this trivariate case is radically faster than Fourier – Chebychev interpolation was not found useful in this case either – from 30 times up to 60 times more efficient.

#### 4.5.3 Higher Dimensions

Figure 4.4 shows the variance-covariance matrix and the resulting risk allocation in a 30-variate case using Monte Carlo, coupled with 15 node Chebychev interpolation when  $\alpha = 1$ . Indeed, the dimension being large, the preprocessing time with Monte Carlo to compute the Chebychev coefficients together with the computational time resulting from the root-finding for the resulting interpolation function is lower than the raw Monte Carlo root finding. The plot shows that the risk allocation depends not only on the variance of the different risk components, but also, in the case where  $\alpha = 1$ , on the corresponding dependence structure. For instance,

$\rho$	Fourier Method					Monte Carlo 2 Mio				
	$m_1^* = m_2^*$		$m_3^*$	$R(X)$	TCP	$m_1^* = m_2^*$		$m_3^*$	$R(X)$	TCP
-0.9	-0.189	$\leq$	0.096	-0.258	2 m 55 s	-0.190	$\leq$	0.095	-0.283	3159 ms
-0.5	-0.135	$\leq$	0.016	-0.253	1 m 39 s	-0.134	$\leq$	0.017	-0.252	2799 ms
-0.2	-0.099	$\leq$	-0.030	-0.229	1 m 32 s	-0.098	$\leq$	-0.030	-0.228	2760 ms
0	-0.076	$\leq$	-0.059	-0.212	2 m 22 s	-0.077	$\leq$	-0.058	-0.212	3188 ms
0.2	-0.053	$\leq$	-0.086	-0.194	1 m 37 s	-0.055	$\leq$	-0.086	-0.195	2741 ms
0.5	-0.020	$\geq$	-0.125	-0.165	1 m 47 s	-0.020	$\geq$	-0.124	-0.164	3358 ms
0.9	0.025	$\geq$	-0.173	-0.121	2 m 07 s	0.026	$\geq$	-0.171	-0.119	2722 ms

TABLE 4.2 – Trivariate case with systemic weight, that is  $\alpha = 1$ . Computed by Fourier.

compare components 28 and 29 in the 30-variate case in Figure 4.4. In the first case we observe that when  $\alpha = 0$ , component 28 contributes more than 29, and conversely when  $\alpha = 1$ . The reason is that even if component 28 has a slightly higher variance, it is relatively less correlated than 29 to the components 2, 3, 6, 20 and 30 that have the highest variance, and thus are the most ‘dangerous’ from the systemic point of view. Hence, component 29 is more exposed than 28 in case of a systemic event.

## 4.6 Empirical Study : Default Fund Allocation

In the sequel we consider loss functions of the type

$$\ell_1(\mathbf{x}) = \sum_k x_k^+ - \frac{1}{2} \sum_k x_k^- \quad (4.6.1)$$

$$\ell_2(\mathbf{x}) = \sum_k x_k^+ - \frac{1}{2} \sum_k x_k^- + \sum_{k \neq j} (x_k + x_j)^+ - \frac{1}{2} \sum_{k \neq j} (x_k + x_j)^- \quad (4.6.2)$$

studied in Example 4.2.4. The first loss function means that a position is acceptable if on average, the losses are compensated by gains twice as large.<sup>11</sup> In this case, the risk assessment of the losses is marginal or component-wise. The second one is similar, however, it also aggregates pairwise losses and gains among the different components. Here the risk assessment considers additionally the pairwise dependence between the losses. Note that each of these loss function is positive homogeneous (hence so is  $R$ ) and permutation invariant.

The default fund of a CCP is a protection against extreme and systemic risk. As of today, it is sized according to the Cover 2 rule, see [European Parliament \(2012b\)](#), article 42, §3, p. 37). In a rough way, this corresponds to the maximal joint loss of two members over their posted collateral (initial margin) in a stressed situation over the last 60 days. The relative contribution of each member to the default fund is proportional to their respective initial margin – that is, the value at risk at a given level of confidence of their loss and profit over a three-day time horizon. Hence, denoting by DF the total size of the default fund and by  $\text{IM}_k(X_k)$  the initial margin of member  $k$ , the contribution of member  $k$  is given by

$$\frac{\text{IM}_k(X_k)}{\sum_j \text{IM}_j(X_j)} \text{DF} \quad (4.6.3)$$

As an alternative, we propose to define the contribution of member  $k$  to the default fund as follows. According to Theorem 4.3.1 there exists a unique optimal capital allocation  $RA(\mathbf{X})$

11. The coefficient  $1/2$  is naturally subject to consensus and can be taken as any real number between 0 and 1.

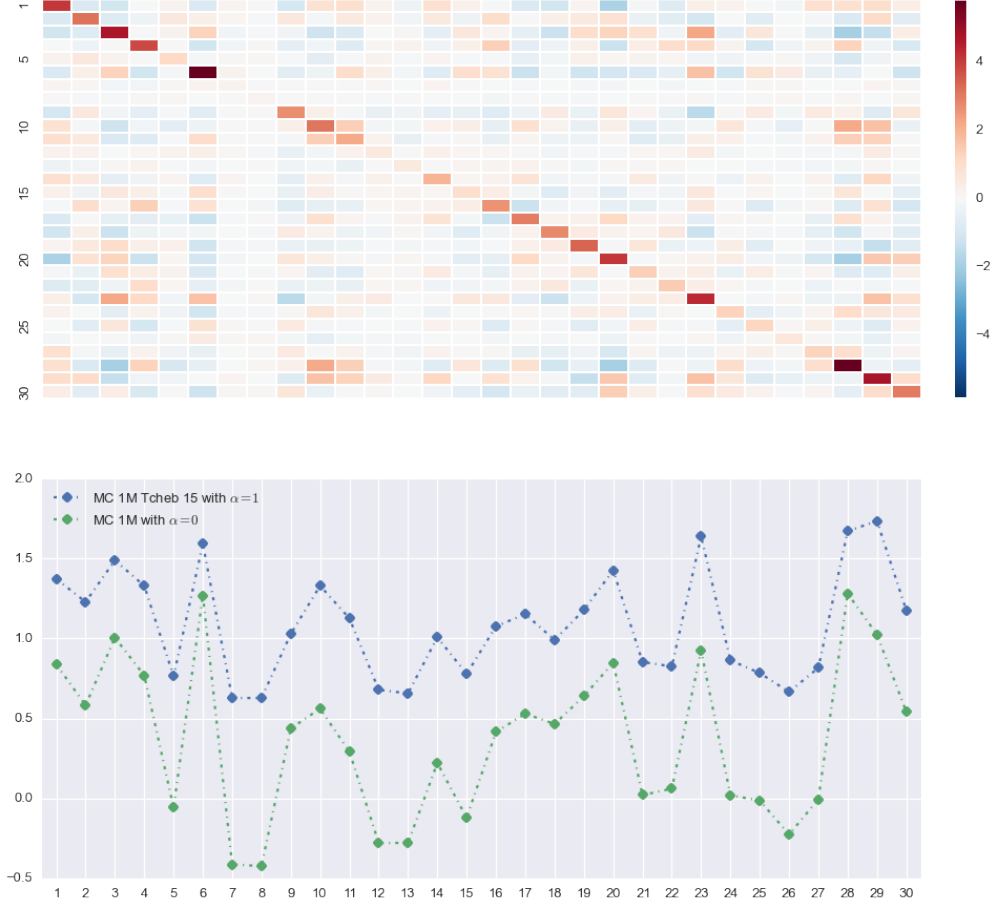


FIGURE 4.4 – Plot showing the variance-covariance matrix together with the respective allocation in the 30-variate case for  $\alpha = 0, 1$ .

for a given loss vector  $\mathbf{X}$ . We define therefore the relative risk contribution of each financial component as

$$RC_k := RC_k(\mathbf{X}) = \frac{RA_k(\mathbf{X})}{\sum_j RA_j(\mathbf{X})} = \frac{RA_k(\mathbf{X})}{R(\mathbf{X})} \quad (4.6.4)$$

The value at risk for the initial margins  $IM_k$ , the overall risk measure  $R$  as well as the optimal capital allocation are all positive homogeneous. It follows that  $RC_k(\lambda \mathbf{X}) = RC_k(\mathbf{X})$  for every  $\lambda > 0$ , that is, the relative risk contribution is scaling invariant as for instance the Sharpe ratio, Minmax ratio or Gini ratio among others, see [Cheridito and Kromer \(2013\)](#). The scaling invariance property allows one to consider the allocation independently of the total size of the default fund. The contribution of member  $k$  is then given as

$$RC_k \times DF \quad (4.6.5)$$

The current practice based on the ratio of initial margins (4.6.3) provides an allocation that only depends on the marginal risk of each member profit and loss  $X_k$ , and does not take their joint dependence into account, that is, the systemic risk component. By contrast, the approach (4.6.5) allows one to take this systemic risk component into account in the allocation of the default fund in the sense of the following proposition already discussed in Section 4.4.

### 4.6.1 Data

In this section we compare a standard IM based allocation of the default fund of a CCP with the multivariate shortfall risk allocation resulting from the use of the loss functions  $\ell_1$  and  $\ell_2$ . This empirical study is based on an LCH real dataset corresponding to the clearing of 74 portfolios of equity derivatives bearing on 90 underlyings. The clearing members have been anonymized and are referenced in the sequel by labels starting by PB plus number (e.g. PB7), whereas the underlying assets are identified by their real tickers, such as FCE for CAC40 index future and AEX for Amsterdam exchange index, which can all be retrieved online. The Jupyter notebook corresponding to this empirical study, including all the data and numerical codes, is publically available at <https://github.com/yarmentti/MSRA>. In order to avoid the repricing of the options, all the derivative positions have been linearized and reformulated in equivalent Delta positions in their underlyings. We denote by  $P$  the  $74 \times 90$  matrix of the positions of the 74 clearing members in the 90 underlyings. As the CCP clears, each column of  $P$  sums up to zero. The vector of the clearing member losses at a three day (3d) horizon is given by

$$\mathbf{X} = -P \times (\mathbf{S}_{3d} - \mathbf{S}_0) \quad (4.6.6)$$

where  $S$  is the vector of the underlying price processes. The vector  $S_0$  is observed and the vector  $S_{3d}$  is simulated in a Student's t model estimated by maximum-likelihood on the underlying return time series, i.e.

$$S_{3d}^i - S_0^i \sim \kappa_i \times T_i^{\nu_i} \times S_0^i, \quad (4.6.7)$$

where  $T_i^{\nu_i}$  is a Student's t random variable with  $\nu_i$  degrees of freedom and where  $\kappa_i$  a calibration fudge coefficient. The dependence between the underlyings is modelled by a Student's t copula with correlation matrix  $\rho$  and  $\nu$  degrees of freedom, that is

$$C_{\rho, \nu}(u_1, \dots, u_n) = F_{\rho}^{\nu} \left( F_{\nu}^{-1}(u_1), \dots, F_{\nu}^{-1}(u_n) \right)$$

Here  $F_{\rho}^{\nu}$  is the cumulative distribution function of the multivariate Student's t distribution with correlation matrix  $\rho$  and  $\nu$  degrees of freedom, and  $F_{\nu}$  is the Student's t cdf with  $\nu$  degrees of freedom.

### 4.6.2 Simulations

The correlation matrix  $\rho$  is estimated empirically on the return time series and the dependence copula parameter is set to  $\nu = 6$ . Each of  $m = 10^5$  realizations of  $\mathbf{S}_{3d}$ , hence of the loss vector  $\mathbf{X}$ , is simulated as follows :

1. Simulate a Gaussian random vector  $\mathbf{G}$  of size 90 with zero-mean and correlation  $\rho$
2. Generate a  $\chi_2$  random variable  $\xi$  with parameter  $\nu$
3. Obtain the Student's t vector  $\mathbf{R} = \sqrt{\nu \xi^{-1}} \mathbf{G}$
4. Transform  $\mathbf{R}$  into uniform coordinates by  $U_i = F_{\nu}(R_i)$  and compute  $T_i^{\nu_i} = F_{\nu_i}^{-1}(U_i)$
5. Compute  $\mathbf{S}_{3d}$  by (4.6.7) and  $\mathbf{X}$  by (4.6.6)

The resulting inputs to the allocation optimization problem are analysed in Appendix 4.7.3. Figure 4.5 shows the correlation matrices of the underlying assets and of the loss vector  $\mathbf{X}$  of the clearing members, in a heatmap representation. In the left panel, which is directly estimated from the data, we see that the underlying assets are all positively correlated, as commonly found in the case of equity derivatives. However, due to positions in opposite directions taken by the clearing members, some of their losses exhibit significant negative correlations, as shown by the blue cells in the right panel.

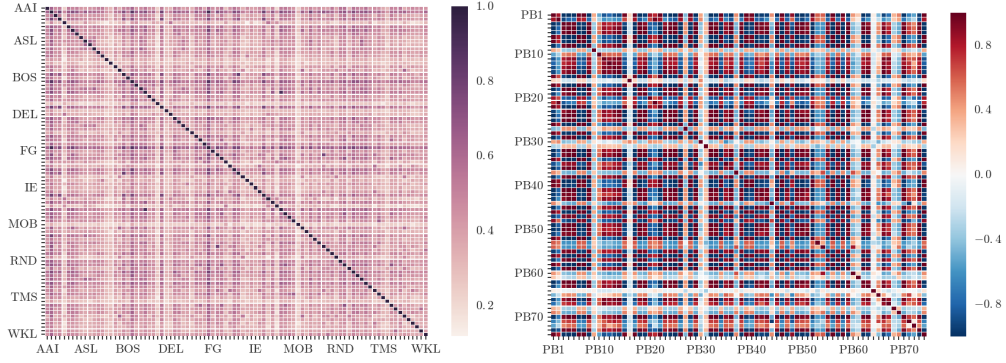


FIGURE 4.5 – Left : Correlation matrix of the underlying assets (ranked by alphabetical order of asset ticker ; one ticker out of ten is displayed along the coordinate axes). Right : Correlation matrix of the loss vector  $\mathbf{X}$  of the clearing members (ranked by alphabetical order of member label ; one label out of ten is displayed along the coordinate axes).

### 4.6.3 Allocation Results

The total size of the default fund as of a standard Cover 2 methodology are shown in Table 4.3, for three values of the dependence copula parameter  $\nu$  and for 99% vs. 99.7% initial margins (IM). Since a Cover 2 default fund is a cushion over IM, its size is directly responsive to the level of the quantile which is used for setting the IM (compare the two lines in Table 4.3). In relative terms the size of the default fund is quite stable with respect to  $\nu$ . However we emphasize that these are monetary amounts, so that the difference between for instance  $6.16 \cdot 10^8$  and  $6.72 \cdot 10^8$  corresponds to  $0.56 \cdot 10^8$ , i.e. more than half a billion of the corresponding currency.

	$\nu = 2$	$\nu = 6$	$\nu = 50$
99 % IM	$6.16 \cdot 10^8$	$6.72 \cdot 10^8$	$6.27 \cdot 10^8$
99.7 % IM	$4.96 \cdot 10^8$	$5.48 \cdot 10^8$	$5.00 \cdot 10^8$

TABLE 4.3 – Size of a Cover 2 default fund for different levels of initial margins and different values of the dependence copula parameter  $\nu$ .

In the sequel we set  $\nu = 6$ , which corresponds to an intermediate level of tail dependence, and we use 99% IM, which corresponds to the EMIR regulatory floor on initial margins.

Figure 4.6 compares the allocation weights implied by the loss function  $\ell_1$  with the ones implied by 99% IM. The allocations are very similar, as confirmed by the examination of the percentage relative differences displayed in the upper panels of Figure 4.6. By contrast, the lower panels of Figure 4.7 show that the allocation weights implied by the loss function  $\ell_1$  and the dependence sensitive loss function  $\ell_2$  differ significantly in relative terms, including for the names with the greatest contributions to the default fund. These results illustrate the impact of the use of a “systemic” loss function on the allocation of the default fund.

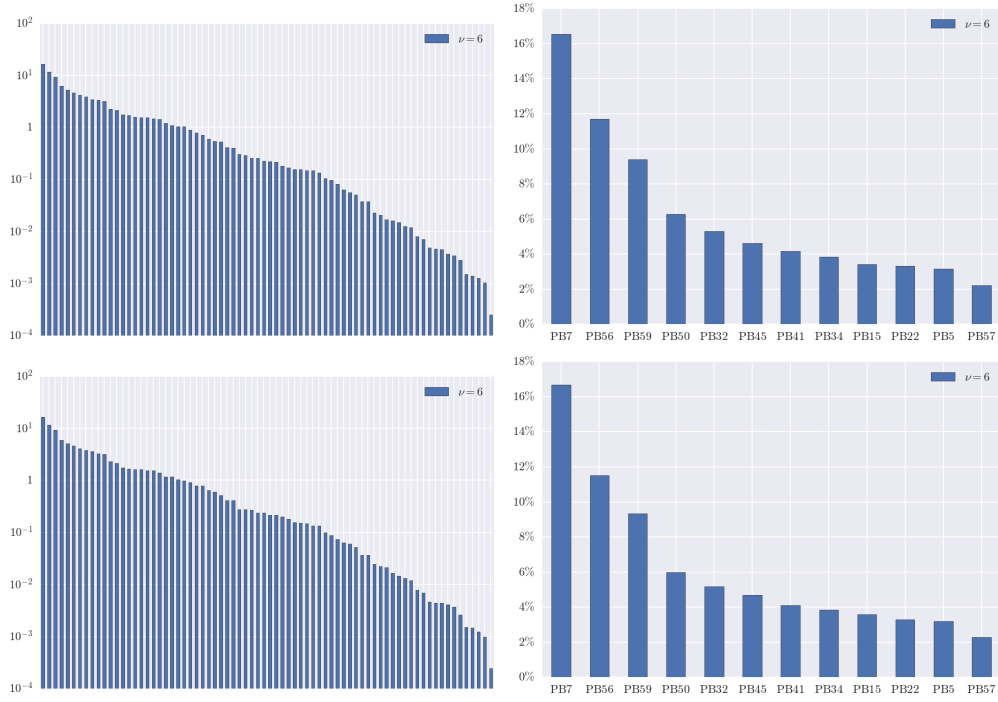


FIGURE 4.6 – Left : Decreasing log-allocation weights implied by the loss function  $\ell_1$  (top) and 99% IM (bottom). Right : Twelve highest allocation weights implied by the loss function  $\ell_1$  (top) and by 99% IM (bottom), with the corresponding member labels.

## 4.7 Appendix

### 4.7.1 Some Classical Facts in Convex Optimization

For an extended real valued function  $f$  on a locally convex topological vector space  $X$ , its convex conjugate is defined as

$$f^*(x^*) = \sup_{x \in X} \{x^* \cdot x - f(x)\}, \quad x^* \in X^*$$

where  $X^*$  is the topological dual of  $X$ . The Fenchel–Moreau theorem states that if  $f$  is lower semi-continuous, convex and proper, then so is  $f^*$ , and it holds

$$f(x) = f^{**}(x) = \sup_{x^* \in X^*} \{x^* \cdot x - f^*(x^*)\}, \quad x \in X$$

Following Rockafellar (1970), for any non-empty set  $C \subseteq \mathbb{R}^d$ , we define its *recession cone*

$$0^+C := \{y \in \mathbb{R}^d : x + \lambda y \in C \text{ for every } x \in C \text{ and } \lambda \in \mathbb{R}_+\}$$

By Rockafellar (1970, Theorem 8.3), if  $C$  is non-empty, closed and convex, then

$$0^+C = \{y \in \mathbb{R}^d : \text{there exists } x \in C \text{ such that } x + \lambda y \in C \text{ for every } \lambda \in \mathbb{R}_+\} \quad (4.7.1)$$

By Rockafellar (1970, Theorem 8.4), a non-empty, closed and convex set  $C$  is compact if and only if  $0^+C = \{0\}$ .

Given a proper, convex and lower semi-continuous function  $f$  on  $\mathbb{R}^d$ , we call  $y \in \mathbb{R}^d$  a *direction of recession* of  $f$  if there exists  $x \in \text{dom}(f)$  such that the map  $\lambda \mapsto f(x + \lambda y)$  is decreasing on

$\mathbb{R}_+$ . We denote by  $f0^+$  the *recession function* of  $f$ , that is, the function with epigraph given as the recession cone of the epigraph of  $f$ , and we call

$$0^+f := \{\mathbf{y} \in \mathbb{R}^d : (f0^+)(\mathbf{y}) \leq 0\}$$

the *recession cone* of  $f$ . The following theorem gathers results from Rockafellar (1970, Theorems 8.5, 8.6, 8.7 and Corollaries pp. 66–70).

**Theorem 4.7.1** Let  $f$  be a proper, closed and convex function on  $\mathbb{R}^d$ .

1. Given  $\mathbf{x}, \mathbf{y}$  in  $\mathbb{R}^d$ , if  $\liminf_{\lambda \rightarrow \infty} f(\mathbf{x} + \lambda \mathbf{y}) < \infty$ , then  $\lambda \mapsto f(\mathbf{x} + \lambda \mathbf{y})$  is decreasing.
2. All the non-empty level sets  $B := \{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \leq \gamma\} \neq \emptyset$  of  $f$  have the same recession cone, namely the recession cone of  $f$ . That is :

$$0^+f = 0^+B, \text{ for every } \gamma \in \mathbb{R} \text{ such that } B \neq \emptyset$$

3.  $f0^+$  is a positively homogeneous, proper, closed and convex function, such that

$$(f0^+)(\mathbf{y}) = \sup_{\lambda > 0} \frac{f(\mathbf{x} + \lambda \mathbf{y}) - f(\mathbf{x})}{\lambda} = \lim_{\lambda \rightarrow \infty} \frac{f(\mathbf{x} + \lambda \mathbf{y}) - f(\mathbf{x})}{\lambda}, \quad \mathbf{y} \in \mathbb{R}^d$$

for every  $\mathbf{x} \in \text{dom}(f)$ .

4. There exists  $\mathbf{x} \in \text{dom}(f)$  such that the map  $\lambda \mapsto f(\mathbf{x} + \lambda \mathbf{y})$  is decreasing on  $\mathbb{R}_+$ , that is,  $\mathbf{y}$  is a direction of recession of  $f$ , if and only if this map is decreasing for every  $\mathbf{x} \in \text{dom}(f)$ , which in turn is equivalent to  $(f0^+)(\mathbf{y}) \leq 0$ .
5. The map  $\lambda \mapsto f(\mathbf{x} + \lambda \mathbf{y})$  is constant on  $\mathbb{R}_+$  for every  $\mathbf{x} \in \text{dom}(f)$  if and only if  $(f0^+)(\mathbf{y}) \leq 0$  and  $(f0^+)(-\mathbf{y}) \leq 0$ .

## 4.7.2 Multivariate Orlicz Spaces

In this appendix we briefly sketch how the classical theory of univariate Orlicz spaces carries over to the  $d$ -variate case without any significant change. We follow the lecture notes by Léonard Léonard (2007), only providing the proofs that differ structurally from the univariate case.

A function  $\theta : \mathbb{R}^d \rightarrow [0, \infty]$  is called a Young function if it is

- convex and lower semi-continuous ;
- such that  $\theta(\mathbf{x}) = \theta(|\mathbf{x}|)$  and  $\theta(\mathbf{0}) = 0$  ;
- non trivial, that is,  $\text{dom}(\theta)$  contains a neighborhood of  $\mathbf{0}$  and  $\theta(\mathbf{x}) \geq a \|\mathbf{x}\| - b$  for some  $a > 0$ .

In particular,  $\theta$  achieves its minimum at  $\mathbf{0}$  and is increasing on  $\mathbb{R}_+^d$ . It is said to be finite if  $\text{dom}(\theta) = \mathbb{R}^d$  and strict if  $\lim_{\mathbf{x} \rightarrow \infty} \frac{\theta(\mathbf{x})}{\|\mathbf{x}\|} = \infty$ .

**Lemma 4.7.1** *The function  $\theta$  is Young if and only if  $\theta^*$  is Young. Furthermore,  $\theta$  is strict if and only if  $\theta^*$  is strict if and only if  $\theta$  and  $\theta^*$  are both finite.*

**Proof:** This follows by application of the Fenchel-Moreau theorem and from the relation  $x \cdot y \leq \theta(x) + \theta^*(y)$ . ■

For  $\mathbf{X} \in L^0$ , the Luxembourg norm of  $\mathbf{X}$  is given as

$$\|\mathbf{X}\|_\theta = \inf \left\{ \lambda \in \mathbb{R} : \lambda > 0 \text{ and } \mathbb{E} \left[ \theta \left( \frac{1}{\lambda} \mathbf{X} \right) \right] \leq 1 \right\},$$

where  $\inf \emptyset = \infty$ . The Orlicz space and heart are respectively defined as

$$L^\theta := \left\{ \mathbf{X} \in L^0 : \|\mathbf{X}\|_\theta < \infty \right\} = \left\{ \mathbf{X} \in L^0 : \mathbb{E} \left[ \theta \left( \frac{1}{\lambda} \mathbf{X} \right) \right] < \infty \text{ for some } \lambda \in \mathbb{R}, \lambda > 0 \right\}$$

$$M^\theta := \left\{ \mathbf{X} \in L^0 : \mathbb{E} \left[ \theta \left( \frac{1}{\lambda} \mathbf{X} \right) \right] < \infty \text{ for all } \lambda \in \mathbb{R}, \lambda > 0 \right\}$$

#### Lemma 4.7.2

1. We have  $\|\mathbf{X}\|_\theta = 0$  if and only if  $\mathbf{X} = \mathbf{0}$ .
2. If  $0 < \|\mathbf{X}\|_\theta < \infty$ , then  $\mathbb{E}[\theta(\frac{1}{\|\mathbf{X}\|_\theta} \mathbf{X})] \leq 1$ . In particular,  $B := \{\mathbf{X} : \|\mathbf{X}\|_\theta \leq 1\} = \{\mathbf{X} : \mathbb{E}[\theta(\mathbf{X})] \leq 1\}$ .
3. The gauge  $\|\cdot\|_\theta$  is a norm both on the Orlicz space  $L^\theta$  and on the Orlicz heart  $M^\theta$ .
4. The following Hölder Inequality holds :

$$\mathbb{E}[|\mathbf{X} \cdot \mathbf{Y}|] \leq \|\mathbf{X}\|_\theta \|\mathbf{Y}\|_{\theta^*}$$

5.  $L^\theta$  is continuously embedded into  $L^1$ , the space of integrable random variables on  $\Omega \times \{1, \dots, d\}$  for the product measure  $\mathbb{P} \otimes \text{Unif}_{\{1, \dots, d\}}$ .<sup>12</sup>
6. The normed spaces  $(L^\theta, \|\cdot\|_\theta)$  and  $(M^\theta, \|\cdot\|_\theta)$  are Banach spaces.

**Proof:** These results can be established along the same lines as in the univariate case, see [Léonard \(2007, Lemmas 1.8 and 1.10 and Propositions 1.11, 1.14, 1.15 and 1.18\)](#), using the Fenchel-Moreau Theorem in  $\mathbb{R}_+^d$ . ■

**Theorem 4.7.2** If  $\theta$  is finite, then the topological dual of  $M^\theta$  is  $L^{\theta^*}$ .

**Proof:** Again, the proof follows the univariate case, see [Léonard \(2007, Proposition 1.20, Theorem 2.2 and Lemmas 2.4 and 2.5\)](#). ■

#### 4.7.3 Data Analysis

Figure 4.8 shows the gross positions (sum of the absolute values of the positions in the underlying asset) per clearing member. Four members concentrate particularly high positions in the CCP. Figure 4.9 shows the gross positions of the CCP per underlying asset (top) and the corresponding underlying asset values (bottom). The largest investment by far of the clearing members is in the asset with ticker FCE (CAC40 index future, with spot value 4463), by a factor about three to the second one AEX (Amsterdam exchange index, with spot value 443.83). The investments of the clearing members in the other assets are comparatively much smaller.

Figure 4.10 shows the signed positions in the underlying assets of the twelve clearing members with the largest gross positions (left) and the signed positions of the clearing members in the nine most traded underlying assets (right), in a heatmap representation. In particular, we observe from the left panel that the biggest players in the CCP, namely the members labeled PB7, PB56, PB59 and PB50, have opposite sign positions in the main asset (the one with ticker FCE). The right panel shows that the dominant asset position in the CCP, i.e. the one in FCE, is shared (with opposite signs) between a significant number of clearing members.

12. The case where  $L^\theta = L^1$  corresponds to  $\theta(x) = \sum_k |x_k|$ .

Figure 4.11 shows the annualized volatilities  $\kappa_i \times \sqrt{\frac{\nu_i}{\nu_i-2}} \times \sqrt{\frac{250}{3}}$  of the underlying assets (cf. (4.6.7)). Most of these volatilities are comprised between 15% and 40%, with two assets, KBC and TMS, spiking over 60% volatility. However, the clearing members are only very marginally invested in these two assets (their tickers do not even appear in the right panel of Figure 4.9).

Figure 4.12 shows the monetary risks (3d volatilities  $\times$  absolute monetary positions) in the underlying assets of the ten clearing members with the largest gross positions. From the right panel we see that the FCE and AEX assets (CAC40 index future FCE and Amsterdam exchange index AEX, two major indices) concentrate most of the risk of the clearing members. The comparison with Figure 4.11 shows that this is not an effect of the volatility of these assets, but of very large monetary positions of the clearing members.

**Acknowledgments** This paper greatly benefited from regular exchanges with the quantitative research team of LCH in Paris : Quentin Archer, Julien Dosseur, Pierre Mouy and Mohamed Selmi. In particular we are grateful to Pierre Mouy for the preparation of the real dataset used for the empirical study of Section 4.6.

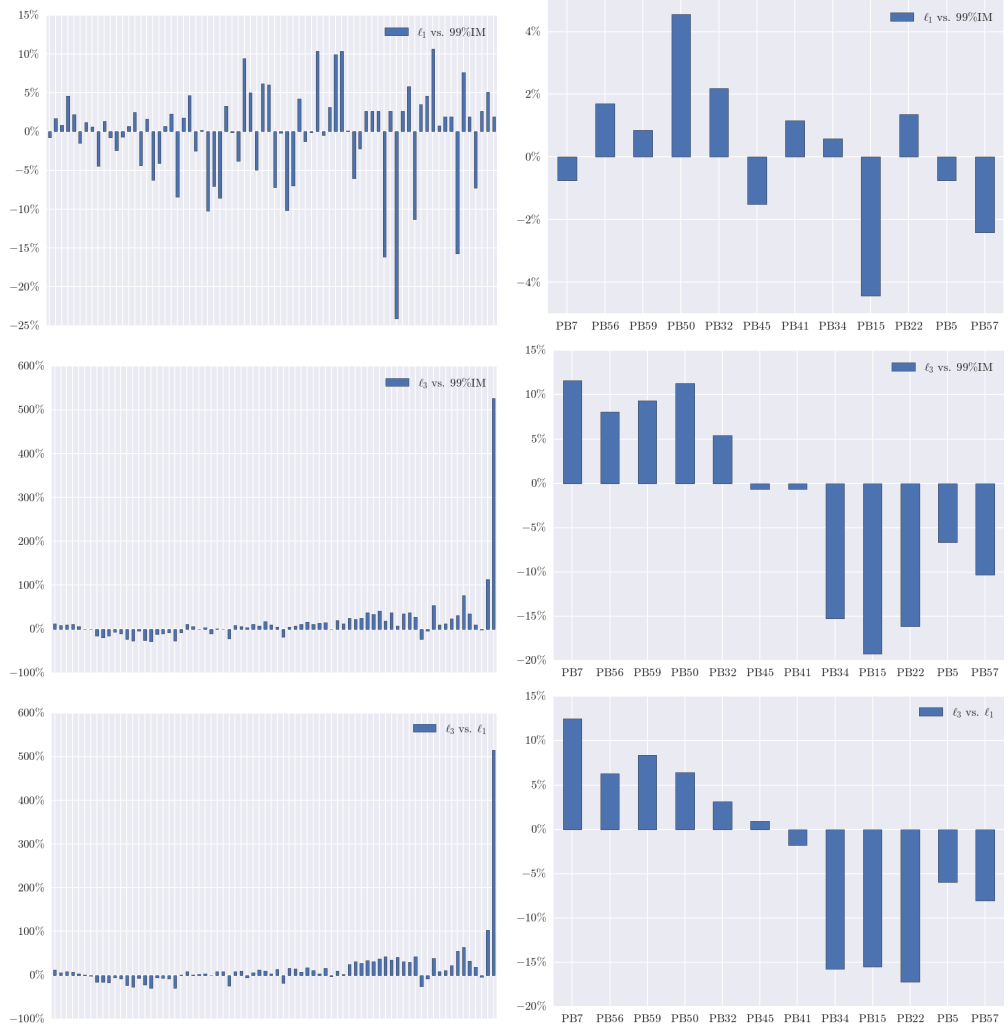


FIGURE 4.7 – Left : Percentage relative differences between the allocation weights implied by the loss function  $\ell_1$  and 99%IM (top), the loss function  $\ell_2$  and 99% IM (middle), and the loss functions  $\ell_1$  and  $\ell_2$  (bottom), ranked by decreasing values of the allocation weights implied by the loss function  $\ell_1$ . Right : Zoom on the left parts of the graphs, with member labels.

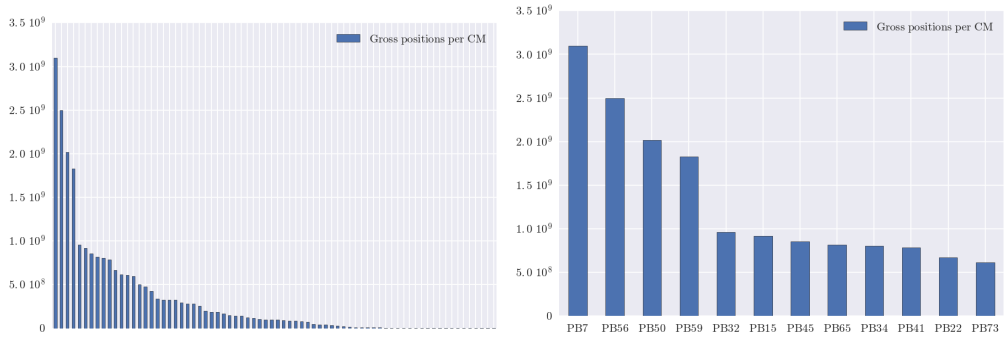


FIGURE 4.8 – Left : Gross positions per clearing member, ranked decreasing. Right : Zoom on the left part of the graph with member labels.

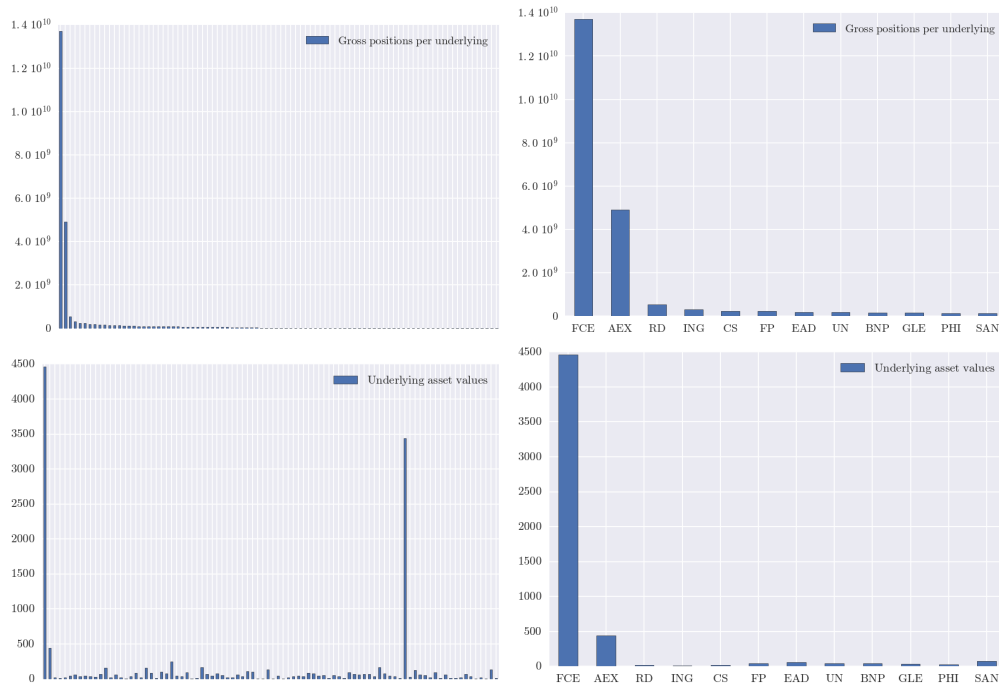


FIGURE 4.9 – Top : Gross positions per underlying, ranked decreasing (left) and zoom on the left part of the graph with tickers (right). Bottom : Spot values of the underlying assets, ranked as above (left) and zoom on the left part of the graph with tickers (right).

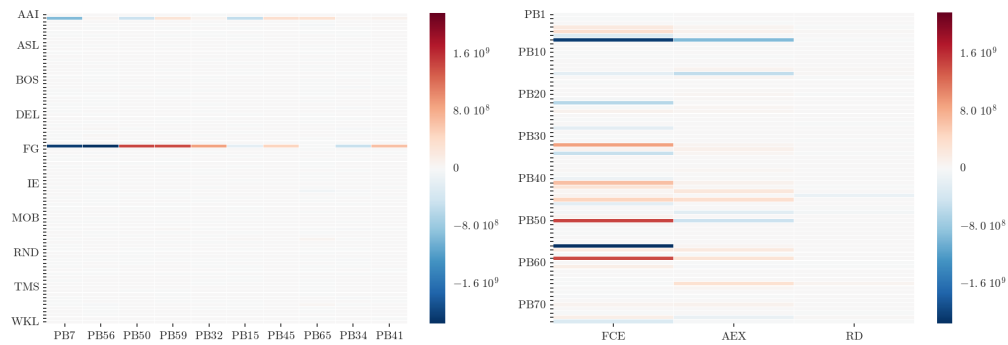


FIGURE 4.10 – Left : Positions in the underlying assets (one ticker out of ten displayed along the  $y$  axis) of the ten clearing members with the largest gross positions, ranked by decreasing gross positions. Right : Positions of the clearing members (one label out of ten displayed along the  $x$  axis) in the three most invested-in underlying assets, ranked by asset gross positions of the CCP.

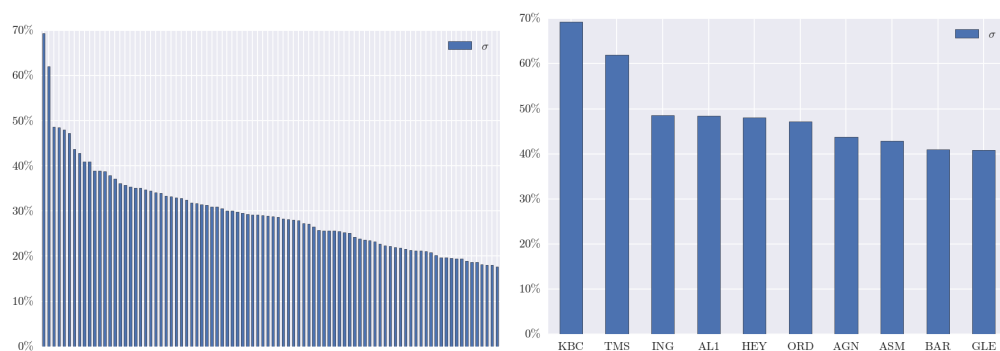


FIGURE 4.11 – Left : Underlying asset volatilities (ranked by decreasing order). Right : Zoom on the left part of the graph with tickers.

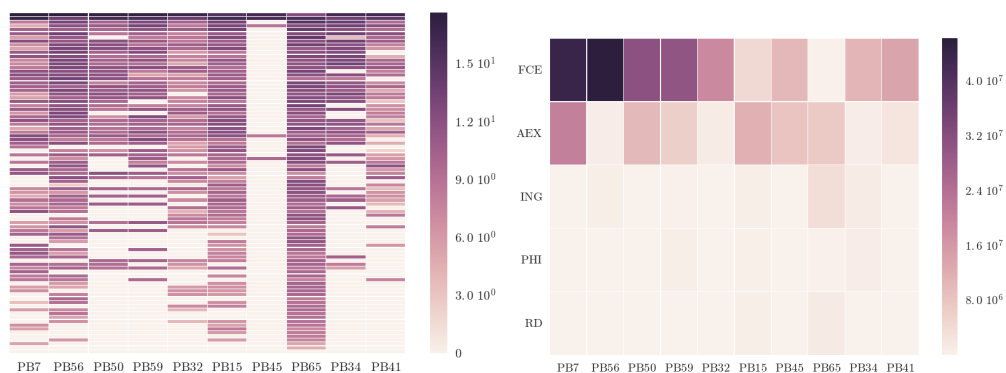


FIGURE 4.12 – Left : Log monetary risks in the underlying assets, ranked by decreasing risk order, of the ten clearing members with the largest gross positions. Right : Monetary risks in the five most invested-in underlying assets of the ten clearing members with the largest gross positions.



# Appendices



# EMIR Regulation

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In this appendix, we provide some references taken from the EMIR regulation concerning the CCPs margin calculation and related liquidation period, sizing of the default fund or calibration of the different parameters.

## A.1 Initial Margins

From [European Parliament \(2012b\)](#), Article 41 §1 to §5) :

### Article 41 - Margin Requirements

1. “A CCP shall impose, call and collect margins to limit its credit exposures from its clearing members and, where relevant, from CCPs with which it has interoperability arrangements. Such margins shall be sufficient to cover potential exposures that the CCP estimates will occur until the liquidation of the relevant positions. They shall also be sufficient to cover losses that result from at least 99% of the exposures movements over an appropriate time horizon and they shall ensure that a CCP fully collateralises its exposures with all its clearing members, and, where relevant, with CCPs with which it has interoperability arrangements, at least on a daily basis. A CCP shall regularly monitor and, if necessary, revise the level of its margins to reflect current market conditions taking into account any potentially procyclical effects of such revisions.
2. A CCP shall adopt models and parameters in setting its margin requirements that capture the risk characteristics of the products cleared and take into account the interval between margin collections, market liquidity and the possibility of changes over the duration of the transaction. The models and parameters shall be validated by the competent authority and subject to an opinion in accordance with Article 19.
3. A CCP shall call and collect margins on an intraday basis, at least when pre-defined thresholds are exceeded.
4. A CCP shall call and collect margins that are adequate to cover the risk stemming from the positions registered in each account kept in accordance with Article 39 with respect to specific financial instruments. A CCP may calculate margins with respect to a portfolio of financial instruments provided that the methodology used is prudent and robust.
5. In order to ensure consistent application of this Article, ESMA shall, after consulting EBA and the ESCB, develop draft regulatory technical standards specifying the appropriate percentage and time horizons for the liquidation period and the calculation of historical volatility, as referred to in paragraph 1, to be considered for the different classes of financial instruments, taking into account the objective to limit procyclicality, and the conditions under which portfolio margining practices referred to in paragraph 4 can be implemented.”

More precisely, in [European Parliament \(2012a, Article 24 §1 to §4\)](#) :

#### Article 24 - Percentage

1. “A CCP shall calculate the initial margins to cover the exposures arising from market movements for each financial instrument that is collateralised on a product basis, over the time period defined in Article 25 and assuming a time horizon for the liquidation of the position as defined in Article 26. For the calculation of initial margins the CCP shall at least respect the following confidence intervals :
  - (a) for OTC derivatives, 99,5 % ;
  - (b) for financial instruments other than OTC derivatives, 99 %.
2. For the determination of the adequate confidence interval for each class of financial instruments it clears, a CCP shall in addition consider at least the following factors :
  - (a) the complexities and level of pricing uncertainties of the class of financial instruments which may limit the validation of the calculation of initial and variation margin ;
  - (b) the risk characteristics of the class of financial instruments, which can include, but are not limited to, volatility, duration, liquidity, non-linear price characteristics, jump to default risk and wrong way risk ;
  - (c) the degree to which other risk controls do not adequately limit credit exposures ;
  - (d) the inherent leverage of the class of financial instruments, including whether the class of financial instrument is significantly volatile, is highly concentrated among a few market players or may be difficult to close out.
3. The CCP shall inform its competent authority and its clearing members on the criteria considered to determine the percentage applied to the calculation of the margins for each class of financial instruments.
4. Where a CCP clears OTC derivatives that have the same risk characteristics as derivatives executed on regulated markets or an equivalent third country market, on the basis of an assessment of the risk factors set out in paragraph 2, the CCP may use an alternative confidence interval of at least 99 % for those contracts if the risks of OTC derivatives contracts it clears are appropriately mitigated using such confidence interval and the conditions in paragraph 2 are respected.”

Then continuing with the calibration of the volatility in [European Parliament \(2012a, Article 25 §1 to §3\)](#) :

#### Article 25 - Time horizon for the calculation of historical volatility

1. “A CCP shall ensure that according to its model methodology and its validation process established in accordance with Chapter XII, initial margins cover at least with the confidence interval defined in Article 24 and for the liquidation period defined in Article 26 the exposures resulting from historical volatility calculated based on data covering at least the latest 12 months.  
A CCP shall ensure that the data used for calculating historical volatility capture a full range of market conditions, including periods of stress.

2. A CCP may use any other time horizon for the calculation of historical volatility provided that the use of such time horizon results in margin requirements at least as high as those obtained with the time period defined in paragraph 1.
3. Margin parameters for financial instruments without a historical observation period shall be based on conservative assumptions. A CCP shall promptly adapt the calculation of the required margins based on the analysis of the price history of the new financial instruments.”

The liquidation period is also specified in [European Parliament \(2012a, Article 26 §1 to §4\)](#) :

Article 26 - Time horizons for the liquidation period

1. “A CCP shall define the time horizons for the liquidation period taking into account the characteristics of the financial instrument cleared, the market where it is traded, and the period for the calculation and collection of the margins. These liquidation periods shall be at least :
  - (a) five business days for OTC derivatives ;
  - (b) two business days for financial instruments other than OTC derivatives.
2. In all cases, for the determination of the adequate liquidation period, the CCP shall evaluate and sum at least the following :
  - (a) the longest possible period that may elapse from the last collection of margins up to the declaration of default by the CCP or activation of the default management process by the CCP ;
  - (b) the estimated period needed to design and execute the strategy for the management of the default of a clearing member according to the particularities of each class of financial instrument, including its level of liquidity and the size and concentration of the positions, and the markets the CCP will use to close-out or hedge completely a clearing member position ;
  - (c) where relevant, the period needed to cover the counterparty risk to which the CCP is exposed.
3. In evaluating the periods defined in paragraph 2, the CCP shall consider at least the factors indicated in Article 24(2) and the time period for the calculation of the historical volatility as defined in Article 25.
4. Where a CCP clears OTC derivatives that have the same risk characteristics as derivatives executed on regulated markets or an equivalent third country market, it may use a time horizon for the liquidation period different from the one specified in paragraph 1, provided that it can demonstrate to its competent authority that :
  - (a) such time horizon would be more appropriate than that specified in paragraph 1 in view of the specific features of the relevant OTC derivatives ;
  - (b) such time horizon is at least two business days.”

The netting of the portfolio is also specified as not reducing the risk higher than 80% of the individual risks taken individually, as stated in [European Parliament \(2012a, Article 27\)](#) :

#### Article 27 - Portfolio margining

1. “A CCP may allow offsets or reductions in the required margin across the financial instruments that it clears if the price risk of one financial instrument or a set of financial instruments is significantly and reliably correlated, or based on equivalent statistical parameter of dependence, with the price risk of other financial instruments.
2. The CCP shall document its approach on portfolio margining and it shall at least provide that the correlation, or an equivalent statistical parameter of dependence, between two or more financial instruments cleared is shown to be reliable over the lookback period calculated in accordance with Article 25 and demonstrates resilience during stressed historical or hypothetical scenarios. The CCP shall demonstrate the existence of an economic rationale for the price relation.
3. All financial instruments to which portfolio margining is applied shall be covered by the same default fund. By way of derogation, if a CCP can demonstrate in advance to its competent authority and to its clearing members how potential losses would be allocated among different default funds and has set out the necessary provisions in its rules, portfolio margining may be applied to financial instruments covered by different default funds.
4. Where portfolio margining covers multiple instruments, the amount of margin reductions shall be no greater than 80 % of the difference between the sum of the margins for each product calculated on an individual basis and the margin calculated based on a combined estimation of the exposure for the combined portfolio. Where the CCP is not exposed to any potential risk from the margin reduction, it may apply a reduction of up to 100 % of that difference.
5. The margin reductions related to portfolio margining shall be subject to a sound stress test programme in accordance with Chapter XII.”

We finally conclude that section concerning Initial Margins by highlighting the management of the margins procyclicality in [European Parliament \(2012a, Article 28 §1 to §2\)](#) :

#### Article 28 - Procyclicality

1. “A CCP shall ensure that its policy for selecting and revising the confidence interval, the liquidation period and the lookback period deliver forward looking, stable and prudent margin requirements that limit procyclicality to the extent that the soundness and financial security of the CCP is not negatively affected. This shall include avoiding when possible disruptive or big step changes in margin requirements and establishing transparent and predictable procedures for adjusting margin requirements in response to changing market conditions. In doing so, the CCP shall employ at least one of the following options :
  - (a) applying a margin buffer at least equal to 25 % of the calculated margins which it allows to be temporarily exhausted in periods where calculated margin requirements are rising significantly ;

- (b) assigning at least 25 % weight to stressed observations in the lookback period calculated in accordance with Article 26 ;
  - (c) ensuring that its margin requirements are not lower than those that would be calculated using volatility estimated over a 10 year historical lookback period.
2. When a CCP revises the parameters of the margin model in order to better reflect current market conditions, it shall take into account any potential pro-cyclical effects of such revision.”

## A.2 Default Fund

As in the previous section, we first introduce the general concept of Default Fund defined in [European Parliament \(2012b\)](#), Article 42 §1 to §5) :

### Article 42 - Default Fund

1. “To limit its credit exposures to its clearing members further, a CCP shall maintain a pre-funded default fund to cover losses that exceed the losses to be covered by margin requirements laid down in Article 41, arising from the default, including the opening of an insolvency procedure, of one or more clearing members. The CCP shall establish a minimum amount below which the size of the default fund is not to fall under any circumstances.
2. A CCP shall establish the minimum size of contributions to the default fund and the criteria to calculate the contributions of the single clearing members. The contributions shall be proportional to the exposures of each clearing member.
3. The default fund shall at least enable the CCP to withstand, under extreme but plausible market conditions, the default of the clearing member to which it has the largest exposures or of the second and third largest clearing members, if the sum of their exposures is larger. A CCP shall develop scenarios of extreme but plausible market conditions. The scenarios shall include the most volatile periods that have been experienced by the markets for which the CCP provides its services and a range of potential future scenarios. They shall take into account sudden sales of financial resources and rapid reductions in market liquidity.
4. A CCP may establish more than one default fund for the different classes of instrument that it clears.
5. In order to ensure consistent application of this Article, ESMA shall, in close cooperation with the ESCB and after consulting EBA, develop draft regulatory technical standards specifying the framework for defining extreme but plausible market conditions referred to in paragraph 3, that should be used when defining the size of the default fund and the other financial resources referred to in Article 43. ESMA shall submit those draft regulatory technical standards to the Commission by 30 September 2012. Power is delegated to the Commission to adopt the regulatory technical standards referred to in the first subparagraph in accordance with Articles 10 to 14 of Regulation (EU) No 1095/2010.”

The Default Fund specifications are completed in [European Parliament \(2012a\)](#), Article 30 §1 to §3) :

### Article 30 - Identifying extreme but plausible market conditions

1. “The framework described in Article 29 shall reflect the risk profile of the CCP, taking account of cross-border and cross-currency exposures where relevant. It shall identify all the market risks to which a CCP would be exposed following the default of one or more clearing member, including unfavourable movements in the market prices of cleared instruments, reduced market liquidity for these instruments, and declines in the liquidation value of collateral. The framework shall also reflect additional risks to the CCP arising from the simultaneous failure of entities in the group of the defaulting clearing member.
2. The framework shall individually identify all the markets to which a CCP is exposed in a clearing member default scenario. For each identified market the CCP shall specify extreme but plausible conditions based at least on :
  - (a) a range of historical scenarios, including periods of extreme market movements observed over the past 30 years, or as long as reliable data have been available, that would have exposed the CCP to greatest financial risk. If a CCP decides that recurrence of a historical instance of large price movements is not plausible, it shall justify its omission from the framework to the competent authority ;
  - (b) a range of potential future scenarios, founded on consistent assumptions regarding market volatility and price correlation across markets and financial instruments, drawing on both quantitative and qualitative assessments of potential market conditions.
3. The framework shall also consider, quantitatively and qualitatively, the extent to which extreme price movements could occur in multiple identified markets simultaneously. The framework shall recognise that historical price correlations may breakdown in extreme but plausible market conditions.”

### A.3 Skin-In-The-Game

To introduce the so-called skin-in-the-game, we first refer to the default waterfall process as defined in [European Parliament \(2012b\)](#), Article 45 §1 to §5) :

#### Article 45 - Default waterfall

1. “A CCP shall use the margins posted by a defaulting clearing member prior to other financial resources in covering losses.
2. Where the margins posted by the defaulting clearing member are not sufficient to cover the losses incurred by the CCP, the CCP shall use the default fund contribution of the defaulting member to cover those losses.
3. A CCP shall use contributions to the default fund of the non-defaulting clearing members and any other financial resources referred to in Article 43(1) only after having exhausted the contributions of the defaulting clearing member.
4. A CCP shall use dedicated own resources before using the default fund contributions of non-defaulting clearing members. A CCP shall not use the margins posted by non-defaulting clearing members to cover the losses resulting from the default of another clearing member.
5. In order to ensure consistent application of this Article, ESMA, shall, after consulting the relevant competent authorities and the members of the ESCB, develop draft regulatory technical standards specifying the methodology for

calculation and maintenance of the amount of the CCP's own resources to be used in accordance with paragraph 4.

ESMA shall submit those draft regulatory technical standards to the Commission by 30 September 2012.

Power is delegated to the Commission to adopt the regulatory technical standards referred to in the first subparagraph in accordance with Articles 10 to 14 of Regulation (EU) No 1095/2010.

The following article illustrates the calculation of this amount of collateral provided by the CCP, [European Parliament \(2012a\)](#), Article 35 §1 to §4) :

Article 35 - Calculation of the amount of the CCP's own resources to be used in the default waterfall

1. "A CCP shall keep, and indicate separately in its balance sheet, an amount of dedicated own resources for the purpose set out in Article 45(4) of Regulation (EU) No 648/2012.
2. A CCP shall calculate the minimum amount referred to in paragraph 1 by multiplying the minimum capital, including retained earnings and reserves, held in accordance with Article 16 of Regulation (EU) No 648/2012 and Commission Delegated Regulation (EU) No 152/2013 (1) by 25 %.  
The CCP shall revise that minimum amount on a yearly basis.
3. Where the CCP has established more than one default fund for the different classes of financial instruments it clears, the total dedicated own resources calculated under paragraph 1 shall be allocated to each of the default funds in proportion to the size of each default fund, to be separately indicated in its balance sheet and used for defaults arising in the different market segments to which the default funds refer to.
4. No resources other than capital, including retained earnings and reserves, as referred to in Article 16 of Regulation (EU) No 648/2012 shall be used to comply with the requirement under paragraph 1."

## A.4 Back and Stress Testing

The following articles describe the procedures of back test and stress tests of the margining framework. In [European Parliament \(2012b\)](#), Article 49 §1 to §4) :

Article 49 - Review of models, stress testing and back testing

1. "A CCP shall regularly review the models and parameters adopted to calculate its margin requirements, default fund contributions, collateral requirements and other risk control mechanisms. It shall subject the models to rigorous and frequent stress tests to assess their resilience in extreme but plausible market conditions and shall perform back tests to assess the reliability of the methodology adopted. The CCP shall obtain independent validation, shall inform its competent authority and ESMA of the results of the tests performed and shall obtain their validation before adopting any significant change to the models and parameters.

The adopted models and parameters, including any significant change thereto, shall be subject to an opinion of the college pursuant to Article 19.

ESMA shall ensure that information on the results of the stress tests is passed on to the ESAs to enable them to assess the exposure of financial undertakings to the default of CCPs.

2. A CCP shall regularly test the key aspects of its default procedures and take all reasonable steps to ensure that all clearing members understand them and have appropriate arrangements in place to respond to a default event.
3. A CCP shall publicly disclose key information on its risk- management model and assumptions adopted to perform the stress tests referred to in paragraph 1.
4. In order to ensure consistent application of this Article, ESMA shall, after consulting EBA, other relevant competent authorities and the members of the ESCB, develop draft regulatory technical standards specifying :
  - (a) the type of tests to be undertaken for different classes of financial instruments and portfolios ;
  - (b) the involvement of clearing members or other parties in the tests ;
  - (c) the frequency of the tests ;
  - (d) the time horizons of the tests ;
  - (e) the key information referred to in paragraph 3.

ESMA shall submit those draft regulatory technical standards to the Commission by 30 September 2012.

Power is delegated to the Commission to adopt the regulatory technical standards referred to in the first subparagraph in accordance with Articles 10 to 14 of Regulation (EU) No 1095/2010.”

The backtesting procedure is finally specified in [European Parliament \(2012a, Article 49 §1 to §6\)](#) :

#### Article 49 - Back testing procedure

1. “A CCP shall assess its margin coverage by performing an ex-post comparison of observed outcomes with expected outcomes derived from the use of margin models. Such back testing analysis shall be performed each day in order to evaluate whether there are any testing exceptions to margin coverage. Coverage shall be evaluated on current positions for financial instruments, clearing members and take into account possible effects from portfolio margining and, where appropriate, interoperable CCPs.
2. A CCP shall consider the appropriate historical time horizons for its back testing programme to ensure that the observation window used is sufficient enough to mitigate any detrimental effect on the statistical significance.
3. A CCP shall consider in its back testing programme, at least, clear statistical tests, and performance criteria to be defined by CCPs for the assessment of back testing results.
4. A CCP shall periodically report its back testing results and analysis in a form that does not breach confidentiality to the risk committee in order to seek their advice in the review of its margin model.

5. Back testing results and analysis shall be made available to all clearing members and, where known to the CCP, clients. For all other clients back testing results and analysis shall be made available by the relevant clearing members on request. Such information shall be aggregated in a form that does not breach confidentiality and clearing members and clients shall only have access to detailed back testing results and analysis for their own portfolios.
6. A CCP shall define the procedures to detail the actions it could take given the results of back testing analysis.”

The all stress testing framework is also defined in [European Parliament \(2012a, Articles 51 to 53\)](#) :

Article 51 - Stress testing procedure

1. “A CCP’s stress tests shall apply stressed parameters, assumptions, and scenarios to the models used for the estimation of risk exposures to make sure its financial resources are sufficient to cover those exposures under extreme but plausible market conditions.
2. A CCP’s stress testing programme shall require the CCP to conduct a range of stress tests on a regular basis that shall consider the CCP’s product mix and all elements of its models and their methodologies and its liquidity risk management framework.
3. A CCP’s stress testing programme shall prescribe that stress tests are performed, using defined stress testing scenarios, on both past and hypothetical extreme but plausible market conditions in accordance with Chapter VII. Past conditions to be used shall be reviewed and adjusted, where appropriate. A CCP shall also consider other forms of appropriate stress testing scenarios including, but not limited to, the technical or financial failure of its settlement banks, nostro agents, custodian banks, liquidity providers, or interoperable CCPs.
4. A CCP shall have the capacity to adapt its stress tests quickly to incorporate new or emerging risks.
5. A CCP shall consider the potential losses arising from the default of a client, where known, which clears through multiple clearing members.
6. A CCP shall periodically report its stress testing results and analysis in a form that does not breach confidentiality to the risk committee in order to seek its advice in the review of its models, its methodologies and its liquidity risk management framework.
7. Stress testing results and analysis shall be made available to all clearing members and, where known to the CCP, clients. For all other clients, back testing results and analysis shall be made available by the relevant clearing members on request. Such information shall be aggregated in a form that does not breach confidentiality and clearing members and clients shall only have access to detailed stress testing results and analysis for their own portfolios.
8. A CCP shall define the procedures to detail the actions it could take given the results of stress testing analysis.”

## Article 52 - Risk factors to stress test

1. "A CCP shall identify, and have an appropriate method for measuring, relevant risk factors specific to the contracts it clears that could affect its losses. A CCP's stress tests shall, at least, take into account risk factors specified for the following type of financial instruments, where applicable :
  - (a) interest rate related contracts : risk factors corresponding to interest rates in each currency in which the CCP clears financial instruments. The yield curve modelling shall be divided into various maturity segments in order to capture variation in the volatility of rates along the yield curve. The number of related risk factors shall depend on the complexity of the interest rate contracts cleared by the CCP. Basis risk, arising from less than perfectly correlated movements between government and other fixed-income interest rates, shall be captured separately ;
  - (b) exchange rate related contracts : risk factors corresponding to each foreign currency in which the CCP clears financial instruments and to the exchange rate between the currency in which margin calls are made and the currency in which the CCP clears financial instruments ;
  - (c) equity related contracts : risk factors corresponding to the volatility of individual equity issues for each of the markets cleared by the CCP and to the volatility of various sectors of the overall equity market. The sophistication and nature of the modelling technique for a given market shall correspond to the CCP's exposure to the overall market as well as its concentration in individual equity issues in that market ;
  - (d) commodity contracts : risk factors that take into account the different categories and sub-categories of commodity contracts and related derivatives cleared by the CCP, including, where appropriate, variations in the convenience yield between derivatives positions and cash positions in the commodity ;
  - (e) credit related contracts : risk factors that consider jump to default risk, including the cumulative risk arising from multiple defaults, basis risk and recovery rate volatility.
2. In its stress tests, a CCP shall also give appropriate consideration at least to the following :
  - (a) correlations, including those between identified risk factors and similar contracts cleared by the CCP ;
  - (b) factors corresponding to the implied and historical volatility of the contract being cleared ;
  - (c) specific characteristics of any new contracts to be cleared by the CCP ;
  - (d) concentration risk, including to a clearing member, and group entities of clearing members ;
  - (e) interdependencies and multiple relationships ;
  - (f) relevant risks including foreign exchange risk ;
  - (g) set exposure limits ;
  - (h) wrong-way risk."

## Article 53 - Stress testing total financial resources

1. “A CCP’s stress-testing programme shall ensure that its combination of margin, default fund contributions and other financial resources are sufficient to cover the default of at least the two clearing members to which it has the largest exposures under extreme but plausible market conditions. The stress testing programme shall also examine potential losses resulting from the default of entities in the same group as the two clearing members to which it has the largest exposures under extreme but plausible market conditions.
2. A CCP’s stress-testing programme shall ensure that its margins and default fund are sufficient to cover at least the default of the clearing member to which it has the largest exposures or of the second and third largest clearing members, if the sum of their exposures is larger in accordance with Article 42 of Regulation (EU) No 648/2012.
3. The CCP shall conduct a thorough analysis of the potential losses it could suffer and shall evaluate the potential losses in clearing member positions, including the risk that liquidating such positions could have an impact on the market and the CCP’s level of margin coverage.
4. A CCP shall, where applicable, consider in its stress tests, the effects of the default of a clearing member that issues financial instruments cleared by the CCP or the underlying of derivatives cleared by the CCP. Where applicable, the effects of a client’s default that issues financial instruments cleared by the CCP or the underlying of derivatives cleared by the CCP shall also be considered.
5. A CCP’s stress tests shall consider the liquidation period as provided for in Article 26.”



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**Titre :** Chambres de compensation : analyse XVA, mesures de risque et applications

**Mots Clefs :** Chambres de compensation (CCP), Risque de crédit et de contrepartie, Collatéral, Analyse XVA

**Résumé :** Cette thèse traite de diverses problématiques ayant trait à la gestion du collatéral dans le contexte du trading centralisé au travers des chambres de compensation. Préliminairement, nous présentons les notions de coûts de capital et coût de financement pour une banque, en les replaçant dans un cadre Black-Scholes élémentaire où le payoff d'un call standard tient lieu d'exposition au défaut d'une contrepartie. On suppose que la banque ne couvre qu'imparfaitement ce call et doit faire face à un coût de financement supérieur au taux sans risque, d'où des corrections de pricing de type FVA et KVA par rapport au prix Black-Scholes. Nous nous intéressons ensuite aux coûts auxquels une banque doit faire face lorsqu'elle trade dans le cadre d'une CCP. À cette fin, nous transposons au trading centralisé le cadre d'analyse XVA du trading bilatéral. Le coût total pour un membre de trader au travers d'une CCP est ainsi décomposé en une CVA correspondant au coût pour le membre de renflouer sa contribution au fonds de garantie en cas de pertes consécutives aux défauts d'autres membres, une MVA correspondant au coût de financement de sa marge initiale et une KVA correspondant au coût du capital mis à risque par le membre sous la forme de sa contribution au fonds de garantie. Nous remettons ensuite en cause les hypothèses réglementaires précédemment utilisées, nous intéressant à des alternatives dans lesquelles les membres auraient recours pour leur marge initiale à une tierce partie, qui posterait la marge à la place du membre en échange d'une rémunération. Nous considérons également un mode de calcul du fonds de garantie et de son allocation qui prennent en compte le risque de la chambre au sens des fluctuations de son P&L sur l'année suivante, tel qu'il résulte de la combinaison des risques de marché et des risques de défaut des membres. Enfin, nous proposons l'application de méthodologies de type mesures de risque multivariées pour le calcul des marges et/ou du fonds de garantie des membres. Nous introduisons une notion de mesures de risque systémiques au sens où elles présentent une sensibilité non seulement aux risques marginaux des composantes d'un système financier (par exemple, mais non nécessairement, les positions des membres d'une CCP), mais aussi à leur dépendance.

**Title :** XVA Analysis, Risk Measures and Applications to Centrally Cleared Trading

**Keywords :** Central Clearing Houses (CCP), Counterparty Credit Risk, Collateral, XVA Analysis

**Abstract :** This thesis deals with various issues related to collateral management in the context of centralized trading through central clearing houses. In the first place, we present the notions of cost of capital and funding cost for a bank, placing them in an elementary Black-Scholes framework where the payoff of a standard call is used as the exposure at default of a counterparty. It is assumed that the bank can't perfectly hedge this call and must face with a funding cost higher than the risk free rate, hence pricing corrections of the FVA and KVA type appear in top of the Black-Scholes price. Then, we look at the different costs that a bank has to face when trading in the CCP context. To this end, we transpose the well-known XVA analysis framework from the bilateral trading world to the central clearing one. The total cost for a member trading through a CCP is thus decomposed into a CVA corresponding to the cost for the member to reimburse its contribution to the guarantee fund in the event of losses due to the defaults of other members, a MVA which is the cost of financing its initial margin and a KVA corresponding to the cost of capital put at risk by the member in the form of its contribution to the guarantee fund. Afterwards, we question the previously used regulatory assumptions, focusing on alternatives in which members would borrow their initial margin to a third party who would post the margin instead of the member himself, and this, in exchange for remuneration. We also consider a method of computing the guarantee fund and its allocation taking into account the risk of the CCP in the sense of fluctuations of its P&L over the following year, as it results from the market risk and the counterparty risk of the members. Finally, we propose the application of multivariate risk measure methodologies for the computation of margins and/or the CCP guarantee fund. We introduce a notion of systemic risk measures in the sense that they are sensitive not only to the marginal risks of the components of a financial system (for example, but not necessarily the positions of the members of a CCP) but also to the dependence of their components.