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Yavar Kian

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Manuscrit

présenté pour l'obtention de

l'Habilitation à Diriger des Recherches

par Yavar Kian
du Centre de Physique Théorique

Étude de problèmes inverses et directs pour différentes équations aux dérivées partielles

Soutenue le 15/11/2017 devant le jury composé de:

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El Maati Ouhabaz	Université de Bordeaux	Rapporteur
Luc Robbiano	Université de Versailles	Examinateur
Serge Troubetzkoy	Université d'Aix-Marseille	Tuteur

Après lecture des rapports de
Colin Guillarmou, El Maati Ouhabaz, Alberto Ruiz

"Trouver quelque chose en mathématiques, c'est vaincre une inhibition et une tradition."
Laurent Schwartz

"Un temps, durant notre enfance,
nous nous voulûmes savant ;
Un temps, de notre science,
nous eûmes contentement.
Mais écoute, maintenant,
ami d'exactes mesures,
La somme de l'aventure :
de l'eau courante et du vent! "
Omar Khayyam

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General introduction in french

Ce mémoire développe deux thèmes principaux: en premier lieu, les problèmes inverses et de façon connectée les problèmes directs, soit l'existence et l'unicité de solutions, pour différentes équations aux dérivées partielles. Ces deux aspects seront liés dans l'ensemble des travaux présentés dans ce manuscrit à l'exception des travaux de la Section 4.3.

Durant ces dernières décennies, les problèmes inverses ont connu un essor important. Ces problèmes prennent des formes variées et sont liés à des applications multiples (imagerie médicale, environnement, sismologie, finance...). Plus généralement, on remarque que ces problèmes apparaissent de façon récurrente au quotidien, comme lorsqu'on se demande d'où vient tel son ou d'où vient telle lumière. Au delà des applications multiples, du fait de leur caractère mal posé et non-linéaire, ces problèmes ont en tant que tel un intérêt mathématique. Dans ce manuscrit, nous nous focaliserons sur les problèmes de détermination d'un coefficient ou d'une source apparaissant dans des équations aux dérivées partielles (respectivement d'un opérateur) à partir d'observations des solutions (respectivement d'informations partielles à propos des données spectrales). Un exemple important de ce type de problèmes provient de la fameuse question de Calderón liée à des méthodes d'imagerie comme l'Electrical Impedance Tomography (EIT). Cette question peut être formulée de la façon suivante: Est il possible de déterminer la conductivité électrique d'un milieu à partir de mesures d'intensité et de courant sur le bord de ce milieu? Mathématiquement parlant, ce problème consiste à déterminer une conductivité apparaissant dans une équation elliptique à partir d'observations sur le bord du domaine. Ce type de problèmes peuvent être considérés pour d'autres équations aux dérivées partielles (hyperboliques, paraboliques, Schrödinger) avec différentes applications (problèmes de transmission de son, de lumière, de chaleur...). À travers les différents travaux que nous présenterons dans ce manuscrit, notre objectif sera d'étudier ce type de problèmes inverses sous des formes variées et pour différentes équations aux dérivées partielles. Les travaux que nous présenterons ici constituent une part importante et significative de mes travaux de recherches de ces six dernières années (soit depuis mon recrutement en qualité de maître de conférences à l'Université d'Aix-Marseille). Ce manuscrit se décompose en quatre parties comprenant: 1) Les problèmes sur un domaine cylindrique non-borné; 2) La détermination de coefficients dépendant du temps et de l'espace pour des équations aux dérivées partielles d'évolution; 3) Problèmes inverses spectraux; 4) Problèmes inverses et directs pour des équations de diffusion fractionnaire en temps. Un chapitre sera dédié à chacun de ces aspects.

Dans le premier chapitre, nous nous intéresserons à la détermination de coefficients ou d'une source pour différentes équations aux dérivées partielles dans un domaine cylindrique non-borné aussi appelé guide d'ondes. Dans ce cadre, nous chercherons à déterminer un coefficient ou une source à partir de mesures sur le bord du domaine. Ces résultats sont bien connus pour un domaine borné. Néanmoins, le passage d'un domaine borné vers un domaine non-borné engendre tout une série de difficultés que nous détaillerons dans le Chapitre 1. Pour contourner ces difficultés, nous proposons deux approches. La première approche consiste à utiliser la géométrie particulière des domaines cylindriques pour étendre les résultats connus pour un domaine borné. L'autre approche consiste à se restreindre à des coefficients périodiques le long de l'axe du cylindre et à utiliser la décomposition de Floquet. Ce procédé, bien connu en théorie spectrale, permet de ramener un problème énoncé sur un domaine non-borné à un problème sur un domaine borné avec des

conditions de quasi-périodicité.

Dans le second chapitre, nous considérerons la détermination de coefficients, dépendant des variables temporelles et spatiales, pour différentes équations d'évolution (hyperboliques, paraboliques, Schrödinger) sur un domaine borné. Pour les équations hyperboliques, notre objectif consistera à déterminer de façon globale une classe importante de coefficients dépendant du temps et de l'espace. Pour cela, nous rappelerons les limites de ce type de résultats qui proviennent d'obstructions à l'unicité que nous détaillerons dans la Sous-section 2.2.2. En tenant compte de ces limitations, nous établirons les données qui permettront de déterminer de façon unique et stable ces coefficients sans faire d'hypothèses importantes sur la forme du domaine ou la régularité des coefficients. Pour les équations paraboliques et de type Schrödinger, nous étudierons la question de la stabilité. Enfin, dans le cas des équations paraboliques, nous montrerons comment ces résultats peuvent être appliqués au problème consistant à déterminer un terme non linéaire à partir d'observations des solutions.

Le troisième chapitre sera consacré aux problèmes inverses spectraux. Plus précisément, nous considérerons des problèmes inverses spectraux de type Borg-Levinson dont le principe consiste à déterminer un opérateur à partir d'informations partielles sur ses données spectrales. Nous commencerons par considérer un opérateur de Schrödinger dans un domaine cylindrique non-borné avec un potentiel périodique le long de l'axe du cylindre. Dans ce cas précis, nous chercherons à déterminer l'opérateur à partir d'informations asymptotiques sur les données spectrales de Floquet de l'opérateur. Par la suite, nous considérerons la détermination d'un opérateur de Schrödinger magnétique sur un domaine borné à partir d'une certaine connaissance asymptotique des valeurs propres et à partir d'informations concernant des vecteurs propres de l'opérateur sur le bord du domaine. Nous finirons par présenter une application des résultats de contrôle aux bords à ce type de problèmes.

Le dernier chapitre sera consacré aux équations de diffusion fractionnaire en temps. Nous commencerons par considérer un problème inverse consistant à déterminer une classe importante de coefficients apparaissant dans ces équations à partir d'observations sur le bord des solutions en un temps fixé. Puis nous établirons des résultats liés au problème direct dans un contexte de régularité faible. Nous appliquerons ce dernier résultat pour déterminer l'existence de solutions pour des équations fractionnaires en temps non linéaires. Enfin, nous considérerons les équations de diffusion fractionnaire en temps dont l'ordre de la dérivation est variable. Pour ce problème, nous commencerons par prouver l'existence et l'unicité des solutions. Puis nous appliquerons ce résultat au problème inverse consistant à déterminer l'ordre de dérivation ainsi que des coefficients apparaissant dans l'équation à partir d'observations sur le bord des solutions.

General introduction in english

The main aspects developed in the present manuscript concern inverse problems. The other aspects are related to forward problems, namely existence and uniqueness of solutions for different partial differential equations. These two aspects will be often connected, except in Section 4.3 where we consider only the forward problem.

During these last decades, inverse problems have been growing in interest. These problems can be formulated in different ways and they have many applications (medical imaging, seismology, finance...). More generally, we often consider inverse problems in our real life, like when we wonder where a light or a sound comes from. Beside these applications, due to their ill-posedness and nonlinearity, inverse problems are also challenging from a pure mathematical point of view. In this manuscript, we focus our attention on the inverse problems of determining sources or coefficients appearing in different partial differential equations (respectively operators) from observations of solutions (respectively partial information about the spectral data). An important example of such a problem comes from the so called question of Calderón related to imaging methods like the Electrical Impedance Tomography (EIT). This question can be formulated in the following way: is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary? From a mathematical point of view, this problem corresponds to the determination of a conductivity appearing in an elliptic equation from observations on the boundary. Such problems can be formulated with other partial differential equations (hyperbolic, parabolic, Schrödinger) for various applications (transmission of light, sound, heat...). Through the contribution presented in this manuscript, we have studied such problems in various context for different partial differential equations. These works have been a significant part of my research for the past six years (since my nomination as assistant professor in Aix-Marseille University). This manuscript is decomposed into four parts: 1) Inverse problems in unbounded cylindrical domains; 2) Determination of coefficients depending on time and space variables for various evolution partial differential equations; 3) Inverse spectral problems; 4) Forward and inverse problems for fractional diffusion equations with time fractional derivative.

In the first chapter, we consider the determination of coefficients or sources for different partial differential equations in an unbounded cylindrical domain also called waveguide. In this context, we study the determination of such parameters from observations of solutions on the boundary of the domain. For bounded domains these results are well known. Nevertheless, the extension to unbounded domains generates some difficulties which will be described in Chapter 1. To overcome these difficulties, we use two strategies. First, we use some geometrical properties of cylindrical domains in order to extend some arguments stated for bounded domains to our case. Second, we consider coefficients periodic along the axis of the cylindrical domain and we use Floquet decomposition in order to transform our problem into a problem in a bounded domain with quasiperiodic boundary conditions.

The second chapter is devoted to the determination of coefficients depending on time and space variables for various evolution partial differential equations (hyperbolic, parabolic, Schrödinger) on a bounded domain. For hyperbolic equations, our goal is to consider the minimal data that allow to recover globally a general class of coefficients depending on time and space variables. For this purpose, we start by recalling some obstructions to our problem (see Section 2.2.2). Then, taking into account these obstructions, we establish the

set of data that allows to recover such coefficients without additional geometrical or smoothness assumptions. For parabolic and Schrödinger equations we focus our attention on the stability issue. Finally, for parabolic equations, we show how such results can be applied to the recovery of a nonlinear term.

In the third chapter, we study some inverse spectral problems. More precisely, we consider Borg-Levinson type of inverse spectral problems which consist in determining Schrödinger operators from partial knowledge of the spectral data. We consider these problems in different contexts. We start by considering the case of Schrödinger operators on an unbounded cylindrical domain with a potential periodic along the axis of the domain. In that case, we look for the recovery of the operator from some asymptotic knowledge of the Floquet spectral data. Next, we study the recovery of magnetic Schrödinger operators from some asymptotic knowledge of the eigenvalues and from observations on the boundary of eigenfunctions of the operator. Finally, we consider an application of the boundary control method to such problems.

The last chapter is devoted to the study of time fractional diffusion equations. We start by considering the inverse problem of determining a large class of coefficients appearing in these equations from observations of the solutions on the boundary of the domain at one fixed time. Then, we consider a forward problem for such equations stated with weak regularity of the data. We apply this last result to prove existence and uniqueness of solutions for time fractional diffusion nonlinear equations. Finally, we study fractional diffusion equations with variable time fractional order. For this problem, we start by considering the forward problem. Then, we apply this result to the problem of determining the fractional order from observations of the solutions at the boundary of the domain.

List of presented publications

You can find below the list of publications presented in this manuscript:

1) Y. Kian, Q. S. Phan, E. Soccorsi, *Carleman estimate for infinite cylindrical quantum domains and application to inverse problems*, Inverse Problems, **30** (2014), 055016.

2) Y. Kian, *Stability of the determination of a coefficient for the wave equation in an infinite wave guide*, Inverse Probl. Imaging, **8** (3) (2014), 713-732.

3) Y. Kian, Q. S. Phan, E. Soccorsi, *Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains*, preprint, J. Math. Anal. Appl, **426** (2015), 194-210.

4) O. Kavian, Y. Kian, E. Soccorsi, *Uniqueness and stability results for an inverse spectral problem in a periodic waveguide*, J. Math. Pures Appl., **104** (2015), 1160-1189.

5) Y. Kian, *Stability in the determination of a time-dependent coefficient for wave equations from partial data*, J. Math. Anal. Appl., **436** (2016), 408-428.

6) M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse stability result for non compactly supported potentials by one arbitrary lateral Neumann observation*, J. Diff. Equat., **260** (2016), 7535-7562.

7) Y. Kian, *Recovery of time-dependent damping coefficients and potentials appearing in wave equations from partial data*, SIAM J. Math. Anal., **48** (6) (2016), 4021-4046.

8) Y. Kian, M. Yamamoto, *On existence and uniqueness of solutions for semilinear fractional wave equations*, Fractional Calculus and Applied Analysis, **20** (1) (2017), 117-138.

9) Y. Kian, *Unique determination of a time-dependent potential for wave equations from partial data*, Annales de l'IHP (C) Nonlinear Analysis, **34** (2017), 973-990.

10) Y. Kian, *A multidimensional Borg-Levinson theorem for magnetic Schrödinger operators with partial spectral data*, to appear in Journal of Spectral Theory, `arXiv:1504.04514`.

11) Y. Kian, L. Oksanen, E. Soccorsi, M. Yamamoto, *Global uniqueness in an inverse problem for time fractional diffusion equations*, J. Diff. Equat., **264** (2018), 1146-1170.

12) M. Choulli, Y. Kian, E. Soccorsi, *Stability result for elliptic inverse periodic coefficient problem by partial Dirichlet-to-Neumann map*, to appear in Journal of Spectral Theory, `arXiv:1601.05355`.

13) M. Choulli, Y. Kian, E. Soccorsi, *On the Calderón problem in periodic cylindrical domain with partial Dirichlet and Neumann data*, Mathematical Methods in the Applied Sciences, **40** (2017), 5959-5974.

14) M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse problem for the magnetic Schrödinger equation in infinite cylindrical domains*, to appear in PRIMS, `arXiv:1605.06599`.

15) M. Choulli, Y. Kian, *Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term*, to appear in J. Math. Pures Appl., `arXiv:1605.08672`.

16) Y. Kian, L. Oksanen, *Recovery of time-dependent coefficient on Riemannian manifold for hyperbolic equations*, to appear in IMRN, `https://doi.org/10.1093/imrn/rnx263`.

17) Y. Kian, D. Sambou, E. Soccorsi, *Logarithmic stability inequality in an inverse source problem for the heat equation on a waveguide*, preprint, `arXiv:1612.07942`.

18) Y. Kian, E. Soccorsi, M. Yamamoto, *On time-fractional diffusion equations with space-dependent variable order*, preprint, `arXiv:1701.04046`.

19) Y. Kian, M. Morancey, L. Oksanen, *Application of the boundary control method to partial data Borg-Levinson inverse spectral problem*, preprint, `arXiv:1703.08832`.

20) Y. Kian, E. Soccorsi, *Hölder stably determining the time-dependent electromagnetic potential of the Schrödinger equation*, preprint, `arXiv:1705.01322`.

Chapter 1

Inverse problems in unbounded cylindrical domains

1.1 Introduction

Inverse coefficient or source problems for partial differential equations have attracted much attention over the last decades. Both results of unique and stable recovery of coefficients or sources have been considered for various partial differential equations (PDE in short) and with various measurements. We focus here our attention on inverse boundary value problems which correspond to recovery of sources or coefficients from boundary measurements of solutions. Such problems are usually stated in a bounded domain Ω of $\mathbb{R}^n$, $n \geqslant 1$, and the measurements are associated with observations of the solutions at the boundary $\partial\Omega$ of the domain. In this context, the recovery of the parameter is considered both from many boundary measurements given by the so called Dirichlet-to-Neumann (DN in short) map associated with the PDE (e.g. [BeliKu92, BuUh, KeSjUh, RakSy, SyUh]) or from a single boundary measurement given by the measure of the flux at $\partial\Omega$ of a suitable solution of the PDE (e.g. [BaPu02, BaPu07, BuKl, ImYa98]). In most of approaches developed for these problems the fact that Ω is a bounded domain plays an important role.

In contrast to the important development of inverse boundary value problems stated in a bounded domain, only a small number of mathematical papers considered such results for unbounded domains. One of them, [Ra93], examines the problem of determining a potential appearing in the wave equation in the half-space. In [Nak], Nakamura extended the work of [Ra93] to more general coefficients. In [Ik] and [SalWa], the authors examined the inverse problem of identifying an embedded object in an infinite slab. Unique determination of compactly supported potentials appearing in the stationary Schrödinger equation in an infinite slab from partial boundary measurements is established in [LiUh]. The same problem is addressed by [KrLaUh] for the stationary magnetic Schrödinger equation, and by [Yan] for bi-harmonic operators with perturbations of order zero or one. More recently, [CaMa] treated the stability issue associated with [LiUh]. The inverse problem of determining the twisting function of an infinite twisted closed waveguide by the DN map, is addressed in [ChSo]. In [ChKiSo152], time-dependent potentials that are periodic in the translational direction of the waveguide, are stably retrieved by the measurements of the Schrödinger equation.

Note that, beside [CaMa, ChKiSo152, ChSo], in all the above mentioned results the authors considered the uniqueness issue stated in a domain corresponding to the half space or a slab. In this section we will introduce the work [BeKiSo1, BeKiSo2, ChKiSo161, ChKiSo162, Ki141, KiPhSo1, KiPhSo2, KiSaSo] where we studied the stability issue for inverse boundary value problems for various PDEs stated on an infinite waveguide taking the form of a cylindrical domain $\Omega = \omega \times \mathbb{R}$ (or $\Omega = \mathbb{R} \times \omega$ in [ChKiSo161, ChKiSo162]) where ω is a bounded domain of $\mathbb{R}^{n-1}$ for $n \geqslant 3$. The main purpose of [BeKiSo1, BeKiSo2, ChKiSo161,

ChKiSo162, Ki141, KiPhSo1, KiPhSo2, KiSaSo] was an extension of results stated in a bounded domain to an infinite waveguide. For this purpose, we considered different approaches in order to overcome the difficulties arising from the consideration of an unbounded domain. For some of these results an additional assumption was required. For instance in [BeKiSo1, KiPhSo1, KiPhSo2], we proved that the Bukgheim-Klibanov strategy stated in [BuKl] can be extended to our setting provided that the unknown coefficient becomes exponentially close to some *a priori* known coefficient along the axis of the waveguide. In [ChKiSo161, ChKiSo162], we have studied the so called Calderón problem in the specific case of coefficients periodic along the axis of the waveguide. In [BeKiSo2], some restrictions on the class of magnetic potentials along the infinite direction have been made. On the other hand, some of our results such as [Ki141, KiSaSo], correspond to the exact statement of the result on bounded domains without any additional assumptions. We mention also that our analysis in [BeKiSo1, BeKiSo2, ChKiSo161, ChKiSo162, Ki141, KiPhSo1, KiPhSo2, KiSaSo] is associated with problems of transmission to long distance or transmission throw nanostructures and periodic structures (see Subsection 1.2.2, 1.3.1 and 1.6.1).

This chapter is organized as follows. In Section 1.2, we describe the result of [Ki141] related to the stable recovery of a time-independent coefficient appearing in a wave equation. Section 1.3 is devoted to [BeKiSo1, KiPhSo1, KiPhSo2] where we have proved the stable recovery of a time-independent coefficient appearing in a Schrödinger equation on a waveguide from a single boundary measurement. In Section 1.4, we consider the stable recovery of an electromagnetic potential from boundary measurements of solutions of a Schrödinger equation on a waveguide stated in [BeKiSo2]. In Section 1.5, we discuss about our results [ChKiSo161, ChKiSo162] related to the Calderón problem stated on an infinite cylindrical domain for periodic coefficients. Finally, in Section 1.6, we introduce our result [KiSaSo] of stable recovery of a time-independent source term from a single measurement of a solution of an associated parabolic equation on an unbounded waveguide.

1.2 Stable recovery of a time-independent potential for a wave equation

This section is devoted to the work [Ki141] where a stable recovery of a time-independent potential appearing in a wave equation on a closed infinite waveguide is stated.

1.2.1 Statement of the problem

In this section, we consider the waveguide $\Omega = \omega \times \mathbb{R}$, where ω is a $\mathcal{C}^\infty$ bounded connected domain of $\mathbb{R}^2$. We set $\Sigma := (0,T) \times \partial\Omega$ and $Q := (0,T) \times \Omega$. We introduce the following initial-boundary value problem (IBVP in short) for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0, & \text{in } Q \\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma. \end{cases} \tag{1.2.1.1}$$

In [Ki141], we study the inverse problem which consists in determining the coefficient of order zero q from the DN map

$$\Lambda_q : \ f \mapsto \partial_\nu u_{|\Sigma}$$

with u the solution of problem (1.2.1.1) in some appropriate space, ν the unit outward normal vector to $\partial\Omega$ and $\partial_\nu = \nu \cdot \nabla$ the normal derivative.

1.2.2 Physical motivation

Physically speaking, this inverse problem consists in determining properties such as density of an inhomogeneous medium by probing it with disturbances generated on the boundary. The data is the response of the

medium to these disturbances, measured on the boundary, and the purpose is to recover the function which measures the property of the medium.

Let us remark that equation (1.2.1.1) describes the propagation of waves, such as electromagnetic waves or sound waves, along the axis of an infinite cylindrical waveguide under the influence of a coefficient of order zero q. Such a model can be considered when a cylindrical waveguide is used for transmitting light, signals, sound or energy to long distance. In these cases the cylindrical waveguide can be supposed infinite. Moreover, the case considered in Theorem 1.2.3 is related to problems of transmission through a periodic structure such as photonic crystal.

1.2.3 Stability estimates

Recall that $\partial\Omega = \partial\omega \times \mathbb{R}$. Since $\partial\Omega$ is not bounded, for all $s > 0$ we give the following definition of the space $H^s(\partial\Omega)$:

$$H^s(\partial\Omega) = H^s(\mathbb{R}_{x_3}; L^2(\partial\omega)) \cap L^2(\mathbb{R}_{x_3}; H^s(\partial\omega)).$$

Then, for $r \geqslant 0$, we introduce the usual space

$$H^{r,s}((0,T) \times X) = H^r(0,T; L^2(X)) \cap L^2(0,T; H^s(X))$$

where $X = \Omega$ or $X = \partial\Omega$. Set the space

$$L = \left\{ f \in H^{\frac{3}{2},\frac{3}{2}}(\Sigma) : f_{|t=0} = 0,\ \partial_t f, \partial_\tau f, \partial_{x_3} f \in L^2\left(\Sigma; \mathrm{d}\sigma(x)\frac{\mathrm{d}t}{t}\right)\right\}$$

with $\|\|_L$ defined by

$$\|f\|_L^2 = \|f\|^2_{H^{\frac{3}{2},\frac{3}{2}}(\Sigma)} + \int_\Sigma \frac{|\partial_t f|^2 + |\partial_\tau f|^2 + |\partial_{x_3} f|^2}{t} \mathrm{d}\sigma(x)\mathrm{d}t.$$

Here, we denote by ∂_τ a tangential derivative with respect to $\partial\omega$. We have considered first the well-posedness of the IBVP (1.2.1.1) given by the following result.

Theorem 1.2.1 ([Ki141], Theorem 3)*Let $q \in L^\infty(\Omega)$ and $f \in L$. Then problem (1.2.1.1) admits a unique solution $u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ such that $\partial_\nu u \in L^2(\Sigma)$. Moreover, this solution u satisfies*

$$\|u\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C \|f\|_L . \tag{1.2.3.1}$$

We recall that for a bounded domain Ω, according to [LaLiTr, Theorem 2.1] (see also [BeChYa, Theorem A. 2]), this result holds true for $f \in H^1(\Sigma)$. Since $\Omega = \omega \times \mathbb{R}$ is an unbounded domain, we can not apply the analysis of [LaLiTr]. Nevertheless, we prove in [Ki141] that (1.2.1.1) can be solved by a classical lifting argument.

In view of Theorem 1.2.1, we can define the hyperbolic DN map

$$\begin{array}{rccl} \Lambda_q : & L & \to & L^2(\Sigma), \\ & f & \mapsto & \partial_\nu u_q \end{array}$$

associated with (1.2.1.1).

For $0 < \alpha < 1$ and $h \in \mathcal{C}(\overline{\Omega})$, we set

$$[h]_\alpha = \sup\left\{ \frac{|h(x) - h(y)|}{|x-y|^\alpha} :\ x, y \in \overline{\Omega},\ x \neq y \right\}$$

and we consider the space

$$\mathcal{C}_b^\alpha(\overline{\Omega}) = \{h \in \mathcal{C}(\overline{\Omega}) \cap L^\infty(\Omega) : \ [h]_\alpha < \infty\}$$

with the norm

$$\|h\|_{\mathcal{C}_b^\alpha(\overline{\Omega})} = \|h\|_{L^\infty(\Omega)} + [h]_\alpha.$$

Our first main result can be stated as follows.

Theorem 1.2.2 ([Ki141], Theorem 1) *Let $M > 0$, $0 < \alpha < 1$ and let B_M be the ball centered at 0 and of radius M of $\mathcal{C}_b^\alpha(\overline{\Omega})$. Then, for $T > Diam(\omega)$ and $q_1, q_2 \in B_M$, we have*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|^d \tag{1.2.3.2}$$

with $d = \frac{\min(2\alpha,1)\alpha}{3(2\alpha+2)(\min(4\alpha,2)+21)}$ and C depending on M, T, α, Ω. Here $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ is the norm of $\Lambda_{q_1} - \Lambda_{q_2}$ with respect to $B\left(L, L^2(\Sigma)\right)$.

Let us remark that in this result we consider the full DN map. Consequently, we determine the coefficient q from measurements on the whole lateral boundary Σ which is an unbounded set. This is due to the fact that we consider a large class of coefficients q without any restriction on their behavior outside a compact set (we only assume that the coefficients are uniformly bounded and Hölderian). In order to extend this result to the determination of q from measurements in a bounded subset of Σ, we need more information about q. Namely, we need that the gap between two coefficients q_1, q_2 reach its maximum in a fixed bounded subset of $\overline{\Omega}$. More precisely, let $R > 0$ and consider the spaces L_R which consists in functions $f \in L$ satisfying

$$f(t, x', x_3) = 0, \quad t \in (0,T), \ x' \in \partial\omega, \ |x_3| \geqslant R.$$

Let us introduce the partial DN map defined by

$$\begin{aligned} \Lambda_q^R : \quad L_R \quad &\to \quad L^2((0,T) \times \partial\omega \times (-R,R)), \\ f \quad &\mapsto \quad \partial_\nu u_{q_{|(0,T)\times\partial\omega\times(-R,R)}}. \end{aligned}$$

Our second result is the following.

Theorem 1.2.3 ([Ki141], Theorem 2)*Let $M > 0$, $0 < \alpha < 1$ and let B_M be the ball centered at 0 and of radius M of $\mathcal{C}_b^\alpha(\overline{\Omega})$. Let $T > Diam(\omega)$, $q_1, q_2 \in B_M$ and assume that there exists $r > 0$ such that*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} = \|q_1 - q_2\|_{L^\infty(\omega\times(-r,r))}. \tag{1.2.3.3}$$

Then, for all $R > r$ we have

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C \left\|\Lambda_{q_1}^R - \Lambda_{q_2}^R\right\|^d \tag{1.2.3.4}$$

with $d = \frac{\min(2\alpha,1)\alpha}{3(2\alpha+2)(\min(4\alpha,2)+21)}$ and C depending on M, T, α, Ω, R. Here $\left\|\Lambda_{q_1}^R - \Lambda_{q_2}^R\right\|$ is the norm of $\Lambda_{q_1}^R - \Lambda_{q_2}^R$ with respect to $B\left(L_R, L^2((0,T) \times \partial\omega \times (-R,R))\right)$.

Clearly condition (1.2.3.3) will be fulfilled if we assume that $q_1 = q_2$ outside $\omega \times (-r,r)$. Let us remark that this condition can also be fulfilled in more general cases. For instance, consider the condition

$$v(x', x_3 + 2r) = v(x', x_3), \quad x' \in \overline{\omega}, \ x_3 \in \mathbb{R}. \tag{1.2.3.5}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a non negative continuous even function which is decreasing in $(0, +\infty)$. Then, condition (1.2.3.3) will be fulfilled if we assume that q_1, q_2 are lying in the set

$$A_g = \{q : \ q(x', x_3) = g(x_3)v(x', x_3), \ v \in \mathcal{C}(\overline{\Omega}) \cap L^\infty(\Omega), \ v \text{ satisfies } (1.2.3.5)\}.$$

Note that the solution of (1.2.1.1) is defined in the unbounded domain $Q = (0,T) \times \Omega$ and condition (1.2.3.3) is fulfilled by coefficients having different types of behavior outside $\omega \times (-r,r)$. Therefore, it seems to us that we can not prove Theorem 1.2.3 from results in bounded domain.

1.2.4 Description of the result

The main ingredients in the proof of the stability estimates (1.2.3.2) and (1.2.3.4) are suitable solutions of problem (1.2.1.1) called "geometric optics" (GO in short) solutions. This approach has been considered by various authors for the recovery of time-independent coefficients in a bounded domain (e.g. [RakSy, BeChYa, BeJeYa1, BeJeYa2, StUh98]) or in an unbounded domain for coefficients constant outside a compact set (e. g. [Ra93, Nak]). In [Ki141], we take into account the cylindrical form of the infinite waveguide in order to extend this approach to the stable recovery of time-independent coefficients in an unbounded domain. For this purpose, we build GO solutions suitably designed for our problem by using a separation of variable argument. More precisely, we present every variable $x \in \Omega$ in the form $x = (x', x_3)$ with $x' = (x_1, x_2) \in \omega$ and $x_3 \in \mathbb{R}$. Using this representation we split the differential operator $\partial_t^2 - \Delta$ defined on Q into the sum of two differential operators $[\partial_t^2 - \Delta_{x'}] + [-\partial_{x_3}^2]$ defined on Q, with $\Delta_{x'} = \partial_{x_1}^2 + \partial_{x_2}^2$ the Laplacian in ω. Applying this decomposition, we prove existence of GO solutions $u^{\pm} \in H^2(Q)$ of the equation $\partial_t^2 u - \Delta u + qu = 0$ on Q which take the form

$$u^{\pm}(t, x', x_3) = \Phi(x' + t\theta)h(x_3)e^{\pm i\rho(x'\cdot\theta + t)} + \Psi^{\pm}(t, x', x_3; \rho),\ t \in (0, T),\ x' \in \omega,\ x_3 \in \mathbb{R} \tag{1.2.4.1}$$

with $h \in \mathcal{S}(\mathbb{R})$, $\Phi \in \mathcal{C}_0^{\infty}(\mathbb{R}^2)$, $\theta \in \mathbb{S}^1 = \{y \in \mathbb{R}^2 :\ |y| = 1\}$, a large parameter $\rho > 1$ and a remainder term $\Psi^{\pm}$ that satisfies the decay

$$\left\|\Psi^{\pm}(.;\rho)\right\|_{L^2(Q)} \leqslant \frac{C}{\rho}$$

with respect to ρ. Combining some properties of X-ray transform of compactly supported functions (e.g. [Nat]), we use the expression Φ and h to prove (1.2.3.2) and (1.2.3.4).

1.3 Stable recovery of an electric potential in a dynamical Schrödinger equation from a single boundary measurement

In this section, we are concerned with [BeKiSo1, KiPhSo1, KiPhSo2] which are joint works with Mourad Bellassoued, Quang Sang Phan and Eric Soccorsi. We introduce the infinite cylindrical domain $\Omega = \omega \times \mathbb{R}$, where ω is a connected bounded open subset of $\mathbb{R}^{n-1}$, $n \geqslant 2$, with C^2-boundary $\partial\omega$. Given $T > 0$ we examine the following initial boundary value problem

$$\left\{\begin{array}{rcll} -i\partial_t u - \Delta u + q(x)u & = & 0, & \text{in } Q := (0,T) \times \Omega, \\ u(0,x) & = & u_0(x), & x \in \Omega, \\ u(t,x) & = & g(t,x), & (t,x) \in \Sigma := (0,T) \times \Gamma. \end{array}\right. \tag{1.3.0.2}$$

Here u_0 (resp., g) is the initial (resp., boundary) condition associated with (1.3.0.2) and q is a function of $x \in \Omega$ only.

Since Γ is unbounded we make the boundary condition in the last line of (1.3.0.2) more precise. Writing $x := (x', x_n)$ with $x' := (x_1, \ldots, x_{n-1}) \in \omega$ for every $x \in \Omega$ we extend the mapping

$$\begin{array}{rcl} C_0^{\infty}((0,T) \times \mathbb{R}; \mathrm{H}^2(\omega)) & \longrightarrow & \mathrm{L}^2((0,T) \times \mathbb{R}; \mathrm{H}^{3/2}(\partial\omega))) \\ v & \mapsto & [(t, x_n) \in (0,T) \times \mathbb{R} \mapsto v(t, \cdot, x_n)_{|\partial\omega}], \end{array} \tag{1.3.0.3}$$

to a bounded operator from $\mathrm{L}^2((0,\mathrm{T}) \times \mathbb{R}; \mathrm{H}^2(\omega))$ into $\mathrm{L}^2((0,\mathrm{T}) \times \mathbb{R}; \mathrm{H}^{3/2}(\partial\omega))$, denoted by γ_0. Then for every $u \in C^0([0,T]; \mathrm{H}^2(\Omega))$ the above mentioned boundary condition reads $\gamma_0 u = g$.

The main purpose of [BeKiSo1, KiPhSo1, KiPhSo2] is to prove stability in the determination of the scalar potential q from one boundary measurement of the normal derivative of the solution u to (1.3.0.2) on some portion of Γ.

1.3.1 Motivations

The problem (1.3.0.2) describe the quantum motion constrained by the waveguide Ω, of a charged particle (in a "natural" system of units where the various physical constants such as the mass and the electric charge are taken equal to one) under the influence of the "electric" potential q. Carbon nanotubes, who have a length-to-diameter ratio up to $10^8/1$, are commonly modelled by infinite cylindrical domains such as $\Omega = \omega \times \mathbb{R}$. These nanostructures exhibit unusual physical properties, which are valuable for electronics, optics and other fields of materials science and technology, but they can be affected by the inevitable presence of electrostatic quantum disorder, see e. g. [ChLi, KaBaFi]. This motivates for a closer look into the inverse problem of estimating the strength of the electric impurity potential p from the (partial) knowledge of the wave function u on the boundary Γ of the infinite carbon nanotube Ω.

1.3.2 Forward problem

We recall that in [BeKiSo1, KiPhSo1, KiPhSo2] we use the so called Bukgheim-Klibanov approach initially introduced in [BuKl] and extended to the Schrödinger equation by [BaPu02]. This approach requires sufficiently smooth solutions of (1.3.0.2). In the context of [BeKiSo1, KiPhSo1, KiPhSo2] where the problem is stated in an unbounded domain, such results need to be clearly stated. We start by recalling the results of [BeKiSo1, KiPhSo1, KiPhSo2] related to existence of sufficiently smooth solutions of (1.3.0.2).

We choose

$$g := \gamma_0 G_0, \text{ with } G_0(t,x) := u_0(x) + it(\Delta - q_0)u_0(x), \ (t,x) \in Q, \tag{1.3.2.1}$$

where $q_0 = q_0(x)$ is a given scalar function we shall make precise below.

Our result related to the forward problem for [KiPhSo1, KiPhSo2] can be stated as follows.

Theorem 1.3.1 (Thorem 1.1, [KiPhSo2]) *Let $k \geqslant 2$, assume that $\partial\omega$ is C^{2k}, and pick*

$$(q_0, u_0) \in \left(W^{2k,\infty}(\Omega) \cap C^{2(k-1)}(\overline{\Omega}; \mathbb{R})\right) \times H^{2(k+1)}(\Omega),$$

such that

$$(-\Delta + q_0)^{2+j} u_0 = 0 \text{ on } \partial\Omega \text{ for all } j \in \{0, 1, \ldots, k-2\}. \tag{1.3.2.2}$$

Then for each $q \in W^{2k,\infty}(\Omega) \cap C^{2(k-1)}(\overline{\Omega})$ obeying the condition

$$\partial_x^m q = \partial_x^m q_0 \text{ on } \partial\Omega \text{ for all } m := (m_j)_{j=1}^n \in \mathbb{N}^n \text{ with } |m| := \sum_{j=1}^n m_j \leqslant 2(k-2), \tag{1.3.2.3}$$

there is a unique solution $u \in \cap_{j=0}^k C^j([0,T]; H^{2(k-j)}(\Omega))$ to the boundary value problem (1.3.0.2) *with g satisfying* (1.3.2.1)*. Moreover, we have the estimate*

$$\sum_{j=0}^k \|u\|_{C^j([0,T]; H^{2(k-j)}(\Omega))} \leqslant C \|u_0\|_{H^{2(k+1)}(\Omega)}, \tag{1.3.2.4}$$

where $C > 0$ is a constant depending only on T, ω, k, and $\max(\|q_0\|_{W^{2k,\infty}(\Omega)}, \|q\|_{W^{2k,\infty}(\Omega)})$.

We now introduce the natural number

$$\ell \in \mathbb{N} \cap (n/4, n/4 + 1]\,. \tag{1.3.2.5}$$

Then, applying Theorem 1.3.1 with $k = \ell + 1$, we get that $u \in C^1([0,T]; H^{2\ell}(\Omega))$ and the estimate $\|u\|_{C^1([0,T]; H^{2\ell}(\Omega))} \leqslant C\|u_0\|_{H^{2(\ell+2)}(\Omega)}$. Since $2\ell > n/2$ then $H^{2\ell}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$. This immediately entails the:

Corollary 1.3.1 *Under the conditions of Theorem 1.3.1 for $k = \ell + 1$, where ℓ is defined by* (1.3.2.5)*, the solution u to* (1.3.0.2) *with g given by* (1.3.2.1) *satisfies* (1.3.2.4) *and the estimate*

$$\|u\|_{C^1([0,T],L^\infty(\Omega))} \leqslant C\|u_0\|_{H^{2(\ell+2)}(\Omega)}.$$

Here $C > 0$ is a constant depending only on T, ω and $\max(\|q_0\|_{W^{2(\ell+1),\infty}(\Omega)}, \|q\|_{W^{2(\ell+1),\infty}(\Omega)})$.

This completes the statement of the forward problem of [KiPhSo1, KiPhSo2].

In [BeKiSo1] we consider problem (1.3.0.2) with $g = 0$. In this context, the forward problem can be stated in the following way.

Corollary 1.3.2 *Assume that the conditions of Theorem 1.3.1 are satisfied with $k = \ell + 1$, $g = 0$ and u_0 satisfying*

$$u_0(x) = (-\Delta + q_0)u_0(x) = 0, \quad x \in \Gamma. \tag{1.3.2.6}$$

Then there exists a positive constant C, depending only on ω, T and M, such that the solution u to (1.3.0.2) *satisfies the estimate:*

$$\|u\|_{\mathcal{C}^1([0,T],L^\infty(\Omega))} \leqslant C\|u_0\|_{H^{2(\ell+1)}(\Omega)}.$$

1.3.3 Stable recovery of the electric potential

In this subsection we introduce [BeKiSo1, KiPhSo1, KiPhSo2] and we briefly comment on them. We start with [KiPhSo1]. From now on we fix ℓ given by (1.3.2.5) and

$$(q_0, u_0) \in \left(W^{2(\ell+1),\infty}(\Omega) \cap C^{2\ell}(\overline{\Omega};\mathbb{R})\right) \times H^{2(\ell+2)}(\Omega)$$

satisfying (1.3.2.2) for $k = \ell+1$. We assume also that $\partial\omega$ is $\mathcal{C}^{2(\ell+1)}$. For $M > 0$ and $q \in W^{2(\ell+1),\infty}(\Omega)$ fixed, we define the set of "admissible scalar potentials" as

$$\mathcal{Q}_M(q_0) := \{q \in W^{2(\ell+1),\infty}(\Omega;\mathbb{R}),\ \|q\|_{W^{2(\ell+1),\infty}(\Omega)} \leqslant M \text{ and } q \text{ satifies (1.3.2.3)}\}.$$

Moreover, for $\mathcal{O}$ a $C^{2(\ell+1)}$-domain obeying $\omega \times (-\ell,\ell) \subset \mathcal{O} \subset \omega \times (-L,L)$, we consider Γ_* satisfying $\Gamma_* \supset \{x \in \partial\mathcal{O},\ (x - x_0)\cdot\nu_{\mathcal{O}}(x) \geqslant 0\} \cap \partial\omega \times (-L,L)$, where $\nu_{\mathcal{O}}$ is the outward unit normal to $\partial\mathcal{O}$ and x_0 is arbitrary in $\mathbb{R}^n \setminus \overline{\mathcal{O}}$. We fix also $\gamma_* \supset \{x' \in \partial\omega,\ (x' - x_0')\cdot\nu'(x') \geqslant 0\}$, where x_0' is arbitrarily fixed in $\mathbb{R}^{n-1} \setminus \overline{\omega}$.

In [KiPhSo1], we are concerned with the stability issue around any $q_1 \in \mathcal{Q}_M(q_0)$, i.e. we want to upper bound the L^2 norm of $q_1 - q_2$ by some increasing function of the difference $u_1 - u_2$. That is to say that $q_2 \in \mathcal{Q}_M(p)$ and the solution u_j, for $j = 1, 2$, to (1.3.0.2), where q_j is substituted for q, are known, while q_1 is unknown.

Theorem 1.3.2 (Theorem 1.1, [KiPhSo1]) *consider $\partial\omega$, q_0, u_0 and g obeying the conditions of Theorem 1.3.1 for $k = \ell + 1$, where ℓ is the same as in* (1.3.2.5)*. Let u_0 satisfy*

$$u_0(x) \geqslant \alpha > 0,\ x \in \omega \times (-\ell,\ell), \tag{1.3.3.1}$$

let $q_j \in \mathcal{Q}_M(q_0)$, $j = 1, 2$, fulfill

$$q_1(x) = q_2(x),\ x \in \omega \times (\mathbb{R} \setminus (-\ell,\ell)), \tag{1.3.3.2}$$

and let u_j denote the $C^1([0,T];\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)) \cap C^2([0,T];\mathrm{L}^2(\Omega))$-solution to (1.3.0.2) *associated with u_0, $g = \gamma_0 G$ and q_j. Then for every $L > \ell$, there exist $\Gamma_* \subset \partial\omega \times (-L,L)$ and a constant $C > 0$ depending only on L, T, M, ω and Γ_*, such that we have*

$$\|q_1 - q_2\|_{\mathrm{L}^2(\Omega)} \leqslant C\left(\|\partial_\nu(\partial_t u_1 - \partial_t u_2)\|_{\mathrm{L}^2((0,T)\times\Gamma_*)} + \|u_1 - u_2\|_{\mathrm{H}^1(0,T;\mathrm{H}^1(\omega\times\mathcal{S}_L))}\right), \tag{1.3.3.3}$$

with $\mathcal{S}_L := (-L, -\ell) \cup (\ell, L)$.
Moreover there is a subboundary $\gamma_* \subset \partial\omega$*, such that the estimate*

$$\|q_1 - q_2\|_{\mathrm{L}^2(\Omega)} \leqslant C' \|\partial_\nu \partial_t u_1 - \partial_\nu \partial_t u_2\|_{\mathrm{L}^2((0,T)\times\gamma_*\times\mathbb{R})}, \tag{1.3.3.4}$$

holds for some positive constant C' *depending only on* ℓ*,* T*,* M*,* ω *and* γ_**.*

Under the prescribed conditions (1.3.2.1), (1.3.3.1) and Theorem 1.3.1, the first statement (1.3.3.3) of Theorem 1.3.2 claims Lipschitz stability in the determination of the scalar potential appearing in the dynamic Schrödinger equation in Ω from two different observations of the solution u to (1.3.0.2). The first one is a lateral measurement on some subboundary of $\partial\omega \times (-L, L)$ of the normal derivative $\partial_\nu u_{|\Sigma}$. The second observation is an internal measurement of u which is performed in each of the two "slices" $\mathcal{S}_L^- := \omega \times (-L, -\ell)$ and $\mathcal{S}_L^+ := \omega \times (\ell, L)$ of Ω.

One way to get rid of both volume observations simultaneously is to use a global Carleman estimate specifically designed for the unbounded quantum waveguide Ω, which is stated in [KiPhSo1, Proposition 3.3]. This yields (1.3.3.4), implying that the electrostatic quantum potential is now Lipschitz stably retrieved in Ω from only one lateral measurement of the normal derivative $\partial_\nu u$ on some subboundary of Σ. This result is similar to the one obtained in a bounded domain by Baudouin and Puel, see [BaPu02][Theorem 1]. It should nevertheless be noticed that, contrarily to (1.3.3.3), and despite of the fact that the scalar potential under identification is assumed to be known outside a compact set, the Neumann data required in the right hand side of (1.3.3.2) is measured on an infinitely extended subboundary of Σ.

In [KiPhSo2] we have extended the result of [KiPhSo1] to the stable recovery of coefficients that are not necessary compactly supported. Indeed, in [KiPhSo2] we aim to retrieve real-valued scalar potentials q verifying

$$|q(x', x_n) - q_0(x', x_n)| \leqslant a e^{-b\langle x_n\rangle^{d_\varepsilon}}, \ (x', x_n) \in \Omega, \tag{1.3.3.5}$$

where $a > 0$, $b > 0$, $\varepsilon > 0$ and $d_\varepsilon \in (2(1+\varepsilon)/3, +\infty)$ are *a priori* fixed constants. Here and henceforth the notation $\langle t\rangle$ stands for $(1+t^2)^{1/2}$, $t \in \mathbb{R}$. Notice that this condition is weaker than the compactness assumption imposed in [KiPhSo1] on the support of q. Namely, we introduce the set of admissible potentials as

$$\mathcal{A}_\varepsilon(q_0) := \{q \in W^{2(\ell+1),\infty}(\Omega) \cap \mathcal{C}^{2\ell}(\overline{\Omega}; \mathbb{R}) \text{ verifying (1.3.2.3) for } k = \ell+1 \text{ and (1.3.3.5)}\}.$$

The main result of [KiPhSo2] on the above mentioned inverse problem is as follows.

Theorem 1.3.3 (Theorem 1.4, [KiPhSo2]) *Let* $\partial\omega$*,* q_0*,* u_0 *and* g *obey the conditions of Theorem 1.3.1 for* $k = \ell + 1$*, where* ℓ *is the same as in* (1.3.2.5)*. Assume moreover that there are two constants* $\upsilon_0 > 0$ *and* $\varepsilon > 0$ *such that we have*

$$|u_0(x', x_n)| \geqslant \upsilon_0 \langle x_n\rangle^{-(1+\varepsilon)/2}, \ (x', x_n) \in \Omega. \tag{1.3.3.6}$$

For $M > 0$ *fixed, we consider two potentials* q_j*,* $j = 1, 2$*, in* $\mathcal{A}_\varepsilon(q_0)$*, such that* $\|q_j\|_{W^{2(\ell+1),\infty}(\Omega)} \leqslant M$*, and we note* u_j *the solution to* (1.3.0.2) *where* q_j *is substituted for* q*, given by Theorem 1.3.1. Then, for all* $\delta \in (0, b)$*, where* b *is the same as in* (1.3.3.5)*, there exists a subboundary* $\gamma_* \subset \partial\omega$ *and a constant* $C > 0$*, depending only on* ω*,* T*,* M*,* $\|u_0\|_{2(\ell+2),\Omega}$*,* δ*,* ε*,* a*,* b *and* υ_0*, such that the estimate*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C \|\partial_\nu \partial_t u_1 - \partial_\nu \partial_t u_2\|^\theta_{L^2(\Sigma_*)}, \tag{1.3.3.7}$$

holds for $\Sigma_* := (0, T) \times \gamma_* \times \mathbb{R}$ *and* $\theta := (b - \delta)/(2b - \delta)$*.*

It is evident that Theorem 1.3.3 yields uniqueness in the identification of the scalar potential in $\mathcal{A}_\varepsilon(q_0)$ from the knowledge of partial Neumann data for the time-derivative of the solution to (1.3.0.2):

$$\forall (q_1, q_2) \in \mathcal{A}_\varepsilon(q_0)^2, \ (\partial_\nu u_1'(t, x) = \partial_\nu u_2'(t, x), \ (t, x) \in \Sigma_*) \Longrightarrow (q_1(x) = q_2(x), \ x \in \Omega).$$

Moreover we stress out that a whole class of potentials q_0 and initial values u_0 fulfilling the conditions of Theorem 1.3.1 and 1.3.3 is exhibited in [KiPhSo2, Subsection 3.4]. To our best knowledge, this result is the first result of stable recovery of a non-compactly supported time-independent coefficient appearing in Schrödinger equation from a single boundary measurement.

In contrast to [KiPhSo1, KiPhSo2], the result of [BeKiSo1] is stated with measurement on a subset of $\partial\Omega$ of the form $S_* \times \mathbb{R}$, where this time S_* is an arbitrary relatively open subset of $\partial\omega$. Like in [BeCh09], such a result requires an extra information about the admissible potential q. Namely, we need to know the coefficient q on an arbitrary neighborhood of the boundary Γ. This extra information is technically restrictive, but it is acceptable from a strict practical viewpoint upon admitting that the electric potential can be measured from outside the domain Ω in the vicinity of the boundary. We consider $q_0 \in W^{2(\ell+1),\infty}(\Omega;\mathbb{R})$ and pick an open subset ω_0 of ω, such that $\partial\omega \subset \overline{\omega_0}$. Given $b > 0$ and $d > 0$, we aim in [BeKiSo1] to retrieve all functions $q : \Omega \to \mathbb{R}$ satisfying

$$\mathcal{N}_{b,d}(q - q_0) := \|e^{b\langle x_n\rangle^d}(q - q_0)\|_{L^\infty(\Omega)} < \infty \text{ and } q(x) = q_0(x) \text{ for } x \in \Omega_0 := \omega_0 \times \mathbb{R}.$$

Further, M being an *a priori* fixed non-negative constant, we define the set of admissible potentials as

$$\mathcal{A}(q_0, \omega_0) := \{q \in W^{2(\ell+1),\infty}(\Omega);\ q = q_0 \text{ in } \Omega_0,\ \|q\|_{W^{2(\ell+1),\infty}(\Omega)} \leqslant M \text{ and } \mathcal{N}_{b,d}(q - q_0) \leqslant M\}.$$

Last, we choose a relatively open subset S_* of $\partial\omega$, put $\Gamma_* := S_* \times \mathbb{R}$, and introduce the norm

$$\|\partial_\nu u\|_* := \|\partial_\nu u\|_{H^1(0,T;L^2(\Gamma_*))},\ u \in \mathcal{H}^2.$$

The main result of [BeKiSo1] is as follows.

Theorem 1.3.4 (Theorem 1.3, [BeKiSo1]) *Let condition* (1.3.2.6) *and the conditions of Theorem 1.3.1 with* $k = \ell + 1$, $q = q_0$ *and* $g = 0$ *be satisfied.*

Assume moreover that u_0 *fulfills* $\|u_0\|_{H^{2(\ell+1)}(\Omega)} \leqslant M'$ *for some constant* $M' > 0$, *and that*

$$\exists \kappa > 0,\ \exists d_0 \in (0, 2d/3),\ |u_0(x', x_n)| \geqslant \kappa\langle x_n\rangle^{-d_0/2},\ (x', x_n) \in \Omega\backslash\Omega_0. \tag{1.3.3.8}$$

For $q_j \in \mathcal{A}(q_0, \omega_0)$, $j = 1, 2$, *we denote by* u_j *the solution to* (1.3.0.2), *where* q_j *is substituted for* q. *Then, for any* $\varepsilon \in (0, \ell/2)$, *there exists a constant* $C = C(\omega, \omega_0, T, M, M', b, d, \varepsilon) > 0$, *such that we have*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C\left(\|\partial_\nu(u_1 - u_2)\|_{H^1(0,T;L^2(\Gamma_*))} + \left|\log \|\partial_\nu(u_1 - u_2)\|_{H^1(0,T;L^2(\Gamma_*))}\right|^{-1}\right)^{\varepsilon}. \tag{1.3.3.9}$$

Let us remark that, we assume in (1.3.3.6) and (1.3.3.8) that $|u(\cdot, 0)| = |u_0| > 0$ in any subset of Ω where the electric potential is retrieved. This is because the uniqueness of the potential is not known in general, without this specific assumption, even in the case where the set $\{x \in \Omega \setminus \Omega_0; u_0(x) = 0\}$ has zero Lebesgue measure. This non-degeneracy condition is very restrictive but it is still an open question to know how it can be weakened in the context of inverse coefficients problems with a finite number of data observations.

1.3.4 Idea of the proof

The occurrence of the internal measurement of u in (1.3.3.3) is due to the unbounded geometry of Ω. More precisely this is a direct consequence of the technique used for the derivation of the stability inequality (1.3.3.3), which is by means of a global Carleman estimate for the Schrödinger equation in a bounded domain established by [BaPu02]. Indeed, this strategy requires a cut off function with first derivative supported in $(-L, -\ell) \cup (\ell, L)$, which gives rise to the measurement of u in $\mathcal{S}_L^{\pm}$. Notice that the use of a Carleman estimate known to be valid in a bounded domain of $\mathbb{R}^n$ only, was made possible here since the difference $q_1 - q_2$ is compactly supported in $\mathbb{R}^n$. A fact that follows from assumption (1.3.3.3) expressing that the

scalar potential to be retrieved is known outside some fixed bounded set. Since the unknown part of q is compactly supported, it seems at first sight quite reasonable to guess that this question could well be answered by adapting the above technique to some suitable truncation of u. Nevertheless we shall prove that such a strategy necessarily adds unexpected "control" terms (i.e. "volume observations" of the solution) in the right hand side of the corresponding stability inequality, and is therefore inaccurate. It turns out that this inconvenience can be avoided upon substituting some specifically designed Carleman estimate for the Schrödinger equation in an unbounded cylindrical domain to the one of [BaPu02][Proposition 3]. In order to introduce this Carleman estimate, we start by considering the function $\widetilde{\beta}$ defined on Ω by

$$\widetilde{\beta}(x) = |x' - x_0'|^2, \quad x = (x', x_n) \in \Omega,\ x' \in \omega,\ x_n \in \mathbb{R}.$$

Further, we put

$$\beta := \widetilde{\beta} + K, \text{ where } K := r\|\tilde{\beta}\|_\infty \text{ for some } r > 1, \tag{1.3.4.1}$$

and define the two following weight functions for $\lambda > 0$:

$$\varphi(t,x) := \frac{\mathrm{e}^{\lambda\beta(x)}}{(T+t)(T-t)} \text{ and } \eta(t,x) := \frac{\mathrm{e}^{2\lambda K} - \mathrm{e}^{\lambda\beta(x)}}{(T+t)(T-t)},\ (t,x) \in (-T,T)\times\Omega. \tag{1.3.4.2}$$

Finally, for all $s > 0$ we introduce the two following operators acting in $(C_0^\infty)'((-T,T)\times\Omega)$:

$$M_1 := i\partial_t + \Delta + s^2|\nabla\eta|^2 \text{ and } M_2 := is\eta' + 2s\nabla_{x'}\eta\cdot\nabla + s(\Delta_{x'}\eta). \tag{1.3.4.3}$$

It is easily seen that M_1 (resp. M_2) is the adjoint (resp. skew-adjoint) part of the operator $\mathrm{e}^{-s\eta}(i\partial_t - \Delta)\mathrm{e}^{s\eta}$. Here $\Delta := \Delta_{x'} + \partial_{x_n}^2$ where $\Delta_{x'} := \Sigma_{j=1}^{n-1}\partial_{x_j}^2$ is the Laplacian in ω, where $\nabla_{x'}$ (resp. $\Delta_{x'}$) stands for the gradient (resp. the Laplacian) operator w.r.t. $x' \in \omega$. Having said that we may now state the following global Carleman estimate

Proposition 1.3.1 (Proposition 3.3, [KiPhSo1]) *There are two constants $s_0 > 0$ and $C > 0$, depending only on T, ω and γ_*, such that the estimate*

$$\begin{aligned} & s\|\mathrm{e}^{-s\eta}\nabla_{x'}v\|_{\mathrm{L}^2(Q)}^2 + s^3\|\mathrm{e}^{-s\eta}v\|_{\mathrm{L}^2(Q)}^2 + \sum_{j=1,2}\|M_j\mathrm{e}^{-s\eta}v\|_{\mathrm{L}^2(Q)}^2 \\ \leqslant\ & C\left(s\|\mathrm{e}^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu v\|_{\mathrm{L}^2((-T,T)\times\gamma_*\times\mathbb{R})}^2 + \|\mathrm{e}^{-s\eta}(i\partial_t - \Delta)v\|_{\mathrm{L}^2(Q)}^2\right), \end{aligned} \tag{1.3.4.4}$$

holds for all $s \geqslant s_0$ and any function $v \in \mathrm{L}^2(-T,T;\mathrm{H}_0^1(\Omega))$ verifying $(i\partial_t - \Delta)v \in \mathrm{L}^2(Q)$ and $\partial_\nu v \in \mathrm{L}^2(-T,T;\mathrm{L}^2(\gamma_*\times\mathbb{R}))$.

This new global Carleman estimate is the main novelty of [KiPhSo1] as it is the main tool for generalizing the Lipschitz stability inequality of [BaPu02][Theorem 1] to the unbounded domain Ω under consideration. The derivation of this Carleman estimate follows from the combination of the Carleman estimate of [BaPu02] for the equation $i\partial_t - \Delta_{x'}$ on the bounded domain $(0,T)\times\omega$ and of some properties of the unitary transform $e^{it\partial_{x_n}^2}$ on $L^2(Q)$. Roughly speaking, we apply the unitary operator $e^{it\partial_{x_n}^2}$ to transform equations of the form $i\partial_t v - \Delta_x v = F$ on Q to equations of the form $i\partial_t w - \Delta_{x'}w = G(\cdot, x_n)$ on $(0,T)\times\omega$ for a.e. $x_n \in \mathbb{R}$. Combining this with [BaPu02][Proposition 3], we derive Proposition 1.3.1.

In [KiPhSo2], we use the Carleman estimate, stated in Proposition 1.3.1, for proving Theorem 1.3.3. In order to extend the result of [KiPhSo1] to the stable recovery of none compactly supported function q we use the decay of the difference of the coefficient with respect to the *a priori* known coefficient q_0 stated in (1.3.3.5). This approach allows to extend the stable recovery of [KiPhSo1] to none compactly supported potential, but Lipschitz stability (1.3.3.4) degenerated to the Hölder stability (1.3.3.7). Notice that in the framework of the Bukhgeim-Klibanov method in a bounded spatial domain Ω, it is crucial that $|u_0|$ be

bounded from below by a positive constant, uniformly in Ω. But since Ω is infinitely extended in [KiPhSo2], such a statement is incompatible with the square integrability property satisfied by u_0 in Ω. Therefore the usual non-degeneracy condition imposed on the initial condition function has to be weakened into (1.3.3.6). In the same spirit we point out that the derivation of a Carleman estimate in an unbounded domain such as Ω is not straightforward and does not directly follows from the corresponding known results in bounded domains.

In [BeKiSo2] we combine the approach developed in [KiPhSo2] with an extension of the approach of [BeCh09] to unbounded cylindrical domain in order to relax the geometrical condition imposed to γ_* and derive estimate (1.3.3.9). For this purpose, we use the so called Fourier-Bros-Iagolnitzer (abbreviated as FBI) transform in order to establish the connection between Schrödinger and parabolic equations. This application of the FBI transform, already used by [Le92, LeRo, Ph, RaTa, Ro91, Ro95, RoZu] for sharp unique continuation results, transform the solution to (1.3.0.2) to a solution of a parabolic equation in the vicinity of the boundary Γ. Then, applying a parabolic Carleman estimate, where no geometric condition is imposed on the control domain, to the FBI transform of the solution of (1.3.0.2) and applying the Carleman estimate 1.3.1 we derive (1.3.3.9).

1.4 Recovery of an electromagnetic potential in a waveguide

1.4.1 Statement of the problem

This section is devoted to [BeKiSo2] which is a joint work with Mourad Bellassoued and Eric Soccorsi. We consider $\Omega = \omega \times \mathbb{R} \subset \mathbb{R}^3$, where ω is a simply connected bounded open subset of $\mathbb{R}^2$ with C^2-boundary $\partial\omega$. Then, we introduce the IBVP

$$\begin{cases} i\partial_t u + \Delta_A u + qu = 0, & \text{in } Q := (0,T) \times \Omega, \\ u(0,\cdot) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma := (0,T) \times \Gamma, \end{cases} \tag{1.4.1.1}$$

where Δ_A is the Laplace operator associated with the magnetic potential $A \in W^{1,\infty}(\Omega)^3$,

$$\Delta_A := \sum_{j=1}^{3} (\partial_{x_j} + ia_j)^2 = \Delta + 2iA \cdot \nabla + i(\nabla \cdot A) - |A|^2, \tag{1.4.1.2}$$

and $q \in L^\infty(\Omega)$. We define the DN map associated with (1.4.1.1), as

$$\Lambda_{A,q}(f) := (\partial_\nu + iA \cdot \nu)u, \ f \in L^2(\Sigma), \tag{1.4.1.3}$$

where u is the solution to (1.4.1.1). In [BeKiSo2] we examine the uniqueness and stability issues in the inverse problem of determining in some suitable sense the electric potential q and the gauge class of A, from the knowledge of $\Lambda_{A,q}$.

1.4.2 Direct problem

We start by examining the well-posedness of the IBVP (1.4.1.1) in the functional space $\mathcal{C}([0,T], H^1(\Omega)) \cap \mathcal{C}^1([0,T], H^{-1}(\Omega))$. Namely, we are aiming for sufficient conditions on the coefficients A, q and the non-homogeneous Dirichlet data f, ensuring that (1.4.1.1) admits a unique solution in the transposition sense. We say that $u \in L^\infty(0,T; H^{-1}(\Omega))$ is a solution to (1.4.1.1) in the transposition sense, if the identity

$$\langle u, F \rangle_{L^\infty(0,T;H^{-1}(\Omega)), L^1(0,T;H_0^1(\Omega))} = \langle f, \partial_\nu v \rangle_{L^2(\Sigma)},$$

holds for any $F \in L^1(0,T;H_0^1(\Omega))$. Here v denotes the unique $\mathcal{C}([0,T],H^1(\Omega))$-solution to the transposition system

$$\begin{cases} (i\partial_t v + \Delta_A + q)v = F, & \text{in } Q, \\ v(T,\cdot) = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \Sigma. \end{cases}$$

Since $\partial\Omega$ is not bounded, we use the notation introduced in Subsection 1.2.3 and we define

$$H_0^{2,1}(\Sigma) := \{f \in H^{2,1}(\Sigma);\ f(0,\cdot) = \partial_t f(0,\cdot) = 0\}$$

and state the existence and uniqueness result of solutions to (1.4.1.1) in the transposition sense, as follows.

Theorem 1.4.1 (Theorem 1.1, [BeKiSo2]) *For $M > 0$, let $A \in W^{1,\infty}(\Omega,\mathbb{R})^3$ and $q \in W^{1,\infty}(\Omega,\mathbb{R})$ satisfy the condition*

$$\|A\|_{W^{1,\infty}(\Omega)^3} + \|q\|_{W^{1,\infty}(\Omega)} \leqslant M. \tag{1.4.2.1}$$

Then, for each $f \in H_0^{2,1}(\Sigma)$, the IBVP (1.4.1.1) admits a unique solution in the transposition sense $u \in H^1(0,T;H^1(\Omega))$, and the estimate

$$\|u\|_{H^1(0,T;H^1(\Omega))} \leqslant C\,\|f\|_{H^{2,1}(\Sigma)}\,, \tag{1.4.2.2}$$

holds for some positive constant C depending only on T, ω and M. Moreover, the normal derivative $\partial_\nu u \in L^2(\Sigma)$, and we have

$$\|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\,\|f\|_{H^{2,1}(\Sigma)}\,. \tag{1.4.2.3}$$

It is clear from the definition (1.4.1.3) and the continuity property (1.4.2.3), that the DN map $\Lambda_{A,q}$ belongs to $\mathcal{B}(H_0^{2,1}(\Sigma),L^2(\Sigma))$, the set of linear bounded operators from $H_0^{2,1}(\Sigma)$ into $L^2(\Sigma)$.

1.4.3 Obstructions to uniqueness

Two magnetic potentials $A_j \in W^{1,\infty}(\Omega)^3$, $j = 1,2$, are said gauge equivalent, if there exists $\Psi \in W^{2,\infty}(\Omega)$ obeying $\Psi|_\Gamma = 0$, such that

$$A_2 = A_1 + \nabla\Psi. \tag{1.4.3.1}$$

There is a natural obstruction to the identification of A by $\Lambda_{A,q}$, arising from the invariance of the DN map under gauge transformation. More precisely, if $\Psi \in W^{2,\infty}(\Omega)$ verifies $\Psi|_\Gamma = 0$, then we have $u_{A+\nabla\Psi} = e^{-i\Psi}u_A$, where u_A (resp., $u_{A+\nabla\Psi}$) denotes the solution to (1.4.1.1) associated with the magnetic potential A (resp., $A+\nabla\Psi$), $q \in L^\infty(\Omega)$ and $f \in H_0^{2,1}(\Sigma)$. Further, as

$$(\partial_\nu + i(A+\nabla\Psi)\cdot\nu)u_{A+\nabla\Psi} = e^{-i\Psi}(\partial_\nu + iA\cdot\nu)u_A = (\partial_\nu + iA\cdot\nu)u_A \text{ on } \Sigma,$$

by direct calculation, we get that $\Lambda_{A,q} = \Lambda_{A+\nabla\Psi,q}$, despite of the fact that the two potentials A and $A+\nabla\Psi$ do not coincide in Ω (unless ψ is uniformly zero).

This shows that the best we can expect from the knowledge of the DN map is to identify (A,q) modulo gauge transformation of A. Assuming that A is known on $\partial\Omega$, this may be equivalently reformulated as to whether the magnetic field defined by the 2-form

$$\mathrm{d}A := \frac{1}{2}\sum_{i,j=1}^{3}(\partial_{x_j}a_i - \partial_{x_i}a_j)\mathrm{d}x_j \wedge \mathrm{d}x_i,$$

and the electric potential q, can be retrieved by $\Lambda_{A,q}$. This is the inverse problem that we examine [BeKiSo2].

1.4.4 Stable recovery of the electromagnetic potential

We define the set of admissible magnetic potentials as

$$\mathcal{A} := \left\{ A = (a_i)_{1\leqslant i\leqslant 3};\ a_1, a_2 \in L^\infty_{x_3}(\mathbb{R}, H^2_0(\omega)) \cap W^{2,\infty}(\Omega) \text{ and } a_3 \in C^3(\overline{\Omega}) \text{ satisfies (1.4.4.1)} - \text{(1.4.4.2)} \right\},$$

where

$$\sup_{x\in\Omega} \left(\sum_{\alpha\in\mathbb{N}^3, |\alpha|\leqslant 3} \langle x_3\rangle^d |\partial_x^\alpha a_3(x)| \right) < \infty \text{ for some } d > 1, \tag{1.4.4.1}$$

and

$$\partial_x^\alpha a_3(x) = 0,\ x \in \partial\Omega,\ \alpha \in \mathbb{N}^3 \text{ such that } |\alpha| \leqslant 2. \tag{1.4.4.2}$$

Here $H^2_0(\omega)$ denotes the closure of $\mathcal{C}^\infty_0(\omega)$ in the $H^2(\omega)$-topology, and $\langle x_3\rangle := (1 + x_3^2)^{1/2}$.

The first result of [BeKiSo2] claims stable determination of the magnetic field $\mathrm{d}A$ and unique identification of electric potential q, from the knowledge of the full data, i.e. the DN map defined by (1.4.1.3), where both the Dirichlet and Neumann measurements are performed on the whole boundary Σ.

Theorem 1.4.2 (Theorem 1.2, [BeKiSo2]) *Fix $A_* := (a_{i,*})_{1\leqslant i\leqslant 3} \in W^{2,\infty}(\Omega;\mathbb{R})^3$, and for $j = 1, 2$, let $q_j \in W^{1,\infty}(\Omega;\mathbb{R})$, and $A_j := (a_{i,j})_{1\leqslant i\leqslant 3} \in A_* + \mathcal{A}$, satisfy the condition:*

$$\sum_{i=1}^{2} \partial_{x_i} \left(\partial_{x_3}(a_{i,1} - a_{i,2}) - \partial_{x_i}(a_{3,1} - a_{3,2}) \right) = 0,\ \text{in } \Omega. \tag{1.4.4.3}$$

Then, $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$ yields $(dA_1, q_1) = (dA_2, q_2)$.
Assume moreover that the estimate

$$\sum_{j=1}^{2} \left(\|A_j\|_{W^{2,\infty}(\Omega)} + \|q_j\|_{W^{1,\infty}(\Omega)} + \|e_j\|_{W^{3,\infty}(\Omega)} \right) + \|A_*\|_{W^{2,\infty}(\Omega)} \leqslant M, \tag{1.4.4.4}$$

holds for some $M > 0$, with

$$e_j(x', x_3) := \int_{-\infty}^{x_3} (a_{3,j}(x', y_3) - a_{3,*}(x', y_3))\, dy_3,\ (x', x_3) \in \Omega.$$

Then there exist two constants $\mu_0 \in (0, 1)$ and $C > 0$, both of them depending only on T, ω and M, such that we have

$$\|dA_1 - dA_2\|_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} \leqslant C \left\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\right\|^{\mu_0}. \tag{1.4.4.5}$$

In (1.4.4.5), $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(H^{2,1}_0(\Sigma), L^2(\Sigma))$. Notice that in Theorem 1.4.2 we make use of the full DN map, as the magnetic field $\mathrm{d}A$ and the electric potential q are recovered by observing the solution to (1.4.1.1) on the entire lateral boundary Σ. In this case we may consider general unknown coefficients, in the sense that the behavior of A and q with respect to the infinite variable is not prescribed (we only assume that these coefficients and their derivatives are uniformly bounded in Ω). In order to achieve the same result by measuring on a bounded subset of Σ only, we need some extra information on the behavior of the unknown coefficients with respect to x_3. Namely, we impose that the strength of the magnetic field generated by $A = (a_i)_{1\leqslant i\leqslant 3}$, reaches its maximum in the bounded subset $(-r, r) \times \omega$ of Ω, for some fixed $r > 0$, i.e.

$$\|\partial_{x_i} a_j - \partial_{x_j} a_i\|_{L^\infty_{x_3}(\mathbb{R}, L^2(\omega))} = \|\partial_{x_i} a_j - \partial_{x_j} a_i\|_{L^\infty_{x_3}(-r,r; L^2(\omega))},\ i, j = 1, 2, 3. \tag{1.4.4.6}$$

Thus, with reference to (1.4.4.6), we set $\Gamma_r := \partial\omega \times (-r, r)$, introduce the space

$$H^{2,1}_0((0, T) \times \Gamma_r) := \{f \in H^{2,1}(\Sigma); f(0, \cdot) = \partial_t f(0, \cdot) = 0 \text{ and supp } f \subset [0, T] \times \partial\omega \times [-r, r]\},$$

and define the partial DN map $\Lambda_{A,q,r}$, by

$$\Lambda_{A,q,r}(f) := (\partial_\nu + iA\cdot\nu)u_{|(0,T)\times\Gamma_r},\ f\in H_0^{2,1}((0,T)\times\Gamma_r),$$

where u denotes the solution to (1.4.1.1). The following result states for each $r>0$, that the magnetic field induced by potentials belonging (up to an additive $W^{2,\infty}(\Omega,\mathbb{R})^3$-term) to

$$\mathcal{A}_r := \{A=(a_i)_{1\leqslant i\leqslant 3}\in\mathcal{A} \text{ satisfying (1.4.4.6)}\},$$

can be retrieved from the knowledge of the partial DN map $\Lambda_{A,q,r'}$, provided we have $r'>r$.

Theorem 1.4.3 (Theorem 1.3, [BeKiSo2]) *For $j=1,2$, let $q_j\in W^{1,\infty}(\Omega;\mathbb{R})$, and let $A_j\in W^{2,\infty}(\Omega;\mathbb{R})^3$ satisfy $A_1-A_2\in\mathcal{A}_r$, for some $r>0$. Suppose that there exists $r'>r$, such that $\Lambda_{A_1,q_1,r'}=\Lambda_{A_2,q_2,r'}$. Then, we have $dA_1=dA_2$. Furthermore, if*

$$\|q_1-q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} = \|q_1-q_2\|_{L^\infty_{x_3}(-r,r;H^{-1}(\omega))},$$

we have in addition $q_1=q_2$.
Assume moreover that (1.4.4.3)-(1.4.4.4) *hold. Then, the estimate*

$$\|dA_1-dA_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))^3} \leqslant C\,\|\Lambda_{A_1,q_1,r'}-\Lambda_{A_2,q_2,r'}\|^{\mu_1}, \tag{1.4.4.7}$$

holds with two constants $C>0$, and $\mu_1\in(0,1)$, depending only on T, ω, M, r and r'.

We stress out that Theorem 1.4.3 applies not only to magnetic (resp., electric) potentials A_j (resp., q_j), $j=1,2$, which coincide outside $\omega\times(-r,r)$, but to a fairly more general class of magnetic potentials, containing, e.g., $2r$-periodic potentials with respect to x_3. More generally, if $g\in W^{2,\infty}(\mathbb{R},\mathbb{R}_+)$ (resp. $g\in W^{1,\infty}(\mathbb{R},\mathbb{R}_+)$) is an even and non-increasing function in $\mathbb{R}_+$, then it is easy to see that potentials of the form $g\times A_j$ (resp., $g\times q_j$), where A_j (resp., q_j) are suitable $2r$-periodic magnetic (resp., electric) potentials with respect to x_3, fulfill the conditions of Theorem 1.4.3.

Notice that the absence of stability for the electric potential q, manifested in both Theorems 1.4.2 and 1.4.3, arises from the infinite extension of the spatial domain Ω in the x_3 direction. Indeed, the usual derivation of a stability equality for q, from estimates such as (1.4.4.5) or (1.4.4.7), requires that the differential operator d be invertible in Ω. Such a property is true in bounded domains (see e.g. [Tz]), but, to the best of our knowledge, it is not known whether it can be extended to unbounded waveguides. One way to overcome this technical difficulty is to impose certain gauge condition on the magnetic potentials, by prescribing their divergence. In this case, we establish in Theorem 1.4.3, below, that the electric and magnetic potentials can be simultaneously and stably determined by the DN map. We first introduce the set of divergence free transverse magnetic potentials,

$$\mathcal{A}_0 := \{A=(a_1,a_2,0);\ a_1,a_2\in L^\infty_{x_3}(\mathbb{R},H_0^2(\omega))\cap W^{2,\infty}(\Omega),\ \partial_{x_1}a_1+\partial_{x_2}a_2=0 \text{ in } \Omega\},$$

in such a way that we have $\nabla\cdot A=\nabla\cdot A_*$ for any $A\in A_*+\mathcal{A}_0$. Here $A_*\in W^{2,\infty}(\Omega)^3$ is an arbitrary fixed magnetic potential. Since identifying $A\in A_*+\mathcal{A}_0$ from the knowledge of the DN map, amounts to determining the magnetic field $\mathrm{d}A$, we have the following result.

Theorem 1.4.4 (Theorem 1.4, [BeKiSo2]) *Let $M>0$, and let $A_*\in W^{2,\infty}(\Omega,\mathbb{R})^3$. For $j=1,2$, let $q_j\in W^{1,\infty}(\Omega,\mathbb{R})$, and let $A_j\in A_*+\mathcal{A}_0$ satisfy* (1.4.4.4)*. Then, there exist two constant $\mu_2\in(0,1)$ and $C=C(T,\omega,M)>0$, such that we have*

$$\|A_1-A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))^3} + \|q_1-q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} \leqslant C\,\|\Lambda_{A_1,q_1}-\Lambda_{A_2,q_2}\|^{\mu_2}. \tag{1.4.4.8}$$

Assume moreover that the two following conditions

$$\|A_1 - A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))^3} = \|A_1 - A_2\|_{L^\infty_{x_3}(-r,r;L^2(\omega))^3}\,, \tag{1.4.4.9}$$

and

$$\|q_1 - q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} = \|q_1 - q_2\|_{L^\infty_{x_3}(-r,r;H^{-1}(\omega))}\,, \tag{1.4.4.10}$$

hold simultaneously for some $r > 0$. Then, for each $r' > r$, we have

$$\|A_1 - A_2\|_{L^\infty_{x_3}(\mathbb{R},L^2(\omega))} + \|q_1 - q_2\|_{L^\infty_{x_3}(\mathbb{R},H^{-1}(\omega))} \leqslant C\,\|\Lambda_{A_1,q_1,r'} - \Lambda_{A_2,q_2,r'}\|^{\mu_2}\,, \tag{1.4.4.11}$$

where C is a positive constant depending only on T, ω, M, r and r'.

1.4.5 Comments

The key ingredient in the analysis of the inverse problem under examination is a suitable set of GO solutions to the magnetic Schrödinger equation appearing in (1.4.1.1). These functions are specifically designed for the waveguide geometry of Ω, in such a way that the unknown coefficients can be recovered by a separation of variables argument. More precisely, we seek GO solutions that are functions of $x = (x', x_3) \in \Omega$, but where the transverse variable $x' \in \omega$ and the translational variable $x_3 \in \mathbb{R}$ are separated. This approach was already used in [Ki141] (see Section 1.2), for determining zero order unknown coefficients of the wave equation. Since we consider first order unknown coefficients in [BeKiSo2], the main issue here is to take into account both the cylindrical shape of Ω and the presence of the magnetic potential, in the design of the GO solutions.

When the domain Ω is bounded, we know from [BeCh10] that the magnetic field $\mathrm{d}A$ is determined by the DN map associated with (1.4.1.1). The main achievement of [BeKiSo2] is to extend the above statement to unbounded cylindrical domains. Actually, we also improve the results of [BeCh10] in two directions. First, we prove simultaneous determination of the magnetic field $\mathrm{d}A$ and the electric potential q. Second, the regularity condition imposed on admissible magnetic potentials entering the Schrödinger equation of (1.4.1.1), is weakened from $W^{3,\infty}(\Omega)$ to $W^{2,\infty}(\Omega)$.

To our best knowledge, [BeKiSo2] is the first mathematical paper claiming identification by boundary measurements, of non-compactly supported magnetic field and electric potential. Moreover, in contrast to the other works [BeKiSo1, KiPhSo1, KiPhSo2] dealing with the stability issue of inverse problems for the Schrödinger equation in an infinite cylindrical domain, available in the mathematics literature, here we no longer require that the various unknown coefficients be periodic, or decay exponentially fast, in the translational direction of the waveguide.

Finally, since the conditions (1.4.4.6) and (1.4.4.9)-(1.4.4.10) are imposed in $\omega \times (-r, r)$ only, and since the solution to (1.4.1.1) lives in the infinitely extended cylinder $(0, T) \times \Omega$, we point out that the results of Theorems 1.4.3 and 1.4.4 cannot be derived from similar statements stated in a bounded domain.

1.5 The Calderón problem in an unbounded cylindrical domain for periodic coefficients

1.5.1 Position of the problem

In this section we introduce the results of [ChKiSo161, ChKiSo162] which correspond to joint works with Mourad Choulli and Eric Soccorsi. Let $\Omega := \mathbb{R} \times \omega$, where ω is a bounded domain of $\mathbb{R}^2$ which contains the origin, with C^2-boundary. Throughout the entire text we denote the generic point $x \in \Omega$ by $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' := (x_2, x_3) \in \omega$. Given $V \in L^\infty(\Omega)$, real-valued and 1-periodic w.r.t. x_1, i.e.

$$V(x_1 + 1, x') = V(x_1, x'),\ x' \in \omega,\ x_1 \in \mathbb{R}, \tag{1.5.1.1}$$

we consider the following boundary value problem (abbreviated as BVP):

$$\left\{ \begin{array}{rcl} (-\Delta + V)v & = & 0, \quad \text{in } \Omega, \\ v & = & f, \quad \text{on } \Gamma := \partial\Omega. \end{array} \right. \tag{1.5.1.2}$$

Since $\Gamma = \mathbb{R} \times \partial\omega$, the outward unit vector ν normal to Γ reads

$$\nu(x_1, x') = (0, \nu'(x')), \ x = (x_1, x') \in \Gamma,$$

where ν' is the outer unit normal vector of $\partial\omega$. Therefore, for notational simplicity, we shall refer to ν for both exterior unit vectors normal to $\partial\omega$ and to Γ. Next for $\xi \in \mathbb{S}^1 := \{y \in \mathbb{R}^2; \ |y| = 1\}$ fixed, we introduce the ξ-illuminated (resp., ξ-shadowed) face of $\partial\omega$, as

$$\partial\omega_\xi^- := \{x \in \partial\omega; \ \xi \cdot \nu(x) \leqslant 0\} \text{ (resp., } \partial\omega_\xi^+ = \{x \in \partial\omega; \ \xi \cdot \nu(x) \geqslant 0\}). \tag{1.5.1.3}$$

Here and in the remaining part of this text, we denote by $x \cdot y := \sum_{j=1}^k x_j y_j$ the Euclidian scalar product of any two vectors $x := (x_1, \ldots, x_k)$ and $y := (y_1, \ldots, y_k)$ of $\mathbb{R}^k$, for $k \in \mathbb{N}^* := \{1, 2, \ldots\}$, and we put $|x| := (x \cdot x)^{1/2}$.

Set $G := \mathbb{R} \times G'$, where G' is an arbitrary closed neighborhood of $\partial\omega_\xi^-$ in $\partial\omega$. In [ChKiSo161, ChKiSo162], we seek stability in the determination of V from the knowledge of the partial DN map

$$\Lambda_V : f \mapsto \partial_\nu v_{|G}, \tag{1.5.1.4}$$

where v is the solution of (1.5.1.2). Otherwise stated we aim for recovering the 1-periodic electric perturbation V of the Dirichlet Laplacian in the waveguide Ω, by probing the system with voltage f at the boundary and measuring the current $\partial_\nu u$ on the sub-part G of Γ. From a physics viewpoint, this amounts to estimating the impurity potential perturbing the guided propagation in periodic media such as crystals.

1.5.2 State of the art

Since the seminal paper [Ca] by Calderón, the electrical impedance tomography problem, or Calderón problem, of retrieving the conductivity from the knowledge of the DN map on the boundary of a bounded domain, has attracted many attention. If the conductivity coefficient is scalar, then the Liouville transform allows us to rewrite the Calderón problem into the inverse problem of determining the electric potential in Laplace operator, from boundary measurements. There is an extensive literature on the Calderón problem. For isotropic conductivities, a great deal of work has been spent to weaken the regularity assumption on the conductivity required by [SyUh], in the study of the uniqueness issue, see e.g. [AsPa, CaRo, HaTa]. In all the above mentioned papers, the full DN map are needed, i.e. lateral observations are performed on the whole boundary. The first uniqueness result from partial data for the Calderón problem, was obtained in dimension 3 or greater, by Bukhgeim and Uhlmann in [BuUh]. Their result, which requires that Dirichlet data be imposed on the whole boundary, and that Neumann data boundary be observed on slightly more than half of the boundary, was improved by Kenig, Sjöstrand and Uhlmann in [KeSjUh], where both input and ouput data are measured on subsets of the boundary. In the two-dimensional case, Imanuvilov, Uhlmann and Yamamoto proved in [ImUhYa10, ImUhYa12] that the partial DN map uniquely determines the conductivity. These last results were extended to Riemann surfaces by Guillarmou, Tzou in [GuTz11-1] who considered also the recovery of a connection in [GuTz11-2]. We also mention that the special case of the Calderón problem in a bounded cylindrical domain of $\mathbb{R}^3$, was treated in [ImYa132].

The stability issue for the Calderón problem was addressed by Alessandrini in [Al]. He proved a log-type stability estimate with respect to the full DN map. Such a result, which is known to be optimal, see [Ma], degenerates to log-log stability with partial Neumann data, see [HeWa]. In [CaDoRu14, CaDoRu16], Caro, Dos Santos Ferreira and Ruiz proved stability results of log-log type, corresponding to the uniqueness results of [KeSjUh] in dimension 3 or greater. We refer to [BaFaRu, San] for stability estimates associated with the two-dimensional Calderón problem, and we point out that both the electric and the magnetic potentials are stably determined by the partial DN map in [Tz, Pot1, Pot2].

1.5.3 Notations and admissible potentials

In this subsection we introduce some basic notations used throughout the section and define the set of admissible potentials under consideration in [ChKiSo161, ChKiSo162].

Let Y be either ω, $\partial\omega$ or G'. For r and s in $\mathbb{R}$, we denote by $\mathcal{H}^{r,s}(\mathbb{R}\times Y)$ the set $H^r(\mathbb{R};H^s(Y))$. Evidently we write $\mathcal{H}^{r,s}(\Omega)$ (resp., $\mathcal{H}^{r,s}(\Gamma)$, $\mathcal{H}^{r,s}(G)$) instead of $\mathcal{H}^{r,s}(\mathbb{R}\times\omega)$ (resp., $\mathcal{H}^{r,s}(\mathbb{R}\times\partial\omega)$, $\mathcal{H}^{r,s}(\mathbb{R}\times G')$). Although this notation is reminiscent of the one used by Lions and Magenes in [LiMa1] for anisotropic Sobolev spaces $H^r(\mathbb{R};L^2(Y))\cap L^2(\mathbb{R};H^s(Y))$, it is worth noticing that they do not coincide with $\mathcal{H}^{r,s}(\mathbb{R}\times Y)$, unless we have $r=s=0$. Next, it is easy to see for each $r>0$ and $s>0$ that $\mathcal{H}^{-r,-s}(\mathbb{R}\times Y)$ is canonically identified with the space dual to $\mathcal{H}_0^{r,s}(\mathbb{R}\times Y)$, with respect to the pivot space $\mathcal{H}^{0,0}(\mathbb{R}\times Y)=L^2(\mathbb{R}\times Y)$. Here we have set $\mathcal{H}_0^{r,s}(\mathbb{R}\times Y):=H^r(\mathbb{R};H_0^s(Y))$, where $H_0^s(Y)$ denotes the closure of $C_0^\infty(Y)$ in the topology of the Sobolev space $H^s(Y)$.

Further, X_1 and X_2 being two Hilbert spaces, we denote by $\mathcal{B}(X_1,X_2)$ the class of bounded operators $T:X_1\to X_2$.

Let us now introduce the set of admissible unknown potentials. To this end we denote by C_ω the Poincaré constant associated with ω, i.e. the largest of those constants $c>0$ such that the Poincaré inequality

$$\|\nabla' u\|_{L^2(\omega)}\geqslant c\|u\|_{L^2(\omega)},\ u\in H_0^1(\omega), \tag{1.5.3.1}$$

holds. Here ∇' stands for the gradient with respect to $x'=(x_2,x_3)$. Otherwise stated, we have

$$C_\omega:=\sup\{c>0 \text{ satisfying (1.5.3.1)}\}. \tag{1.5.3.2}$$

For $M_-\in(0,C_\omega)$ and $M_+\in[M_-,+\infty)$, we define the set of admissible unknown potentials as

$$\mathscr{V}_\omega(M_\pm):=\{V\in L^\infty(\Omega;\mathbb{R}) \text{ satisfying (1.5.1.1)},\ \|V\|_{L^\infty(\Omega)}\leqslant M_+ \text{ and } \|\max(0,-V)\|_{L^\infty(\Omega)}\leqslant M_-\}. \tag{1.5.3.3}$$

Notice that the constraint $\|\max(0,-V)\|_{L^\infty(\Omega)}\leqslant M_-$, imposed on admissible potentials V in $\mathscr{V}_\omega(M_\pm)$, guarantees that the perturbation by V of the Dirichlet Laplacian in Ω, is boundedly invertible in $L^2(\Omega)$, with norm not greater than $(C_\omega-M_-)^{-1}$. This condition could actually be weakened by only requiring that the distance of the spectrum of this operator to zero, be positive. Nevertheless, since the above mentioned condition on V is more explicit than this latter, we stick with the definition (1.5.3.3) in the remaining part of this text.

1.5.4 Statement of the main results

We start with the result of [ChKiSo161]. Prior to stating the main results of this article we first examine in Proposition 1.5.1 below, the well-posedness of the BVP (1.5.1.2) in the space $H_\Delta(\Omega):=\{u\in L^2(\Omega);\ \Delta u\in L^2(\Omega)\}$ endowed with the norm

$$\|u\|^2_{H_\Delta(\Omega)}:=\|u\|^2_{L^2(\Omega)}+\|\Delta u\|^2_{L^2(\Omega)},$$

for suitable non-homogeneous Dirichlet boundary data f. Second, we rigorously define the DN map Λ_V expressed in (1.5.1.4) and describe its main properties.

As a preamble, we introduce the two following trace maps by adapting the derivation of [LiMa1, Section 2, Theorem 6.5]. Namely, since $C_0^\infty(\overline{\Omega}):=\{u_{|\overline{\Omega}},\ u\in C_0^\infty(\mathbb{R}^3)\}$ is dense in $H_\Delta(\Omega)$, by [ChKiSo161, Lemma 2.1], we extend the mapping

$$\mathcal{T}_0u:=u_{|\Gamma}\ (\text{resp., } \mathcal{T}_1u:=\partial_\nu u_{|\Gamma}),\ u\in C_0^\infty(\overline{\Omega}),$$

into a continuous function $\mathcal{T}_0:H_\Delta(\Omega)\to\mathcal{H}^{-2,-\frac12}(\Gamma)$ (resp., $\mathcal{T}_1:H_\Delta(\Omega)\to\mathcal{H}^{-2,-\frac32}(\Gamma)$). We refer to [ChKiSo161, Lemma 2.2] and its proof, for more details.

Next we consider the space

$$\mathscr{H}(\Gamma) := \mathcal{T}_0 H_\Delta(\Omega) = \{\mathcal{T}_0 u;\ u \in H_\Delta(\Omega)\},$$

and notice from [ChKiSo161, Lemma 2.3] that $\mathcal{T}_0$ is bijective from $B := \{u \in L^2(\Omega);\ \Delta u = 0\}$ onto $\mathscr{H}(\Gamma)$. Therefore, with reference to [BuUh, NaSt], we put

$$\|f\|_{\mathscr{H}(\Gamma)} := \left\|\mathcal{T}_0^{-1} f\right\|_{H_\Delta(\Omega)} = \left\|\mathcal{T}_0^{-1} f\right\|_{L^2(\Omega)}, \tag{1.5.4.1}$$

where $\mathcal{T}_0^{-1}$ denotes the operator inverse to $\mathcal{T}_0 : B \to \mathscr{H}(\Gamma)$.

We have the following existence and uniqueness result for the BVP (1.5.1.2).

Proposition 1.5.1 (Proposition 1.1, [ChKiSo161]) *Pick $V \in \mathscr{V}_\omega(M_\pm)$, where $M_- \in (0, C_\omega)$ and $M_+ \in [M_-, +\infty)$ are fixed.*
1) *Then, for any $f \in \mathscr{H}(\Gamma)$, there exists a unique solution $v \in L^2(\Omega)$ to* (1.5.1.2)*, such that the estimate*

$$\|v\|_{L^2(\Omega)} \leqslant C \|f\|_{\mathscr{H}(\Gamma)}, \tag{1.5.4.2}$$

holds for some constant $C > 0$ depending only on ω and $M_\pm$.
2) *The DN map $\Lambda_V : f \mapsto \mathcal{T}_1 v_{|G}$ is a bounded operator from $\mathscr{H}(\Gamma)$ into $\mathcal{H}^{-2,-\frac{3}{2}}(G)$.*
3) *Moreover, for each $W \in \mathscr{V}_\omega(M_\pm)$, the operator $\Lambda_V - \Lambda_W$ is bounded from $\mathscr{H}(\Gamma)$ into $L^2(G)$.*

Put $\check{\Omega} := (0,1) \times \omega$. In view of Proposition 1.5.1, we now state the main result of [ChKiSo161].

Theorem 1.5.1 (Theorem 1.2, [ChKiSo161]) *Given $M_- \in (0, C_\omega)$ and $M_+ \in [M_-, +\infty)$, let $V_j \in \mathscr{V}_\omega(M_\pm)$ for $j = 1, 2$. Then, there exist two constants $C > 0$ and $\gamma_* \in (0,1)$, both of them depending only on ω, $M_\pm$ and G', such that the estimate*

$$\|V_1 - V_2\|_{H^{-1}(\check{\Omega})} \leqslant C\Phi\left(\|\Lambda_{V_1} - \Lambda_{V_2}\|\right), \tag{1.5.4.3}$$

holds for

$$\Phi(\gamma) := \begin{cases} \gamma & \text{if } \gamma \geqslant \gamma^*, \\ (\ln|\ln\gamma|)^{-1} & \text{if } \gamma \in (0, \gamma^*), \\ 0 & \text{if } \gamma = 0. \end{cases} \tag{1.5.4.4}$$

Here $\|\Lambda_{V_1} - \Lambda_{V_2}\|$ denotes the norm of $\Lambda_{V_1} - \Lambda_{V_2}$ in $\mathcal{B}(\mathscr{H}(\Gamma), L^2(G))$.

In [ChKiSo162] we extend the result of Theorem 1.5.1 by considering additional restriction on the support of the Dirichlet inputs. In order to introduce the result of [ChKiSo162], we fix F' an arbitrary closed neighborhood of $\partial\omega_\xi^+$ in $\partial\omega$, $F = \mathbb{R} \times F'$ and we consider Dirichlet data in $\mathscr{H}(\Gamma)$ which are supported in F, i.e. input functions belonging to

$$\mathscr{H}_c(F) := \{f \in \mathscr{H}(\Gamma);\ \mathrm{supp} f \subset F\}.$$

To any $f \in \mathscr{H}_c(F)$, we associate the unique solution $u \in H_\Delta(\Omega)$ to (1.5.1.2), given by Proposition 1.5.1, and define the partial DN map associated with (1.5.1.2), as

$$\Lambda_V^* : f \in \mathscr{H}_c(F) \mapsto \mathcal{T}_1 u_{|G}. \tag{1.5.4.5}$$

We recall from Proposition 1.5.1 that

$$\Lambda_V^* \in \mathcal{B}(\mathscr{H}_c(F), H^{-2}(\mathbb{R}, H^{-\frac{3}{2}}(G'))) \text{ and } \Lambda_V^* - \Lambda_W^* \in \mathcal{B}(\mathscr{H}_c(F), L^2(G)),\ V,\ W \in \mathscr{V}_\omega(M_\pm). \tag{1.5.4.6}$$

The main result of [ChKiSo162], which claims that unknown potentials of $\mathscr{V}_\omega(M_\pm)$ are stably determined in the elementary cell $\check{\Omega} := (0,1) \times \omega$, by the partial DN map, is stated as follows.

Theorem 1.5.2 (Theorem 1.1, [ChKiSo162]) *Let $V_j \in \mathscr{V}_\omega(M_\pm)$, $j = 1, 2$, where $M_+ \in [M_-, +\infty)$, $M_- \in (0, C_\omega)$, and C_ω is defined by* (1.5.3.2)*. Then, there exist two constants $C > 0$ and $\gamma_* > 0$, both of them depending only on ω, $M_\pm$, F', and G', such that the estimate*

$$\|V_1 - V_2\|_{H^{-1}(\check{\Omega})} \leqslant C\Phi\left(\left\|\Lambda^*_{V_1} - \Lambda^*_{V_2}\right\|\right), \tag{1.5.4.7}$$

holds with Φ given by (1.5.4.4)*. Here $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(\mathscr{H}_c(F), L^2(G))$.*

The statement of Theorem 1.5.1 and 1.5.2 remain valid for any periodic potential $V \in L^\infty(\Omega)$, provided 0 is in the resolvent set of A_V, the self-adjoint realization in $L^2(\Omega)$ of the Dirichlet Laplacian $-\Delta + V$. In this case, the multiplicative constants C and γ_*, appearing in (1.5.4.3)-(1.5.4.4) and (1.5.4.7), depend on (the inverse of) the distance $d > 0$, between 0 and the spectrum of A_V. In the particular case where $V \in \mathscr{V}_\omega(M_\pm)$, with $M_- \in (0, C_\omega)$, we have $d \geqslant C_\omega - M_-$, and the implicit condition $d > 0$ imposed on V, can be replaced by the explicit one on the negative part of the potential, i.e. $\|\max(0, -V)\|_{L^\infty(\Omega)} \leqslant M_-$.

1.5.5 Application to the Calderón Problem

The inverse problem addressed in Subsection 1.5.4 is closely related to the periodic Calderón problem in Ω, i.e. the inverse problem of determining the conductivity coefficient a, obeying

$$a(x_1 + 1, x') = a(x_1, x'),\ x' \in \omega,\ x_1 \in \mathbb{R}, \tag{1.5.5.1}$$

from partial boundary data of the BVP in the divergence form

$$\begin{cases} -\mathrm{div}(a\nabla u) &= 0, \quad \text{in } \Omega, \\ u &= f, \quad \text{on } \Gamma. \end{cases} \tag{1.5.5.2}$$

Let $\mathcal{T}_0$ denote the trace operator $u \mapsto u_{|\Gamma}$ on $H^1(\Omega)$. We equip the space $\mathscr{K}(\Gamma) := \mathcal{T}_0(H^1(\Omega))$ with the norm

$$\|f\|_{\mathscr{K}(\Gamma)} := \inf\{\|u\|_{H^1(\Omega)};\ \mathcal{T}_0 u = f\},$$

and recall for any $a \in C^1(\overline{\Omega})$ satisfying the ellipticity condition

$$a(x) \geqslant a_* > 0,\ x \in \Omega, \tag{1.5.5.3}$$

for some fixed positive constant a_*, that the BVP (1.5.5.2) admits a unique solution $u \in H^1(\Omega)$ for each $f \in \mathscr{K}(\Gamma)$. Moreover, the full DN map associated with (1.5.5.2), defined by $f \mapsto a\mathcal{T}_1 u$, where $\mathcal{T}_1 u := \partial_\nu u_{|\Gamma}$, is a bounded operator from $\mathscr{K}(\Gamma)$ to $H^{-1}(\mathbb{R}; H^{-\frac{1}{2}}(\partial\omega))$. Here, we rather consider the partial DN map,

$$\Sigma_a : f \in \mathscr{K}(\Gamma) \cap a^{-\frac{1}{2}}(\mathscr{H}_c(F)) \mapsto a\mathcal{T}_1 u_{|G}, \tag{1.5.5.4}$$

where $a^{-\frac{1}{2}}(\mathscr{H}_c(F)) := \{a^{-\frac{1}{2}} f;\ f \in \mathscr{H}_c(F)\}$.

Further, since the BVP (1.5.5.2) is brought by the Liouville transform into the form (1.5.1.2), with $V_a := a^{-\frac{1}{2}}\Delta a^{\frac{1}{2}}$, then, with reference to (1.5.3.3), we impose that V_a be bounded in Ω and satisfies the following conditions

$$\|V_a\|_{L^\infty(\Omega)} \leqslant M_+ \text{ and } \|\max(0, -V_a)\|_{L^\infty(\Omega)} \leqslant M_-, \tag{1.5.5.5}$$

where $M_- \in (0, C_\omega)$ and $M_+ \in [M_-, +\infty)$ are *a priori* arbitrarily fixed constants. Namely, we introduce the set of admissible conductivities, as

$$\mathscr{A}_\omega(a_*, M_\pm) := \left\{a \in C^1(\overline{\Omega}; \mathbb{R}) \text{ satisfying } \Delta a \in L^\infty(\Omega),\ \|a\|_{W^{1,\infty}(\Omega)} \leqslant M_+, (1.5.5.1), (1.5.5.3), \text{and } (1.5.5.5)\right\}. \tag{1.5.5.6}$$

We check by standard computations that the condition (1.5.5.5) is automatically verified, provided the conductivity $a \in \mathscr{A}_\omega(a_*, M_\pm)$ is taken so small that $\|a\|^2_{W^{1,\infty}(\Omega)} + 2a_*\|\Delta a\|_{L^\infty(\Omega)} \leqslant 4M_- a_*^2$, or even that

$$\|a\|_{W^{2,\infty}(\Omega)} \leqslant \frac{4M_-}{(4M_- + 1)^{\frac{1}{2}} + 1} a_*,$$

in the particular case where $a \in W^{2,\infty}(\Omega)$.

The main result of this subsection claims stable determination of such admissible conductivities a, from the knowledge of Σ_a. It is stated as follows.

Corollary 1.5.1 (Corollary 1.2, [ChKiSo162]) *Fix $a_* > 0$, and let $M_\pm$ be as in Theorem 1.5.2. Pick $a_j \in \mathscr{A}_\omega(a_*, M_\pm)$, for $j = 1, 2$, obeying*

$$a_1(x) = a_2(x), \ x \in \partial\Omega \tag{1.5.5.7}$$

and

$$\partial_\nu a_1(x) = \partial_\nu a_2(x), \ x \in F \cap G. \tag{1.5.5.8}$$

Then $\Sigma_{a_1} - \Sigma_{a_2}$ is extendable to a bounded operator from $a_1^{-\frac{1}{2}}(\mathscr{H}_c(F))$ into $L^2(G)$. Moreover, there exists two constant $C > 0$ and $\gamma_* > 0$, both of them depending only on ω, $M_\pm$, a_*, F', and G', such that we have

$$\|a_1 - a_2\|_{H^1(\check{\Omega})} \leqslant C\Phi\left(a_*^{-\frac{1}{2}} \|\Sigma_{a_1} - \Sigma_{a_2}\|\right), \tag{1.5.5.9}$$

where Φ is the same as in Theorem 1.5.1. Here $\|\cdot\|$ denotes the usual operator norm in $\mathcal{B}(a_1^{-\frac{1}{2}}(\mathscr{H}_c(F)), L^2(G))$.

1.5.6 Floquet decomposition

In this subsection, we reformulate the inverse problem presented in Subsection 1.5.4 into a family of inverse coefficients problems associated with the BVP

$$\begin{cases} (-\Delta + V)v &= 0, \quad \text{in } \check{\Omega} := (0,1) \times \omega, \\ v &= g, \quad \text{on } \check{\Gamma} := (0,1) \times \partial\omega, \\ v(1,\cdot) - e^{i\theta} v(0,\cdot) &= 0, \quad \text{in } \omega, \\ \partial_{x_1} v(1,\cdot) - e^{i\theta}\partial_{x_1} v(0,\cdot) &= 0, \quad \text{in } \omega, \end{cases} \tag{1.5.6.1}$$

for $\theta \in [0, 2\pi)$, and suitable Dirichlet data g. This is by means of the Floquet-Bloch-Gel'fand (FBG) transform introduced in [ChKiSo161, Section 3.1]. We stick with the notations of [ChKiSo161, Section 3.1], and, for Y being either ω of $\partial\omega$, we denote by $\mathcal{U}$ the FBG transform from $L^2(\mathbb{R} \times Y)$ onto $\int_{(0,2\pi)}^{\oplus} L^2((0,1) \times Y)\frac{d\theta}{2\pi}$. That is to say, the FBG transform $\mathcal{U}$ maps $L^2(\Omega)$ onto $\int_{(0,2\pi)}^{\oplus} L^2(\check{\Omega})\frac{d\theta}{2\pi}$ if $Y = \omega$, and $L^2(\Gamma)$ onto $\int_{(0,2\pi)}^{\oplus} L^2(\check{\Gamma})\frac{d\theta}{2\pi}$ when $Y = \partial\omega$. We recall that the operator $\mathcal{U}$ is unitary in both cases. We start by introducing several functional spaces and trace operators that are needed by the analysis of the inverse problem associated with (1.5.6.1).

Functional spaces and trace operators

Fix $\theta \in [0, 2\pi)$. With reference to [ChKiSo14, Section 6.1] or [ChKiSo161, Section 3.1], we set for each $n \in \mathbb{N} \cup \{\infty\}$,

$$\mathcal{C}^n_\theta([0,1] \times \overline{\omega}) := \left\{u \in \mathcal{C}^n([0,1] \times \overline{\omega}); \ \partial^j_{x_1} u(1,\cdot) - e^{i\theta}\partial^j_{x_1} u(0,\cdot) = 0 \text{ in } \omega, \ j \leqslant n\right\},$$

and for Y being either ω or $\partial\omega$, we put

$$\mathcal{H}^s_\theta((0,1)\times Y) := \left\{ u \in H^s((0,1)\times Y);\ \partial^j_{x_1} u(1,\cdot) - e^{i\theta}\partial^j_{x_1} u(0,\cdot) = 0 \text{ in } \omega,\ j < s - \frac{1}{2} \right\} \text{ if } s > \frac{1}{2},$$

and

$$\mathcal{H}^s_\theta((0,1)\times Y) := H^s((0,1)\times Y) \text{ if } s \in \left[0, \frac{1}{2}\right].$$

Moreover, if X is a Banach space for the norm $\|\cdot\|_X$, we define $H^s_\theta(0,1;X)$, for $s \in \mathbb{R}$, as the set of functions

$$t \in (0,1) \mapsto \varphi(t) := \sum_{k\in\mathbb{Z}} \varphi_k e^{i(\theta+2\pi k)t} \text{ associated with } (\varphi_k)_k \in X^{\mathbb{Z}} \text{ obeying } \sum_{k\in\mathbb{Z}} (1+k^2)^s \|\varphi_k\|^2_X < \infty.$$

Further, we recall from [ChKiSo161, Eq. (3.29)] that $\mathcal{U}H_\Delta(\Omega) = \int^{\oplus}_{(0,2\pi)} H_{\Delta,\theta}(\check{\Gamma})\frac{d\theta}{2\pi}$, where

$$H_{\Delta,\theta}(\check{\Omega}) := \{u \in L^2(\check{\Omega});\ \Delta u \in L^2(\check{\Omega}) \text{ and } u(1,\cdot) - e^{i\theta}u(0,\cdot) = \partial_{x_1}u(1,\cdot) - e^{i\theta}\partial_{x_1}u(0,\cdot) = 0 \text{ in } \omega\}.$$

Moreover, the space $\mathcal{C}^\infty_\theta([0,2\pi]\times\overline{\omega})$ is dense in $H_{\Delta,\theta}(\check{\Omega})$, and we have $\mathcal{U}\mathcal{T}_j\mathcal{U}^{-1} = \int^{\oplus}_{(0,2\pi)} \mathcal{T}_{j,\theta}\frac{d\theta}{2\pi}$ for $j = 0,1$, where the linear bounded operator

$$\mathcal{T}_{j,\theta} : H_{\Delta,\theta}(\check{\Omega}) \to H^{-2}_\theta(0,1;H^{-\frac{2j+1}{2}}(\partial\omega)),$$

fulfills $\mathcal{T}_{0,\theta}u = u_{|\check{\Gamma}}$ if $j = 0$, and $\mathcal{T}_{1,\theta}u = \partial_\nu u_{|\check{\Gamma}}$ if $j = 1$, provided $u \in \mathcal{C}^\infty_\theta([0,1]\times\overline{\omega})$. Therefore, putting

$$\mathscr{H}_\theta(\check{\Gamma}) := \{\mathcal{T}_{0,\theta}u;\ u \in H_{\Delta,\theta}(\check{\Omega})\},\ \text{and } \mathscr{H}_{c,\theta}(\check{F}) := \{f \in \mathscr{H}_\theta(\check{\Gamma}),\ \text{supp } f \subset \check{F}\},$$

we get that $\mathcal{U}\mathscr{H}(\Gamma) = \int^{\oplus}_{(0,2\pi)} \mathscr{H}_\theta(\check{\Gamma})\frac{d\theta}{2\pi}$ and $\mathcal{U}\mathscr{H}_c(F) = \int^{\oplus}_{(0,2\pi)} \mathscr{H}_{c,\theta}(\check{F})\frac{d\theta}{2\pi}$. As in [ChKiSo161, Eq. (3.30)], the space $\mathscr{H}_\theta(\check{\Gamma})$ is endowed with the norm $\|g\|_{\mathscr{H}_\theta(\check{\Gamma})} := \|v_g\|_{L^2(\check{\Omega})}$, where v_g denotes the unique $L^2(\check{\Omega})$-solution to (1.5.6.1) with $V = 0$, given by Proposition 1.5.1.

Inverse fibered problems

Let $V \in \mathscr{V}_\omega(M_\pm)$, where $M_\pm$ are as in Theorem 1.5.1. Then, for any f be in $\mathscr{H}_c(F)$, u is the $H_\Delta(\Omega)$-solution to (1.5.1.2), if and only if, for almost every $\theta \in [0,2\pi)$, $(\mathcal{U}u)_\theta$ is the $H_\Delta(\check{\Omega})$-solution to (1.5.6.1), associated with $g = (\mathcal{U}f)_\theta \in \mathscr{H}_{c,\theta}(\check{F})$. The corresponding partial DN map, defined by $\Lambda_{V,\theta} : g \in \mathscr{H}_{c,\theta}(\check{F}) \mapsto \mathcal{T}_{1,\theta}v_{|\check{G}}$, where v is the unique $H_\Delta(\check{\Omega})$-solution to (1.5.6.1), is a bounded operator from $\mathscr{H}_{c,\theta}(\check{F})$ into $H^{-2}_\theta(0,1;H^{-\frac{3}{2}}(G'))$, and we have

$$\mathcal{U}\Lambda_V\mathcal{U}^{-1} = \int^{\oplus}_{(0,2\pi)} \Lambda_{V,\theta}\frac{d\theta}{2\pi}, \tag{1.5.6.2}$$

according to [ChKiSo161, Proposition 7.1]. Further, if V_1 and V_2 are two potentials lying in $\mathscr{V}_\omega(M_\pm)$, then $\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta} \in \mathcal{B}(\mathscr{H}_{c,\theta}(\check{F}), L^2(\check{G}))$, for each $\theta \in [0,2\pi)$, by (1.5.4.6) and (1.5.6.2). Moreover, $\Lambda_{V_1} - \Lambda_{V_2}$ being unitarily equivalent to the family of partial DN maps $\{\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta},\ \theta \in [0,2\pi)\}$, it holds true that

$$\|\Lambda_{V_1} - \Lambda_{V_2}\|_{\mathcal{B}(\mathscr{H}_c(F);L^2(G))} = \sup_{\theta\in[0,2\pi)} \|\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta}\|_{\mathcal{B}(\mathscr{H}_{c,\theta}(\check{F}),L^2(\check{G}))}. \tag{1.5.6.3}$$

Therefore, it is clear from (1.5.6.3) that Theorem 1.5.1 and 1.5.2 are a byproduct of the following statement.

Theorem 1.5.3 (Theorem 1.3, [ChKiSo162]) *Let $M_\pm$ and V_j, $j = 1,2$, be as in Theorem 1.5.2. Fix $\theta \in [0,2\pi)$. Then, there exist two constants $C_\theta > 0$ and $\gamma_{\theta,*} > 0$, both of them depending only on ω, $M_\pm$, F', and G', such that we have*

$$\|V_1 - V_2\|_{H^{-1}(\check{\Omega})} \leqslant C_\theta \Phi_\theta\left(\|\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta}\|\right). \tag{1.5.6.4}$$

Here, Φ_θ is the function defined in Theorem 1.5.2 and $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(\mathscr{H}_{c,\theta}(\check{F}), L^2(\check{G}))$.

We notice that the constants C_θ and $\gamma_{\theta,*}$ of Theorem 1.5.3, may possibly depend on θ. Nevertheless, we infer from (1.5.6.3) that this is no longer the case for C and γ, appearing in the stability estimate (1.5.4.7) of Theorem 1.5.2, as we can choose $C = C_\theta$ and $\gamma = \gamma_{\theta,*}$ for any arbitrary $\theta \in [0, 2\pi)$. Therefore, we may completely leave aside the question of how C_θ and $\gamma_{\theta,*}$ depend on θ. For this reason, we shall not specify the possible dependence with respect to θ of the various constants appearing in the remaining part of this text. Finally, we stress out that the function Φ_θ does actually depend on θ through the constant γ_θ, as it is obtained by substituting γ_θ for θ in the definition (1.5.4.4).

1.5.7 Description of the result

The proofs of Theorem 1.5.1, 1.5.2, 1.5.3 rely on two different types of complex geometric optics (CGO) solutions to the quasi-periodic Laplace equation in $(0,1) \times \omega$. These functions are built by means of an extension arguments by periodicity and by means of a suitable Carleman estimate. This technique is inspired by [Ha, KeSjUh], but, in contrast to [CaDoRu14, CaDoRu16], due to the quasi-periodic boundary conditions imposed on the CGO solutions, we cannot apply the Carleman estimate of [KeSjUh]. Instead we establish a Carleman estimate with linear weight taking the form

Proposition 1.5.2 (Proposition 5.1, [ChKiSo161]) *Let $\xi \in \mathbb{S}^1$ and pick a, b in $\mathbb{R}$, with $a < b$, in such a way that we have*

$$\omega \subset \{x' \in \mathbb{R}^2;\ \xi \cdot x' \in (a,b)\}.$$

Put $d := b - a$. Then for all $\theta \in [0, 2\pi)$ and all $\tau > 0$, the estimate

$$\begin{aligned} & \frac{8\tau^2}{d}\|e^{-\tau\xi\cdot x'}u\|^2_{L^2(\check{\Omega})} + 2\tau\|e^{-\tau\xi\cdot x'}(\xi\cdot\nu)^{1/2}\partial_\nu u\|^2_{L^2(\check{\Gamma}^+_\xi)} \\ \leqslant\ & \|e^{-\tau\xi\cdot x'}\Delta u\|^2_{L^2(\check{\Omega})} + 2\tau\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu|^{1/2}\partial_\nu u\|^2_{L^2(\check{\Gamma}^-_\xi)}, \end{aligned} \tag{1.5.7.1}$$

holds for every $u \in \mathcal{C}^2_\theta([0,1]\times\overline{\omega})$ satisfying $u_{|\check{\Gamma}} = 0$. Here we used the notations $\check{\Gamma}^\pm_\xi := (0,1)\times\partial\omega^\pm_\xi$.

In [ChKiSo161], we build CGO solutions to the system

$$\left\{\begin{array}{rcll} (-\Delta + V)u & = & 0, & \text{in } \check{\Omega}, \\ u(1,\cdot) - e^{i\theta}u(0,\cdot) & = & 0, & \text{in } \omega, \\ \partial_{x_1}u(1,\cdot) - e^{i\theta}\partial_{x_1}u(0,\cdot) & = & 0, & \text{in } \omega, \end{array}\right. \tag{1.5.7.2}$$

associated with $V \in L^\infty(\check{\Omega};\mathbb{R})$ and $\theta \in [0, 2\pi)$. Namely, given a sufficiently large $\tau > 0$, we seek solutions of the form

$$u(x) = (1 + w(x))\, e^{\zeta\cdot x},\ x \in \check{\Omega}, \tag{1.5.7.3}$$

to (1.5.7.2), where $\zeta \in i(\theta + 2\pi\mathbb{Z}) \times \mathbb{C}^2$ is chosen in such a way that $\Delta e^{\zeta\cdot x} = 0$ for every $x \in \check{\Omega}$, and $w \in \mathcal{H}^2_0(\check{\Omega})$ satisfies the estimate

$$\|w\|_{H^s(\check{\Omega})} \leqslant C\tau^{s-1},\ s \in [0,2], \tag{1.5.7.4}$$

for some positive constant C, independent of τ. Then combining these solutions with the Carleman estimate (1.5.7.1) and a result of stability of the unique continuation for analytic functions we complete in [ChKiSo161] the proof of Theorem 1.5.1. In order to state the result of [ChKiSo161] with additional restriction on the Dirichlet input, we build in [ChKiSo162] CGO solutions to the BVP

$$\left\{\begin{array}{rcll} (-\Delta + V)u & = & 0, & \text{in } \check{\Omega}, \\ u(1,\cdot) & = & e^{i\theta}u(0,\cdot), & \text{on } \omega, \\ \partial_{x_1}u(1,\cdot) & = & e^{i\theta}\partial_{x_1}u(0,\cdot), & \text{on } \omega, \\ u & = & 0, & \text{on } \check{\Gamma}^+_{\frac{\varepsilon}{2},-\xi} \end{array}\right. \tag{1.5.7.5}$$

taking the form (1.5.7.3). Here for $\varepsilon > 0$, we set

$$\partial\omega^+_{\varepsilon,\xi} := \{x' \in \partial\omega;\ \xi \cdot \nu'(x') > \varepsilon\} \text{ and } \partial\omega^-_{\varepsilon,\xi} := \{x' \in \partial\omega;\ \xi \cdot \nu'(x') \leqslant \varepsilon\}, \tag{1.5.7.6}$$

and we write $\check{\Gamma}^{\pm}_{\varepsilon,\xi}$ instead of $(0,1) \times \partial\omega^{\pm}_{\varepsilon,\xi}$. We build the CGO solutions of (1.5.7.5) by using the Carleman estimate (1.5.7.1) and by applying the Hahn-Banach theorem. This argument is inspired by [KeSjUh]. Nevertheless, in contrast to [KeSjUh], in our context we need to prove that our construction allows to preserve at the same time the quasiperiodic condition at $\{0\} \times \omega$, $\{1\} \times \omega$, and the homogeneous Dirichlet boundary condition on $\check{\Gamma}^{+}_{\frac{\varepsilon}{2},-\xi}$. Actually, these is the main difficulty in the proof of Theorem 1.5.3 and, in some sense, it makes an important difference with the usual application of this duality argument. To complete the construction of the CGO solutions of (1.5.7.5), we start by using the Carleman estimate (1.5.7.1) and arguments similar to [KeSjUh] in order to build CGO solutions satisfying only the homogeneous Dirichlet boundary condition on $\check{\Gamma}^{+}_{\frac{\varepsilon}{2},-\xi}$. Then, we prove that this solutions lies in some suitable space satisfying the quasiperiodic condition on $\{0\} \times \omega$, $\{1\} \times \omega$. For this purpose, we use the expansion in Fourier series of these solutions and we consider some duality arguments that allow to identify the quasiperiodic condition on $\{0\} \times \omega$, $\{1\} \times \omega$.

1.6 Stable recovery of a source term for the heat equation

1.6.1 Statement and origin of the problem

In this section consider [KiSaSo] which corresponds to a joint work with Diomba Sambou and Eric Soccorsi. Let $\omega \subset \mathbb{R}^{n-1}$, $n \geqslant 2$, be open and connected, with $\mathcal{C}^4$ boundary $\partial\omega$. Set $\Omega := \omega \times \mathbb{R}$ and $\Gamma := \partial\omega \times \mathbb{R}$. For $T \in (0, +\infty)$ fixed, we consider the parabolic initial boundary value problem (IBVP)

$$\begin{cases} \partial_t u - \Delta u = F(t,x) & \text{in } Q := (0,T) \times \Omega, \\ u(0,\cdot) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma := (0,T) \times \Gamma, \end{cases} \tag{1.6.1.1}$$

with source term $F \in L^2(Q)$. In [KiSaSo], we examine the inverse problem of determining F from a single Neumann boundary measurement of the solution u to (1.6.1.1).

Let us first notice that there is a natural obstruction to uniqueness in this problem. This can be easily understood from the identity $\partial_\nu u = 0$ on Σ, verified by any $u \in \mathcal{C}^\infty_0(Q)$, despite of the fact that the function $F := (\partial_t - \Delta)u$ may well be non uniformly zero in Q. Otherwise stated, the observation of $\partial_\nu u$ on Σ may be unchanged, whereas F is modified. To overcome this problem, different lines of research can be pursued. One of them is to extend the set of data available in such a way that F is uniquely determined by these observations. Another direction is the one of assuming that the source term F is *a priori* known to have the structure

$$F(t,x) = \sigma(t)\beta(x), \quad (t,x) \in Q, \tag{1.6.1.2}$$

where $t \mapsto \sigma(t)$ is a known function, and then proving that $\partial_\nu u$ uniquely determines β. In [KiSaSo] we investigate the second direction. Namely, we examine the stability issue in the identification of the time-independent part β of the source, from partial observation on Σ of the flux $\partial_\nu u$ induced by the solution u to (1.6.1.1).

Source terms of the form (1.6.1.2) are commonly associated with the reaction term in linear reaction diffusion equations. These equations arise naturally in various fields of application, investigating systems made of several interacting components, such as population dynamics [Mu], fluid dynamics [Bea], or heat conduction [BeBlCl]. More precisely, when $\sigma(t) := e^{-\mu t}$, where μ a positive constant, the system (1.6.1.1)-(1.6.1.2) describes the diffusion in transmission lines or cooling pipes with significantly large length-to-diameter ratio, of decay heat, that is the heat released as a result of radioactive decay. In this particular

case, (1.6.1.2) models a heat source produced by the decay of a radioactive isotope, and β is the spatial density of the isotope. From a practical viewpoint, the rate of decay μ of the isotope inducing the decay heat diffusion process, is known, and therefore the same is true for the function σ, while the density function β is generally unknown. This motivates for a closer look into the inverse problem under investigation in [KiSaSo].

1.6.2 Existing papers: a short review

Inverse source problems have been extensively studied over the last decades. We refer to [Is90] for a more general overview of this topic than the one presented in this subsection, where we solely focus on parabolic inverse source problems consisting in determining a source term by boundary measurements of the solution to a parabolic equation. Conditional stability for this problem was derived in [ChYa04, Ya93, Ya94]. In [ImYa98], inspired by the Bukhgeim-Klibanov approach introduced in [BuKl], Imanuvilov and Yamamoto proved Lipschitz stability of the source with respect to one Neumann boundary measurement of the solution to a parabolic equation with non-degenerate initial data, and partial Dirichlet data supported on arbitrary subregions of the boundary. In [ChYa06], Choulli and Yamamoto established a log-type stability estimate for the time-independent source term β, appearing in (1.6.1.2), by a single Neumann observation of the solution on an arbitrary sub-boundary.

All the above mentioned results are stated in a bounded spatial domain. But, to the best of our knowledge, there is no result available in the mathematical literature, dealing with the recovery of a non-compactly supported unknown source function, appearing in a parabolic equation, by boundary measurements of the solution. This is the starting point of [KiSaSo], in the sense that we aim for extending the stability result of [ChYa06], which is valid in bounded spatial domains only, to the framework of infinite cylindrical domains.

1.6.3 Well-posedness

Prior to describing the main achievement of [KiSaSo], we briefly investigate the well-posedness of the IBVP (1.6.1.1). Actually, we start by examining the forward problem associated with the IBVP

$$\begin{cases} \partial_t v - \Delta v = f & \text{in } Q, \\ v(0,\cdot) = v_0 & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma, \end{cases} \tag{1.6.3.1}$$

for suitable source term f and initial data v_0. More precisely, we seek an existence, uniqueness and (improved) regularity result for the solution to the above system, as well as a suitable energy estimate. Such results are rather classical in the case of bounded spatial domains, but it turns out that they are not so well-documented for unbounded domains such as Ω. Therefore, for the sake of completeness, we shall establish Theorem 1.6.1, presented below.

Theorem 1.6.1 (Theorem 1.1, [KiSaSo]) *Let $v_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;H_0^1(\Omega))$. Then, there exists a unique solution $v \in H^{1,2}(Q) \cap \mathcal{C}([0,T];H_0^1(\Omega))$ to the IBVP* (1.6.3.1)*, such that*

$$\|v(t)\|_{H^1(\Omega)} \leqslant \|v_0\|_{H^1(\Omega)} + T^{\frac{1}{2}} \|f\|_{L^2(0,T;H^1(\Omega))}, \ t \in [0,T]. \tag{1.6.3.2}$$

Theorem 1.6.1 is a crucial step in the derivation of the observability inequality (1.6.5.3) stated below, which is a cornerstone in the analysis of the inverse problem under investigation. But, just as important is the following consequence of Theorem 1.6.1, which enables us to define properly the boundary data used by the identification of the unknown function β in Theorem 1.6.2, below.

Corollary 1.6.1 (Corollary 1.1, [KiSaSo]) *Let F be defined by* (1.6.1.2)*, where $\sigma \in \mathcal{C}^1([0,T])$ and $\beta \in H_0^1(\Omega)$. Then, the IBVP* (1.6.1.1) *admits a unique solution $u \in H^{1,2}(Q)$. Moreover, we have $\partial_t u \in H^{1,2}(Q) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$, and the following estimate holds:*

$$\|\partial_t u(t)\|_{H^1(\Omega)} \leqslant (1+T)\|\sigma\|_{\mathcal{C}^1([0,T])}\|\beta\|_{H^1(\Omega)},\ t \in [0,T]. \tag{1.6.3.3}$$

1.6.4 Stable recovery of the source

For $M \in (0,+\infty)$ fixed, we introduce the set of admissible unknown source functions, as

$$B(M) := \left\{\varphi \in H_0^1(\Omega); \|\varphi\|_{H^1(\Omega)} \leqslant M\right\}. \tag{1.6.4.1}$$

Then, the main result of [KiSaSo] can be stated as follows.

Theorem 1.6.2 (Theorem 1.2, [KiSaSo]) *Put $\gamma := \gamma' \times \mathbb{R}$, where γ' is an arbitary closed subset of the boundary $\partial\omega$, with non empty interior, and let $\sigma \in \mathcal{C}^1([0,T])$ satisfy $\sigma(0) \neq 0$. For $M \in (0,+\infty)$, pick $\beta \in B(M)$, and let u be the $H^{1,2}(Q)$-solution to the IBVP* (1.6.1.1)*, associated with*

$$F(t,x) = \sigma(t)\beta(x),\ (t,x) \in Q,$$

which is given by Corollary 1.6.1. Then, there exists a constant $C > 0$, depending only on ω, σ, T, M and γ', such that the estimate

$$\|\beta\|_{L^2(\Omega)} \leqslant C\Phi\left(\|\partial_\nu u\|_{H^1(0,T;L^2(\gamma))}\right), \tag{1.6.4.2}$$

holds with

$$\Phi(r) := \begin{cases} r^{1/2} + |\ln r|^{-1/2} & \text{if } r \in (0,+\infty) \\ 0 & \text{if } r = 0. \end{cases} \tag{1.6.4.3}$$

Notice that we have $u \in H^1(0,T;H^2(\Omega))$ from Corollary 1.6.1, which guarantees that the trace $\partial_\nu u$ appearing in the right hand side of the stability estimate (1.6.4.2) is well-defined in $H^1(0,T;L^2(\gamma))$.

To the best of our knowledge, Theorem 1.6.2 is the first stability result in the identification of the non-compactly supported source term β, appearing in a parabolic equation, by a single partial boundary observation of the solution. A similar statement was actually derived in [ChYa06, Theorem 2.2] (see also [Ch09, Theorem 3.4]) when the domain Ω is bounded, so Theorem 1.6.2 extends this result to the case of infinite cylindrical domains.

Notice that the statement of Theorem 1.6.2 is valid in absence of any assumption on the behavior of the source term β outside a compact subset of the infinite cylindrical domain $\Omega = \omega \times \mathbb{R}$. Another remarkable feature of the result of Theorem 1.6.2 is that the logarithmic dependency of the space-varying source term, with respect to the boundary data, manifested in [Ch09, Theorem 2.2] for a bounded domain, is preserved by the stability estimate (1.6.4.2). Otherwise stated, the stability of the reconstruction of β by a single boundary observation of the solution, is not affected by the infinite extension of the support of the unknown coefficient. This phenomenon is in sharp contrast with the one observed for the determination of the electric potential appearing in the Schrödinger equation, by a finite number of Neuman data, where Lipschitz stability (see [BaPu02, Theorem 1] and [BaPu07, Theorem 1]) degenerates to Hölder (see [KiPhSo2, Theorem 1.4]), as the support of the unknown potential becomes infinite.

1.6.5 Information about the proof

The proof of Theorem 1.6.2 is by means of a Carleman inequality specifically designed for the heat operator in the unbounded cylindrical domain Ω. The derivation of this estimate is inspired by the approach used in this particular framework by [BeKiSo1, KiPhSo1, KiPhSo2] for the Schrödinger equation. In order to present this Carleman estimate, we pick a function $\psi_0 \in \mathcal{C}^4(\overline{\omega})$, such that

(c.i) $\psi_0(x') > 0$ for all $x' \in \overline{\omega}$;

(c.ii) $\exists \alpha_0 > 0$ such that $|\nabla' \psi_0(x')| \geqslant \alpha_0$ for all $x' \in \omega$;

(c.iii) $\partial_{\nu'} \psi_0(x') \leq 0$ for all $x' \in \partial\omega \backslash \gamma'$.

Here, ∇' denotes the gradient with respect to $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, i.e. $\nabla' f := (\partial_{x_1} f, \ldots, \partial_{x_{n-1}} f)$, and $\partial_{\nu'}$ is the normal derivative with respect to $\partial\omega$, that is $\partial_{\nu'} := \nu' \cdot \nabla'$, where ν' stands for the outward normal vector to $\partial\omega$.

Thus, putting $\psi(x) = \psi(x', x_n) := \psi_0(x')$ for all $x = (x', x_n) \in \overline{\Omega}$, it is apparent that the function $\psi \in \mathcal{C}^4(\overline{\Omega}) \cap W^{4,\infty}(\Omega)$ satisfies the three following conditions:

(C.i) $\inf_{x \in \Omega} \psi(x) > 0$;

(C.ii) $|\nabla \psi(x)| \geqslant \alpha_0 > 0$ for all $x \in \overline{\Omega}$;

(C.iii) $\partial_\nu \psi(x) \leqslant 0$ for all $x \in \Gamma \backslash \gamma$.

Next, for each $\rho \in (0, +\infty)$, we introduce the following weight function

$$\Phi_\rho(t, x) = \Phi_\rho(t, x') := g(t) \left(e^{\rho \psi(x')} - e^{2\rho \|\psi\|_{L^\infty(\Omega)}} \right) \text{ with } g(t) := \frac{1}{t(T-t)}, \ (t, x) \in Q, \tag{1.6.5.1}$$

Now, with reference to (1.6.1.1), we may state the Carleman estimate for the operator $P = \partial_t - \Delta$, as follows.

Theorem 1.6.3 (Theorem 3.1, [KiSaSo]) *Let $u \in H^{1,2}(Q) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$ be real valued. Then, there exists $\rho_0 \in (0, +\infty)$, such that for all $\rho \in [\rho_0, +\infty)$, there is $\lambda_0 = \lambda_0(\rho) \in (0, +\infty)$, depending only α_0, ω, γ', T and ρ, such that the estimate*

$$\begin{aligned} &\left\| e^{\lambda \Phi_\rho} (\lambda g)^{-1/2} \Delta u \right\|_{L^2(Q)} + \left\| e^{\lambda \Phi_\rho} (\lambda g)^{-1/2} \partial_t u \right\|_{L^2(Q)} + \left\| e^{\lambda \Phi_\rho} (\lambda g)^{1/2} |\nabla u| \right\|_{L^2(Q)} \\ &+ \left\| e^{\lambda \Phi_\rho} (\lambda g)^{\frac{3}{2}} u \right\|_{L^2(Q)} \leq C \left(\left\| e^{\lambda \Phi_\rho} P u \right\|_{L^2(Q)} + \left\| e^{\lambda \Phi_\rho} (\lambda g)^{1/2} \partial_\nu u \right\|_{L^2((0,T) \times \gamma)} \right), \end{aligned} \tag{1.6.5.2}$$

holds for all $\lambda \in [\lambda_0, +\infty)$ and some positive constant C, which depends only on α_0, ω, γ', T, ρ and λ_0.

Applying this Carleman estimate we derive the observability inequality.

Proposition 1.6.1 (Proposition 4.1, [KiSaSo]) *Let γ' and γ be the same as in Theorem 1.6.2. For $v_0 \in H_0^1(\Omega)$, let v be the $H^{1,2}(Q) \cap \mathcal{C}([0,T], H_0^1(\Omega))$-solution given by Theorem 1.6.1, to the IBVP* (1.6.3.1) *associated with $f = 0$. Then, there exists a constant $C > 0$, depending only on α_0, ω, γ', and T, such that we have*

$$\|v(T, \cdot)\|_{H^1(\Omega)} \leqslant C \|\partial_\nu v\|_{L^2((0,T) \times \gamma)}. \tag{1.6.5.3}$$

Combining this result with a decomposition in high and low frequency of the solution of (1.6.1.1) we complete the proof of Theorem 1.6.2.

Chapter 2

Recovery of time-dependent coefficients for evolution PDEs

2.1 Introduction

As mentioned in Section 1.1, the recovery of time-independent coefficients appearing in evolution PDEs has been intensively studied over the last decades. Such results have been considered with both single and many boundary measurements. Sometimes, the arguments used for these results can not be applied to the recovery of time-dependent coefficients. There is even counter-examples (see for instance Subsection 2.2.2 below for the obstruction to unique recovery of time-dependent coefficients from the hyperbolic DN map). This chapter is devoted to some results related to this issue. That is the unique and stable recovery of time-dependent coefficients appearing in different evolution PDEs (hyperbolic, parabolic and Schrödinger equations). Beside their own mathematical and physical interests, these problems are related to the recovery of nonlinear terms from boundary measurements. We discuss about this connection in Section 2.6.

We recall that, several authors considered the problem of determining time-dependent coefficients for evolution PDEs. These results concern in many case hyperbolic equations. For instance, we can mention [Stef89] dealing with the recovery of a time-dependent potential appearing in the wave equation from the knowledge of scattering data, [RaSj, Sala] dealing with recovery of time-dependent coefficients from measurements of forward solutions of wave equations on the infinite time-space cylindrical domain, [Es07, Es16] treating the specific case of time-dependent coefficients analytic with respect to the time variable and [BeBen, Ben15, RakRam] considering the recovery of some restriction of time-dependent coefficients. To our best knowledge, [Is911, Theorem 4.2] is the first result of global unique recovery of general time-dependent coefficients. Despite the general statement of [Is911, Theorem 4.2], this result requires important measurements including information at the initial and the final time. In this context, the main goal of the paper [Ki161, Ki162, Ki164, KiOk] was to preserve the general statement of [Is911, Theorem 4.2] but with less information. Namely, the main problem that we have considered in [Ki161, Ki162, Ki164, KiOk] concerns the minimal data that allow the global recovery of general time-dependent coefficients. We have considered these problems with different settings (bounded domain, Riemannian manifold), different coefficients (potential, damping coefficient) with both results of uniqueness and stability.

Some authors considered also the recovery of time-dependent coefficients appearing in parabolic equations [CaEs862, Ch911, ChKi13] and Schrödinger equations [ChKiSo14, ChKiSo152, Es08]. In contrast to hyperbolic equations, according to [Ch09, Es08], one can recover uniquely some time-dependent coefficients from measurements restricted to the lateral boundary Σ of the time-space cylindrical domain Q. In this context, we have considered in [ChKi16, KiSo] the stability issue for these problems.

This chapter is organized as follows. In Section 2.2 we introduce the work [Ki161, Ki162, Ki164] concern-

ing the recovery of time-dependent coefficients for hyperbolic equations in a bounded domain. In Section 2.3, we describe the extension of these results to hyperbolic equations in Riemannian manifolds stated in [KiOk]. In Section 2.4, we consider the stable recovery of electromagnetic potentials from boundary measurements of solutions of Schrödinger equations. In Section 2.5, we consider the stable recovery of zero order coefficients from full and partial boundary measurements for parabolic equations stated in [ChKi16]. Finally, in Section 2.6, we discuss about the application of the results introduced in Section 2.5 to the stable recovery of a nonlinear term appearing in a parabolic equation.

2.2 Recovery of time-dependent coefficients for hyperbolic equations

2.2.1 Statement

This section is devoted to the work [Ki161, Ki162, Ki164]. Let Ω be a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and fix $0 < T < \infty$. We consider the wave equation

$$\partial_t^2 u - \Delta u + a(t,x)\partial_t u + q(t,x)u = 0, \quad (t,x) \in Q, \tag{2.2.1.1}$$

where the damping coefficient $a \in L^\infty(Q)$ and the potential $q \in L^\infty(Q)$ are real valued. In [Ki161, Ki162, Ki164], we seek uniqueness and stability in the determination of both a and q from observations of solutions of (2.2.1.1) on $\partial Q = \Sigma \cup (\{0\} \times \overline{\Omega}) \cup (\{T\} \times \overline{\Omega})$.

2.2.2 Obstruction to uniqueness

Let $\Box$ and $L_{a,q}$ be the differential operators $\Box := \partial_t^2 - \Delta$, $L_{a,q} := \Box + a\partial_t + q$. It has been proved by [RakSy], that for $T > \mathrm{Diam}(\Omega)$ the data

$$\mathcal{A}_{a,q} = \{(u_{|\Sigma}, \partial_\nu u_{|\Sigma}) : \ u \in H^1(0,T;L^2(\Omega)), \ \Box u + a\partial_t u + qu = 0, \ u_{|t=0} = \partial_t u_{|t=0} = 0\} \tag{2.2.2.1}$$

determines uniquely a time-independent potential q when $a = 0$. The result of [RakSy] has been extended to the recovery of a time-independent damping coefficient a by [Is912]. Contrary to time-independent coefficients, due to domain of dependence arguments there is no hope to recover the restriction of a general time-dependent coefficient to the set

$$D = \{(t,x) \in Q : \ 0 < t < \mathrm{Diam}(\Omega)/2, \ \mathrm{dist}(x, \partial\Omega) > t\}$$

from the data $\mathcal{A}_{a,q}$. Indeed, assume that $\Omega = \{x \in \mathbb{R}^n : \ |x| < R\}$, $T > R > 0$. Now let u solve

$$\partial_t^2 u - \Delta u = 0, \ u_{|\Sigma} = f, \quad u_{|t=0} = \partial_t u_{|t=0} = 0$$

with $f \in H^1(\Sigma)$ satisfying $f_{|t=0} = 0$. Since $u_{|t=0} = \partial_t u_{|t=0} = 0$, the finite speed of propagation implies that $u_{|D} = 0$. Therefore, for any $a, q \in \mathcal{C}_0^\infty(D)$, we have $a\partial_t u + qu = 0$ and u solves

$$L_{a,q} u = 0, \ u_{|\Sigma} = f, \quad u_{|t=0} = \partial_t u_{|t=0} = 0.$$

This last result implies that for any $a, q \in \mathcal{C}_0^\infty(D)$ we have $\mathcal{A}_{a,q} = \mathcal{A}_{0,0}$ where $\mathcal{A}_{0,0}$ stands for $\mathcal{A}_{a,q}$ when $a = q = 0$.

Facing this obstruction to uniqueness, it appears that four different approaches have been considered so far to solve this problem when $a = 0$:
1) Considering the equation (2.2.1.1) for any time $t \in \mathbb{R}$ instead of $0 < t < T$ (e.g. [RaSj], [Sala]).
2) Recovering the restriction on a subset of Q of a time-dependent potential q from the data $\mathcal{A}_q$ (e.g.

[RakRam]).
3) For $a = 0$, recovering a time-dependent potential q from the extended data $C_{a,q}$ (e.g. [Is911]) given by

$$C_{a,q} = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|\Sigma}, u_{|t=T}, \partial_t u_{|t=T}) : \ u \in L^2(Q), \ L_{a,q} u = 0\}. \tag{2.2.2.2}$$

4) Recovering time-dependent coefficients that are analytic with respect to the t variable (e.g. [Es07]).

Therefore, it seems that the only results of unique global determination of a time-dependent potential q proved so far (at finite time) involve strong smoothness assumptions such as analyticity with respect to the t variable or the important set of data $C_{a,q}$. In [Ki161, Ki162, Ki164], we investigate some conditions that guaranty unique determination of general time-dependent potentials without involving an important set of data. More precisely, the goal of [Ki161, Ki162, Ki164] is to prove unique and stable global determination of a general time-dependent damping coefficient a and the potential q from partial knowledge of the set of data $C_{a,q}$.

2.2.3 Motivations

Physically speaking, our inverse problem can be stated as the determination of physical properties such as the time evolving density of an inhomogeneous medium by probing it with disturbances generated on some parts of the boundary and at initial time. The data is the response of the medium to these disturbances, measured on some parts of the boundary and at the end of the experiment, and the purpose is to recover the function q which measures the property of the medium. Note also that the determination of time-dependent potentials can be associated with models where it is necessary to take into account the evolution in time of the perturbation.

We also mention that time-dependent coefficients appear often due to mathematical reductions of nonlinear problems. For instance, in [Is93] Isakov applied results on inverse boundary value problems with time-dependent coefficients in order to prove unique recovery of a general semilinear term appearing in a nonlinear parabolic equation from traces of all the solutions to the equation. More recently, applying their results of stable recovery of time-dependent coefficients from the parabolic Dirichlet-to-Neumann map, [ChKi16] treated the stability issue for this problem (see Section 2.4 below). In the same spirit the inverse problem of [Ki161, Ki162, Ki164] can be a tool for the problem of determining a semilinear term appearing in a nonlinear wave equation from observations given by traces of the solutions. We point out that with this application in mind, it is important to treat recovery of non-smooth coefficients.

2.2.4 Known results

The determination of coefficients for hyperbolic equations from boundary measurements has attracted many attention in recent years. Many authors considered the recovery of time-independent potentials from observations given by the set $\mathcal{A}_{a,q}$ defined by (2.2.2.1) for $a = 0$. In [RakSy], the authors proved that, for $a = 0$, $\mathcal{A}_{a,q}$ determines uniquely a time-independent potential q. The uniqueness by partial boundary observations has been considered in [Es06]. We also mention that the stability issue for this problem has been studied by [BelDo11, BeJeYa1, Mo, StUh98, StUh05].

Some authors treated the recovery of both time-independent damping coefficients and potentials from boundary measurements. In [Is912], Isakov extended the result of [RakSy], to the recovery of both damping coefficients and potentials from the data $\mathcal{A}_{a,q}$. For $n = 3$, [IsSu] proved stable recovery of the restriction of both time-independent damping coefficients and potentials on the intersection of the domain and a half-space from measurements on the intersection of the boundary of the domain and the same half-space. Following the strategy set by [BuKl], [BuChIsYa, LiTr111, LiTr112] proved uniqueness and stability in the recovery of both damping coefficients and potentials from a single boundary measurements. In some recent work, [AmCh] proved a log-type stability estimate in the recovery of time-independent damping coefficients and potentials appearing in a dissipative wave equation from the initial boundary map.

All the above mentioned results are concerned with time-independent coefficients. Several authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [Stef89], Stefanov proved the recovery of a time-dependent potential appearing in the wave equation from the knowledge of scattering data by using some properties of the light-ray transform. In [RaSj], Ramm and Sjöstrand considered the determination of a time-dependent potential q from the data $(u_{|\mathbb{R}\times\partial\Omega}, \partial_\nu u_{|\mathbb{R}\times\partial\Omega})$ of forward solutions of (2.2.1.1) with $a = 0$ on the infinite time-space cylindrical domain $\mathbb{R}_t \times \Omega$ instead of Q ($t \in \mathbb{R}$ instead of $0 < t < T < \infty$). Rakesh and Ramm [RakRam] treated this problem at finite time on Q, with $T > \mathrm{Diam}(\Omega)$, and they determined uniquely q restricted to some subset of Q from $\mathcal{A}_{a,q}$ with $a = 0$. Isakov established in [Is911, Theorem 4.2] unique determination of general time-dependent potentials on the whole domain Q from the extended data $C_{a,q}$ given by (2.2.2.2) with $a = 0$. Using a result of unique continuation borrowed from [Ta95], Eskin [Es08] proved unique recovery of time-dependent coefficients analytic with respect to the time variable t from partial knowledge of the data $\mathcal{A}_{a,q}$. Salazar [Sala] extended the result of [RaSj] to more general coefficients. Moreover, [Wa] stated stability in the recovery of X-ray transforms of time-dependent potentials on a manifold and [Ben15] proved log-type stability in the determination of time-dependent potentials from the data considered by [Is911] and [RakRam]. We mention also the recent work of [BeBen] where the authors have extended the results of [Ben15] to the recovery of both time-dependent damping coefficients and potentials.

2.2.5 Unique and stable recovery of the coefficients

In order to state the main results of [Ki161, Ki162, Ki164], we first introduce some intermediate tools and notations. For all $\omega \in \mathbb{S}^{n-1} := \{y \in \mathbb{R}^n : \ |y| = 1\}$ we introduce the ω-shadowed and ω-illuminated faces

$$\partial\Omega_{+,\omega} = \{x \in \partial\Omega : \ \nu(x) \cdot \omega \geqslant 0\}, \quad \partial\Omega_{-,\omega} = \{x \in \partial\Omega : \ \nu(x) \cdot \omega \leqslant 0\}$$

of $\partial\Omega$. We consider also the parts of the lateral boundary Σ given by

$$\Sigma_{+,\omega} = \{(t,x) \in \Sigma : \ \nu(x) \cdot \omega > 0\}, \quad \Sigma_{-,\omega} = \{(t,x) \in \Sigma : \ \nu(x) \cdot \omega \leqslant 0\}.$$

From now on we fix $\omega_0 \in \mathbb{S}^{n-1}$ and we consider $F = [0,T] \times F'$ (resp $G = (0,T) \times G'$) with F' (resp G') an open neighborhood of $\partial\Omega_{+,\omega_0}$ (resp $\partial\Omega_{-,\omega_0}$) in $\partial\Omega$.

We start by recalling the result of [Ki161] where we have treated the case $a = 0$. For this purpose, we consider the set of data

$$C_q^* = \{(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u_{|G}, u_{|t=T}) : \ u \in L^2(Q), \ \Box u + qu = 0, \ u_{|t=0} = 0, \ \mathrm{supp} u_{|\Sigma} \subset F\}$$

defined rigorously in [Ki161, Section 2]. The main result of [Ki161] can be stated as follows.

Theorem 2.2.1 (Theorem 1.1, [Ki161]) *Let $q_1, \ q_2 \in L^\infty(Q)$. Assume that $C_{q_1}^* = C_{q_2}^*$. Then $q_1 = q_2$.*

Note that this uniqueness result is stated for bounded potentials with, roughly speaking, half of the data (2.2.2.2) considered in [Is911, Theorem 4.2] which seems to be, with [Ben15], the only result of unique global determination of general time-dependent coefficients for the wave equation, at finite time, in the mathematical literature. More precisely, we consider $u \in L^2(Q)$ solutions of $(\partial_t^2 - \Delta + q)u = 0$, in Q, with initial condition $u_{|t=0} = 0$ and Dirichlet boundary condition $u_{|\Sigma}$ supported on F (which, roughly speaking, corresponds to half of the boundary). Moreover, we exclude the data $\partial_t u_{|t=T}$ and we consider the Neumann data $\partial_\nu u$ only on G (which, roughly speaking, corresponds to the other half of the boundary). We also mention that in contrast to [Es08], we do not use results of unique continuation where the analyticity of the coefficients with respect to t is required. To our best knowledge condition (2.2.1) is the weakest condition that guaranties global uniqueness of general time-dependent potentials.

Let us also mention that, according to the obstruction to uniqueness given by domain of dependence arguments (see Subsection 2.2.2), even for large values of T, there is no hope to remove all the information

on $\{t=0\}$ and $\{t=T\}$ for the global recovery of general time-dependent coefficients. Thus, for our problem the data $\partial_t u_{|t=0}$ and $u_{|t=T}$, of solutions u of (2.2.1.1), can not be removed.

In [Ki162], we have established a result of stability associated with the uniqueness result of Theorem 2.2.1. For this purpose, we consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(t,x)u = 0, & \text{in } Q, \\ u(0,\cdot) = v_0, \quad \partial_t u(0,\cdot) = v_1, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma, \end{cases} \tag{2.2.5.1}$$

and we examine the well-posedness of this IBVP. We introduce the space

$$J = \{u \in L^2(Q) : \ (\partial_t^2 - \Delta)u = 0\}$$

and topologize it as a closed subset of $L^2(Q)$. We work with the space

$$H_\Box(Q) = \{u \in L^2(Q) : \ \Box u = (\partial_t^2 - \Delta)u \in L^2(Q)\},$$

with the norm

$$\|u\|^2_{H_\Box(Q)} = \|u\|^2_{L^2(Q)} + \left\|(\partial_t^2 - \Delta)u\right\|^2_{L^2(Q)}.$$

Repeating some arguments of [LiMa1, Chapter 2, Theorem 6.4] we prove in [Ki161, Theorem A.1] that $H_\Box(Q)$ is embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ in the space

$$K_\Box(Q) = \{u \in H^{-1}(0,T;L^2(\Omega)) : \ \Box u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

topologized by the norm

$$\|u\|^2_{K_\Box(Q)} = \|u\|^2_{H^{-1}(0,T;L^2(\Omega))} + \left\|(\partial_t^2 - \Delta)u\right\|^2_{L^2(Q)}.$$

Then, following [LiMa1, Chapter 2, Theorem 6.5], we prove in [Ki161, Proposition A.1] that the maps

$$\tau_0 w = (w_{|\Sigma}, w_{|t=0}, \partial_t w_{|t=0}), \quad \tau_1 w = (\partial_\nu w_{|\Sigma}, w_{|t=T}, \partial_t w_{|t=T}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0 : H_\Box(Q) \to H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\tau_1 : H_\Box(Q) \to H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$. Here for all $w \in \mathcal{C}^\infty(\overline{Q})$ we set

$$\tau_0 w = (\tau_{0,1}w, \tau_{0,2}w, \tau_{0,3}w), \quad \tau_1 w = (\tau_{1,1}w, \tau_{1,2}w, \tau_{1,3}w),$$

where

$$\tau_{0,1}w = w_{|\Sigma}, \ \tau_{0,2}w = w_{|t=0}, \ \tau_{0,3}w = \partial_t w_{|t=0}, \ \tau_{1,1}w = \partial_\nu w_{|\Sigma}, \ \tau_{1,2}w = w_{|t=T}, \ \tau_{1,3}w = \partial_t w_{|t=T}.$$

Therefore, we can introduce

$$\mathcal{H}(\partial Q) = \{\tau_0 u : \ u \in H_\Box(Q)\} \subset H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega).$$

Following [BuUh] and [NaSt], in order to define an appropriate topology on $\mathcal{H}(\partial Q)$ we consider the restriction of τ_0 to the space J.

Proposition 2.2.1 (Proposition 2.1, [Ki161]) *The restriction of τ_0 to J, that maps J onto $\mathcal{H}(\partial Q)$, is one to one and onto.*

From now on, we set $\mathcal{P}_0$ the inverse of $\tau_0 : J \to \mathcal{H}(\partial Q)$ and define the norm of $\mathcal{H}(\partial Q)$ by

$$\|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} = \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}, \quad (g, v_0, v_1) \in \mathcal{H}(\partial Q).$$

In the same way, we introduce the space $\mathcal{H}_F(\partial Q)$ defined by

$$\mathcal{H}_F(\partial Q) = \{(\tau_{0,1}h, \tau_{0,3}h) : \ h \in H_\square(Q), \ \tau_{0,2}h = 0, \ \mathrm{supp}(\tau_{0,1}h) \subset F\}$$

with the associated norm given by

$$\|(g, v_1)\|_{\mathcal{H}_F(\partial Q)} = \|(g, 0, v_1)\|_{\mathcal{H}(\partial Q)}, \quad (g, v_1) \in \mathcal{H}_F(\partial Q).$$

One can easily check that the space $\mathcal{H}_F(\partial Q)$ is embedded continuously into $\mathcal{H}(\partial Q)$. We are now in position to state existence and uniqueness of solutions of the IBVP (2.2.5.1) for $(g, v_1) \in \mathcal{H}_F(\partial Q)$.

Proposition 2.2.2 (Proposition 2.2, [Ki161]) *Let $(g, v_1) \in \mathcal{H}_F(\partial Q)$ and $q \in L^\infty(Q)$. Then, the IBVP* (2.2.5.1) *admits a unique weak solution $u \in L^2(Q)$ satisfying*

$$\|u\|_{L^2(Q)} \leqslant C \, \|(g, v_1)\|_{\mathcal{H}_F(\partial Q)}$$

and the boundary operator $B_q : (g, v_1) \mapsto (\tau_{1,1}u_{|G}, \tau_{1,2}u)$ is a bounded operator from $\mathcal{H}_F(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(G')) \times H^{-2}(\Omega)$.

We have established also a smoothing result for the difference of two boundary operators B_q given by.

Proposition 2.2.3 (Proposition 1, [Ki162])*Let $q_1, \ q_2 \in L^\infty(Q)$. Then, the operator $B_{q_1} - B_{q_2}$ is a bounded operator from $\mathcal{H}_F(\partial Q)$ to $L^2(G) \times H^1(\Omega)$.*

Using this smoothing property we can state the main result of [Ki162] in the following way.

Theorem 2.2.2 (Theorem 1, [Ki162]) *Let $p > n + 1$ and $q_1, \ q_2 \in W^{1,p}(Q)$. Assume that the conditions*

$$q_1(t, x) = q_2(t, x), \quad (t, x) \in \Sigma, \tag{2.2.5.2}$$

$$\|q_1\|_{W^{1,p}(Q)} + \|q_2\|_{W^{1,p}(Q)} \leqslant M$$

are fulfilled. Then, there exists a constant $C > 0$ depending on n, p, M, T, Ω, F', G', such that

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant C \left(\|B_{q_1} - B_{q_2}\| + \left| \ln |\ln (\|B_{q_1} - B_{q_2}\|)| \right|^{-1} \right). \tag{2.2.5.3}$$

Here $\|B_{q_1} - B_{q_2}\|$ stands for the norm of $B_{q_1} - B_{q_2}$ as an element of $\mathcal{B}(\mathcal{H}_F(\partial Q); L^2(G) \times H^1(\Omega))$.

Let us observe that this stability estimate is the first that is stated with the data considered in [Ki161], where uniqueness is proved with conditions that seems to be one of the weakest so far. Moreover, it appears that with the paper of [Ben15], [Ki162] is the first where stability is stated for global determination of general time-dependent potentials appearing in a wave equation from boundary measurements.

The result of [Ki161] has been extended in [Ki164] to the unique simultaneous recovery of the damping coefficient a and the potential q appearing in (2.2.1.1) from the data

$$C^*_{a,q} = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|F}, u_{|t=T}) : \ u \in H^1(0, T; L^2(\Omega)), \ L_{a,q}u = 0\}.$$

The main result of [Ki164] can be stated as follows.

Theorem 2.2.3 (Theorem 1.1, [Ki164]) *Let $q_1,\ q_2 \in L^\infty(Q)$ and let $a_1, a_2 \in W^{1,p}(Q)$ with $p > n+1$. Assume that*

$$a_1(t,x) = a_2(t,x), \quad (t,x) \in \partial Q. \tag{2.2.5.4}$$

Then, the condition

$$C^*_{a_1,q_1} = C^*_{a_2,q_2} \tag{2.2.5.5}$$

implies that $a_1 = a_2$ and $q_1 = q_2$.

To our best knowledge [Ki164] is the first paper treating uniqueness in the recovery of time-dependent damping coefficients. Moreover, it seems that, with [Es08, Es16, Sala], [Ki164] is the first paper considering recovery of time-dependent coefficients of order one and it appears that this work is the first treating this problem for general coefficients at finite time ([Es08, Es16] proved recovery of coefficients analytic with respect to the time variable t, [Sala] considered the problem for all time $t \in \mathbb{R}$). We point out that Theorem 2.2.3 is stated for general coefficients with observations close to the one considered in Theorem 2.2.1. Indeed, the only difference between Theorem 2.2.1 and Theorem 2.2.3 comes from the restriction on the Dirichlet boundary condition and the initial value.

Note that condition (2.2.5.4) is meaningful for damping coefficients that actually depend on the time variable t ($\partial_t a_j \neq 0$, $j = 1, 2$). Indeed, for time-independent damping coefficients a_1, a_2, (2.2.5.4) implies that $a_1 = a_2$. However, by modifying the argumentation of [Ki164] in accordance with [Is911], for $T > \mathrm{Diam}(\Omega)$ we believe that we can restrict condition (2.2.5.4) to the knowledge of time-independent damping coefficients on $\partial\Omega$ ($a_1 = a_2$ on $\partial\Omega$ instead of (2.2.5.4)).

We believe that, with some suitable modifications, the approach developed in [Ki164] can be used for proving recovery of more general time-dependent coefficients of order one including a magnetic field associated with a time-dependent magnetic potential.

2.2.6 Description of the results

The main tools in the analysis of [Ki161, Ki162, Ki164] are GO solutions and Carleman estimates. Following an approach used for elliptic equations (e.g. [BuUh, ChKiSo151, KeSjUh, NaSt]) and for the determination of time-independent potentials by [BeJeYa1], we construct two kind of GO solutions: exponentially growing GO solutions and exponentially decaying GO solutions. Using these solutions and some Carleman estimates with linear weight, we prove the recovery of the coefficient a and q stated in Theorem 2.2.1, 2.2.2 and 2.2.3. Our GO solutions differ from the one of [Es08, Is912, RakSy, RaSj, Sala] and, combined with our Carleman estimate, they make it possible to prove global recovery of time-dependent coefficients from partial knowledge of the set $C_{a,q}$ without using additional smoothness or geometrical assumptions.

In [Ki161] we consider exponentially decaying GO solutions lying in $H^1(Q)$ without condition on ∂Q and exponentially growing GO solutions associated with (2.2.1.1) for $a = 0$ that vanish on parts of ∂Q. With these solutions and some Carleman estimates with linear weight, we prove Theorem 2.2.1. Using some properties of fundamental solutions for PDEs with constant coefficients stated in [Ch09, Ho1, Ho2], we build exponentially decaying GO solutions $u \in H^1(Q)$ associated with the equation

$$\partial_t^2 u - \Delta u + q(t,x)u = 0 \quad \text{on } Q. \tag{2.2.6.1}$$

More precisely, for $\lambda > 1$ a large parameter and for $\omega \in \mathbb{S}^{n-1}$, $\xi \in \mathbb{R}^{1+n}$ satisfying $\xi \cdot (1, -\omega) = 0$, we consider solutions of (2.2.1.1) with $a = 0$ of the form

$$u(t,x) = e^{-\lambda(t+x\cdot\omega)}(e^{-i\xi\cdot(t,x)} + w(t,x)), \quad (t,x) \in Q. \tag{2.2.6.2}$$

Here $w \in H^1(Q)$ fulfills

$$\|w\|_{L^2(Q)} \leqslant \frac{C}{\lambda},$$

with $C > 0$ independent of λ. To obtain parts of the restriction on the set of data $C_{a,q}$ stated in Theorem 2.2.1, we consider also exponentially growing GO solutions associated with (2.2.1.1) for $a = 0$ that vanish on parts of ∂Q. The construction of such solutions requires the following Carleman estimates.

Theorem 2.2.4 (Theorem 4.1, [Ki161]) *Let $q \in L^\infty(Q)$ and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition $u_{|\Sigma} = 0, \quad u_{|t=0} = \partial_t u_{|t=0} = 0$, then there exists $\lambda_1 > 1$ depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$ such that the estimate*

$$\begin{aligned}
&\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\partial_t u(T,x)\right|^2 dx + \lambda \int_{\Sigma_{+,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 \left|\omega\cdot\nu(x)\right| d\sigma(x)dt + \lambda^2 \int_Q e^{-2\lambda(t+\omega\cdot x)} \left|u\right|^2 dxdt \\
&\leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta + q)u\right|^2 dxdt + \lambda^3 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|u(T,x)\right|^2 dx\right) \\
&\quad + C\left(\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\nabla u(T,x)\right|^2 dx + \lambda \int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 \left|\omega\cdot\nu(x)\right| d\sigma(x)dt\right)
\end{aligned} \tag{2.2.6.3}$$

holds true for $\lambda \geqslant \lambda_1$ with C depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$. If u satisfies the condition $u_{|\Sigma} = 0, \quad u_{|t=T} = \partial_t u_{|t=T} = 0$, then the estimate

$$\begin{aligned}
&\lambda \int_\Omega e^{2\lambda\omega\cdot x} \left|\partial_t u_{|t=0}\right|^2 dx + \lambda \int_{\Sigma_{-,\omega}} e^{2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 \left|\omega\cdot\nu(x)\right| d\sigma(x)dt + \lambda^2 \int_Q e^{2\lambda(t+\omega\cdot x)} \left|u\right|^2 dxdt \\
&\leqslant C\left(\int_Q e^{2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta + q)u\right|^2 dxdt + \lambda^3 \int_\Omega e^{2\lambda\omega\cdot x} \left|u(0,x)\right|^2 dx + \lambda \int_\Omega e^{2\lambda\omega\cdot x} \left|\nabla u(0,x)\right|^2 dx\right) \\
&\quad + C\lambda \int_{\Sigma_{+,\omega}} e^{2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 \left|\omega\cdot\nu(x)\right| d\sigma(x)dt
\end{aligned} \tag{2.2.6.4}$$

holds true for $\lambda \geqslant \lambda_1$.

For all $y \in \mathbb{S}^{n-1}$ and all $r > 0$, we set

$$\partial\Omega_{+,r,y} = \{x \in \partial\Omega : \ \nu(x)\cdot y > r\}, \quad \partial\Omega_{-,r,y} = \{x \in \partial\Omega : \ \nu(x)\cdot y \leqslant r\}$$

and $\Sigma_{\pm,r,y} = (0,T) \times \partial\Omega_{\pm,r,y}$. Here and in the remaining of this text we always assume, without mentioning it, that y and r are chosen in such way that $\partial\Omega_{\pm,r,\pm y}$ contain a non-empty relatively open subset of $\partial\Omega$. Without lost of generality we assume that there exists $0 < \varepsilon < 1$ such that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,-\omega} \subset F'$. Using the Carleman estimate (2.2.6.4), we build solutions $u \in H_\square(Q)$ to

$$\begin{cases} (\partial_t^2 - \Delta + q(t,x))u = 0 \ \text{ in } Q, \\ u_{|t=0} = 0, \\ u = 0, \quad \text{ on } \Sigma_{+,\varepsilon/2,-\omega}, \end{cases} \tag{2.2.6.5}$$

of the form

$$u(t,x) = e^{\lambda(t+\omega\cdot x)}\left(1 + z(t,x)\right), \quad (t,x) \in Q.$$

Here $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$, $z \in e^{-\lambda(t+\omega\cdot x)} H_\square(Q)$ fulfills: $z(0,x) = -1$, $x \in \Omega$, $z = -1$ on $\Sigma_{+,\varepsilon/2,-\omega}$ and

$$\|z\|_{L^2(Q)} \leqslant C\lambda^{-\frac{1}{2}}$$

with C depending on F', Ω, T and any $M \geqslant \|q\|_{L^\infty(Q)}$. Since $\Sigma \setminus F \subset \Sigma \setminus \Sigma_{-,\varepsilon,-\omega} = \Sigma_{+,\varepsilon,-\omega}$ and since $\Sigma_{+,\varepsilon/2,-\omega}$ is a neighborhood of $\Sigma_{+,\varepsilon,-\omega}$ in Σ, it is clear that condition (2.2.6.5) implies $(\tau_{0,1}u, \tau_{0,3}u) \in \mathcal{H}_F(\partial Q)$ (recall that for $v \in \mathcal{C}^\infty(\overline{Q})$, $\tau_{0,1}v = v_{|\Sigma}$, $\tau_{0,3}v = \partial_t v_{|t=0}$). Using these solutions and the Carleman estimate (2.2.6.3), we prove in [Ki161] the unique recovery of q from the data C_q^* by mean of the Fourier transform of q.

The strategy for proving the stability estimate (2.2.5.3) is different and requires different assumptions such as the knowledge of the potential on the lateral boundary Σ stated in (2.2.5.2). More precisely, in order

to derive a stability result associated with the uniqueness stated in Theorem 2.2.1, for $\lambda > 0$, $\omega \in \mathbb{S}^{n-1}$, $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, we consider in [Ki162] GO solutions of the form

$$u = e^{\pm\lambda(t+x\cdot\omega)}(\chi(t,x) + w(t,x)),$$

where $\chi(t,x) = \varphi(x+t\omega)$. Here, in contrast to [Ki161], we need exponentially decaying GO solutions lying in $H^2(Q)$ and exponentially growing GO solutions satisfying (2.2.6.5). Indeed, in contrast to uniqueness result, for the stability issue we need to consider traces of GO solutions on ∂Q lying in some suitable spaces and we need estimates of these terms.

Let $q = q_1 - q_2$ be extended to $\mathbb{R}^{1+n}$ by 0. We consider the light-ray transform of q (see [RaSj, Stef89]) given by

$$\mathcal{R}q(x,\omega) = \int_{\mathbb{R}} q(t, x+t\omega)dt, \quad x \in \mathbb{R}^n,\ \omega \in \mathbb{S}^{n-1}.$$

Repeating some of the arguments already used in Theorem 2.2.1, with additional cares about the dependence of the constant, we start by proving the stable recovery of the light-ray $\mathcal{R}q(\cdot,\omega)$ for ω on a small neighborhood of ω_0 in $\mathbb{S}^{n-1}$. Then, combining this with a result of stability in the analytic continuation problem that follows from [ApEs, Theorem 3] (see also [Ve]), we complete the proof of the stability estimate (2.2.5.3).

Now let us consider the result of [Ki164] stated in Theorem 2.2.3. This time we want to recover simultaneously the first order coefficient a and the zero order coefficient q appearing in (2.2.1.1). Even if the general strategy of [Ki164] is inspired by [BeJeYa1, Ki161, Ki162] (see also [BuUh, KeSjUh] for the original idea in the case of elliptic equations), due to the presence of a variable coefficient of order one in (2.2.1.1), our approach in [Ki164] differs from [BeJeYa1, Ki161, Ki162] in many aspects. Indeed, to prove our Carleman estimate we perturb the linear weight and we prove this estimate by using a convexity argument that allows us to absorb the damping coefficient. Moreover, in contrast to [Ki161, Ki162] our GO are designed for the recovery of the damping coefficient and we can not construct them by applying properties of solutions of PDEs with constant coefficients. We remedy to this by considering Carleman estimates in Sobolev space of negative order and by using these estimates to build our GO solutions. This construction is inspired by the one used in [DoKeSjUh, KeSjUh] for the recovery of Schrödinger operators from partial boundary measurements. The Carleman estimate of [Ki164] takes the following form.

Theorem 2.2.5 (Theorem 3.1, [Ki164]) *Let $\omega \in \mathbb{S}^{n-1}$, $a,\ q \in L^\infty(Q)$ and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition*

$$u_{|\Sigma} = 0, \quad u_{|t=0} = \partial_t u_{|t=0} = 0,$$

then there exists $\lambda_1 > 1$ depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)} + \|a\|_{L^\infty(Q)}$ such that the estimate

$$\begin{aligned}
&\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\partial_t u_{|t=T}\right|^2 dx + \lambda \int_{\Sigma_{+,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt + \lambda^2 \int_Q e^{-2\lambda(t+\omega\cdot x)} |u|^2 dxdt \\
&+ \int_Q e^{-2\lambda(t+\omega\cdot x)}(|\nabla u|^2 + |\partial_t u|^2)dxdt \leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)} |L_{a,q}u|^2 dxdt + \lambda^3 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|u_{|t=T}\right|^2 dx\right) \\
&\quad + C\left(\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\nabla u_{|t=T}\right|^2 dx + \lambda \int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\right)
\end{aligned} \tag{2.2.6.6}$$

holds true for $\lambda \geqslant \lambda_1$ with C depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)} + \|a\|_{L^\infty(Q)}$.

The main difference between the Carleman estimate (2.2.6.3) and (2.2.6.6), comes from the expression involving $\partial_t u$ and ∇u. Due to the presence of a variable coefficient of order one this expressions are required for the construction of our GO solutions. In order to establish this Carleman estimate, in a similar way to [DoKeSjUh, KeSjUh], we need to perturb our linear weight in order to absorb the damping coefficient. Namely, we introduce a new parameter s independent of λ, that will be precised later, and we consider, for $\lambda > s > 1$, the perturbed weight

$$\varphi_{\pm\lambda,s}(t,x) := \pm\lambda(t+\omega\cdot x) - \frac{st^2}{2}.$$

Then, for a suitable value of s depending on the damping coefficient a and for $\lambda > \lambda_0(s)$ where $\lambda_0(s)$ depends on s, we obtain (2.2.6.6) (with a constant depending on s). In order to construct our GO solutions we need also an extension of our Carleman estimate to negative order Sobolev space. For this purpose, for all $m \in \mathbb{R}$, we introduce the space $H^m_\lambda(\mathbb{R}^{1+n})$ defined by

$$H^m_\lambda(\mathbb{R}^{1+n}) = \{u \in \mathcal{S}'(\mathbb{R}^{1+n}) : \ (|(\tau,\xi)|^2 + \lambda^2)^{\frac{m}{2}} \hat{u} \in L^2(\mathbb{R}^{1+n})\},$$

with the norm

$$\|u\|^2_{H^m_\lambda(\mathbb{R}^{1+n})} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} (|(\tau,\xi)|^2 + \lambda^2)^m |\hat{u}(\tau,\xi)|^2 d\xi d\tau.$$

Here for all tempered distribution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$, we denote by $\hat{u}$ the Fourier transform of u which, for $u \in L^1(\mathbb{R}^{1+n})$, is defined by

$$\hat{u}(\tau,\xi) := \mathcal{F}u(\tau,\xi) := (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{1+n}} e^{-it\tau - ix\cdot\xi} u(t,x) dt dx.$$

Then, in a similar way, combining the arguments used in Theorem 2.2.5 with properties of pseudodifferential operator stated in [Ho3, Volume 18], for

$$P_{a,\omega,\pm\lambda} := e^{\mp\lambda(t+x\cdot\omega)}(L_{a,q} - q)e^{\pm\lambda(t+x\cdot\omega)},$$

we prove two Carleman estimate in negative Sobolev space taking the form

Lemma 2.2.1 (Lemma 5.1, [Ki164]) *Let $a \in W^{1,p}(Q)$. Then, there exists $\lambda_2' > \lambda_1$ such that*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leqslant C \|P_{a,\omega,\lambda} v\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}_0^\infty(Q), \quad \lambda > \lambda_2', \tag{2.2.6.7}$$

with $C > 0$ independent of v and λ.

Lemma 2.2.2 (Lemma 5.4, [Ki164]) *Let $a \in W^{1,p}(Q)$. Then, there exists $\lambda_3' > 0$ such that for $\lambda > \lambda_3'$, we have*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leqslant C \|P_{-a,\omega,-\lambda} v\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}_0^\infty(Q), \quad \lambda > \lambda_3', \tag{2.2.6.8}$$

with $C > 0$ independent of v and λ.

Combining Carleman estimates (2.2.6.7)-(2.2.6.8), we construct by duality exponentially growing and decaying solutions of the form

$$u_1(t,x) = e^{-\lambda(t+x\cdot\omega)}(b_{1,\lambda}(t,x) + w_1(t,x)), \quad u_2(t,x) = e^{\lambda(t+x\cdot\omega)}(b_{2,\lambda}(t,x) + w_2(t,x)) \tag{2.2.6.9}$$

with $w_j \in H^1(Q)$, $j = 1,2$, satisfying the decay

$$\|w_j\|_{H^1(Q)} + \lambda \|w_j\|_{L^2(Q)} \leqslant C\lambda^{\frac{3-\alpha}{3}}.$$

Here the expression $b_{1,\lambda}$, $b_{2,\lambda}$, are respectively a solution of a transport equation and they will be used for the recovery of the damping coefficient. In order to relax the assumptions imposed on the set of admissible coefficients, in a similar way to [Ki163, Salo04], we consider expressions $b_{1,\lambda}$, $b_{2,\lambda}$ depending on some smooth approximations of the damping coefficients instead of the damping coefficients themselves.

2.3 Recovery of time-dependent coefficients on Riemannian manifold for hyperbolic equations

2.3.1 Formulation of the problem

Let us observe that in [Ki161, Ki162, Ki164] we have considered the recovery of time-dependent potential q in the flat case for operators whose principal part is characterized by constant coefficients. This allows us to conclude through an argument using the Fourier transform of the potential q. The same problem stated on a manifold with boundaries should be carry out differently. This section is devoted to the presentation of the extension of [Ki161] to Riemannian manifold stated in [KiOk] which is a joint work with Lauri Oksanen. Let (M, g) be a smooth Riemannian manifold with boundary of dimension $n \geqslant 2$ and let $T > 0$. We introduce the Laplace and wave operators

$$\Delta_g u = |g|^{-1/2} \sum_{j,k=1}^{n} \partial_{x_j} \left(g^{jk} |g|^{1/2} \partial_{x_k} u \right), \quad \Box_g = \partial_t^2 - \Delta_g, \tag{2.3.1.1}$$

where $|g|$ and g^{jk} denote the absolute of value of the determinant and the inverse of g in local coordinates, and consider the wave equation

$$\Box_g u + q(t,x)u = 0, \quad (t,x) \in (0,T) \times M, \tag{2.3.1.2}$$

with $q \in L^\infty((0,T) \times M)$. Let ν be the outward unit normal vector to ∂M with respect to the metric g and let ∂_ν be the corresponding normal derivative. We define $\partial_{\overline{\nu}} = \partial_\nu$ on the lateral surface $(0,T) \times \partial M$, $\partial_{\overline{\nu}} = \partial_t$ on the top surface $\{T\} \times M$ and $\partial_{\overline{\nu}} = -\partial_t$ on the bottom surface $\{0\} \times M$, and consider the Cauchy data set on the boundary of the cylinder $\overline{M} = (0,T) \times M$,

$$\mathcal{C}_q = \{(u_{|\partial\overline{M}}, \partial_{\overline{\nu}} u_{|\partial\overline{M}}) : \ u \in L^2(\overline{M}), \ \Box_g u + qu = 0\}. \tag{2.3.1.3}$$

In [KiOk] we study the inverse boundary value problem to recover the time-dependent zeroth order term q appearing in (2.3.1.2) from partial knowledge of the set $\mathcal{C}_q$.

There are several previous results on this problem, however to our knowledge all of them assume either that (M, g) is a domain in $\mathbb{R}^n$ with the Euclidean metric or that q is time-independent.

In the case of time-independent potential q it is enough to know the following lateral restriction of $\mathcal{C}_q$,

$$\mathcal{C}_q^{\mathrm{Lat}} = \{(u_{|(0,T)\times\partial M}, \partial_\nu u_{|(0,T)\times\partial M}) : \ u \in L^2(\overline{M}), \ \Box_g u + qu = 0, \ u_{|t=0} = \partial_t u_{|t=0} = 0\},$$

for sufficiently large $T > 0$, in order to determine $q(x)$ for all $x \in M$, see [BelDo11, KaKuLa04, Mo, RakSy]. However, if q depends on time, due to domain of dependence argument stated in Subsection 2.2.2, the data $\mathcal{C}_q^{\mathrm{Lat}}$ contains no information on the restriction of q on the set

$$\{(t,x) \in \overline{M} : \ \mathrm{dist}(x, \partial M) > t \text{ or } \mathrm{dist}(x, \partial M) > T - t\}. \tag{2.3.1.4}$$

Here $\mathrm{dist}(\cdot,\cdot)$ is the distance function on (M, g). Indeed, the finite speed of propagation property for the wave equation (2.3.1.2), see e.g. [KaKuLa01, Theorem 2.47], implies that $u(t,x)$, satisfying the equations in the definition of $\mathcal{C}_q^{\mathrm{Lat}}$, vanishes when $\mathrm{dist}(x, \partial M) > t$, and therefore changing q in this cone does not affect $\mathcal{C}_q^{\mathrm{Lat}}$. Moreover, again due to the finite speed of propagation, changing q in the cone $\mathrm{dist}(x, \partial M) > T - t$ causes u to change only in the same cone, but this cone does not intersect the lateral boundary $(0,T) \times \partial M$.

Facing this obstruction to the uniqueness, all the results of [KiOk] assume some information on the top $\{T\} \times M$ and bottom $\{0\} \times M$ surfaces. In particular, under the assumption that (M, g) is a simple manifold, see Definition 2.3.1 below, we show that the full Cauchy data set $\mathcal{C}_q$ determines q uniquely.

2.3.2 Physical and mathematical motivations

Note first that this problem has the same physical motivations as [Ki161, Ki162, Ki164]. Beside these motivations, this problem has an additional mathematical motivation: the problem to determine q given $\mathcal{C}_q$ can be seen as a hyperbolic analogy of the Calderón problem on a cylinder as stated in [DoKeSjUh]. Indeed, denoting by $dt^2 - g$ the product Lorentzian metric on $\overline{M}$, the wave operator $\Box_g$ coincides with the Laplace operator on $(\overline{M}, dt^2 - g)$. On the other hand, denoting by $\overline{g} = dt^2 + g$ the Riemannian product metric on $\overline{M}$, and choosing a smooth domain $\Omega \subset \overline{M}$, we can formulate the the Calderón problem on a cylinder as follows: given the elliptic Cauchy data set

$$\mathcal{C}_q^{\mathrm{Ell}} = \{(u_{|\partial\Omega}, \partial_{\overline{\nu}} u_{|\partial\Omega}) : \ u \in L^2(\Omega), \ \Delta_{\overline{g}} u + qu = 0\}$$

determine q (here $\overline{\nu}$ is the outward unit normal vector to $\partial\Omega$). In [DoKeSjUh] this problem was solved under the assumption that (M, g) is a simple manifold.

One reason to study these problems is to gain some understanding of the fundamental problem to determine, up to an isometry, a smooth Riemannian or Lorentzian manifold $(\Omega, \overline{g})$ with boundary given the set of Cauchy data

$$\mathcal{C}(\overline{g}) = \{(u|_{\partial\Omega}, \partial_{\overline{\nu}} u|_{\partial\Omega}); \ u \in L^2(\Omega), \ \Delta_{\overline{g}} u = 0\}.$$

Excluding results where full or partial real analyticity is assumed, this problem is open in dimensions three or higher, in both the elliptic and hyperbolic cases. The relation to the present problem to determine q given $\mathcal{C}_q$ is as follows. In the case when $(\Omega, \overline{g})$ is a subset of the conformal cylinder

$$\overline{M} = (0, T) \times M, \quad \overline{g} = c(dt^2 + g), \tag{2.3.2.1}$$

where only the positive conformal factor $c \in C^2(\overline{M})$ is assumed to be unknown, the problem to determine c given $\mathcal{C}(\overline{g})$ can be reduced to the problem to determine q given $\mathcal{C}_q^{\mathrm{Ell}}$ via a gauge transformation. Indeed, as explained e.g. in [DoKuLaSa], the function $v = c^{(n-1)/4} u$ satisfies $\Delta_{\overline{g}} v + q_c v = 0$ if the function u satisfies $\Delta_{\overline{g}} u = 0$, where d is the dimension of M and

$$q_c = c^{-(n-1)/4} \Delta_{\overline{g}} c^{(n-1)/4}.$$

This allows us to first determine $\mathcal{C}_{q_c}^{\mathrm{Ell}}$ given $\mathcal{C}(\overline{g})$, then to solve the inverse boundary value problem for q_c, and finally determine c given q_c. The argument can be adapted also to the hyperbolic case.

2.3.3 Previous literature

The recovery of coefficients appearing in hyperbolic equations is a topic that has attracted considerable attention. Several authors have treated the determination of time-independent coefficients from Cauchy data analogous to $\mathcal{C}_q^{\mathrm{Lat}}$ above. In this case, the Boundary Control method, originating from [Beli87], gives very general uniqueness results when combined with the time-sharp unique continuation theorem [RoZu, Ta95]. We refer to [KuOkPa, LaOk141] for state-of-the-art results and to [KaKuLa01] for reviews. However, as shown in [Al, AlBa], unique continuation analogous to [RoZu, Ta95] may fail in the presence of time-dependent zeroth order terms, and the Boundary Control method generalizes only to the case where the dependence on time is real analytic [Es08, Es16].

Let us now turn to the approach underpinning most of the results in the time-dependent case, including the results in [KiOk], that is, the use of GO solutions. This approach is widely applied also to time-independent case, and the data used then is typically the same as in the case of the Boundary Control method, that is, $\mathcal{C}_q^{\mathrm{Lat}}$. Although the GO approach gives less sharp uniqueness results in terms of geometrical assumptions than the Boundary Control method, the advantage of the former is that it yields stronger stability results.

Apart from [Es08, Es16], all the above results are concerned with time-independent coefficients. For time-dependent coefficient we refer to the results mentioned in Subsection 2.2.4. All the results [Ben15, Is911, Ki161, Ki162, Ki164, RakRam, RakSy, Sala, Ste], introduced in Subsection 2.2.4, assume that the leading order coefficients in the wave equation are constant. The main contribution of [KiOk] is to consider the recovery of a time-dependent potential in the case of non-constant leading order coefficients.

2.3.4 Unique recovery of q

We prove two results on unique determination of the potential q. In the first result we assume that the Cauchy data set $\mathcal{C}_q$ is fully known on the lateral boundary $(0,T)\times\partial M$ and partly restricted on the top and bottom. In the second result we restrict the data also on the lateral boundary. In both results we impose geometric conditions on the manifold (M,g), the conditions being more stringent in the second case. In the first case, we make the typical assumption that (M,g) is simple in the sense of the following definition.

Definition 2.3.1 *A compact smooth Riemannian manifold with boundary (M,g) is simple if it is simply connected, the boundary ∂M is strictly convex in the sense of the second fundamental form, and M has no conjugate points.*

We consider the restricted version of $\mathcal{C}_q$,

$$\mathcal{C}(q,0)=\{(u|_{\partial\overline{M}\setminus(\{0\}\times M)},\partial_{\overline{\nu}}u|_{\partial\overline{M}});\ u\in L^2(\overline{M}),\ \Box_g u+qu=0,\ u_{|t=0}=0\},$$

and formulate our first result.

Theorem 2.3.1 (Thorem 1.2, [KiOk]) *Suppose that (M,g) is a simple manifold. Let $T>0$ and let q_1, $q_2\in L^\infty((0,T)\times M)$. Then*

$$\mathcal{C}(q_1,0)=\mathcal{C}(q_2,0) \tag{2.3.4.1}$$

implies that $q_1=q_2$.

Let us point out that an analogous result holds with the data restricted on the top $\{T\}\times M$ rather than on the bottom $\{0\}\times M$, and also with the time derivative $\partial_t u|_{t=T}$ vanishing instead of $u|_{t=0}$. Moreover, we prove also a variation of Theorem 2.3.1 using the data

$$\mathcal{C}(q,0,T)=\{(u|_{\partial\overline{M}\setminus(\{0\}\times M)},\partial_{\overline{\nu}}u|_{\partial\overline{M}\setminus(\{T\}\times M)});\ u\in L^2(\overline{M}),\ \Box_g u+qu=0,\ u_{|t=0}=0\}.$$

Theorem 2.3.2 (Thorem 1.3, [KiOk]) *Let (M,g) be simple. Let $T>Diam(M)$ and let q_1, $q_2\in L^\infty((0,T)\times M)$. Then*

$$\mathcal{C}(q_1,0,T)=\mathcal{C}(q_2,0,T) \tag{2.3.4.2}$$

implies that $q_1=q_2$.

In order to restrict the data also on the lateral part of the boundary, we make the assumption that (M,g) is contained in a conformal cylinder of the form (2.3.2.1), that is, we assume that it satisfies the geometric assumption introduced in [DoKeSaUh] in the context of the Calderón problem. Furthermore, we assume that also the time direction is multiplied by the same conformal factor, which amounts to assuming, after the gauge transformation discussed in Section 2.3.2, that the wave equation has two Euclidean directions, one of them being the time direction.

More precisely, we assume that (M_0,g') is a simple Riemannian manifold of dimension $n-1\geqslant 2$, $M\subset\mathbb{R}\times int(M_0)$ is a compact domain with smooth boundary, and that $g=a(e\oplus g')$ where e is the euclidean metric on $\mathbb{R}$ and $a\in C^\infty(M)$ is positive, and consider the wave operator

$$\Box_{a,g}=a^{-1}\partial_t^2-\Delta_g. \tag{2.3.4.3}$$

Let us now describe the restriction of $\mathcal{C}_q$ considered in our second result. To every variable $x \in M$ we associate the coordinate $x_1 \in \mathbb{R}$ and $x' \in M_{x_1} = \{x' \in M_0 : \ (x_1, x') \in M\}$ such that $x = (x_1, x')$. We define $\varphi(x) = x_1$,

$$\partial M_\pm = \{x \in \partial M : \ \pm\partial_\nu \varphi(x) \geqslant 0\},$$

and $\Sigma_\pm = (0,T) \times int(\partial M_\pm)$. We consider $U = [0,T] \times U'$ (resp $V = (0,T) \times V'$) with U' (resp V') a closed neighborhood of ∂M_+ (resp ∂M_-) in ∂M, and define the following restriction of $\mathcal{C}_q$,

$$\mathcal{C}_{q,*} = \{(u_{|U}, \partial_t u_{|t=0}, \partial_\nu u_{|V}, u_{|t=T}) : \ u \in L^2(\overline{M}), \ (\Box_{a,g} + q)u = 0, \ u_{|t=0} = 0, \ \mathrm{supp} u_{|(0,T)\times\partial M} \subset U\}.$$

Our second result is stated as follows.

Theorem 2.3.3 (Theorem 1.4, [KiOk]) *Suppose that the leading part of the wave operator is of the form (2.3.4.3). Let $T > 0$ and let $q_1, \ q_2 \in L^\infty((0,T) \times M)$. Then $\mathcal{C}_{q_1,*} = \mathcal{C}_{q_2,*}$ implies that $q_1 = q_2$.*

2.3.5 Remarks about the proofs of the main results

As indicated above, the proofs of Theorems 2.3.1 and 2.3.2 are based on the use of GO solutions. In the case of the former, we use the oscillating solutions of the form

$$u(t,x) = \sum_{j=1}^{k} a_j(t,x) e^{i\sigma\psi_j(t,x)} + R_\sigma(t,x), \quad (t,x) \in (0,T) \times M, \tag{2.3.5.1}$$

with $\sigma \in \mathbb{R}$ a parameter, R_σ a term that admits a decay with respect to the parameter $|\sigma|$ and ψ_j, $j = 1, .., k$, real valued. Inspired by the elliptic result [DoKeSaUh], we use these solutions to prove that the hyperbolic inverse boundary value problem reduces to the problem to invert a weighted geodesic ray transform on (M, g). The assumption that (M, g) is simple guarantees that this transform is indeed invertible.

For our purposes it is enough to take $k = 2$ in (2.3.5.1), and in the case of full data $\mathcal{C}_q$ already $k = 1$ is enough. In the case of data sets $\mathcal{C}(q, 0)$ and $\mathcal{C}(q, 0, T)$, the second term is needed in order to be able to restrict the data while avoiding a "reflection". Similar construction is likely to work also on the lateral boundary, and one may hope that this could be used to reduce the amount of lateral data. In fact, this type of argument was used in the elliptic case in [KeSa]. There it was assumed that the part of the lateral boundary lacking data, that is, the inaccessible part, satisfies a (conformal) flatness condition in one direction, and the elliptic inverse boundary value problem was reduced to the invertibility of a broken geodesic ray transform. The geodesics used in the transform break via the normal reflection when they hit the inaccessible part of the boundary. However, barring some special cases, it is not known if such a transform is invertible, and moreover, there are also counter-examples to invertibility in general. We refer to [Il] for a discussion of both positive results and counter-examples. In [KiOk] we do not pursue a lateral reflection type argument.

We recall that the result of several authors, that treated our problem for wave equations with constant leading order coefficients (e.g. [Ben15, RakRam, RaSj, Sala]), is based on the use of solutions of form (2.3.5.1). There (M, g) is a domain with the Euclidean geometry, and the inverse boundary value problem is reduced to the problem to invert the light-ray transform in the Minkowski space. An analogous reduction is possible also in the case of more complicated geometry [Wa], however, the invertibility of the light-ray transform on a Lorentzian manifold of the product form $((0,T) \times M, dt^2 - g)$, where (M, g) is simple, was an open question during the preparation of the paper [KiOk]. Therefore, in [KiOk], as having (restricted) data on the top and bottom allows for a reduction to the well-understood problem to invert a weighted geodesic ray transform, rather than the light-ray transform. More precisely, we start by extending M to a simple manifold M_1. Then, for all $x \in \partial M_1$ denoting by $\tau_+(x, \theta)$ the time of existence in M_1 of the maximal geodesic $\gamma_{x,\theta}$ satisfying $\gamma_{x,\theta}(0) = x$ and $\gamma'_{x,\theta}(0) = \theta$, we introduce the attenuated geodesic ray transform I_μ on the

inward pointing boundary of the unit sphere bundle $\partial_+ SM_1 = \{(x,\theta) \in SM_1 : \ x \in \partial M_1, \ \langle \theta, \nu(x) \rangle_g < 0\}$, defined by

$$I_\mu f(x,\theta) = \int_0^{\tau_+(x,\theta)} f(\gamma_{x,\theta}(r)) e^{-\mu r} dr, \quad (x,\theta) \in \partial_+ SM_1, \ f \in C^\infty(M_1).$$

Here $\mu > 0$ gives constant attenuation and we recall that $SM_1 = \{(x,\theta) \in TM : \ |\theta|_{g(x)} = 1\}$. The map I_μ admits a unique continuous extension to the distributions on M_1. We denote by $\mathcal{L}_\mu$ the Laplace transform with respect to $t \in (0,+\infty)$, that is,

$$\mathcal{L}_\mu f = \int_0^{+\infty} f(t) e^{-\mu t} dt, \quad f \in L^1(0,+\infty).$$

Using solutions of the form (2.3.5.1), we prove that for $q = q_1 - q_2$ extended by 0 to $(0,+\infty) \times M_1$, we have $I_\mu \mathcal{L}_\mu q = 0$, $\mu > 0$, in the sense of distributions on $\partial_+ SM_1$. Combining this with some properties of weighted geodesic transform stated in [FrStUh, DoKeSaUh] we complete the proof of Theorem 2.3.1 and 2.3.2. More recently, after a remark of Yaroslav Kurylev, we realized that the injectivity of the light-ray transform on a Lorentzian manifold of the product form $(\mathbb{R} \times M, dt^2 - g)$ can be deduced from the injectivity of the geodesic ray transform on M. Thus, the proof of Theorem 2.3.1 and 2.3.2 can be refactored to give invertibility of the light ray transform in the product geometry case.

For Theorem 2.3.3, inspired by [Ki161, Ki162, Ki164], we replace the oscillating solutions (2.3.5.1) by exponentially growing or decaying solutions of the form

$$u(t,x) = e^{\sigma(\beta t + \varphi(x))}(a_\sigma(t,x) + R_\sigma(t,x)), \quad (t,x) \in (0,T) \times M, \tag{2.3.5.2}$$

with $\sigma \in \mathbb{R}$ a parameter, $\beta \in [1/2,1]$, R_σ a term that admits a decay with respect to the parameter $|\sigma|$ and φ a limiting Carleman weight for elliptic equations as defined in [DoKeSaUh]. In a similar way to [Ki161], we use two different approaches for the construction of exponentially growing and decaying solutions of the form (2.3.5.2). We start with a construction of exponentially decaying solutions $u_1 \in H^1((0,T) \times M)$ of $\Box_{a,g} u_1 + q_1 u_1 = 0$ taking the form

$$u_1(t,x) = e^{-\sigma(\beta t + \varphi(x))}(a_{1,\sigma}(t,x) + R_1(t,x)), \tag{2.3.5.3}$$

where $\beta \in [1/2,1]$, $\sigma > 0$. For this purpose, we extend our manifold M into a cylindrical manifold and we consider the restriction on $(0,T) \times M$ of exponentially decaying solutions on the extended domain.[1/2,1] More precisely, using the reduction

$$(a^{-1}\partial_t^2 - \Delta_g + q)\left(a^{-\frac{n-2}{4}} v\right) = a^{-\frac{n+2}{4}}\left(\partial_t^2 v - \Delta_{e \oplus g'} v + q_a v\right), \tag{2.3.5.4}$$

with $q_a = aq + a^{\frac{n+2}{4}} \Delta_g \left(a^{-\frac{n-2}{4}}\right)$, we consider $q_{1,a} = aq_1 + a^{\frac{n+2}{4}} \Delta_g \left(a^{-\frac{n-2}{4}}\right)$ and $u_1 = a^{-\frac{n-2}{4}} v$ where v is a solution of

$$\partial_t^2 v - \Delta_{e \oplus g'} v + q_{1,a}(t,x) v = 0 \quad \text{on } (0,T) \times (-R,R) \times M_1, \tag{2.3.5.5}$$

with $q_{1,a}$ is extended by zero to a function lying in $L^\infty((0,T) \times (-R,R) \times M_0)$. Here, for $\sigma > 1$, $\beta \in [1/2,1]$, v takes the form

$$v(t,x) = e^{-\sigma(\beta t + x_1)}(k(t,x) + w(t,x)), \quad (t,x) \in (0,T) \times (-R,R) \times M_1 \tag{2.3.5.6}$$

with $w \in H^1((0,T) \times (-R,R) \times M_1)$ satisfying

$$\|w\|_{L^2((0,T) \times (-R,R) \times M_1)} \leqslant \frac{C}{\sigma}.$$

Here the expression k is suitably chosen for the uniqueness result of Theorem 2.3.3. For the construction of the remainder term w appearing in (2.3.5.6), we combine a seperation of variable argument with properties of

fundamental solutions of PDEs with constant coefficients. We consider also exponentially growing solutions $u_2 \in L^2((0,T)\times M)$ taking the form

$$u_2(t,x) = e^{\sigma(\beta t+\varphi(x))}(a_{2,\sigma}(t,x) + R_2(t,x)) \tag{2.3.5.7}$$

and satisfying the additional condition

$$u_2(t,x) = 0, \quad (t,x) \in (\{0\}\times M)\cup U_1,$$

where U_1 is a neighborhood of $((0,T)\times\partial M)\setminus U$ in $(0,T)\times\partial M$. The construction of the solutions (2.3.5.7) requires the following Carleman estimate.

Theorem 2.3.4 (Theorem 4.1, [KiOk]) *Let $q \in L^\infty((0,T)\times M)$, $\beta \in [1/2,1]$ and $u \in C^2([0,T]\times M)$. We use the following notation $s_- = 0$, $s_+ = T$, $\psi(x,t) = \beta t + x_1$, $\psi_-(x_1) = -\beta T - x_1$ and $\psi_+(x_1) = x_1$. If u satisfies the condition*

$$u_{|(0,T)\times\partial M} = 0, \quad u_{|t=s_\pm} = \partial_t u_{|t=s_\pm} = 0,$$

then there exist constants $\sigma_1 > 1$ and $C > 0$ depending only on M, T and $\|q\|_{L^\infty((0,T)\times M)}$ such that the estimate

$$\begin{aligned}
&\sigma \int_M e^{2\sigma\psi_\pm} |\partial_t u(s_\mp,x)|^2 \, dV_g(x) \\
&+\sigma \int_{\Sigma_\mp} e^{\pm 2\sigma\psi} |\partial_\nu u|^2 |\partial_\nu\varphi| \, d\sigma_g(x)dt + \sigma^2 \int_{(0,T)\times M} e^{\pm 2\sigma\psi} |u|^2 \, dV_g(x)dt \\
&\leqslant C\left(\int_{(0,T)\times M} e^{\pm 2\sigma\psi} |(\Box_{a,g}+q)u|^2 \, dxdt + \sigma^3 \int_M e^{2\sigma\psi_\pm} |u(s_\mp,x)|^2 \, dV_g(x)\right) \\
&+C\left(\sigma \int_M e^{2\sigma\psi_\pm} |\nabla_g u(s_\mp,x)|_g^2 \, dV_g(x) + \sigma \int_{\Sigma_\pm} e^{\pm 2\sigma\psi} |\partial_\nu u|^2 |\partial_\nu\varphi| \, d\sigma_g(x)dt\right)
\end{aligned}$$

holds true for $\sigma \geqslant \sigma_1$.

In a similar way to [Ki161], using these Carleman estimates we construct solutions of the form (2.3.5.7). Combining these two constructions with the Carleman estimates stated in Theorem 2.3.4, we complete the proof of Theorem 2.3.3 by using the injectivity of geodesic ray transform on simple manifolds.

2.4 Stable determination of an electromagnetic potential appearing in a Schrödinger equation

2.4.1 Statement of the problem

This section is devoted to the result [KiSo] which is a joint work with Eric Soccorsi. Let Ω be a bounded and simply connected domain of $\mathbb{R}^n$, $n \geqslant 2$, with C^3 boundary $\partial\Omega$. For $T > 0$, we consider the initial boundary value problem (IBVP)

$$\begin{cases} (i\partial_t + \Delta_{A(t)} + q(t))u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma, \end{cases} \tag{2.4.1.1}$$

where $\Delta_{A(t)}$ is the Laplace operator associated with the magnetic potential $A \in W^{2,\infty}(Q)^n$,

$$\Delta_{A(t)} := \sum_{j=1}^n (\partial_{x_j} + ia_j(t,x))^2 = \Delta + 2iA(t,x)\cdot\nabla + i(\nabla\cdot A(t,x)) - |A(t,x)|^2,$$

and $q \in W^{1,\infty}(Q)$. We associate with (2.4.1.1) the DN map

$$\Lambda_{A,q}(g) := (\partial_\nu + iA\cdot\nu)u. \tag{2.4.1.2}$$

The goal of [KiSo], is to prove, in some suitable sense, the stable recovery of the coefficients (A,q) from the DN map $\Lambda_{A,q}$.

2.4.2 Direct problem

We introduce the spaces $\mathcal{H}(\Sigma)$ defined by

$$g \in \mathcal{H}(\Sigma) := \left\{ g \in H^{\frac{5}{2},\frac{5}{2}}(\Sigma);\ g(0,\cdot) = \partial_t g(0,\cdot) = 0 \text{ on } \partial\Omega \right\}$$

Then, for all $g \in \mathcal{H}(\Sigma)$, we establish below, that there exists a unique solution $u_g \in H^{1,2}(Q)$ to (2.4.1.1) and that the mapping $g \mapsto u_g$ is continuous.

Proposition 2.4.1 (Proposition 2.1, [KiSo]) *Let $A \in W^{2,\infty}(Q)$, $q \in W^{1,\infty}(Q)$ and $g \in \mathcal{H}(\Sigma)$ such that*

$$\|A\|_{W^{2,\infty}(Q)} + \|q\|_{W^{1,\infty}(Q)} \leqslant M.$$

Then, problem (2.4.1.1) *admits a unique solution $u \in H^{1,2}(Q)$ satisfying*

$$\|u\|_{H^{1,2}(Q)} \leqslant C\|g\|_{\mathcal{H}(\Sigma)}.$$

with C depending on M, T and Ω. Therefore, the DN map $\Lambda_{A,q}$ is well defined and it is a bounded operator from $\mathcal{H}(\Sigma)$ to $L^2(\Sigma)$.

Using this result we can prove that $\Lambda_{A,q}$ is a bounded operator from $\mathcal{H}(\Sigma)$ to $L^2(\Sigma)$.

2.4.3 Obstruction to the unique recovery

For $j = 1, 2$, let $(A_j, q_j) \in W^{2,\infty}(Q)^n \times W^{1,\infty}(Q)$. We say that (A_1, q_1) and (A_2, q_2) are gauge equivalent if there exists $\varphi \in W^{3,\infty}(Q)$ such that $\varphi_{|(0,T)\times\partial\Omega} = 0$, $A_2 = A_1 + \nabla\varphi$ and $q_2 = q_1 - \partial_t\varphi$. We recall that the DN map (2.4.1.2) is invariant by this gauge transformation. Namely, let $\varphi \in W^{3,\infty}(Q)$ be such that $\varphi_{|(0,T)\times\partial\Omega} = 0$, $A_2 = A_1 + \nabla\varphi$, $q_2 = q_1 - \partial_t\varphi$ and, for $j = 1, 2$, let u_j be the solution (2.4.1.1) with $A = A_j$, $q = q_j$. Then, we have

$$(i\partial_t + \Delta_{A_1(t)}u + q_1(t))e^{i\varphi}u_2 = e^{i\varphi}(i\partial_t + \Delta_{A_2(t)}u + q_2(t))u_2 = 0$$

and we deduce that $e^{i\varphi}u_2 = u_1$ and

$$(\partial_\nu + iA_1\nu)u_1 = (\partial_\nu + i(A_1 + \nabla\varphi)\cdot\nu)u_2 = (\partial_\nu + iA_2\nu)u_2.$$

This implies that $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$ but $(A_1, q_1) \neq (A_2, q_2)$ as soon as $\varphi \neq 0$. Taking into account this obstruction to uniqueness, in [KiSo], we study the recovery of the time-dependent electromagnetic potential (A, q) modulo gauge invariance from the DN map $\Lambda_{A,q}$. This problem is equivalent to the determination of (A, q), with $\mathrm{div}_x(A)$ fixed, from the DN map $\Lambda_{A,q}$. The goal of [KiSo] is to treat the stability issue for this last problem.

2.4.4 Hölder stability estimate

Theorem 2.4.1 (Theorem 1, [KiSo]) *Fix $M \in (0, +\infty)$ and for $j = 1, 2$, let $A_j \in W^{5,\infty}(Q)^n \cap H^6(Q)^n$ and $q_j \in W^{4,\infty}(Q)$ satisfy the three following conditions:*

$$\partial_x^\alpha A_1(t,x) = \partial_x^\alpha A_2(t,x),\ (t,x) \in \Sigma,\ \alpha \in \mathbb{N}^n,\ |\alpha| \leqslant 5, \tag{2.4.4.1}$$

$$\mathrm{div}_x A_1(t,x) = \mathrm{div}_x A_2(t,x),\ (t,x) \in Q \tag{2.4.4.2}$$

and

$$\sum_{j=1}^{2} \left(\|A_j\|_{W^{5,\infty}(Q)^n} + \|A_j\|_{H^6(Q)^n} + \|q_j\|_{W^{4,\infty}(Q)} \right) \leqslant M. \tag{2.4.4.3}$$

Then, there exist three positive constants, r and s, depending only on n, and C, depending only on n, T, Ω and M, such that we have

$$\|A_1 - A_2\|_{L^2(0,T;H^5(\Omega))} \leqslant C \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^r \tag{2.4.4.4}$$

and

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant C \|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|^s . \tag{2.4.4.5}$$

Here $\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|$ denotes the norm of $\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}$ as an element of $\mathcal{B}(\mathcal{H}(\Sigma), L^2(\Sigma))$.

2.4.5 Known results and comments

Only few authors considered the problem of determining time-dependent coefficients for Schrödinger equations. The recovery of time-dependent electromagnetic potentials has been considered by [Es08], who proved uniqueness modulo gauge invariance from the DN map. The stability issue for this problem was treated by [ChKiSo152] who established a log-type stability estimate in the recovery of a time-dependent electric potential from boundary operator including the DN map and information at initial and final time of the solutions. More recently [Ben17] extended the work of [ChKiSo152] to the stable recovery of electromagnetic potential with a magnetic potential which is time-independent and sufficiently small. To our best knowledge [Ben17, ChKiSo152] are the only papers dealing with stable recovery of time-dependent coefficients for Schrödinger equations.

Let us observe that, to our best knowledge, this result is the first result of stability in the recovery of general time-dependent electromagnetic potentials, with both electric and magnetic potential that dependent on time and space variables, appearing in a Schrödinger equation from boundary measurements. Moreover, not only the stability estimates (2.4.4.4)-(2.4.4.5) restrict the data considered by [Ben17, ChKiSo152], which seem to be the only other results of stable recovery of time-dependent coefficients for Schrödinger equations, but they also improve the log-type stability of [Ben17, ChKiSo152] to a Hölder stability. Actually, our result seems to be the first result of Hölder stability for the recovery of general coefficients, depending on time and space variables, for evolution PDEs (see the log-type stability estimate derived in [Ben15, ChKi16] for hyperbolic and parabolic equations).

Like many results related to this problem (e.g. [Es08, BeCh10, BelDo10]) our result is based on construction of suitable GO solutions associated with (2.4.1.1). In [KiSo], we purpose a new construction that allows to reduce the regularity assumptions of [Es08] and derive Hölder stability estimate for time-dependent coefficients. More precisely, we consider GO solutions of the form

$$u(t,x) = e^{i\sigma(-\sigma t + x\cdot\omega)} \left(\sum_{k=1}^{N} \frac{b_k(t,x)}{\sigma^{k-1}} \right) + R_\sigma, \tag{2.4.5.1}$$

with $\omega \in \mathbb{S}^{n-1}$ and a reminder term R_σ satisfying

$$\|R_\sigma\|_{L^2(0,T;H^1(\Omega))} + \sigma \|R_\sigma\|_{L^2(Q)} \leqslant C\sigma^{-1}.$$

In [Es08], Eskin considered such solutions for large value of N for proving unique recovery of infinitely smooth time-dependent electromagnetic coefficients. By using the scaling $t \leftrightarrow \sigma t$, [Bel17, BeCh10] considered solutions of the form (2.4.5.1) with $N = 1$. In [KiSo], we use solutions of the form (2.4.5.1) for $N = 2$. This approach allows us at the same time to reduce the strong regularity assumption of [Es08] and to derive in a straightforward way the stability estimates (2.4.4.4)-(2.4.4.5). Note also that, by extending the asymptotic expansion of the principal part of the GO solutions from order 1 to order 2, with respect to $\sigma \to +\infty$, we manege to extend the stability estimate of [BeCh10] to time-dependent coefficients by "*freezing*" the time variable in the expression b_1, b_2 appearing in (2.4.5.1) for $N = 2$.

2.5 Stable recovery of time-dependent coefficients for parabolic equations

2.5.1 Statement

This section is devoted to the inverse parabolic problem of [ChKi16] which is a joint work with Mourad Choulli. We fix Ω a $\mathcal{C}^2$ bounded domain. For $X = \Omega$ or $X = \partial\Omega$ and $r, s > 0$ we introduce the Sobolev spaces

$$H^{r,s}((0,T) \times X) = H^r(0,T; L^2(X)) \cap L^2(0,T; H^s(X)).$$

Following [LiMa2, Section 8, Chapter 4], we define also the space $H^{-r,-s}((0,T) \times X)$ to be the dual space of the $H^{r,s}((0,T) \times X)$ closure of $\mathcal{C}_0^\infty((0,T) \times X)$. In particular, in view of [LiMa1, Theorem 11.1, Chapter 1], for $r \in (0, \frac{1}{2})$, $H^{-r,-s}(\Sigma)$ is the dual space of $H^{r,s}(\Sigma)$. We consider the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(t,x)u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma. \end{cases} \tag{2.5.1.1}$$

We prove in [ChKi16, Proposition 2.3] that for $q \in L^\infty(Q)$ and $g \in H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma)$ this problem admits a unique weak solution $u \in L^2(Q)$. Moreover, the DN map $\Lambda_q : H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma) \ni g \mapsto \partial_\nu u \in H^{-\frac{3}{4},-\frac{3}{2}}(\Sigma)$ is bounded. In [ChKi16] we consider the problem of determining a time-dependent potential q from partial knowledge of Λ_q. More precisely we look for a stability estimate associated with this inverse problem.

2.5.2 Motivations

The IBVP (2.5.1.1) is for instance a typical model of the propagation of the heat through a time-evolving homogeneous body. The goal is to determine the coefficient q, who contains some properties of the body, by applying a heat source on some part of the boundary of the body and measuring the temperature on another part of the boundary of the body. Moreover, in many applications we are often lead to determine physical quantities via parabolic IBVP's including nonlinear terms from boundary measurements. For instance such kind of problems appears in reservoir simulation, chemical kinetics and aerodynamics. Considering time-dependent unknown coefficients in parabolic equations is very useful when treating the determination of the nonlinear term appearing in a semilinear parabolic equation. We discuss this topic in Section 2.4.

2.5.3 Known results

There is a wide literature devoted to inverse parabolic problems and specifically the determination of time-dependent coefficients. We just present briefly some typical results. Canon and Esteva [CaEs861] proved a logarithmic stability estimate for the determination of the support of a source term in a one dimension parabolic equation from a boundary measurement. This result was extended to three dimension heat equation in [CaEs862]. The case of a non local measurement was considered by Canon and Lin in [CaLi88, CaLi90]. In [Ch911], Choulli proved existence, uniqueness and Lipschitz stability for the determination of a time-dependent coefficient appearing in an abstract integro-differential equation, extending earlier results in [Ch912]. Choulli and Yamamoto established in [ChYa06] a stability estimate for the inverse problem of determining a source term appearing in a heat equation from Neumann boundary measurements. In [Is911], Isakov extended the construction of CGO solutions, introduced in [SyUh], to various PDE's including hyperbolic and parabolic equations to prove the density of products of solutions. One can get from the results in [Is911] the unique determination of q from the measurements on the lateral boundary together with data at the final time and all input at initial time. When the space domain is cylindrical, adopting the strategy introduced in [BuKl], [GaKi] proved that the time-dependent zero order coefficient can be recovered uniquely from a single boundary measurement. Based on properties of fundamental solutions of parabolic

equations, [ChKi13] proved Lipschitz stability for the determination of the time-dependent part of a zero order coefficient in a parabolic IBVP from a single boundary measurement.

2.5.4 Direct problem

In this subsection we introduce the result of [ChKi16] related to the well-posedness of (2.5.1.1) and definition of the DN map Λ_q. For $\varepsilon = \pm$, $\Omega_+ = \{0\} \times \Omega$, $\Omega_- = \{T\} \times \Omega$, we consider the IBVP

$$\begin{cases} (\varepsilon\partial_t - \Delta + q(t,x))u = 0 \quad \text{in } Q, \\ u_{|\Omega_\varepsilon} = 0, \\ u_{|\Sigma} = g. \end{cases} \tag{2.5.4.1}$$

Our first result is given by the following.

Proposition 2.5.1 (Proposition 2.3, [ChKi16]) *For $m > 0$, $g \in H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma)$ and $q \in L^\infty(Q)$, with $\|q\|_{L^\infty(Q)} \leqslant m$, the IBVP* (2.5.4.1) *admits a unique transposition solution $u^\varepsilon_{q,g} \in L^2(Q)$ satisfying*

$$\left\|u^\varepsilon_{q,g}\right\|_{L^2(Q)} \leqslant C \|g\|_{H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma)}, \tag{2.5.4.2}$$

where the constant C depends only on Q and m. Additionally the parabolic DN map

$$\Lambda_q : g \mapsto \partial_\nu u^+_{q,g}$$

defines a bounded operator from $H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma)$ into $H^{-\frac{3}{4},-\frac{3}{2}}(\Sigma)$.

We prove an additional result of smoothing stated in

Proposition 2.5.2 (Proposition 2.4, [ChKi16]) *Let $q, \widetilde{q} \in L^\infty(Q)$. Then $\Lambda_q - \Lambda_{\widetilde{q}}$ is a bounded operator from $H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma)$ into $H^{\frac{1}{4},\frac{1}{2}}(\Sigma)$ and*

$$\langle(\Lambda_q - \Lambda_{\widetilde{q}})g, h\rangle = \int_Q (q - \widetilde{q}) u^+_{q,g} u^-_{\widetilde{q},h} dxdt, \quad g, h \in H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma). \tag{2.5.4.3}$$

2.5.5 Main results and comments

The unit ball of a Banach space X will be denoted in the sequel by B_X. For $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$, set

$$\Psi_s(\rho) = \rho + |\ln \rho|^{-\frac{1-2s(n+1)}{8}}, \quad \rho > 0,$$

extended by continuity at $\rho = 0$ by setting $\Psi_s(0) = 0$.

Theorem 2.5.1 (Theorem 1.1, [ChKi16]) *Fix $m > 0$ and $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$. There exists a constant $C > 0$, that can depend only on m, Q and s, so that, for any $q, \widetilde{q} \in L^\infty(Q)$ satisfying*

$$\max\left(\|q\|_{L^\infty(Q)}, \|q\|_{L^\infty(Q)}\right) \leqslant m,$$

we have

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant C\Psi_s\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|\right). \tag{2.5.5.1}$$

Here $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ stands for the norm of $\Lambda_{q_1} - \Lambda_{q_2}$ in $\mathscr{B}(H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma); H^{\frac{1}{4},\frac{1}{2}}(\Sigma))$.

In the case of the infinite cylindrical domain $Q = (0, +\infty) \times \Omega$, Isakov [Is95] got a stability estimate of determining $q = q(x)$ from the full parabolic DN map by combining the decay in time of solutions of parabolic equations and the stability estimate in [Al] concerning the problem of determining the zero order coefficient in a elliptic BVP from a full DN map. For finite cylindrical domain Q, to our knowledge, even for time-independent coefficients, there is no result in the literature dealing with the stability issue of recovering q from measurements only on the lateral boundary Σ. All such results require also volume measurements at some fix time $t_0 \in [0, T]$.

For $\omega \in \mathbb{S}^{n-1}$, set

$$\Gamma_{\pm,\omega} = \{x \in \Gamma;\ \pm\nu(x) \cdot \omega > 0\}$$

and $\Sigma_{\pm,\omega} = \Gamma_{\pm,\omega} \times (0, T)$. Fix $\omega_0 \in \mathbb{S}^{n-1}$, $\mathcal{U}_\pm$ a neighborhood of $\Gamma_{\pm,\omega_0}$ in Γ and set $\mathcal{V}_+ = \mathcal{U}_+ \times [0, T]$, $\mathcal{V}_- = \mathcal{U}_- \times (0, T)$. Define then the partial parabolic DN map

$$\begin{aligned} \widehat{\Lambda}_q : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \cap \mathscr{E}'(\mathcal{V}_+) &\to H^{-\frac{3}{2},-\frac{3}{4}}(\mathcal{V}_-) \\ g &\mapsto \partial_\nu u_{q,g|\mathcal{V}_-}. \end{aligned}$$

Note that Theorem 2.5.1 is obtained as by-product of the analysis we developed to derive a logarithmic stability estimate for the problem of determining q from the partial parabolic DN map $\widehat{\Lambda}_q$. This result is stated in the following theorem, where

$$\Phi_s(\rho) = \rho + |\ln|\ln \rho||^{-s}, \quad \rho > 0,\ s > 0, \tag{2.5.5.2}$$

extended by continuity at $\rho = 0$ by setting $\Phi_s(0) = 0$.

Theorem 2.5.2 (Theorem 1.2, [ChKi16]) *Let $m > 0$, there exist two constants $C > 0$ and $s \in (0, 1/2)$, that can depend only on m, Q and $\mathcal{V}_\pm$, so that, for any $q, \widetilde{q} \in mB_{L^\infty(Q)}$,*

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant C\Phi_s\left(\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\|\right). \tag{2.5.5.3}$$

Here $\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}\|$ denotes the norm of $\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}$ in $\mathscr{B}(H^{-\frac{1}{4},-\frac{1}{2}}(\Sigma); H^{\frac{1}{4},\frac{1}{2}}(\mathcal{V}_-))$.

It is worth mentioning that the uniqueness holds for the problem of determining q from the partial DN operator that maps the boundary condition g supported on $\gamma_0 \times (0, T)$ into $\partial_\nu u_{q,g}$ restricted to $\gamma_0 \times (0, T)$, where γ_0 is an arbitrary nonempty open subsets of Γ. This result is stated in [Ch09, Theorem 3.27]. We note that the stability estimate corresponding to this uniqueness result remains an open problem.

2.5.6 Idea of the proof

Like [Ki161, Ki162, Ki164], the main tool in the analysis of [ChKi16] are suitable solutions depending explicitly on some Carleman weight for the parabolic equation and its adjoint equation. In contrast to the application of such results for elliptic equations by [BuUh, KeSjUh], the main contribution of these solutions is not so much the restriction of the data on the lateral boundary Σ but on the top and on the bottom of the time-space cylindrical domain corresponding to $\{0\} \times \Omega$ and $\{T\} \times \Omega$. In contrast to the result for hyperbolic equations [Ki161, Ki162, Ki164], we actually manage to get rid of the data at $\{0\} \times \Omega$ and $\{T\} \times \Omega$ for the stable recovery of a time-dependent coefficient. To our best knowledge this is the first result of stability of this kind for general recovery of general time-dependent coefficients appearing in a parabolic equation.

We fix $q_j \in L^\infty(Q)$, $j = 1, 2$, and we construct in [ChKi16] two sets of exponentially growing and decaying solutions $u_1 \in L^2(Q)$ and $u_2 \in L^2(Q)$ of the equations

$$\begin{cases} \partial_t u_1 - \Delta u_1 + q_1(t, x)u_1 = 0, & \text{in } Q, \\ u_1(0, \cdot) = 0, & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} -\partial_t u_2 - \Delta u_2 + q_2(t,x)u_2 = 0, & \text{in } Q, \\ u_2(T,\cdot) = 0, & \text{in } \Omega. \end{cases}$$

In order to introduce the general form of these solutions we consider, for all $y \in \mathbb{S}^{n-1}$ and all $r > 0$, the parts of the boundary

$$\partial\Omega_{+,r,y} = \{x \in \partial\Omega : \ \nu(x)\cdot y > r\}, \quad \partial\Omega_{-,r,y} = \{x \in \partial\Omega : \ \nu(x)\cdot y \leqslant r\}$$

and $\Sigma_{\pm,r,y} = (0,T) \times \partial\Omega_{\pm,r,y}$. Then, we fix $\omega \in \mathbb{S}^{n-1}$, $\xi \in \mathbb{R}^n$ with $\xi\cdot\omega = 0$, $\tau \in \mathbb{R}$ and $\rho > \rho_1$. We build solutions u_1 of the form

$$u_1(t,x) = e^{\rho^2 t + \rho\omega\cdot x}\left[\left(1 - e^{-\rho^{\frac{3}{4}}t}\right)e^{-it\tau - ix\cdot\xi} + w_1(t,x)\right]$$

and solutions u_2 of the form

$$u_2(t,x) = e^{-\rho^2 t - \rho\omega\cdot x}\left[\left(1 - e^{-\rho^{\frac{3}{4}}(T-t)}\right) + w_2(t,x)\right].$$

Here the function w_j, $j = 1,2$, are chosen in such a way that for some $\varepsilon \in (0,1)$ we have

$$\begin{cases} w_1 \in L^2(Q), \\ \|w_1\|_{L^2(Q)} \leqslant C(\rho^{-\frac{1}{4}} + \rho^{-1}\langle(\tau,\xi)\rangle^2), \\ w_1(0,x) = 0, \quad x \in \Omega, \\ w_1(t,x) = -\left(1 - e^{-\rho^{\frac{3}{4}}t}\right)e^{-it\tau - ix\cdot\xi}, \quad (t,x) \in \Sigma_{+,\varepsilon/2,-\omega}, \end{cases} \tag{2.5.6.1}$$

$$\begin{cases} w_2 \in L^2(Q), \\ \|w_2\|_{L^2(Q)} \leqslant C(\rho^{-\frac{1}{4}} + \rho^{-1}\langle(\tau,\xi)\rangle^2), \\ w_2(T,x) = 0, \quad x \in \Omega, \\ w_2(t,x) = -\left(1 - e^{-\rho^{\frac{3}{4}}(T-t)}\right), \quad (t,x) \in \Sigma_{+,\varepsilon/2,\omega}. \end{cases} \tag{2.5.6.2}$$

In all these estimates C is a constant depending on Ω, T, ε, $M \geqslant \|q_1\|_{L^\infty(Q)} + \|q_2\|_{L^\infty(Q)}$. Moreover, $\langle\xi\rangle$, $\langle(\tau,\xi)\rangle$ denote respectively the quantity

$$\langle\xi\rangle := \sqrt{1+|\xi|^2}, \quad \langle(\tau,\xi)\rangle := \sqrt{1+|\xi|^2+\tau^2}.$$

The main point in the construction of these solutions comes from the expression w_j, $j = 1,2$, satisfying (2.5.6.1)-(2.5.6.2). The construction of this expression is by means of the following Carleman estimate.

Theorem 2.5.3 (Theorem 3.1, [ChKi16]) *Let $q \in L^\infty(Q)$ and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition*

$$u_{|\Sigma} = 0, \quad u_{|t=0} = 0$$

then there exists $\rho_1 > 1$ depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$ such that the estimate

$$\begin{aligned} &\int_\Omega e^{-2(\rho^2 T + \rho\omega\cdot x)}|u(T,x)|^2\,dx + \rho\int_{\Sigma_{+,\omega}} e^{-2(\rho^2 t+\rho\omega\cdot x)}|\partial_\nu u|^2|\omega\cdot\nu(x)|\,d\sigma(x)dt + \rho^2\int_Q e^{-2(\rho^2 t+\rho\omega\cdot x)}|u|^2\,dxdt \\ &\leqslant C\left(\int_Q e^{-2(\rho^2 t+\rho\omega\cdot x)}|(\partial_t - \Delta + q)u|^2\,dxdt + \rho\int_{\Sigma_{-,\omega}} e^{-2(\rho^2 t+\rho\omega\cdot x)}|\partial_\nu u|^2|\omega\cdot\nu(x)|\,d\sigma(x)dt\right) \end{aligned}$$

holds true for $\rho \geqslant \rho_1$ with C depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$. If u satisfies the condition

$$u_{|\Sigma} = 0, \quad u_{|t=T} = 0$$

then the estimate

$$\int_\Omega e^{2\rho\omega\cdot x}\left|u(0,x)\right|^2 dx+\rho\int_{\Sigma_{-,\omega}} e^{2(\rho^2 t+\rho\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right| d\sigma(x)dt+\rho^2\int_Q e^{2(\rho^2 t+\rho\omega\cdot x)}\left|u\right|^2 dxdt$$
$$\leqslant C\left(\int_Q e^{2(\rho^2 t+\rho\omega\cdot x)}\left|(-\partial_t-\Delta+q)u\right|^2 dxdt+\rho\int_{\Sigma_{+,\omega}} e^{2(\rho^2 t+\rho\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right| d\sigma(x)dt\right)$$

holds true for $\rho\geqslant\rho_1$.

Once the construction of our sets of solutions is completed we derive our stability result by recovering the Fourier transform of our coefficients.

2.6 Application to the recovery of a nonlinear term

This section is devoted to the application of the result of stable recovery of a time-dependent coefficient for a parabolic equation to the stable recovery of a nonlinear term. This result is stated in the last section of [ChKi16] which is a joint work with Mourad Choulli. The objective is the derivation of a stability estimate for the problem of determining the nonlinear term in a semilinear parabolic IBVP from the corresponding "linearized" DN map. We will give the precise definition of the "linearized" DN map later in this section. The results of this section are obtained as a consequence of Theorem 2.5.1. The linearization procedure, that we use, requires existence, uniqueness and *a priori* estimate of solutions of IBVP's under consideration. We preferred to work in the Hölder space setting for which we have a precise literature devoted to these aspects of solutions. However we are convinced that the same analysis can be achieved in the Sobolev space setting. But in that case this analysis seems to be more delicate. We mention also that this linearization process is inspired by the strategy set by Isakov in the papers [Is93, Is011, Is012] for results of uniqueness. In this section Ω is of class $C^{2+\alpha}$ for some $0<\alpha<1$. The parabolic boundary of Q is denoted by $\Sigma_p=\Sigma\cup(\{0\}\times\Omega)$. Consider the semilinear IBVP for the heat equation

$$\begin{cases}\partial_t u-\Delta u+a(t,x,u)=0 & \text{in } Q,\\ u=g & \text{on } \Sigma,\\ u(0,x)=u_0(x) & x\in\Omega.\end{cases}\tag{2.6.0.3}$$

We introduce some notations. We denote by $\mathscr{A}_0$ the set of functions from $C^1(\overline{Q}\times\mathbb{R})$ satisfying one of the following conditions:

(i) There exist two non negative constants c_0 and c_1 so that

$$ua(x,t,u)\geq -c_0u^2-c_1,\ \ (t,x,u)\in\overline{Q}\times\mathbb{R}.\tag{2.6.0.4}$$

(ii) There exist a non negative constant c_2 and a non decreasing positive function Φ of $\tau\geq 0$ satisfying

$$\int_0^\infty\frac{d\tau}{\Phi(\tau)}=\infty$$

such that

$$ua(x,t,u)\geq -|u|\Phi(|u|)-c_2,\ \ (t,x,u)\in\overline{Q}\times\mathbb{R}.\tag{2.6.0.5}$$

Set $X=C^{2+\alpha,1+\alpha/2}(\overline{Q})$ and let $X_0=\{(G_{|\overline{\Sigma}},G_{|t=0});\ \text{for some } G\in X\}$. If $\|\cdot\|_X$ denotes the natural norm on X we equip X_0 with the quotient norm

$$\|H\|_{X_0}=\inf\{\|G\|_X;\ (G_{|\overline{\Sigma}},G_{|t=0})=H\}.$$

By [LaSoUr, Theorem 6.1, page 452], for any $a\in\mathscr{A}_0$ and $G=(g,u_0)\in X_0$, the IBVP (2.6.0.3) has a unique solution $u_{a,G}\in X$. Additionally, according to [LaSoUr, Theorem 2.9, page 23], there exists a constant C that can depend only on Q, $\mathscr{A}_0$ and $\max\limits_{\overline{\Sigma_p}}|G|$ such that

$$\max_{\overline{Q}}|u_{a,G}|\leq C.\tag{2.6.0.6}$$

A quick inspection of [LaSoUr, inequalities (2.31) and (2.34), page 23] shows that

$$\max_{\overline{\Sigma}_p}|G| \to C = C(\max_{\overline{\Sigma}_p}|G|)$$

is non decreasing. Define the parabolic DN map N_a associated with $a \in \mathscr{A}_0$ by

$$N_a : G \in X_0 \longrightarrow \partial_\nu u_{a,G} \in Y = C^{1+\alpha,(1+\alpha)/2}(\overline{\Sigma}).$$

Note that, contrary to Section 2.5, the DN map N_a is no longer linear. The linearization procedure consists then in computing the Fréchet derivative of N_a. Let $\mathscr{A}$ be the subset of $\mathscr{A}_0$ of those functions a satisfying $\partial_u a \in C^2(\overline{Q} \times \mathbb{R})$. For $a \in \mathscr{A}$ and $H \in X_0$, consider the IBVP

$$\begin{cases} (\partial_t - \Delta)v + \partial_u a(x,t,u_{a,G}(x,t))v = 0 & \text{in } Q, \\ v = H & \text{on } \Sigma_p. \end{cases}$$

In light of [LaSoUr, Theorem 5.4, page 322] this IBVP has a unique solution $v = v_{a,G,H} \in X$ satisfying

$$\|v_{a,G,H}\|_X \leq c\|H\|_{X_0}$$

for some constant c depending only on Q, a and G. In particular $H \in X_0 \to v_{a,G,H} \in X$ defines a bounded operator. The Fréchet derivative of N_a is given by the following.

Proposition 2.6.1 (Proposition 6.1, [ChKi16]) *For each $a \in \mathscr{A}$, N_a is continuously Fréchet differentiable and*

$$N_a'(G)H = \partial_\nu v_{a,G,H} \in Y, \;\; G, H \in X_0.$$

In order to handle the inverse problem corresponding to the semi-linear IBVP (2.6.0.3), we need to extend the operator Λ_q introduced in Section 2.5 by varying also the initial condition. To do that we start by considering the IBVP

$$\begin{cases} (\partial_t - \Delta + q(t,x))u = 0 \;\text{ in } Q, \\ u(0,x) = u_0, \quad x \in \Omega \\ u_{|\Sigma} = g. \end{cases} \tag{2.6.0.7}$$

Then, we prove in [ChKi16] that the extended parabolic DN map

$$\begin{aligned} \Lambda_q^e : X_+ \oplus H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) &\to H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma) \\ (u_0, g) &\mapsto \partial_\nu u_{q,u_0,g} \end{aligned}$$

defines a bounded operator. Here X_+ is a subspace of $H^{-1}(\Omega)$ with a suitable topology (see [ChKi16, page 4] and [ChKi16, page 16]). For $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$, we define

$$\Theta_s(\rho) = |\ln \rho|^{-\frac{1-2s(n+1)}{n+3}} + \rho, \;\; \rho > 0, \tag{2.6.0.8}$$

extended by continuity at $\rho = 0$ by setting $\Theta_s(0) = 0$.

Fix $\lambda > 0$. There exists a constant $c_\lambda > 0$ so that

$$\max_{\overline{Q}} |u_{a,G}| \leq c_\lambda, \;\; a \in \mathscr{A}_0, \; \max_{\overline{\Sigma}_p} |G| \leq \lambda.$$

For fixed $\delta > 0$, consider

$$\widehat{\mathscr{A}} = \{a = a(x,u) \in \mathscr{A};\; \|\partial_u a\|_{C(\overline{\Omega}\times[-c_\lambda,c_\lambda])} \leq \delta\}.$$

To $a \in \widehat{\mathscr{A}}$ and $g \in X_0$ we associate

$$p_{a,G}(t,x) = \partial_u a(x, u_{a,G}(t,x)), \quad (t,x) \in \overline{Q}.$$

It is straightforward to check that

$$N_a'(G) = \Lambda^e_{p_{a,G}|X_0}.$$

From now on $N_a'(G) - N_{\widetilde{a}}'(G)$ is considered as a bounded operator from X_0 endowed with norm of $X_+ \oplus H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ into $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Since $\|p_{a,G}\|_{L^\infty(Q)} \leq \delta$ for any $a \in \widehat{\mathscr{A}}$ and $G \in X_0$ so that $\max_{\overline{\Sigma}_p} |G| \leq \lambda$, we get as a consequence of Proposition 2.6.1

$$\sup\{\|N_a'(G) - N_{\widetilde{a}}'(G)\|;\ a \in \widehat{\mathscr{A}},\ G \in X_0 \text{ and } \max_{\overline{\Sigma}_p} |G| \leq \lambda\} < \infty.$$

Moreover,

$$\|N_a'(G) - N_{\widetilde{a}}'(G)\| = \|\Lambda^e_{p_{a,G}} - \Lambda^e_{p_{\widetilde{a},G}}\|. \tag{2.6.0.9}$$

Pick $a_0 \in C^1(\overline{\Omega})$ and set

$$\widehat{\mathscr{A}_0} = \{a \in \widehat{\mathscr{A}};\ a(\cdot,0) = a_0\}.$$

We note that when $G \equiv s$, $|s| \leqslant \lambda$, we have

$$p_{a,G}(0,x) = \partial_u a(x, u_{a,G}(0,x)) = \partial_u a(x,s), \quad x \in \overline{\Omega}.$$

In light of this identity, combining Theorem 2.5.1 with Proposition 2.6.1, we prove the stable recovery of the nonlinear term a from N_a'.

Theorem 2.6.1 (Theorem 6.2, [ChKi16]) *Fix $\lambda > 0$ and $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$. There exists a constant $C > 0$, that can depend only on δ, λ, s, Q and $\widehat{\mathscr{A}_0}$, so that for any $a, \widetilde{a} \in \widehat{\mathscr{A}_0}$,*

$$\|a - \widetilde{a}\|_{C(\overline{\Omega}\times[-\lambda,\lambda])} \leqslant C\Theta_s \left(\sup_{g \in X_{0,\lambda}} \|N_a'(g) - N_{\widetilde{a}}'(g)\| \right).$$

Here $X_{0,\lambda} = \{G \in X_0;\ \max_{\overline{\Sigma}_p} |G| \leq \lambda\}$ and $\|N_a'(G) - N_{\widetilde{a}}'(G)\|$ stands for the norm of $N_a'(G) - N_{\widetilde{a}}'(G)$ in $\mathscr{B}(X_+ \oplus H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma); H^{\frac{1}{2},\frac{1}{4}}(\Sigma))$.

Remark 2.6.1 *From our approach, one can derive many other stability results. We just mention one of them. To this end, let $\widehat{\mathscr{A}_0}$ be defined as before with the only difference that we actually permit to functions of $\widehat{\mathscr{A}_0}$ to depend also on the time variable t. Let $a, \widetilde{a} \in \widehat{\mathscr{A}_0}$ and pick $(t_0, x_0, u_0) \in \Gamma \times (0,T) \times [-\lambda, \lambda]$ so that*

$$|(a - \widetilde{a})(t_0, x_0, u_0)| = \frac{1}{2}\|a - \widetilde{a}\|_{C([0,T]\times\Gamma\times[-\lambda,\lambda])}. \tag{2.6.0.10}$$

Let $\varepsilon = \min(t_0, T - t_0)$ and $G \in X_{0,\lambda}$ so that $G = s$ on $[\varepsilon, T-\varepsilon] \times \Gamma$ for some $|s| \leq \lambda$. We proceed as in the proof of Theorem 2.6.1 in order to derive

$$\|a - \widetilde{a}\|_{C([\varepsilon,T-\varepsilon]\times\Gamma\times[-\lambda,\lambda])} \leq C\Theta_s \left(\sup_{G \in X_{0,\lambda}} \|N_a'(G) - N_{\widetilde{a}}'(G)\| \right),$$

where the constant C depends only on λ, s, Q and $\widehat{\mathscr{A}_0}$. In light of (2.6.0.10) this estimate yields

$$\|a - \widetilde{a}\|_{C([0,T]\times\Gamma\times[-\lambda,\lambda])} \leq C\Theta_s \left(\sup_{G \in X_{0,\lambda}} \|N_a'(G) - N_{\widetilde{a}}'(G)\| \right).$$

Let us mention that uniqueness results for such kind of inverse semilinear parabolic problems was already established by Isakov [Is93, Is011, Is012]. Stability estimates and uniqueness in the case of a single boundary lateral measurement has been proved in [ChOuYa] for a restricted class of unknown nonlinearities. To our best knowledge, this result is the first stability estimate for the recovery of general non-linear term. Even in the context of uniqueness, Theorem 2.6.1 improves the work of [Is93] related to this problem.

Chapter 3

Inverse spectral problems

3.1 Introduction

The study of inverse spectral problems, which were one of the first mathematical formulation of inverse problems, goes back to Ambarzumian [Am] who investigated in 1929 the inverse spectral problem of determining the real potential V appearing in the Sturm-Liouville operator $A = -\partial_{xx} + V$, acting in $L^2(0, 2\pi)$ with Neumann boundary condition, from partial spectral data of A. For the same operator acting on $L^2(0, \pi)$, but endowed with homogeneous Dirichlet boundary conditions, Borg [Bo46] and Levinson [Le] established that while the Dirichlet spectrum does not uniquely determine V, assuming that $\varphi_k'(0) = 1$ for $k \geqslant 1$, additional spectral data, namely $\{\|\varphi_k\|_{L^2(0,\pi)} \; ; \; k \in \mathbb{N}^*\}$ is needed, where $\{\varphi_k \; ; \; k \in \mathbb{N}^*\}$ is an $L^2(0, \pi)$-orthogonal basis of eigenfunctions of A. Gel'fand and Levitan [GeLe] proved that uniqueness is still valid upon substituting $\varphi_k'(\pi)$ for $\|\varphi_k\|_{L^2(0,\pi)}$ in the one-dimensional Borg and Levinson theorem. In 1998, the case where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, was treated by Nachman, Sylvester and Uhlmann [NaSyUh], and by Novikov [Nov]. Inspired by [GeLe], these authors proved that the boundary spectral data $\{(\lambda_k, \partial_\nu \varphi_{k|\partial\Omega}) \; ; \; k \in \mathbb{N}^*\}$, where (λ_k, φ_k) is the kth eigenpair of A, uniquely determines the Dirichlet realization of the operator A. This result has been improved in several ways by various authors. For instance, Isozaki [Is] (see also Choulli [Ch09]) extended the result of [NaSyUh] when finitely many eigenpairs remain unknown, and, recently, Choulli and Stefanov [ChSt] proved uniqueness in the determination of V from the asymptotic behaviour of $(\lambda_k, \partial_\nu \varphi_{k|\partial\Omega})$ as $k \to \infty$. Moreover, Canuto and Kavian [CaKa01, CaKa04] proved that both the electric potential V and the conductivity c are uniquely determined from the boundary spectral data of the operator $u \mapsto -\nabla \cdot (c\nabla u) + Vu$ endowed with either Dirichlet or Neumann boundary conditions. We mention also the work of [Ou] related to these problems.

In this chapter, we will introduce our contributions to this field corresponding to the papers [KaKiSo, Ki163, KiMoOk]. We have considered first a formulation in the context of a Schrödinger operator in an infinite cylindrical domain with periodic coefficients. In this context the discrete spectrum is replaced by absolutely continuous spectrum. Nevertheless, by mean of Floquet theory we have established in [KaKiSo] a formulation of a Borg-Levinson type of result similar to [NaSyUh] for such operators. In our analysis, we have not only extended the result of [NaSyUh] to a class of operators with absolute continuous spectrum but we have also improved the result of [ChSt] by showing that a more precise asymptotic behavior of the boundary spectral data determines the Schrödinger operator. In our other contribution, we have considered the case of magnetic Schrödinger operators on a bounded domain. Namely, we have proved in [Ki163] that an asymptotic behavior of the boundary spectral data, similar to [KaKiSo], determines uniquely a magnetic Schrödinger operator modulo gauge invariance. Finally, in [KiMoOk] we have considered the problem of determining a Schrödinger operator from boundary spectral data restricted to an arbitrary portion of the boundary. For this last result, we apply the so called boundary control method introduced by Bellishev

[Beli87] and improved in various way by [BeliKu92, KaKuLa01, KuOkPa, LaOk10, LaOk141]. The first contribution of [KiMoOk] corresponds to an extension of such results to the case of non-smooth coefficients. Moreover, by considering a convex domain instead of a general Riemannian manifold, we have tried in [KiMoOk] to emphasize the main idea of the boundary control method.

3.2 Inverse spectral problem on an infinite waveguide

This section is devoted to our work [KaKiSo], which is a joint work with Otared Kavian and Eric Soccorsi related to two inverse spectral problems in which a potential is identified through an incomplete boundary spectral data.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain. On the one hand set $Y := \omega \times (0, 2\pi)$ and $\Gamma := \partial\omega \times [0, 2\pi]$; on the other hand consider an infinite waveguide $\Omega = \omega \times \mathbb{R}$, and $\partial\Omega = \partial\omega \times \mathbb{R}$. We may assume, without loss of generality, that the cross section ω of the waveguide contains the origin $0_{\mathbb{R}^2}$ of $\mathbb{R}^2$. For simplicity we assume that ω is $\mathcal{C}^{1,1}$ domain. For the sake of brevity of notations we write $x = (x', x_3)$ with $x' = (x_1, x_2) \in \omega$ for every $x = (x_1, x_2, x_3) \in \Omega$.

The main problem we study, and whose solution is a consequence of a result presented a few lines below, concerns an inverse spectral problem in a waveguide given by $\Omega = \omega \times \mathbb{R}$. We consider a real valued bounded electric potential $V \in L^\infty(\Omega; \mathbb{R})$ which is 2π-periodic with respect to the infinite variable x_3. Namely, we assume that $V \in L^\infty(\Omega; \mathbb{R})$, satisfies

$$V(x', x_3 + 2\pi) = V(x', x_3), \quad \forall\, x_3 \in \mathbb{R}, \tag{3.2.0.1}$$

and then we define the self-adjoint operator $(A, D(A))$ acting in $L^2(\Omega)$ by

$$Au := -\Delta u + Vu, \quad \text{for } \ u \in D(A) \tag{3.2.0.2}$$

with its domain

$$D(A) := \left\{ u \in H_0^1(\Omega) \ ; \ -\Delta u + Vu \in L^2(\Omega) \right\}.$$

We are interested in the problem of determining V from the partial knowledge of the spectral data associated with A. However, the operator $(A, D(A))$ being self-adjoint and its resolvent not being compact, it may have a continuous spectrum contained in an interval of type $[\lambda_*, +\infty)$: thus in the first place one should state precisely what is meant by an inverse spectral problem. To make this statement more precise we are going to recall the definition of the (full) spectral data associated with the operator A, but before doing so we state another result closely related to the above problem.

This result concerns the following inverse spectral problem: let Y be as above and consider a real valued potential $V \in L^\infty(Y)$, and for a given fixed $\theta \in [0, 2\pi)$ let $(\lambda_j(\theta), \varphi_{\theta,j})_{j \geqslant 1}$ be the eigenvalues and normalized eigenfunctions of the realization of the operator $-\Delta + V$ with quasi-periodic and Dirichlet boundary conditions, more precisely those eigenvalues and eigenfunctions given by

$$\left\{ \begin{array}{rcll} -\Delta\varphi_{\theta,j} + V\varphi_{\theta,j} & = & \lambda_j(\theta)\varphi_{\theta,j} & \text{in } \ Y, \\ \varphi_{\theta,j}(\sigma) & = & 0, & \sigma \in \Gamma, \\ \varphi_{\theta,j}(x', 2\pi) & = & \mathrm{e}^{\mathrm{i}\theta}\varphi_{\theta,j}(x', 0), & x' \in \omega, \\ \partial_3\varphi_{\theta,j}(x', 2\pi) & = & \mathrm{e}^{\mathrm{i}\theta}\partial_3\varphi_{\theta,j}(x', 0), & x' \in \omega. \end{array} \right. \tag{3.2.0.3}$$

Then we show that if $N \geqslant 1$ is a given integer, knowledge of

$$\lambda_j(\theta), \ \partial_\nu\varphi_{\theta,j\,|\Gamma} \quad \text{for } \ j \geqslant N + 1, \quad \text{with } \ \Gamma := \partial\omega \times [0, 2\pi],$$

allows us to identify the potential V in $Y = \omega \times (0, 2\pi)$. More precisely we show the following:

Theorem 3.2.1 (Theorem 1.1, [KaKiSo]) *Denote* $\Gamma := \partial\omega \times [0, 2\pi]$. *Let* $\theta \in [0, 2\pi)$ *and, for* $m = 1, 2$, *let* $V_m \in L^\infty(Y; \mathbb{R})$. *We denote by* $(\lambda_{m,k}(\theta), \varphi_{m,\theta,k})_{k\geqslant 1}$ *the eigenvalues and normalized eigenfunctions given by the eigenvalue problem* (3.2.0.3) *where* $V := V_m$, *for* $m = 1$ *or* $m = 2$. *Let* $N \geqslant 1$ *be an integer such that the following two conditions*

$$\forall\, k \geqslant N+1, \qquad \lambda_{1,\theta,k}(\theta) = \lambda_{2,k}(\theta), \tag{3.2.0.4}$$

and

$$\forall\, k \geqslant N+1, \qquad \partial_\nu \varphi_{1,\theta,k} = \partial_\nu \varphi_{2,\theta,k} \qquad \text{on } \Gamma, \tag{3.2.0.5}$$

hold simultaneously. Then we have $V_1 \equiv V_2$.

In order to explain and state our main result concerning waveguides in the next subsection we recall what is meant by boundary spectral data for a waveguide.

3.2.1 The spectral data of the operator A

We recall (see [ReSi4, §XIII.16] for more details) that the operator A given by (3.2.0.2), admits the following Floquet decomposition into the direct some

$$\mathcal{U} A \mathcal{U}^{-1} = \int_{(0,2\pi)}^{\oplus} A_\theta \frac{d\theta}{2\pi}. \tag{3.2.1.1}$$

Here, $\mathcal{U}$ is the FBG transform already considered in Subsection 1.5.6. Moreover, for each fixed $\theta \in [0, 2\pi)$ the operator A_θ acts in $L^2(Y)$ as $-\Delta + V$ on its domain, composed of functions $\psi \in H^2(Y)$ such that

$$\forall\, \sigma' \in \partial\omega, \quad \forall\, x_3 \in (0, 2\pi), \qquad \psi(\sigma', x_3) = 0, \tag{3.2.1.2}$$

and

$$\psi(\cdot, 2\pi) - \mathrm{e}^{\mathrm{i}\theta}\psi(\cdot, 0) = \partial_3\psi(\cdot, 2\pi) - \mathrm{e}^{\mathrm{i}\theta}\partial_3\psi(\cdot, 0) = 0 \quad \text{in } \omega. \tag{3.2.1.3}$$

Thus, the operator A_θ is defined by

$$A_\theta \psi := -\Delta\psi + V\psi,$$

for $\psi \in D(A_\theta)$ defined to be

$$D(A_\theta) := \left\{\psi \in H^1(Y)\ ;\ \Delta\psi \in L^2(Y),\ \psi \text{ satisfies (3.2.1.2) and (3.2.1.3)}\right\}.$$

It is clear that for each $\theta \in [0, 2\pi)$ the operator A_θ is self-adjoint, and that the imbedding of $D(A_\theta)$ (endowed with its graph norm) into $L^2(Y)$ is compact: this means that A_θ has a compact resolvent and thus its spectrum is composed of a sequence of real numbers $\{\lambda_k(\theta)\ ;\ k \in \mathbb{N}^*\}$, where these numbers are assumed to be ordered in a non-decreasing order, and $\lambda_k(\theta) \to +\infty$ as $k \to +\infty$. Actually the spectrum of A is determined in terms of the spectrums of $(A_\theta)_{\theta\in[0,2\pi)}$, by the relation:

$$\mathrm{sp}(A) = \overline{\bigcup_{k\in\mathbb{N}^*} \lambda_k([0, 2\pi))}. \tag{3.2.1.4}$$

Moreover, the spectrum of A is purely absolutely continuous (cf. Filonov and Kachkovskii [FiKa, Theorem 2.1]), which amounts to saying that the so-called band functions $\theta \mapsto \lambda_j(\theta)$, $j \in \mathbb{N}^*$, are non constant.

To go further and say a few words about the *generalized eigenfunctions* of A, we introduce a family $\{\varphi_{\theta,k}\ ;\ k \in \mathbb{N}^*\}$ of eigenfunctions of the operator A_θ, which satisfy

$$A_\theta \varphi_{\theta,k} = \lambda_k(\theta)\varphi_{\theta,k} \quad \text{in } Y,$$

and form an orthonormal basis of $L^2(Y)$. For all $k \in \mathbb{N}^*$ and $\theta \in [0, 2\pi)$, we set

$$\Phi_{\theta,k}(x', x_3 + 2n\pi) := \mathrm{e}^{in\theta}\varphi_{\theta,k}(x), \qquad \text{for } x = (x', x_3) \in Y, \quad n \in \mathbb{Z}, \tag{3.2.1.5}$$

so that for any $\chi \in C_c^\infty(\mathbb{R})$, the function $x \mapsto \chi(x_3)\,\Phi_{\theta,k}(x)$ belongs to the domain $D(A)$. For any $k \in \mathbb{N}^*$ and $\theta \in [0, 2\pi)$, it is easy to check that

$$(-\Delta + V)\Phi_{\theta,k} = \lambda_k(\theta)\Phi_{\theta,k} \qquad \text{in } \Omega,$$

in the distributional sense. Therefore, for any given $k \in \mathbb{N}^*$ and $\theta \in [0, 2\pi)$, each $\Phi_{\theta,k}$ is a generalized eigenfunction of A associated with the generalized eigenvalue $\lambda_k(\theta)$. Furthermore, the family $\{\Phi_{\theta,k}\ ;\ k \in \mathbb{N}^*,\ \theta \in [0, 2\pi)\}$ is a complete system of generalized eigenfunctions of A, in the sense that upon setting

$$u_k(\theta) := \int_Y u(x', x_3)\overline{\Phi_{\theta,k}(x', x_3)}dx'dx_3,$$

for $u \in L^2(\Omega)$, the mapping

$$u \mapsto \{u_k(\theta)\ ;\ k \in \mathbb{N}^*,\ \theta \in [0, 2\pi)\},$$

defines a unitary operator from $L^2(\Omega)$ onto $\bigoplus_{k\in\mathbb{N}^*} L^2(0, 2\pi)$, that is for any $u, v \in L^2(\Omega)$ we have:

$$(u|v)_{L^2(\Omega)} = \sum_{k\geqslant 1} \int_0^{2\pi} u_k(\theta)\overline{v_k(\theta)}\,\frac{d\theta}{2\pi}\,.$$

Now, the (full) Floquet spectral data associated with the operator A is defined as the set

$$\{(\lambda_k(\theta), \mathrm{span}(\Phi_{\theta,k}))\ ;\ k \in \mathbb{N}^*,\ \theta \in [0, 2\pi)\}\,.$$

Often, with two abuses of notations, we shall denote the above full Floquet spectral data set as

$$\mathrm{FSD}(V) := \{(\lambda_k(\theta), \varphi_{\theta,k})\ ;\ k \in \mathbb{N}^*,\ \theta \in [0, 2\pi)\}\,. \tag{3.2.1.6}$$

that is in the first place we use the eigenfunctions $\varphi_{\theta,k}$ defined on $\omega \times (0, 2\pi)$ instead of $\Phi_{\theta,k}$: clearly this does not create any ambiguity since $\Phi_{\theta,k}$ is known in a unique manner through the definition (3.2.1.5). The next abuse of notations is owed to the fact that we omit to say that what is indeed important is the eigenspace $\mathrm{span}(\Phi_{\theta,k})$, or $\mathrm{span}(\varphi_{\theta,k})$, rather than each eigenfunction $\Phi_{\theta,k}$ or $\varphi_{\theta,k}$, in particular when the Floquet eigenvalue $\lambda_k(\theta)$ is a multiple eigenvalue.

Moreover, in accordance with Eskin, Ralston and Trubowitz [EsRaTr, §I.6], for any $\theta \in [0, 2\pi)$ fixed, the set $\{(\lambda_j(\theta), \Phi_{\theta,j})\ ;\ j \in \mathbb{N}^*\}$ will be referred to as the *Floquet spectral data* (or equivalently, the *Floquet eigenpairs*) associated with the operator A at $\theta \in [0, 2\pi)$.

3.2.2 Main results in an infinite waveguide

We consider two potentials $V_m \in L^\infty(\Omega; \mathbb{R})$, $m = 1, 2$, that are 2π-periodic with respect to x_3,

$$V_m(x', x_3 + 2\pi) = V_m(x', x_3),\ x' \in \omega,\ x_3 \in \mathbb{R}, \tag{3.2.2.1}$$

and we call A_m (resp. $A_m(\theta)$ for all $\theta \in [0, 2\pi)$) the operator obtained by substituting V_m for V in the definition of the operator A (resp. $A(\theta)$), so that we have:

$$UA_mU^{-1} = \int_{(0,2\pi)}^{\oplus} A_m(\theta)\,\frac{d\theta}{2\pi}, \qquad \text{for } m = 1, 2. \tag{3.2.2.2}$$

Further, we note $\{(\lambda_{m,k}(\theta), \varphi_{m,\theta,k})\ ;\ k \in \mathbb{N}^*,\ \theta \in [0, 2\pi)\}$ the full spectral data associated with A_m, for $m = 1, 2$, as defined in (3.2.1.6). The main result of [KaKiSo] is the following uniqueness result.

Theorem 3.2.2 (Theorem 1.2, [KaKiSo]) *Denote* $\Gamma := \partial\omega \times [0, 2\pi]$. *For* $m = 1, 2$ *let* $V_m \in L^\infty(\Omega; \mathbb{R})$ *fulfill* (3.2.2.1). *Assume that for some* $\theta_0 \in [0, 2\pi)$ *and some integer* $N \geqslant 1$ *the following two conditions*

$$\forall\, k \geqslant N + 1, \qquad \lambda_{1,k}(\theta_0) = \lambda_{2,k}(\theta_0), \tag{3.2.2.3}$$

and

$$\forall\, k \geqslant N + 1, \qquad \partial_\nu \varphi_{1,\theta_0,k} = \partial_\nu \varphi_{2,\theta_0,k} \qquad \text{on } \Gamma, \tag{3.2.2.4}$$

hold simultaneously. Then we have $V_1 \equiv V_2$.

Theorem 3.2.2 yields that the knowledge of the Floquet spectral data (with the possible exception of finitely many generalized eigenpairs) at one arbitrary $\theta_0 \in [0, 2\pi)$, uniquely determines the operator A. The claim seems quite surprising at first sight, since the full spectral data of A is the collection of the Floquet data at θ for θ evolving in $[0, 2\pi)$. Nevertheless, we point out that this result is in accordance with Eskin, Ralston and Trubowitz [EsRaTr, I,Theorem 6.2], where Floquet isospectrality at $\theta = 0$ for Schrödinger operators with analytic periodic potential in $\mathbb{R}^n$, $n \geqslant 2$, implies Floquet isospectrality for all $\theta \in [0, 2\pi)$.

As a matter of fact, we show the stability result stated in Theorem 3.2.3 below, which yields a much stronger uniqueness result. Indeed, notwithstanding the fact that the main interest of Theorem 3.2.2 lies in its simplicity, notice that under the assumptions (3.2.2.3)-(3.2.2.4) one has also

$$\sum_{k=1}^{\infty} \|\partial_\nu \varphi_{1,\theta_0,k} - \partial_\nu \varphi_{2,\theta_0,k}\|^2_{L^2(\Gamma)} < \infty. \tag{3.2.2.5}$$

Actually, the above condition is sufficient to state a stability result in terms of the asymptotic distance between the eigenvalues $|\lambda_{1,k}(\theta_0) - \lambda_{2,k}(\theta_0)|$, as stated in the following:

Theorem 3.2.3 (Theorem 1.3, [KaKiSo]) *Let* $M > 0$ *be fixed and let* V_m, $m = 1, 2$, *be the same as in Theorem 3.2.2 and let* $\max_{m=1,2} |V_m| \leqslant M$. *Suppose that* (3.2.2.5) *is fulfilled for some* $\theta_0 \in [0, 2\pi)$. *Fix* V *the function defined on* $\mathbb{R}^3$ *by* $V = V_1 - V_2$ *and extended by* 0 *outside of* Ω *and let* $\hat{V}$ *be the Fourier transform of* V *defined by*

$$\hat{V}(\xi', j) := (2\pi)^{-\frac{3}{2}} \int_0^{2\pi} \int_{\mathbb{R}^2} V(x', x_3) e^{-i(\xi' \cdot x' + j x_3)} dx' dx_3, \quad \xi' \in \mathbb{R}^2,\ j \in \mathbb{Z}.$$

Then there exists a positive constant c *depending only on* ω *and* M *such that the following stability estimate holds*

$$\left|\hat{V}(\xi', j)\right| \leqslant c \limsup_{k \to +\infty} |\lambda_{1,k}(\theta_0) - \lambda_{2,k}(\theta_0)|, \quad (\xi', j) \in \mathbb{R}^2 \times \mathbb{Z}. \tag{3.2.2.6}$$

Since one can easily see that in general one has

$$|\lambda_{1,k}(\theta_0) - \lambda_{2,k}(\theta_0)| \leqslant \|V_1 - V_2\|_{L^\infty(\Omega)} = \|V\|_{L^\infty(\Omega)}$$

the above stability estimate is, in some loose sense, optimal. Actually, from estimate (3.2.2.6) one can deduce estimates of $\|V\|_{H^{-1}(\Omega)}$ with respect to δ. Also, with some additional assumption one can get estimates of $\|V\|$ in some suitable spaces ($L^2(\Omega)$, $L^\infty(\Omega)$,...). Here, in order to preserve some generality, we do not consider such application. We only introduce a stability estimate in a general setting with as little assumption as possible.

3.2.3 Inverse spectral theory in unbounded domain and results of stability

Note that all the results mentioned in Section 3.1 were obtained when Ω is a bounded domain and thus the operator A has a purely discrete spectrum. Borg [Bo52] and Marchenko [Mar] independently examined the uniqueness issue in the inverse problem of determining the electric potential of $-\partial_{xx} + V$ in $\Omega = \mathbb{R}_+^*$, with Fourier flux boundary condition $\alpha\psi(0) - \psi'(0) = 0$ at $x = 0$. They proved that when there is no continuous spectrum, two sets of discrete spectra associated with two distinct boundary conditions at $x = 0$ uniquely determine the potential and the two boundary conditions. Gesztesy and Simon [GeSi96, GeSi97, GeSi00] and Aktosun and Weder [AkWe] extended the Borg-Marchenko result in presence of a continuous spectrum, where either the Krein's spectral shift function, or an appropriate set containing the discrete eigenvalues and the continuous part of the spectral measure, are used as the known data. To the best of our knowledge, there is only one multi-dimensional Borg-Marcheko uniqueness result available in the mathematical literature, that of Gesztesy and Simon [GeSi96, Theorem 2.6], where the special case of three-dimensional Schrödinger operators with spherically symmetric potentials is studied.

Finally, let us mention for the sake of completeness that the stability issue in the context of inverse spectral problems has been examined by Alessandrini and Sylvester [AlSy], Bellassoued, Choulli and Yamamoto [BeChYa], Bellassoued and Dos Santos Ferreira [BelDo11], Choulli [Ch09], Choulli and Stefanov [ChSt], that inverse spectral problems stated on Riemannian manifolds have been investigated in Bellassoued and Dos Santos Ferreira [BelDo11], and in Kurylev, Lassas and Weder [KuLaWe], and that isospectral sets of Schrödinger operators with periodic potentials or Schrödinger operators defined on a torus, have been characterized in Eskin [Es89], Eskin, Ralston and Trubowitz [EsRaTr], V. Guillemin [Gui].

We should point out that the problem under examination in [KaKiSo] is a three-dimensional Borg-Levinson inverse problem, stated on the infinitely extended cylindrical domain $\Omega = \omega \times \mathbb{R}$, associated with an operator $A = -\Delta + V$ of (as already mentioned in subsection 3.2.1) purely absolutely continuous spectral type. As far as we know, Theorem 3.2.3 is the only multi-dimensional Borg-Levinson uniqueness result for an operator with continuous spectrum.

3.2.4 Brief summary of the proof

The main point in the proof of Theorem 3.2.2, 3.2.3 is contained in the proof of Theorem 3.2.1. For this reason, we will only give an idea of the proof of Theorem 3.2.1. We start with three intermediate results of [KaKiSo]. We denote by $\langle f, \psi\rangle$ the duality between $\psi \in H^{1/2}(\Gamma)$ and f belonging to the dual of $H^{1/2}(\Gamma)$. However, when in $\langle f, \psi\rangle$ both f and ψ belong to $L^2(\Gamma)$, to make things simpler $\langle \cdot, \cdot\rangle$ can be interpreted as the scalar product of $L^2(\Gamma)$, namely

$$\langle f, \psi\rangle = \int_\Gamma \psi(\sigma)\, \overline{f(\sigma)}\, d\sigma.$$

Recall that the trace operator $\gamma_0 : C^1(\overline{Y}) \longrightarrow C(\partial Y)$ defined by $\gamma_0(\varphi) := \varphi_{|\partial Y}$ can be extended to $H^1(Y)$. For $\theta \in [0, 2\pi)$ fixed, we denote by $H^1_\theta(Y)$ the closed subspace of those functions $u \in H^1(Y)$ satisfying in the sense of traces

$$u(x', 2\pi) = \mathrm{e}^{\mathrm{i}\theta} u(x', 0) \quad \text{for } x' \in \omega,$$

and we shall set

$$H^{1/2}_\theta(\partial Y) := \gamma_0(H^1_\theta(Y)).$$

The space $H^1_{0,\theta}(Y)$ denotes the closed subspace of those functions $u \in H^1_\theta(Y)$ satisfying in the sense of traces

$$u(\sigma', x_3) = 0 \quad \text{for } (\sigma', x_3) \in \Gamma.$$

We start by considering the following BVP

$$\begin{cases} -\Delta u + Vu - \lambda u &= 0, & \text{in } Y, \\ u(\sigma) &= f(\sigma), & \sigma \in \Gamma, \\ u(x', 2\pi) &= \mathrm{e}^{\mathrm{i}\theta} u(x', 0), & x' \in \omega, \\ \partial_3 u(x', 2\pi) &= \mathrm{e}^{\mathrm{i}\theta} \partial_3 u(x', 0), & x' \in \omega. \end{cases} \tag{3.2.4.1}$$

Lemma 3.2.1 (Lemma 2.3, [KaKiSo]) *For any $f \in H^{1/2}_\theta(\partial Y)$ and $\lambda \in \mathbb{C} \setminus \mathrm{sp}(A_\theta)$, there exists a unique solution $u \in H^2_\theta(Y)$ to the BVP* (3.2.4.1) *which can be written as*

$$u_\lambda := u = \sum_{k \geqslant 1} \frac{\alpha_k}{\lambda - \lambda_k(\theta)} \varphi_{\theta,k}, \tag{3.2.4.2}$$

where for convenience we set

$$\psi_{k,\theta} := \partial_\nu \varphi_{\theta,k}, \qquad \text{and} \qquad \alpha_k := \alpha_k(\theta, f) := \langle \psi_{k,\theta}, f \rangle.$$

Moreover

$$\|u_\lambda\|^2_{L^2(Y)} = \sum_{k \geqslant 1} \frac{|\alpha_k|^2}{|\lambda - \lambda_k(\theta)|^2} \to 0 \qquad \text{as } \lambda \to -\infty.$$

It is clear that the series (3.2.4.2) giving u_λ in terms of $\alpha_k, \lambda_k(\theta)$ and $\varphi_{\theta,k}$, converges only in $L^2(Y)$ and thus we cannot deduce an expression of the normal derivative $\partial_\nu u_\lambda$ in terms of $\alpha_k, \lambda_k(\theta)$ and ψ_k. To avoid this difficulty we have the following lemma:

Lemma 3.2.2 (Lemma 2.4, [KaKiSo]) *Let $f \in H^{1/2}_\theta(\partial Y)$ be fixed and for $\lambda, \mu \in \mathbb{C} \setminus \mathrm{sp}(A_\theta)$ let u_λ and u_μ be the solutions given by Lemma 3.2.1. If we set $v := v_{\lambda,\mu} := u_\lambda - u_\mu$, then*

$$\partial_\nu v = \sum_k \frac{(\mu - \lambda)\alpha_k}{(\lambda - \lambda_k(\theta))(\mu - \lambda_k(\theta))} \psi_{k,\theta}\,, \tag{3.2.4.3}$$

the convergence taking place in $H^{1/2}(\Gamma)$

The next lemma states essentially that if for $m = 1$ or $m = 2$ we have two potentials V_m and $u_m := u_{m,\mu}$ solves

$$\begin{cases} -\Delta u_m + V_m u_m - \mu u_m &= 0, & \text{in } Y, \\ u_m(\sigma) &= f(\sigma), & \sigma \in \Gamma, \\ u_m(x', 2\pi) &= \mathrm{e}^{\mathrm{i}\theta} u_m(x', 0), & x' \in \omega, \\ \partial_3 u_m(x', 2\pi) &= \mathrm{e}^{\mathrm{i}\theta} \partial_3 u_m(x', 0), & x' \in \omega, \end{cases} \tag{3.2.4.4}$$

then $u_{1,\mu}$ and $u_{2,\mu}$ are *close* as $\mu \to -\infty$: in some sense the influence of the potentials V_m is dimmed when $\mu \to -\infty$. More precisely, we have:

Lemma 3.2.3 (Lemma 2.5, [KaKiSo]) *Let $V_m \in L^\infty(Y, \mathbb{R})$ be given for $m = 1$ or $m = 2$, and denote by $A_m(\theta)$ the corresponding operator defined in the beginning of Subsection 3.2.2. For $f \in H^{1/2}_\theta(\partial Y)$ and $\mu \in \mathbb{C}$ and $\mu \notin \mathrm{sp}(A_{1,\theta}) \cup \mathrm{sp}(A_{2,\theta})$, let $u_{m,\mu} := u_m$ be the solution of* (3.2.4.4)*. Then if $z_\mu := u_{1,\mu} - u_{2,\mu}$ we have*

$$\|z_\mu\| + \|\nabla z_\mu\| + \|\Delta z_\mu\| \to 0 \qquad \text{as } \mu \to -\infty.$$

In particular $\partial_\nu z_\mu \to 0$ in $L^2(\Gamma)$ as $\mu \to -\infty$.

In order to prove Theorem 3.2.1, we will combine the properties stated in Lemma 3.2.1, 3.2.2 and 3.2.3 with a suitable representation formula. In his paper going back to 1991, Isozaki [Is], gives a simple representation formula which, in some sense, allows to express the potential V in terms of the DN operator. More precisely, adapting the argument to fit our aim in [KaKiSo], let $\lambda \notin \mathrm{sp}(A_\theta)$ and denote by $\Lambda_{\theta,V-\lambda}$ the DN map defined by

$$f \mapsto \partial_\nu u \quad \text{on } \Gamma,$$

where u is the solution of equation (3.2.4.1). For $\zeta = \mathrm{i}\xi + \eta \in \mathbb{C}^3$, where $\xi, \eta \in \mathbb{R}^3$, we shall denote

$$\zeta \cdot \zeta := -|\xi|^2 + |\eta|^2 + 2\mathrm{i}\xi \cdot \eta,$$

where $\xi \cdot \eta$ denotes the usual scalar product of ξ and η in $\mathbb{R}^3$. Then, for $p = 0$ or $p = 1$ consider

$$\zeta_p \in \mathbb{C}^3, \quad \zeta_p \cdot \zeta_p = -\lambda, \quad e_\zeta(x) := \exp(\zeta \cdot x), \quad e_{*\zeta}(x) := \exp(\overline{\zeta} \cdot x). \tag{3.2.4.5}$$

Definition 3.2.1 *Assume that $\zeta_p \in \mathbb{C}^3$ for $p = 0, 1$ satisfy* (3.2.4.5) *and are such that $e_{\zeta_0} \in H^1_\theta(Y)$ and $e_{*\zeta_1} \in H^1_\theta(Y)$. Then, following Isozaki, we set*

$$S_{\theta,V}(\lambda, \zeta_0, \zeta_1) := \int_\Gamma \Lambda_{\theta,V-\lambda}(e_{\zeta_0})(\sigma)\, e_{\zeta_1}(\sigma)\, d\sigma = \langle e_{*\zeta_1}, \Lambda_{\theta,V-\lambda}(e_{\zeta_0})\rangle.$$

Combining the result of [KaKiSo, Lemma 3.3, 3.4, 3.5, 3.6], for all $\xi \in \mathbb{R}^2 \times \mathbb{Z}$ and all $t > 0$, we define $\lambda(t,\xi)$, $\zeta_0(t,\xi)$ and $\zeta_1(t,\xi)$ such that the following property is fulfilled.

Theorem 3.2.4 (Theorem 3.7, [KaKiSo]) *For all $\xi \in \mathbb{R}^2 \times \mathbb{Z}$, we have*

$$\int_Y V(x)\, \mathrm{e}^{-\mathrm{i}\,\xi\cdot x}\, dx = \lim_{t\to+\infty} S_{\theta,V}(\lambda(t,\xi), \zeta_0(t,\xi), \zeta_1(t,\xi)) + \frac{|\xi|^2}{2} \int_Y \mathrm{e}^{-\mathrm{i}\,\xi\cdot x}\, dx. \tag{3.2.4.6}$$

For $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{R}$, such that $\lambda, \mu \notin \mathrm{sp}(A_{1,\theta}) \cup \mathrm{sp}(A_{2,\theta})$, and $f \in H^{1/2}_\theta(\partial Y)$ consider $u_{m,\lambda}$ the solution to the equation (3.2.4.1) where $V := V_m$, and also denote

$$\psi_{m,k} := \psi_{m,\theta,k} := \partial_\nu \varphi_{m,\theta,k}, \qquad \alpha_{m,k} := \langle \psi_{m,\theta,k}, f\rangle.$$

We shall split $\mathbb{N}^*$ into two subsets of integers $k \geqslant 1$, according to whether $(\lambda_{1,\theta,k}, \partial_\nu\varphi_{1,\theta,k}) = (\lambda_{2,\theta,k}, \partial_\nu\varphi_{2,\theta,k})$ or not: more precisely we set

$$\mathbb{K}_1 := \{k \geqslant 1\ ;\ (\lambda_{1,\theta,k}, \partial_\nu\varphi_{1,\theta,k}) = (\lambda_{2,\theta,k}, \partial_\nu\varphi_{2,\theta,k})\},$$

and then

$$\mathbb{K}_0 := \mathbb{N}^* \setminus \mathbb{K}_1.$$

Moreover, when $k \in \mathbb{K}_1$ we drop the index $m = 1$ or $m = 2$, that is we denote by $\lambda_k(\theta)$ and ψ_k, as well as α_k, the common value of these entities. We set

$$F_m(\lambda, \mu, f) := \sum_{k\in\mathbb{K}_0} \frac{(\mu-\lambda)\alpha_{m,k}}{(\lambda - \lambda_{m,k}(\theta))(\mu - \lambda_{m,k}(\theta))}\, \psi_{m,k}$$

and analogously set

$$G(\lambda, \mu, f) := \sum_{k\in\mathbb{K}_1} \frac{(\mu-\lambda)\alpha_k}{(\lambda - \lambda_k(\theta))(\mu - \lambda_k(\theta))}\, \psi_k.$$

We obtain

$$\partial_\nu v_{m,\lambda,\mu} = F_m(\lambda, \mu, f) + G(\lambda, \mu, f). \tag{3.2.4.7}$$

Recalling that in Lemma 3.2.3 we have set $z_\mu = u_{1,\mu} - u_{2,\mu}$, writing the above identity (3.2.4.7) for $m = 1$ and $m = 2$, and then subtracting the resulting equations, we end up with a new relation, namely

$$\partial_\nu u_{1,\lambda} - \partial_\nu u_{2,\lambda} = \partial_\nu z_\mu + F_1(\lambda, \mu, f) - F_2(\lambda, \mu, f). \tag{3.2.4.8}$$

In view of Lemma 3.2.3, upon letting $\mu \to -\infty$, we prove in [KaKiSo, Lemma 4.3] that

$$\partial_\nu u_{1,\lambda} - \partial_\nu u_{2,\lambda} = F_{*1}(\lambda, f) - F_{*2}(\lambda, f), \tag{3.2.4.9}$$

where for convenience we have set

$$F_{*m}(\lambda, f) := \sum_{k \in \mathbb{K}_0} \frac{\alpha_{m,k}}{\lambda - \lambda_{m,k}(\theta)} \psi_{m,k}.$$

Now choose $\lambda(t, \xi), \zeta_0(t, \xi), \zeta_1(t, \xi)$ as in Theorem 3.2.4, and $f = e_{\zeta_0}$. Then, the identity given by (3.2.4.9) and the Definition 3.2.1 yield

$$S_{\theta,V_1}(\lambda(t,\xi), \zeta_0(t,\xi), \zeta_1(t,\xi)) - S_{\theta,V_2}(\lambda(t,\xi), \zeta_0(t,\xi), \zeta_1(t,\xi)) = \langle e_{*\zeta_1}, F_{*1}(\lambda, e_{\zeta_0}) - F_{*2}(\lambda, e_{\zeta_0})\rangle.$$

Moreover, we prove in [KaKiSo, Lemma 4.4, 4.5] that

$$\lim_{t \to +\infty} \langle e_{*\zeta_1}(t,\xi), F_{*1}(\lambda(t,\xi), e_{\zeta_0}(t,\xi)) - F_{*2}(\lambda(t,\xi), e_{\zeta_0(t,\xi)})\rangle = 0.$$

Therefore, Theorem 3.2.4 shows that we have

$$\int_Y (V_1(x) - V_2(x))\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, dx = \lim_{t\to\infty} (S_{\theta,V_1}(\lambda(t,\xi), \zeta_0(t,\xi), \zeta_1(t,\xi)) - S_{\theta,V_2}(\lambda(t,\xi), \zeta_0(t,\xi), \zeta_1(t,\xi))) = 0$$

and the proof of Theorem 3.2.1 is completed. Note that the strategy described above can be used for Theorem 3.2.3 where the incomplete Floquet boundary spectral data is replaced by the asymptotic properties given by (3.2.2.5) and $\limsup_{k\to+\infty} |\lambda_{1,k}(\theta_0) - \lambda_{2,k}(\theta_0)|$.

3.3 Borg-levinson theorem for magnetic Schrödinger operators

3.3.1 Our goal

This section is devoted to the result of [Ki163]. We consider $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, a $\mathcal{C}^{1,1}$ bounded and connected domain such that $\mathbb{R}^n \setminus \Omega$ is also connected. We set $\Gamma = \partial\Omega$. Let $A \in W^{1,\infty}(\Omega; \mathbb{R})^n$, $V \in L^\infty(\Omega; \mathbb{R})$ and consider the magnetic Schrödinger operator $H = -\Delta_A + V$, where Δ_A denote the magnetic Laplacian already defined in (1.4.1.2), acting on $L^2(\Omega)$ with domain $D(H) = \{v \in H_0^1(\Omega) : \ \Delta_A v \in L^2(\Omega)\}$.

It is well known that H is a selfadjoint operator. By the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the spectrum of H is purely discrete. We note $\{\lambda_k : \ k \in \mathbb{N}^*\}$ the non-decreasing sequence of eigenvalues of H and $\{\varphi_k : \ k \in \mathbb{N}^*\}$ an associated Hilbertian basis of eigenfunctions. In [Ki163], we consider the Borg-Levinson inverse spectral problem of determining uniquely H in some suitable sense.

3.3.2 Obstruction to uniqueness

Let us remark that there is an obstruction to our problem given by the gauge invariance of boundary spectral data for magnetic Shrödinger operators. More precisely, for $j = 1, 2$, fix $A_j \in W^{1,\infty}(\Omega; \mathbb{R})^n$ and assume that there exists $p \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}) \setminus \{0\}$ such that $A_1 = \nabla p + A_2 \neq A_2$, $V_1 = V_2$. Now, for $j = 1, 2$, fix H_j the Dirichlet realization of $-\Delta_{A_j} + V_j$ and let $\{\lambda_{1,k} : \ k \in \mathbb{N}^*\}$ be the non-decreasing sequence of eigenvalues of H_1 with

$\{\varphi_{1,k} : k \in \mathbb{N}^*\}$ an associated Hilbertian basis of eigenfunctions. For all $k \in \mathbb{N}^*$, we fix $\varphi_{2,k} = e^{ip}\varphi_{1,k}$ and one can easily check that $\{\varphi_{2,k} : k \in \mathbb{N}^*\}$ is an Hilbertian basis of $L^2(\Omega)$. Moreover, we have

$$\Delta_{A_2}\varphi_{2,k} = e^{ip}(\Delta_{A_2}\varphi_{1,k} + 2i\nabla p \cdot \nabla\varphi_{1,k} + (i\Delta p - 2A_2 \cdot \nabla p - |\nabla p|^2)\varphi_{1,k}) = e^{ip}\Delta_{A_1}\varphi_{1,k}, \quad k \in \mathbb{N}^*.$$

It follows

$$H_2\varphi_{2,k} = \lambda_{1,k}\varphi_{2,k}, \quad k \in \mathbb{N}^*$$

and, since $\{\varphi_{2,k} : k \in \mathbb{N}^*\}$ is an Hilbertian basis of $L^2(\Omega)$, we deduce that $\{\lambda_{1,k} : k \in \mathbb{N}^*\}$ corresponds to the non-decreasing sequence of eigenvalues of H_2 with $\{\varphi_{2,k} : k \in \mathbb{N}^*\}$ an associated Hilbertian basis of eigenfunctions. In addition, we have

$$\partial_\nu\varphi_{2,k|\Gamma} = (i\partial_\nu p)\varphi_{2,k|\Gamma} + e^{ip}\partial_\nu\varphi_{1,k|\Gamma} = \partial_\nu\varphi_{1,k|\Gamma}.$$

Therefore, the boundary spectral data $\{(\lambda_{1,k}, \partial_\nu\varphi_{1,k|\partial\Omega}) : k \geqslant 1\}$ of H_1 coincides with the boundary spectral data $\{(\lambda_{1,k}, \partial_\nu\varphi_{2,k|\partial\Omega}) : k \geqslant 1\}$ of H_2 but $H_1 \neq H_2$.

Taking into account this obstruction to uniqueness we are restricted to the recovery of the magnetic Schrödinger operator modulo the gauge invariance given by: H_1 and H_2 are gauge equivalent if there exists $p \in W^{2,\infty}(\Omega, \mathbb{R}) \cap H_0^1(\Omega)$ such that $H_2 = e^{-\mathrm{i}p}H_1e^{\mathrm{i}p}$. Assuming A is known on Γ, the recovery of the operator H modulo gauge invariance is equivalent to the recovery of V and of the 2-form dA of the vector valued function $A = (a_1, \ldots, a_n)$ defined by

$$dA = \frac{1}{2}\sum_{i,j=1}^{n}(\partial_{x_j}a_i - \partial_{x_i}a_j)dx_j \wedge dx_i.$$

In [Ki163], we study this problem.

3.3.3 Unique recovery modulo gauge invariance

Let $A_j \in W^{1,\infty}(\Omega; \mathbb{R})^n$, $V_j \in L^\infty(\Omega; \mathbb{R})$ and consider the magnetic Schrödinger operators $H_j = H$ for $A = A_j$ and $V = V_j$, $j = 1, 2$. Further, we note $(\lambda_{j,k}, \varphi_{j,k})$, $k \geqslant 1$, the kth eigenpair of H_j, for $j = 1, 2$. Our main result can be stated as follows.

Theorem 3.3.1 (Theorem 1.1, [Ki163]) *We fix Ω_1 an arbitrary open neighborhood of Γ in Ω ($\Gamma \subset \overline{\Omega_1}$ and $\Omega_1 \subsetneq \Omega$). For $j = 1, 2$, let $V_j \in L^\infty(\Omega; \mathbb{R})$ and let $A_j \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R})^n$ fulfill*

$$A_1(x) = A_2(x), \quad x \in \Omega_1. \tag{3.3.3.1}$$

Assume that the conditions

$$\lim_{k\to+\infty}|\lambda_{1,k} - \lambda_{2,k}| = 0, \quad \sum_{k=1}^{+\infty}\|\partial_\nu\varphi_{1,k} - \partial_\nu\varphi_{2,k}\|^2_{L^2(\Gamma)} < \infty \tag{3.3.3.2}$$

hold simultaneously. Then, we have $dA_1 = dA_2$ and $V_1 = V_2$.

Note that condition (3.3.3.1) corresponds to the knowledge of the magnetic potential on a neighborhood of the boundary.

Let us observe that, as mentioned by [ChSt, KaKiSo], Theorem 3.3.1 can be considered as a uniqueness theorem under the assumption that the spectral data are asymptotically "very close". Conditions (3.3.3.2) are similar to the one considered by [KaKiSo] and they are weaker than the requirement that

$$|\lambda_{1,k} - \lambda_{2,k}| \leqslant Ck^{-\alpha}, \quad \|\partial_\nu\varphi_{1,k} - \partial_\nu\varphi_{2,k}\|_{L^2(\Gamma)} \leqslant Ck^{-\beta}$$

for some $\alpha > 1$ and $\beta > 1 - \frac{1}{2n}$, considered in [ChSt, Theorem 2.1]. Note also that conditions (3.3.3.2) are weaker than the knowledge of the boundary spectral data with a finite number of data missing considered by [Is].

We stress out that this problem is a Borg-Levinson inverse problem for the magnetic Schrödinger operator $H = -\Delta_A + V$. To our best knowledge, there are only two multi-dimensional Borg-Levinson uniqueness result for magnetic Schrödinger operators available in the mathematical literature, [KaKu98, Theorem B] and [Ser, Theorem 3.2]. In [KaKu98], the authors considered general magnetic Schrödinger operators with smooth coefficients on a smooth connected Riemannian manifold and they proved unique determination of these operators modulo gauge invariance from the knowledge of the boundary spectral data with a missing finite number of data. In [Ser], Serov treated this problem on a bounded domain of $\mathbb{R}^n$, and he proved that, for $A \in W^{1,\infty}(\Omega;\mathbb{R})^n$ and $V \in L^\infty(\Omega;\mathbb{R})$, the full boundary spectral data $\{(\lambda_k, \partial_\nu \varphi_{k|\Gamma}) : \ k \in \mathbb{N}^*\}$ determines uniquely dA and V. In contrast to [KaKu98, Ser], in [Ki163] we prove that the asymptotic knowledge of the boundary spectral data, given by the conditions (3.3.3.2), is sufficient for the unique determination of dA and V. To our best knowledge, conditions (3.3.3.2) are the weakest conditions on boundary spectral data that guaranty uniqueness of magnetic Schrödinger operators modulo gauge transformation. Moreover, our uniqueness result is stated with conditions similar to [KaKiSo, Theorem 1.4], which seems to be the most precise Borg-Levinson uniqueness result so far for Schrödinger operators without magnetic potential ($A = 0$).

3.3.4 Idea of the proof

An important ingredient in the analysis of [Ki163] is a suitable representation that allows to express the magnetic potential A and the electric potential V in terms of the DN map associated with the equations $-\Delta_A u + Vu - \lambda u = 0$ for some $\lambda \in \mathbb{C}$. In [Is], Isozaki applied a similar approach to the Schrödinger operator $-\Delta + V$ with Dirichlet boundary condition and [ChSt, KaKiSo] applied the representation formulas of [Is]. Inspired by the construction of CGO solutions of [BeCh10, DoKeSjUh, Salo04, Su] we prove that the approach of [ChSt, Is, KaKiSo] can be extended to magnetic Schrödinger operators. More precisely, we derive two representation formulas that allow to recover both the magnetic field and the electric potential of magnetic Schrödinger operators which means recovery of both coefficients of order one and zero in contrast to [ChSt, Is, KaKiSo], where only determination of coefficients of order zero is considered. It seems that [Ki163] is the first paper where the extension of the approach developed by [Is] to more general coefficients than coefficients of order zero is considered. Note also that our approach makes it possible to prove this extension without imposing important assumptions of regularity on the admissible coefficients.

For this representation, we set $A_j \in \mathcal{C}^1(\overline{\Omega};\mathbb{R})^n$, $V_j \in L^\infty(\Omega;\mathbb{R})$, $j = 1, 2$, and we assume that condition (3.3.3.1) is fulfilled. For $j = 1, 2$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we associate with the problem

$$\left\{ \begin{array}{rcll} -\Delta_{A_j} u_j + V_j u_j - \lambda u_j & = & 0, & \text{in } \Omega, \\ u_j(x) & = & f(x), & x \in \Gamma \end{array} \right. \tag{3.3.4.1}$$

the DN map

$$\Lambda_{j,\lambda} : H^{\frac{1}{2}}(\partial\Omega) \ni f \mapsto (\partial_\nu + \mathrm{i} A_j \cdot \nu) u_{j,\lambda|\Gamma},$$

where $u_{j,\lambda}$ solves (3.3.4.1). Then, we apply the DN maps $\Lambda_{j,\lambda}$ to some suitable ansatzs associated with (3.3.4.1) in order to get two representation formulas involving the magnetic potentials A_j and the electric potentials V_j, $j = 1, 2$. The idea is to establish the link between the electric and magnetic potentials and the boundary spectral data by mean of an expression involving the DN maps $\Lambda_{1,\lambda}$, $\Lambda_{2,\lambda}$. In contrast to [ChSt, Is, KaKiSo], in [Ki163] we need to extend this strategy to Schrödinger operators with both magnetic and electric potentials, which means an extension to Schrödinger operators with variable coefficients of order zero and one. In addition, we need to consider ansatzs that allow to recover both the magnetic field and the electric potential. Therefore, we fix a large parameter $\tau > 1$ and two vectors $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{S}^{n-1}$ satisfying

$\xi \cdot \eta = 0$. We define $\lambda(\tau) \in \mathbb{C} \setminus \mathbb{R}$ depending on τ and $\eta_1(\tau, \eta, \xi), \eta_2(\tau, \eta, \xi) \in \mathbb{S}^{n-1}$ depending on τ, ξ and η. Then, we consider some ansatzs, associated with (3.3.4.1), of the form

$$\Phi_{1,\tau,\eta,\xi}(x) = e^{\mathrm{i}\sqrt{\lambda(\tau)}\eta_1(\tau,\eta,\xi)\cdot x} g_1(x), \quad , \Phi_{2,\tau,\eta,\xi}(x) = e^{-\mathrm{i}\sqrt{\lambda(\tau)}\eta_2(\tau,\eta,\xi)\cdot x} g_2(x), \quad x \in \Omega,\ j = 1, 2. \tag{3.3.4.2}$$

For $\lambda > 0$, these ansatzs are the principal term of the standard GO solutions. In contrast to the CGO solutions taking the form $e^{\zeta_j \cdot x} h_j(x)$, $j = 1, 2$, where the entire vector $\zeta_j \in \mathbb{C}^n$ is complex valued, here we consider these ansatzs only with complex frequency. The expression g_1 and g_2 appearing in (3.3.4.2), are respectively a solution of the transport equations

$$\mathrm{i}\eta_1 \cdot \nabla g_1 - (\eta_1 \cdot A_{1,\sharp}) g_1 = 0, \quad \mathrm{i}\eta_2 \cdot \nabla g_2 + (\eta_2 \cdot A_{2,\sharp}) g_2 = 0, \tag{3.3.4.3}$$

with $A_{j,\sharp}$ some smooth function close to the magnetic potential A_j, $j = 1, 2$. More precisely, we define $A_{j,\sharp} \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})^n$, $j = 1, 2$, some smooth approximations on $\overline{\Omega}$ of A_j. Then, we consider solutions of the transport equations (3.3.4.3) given by

$$g_1(x) := e^{\mathrm{i}\psi_1(x)}, \quad g_2(x) := b_2(x) e^{-\mathrm{i}\psi_2(x)}, \quad \psi_j(x) := -\int_{-\infty}^0 \eta_j \cdot A_{j,\sharp}(x + s\eta_j) ds, \quad \eta_2 \cdot \nabla b_2(x) = 0, \quad x \in \mathbb{R}^n.$$

Therefore, we consider ansatzs associated with (3.3.4.1) taking the form

$$\Phi_{1,\tau,\eta,\xi}(x) := e^{\mathrm{i}\sqrt{\lambda}\eta_1 \cdot x} e^{\mathrm{i}\psi_1(x)}, \quad \Phi_{2,\tau,\eta,\xi}(x) := e^{-\mathrm{i}\sqrt{\lambda}\eta_2 \cdot x} b_2(x) e^{-\mathrm{i}\psi_2(x)}, \quad x \in \Omega. \tag{3.3.4.4}$$

We assume in addition that $b_2 \in W^{2,\infty}(\mathbb{R}^n)$ and we recall that ψ_j solves the equation

$$\eta_j \cdot \nabla \psi_j(x) = -\eta_j \cdot A_{j,\sharp}, \quad j = 1, 2, \quad x \in \mathbb{R}^n.$$

In the construction of our ansatzs we consider some smooth approximations of the magnetic potentials instead of the magnetic potentials to obtain sufficiently smooth functions $\Phi_{j,\tau,\eta,\xi}$, $j = 1, 2$. Using this approach, we can weaken the regularity assumption imposed on admissible magnetic potential from $W^{3,\infty}(\Omega)^n$ to $\mathcal{C}^1(\overline{\Omega})^n$. Further, for $j = 1, 2$, we put

$$S_j(\tau, \eta, \xi) = \int_\Gamma (\Lambda_{j,\lambda(\tau)} \Phi_{1,\tau,\eta,\xi}) \Phi_{2,\tau,\eta,\xi}(x) d\sigma(x). \tag{3.3.4.5}$$

In other words, we apply $\Lambda_{j,\lambda(\tau)}$, $j = 1, 2$, to ansatzs of the form (3.3.4.2) with $g_1 = e^{\mathrm{i}\psi_1(x)}$ and $g_2 = b_2(x) e^{-\mathrm{i}\psi_2(x)}$. From some asymptotic properties of $S_1(\tau, \eta, \xi) - S_2(\tau, \eta, \xi)$ as $\tau \to +\infty$ we derive the two following conditions that guaranty the unique recovery of the magnetic field and the electric potential.

Lemma 3.3.1 (Lemma 4.1, [Ki163]) *Assume that for all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{S}^{n-1}$ satisfying $\xi \cdot \eta = 0$, the condition*

$$\lim_{\tau \to +\infty} \frac{S_1(\tau, \eta, \xi) - S_2(\tau, \eta, \xi)}{\sqrt{\lambda(\tau)}} = 0 \tag{3.3.4.6}$$

is fulfilled. Then, we have $dA_1 = dA_2$.

Lemma 3.3.2 (Lemma 4.2, [Ki163]) *Let $A_1 = A_2$. Assume that for all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{S}^{n-1}$ satisfying $\xi \cdot \eta = 0$, the condition*

$$\lim_{\tau \to +\infty} S_1(\tau, \eta, \xi) - S_2(\tau, \eta, \xi) = 0 \tag{3.3.4.7}$$

is fulfilled. Then, we have $V_1 = V_2$.

The remaining part of [Ki163], consists in proving that conditions (3.3.3.1)-(3.3.3.2) imply (3.3.4.6)-(3.3.4.7). We prove this implication by adapting to magnetic Schrödinger operators some arguments of [KaKiSo].

3.4 Borg-levinson theorem with measurements on an arbitrary portion

3.4.1 Statement of the problem

This section is devoted to [KiMoOk] which is a joint work with Morgan Morancey and Lauri Oksanen. We fix Ω a $\mathcal{C}^2$ bounded and connected domain of $\mathbb{R}^n$, $n \geqslant 2$ and we set γ a non empty open set of $\Gamma = \partial\Omega$. We consider the Schrödinger operator $A_q = -\Delta + q$ acting on $L^2(\Omega)$ with Dirichlet boundary condition and $q \in L^\infty(\Omega)$ real valued. The spectrum of A_q consists in a non decreasing sequence of eigenvalues $\{\lambda_k : \ k \in \mathbb{N}^*\}$ to which we associate the Hilbertian basis of eigenfunctions $\{\varphi_k : \ k \in \mathbb{N}^*\}$. Then, we introduce the boundary spectral data restricted to the portion γ given by

$$\mathrm{BSD}(q,\gamma) := \left\{ (\lambda_k, \partial_\nu \varphi_{k|\overline{\gamma}}) : \ k \in \mathbb{N}^* \right\}.$$

The main goal of [KiMoOk] is to prove uniqueness in the recovery of q from the data $\mathrm{BSD}(q,\gamma)$.

3.4.2 Main results

For $j = 1, 2$, we fix $q_j \in L^\infty(\Omega)$. To every operator A_{q_j} we associate the non decreasing sequence of eigenvalues $\{\lambda_{j,k} : \ k \in \mathbb{N}^*\}$ to which we associate the Hilbertian basis of eigenfunctions $\{\varphi_{j,k} : \ k \in \mathbb{N}^*\}$. Our main result is the following

Theorem 3.4.1 (Theorem 1.1, [KiMoOk]) *Assume that Ω is convex and let $q_j \in L^\infty(\Omega)$, $j = 1, 2$. Let the conditions*

$$\lambda_{1,k} = \lambda_{2,k} = \lambda_k, \quad k \geqslant 1 \tag{3.4.2.1}$$

$$\partial_\nu \varphi_{1,k}(x) = \partial_\nu \varphi_{2,k}(x) = \psi_k(x), \quad x \in \overline{\gamma}, \ k \geqslant 1 \tag{3.4.2.2}$$

be fulfilled. Then $q_1 = q_2$.

This result will be proved by applying the so called boundary control method that we adapt to the particular setting of a convex domain. More precisely, we consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x) u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma. \end{cases} \tag{3.4.2.3}$$

Then, we prove Theorem 3.4.1 by applying the connection between the boundary spectral data $\mathrm{BSD}(q,\gamma)$ and the solution of (3.4.2.3) when $\mathrm{supp}(f) \subset (0,T] \times \gamma$. According to [LaLiTr, Theorem 2.1], for $f \in H^1(\Sigma)$, the problem (3.4.2.3) admits a unique solution $u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ which satisfies $\partial_\nu u \in L^2(\Sigma)$. Thus, we can define the partial hyperbolic DN map given by

$$\Lambda_q : \mathcal{C}_0^\infty((0,T] \times \gamma) \ni f \mapsto \partial_\nu u_{|(0,T)\times\gamma}.$$

Repeating the argumentation of Theorem 3.4.1, we can prove that Λ_q determines q. This result can be stated in the following way.

Theorem 3.4.2 (Theorem 1.2, [KiMoOk]) *Assume that Ω is convex, $T > 2Diam(\Omega)$ and let $q_j \in L^\infty(\Omega)$, $j = 1, 2$. Then the condition $\Lambda_{q_1} = \Lambda_{q_2}$ implies that $q_1 = q_2$.*

Let us mention that the boundary control method considered in the [KiMoOk] was initially introduced by [Beli87] and extended by [BeliKu92] to the recovery of a Riemannian manifold up to an isometry from the boundary spectral data $\mathrm{BSD}(q, \partial\Omega)$. The result of [BeliKu92] has been extended by [KaKu98] who proved

that this result is still true if one consider the data $\mathrm{BSD}(q, \partial\Omega)$ with the exception of finitely many eigenpairs and [KaKuLa01] proved that the uniqueness remains true if one consider the partial boundary spectral data $\mathrm{BSD}(q, \gamma)$ with γ an arbitrary portion of the boundary. We mention also [KuOkPa, LaOk10, LaOk141] who considered hyperbolic inverse problems based on the boundary control method and related to the present Borg-Levinson inverse spectral problem.

Let us observe that results similar to Theorem 3.4.1 and 3.4.2 have been proved by [KaKuLa01], and an improvement of Theorem 3.4.2 has been established by [LaOk10] in the specific case of smooth coefficients on a smooth Riemannian manifolds. More precisely, in the context of Theorem 3.4.1 and 3.4.2, the result of [KaKuLa01, LaOk10] are stated with Ω a $\mathcal{C}^\infty$ domain and coefficient $q_j \in \mathcal{C}^\infty(\overline{\Omega})$, $j = 1, 2$. To our best knowledge [KiMoOk] is the first one dealing with the recovery of a general bounded potential from boundary spectral data or boundary measurements on an arbitrary portion of the boundary.

Let us remark that all the results of [KiMoOk] can be extended to the recovery of more general coefficients on a general bounded and connected domain Ω by changing some intermediate tools and transforming the last part of the proof into an iterative process described in [KuOkPa, Subsection 4.2]. The assumption of convexity allows to simplify in various way the exposition in order to emphasize the main idea of the boundary control method. Indeed, for convex domain we can consider Euclidean distance and replace general geodesic by lines. Moreover, in contrast to [KaKuLa01, LaOk10] who considered this problem in a more general setting, our result can be proved only in two steps. Note also that in [KiMoOk] we focus our attention on the analytic rather than geometric aspects of the boundary control method. For these reasons [KiMoOk] can also be considered as an introduction to the boundary control method.

3.4.3 Description of the proof

Let us first recall the definition of domain of influence at a time $t = T_0$.

Definition 3.4.1 *For every $T_0 > 0$ and every open subset S of Γ we define the subset $\Omega(S, T_0)$ of $\overline{\Omega}$ given by*

$$\Omega(S, T_0) := \{x \in \overline{\Omega} : \ dist(x, S) \leqslant T_0\}.$$

The set $\Omega(S, T_0)$ is called the domain of influence of S at time $t = T_0$.

The proof of Theorem 3.4.1 is divided into two steps. The local step and the global step. In the local step we prove the following local recovery by mean of the boundary spectral data.

Theorem 3.4.3 (Theorem 3.1, [KiMoOk]) *Let $q_j \in L^\infty(\Omega)$, $j = 1, 2$, and let the conditions* (3.4.2.1)-(3.4.2.2) *be fulfilled. Then there exists $\tau \in (0, +\infty)$ and an open set γ' of $\partial\Omega$ such that $\gamma' \subset \gamma$ and*

$$q_1(x) = q_2(x), \quad x \in \Omega(\gamma', \tau), \tag{3.4.3.1}$$

In the global step we apply (3.4.3.1), in order to replace the boundary data into internal data corresponding to some source to solution map. Namely, we fix B a small ball contained into $\Omega(\gamma', \tau)$ and we consider the following map

$$L_{j,B}F := v_{j,F\,|[0,2T]\times B}, \quad F \in \mathcal{C}_0^\infty((\tau, 2T) \times B)$$

with $v_{j,F}$ solving

$$\begin{cases} \partial_t^2 v_j - \Delta v_j + q_j(x) v_j = F, & \text{in } (0, 2T) \times \Omega, \\ v_j(0, \cdot) = 0, \quad \partial_t v_j(0, \cdot) = 0, & \text{in } \Omega, \\ v_j = 0, & \text{on } (0, 2T) \times \partial\Omega. \end{cases}$$

Then applying (3.4.3.1), we prove that $L_{1,B} = L_{2,B}$ and we complete the proof of Theorem 3.4.1 by mean of this internal measurements.

The main points in the proof of the local recovery stated in Theorem 3.4.3 are both the connection between boundary spectral data and solutions of (3.4.2.3), and a result of density of solutions restricted to a domain of influence at a fixed time often called approximate controllability. The link between boundary spectral data and solutions of (3.4.2.3), is given by the following.

Lemma 3.4.1 (Lemma 3.1, [KiMoOk]) *Assume that conditions* (3.4.2.1)-(3.4.2.2) *are fulfilled. Let* $\gamma' \subset \gamma$ *be an open set of* $\partial\Omega$ *and let* $f \in H^1(\Sigma)$ *satisfy* $f_{|t=0} = 0$, *supp*$(f) \subset [0,T] \times \gamma'$ *and, for* $j = 1,2$, *let* u_j^f *be the solution of* (3.4.2.3) *with* $q = q_j$. *Then, we have*

$$\left\langle u_1^f(t), \varphi_{1,k} \right\rangle_{L^2(\Omega)} = \left\langle u_2^f(t), \varphi_{2,k} \right\rangle_{L^2(\Omega)} = v_k^f(t) := \int_0^t \int_{\gamma'} f(t,x) s_k(t-s) \overline{\psi_k(x)} d\sigma(x), \tag{3.4.3.2}$$

with $s_k(t)$ *which is given by*

$$s_k(t) := \begin{cases} \frac{\sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} & \text{if } \lambda_k > 0, \\ t & \text{if } \lambda_k = 0, \\ \frac{\sin(\sqrt{|\lambda_k|} t)}{\sqrt{|\lambda_k|}} & \text{if } \lambda_k < 0 \end{cases}$$

The result of density that we need is given by

Theorem 3.4.4 (Corollary 2.1, [KiMoOk]) *Let* $S \subset \gamma$ *be an open set* $\partial\Omega$, $T' \in (0,T]$ *and let* $q_j \in L^\infty(\Omega)$, $j = 1,2$. *For* $j = 1,2$, *let* u_j^f *be the solution of* (3.4.2.3) *with* $q = q_j$. *Then, the set*

$$\{u_j^f(T', \cdot)_{|\Omega(S,T')} : \ f \in \mathcal{C}_0^\infty((0,T'] \times S)\} \tag{3.4.3.3}$$

is dense in $L^2(\Omega(S,T'))$.

Theorem 3.4.4 follows from a global Holmgren-John unique continuation, derived from results of unique continuation such as [RoZu, Ta95], and a duality argument.

Combining Lemma 3.4.1 and Theorem 3.4.4 with a suitable application of the Lebesgue differentiation measure, we recover the restriction of products of solutions (3.4.2.3), for all input f supported on $(0,T] \times \gamma$, to a neighborhood of γ'. Then, with suitable choice of the input f in (3.4.2.3) to one of the two solutions appearing in the product, we derive the unique recovery of the restriction of products of solutions (3.4.2.3), for all input f supported on $(0,T] \times \gamma$, to a neighborhood of γ'. Finally, we complete the proof of Theorem 3.4.3 by applying again the density result stated in Theorem 3.4.4.

The global step requires arguments similar to the local step but for internal measurements.

Chapter 4

Inverse and direct problems for fractional diffusion equations

4.1 Introduction

This chapter is devoted to our work stated in [KiOkSoYa, KiYa, KiSoYa] related to fractional diffusion equations with time fractional derivatives. These equations take the form

$$\rho(x)\partial_t^\alpha u - \sum_{i,j=1}^n \partial_{x_i}(a_{i,j}(x)\partial_{x_j}u) + q(x)u = F(t,x) \tag{4.1.0.1}$$

where $\alpha \in (0,1)\cup(1,2)$ is not an integer and the fractional derivative ∂_t^α considered in the Caputo sense is defined by

$$\partial_t^\alpha u(t,x) := \frac{1}{\Gamma(m+1-\alpha)}\int_0^t (t-s)^{m-\alpha}\partial_s^{m+1}u(s,x)ds, \tag{4.1.0.2}$$

with m the integer part of α. Here Γ is the usual Gamma function expressed as

$$\Gamma(z) := \int_0^{+\infty} e^{-t}t^{z-1}dt, \quad z \in \{\xi \in \mathbb{C} : \ \mathrm{Re}\xi > 0\}.$$

In [KiOkSoYa, KiYa, KiSoYa] we have studied both forward and inverse problems for these equations.

Recall that fractional diffusion equations with time fractional derivatives of the form (4.1.0.1) describe several physical phenomena related to anomalous diffusion such as diffusion of substances in heterogeneous media, diffusion of fluid flow in inhomogeneous anisotropic porous media, turbulent plasma, diffusion of carriers in amorphous photoconductors, diffusion in a turbulent flow, a percolation model in porous media, fractal media, various biological phenomena and finance problems (see [CaSaLuGa]). In particular, it is known (e.g., [AdGe]) that the classical diffusion-advection equation does not often interpret field data of diffusion of substances in the soil, and as one model equation, the fractional diffusion equation is used. The diffusion equation with time fractional derivative is a corresponding macroscopic model equation to the continuous-time random walk (CTRW in short) and is derived from the CTRW (e.g., [MetKl, RoAl]). Note also that fractional diffusion equations with fractional variable power α, instead of a constant, is suitable for some complex media where the presence of heterogeneous regions causes variations of the permeability in different spatial positions.

The main purpose of [KiOkSoYa, KiYa, KiSoYa] was the study of these equations from different aspects. In [KiOkSoYa], we have studied the inverse problem of determining different coefficients appearing in the

equation (4.1.0.1) from measurements of the solutions on the boundary of the domain. In [KiYa] we have been interested by the forward problem for (4.1.0.1) when $\alpha \in (1,2)$. This corresponds to the super-diffusive case where the equation is called fractional wave equation. For these equations, we have introduced a new definition of weak solutions, by mean of their Laplace transform, that extended other known works to more general conditions. In addition, in [KiYa] we have established some L^pL^q estimates, also called Strichartz estimates, that we have used for proving existence and uniqueness of solutions for such equations with a nonlinear term. In [KiSoYa], we have considered both the forward and inverse problem for equations of the form (4.1.0.1) in the specific case where the fractional power α is a function of x. After proving existence of solutions, with a definition of weak solutions that extends the one of [KiYa], with suitable properties of analiticity, we have considered the inverse problem of recovering the power α as well as some coefficients appearing in the equation.

4.2 An inverse problem for fractional diffusion equations

4.2.1 The inverse problem

This section is devoted to [KiOkSoYa] which is a joint work with Lauri Oksanen, Eric Soccorsi and Masahiro Yamamoto. Let (M,g) be a compact connected Riemannian manifold of dimension $n \geqslant 2$, with boundary ∂M. For a positive function μ we consider the weighted Laplace-Beltrami operator

$$\Delta_{g,\mu} := \mu^{-1}\mathrm{div}_g\, \mu\, \nabla_g,$$

where div_g (resp., ∇_g) denotes the divergence (resp., gradient) operator on (M,g), and $\mu^{\pm 1}$ stands for the multiplier by the function $\mu^{\pm 1}$. If μ is identically 1 in M then $\Delta_{g,\mu}$ coincides with the usual Laplace-Beltrami operator on (M,g). In local coordinates, we have

$$\Delta_{g,\mu}u = \sum_{i,j=1}^{n} \mu^{-1}|g|^{-1/2}\partial_{x_i}(\mu|g|^{1/2}g^{ij}\partial_{x_j}u), \quad u \in C^\infty(M),$$

where $g^{-1} := (g^{ij})_{1\leqslant i,j\leqslant n}$ and $|g| := \det g$. For $\alpha \in (0,2)\setminus\{1\}$ we consider the IBVP

$$\left\{\begin{array}{rcll} \partial_t^\alpha u - \Delta_{g,\mu}u + qu &=& 0, & \text{in } (0,T)\times M, \\ u &=& f, & \text{on } (0,T)\times \partial M, \\ \partial_t^k u(0,\cdot) &=& 0, & \text{in } M,\ k=0,...,m, \end{array}\right. \tag{4.2.1.1}$$

with non-homogeneous Dirichlet data f. Here $m := [\alpha]$ denotes the integer part of α and ∂_t^α is the Caputo fractional derivative of order α with respect to t, defined by (4.1.0.2).

The system (4.2.1.1) models anomalous diffusion phenomena. In the sub-diffusive case $\alpha \in (0,1)$, the first line in (4.2.1.1) is usually named fractional diffusion equation, while in the super-diffusive case $\alpha \in (1,2)$, it is referred as fractional wave equation.

Given two non empty open subsets S_{in} and S_{out} of ∂M, $T_0 \in (0,T)$ and $\alpha \in (0,2)\setminus\{1\}$, we introduce the function space

$$\mathcal{H}_{\mathrm{in},\alpha,T_0} := \{f \in C^{[\alpha]+1}([0,T], H^{\frac{3}{2}}(\partial M));\ \mathrm{supp}(f) \subset (0,T_0)\times S_{in}\},$$

where we recall that $[\alpha]$ stands for the integer part of α. As established in [KiOkSoYa, Section 2], problem (4.2.1.1) associated with $f \in \mathcal{H}_{\mathrm{in},\alpha,T_0}$ is well posed and the partial DN map

$$\Lambda_{M,g,\mu,q}:\ \mathcal{H}_{\mathrm{in},\alpha,T_0} \ni f \mapsto \partial_\nu u(T_0,\cdot)_{|S_{out}}, \tag{4.2.1.2}$$

where u denotes the solution to (4.2.1.1) and ν is the outward unit normal vector with respect to g field along the boundary ∂M, is linear bounded from $\mathcal{H}_{\mathrm{in},\alpha,T_0}$ into $L^2(S_{out})$.

In [KiOkSoYa], we examine the problem whether knowledge of $\Lambda_{M,g,\mu,q}$ determines the Riemannian manifold (M,g), and the functions μ and q, uniquely.

4.2.2 Physical motivations

Recall that fractional diffusion equations with time fractional derivatives of the form (4.2.1.1) describe several physical phenomena introduced in Section 4.1. In particular, in the case where we consider fractional diffusion equations describing the diffusion of contaminants in a soil, we cannot *a priori* know governing parameters in (4.2.1.1) such as reaction rate of pollutants. Thus, for prediction of contamination, we need to discuss our inverse problem of determining these parameters from measurements of the flux on S_{out} at a fixed time $t = T_0$ associated with Dirichlet inputs at S_{in}.

4.2.3 State of the art

Fractional derivative, ordinary and partial, differential equations have attracted attention over the two last decades. See [MiRo, SaKiMa, Pod] regarding fractional calculus, and [Ag, GoMa], and references therein, for studies of partial differential equations with time fractional derivatives. More specifically, the well-posedness of problem (4.2.1.1) with time-independent coefficients is examined in [BekYa, SaYa].

There is a wide mathematical literature for inverse coefficients problems associated with the system (4.2.1.1) when $\alpha = 1$ or 2. Without being exhaustive, we refer to [BuKl, CaKa01, Ch09, ChKi13, ChYa06, KaKuLa04] for the parabolic case $\alpha = 1$ and to [Beli87, BeliKu92, BeChYa, BelDo11, BeJeYa1, LaOk10, LaOk141, LaOk142] for the hyperbolic case $\alpha = 2$. In contrast to parabolic or hyperbolic inverse coefficient problems, there is only a few mathematical papers dealing with inverse problems associated with (4.2.1.1) when $\alpha \in (0, 1) \cup (1, 2)$. In the one-dimensional case, [ChNaYaYa] proved unique determination of the fractional order α and a time-independent coefficient, by Dirichlet boundary measurements. For $n \geqslant 2$, the fractional order α is recovered in [HaNaWaYa] from pointwise measurements of the solution over the entire time span. In [SaYa], the authors prove stable determination of the time-dependent prefactor of the source term. In the particular case where $n = 1$ and $\alpha = 1/2$, using a specifically designed Carleman estimate for (4.2.1.1), [ChXaYa, YaZh] derive a stability estimate of a zero order time-independent coefficient, with respect to partial internal observation of the solution. In [LiImYa], time-independent coefficients are uniquely identified by the DN map obtained by probing the system with inhomogeneous Dirichlet boundary conditions of the form $\lambda(t)g(x)$, where λ is a fixed real-analytic positive function of the time variable. Recently, [FuKi] proved unique determination of a time-dependent parameter appearing in the source term or in a zero order coefficient, from pointwise measurements of the solution over the whole time interval.

4.2.4 Unique recovery of manifolds and coefficients

The paper [KiOkSoYa] contains two main results. Both of them are uniqueness results for inverse coefficients problems associated with (4.2.1.1), but related to two different settings. In the first one, (M, g) is a known compact subset of $\mathbb{R}^d$, while in the second one, (M, g) is an unknown Riemannian manifold to be determined. The first setting is not contained in the second one, however, in the second case, (M, g) and all the other unknown coefficients are assumed to be smooth, while in the first case the regularity assumptions are relaxed considerably.

We begin by considering the case of a connected bounded domain Ω in $\mathbb{R}^n$, $n \geqslant 2$, with $C^{1,1}$ boundary $\partial\Omega$. Let $\rho \in C(\overline{\Omega})$, $V \in L^\infty(\Omega)$ and $a \in C^1(\overline{\Omega})$ fulfill the condition

$$\rho(x) \geqslant c,\ a(x) \geqslant c,\ V(x) \geqslant 0,\ x \in \Omega, \tag{4.2.4.1}$$

for some positive constant c. For $M := \overline{\Omega}$, put

$$g := \rho a^{-1} I_n,\ \mu := \rho^{1-n/2}|a|^{1/2},\ \text{and } q := \rho^{-1}V, \tag{4.2.4.2}$$

in the first line of (4.2.1.1), where I_n denotes the identity matrix in $\mathbb{R}^{n^2}$. Since (M, g) is a Riemannian manifold with boundary such that $\mu|g|^{1/2} = \rho$, $g^{ij} = 0$ if $i \neq j$, and $g^{ii} = \rho^{-1}a$ for $i, j \in \{1, \ldots, n\}$, we have

$$\Delta_{g,\mu} u = \rho^{-1}\mathrm{div}(a\nabla u),\ u \in C^\infty(\overline{\Omega}).$$

Therefore, (4.2.1.1) can be equivalently rewritten as

$$\begin{cases} \rho\partial_t^\alpha u - \mathrm{div}(a\nabla u) + Vu &= 0, \quad \text{in } Q := (0,T)\times\Omega, \\ u &= f, \quad \text{on } \Sigma := (0,T)\times\partial\Omega, \\ \partial_t^k u(0,\cdot) &= 0, \quad \text{in } \Omega,\ k = 0,\dots,m. \end{cases} \tag{4.2.4.3}$$

In [KiOkSoYa, Proposition 3.1], we prove that for any arbitrary $\alpha \in (0,1)\cup(1,2)$ and $T_0 \in (0,T)$, the partial DN map

$$\Lambda_{\rho,a,V} :\ \mathcal{H}_{\mathrm{in},\alpha,T_0} \ni f \mapsto a\partial_\nu u(T_0,\cdot)_{|S_{out}}, \tag{4.2.4.4}$$

where u is the solution to (4.2.4.3), is bounded from $\mathcal{H}_{\mathrm{in},\alpha,T_0}$ into $L^2(S_{out})$. Our first result claims that knowledge of $\Lambda_{\rho,a,V}$ uniquely determines two out of the three coefficients ρ, a, and V, which are referred as, respectively, the density, the conductivity, and the (electric) potential.

Theorem 4.2.1 (Theorem 1.1, [KiOkSoYa]) *Assume that $S_{in}\cap S_{out} \neq \emptyset$ and that $S_{in}\cup S_{out} = \partial\Omega$. For $j = 1,2$, let $\rho_j \in L^\infty(\Omega)$, $a_j \in W^{2,\infty}(\Omega)$, and $V_j \in L^\infty(\Omega)$ satisfy (4.2.4.1) with $\rho = \rho_j$, $a = a_j$, $V = V_j$. Moreover, let either of the three following conditions be fulfilled:*

(i) $\rho_1 = \rho_2$ and

$$\nabla a_1(x) = \nabla a_2(x),\ x \in \partial\Omega. \tag{4.2.4.5}$$

(ii) $a_1 = a_2$ and

$$\exists C > 0,\ |\rho_1(x) - \rho_2(x)| \leqslant C\mathrm{dist}(x,\partial\Omega)^2,\ x \in \Omega. \tag{4.2.4.6}$$

(iii) $V_1 = V_2$ and (4.2.4.5)-(4.2.4.6) hold simultaneously true.

Then, $\Lambda_{\rho_1,a_1,V_1} = \Lambda_{\rho_2,a_2,V_2}$ yields $(\rho_1,a_1,V_1) = (\rho_2,a_2,V_2)$.

The second result describes the identifiability properties of the Riemannian manifold (M,g) and the functions $\mu \in \mathcal{C}^\infty(M)$ and $q \in \mathcal{C}^\infty(M)$, appearing in the first line of the IBVP (4.2.1.1), that can be inferred from $\Lambda_{M,g,\mu,q}$. It is well known that the DN map is invariant under isometries fixing the boundary. Moreover, gauge equivalent coefficients (μ,q) cannot be distinguished by the DN map either. Here and henceforth, (μ_1,q_1) and (μ_2,q_2) are said gauge equivalent if there exists a strictly positive valued function $\kappa \in C^\infty(M)$ satisfying

$$\kappa(x) = 1 \text{ and } \partial_\nu\kappa(x) = 0,\ x \in \partial M \tag{4.2.4.7}$$

such that

$$\mu_2 = \kappa^{-2}\mu_1, \quad q_2 = q_1 - \kappa\Delta_{g,\mu_1}\kappa^{-1}. \tag{4.2.4.8}$$

The second result of [KiOkSoYa] can be stated as follows.

Theorem 4.2.2 (Theorem 1.2, [KiOkSoYa]) *For $j = 1,2$, let (M_j,g_j) be two compact and smooth connected Riemannian manifolds of dimension $n \geqslant 2$ with the same boundary, and let $\mu_j \in C^\infty(M_j)$ and $q_j \in C^\infty(M_k)$ satisfy $\mu_j(x) > 0$ and $q_j(x) \geq 0$ for all $x \in M_j$. Let $S_{in}, S_{out} \subset \partial M_1$ be relatively open and suppose that $\overline{S_{in}} \cap \overline{S_{out}} \neq \emptyset$. Suppose, moreover, that $g_1 = g_2$, $\mu_1 = \mu_2 = 1$ and $\partial_\nu\mu_1 = \partial_\nu\mu_2 = 0$ on ∂M_1. Then, $\Lambda_{M_1,g_1,\mu_1,q_1} = \Lambda_{M_2,g_2,\mu_2,q_2}$ yields that (M_1,g_1) and (M_2,g_2) are isometric and that (μ_1,q_1) and (μ_2,q_2) are gauge equivalent.*

4.2.5 Obstruction to uniqueness

Notice that the absence of global uniqueness result manifested in Theorems 4.2.1 (in the sense that only two among the three coefficients ρ, a, and V, are recovered) and 4.2.2 (where the metric g is determined up to an isometry and (μ, q) are identified modulo gauge transformation) arises from one or several natural obstructions to identification in the system under investigation, each of them being induced by an invariance property satisfied by (4.2.1.1).

The first obstruction, concerns the recovery of the coefficients (ρ, a, q). Namely, fix (ρ_1, a_1, q_1) be defined in a similar way than in Theorems 4.2.1 and, for any positive function $\kappa \in C^2(\overline{\Omega}) \setminus \{1\}$ satisfying (4.2.4.7), we assume that (ρ_2, a_2, q_2) are given by

$$\rho_2 = \kappa^2 \rho_1, \quad a_2 = \kappa^2 a_1, \quad V_2 = V_1 \kappa^2 - \kappa \mathrm{div}(a_1 \nabla \kappa).$$

Then, we have

$$\begin{aligned}(\rho_1 \partial_t^\alpha - \mathrm{div}(a_1 \nabla \cdot) + V_1)\kappa u &= \kappa \rho_1 \partial_t^\alpha u - \kappa^{-1} \mathrm{div}(a_1 \kappa^2 \nabla u) + (V_1 \kappa - \mathrm{div}(a_1 \nabla \kappa))u \\ &= \kappa^{-1} \left(\rho_2 \partial_t^\alpha u - \mathrm{div}(a_2 \nabla u) + V_2 u\right)\end{aligned}.$$

Thus, for u_j, $j = 1, 2$, the solution of (4.2.4.3) with $\rho = \rho_j$, $a = a_j$ and $V = V_j$, we have $u_1 = \kappa u_2$. This, in particular means that $\Lambda_{\rho_1, a_1, V_1} = \Lambda_{\rho_2, a_2, V_2}$ but $(\rho_1, a_1, V_1) \neq (\rho_2, a_2, V_2)$. Therefore, the DN map is invariant under the group of gauge transformations

$$(\rho, a, V) \mapsto (\kappa^2 \rho, \kappa^2 a, V \kappa^2 - \kappa \mathrm{div}(a \nabla \kappa))$$

parametrized by positive functions $\kappa \in C^\infty(\overline{\Omega})$ satisfying (4.2.4.7) for $M = \overline{\Omega}$.

A similar obstruction can be found in Theorem 4.2.2. This obstruction is due to the invariance of (4.2.1.1) under the group of gauge transformations given by (4.2.4.8). Indeed, given a strictly positive function $\kappa \in C^\infty(M)$ satisfying (4.2.4.7), we observe for any (μ_1, q_1) and (μ_2, q_2) obeying (4.2.4.8), that

$$\Delta_{g,\mu_2}(\kappa w) = \kappa \Delta_{g,\mu_1} w + \delta \kappa w, \ w \in C^\infty(M),$$

where $\delta := \kappa^{-1} \Delta_{g,\mu_1} \kappa - 2\kappa^{-2} (\nabla_g \kappa, \nabla_g \kappa)_g$, and $(\cdot, \cdot)_g$ denotes the inner product on (M, g). In particular, taking $w = \kappa^{-1}$ we get the simpler expression $\delta = -\kappa \Delta_{g,\mu_1} \kappa^{-1}$. Finally, taking $w = u$, where u is the solution to (4.2.1.1) associated with $\mu = \mu_1$ and $q = q_1$, we find that

$$(\partial_t^\alpha - \Delta_{g,\mu_2} + q_2)(\kappa u) = \kappa(\partial_t^\alpha - \Delta_{g,\mu_1} + q_1)u = 0.$$

Since our assumptions (4.2.4.7) on κ imply that $\partial_\nu(\kappa u) = \partial_\nu u$ and $\kappa u = u$ on $(0, T) \times \partial M$, we find that $\Lambda_{M,g,\mu_1,q_1} = \Lambda_{M,g,\mu_2,q_2}$. This proves that the DN map is invariant under the group of gauge transformations

$$(\mu, q) \mapsto (\kappa^{-2} \mu, q - \kappa \Delta_{g,\mu} \kappa^{-1})$$

parametrized by strictly positive functions $\kappa \in C^\infty(M)$ satisfying (4.2.4.7).

The last obstruction arises from the fact that (4.2.1.1) is invariant with respect to changes of coordinates. That is, if $\Phi : M \to M$ is a diffeomorphism fixing the boundary ∂M then $\Lambda_{M,g,\mu,q} = \Lambda_{M,\Phi^* g,\mu \circ \Phi, q \circ \Phi}$ where $\Phi^* g$ is the pullback of g by Φ.

4.2.6 Comments about our results

To our best knowledge, the results of [KiOkSoYa] are the most precise so far, about the recovery of coefficients appearing in a time fractional diffusion equation from boundary measurements. We prove recovery of a wide class of coefficients from partial boundary measurements that consist in an input on the part S_{in} of the boundary and observation of the flux at the part S_{out} for one fixed time $t = T_0 \in (0, T)$. Our results extend

the ones contained in the previous works [ChNaYaYa, ChXaYa, LiImYa, YaZh] related to this problem. Another benefit of our approach is its generality, which makes it possible to treat the case of a smooth Riemannian manifold, and the one of a bounded domain with weak regularity assumptions on the coefficients.

Notice that (4.2.4.3) associated with $\alpha = 1$ is the usual heat equation, in which case Theorem 4.2.1 is contained in [CaKa01, CaKa04]. We point out that the strategy used in [CaKa01, CaKa04] for the derivation of Theorem 4.2.1 with $\alpha = 1$, cannot be adapted to the framework of time fractional derivative diffusion equations of order $\alpha \in (0,1) \cup (1,2)$. This is due to the facts that a solution to a time fractional derivative equation is not described by a semi-group, and that there is only limited smoothing property, and no integration by parts formula or Leibniz rule, with respect to the time variable, in this context. As a consequence, the analysis developed in this text is quite different from the one carried out by [CaKa01, CaKa04].

Notice from Theorem 4.2.2 that the statement of Theorem 4.2.1 still holds true for smooth coefficients in a smooth domain, under the weaker assumption $\overline{S_{in}} \cap \overline{S_{out}} \neq \emptyset$. Nevertheless, in contrast to Theorem 4.2.2 where we focus on the recovery of the Riemanniann manifold and the metric, the main interest of Theorem 4.2.1 lies in the weak regularity assumptions imposed on the unknown coefficients of the inverse problem under consideration. In the same spirit, we point out with Theorem 4.2.4 below, that the result of Theorem 4.2.2 remains valid when $\overline{S_{in}} \cap \overline{S_{out}} = \emptyset$, in the special case where $\mu = 1$ and $q = 0$, and assuming a Hassell-Tao type inequality [HasTao].

4.2.7 Idea of the proof and extension

The key idea to our proof is the connection between the DN map associated with (4.2.1.1) and the boundary spectral data of the corresponding elliptic Schrödinger operator. This ingredient has already been used by several authors in the context of hyperbolic (see e.g. [KaKuLa01, KaKuLa04, LaOk10, LaOk141]), parabolic (see e.g. [CaKa04, KaKuLa04]), and dynamical Schrödinger (see e.g. [KaKuLa04]) equations. Nevertheless, to our best knowledge, there is no such approach for time fractional diffusion equations, available in the mathematical literature.

Let us start by considering Theorem 4.2.1. Given a positive constant c, we assume that $\rho \in L^\infty(\Omega)$ satisfies $\rho(x) \geqslant c > 0$ for a.e. $x \in \Omega$, so the scalar product

$$\langle u, v \rangle_\rho := \int_\Omega \rho(x) u(x) v(x) dx, \ u, v \in L^2(\Omega),$$

is equivalent to the usual one in $L^2(\Omega)$. We denote by $L^2_\rho(\Omega)$ the Hilbertian space $L^2(\Omega)$ endowed with $\langle \cdot, \cdot \rangle_\rho$.

Next, for a nonnegative $V \in L^\infty(\Omega)$, and for $a \in C^1(\overline{\Omega})$ fulfilling $a(x) \geqslant c > 0$ for every $x \in \Omega$, we introduce the quadratic form

$$h[u] := \int_\Omega \left(a(x) |\nabla u(x)|^2 + V(x) |u(x)|^2 \right) dx, \ u \in \mathrm{Dom}(h) := H^1_0(\Omega),$$

and consider the operator H generated by h in $L^2_\rho(\Omega)$. Since $\partial\Omega$ is $C^{1,1}$, H is self-adjoint in $L^2_\rho(\Omega)$ and acts on its domain as

$$Hu := \rho^{-1} \left(\mathrm{div}(a \nabla u) + V u \right), \ u \in \mathrm{Dom}(H) := H^1_0(\Omega) \cap H^2(\Omega), \tag{4.2.7.1}$$

according to [Gr, Theorem 2.2.2.3].

By the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^2_\rho(\Omega)$, the spectrum $\sigma(H)$ of the operator H is purely discrete. Let $\{\lambda_k; \, k \in \mathbb{N}^*\}$ be the non-decreasing sequence of the eigenvalues of H. Furthermore, we introduce a family $\{\varphi_k; \ k \in \mathbb{N}^*\}$ of eigenfunctions of the operator H, which satisfy

$$H \varphi_k = \lambda_k \varphi_k, \ k \in \mathbb{N}^*, \tag{4.2.7.2}$$

and form an orthonormal basis in $L^2_\rho(\Omega)$. Notice that each φ_k is a solution to the following Dirichlet problem:

$$\left\{\begin{array}{rcll} -\mathrm{div}(a\nabla\varphi_k)+V\varphi_k & = & \lambda_k\rho\varphi_k, & \text{in } \Omega, \\ \varphi_k & = & 0, & \text{on } \partial\Omega, \\ \int_\Omega \rho(x)\left|\varphi_k(x)\right|^2 dx & = & 1, & \end{array}\right. \tag{4.2.7.3}$$

Put $\psi_k := (a\partial_\nu\varphi_k)_{|\partial\Omega}$ for every $k\in\mathbb{N}^*$. Following [CaKa01, KaKuLa01], we define the boundary spectral data associated with (ρ,a,V), as

$$\mathrm{BSD}(\rho,a,V) := \{(\lambda_k,\psi_k);\ k\geqslant 1\}.$$

We recall from [CaKa04, Corollaries 1.5, 1.6 and 1.7] the following Borg-Levinson type theorem.

Proposition 4.2.1 *Under the conditions of Theorem 4.2.1, assume that either of the three assumptions (i), (ii) or (iii) is verified. Then* $\mathrm{BSD}(\rho_1,a_1,q_1)=\mathrm{BSD}(\rho_2,a_2,q_2)$ *entails that* $(\rho_1,a_1,q_1)=(\rho_2,a_2,q_2)$.

In view of the inverse spectral result stated in Proposition 4.2.1, we may derive the claim of Theorem 4.2.1 upon showing that two sets of admissible coefficients (ρ_j,a_j,V_j), $j=1,2$, have same boundary spectral data, provided their boundary operators Λ_{ρ_j,a_j,V_j} coincide. Otherwise stated, the proof of Theorem 4.2.1 is a consequence of Proposition 4.2.1 combined with the coming result:

Theorem 4.2.3 (Theorem 2.2, [KiOkSoYa]) *For $j=1,2$, let $V_j\in L^\infty(\Omega)$, $\rho_j\in L^\infty(\Omega)$ and $a_j\in\mathcal{C}^1(\overline{\Omega})$ satisfy* (2.3.4.1) *with $\rho=\rho_j$, $a=a_j$, $V=V_j$. Then $\Lambda_{\rho_1,a_1,V_1}=\Lambda_{\rho_2,a_2,V_2}$ implies BSD$(\rho_1,a_1,V_1)=$ BSD(ρ_2,a_2,V_2), up to an appropriate choice of the eigenfunctions of the operator H_1 defined in* (4.2.7.1) *and associated with $(\rho,a,V)=(\rho_1,a_1,V_1)$.*

Therefore, we are left with the task of proving Theorem 4.2.3. To do so, we establish a representation of solutions of (4.2.4.3) involving the Mittag-Leffler functions, given by

$$E_{r,s}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(rk+s)},\quad r,s>0,\ z\in\mathbb{C},$$

and the boundary spectral data of the operator (4.2.7.1). Combining this representation with different properties of solutions of (4.2.4.3), we derive (4.2.3).

For more general Riemannian manifold, Theorem 4.2.3 can be replaced by similar results related to the boundary control method (e.g. [KaKuLa01]) where the condion $S_{in}\cup S_{out}=\partial M$ is not required. The main task of Theorem 4.2.2 consisting of proving the recovery of the boundary spectral data restricted to an arbitrary portion of the boundary in the specific case $S_{in}\cap S_{out}=\emptyset$, $\overline{S_{in}}\cap\overline{S_{out}}\neq\emptyset$. This can be done by using some intermediate data and by applying some results of [LaOk10].

In the specific case when $\mu=1$, $q=0$, we can even extend our result to the recovery of the manifold (M,g) up to an isometry from the DN map $\Lambda_{M,g,1,0}$ when $\overline{S_{in}}\cap\overline{S_{out}}=\emptyset$. In order to introduce this result, we fix (M_k,g_k), $k=1,2$, two compact and smooth connected Riemannian manifolds of dimension $n\geqslant 2$ with the same boundary. We associate with (M_k,g_k), $k=1,2$, the sequence of increasing eigenvalues $\{\lambda_{k,\ell};\ell\in\mathbb{N}^*\}$. For each $\ell\in\mathbb{N}^*$, we denote by $m_{k,\ell}\in\mathbb{N}^*$ the algebraic multiplicity of the eigenvalue $\lambda_{k,\ell}$ and we introduce a family $\{\varphi_{k,\ell,p};\ p=1,\ldots,m_{k,\ell}\}$ of eigenfunctions of H_k, which satisfy

$$H_k\varphi_{k,\ell,p}=\lambda_{k,\ell}\varphi_{k,\ell,p},$$

and form an orthonormal basis in $L^2(M_k)$ of the algebraic eigenspace of H_k associated with $\lambda_{k,\ell}$ (i.e. the linear sub-space of $L^2(M_k)$ spanned by $\{\varphi_{k,\ell,p},\ p=1,\ldots,m_{j,\ell}\}$). Then, we introduce the following spectral inequality

$$\lambda_{k,\ell}\leq C\|\partial_\nu\varphi_{k,\ell,p}\|^2_{L^2(S_{in})}, \tag{4.2.7.4}$$

where the constant $C > 0$ is independent of ℓ and p. We recall that Hassell and Tao [HasTao] showed that all non-trapping Riemannian manifolds (M_k, g_k) satisfy (4.2.7.4) when S_{in} is replaced by ∂M_k. Moreover, (4.2.7.4) follows from (and is strictly weaker than) the geometric control condition by Bardos, Lebeau and Rauch [BaLeRa92] (see [LaOk141]). Using the spectral condition (4.2.7.4), we prove the following extension of our result to measurements on disjoint portions.

Theorem 4.2.4 (Theorem 5.3, [KiOkSoYa]) *Let (M_k, g_k), $k = 1, 2$, be two compact and smooth connected Riemannian manifolds of dimension $n \geqslant 2$ with the same boundary. Let $S_{in}, S_{out} \subset \partial M_1$ be relatively open, and suppose that $g_1 = g_2$ on ∂M_1. Suppose, moreover, that both (M_k, g_k), $k = 1, 2$, satisfy the spectral inequality* (4.2.7.4)*. Then, the condition $\Lambda_{M_1,g_1,1,0} = \Lambda_{M_2,g_2,1,0}$ implies that (M_k, g_k), $k = 1, 2$, are isometric.*

We prove this extension by considering first the recovery of a partial hyperbolic DN map associated with the operators H_k, $k = 1, 2$. Then, we apply the result of [LaOk141]. We do not know if Theorem 4.2.4 holds for operators with varying μ and q, see the discussion in [LaOk141, pp. 7-8].

4.3 Well posedness for semilinear fractional wave equations

4.3.1 Our objective

In this section we will introduce [KiYa] which is a joint work with Masahiro Yamamoto. Let Ω be a $\mathcal{C}^2$-bounded domain of $\mathbb{R}^n$ with $n = 2, 3$. In what follows, we define $\mathcal{A}$ by the differential operator

$$\mathcal{A}u(x) = -\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) + V(x)u(x), \quad x \in \Omega,$$

where $a_{ij} = a_{ji} \in \mathcal{C}^1(\overline{\Omega})$ and $V \in L^\kappa(\Omega)$, for some $\kappa > n$, satisfy

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant c|\xi|^2, \quad x \in \overline{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$$

and $V \geqslant 0$ a.e. in Ω.

We consider the following IBVP for the fractional semilinear wave equation

$$\begin{cases} \partial_t^\alpha u + \mathcal{A}u = f_b(u), & (t,x) \in Q, \\ u(t,x) = 0, & (t,x) \in \Sigma, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{4.3.1.1}$$

where $1 < \alpha < 2$, ∂_t^α denotes the Caputo fractional derivative with respect to t,

$$\partial_t^\alpha u(t,x) := \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}\partial_s^2 u(s,x)ds,\ (t,x) \in Q,$$

$b > 1$ and $f_b \in \mathcal{C}^1(\mathbb{R})$ satisfies $f_b(0) = 0$ and

$$|f_b'(u)| \leqslant C\,|u|^{b-1}, \quad u \in \mathbb{R}.$$

The main purpose of [KiYa] is to give a suitable definition of solutions of (4.3.1.1) and to study the well-posedeness of this problem.

4.3.2 Physical motivations and known results

Recall that equation (4.3.1.1) is associated with anomalous diffusion phenomenon. More precisely, for $1 < \alpha < 2$, the linear part of equation (4.3.1.1) is frequently used for super-diffusive model of anomalous diffusion such as diffusion in heterogeneous media. In particular, in the linear case (i.e., $f_b \equiv 0$), some physical background is found in Sokolov, Klafter and Blumen [SoKlBl]. As for analytical results in the case of $1 < \alpha < 2$, we refer to Mainardi [Ma] as one early work, and also to §6.1 in Kilbas, Srivastava and Trujillo [KilSrTr], §10.10 in Podlubny [Pod]. For $0 < \alpha < 1$, there are works in view of the theory of partial differential equations (e.g., Beckers and Yamamoto [BekYa], Sakamoto and Yamamoto [SaYa]). Such researches are rapidly developing and here we do not intend to give any comprehensive lists of references.

In contrast to the wave equation, even linear fractional wave equations are not well studied. In fact, few authors treated the well-posedness of the linear IBVP associated with (4.3.1.1) and to our best knowledge even the definition of weak solutions does not allow source term with low regularity. For a general study of the linear fractional wave equation and the regularity of solutions we refer to [SaYa]. When we consider e.g., reaction effects in a super-diffusive model, we have to introduce a semilinear term.

To the best knowledge of the authors, there are no publications on fractional semilinear wave equations by the Strichartz estimate which is a common technique for semilinear wave and Schrödinger equations. For the wave equation ($\alpha = 2$), the well-posedness of problem (4.3.1.1) has been studied by various authors. In the case $\Omega = \mathbb{R}^k$ with $k \geqslant 3$ and $\mathcal{A} = -\Delta$, the global well-posedness has been proved both in the subcritical case $1 < b < 1 + \frac{4}{k-2}$ by Ginibre and Velo [GiVe85], and in the critical case $b = 1 + \frac{4}{k-2}$ by Shatah and Struwe [ShSt]. For $\Omega = \mathbb{R}^2$, Nakamura and Ozawa [NaOz991, Naoz992] proved global well-posedness with exponentially growing nonlinearity. Without being exhaustive, for other results related to regularity of solutions or existence of solutions for more general semilinear hyperbolic equations we refer to [Kap]. In the case of Ω a smooth bounded domain of $\mathbb{R}^3$, [BuLePl] proved the global well-posedness in the critical case $b = 5$. In addition, following the strategy set by [BuLePl], [IbJr] treated the case of Ω a smooth bounded domain of $\mathbb{R}^2$ with exponentially growing nonlinearity.

4.3.3 Well-posedness and Strichartz estimates

In order to give a suitable definition of solutions of (4.3.1.1) we first need to consider the IBVP associated with the linear fractional wave equation

$$\begin{cases} \partial_t^\alpha u + \mathcal{A}u = f(t,x), & (t,x) \in Q, \\ u(t,x) = 0, & (t,x) \in \Sigma, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{4.3.3.1}$$

The paper [KiYa] contains three main results. Our two first main results are related to properties of solutions of (4.3.3.1), while our last result concerns the nonlinear problem (4.3.1.1).

Let us first remark that in contrast to usual derivatives, there is no exact integration by parts formula for fractional derivatives. Therefore, it is difficult to introduce the definition of weak solutions of (4.3.3.1) in the sense of distributions. To overcome this gap we give the following definition of weak solutions of (4.3.3.1). Let $\mathbb{1}_{(0,T)}(t)$ be the characteristic function of $(0,T)$.

Definition 4.3.1 *Let $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$. We say that problem (4.3.3.1) admits a weak solution u if there exists $v \in L^\infty_{loc}(\mathbb{R}^+;L^2(\Omega))$ such that:*
1) $v_{|Q} = u$ and $\inf\{\varepsilon > 0 :\ e^{-\varepsilon t} v \in L^1(\mathbb{R}^+;L^2(\Omega))\} = 0$,
2) for all $p > 0$ the Laplace transform $V(p) = \int_0^{+\infty} e^{-pt} v(t,.)dt$ with respect to t of v solves

$$\begin{cases} (\mathcal{A} + p^\alpha)V(p) = F(p) + p^{\alpha-1}u_0 + p^{\alpha-2}u_1, & \text{in } \Omega, \\ V(p) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $F(p) = \mathcal{L}[f(t,.)\mathbb{1}_{(0,T)}(t)](p) = \int_0^T e^{-pt} f(t,.)dt$.

Remark 4.3.1 *Recall* (e.g. formula (2.140) page 80 of [Pod]) *that for* $h \in \mathcal{C}^2(\mathbb{R}^+)$ *satisfying* $\inf\{\varepsilon > 0 : e^{-\varepsilon t} h^{(k)} \in L^1(\mathbb{R}^+),\ k = 0,1,2\} = \varepsilon_0$ *we have*

$$\mathcal{L}[\partial^\alpha h](p) = p^\alpha H(p) - p^{\alpha-1} h(0) - p^{\alpha-2} h'(0), \quad p > \varepsilon_0,$$

where $H(p) = \mathcal{L}[h](p) = \int_0^{+\infty} e^{-pt} h(t)dt$. *Therefore, for sufficiently smooth data* u_0, u_1, f *(e.g. [SaYa]) one can check that problem* (4.3.3.1) *admits a unique strong solution which is also a weak solution of* (4.3.3.1).

Consider the operator A acting on $L^2(\Omega)$ with domain $D(A) = \{g \in H_0^1(\Omega) :\ \mathcal{A}g \in L^2(\Omega)\}$ defined by $Au = \mathcal{A}u$, $u \in D(A)$. Recall that in view of the Sobolev embedding theorem (e.g. [Gr, Theorem 1.4.4.1]) the multiplication operator $u \mapsto Vu$ is bounded from $H^1(\Omega)$ to $L^2(\Omega)$. Thus, we have $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, by $V \geq 0$ in Ω, the operator A is a positive selfadjoint operator with a compact resolvent. Therefore, the spectrum of A consists in a non-decreasing sequence of positive eigenvalues $(\lambda_k)_{k\geqslant 1}$. Let us also introduce an orthonormal basis in the Hilbert space $L^2(\Omega)$ of eigenfunctions $(\varphi_k)_{k\geqslant 1}$ of A associated with the non-decreasing sequence of eigenvalues $(\lambda_k)_{k\geqslant 1}$. From now on, by $\langle\cdot,\cdot\rangle$, we denote the scalar product of $L^2(\Omega)$. For all $s \geqslant 0$, we denote by A^s the operator defined by

$$A^s h = \sum_{k=1}^{+\infty} \langle h, \varphi_k\rangle \lambda_k^s \varphi_k, \quad h \in D(A^s) = \left\{ h \in L^2(\Omega) :\ \sum_{k=1}^{+\infty} |\langle h, \varphi_k\rangle|^2 \lambda_k^{2s} < \infty \right\}$$

and consider on $D(A^s)$ the norm

$$\|h\|_{D(A^s)} = \left(\sum_{k=1}^{+\infty} |\langle h, \varphi_k\rangle|^2 \lambda_k^{2s} \right)^{\frac{1}{2}}, \quad h \in D(A^s).$$

By duality, we can also set $D(A^{-s}) = D(A^s)'$ by identifying $L^2(\Omega)' = L^2(\Omega)$ which is a Hilbert space with the norm

$$\|h\|_{D(A^{-s})} = \left(\sum_{k=1}^{\infty} \langle h, \varphi_k\rangle_{-2s} \lambda_k^{-2s} \right)^{\frac{1}{2}}.$$

Here $\langle\cdot,\cdot\rangle_{-2s}$ denotes the duality bracket between $D(A^{-s})$ and $D(A^s)$. Since $D(A^{1/2}) = H_0^1(\Omega)$, we identify $H^{-1}(\Omega)$ with $D(A^{-1/2})$.

Using eigenfunction expansions we show our first main result where we state existence and uniqueness of weak solutions of (4.3.3.1).

Theorem 4.3.1 (Theorem 1.2, [KiYa]) *Let* $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega) = D(A^{-\frac{1}{2}})$, $f \in L^1(0,T;L^2(\Omega))$. *Then, problem* (4.3.3.1) *admits a unique weak solution* $u \in \mathcal{C}([0,T];L^2(\Omega))$ *satisfying*

$$\|u\|_{\mathcal{C}([0,T];L^2(\Omega))} \leqslant C(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}).$$

Moreover, assuming that there exists $0 < r < \frac{1}{4}$ *such that* $u_0 \in H^{2r}(\Omega)$, *we have* $u \in W^{1,1}(0,T;L^2(\Omega))$ *and*

$$\|u\|_{W^{1,1}(0,T;L^2(\Omega))} \leqslant C(\|u_0\|_{H^{2r}(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}).$$

Recall that for $\gamma, r, s \geqslant 0$, $1 \leqslant p, q, \tilde{p}, \tilde{q} \leqslant \infty$, Strichartz estimates for solutions u of (4.3.1.1) denotes estimates of the form

$$\|u\|_{\mathcal{C}([0,T];H^{2r}(\Omega))} + \|u\|_{L^p(0,T;L^q(\Omega))} \leqslant C(\|u_0\|_{H^{2\gamma}(\Omega)} + \|u_1\|_{H^{2s}(\Omega)} + \|f\|_{L^{\tilde{p}}(0,T;L^{\tilde{q}}(\Omega))}).$$

It is well known that these estimates, introduced by [Str70, Str77] and extended to the endpoints by [KeTa] (as well as on manifolds with hyperbolic trapped geodesics by [BuGuHa]) for both wave and Schrödinger equations, are important tools in the study of well-posedness of nonlinear equations (e.g. [Kap, BuLePl]). In [KiYa], we prove these estimates for solutions of (4.3.3.1). For this purpose, we consider $1 \leqslant p, q \leqslant \infty$ and $0 < \gamma < 1$ satisfying:

$$\begin{array}{ll} 1) & q = \infty, \quad \text{for } \frac{n}{4} < \gamma < 1, \\ 2) & 2 < q < \infty, \quad \text{for } \gamma = \frac{n}{4}, \\ 3) & q = \frac{2n}{n-4\gamma}, \quad \text{for } 0 < \gamma < \frac{n}{4}. \end{array} \tag{4.3.3.2}$$

$$\begin{array}{ll} 1) & p < \frac{1}{1-\alpha(1-\gamma)}, \quad \text{for } \gamma > 1 - \frac{1}{\alpha}, \\ 2) & p = \infty, \quad \text{for } \gamma \leqslant 1 - \frac{1}{\alpha}. \end{array} \tag{4.3.3.3}$$

Then, our second main result can be stated as follows.

Theorem 4.3.2 (Theorem 1.3, [KiYa]) *(Strichartz estimates) Assume that* $1 \leqslant p, q \leqslant \infty$ *and* $0 < \gamma < 1$ *fulfill* (4.3.3.2), (4.3.3.3) *and set*

$$s = \max\left(0, \gamma - \frac{1}{\alpha}\right), \quad r = \min\left(1 - \frac{1}{\alpha}, \gamma\right).$$

Let $u_0 \in D(A^\gamma)$, $u_1 \in D(A^s)$, $f \in L^1(0,T;L^2(\Omega))$. *Then, the unique weak solution* u *of problem* (4.3.3.1) *is lying in* $L^p(0,T;L^q(\Omega)) \cap \mathcal{C}([0,T];H^{2r}(\Omega))$ *and fulfills estimate*

$$\|u\|_{\mathcal{C}([0,T];H^{2r}(\Omega))} + \|u\|_{L^p(0,T;L^q(\Omega))} \leqslant C(\|u_0\|_{H^{2\gamma}(\Omega)} + \|u_1\|_{H^{2s}(\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}). \tag{4.3.3.4}$$

Here the constant C *takes the form*

$$C = C_0(1+T)^\delta, \tag{4.3.3.5}$$

where

$$\delta = \begin{cases} \max\left(\alpha(1-\gamma)-1, 1-\alpha(\gamma-s), 1-\alpha(r-s), \alpha(1-r)-1\right), & \text{for } p = \infty, \\ \max\left(\frac{1}{p}, 1-\alpha(\gamma-s)+\frac{1}{p}, 1-\alpha(r-s), \alpha(1-r)-1, \alpha(1-\gamma)-1+\frac{1}{p}\right), & \text{for } p < \infty \end{cases} \tag{4.3.3.6}$$

and C_0 *depends only on* Ω, γ, n, α, p, $\mathcal{A}$.

In the last section of [KiYa], we apply estimates (4.3.3.4) to prove the last result of this paper which is related to the existence and uniqueness of local solutions of (4.3.1.1). For this purpose, we first need to define local solutions of (4.3.1.1). In [KiYa, Section 2] (see also [SaYa]), using the eigenfunction expansions we introduce the operators

$$S_1(t)h = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k t^\alpha) \langle h, \varphi_k \rangle \varphi_n, \quad h \in L^2(\Omega),$$

$$S_2(t)h = \sum_{k=1}^{\infty} t E_{\alpha,2}(-\lambda_k t^\alpha) \langle h, \varphi_k \rangle \varphi_n, \quad h \in L^2(\Omega),$$

$$S_3(t)h = \sum_{k=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha) \langle h, \varphi_k \rangle \varphi_k, \quad h \in L^2(\Omega),$$

where for all $\alpha > 0$, $\beta \in \mathbb{R}$, $E_{\alpha,\beta}$ denotes the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

It is well known (e.g. [BekYa, Pod, SaYa]) that for all $t > 0$ we have $S_j(t) \in B(L^2(\Omega))$, $j = 1, 2, 3$. Moreover, according to [KiYa, Theorem 1.2], for $u_0, u_1 \in L^2(\Omega)$ and $f \in L^1(0, T; L^2(\Omega))$, the unique weak solution of (4.3.1.1) is given by

$$u(t) = S_1(t)u_0 + S_2(t)u_1 + \int_0^t S_3(t-s)f(s)ds. \tag{4.3.3.7}$$

For all $T > 0$, we introduce the space

$$X_T = \mathcal{C}([0, T]; L^2(\Omega)) \cap L^b(0, T; L^{2b}(\Omega))$$

with the norm

$$\|v\|_{X_T} = \|v\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|v\|_{L^b(0,T;L^{2b}(\Omega))}.$$

Recall that, by applying the Hölder inequality, one can check that for all $u, v \in X_T$ we have $f_b(u), f_b(v) \in L^1(0, T; L^2(\Omega))$ with

$$\|f_b(u)\|_{L^1(0,T;L^2(\Omega))} \leqslant C_b \|u\|^b_{L^b(0,T;L^{2b}(\Omega))} \leqslant C_b \|u\|^b_{X_T} \tag{4.3.3.8}$$

and

$$\|f_b(u) - f_b(v)\|_{L^1(0,T;L^2(\Omega))} \leqslant C_b \|u - v\|_{X_T} (\|u\|^{b-1}_{X_T} + \|v\|^{b-1}_{X_T}), \tag{4.3.3.9}$$

where the constant $C_b > 0$ depends only on b, f_b. Therefore, in view of Theorem 4.3.1, the map $\mathcal{H}_b$ defined by

$$\mathcal{H}_b u(t) = \int_0^t S_3(t-s)f_b(u(s))ds, \quad u \in X_T$$

is locally Lipschitz from X_T to $\mathcal{C}([0, T]; L^2(\Omega))$.

Definition 4.3.2 *Let $u_0, u_1 \in L^2(\Omega)$ and $T > 0$. We say that* (4.3.1.1) *admits a weak solution on* $(0, T)$ *if the map $\mathcal{G}_b : X_T \to \mathcal{C}([0, T]; L^2(\Omega))$ defined by*

$$\mathcal{G}_b u(t) = S_1(t)u_0 + S_2(t)v_2 + \int_0^t S_3(t-s)f_b(u(s))ds$$

admits a fixed point $u \in X_T$. Such a fixed point $u \in X_T$ is called a weak solution to (4.3.1.1) *on* $(0, T)$*. We say that problem* (4.3.1.1) *admits a local weak solution if there exists $T > 0$, depending on u_0, u_1, such that problem* (4.3.1.1) *admits a weak solution on* $(0, T)$*.*

Now we can state our result of existence and uniqueness of local solutions for (4.3.1.1). We recall that $\delta > 0$ is given in (4.3.3.6).

Theorem 4.3.3 (Theorem 1.5, [KiYa]) *Let $b > 1$ satisfy*

$$\frac{n\alpha}{n\alpha + 4(1-\alpha)} < b < \frac{n\alpha + 4}{n\alpha + 4(1-\alpha)} \tag{4.3.3.10}$$

and let

$$\gamma = \frac{n(b-1)}{4b}, \ q = 2b, \ s = \max(0, \gamma - \frac{1}{\alpha}), \ r = \min(1 - \frac{1}{\alpha}, \gamma), \ 1 \leqslant \ell < \frac{1}{2-\alpha}. \tag{4.3.3.11}$$

Then, we have $b < \frac{1}{1-\alpha(1-\gamma)}$ and for any $p \in \left(b, \frac{1}{1-\alpha(1-\gamma)}\right)$, $u_0 \in D(A^\gamma)$, $u_1 \in D(A^s)$, $T_0 > 0$, there exists $T \leqslant T_0$ that takes the form

$$T = \min\left[\left(\tilde{C}(\|u_0\|_{H^{2\gamma}(\Omega)} + \|u_1\|_{H^{2s}(\Omega)})\right)^{-\frac{p(b-1)}{p-b}}, T_0\right], \tag{4.3.3.12}$$

such that problem (4.3.1.1) *admits a weak solution* u *on* $(0,T)$ *lying in* $L^p(0,T;L^q(\Omega)) \cap \mathcal{C}([0,T];H^{2r}(\Omega)) \cap W^{1,\ell}(0,T;L^2(\Omega))$. *Note that in* (4.3.3.12) *the constant* $\tilde{C}$ *takes the form*

$$\tilde{C} = \tilde{C}_0(1+T_0)^{\frac{\delta}{b-1}}, \tag{4.3.3.13}$$

with $\tilde{C}_0$ *depending only on* f_b, Ω, α, b, p, n, $\mathcal{A}$. *Moreover, this local weak solution* u *is the unique weak solution of* (4.3.1.1) *on* $(0,T)$ *lying in* $L^p(0,T;L^q(\Omega))$ *and* u *satisfies*

$$\|u\|_{\mathcal{C}([0,T];H^{2r}(\Omega))} + \|u\|_{L^p(0,T;L^q(\Omega))} + \|u\|_{W^{1,\ell}(0,T;L^2(\Omega))} \leqslant C(\|u_0\|_{H^{2\gamma}(\Omega)} + \|u_1\|_{H^{2s}(\Omega)}). \tag{4.3.3.14}$$

Here the constant $C>0$ *depends on* n, Ω, f_b, b, T_0, p, α, $\mathcal{A}$.

A direct consequence of Theorem 4.3.3 is the following.

Corollary 4.3.1 (Corollary 1.6, [KiYa]) *Assume that conditions* (4.3.3.10) *and* (4.3.3.11) *are fulfilled. Let* $u_0 \in D(A^\gamma)$, $u_1 \in D(A^s)$ *satisfy*

$$\left[\tilde{C}_0(\|u_0\|_{H^{2\gamma}(\Omega)} + \|u_1\|_{H^{2s}(\Omega)})\right]^{-\frac{p(b-1)}{p(1+\delta)-b}} > 1$$

for some $b < p < \frac{1}{1-\alpha(1-\gamma)}$, *where the constant* $\tilde{C}_0$ *is introduced in* (4.3.3.13). *Then, for any* $T>0$ *satisfying*

$$T < \left[\tilde{C}_0(\|u_0\|_{H^{2\gamma}(\Omega)} + \|u_1\|_{H^{2s}(\Omega)})\right]^{-\frac{p(b-1)}{p(1+\delta)-b}},$$

problem (4.3.1.1) *admits a unique weak solution* u *on* $(0,T)$ *lying in* $L^p(0,T;L^q(\Omega)) \cap \mathcal{C}([0,T];H^{2r}(\Omega)) \cap W^{1,\ell}(0,T;L^2(\Omega))$.

This last result means that for smaller initial data we obtain longer time of existence of weak solutions.

Let us remark that, [KiYa] seems to be the first paper where the Definition 4.3.1 of weak solutions of (4.3.3.1) is considered. The main contribution of Definition 4.3.1 comes from the fact that it allows well-posedness of (4.3.3.1) with weak conditions. Indeed, in contrast to other definitions of weak solutions for (4.3.3.1) (e.g. [SaYa, Definition 2.1] used by [SaYa] to prove existence of weak solutions of (4.3.3.1) with $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$, $u_1 = 0$ in [SaYa, Corollary 2.5, 2.6]), applying Definition 4.3.1 we can show well-posedness of (4.3.3.1) with $f \in L^1(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and $u_1 \in H^{-1}(\Omega)$. The choice of Definition 4.3.1 is inspired both by the analysis of [Pod] and the connection between elliptic equations and fractional diffusion equations used by [LiImYa]. Note also that Definition 4.3.1 plays an important role in the Definition 4.3.2 of weak solutions of (4.3.1.1).

Let us observe that in contrast to the wave equation the solution of (4.3.3.1) are not described by a semigroup. Therefore, we can not apply many arguments that allow to improve the Strichartz estimates (4.3.3.4) such as the TT^* method of [KeTa]. Nevertheless, we prove local existence of solution of (4.3.1.1) with estimates (4.3.3.4). Note also that estimates (4.3.3.4) are derived from suitable estimates of Mittag-Leffler functions.

To our best knowledge [KiYa] is the first work treating well-posedness for semilinear fractional wave equations. Contrary to semilinear wave equations, it seems difficult to give a suitable definition of the energy for (4.3.1.1). This is mainly due to the fact that, once again, there is no exact integration by parts formula for fractional derivatives as well as properties of composition and conjugation of the fractional Caputo derivative ∂_t^α (e.g. [Pod, Section 2]). For this reason, it seems complicate to derive global well-posedness from local well-posedness. However, using the explicit dependence with respect to T of the constant in (4.3.3.4) we can establish an explicit dependence of the time of existence T of (4.3.1.1) with respect to the initial conditions u_0, u_1. From this result, we prove long time of existence for small initial data (see Corollary 4.3.1).

4.4 Fractional diffusion equations with variable order

4.4.1 Statement of the problem

This section is devoted to the work [KiSoYa], which is a joint work with Eric Soccorsi and Masahiro Yamamoto. Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz continuous boundary $\partial\Omega$, and let $(a_{i,j})_{1\leqslant i,j\leqslant n} \in L^\infty(\Omega;\mathbb{R}^{n^2})$ be symmetric, i.e., fulfill $a_{i,j} = a_{j,i}$ a.e. in Ω, for $i,j = 1,\ldots,n$, and satisfy the ellipticity condition

$$\exists c > 0,\ \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geqslant c|\xi|^2,\ x \in \Omega,\ \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n. \tag{4.4.1.1}$$

For $\kappa \in (n,+\infty]$ and $q \in L^\kappa(\Omega;\mathbb{R}_+)$, we introduce the formal differential operators

$$\mathcal{A}_0 u(x) = -\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{i,j}(x)\partial_{x_j}u(x)\right) \text{ and } \mathcal{A}_q u(x) := \mathcal{A}_0 u(x) + q(x)u(x),\ x \in \Omega.$$

Given $T \in (0,+\infty]$ and two functions $\alpha \in L^\infty(\Omega)$ and $\rho \in L^\infty(\Omega)$, such that

$$0 < \alpha_0 \leqslant \alpha(x) \leqslant \alpha_M < 1 \text{ and } 0 < \rho_0 \leqslant \rho(x) \leqslant \rho_M < +\infty,\ x \in \Omega, \tag{4.4.1.2}$$

we consider the initial boundary value problem for space-dependent variable order fractional diffusion equations

$$\left\{\begin{array}{rcll} (\rho(x)\partial_t^{\alpha(x)} + \mathcal{A}_q)u(t,x) & = & f(t,x), & (t,x) \in Q := (0,T)\times\Omega, \\ u(t,x) & = & 0, & (t,x) \in \Sigma := (0,T)\times\partial\Omega, \\ u(0,x) & = & u_0(x), & x \in \Omega. \end{array}\right. \tag{4.4.1.3}$$

Here and below, $\partial_t^{\alpha(x)}$ denotes the Caputo fractional derivative of order $\alpha(x)$ with respect to t, defined by

$$\partial_t^{\alpha(x)}u(t,x) := \frac{1}{\Gamma(1-\alpha(x))}\int_0^t (t-s)^{-\alpha(x)}\partial_s u(s,x)ds,\ (t,x) \in Q,$$

where Γ is the usual Gamma function.

In [KiSoYa] we pursue two goals. The first one is to establish the well-posedness of the IBVP (4.4.1.3) for suitable source terms f and initial data u_0. The second one is to analyse the uniqueness issue in the inverse problem of determining simultaneously the fractional order α, the density ρ, and the electric potential q, entering the diffusion equation appearing in (4.4.1.3), by partial Neuman data.

4.4.2 Physical motivations

Anomalous diffusion in complex media is a rapidly growing field of academic research with multiple engineering applications in geophysics, environmental science and biology. The diffusion properties of homogeneous porous media are currently modeled, see e.g. [AdGe, CaSaLuGa], by constant order time-fractional diffusion processes (that is by (4.4.1.3) where the mapping $x \mapsto \alpha(x)$ is maintained constant over Ω). But in complex media, the presence of heterogeneous regions causes variations of the permeability in different spatial positions, and in this case, the variable order time-fractional model is more relevant for describing the diffusion process, see [SuChCh]. This justifies for a closer look into the analysis of variable order time-fractional diffusion equations.

4.4.3 Existence of solutions and identification

The first result of [KiSoYa], which is stated in the framework of Lipschitz continuous bounded domains Ω, is concerned with the forward problem associated with (4.4.1.3), that is with the existence, uniqueness and some regularity properties of the weak solution to the Cauchy problem (4.4.1.3). Namely, Theorem 4.4.1 states for all $T \in (0, +\infty]$, all bounded functions α and ρ fulfilling (4.4.1.2), and all non-negative potentials $q \in L^\kappa(\Omega)$ with $\kappa \in (d, +\infty]$, that the IBVP (4.4.1.3) admits a unique solution within the class $\mathcal{C}((0,T]; L^2(\Omega))$, provided the initial data u_0 is taken in $L^2(\Omega)$ and the source terms f belonging to $\mathcal{C}((0,T]; L^2(\Omega))$ and satisfying the condition $t \mapsto (1+t)^{-m} f(t,\cdot) \in L^\infty((0,T); L^2(\Omega)$ for some $m \in \mathbb{N}$. Here and in the remaining part of this section, the interval $(0,T]$ should be understood as $(0,+\infty)$ in the particular case where $T = +\infty$. In order to introduce this first result, we start by considering a definition of weak solutions close to the one introduced in the last section but extended to more general equations.

Let $\mathcal{S}'(\mathbb{R}; L^2(\Omega)) := \mathcal{B}(\mathcal{S}(\mathbb{R}; L^2(\Omega)); \mathbb{R})$ be the space dual to $\mathcal{S}(\mathbb{R}; L^2(\Omega))$. We denote by $\mathcal{S}'(\mathbb{R}_+; L^2(\Omega)) := \{v \in \mathcal{S}'(\mathbb{R}; L^2(\Omega)) : \text{ supp } (v) \subset [0,+\infty) \times \overline{\Omega}\}$ the set of distributions in $\mathcal{S}'(\mathbb{R}; L^2(\Omega))$, supported in $[0,+\infty) \times \overline{\Omega}$. Otherwise stated, $v \in \mathcal{S}'(\mathbb{R}; L^2(\Omega))$ lies in $\mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$ if and only if $\langle v, \varphi \rangle_{\mathcal{S}'(\mathbb{R}; L^2(\Omega)), \mathcal{S}(\mathbb{R}; L^2(\Omega))} = 0$, whenever $\varphi \in \mathcal{S}(\mathbb{R}; L^2(\Omega))$ vanishes in $\mathbb{R}_+ \times \overline{\Omega}$. As a consequence, for a.e. $x \in \Omega$, we have

$$\langle v(\cdot, x), \varphi \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} = \langle v(\cdot, x), \psi \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})}, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}), \tag{4.4.3.1}$$

provided $\varphi = \psi$ in $\mathbb{R}_+$. Further, we say that $\varphi \in \mathcal{S}(\mathbb{R}_+)$ if φ is the restriction to $\mathbb{R}_+$ of a function $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$. Then, we set

$$x \mapsto \langle v(\cdot, x), \varphi \rangle_{\mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)} := x \mapsto \langle v(\cdot, x), \tilde{\varphi} \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})}, \ v \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega)).$$

Notice from (4.4.3.1) that $\tilde{\varphi}$ may be any function in $\mathcal{S}(\mathbb{R})$ such that $\tilde{\varphi}(t) = \varphi(t)$ for all $t \in \mathbb{R}_+$. For $p \in \mathbb{C}_+ := \{z \in \mathbb{C} : \text{ Re} z > 0\}$, we put

$$e_p(t) := \exp(-pt), \ t \in \mathbb{R}_+.$$

Evidently, e_p lies in $\mathcal{S}(\mathbb{R}_+)$. For $v \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$, we define the Laplace transform $\mathcal{L}[v]$ in t of v, by

$$\mathcal{L}[v](p) := x \mapsto \langle v(\cdot, x), e_p \rangle_{\mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)}, \ p \in \mathbb{C}_+,$$

and notice that $p \mapsto \mathcal{L}[v](p) \in \mathcal{C}^\infty(\mathbb{C}_+; L^2(\Omega))$. Having seen this, we define the weak solution to (4.4.1.3) as follows.

Definition 4.4.1 *Let $u_0 \in L^2(\Omega)$. For $T < +\infty$, we assume that $f \in L^1(0,T; L^2(\Omega))$ and, for $T = +\infty$, we assume that there exists $m \in \mathbb{N}$ such that $(1+|t|)^{-m} f \in L^1(\mathbb{R}_+; L^2(\Omega)$. We say that u is a weak solution to (4.4.1.3) if u is the restriction to Q of a distribution $v \in \mathcal{S}'(\mathbb{R}_+; L^2(\Omega))$, i.e. $u = v_{|Q}$, whose Laplace transform $V := \mathcal{L}[v]$ solves, for all $p \in (0,+\infty)$, the boundary value problem:*

$$\left\{ \begin{array}{rcll} \mathcal{A}_q V + \rho(x) p^{\alpha(x)} V & = & F(p) + \rho(x) p^{\alpha(x)-1} u_0, & x \in \Omega, \\ V & = & 0, & x \in \partial\Omega, \end{array} \right. \tag{4.4.3.2}$$

Here $F(p) := \mathcal{L}[f(t,.)\mathbb{1}_{(0,T)}(t)](p) = \int_0^T e^{-pt} f(t,.) dt$.

For $\theta \in (\pi/2, \pi)$ and $\varepsilon \in (0,1)$, we introduce the contour

$$\gamma(\varepsilon, \theta) := \gamma_-(\varepsilon, \theta) \cup \gamma_0(\varepsilon, \theta) \cup \gamma_+(\varepsilon, \theta), \tag{4.4.3.3}$$

in the complex plane $\mathbb{C}$, where

$$\gamma_0(\varepsilon, \theta) := \{\varepsilon e^{i\beta}; \ \beta \in [-\theta, \theta]\} \text{ and } \gamma_\pm(\varepsilon, \theta) := \{s e^{\pm i\theta}; \ s \in [\varepsilon, +\infty)\}. \tag{4.4.3.4}$$

Further, we denote by A_q the self-adjoint realization in $L^2(\Omega)$, of the operator $\mathcal{A}_q$ with homogeneous Dirichlet boundary conditions, and by $(A_q + \rho(x) p^{\alpha(x)})^{-1}$, for $p \in \mathbb{C} \setminus \mathbb{R}_-$, the resolvent operator of $A_q + \rho(x) p^{\alpha(x)}$.

Then, the existence and uniqueness result of a weak solution to the IBVP (4.4.1.3) is as follows.

Theorem 4.4.1 (Theorem 1.1, [KiSoYa]) *Suppose that* (4.4.1.1) *and* (4.4.1.2) *are fulfilled. Let* $u_0 \in L^2(\Omega)$ *and, depending on whether* $T \in (0,+\infty)$ *or* $T = +\infty$*, assume either that* $f \in L^\infty(0,T;L^2(\Omega)) \cap \mathcal{C}((0,T];L^2(\Omega))$ *or that* $f \in \mathcal{C}((0,+\infty);L^2(\Omega))$ *satisfies* $(1+|t|)^{-\zeta} f \in L^\infty(\mathbb{R}_+;L^2(\Omega))$ *for some* $\zeta \in \mathbb{R}_+$*. Then, there exists a unique weak solution* $u \in \mathcal{C}((0,T];L^2(\Omega))$ *to* (4.4.1.3)*, which is expressed by*

$$u(t) = S_0(t)u_0 + \int_0^t S_1(t-\tau)f(\tau)d\tau + S_2 f(t),\ t \in (0,T], \tag{4.4.3.5}$$

where we have set for all $\psi \in L^2(\Omega)$*,*

$$S_0(t)\psi := \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} e^{tp}(A_q + \rho(x)p^{\alpha(x)})^{-1}\rho(x)p^{\alpha(x)-1}\psi dp,$$

$$S_1(t)\psi := \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} e^{tp}(A_q + \rho(x)p^{\alpha(x)})^{-1}\psi dp,$$

and

$$S_2\psi := \frac{1}{2i\pi}\int_{\gamma(\varepsilon,\theta)} p^{-1}(A_q + \rho(x)p^{\alpha(x)})^{-1}\psi dp,$$

the three above integrals being independent of the choice of $\varepsilon \in (0,1)$ *and* $\theta \in \left(\frac{\pi}{2},\pi\right)$*.*

Moreover, in the particular case where $f = 0$*, the mapping* $t \mapsto u(t)$ *is analytic in* $(0,T)$*.*

Remark 4.4.1 (Remark 1, [KiSoYa]) *We point out for all* $\alpha_0 \in \left(0,\frac{1}{2}\right)$*, that the operator* S_2 *is identically zero, provided we have* $\alpha_M \in (\alpha_0, 2\alpha_0)$*. Therefore,* (4.4.3.5) *reduces to the "classical" Duhamel formula in this case:*

$$u(t) = S_0(t)u_0 + \int_0^t S_1(t-\tau)f(\tau)d\tau,\ t \in (0,T]. \tag{4.4.3.6}$$

The second result of [KiSoYa] deals with the inverse problem of determining the unknown coefficients α, ρ, q, entering the time fractional diffusion equation appearing in (4.4.1.3), by partial boundary measurements of the solution. More precisely, assuming that $\partial\Omega$ is $\mathcal{C}^{1,1}$, $\ell \in \mathbb{N} \setminus \{0,1\}$, we probe the following system

$$\left\{\begin{array}{rcll} (\rho(x)\partial_t^{\alpha(x)} - \Delta + q(x))u(t,x) & = & 0, & (t,x) \in (0,+\infty)\times\Omega, \\ u(t,x) & = & t^\ell g(x), & (t,x) \in (0,+\infty)\times\partial\Omega, \\ u(0,x) & = & 0, & x \in \Omega, \end{array}\right. \tag{4.4.3.7}$$

with suitable Dirichlet data g. Given two non empty subsets of $\partial\Omega$, S_{in} and S_{out}, we then introduce for all $t \in (0,+\infty)$, the following boundary operator

$$\mathcal{N}_{\alpha,\rho,q}(t) : \mathcal{H}_{in} \ni g \mapsto \partial_\nu u_g(t,\cdot)_{|S_{out}}, \tag{4.4.3.8}$$

where $\mathcal{H}_{in} := \{g \in H^{3/2}(\partial\Omega) :\ \text{supp}\ (g) \subset \overline{S_{in}}\}$. Here, u_g denotes the unique $\mathcal{C}([0,+\infty);H^2(\Omega))$-solution to (4.4.3.7), whose existence is guaranteed by Theorem 4.4.1, stated in Proposition 4.4.2 below.

We examine the uniqueness issue in the inverse problem of determining the coefficients (α,ρ,q) from the knowledge of the boundary operators $\{\mathcal{N}_{\alpha,\rho,q}(t_k),\ k \in \mathbb{N}\}$ associated with a time-sequence $\{t_k,\ k \in \mathbb{N}\} \in (0,+\infty)^{\mathbb{N}}$, fulfilling

$$\exists \tau \in (0,+\infty),\ \tau \text{ is an accumulation point of } \{t_k,\ k \in \mathbb{N}\}, \tag{4.4.3.9}$$

and suitable observation regions S_{in} and S_{out}. Namely, we assume that the domain Ω and the input and output regions satisfy the following conditions.

(i) If $n = 2$, it is required that Ω be connected and that $\partial\Omega = \bigcup_{k=1}^{N} \gamma_k$ be made of a finite number of smooth closed contours γ_k, $k = 1, \ldots, N$. In this case, we choose $S_{in} = S_{out} := \gamma$, where γ is any arbitrary non-empty open subset of $\partial\Omega$, and the set of admissible unknown coefficients reads

$$\begin{aligned}\mathcal{E}_2 \quad := \quad & \big\{(\alpha, \rho, q) : \ \alpha \in W^{1,r}(\Omega) \text{ and } \rho \in W^{1,r}(\Omega) \text{ fulfill (4.4.1.2), and} \\ & q \in W^{1,r}(\Omega; \mathbb{R}_+) \text{ with } r \in (2, +\infty)\big\}.\end{aligned}$$

(ii) If $n \geqslant 3$, we assume that the domain Ω is smooth and connected. We pick $x_0 \in \mathbb{R}^n$ outside the convex hull of $\overline{\Omega}$, impose that

$$\{x \in \partial\Omega : \ (x - x_0) \cdot \nu \geqslant 0\} \subset S_{in} \text{ and } \{x \in \partial\Omega : \ (x - x_0) \cdot \nu \leqslant 0\} \subset S_{out},$$

and define the set of admissible unknown coefficients by

$$\mathcal{E}_n := \{(\alpha, \rho, q); \ \alpha \in L^\infty(\Omega) \text{ and } \rho \in L^\infty(\Omega) \text{ fulfill (4.4.1.2), and } q \in L^\infty(\Omega; \mathbb{R}_+)\}.$$

The uniqueness result for the above described inverse coefficients problem is as follows.

Theorem 4.4.2 (Theorem 1.2, [KiSoYa]) *Let $\{t_k, \ k \in \mathbb{N}\} \in (0, +\infty)^{\mathbb{N}}$ fulfill (4.4.3.9) and assume that either (i) or (ii) is satisfied, depending on whether $d = 2$ or $d \geqslant 3$. Pick $(\alpha_j, \rho_j, q_j) \in \mathcal{E}_n$, $j = 1, 2$, such that*

$$\mathcal{N}_{\alpha_1,\rho_1,q_1}(t_k) = \mathcal{N}_{\alpha_2,\rho_2,q_2}(t_k), \ k \in \mathbb{N}. \tag{4.4.3.10}$$

Then, we have $(\alpha_1, \rho_1, q_1) = (\alpha_2, \rho_2, q_2)$.

4.4.4 Comments and outline

As the Laplace transform of a solution to constant order time-fractional diffusion equations is expressed in terms of Mittag-Leffler functions, most of its features are inherited from the well known properties of these special functions. This is no longer the case when the fractional order of the time-fractional diffusion equation depends on the space variable, which makes for a more challenging analysis of the well-posedness of these systems. This new technical difficulty translates in particular into the definition of a weak solution to variable order time-fractional diffusion equations, which extends the one of a weak solution to constant order time-fractional diffusion equations.

4.4.5 Description of the proof

We start with Theorem 4.4.1. For this purpose, we remark that problem (4.4.3.2), admits a unique solution if and only if the the operator $A_q + \rho p^{\alpha(x)}$ is invertible. Moreover, if $A_q + \rho p^{\alpha(x)}$ is invertible, the unique solution of (4.4.3.2) is given by

$$V(p) = (A_q + \rho p^{\alpha(x)})^{-1} F(p) + (A_q + \rho p^{\alpha(x)})^{-1} \rho(x) p^{\alpha(x)-1} u_0.$$

Combining this remark with properties of inversion of Laplace transform (e.g. [Ru, Theorem 19.2]), we need to consider some properties of the family of operators $(A_q + \rho p^{\alpha(x)})^{-1}$ for a complex valued parameter p. We remark that for $p \in \mathbb{C} \setminus \mathbb{R}$ the operator $A_q + \rho p^{\alpha(x)}$ is no longer selfadjoint and some specific treatment should be used for the inversion of such operators. We prove the following properties of this family of operators.

Proposition 4.4.1 (Proposition 2.1, [KiSoYa]) *For all $p \in \mathbb{C} \setminus \mathbb{R}_-$, the operator $A_q + \rho(x) p^{\alpha(x)}$ is boundedly invertible in $L^2(\Omega)$ and $(A_q + \rho(x) p^{\alpha(x)})^{-1}$ maps $L^2(\Omega)$ into $D(A_0)$. Moreover, the following estimate holds for all $r \in (0, +\infty)$,*

$$\left\| (A_q + \rho(x) r^{\alpha(x)} e^{i\beta\alpha(x)})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \leqslant C_\beta \max_{j=0,M} r^{-\alpha_j}, \ \beta \in (-\pi, \pi),$$

with

$$C_\beta := \begin{cases} 2\rho_0^{-1}, & \text{if } |\beta| \leqslant \theta_*(r) := \alpha_M^{-1} \min\limits_{\sigma=\pm 1} \arctan\left(\frac{\rho_0}{3\rho_M} r^{\sigma(\alpha_M-\alpha_0)}\right) \\ \rho_0^{-1} \max_{j=0,M} |\sin(\alpha_j\beta)|^{-1}, & \text{otherwise.} \end{cases} \tag{4.4.5.1}$$

Furthermore, the mapping $p \mapsto (A_q + \rho(x)p^{\alpha(x)})^{-1}$ *is bounded holomorphic in* $\mathbb{C} \setminus \mathbb{R}_-$.

Using these properties we prove Theorem 4.4.1. In contrast to Theorem 4.4.1, Theorem 4.4.2 requires additional regularity and some properties of analyticity of the DN map. In order to define the DN map we consider first the following result.

Proposition 4.4.2 (Proposition 3.1, [KiSoYa]) *Let* α, ρ *and* q *be the same as in Theorem 4.4.1. Then, for all* $g \in H^{3/2}(\partial\Omega)$, *the IBVP* (4.4.3.7) *admits a unique weak solution in* $\mathcal{C}([0,+\infty); H^2(\Omega))$.

Using the representation of the solution given in Proposition 4.4.2, we establish the analiticity of the DN map associated with (4.4.3.7) as follows.

Lemma 4.4.1 (Lemme 3.2, [KiSoYa]) *Let* $g \in H^{3/2}(\partial\Omega)$ *and let* u *be the* $\mathcal{C}([0,+\infty); H^2(\Omega))$*-solution to the IBVP* (4.4.3.7) *associated with* g, *given by Proposition 4.4.2. Then the mapping* $t \mapsto \partial_\nu u(t,\cdot)_{|\partial\Omega}$ *lies in* $\mathcal{A}((0,+\infty); L^2(\partial\Omega))$. *Here* $\mathcal{A}((0,+\infty); L^2(\partial\Omega))$ *denotes the space of function analytic on* $(0,+\infty)$ *taking value in* $L^2(\partial\Omega)$.

In view of this result, the condition (4.4.3.10) implies

$$\mathcal{N}_{\alpha_1,\rho_1,q_1}(t) = \mathcal{N}_{\alpha_2,\rho_2,q_2}(t), \quad t \in (0,+\infty).$$

Using this identity with some additional estimates, we can transform the inverse problem stated in Theorem 4.4.2 to the inverse problem of determining the coefficient (α, ρ, q) appearing in the family of elliptic boundary value problems

$$\begin{cases} -\Delta U(p) + [\rho(x)p^{\alpha(x)} + q(x)]U(p) &= 0, & x \in \Omega,\ p \in \mathbb{R}_+, \\ U(p)(x) &= g(x), & x \in \partial\Omega \end{cases}$$

from the knowledge of the associated elliptic partial DN map given by

$$\Lambda_{\rho p^\alpha + q} : \mathcal{H}_{in} \ni g \mapsto \partial_\nu U(p)_{|S_{out}}.$$

Then, we complete the proof of Theorem 4.4.2 by applying [ImYa131, Theorem 7] (see also [ImYa12]) in the particular case where $n = 2$, and from [KeSjUh, Theorem 1.2] when $n \geqslant 3$.

Chapter 5

Curiculum Vitae

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2. Problèmes inverses consistant à déterminer des coefficients dépendant du temps ou du temps et de l'espace pour différentes équations aux dérivées partielles d'évolution (Schrödinger, hyperbolique, parabolique).
3. Problèmes inverses spectraux de type Borg-Levinson et autres.
4. Stabilité pour le problème de Calderón dans un domaine borné ou dans un guide d'onde prenant la forme d'un cylindre infini.
5. Problèmes inverses et directs pour des équations de diffusion fractionnaire en temps.
6. Problèmes inverses pour des équations aux dérivées partielles non linéaires (résultats obtenus pour le moment uniquement pour les EDP paraboliques).
7. Problèmes inverses pour des équations aux dérivées partielles et problèmes inverses spectraux sur des variétés riemanniennes.
8. Estimations (estimations dispersives et estimations de Strichartz) et étude du comportement asymptotique par rapport au temps de solutions des équations des ondes avec la présence de perturbations, dépendantes du temps et de l'espace.

Mots-clés: Problèmes inverses, Problèmes inverses spectraux, Estimation de stabilité pour des problèmes inverses, Estimation de Carleman dans un guide d'onde, Problème de Calderón, Équations de diffusion fractionnaires en temps, Équation des ondes, Équation de Schrödinger, Équation de la chaleur, Estimations de Strichartz locales et globales.

Langues parlées

Français, anglais, persan, notions d'espagnol.

Outils informatiques utilisés

Latex, Maple...

Publications et prépublications

Production scientifique

26 articles publiés ou à paraître dans des revues avec comité de lecture.

9 articles soumis et prépublications.
1 chapitre de livre

Publications et prépublication accépté dans des revues internationales

1) Y. Kian, *Global Strichartz estimates for the wave equation with a time-periodic non-trapping metric*, Asymptotic Analysis, **68** (2010), 41-76.

2) Y. Kian, *Cauchy problem for semilinear wave equation with time-dependent metrics*, Nonlinear Analysis: Theory, Methods & Applications, **73** (2010), 2204-2212.

3) Y. Kian, *Local energy decay in even dimensions for the wave equation with a time-periodic non-trapping metric and applications to Strichartz estimates*, Serdica Math. J., **36** (2010), 329-370.

4) Y. Kian, *Local energy decay for the wave equation with a time-periodic non-trapping metric and moving obstacle*, Cubo, **14** (2012), 153-173.

5) M. Choulli, Y. Kian, *Stability of the determination of a time-dependent coefficient in parabolic equations*, Math. Control Relat. Fields, **3** (2013), 143-160.

6) P. Gaitan, Y. Kian, *A stability result for a time-dependent potential in a cylindrical domain*, Inverse Problems, **29** (2013), 065006 (18pp).

7) Y. Kian, Q. S. Phan, E. Soccorsi, *Carleman estimate for infinite cylindrical quantum domains and application to inverse problems*, Inverse Problems, **30** (2014), 055016. doi:10.1088/0266-5611/30/5/055016.

8) Y. Kian, *Stability of the determination of a coefficient for the wave equation in an infinite wave guide*, Inverse Probl. Imaging, **8** (3) (2014), 713-732.

9) Y. Kian, Q. S. Phan, E. Soccorsi, *Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains*, preprint, Journal of Mathematical Analysis and Applications, **426** (2015), 194-210.

10) M. Choulli, L. Kayser, Y. Kian, E. Soccorsi, *Heat trace asymptotics and compactness of isospectral potentials for the Dirichlet Laplacian*, Asymptotic Analysis, **92** (2015), 259-278.

11) O. Kavian, Y. Kian, E. Soccorsi, *Uniqueness and stability results for an inverse spectral problem in a periodic waveguide*, J. Math. Pures Appl., **104** (2015), 1160-1189.

12) M. Choulli, Y. Kian, E. Soccorsi, *Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide*, SIAM J. Math. Anal., **47** (6) (2015), 4536-4558.

13) M. Choulli, Y. Kian, E. Soccorsi, *Double logarithmic stability estimate in the identification of a scalar potential by a partial elliptic Dirichlet-to-Neumann map*, In memory of Alfredo Lorenzi, Bulletin SUSU MMCS, **8** (3) (2015), 78-94.

14) K. Fujishiro and Y. Kian, *Determination of time dependent factors of coefficients in fractional diffusion equations*, Math. Control Relat. Fields, **6** (2016), 251-269.

15) Y. Kian, *Unique determination of a time-dependent potential for wave equations from partial data*, Annales de l'IHP (C) Nonlinear Analysis, **34** (2017), 973-990.

16) Y. Kian, *Stability in the determination of a time-dependent coefficient for wave equations from partial data*, Journal of Mathematical Analysis and Applications, **436** (2016), 408-428.

17) M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse stability result for non compactly supported potentials by one arbitrary lateral Neumann observation*, J. Diff. Equat., **260** (2016), 7535-7562.

18) Y. Kian, *Recovery of time-dependent damping coefficients and potentials appearing in wave equations from partial data*, SIAM J. Math. Anal., **48** (6) (2016), 4021-4046.

19) Y. Kian, M. Yamamoto, *On existence and uniqueness of solutions for semilinear fractional wave equations*, Fractional Calculus and Applied Analysis, **20** (1) (2017), 117-138.

20) Y. Kian, *A multidimensional Borg-Levinson theorem for magnetic Schrödinger operators with partial spectral data*, à paraître dans Journal of Spectral Theory, `arXiv:1504.04514`.

21) M. Choulli, Y. Kian, E. Soccorsi, *Stability result for elliptic inverse periodic coefficient problem by partial Dirichlet-to-Neumann map*, à paraître dans Journal of Spectral Theory, `arXiv:1601.05355`.

22) M. Choulli, Y. Kian, E. Soccorsi, *On the Calderón problem in periodic cylindrical domain with partial Dirichlet and Neumann data*, à paraître dans Mathematical Methods in the Applied Sciences, `DOI: 10.1002/mma.4446`.

23) Y. Kian, L. Oksanen, E. Soccorsi, M. Yamamoto, *Global uniqueness in an inverse problem for time fractional diffusion equations*, J. Diff. Equat., **264** (2018), 1146-1170.

24) Y. Kian, L. Oksanen, *Recovery of time-dependent coefficient on Riemannian manifold for hyperbolic equations*, à paraître dans IMRN, `https://doi.org/10.1093/imrn/rnx263`.

25) M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse problem for the magnetic Schrödinger equation in infinite cylindrical domains*, à paraître dans PRIMS, `arXiv:1605.06599`.

26) M. Choulli, Y. Kian, *Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term*, à paraître dans J. Math. Pures Appl., `arXiv:1605.08672`.

Chapitre de livre

1) Stable Determination of Time-Dependent Scalar Potential From Boundary Measurements in a Periodic Quantum Waveguide, (with Mourad Choulli and Eric Soccorsi), New Prospects in Direct, Inverse and Control Problems for Evolution Equations, A. Favini, G. Fragnelli and R. M. Mininni (Eds), Springer-INdAM, Roma, 2014, 93-105.

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1) Y. Kian, *On the meromorphic continuation of the resolvent for the wave equation with time-periodic perturbations and applications*, preprint, `arXiv:1103.2530v2`.

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7) G. Bao, G. Hu, Y. Kian, T. Yin, *Inverse source problems in elastodynamics*, preprint, arXiv:1708.04684.

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9) Z.Li, Y. Kian, E. Soccorsi, *Initial-boundary value problem for distributed order time-fractional diffusion equations*, preprint, `arXiv:1709.06823`.

Exposés aux séminaires et conférences:

1) Novembre 2008: Exposé au séminaire de Physique Mathématique de Bordeaux.

2) Novembre 2009: Exposé au séminaire d'Analyse de l'université de Nantes.

3) Décembre 2009: Exposé à la rencontre conférence de l'ANR " Opérateurs non-autoadjoints et analyse semi-classique" (troisième rencontre de l'ANR blanc NONAa).

4) Novembre 2010: Exposé au séminaire d'analyse appliquée du LATP à l'université Aix-Marseille 1.

5) Fevrier 2011: Exposé à la troisième rencontre conférence du GDR "Dynamique Quantique" à Orléans.

6) Mars 2011: Exposé au séminaire de Dynamique Quantique et Classique au CPT (centre de physique théorique) à Marseille.

7) Avril 2011: Exposé au groupe de travail Scattering et Stabilité à l'université Bordeaux 1

8) Octobre 2011: Groupe de travail de Problèmes inverses et contrÃťle au LATP (Laboratoire d'Analyse, Topologie, Probabilités), Marseille.

9) Octobre 2011: Séminaire d'intérêt général: journée des nouveaux entrants du CPT, Marseille.

10) Novembre 2011: Séminaire de dynamique quantique et classique du CPT, Marseille.

11) Décembre 2011: Séminaire d'analyse et géometrie du LATP, Marseille.

12) Janvier 2012: Journées de Metz 2012 de l'Université de Lorraine, Metz.

13) Septembre 2013: Séminaire contrôle-EDP de Monastir, Tunisie.

14) Novembre 2013: Séminaire du groupe guide d'ondes et milieux starifiés (GOMS), Marseille.

15) Decembre 2013: Séminaire d'analyse et géometrie du LATP, Marseille.

16) Mai 2014: conférence au CIRM "Progrès récents dans l'analyse mathématique et numérique des problèmes inverses", Marseille.

17) Octobre 2014: Séminaire Géométrie, EDP et Physique Mathématique, Cergy.

18) Mai 2015: Conférence "Spectral and analytic inverse problems" lors du trimestre lié au programme sur les problèmes inverses à l'IHP, Paris.

19) Mai 2015: Séminaire de Physique Mathématique, Bordeaux.

20) Mai 2015: Journées du CPT, Toulon.

21) Juin 2015: Workshop "Inverse Problems for PDEs", Reims.

22) Octobre 2015: Séminaire problèmes inverses de l'Unversity College London, Londre, Royaume-Uni.

23) Avril 2016: FMSP lecture, University of Tokyo, Tokyo, Japon.

24) Mai 2016: Conference "Frontieres in mathematical sciences", Sharif University of technology, Téhéran, Iran.

25) Novembre 2016, conférence "Carleman estimates, unique continuation and applications", Unversity College London, Londre, Royaume-Uni.

26) Decembre 2016: Séminaire d'analyse et géometrie de l'I2M, Marseille.

27) Février 2017: Workshop on Fractional Differential Equations and Mathematical, University of Tokyo, Tokyo, Japon.

28) Mars 2017: Workshop on inverse problems, Universidad Autonoma de Madrid, Madrid, Espagne.

29) Avril 2017: Journées scientifiques de l'Université de Toulon, Toulon.

30) Mai 2017: Séminaire, Beijing Computational Science Research Center, Pékin, Chine.

31) Juin 2017: Séminaire de l'équipe physique mathématique, Institut Joseph Fourier, Grenoble.

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