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THÈSE

Pour obtenir le grade de

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préparée au sein du **l'Institut Fourier**
dans **l'École Doctorale MSTII**

Géométrie des variétés de Fano singulières et des fibrés projectifs sur une courbe

Thèse soutenue publiquement le **11 octobre 2017**,
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Abstract. This thesis is devoted to the geometry of Fano varieties and projective vector bundles over a smooth projective curve.

In the first part we study the geometry of mildly singular Fano varieties on which there is a prime divisor of Picard number 1. By studying the contractions associated to extremal rays in the Mori cone of these varieties, we provide a structure theorem in dimension 3 for varieties with maximal Picard number. Afterwards, we address the case of toric varieties and we extend the structure theorem to toric varieties of dimension greater than 3 and with maximal Picard number. Finally, we treat the lifting of extremal contractions to universal covering spaces in codimension 1.

In the second part we study Newton-Okounkov bodies on projective vector bundles over a smooth projective curve. Inspired by Wolfe's estimates used to compute the volume function on these varieties, we compute all Newton-Okounkov bodies with respect to linear flags and we study how these bodies depend on the Schubert cell decomposition with respect to linear flags which are compatible with the Harder-Narasimhan filtration of the bundle. Moreover, we characterize semi-stable vector bundles over smooth projective curves via Newton-Okounkov bodies.

Résumé. Cette thèse est consacrée à la géométrie des variétés de Fano et des fibrés projectifs sur une courbe projective lisse.

Dans la première partie on étudie la géométrie des variétés de Fano pas trop singulières admettant un diviseur premier de nombre de Picard 1. En étudiant les contractions associées aux rayons extrémaux dans le cône de Mori de ces variétés nous fournissons un théorème de structure en dimension 3 pour les variétés dont le nombre de Picard est maximal. Ensuite, nous traitons le cas des variétés toriques et nous étendons le théorème de structure aux variétés toriques de dimension supérieure à 3 dont le nombre de Picard est maximal. Enfin, nous traitons les relèvements des contractions extrémales aux espaces de revêtement universels en codimension 1.

Dans la deuxième partie on étudie les corps de Newton-Okounkov sur les fibrés projectifs sur une courbe projective lisse. En nous inspirant des estimations de Wolfe utilisées pour calculer la fonction de volume sur ces variétés, nous calculons tous les corps de Newton-Okounkov par rapport aux drapeaux linéaires et nous étudions comment ces corps dépendent de la décomposition en cellules de Schubert par rapport aux drapeaux linéaires compatibles avec la filtration de Harder-Narasimhan du fibré. De plus, nous caractérisons les fibrés vectoriels semi-stables sur une courbe projective lisse à l'aide des corps de Newton-Okounkov.

Mots-clés. Variétés de Fano, Programme de Modèles Minimaux (MMP), Variétés toriques, Schéma de Hilbert, Revêtements quasi-étales, Corps de Newton-Okounkov, Semi-stabilité de fibrés vectoriels, Variétés de drapeaux.

Classification Mathématique. 14C20, 14E30, 14H30, 14H60, 14J45, 14M25, 14M99.

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INTRODUCTION (VERSION FRANÇAISE)

Un des objectifs principaux de la géométrie algébrique est la classification des variétés projectives. Dans le cas d'une courbe X , nous distinguons les cas fonction du *genre* : $g(X) = 0$ ($X \cong \mathbb{P}^1$ et $-K_X$ est ample), $g = 1$ (X est une courbe elliptique et K_X est trivial) et $g \geq 2$ (X est une courbe générale et K_X est ample). En dimension 2, des résultats similaires peuvent être énoncés en regardant la classification de Enriques-Kodaira qui prend en compte la négativité (resp. trivialité, resp. positivité) de la classe canonique K_X d'une surface X . En dimension supérieure, le principe général pour la classification des variétés projectives lisses devrait être de regarder si la classe canonique K_X est négative, triviale ou positive. Cela s'inscrit dans le contexte de la théorie de Mori ou Programme de Modèles Minimaux (MMP).

Les principaux objets d'étude de cette thèse sont des exemples de variétés projectives de dimension de Kodaira négative définies sur un corps k algébriquement clos non-dénombrable de caractéristique zéro⁽¹⁾. Celle-ci est une version générale du cas négatif $K_X < 0$. Le Programme de Modèles Minimaux pour une variété de dimension de Kodaira négative devrait se terminer par un morphisme de type fibration dont les fibres générales sont des variétés (éventuellement singulières) à fibré anti-canonique ample.

La première partie des travaux de l'auteur porte sur les variétés projectives à fibré anti-canonique ample. Ces variétés sont appelées *variétés de Fano* et elles ont une géométrie très riche. Par exemple, le nombre de Picard de X qui est défini comme $\rho_X = \dim_{\mathbb{R}} N_1(X)_{\mathbb{R}}$, la dimension de l'espace de classes

1. La plupart des résultats restent vrais sur un corps algébriquement clos de caractéristique zéro quelconque, mais les preuves du Corollaire 2.3.2 et [LM09, §4.3] (utilisée implicitement dans la preuve du Théorème H) reposent sur le fait que le corps de base est non-dénombrable.

d'équivalence numérique de 1-cycles dans X à coefficients réels, coïncide avec le second nombre de Betti de X et donc c'est un invariant topologique. De plus, pour tout $n \geq 1$ il y a qu'un nombre fini de familles de variétés de Fano à déformation près : ceci a été établi dans le cas lisse par Kollár, Miyaoka et Mori dans son célèbre article [KMM92], tandis que le cas singulier (Conjecture BAB) a été traité dans la pré-publication récente [Bir16] de Birkar.

Cependant, même dans le cas lisse, la classification de toutes les familles à déformation près est loin d'être complète. En dimension 1, la seule courbe de Fano est \mathbb{P}^1 . En dimension 2, les surfaces de Fano lisses sont appelées surfaces de *del Pezzo* et elles sont isomorphes à l'une des 10 surfaces suivantes : $\mathbb{P}^1 \times \mathbb{P}^1$ ou l'éclatement de \mathbb{P}^2 en $0 \leq r \leq 9$ points généraux. En dimension supérieure, la classification est beaucoup plus compliquée. Par exemple, les variétés de Fano lisses de dimension 3 de nombre de Picard 1 ont été classifiées par Iskovskikh dans la série d'articles [Isk77, Isk78, Isk80], et celles pour lesquelles $\rho_X \geq 2$ ont été classifiées par Mori et Mukai dans [MM81, MM03]; cette classification donne 105 familles à déformation près de variétés de Fano lisses de dimension trois. La classification de variétés de Fano lisses de dimension supérieure à 4 reste ouverte. Pourtant, il est connu d'après l'article de Birkar, Cascini, Hacon et McKernan [BCHM10] que la Théorie de Mori s'applique très bien aux variétés de Fano.

Le premier objectif de cette thèse est l'étude de la géométrie de variétés de Fano pas trop singulières sur lesquelles il existe un diviseur de nombre de Picard 1. Les résultats suivants peuvent aussi être trouvés dans la pré-publication en ligne [Mon16].

Un premier résultat relié à ce problème a été établi par Bonavero, Campana et Wiśniewski dans les articles [Bon02] et [BCW02], où les auteurs ont classifié les variétés de Fano (toriques) de dimension $n \geq 3$ sur lesquelles il existe un diviseur isomorphe à l'espace projectif \mathbb{P}^{n-1} , puis ils ont utilisé ce résultat pour étudier des variétés (toriques) dont l'éclatement en un point est de Fano. Par exemple, dans le cas torique ils démontrent le résultat suivant.

Théorème 0.0.1 ([Bon02, Theo. 2]). — *Soit X une variété de Fano torique de dimension $n \geq 3$. Alors il existe un diviseur torique D dans X isomorphe à \mathbb{P}^{n-1} si et seulement si l'une des situations suivantes se produit :*

1. $X \cong \mathbb{P}^n$ et D est un sous-espace linéaire de codimension 1 dans X .

2. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}) \cong \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$ et D est une fibre de la projection vers \mathbb{P}^1 .
3. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a))$, où $0 \leq a \leq n-1$, et D est soit le diviseur $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$, soit le diviseur $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(a))$.
4. X est isomorphe à l'éclatement de $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a+1))$ le long d'un sous-espace linéaire \mathbb{P}^{n-2} contenu dans le diviseur $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$, où $0 \leq a \leq n-2$, et D est soit la transformée stricte du diviseur $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$, soit la transformée stricte du diviseur $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(a+1))$.

En particulier, cette classification donne $\rho_X \leq 3$. Quelques années plus tard, Tsukioka a utilisé dans [Tsu06] quelques arguments inspirés de [And85] et [BCW02] pour généraliser ces résultats et démontrer que toute variété de Fano lisse X de dimension $n \geq 3$ contenant un diviseur effectif premier de nombre de Picard 1 doit satisfaire $\rho_X \leq 3$.

La majoration $\rho_X \leq 3$ a été récemment établie par Della Noce dans [Del14, Rema. 5.5] lorsqu'on suppose que X est une variété de Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques, ayant au plus un nombre fini de points non-terminaux, et sous l'hypothèse plus générale de l'existence d'un diviseur effectif premier $D \subseteq X$ tel que l'espace vectoriel réel $N_1(D, X) := \text{Im}(N_1(D)_{\mathbb{R}} \rightarrow N_1(X)_{\mathbb{R}})$ de classes numériques de 1-cycles dans X qui sont équivalents à 1-cycles contenus dans D , soit de dimension 1.

Dans le cas lisse, Casagrande et Druel fournissent dans [CD15] une classification (et des exemples) de tous les cas de nombre de Picard maximal $\rho_X = 3$.

Théorème 0.0.2 ([CD15, Theo. 3.8]). — *Soit X une variété de Fano lisse de dimension $n \geq 3$ telle que $\rho_X = 3$. Soit $D \subseteq X$ un diviseur premier tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$. Alors X est isomorphe à l'éclatement d'une variété de Fano lisse $Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$ le long d'une sous-variété irréductible de dimension $n-2$ contenue dans une section du \mathbb{P}^1 -fibré $\pi : Y \rightarrow Z$, où Z est une variété de Fano lisse de dimension $n-1$ telle que $\rho_Z = 1$.*

Premièrement, nous rappelons dans la section §2.1 qu'une variété de Fano X , pas trop singulière, admet toujours un rayon extrémal $R \subseteq \overline{NE}(X)$ dont l'intersection avec un diviseur effectif donné est positive. Le reste de la section §2.1 est consacré à l'étude de ces contractions extrémales dans le cas où le diviseur donné est de nombre de Picard 1. Cela nous permet de prouver le résultat suivant dans la section §2.2.

Théorème A. — *Soit X une variété de Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques, ayant au plus un nombre fini de*

points non-terminaux. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 3$. Alors il existe un diagramme commutatif

$$\begin{array}{ccc}
 & X & \\
 \hat{\sigma} \swarrow & & \searrow \sigma \\
 \hat{Y} & & Y \\
 \hat{\pi} \searrow & & \swarrow \pi \\
 & Z &
 \end{array}$$

où σ (resp. $\hat{\sigma}$) correspond à une contraction divisorielle d'un rayon extrémal $R \subseteq \overline{NE}(X)$ (resp. $\hat{R} \subseteq \overline{NE}(X)$) qui est donnée par l'éclatement en codimension 2 d'une sous-variété irréductible de dimension $n - 2$, et φ est une contraction de type fibration, finie sur D , qui correspond à la face extrémale $R + \hat{R} \subseteq \overline{NE}(X)$. De plus, $D \cdot R > 0$, Y et \hat{Y} sont variétés \mathbb{Q} -factorielles à singularités canoniques, ayant au plus un nombre fini de points non-terminaux, Y est une variété de Fano et Z est une variété de Fano \mathbb{Q} -factorielle à singularités klt. En particulier, les singularités de Z sont rationnelles.

Les résultats de Cutkosky [Cut88] sur les contractions extrémales de variétés de dimension 3 à singularités Gorenstein terminales, ainsi que le résultat précédent, impliquent le corollaire suivant.

Corollaire B. — Soit X une variété de Fano Gorenstein \mathbb{Q} -factorielle de dimension 3 à singularités terminales. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 3$. Alors X est factorielle et est isomorphe à l'éclatement d'une variété de Fano lisse Y le long d'une courbe localement intersection complète $C \subseteq Y$. De plus, Y est isomorphe à $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$, où $0 \leq a \leq 2$.

Dans le cas $\rho_X = 2$, nous obtenons dans la section §2.3 la généralisation suivante de [CD15, Rema. 3.2, Prop. 3.3] dans le cas des variétés de Fano pas trop singulières X telles que $\rho_X = 2$ et dans lesquelles il existe un diviseur premier de nombre de Picard 1.

Théorème C. — Soit X une variété de Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques, ayant au plus un nombre fini de points non-terminaux. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 2$. Alors il existe deux possibilités :

1. Si D n'est pas nef, alors il existe une contraction extrême qui envoie D sur un point.
2. Si D est nef, alors $S = D^\perp \cap \text{NE}(X)$ est un rayon extrême. L'une des assertions suivantes doit être satisfaite :
 - a) cont_S est de type fibration vers \mathbb{P}^1 , et D est une fibre.
 - b) cont_S est une contraction divisorielle qui envoie son diviseur exceptionnel G sur un point, et telle que $G \cap D = \emptyset$.
 - c) cont_S est une petite contraction et il existe un flip $X \dashrightarrow X'$ et une contraction de type fibration $\psi : X' \rightarrow Y'$ dont la fibre générale est isomorphe à \mathbb{P}^1 , de degré anti-canonique égal à 2. De plus, ψ est finie sur la transformée stricte de D dans X' .

Afin d'étendre les résultats de classification aux variétés de dimension supérieure, nous allons nous restreindre au cas des variétés toriques. Dans ce cas, la description combinatoire du MMP pour les variétés toriques, traité dans la section §2.4, ainsi que certaines propriétés particulières aux variétés toriques, nous permettront de démontrer le résultat suivant dans la section §2.5.

Théorème D. — Soit X une variété torique qui est Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques, ayant au plus un nombre fini de points non-terminaux. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 3$. Alors il existe deux variétés toriques Y et Z qui sont Fano Gorenstein \mathbb{Q} -factorielles à singularités terminales et telles que :

1. $X \cong \overline{\text{Bl}}_A(Y)$, l'éclatement normalisé d'une sous-variété invariante $A \subseteq Y$ de dimension $n - 2$;
2. $Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$ avec $0 \leq a \leq i_Z - 1$, où i_Z est l'indice de Fano de Z et $\mathcal{O}_Z(1)$ est le générateur ample de $\text{Pic}(Z)$.

Si $\dim X \leq 4$, alors X est lisse et nous sommes dans le cadre du Théorème 0.0.1, cas (4).

Dans le cadre torique, nous obtenons dans la section §2.6 des résultats qui généralisent la description due à Bonavero des contractions extrêmes dans le cas $\rho_X = 2$ au cas des variétés toriques de Fano pas trop singulières. Si on suppose que X a des singularités canoniques isolées alors nous obtenons la classification suivante.

Théorème E. — Soit X une variété torique qui est Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques isolées. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 2$. Alors l'une des assertions suivantes doit être satisfaite :

1. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a))$, avec $0 \leq a \leq n - 1$. Autrement dit, nous sommes dans le cadre du Théorème 0.0.1, cas (3).
2. X est isomorphe à l'éclatement d'une variété torique Y le long d'une sous-variété invariante $A \subseteq Y$ de dimension $n - 2$, contenue dans le lieu lisse de Y . De plus, Y est isomorphe à l'une des variétés toriques suivantes :
 - (a) \mathbb{P}^n ,
 - (b) $\mathbb{P}(1^{n-1}, 2, n + 1)$ si n est pair, ou
 - (c) $\mathbb{P}(1^{n-1}, a, b)$, où $1 \leq a < b \leq n$ sont deux nombres premiers entre eux tels que $a|(n - 1 + b)$ et $b|(n - 1 + a)$.

En particulier, Y est une variété torique qui est Fano Gorenstein \mathbb{Q} -factorielle telle que $\rho_Y = 1$, et elle a au plus deux points singuliers. Réciproquement, l'éclatement d'une des variétés dans la liste précédente le long d'une sous-variété invariante $A \subseteq Y$ de dimension $n - 2$ contenue dans le lieu lisse de Y fournit une variété torique X qui satisfait les hypothèses précédentes.

De plus, dans le cas où la contraction est de type fibration nous obtenons le résultat suivant sans avoir besoin de l'hypothèse de singularités isolées.

Proposition F. — Soit X une variété torique qui est Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques, ayant au plus un nombre fini de points non-terminaux. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 2$. Soit $R \subseteq \overline{NE}(X)$ un rayon extrémal tel que $D \cdot R > 0$ et supposons que la contraction extrémale correspondante $\pi : X \rightarrow Y$ soit de type fibration. Alors $X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$. De plus, Y est une variété torique qui est Fano Gorenstein \mathbb{Q} -factorielle de dimension $n - 1$ à singularités terminales et d'indice de Fano i_Y , où $0 \leq a \leq i_Y - 1$. En particulier, X est à singularités terminales.

Enfin, la section §2.7 est consacrée à la démonstration du fait que les contractions extrémales étudiées dans la section §2.6 se relèvent aux revêtements universels quasi-étales, introduits par Buczyńska dans [Buc08] lorsqu'elle étudiait des variétés toriques de nombre de Picard 1. Voir Définition 2.7.8 pour la notion de Poly Weighted Space (PWS), introduite par Rossi et Terracini dans

[RT16] et le fait qu'elle coïncide avec celle de revêtement universel quasi-étale pour les variétés toriques \mathbb{Q} -factorielles de nombre de Picard arbitraire.

En particulier, nous obtenons la description suivante pour les contractions divisorielles des variétés toriques de Fano pas trop singulières de nombre de Picard 2. Il mérite d'être noté que même si la description combinatoire de ces contractions divisorielles est très simple (voir Lemme 2.4.1) et qu'elle coïncide avec celle de l'éclatement d'une sous-variété invariante de dimension $n - 2$ dans le cas lisse, il se peut que les morphismes ne soient pas globalement l'éclatement d'un faisceau cohérent d'idéaux associé à une sous-variété (réduite et irréductible), mais qu'ils soient seulement un éclatement en codimension 2 si les singularités ne sont pas isolées (voir Exemple 2.6.2).

Proposition G. — *Soit X une variété torique qui est Fano Gorenstein \mathbb{Q} -factorielle de dimension $n \geq 3$ à singularités canoniques, ayant au plus un nombre fini de points non-terminaux. Supposons qu'il existe un diviseur premier $D \subseteq X$ tel que $\dim_{\mathbb{R}} N_1(D, X) = 1$ et que $\rho_X = 2$. Soit $R \subseteq \overline{NE}(X)$ un rayon extrémal tel que $D \cdot R > 0$ et supposons que la contraction extrémale correspondante $\pi : X \rightarrow Y$ soit birationnelle. Alors il existe des poids $\lambda_0, \dots, \lambda_n \in \mathbb{Z}_{>0}$ et un carré cartésien de variétés toriques*

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\pi}} & \mathbb{P}(\lambda_0, \dots, \lambda_n) \\ \pi_X \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{\pi} & Y \end{array}$$

où les flèches verticales désignent les revêtements universels quasi-étales canoniques correspondants, et \widehat{X} est un PWS qui est Fano Gorenstein à singularités canoniques, ayant au plus un nombre fini de points non-terminaux, tel que $\rho_{\widehat{X}} = 2$. De plus, $\widehat{\pi} : \widehat{X} \rightarrow \mathbb{P}(\lambda_0, \dots, \lambda_n)$ est une contraction divisorielle qui envoie son diviseur exceptionnel $\widehat{E} \subseteq \widehat{X}$ sur une sous-variété invariante $\widehat{A} \subseteq \mathbb{P}(\lambda_0, \dots, \lambda_n)$ de dimension $n - 2$.

La deuxième partie des travaux de l'auteur porte sur des fibrés projectifs. Autrement dit, des variétés projectives lisses qui sont obtenues comme la projectivisation d'un fibré vectoriel E de rang $r \geq 2$ sur une variété de base S . Ce sont des exemples de variétés de dimension de Kodaira négative. Nous allons nous concentrer sur le cas le plus simple où la variété de base est une courbe projective lisse.

Le deuxième objectif de cette thèse est l'étude des corps de Newton-Okounkov sur les fibrés projectifs sur une courbe. Les résultats suivants peuvent aussi être trouvés dans la pré-publication en ligne [Mon17].

Soit C une courbe projective lisse et soit E un fibré vectoriel sur C de rang $r \geq 2$. Il est bien connu d'après le travail de Hartshorne [Har71] que l'information numérique provenant des propriétés de semi-stabilité de E peut être traduite en conditions de positivité. Plus précisément, Hartshorne a démontré qu'un fibré E est ample si et seulement si la pente minimale $\mu_{\min}(E)$ de E est positive. La forme du cône nef du fibré projectif $\mathbb{P}(E)$ peut être déduite de ce résultat (voir [Laz04, §6.4.B], par exemple).

De même, Miyaoka a prouvé dans [Miy87, Theo. 3.1] le résultat suivant, qui a été généralisé par Fulger dans [Ful11].

Théorème 0.0.3 (Miyaoka). — *Soient C et E comme ci-dessus. Les conditions suivantes sont équivalentes :*

- (1) E est semi-stable.
- (2) $\text{Nef}(\mathbb{P}(E)) = \text{Psef}(\mathbb{P}(E))$.

En général, la forme du cône pseudo-effectif de $\mathbb{P}(E)$ a été déterminée par Nakayama dans [Nak04, Chapter IV] au cours de la preuve du fait qu'il existe une décomposition de Zariski faible pour les diviseurs pseudo-effectifs dans $\mathbb{P}(E)$ (voir aussi [MDS15]). Ce cône est engendré par les classes numériques f et $\xi - \mu_{\max}(E)f$, où ξ est la classe du fibré en droites tautologique $\mathcal{O}_{\mathbb{P}(E)}(1)$, f est la classe d'une fibre de $\pi : \mathbb{P}(E) \rightarrow C$ et $\mu_{\max}(E)$ est la pente maximale de E .

Indirectement, le cône pseudo-effectif peut également être déduit des travaux de Wolfe [Wol05] et Chen [Che11], qui ont calculé explicitement la fonction de volume sur $\mathbb{P}(E)$. En fait, ils démontrent que pour tout $t \in \mathbb{R}$, le volume de la classe numérique $\xi - tf$ sur $\mathbb{P}(E)$ peut être exprimé en termes de l'information numérique provenant de la filtration de Harder-Narasimhan de E . Plus précisément, leurs résultats peuvent être résumés de la façon suivante.

Théorème 0.0.4. — *Soient C et E comme ci-dessus. Soit*

$$\text{HN}_{\bullet}(E) : 0 = E_{\ell} \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

la filtration de Harder-Narasimhan de E , avec quotients successifs semi-stables $Q_i = E_{i-1}/E_i$ de rang r_i et pente μ_i . Alors,

$$\mathrm{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{r-1}} \max \left\{ \sum_{j=1}^r s_j \lambda_j - t, 0 \right\} d\lambda,$$

où $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ est le simplexe standard de dimension $r - 1$ avec coordonnées $\lambda_1, \dots, \lambda_r$, $d\lambda$ est la mesure de Lebesgue standard telle que le volume de $\widehat{\Delta}_{r-1}$ soit $\frac{1}{(r-1)!}$, et $\mathbf{s} = (s_1, \dots, s_r)$ est le vecteur dans \mathbb{R}^r tel que la valeur μ_i apparait exactement r_i fois comme coordonnée de \mathbf{s} , et ces coordonnées apparaissent par ordre croissant.

Compte tenu du principe que l'information numérique encodée par la filtration de Harder-Narasimhan de E doit être reliée aux invariants numériques asymptotiques de $\mathbb{P}(E)$, nous étudions la géométrie des corps de Newton-Okounkov sur $\mathbb{P}(E)$. Ces corps convexes et compacts ont été introduits par Okounkov dans son article [Oko96] et ils ont été étudiés plus tard par Kaveh et Khovanskii [KK12] et par Lazarsfeld et Mustața [LM09], qui associent à chaque diviseur big sur une variété projective normale X de dimension r , et toute drapeau complet de sous-variétés Y_\bullet dans X satisfaisant certaines conditions, un corps convexe $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^r$ dépendant uniquement de la classe numérique du diviseur D . De plus, il existe un *corps de Newton-Okounkov global* $\Delta_{Y_\bullet}(X) \subseteq \mathbb{R}^r \times \mathbb{N}^1(X)_{\mathbb{R}}$ dont la fibre de $\Delta_{Y_\bullet}(X)$ au-dessus de toute classe rationnelle big $\eta \in \mathbb{N}^1(X)_{\mathbb{Q}}$ est donnée par $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^r \times \{\eta\}$.

Les corps de Newton-Okounkov des diviseurs big sur une surface réglée par rapport à des *drapeaux linéaires* (voir Définition 3.3.1) peuvent être calculés en utilisant la décomposition de Zariski (voir Exemple 3.3.2). En dimension supérieure, un calcul analogue nous permet de calculer ces corps sur $\mathbb{P}(E)$ lorsque le cône des diviseurs mobiles coïncide avec le cône des diviseurs nef (voir Remarque 3.3.5). En général, nous utiliserons des méthodes similaires à celles utilisées par Wolfe pour calculer la fonction de volume dans [Wol05]. Plus précisément, nous devons comprendre la filtration de Harder-Narasimhan des puissances symétriques $S^m E$ pour $m \geq 1$ et puis considérer certaines sous-filtrations.

Soit Y_\bullet^{HN} un *drapeau linéaire compatible avec la filtration de Harder-Narasimhan de E* (voir Définition 3.3.8) de composante divisorielle $\pi^{-1}(q) \cong \mathbb{P}^{r-1}$ et notons \mathbb{F}_r la variété de drapeaux complets qui paramètre les drapeaux complets de sous-espaces linéaires de \mathbb{P}^{r-1} . Le drapeau linéaire Y_\bullet^{HN} détermine

une décomposition de \mathbb{F}_r en *cellules de Schubert* (Voir §2 et Convention 3.3.9)

$$\mathbb{F}_r = \coprod_{w \in \mathfrak{S}_r} \Omega_w.$$

Avec les notations du Théorème 0.0.4 ci-dessus, considérons

$$\sigma = \underbrace{(\mu_\ell, \dots, \mu_\ell)}_{r_\ell \text{ fois}}, \underbrace{(\mu_{\ell-1}, \dots, \mu_{\ell-1})}_{r_{\ell-1} \text{ fois}}, \dots, \underbrace{(\mu_1, \dots, \mu_1)}_{r_1 \text{ fois}} \in \mathbb{Q}^r$$

et définissons pour chaque permutation $w \in \mathfrak{S}_r$ et chaque nombre réel $t \in \mathbb{R}$ le polytope suivant contenu dans le simplexe standard Δ_{r-1} de dimension $r-1$ dans \mathbb{R}^{r-1}

$$\square_t^w = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{w(i-1)} \nu_i + \sigma_{w(r)} \left(1 - \sum_{i=2}^r \nu_i \right) \geq t \right\}.$$

Alors nous démontrons le résultat suivant.

Théorème H. — Soient C une courbe projective lisse et E un fibré vectoriel sur C de rang $r \geq 2$. Alors pour tout drapeau linéaire Y_\bullet sur $\mathbb{P}(E)$ qui appartient à la cellule de Schubert Ω_w et pour toute classe rationnelle big $\eta = a(\xi - \mu_\ell f) + bf$ on a

$$\Delta_{Y_\bullet}(\eta) = \left\{ (\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq b, (\nu_2, \dots, \nu_r) \in a \square_{\mu_\ell - \frac{1}{a}(b - \nu_1)}^w \right\},$$

et donc le corps de Newton-Okounkov global de $\mathbb{P}(E)$ par rapport à Y_\bullet est donné par

$$\Delta_{Y_\bullet}(\mathbb{P}(E)) = \left\{ ((a(\xi - \mu_\ell f) + bf), (\nu_1, \dots, \nu_r)) \in \mathbb{N}^1(\mathbb{P}(E))_{\mathbb{R}} \times \mathbb{R}^r \text{ tels que } \right. \\ \left. 0 \leq \nu_1 \leq b \text{ et } (\nu_2, \dots, \nu_r) \in a \square_{\mu_\ell - \frac{1}{a}(b - \nu_1)}^w \right\}.$$

En particulier, le corps de Newton-Okounkov global $\Delta_{Y_\bullet}(\mathbb{P}(E))$ est un cône rationnel polyédral et il dépend uniquement du fibré vectoriel gradué $\text{gr}(\text{HN}_\bullet(E))$ associé à la filtration de Harder-Narasimhan de E .

De plus, nous obtenons la caractérisation suivante de la semi-stabilité en termes des corps de Newton-Okounkov.

Proposition I. — Soient C une courbe projective lisse et E un fibré vectoriel sur C de rang $r \geq 2$. Les conditions suivantes sont équivalentes :

- (1) E est semi-stable.
- (2) Pour toute classe rationnelle big $\eta = a(\xi - \mu_\ell f) + bf$ sur $\mathbb{P}(E)$ et pour tout drapeau linéaire Y_\bullet sur $\mathbb{P}(E)$ on a $\Delta_{Y_\bullet}(\eta) = [0, b] \times a \Delta_{r-1} \subseteq \mathbb{R}^r$.

Ici $a\Delta_{r-1} = \{(\nu_2, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^{r-1} \mid \sum_{i=2}^r \nu_i \leq a\}$ désigne le simplexe standard de dimension $r - 1$ dans \mathbb{R}^{r-1} de côté de longueur a .

Le schéma de la deuxième partie de cette thèse est le suivant. Tout d'abord, nous rappelons dans section §3.1 quelques définitions et des résultats bien connus sur les corps de Newton-Okounkov et sur la semi-stabilité de fibrés vectoriels sur une courbe. Ensuite, la section §3.2 est consacrée aux différents cônes des diviseurs sur $\mathbb{P}(E)$ ainsi qu'à des résultats concernant leur volume et leur volume restreint. Enfin, nous démontrons le Théorème H et la Proposition I dans la section §3.3.

INTRODUCTION

One of the main purposes of algebraic geometry is the classification of projective varieties. In the case of curves, there is a clear distinction in terms of the *genus* of a given curve X , which leads to distinguish between the case $g(X) = 0$ ($X \cong \mathbb{P}^1$ and $-K_X$ is ample), $g = 1$ (X is an elliptic curve and K_X is trivial) and $g \geq 2$ (X is a general curve and K_X is ample). In dimension 2, similar results can be stated in regard of the so called Enriques-Kodaira classification which takes account on the negativity (resp. triviality, resp. positivity) of the canonical class K_X of surfaces X . In higher dimension, the general principle is the classification of projective manifolds should be carried out by looking on whether the canonical class K_X is negative, trivial or positive. This fits in the context of the so called Mori theory or Minimal Model Program (MMP).

The main objects of study of this thesis are some particular projective algebraic varieties with negative Kodaira dimension defined over an uncountable algebraically closed field k of characteristic zero⁽¹⁾. This is a general version of the negative case $K_X < 0$ above. The Minimal Model Program for a variety with negative Kodaira dimension is expected to end up with a fiber type morphism whose general fibers are possibly singular varieties with ample anti-canonical class.

The first part of author's work concerns projective varieties with $-K_X$ ample. These varieties are called *Fano varieties*, and they have a very rich geometry. For instance, the Picard number of X which is defined to be $\rho_X = \dim_{\mathbb{R}} N_1(X)_{\mathbb{R}}$, the real dimension of numerical equivalence classes of

1. Most of the results are true over an arbitrary algebraically closed field of characteristic zero, but for Corollary 2.3.2 and [LM09, §4.3] (implicitly used in the proof of Theorem H) to hold, the field must be also uncountable.

1-cycles on X with real coefficients, coincides with the second Betti number of X and hence it is a topological invariant. Moreover, for every $n \geq 1$ there are only finitely many deformation families of Fano varieties: this was established in the smooth case by Kollár, Miyaoka and Mori in their celebrated article [KMM92], while the singular case (BAB Conjecture) was treated in the recent pre-publication [Bir16] by Birkar.

However, even in the smooth case, the classification of all deformation families is far from being complete. In dimension 1, the only Fano curve is \mathbb{P}^1 . In dimension 2, smooth Fano surfaces are commonly called *del Pezzo* surfaces and they are isomorphic to one of the 10 following surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of \mathbb{P}^2 in $0 \leq r \leq 9$ general points. In higher dimension the classification is much more involved. For instance, smooth Fano threefolds with Picard number 1 were classified by Iskovskikh in the series of articles [Isk77, Isk78, Isk80], while smooth Fano threefolds with $\rho_X \geq 2$ were classified by Mori and Mukai in [MM81, MM03]; this classification leads to 105 deformation families of Fano manifolds in dimension 3. The classification of Fano manifolds of dimension ≥ 4 is still open. However, it follows from the work of Birkar, Cascini, Hacon and McKernan [BCHM10] that Fano varieties are very well-behaved from the Mori theory point of view.

The first aim of this thesis is to study the geometry of mildly singular Fano varieties on which there is a prime divisor of Picard number 1. This can be found in author's electronic pre-publication [Mon16].

A first related result is given by Bonavero, Campana and Wiśniewski in the sequel of articles [Bon02] and [BCW02], where the authors classified (toric) Fano varieties of dimension $n \geq 3$ on which there is a divisor isomorphic to \mathbb{P}^{n-1} and later used these results to study (toric) varieties whose blow-up at a point is Fano. For instance, in the toric case we have the following result.

Theorem 0.0.1 ([Bon02, Theo. 2]). — *Let X be a smooth toric Fano variety of dimension $n \geq 3$. Then, there exists a toric divisor D of X isomorphic to \mathbb{P}^{n-1} if and only if one of the following situations occurs:*

1. $X \cong \mathbb{P}^n$ and D is a linear codimension 1 subspace of X .
2. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}) \cong \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$ and D is a fiber of the projection on \mathbb{P}^1 .
3. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a))$, where $0 \leq a \leq n-1$, and D is either the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$ or the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(a))$.

4. X is isomorphic to the blow-up of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a+1))$ along a linear \mathbb{P}^{n-2} contained in the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$, where $0 \leq a \leq n-2$, and D is either the strict transform of the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$ or the strict transform of the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(a+1))$.

In particular, this classification leads $\rho_X \leq 3$. Some years later, Tsukioka in [Tsu06] used some arguments from [And85] and [BCW02] to generalize these results and proved that a smooth Fano variety X of dimension $n \geq 3$ containing an effective prime divisor of Picard number 1 must satisfy $\rho_X \leq 3$.

The bound $\rho_X \leq 3$ was recently proved by Della Noce in [Del14, Rema. 5.5], when X is supposed to be a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities, with at most finitely many non-terminal points, and under the more general assumption of the existence of an effective prime divisor $D \subseteq X$ such that the real vector space $N_1(D, X) := \text{Im}(N_1(D)_{\mathbb{R}} \rightarrow N_1(X)_{\mathbb{R}})$ of numerical classes of 1-cycles on X that are equivalent to 1-cycles on D , is one-dimensional.

In the smooth case, Casagrande and Druel provide in [CD15] a classification (and examples) of all cases with maximal Picard number $\rho_X = 3$.

Theorem 0.0.2 ([CD15, Theo. 3.8]). — *Let X be a Fano manifold of dimension $n \geq 3$ and $\rho_X = 3$. Let $D \subseteq X$ be a prime divisor with $\dim_{\mathbb{R}} N_1(D, X) = 1$. Then X is isomorphic to the blow-up of a Fano manifold $Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$ along an irreducible subvariety of dimension $n-2$ contained in a section of the \mathbb{P}^1 -bundle $\pi : Y \rightarrow Z$, where Z is a Fano manifold of dimension $n-1$ and $\rho_Z = 1$.*

Firstly, we recall in section §2.1 that a mildly singular Fano variety X always has an extremal ray $R \subseteq \overline{NE}(X)$ whose intersection with a given effective divisor is positive. The rest of section §2.1 is devoted to the study of these extremal contractions in the case that the given divisor has Picard number 1. This allows us to prove the following result in section §2.2.

Theorem A. — *Let X be a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that*

$\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 3$. Then, there is a commutative diagram

$$\begin{array}{ccc}
 & X & \\
 \hat{\sigma} \swarrow & & \searrow \sigma \\
 \hat{Y} & & Y \\
 \hat{\pi} \searrow & \downarrow \varphi & \swarrow \pi \\
 & Z &
 \end{array}$$

where σ (resp. $\hat{\sigma}$) corresponds to a divisorial contraction of an extremal ray $R \subseteq \overline{\text{NE}}(X)$ (resp. $\hat{R} \subseteq \overline{\text{NE}}(X)$) which is the blow-up in codimension 2 of an irreducible subvariety of dimension $n - 2$, and φ is a contraction of fiber type, finite over D , corresponding to the face $R + \hat{R} \subseteq \overline{\text{NE}}(X)$. Moreover, $D \cdot R > 0$, Y and \hat{Y} are \mathbb{Q} -factorial varieties with canonical singularities and with at most finitely many non-terminal points, Y is Fano and Z is a \mathbb{Q} -factorial klt Fano variety. In particular, Z has only rational singularities.

The results of Cutkosky on the contractions of terminal Gorenstein threefolds [Cut88], together with the previous result imply the following corollary.

Corollary B. — *Let X be a \mathbb{Q} -factorial Gorenstein Fano threefold with terminal singularities. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 3$. Then, X is factorial and it can be realized as the blow-up of a smooth Fano threefold Y along a locally complete intersection curve $C \subseteq Y$. Moreover, Y is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$, where $0 \leq a \leq 2$.*

In the case $\rho_X = 2$, we obtain in section §2.3 the following extension of [CD15, Rema. 3.2, Prop. 3.3] to mildly singular Fano varieties X with $\rho_X = 2$, on which there is an effective prime divisor of Picard number 1.

Theorem C. — *Let X be a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 2$. There are two possibilities:*

1. *If D is not nef, then there is an extremal contraction sending D to a point.*
2. *If D is nef, then $S = D^\perp \cap \text{NE}(X)$ is an extremal ray. One of the following assertions must hold:*
 - a) *cont_S is of fiber type onto \mathbb{P}^1 , and D is a fiber.*

- b) cont_G is a divisorial contraction sending its exceptional divisor G to a point, and such that $G \cap D = \emptyset$.
- c) cont_G is a small contraction and there is a flip $X \dashrightarrow X'$ and a contraction of fiber type $\psi : X' \rightarrow Y'$ such that the general fiber is isomorphic to \mathbb{P}^1 , with anticanonical degree 2. Moreover, ψ is finite over the strict transform of D in X' .

In order to extend the classification results to higher dimensions, we will restrict ourselves to the case of toric varieties. In that case, the combinatorial description of the MMP for toric varieties treated in section §2.4, as well as some particular properties of them, will allow us to prove the following result in section §2.5.

Theorem D. — *Let X be a \mathbb{Q} -factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 3$. Then, there exist \mathbb{Q} -factorial Gorenstein toric Fano varieties Y and Z , with terminal singularities, such that*

1. $X \cong \overline{\text{Bl}}_A(Y)$, the normalized blow-up of an invariant toric subvariety $A \subseteq Y$ of dimension $n - 2$;
2. $Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$ with $0 \leq a \leq i_Z - 1$, where i_Z is the Fano index of Z and $\mathcal{O}_Z(1)$ is the ample generator of $\text{Pic}(Z)$.

If $\dim X \leq 4$, then X is smooth and we are in the situation of Theorem 0.0.1, case (4).

In the toric setting, we obtain in section §2.6 results that extend Bonavero's description of the extremal contractions in the case $\rho_X = 2$ to mildly singular toric Fano varieties. If X is supposed to have isolated canonical singularities then we obtain the following classification.

Theorem E. — *Let X be a \mathbb{Q} -factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with isolated canonical singularities. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 2$. Then, either*

1. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a))$, with $0 \leq a \leq n - 1$. In other words, we are in the situation of Theorem 0.0.1, case (3).
2. X is isomorphic to the blow-up of a toric variety Y along an invariant subvariety $A \subseteq Y$ of dimension $n - 2$, contained in the smooth locus of Y . Moreover, Y is isomorphic to either

- (a) \mathbb{P}^n ,
- (b) $\mathbb{P}(1^{n-1}, 2, n+1)$ if n is even, or
- (c) $\mathbb{P}(1^{n-1}, a, b)$, where $1 \leq a < b \leq n$ are two relatively prime integers such that $a|(n-1+b)$ and $b|(n-1+a)$.

In particular, Y is a \mathbb{Q} -factorial Gorenstein Fano variety with $\rho_Y = 1$ and it has at most two singular points. Conversely, the blow-up of any of the listed varieties Y along an invariant irreducible subvariety $A \subseteq Y$ of dimension $n-2$ and contained in the smooth locus of Y , leads to a toric variety X satisfying the hypothesis.

Moreover, in the case of contractions of fiber type we obtain the following result without the assumption of isolated singularities.

Proposition F. — *Let X be a \mathbb{Q} -factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 2$. Let $R \subseteq \overline{NE}(X)$ be an extremal ray such that $D \cdot R > 0$ and assume that the corresponding extremal contraction $\pi : X \rightarrow Y$ is of fiber type. Then, $X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$. Moreover, Y is a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n-1$ with terminal singularities and Fano index i_Y , and $0 \leq a \leq i_Y - 1$. In particular, X has only terminal singularities.*

Finally, section §2.7 is devoted to show that the extremal contractions studied in section §2.6 lift to *quasi-étale universal covers*, introduced by Buczyńska in [Buc08] in order to study toric varieties of Picard number 1. See Definition 2.7.8 for the notion of Poly Weighted Space (PWS), introduced by Rossi and Terracini in [RT16] and proved to be universal covering spaces in codimension 1 for \mathbb{Q} -factorial toric varieties of arbitrary Picard number.

In particular, we obtain the following description of divisorial contractions of toric mildly Fano varieties with Picard number 2. It should be noticed that even if the combinatorial description of these divisorial contractions is very simple (see Lemma 2.4.1) and it coincides with the one of the blow-up of a subvariety of dimension $n-2$ in the smooth case, it may happen that the morphisms are not globally a blow-up of the coherent sheaf of ideals of a (irreducible and reduced) subvariety but only a blow-up in codimension 2 if the singularities are not isolated (see Example 2.6.2).

Proposition G. — *Let X be a \mathbb{Q} -factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many*

non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 2$. Let $R \subseteq \overline{NE}(X)$ be an extremal ray such that $D \cdot R > 0$ and let us denote by $\pi : X \rightarrow Y$ the corresponding extremal contraction. Assume that π is birational. Then there exist weights $\lambda_0, \dots, \lambda_n \in \mathbb{Z}_{>0}$ and a cartesian diagram of toric varieties

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\pi}} & \mathbb{P}(\lambda_0, \dots, \lambda_n) \\ \pi_X \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{\pi} & Y \end{array}$$

where vertical arrows denote the corresponding canonical quasi-étale universal covers, and \widehat{X} is a Gorenstein Fano PWS with canonical singularities and with at most finitely many non-terminal points such that $\rho_{\widehat{X}} = 2$. Moreover, $\widehat{\pi} : \widehat{X} \rightarrow \mathbb{P}(\lambda_0, \dots, \lambda_n)$ is a divisorial contraction sending its exceptional divisor $\widehat{E} \subseteq \widehat{X}$ onto an invariant subvariety $\widehat{A} \subseteq \mathbb{P}(\lambda_0, \dots, \lambda_n)$ of dimension $n - 2$.

The second part of author's work concerns projective manifolds X which are projective vector bundles. Namely, obtained as the projectivization of a vector bundle E of rank $r \geq 2$ over a base variety S . They are examples of varieties of negative Kodaira dimension. We will focus our attention in simplest case where the base variety is a smooth projective curve.

The second aim of this thesis is to study Newton-Okounkov bodies on projective vector bundles over curves. This can be found in author's electronic pre-publication [Mon17].

Let C be a smooth projective curve and let E be vector bundle over C of rank $r \geq 2$. It is well-known since Hartshorne's work [Har71] that numerical information coming from semi-stability properties of E can be translated into positivity conditions. Namely, Hartshorne proved that E is an ample vector bundle if and only if $\mu_{\min}(E)$, the minimal slope of E , is strictly positive. The shape of the nef cone of the projective vector bundle $\mathbb{P}(E)$ can be deduced from this result (see [Laz04, §6.4.B], for instance).

Similarly, Miyaoka proved in [Miy87, Theo. 3.1] the following result, which was generalized by Fulger in [Ful11].

Theorem 0.0.3 (Miyaoka). — *Let C and E be as above. The following conditions are equivalent:*

- (1) E is semi-stable.

(2) $\text{Nef}(\mathbb{P}(E)) = \text{Psef}(\mathbb{P}(E))$.

In general, the shape of the pseudo-effective cone of $\mathbb{P}(E)$ was determined by Nakayama in [Nak04, Chapter IV] in the course of the proof of the fact that exists a weak Zariski decomposition of pseudo-effective divisors on $\mathbb{P}(E)$ (see also [MDS15]). This cone is spanned by the numerical classes f and $\xi - \mu_{\max}(E)f$, where ξ is the class of the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, f is the class of a fiber of $\pi : \mathbb{P}(E) \rightarrow C$ and $\mu_{\max}(E)$ is the maximal slope of E .

Indirectly, the pseudo-effective cone can be also deduced from the work of Wolfe [Wol05] and Chen [Che11], who explicitly computed the volume function on $\mathbb{P}(E)$. In fact, they showed that for every $t \in \mathbb{R}$, the volume of the numerical class $\xi - tf$ on $\mathbb{P}(E)$ can be expressed in terms of numerical information coming from the Harder-Narasimhan filtration of E . More precisely, their results can be summarized as follows.

Theorem 0.0.4. — *Let C and E as above. Let*

$$\text{HN}_{\bullet}(E) : 0 = E_{\ell} \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

be the Harder-Narasimhan filtration of E with successive semi-stable quotients $Q_i = E_{i-1}/E_i$ of rank r_i and slope μ_i . Then,

$$\text{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{r-1}} \max \left\{ \sum_{j=1}^r s_j \lambda_j - t, 0 \right\} d\lambda,$$

where $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ is the standard $r-1$ -simplex with coordinates $\lambda_1, \dots, \lambda_r$, $d\lambda$ is the standard induced Lebesgue measure for which $\widehat{\Delta}_{r-1}$ has volume $\frac{1}{(r-1)!}$, and $\mathbf{s} = (s_1, \dots, s_r)$ is a vector in \mathbb{R}^r such that the value μ_i appears exactly r_i times in the coordinates of \mathbf{s} , and their appear in increasing order.

Following the idea that numerical information encoded by the Harder-Narasimhan filtration of E should be related to asymptotic numerical invariants of $\mathbb{P}(E)$, we study the geometry of Newton-Okounkov bodies on $\mathbb{P}(E)$. These compact convex bodies were introduced by Okounkov in his original article [Oko96] and they were studied later on by Kaveh and Khovanskii [KK12] and Lazarsfeld and Mustața [LM09], who associated to any big divisor D on a normal projective variety X of dimension r , and any complete flag of subvarieties Y_{\bullet} on X satisfying suitable conditions, a convex body $\Delta_{Y_{\bullet}}(D) \subseteq \mathbb{R}^r$ depending only on the numerical equivalence class of D . Moreover, there exists a *global Newton-Okounkov body* $\Delta_{Y_{\bullet}}(X) \subseteq \mathbb{R}^r \times \mathbb{N}^1(X)_{\mathbb{R}}$

such that the slice of $\Delta_{Y_\bullet}(X)$ over any big rational class $\eta \in N^1(X)_{\mathbb{Q}}$ is given by $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^r \times \{\eta\}$.

Newton-Okounkov bodies of big divisors on ruled surfaces with respect to *linear flags* (see Definition 3.3.1) can be computed via Zariski decomposition (see Example 3.3.2). In higher dimension, an analogous computation allow us to compute these bodies on $\mathbb{P}(E)$ whenever the cone of movable divisors and the cone of nef divisors coincide (see Remark 3.3.5). In general, we will use methods similar to those used by Wolfe to compute the volume function in [Wol105]. More precisely, we will need to understand the Harder-Narasimhan filtration of the symmetric products $S^m E$ for $m \geq 1$ and then to consider suitable refinements of these filtrations.

Let Y_\bullet^{HN} be a *linear flag which is compatible with the Harder-Narasimhan filtration of E* (see Definition 3.3.8) with divisorial component $\pi^{-1}(q) \cong \mathbb{P}^{r-1}$ and denote by \mathbb{F}_r be the full flag variety parametrizing complete flags of linear subspaces of \mathbb{P}^{r-1} . The linear flag Y_\bullet^{HN} determines a decomposition of \mathbb{F}_r into *Schubert cells* (see §2 and Convention 3.3.9)

$$\mathbb{F}_r = \coprod_{w \in \mathfrak{S}_r} \Omega_w.$$

With the notation of Theorem 0.0.4 above, let us consider

$$\sigma = \underbrace{(\mu_\ell, \dots, \mu_\ell)}_{r_\ell \text{ times}}, \underbrace{(\mu_{\ell-1}, \dots, \mu_{\ell-1})}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{(\mu_1, \dots, \mu_1)}_{r_1 \text{ times}} \in \mathbb{Q}^r$$

and define for each permutation $w \in \mathfrak{S}_r$ and each real number $t \in \mathbb{R}$ the following polytope inside the full dimensional standard simplex Δ_{r-1} in \mathbb{R}^{r-1}

$$\square_t^w = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{w(i-1)} \nu_i + \sigma_{w(r)} \left(1 - \sum_{i=2}^r \nu_i \right) \geq t \right\}.$$

Then, we prove the following result.

Theorem H. — *Let C be a smooth projective curve and let E be a vector bundle over C of rank $r \geq 2$. Then, for every linear flag Y_\bullet on $\mathbb{P}(E)$ that belongs to the Schubert cell Ω_w and every big rational class $\eta = a(\xi - \mu_\ell f) + bf$ we have that*

$$\Delta_{Y_\bullet}(\eta) = \left\{ (\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq b, (\nu_2, \dots, \nu_r) \in a \square_{\mu_\ell - \frac{1}{a}(b - \nu_1)}^w \right\},$$

and hence the global Okounkov body of $\mathbb{P}(E)$ with respect to Y_\bullet is given by

$$\Delta_{Y_\bullet}(\mathbb{P}(E)) = \left\{ ((a(\xi - \mu_\ell f) + bf), (\nu_1, \dots, \nu_r)) \in N^1(\mathbb{P}(E))_{\mathbb{R}} \times \mathbb{R}^r \text{ such that } \right. \\ \left. 0 \leq \nu_1 \leq b \text{ and } (\nu_2, \dots, \nu_r) \in a \square_{\mu_\ell - \frac{1}{a}(b - \nu_1)}^w \right\}.$$

In particular, the global Okounkov body $\Delta_{Y_\bullet}(\mathbb{P}(E))$ is a rational polyhedral cone and it depends only on $\text{gr}(\text{HN}_\bullet(E))$, the graded vector bundle associated to the Harder-Narasimhan filtration of E .

Moreover, we obtain the following characterization of semi-stability in terms of Newton-Okounkov bodies.

Proposition I. — *Let C be a smooth projective curve and let E be a vector bundle over C of rank $r \geq 2$. The following conditions are then equivalent:*

- (1) *E is semi-stable.*
- (2) *For every big rational class $\eta = a(\xi - \mu_\ell f) + bf$ on $\mathbb{P}(E)$ and every linear flag Y_\bullet on $\mathbb{P}(E)$ we have that $\Delta_{Y_\bullet}(\eta) = [0, b] \times a\Delta_{r-1} \subseteq \mathbb{R}^r$.*

Here, $a\Delta_{r-1} = \{(\nu_2, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^{r-1} \mid \sum_{i=2}^r \nu_i \leq a\}$ is the full dimensional standard $(r-1)$ -simplex with side length a .

The outline of the second part of this thesis is as follows. First of all, we recall in section §3.1 some definitions and well-known results about Newton-Okounkov bodies and semi-stability of vector bundles over curves. Secondly, section §3.2 is devoted to the different cones of divisors on $\mathbb{P}(E)$ as well as results concerning their volume and restricted volume. Finally, we prove both Theorem H and Proposition I in section §3.3.

CHAPTER 1

NOTATION AND PRELIMINARY RESULTS

This chapter is devoted to establish the notation that we will use through this thesis and to recall some previous results.

For us, all varieties will be assumed to be reduced and irreducible schemes of finite type over an uncountable algebraically closed field k of characteristic zero. The smooth locus of an algebraic variety X will be denoted by $X_{\text{reg}} \subseteq X$, while $\text{Sing}(X) = X \setminus X_{\text{reg}}$ denotes its singular locus.

By a \mathbb{P}^1 -bundle we mean a smooth morphism all of whose fibers are isomorphic to \mathbb{P}^1 .

Let E be a locally free sheaf on a variety X . We follow Grothendieck's convention and we define the *projectivization* $\mathbb{P}_X(E) = \mathbb{P}(E)$ of E to be $\mathbf{Proj}_{\mathcal{O}_X} \bigoplus_{m \geq 0} S^m E$, where $S^m E$ denotes the m th symmetric power of E . This variety is endowed with a natural projection $\pi : \mathbb{P}(E) \rightarrow X$ and a tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$.

1.1. Cones of curves and divisors

Let X be a normal projective variety. For us, a *divisor* will always be a Weil divisor. A divisor is called \mathbb{Q} -Cartier if there exists a positive multiple which is Cartier, and we say that X is \mathbb{Q} -factorial if every divisor is \mathbb{Q} -Cartier. The class group of X , denoted by $\text{Cl}(X)$, is the group of Weil divisors on X modulo linear equivalence. Similarly, the Picard group of X , denoted by $\text{Pic}(X)$, is the group of Cartier divisors on X modulo linear equivalence. For a normal projective variety, there is a natural inclusion

$$\text{Pic}(X) \longrightarrow \text{Cl}(X),$$

which is not an isomorphism in general. If X is smooth then $\text{Pic}(X) \cong \text{Cl}(X)$, in which case we will not distinguish between Weil and Cartier divisors.

If C is a projective curve on X and $D \in \text{Pic}(X)$, then we define the intersection number of D and C to be $\deg(\nu^*D) \in \mathbb{Z}$, where $\nu : \bar{C} \rightarrow C$ is the normalization of C . Let us denote by $Z_1(X)$ the free abelian group generated by projective curves on X and let us call the elements in this group *1-cycles*. Then, we can extend by linearity the intersection product in order to obtain a pairing

$$\begin{aligned} \text{Pic}(X) \times Z_1(X) &\longrightarrow \mathbb{Z} \\ (D, C) &\longmapsto D \cdot C \end{aligned}$$

We say that two Cartier divisors D_1 and D_2 are numerically equivalent if $D_1 \cdot C = D_2 \cdot C$ for every 1-cycle C , and we will use the notation $D_1 \equiv D_2$ in that case. In the same way, we say that two 1-cycles C_1 and C_2 are numerically equivalent if $D \cdot C_1 = D \cdot C_2$ for every Cartier divisor D , and we will use the notation $C_1 \equiv C_2$ in that case. We denote by $N^1(X)$ the group of numerical equivalence classes of divisors on X and we define $N^1(X)_{\mathbb{k}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{k}$, for $\mathbb{k} = \mathbb{Q}$ or \mathbb{R} , to be the \mathbb{k} -vector space spanned by \mathbb{Q} -Cartier divisors modulo numerical equivalence. Dually, we denote by $N_1(X)$ the group of numerical equivalence classes of 1-cycles on X and $N_1(X)_{\mathbb{k}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{k}$, the \mathbb{k} -vector space spanned by 1-cycles modulo numerical equivalence, for $\mathbb{k} = \mathbb{Q}$ or \mathbb{R} .

The intersection pairing induce therefore a non-degenerated bilinear form

$$\begin{aligned} N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ ([D], [C]) &\longmapsto [D] \cdot [C] \end{aligned}$$

where $[C]$ (resp. $[D]$) denote the numerical class of C (resp. D) in $N_1(X)_{\mathbb{R}}$ (resp. $N^1(X)_{\mathbb{R}}$). We will only insist on the fact that we are considering numerical classes of 1-cycles and hence $[D] \cdot [C]$ will be written as $D \cdot [C]$. In particular, $\dim_{\mathbb{R}} N_1(X)_{\mathbb{R}} = \dim_{\mathbb{R}} N^1(X)_{\mathbb{R}}$. This dimension is called the Picard number of X , denoted by ρ_X , and it is a classical fact that it is finite.

Let $Z \subseteq X$ be a closed subset and $\iota : Z \rightarrow X$ be the natural inclusion, we define

$$N_1(Z, X) = \iota_* N_1(Z)_{\mathbb{R}} \subseteq N_1(X)_{\mathbb{R}}.$$

Let $\text{NE}(X) \subseteq N_1(X)_{\mathbb{R}}$ be the convex cone generated by the classes of effective 1-cycles, i.e., 1-cycles with non-negative coefficients, and $\overline{\text{NE}}(X)$ be its topological closure. The cone $\overline{\text{NE}}(X)$ is often called the *Mori cone* of X .

Definition 1.1.1 (Nef divisor). — A \mathbb{Q} -Cartier divisor D on X is said to be *numerically eventually free* (or simply, *nef*) if $D \cdot [C] \geq 0$ for every $[C] \in \overline{\text{NE}}(X)$.

We define $\text{Nef}(X)$ (resp. $\text{Amp}(X)$) to be the convex cone in $N^1(X)_{\mathbb{R}}$ generated by the classes of nef divisors (resp. ample divisors). These cones are related by Kleiman's criterion of ampleness.

Theorem 1.1.2 ([Kle66, Prop. IV 2.2], [Laz04, Theo. 1.4.23])

A Cartier divisor D is ample if and only if $D \cdot [C] > 0$ for every class $[C] \in \overline{\text{NE}}(X) \setminus \{0\}$. In particular,

$$\text{int}(\text{Nef}(X)) = \text{Amp}(X) \quad \text{and} \quad \overline{\text{Amp}(X)} = \text{Nef}(X).$$

Definition 1.1.3 (Big divisor). — A \mathbb{Q} -Cartier divisor D on X is said to be *big* if there exists an effective \mathbb{Q} -divisor E such that $D - E$ is ample.

We denote by $\text{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$ the open convex cone of big numerical classes.

Definition 1.1.4 (Pseudo-effective divisor). — A numerical class η in $N^1(X)_{\mathbb{R}}$ is called *pseudo-effective* if it can be written as the limit of classes of effective \mathbb{R} -divisors.

The pseudo-effective cone is the closure of the big cone: $\overline{\text{Big}}(X) = \text{Psef}(X)$ (see [Laz04, Theo. 2.2.26], for instance).

One last important cone will be the *movable cone* (cf. *modified nef cone* in [Bou04, §2.7]) defined by Y. Kawamata in [Kaw88, §2], and independently used by S. Boucksom [Bou04] and N. Nakayama [Nak04] to introduce the notion of divisorial Zariski decomposition or σ -decomposition. We follow N. Nakayama's presentation from [Nak04, Chapter III].

Definition 1.1.5 (Movable divisor). — Let X be a normal projective variety. Let $\text{Mov}'(X)$ be the convex cone in $N^1(X)_{\mathbb{R}}$ generated by the classes $c_1(L)$ of all fixed part free line bundles L on X (namely, whose base locus has codimension at least two). We denote its closure by $\overline{\text{Mov}}(X)$, which is called the *movable cone*⁽¹⁾ of X .

We can also define ampleness in the relative setting. Let $f : X \rightarrow Y$ be a projective morphism and let $N_1(X \setminus Y)_{\mathbb{R}}$ be the real vector space generated by irreducible curves $C \subseteq X$ which are contracted to a point by f , modulo

1. The notation $\text{Mov}(X)$ is often reserved to the interior of $\overline{\text{Mov}}(X)$, which is called the *strictly movable cone* of X .

numerical equivalence. Let $\overline{NE}(X \setminus Y)$ be the convex closed cone in $N_1(X \setminus Y)_{\mathbb{R}}$ generated by the classes of effective curves which are contracted by f . We say that a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on X is *f-ample* if the restriction of D to every irreducible component of each fiber of f is an ample divisor. Equivalently, the Relative Kleiman's criterion of ampleness [KM98, Theo. 1.44] says that D is *f-ample* if and only if $D \cdot [C] > 0$ for every $[C] \in \overline{NE}(X \setminus Y) \setminus \{0\}$.

Let us recall that the *stable base locus* of a \mathbb{Q} -divisor D on X is the closed set

$$\mathbf{B}(D) = \bigcap_{m>0} \text{Bs}(mD)$$

where $\text{Bs}(mD)$ is the base locus of the complete linear system $|mD|$. Following [ELM⁺06], we define the *augmented base locus* of D to be the closed set

$$\mathbf{B}_+(D) = \bigcap_A \mathbf{B}(D - A),$$

where the intersection runs over all ample \mathbb{Q} -divisors A . Similarly, the *restricted base locus* of D is defined by

$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}(D + A),$$

where the union runs over all ample \mathbb{Q} -divisors A . By [ELM⁺06, Prop. 1.4, Exam. 1.8, Prop. 1.15, Exam. 1.16], both $\mathbf{B}_-(D)$ and $\mathbf{B}_+(D)$ depend only on the numerical class of D , there is an inclusion $\mathbf{B}_-(D) \subseteq \mathbf{B}_+(D)$, and for any rational number $c > 0$ we have $\mathbf{B}_-(cD) = \mathbf{B}_-(D)$ and $\mathbf{B}_+(cD) = \mathbf{B}_+(D)$. Moreover by [ELM⁺06, Exam. 1.7, Exam. 1.18] we have that $\mathbf{B}_+(D) = \emptyset$ if and only if D is ample, and that $\mathbf{B}_-(D) = \emptyset$ if and only if D is nef.

Let $f : Y \rightarrow X$ be a morphism between normal projective varieties and let $C \subseteq Y$ be an irreducible projective curve. We define the 1-cycle f_*C to be zero if C is contracted to a point by f , and to be $\deg(f|_C : C \rightarrow f(C)) f(C)$ if $f(C)$ is a curve on X . If D is a Cartier divisor on X , we have the projection formula

$$f^*D \cdot [C] = D \cdot [\pi_*C]$$

that extends by linearity to \mathbb{Q} -Cartier divisors and 1-cycles.

1.2. The Minimal Model Program (MMP)

We will use the notation and results of the Minimal Model Program (MMP for short) in [KM98].

1.2.1. Singularities of the MMP. — Let X be a normal projective variety. In this case, we can define the class of a canonical divisor $K_X \in \text{Cl}(X)$. See [Rei87, §1] for details.

The variety X is said to be Gorenstein if K_X is a Cartier divisor and its singularities are Cohen-Macaulay. The property of being Gorenstein is local and open, so the Gorenstein locus of X is the open subset containing all the Gorenstein points of X (it contains X_{reg} , in particular).

We follow the usual convention, and we say that X is a \mathbb{Q} -Gorenstein variety if some positive multiple of K_X is a Cartier divisor; we do not require Cohen-Macaulay singularities. In this case, the Gorenstein index of X is the smallest positive integer $\ell \in \mathbb{Z}_{>0}$ such that ℓK_X is a Cartier divisor.

Let us recall the notion of singularities of pairs for \mathbb{Q} -factorial varieties. See [KM98, §2.3] for details.

Definition 1.2.1 (Singularities of pairs). — Let X be a \mathbb{Q} -factorial normal projective variety, and $\Delta = \sum a_i \Delta_i$ an effective \mathbb{Q} -divisor on X . Let $f : Y \rightarrow X$ be a log-resolution of the pair (X, Δ) , i.e., a birational projective morphism f whose exceptional locus $\text{Exc}(f)$ is the union of the effective prime divisors E_i 's and such that $\Delta_Y + \sum E_i$ is a simple normal crossing divisor, where Δ_Y is the strict transform of Δ in Y by f . Using numerical equivalence, we have

$$K_Y + \Delta_Y \cong f^*(K_X + \Delta) + \sum_{E_i} a(E_i; X, \Delta) E_i.$$

The numbers $a(E_i; X, \Delta) \in \mathbb{Q}$ are independent of the log-resolution and depend only on the discrete valuation that corresponds to E_i . The *discrepancy* of the pair (X, Δ) is given by

$$\text{discrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor of a resolution of } X\}.$$

Suppose that all $a_i \leq 1$ and that f is a log-resolution of (X, Δ) , we say that the pair (X, Δ) is

$$\begin{cases} \text{terminal} & \text{discrep}(X, \Delta) > 0 \\ \text{canonical} & \text{if } \text{discrep}(X, \Delta) \geq 0 \\ \text{klt} & \text{discrep}(X, \Delta) > -1 \text{ and all } a_i < 1 \end{cases}$$

Here klt means ‘‘Kawamata log terminal’’. If the conditions above hold for one log-resolution of (X, Δ) , then they hold for every log-resolution of (X, Δ) .

We say that X is terminal (canonical,...) or that it has terminal (canonical,...) singularities if $(X, 0)$ is a terminal (canonical,...) pair.

An important class of examples is given by quotient singularities.

Definition 1.2.2 (Quotient singularity). — Let $x \in X$ be a germ of a complex analytic space. We say that X has a quotient singularity if there is a smooth germ $0 \in Y$ and a finite group G acting on $0 \in Y$ such that $(x \in X) \cong (0 \in Y)/G$. We say that a complex algebraic variety has quotient singularities if its associated complex analytic space has only quotient singularities at every point.

Proposition 1.2.3 ([KM98, Prop. 5.15], [Kol13, §3.18])

Let X be a complex projective algebraic variety with quotient singularities. Then X is \mathbb{Q} -factorial and X has klt singularities.

Remark 1.2.4. — By definition, if X has Gorenstein klt singularities then it has canonical singularities. In particular, Proposition 1.2.3 says that Gorenstein quotient singularities are canonical.

The singularities of the Minimal Model Program behave well by taking hyperplane sections. We have the following Bertini type result.

Lemma 1.2.5 ([KM98, Lemm. 5.17]). — *Let (X, Δ) be a pair and $|H|$ be a free linear system on X . Then,*

$$\text{discrep}(X, \Delta) \leq \text{discrep}(H_g, \Delta|_{H_g})$$

for a general member $H_g \in |H|$.

We say that an algebraic variety X is *smooth in codimension k* if $\text{codim}_X \text{Sing}(X) \geq k + 1$. From the previous result and the description of terminal surface singularities in [KM98, Theo. 4.5], we can deduce that terminal varieties are smooth in codimension 2.

Proposition 1.2.6 ([KM98, Cor. 5.18]). — *Let X be a normal projective variety with terminal singularities. Then, X is smooth in codimension 2.*

For varieties with terminal Gorenstein quotient singularities the singular locus is even smaller.

Proposition 1.2.7 ([CK99, Prop. A.2.2]). — *Let X be a normal projective variety with terminal Gorenstein quotient singularities. Then, X is smooth in codimension 3.*

An other important class is given by rational singularities.

Definition 1.2.8 (Rational singularity). — Let X be a normal algebraic variety and $f : Y \rightarrow X$ a resolution of singularities. We say that f is a rational resolution if

$$R^i f_* \mathcal{O}_X = 0 \text{ for } i > 0.$$

We say that X has rational singularities if every resolution of singularities is rational.

In fact, if an algebraic variety X has a rational resolution then every resolution will be rational (see [KM98, Theo. 5.10] for instance). Kawamata log terminal pairs provide examples of rational singularities.

Theorem 1.2.9 ([KMM87, Theo. 1-3-6]). — *Let (X, Δ) be a klt pair. Then X has rational singularities.*

Moreover, Serre's duality applies for rational singularities and hence for klt pairs.

Proposition 1.2.10 ([KM98, Theo. 5.10]). — *Let X be a variety with rational singularities. Then X has Cohen-Macaulay singularities.*

Let us recall the following version of Kawamata-Viehweg Vanishing Theorem.

Theorem 1.2.11 ([KM98, Theo. 2.70]). — *Let (X, Δ) be a klt pair. Let N be a \mathbb{Q} -Cartier divisor on X such that $N \equiv M + \Delta$, where M is a nef and big \mathbb{Q} -Cartier \mathbb{Q} -divisor. Then*

$$H^i(X, \mathcal{O}_X(-N)) = 0 \text{ for } i < \dim X.$$

1.2.2. Contraction of extremal rays. — Let X be a normal projective variety. A *contraction* of X is a projective surjective morphism $\varphi : X \rightarrow Y$ with connected fibers, where Y is a normal projective variety. In particular, $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$.

Definition 1.2.12 (Extremal face). — Let V be a real vector space of finite dimension. A subset $\sigma \subseteq V$ is called a *cone* if $0 \in \sigma$ and σ is closed under multiplication by positive scalars. We say that a cone is *convex* if for every $u, v \in \sigma$ we have that $u + v \in \sigma$. A *extremal face* τ of a cone σ is a non-empty subset of σ which is convex and that for every $u, v \in \sigma$, if $tu + (1 - t)v \in \tau$ for some $t \in]0, 1[$ then $tu + (1 - t)v \in \tau$ for every $t \in [0, 1]$. A one-dimensional extremal face is called an *extremal ray*.

If D is a \mathbb{Q} -Cartier divisor and $F \subseteq \overline{\text{NE}}(X)$ is an extremal face, then the sign $D \cdot F > 0$ (resp. $=, <$) is well defined. We can therefore define the sets $D_{\geq 0} = \{[C] \in N_1(X)_{\mathbb{R}} \mid D \cdot [C] \geq 0\}$ (similarly for $> 0, \leq 0, < 0$) and $D^{\perp} = \{[C] \in N_1(X)_{\mathbb{R}} \mid D \cdot [C] = 0\}$. We define

$$\overline{\text{NE}}(X)_{D \geq 0} = \overline{\text{NE}}(X) \cap D_{\geq 0},$$

similarly for $> 0, \leq 0, < 0$.

Let $R \subseteq \overline{\text{NE}}(X)$ be an extremal ray. We define the *locus* of R to be

$$\text{Locus}(R) = \bigcup_{[C] \in R} C \subseteq X.$$

One of the main tool in the study of extremal contractions is the Cone Theorem (see [KM98, Theo. 3.7]).

Theorem 1.2.13 (Cone Theorem). — *Let (X, Δ) be a klt pair with X a projective \mathbb{Q} -factorial variety. Then*

1. *There are countably many rational curves $\{C_i\}_{i \in I} \subseteq X$ such that $(K_X + \Delta) \cdot [C_i] < 0$ for every $i \in I$, and*

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + \Delta) \geq 0} + \sum_{i \in I} \mathbb{R}_{\geq 0}[C_i].$$

2. *For any $\varepsilon > 0$ and ample \mathbb{Q} -divisor H , there exists a finite subset $I_0 \subseteq I$ such that*

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + \Delta + \varepsilon H) \geq 0} + \sum_{i \in I_0} \mathbb{R}_{\geq 0}[C_i].$$

3. *Let $F \subseteq \overline{\text{NE}}(X)_{(K_X + \Delta) < 0}$ be an extremal face. Then, there exists a unique morphism $\text{cont}_F : X \rightarrow X_F$ with connected fibers, called the extremal contraction of F , from X onto a normal projective variety X_F such that the irreducible curves on X contracted by cont_F to points in X_F are exactly the curves whose classes in $N_1(X)$ belongs to F .*
4. *Let $F \subseteq \overline{\text{NE}}(X)_{(K_X + \Delta) < 0}$ be an extremal face with extremal contraction $\text{cont}_F : X \rightarrow X_F$. Let D be a Cartier divisor on X such that $D \cdot F = 0$. Then, there exists a Cartier divisor D_F on X_F such that $D \sim \text{cont}_F^* D_F$.*

Let (X, Δ) be a klt pair with X a projective \mathbb{Q} -factorial variety and let $R \subseteq \overline{\text{NE}}(X)_{(K_X + \Delta) < 0}$ be an extremal ray. We denote by $\text{Exc}(\text{cont}_R)$ the exceptional locus of cont_R , i.e., the subset of X where cont_R is not an isomorphism. Following [KM98, Prop. 2.5], there are three possibilities:

1. $\dim(X_R) < \dim(X)$; we say that cont_R is of *fiber type*.
2. cont_R is birational and its exceptional locus is an effective prime divisor E such that $E \cdot R < 0$; we say that cont_R is a *divisorial contraction*.
3. cont_R is birational and its exceptional locus has codimension > 1 in X ; we say that cont_R is a *small contraction*.

In order to study small contractions, the notion of *flip* was introduced. See [KM98, Definition 3.33 and §6.1] for details.

Definition 1.2.14 (Flip). — Let (X, Δ) be a klt pair with X a projective \mathbb{Q} -factorial variety. Let $f : X \rightarrow Y$ be a birational morphism onto a projective normal variety such that $\text{codim}_X \text{Exc}(f) \geq 2$ and that $-(K_X + \Delta)$ is f -ample. A normal projective variety X^+ together with a birational morphism $f^+ : X^+ \rightarrow Y$ is called a $(K_X + \Delta)$ -flip of f if

1. $(K_{X^+} + \Delta^+)$ is \mathbb{Q} -Cartier, where Δ^+ is the strict transform of Δ on X^+ ,
2. $(K_{X^+} + \Delta^+)$ is f^+ -ample,
3. $\text{codim}_{X^+} \text{Exc}(f^+) \geq 2$.

By an abuse of terminology, we say that the induced rational map $X \dashrightarrow X^+$ is a $(K_X + \Delta)$ -flip.

As a consequence of the Cone Theorem and Negativity Lemma [KM98, Lemm. 3.39] we obtain the following results.

Proposition 1.2.15 ([KM98, Prop. 3.36, Cor. 3.43])

Let (X, Δ) be a klt pair with X a projective \mathbb{Q} -factorial variety. Let $f : X \rightarrow Y$ be the divisorial contraction of a $(K_X + \Delta)$ -negative extremal ray with exceptional divisor E . Then

1. Y is \mathbb{Q} -factorial.
2. $\rho_Y = \rho_X - 1$.
3. (Y, Δ_Y) is a klt pair, where Δ_Y is the strict transform of Δ on Y .
4. If (X, Δ) is terminal (resp. canonical) and $E \not\subseteq \text{Supp } \Delta$, then (Y, Δ_Y) is also terminal (resp. canonical).

Proposition 1.2.16 ([KM98, Prop. 3.37, Cor. 3.42])

Let (X, Δ) be a klt (resp. terminal, canonical) pair with X a projective \mathbb{Q} -factorial variety. Let $f : X \rightarrow Y$ be the small contraction of a $(K_X + \Delta)$ -negative extremal ray and X^+ be a $(K_X + \Delta)$ -flip of f . Then

1. X^+ is \mathbb{Q} -factorial.

2. $\rho_{X^+} = \rho_X$.
3. (X^+, Δ_{X^+}) is a klt (resp. terminal, canonical) pair, where Δ_{X^+} is the strict transform of Δ on X^+ .

Contractions of fiber type are more subtle.

Proposition 1.2.17 ([KM98, Prop. 3.36], [Fuj99, Cor. 4.5])

Let (X, Δ) be a klt pair with X a projective \mathbb{Q} -factorial variety. Let $f : X \rightarrow Y$ be the contraction of a $(K_X + \Delta)$ -negative extremal ray of fiber type. Then

1. Y is \mathbb{Q} -factorial.
2. $\rho_Y = \rho_X - 1$.
3. There exists an effective \mathbb{Q} -divisor Δ_Y on Y such that (Y, Δ_Y) is klt. In particular, Y has only rational singularities.

1.2.3. MMP for Fano varieties. — Fano varieties have a very rich geometry and give rise to more precise descriptions.

Definition 1.2.18 (Fano variety). — A complete normal variety X is said to be a Fano variety if there exists a positive multiple of $-K_X$ which is Cartier and ample. A Fano variety is therefore always projective.

Thanks to the Cone Theorem we have the following description of the Mori cone of Fano varieties.

Proposition 1.2.19. — Let X be a klt Fano variety. Then, $\overline{\text{NE}}(X)$ is a polyhedral cone generated by finitely many classes of rational curves on X . In particular,

$$\overline{\text{NE}}(X) = \text{NE}(X).$$

Remark 1.2.20. — As a consequence of Proposition 1.2.19 and Kleiman's criterion of ampleness we have that if D is a \mathbb{Q} -Cartier divisor on a klt Fano variety X , then D is ample if and only if $D \cdot [C] > 0$ for every irreducible curve on X . This is not true in general (see [Laz04, Exam. 1.5.2], for instance).

Kawamata-Viehweg Vanishing Theorem implies the following well known result in this case.

Proposition 1.2.21. — Let X be a klt Fano variety. Then,

$$H^i(X, \mathcal{O}_X) = \{0\} \text{ for every } i > 0.$$

In particular, finite connected étale covers of klt Fano varieties are trivial.

Proof. — We apply Theorem 1.2.11 to the ample divisor $N = -K_X$. Serre's duality leads the first statement. In particular, we have that $\chi(X, \mathcal{O}_X) = 1$. If $\pi : \tilde{X} \rightarrow X$ is any connected finite étale cover we have that $\pi^*K_X = K_{\tilde{X}}$ and that \tilde{X} has klt singularities, by [Kol97, Prop. 3.16]. \tilde{X} is therefore a klt Fano variety and hence $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$. By Hirzebruch-Riemann-Roch Theorem we have that $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \deg(\pi)\chi(X, \mathcal{O}_X)$ hence π is an isomorphism. \square

Remark 1.2.22. — By Zariski-Nagata Purity Theorem [SGAI, Exposé X, Théorème 3.1] we have that if X is a smooth Fano variety and $\pi : \tilde{X} \rightarrow X$ is a finite connected covering which is étale in codimension 1 (see Definition 2.7.1) then π is étale and hence trivial. This may not be true for singular Fano varieties as we will discuss in §2.7.

After the works of Birkar, Cascini, Hacon and McKernan, if X is a klt Fano variety, then X is a Mori Dream Space (see [BCHM10, Cor. 1.3.2] and [HK00]). We will be specially interested in the case when X has canonical singularities.

Theorem 1.2.23 ([Del14, Theo. 3.1]). — *Let X be a \mathbb{Q} -factorial Fano variety with canonical singularities. Then, for any prime divisor $D \subseteq X$, there exists a finite sequence (called a special Mori program for the divisor $-D$)*

$$X = X_0 \xrightarrow{\sigma_0} X_1 \dashrightarrow \cdots \dashrightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k \xrightarrow{\pi} Y$$

such that, if $D_i \subseteq X_i$ is the transform of D for $i = 1, \dots, k$ and $D_0 := D$, the following hold:

1. X_1, \dots, X_k and Y are \mathbb{Q} -factorial projective varieties and X_1, \dots, X_k have canonical singularities.
2. for every $i = 0, \dots, k$, there exists an extremal ray Q_i of X_i with $D_i \cdot Q_i > 0$ and $-K_{X_i} \cdot Q_i > 0$ such that:
 - (a) for $i = 1, \dots, k-1$, $\text{Locus}(Q_i) \subsetneq X_i$, and σ_i is either the contraction of Q_i (if Q_i is divisorial), or its flip (if Q_i is small);
 - (b) the morphism $\pi : X_k \rightarrow Y$ is the contraction of Q_k and π is a fiber type contraction.

Let us finish with the following result concerning extremal contractions of smooth Fano varieties having at most one-dimensional fibers. We will address the mildly singular case in §2.1.

Theorem 1.2.24 ([Wis91, Theo. 1.2], [And85, Theo. 2.3, Theo. 3.1])

Let X be a smooth Fano variety and let $\pi : X \rightarrow Y$ be the contraction of an extremal ray $R \subseteq \overline{\text{NE}}(X)$. If every fiber of π is of dimension ≤ 1 then Y is smooth and either

1. $\pi : X \rightarrow Y$ is a (possibly singular) conic bundle.
2. $\pi : X \rightarrow Y$ is the blow-up of Y along a smooth subvariety $A \subseteq Y$ of codimension 2.

1.3. Toric varieties

We may refer the reader to [CLS11] for the general theory of toric varieties and to [Mat02] for details of the toric MMP. We will keep the same notation as [CLS11].

Let $N \cong \mathbb{Z}^n$ be a lattice, $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice and let $N_{\mathbb{R}}$ (resp. $N_{\mathbb{Q}}$) and $M_{\mathbb{R}}$ (resp. $M_{\mathbb{Q}}$) be their real scalar (resp. rational scalar) extensions. Let us denote by $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ the natural \mathbb{R} -bilinear pairing.

Let $\Delta_X \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ be a fan. As we will see, most of the properties of our interest in the context of the MMP of the associated toric variety $X = X(\Delta_X)$, can be translated into combinatorial properties of the fan Δ_X .

Sometimes we will write N_X instead of N in order to emphasize the dependence of $X(\Delta_X)$ on the lattice where primitive generators of Δ_X belong.

Let $\Delta_X(k)$ be the set of k dimensional cones in Δ_X . In the same way, if $\sigma \in \Delta_X$ is a cone, we will denote by $\sigma(k)$ the set of its k -dimensional faces. Usually, we will not distinguish between 1-dimensional cones $\rho \in \Delta_X(1)$ (or 1-dimensional faces $\rho \in \sigma(1)$) and the primitive vector $u_{\rho} \in N$ generating them.

If $\sigma \in \Delta_X(k)$ we will denote by U_{σ} the associated affine toric variety, and by $V(\sigma) \subseteq X(\Delta_X)$ the closed invariant subvariety of codimension k . In particular, each $\rho \in \Delta_X(1)$ corresponds to an invariant Weil divisor $V(\rho)$ on X (also noted $V(u_{\rho})$); such a cone is called a *ray*. Similarly, each cone of codimension one $\omega \in \Delta_X(n-1)$ corresponds to an invariant rational curve on X ; such a cone is called a *wall*.

Let us recall that an invariant Weil divisor $D = \sum_{\rho \in \Delta_X(1)} a_{\rho} V(\rho)$ is Cartier if and only if for each $\sigma \in \Delta_X$ there is $m_{\sigma} \in M$ such that $\langle m_{\sigma}, u_{\rho} \rangle = -a_{\rho}$ for all $\rho \in \sigma(1)$. We call $\{m_{\sigma}\}_{\sigma \in \Delta_X}$ a *Cartier data* of D . See [CLS11, Theo. 4.2.8] for details.

It is a classical fact that every Weil divisor on a toric variety X is linearly equivalent to an invariant toric divisor (see [CLS11, Theo. 4.1.3]). The same holds for effective curves on complete toric varieties.

Theorem 1.3.1 ([Rei83, Prop. 1.6]). — *Let $X = X(\Delta_X)$ be a complete toric variety of dimension n . Then*

$$\mathrm{NE}(X) = \sum_{\omega \in \Delta_X(n-1)} \mathbb{R}_{\geq 0}[V(\omega)].$$

In particular, $\mathrm{NE}(X)$ is a closed rational polyhedral cone and it is strictly convex if and only if X is a projective variety.

Finally, we recall that if $\lambda_0, \dots, \lambda_n$ are positive integers with $\mathrm{gcd}(\lambda_0, \dots, \lambda_n) = 1$ then we define the associated Weighted Projective Space (WPS) to be

$$\mathbb{P}(\lambda_0, \dots, \lambda_n) = (\mathbb{A}^{n+1} \setminus \{0\}) / \sim,$$

where \sim is the equivalence relation

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \Leftrightarrow x_i = \varepsilon^{\lambda_i} y_i, i = 0, \dots, n \text{ for some } \varepsilon \in \mathbb{G}_m.$$

Moreover, $\mathbb{P}(\lambda_0, \dots, \lambda_n)$ is a toric variety with Picard number one and torsion-free class group. In general, we say that a \mathbb{Q} -factorial complete toric variety is a Poly Weighted Space (PWS) if its class group is torsion-free (see also Definition 2.7.8).

1.3.1. Toric singularities. — All affine toric varieties associated to strongly convex rational polyhedral cones are normal (see [CLS11, Theo. 1.3.5]). Thus, a toric variety X associated to a fan Δ_X is also normal. Moreover, we can check if a toric variety associated to a fan Δ_X is smooth or not by looking at all the cones $\sigma \in \Delta_X$. In fact, if we define that a cone $\sigma \in \Delta_X$ is smooth if and only if the associated affine toric variety U_σ is smooth, then we have the following result.

Proposition 1.3.2 ([CLS11, Prop. 11.1.2, Prop. 11.1.8])

Let X be the toric variety associated to the fan Δ_X . Then,

$$\mathrm{Sing}(X) = \bigcup_{\sigma \text{ not smooth}} V(\sigma)$$

and

$$X_{\mathrm{reg}} = \bigcup_{\sigma \text{ smooth}} U_\sigma.$$

Moreover, given a d -dimensional simplicial cone $\sigma \subseteq N_{\mathbb{R}}$ with generators $u_1, \dots, u_d \in N$, let $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and define the multiplicity of σ by

$$\mathrm{mult}(\sigma) = [N_\sigma : \mathbb{Z}u_1 + \dots + \mathbb{Z}u_d].$$

Then,

1. σ is smooth if and only if $\mathrm{mult}(\sigma) = 1$.

2. $\text{mult}(\sigma)$ is the number of points of $P_\sigma \cap N$, where

$$P_\sigma = \left\{ \sum_{i=1}^d \lambda_i u_i \mid 0 \leq \lambda_i < 1 \right\}.$$

3. Let e_1, \dots, e_d be a basis of N_σ and write $u_i = \sum_{j=1}^d a_{ij} e_j$. Then,

$$\text{mult}(\sigma) = |\det(a_{ij})|.$$

4. If $\tau \preceq \sigma$ is a face of σ , then

$$\text{mult}(\sigma) = \text{mult}(\tau)[N_\sigma : N_\tau + \mathbb{Z}u_1 + \dots + \mathbb{Z}u_d].$$

In particular, $\text{mult}(\tau) \leq \text{mult}(\sigma)$ whenever $\tau \preceq \sigma$.

Remark 1.3.3. — Let X be a n -dimensional toric variety associated to a simplicial fan Δ_X , i.e., a fan whose cones are all simplicial, admitting a cone σ of full dimension n . If X is smooth in codimension k , namely the closed invariant subset $\text{Sing}(X) \subseteq X$ is such that $\text{codim}_X \text{Sing}(X) \geq k + 1$, then we can choose a basis of $\mathbb{Z}^n \cong N$ in such a way the first k generators of the cone σ corresponds to the first k elements of the canonical basis of \mathbb{Z}^n .

In general, most of the interesting kind of singularities can also be characterized by looking at the (maximal) cones belonging to the fan.

Theorem 1.3.4. — Let σ be a strongly convex rational polyhedral cone and let U_σ be the corresponding affine toric variety, then the following hold.

1. U_σ is Cohen-Macaulay.
2. U_σ is \mathbb{Q} -factorial if and only if σ is simplicial.
3. U_σ is \mathbb{Q} -Gorenstein if and only if there exists $m_\sigma \in M_\mathbb{Q}$ such that $\langle m_\sigma, u_\rho \rangle = 1$, for every ray $\rho \in \sigma(1)$. In this case, the Gorenstein index of U_σ is the smallest positive integer $\ell \in \mathbb{Z}_{>0}$ such that $\ell m_\sigma \in M$.
4. If U_σ is \mathbb{Q} -Gorenstein then U_σ has klt singularities.
5. If U_σ is \mathbb{Q} -Gorenstein then U_σ has terminal singularities of Gorenstein index ℓ if and only if there exists $m_\sigma \in M$ such that

$$\langle m_\sigma, u_\rho \rangle = \ell \text{ for all } u_\rho \in \text{Gen}(\sigma) \text{ and}$$

$$\langle m_\sigma, u_\rho \rangle > \ell \text{ for all } u_\rho \in \sigma \cap N \setminus (\{0\} \cup \text{Gen}(\sigma)).$$

The element m_σ is uniquely determined whenever σ is of maximal dimension in the fan.

6. If U_σ is \mathbb{Q} -Gorenstein then U_σ has canonical singularities of Gorenstein index ℓ if and only if there exists $m_\sigma \in M$ such that

$$\langle m_\sigma, u_\rho \rangle = \ell \text{ for all } u_\rho \in \text{Gen}(\sigma) \text{ and}$$

$$\langle m_\sigma, u_\rho \rangle \geq \ell \text{ for all } u_\rho \in \sigma \cap N \setminus (\{0\} \cup \text{Gen}(\sigma)).$$

The element m_σ is uniquely determined whenever σ is of maximal dimension in the fan.

7. If U_σ is Gorenstein then U_σ has canonical singularities.

Here, $\text{Gen}(\sigma)$ denotes the set of primitive vectors u_ρ generating all the rays $\rho = \{\lambda u_\rho \mid \lambda \geq 0\} \in \sigma(1)$

Proof. — We may refer the reader to the survey [Dai02] for proofs or references to proofs. \square

Remark 1.3.5. — By Theorem 1.3.4 above, if X is a \mathbb{Q} -factorial toric variety with canonical singularities then we have the decomposition

$$\text{Sing}(X) = \bigcup_{\substack{\sigma \text{ canonical} \\ \text{non-terminal}}} V(\sigma) \cup \bigcup_{\substack{\sigma \text{ terminal} \\ \text{non-smooth}}} V(\sigma).$$

Therefore, if X is a \mathbb{Q} -factorial toric variety with canonical singularities and with at most finitely many non-terminal points, then the (finite) set of canonical points is made up by some invariant points $V(\sigma)$, where σ are of maximal dimension in the fan.

1.3.2. Toric MMP. — Let us introduce now the necessary elements to run the Toric MMP.

Definition 1.3.6 (Degenerate fan). — Let $U \subseteq N_{\mathbb{R}}$ be a rational vector subspace; a collection of cones Δ^* is a degenerate fan with vertex U if it satisfies the usual conditions of a fan with strict convexity of cones replaced by

$$\forall \sigma \in \Delta^*, \sigma \cap -\sigma = U.$$

This coincides with the usual notion of a fan $\Delta = \Delta^*/U$ in the quotient space $N_{\mathbb{R}}/U$.

In our setting, we will always deal with \mathbb{Q} -factorial complete toric varieties, i.e., toric varieties having a simplicial fan whose support is the whole space $N_{\mathbb{R}}$, by Theorem 1.3.4 above and [CLS11, Theo. 3.4.1]. In the toric case, every extremal ray $R \subseteq \overline{\text{NE}}(X)$ of such a variety will correspond to a invariant curve C_ω such that $R = \mathbb{R}_{\geq 0}[C_\omega]$ or, equivalently, to a wall $\omega \in \Delta_X(n-1)$.

Let us suppose that $\omega = \text{cone}(u_1, \dots, u_{n-1})$, where u_i are primitive vectors. Since Δ_X is a simplicial fan, ω separates two maximal cones $\sigma = \text{cone}(u_1, \dots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, u_{n+1})$, where u_n and u_{n+1} are primitive on rays on opposite sides of ω . The $n+1$ vectors u_1, \dots, u_{n+1} are linearly dependent. Hence, they satisfy a so called *wall relation*:

$$b_n u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n, b_{n+1} \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{Z}$ for $i = 1, \dots, n-1$. By reordering if necessary, we can assume that

$$\begin{aligned} b_i < 0 & \quad \text{for} \quad 1 \leq i \leq \alpha \\ b_i = 0 & \quad \text{for} \quad \alpha + 1 \leq i \leq \beta \\ b_i > 0 & \quad \text{for} \quad \beta + 1 \leq i \leq n + 1. \end{aligned}$$

Let us introduce the notation

$$\Delta(\omega) = \sigma + \sigma' = \text{cone}(u_1, \dots, u_{n+1})$$

and

$$U(\omega) = \text{cone}(u_1, \dots, u_\alpha, u_{\beta+1}, \dots, u_{n+1}).$$

This wall relation and the signs of the coefficients involved allow us to describe the nature of the associated contraction.

Theorem 1.3.7 ([Rei83, Theo. 2.4, Cor. 2.5]). — *Let X be a \mathbb{Q} -factorial complete toric variety of dimension n associated to the fan $\Delta_X \subseteq N_{\mathbb{R}}$, and suppose that $R \subseteq \overline{NE}(X)$ is an extremal ray of X . Let us remove from $\Delta_X(n-1)$ all the walls ω associated to curves from R and for each such ω replace the two adjacent maximal cones σ and σ' from $\Delta_X(n)$ by the cone $\Delta(\omega)$. Then, α, β and $U_R = U(\omega)$ are independent of ω and, by taking respectively their faces in $\Delta_X(i)$, where $i \leq n-2$, we get a complete fan $\Delta_R^* \subseteq N_{\mathbb{R}}$, degenerate with vertex U_R if $\alpha = 0$, non-degenerate if $\alpha > 0$. Moreover, if $\alpha = 0$ then $\Delta_R := \Delta_R^*/U_R$ is a complete simplicial fan. If $\alpha = 1$, then $\Delta_R := \Delta_R^*$ is simplicial.*

Furthermore, the induced morphism of toric varieties $\varphi_R : X(\Delta_X) \rightarrow X(\Delta_R)$ is the contraction of the extremal ray R in the sense of Mori theory. Moreover, the exceptional locus of φ_R corresponds to the irreducible closed invariant subvariety $A \subseteq X$ associated to the cone $\text{cone}(u_1, \dots, u_\alpha) \in \Delta_X(\alpha)$, which is contracted onto the irreducible closed invariant subvariety $B \subseteq X_R = X(\Delta_R)$ corresponding to the cone $U_R \in \Delta_X^*(n-\beta)$ (if $\alpha = 0$ then both are equal to

the whole variety X and X_R , respectively). Therefore, $\dim A = n - \alpha$ and $\dim B = \beta$.

In general, if $\omega \in \Delta_X(n-1)$ is any wall of Δ_X (not necessarily corresponding to an extremal ray), we will also have a wall relation allowing us to compute the intersection number of the curve C_ω with every invariant divisor D_ρ , $\rho \in \Delta_X(1)$.

Proposition 1.3.8 ([CLS11, Prop. 6.4.4]). — *Let Δ_X be a simplicial fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$ and $\omega = \text{cone}(u_1, \dots, u_{n-1}) \in \Delta_X(n-1)$ be a wall, separating the two maximal cones $\sigma = \text{cone}(u_1, \dots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, u_{n+1})$, satisfying the wall relation*

$$b_n u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n, b_{n+1} \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{Z}$ for $i = 1, \dots, n-1$. Then, if we denote by $V(u_\rho) := V(\rho)$ the invariant Weil divisor associated to the primitive vector $u_\rho \in N$ corresponding to $\rho \in \Delta_X(1)$ and by C the invariant curve associated to the wall ω , we have that

1. $V(u) \cdot C = 0$ for all $u_\rho \notin \{u_1, \dots, u_n, u_{n+1}\}$.
2. $V(u_n) \cdot C = \frac{\text{mult}(\omega)}{\text{mult}(\sigma)}$ and $V(u_{n+1}) \cdot C = \frac{\text{mult}(\omega)}{\text{mult}(\sigma')}$.
3. $V(u_i) \cdot C = \frac{b_i \text{mult}(\omega)}{b_n \text{mult}(\sigma)} = \frac{b_i \text{mult}(\omega)}{b_{n+1} \text{mult}(\sigma')}$ for $i \in \{2, \dots, n-1\}$.

1.4. Flag varieties and Schubert cell decomposition

We refer the reader to [Bri05] and [Man98] for the geometry and combinatorics of flag varieties. We will keep the same notation as [Man98, §3.6].

Let \mathbb{F}_r be the set of complete flags of linear projective subspaces of \mathbb{P}^{r-1} ,

$$Y_\bullet : Y_1 = \mathbb{P}^{r-1} \supseteq Y_2 \supseteq \dots \supseteq Y_r,$$

where $Y_i \cong \mathbb{P}^{r-i}$ for every $i = 1, \dots, r$. The set \mathbb{F}_r can be endowed with a structure of algebraic variety over k and it is shown in [Man98, §3.6.1] that it is a smooth projective variety of dimension $r(r-1)/2$. The variety \mathbb{F}_r is called the *full flag variety*.

Let us fix a reference complete linear flag Y_\bullet on \mathbb{P}^{r-1} . Then, there is a decomposition of \mathbb{F}_r into *Schubert cells*

$$\mathbb{F}_r = \coprod_{w \in \mathfrak{S}_r} \Omega_w.$$

Moreover, there is a combinatorial description of each Schubert cell in terms of incidence conditions with respect to the reference flag Y_\bullet . See [Man98, §3.6.2] for details on this point of view.

We will consider the following alternative description of the Schubert cells, which is just a reformulation of [Bri05, Prop. 1.2.1].

Let us choose homogeneous coordinates $[x_1 : \dots : x_r]$ on \mathbb{P}^{r-1} such that for every $i = 1, \dots, r-1$ we have

$$Y_i = \{x_1 = \dots = x_i = 0\} \subseteq \mathbb{P}^{r-1}$$

and regard the permutation group \mathfrak{S}_r as a subgroup of $\mathrm{PGL}_r(k)$ via its natural action on the standard basis points $e_1, \dots, e_r \in \mathbb{P}^{r-1}$. Then, Ω_w is the orbit $B \cdot W_\bullet^w$, where B denotes the (Borel) subgroup of $\mathrm{PGL}_r(k)$ that fixes the flag Y_\bullet and W_\bullet^w is the complete linear flag such that for every $i = 1, \dots, r-1$ we have

$$W_i^w = \{x_{w(1)} = \dots = x_{w(i)} = 0\} \subseteq \mathbb{P}^{r-1}.$$

CHAPTER 2

FANO VARIETIES WITH A DIVISOR OF PICARD NUMBER ONE

2.1. Study of extremal contractions

In this section we study extremal contractions of mildly singular Fano varieties X that admits an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$.

Firstly, notice that klt Fano varieties always have an extremal ray whose intersection with a given effective prime divisor is positive (cf. [BCW02, Lemm. 2]).

Lemma 2.1.1. — *Let X be a \mathbb{Q} -factorial klt Fano variety and let $D \subseteq X$ be an effective prime divisor. Then, there exists an extremal ray $R \subseteq \overline{NE}(X)$ such that $D \cdot R > 0$.*

Proof. — The Cone Theorem 1.2.13 implies that $\overline{NE}(X) = NE(X)$ is a rational polyhedral cone generated by a finite number of extremal rays R_1, \dots, R_s . Let $C \subseteq X$ be any curve such that $D \cdot [C] > 0$. Since C is numerically equivalent to a positive sum of extremal curves, $[C] = \sum_{i=1}^s a_i [C_i]$ with $a_i \geq 0$ and $[C_i] \in R_i$, we can pick one such that $D \cdot R_i > 0$. \square

Secondly, we have that the contraction of an extremal ray whose intersection with an effective prime divisor of Picard number one is positive has at most one-dimensional fibers.

Lemma 2.1.2. — *Let X be a \mathbb{Q} -factorial klt Fano variety and $D \subseteq X$ be an effective prime divisor such that $\dim_{\mathbb{R}} N_1(D, X) = 1$. Let us suppose that $\rho_X > 1$ and let $R \subseteq \overline{NE}(X)$ be an extremal ray such that $D \cdot R > 0$. Then, $R \not\subseteq N_1(D, X)$. In particular, the extremal contraction cont_R is finite on D and all the fibers of cont_R are at most of dimension 1.*

Proof. — We follow the proof of [CD15, Lemma 3.1]. Assume, to the contrary, that $R \subseteq N_1(D, X)$. The numerical class of every irreducible curve $C \subseteq D$ must belong to R , since $\dim_{\mathbb{R}} N_1(D, X) = 1$. Thus, cont_R sends D to a point and $D \subseteq \text{Locus}(R)$. If cont_R is birational, then we would have that $-D$ is an effective divisor by the negativity lemma, a contradiction. The contraction cont_R is then of fiber type and $\text{Locus}(R) = X$. As $\rho_X > 1$ and $\text{cont}_R(D)$ is a point, there exists a non-trivial fiber of cont_R disjoint of D , and then $D \cdot R = 0$, a contradiction.

As a consequence we have that all the fibers of cont_R are at most of dimension 1. In fact, $\text{cont}_R|_D$ is a finite morphism: if there exists a curve $C \subseteq D$ contained in a fiber, we would have that $[C] \in R$ and therefore $R \subseteq N_1(D, X)$. \square

G. Della Noce proved in [Del14, Theo. 2.2] that a mildly singular projective variety (not necessarily Fano) has no small K -negative extremal contractions having all the fibers of dimension at most 1 (cf. [Del14, Exam. 2.11] and Theorem 1.2.24). In our context we have the following result.

Proposition 2.1.3. — *Let X be a \mathbb{Q} -factorial klt Fano variety of dimension $n \geq 3$. Let us suppose that $\rho_X > 1$ and that there exists an effective prime divisor such that $\dim_{\mathbb{R}} N_1(D, X) = 1$. Let $R \subseteq \overline{NE}(X)$ be an extremal ray such that $D \cdot R > 0$ and let us denote by $\varphi_R : X \rightarrow X_R$ the corresponding extremal contraction. Then,*

1. *If φ_R is of fiber type, then X_R is a \mathbb{Q} -factorial klt Fano variety of dimension $(n - 1)$ such that $\rho_{X_R} = 1$. In particular, X_R has only rational singularities.*
2. *If φ_R is birational and we suppose that X has Gorenstein canonical singularities with at most finitely many non-terminal points, then φ_R is a divisorial contraction and there exists a closed subset $S \subseteq X_R$ with $\text{codim}_{X_R}(S) \geq 3$ such that $X_R \setminus S \subseteq X_{R,\text{reg}}$, $\text{codim}_X \varphi_R^{-1}(S) \geq 2$, $X \setminus \varphi_R^{-1}(S) \subseteq X_{\text{reg}}$ and*

$$\varphi_R|_{X \setminus \varphi_R^{-1}(S)} : X \setminus \varphi_R^{-1}(S) \longrightarrow X_R \setminus S$$

is the blow-up of a $(n - 2)$ -dimensional smooth subvariety in $X_R \setminus S$. Moreover, X_R is a \mathbb{Q} -factorial Fano variety with canonical singularities with at most finitely many non-terminal points. In particular,

$$K_X \cdot [F] = E \cdot [F] = -1,$$

for every irreducible curve F such that $[F] \in R$, where $E = \text{Exc}(\varphi_R)$ is the exceptional divisor of φ_R .

Proof. — Let us suppose that $\varphi_R : X \rightarrow X_R$ is of fiber type. Then, $\varphi_R|_D : D \rightarrow X_R$ is a finite morphism, by Lemma 2.1.2, and thus $\dim X_R = n - 1$. Since X_R is a \mathbb{Q} -factorial projective variety, the projection formula and the fact that $\dim_{\mathbb{R}} N_1(D, X) = 1$ imply that $\rho_{X_R} = 1$. In particular, every big divisor on X_R is ample.

Let us prove that X_R is a klt Fano variety in this case. Since X is klt and Fano, there exists an effective \mathbb{Q} -divisor Δ_{X_R} on X_R such that (X_R, Δ_{X_R}) is klt and such that $-(K_{X_R} + \Delta_{X_R})$ is ample, by [PS09, Lemm. 2.8]. In particular X_R has rational singularities, by Theorem 1.2.9, and $-K_{X_R}$ is a big divisor, and therefore ample. Finally, let us remark that since (X_R, Δ_{X_R}) is klt and Δ_{X_R} is an effective \mathbb{Q} -divisor then it follows from [KM98, Cor. 2.35] that $(X_R, 0)$ is klt as well.

Let us suppose now that $\varphi_R : X \rightarrow X_R$ is a birational contraction. Then, it follows from [Del14, Theo. 2.2] that φ_R is a divisorial contraction given by the blow-up in codimension two of an irreducible subvariety of dimension $(n - 2)$ on X_R , and that X_R is a \mathbb{Q} -factorial Fano variety with canonical singularities with at most finitely many non-terminal points. Finally, the ampleness of the anti-canonical divisor $-K_{X_R}$ follows verbatim the proof in the smooth case given in [CD15, Lemma 3.1]. \square

Remark 2.1.4. — Let X be a Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities with at most finitely many non-terminal points, and let $D \subseteq X$ be an effective prime divisor such that $\dim_{\mathbb{R}} N_1(D, X) = 1$. Then, $\rho_X \leq 3$, by [Del14, Rema. 5.5].

Remark 2.1.5 (Generalized conic bundles). — Following G. Della Noce [Del14, p. 984], we say that a morphism $\varphi : X \rightarrow Y$ between \mathbb{Q} -factorial normal projective varieties is a *generalized conic bundle* if all its fibers are one-dimensional, the general fiber F_g is isomorphic to \mathbb{P}^1 with anti-canonical degree $-K_X \cdot [F_g] = 2$ and if F is an arbitrary fiber of φ then either:

1. F is an irreducible and generically reduced rational curve such that $F_{\text{red}} \cong \mathbb{P}^1$ and that $-K_X \cdot [F] = 2$.
2. $[F] = 2[C]$ as 1-cycles, where C is an irreducible and generically reduced rational curve such that $C_{\text{red}} \cong \mathbb{P}^1$ and that $-K_X \cdot [C] = 1$.
3. $F = C \cup C'$, with $C \neq C'$ irreducible and generically reduced rational curves such that $C_{\text{red}} \cong C'_{\text{red}} \cong \mathbb{P}^1$ and that $-K_X \cdot [C] = -K_X \cdot [C'] = 1$.

Moreover, it follows from [Kol96, Theo. II.2.8] that if all the fibers are of type (1) then φ is in fact a \mathbb{P}^1 -bundle. The main difference with the classical conic

bundle case is that φ might not be flat. As observed in [Del14, p. 984] if X is a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points and

$$X = X_0 \xrightarrow{\sigma_0} X_1 \rightarrow \cdots \rightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k \xrightarrow{\pi} Y$$

is a sequence of $k \geq 0$ elementary divisorial contractions $\sigma_i : X_i \rightarrow X_{i+1}$ sending its corresponding exceptional divisor onto a subvariety of codimension two (cf. [Del14, Definition 2.10]) followed by a contraction of fiber type $\pi : X_k \rightarrow Y$ all whose fibers are one-dimensional, then π is a generalized conic bundle in the above sense and, moreover, for every $i = 0, \dots, k-1$, the composition

$$\varphi_i : X_i \xrightarrow{\sigma_i} X_{i+1} \rightarrow \cdots \rightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k \xrightarrow{\pi} Y$$

is a generalized conic bundle. We will be mainly interested in the cases $k = 0, 1$.

The following result is a particular case of [Del14, Lemm. 3.3]. We include the statement with our notation for completeness.

Lemma 2.1.6. — *Let X_0 be a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Let $D_0 \subseteq X_0$ be an effective prime divisor and let us suppose that there is a diagram*

$$X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} X_2,$$

where σ_i is the birational map associated to the contraction of an extremal ray $Q_i \subseteq \overline{NE}(X_i)$ such that $D_i \cdot Q_i > 0$, for $i = 0, 1$; as in Theorem 1.2.23. If $Q_i \not\subseteq N_1(D_i, X_i)$ for $i = 0, 1$, then both σ_0 and σ_1 are divisorial contractions, $\text{Exc}(\sigma_i)$ is contained in the Gorenstein locus of X_i and $\text{Exc}(\sigma_0)$ is disjoint from the transform of $\text{Exc}(\sigma_1)$ in X_0 .

2.2. The extremal case $\rho_X = 3$

In this section we study the extremal contractions of mildly singular Fano varieties X on which there is an effective prime divisor of Picard number one and such that $\rho_X = 3$. As we pointed out in Remark 2.1.4, this is the largest possible Picard number for such varieties. Compare with the smooth case [CD15, Lemm. 3.1, Theo. 3.8].

Let us begin with the proof of Theorem A.

Proof of Theorem A. — Since X is a \mathbb{Q} -factorial klt Fano variety, there is an extremal ray $R \subseteq \overline{NE}(X)$ such that $D \cdot R > 0$, by Lemma 2.1.1. We denote

by $\sigma := \varphi_R : X \rightarrow Y$ the associated extremal contraction. We note that Proposition 2.1.3 implies that σ is a divisorial contraction sending the effective prime divisor $E = \text{Exc}(\sigma)$ onto a subvariety $A \subseteq Y$ of dimension $(n - 2)$. Moreover, Y is a Fano variety with canonical singularities and with at most finitely many non-terminal points, such that if we denote by D_Y the image of D by σ , then we have that $A \subseteq D_Y$ and that $\dim_{\mathbb{R}} N_1(D_Y, Y) = 1$.

Since Y is \mathbb{Q} -factorial klt Fano variety, there is an extremal ray $Q \subseteq \overline{\text{NE}}(Y)$ such that $D_Y \cdot Q > 0$, by Lemma 2.1.1. We denote by $\pi := \varphi_Q : Y \rightarrow Z$ the associated extremal contraction. Let us prove that π is of fiber type. Assume, to the contrary, that π is a birational contraction. Hence, Lemma 2.1.6 implies that both σ and π are divisorial contractions, and that the exceptional locus $\text{Exc}(\pi \circ \sigma)$ consists of two disjoint effective prime divisors. Since $D_Y \cdot Q > 0$ we have that $\text{Exc}(\pi) \cdot [C] > 0$ for every irreducible curve $C \subseteq D_Y$. In particular, $\text{Exc}(\pi) \cap A \neq \emptyset$, as $\dim A = n - 2 \geq 1$, contradicting the fact that the exceptional divisors are disjoint.

Let $\widehat{R} \subseteq \overline{\text{NE}}(X)$ be the extremal ray such that $\text{cont}_{R+\widehat{R}} = \varphi := \pi \circ \sigma$. Then, φ can be factorized as $\varphi = \widehat{\pi} \circ \widehat{\sigma}$, where $\widehat{\sigma} := \text{cont}_{\widehat{R}} : X \rightarrow \widehat{Y}$ and $\widehat{\pi} := \text{cont}_{\widehat{\pi}(R)} : \widehat{Y} \rightarrow Z$. Since φ has fibers of dimension 1, both contractions must have fibers of dimension at most 1. Notice that the general fiber of φ is not contracted by $\widehat{\sigma}$, hence $\widehat{\sigma}$ must be divisorial and $\widehat{\pi}$ a contraction of fiber type, by the same arguments as above. \square

The results of Cutkosky on the contractions of terminal Gorenstein threefolds and Theorem A imply Corollary B.

Proof of Corollary B. — By Theorem A, there is a diagram

$$X \begin{array}{c} \xrightarrow{\sigma} \\ \searrow \varphi \\ \xrightarrow{\pi} \end{array} Y \rightarrow Z ,$$

where $\sigma : X \rightarrow Y$ is a divisorial contraction sending a prime divisor $E = \text{Exc}(\sigma) \subseteq X$ onto a curve $C \subseteq Y$, and π and φ are both extremal contractions of fiber type whose fibers are of dimension 1. All these varieties are \mathbb{Q} -factorial Fano varieties, and Y has terminal singularities. Moreover, X is factorial by [Cut88, Lemm. 2].

By [Cut88, Lemm. 3, Theo. 4], $C \subseteq Y$ is an irreducible reduced curve which is a locally complete intersection, Y is a factorial threefold which is smooth near the curve C , and $\sigma : X \rightarrow Y$ is the blow-up of the ideal sheaf \mathcal{I}_C . In particular, Y is a Gorenstein Fano threefold with terminal singularities and therefore [Cut88, Theo. 7] implies that Z is a smooth surface and $\pi : Y \rightarrow Z$

is a (possibly singular) conic bundle over Z . We note that Z is a rationally connected (and therefore rational) surface with $\rho_Z = 1$, hence $Z \cong \mathbb{P}^2$.

Let $H = \sigma(C) \subseteq Z$. Since π is finite on $D_Y = \sigma(D)$ and $C \subseteq D_Y$, we have that H is an effective prime divisor on \mathbb{P}^2 , which is therefore ample. Let us denote by S_π the locus of points of \mathbb{P}^2 over which π is not a smooth morphism. By [SGAI, Prop. II.1.1], is a closed subset of \mathbb{P}^2 . Then, S_π has pure codimension 1 on \mathbb{P}^2 or $S_\pi = \emptyset$, by [AR14, Theo. 3].

Let us suppose that S_π is not empty. If we take $z \in H \cap S_\pi$ and we denote by $F_z \subseteq Y$ its fiber by π , then we have that $F_z \cap C \neq \emptyset$ (as $z \in H$) and that the 1-cycle on Y associated to F_z is of the form $[F_z] = [C] + [C']$, where C and C' are (possibly equal) irreducible and generically reduced rational curves such that $C_{\text{red}} \cong C'_{\text{red}} \cong \mathbb{P}^1$ (as $z \in S_\pi$), by Remark 2.1.5. Thus, if we denote by $\tilde{F}_z \subseteq X$ the strict transform of F_z on X by σ , we will have $-K_X \cdot [\tilde{F}_z] \geq 3$, contradicting the fact that the anti-canonical degree of every fiber of $\varphi = \pi \circ \sigma$ is 2 (see Remark 2.1.5). We conclude in this way that $\pi : Y \rightarrow \mathbb{P}^2$ is a \mathbb{P}^1 -bundle, and then Y is a smooth threefold by [AR14, Theo. 5].

Finally, let us notice that if D_Y is not nef then there is a birational contraction $Y \rightarrow Y_0$ sending D_Y to a point, by [CD15, Rema. 3.2], and hence [CD15, Lemm. 3.9] implies that $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$, with $0 \leq a \leq 2$ because Y is Fano. On the other hand, if D_Y is nef then we apply [CD15, Prop. 3.3] to conclude that either there is a divisorial contraction $Y \rightarrow Y_0$ sending an effective prime divisor $G_Y \subseteq Y$ to a point, or there is a contraction of fiber type $Y \rightarrow \mathbb{P}^1$; small contractions are excluded since Y is a smooth Fano threefold (see for instance [Cut88] or [KM98, Theo. 1.32]). In the first case [CD15, Lemm. 3.9] allows us to conclude, while in the second case we have that $Y \cong \mathbb{P}^2 \times \mathbb{P}^1$, by [Cas09, Lemm. 4.9]. \square

2.3. The case $\rho_X = 2$

In the case $\rho_X = 2$ we can describe the extremal contraction associated to the other extremal ray in the Mori cone of X (compare with [CD15, Prop. 3.3]). We will need the following result of Ravindra and Srinivas (see [RS06]).

Theorem 2.3.1. — *Let X be a normal projective variety and let \mathcal{L} be an ample and globally generated line bundle over X . Then there is a dense Zariski open set of divisors $E \in |\mathcal{L}|$ such that the restriction map*

$$\text{Cl}(X) \rightarrow \text{Cl}(E)$$

is an isomorphism, if $\dim X \geq 4$, and is injective, with finitely generated cokernel, if $\dim X = 3$.

Corollary 2.3.2. — *Let X be a \mathbb{Q} -factorial Fano variety of dimension $n \geq 3$ with klt singularities. If $\rho_X > 1$ and D is an effective prime divisor such that $\dim_{\mathbb{R}} N_1(D, X) = 1$, then D is not an ample divisor.*

Proof. — Assume, to the contrary, that D is an ample divisor. Let $m \in \mathbb{Z}_{>0}$ such that mD is a very ample Cartier divisor on X and use the complete linear system $|mD|$ to embed $X \hookrightarrow \mathbb{P}(H^0(X, mD)) = \mathbb{P}^N$. Let us define the projective incidence variety

$$\mathfrak{D} = \{(x, [E]) \in X \times |mD| \mid x \in E\} \subseteq X \times |mD| = X \times (\mathbb{P}^N)^*,$$

and let $\pi : \mathfrak{D} \rightarrow (\mathbb{P}^N)^*$ be the second projection.

Let \mathcal{H} be the relative Hilbert scheme of curves associated to the morphism $\pi : \mathfrak{D} \rightarrow (\mathbb{P}^N)^*$, which is a projective scheme with countably many irreducible components. Let us denote by $Z_i \subseteq (\mathbb{P}^N)^*$ the image of the components of \mathcal{H} that do not dominate $(\mathbb{P}^N)^*$. They are closed subsets of $(\mathbb{P}^N)^*$.

Thus, if we take $[E] \in (\mathbb{P}^N)^* - \cup Z_i$ (a very general point on $(\mathbb{P}^N)^*$), then for every curve C_E on E there is a dominant component of \mathcal{H} such that C_E is one the curves parametrized by this component. Since the image of this dominant component is in fact the whole projective space $(\mathbb{P}^N)^*$, then we have that there is a curve C_D on D which is also parametrized for this component. In particular, C_E and C_D are numerically equivalent. But, since there is only one curve on D up to numerical equivalence, we will have $\dim_{\mathbb{R}} N_1(E, X) = 1$.

On the other hand, Theorem 2.3.1 implies that we can also suppose that this very general divisor $E \in |mD|$ is chosen in such a way the restriction

$$\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(E)$$

is injective. Since $\rho_X > 1$ and $\dim_{\mathbb{R}} N_1(E, X) = 1$, we have that the inclusion $N_1(E)_{\mathbb{R}} \rightarrow N_1(X)_{\mathbb{R}}$ is not surjective. Therefore, the induced map on the dual spaces obtained by restriction $N^1(X)_{\mathbb{R}} \rightarrow N^1(E)_{\mathbb{R}}$ is not injective.

Since X is a \mathbb{Q} -factorial klt Fano variety, we have that numerical and linear equivalence coincide, by [AD14, Lemm. 2.5]. Hence, we have the following diagram

$$\begin{array}{ccccc} \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{R} & \xleftarrow{\cong} & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\cong} & N^1(X)_{\mathbb{R}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cl}(E) \otimes_{\mathbb{Z}} \mathbb{R} & \xleftarrow{\quad} & \mathrm{Pic}(E) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\quad} & N^1(E)_{\mathbb{R}} \end{array}$$

Therefore, $N^1(X)_{\mathbb{R}} \rightarrow N^1(E)_{\mathbb{R}}$ is not injective if and only if $\text{Pic}(E) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow N^1(E)_{\mathbb{R}}$ is not injective.

Let us consider a line bundle \mathcal{L} on E such that $\mathcal{L} \equiv 0$. Notice that E has klt singularities since it is a general member of the ample linear system $|mD|$, by [KM98, Lemm. 5.17]. In particular, E has rational singularities and thus if we consider a resolution of singularities $\varepsilon : \tilde{E} \rightarrow E$, then the Leray spectral sequence leads to $h^i(\tilde{E}, \mathcal{O}_{\tilde{E}}) = h^i(E, \mathcal{O}_E)$ and $h^i(E, \mathcal{L}) = h^i(\tilde{E}, \varepsilon^* \mathcal{L})$ for all $i \geq 0$. On the other hand, the short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-E) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0$$

and the vanishing $h^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$, give us $h^1(E, \mathcal{O}_E) = h^2(X, \mathcal{O}_X(-E))$. Since X is a normal variety with Cohen-Macaulay singularities, Serre's duality implies

$$H^2(X, \mathcal{O}_X(-E)) \cong H^2(X, \mathcal{O}_X(-mD)) \cong H^{n-2}(X, \mathcal{O}_X(K_X + mD))^{\vee}.$$

So, by taking m large enough at the beginning if necessary, we can suppose that $h^1(E, \mathcal{O}_E) = 0$ by the Kawamata-Viehweg vanishing theorem.

We get that $h^1(\tilde{E}, \mathcal{O}_{\tilde{E}}) = 0$ and hence an inclusion $\text{Pic}(\tilde{E}) \hookrightarrow H^2(\tilde{E}, \mathbb{Z})$. Clearly $\varepsilon^* \mathcal{L} \equiv 0$ and thus $\varepsilon^* \mathcal{L} \cong \mathcal{O}_{\tilde{E}}$. By the projection formula, $\mathcal{L} \cong \mathcal{O}_E$, a contradiction. \square

We end this section by proving Theorem C, which is an extension of [CD15, Rema. 3.2, Prop. 3.3] to mildly singular Fano varieties X with $\rho_X = 2$, on which there is an effective prime divisor of Picard number one.

Proof of Theorem C. — If D is not nef, then there exists an extremal ray $R \subseteq \text{NE}(X)$ such that $D \cdot R < 0$, and therefore $\text{Exc}(\text{cont}_R) \subseteq D$. Since $\dim_{\mathbb{R}} N_1(D, X) = 1$ we must have that $\text{Exc}(\text{cont}_R) = D$ and that $\text{cont}_R(D)$ is a point.

If D is nef, then it is not ample by Corollary 2.3.2, and thus $S = D^{\perp} \cap \text{NE}(X)$ is an extremal ray, as $\rho_X = 2$. If we denote $G = \text{Locus}(S) \subseteq X$, the proof follows almost verbatim the proof in the smooth case given in [CD15, Prop. 3.3]. For reader's convenience, let us briefly sketch the proof:

If $S \subseteq N_1(D, X)$ then cont_S sends D to a point. On the other hand, $D \cdot S = 0$ and hence D is the pullback of a divisor via cont_S , from which we conclude that the target of cont_S is \mathbb{P}^1 and hence that we are in case (a).

If $S \not\subseteq N_1(D, X)$ we verify that $G \cap D = \emptyset$ and hence cont_S is birational. Moreover $N_1(G, X) \subseteq D^{\perp}$ has dimension 1 and hence cont_S sends G to points. If cont_S is a divisorial contraction then we are in case (b).

If cont_S is a small contraction, we consider the associated flip $X \dashrightarrow X'$ and we denote by D' the strict transform of D in X' and by S' the corresponding small extremal ray on X' with exceptional locus $G' \subseteq X'$ and satisfying $K_{X'} \cdot S' > 0$. Since X is Fano and $G \cap D = \emptyset$ we have that X' has an extremal ray $T \subseteq \overline{NE}(X')$ such that $-K_{X'} \cdot T > 0$ and $D' \cdot T > 0$, from which we deduce that if we denote $\text{cont}_T =: \psi : X' \rightarrow Y'$ then $T \not\subseteq N_1(D', X')$ and hence ψ is finite both on D' and G' . In particular, every non trivial fiber of ψ has dimension 1, since $D' \cdot T > 0$. It only remains to prove that ψ is of fiber type. Suppose, to the contrary, that ψ is a birational contraction. If we suppose that $\text{Exc}(\psi) \cap G \neq \emptyset$ and we consider F_0 to be an irreducible component of a fiber of ψ that intersects G' then it follows from the fact that X is Fano Gorenstein that $-K_{X'} \cdot [F_0] > 1$, by [Del14, Lemm. 3.2] (which is the singular version of [Cas09, Lemm. 3.8], used in step (3.3.5) of the proof of [CD15, Prop. 3.3]). On the other hand, we notice that $F_0 \not\subseteq G'$ since ψ is finite on G' and hence [Ish91, Lemm. 1.1] can be applied (since F_0 contains a Gorenstein point of X') in order to deduce that $-K_{X'} \cdot [F_0] \leq 1$, a contradiction. We obtain therefore that $\text{Exc}(\psi) \cap G' = \emptyset$, so that $\text{Exc}(\psi)$ is contained in the Gorenstein locus of X' and therefore [Del14, Theo. 2.2] implies that $\text{Exc}(\psi)$ is a divisor. But in this case we would have that $\text{Exc}(\psi) \cdot S' = 0$ and $\text{Exc}(\psi) \cdot T < 0$ implying that $-\text{Exc}(\psi)$ is nef, a contradiction. We conclude therefore that ψ is of fiber type. \square

2.4. Study of toric extremal contractions

In the setting of toric varieties, we will be interested in analyzing the extremal contractions appearing in §2.1 in terms of the description given by Theorem 1.3.7. Let us begin by the birational case.

Lemma 2.4.1. — *Let $X = X(\Delta_X)$ be a \mathbb{Q} -factorial Gorenstein toric variety of dimension $n \geq 3$. Let $\varphi_R : X \rightarrow X_R$ be a divisorial contraction with exceptional divisor $\text{Exc}(\varphi_R) = E$ such that*

1. $A = \varphi_R(E)$ is an invariant subvariety of codimension two.
2. $E \cdot [F] = K_X \cdot [F] = -1$ for every non-trivial fiber F of φ_R .

Let us suppose that $\varphi_R : X \rightarrow X_R$ is defined by the contraction of the wall $\omega = \text{cone}(u_1, \dots, u_{n-1})$ that separates the maximal cones $\sigma = \text{cone}(u_1, \dots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, u_{n+1})$. Then, up to reordering if necessary, the wall relation satisfied by these cones is of the form

$$u_n + u_{n+1} = u_1.$$

If moreover X has isolated singularities, then X_R has isolated Gorenstein singularities, A is contained in the smooth locus of X_R , and $X \cong \text{Bl}_A(X_R)$.

Proof. — By Theorem 1.3.7, up to reordering if necessary, we can suppose that $\varphi_R : X \rightarrow X_R$ is defined by the relation

$$(1) \quad \alpha u_n + \lambda u_1 + \beta u_{n+1} = 0,$$

where $\alpha, \beta \in \mathbb{Z}_{>0}$, $\lambda \in \mathbb{Z}_{<0}$ and $E = V(u_1)$. Let us denote by $C = V(\omega)$ the invariant curve associated to the wall ω .

By Proposition 1.3.8, $V(u) \cdot [C] = 0$ for $u \notin \{u_1, \dots, u_{n+1}\}$. Moreover, the wall relation gives us $V(u_i) \cdot [C] = 0$ for $i \in \{2, \dots, n-1\}$ and

$$V(u_1) \cdot [C] = \frac{\lambda}{\alpha} V(u_n) \cdot [C] = \frac{\lambda}{\beta} V(u_{n+1}) \cdot [C].$$

By hypothesis, $V(u_1) \cdot [C] = -1$ and thus $V(u_n) \cdot [C] = -\frac{\alpha}{\lambda}$ and $V(u_{n+1}) \cdot [C] = -\frac{\beta}{\lambda}$.

It is well known that for a toric variety $X = X(\Delta_X)$ we have that $K_X = -\sum_{\rho \in \Delta_X(1)} V(\rho)$ is an invariant canonical divisor on X (see [CLS11, Theo. 8.2.3]), and thus the condition $-K_X \cdot [C] = 1$ can be translated into

$$(2) \quad -1 + \left(-\frac{\alpha}{\lambda}\right) + \left(-\frac{\beta}{\lambda}\right) = 1 \Leftrightarrow \alpha + \beta = -2\lambda.$$

Let us note that we can suppose that $\gcd(\lambda, \alpha) = \gcd(\lambda, \beta) = \gcd(\alpha, \beta) = 1$. In fact, if $\gcd(\alpha, \beta) = d > 1$, then the equation (1) implies $d|\lambda$, as u_1 is a primitive vector. The same argument applies to the other two pairs.

By assumption, $K_X = -\sum_{\rho \in \Delta_X(1)} V(\rho)$ is a Cartier divisor, i.e., for each maximal cone $\sigma \in \Delta_X(n)$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho \rangle = 1$ for all $\rho \in \sigma(1)$. In our setting, this condition applied to the two maximal cones σ and σ' tells us that there exists two elements $m, m' \in M$ such that

$$\begin{aligned} \langle m, u_i \rangle &= 1 & \text{for } i &\in \{1, \dots, n\}, \\ \langle m', u_i \rangle &= 1 & \text{for } i &\in \{1, \dots, n-1, n+1\}. \end{aligned}$$

From the equation (1) we obtain

$$(3) \quad \alpha + \lambda + \beta \langle m, u_{n+1} \rangle = 0, \text{ and}$$

$$(4) \quad \alpha \langle m', u_n \rangle + \lambda + \beta = 0$$

By using the equation (2) and (3) we obtain that $\lambda = \beta(\langle m, u_{n+1} \rangle - 1)$ and thus $\beta = 1$, as $\beta \in \mathbb{Z}_{>0}$ and $\gcd(\lambda, \beta) = 1$. In the same way, by using the equation (2) and (4), we deduce that $\alpha = 1$ and hence $\lambda = -1$. Finally, we get the relation

$$u_n + u_{n+1} = u_1,$$

and $\langle m, u_{n+1} \rangle = \langle m', u_n \rangle = 0$.

Let us suppose now that X has isolated singularities. We note that there are exactly $n - 1$ walls satisfying this relation (corresponding to the fibers over $n - 1$ invariant points of A), ω_i with $i = 1, \dots, n - 1$, each of them separating two maximal cones σ_i and σ'_i . It follows from the wall relation above and Proposition 1.3.8 that

$$E \cdot [C_{\omega_i}] = -\frac{\text{mult}(\omega_i)}{\text{mult}(\sigma_i)} = -\frac{\text{mult}(\omega_i)}{\text{mult}(\sigma'_i)} = -1,$$

and

$$-K_X \cdot [C_{\omega_i}] = \frac{\text{mult}(\omega_i)}{\text{mult}(\sigma_i)} = \frac{\text{mult}(\omega_i)}{\text{mult}(\sigma'_i)} = 1.$$

Therefore, $\text{mult}(\omega_i) = \text{mult}(\sigma_i) = \text{mult}(\sigma'_i)$.

On the other hand, Remark 1.3.3 implies that $\text{mult}(\omega_i) = 1$ since X has isolated singularities. Hence, both σ_i and σ'_i are smooth cones in $\Delta_X(n)$. We get that the associated maximal cone $\tau_i \in \Delta_{X_R}(n)$ obtained from σ_i and σ'_i by removing the ray corresponding to the exceptional divisor $E = \text{Exc}(\varphi_R)$ is also smooth, for $i = 1, \dots, n - 1$. Thus, $A = V(u_n, u_{n+1})$ is contained in the smooth locus of X_R . The isomorphism $X \cong \text{Bl}_A(X_R)$ follows from the description of the blow-up of a smooth toric variety along an irreducible invariant smooth subvariety (see [CLS11, Definition 3.3.17]). A posteriori, we note that X_R also have isolated Gorenstein singularities since A is contained in the smooth locus of X_R . \square

Let us consider now the case when $\varphi_R : X \rightarrow X_R$ is a contraction of fiber type.

Remark 2.4.2 (Flatness of toric fibrations). — Contrary to what was originally written in [Rei83, Cor. 2.5], the restriction of an extremal contraction to its exceptional locus may not be flat in general, as Miles Reid communicated to the author. See [AR14, Exam. 13] for a local counterexample that can be compactified by [Fuj06]. We refer the interested reader to Kato's criterion of flatness for toric morphisms [Kat89, Prop. 4.1].

We will need the following result concerning extremal contractions of fiber type.

Lemma 2.4.3. — *Let X be a \mathbb{Q} -factorial Gorenstein toric variety of dimension $n \geq 3$ and let $\varphi_R : X \rightarrow X_R$ be a proper toric morphism given by the contraction of an extremal ray $R \subseteq \overline{\text{NE}}(X)$ of fiber type. Suppose that all the fibers of φ_R are of dimension 1. If φ_R has a non-reduced fiber then $\text{Sing}(X)$*

has an irreducible component of codimension 2. In particular, if X has at most finitely many non-terminal points then φ_R is a \mathbb{P}^1 -bundle and there exists a split vector bundle E of rank 2 on X_R and an isomorphism $X \cong \mathbb{P}_{X_R}(E)$ over X_R .

Proof. — The statement is local on X_R so we may suppose that $X_R = U_\tau$ is an affine simplicial toric variety. It follows from Theorem 1.3.7 that if $\varphi_R : X \rightarrow X_R$ is an extremal contraction of pure relative dimension one then all the fibers of φ_R are irreducible, the fan of $X = \varphi_R^{-1}(U_\tau)$ has two maximal cones $\sigma = \text{cone}(u_1, \dots, u_n)$ and $\sigma' = \text{cone}(u_0, \dots, u_{n-1})$ such that $u_0 = -u_n$, and φ_R is induced by the projection $\Phi : N_X \rightarrow N_{X_R} = N_X / (\text{Span}(u_n) \cap N_X)$ onto $\tau = \text{cone}(t_1, \dots, t_{n-1})$. Since X is smooth in codimension one, we can always suppose that $u_n = e_n$ and hence that φ_R is induced by the projection $\Phi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, 0)$.

It follows from [Kar99, Ch.2, Lemm. 5.2] that if φ_R has a non-reduced fiber over U_τ then there exists u_i such that $\Phi(u_i)$ is not primitive on N_{X_R} for some $i = 1, \dots, n-1$. Let us suppose therefore that $\Phi(u_i) = \lambda_i t_i$ with $t_i \in \mathbb{Z}^{n-1} \times \{0\}$ primitive lattice vector and $\lambda_i \geq 2$. Let us suppose that $u_i = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and hence that $\Phi(u_i) = (a_1, \dots, a_{n-1}, 0) = \lambda_i(\bar{a}_1, \dots, \bar{a}_{n-1}, 0)$ with $t_i = (\bar{a}_1, \dots, \bar{a}_{n-1}, 0)$ primitive lattice vector.

Since X is Gorenstein there are integer (dual) vectors $m_\sigma = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and $m'_\sigma = (m'_1, \dots, m'_n) \in \mathbb{Z}^n$ such that $\langle m_\sigma, u_i \rangle = 1$ for $i = 1, \dots, n$ and $\langle m'_\sigma, u_i \rangle = 1$ for $i = 0, \dots, n-1$, by Theorem 1.3.4. In particular we have that $m_n = 1$, $m'_n = -1$ and

$$(5) \quad \langle m_\sigma, u_i \rangle = m_1 a_1 + \dots + m_{n-1} a_{n-1} + a_n = 1,$$

$$(6) \quad \langle m'_\sigma, u_i \rangle = m'_1 a_1 + \dots + m'_{n-1} a_{n-1} - a_n = 1.$$

By adding equation (1) and (2) above, we get that $\lambda_i \langle m_\sigma + m'_\sigma, t_i \rangle = 2$ and hence $\lambda_i = 2$. We can write therefore $u_i = (2\bar{a}_1, \dots, 2\bar{a}_{n-1}, a_n) \in \mathbb{Z}^n$ with a_n odd number and $t_i = (\bar{a}_1, \dots, \bar{a}_{n-1}, 0)$.

Let us produce now a lattice vector $\bar{u}_i \in \mathbb{Z}^n$ lying over t_i and belonging to the hyperplane $\{u \in \mathbb{R}^n \mid \langle m_\sigma, u \rangle = 1\}$. We have that

$$\begin{aligned} 1 &= m_1 a_1 + \dots + m_{n-1} a_{n-1} + a_n \\ &= m_1 2\bar{a}_1 + \dots + m_{n-1} 2\bar{a}_{n-1} + a_n \\ &= m_1 \bar{a}_1 + \dots + m_{n-1} \bar{a}_{n-1} + b = \langle m_\sigma, \bar{u}_i \rangle, \end{aligned}$$

where $\bar{u}_i = (\bar{a}_1, \dots, \bar{a}_{n-1}, b)$ and $b = \bar{a}_1 + \dots + m_{n-1}\bar{a}_{n-1} + a_n \in \mathbb{Z}$. We note that $2\bar{u}_i = u_i + u_n$ and thus $\bar{u}_i \in \text{cone}(u_i, u_n)$. We conclude from Proposition 1.3.2 and Theorem 1.3.4 that X has canonical singularities along the codimension two closed subvariety $V(u_i, u_n)$, which are therefore non terminal by codimension reasons.

Finally, if X has at most finitely many non-terminal points we conclude therefore that all the fibers of φ_R are irreducible and reduced and thus φ_R is a \mathbb{P}^1 -bundle. Moreover, the invariant divisors $V(u_0)$ and $V(u_n)$ correspond to two disjoint invariant sections $s_0 : X_R \rightarrow X$ and $s_\infty : X_R \rightarrow X$ passing through the two invariant points of all the fibers of φ . Hence, [AR14, Rema. 8] implies that there exists a rank 2 split vector bundle E such that $X \cong \mathbb{P}_{X_R}(E)$. \square

2.5. The extremal case $\rho_X = 3$ for toric varieties

We are now able to prove the structure theorem for toric varieties with Picard number 3.

Proof of Theorem D. — By Theorem A, we obtain a diagram

$$\begin{array}{ccccc} X & \xrightarrow{\sigma} & Y & \xrightarrow{\pi} & Z, \\ & & \searrow & \swarrow & \\ & & & & \varphi \end{array}$$

where $\sigma : X \rightarrow Y$ is a divisorial contraction sending a toric prime divisor E onto an invariant subvariety A of codimension two on Y , and π and φ are both extremal contractions of fiber type whose fibers are of dimension 1. All these varieties are \mathbb{Q} -factorial toric Fano varieties.

Let us first prove that $\pi : Y \rightarrow Z$ is a \mathbb{P}^1 -bundle. It follows from [SGAI, Prop. II. 1.1] that S_π , the locus of points of Z over which π is not a smooth morphism, is a closed subset of Z . Moreover, as we noticed in Remark 2.1.5, we have that if X is a \mathbb{Q} -factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points then π and φ are *generalized conic bundles* and therefore the 1-cycles associated to singular fibers of π are of the form $[F] = [C] + [C']$ with C and C' (eventually coincident) irreducible and generically reduced rational curves on Y such that $C_{\text{red}} \cong C'_{\text{red}} \cong \mathbb{P}^1$ and $-K_X \cdot [\tilde{C}] = -K_X \cdot [\tilde{C}'] = -K_Y \cdot [C] = -K_Y \cdot [C'] = 1$, where \tilde{C} (resp. \tilde{C}') is the total transform of C (resp. C') on X via σ . In particular, we observe as in the Proof of Corollary B that the singular fibers of π must be disjoint from $A \subseteq Y$ and hence they are contained in the Gorenstein locus of Y . We conclude from Lemma 2.4.3 that $\pi : Y \rightarrow Z$ is a \mathbb{P}^1 -bundle isomorphic to the projectivization of the rank 2 split vector bundle $E = \mathcal{L}' \oplus \mathcal{L}$.

As $\mathbb{P}_Z(E) \cong \mathbb{P}_Z(E \otimes \mathcal{M})$ for any line bundle $\mathcal{M} \in \text{Pic}(Z)$, we can suppose that $\mathcal{L}' \cong \mathcal{O}_Z$. On the other hand, since $\text{Pic}(Z)$ is isomorphic to \mathbb{Z} , we can consider an ample generator $\mathcal{O}_Z(1)$ of $\text{Pic}(Z)$ and an integer $a \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}_Z(a)$. Up to tensor by \mathcal{L}^\vee , we can always suppose that $a \geq 0$. In particular, both Y and Z must have at most terminal singularities, since Y has at most a finite number of canonical singularities and π is locally trivial. Moreover, $0 \leq a \leq i_Z - 1$ by Proposition F below.

It should be noticed that in this case $A \subseteq Y$ is contained in one of the two disjoint invariant sections associated to the \mathbb{P}^1 -bundle $\pi : Y \rightarrow Z$. In fact, by Theorem 1.3.7 (taking $\alpha = 0$ and $\beta = n - 1$) we have that $\pi : Y \rightarrow Z$ is defined by a wall relation of the form

$$b_n u_n + \sum_{i=1}^{n-1} 0 \cdot u_i + b_{n+1} u_{n+1} = 0,$$

where $\omega = \text{cone}(u_1, \dots, u_{n-1})$ is a wall generating the extremal ray associated to this contraction, which separates the two maximal cones $\sigma = \text{cone}(u_1, \dots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, u_{n+1})$.

Thus $U(\omega) = \text{cone}(u_n, u_{n+1}) = \text{Span}(u_n) = \text{Span}(u_{n+1}) \cong \mathbb{R}$ and $\pi : Y \rightarrow Z$ is induced by the quotient $N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/U(\omega)$. Since $A \subseteq D_Y = \sigma(D)$ and $\pi|_{D_Y}$ is a finite morphism, A is sent onto an invariant subvariety of dimension $n - 2$, a divisor on Z . Thus, $A \subseteq D(u_n) := D_0$ or $A \subseteq D(u_{n+1}) := D_\infty$, otherwise it will be sent onto a subvariety of dimension $n - 3$.

If we suppose that $A \subseteq D_\infty$, where $D_\infty \cong Z \subseteq Y$ is one of these disjoint invariant sections, then for any invariant affine open subset $U \subseteq Z$ such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$ we will have

$$\pi^{-1}(U) \setminus (\pi^{-1}(U) \cap D_\infty) \cong U \times \mathbb{A}^1.$$

The open set $U \times \mathbb{A}^1$ is therefore isomorphic to an open set on Y contained in the locus where $\sigma^{-1} : Y \dashrightarrow X$ is an isomorphism. In particular, $U \times \mathbb{A}^1$ is an affine Gorenstein toric variety and thus U is also Gorenstein. We conclude in this way that both Y and Z are Gorenstein varieties.

As a consequence of the formula $K_X = \sigma^*(K_Y) + E$ we have that $E = \text{Exc}(\sigma)$ is a Cartier divisor. Let us prove that $X \rightarrow Y$ verifies the universal property of the blow-up. The short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-E) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0$$

gives

$$0 \rightarrow \sigma_* \mathcal{O}_X(-E) \rightarrow \sigma_* \mathcal{O}_X \rightarrow (\sigma|_E)_* \mathcal{O}_E \rightarrow R^1 \sigma_* \mathcal{O}_X(-E) \rightarrow \dots,$$

where $\sigma_*\mathcal{O}_X = \mathcal{O}_Y$ since $\sigma : X \rightarrow Y$ is a contraction.

Notice that $\sigma|_E : E \rightarrow A$ is a \mathbb{P}^1 -bundle. In fact, since K_X is a Cartier divisor and $-K_X \cdot [F] = 1$ for any non-trivial fiber F of σ , it follows that the scheme theoretic fiber F is an irreducible and generically reduced rational curve on X . Then, by [Kol96, Theo. II.2.8], $\sigma|_E : E \rightarrow A$ is a \mathbb{P}^1 -bundle and thus $(\sigma|_E)_*\mathcal{O}_E = \mathcal{O}_A$.

On the other hand, the Cartier divisor $-(K_X + E)$ is σ -ample and therefore $R^i\sigma_*\mathcal{O}_X(-E) = 0$ for $i > 0$, by [AW97, Vanishing Theorem 1.1]. Hence, the above long exact sequence becomes

$$0 \rightarrow \sigma_*\mathcal{O}_X(-E) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_A \rightarrow 0,$$

and thus $\mathcal{I}_A \cong \sigma_*\mathcal{O}_X(-E)$.

Let us follow [AW93] and notice that $\sigma : X \rightarrow Y$ is a local contraction supported by the Cartier divisor $K_X - E$. Let F be any non-trivial fiber of σ . Then, by [AW93, Theo. 5.1], the evaluation morphism

$$\sigma^*\sigma_*\mathcal{O}_X(-E) \rightarrow \mathcal{O}_X(-E)$$

is surjective at every point of F . On the other hand, $\sigma_*\mathcal{O}_X(-E) \cong \mathcal{I}_A$ and $\sigma^{-1}\mathcal{I}_A \cdot \mathcal{O}_X$ is defined to be the image of $\sigma^*\mathcal{I}_A \rightarrow \mathcal{O}_X(-E)$. Thus, $\sigma^{-1}\mathcal{I}_A \cong \mathcal{O}_X(-E)$ is an invertible sheaf.

Then, by the universal property of the normalized blow-up, σ factorizes as

$$X \begin{array}{c} \xrightarrow{\tau} \overline{\text{Bl}}_A(Y) \xrightarrow{\nu \circ \varepsilon} Y \\ \searrow \sigma \nearrow \end{array}$$

where $\varepsilon : \overline{\text{Bl}}_A(Y) \rightarrow Y$ is the blow-up of the coherent sheaf of ideals \mathcal{I}_A and $\nu : \overline{\text{Bl}}_A(Y) \rightarrow \text{Bl}_A(Y)$ its normalization.

Since σ contracts only the irreducible divisor E , τ contracts no divisor. If τ is not finite, it is a small contraction sending a curve $C \subseteq E$ to a point. The rigidity lemma [KM98, Lemm. 1.6] applied to the \mathbb{P}^1 -bundle $E \rightarrow A$ and the morphism $\tau(E) \rightarrow A$ implies that τ contracts the divisor E , a contradiction. Hence τ is a finite and birational morphism onto a normal variety, and therefore τ is an isomorphism by Zariski's Main Theorem.

Finally, it follows from [CLS11, Prop. 11.4.22] that if $X = X(\Delta_X)$ is a \mathbb{Q} -factorial toric variety with Gorenstein terminal singularities, then $\text{codim}_X \text{Sing}(X) \geq 4$ (see also Proposition 1.2.7). Therefore, if we start with a toric variety X of dimension 3 or 4 satisfying the hypothesis of Theorem D, we will obtain that Z is a smooth toric variety, implying that Y and X must be both smooth toric varieties too. \square

2.6. The case $\rho_X = 2$ for toric varieties

In this section we study the extremal contractions described in §2.1 for toric varieties with Picard number 2. Let us begin with the proof of Proposition F, which describes extremal contractions of fiber type.

Proof of Proposition F. — The isomorphism $X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$, with $a \geq 0$, follows from Lemma 2.4.3 and the fact that $\rho_Y = 1$. On the other hand, the condition $0 \leq a \leq i_Y$ is in fact equivalent to the condition of X being Fano. To see this, let us recall that the adjunction formula for projectivized vector bundles [BS95, §1.1.7] gives

$$K_X = \pi^* \mathcal{O}_Y(a - i_Y) - 2\xi,$$

where ξ is the tautological divisor on $\mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$ and i_Y is defined in such a way $\mathcal{O}_Y(-K_Y) \cong \mathcal{O}_Y(i_Y)$. Now, we notice that if F is any fiber π , then

$$-K_X \cdot [F] = 2\xi \cdot [F] = 2 > 0.$$

On the other hand, if $C \subseteq Y$ is an irreducible reduced curve and $C_X \subseteq X$ is the image of C by the section associated to the quotient $\mathcal{O}_Y \oplus \mathcal{O}_Y(a) \rightarrow \mathcal{O}_Y$, then

$$-K_X \cdot [C_X] = \deg(C) \cdot (i_Y - a).$$

Hence, X is Fano if and only if $0 \leq a \leq i_Y - 1$, since the numerical classes of these curves above generates the Mori cone of X . Finally, let us note that the two invariant sections of π , $s_0 : Y \rightarrow X$ and $s_\infty : Y \rightarrow X$, provide divisors $D_0 = s_0(Y)$ and $D_\infty = s_\infty(Y)$ on X such that $D_0 \cong D_\infty \cong Y$ and hence $\dim_{\mathbb{R}} N_1(D_0, X) = \dim_{\mathbb{R}} N_1(D_\infty, X) = 1$. \square

If X is supposed to have isolated canonical singularities then we obtain a precise classification in Theorem E.

Proof of Theorem E. — Let $R \subseteq \overline{\text{NE}}(X)$ be an extremal ray such that $D \cdot R > 0$ and let $\pi : X \rightarrow Y$ be the corresponding extremal contraction. If π is of fiber type then Proposition F provides an isomorphism $X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$ for some toric variety Y . Moreover, Y is a \mathbb{Q} -factorial Gorenstein toric Fano variety of dimension $(n - 1)$ with terminal singularities and Fano index i_Y , and $0 \leq a \leq i_Y - 1$. Since X has isolated singularities, and $\pi : X \rightarrow Y$ is a locally trivial \mathbb{P}^1 -bundle, both X and Y are smooth in this case. It follows that $Y \cong \mathbb{P}^{n-1}$ and $i_Y = n$, which leads us to the first case.

Let us suppose that $\pi : X \rightarrow Y$ is a birational contraction. It follows from Proposition 2.1.3 that $\pi : X \rightarrow Y$ is a divisorial contraction sending an irreducible invariant divisor $E = V(u_E)$ onto a codimension two subvariety $A \subseteq Y$,

and Y is a \mathbb{Q} -factorial toric Fano variety. Moreover, $E \cdot [F] = K_X \cdot [F] = -1$ for every non-trivial fiber F of π . In particular, Lemma 2.4.1 implies that Y has isolated Gorenstein singularities, A is contained in the smooth locus of Y , and $X \cong \text{Bl}_A(Y)$. Additionally, it follows from the proof of Lemma 2.4.1 that Y contains $n - 1$ smooth maximal cones, each of them containing the cone of dimension two defining the subvariety A .

Since Y is a complete and simplicial toric variety of dimension n and Picard number one, the fan of Y contains exactly $n + 1$ maximal cones (corresponding to $n + 1$ invariant points), Y has at most 2 singular points and they are outside $A \subseteq Y$.

Let us denote by $u_1, \dots, u_{n+1} \in N$ the primitive lattice vectors generating the rays in the fan of Y and suppose that $A = \text{cone}(u_1, u_2)$, and thus that the extremal contraction π is defined by the wall relation $u_E = u_1 + u_2$. There are exactly $n - 1$ walls in Δ_X satisfying this relation (corresponding to the fibers over the $n - 1$ invariant points of A). Namely, the walls

$$\omega_i = \text{cone}(u_E, u_3, \dots, \widehat{u}_i, \dots, u_{n+1}) \quad \text{with } i \in \{3, \dots, n + 1\},$$

separating the two maximal cones $\sigma_i = \text{cone}(u_E, u_3, \dots, \widehat{u}_i, \dots, u_{n+1}, u_1)$ and $\sigma'_i = \text{cone}(u_E, u_3, \dots, \widehat{u}_i, \dots, u_{n+1}, u_2)$.

Let us prove that Y is isomorphic to one of the listed varieties. Since the fan of Y contains a smooth maximal cone, we can suppose that the vectors u_1, \dots, u_n correspond to the first n elements of the canonical basis of \mathbb{Z}^n , by Remark 1.3.3.

Let us write $u_{n+1} = (-a_1, \dots, -a_n)$, with $a_i \in \mathbb{Z}_{>0}$ for $i \in \{1, \dots, n\}$. For each $i \in \{3, \dots, n + 1\}$ we have that $\text{mult}(\sigma_i) = 1$. Therefore, Proposition 1.3.2 leads $\text{mult}(\sigma_i) = |\det(e_1, e_1 + e_2, e_3, \dots, \widehat{e}_i, \dots, e_n, u_{n+1})| = a_i = 1$, for $i \in \{3, \dots, n\}$. Hence, we can write $u_{n+1} = (-a, -b, -1, \dots, -1)$, with $a, b \in \mathbb{Z}_{>0}$. It should be noticed that Y has isolated singularities if and only if $\text{gcd}(a, b) = 1$.

Thus, $Y \cong \mathbb{P}(1^{n-1}, a, b)$ with $a, b \in \mathbb{Z}_{>0}$ relatively prime integers. Now, Y is a Gorenstein Weighted Projective Space if and only if $a|(n - 1 + a + b)$ and $b|(n - 1 + a + b)$, by [CK99, Lemm. 3.5.6]. Equivalently, $a|(n - 1 + b)$ and $b|(n - 1 + a)$. If $a = b$ the only possibility is $a = b = 1$, leading to (a): $Y \cong \mathbb{P}^n$.

Let us suppose that $1 \leq a < b$ and notice that $ab|(n-1+a+b)$ since $\gcd(a, b) = 1$. On the other hand,

$$\begin{aligned} \frac{n-1+a+b}{ab} = 1 &\Leftrightarrow \frac{n-1+a+b}{ab} < 2 \\ &\Leftrightarrow (2b-1) \left(a - \frac{1}{2} \right) - n + \frac{1}{2} > 0. \end{aligned}$$

Since $a \geq 1$, this condition is fulfilled when $(2b-1) \cdot \frac{1}{2} - n + \frac{1}{2} > 0 \Leftrightarrow b > n$.

Therefore, $b \geq n+1$ implies that $n-1+a+b = ab$ or, equivalently, $n = (a-1)(b-1)$. This leads to $a = 2$, $b = n+1$ and hence to (b): $Y \cong \mathbb{P}(1^{n-1}, 2, n+1)$ and n must be even. Finally, if $1 \leq a < b \leq n$ we get the last case (c): $Y \cong \mathbb{P}(1^{n-1}, a, b)$.

Conversely, given one of these listed varieties Y with their fans as above and considering X to be the blow-up of Y along $A = V(e_1, e_2)$, we obtain a projective toric variety satisfying the hypothesis.

In fact, since A is contained in the smooth locus of Y we obtain that X is a \mathbb{Q} -factorial Gorenstein toric variety with isolated canonical singularities. In order to prove that X is Fano we need to analyze the second extremal contraction that corresponds to the wall relation on X given by

$$bu_E + (a-b)e_1 + e_3 + \cdots + e_n + u_{n+1} = 0.$$

In any of the three listed cases we will obtain

$$-K_X \cdot [C_\omega] = \frac{n-1+a}{b} \in \mathbb{Z}_{>0},$$

proving that X is a Fano variety, by the Cone Theorem and Kleiman's criterion of ampleness. Finally, let us note that $A \subseteq Y$ is the intersection of two invariant prime divisors $D_1 = V(u_1)$ and $D_2 = V(u_2)$, whose strict transforms \widehat{D}_1 and \widehat{D}_2 in X satisfy $\dim_{\mathbb{R}} N_1(\widehat{D}_1, X) = \dim_{\mathbb{R}} N_1(\widehat{D}_2, X) = 1$, which follows from the wall relation $u_E = u_1 + u_2$ and the description of the fans defining these divisors as toric varieties (see for instance [CLS11, Prop. 3.2.7]). \square

As a consequence we obtain the following list of possible admissible weights $(a, b) \in \mathbb{Z}_{>0}^2$ that corresponds to varieties $Y \cong \mathbb{P}(1^{n-1}, a, b)$ as in Theorem E.2. Compare with [Kas13] and [Mir85].

TABLE 1. Admissible $Y \cong \mathbb{P}(1^{n-1}, a, b)$ as in Theorem E.2

n	Weights $(a, b) \in \mathbb{Z}_{>0}^2$
3	(1, 1), (1, 3)
4	(1, 1), (1, 2), (1, 4), (2, 5)
5	(1, 1), (1, 5)
6	(1, 1), (1, 3), (1, 6), (2, 7), (3, 4)
7	(1, 1), (1, 7)
8	(1, 1), (1, 2), (1, 4), (1, 8), (2, 3), (2, 9), (3, 5)
9	(1, 1), (1, 3), (1, 9)
10	(1, 1), (1, 2), (1, 5), (1, 10), (2, 11)

Remark 2.6.1. — From the wall relation

$$bu_E + (a - b)e_1 + e_3 + \cdots + e_n + u_{n+1} = 0$$

we can deduce the nature of the second extremal contraction $\varphi : X \rightarrow W$. In the smooth case it is a contraction of fiber type onto $W \cong \mathbb{P}^1$. In the singular case, it is a divisorial contraction sending its exceptional locus onto a point. Moreover, $W \cong \mathbb{P}(1^{n-1}, a, b - a)$ since

$$au_E + (b - a)e_2 + e_3 + \cdots + e_n + u_{n+1} = 0.$$

We conclude with an example showing that the hypothesis of isolated singularities in Theorem E.2 cannot be dropped. We exhibit an example of a \mathbb{Q} -factorial Gorenstein toric Fano fivefold X with terminal singularities, whose singular locus is one-dimensional and that admits a birational extremal contraction $\pi : X \rightarrow Y$ which is not a blow-up, but only a blow-up in codimension two, and where Y is a non-Gorenstein \mathbb{Q} -factorial toric Fano fivefold.

Example 2.6.2. — Let us consider the fan $\Delta_X \subseteq \mathbb{R}^5$ generated by the vectors

$$e_1, e_2, e_3, e_4, e_5, u_6, u_E,$$

where $\{e_1, \dots, e_5\}$ is the canonical basis of \mathbb{R}^5 , $u_6 = (-1, -1, -1, -2, -3)$ and $u_E = (-1, -1, -1, -2, -2)$.

It can be checked by using [Macaulay2] that X is a \mathbb{Q} -factorial Gorenstein Fano fivefold. Moreover, it can be checked by hand that its singular locus is

one-dimensional, consisting only of terminal points, and given by

$$\text{Sing}(X) = V(e_1, e_2, e_3, u_E) \cup V(e_1, e_2, e_3, e_4, u_6).$$

The wall relation $e_5 + (-1)u_E + u_6 = 0$ determines an extremal contraction $\pi : X \rightarrow Y$ sending the Weil divisor $V(u_E) \subseteq X$ (which is not Cartier) onto $A = V(e_5, u_6) \subseteq Y$. Finally, the relation

$$e_1 + e_2 + e_3 + 2e_4 + 3e_5 + u_6 = 0$$

implies that $Y \cong \mathbb{P}(1^4, 2, 3)$, which is not Gorenstein, by [CK99, Lemm. 3.5.6].

2.7. Toric universal coverings in codimension 1

Let us recall that, every complete \mathbb{Q} -factorial toric variety of Picard number one can be seen as the quotient of a Weighted Projective Space by the action of a finite group such that the corresponding quotient map is *quasi-étale*, i.e., a finite surjective morphism which is étale in codimension 1 (see Definition 2.7.1 and Example 2.7.6). In this sense, Weighted Projective Spaces are the simplest singular \mathbb{Q} -factorial toric varieties of Picard number one. In the context of classification of toric Fano varieties with mild singularities, we would like to consider similar covers, but without the restriction on the Picard number. This is because quasi-étale morphisms have no ramification divisor and thus we can easily show that the resulting covering variety will be Fano and will have mild singularities (see Proposition 2.7.3). Therefore, in principle, it would be enough to consider the simplest toric varieties from the point of view of these quasi-étale morphisms in order to give a classification for the general case. We also refer the reader to the recent article of Greb, Kebekus and Peternell [GKP16] where they constructed and studied these spaces for klt pairs.

The aim of this section is therefore to recall some recent results due to Rossi and Terracini concerning the construction and combinatorics of these quasi-étale universal covers in the toric setting and to describe the lifting of the extremal contractions appearing in the previous sections to these spaces.

Let us recall some of the results and definitions introduced in [Buc08]. We will follow the terminology of *quasi-étale morphisms*, introduced by Catanese [Cat07].

Definition 2.7.1 (Quasi-étale morphism). — Let X be a normal algebraic variety. A *quasi-étale* morphism (or a *1-covering*) is a finite surjective morphism $\varphi : Y \rightarrow X$ which is unramified in codimension 1. Namely,

there exists a subvariety $V \subseteq X$ of codimension $\text{codim}_X(V) \geq 2$ such that $\varphi|_{\varphi^{-1}(X \setminus V)} : \varphi^{-1}(X \setminus V) \rightarrow X \setminus V$ is étale.

Moreover, a *universal quasi-étale morphism* is a quasi-étale morphism $\varphi : Y \rightarrow X$ which is universal in the sense that for any quasi-étale morphism $f : Z \rightarrow X$ there exists a (not necessarily unique) quasi-étale morphism $g : Y \rightarrow Z$ such that $\varphi = f \circ g$.

Proposition 2.7.2 ([Buc08, Cor. 3.10, Rem. 3.14]). — *A quasi-étale morphism $\varphi : Y \rightarrow X$ is universal if and only if $\pi_1(Y_{\text{reg}})$ is trivial.*

Proposition 2.7.3. — *Let $\varphi : Y \rightarrow X$ be a quasi-étale morphism between normal projective varieties. Then,*

1. *If K_X is a Cartier divisor, then K_Y is a Cartier divisor.*
2. *If X is a Fano variety, then Y is a Fano variety.*
3. *If X has terminal (resp. canonical) singularities, then Y also has terminal (resp. canonical) singularities.*

Proof. — As $\varphi : Y \rightarrow X$ is unramified in codimension 1, there is no ramification divisor and hence $\varphi^*K_X = K_Y$, implying (1). The point (2) follows from [EGAII, Prop. 5.1.12], while (3) follows from [Kol97, Prop. 3.16]. \square

In the case of toric varieties we can describe the ramification divisor of a toric finite surjective morphism.

Lemma 2.7.4 ([AP13, Lemm. 3.3]). — *Let $\varphi : Y \rightarrow X$ be a finite morphism between toric varieties that corresponds to the map of fans $\Phi : (N_Y, \Delta_Y) \rightarrow (N_X, \Delta_X)$ given by the inclusion of lattices $N_Y \subseteq N_X$ of finite index, so that $N_X \otimes_{\mathbb{Z}} \mathbb{R} = N_Y \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Delta_X = \Delta_Y$. Then,*

1. *φ is equivariant with respect to the homomorphism of tori $T_Y \rightarrow T_X$.*
2. *φ is an abelian cover with Galois group $G = \ker(T_Y \rightarrow T_X) \cong N_X/N_Y$.*
3. *The ramification divisor $\text{Ram}(\varphi)$ is supported on the torus invariant divisors $V(\rho)$, with multiplicities $d_\rho \geq 1$ defined by the condition that the integral generator of $N_Y \cap \mathbb{R}_{\geq 0}u_\rho$ is $d_\rho u_\rho$, for every ray $\rho = \mathbb{R}_{\geq 0}u_\rho \in \Delta_X(1)$.*

Moreover, we have the following theorem characterizing the fundamental group of the smooth locus for \mathbb{Q} -factorial toric varieties.

Theorem 2.7.5 ([Buc08, Cor. 3.10, Theo. 4.8] and [RT16, Theo. 2.4])

Let X be a \mathbb{Q} -factorial toric variety defined by the fan $\Delta_X \subseteq N_{\mathbb{R}}$ and let $N_{\Delta_X(1)} \subseteq N$ be the sub-lattice of N generated by the primitive lattice generators $u_\rho \in N$ of all the rays $\rho \in \Delta_X(1)$. Then,

$$\pi_1(X_{\text{reg}}) \cong N/N_{\Delta_X(1)} \cong \text{Tors}(\text{Cl}(X)).$$

Example 2.7.6. — Let $X = X(\Delta_X)$ be a complete \mathbb{Q} -factorial toric variety of dimension n such that $\rho_X = 1$. Then, we will say that X is a Fake Weighted Projective Space.

This name comes from the following observation: the fan Δ_X has cone generators $u_0, \dots, u_n \in \Delta_X(1)$, and the maximal cones of Δ_X are generated by the n -element subsets of $\{u_0, \dots, u_n\} \subseteq N \cong \mathbb{Z}^n$. As they are linearly dependent,

$$\sum_{i=0}^n \lambda_i u_i = 0,$$

for some $\lambda_0, \dots, \lambda_n \in \mathbb{Z}_{\geq 1}$. Therefore, $\pi_1(X_{\text{reg}}) = \{0\}$ if and only if the primitive lattice vectors $u_0, \dots, u_n \in N$ generate the lattice N . If it is the case we will have that $X \cong \mathbb{P}(\lambda_0, \dots, \lambda_n)$, by [CLS11, Exam. 5.1.14].

From Lemma 2.7.4 and Theorem 2.7.5 we can deduce the following structure theorem for \mathbb{Q} -factorial toric complete varieties of Picard number one.

Theorem 2.7.7 ([Buc08, Theo. 6.4]). — *Let X be a Fake Weighted Projective Space of dimension n . There exists a unique universal quasi-étale morphism $\varphi : \mathbb{P}(\lambda_0, \dots, \lambda_n) \rightarrow X$, canonically identifying X as a finite geometric quotient of $\mathbb{P}(\lambda_0, \dots, \lambda_n)$ by the torus-equivariant action of $\pi_1(X_{\text{reg}}) \cong \text{Tors}(\text{Cl}(X))$.*

Following [RT16] and [RT17], it is natural to consider \mathbb{Q} -factorial complete toric varieties with torsion-free class group as analogs of Weighted Projective Spaces.

Definition 2.7.8 (Poly Weighted Space). — Let $X = X(\Delta_X)$ be a \mathbb{Q} -factorial complete toric variety of dimension n . We define the *canonical quasi-étale universal cover* of X to be the quasi-étale morphism $\pi_X : \widehat{X} \rightarrow X$ corresponding to the map of fans

$$\Pi_X : (N_{\Delta_X(1)}, \Delta_X) \rightarrow (N_X, \Delta_X).$$

Moreover, we say that X is a Poly Weighted Space (PWS) if

$$\pi_1(X_{\text{reg}}) \cong N/N_{\Delta_X(1)} \cong \text{Tors}(\text{Cl}(X)) \cong \{0\}.$$

After the recent works of Rossi and Terracini, there is an explicit combinatorial construction (via Gale duality) of the canonical quasi-étale universal cover of any \mathbb{Q} -factorial complete toric variety. This extends Example 2.7.6 and Theorem 2.7.7 to higher class group rank varieties (see [RT17, Theo. 2.2] for details).

The remaining of the section will be devoted to study contractions $X \rightarrow Y$ of extremal rays as in the previous section via universal quasi-étale morphisms, and without the assumption of isolated singularities in the divisorial case.

The case of extremal contractions of fiber type was studied by Y. Kawamata in [Kaw06, Lemm. 4.1] and it is commonly known as *Kawamata's covering trick*.

Let X be a \mathbb{Q} -factorial projective toric variety and let $R = \mathbb{R}_{\geq 0}[C_\omega] \subseteq \overline{\text{NE}}(X)$ be an extremal ray defining a contraction of fiber type $\varphi_R : X \rightarrow X_R$, where $C_\omega = V(u_1, \dots, u_{n-1})$ is an invariant curve contracted by φ_R . Then, the wall $\omega = \text{cone}(u_1, \dots, u_{n-1})$ separates two maximal cones $\sigma = \text{cone}(u_1, \dots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, u_{n+1})$.

Following the same notation as in Theorem 1.3.7, we have that (up to re-ordering, if necessary) the contraction of R is defined by the projection

$$N := N_X \xrightarrow{\Phi} N_{X_R} := N_X / (\text{Span}(u_{\beta+1}, \dots, u_{n+1}) \cap N_X).$$

Write $\Phi(u_i) = d_i \bar{u}_i$ for primitive vectors \bar{u}_i in N_{X_R} and positive integers d_i , where $1 \leq i \leq \beta$. Then, this \bar{u}_i define a β -dimensional cone $\sigma_0 \in \Delta_{X_R}$ which is of maximal dimension by Theorem 1.3.7 and hence it corresponds to an invariant open affine subset $X_{R,0}$ of X_R .

In this setting, we have that the contraction of fiber type $\varphi_R : X \rightarrow X_R$ becomes (locally) trivial over the invariant open affine subset $X_{R,0} \subseteq X_R$ after a finite morphism of toric varieties (possibly ramified in codimension 1).

Lemma 2.7.9 ([Kaw06, Lemm. 4.1]). — *Let $X_{R,0} \subseteq X_R$ be an invariant open affine subset and let $X_0 = \varphi_R^{-1}(X_{R,0}) \subseteq X$. Then, there is a commutative diagram of toric morphisms*

$$\begin{array}{ccc}
 \widehat{X}_0 & \xrightarrow{\widehat{\varphi}_R} & \widetilde{X}_{R,0} \\
 \downarrow \pi_{X_0} & & \searrow \mu \\
 & & \widehat{X}_{R,0} \\
 & \nearrow \pi_{X_{R,0}} & \\
 X_0 & \xrightarrow{\varphi_R} & X_{R,0}
 \end{array}$$

that satisfies the following conditions:

- (a) π_{X_0} and $\pi_{X_{R,0}}$ are the corresponding canonical quasi-étale universal covers.
- (b) μ is a finite surjective morphism such that $\pi_{X_{R,0}} \circ \mu$ has ramification order d_i over $V(\bar{u}_i) \subseteq X_{R,0}$.
- (c) $\widehat{\varphi}_R$ is a trivial fibration, whose fiber is a WPS.

In our context, Proposition F and Kawamata’s covering trick specialize to the following result.

Proposition 2.7.10. — *Let X be a \mathbb{Q} -factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} N_1(D, X) = 1$ and that $\rho_X = 2$. Let $R \subseteq \overline{NE}(X)$ be an extremal ray such that $D \cdot R > 0$ and let us denote by $\pi : X \rightarrow Y$ the corresponding extremal contraction. Assume that π is of fiber type. Then there exist weights $\lambda_0, \dots, \lambda_{n-1} \in \mathbb{Z}_{>0}$ and a cartesian diagram of toric varieties*

$$\begin{array}{ccc}
 \widehat{X} & \xrightarrow{\widehat{\pi}} & \mathbb{P}(\lambda_0, \dots, \lambda_{n-1}) \\
 \pi_X \downarrow & & \downarrow \pi_Y \\
 X & \xrightarrow{\pi} & Y
 \end{array}$$

where vertical arrows denote the corresponding canonical quasi-étale universal covers, and \widehat{X} is a Gorenstein Fano PWS with terminal singularities such that $\rho_{\widehat{X}} = 2$. Moreover, $\widehat{\pi} : \widehat{X} \rightarrow \mathbb{P}(\lambda_0, \dots, \lambda_{n-1})$ leads to an isomorphism

$$\widehat{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})} \oplus \mathcal{O}_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})}(a)).$$

Proof. — By Proposition F, $\pi : X \rightarrow Y$ is a \mathbb{P}^1 -bundle and X is isomorphic to $\mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$, where Y is a \mathbb{Q} -factorial Gorenstein Fano variety with terminal singularities. Moreover, Theorem 1.3.7 implies that we can always suppose that π is induced by the projection $\Pi : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1} \times \{0\}$, $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, 0)$.

Since π is locally trivial with reduced fibers, [Cd08, Rem. 3.3, Rem. 3.8] imply that if $\sigma = \text{cone}(u_1, \dots, u_{n-1}, e_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, -e_n)$ are maximal cones in $\Delta_X(n)$ which are sent by Π onto a maximal cone $\tau = \text{cone}(t_1, \dots, t_{n-1}) \in \Delta_Y(n-1)$ then we have that, up reordering if necessary, $\Pi(u_i) = t_i$ for $i = 1, \dots, n-1$.

This holds for every invariant open affine open subset of Y and hence it follows that there is an induced morphism between the canonical quasi-étale universal covers, $\widehat{\pi} : \widehat{X} \rightarrow \widehat{Y}$ (cf. Lemma 2.7.9), which is an extremal contraction of fiber type. Moreover, the induced commutative diagram is cartesian in the category of schemes by [Mol16, Lemm. 2.2.7].

Finally, we have $\widehat{Y} \cong \mathbb{P}(\lambda_0, \dots, \lambda_{n-1})$ for some weights $\lambda_0, \dots, \lambda_{n-1} \in \mathbb{Z}_{>0}$ since $\rho_Y = 1$, while Proposition 2.7.3 and Proposition F imply that \widehat{X} and \widehat{Y} are Fano Gorenstein varieties with terminal singularities and that there is an isomorphism $\widehat{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})} \oplus \mathcal{O}_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})}(a))$. \square

Example 2.7.11. — Let X as in Proposition 2.7.10 and suppose that $\text{Tors}(\text{Cl}(X)) \cong \{0\}$, i.e., that $X \cong \widehat{X}$. The extremal contraction of fiber type

$$X \rightarrow \mathbb{P}(\lambda_0, \dots, \lambda_{n-1})$$

leads to an isomorphism $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})} \oplus \mathcal{O}_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})}(a))$. Then,

- (a) X is Gorenstein $\Leftrightarrow \mathbb{P}(\lambda_0, \dots, \lambda_{n-1})$ is Gorenstein $\Leftrightarrow \lambda_i | h$ for every $i \in \{0, \dots, n-1\}$, by [CK99, Lemm. 3.5.6].
- (b) X is terminal $\Leftrightarrow \mathbb{P}(\lambda_0, \dots, \lambda_{n-1})$ is terminal $\Leftrightarrow \sum_{i=0}^{n-1} \{\lambda_i \kappa / h\} \in \{2, \dots, n-1\}$ for each $\kappa \in \{2, \dots, h-2\}$, by [Kas13, Prop. 2.3].
- (c) X is Fano $\Leftrightarrow 0 \leq a \leq i_{\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})} - 1 = h - 1$, by Proposition F and the formula [Mor75, Prop. 2.3] for the canonical divisor of $\mathbb{P}(\lambda_0, \dots, \lambda_{n-1})$.

Here, $h = \sum_{i=0}^{n-1} \lambda_i$ and $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.

The case of divisorial extremal contractions follows in a similar way.

Lemma 2.7.12. — Let X be a \mathbb{Q} -factorial projective toric variety and $R \subseteq \overline{\text{NE}}(X)$ an extremal ray defining a divisorial contraction $\varphi_R : X \rightarrow X_R$.

Then, there is a commutative diagram of toric morphisms

$$\begin{array}{ccc}
 \widehat{X} & \xrightarrow{\widehat{\varphi}_R} & \widetilde{X}_R \\
 \pi_X \downarrow & & \searrow \mu \\
 & & \widehat{X}_R \\
 & & \swarrow \pi_{X_R} \\
 X & \xrightarrow{\varphi_R} & X_R
 \end{array}$$

that satisfies the following conditions:

- (a) π_X and π_{X_R} are the corresponding canonical quasi-étale universal covers.
- (b) μ is a quasi-étale morphism given by the inclusion of lattices $N_{\Delta_{X_R}(1)} \subseteq N_{\Delta_X(1)}$ of index $d_E \geq 1$, where d_E is defined by the condition that the integral generator of $\mathbb{R}_{\geq 0}u_E \cap N_{\Delta_{X_R}(1)}$ is $d_E u_E$, where $V(u_E)$ is the exceptional divisor of φ_R .

Moreover,

- (c) $\widehat{\varphi}_R : \widehat{X} \rightarrow \widetilde{X}_R$ is a divisorial contraction with $(\pi_X)_* \text{Exc}(\widehat{\varphi}_R) = \text{Exc}(\varphi_R)$ and $(\pi_{X_R} \circ \mu)_* \widehat{\varphi}_R(\text{Exc}(\widehat{\varphi}_R)) = \varphi_R(\text{Exc}(\varphi_R))$.
- (d) If X (resp. X_R) is a Fano variety then \widehat{X} (resp. \widetilde{X}_R and \widehat{X}_R) is.
- (e) If X (resp. X_R) has Gorenstein singularities then \widehat{X} (resp. \widetilde{X}_R and \widehat{X}_R) does.
- (f) If X has terminal (resp. canonical) singularities, then all varieties in the diagram have terminal (resp. canonical) singularities.

Proof. — Let us suppose that $\varphi_R : X \rightarrow X_R$ is given by the contraction of the wall $\omega = \text{cone}(u_1, \dots, u_{n-1})$ separating the maximal cones $\sigma = \text{cone}(u_1, \dots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \dots, u_{n-1}, u_{n+1})$. Then, the wall relation satisfied by these cones (defining the contraction) is given by

$$b_u u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n, b_{n+1} \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{Z}$.

Since φ_R is a divisorial contraction we can suppose that (up to reordering, if necessary) $b_1 < 0$ and $b_2, \dots, b_{n-1} \geq 0$, by Theorem 1.3.7. Thus, $E = \text{Exc}(\varphi_R) = V(u_1)$ and the contraction corresponds to the stellar subdivision of the cone

$$\sigma = \text{cone}(u_2, \dots, u_n, u_{n+1}) \in \Delta_{X_R}(n)$$

with respect to the primitive lattice vector u_1 satisfying the wall relation above.

The canonical quasi-étale universal cover of X (resp. X_R) is given by the fan Δ_X (resp. Δ_{X_R}) but seen in the sub-lattice $N_{\Delta_X(1)}$ (resp. $N_{\Delta_{X_R}(1)}$) of N .

Clearly we have the inclusion of lattices $N_{\Delta_{X_R}(1)} \subseteq N_{\Delta_X(1)}$, which is of finite index since $(-b_1)u_1 \in N_{\Delta_{X_R}(1)}$, by the wall relation above. Hence, we obtain an induced quasi-étale morphism $\mu : \widehat{X}_R \rightarrow \widehat{X}$, by Lemma 2.7.4.

Now, the fan of \widehat{X} is obtained by the stellar subdivision of the fan of \widehat{X}_R with respect to the primitive vector u_1 satisfying the wall relation above, obtaining the desired commutative diagram that satisfies (a), (b) and (c) by construction.

Finally, the last three assertions follows from Proposition 2.7.3 together with [KM98, Cor. 3.43]. \square

We can now give the proof of Proposition G describing divisorial contractions of toric mildly Fano varieties with Picard number two.

Proof of Proposition G. — The situation in Lemma 2.7.12 above becomes simpler in this case because the extremal contraction $\pi : X \rightarrow Y$ is induced by a wall relation of the form $u_n + u_{n+1} = u_1$, by Lemma 2.4.1. Therefore, we have that $N_{\Delta_Y(1)} = N_{\Delta_X(1)}$ and hence $\widetilde{Y} \cong \widehat{Y}$ with the notation as in Lemma 2.7.12. In other words, there is an induced morphism between the canonical quasi-étale universal covers, $\widehat{\pi} : \widehat{X} \rightarrow \widehat{Y}$, which is a divisorial extremal contraction. Moreover, the induced commutative diagram is cartesian in the category of schemes by [Mol16, Lemm. 2.2.7].

Finally, we have that $\widehat{Y} \cong \mathbb{P}(\lambda_0, \dots, \lambda_n)$ for some weights $\lambda_0, \dots, \lambda_n \in \mathbb{Z}_{>0}$, by Example 2.7.6. The result follows now directly from the second part of Lemma 2.7.12. \square

Example 2.7.13. — Let X as in Proposition G and let us suppose that $\text{Tors}(\text{Cl}(X)) \cong \{0\}$, i.e., that $X \cong \widehat{X}$. The extremal divisorial contraction

$$\pi : X \rightarrow \mathbb{P}(\lambda_0, \dots, \lambda_n)$$

determines the shape of the fan of X in terms of the fan of $\mathbb{P}(\lambda_0, \dots, \lambda_n)$ (which is well known):

The fan of $\mathbb{P}(\lambda_0, \dots, \lambda_n)$ is given by $n+1$ lattice primitive vectors u_0, \dots, u_n that generates the lattice N and that satisfy the relation

$$\sum_{i=0}^n \lambda_i u_i = 0.$$

Let us suppose that π contracts $E = V(u_E) \subseteq X$ onto the invariant subvariety $A = V(u_i, u_j) \subseteq \mathbb{P}(\lambda_0, \dots, \lambda_n)$, of codimension two. Then, $u_E = u_i + u_j$, by Lemma 2.4.1.

In particular, the same computation used to prove [CK99, Lemm. 3.5.6] shows that X is a Fano Gorenstein variety if and only if $\lambda_i|h$, $\lambda_j|h$, $\lambda_k|(h - \lambda_i)$ and $\lambda_k|(h - \lambda_j)$ for every $k \neq i, j$, where $h = \sum_{i=0}^n \lambda_i$.

However, to the best of the author's knowledge, the characterization such X having terminal singularities is more subtle. Indeed, if X is a \mathbb{Q} -factorial Fano Gorenstein toric variety then it corresponds to a simplicial reflexive lattice polytope $P \subseteq N_{\mathbb{R}}$ (see [Bat94] for details). In [Nil05, Cor. 3.7], Nill characterizes all polytopes among these ones that correspond to varieties with only terminal singularities, but it does not seem easy to translate this characterization into a function of the weights $\lambda_0, \dots, \lambda_n$.

Finally, it should be noticed that if one of the weights is equal to 1, say $\lambda_0 = 1$, then we have a coordinate-wise description of the primitive vectors defining the fan of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$. Namely, the canonical basis of \mathbb{Z}^n together with the vector $(-\lambda_1, \dots, -\lambda_n) \in \mathbb{Z}^n$. In this case, we can explicitly compute the Cartier data $\{m_\sigma\}_{\sigma \in \Delta_X(n)} \subseteq M$ of K_X , allowing us to decide whether the singularities of X are terminal or not.

In particular, we compute that X has only Gorenstein terminal singularities if all the integers

$$\frac{h}{\lambda_i}, \frac{h}{\lambda_j}, \frac{h - \lambda_i}{\lambda_k}, \frac{h - \lambda_j}{\lambda_k}$$

considered before, are equal or greater than 3. The variety defined in Example 2.6.2 satisfy this condition.

Let us end by showing up the computations justifying these assertions. First of all, we can suppose for simplicity that $u_E = u_0 + u_n$ in order to check the Gorenstein condition. If it is the case, then $\Delta_X(n)$ is composed by the following $2n$ cones of maximal dimension:

$$\begin{aligned} \sigma_0 &= \text{cone}(u_1, \dots, u_n), \\ \sigma_n &= \text{cone}(u_0, \dots, u_{n-1}), \\ \sigma'_i &= \text{cone}(u_0, u_E, u_1, \dots, \hat{u}_i, \dots, u_{n-1}) \\ \text{and } \sigma''_i &= \text{cone}(u_n, u_E, u_1, \dots, \hat{u}_i, \dots, u_{n-1}), \end{aligned}$$

for $i = 1, \dots, n - 1$.

Given $\sigma \in \Delta_X(n)$ as above, let $m_\sigma \in M_{\mathbb{Q}}$ be the unique element such that $\langle m_\sigma, u \rangle = -1$ for every $u \in \sigma(1)$; such $\{m_\sigma\}_{\sigma \in \Delta_X(n)} \subseteq M_{\mathbb{Q}}$ correspond to the Cartier data for the toric Weil divisor $-K_X$. Then, X is Gorenstein if and only if $m_\sigma \in M$ for all $\sigma \in \Delta_X(n)$.

On the other hand, the relations $u_E = u_0 + u_n$ and $\sum_{i=0}^n \lambda_i u_i = 0$ implies that

$$\begin{aligned} \langle m_{\sigma_0}, u_0 \rangle &= \frac{h}{\lambda_0} - 1, \\ \langle m_{\sigma_n}, u_n \rangle &= \frac{h}{\lambda_n} - 1, \\ \langle m_{\sigma'_i}, u_i \rangle &= \frac{h - \lambda_n}{\lambda_i} - 1 \\ \text{and } \langle m_{\sigma''_i}, u_i \rangle &= \frac{h - \lambda_0}{\lambda_i} - 1, \end{aligned}$$

for $i = 1, \dots, n-1$. Since u_0, \dots, u_n generate the lattice N , we get the desired equivalence. From this computation, we get directly that $-K_X$ is an ample divisor, since $\langle m_\sigma, u_\rho \rangle > -1$ for every ray $\rho \not\subseteq \sigma$, and every $\sigma \in \Delta_X(n)$ (see [CLS11, Lemm. 6.1.13, Theo. 6.1.14]).

Let us look now to (Gorenstein) terminality of singularities in the case when one weight equals to 1, say $\lambda_0 = 1$. It should be noticed that in this case the fan of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$ is given explicitly by the canonical basis of \mathbb{Z}^n , together with the primitive vector $u_0 = (-\lambda_1, \dots, -\lambda_n) \in \mathbb{Z}^n$. We will illustrate how the computations are made in the case where $u_E = u_0 + u_n$; the case $u_E = u_i + u_j$ with both $i, j \geq 1$ is similar but simpler.

By Theorem 1.3.4, we need to find $\{m_\sigma\}_{\sigma \in \Delta_X(n)} \subseteq M$ such that $\langle m_\sigma, u \rangle = 1$ for all $u \in \sigma(1)$ and such that $\langle m_\sigma, u \rangle > 1$ for all others lattice elements $u \in \sigma \setminus \sigma(1)$, $u \neq 0$. Notice that these $\{m_\sigma\}_{\sigma \in \Delta_X(n)} \subseteq M$ correspond to the Cartier data for the toric divisor K_X .

For $k = 1, \dots, n$, let us denote by $(m_\sigma)_k$ the k -th coordinate of m_σ with respect to the canonical dual basis. Then, by the previous computations (but with opposite sign) we have that:

$$\begin{aligned} (m_{\sigma_0})_k &= 1 \text{ for all } k, \\ (m_{\sigma_n})_k &= \begin{cases} 1 - \frac{h}{\lambda_n} & \text{if } k = n \\ 1 & \text{else} \end{cases}, \\ (m_{\sigma'_i})_k &= \begin{cases} 1 - \frac{h - \lambda_n}{\lambda_i} & \text{if } k = i \\ 0 & \text{if } k = n \\ 1 & \text{else} \end{cases}, \\ (m_{\sigma''_i})_k &= \begin{cases} 1 - \frac{h - \lambda_0}{\lambda_i} & \text{if } k = i \\ 1 & \text{else} \end{cases}, \end{aligned}$$

for $i = 1, \dots, n - 1$.

We can check now terminality by definition. Notice that σ_0 is a smooth cone so there is nothing to prove. Clearly $\frac{h}{\lambda_0} = h \geq 3$.

For σ_n it should be noticed that $\tau_n = \text{cone}(u_1, \dots, u_{n-1}) \subseteq \sigma_n$ is such that $\text{mult}(\tau_n) = 1$ (smooth cone), while $\text{mult}(\sigma_n) = \lambda_n = \#(P_{\sigma_n} \cap N)$, where P_{σ_n} is the fundamental lattice domain defined in Proposition 1.3.2. Hence, we only need to check the terminality condition for non-zero lattice elements $u \in \sigma_n \setminus \sigma_n(1)$ that are not contained in τ_n . We can easily check that all non-zero elements in P_{σ_n} are positive multiples of $u = -u_n$, while

$$\langle m_{\sigma_n}, u \rangle = \frac{h}{\lambda_n} - 1 > 1 \Leftrightarrow \frac{h}{\lambda_n} \in \mathbb{Z}_{\geq 3}.$$

For σ'_i we consider the cones $\tau'_i = \text{cone}(u_1, \dots, \hat{u}_i, \dots, u_{n-1}) \subseteq \sigma'_i$ that satisfies $\text{mult}(\tau'_i) = 1$ (smooth cones), while $\text{mult}(\sigma'_i) = \lambda_i = \#(P_{\sigma'_i} \cap N)$. As before, we only need to check terminality for lattice elements $u \in P_{\sigma'_i} \setminus \{0\}$ that are not contained in τ'_i . Again, we check that such an element u is of the form

$$u = (0, \dots, 0, -a, 0, \dots, 0, -b),$$

where $a, b \in \mathbb{Z}$ are coprime positive integers appearing in the i -th and last coordinate of u . The condition $\langle m_{\sigma'_i}, u \rangle > 1$ is therefore fulfilled if $\frac{h-\lambda_n}{\lambda_i} \geq 3$.

For σ''_i the same computation with $\tau''_i = \text{cone}(u_1, \dots, \hat{u}_i, \dots, u_n) \subseteq \sigma''_i$ yields that the terminality condition is fulfilled if and only if $\frac{h-\lambda_0}{\lambda_i} \geq 3$.

CHAPTER 3

NEWTON-OKOUNKOV BODIES ON PROJECTIVE VECTOR BUNDLES OVER CURVES

3.1. Newton-Okounkov bodies and Semi-stability

3.1.1. Newton-Okounkov bodies. — Let X be a smooth projective variety of dimension n and let L be a big line bundle on X . A full flag of closed subvarieties of X centered at the point $p \in X$

$$Y_{\bullet} : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{p\}$$

is an *admissible flag* if $\text{codim}_X(Y_i) = i$, and each Y_i is smooth at the point p . In particular, Y_{i+1} defines a Cartier divisor on Y_i in a neighborhood of the point p . Following the work of Okounkov [Oko96, Oko03], Kaveh and Khovanskii [KK12] and Lazarsfeld and Mustața [LM09] independently associated to L and Y_{\bullet} a convex body $\Delta_{Y_{\bullet}}(X, L) \subseteq \mathbb{R}^n$ encoding the asymptotic properties of the complete linear series $|L^{\otimes m}|$. We will follow the presentation of [LM09] and we refer the interested reader to the survey [Bou12] for a comparison of both points of view.

Let D be any divisor on X and let $s = s_1 \in H^0(X, \mathcal{O}_X(D))$ be a non-zero section. We shall compute successive vanishing orders of global sections in the following manner: let $D_1 = D + \text{div}(s_1)$ be the effective divisor in the linear system $|D|$ defined by s_1 and set $\nu_1(s) = \text{ord}_{Y_1}(D_1)$ the coefficient of Y_1 in D_1 . Then $D_1 - \nu_1(s)Y_1$ is an effective divisor in the linear system $|D - \nu_1(s)Y_1|$, and does not contain Y_1 in its support, so we can define $D_2 = (D_1 - \nu_1(s)Y_1)|_{Y_1}$ and set $\nu_2(s) = \text{ord}_{Y_2}(D_2)$. We proceed inductively in order to get $\nu_{Y_{\bullet}}(s) = (\nu_1(s), \dots, \nu_n(s)) \in \mathbb{N}^d$. This construction leads to a valuation-like function

$$\nu_{Y_{\bullet}} : H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \rightarrow \mathbb{Z}^n, s \mapsto (\nu_1(s), \dots, \nu_n(s)).$$

We then define the *graded semigroup* of D to be the sub-semigroup of $\mathbb{N}^n \times \mathbb{N}$ defined by

$$\Gamma_{Y_\bullet}(D) = \{(\nu_{Y_\bullet}(s), m) \in \mathbb{N}^n \times \mathbb{N} \mid 0 \neq s \in H^0(X, \mathcal{O}_X(mD))\}.$$

Finally, we define the *Newton-Okounkov body of D with respect to Y_\bullet* to be

$$\Delta_{Y_\bullet}(D) = \text{cone}(\Gamma_{Y_\bullet}(D)) \cap (\mathbb{R}^n \times \{1\}),$$

where $\text{cone}(\Gamma_{Y_\bullet}(D))$ denotes the closed convex cone in $\mathbb{R}^n \times \mathbb{R}$ spanned by $\Gamma_{Y_\bullet}(D)$.

These convex sets $\Delta_{Y_\bullet}(D)$ are compact and they have non-empty interior whenever D is big. Moreover, by [LM09, Theo. A], we have the following identity

$$\text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \frac{1}{n!} \cdot \text{vol}_X(D),$$

where $\text{vol}_X(D) = \lim_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}$. In particular, if D is big and nef, then $\text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \frac{1}{n!} D^n$, by the Asymptotic Riemann-Roch theorem.

The Newton-Okounkov bodies of big divisors depend only on numerical classes: if $D \equiv_{\text{num}} D'$ are big divisors then $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$ for every admissible flag Y_\bullet on X , by [LM09, Prop. 4.1] (see [Jow10, Theo. A] for the converse). This fact, along with the identity $\Delta_{Y_\bullet}(pD) = p \cdot \Delta_{Y_\bullet}(D)$ for every positive integer p , enables us to define an Newton-Okounkov body $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^n$ for every big rational class $\eta \in \text{Big}(X) \cap N^1(X)_{\mathbb{Q}}$. Moreover, by [LM09, Theo. B], there exists a *global Newton-Okounkov body*: a closed convex cone

$$\Delta_{Y_\bullet}(X) \subseteq \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$$

such that for each big rational class $\eta \in \text{Big}(X)_{\mathbb{Q}} = \text{Big}(X) \cap N^1(X)_{\mathbb{Q}}$ the fiber of the second projection over η is $\Delta_{Y_\bullet}(\eta)$. This enables us to define Newton-Okounkov bodies for big real classes by continuity.

The above construction works for *graded linear series* \mathbf{a}_\bullet associated to a big divisor D on X . A graded linear series is a collection of subspaces $\mathbf{a}_m \subseteq H^0(X, \mathcal{O}_X(mD))$ such that $\mathbf{a}_\bullet = \bigoplus_{m \geq 0} \mathbf{a}_m$ is a graded subalgebra of the section ring $R(D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$. The construction enables us to attach to any graded linear series \mathbf{a}_\bullet a closed and convex set $\Delta_{Y_\bullet}(\mathbf{a}_\bullet) \subseteq \mathbb{R}^n$. This set $\Delta_{Y_\bullet}(\mathbf{a}_\bullet)$ will be compact and will compute the volume of the linear series under some mild conditions listed in [LM09, § 2.3]. We will be specially interested on *restricted complete linear series* of a big divisor D , namely graded linear series of the form

$$\mathbf{a}_m = H^0(X|F, \mathcal{O}_X(mD)) = \text{Im} \left(H^0(X, \mathcal{O}_X(mD)) \xrightarrow{\text{rest}} H^0(F, \mathcal{O}_F(mD)) \right)$$

where $F \subseteq X$ is an irreducible subvariety of dimension $d \geq 1$. Under the hypothesis that $F \not\subseteq \mathbf{B}_+(D)$, the conditions listed in [LM09, § 2.3] are satisfied by [LM09, Lemm. 2.16]. Therefore, the Newton-Okounkov body associated to \mathfrak{a}_\bullet above, the *restricted Newton-Okounkov body* (with respect to a fixed admissible flag)

$$\Delta_{X|F}(D) \subseteq \mathbb{R}^d,$$

is compact and

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta_{X|F}(D)) = \frac{1}{d!} \mathrm{vol}_{X|F}(D),$$

where

$$\mathrm{vol}_{X|F}(D) = \lim_{m \rightarrow \infty} \frac{\dim_k \mathfrak{a}_m}{m^d/d!}$$

is the *restricted volume* on F of the divisor D . In particular, if D is big and nef, then $\mathrm{vol}_{X|F}(D) = (D^d \cdot F)$, by [ELM⁺09, Cor. 2.17]. Restricted Newton-Okounkov bodies depend only on numerical classes (see [LM09, Rema. 4.25]), so it is meaningful to consider $\Delta_{X|F}(\eta)$ for every big rational class η such that $F \not\subseteq \mathbf{B}_+(\eta)$.

As before, there exists a global Newton-Okounkov body $\Delta_{Y_\bullet}(X|F)$ that enables us to define, by continuity, $\Delta_{X|F}(\eta)$ for any big real numerical class η such that $F \not\subseteq \mathbf{B}_+(\eta)$. See [LM09, Exam. 4.24] for details.

Restricted Newton-Okounkov bodies can be used to describe slices of Newton-Okounkov bodies.

Theorem 3.1.1 ([LM09, Theo. 4.26, Cor. 4.27]). — *Let X be a normal projective variety of dimension n , and let $F \subseteq X$ be an irreducible and reduced Cartier divisor on X . Fix an admissible flag*

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{p\}$$

with divisorial component $Y_1 = F$. Let $\eta \in \mathrm{Big}(X)_\mathbb{Q}$ be a rational big class, and consider the Newton-Okounkov body $\Delta_{Y_\bullet}(\eta) \subseteq \mathbb{R}^n$. Write $\mathrm{pr}_1 : \Delta_{Y_\bullet}(\eta) \rightarrow \mathbb{R}$ for the projection onto the first coordinate, and set

$$\Delta_{Y_\bullet}(\eta)_{\nu_1=t} = \mathrm{pr}_1^{-1}(t) \subseteq \{t\} \times \mathbb{R}^{n-1}$$

$$\Delta_{Y_\bullet}(\eta)_{\nu_1 \geq t} = \mathrm{pr}_1^{-1}([t, +\infty)) \subseteq \mathbb{R}^n$$

Assume that $F \not\subseteq \mathbf{B}_+(\eta)$ and let

$$\tau_F(\eta) = \sup\{s > 0 \mid \eta - s \cdot f \in \mathrm{Big}(X)\},$$

where $f \in \mathbf{N}^1(X)$ is the numerical class of F . Then, for any $t \in \mathbb{R}$ with $0 \leq t < \tau_F(\eta)$ we have

1. $\Delta_{Y_\bullet}(\eta)_{\nu_1 \geq t} = \Delta_{Y_\bullet}(\eta - tf) + t \cdot \vec{e}_1$, where $\vec{e}_1 = (1, 0, \dots, 0) \in \mathbb{N}^n$ is the first standard unit vector.⁽¹⁾
2. $\Delta_{Y_\bullet}(\eta)_{\nu_1 = t} = \Delta_{X|F}(\eta - tf)$.
3. The function $t \mapsto \text{vol}_X(\eta + tf)$ is differentiable at $t = 0$, and

$$\frac{d}{dt} (\text{vol}_X(\eta + tf))|_{t=0} = n \cdot \text{vol}_{X|F}(\eta).$$

Let us finish this section with the case of Newton-Okounkov bodies on surfaces (see [LM09, §6.2] for details). We will use this description in Example 3.3.2 in order to illustrate the shape of Newton-Okounkov bodies on ruled surfaces.

Example 3.1.2 (Surfaces). — Let S be a smooth projective surface together with a flag $Y_\bullet : Y_0 = S \supseteq Y_1 = C \supseteq Y_2 = \{p\}$, where $C \subseteq S$ is a smooth curve and $p \in C$.

Let D be a big \mathbb{Q} -divisor on S . Any such divisor admits a *Zariski decomposition*, that is we can uniquely write D as a sum

$$D = P(D) + N(D)$$

of \mathbb{Q} -divisors, with $P(D)$ nef and $N(D)$ either zero or effective with negative definite intersection matrix. Moreover, $P(D) \cdot \Gamma = 0$ for every irreducible component Γ of $N(D)$ and for all $m \geq 0$ there is an isomorphism $H^0(S, \mathcal{O}_S([mP(D)])) \cong H^0(S, \mathcal{O}_S([mD]))$. In this decomposition $P(D)$ is called the *positive part* and $N(D)$ the *negative part*. See [Laz04, §2.3.E] and references therein for proofs and applications.

With the above notation, we have that

$$\Delta_{Y_\bullet}(D) = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \tau_C(D), \alpha(t) \leq t \leq \beta(t)\}$$

where

1. $\nu \in \mathbb{Q}$ the coefficient of C in $N(D)$,
2. $\tau_C(D) = \sup\{t > 0 \mid D - tC \text{ is big}\}$,
3. $\alpha(t) = \text{ord}_p(N_t \cdot C)$,
4. $\beta(t) = \text{ord}_p(N_t \cdot C) + P_t \cdot C$,

where $D - tC = P_t + N_t$ is a Zariski decomposition, P_t being the positive and N_t the negative part. Moreover, these bodies are finite polygons, by [KLM12, Theo. B].

1. In fact, this statement remains true even if we do not assume that $E \not\subseteq \mathbf{B}_+(\eta)$. See [KL15, Prop. 1.6].

3.1.2. Semi-stability and Harder-Narasimhan filtrations. — Throughout this section, C is a smooth projective curve and E is a locally free sheaf on C of rank $r > 0$ and degree $d = \deg(E) = \deg(c_1(E))$. Given such a bundle we call the rational number

$$\mu(E) = \frac{d}{r}$$

the *slope* of E .

Definition 3.1.3 (Semi-stability). — Let E be a vector bundle on C of slope μ . We say that E is *semi-stable* if for every non-zero sub-bundle $S \subseteq E$, we have $\mu(S) \leq \mu$. Equivalently, E is semi-stable if for every locally-free quotient $E \twoheadrightarrow Q$ of non-zero rank, we have $\mu \leq \mu(Q)$.

Following [LP97, Prop. 5.4.2], there is a canonical filtration of E with semi-stable quotients.

Proposition 3.1.4. — *Let E be a vector bundle on C . Then E has an increasing filtration by sub-bundles*

$$\mathrm{HN}_\bullet(E) : 0 = E_\ell \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

where each of the quotients E_{i-1}/E_i satisfies the following conditions:

1. Each quotient E_{i-1}/E_i is a semi-stable vector bundle;
2. $\mu(E_{i-1}/E_i) < \mu(E_i/E_{i+1})$ for $i = 1, \dots, \ell - 1$.

This filtration is unique.

The above filtration is called the *Harder-Narasimhan filtration* of E .

Notation 3.1.5. — *Let E be a vector bundle on a smooth projective curve C . We will denote by $Q_i = E_{i-1}/E_i$ the semi-stable quotients of the Harder-Narasimhan filtration of E , each one of rank $r_i = \mathrm{rank}(Q_i)$, degree $d_i = \deg(c_1(Q_i))$ and slope $\mu_i = \mu(Q_i) = d_i/r_i$. With this notation, μ_1 and μ_ℓ correspond to the minimal and maximal slopes, $\mu_{\min}(E)$ and $\mu_{\max}(E)$, respectively.*

From a cohomological point of view, semi-stable vector bundles can be seen as the good higher-rank analogue of line bundles. For instance, we have the following classical properties (see [RR84] or [But94, Lemm. 1.12, Lemm. 2.5]).

Lemma 3.1.6. — *Let E and F be vector bundles on C and $m \in \mathbb{N}$. Then,*

1. $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$.

2. $\mu_{\min}(E \otimes F) = \mu_{\min}(E) + \mu_{\min}(F)$.
3. $\mu_{\max}(S^m E) = m\mu_{\max}(E)$.
4. $\mu_{\min}(S^m E) = m\mu_{\min}(E)$.
5. If $\mu_{\max}(E) < 0$, then $\dim_k H^0(C, E) = 0$.
6. If $\mu_{\min}(E) > 2g - 2$, then $\dim_k H^1(C, E) = 0$.

In particular, if E and F are semi-stable then $S^m E$ and $E \otimes F$ are semi-stable.

If E_1, \dots, E_ℓ are vector bundles on C and m_1, \dots, m_ℓ be non-negative integers. By the splitting principle [Ful84, Rema. 3.2.3] we can prove the following formula:

$$\mu(S^{m_1} E_1 \otimes \cdots \otimes S^{m_\ell} E_\ell) = \sum_{i=1}^{\ell} m_i \mu(E_i).$$

Moreover, we have that for every $m \geq 1$ the Harder-Narasimhan filtration of the symmetric product $S^m E$ can be computed in terms of the one for E (see [Che11, Prop. 3.4] and [Wol05, Prop. 5.10], for instance).

Proposition 3.1.7. — *Let E be a vector bundle on C with Harder-Narasimhan filtration*

$$\text{HN}_\bullet(E) : 0 = E_\ell \subseteq E_{\ell-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$

and semi-stable quotients $Q_i = E_{i-1}/E_i$ with slopes $\mu_i = \mu(Q_i)$, for $i = 1, \dots, \ell$. For every positive integer $m \geq 1$, let us consider the vector bundle $S^m E$ with Harder-Narasimhan filtration

$$\text{HN}_\bullet(S^m E) : 0 = W_M \subseteq W_{M-1} \subseteq \cdots \subseteq W_1 \subseteq W_0 = S^m E$$

and semi-stable quotients W_{j-1}/W_j with slopes $\nu_j = \mu(W_{j-1}/W_j)$, for $j = 1, \dots, M$. Then, for every $j = 1, \dots, M$ we have that

$$W_j = \sum_{\sum_i m_i \mu_i \geq \nu_{j+1}} S^{m_1} E_0 \otimes \cdots \otimes S^{m_\ell} E_{\ell-1}$$

and

$$W_{j-1}/W_j \cong \bigoplus_{\sum_i m_i \mu_i = \nu_j} S^{m_1} Q_1 \otimes \cdots \otimes S^{m_\ell} Q_\ell,$$

where the sums are taken over all partitions $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m , and $S^{m_1} E_0 \otimes \cdots \otimes S^{m_\ell} E_{\ell-1}$ denotes the image of the composite

$$E_0^{\otimes m_1} \otimes \cdots \otimes E_{\ell-1}^{\otimes m_\ell} \rightarrow E^{\otimes m} \rightarrow S^m E.$$

In particular, there is a refinement F_\bullet of $\mathrm{HN}_\bullet(S^m E)$ of length $L = L(m)$ and whose respective successive quotients are of the form

$$F_{i-1}/F_i \cong Q_{\mathbf{m}(i)} = S^{m_1} Q_1 \otimes \cdots \otimes S^{m_\ell} Q_\ell$$

for some partition $\mathbf{m}(i) = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m , and such that for every $i \in \{1, \dots, L\}$ we have $\mu(Q_{\mathbf{m}(i)}) \leq \mu(Q_{\mathbf{m}(i+1)})$. Moreover, given any partition $\mathbf{m} \in \mathbb{N}^\ell$ of m , there is one and only one $i \in \{1, \dots, L\}$ such that $\mathbf{m}(i) = \mathbf{m}$.

3.2. Divisors on projective bundles over curves

Let E be a vector bundle on a smooth projective curve C , of rank $r \geq 2$ and degree d . In this section we study divisors on the projective bundle $\pi : \mathbb{P}(E) \rightarrow C$ of one-dimensional quotients. Let us recall that in this case the Néron-Severi group of $\mathbb{P}(E)$ is of the form

$$\mathrm{N}^1(\mathbb{P}(E)) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \xi,$$

where f is the numerical class of a fiber of π and $\xi = \xi_E$ is the numerical class of a divisor representing the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. Moreover, if $[\mathrm{pt}]$ denotes the class of a point in the ring $\mathrm{N}^*(\mathbb{P}(E))$ then we have the following relations:

$$f^2 = 0, \quad \xi^{r-1} f = [\mathrm{pt}], \quad \xi^r = d \cdot [\mathrm{pt}].$$

The cone of nef divisors can be described via Hartshorne's characterization of ample vector bundles over curves [Har71, Theo. 2.4] (cf. [Ful11, Lemm. 2.1]).

Lemma 3.2.1. — $\mathrm{Nef}(\mathbb{P}(E)) = \langle \xi - \mu_{\min} f, f \rangle$.

The cone of pseudo-effective divisors was obtained by Nakayama in [Nak04, Cor. IV.3.8], and it was indirectly computed by Wolfe [Wol05] and Chen [Che11] who independently obtained the volume function $\mathrm{vol}_{\mathbb{P}(E)}$ on $\mathrm{N}^1(\mathbb{P}(E))_{\mathbb{R}}$. A more general result on the cone of effective cycles of arbitrary codimension can be found in [Ful11, Theo. 1.1].

Lemma 3.2.2. — $\mathrm{Psef}(\mathbb{P}(E)) = \langle \xi - \mu_{\max} f, f \rangle$.

In particular, we recover a result of Miyaoka [Miy87, Theo. 3.1] on semi-stable vector bundles over curves that was generalized by Fulger in [Ful11, Prop. 1.5].

Corollary 3.2.3. — *A vector bundle E on a smooth projective curve C is semi-stable if and only if $\mathrm{Nef}(\mathbb{P}(E)) = \mathrm{Psef}(\mathbb{P}(E))$.*

Fulger and Lehmann computed the movable cone of $\mathbb{P}(E)$ in [FL, Prop. 7.1 and Cor. 7.2].

Lemma 3.2.4. — *With Notation 3.1.5, $\overline{\text{Mov}}(\mathbb{P}(E)) = \langle \xi - \mu_{\ell-1}f, f \rangle$ if $r_\ell = 1$, and $\overline{\text{Mov}}(\mathbb{P}(E)) = \text{Psef}(\mathbb{P}(E))$ otherwise.*

For the sake of completeness, we finish this section by including A. Wolfe's computation of the volume function on $N^1(\mathbb{P}(E))$. See also [Che11, Theo. 1.2].

Notation 3.2.5. — *Let $d \geq 1$ be an integer. We define the standard d -simplex $\widehat{\Delta}_d$ with $d+1$ vertices in \mathbb{R}^{d+1} to be*

$$\widehat{\Delta}_d = \left\{ (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}.$$

By projecting $\widehat{\Delta}_d$ onto the hyperplane $x_1 = 0$, we can identify $\widehat{\Delta}_d$ with the full dimensional standard d -simplex (or just d -simplex) in \mathbb{R}^d given by

$$\Delta_d = \left\{ (x_2, \dots, x_{d+1}) \in \mathbb{R}^d \mid \sum_{i=2}^{d+1} x_i \leq 1 \text{ and } x_i \geq 0 \text{ for all } i \right\}.$$

Via the previous identification, we will denote by λ the Lebesgue measure on $\widehat{\Delta}_d$ induced by the standard Lebesgue measure on $\Delta_d \subseteq \mathbb{R}^d$. In particular, we will have $\lambda(\widehat{\Delta}_d) = \frac{1}{d!}$.

Given a positive real number $a > 0$, we define the d -simplex with side length a by $a\Delta_d = \left\{ (x_2, \dots, x_{d+1}) \in \mathbb{R}^d \mid \sum_{i=2}^{d+1} x_i \leq a \text{ and } x_i \geq 0 \text{ for all } i \right\}$. Similar for $a\widehat{\Delta}_d \subseteq \mathbb{R}^{d+1}$, the standard d -simplex with side length a .

Theorem 3.2.6 ([Wol05, Theo. 5.14]). — *Let E be a vector bundle with Harder-Narasimhan filtration of length ℓ and semi-stable quotients Q_i of ranks r_i and slopes μ_i . Then, for any $t \in \mathbb{R}$*

$$\text{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{\ell-1}} \max \left\{ \sum_{i=1}^{\ell} \mu_i \beta_i - t, 0 \right\} \frac{\beta_1^{r_1-1} \cdots \beta_\ell^{r_\ell-1}}{(r_1-1)! \cdots (r_\ell-1)!} d\beta$$

where $\widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^\ell$ is the standard $(\ell-1)$ -simplex with coordinates $\beta_1, \dots, \beta_\ell$, and β be the standard induced Lebesgue measure.

Let us first recall the following inequality stated in [Wol05, Prop. 5.11].

Proposition 3.2.7 (Wolfe). — *Let $E = E_\bullet$ be a filtered vector bundle on C with semi-stable quotients Q_1, \dots, Q_ℓ such that $\mu(Q_1) \leq \dots \leq \mu(Q_\ell)$. Then*

$$\sum_{\mu(Q_i) > 2g-2} h^0(C, Q_i) \leq h^0(C, E) \leq \sum_{\mu(Q_i) \geq 0} h^0(C, Q_i).$$

Proof. — Let K^\bullet be the associated filtered complex of Čech cochains over some open cover \mathfrak{U} computing the cohomology of each E_i :

$$0 = \mathcal{C}^\bullet(\mathfrak{U}, E_\ell) \subseteq \cdots \subseteq \mathcal{C}^\bullet(\mathfrak{U}, E_1) \subseteq \mathcal{C}^\bullet(\mathfrak{U}, E).$$

There is a spectral sequence associated to this filtered complex, whose first page terms are

$$E_1^{p,q} = \mathbb{H}^{p+q}(\mathrm{Gr}^p K^\bullet) = \mathbb{H}^{p+q}(\mathcal{C}^\bullet(\mathfrak{U}, Q_{p+1})) = \mathbb{H}^{p+q}(C, Q_{p+1}).$$

This spectral sequence converges to $E_\infty^{\bullet,\bullet}$ where $E_\infty^{p,q} = \mathrm{Gr}^p(\mathbb{H}^{p+q}(K^\bullet))$ and hence

$$\mathbb{H}^0(C, E) = \sum_{p=0}^{\ell-1} \mathrm{Gr}^p(\mathbb{H}^0(K^\bullet)) = \sum_{p=0}^{\ell-1} E_\infty^{p,-p}.$$

Since C is a projective curve and the length of the filtration is ℓ , $E_1^{p,q} = 0$ unless $p+q \in \{0, 1\}$ and $0 \leq p \leq \ell-1$. On the other hand, $h^1(C, Q_{p+1}) = 0$ if $\mu(Q_{p+1}) > 2g-2$ and then one sees that $E_1^{p,-p} = E_\infty^{p,-p}$ if $\mu(Q_{p+1}) > 2g-2$. \square

We will need the following estimate due to Wolfe (see [Wol105, Prop. 5.12]).

Lemma 3.2.8 (Wolfe). — *Let C be a smooth projective curve, let E be a non semi-stable vector bundle over C with successive semi-stable quotients Q_1, \dots, Q_ℓ , let $a, b \in \mathbb{Q}$ be two rational numbers such that $a < b$ and let $t \in \mathbb{Z}$ be any integer. Then, for every positive integer $m \geq 1$ and every point $q \in C$, the Harder-Narasimhan filtration of $S^m E \otimes \mathcal{O}_C(-mt \cdot q)$ admits a refinement F_\bullet of length L whose successive quotients are of the form*

$$F_{i-1}/F_i \cong Q_{\mathbf{m}(i),t} = S^{m_1} Q_1 \otimes \cdots \otimes S^{m_\ell} Q_\ell \otimes \mathcal{O}_C(-mt \cdot q)$$

for some partition $\mathbf{m}(i) = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m , and such that for every $i \in \{1, \dots, L\}$ we have $\mu(Q_{\mathbf{m}(i),t}) \leq \mu(Q_{\mathbf{m}(i+1),t})$. Moreover, the number of vector bundles of the form $Q_{\mathbf{m},t}$ appearing as successive quotients with slopes between a and b is $O(m^{\ell-2})$.

Proof. — The existence of such a refinement follows from Proposition 3.1.7 applied to the bundle $E \otimes \mathcal{O}_C(-t \cdot q)$. For the last assertion we follow almost verbatim Wolfe's argument in the proof of [Wol105, Prop. 5.12].

Following Notation 3.2.5, let $\widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^\ell$ be the standard $(\ell-1)$ -simplex with coordinates (x_1, \dots, x_ℓ) and let $\Delta_{\ell-1} \subseteq \{0\} \times \mathbb{R}^{\ell-1} \cong \mathbb{R}^{\ell-1}$ be the full dimensional $(\ell-1)$ -simplex obtained by projecting $\widehat{\Delta}_{\ell-1}$ onto the hyperplane $x_1 = 0$. Then, partitions $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m correspond to integer coordinate points $(m_1, \dots, m_\ell) \in m\widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^\ell$ and hence to integer coordinate points in $m\Delta_{\ell-1} \subseteq \mathbb{R}^{\ell-1}$.

We note that

$$|m\widehat{\Delta} \cap \mathbb{Z}^\ell| = \binom{m + \ell - 1}{\ell - 1} = O(m^{\ell-1}).$$

If we consider the function $M(x_1, \dots, x_\ell) = \sum_{i=1}^\ell \mu_i x_i - mt$, then $\mu(Q_{\mathbf{m},t}) = M(m_1, \dots, m_\ell)$ and hence vector bundles $Q_{\mathbf{m},t}$ with given slope μ_0 correspond to integer coordinate points on the hypersurface $H_{\mu_0} = M^{-1}(\{\mu_0\})$ in $\widehat{\Delta}_{\ell-1}$. We claim that

$$|m\widehat{\Delta}_{\ell-1} \cap H_{\mu_0} \cap \mathbb{Z}^\ell| \leq |m\widehat{\Delta}_{\ell-2} \cap \mathbb{Z}^{\ell-1}| = O(m^{\ell-2}),$$

where

$$m\widehat{\Delta}_{\ell-2} = \left\{ (0, x_2, \dots, x_\ell) \in \{0\} \times \mathbb{R}^{\ell-1} \mid \sum_{i=2}^\ell x_i = m \right\} \subseteq m\Delta_{\ell-1}.$$

In fact, if we consider the restriction to $m\widehat{\Delta}_{\ell-1} \cap H_{\mu_0}$ of the composite function $\varphi = \partial \circ p$ below

$$\begin{array}{ccc} m\widehat{\Delta}_{\ell-1} & & \\ p \downarrow & \searrow \varphi & \\ m\Delta_{\ell-1} & \xrightarrow{\partial} & m\widehat{\Delta}_{\ell-2} \end{array}$$

where

$$p(x_1, x_2, \dots, x_\ell) = (0, x_2, \dots, x_\ell)$$

and

$$\partial(0, x_2, x_3, \dots, x_\ell) = (0, m - \sum_{i=3}^\ell x_i, x_3, \dots, x_\ell),$$

then a straightforward computation shows that the equality

$$\varphi(x_1, \dots, x_\ell) = \varphi(y_1, \dots, y_\ell)$$

for $(x_1, \dots, x_\ell), (y_1, \dots, y_\ell) \in m\widehat{\Delta}_{\ell-1} \cap H_{\mu_0}$ is equivalent to the equations $x_i = y_i$ for $i \geq 3$,

$$\mu_1 x_1 + \mu_2 x_2 = \mu_1 y_1 + \mu_2 y_2$$

and

$$x_1 + x_2 = y_1 + y_2,$$

from which we deduce the injectivity of φ , since $\mu_1 < \mu_2$.

We note that there are only finite different values of $M(m_1, \dots, m_\ell)$ between a and b for $(m_1, \dots, m_\ell) \in m\widehat{\Delta} \cap \mathbb{Z}^\ell$ since every partition of m appears only one time in the filtration and each increase in the value of M is at least $1/\min\{k \in \mathbb{N}_{\geq 1} \mid k\mu_j \in \mathbb{Z} \text{ for all } 1 \leq j \leq \ell\} \in \mathbb{Q}_{>0}$, and then only finitely many hypersurfaces contains the integer coordinate points corresponding to

vector bundles $Q_{\mathbf{m},t}$ with slopes between a and b . There are therefore at most $O(m^{\ell-2})$ vector bundles $Q_{\mathbf{m},t}$ with slopes between a and b . \square

For the convenience of the reader we include the proof of Theorem 3.2.6.

Proof of Theorem 3.2.6. — We follow A. Wolfe's proof. For the reader's convenience, the proof is subdivided into several steps. Let $F = \pi^{-1}(q)$ be any fiber. As both sides of the volume formula are continuous in t we may (and shall) assume that $t \in \mathbb{Q}$. Let $m \in \mathbb{N}$ be such that $mt \in \mathbb{Z}$. By the projection formula,

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes \mathcal{O}_{\mathbb{P}(E)}(-mt \cdot F)) \cong H^0(C, S^m \otimes \mathcal{O}_C(-mt \cdot q)).$$

Step 1. Bounds for $h^0(C, S^m E \otimes \mathcal{O}_C(-mt \cdot q))$. Given $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ a partition of m , we define

$$Q_{\mathbf{m},t} = S^{m_1} Q_1 \otimes \dots \otimes S^{m_\ell} Q_\ell \otimes \mathcal{O}_C(-mt \cdot q).$$

We consider the sets

$$\begin{aligned} \mathcal{A} &= \{Q_{\mathbf{m},t} \mid 0 \leq \mu(Q_{\mathbf{m},t}) \leq 2g - 2\}, \\ \mathcal{B} &= \{Q_{\mathbf{m},t} \mid \mu(Q_{\mathbf{m},t}) > 2g - 2\}. \end{aligned}$$

We claim that

$$\sum_{\mathcal{B}} h^0(C, Q_{\mathbf{m},t}) \leq h^0(C, S^m E \otimes \mathcal{O}_C(-mt \cdot q)) \leq \sum_{\mathcal{A} \cup \mathcal{B}} h^0(C, Q_{\mathbf{m},t}).$$

In fact, applying Proposition 3.1.7 to the bundle $S^m E \otimes \mathcal{O}_C(-mt \cdot q)$ we get that the semi-stable quotients in the corresponding Harder-Narasimhan filtration are direct sums of bundles of the form $Q_{\mathbf{m},t}$, and all such bundles appears as direct summands of the semi-stable quotients. Therefore, by refining the Harder-Narasimhan filtration if necessary, we get a filtration

$$0 = F_L \subseteq F_{L-1} \subseteq \dots \subseteq F_1 \subseteq F_0 = S^m E \otimes \mathcal{O}_C(-mt \cdot q)$$

whose successive quotients have the form

$$F_{j-1}/F_j \cong Q_{\mathbf{m}(j),t},$$

for some partition $\mathbf{m}(j)$ of m , and $\mu(Q_{\mathbf{m}(j),t}) \leq \mu(Q_{\mathbf{m}(j+1),t})$ for every $j \in \{1, \dots, L\}$. Moreover, given any partition $\mathbf{m} \in \mathbb{N}^\ell$ of m , there is $j \in \{1, \dots, L\}$ such that $\mathbf{m}(j) = \mathbf{m}$. The claim follows therefore from Proposition 3.2.7 applied to the filtration F_\bullet .

Step 2. Cohomology growth. Let us prove that for a given partition $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ of m we have that $h^i(C, Q_{\mathbf{m},t}) = O(m^{r-\ell+1})$ for $i = 0, 1$.

We consider the fiber product $Z = \mathbb{P}(Q_1) \times_C \cdots \times_C \mathbb{P}(Q_\ell)$ with projections $p : Z \rightarrow C$ and $p_i : Z \rightarrow \mathbb{P}(Q_i)$. Then

$$\begin{aligned} H^0(C, Q_{\mathbf{m},t}) &= H^0(Z, \mathcal{O}_Z(m_1, \dots, m_\ell) \otimes p^* \mathcal{O}_C(-mt \cdot q)), \\ H^1(C, Q_{\mathbf{m},t}) &= H^1(Z, \mathcal{O}_Z(m_1, \dots, m_\ell) \otimes p^* \mathcal{O}_C(-mt \cdot q)) \\ &= H^0(Z, \omega_Z \otimes \mathcal{O}_Z(-m_1, \dots, -m_\ell) \otimes p^* \mathcal{O}_C(mt \cdot q)), \end{aligned}$$

where $\mathcal{O}_Z(m_1, \dots, m_\ell) = p_1^* \mathcal{O}_{\mathbb{P}(Q_1)}(m_1) \otimes \cdots \otimes p_\ell^* \mathcal{O}_{\mathbb{P}(Q_\ell)}(m_\ell)$ and ω_Z is the dualizing sheaf of Z .

Write $t = u/v \in \mathbb{Q}$ with $u, v \in \mathbb{Z}$ and let $\mathcal{L}_1 = \mathcal{O}_Z(A_1)$ be an ample line bundle on Z such that $\mathcal{L}_1 \otimes p_i^* \mathcal{O}_{\mathbb{P}(Q_i)}(-1)$ is generated by global sections for all $i \in \{1, \dots, \ell\}$ and such that $\mathcal{L}_1^{\otimes v} \otimes p^* \mathcal{O}_C(u \cdot q)$ is generated by global sections. Then we have that $H^0(Z, \mathcal{L}_1^{\otimes v} \otimes p^* \mathcal{O}_C(u \cdot q)) \neq \{0\}$ and that $H^0(Z, \mathcal{L}_1^{\otimes m} \otimes p^* \mathcal{O}_C(mt \cdot q)) \neq \{0\}$, so we can find a non-zero section $s \in H^0(Z, \mathcal{L}_1^{\otimes 2m} \otimes \mathcal{O}_Z(-m_1, \dots, -m_\ell) \otimes p^* \mathcal{O}_C(mt \cdot q))$ leading to an inclusion

$$H^0(Z, \mathcal{O}_Z(m_1, \dots, m_\ell) \otimes p^* \mathcal{O}_C(-mt \cdot q)) \subseteq H^0(Z, \mathcal{L}_1^{\otimes 2m})$$

and hence to

$$h^0(C, Q_{\mathbf{m},t}) \leq h^0(Z, \mathcal{L}_1^{\otimes 2m}) \leq C_1 (2mA_1)^{\dim Z} = C_1 2^{r-\ell+1} A^{r-\ell+1} m^{r-\ell+1}$$

for some constant $C_1 = C_1(A_1)$. Similarly, we can choose an ample line bundle $\mathcal{L}_2 = \mathcal{O}_Z(A_2)$ on Z such that we have an inclusion

$$H^0(Z, \omega_Z \otimes \mathcal{O}_Z(-m_1, \dots, -m_\ell) \otimes p^* \mathcal{O}_C(mt \cdot q)) \subseteq H^0(Z, \mathcal{L}_2^{\otimes (2m+1)})$$

and hence

$$\begin{aligned} h^1(C, Q_{\mathbf{m},t}) &\leq h^0(Z, \mathcal{L}_2^{\otimes (2m+1)}) \leq C_2 ((2m+1)A_2)^{\dim Z} \\ &\leq C_2 3^{r-\ell+1} A_2^{r-\ell+1} m^{r-\ell+1} \end{aligned}$$

for some constant $C_2 = C_2(A_2)$.

Step 3. Asymptotic estimate. We claim that the following estimate holds

$$h^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes \mathcal{O}_{\mathbb{P}(E)}(-mt \cdot F)) = \sum_{\mu(Q_{\mathbf{m},t}) \geq 0} \chi(Q_{\mathbf{m},t}) + O(m^{r-1}),$$

where the sum runs over all partitions $\mathbf{m} \in \mathbb{N}^\ell$ of m .

To prove this we define $h(m) = h^0(C, S^m E \otimes \mathcal{O}_C(-mt \cdot q))$ and we consider the difference

$$\begin{aligned} \left| h(m) - \sum_{\mu(Q_{\mathbf{m},t}) \geq 0} \chi(Q_{\mathbf{m},t}) \right| &= |h(m) - \sum_{\mathcal{A}} \chi(Q_{\mathbf{m},t}) - \sum_{\mathcal{B}} \chi(Q_{\mathbf{m},t})| \\ &\leq |h(m) - \sum_{\mathcal{B}} h^0(C, Q_{\mathbf{m},t})| + |\sum_{\mathcal{A}} \chi(Q_{\mathbf{m},t})| \\ &\leq \sum_{\mathcal{A}} (2h^0(C, Q_{\mathbf{m},t}) + h^1(C, Q_{\mathbf{m},t})), \end{aligned}$$

where the last inequality is obtained from *Step 1*. It follows from *Step 2* that both $h^0(C, Q_{\mathbf{m},t})$ and $h^1(C, Q_{\mathbf{m},t})$ grow at most as $O(m^{r-\ell+1})$, while Lemma 3.2.8 applied to the integers 0 and $2g-2$ implies that there are at most $O(m^{\ell-2})$ elements in \mathcal{A} . It follows that the difference above is at most $O(m^{r-1})$, which implies the desired asymptotic estimate.

Step 4. Volume computation. Let $t \in \mathbb{Q}$ and consider the limit

$$\begin{aligned} \text{vol}_{\mathbb{P}(E)}(\xi - tf) &= \lim_{m \rightarrow +\infty} \frac{h^0(C, S^m E \otimes \mathcal{O}_C(-mt \cdot q))}{m^r/r!} \\ &= \lim_{m \rightarrow +\infty} \frac{r!}{m^r} \sum_{\mu(Q_{\mathbf{m},t}) \geq 0} \chi(Q_{\mathbf{m},t}), \end{aligned}$$

where the last equality follows from *Step 4*. On the other hand, the Hirzebruch-Riemann-Roch theorem asserts that

$$\chi(Q_{\mathbf{m},t}) = \left(\sum_{i=1}^{\ell} m_i \mu_i - mt + 1 - g \right) \text{rank}(Q_{\mathbf{m},t}).$$

We put $\beta_i = m_i/m$, and we notice that there are at most $O(m^{\ell-2})$ summands with $\beta_i = 0$ (cf. Lemma 3.2.8). If none of the β_i is zero, then we have that

$$\text{rank}(Q_{\mathbf{m},t}) = \prod_{i=1}^{\ell} \binom{m_i + r_i - 1}{m_i} = m^{r-\ell} \frac{\beta_1^{r_1-1} \cdots \beta_{\ell}^{r_{\ell}-1}}{(r_1-1)! \cdots (r_{\ell}-1)!} + O(m^{r-\ell-1}).$$

We obtain therefore that $\text{vol}_{\mathbb{P}(E)}(\xi - tf)$ is computed via the limit

$$\lim_{m \rightarrow +\infty} \frac{r!}{m^{\ell}} \sum_{\mu(Q_{\mathbf{m},t}) > 0} \left(\sum_{i=1}^{\ell} m_i \mu_i - mt + 1 - g \right) \frac{\beta_1^{r_1-1} \cdots \beta_{\ell}^{r_{\ell}-1}}{(r_1-1)! \cdots (r_{\ell}-1)!}.$$

Following Notation 3.2.5, we consider the $(\ell-1)$ -standard simplex $\widehat{\Delta} = \widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^{\ell}$ with coordinates $\beta_1, \dots, \beta_{\ell}$ and we define the continuous function $f : \widehat{\Delta} \rightarrow \mathbb{R}$ by

$$f(\beta_1, \dots, \beta_{\ell}) = \max \left\{ \sum_{i=1}^{\ell} \mu_i \beta_i - t, 0 \right\} \frac{\beta_1^{r_1-1} \cdots \beta_{\ell}^{r_{\ell}-1}}{(r_1-1)! \cdots (r_{\ell}-1)!}.$$

Finally, we have that

$$\mathrm{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \lim_{m \rightarrow +\infty} \frac{1}{m^{\ell-1}} \sum_{\beta \in \widehat{\Delta} \cap \frac{1}{m} \mathbb{Z}^\ell} f(\beta_1, \dots, \beta_\ell) = r! \cdot \int_{\widehat{\Delta}} f d\beta,$$

where the last equality follows for instance from [Bou12, Théorème 1.12] or [Wol05, Lemma 5.15]. \square

Remark 3.2.9. — Alternatively, Chen computed in [Che11, Theo. 1.2] a similar volume formula, but slightly simplified by integrating in \mathbb{R}^r instead of \mathbb{R}^ℓ (cf. [Che11, Prop. 3.5]). More precisely, with the same notation as above

$$\mathrm{vol}_{\mathbb{P}(E)}(\xi - tf) = r! \cdot \int_{\widehat{\Delta}_{r-1}} \max \left\{ \sum_{j=1}^r s_j \lambda_j - t, 0 \right\} d\lambda$$

where $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ is the standard $(r-1)$ -simplex with coordinates $\lambda_1, \dots, \lambda_r$, $d\lambda$ is the standard induced Lebesgue measure⁽²⁾, and $\mathbf{s} = (s_1, \dots, s_r)$ is a vector in \mathbb{R}^r such that the value μ_i appears exactly r_i times in the coordinates of \mathbf{s} as in Notation 3.2.10 below.

Notation 3.2.10. — Fix $\ell \geq 1$ and $r \geq 1$ two integers, $(r_1, \dots, r_\ell) \in \mathbb{N}^\ell$ a partition of r and $t \in \mathbb{R}$. We define for $(\mu_1, \dots, \mu_\ell) \in \mathbb{Q}^\ell$ the following polytopes:

$$\widehat{\square}_t = \left\{ (\beta_1, \dots, \beta_\ell) \in \widehat{\Delta}_{\ell-1} \subseteq \mathbb{R}^\ell \mid \sum_{i=1}^{\ell} \mu_i \beta_i \geq t \right\}$$

and

$$\square_t = \left\{ (\lambda_1, \dots, \lambda_r) \in \widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r \mid \sum_{i=1}^r s_i \lambda_i \geq t \right\},$$

where

$$\mathbf{s} = (\underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}, \underbrace{\mu_2, \dots, \mu_2}_{r_2 \text{ times}}, \dots, \underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}) \in \mathbb{Q}^r.$$

Similarly, for every permutation $w \in \mathfrak{S}_r$ we define

$$\square_t^w = \left\{ (\lambda_1, \dots, \lambda_r) \in \widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r \mid \sum_{i=2}^r \sigma_{w(i-1)} \lambda_i + \sigma_{w(r)} \lambda_1 \geq t \right\},$$

where

$$\boldsymbol{\sigma} = (\underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}, \underbrace{\mu_{\ell-1}, \dots, \mu_{\ell-1}}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}) \in \mathbb{Q}^r.$$

2. Unlike Chen, we do not normalize the measure in order to have $\lambda(\Delta_{r-1}) = 1$.

By abuse of notation, we will also denote by \square_t and \square_t^w the full dimensional polytopes in \mathbb{R}^{r-1} obtained via the projection of $\widehat{\Delta}_{r-1} \subseteq \mathbb{R}^r$ onto $\Delta_{r-1} \subseteq \mathbb{R}^{r-1}$. Explicitly, if (ν_2, \dots, ν_r) are coordinates in \mathbb{R}^{r-1} then

$$\square_t = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \subseteq \mathbb{R}^{r-1} \mid s_1 \left(1 - \sum_{i=2}^r \nu_i \right) + \sum_{i=2}^r s_i \nu_i \geq t \right\}$$

and

$$(\star) \quad \square_t^w = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{w(i-1)} \nu_i + \sigma_{w(r)} \left(1 - \sum_{i=2}^r \nu_i \right) \geq t \right\}.$$

Corollary 3.2.11. — For any admissible flag Y_\bullet on $\mathbb{P}(E)$ and any big rational class $\xi - tf$ on $\mathbb{P}(E)$ we have that

$$\text{vol}_{\mathbb{R}^r}(\Delta_{Y_\bullet}(\xi - tf)) = \int_{\widehat{\square}_t} \left(\sum_{i=1}^{\ell} \mu_i \beta_i - t \right) \frac{\beta_1^{r_1-1} \dots \beta_\ell^{r_\ell-1}}{(r_1-1)! \dots (r_\ell-1)!} d\beta,$$

for $\widehat{\square}_t \subseteq \mathbb{R}^\ell$ as in Notation 3.2.10.

Remark 3.2.12. — Let $F = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$ be any fiber of $\pi : \mathbb{P}(E) \rightarrow C$. Then, for any big \mathbb{R} -divisor D on $\mathbb{P}(E)$ we have that $F \not\subseteq \mathbf{B}_+(D)$. In fact, if $D \sim_{\mathbb{R}} A + E$, with A ample \mathbb{R} -divisor and E effective \mathbb{R} -divisor, and if $F \subseteq \text{Supp}(E)$, then we can write $E = aF + E'$ with $a > 0$ and $F \not\subseteq \text{Supp}(E')$, which implies that $F \not\subseteq \mathbf{B}_+(D)$ since $A + aF$ is ample.

As a direct consequence, we compute the restricted volume function on a fiber $F = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$.

Corollary 3.2.13. — Let F be a fiber of $\pi : \mathbb{P}(E) \rightarrow C$ and let $\xi - tf$ be a big rational class. Then,

$$\text{vol}_{\mathbb{P}(E)|_F}(\xi - tf) = (r-1)! \cdot \int_{\widehat{\square}_t} \frac{\beta_1^{r_1-1} \dots \beta_\ell^{r_\ell-1}}{(r_1-1)! \dots (r_\ell-1)!} d\beta,$$

for $\widehat{\square}_t \subseteq \mathbb{R}^\ell$ as in Notation 3.2.10. In particular, if $0 \leq \tau \leq \mu_\ell - t$ then we have that

$$\text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet}(\xi - tf)_{\nu_1=\tau}) = \int_{\widehat{\square}_{t+\tau}} \frac{\beta_1^{r_1-1} \dots \beta_\ell^{r_\ell-1}}{(r_1-1)! \dots (r_\ell-1)!} d\beta,$$

where Y_\bullet is any admissible flag on $\mathbb{P}(E)$ with divisorial component $Y_1 = F$. In particular, these volumes depend only on $\text{gr}(\text{HN}_\bullet(E))$, the graded vector bundle associated to the Harder-Narasimhan filtration of E .

Proof. — We consider

$$v(\tau) = \text{vol}_{\mathbb{P}(E)}(\xi - tf + \tau f) = r! \cdot \int_{\widehat{\square}_{t-\tau}} \left(\sum_{i=1}^{\ell} \mu_i \beta_i - t + \tau \right) \frac{\beta_1^{r_1-1} \cdots \beta_{\ell}^{r_{\ell}-1}}{(r_1-1)! \cdots (r_{\ell}-1)!} d\beta.$$

Since $F \not\subseteq \mathbf{B}_+(\xi - tf)$ by Remark 3.2.12, Theorem 3.1.1 and the Leibniz integral rule give

$$\text{vol}_{\mathbb{P}(E)|F}(\xi - tf) = \frac{1}{r} \cdot \frac{d}{d\tau} v(\tau) \Big|_{\tau=0} = (r-1)! \cdot \int_{\widehat{\square}_t} \frac{\beta_1^{r_1-1} \cdots \beta_{\ell}^{r_{\ell}-1}}{(r_1-1)! \cdots (r_{\ell}-1)!} d\beta. \quad \square$$

Lemma 3.2.14. — *Following Notation 3.2.10 (\star), we have that*

$$\text{vol}_{\mathbb{R}^r}(\Delta_{Y_{\bullet}}(\xi - tf)) = \int_{\square_t} \left(\sum_{j=1}^r s_j \lambda_j - t \right) d\lambda$$

and

$$\text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_{\bullet}}(\xi - tf)_{\nu_1=\tau}) = \int_{\square_{t+\tau}} d\lambda = \text{vol}_{\mathbb{R}^{r-1}}(\square_{t+\tau}).$$

Moreover, $\text{vol}_{\mathbb{R}^{r-1}}(\square_{t+\tau}) = \text{vol}_{\mathbb{R}^{r-1}}(\square_{t+\tau}^w)$ for every $w \in \mathfrak{S}_r$.

Proof. — The first two equalities follow from Remark 3.2.9. For the last assertion consider $\square_t \subseteq \mathbb{R}^r$ and $\square_t^w \subseteq \mathbb{R}^r$ as in Notation 3.2.10. Then, there is a linear transformation $T : \mathbb{R}^r \rightarrow \mathbb{R}^r$, whose associated matrix in the canonical basis of \mathbb{R}^r is given by a permutation matrix, such that $T(\square_t) = \square_t^w$ and $|\det(T)| = 1$. \square

3.3. Newton-Okounkov bodies on projective bundles over curves

Let E be a vector bundle on a smooth projective curve C , of rank $r \geq 2$ and degree d . In this section we study the geometry of Newton-Okounkov bodies of rational big classes in $N^1(\mathbb{P}(E))$ in terms of the numerical information of the Harder-Narasimhan filtration of E . In particular, Theorem H will be a consequence of Lemma 3.3.4 and Theorem 3.3.10. We follow Notation 3.1.5.

Definition 3.3.1 (Linear flag). — A complete flag of subvarieties Y_{\bullet} on the projective vector bundle $\mathbb{P}(E) \xrightarrow{\pi} C$ is called a *linear flag centered at* $p \in \mathbb{P}(E)$, *over the point* $q \in C$ (or simply a *linear flag*) if $Y_0 = \mathbb{P}(E)$, $Y_1 = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$ and $Y_i \cong \mathbb{P}^{r-i}$ is a linear subspace of Y_{i-1} for $i = 2, \dots, r$, with $Y_r = \{p\}$.

Let us begin with the following example that illustrates the general case. Namely, that the shape of Newton-Okounkov bodies on $\mathbb{P}(E)$ will depend on the semi-stability of E .

Example 3.3.2 (Ruled surfaces). — Suppose that $\text{rank}(E) = 2$ and let $\eta = a(\xi - \mu_\ell f) + bf \in N^1(\mathbb{P}(E))_{\mathbb{Q}}$ be a big class. In other words, $a, b \in \mathbb{Q}_{>0}$. Let

$$Y_\bullet : \mathbb{P}(E) \supseteq F = \pi^{-1}(q) \supseteq \{p\}$$

be the linear flag centered at $p \in \mathbb{P}(E)$, over $q \in C$. The Newton-Okounkov body of η can be computed by applying [LM09, Theo. 6.4] (see Example 3.1.2)

Semi-stable case. If E is semi-stable then Corollary 3.2.3 implies that every big class is ample. In particular, for every $\eta \in \text{Big}(X)_{\mathbb{Q}}$ and every \mathbb{Q} -divisor D_η with numerical class η , we have that $N(D_\eta) = 0$ and $P(D_\eta) = D_\eta$. It follows that, with the notation as in Example 3.1.2, $\nu = 0$, $\tau_F(\eta) = b$, $\alpha(t) = 0$ for every $t \in [0, b]$ and $\beta(t) = a$ for every $t \in [0, b]$. The Newton-Okounkov bodies are given by rectangles in this case.

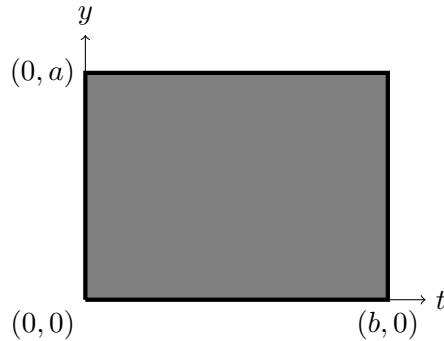


FIGURE 1. Newton-Okounkov body $\Delta_{Y_\bullet}(\eta)$ for E semi-stable

Non semi-stable case. If E is not semi-stable we consider its Harder-Narasimhan filtration

$$0 \rightarrow E_1 \rightarrow E \rightarrow Q_1 \rightarrow 0,$$

and we note that in this case $E_1 = Q_\ell$. The quotient $E \rightarrow Q_1 \rightarrow 0$ corresponds to a section $s : C \rightarrow \mathbb{P}(E)$ with $[s(C)] = \xi - \mu_\ell f$ in $N^1(\mathbb{P}(E))$. The curve $s(C)$ is the only irreducible curve on $\mathbb{P}(E)$ with negative self-intersection. On the other hand, if D_η is any \mathbb{Q} -divisor with numerical class η then, either η is inside

the nef cone of $\mathbb{P}(E)$ and thus $P(D_\eta) = D_\eta$ and $N(D_\eta) = 0$, or η is big and not nef in which case we compute that

$$[P(D_\eta)] = \frac{b}{\mu_\ell - \mu_1}(\xi - \mu_1 f) \quad \text{and} \quad [N(D_\eta)] = \left(\frac{a(\mu_\ell - \mu_1) - b}{\mu_\ell - \mu_1} \right) (\xi - \mu_\ell f)$$

in $N^1(\mathbb{P}(E))_{\mathbb{Q}}$.

We notice that $N(D_\eta) = \left(\frac{a(\mu_\ell - \mu_1) - b}{\mu_\ell - \mu_1} \right) \cdot s(C)$ as \mathbb{Q} -divisor by minimality of the negative part and by the fact that the negative part is unique in its numerical equivalence class. See [Băd01, Lemm. 14.10, Cor. 14.13] for details.

Let $t^* = b - a(\mu_\ell - \mu_1)$. We have that η is big and nef if and only if $t^* \geq 0$, in which case the class $\eta - tf$ is big and nef for $0 \leq t \leq t^*$. For $t^* \leq t \leq b$, the same computation above enables us to find the Zariski decomposition of $D_\eta - tF$, which is big and not nef.

We notice that, with the notation as in Example 3.1.2, $\nu = 0$. On the other hand, the functions $\alpha(t)$ and $\beta(t)$ will depend on whether or not we have $\{p\} = s(C) \cap F$. A straightforward computation shows that the Newton-Okounkov bodies are given by the following finite polygons in \mathbb{R}^2 .

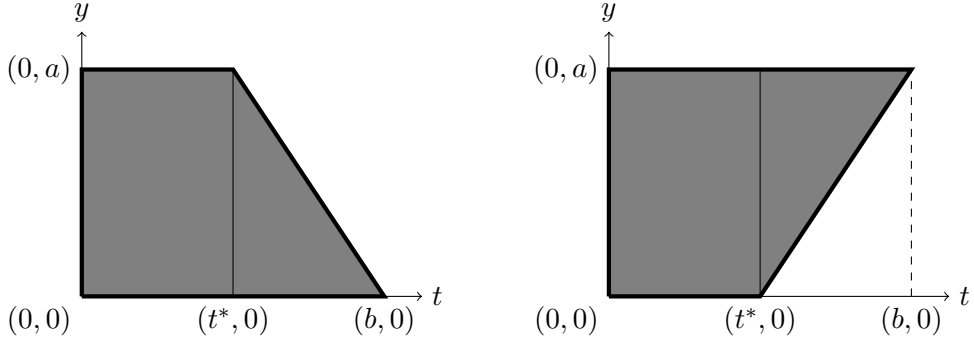


FIGURE 2. $\Delta_{Y_\bullet}(\eta)$ for E not semi-stable and η big and nef

(a) if $\{p\} \neq s(C) \cap F$

(b) if $\{p\} = s(C) \cap F$

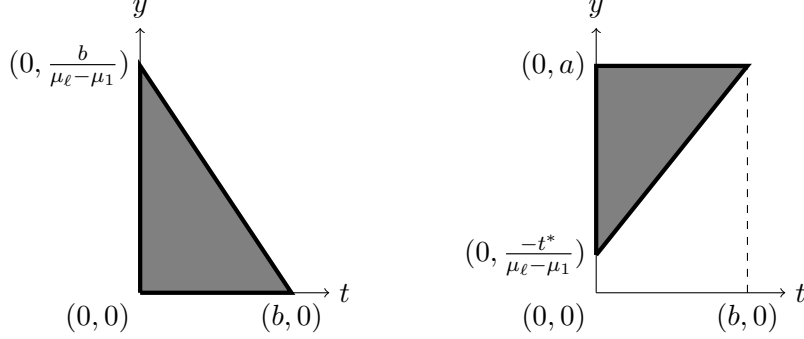


FIGURE 3. $\Delta_{Y_\bullet}(\eta)$ for E not semi-stable and η big and not nef
 (a) if $\{p\} \neq s(C) \cap F$ (b) if $\{p\} = s(C) \cap F$

We notice that Figure 3 provides examples of big and not nef divisors classes η such that the origin $\vec{0} \in \Delta_{Y_\bullet}(\eta)$ for almost every linear flag Y_\bullet except for one. This shows in particular that condition (2) in the characterization of nefness given in [KL15, Cor. 2.2] has to be checked for *all* linear flags.

We first observe that Proposition I states that the shape of Newton-Okounkov bodies on projective semi-stable vector bundles will be similar as in Example 3.3.2 above. Moreover, this provides a characterization of semi-stability in terms of Newton-Okounkov bodies.

We will need the following observation by Küronya, Lozovanu and Maclean.

Proposition 3.3.3 ([KLM12, Prop. 3.1]). — *Let X be a normal projective variety together with an admissible flag Y_\bullet . Suppose that D is a big \mathbb{Q} -divisor such that $Y_1 \not\subseteq \mathbf{B}_+(D)$ and that $D - tY_1$ is ample for some $0 \leq t < \tau_{Y_1}(D)$, where $\tau_{Y_1}(D) = \sup\{s > 0 \mid D - sY_1 \text{ is big}\}$. Then*

$$\Delta_{Y_\bullet}(X, D)_{\nu_1=t} = \Delta_{Y_\bullet|Y_1}(Y_1, (D - tY_1)|_{Y_1}),$$

where $Y_\bullet|Y_1 : Y_1 \supseteq Y_2 \supseteq \dots \supseteq Y_n$ is the induced admissible flag on Y_1 .

In particular, if $\text{Psef}(X) = \text{Nef}(X)$ then we have that the Okounkov body $\Delta_{Y_\bullet}(D)$ is the closure in \mathbb{R}^n of the following set

$$\{(t, \nu_2, \dots, \nu_n) \in \mathbb{R}^n \mid 0 \leq t < \tau_{Y_1}(D), (\nu_2, \dots, \nu_n) \in \Delta_{Y_\bullet|Y_1}(Y_1, (D - tY_1)|_{Y_1})\}.$$

Proof of Proposition I. — (1) \Rightarrow (2). Let $\eta = a(\xi - \mu_\ell f) + bf$ be a big rational class on $\mathbb{P}(E)$ and $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots \supseteq Y_{r-1} \supseteq Y_r = \{p\}$ be a linear flag centered at $p \in \mathbb{P}(E)$, over the point $q \in C$. Since E is semi-stable we

have that $\text{Big}(\mathbb{P}(E)) = \text{Amp}(\mathbb{P}(E))$ by Corollary 3.2.3. Equivalently, we have that $\mathbf{B}_+(\eta) = \emptyset$ for every big rational class η .

We notice that $\tau_F(\eta) = \sup\{s > 0 \mid \eta - sf \in \text{Big}(\mathbb{P}(E))\} = b$. The implication follows from Proposition 3.3.3 and [Bou12, Cor. 4.11], by noting that if D_η is a \mathbb{Q} -divisor with numerical class η then

$$\Delta_{Y_\bullet}(\eta)_{\nu_1=t} = \Delta_{Y_\bullet|F}(F, (D_\eta - tF)|_F) = a\Delta_{Y_\bullet|F}(F, H) = a\Delta_{r-1} \subseteq \mathbb{R}^{r-1},$$

where $H \subseteq F \cong \mathbb{P}^{r-1}$ is an hyperplane section.

(2) \Rightarrow (1). We notice that if for all linear flags Y_\bullet the Newton-Okounkov body of $\eta = a(\xi - \mu_\ell f) + bf$ is given by $\Delta_{Y_\bullet}(\eta) = [0, b] \times a\Delta_{r-1} \subseteq \mathbb{R}^r$ then η is a big and nef class, by [KL15, Cor. 2.2]. We can therefore compute the volume of η via the top self-intersection $\text{vol}_{\mathbb{P}(E)}(\eta) = \eta^r = ra^{r-1}(b - a(\mu_\ell - \mu(E)))$. On the other hand, we have that

$$\text{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\eta)) = \text{vol}_{\mathbb{R}^r}([0, b] \times a\Delta_{r-1}) = \frac{a^{r-1}}{(r-1)!}b,$$

and hence [LM09, Theo. A] leads to $\mu_\ell = \mu(E)$, implying the result. \square

We will first reduce our problem to computing the Newton-Okounkov body of the big and nef divisor class $\xi - \mu_1 f$.

Lemma 3.3.4. — *Let E be a not semi-stable vector bundle on a smooth projective curve C , with $\text{rank}(E) = r \geq 2$. Then the Newton-Okounkov body of any big rational class $\eta \in \text{Big}(\mathbb{P}(E))_{\mathbb{Q}}$ can be expressed in terms of the Newton-Okounkov body $\Delta_{Y_\bullet}(\xi - \mu_1 f) \subseteq \mathbb{R}^r$. More precisely, if $\eta = a(\xi - \mu_\ell f) + bf$ is a big rational class and Y_\bullet is a linear flag, then*

(1) $\Delta_{Y_\bullet}(\eta) = ([0, t^*] \times a\Delta_{r-1}) \cup (a\Delta_{Y_\bullet}(\xi - \mu_1 f) + t^*\vec{e}_1)$ if η is big and nef;

(2) $\Delta_{Y_\bullet}(\eta) = a\Delta_{Y_\bullet}(\xi - \mu_1 f)_{\nu_1 \geq -t^*} + t^*\vec{e}_1$ if η is big and not nef.

Here $t^* = b - a(\mu_\ell - \mu_1)$ and $a\Delta_{r-1} \subseteq \mathbb{R}^{r-1}$ is the $(r-1)$ -simplex with side length a .

Proof. — If η is an ample rational class then $\Delta_{Y_\bullet}(\eta)_{\nu_1=t} = a\Delta_{r-1}$ for $0 \leq t \leq t^*$, by Proposition 3.3.3 and [LM09, Exam. 1.1]. On the other hand, we have that $\Delta_{Y_\bullet}(\eta)_{\nu_1 \geq t^*} = a\Delta_{Y_\bullet}(\xi - \mu_1 f) + t^*\vec{e}_1$ by Theorem 3.1.1.

If η is big and not nef then $t^* < 0$. Theorem 3.1.1 implies therefore that $a\Delta_{Y_\bullet}(\xi - \mu_1 f)_{\nu_1 \geq -t^*} = \Delta_{Y_\bullet}(\eta) - t^*\vec{e}_1$, which leads to (2). \square

As the following remark points out, Okounkov bodies can be computed via divisorial Zariski decomposition whenever $\overline{\text{Mov}}(\mathbb{P}(E)) = \text{Nef}(\mathbb{P}(E))$.

Remark 3.3.5 (σ -decomposition). — Let E be an unstable vector bundle over C and let D_η be a big \mathbb{Q} -divisor such that $[D_\eta] = \eta = a(\xi - \mu_\ell f) + bf$. If we define for each $\Gamma \subseteq \mathbb{P}(E)$ prime divisor

$$\sigma_\Gamma(D_\eta) = \inf\{\text{mult}_\Gamma \Delta \mid \Delta \text{ effective such that } \Delta \equiv_{\text{num}} D_\eta\}$$

then, it follows from [Nak04, III.1.11] that $\sigma_\Gamma(D_\eta) > 0$ only for finitely many Γ . N. Nakayama defined the σ -decomposition or *divisorial Zariski decomposition* of D_η to be $D_\eta = P_\sigma(D_\eta) + N_\sigma(D_\eta)$, where

$$N_\sigma(D_\eta) = \sum_{\Gamma \text{ prime}} \sigma_\Gamma(D_\eta) \Gamma \quad \text{and} \quad P_\sigma(D_\eta) = D_\eta - N_\sigma(D_\eta).$$

It follows from [Nak04, III.1.14 and V.1.3] that

1. $P_\sigma(D_\eta)$ is movable,
2. $N_\sigma(D_\eta) = 0$ if and only if D_η is movable, and
3. $\text{Supp}(N_\sigma(D_\eta))$ coincides with the divisorial part of $\mathbf{B}_-(\eta)$.

Moreover, if Y_\bullet is a linear flag on $\mathbb{P}(E)$ centered at $p \in \mathbb{P}(E)$ then $\Delta_{Y_\bullet}(D_\eta) = \Delta_{Y_\bullet}(P_\sigma(D_\eta))$ whenever $p \notin \text{Supp}(N_\sigma(D_\eta))$, by [KL15, Theo. C].

On the other hand, Lemma 3.2.4 implies that the σ -decomposition of D_η is non-trivial only if $r_\ell = 1$ and $m^* := b - a(\mu_\ell - \mu_{\ell-1}) < 0$, in which case we have that

$$[P_\sigma(D_\eta)] = \frac{b}{\mu_\ell - \mu_{\ell-1}}(\xi - \mu_{\ell-1}f) \quad \text{and} \quad [N_\sigma(D_\eta)] = \frac{-m^*}{\mu_\ell - \mu_{\ell-1}}(\xi - \mu_\ell f)$$

in $N^1(\mathbb{P}(E))_{\mathbb{Q}}$, by [FL, Cor. 7.3].

In particular, if $\overline{\text{Mov}}(\mathbb{P}(E)) = \text{Nef}(\mathbb{P}(E))$ (i.e. if $r_\ell = 1$ and $\ell = 2$) and Y_\bullet is a linear flag centered at a general⁽³⁾ point $p \in \mathbb{P}(E)$ then we have that for $0 \leq t \leq \mu_\ell - \mu_1$ the slice at $\nu_1 = t$ is given by

$$\begin{aligned} \Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t} &= \Delta_{\mathbb{P}(E)|F}(\xi - \mu_1 - tf) && \text{by Theorem 3.1.1} \\ &= \Delta_{Y_\bullet}(\xi - \mu_1 f - tf)|_{\nu_1=0} \\ &= \Delta_{Y_\bullet}(P_\sigma(\xi - \mu_1 f - tf))|_{\nu_1=0} && \text{as } p \text{ is general} \\ &= \left(1 - \frac{t}{\mu_\ell - \mu_1}\right) \cdot \Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=0}. \end{aligned}$$

Lemma 3.3.4 and [KL15, Lemm. 1.7] lead to $\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=0} = \Delta_{r-1}$ and hence

$$\Delta_{Y_\bullet}(\xi - \mu_1 f) = \left\{ (\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq \mu_\ell - \mu_1, \sum_{i=2}^r \nu_i \leq 1 - \frac{\nu_1}{\mu_\ell - \mu_1} \right\}.$$

3. We need to assume that p does not belong to the divisorial part of $\mathbf{B}_-(\xi - \mu_\ell f)$.

We will need the following result concerning the Newton-Okounkov bodies of some special graded algebras.

Lemma 3.3.6. — Fix $A \in \mathbb{R}$, an integer $r \geq 2$, $\sigma = (\sigma_1, \dots, \sigma_r) \in \mathbb{R}^r$ and homogeneous coordinates $[x_1 : \dots : x_r]$ on \mathbb{P}^{r-1} . Put $\mathfrak{b}_0 = k$ and consider for every integer $m \geq 1$ the vector subspace $\mathfrak{b}_m \subseteq H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(m))$ generated by monomials $x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ of total degree $|\alpha| = m$ such that $\sum_{i=1}^r \sigma_i \alpha_i > A$. Suppose that $A \geq 0$ and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$. Then

1. $\mathfrak{b}_\bullet = \bigoplus_{m \geq 0} \mathfrak{b}_m$ is a graded subalgebra of the coordinate ring $k[x_1, \dots, x_r]$.
2. Let us denote by V_\bullet the flag of linear subspaces

$$V_\bullet : V_0 = \mathbb{P}^{r-1} \supseteq V_1 \supseteq \dots \supseteq V_{r-1},$$

where $V_i = \{x_1 = \dots = x_i = 0\}$ for $i = 1, \dots, r-1$ and consider the Schubert cell decomposition of the full flag variety parametrizing complete linear flag on \mathbb{P}^{r-1}

$$\mathbb{F}_r = \coprod_{w \in \mathfrak{S}_r} \Omega_w,$$

with respect to V_\bullet . Fix a permutation $w \in \mathfrak{S}_r$ and let $Y_\bullet \in \Omega_w$. Then $\Delta_{Y_\bullet}(\mathfrak{b}_\bullet)$ is given by the projection of the polytope

$$\left\{ (\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid \sum_{i=1}^r \nu_i = 1 \text{ and } \sum_{i=1}^r \sigma_{w(i)} \nu_i \geq 0 \right\}$$

onto the hyperplane $\{\nu_r = 0\}$.

Proof. — Let us suppose that $x^\alpha \in \mathfrak{b}_m$ and $x^{\alpha'} \in \mathfrak{b}_{m'}$, then $x^{\alpha+\alpha'}$ belongs to $\mathfrak{b}_{m+m'}$ since we have $\sum_{i=1}^r \sigma_i(\alpha_i + \alpha'_i) > 2A \geq A$ as long as $A \geq 0$. This proves (1).

In order to prove (2) we will first show that every automorphism φ of \mathbb{P}^{r-1} that fixes the flag V_\bullet induces an automorphism of the graded algebra \mathfrak{b}_\bullet . For this, we remark that automorphisms φ of \mathbb{P}^{r-1} fixing the flag V_\bullet correspond to lower triangular matrices in $\text{PGL}_r(k)$ and hence, given $\varphi = (\varphi_{i,j})_{1 \leq i, j \leq r}$ such a matrix, the image of the monomial $x^\alpha \in \mathfrak{b}_m$ via the induced action on $k[x_1, \dots, x_r]$ is given by

$$\varphi(x^\alpha) = \prod_{i=1}^r (\varphi_{i,1}x_1 + \dots + \varphi_{i,i}x_i)^{\alpha_i}.$$

The above product can be written as a linear combination of monomials of the form $x^{\alpha'} = x^{k_1+\dots+k_r}$ where $k_i = (k_{i,1}, \dots, k_{i,i}, 0, \dots, 0) \in \mathbb{N}^r$ is such that

$|k_i| = \alpha_i$. Let us prove that all these monomials belong to \mathfrak{b}_m as well. In fact, we have that $\alpha'_i = \sum_{j=i}^r k_{j,i}$ and hence

$$\begin{aligned} \sum_{i=1}^r \sigma_i \alpha'_i &= \sum_{i=1}^r \sum_{j=i}^r \sigma_i k_{j,i} = \sum_{i=1}^r \sum_{j=1}^i \sigma_j k_{i,j} \\ &\geq \sum_{i=1}^r \sum_{j=1}^i \sigma_i k_{i,j} && \text{since } \sigma_1 \geq \dots \geq \sigma_r \\ &= \sum_{i=1}^r \sigma_i |k_i| = \sum_{i=1}^r \sigma_i \alpha_i > A. \end{aligned}$$

It follows that φ induces an automorphism of the graded algebra \mathfrak{b}_\bullet .

In order to compute Newton-Okounkov bodies with respect to linear flags on a given Schubert cell, we note that [Bri05, Prop. 1.2.1] implies that given a permutation $w \in \mathfrak{S}_r$ and a linear flag $Y_\bullet \in \Omega_w$ there exists an automorphism φ of \mathbb{P}^{r-1} that fixes the reference flag V_\bullet and such that the image of the flag W_\bullet^w via the induced action of φ on \mathbb{F}_r is Y_\bullet , where W_\bullet^w is the linear flag such that for every $i = 1, \dots, r-1$ we have

$$W_i^w = \{x_{w(1)} = \dots = x_{w(i)} = 0\} \subseteq \mathbb{P}^{r-1}.$$

It follows from the previous paragraph that φ induces an automorphism of the graded algebra \mathfrak{b}_\bullet and then for every $m \geq 1$ and every $P \in \mathfrak{b}_m$ we have that

$$\nu_{Y_\bullet}(\varphi(P)) = \nu_{\varphi(W_\bullet^w)}(\varphi(P)) = \nu_{W_\bullet^w}(P).$$

In particular, we have $\{\nu_{Y_\bullet}(P)\}_{P \in \mathfrak{b}_m} = \{\nu_{W_\bullet^w}(P)\}_{P \in \mathfrak{b}_m} \subseteq \mathbb{N}^{r-1}$ and consequently $\Delta_{Y_\bullet}(\mathfrak{b}_\bullet) = \Delta_{W_\bullet^w}(\mathfrak{b}_\bullet)$. Therefore, we can suppose that $Y_\bullet = W_\bullet^w \in \Omega_w$ in order to prove (2).

Since \mathfrak{b}_m is generated by monomials we have that

$$\Delta_{W_\bullet^w}(\mathfrak{b}_\bullet) = \bigcup_{m \geq 1} \left\{ \frac{\nu_{W_\bullet^w}(x^\alpha)}{m} \mid x^\alpha \in \mathfrak{b}_m \right\}.$$

We compute for every monomial $x^\alpha = x_1^{\alpha_1} \dots x_r^{\alpha_r} \in \mathfrak{b}_m$ that

$$\nu_{W_\bullet^w}(x^\alpha) = (\alpha_{w(1)}, \alpha_{w(2)}, \dots, \alpha_{w(r-1)}) \in \mathbb{N}^{r-1}$$

and hence for every $m \geq 1$ the set $\left\{ \frac{\nu_{W_\bullet^w}(x^\alpha)}{m} \mid x^\alpha \in \mathfrak{b}_m \right\}$ is given by the set of points of the form $\left(\frac{\alpha_{w(1)}}{m}, \dots, \frac{\alpha_{w(r-1)}}{m} \right) \in \mathbb{Q}_{\geq 0}^{r-1}$ where $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ is such that $|\alpha| = \alpha_1 + \dots + \alpha_r = m$ and $\sum_{i=1}^r \sigma_i \alpha_i > A$. Equivalently, is given by the set of points $\left(\frac{\alpha_{w(1)}}{m}, \dots, \frac{\alpha_{w(r-1)}}{m} \right) \in \mathbb{Q}_{\geq 0}^{r-1}$ such that

$$\begin{aligned} \circ \quad & \frac{\alpha_{w(1)}}{m} + \dots + \frac{\alpha_{w(r-1)}}{m} \leq 1 \text{ and} \\ \circ \quad & \sum_{i=1}^r \sigma_i \frac{\alpha_i}{m} = \sum_{i=1}^r \sigma_{w(i)} \frac{\alpha_{w(i)}}{m} \\ & = \sum_{i=1}^{r-1} \sigma_{w(i)} \frac{\alpha_{w(i)}}{m} + \sigma_{w(r)} \left(1 - \sum_{i=1}^{r-1} \frac{\alpha_{w(i)}}{m} \right) > \frac{A}{m}. \end{aligned}$$

We conclude therefore that

$$\Delta_{W^\bullet}(\mathbf{b}_\bullet) = \left\{ (\nu_1, \dots, \nu_{r-1}) \in \Delta_{r-1} \left| \sum_{i=1}^{r-1} \sigma_{w^{(i)}} \nu_i + \sigma_{w^{(r)}} \left(1 - \sum_{i=1}^{r-1} \nu_i \right) \geq 0 \right. \right\},$$

from which (2) follows. \square

Remark 3.3.7. — The assumption $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$ in Lemma 3.3.6 can always be fulfilled by considering an automorphism of \mathbb{P}^{r-1} permuting the chosen homogeneous coordinates.

Let us note that given $q \in C$, the Harder-Narasimhan filtration

$$\text{HN}_\bullet(E) : 0 = E_\ell \subseteq E_{\ell-1} \subseteq \dots \subseteq E_1 \subseteq E_0 = E$$

induces a (not necessarily complete) flag on $\mathbb{P}(E)$ in the following way: let $Y_0 = \mathbb{P}(E)$ and for every $i = 2, \dots, \ell$ the exact sequence

$$0 \rightarrow Q_i \rightarrow E/E_i \rightarrow E/E_{i-1} \rightarrow 0$$

induces an inclusion $\mathbb{P}((E/E_{i-1})|_q) \hookrightarrow \mathbb{P}((E/E_i)|_q)$. We obtain therefore the following (possibly partial) flag of linear subvarieties

$$\mathbb{P}((E/E_1)|_q) \subseteq \mathbb{P}((E/E_2)|_q) \subseteq \dots \subseteq \mathbb{P}((E/E_{\ell-1})|_q) \subseteq \mathbb{P}(E|_q) = \pi^{-1}(q) \subseteq \mathbb{P}(E)$$

with $\text{codim}_{\mathbb{P}(E)} \mathbb{P}((E/E_i)|_q) = \text{rank}(E_i) + 1$. We also note that this flag is complete if and only if all the semi-stable quotients Q_i are line bundles. In general, it will be necessary to choose a complete linear flag on each $\mathbb{P}((E_{i-1}/E_i)|_q) = \mathbb{P}(Q_i|_q)$ in order to complete the flag above.

We shall consider linear flags that are compatible with the Harder-Narasimhan filtration of E in the sense that they complete the previous flag.

Definition 3.3.8 (Compatible linear flag). — A linear flag Y_\bullet on $\mathbb{P}(E)$ over $q \in C$ is said to be *compatible with the Harder-Narasimhan filtration of E* if

$$Y_{\text{rank } E_i + 1} = \mathbb{P}((E/E_i)|_q) \cong \mathbb{P}^{r - \text{rank } E_i - 1} \subseteq \mathbb{P}(E)$$

for every $i = 1, \dots, \ell$.

We will adopt the following convention.

Convention 3.3.9. — Let V_\bullet be a fixed linear flag on $\mathbb{P}(E)$ over $q \in C$ and consider the corresponding Schubert cell decomposition

$$\mathbb{F}_r = \coprod_{w \in \mathfrak{S}_r} \Omega_w,$$

where \mathbb{F}_r is the full flag variety parametrizing complete flags of linear subspaces of $V_1 = \pi^{-1}(q) \cong \mathbb{P}^{r-1}$. We say that a linear flag Y_\bullet on $\mathbb{P}(E)$ over $q \in C$ belongs to a Schubert cell Ω_w if the induced linear flag $Y_\bullet|Y_1$ belongs to Ω_w .

We can prove now the main reduction step. Namely, compute the Newton-Okounkov body of the big and nef class $\xi - \mu_1 f$ with respect to any linear flag.

Theorem 3.3.10. — *Let C be a smooth projective curve and let E be a vector bundle over C of rank $r \geq 2$. Fix a linear flag Y_\bullet^{HN} on $\mathbb{P}(E)$ over $q \in C$ which is compatible with the Harder-Narasimhan filtration of E and let*

$$\mathbb{F}_r = \coprod_{w \in \mathfrak{S}_r} \Omega_w$$

be the corresponding Schubert cell decomposition of the full flag variety \mathbb{F}_r parametrizing linear flags on $\pi^{-1}(q) \cong \mathbb{P}^{r-1}$. Then, for every linear flag Y_\bullet on $\mathbb{P}(E)$ over $q \in C$ that belongs to Ω_w we have that

$$\Delta_{Y_\bullet}(\xi - \mu_1 f) = \{(\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq \mu_\ell - \mu_1, (\nu_2, \dots, \nu_r) \in \square_{\mu_1 + \nu_1}^w\},$$

where $\square_{\mu_1 + \nu_1}^w \subseteq \mathbb{R}^{r-1}$ is the full dimensional polytope defined in Notation 3.2.10 (\star), with (μ_1, \dots, μ_ℓ) and (r_1, \dots, r_ℓ) given by the Harder-Narasimhan filtration of E as in Notation 3.1.5.

Proof. — We follow Notation 3.1.5. We first note that if E is semi-stable then $\mu_1 = \mu_\ell = \mu(E)$ and hence the Theorem follows from Proposition I. Let us suppose from now on that E is not semi-stable.

It follows from Theorem 3.1.1 and Lemma 3.2.2 that the projection of the Newton-Okounkov body of $\xi - \mu_1 f$ onto the first coordinate is given by

$$\text{pr}_1(\Delta_{Y_\bullet}(\xi - \mu_1 f)) = [0, \mu_\ell - \mu_1].$$

By continuity of slices of Newton-Okounkov bodies (cf. [KL15, Lemm. 1.7]), it suffices to consider a fixed $t \in]0, \mu_\ell - \mu_1[\cap \mathbb{Q}$ and show that the slice of $\Delta_{Y_\bullet}(\xi - \mu_1 f)$ at $t \in]0, \mu_\ell - \mu_1[\cap \mathbb{Q}$ is given by

$$\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t} = \square_{\mu_1+t}^w \subseteq \mathbb{R}^{r-1}$$

for linear flags Y_\bullet on $\mathbb{P}(E)$ that belong to the Schubert cell Ω_w (see Convention 3.3.9).

Let us fix from now on a permutation $w \in \mathfrak{S}_r$ and a linear flag Y_\bullet on $\mathbb{P}(E)$, over $q \in C$, and let us denote by $Y_\bullet|F$ the induced flag on F . Suppose that $Y_\bullet|F$ belongs to the Schubert cell Ω_w with respect to the reference flag $Y_\bullet^{\text{HN}}|F$.

Consider a \mathbb{Q} -divisor D such that $[D] = \xi - \mu_1 f$ in $N^1(\mathbb{P}(E))_{\mathbb{Q}}$. For every integer $m \geq 1$, let us define the subspace

$$\begin{aligned} \mathfrak{a}_m = \mathfrak{a}_{m,t} &= H^0(\mathbb{P}(E)|_F, \mathcal{O}_{\mathbb{P}(E)}(\lfloor m(D - tF) \rfloor)) \\ &= \text{Im}(H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(\lfloor m(D - tF) \rfloor)) \xrightarrow{\text{rest}} H^0(F, \mathcal{O}_F(m))) \\ &\subseteq H^0(F, \mathcal{O}_F(m)). \end{aligned}$$

It follows from [LM09, Prop. 4.1, Rem. 4.25] that the restricted algebra \mathfrak{a}_{\bullet} above computes the desired slice

$$\Delta_{Y_{\bullet}|F}(\mathfrak{a}_{\bullet}) = \Delta_{Y_{\bullet}}(\xi - \mu_1 f)|_{\nu_1=t},$$

and that for every integer $n \geq 1$ we have

$$\Delta_{Y_{\bullet}}(nD)|_{\nu_1=nt} = \Delta_{Y_{\bullet}|F}(n(D - tF)) = n\Delta_{Y_{\bullet}|F}(D - tF) = n\Delta_{Y_{\bullet}}(D)|_{\nu_1=t}.$$

We will consider $\mathfrak{a}_{n\bullet} = \{a_{nm}\}_{m \geq 0}$ instead of $\mathfrak{a}_{\bullet} = \{a_m\}_{m \geq 0}$, for n fixed and divisible enough such that $n\mu_1 \in \mathbb{Z}$ and $nt \in \mathbb{Z}$. We also note that for $0 < t < \mu_{\ell} - \mu_1$ we have that

$$\mu_{\max}(S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q)) = m(\mu_{\ell} - \mu_1 - t) > 0$$

and

$$\mu_{\min}(S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q)) = -mt < 0.$$

Therefore, by considering n above large enough we may also assume that $\mu_{\max}(S^{nm} E \otimes \mathcal{O}_C(-nm(\mu_1 + t) \cdot q)) > 2g - 1$ for every $m \geq 1$.

Let $[x_1 : \cdots : x_r]$ be homogeneous coordinates on $F \cong \mathbb{P}^{r-1}$. Since Y_{\bullet} is a linear flag on $\mathbb{P}(E)$, there is an isomorphism of graded algebras

$$\phi : \bigoplus_{m \geq 0} H^0(F, \mathcal{O}_F(m)) \rightarrow k[x_1, \dots, x_r]$$

such that \mathfrak{a}_m can be regarded as a subspace of $k[x_1, \dots, x_r]_m$, the k -vector space of homogeneous polynomials of degree m in the variables x_1, \dots, x_r , for all $m \geq 0$. Via this identification, \mathfrak{a}_{\bullet} can be seen as a graded subalgebra of $k[x_1, \dots, x_r]$. Moreover, the projection formula implies that we can identify \mathfrak{a}_m with

$$\text{Im}\left(H^0(C, S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q)) \xrightarrow{\text{rest}} H^0(C, (S^m E \otimes \mathcal{O}_C(-m(\mu_1 + t) \cdot q))|_q)\right).$$

We shall define a graded subalgebra $\mathfrak{b}_{n\bullet} \subseteq \mathfrak{a}_{n\bullet}$ for which we can explicitly compute that

$$\Delta_{Y_{\bullet}|F}(\mathfrak{b}_{n\bullet}) = n\Box_{\mu_1+t}^w,$$

and we will prove that $\Delta_{Y_{\bullet}|F}(\mathfrak{b}_{n\bullet}) = \Delta_{Y_{\bullet}|F}(\mathfrak{a}_{n\bullet}) = n\Delta_{Y_{\bullet}|F}(\mathfrak{a}_{\bullet})$.

In order to construct $\mathfrak{b}_{n\bullet}$ let us note that Proposition 3.1.7 implies that for every $m \geq 1$ there is a filtration

$$F_\bullet : 0 = F_L \subseteq F_{L-1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = S^{nm}E \otimes \mathcal{O}_C(-nm(\mu_1 + t) \cdot q)$$

whose successive quotients have the form

$$F_{j-1}/F_j \cong Q_{\mathbf{m}(j), \mu_1+t} = S^{m_1}Q_1 \otimes \cdots \otimes S^{m_\ell}Q_\ell \otimes \mathcal{O}_C(-nm(\mu_1 + t) \cdot q)$$

for some partition $\mathbf{m}(j) \in \mathbb{N}^\ell$ of nm , and $\mu(Q_{\mathbf{m}(j), \mu_1+t}) \leq \mu(Q_{\mathbf{m}(j+1), \mu_1+t})$ for every $j \in \{1, \dots, L\}$.

Let us define $J = J(m) \in \{1, \dots, L\}$ to be the largest index such that $\mu(Q_{\mathbf{m}(J), \mu_1+t}) \leq 2g - 1$. We have that for every $j \in \{J, \dots, L - 1\}$, the short exact sequence

$$0 \rightarrow F_{j+1} \otimes \mathcal{O}_C(-q) \rightarrow F_{j+1} \rightarrow F_{j+1}|_q \rightarrow 0$$

gives an exact sequence in cohomology

$$0 \rightarrow \mathbf{H}^0(C, F_{j+1} \otimes \mathcal{O}_C(-q)) \rightarrow \mathbf{H}^0(C, F_{j+1}) \rightarrow \mathbf{H}^0(C, F_{j+1}|_q) \rightarrow 0,$$

since we have that $h^1(C, F_{j+1} \otimes \mathcal{O}_C(-q)) = 0$, by Lemma 3.1.6. In particular, we get for every $j \in \{J, \dots, L - 1\}$ a surjection $\mathbf{H}^0(C, F_{j+1}) \rightarrow \mathbf{H}^0(C, F_{j+1}|_q)$. Therefore, let us consider the subspaces

$$\mathfrak{b}_{nm} = \text{Im}(\mathbf{H}^0(C, F_{J+1}) \xrightarrow{\text{rest}} \mathbf{H}^0(C, F_{J+1}|_q)) = \mathbf{H}^0(C, F_{J+1}|_q) \subseteq \mathfrak{a}_{nm}.$$

Let us choose homogeneous coordinates $[x_1 : \dots : x_r]$ on F such that

$$Y_{i+1}^{\text{HN}} = \{x_1 = \dots = x_i = 0\} \subseteq \mathbb{P}(E)$$

for $i = 1, \dots, r - 1$. In particular, we have that

$$Y_{\text{rank } E_i + 1}^{\text{HN}} = \mathbb{P}((E/E_i)|_q) = \{x_1 = \dots = x_{\text{rank } E_i} = 0\} \subseteq \mathbb{P}(E)$$

for $i = 1, \dots, \ell$ and therefore the degree 1 part of the isomorphism ϕ ,

$$\phi_1 : \mathbf{H}^0(F, \mathcal{O}_F(1)) \cong \mathbf{H}^0(C, (E \otimes \mathcal{O}_C(-(\mu_1 + t) \cdot q))|_q) \rightarrow k[x_1, \dots, x_r]_1,$$

is such that for every $i = 0, \dots, \ell - 1$ the image of the subspace

$$\mathbf{H}^0(C, (E_i \otimes \mathcal{O}_C(-(\mu_1 + t) \cdot q))|_q) \subseteq \mathbf{H}^0(C, (E \otimes \mathcal{O}_C(-(\mu_1 + t) \cdot q))|_q)$$

via ϕ_1 coincide with the subspace generated by the variables $x_1, \dots, x_{\text{rank } E_i}$. By taking symmetric powers it follows from Proposition 3.1.7 that for each $m \geq 1$ we have that $\mathfrak{b}_{nm} \subseteq k[x_1, \dots, x_r]_{nm}$, the image of the subspace $\mathbf{H}^0(C, F_{J+1}|_q) \subseteq \mathbf{H}^0(C, F_0|_q)$, corresponds to the subspace of homogeneous polynomials of degree nm generated by polynomials of the form

$$P(\mathbf{x}) = P_1(\mathbf{x}_1) \cdots P_\ell(\mathbf{x}_\ell)$$

where P_i is an homogeneous polynomial of degree $m_i \geq 0$ in the variables $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,r_i})$, where $(x_1, \dots, x_r) = (\mathbf{x}_\ell, \dots, \mathbf{x}_1)$, and the m_i are such that $m_1 + \dots + m_\ell = nm$ and

$$\mu_1 m_1 + \dots + \mu_\ell m_\ell > nm(\mu_1 + t) + 2g - 1.$$

In other words, \mathfrak{b}_{nm} is the subspace generated by monomials $x^\alpha = x_1^{\alpha_1} \dots x_r^{\alpha_r}$ of total degree $|\alpha| = nm$ such that $\sum_{i=1}^r (\sigma_i - \mu_1 - t)\alpha_i > 2g - 1$, where $\sigma = (\underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}, \underbrace{\mu_{\ell-1}, \dots, \mu_{\ell-1}}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}) \in \mathbb{Q}^r$.

Then, it follows from Lemma 3.3.6 applied ⁽⁴⁾ to the collection of subspaces $\{\mathfrak{b}_{nm}\}_{m \geq 1}$ that $\mathfrak{b}_{n\bullet}$ is a graded subalgebra of $k[x_1, \dots, x_r]$ whose Newton-Okounkov body, with respect to a linear flag Y_\bullet that belongs to the Schubert cell Ω_w (see Convention 3.3.9) is given by

$$\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) = n\Box_{\mu_1+t}^w,$$

where

$$\Box_{\mu_1+t}^w = \left\{ (\nu_2, \dots, \nu_r) \in \Delta_{r-1} \mid \sum_{i=2}^r \sigma_{w(i-1)} \nu_i + \sigma_{w(r)} (1 - \sum_{i=2}^r \nu_i) \geq \mu_1 + t \right\}$$

and $\sigma = (\underbrace{\mu_\ell, \dots, \mu_\ell}_{r_\ell \text{ times}}, \underbrace{\mu_{\ell-1}, \dots, \mu_{\ell-1}}_{r_{\ell-1} \text{ times}}, \dots, \underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}) \in \mathbb{Q}^r$. Finally, we have

that

$$\begin{aligned} \text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet})) &= \text{vol}_{\mathbb{R}^{r-1}}(n\Box_{\mu_1+t}^w) \\ &= \text{vol}_{\mathbb{R}^{r-1}}(n\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t}) \text{ by Lemma 3.2.14} \\ &= \text{vol}_{\mathbb{R}^{r-1}}(\Delta_{Y_\bullet|F}(\mathfrak{a}_{n\bullet})) \end{aligned}$$

and hence the inclusion $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) \subseteq \Delta_{Y_\bullet|F}(\mathfrak{a}_{n\bullet})$ leads to the equality $\Delta_{Y_\bullet|F}(\mathfrak{b}_{n\bullet}) = \Delta_{Y_\bullet|F}(\mathfrak{a}_{n\bullet})$, as the two convex bodies have equal volume. From this we conclude that

$$\Delta_{Y_\bullet}(\xi - \mu_1 f)|_{\nu_1=t} = \Box_{\mu_1+t}^w \subseteq \mathbb{R}^{r-1}.$$

□

We notice that Theorem H is an immediate consequence of Theorem 3.3.10.

4. We note that if $C \cong \mathbb{P}^1$ then all the slopes $\mu_i = \mu(Q_i)$ are integer numbers and hence the inequality “ $> 2g - 1$ ” becomes “ ≥ 0 ”.

Proof of Theorem H. — We note that if $\eta = a(\xi - \mu_\ell f) + bf$ is a big rational class on $\mathbb{P}(E)$ and the induced linear flag Y_\bullet belongs to the Schubert cell Ω_w with respect to a reference flag Y_\bullet^{HN} , then Theorem 3.3.10 gives

$$\Delta_{Y_\bullet}(\xi - \mu_1 f) = \{(\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq \mu_\ell - \mu_1, (\nu_2, \dots, \nu_r) \in \square_{\mu_1 + \nu_1}^w\}$$

and hence

$$a\Delta_{Y_\bullet}(\xi - \mu_1 f) = \left\{ (\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq a(\mu_\ell - \mu_1), (\nu_2, \dots, \nu_r) \in a\square_{\mu_1 + \frac{1}{a}\nu_1}^w \right\}.$$

We compute that for $t^* = b - a(\mu_\ell - \mu_1)$ we have

$$a\Delta_{Y_\bullet}(\xi - \mu_1 f) + t^* \vec{e}_1 = \left\{ (\nu_1, \dots, \nu_r) \in \mathbb{R}_{\geq 0}^r \mid 0 \leq \nu_1 \leq b, (\nu_2, \dots, \nu_r) \in a\square_{\mu_1 + \frac{1}{a}(\nu_1 - t^*)}^w \right\}$$

with $\mu_1 + \frac{1}{a}(\nu_1 - t^*) = \mu_\ell - \frac{1}{a}(b - \nu_1)$. The result follows from Lemma 3.3.4. \square

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