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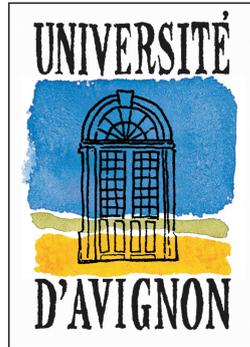
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ACADÉMIE D'AIX MARSEILLE
UNIVERSITÉ D'AVIGNON ET DES PAYS DE VAUCLUSE



UNIVERSITÉ D'AVIGNON
ET DES PAYS DE VAUCLUSE

UFR SCIENCES
ECOLE DOCTORALE "AGROSCIENCES ET SCIENCES"

Variational approximation of interface energies and applications

Approximation variationnelle d'énergies d'interface et applications

PH.D THESIS IN MATHEMATICS

Mohammed ZABIBA

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Abstract

Minimal partition problems consist in finding a partition of a domain into a given number of components in order to minimize a geometric criterion. In applicative fields such as image processing or continuum mechanics, it is standard to incorporate in this objective an interface energy that accounts for the lengths of the interfaces between components. The present work is focused on the theoretical and numerical treatment of minimal partition problems with interface energies. The considered approach is based on a Γ -convergence approximation and duality techniques.

Résumé

Les problèmes de partition minimale consistent à déterminer une partition d'un domaine en un nombre donné de composantes de manière à minimiser un critère géométrique. Dans les champs d'application tels que le traitement d'images et la mécanique des milieux continus, il est courant d'incorporer dans cet objectif une énergie d'interface qui prend en compte les longueurs des interfaces entre composantes. Ce travail est focalisé sur le traitement théorique et numérique de problèmes de partition minimale avec énergie d'interface. L'approche considérée est basée sur une approximation par Γ -convergence et des techniques de dualité.

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Résumé en français

Description du problème

Considérons une partition d'un domaine borné D de \mathbb{R}^2 par des ensembles relativement fermés $\Omega_1, \dots, \Omega_N$ appelés *phases* qui peuvent s'intersecter seulement à leurs frontières:

$$D = \cup_{j=1}^N \Omega_j, \text{ et } \Omega_i \cap \Omega_j = \partial\Omega_i \cap \partial\Omega_j \cap D \text{ pour } i \neq j.$$

On appelle Γ_{ij} l'interface séparant Ω_i et Ω_j :

$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \cap D \text{ pour } i \neq j,$$

avec la convention supplémentaire $\Gamma_{ii} = \emptyset$ (voir Figure 0.1 pour une illustration).

Le problème modèle de partition minimale que nous étudions, défini sur les partitions de D , est:

$$\min_{\Omega_1, \dots, \Omega_N} \sum_{i=1}^N \int_{\Omega_i} g_i(x) dx + E(\Omega_1, \dots, \Omega_N), \quad (0.0.1)$$

où $g_1, \dots, g_N \in L^1(D)$ et $E(\Omega_1, \dots, \Omega_N)$ est l'énergie totale d'interface. Cette énergie est choisie de la manière suivante:

$$E(\Omega_1, \dots, \Omega_N) = \frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \ell(\Gamma_{ij}), \quad (0.0.2)$$

où α_{ij} est un coefficient appelé *tension de surface* associé à Γ_{ij} pour $i, j = 1, \dots, N$, et $\ell(\Gamma_{ij})$ est la longueur de Γ_{ij} . Il est raisonnable de supposer que les tensions de surface satisfont $\alpha_{ij} = \alpha_{ji} > 0$ lorsque $i \neq j$ et $\alpha_{ii} = 0$. Nous noterons par la suite

$$S_N = \{(\alpha_{ij}) \in \mathbb{R}^{N \times N}, \alpha_{ij} = \alpha_{ji} > 0 \text{ si } i \neq j \text{ et } \alpha_{ii} = 0\}.$$

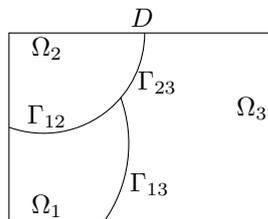


Figure 0.1: Partitionnement d'un domaine par des ensembles Ω_j qui s'intersectent seulement à leurs frontières. L'interface Γ_{ij} sépare Ω_i de Ω_j .

Afin de garantir la semi-continuité inférieure de l'énergie d'interface, il est nécessaire et suffisant de supposer que les tensions de surface satisfont l'inégalité triangulaire [7]:

$$\alpha_{ij} \leq \alpha_{ik} + \alpha_{kj} \quad \forall i, j, k. \quad (0.0.3)$$

Cette condition est aussi discutée dans [19, 21, 35]. Nous travaillerons donc surtout dans la classe de tensions de surface:

$$T_N = \{(\alpha_{ij}) \in S_N : \alpha_{ij} \leq \alpha_{ik} + \alpha_{kj} \quad \forall i, j, k\}.$$

Du point de vue mathématique, l'étude de la semi-continuité inférieure de (0.0.2) nécessite de se placer dans le cadre des ensembles de périmètre fini [9, 38]. L'énergie totale d'interface s'écrit alors

$$\frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D), \quad (0.0.4)$$

où $\Omega_1, \dots, \Omega_N$ sont maintenant supposés être des ensembles de périmètre fini dans D tels que $D = \cup_{i=1}^N \Omega_i$ à un ensemble Lebesgue négligeable près, $|\Omega_i \cap \Omega_j| = 0$ pour tout $i \neq j$ ($|\cdot|$ désigne la mesure de Lebesgue), $\Gamma_{ij} = \partial_M \Omega_i \cap \partial_M \Omega_j \cap D$ pour tout i, j , avec ∂_M la frontière au sens de la théorie géométrique de la mesure (ou frontière essentielle) de Ω_i dans D , et \mathcal{H}^1 est la mesure de Hausdorff (unidimensionnelle) sur \mathbb{R}^2 .

Nous noterons Per le périmètre BV , i.e., si $A \subset D$ est de périmètre fini dans D , $\text{Per}(A) = \mathcal{H}^1(\partial_M A \cap D)$.

Motivations

Traitement d'images

Le traitement d'image concerne principalement deux types de problèmes: la *restauration* et la *segmentation*. Dans ce deuxième cas nous voulons identifier les composantes d'une image, telles que des régions présentant une homogénéité de texture, d'intensité, de couleur, etc. Dans la classe de la segmentation d'image, les problèmes de *classification*, où un nombre donné de composantes doit être identifié, sont étroitement liés au problème (0.0.1).

Science des matériaux

Un matériau *polycristallin* est constitué d'une réunion de grains, qui sont de petits morceaux monocristallins. Beaucoup de métaux et céramiques sont de ce type. Entre deux grains Ω_i et Ω_j , le décalage entre les orientations cristallographiques peut être modélisé par une énergie du type (0.0.2).

Contexte

Dans [12, 13], une approximation du périmètre est obtenue à l'aide de la famille de fonctions:

$$\begin{aligned} \tilde{F}_\varepsilon : L^\infty(D, [0, 1]) &\rightarrow \mathbb{R} \\ u &\mapsto \frac{1}{\varepsilon} \int_D (L_\varepsilon u)(1 - u) dx, \end{aligned} \quad (0.0.5)$$

où $L_\varepsilon u$ est la solution (faible) du problème aux limites d'inconnue $v \in H^1(D)$:

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon &= u & \text{dans } D \\ \partial_n v_\varepsilon &= 0 & \text{sur } \partial D. \end{cases} \quad (0.0.6)$$

Les auteurs prouvent que \tilde{F}_ε Γ -converge, quand ε tend vers 0, vers la fonctionnelle

$$\tilde{F}(u) = \begin{cases} \frac{1}{2} \text{Per}_D(\{u = 1\}) & \text{si } u \in BV(D, \{0, 1\}) \\ +\infty & \text{sinon,} \end{cases}$$

fortement dans $L^1(D)$.

De plus, par transformation de Legendre-Fenchel, on obtient la formulation

$$\tilde{F}_\varepsilon(u) = \inf_{u \in H^1(D)} \left\{ \varepsilon \|\nabla v\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v\|_{L^2(D)}^2 + \int_D u(1 - 2L_\varepsilon u) dx \right) \right\}$$

qui convient bien à l'utilisation d'algorithmes de minimisations alternées. La fonctionnelle \tilde{F}_ε est utilisé dans [12] pour le partitionnement optimal multiphase avec des tensions superficielles uniformes, i.e., $\alpha_{ij} = \alpha$. Dans ce cadre, les auteurs prouvent des résultats de convergence et développent des algorithmes d'optimisation pour la classification et le défloutage d'images.

Travail effectué

Le but de cette thèse est d'étendre les résultats et les algorithmes de [12] au cas de tensions de surface non uniformes. On définit la fonctionnelle $G_\varepsilon : L^\infty(D, [0, 1]) \times L^\infty(D, [0, 1]) \rightarrow \mathbb{R}$ par

$$G_\varepsilon(u_i, u_j) = \frac{1}{\varepsilon} \int_D (L_\varepsilon u_i) u_j dx \quad \forall i \neq j,$$

où $L_\varepsilon u_i$ est la solution faible de (0.0.6). On définit aussi la fonctionnelle

$$G(u_i, u_j) = \begin{cases} \frac{1}{2} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) & \text{si } u_i, u_j \in BV(D, \{0, 1\}), u_i = \chi_{\Omega_i}, u_j = \chi_{\Omega_j}, \\ +\infty & \text{sinon.} \end{cases}$$

Les problèmes mathématiques abordés sont:

- la convergence ponctuelle de G_ε vers G ,
- la semi-continuité inférieure de

$$\frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D)$$

sous la condition (0.0.3) et sa relaxation,

- la Γ -convergence, i.e.,

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i, u_j) \xrightarrow{\Gamma} \frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} G(u_i, u_j),$$

- l'équi-coercivité de

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon).$$

Sur le plan numérique sont considérées:

- la résolution numérique efficace de (0.0.6),
- la conception d'algorithmes d'optimisation pour résoudre

$$\min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i, u_j) \right\}, \quad (0.0.7)$$

avec $\tilde{\mathcal{E}}_N$ défini par

$$\tilde{\mathcal{E}}_N = \left\{ (u_1, \dots, u_N) \in L^\infty(D, [0, 1])^N, \sum_{i=1}^N u_i = 1 \right\}.$$

Des applications variées sont étudiées, en particulier, nous examinons des problèmes de partitionnement minimal multiphase, y compris la classification supervisée ou automatique d'images ou le défloutage.

Chapter 1

Introduction

1.1 Problem description

Consider a partition of a bounded domain D of \mathbb{R}^2 into relatively closed subsets $\Omega_1, \dots, \Omega_N$ called *phases* that may intersect only through their boundaries:

$$D = \cup_{j=1}^N \Omega_j, \text{ and } \Omega_i \cap \Omega_j = \partial\Omega_i \cap \partial\Omega_j \cap D \text{ for } i \neq j.$$

Denote the interface separating Ω_i and Ω_j by Γ_{ij} :

$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \cap D \text{ for } i \neq j,$$

with the additional convention $\Gamma_{i,i} = \emptyset$ (see Figure 1.1 for an illustration).

The model problem of minimal partition we study, defined on partitions of D , is:

$$\min_{\Omega_1, \dots, \Omega_N} \sum_{i=1}^N \int_{\Omega_i} g_i(x) dx + E(\Omega_1, \dots, \Omega_N), \quad (1.1.1)$$

where $g_1, \dots, g_N \in L^1(D)$, and $E(\Omega_1, \dots, \Omega_N)$ is the total interface energy. This energy is chosen in the following way:

$$E(\Omega_1, \dots, \Omega_N) = \frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \text{Length}(\Gamma_{ij}), \quad (1.1.2)$$

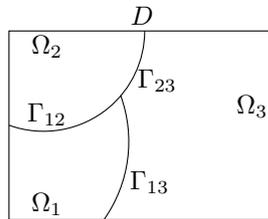


Figure 1.1: A partition of a domain into sets Ω_j that intersect only at their boundaries. Interface Γ_{ij} separates Ω_i from Ω_j .

1.2. Motivation

where α_{ij} is a coefficient called *surface tension* associated with Γ_{ij} for $i, j = 1, \dots, N$. It is reasonable to assume that the surface tensions satisfy $\alpha_{ij} = \alpha_{ji} > 0$ whenever $i \neq j$ and $\alpha_{ii} = 0$. We will denote in the sequel

$$S_N = \{(\alpha_{ij}) \in \mathbb{R}^{N \times N}, \alpha_{ij} = \alpha_{ji} > 0 \text{ if } i \neq j \text{ and } \alpha_{ii} = 0\}.$$

In order to guarantee the lower semicontinuity of the N -phase perimeter, it is necessary and sufficient to assume that the surface tensions satisfy the triangle inequality [7]:

$$\alpha_{ij} \leq \alpha_{ik} + \alpha_{kj} \quad \forall i, j, k. \quad (1.1.3)$$

This condition is also discussed in [19, 21, 35]. We will therefore mostly work with the class of surface tensions:

$$T_N = \{(\alpha_{ij}) \in S_N : \alpha_{ij} \leq \alpha_{ik} + \alpha_{kj} \quad \forall i, j, k\}.$$

From the mathematical viewpoint, the study of the lower semicontinuity of (1.1.2) requires to rephrase it in a suitable setting, namely the space of sets of finite perimeter [9, 38]. In this setting, the total interface energy can be written as

$$\frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D), \quad (1.1.4)$$

where $\Omega_1, \dots, \Omega_N$ are now assumed to be sets of finite perimeter in D such that $D = \cup_{i=1}^N \Omega_i$ up to a Lebesgue negligible set, $|\Omega_i \cap \Omega_j| = 0$ for all $i \neq j$ (denoting as $|\cdot|$ the Lebesgue measure), $\Gamma_{ij} = \partial_M \Omega_i \cap \partial_M \Omega_j \cap D$ for all i, j , with ∂_M is the measure theoretical (or essential) boundary of Ω_i in D , and \mathcal{H}^1 is the one-dimensional Hausdorff measure on \mathbb{R}^2 . We refer to [9, 38] for details on functions of bounded variation BV and sets of finite perimeter. We shall denote as Per the BV perimeter, i.e., if $A \subset D$ has finite perimeter in D we denote $\text{Per}(A) = \mathcal{H}^1(\partial_M A \cap D)$. In the BV context, the lower semicontinuity of the perimeter holds with respect to the strong convergence in L^1 of characteristic functions of sets.

1.2 Motivation

1.2.1 Image processing

Image processing concerns two main types of problems. Firstly, one is interested in image *restoration*, in order to remove the causes of the deterioration of an image. Besides, questions regarding image *segmentation* are addressed: we want to identify the components of an image, such as texture regions, intensities, colors and so on.

Within the class of image segmentation, we recall the problems of image *classification*, where a given number of components have to be identified.

In the standard greyscale image processing problem, one looks at $f = A\bar{u} + \nu$, assuming $f : \Omega \rightarrow [0, 1]$ as the observed image, \bar{u} as the undamaged one, A as a mask operator (blur kernel or a projection operator away from the missing parts of \bar{u}), ν as the noise, and one would like to obtain a segmented version or a continuous restoration of the original image \bar{u} . These classes of issues are linked to ill-posed minimization problems of form

$$\min_{u \in H(\Omega)} J(u) := \|Au - f\|_{H(\Omega)},$$

where the difficulties arise both since unbounded variations in the solution could arise from small perturbations in the data and since the convexity of the problem is not guaranteed. Complications

1.3. Mathematical and numerical methods

are increased if we are interested in a simultaneous segmentation and restoration problem. We remark that an objective assessment of segmentation algorithms is difficult to find. The main reason is that there is no unique ground-truth classification of an image with respect to which the output an algorithm can be compared. In the literature many models for restoration and/or segmentation of image have been developed, such as the Mumford-Shah [9, 42] and the TV- L^2 and TV- L^1 functionals [22, 48].

Nowadays, the problem

$$\min_{u \in H(\Omega)} J(u) + \alpha |Du|(\Omega),$$

is considered to obtain solutions which preserve the edges and are also smooth enough. The notation $|Du|(\Omega)$ refers to the total variation of u in Ω , where Du is its distributional derivative, that is to say a measure made of a concentrated part on the edges and of a diffuse one (that is ∇u) outside them, while α provides a weight on the total variation of the image. We refer to [9] for more details.

There are methods which use anisotropic variants of total variation [28, 29, 34], while others study segmentation problems through moving interfaces, like level-sets [24, 36, 44] or snakes [5]. A still different approach is developed by interpreting the image as a graph on which a minimal cut problem is faced [26, 49, 51].

1.2.2 Material sciences

A *polycrystalline* material is made up of a union of lots of *grains*, which are small single crystal pieces. Lots of metals and ceramics are of this type. In model (1.1.2), single grains are presented as connected components of Ω_i . Looking at two grains Ω_i and Ω_j , the mismatch between the crystallographic orientations of them influences α_{ij} which is the surface tension of the interface Γ_{ij} between the two grains. Actually, also the normal n_{ij} to Γ_{ij} affects the grains boundary energy, even if throughout we will ignore this dependence. The grain boundary network of polycrystalline materials determines important physical features of them such as conductivity and yield strength. Hence, the simulation of these boundaries subjected to industrial processes is an interesting issue. In some framework, a constant dependence of the energy density from misorientation has arisen, for large enough misorientations. In such cases, a good description of the movement of the grain boundary is achieved in model (1.1.2) taking all equal surface tensions $\alpha_{ij} = 1$. Anyway, other phenomena require the full generality of model (1.1.2), such as *grain boundary character distribution* [47]. Some authors [31–33], in the equal surface tension case, managed to obtain large scale simulations of grain growth and recrystallization in 3D through a diffusion generated motion, using signed distance functions to represent phases.

In [19, 35] a simulation of the evolution of the grain boundary is realized allowing the surface tension parameters α_{ij} to be different.

1.3 Mathematical and numerical methods

1.3.1 Relaxation

In the relaxation framework, we have an optimization problem such as

$$\inf_{u \in X} J(u) \tag{1.3.1}$$

for which a solution may not exist. Hence we enlarge the set X into a bigger one X^* and we introduce the functional J^* on X^* such that $J^*|_X = J$ and also:

1. The "relaxed" problem $\inf_{u^* \in X^*} J^*(u^*)$ has a solution.

2. $\inf_X J = \min_{X^*} J^*$.
3. The minimization sequences of the problem (1.3.1) accumulate towards solutions of the "relaxed" problem.
4. If the initial problem has a solution, it is also a solutions of the "relaxed" problem.

We usually enlarge X to X^* in a way such that X is dense in X^* is the smallest possible space with these properties. In order to do that, we have to well understand the behaviour of the minimizing sequences of the initial problem.

To be sure of having a solution, we ask that X^* is compact and J^* is lower semicontinuous. To gain other properties we also demand that if $(u_n) \subset X$ converge to $u^* \in X^*$ then

$$J^*(u^*) \leq \liminf J(u_n) \tag{1.3.2}$$

and that for every $u^* \in X^*$ there exist a sequence $(u_n) \subset X$ converging to u^* , so that

$$J^*(u^*) = \liminf J(u_n). \tag{1.3.3}$$

It can be remarked that, actually, we do not impose that J^* coincides with J on X and this enables us to consider as J^* the lower semicontinuous envelope (or closure) of J , i.e.

$$J^*(u^*) := \inf\{\liminf J(u_n), u_n \rightarrow u\} = clJ(u). \tag{1.3.4}$$

This choice clearly implies the previous properties (1.3.2) and (1.3.2). For more details, see [14,37].

1.3.2 Convexification

The convex envelope, denoted by $\text{conv } f$, of a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, is defined as the greatest convex function majorized by f , or equivalently as the greatest convex function whose epigraph contains $\text{conv}(\text{epi}f)$, i.e.,

$$(\text{conv } f)(x) = \inf\{a : (x, a) \in \text{conv}(\text{epi}f)\}.$$

The link between the convex envelope of a function f and its Legendre-Fenchel transform is made explicit by Theorem 2.42 (Fenchel-Moreau-Rockafellar theorem), detailed in Section 2.4.

In [23] Chambolle, Cremers, and Pock define the following (convex) function

$$\begin{aligned} \mathcal{F} : BV(D, \mathbb{R}^N) &\rightarrow [0, +\infty) \\ v &\mapsto \frac{1}{2} \sum_{i=1}^N \int_D |Dv_i|, \end{aligned}$$

and they extend it to $L^2(D, \mathbb{R}^N)$ by imposing $\mathcal{F} = +\infty$ if $v \notin BV(D, \mathbb{R}^N)$. Then, they introduce the function $\mathcal{J} : L^2(D, \mathbb{R}^N) \rightarrow [0, +\infty]$ as

$$\mathcal{J}(v) = \begin{cases} \mathcal{F}(v) & \text{if } v \in BV(D, \{0, 1\}), \sum_{i=1}^N v_i = 1 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

A reformalized version of the minimal partition problem is

$$\min_{v \in L^2(D, \mathbb{R}^N)} \mathcal{J}(v) + \int_D v(x) \cdot g(x) dx, \tag{1.3.5}$$

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where $g = (g_1, \dots, g_N) \in L^1(D, \mathbb{R}_+^N)$, which is nonconvex, because, even if the function \mathcal{F} is convex, the domain of \mathcal{J} is not convex.

Anyway the authors already know that problem (1.3.5) has a solution. The strategy is to look at the convex envelope of \mathcal{J} (and so also of (1.3.5) because of the linearity of the other term).

By letting, for $v \in L^2(D, \mathbb{R}^N)$

$$\mathcal{J}^*(w) = \sup_{v \in L^2(D, \mathbb{R}^N)} \int_D v(x) \cdot w(x) dx - \mathcal{J}(v)$$

be the Legendre-Fenchel conjugate of \mathcal{J} , and then, again, $v \in L^2(D, \mathbb{R}^N)$

$$\mathcal{J}^{**}(v) = \sup_{w \in L^2(D, \mathbb{R}^N)} \int_D v(x) \cdot w(x) dx - \mathcal{J}^*(w)$$

be the Legendre-Fenchel conjugate of \mathcal{J}^* , the function \mathcal{J}^{**} is the convex, lower-semicontinuous envelope of \mathcal{J} (see [30, 46]).

On the one hand, they remark that the minimizers of problem (1.3.5) are also minimizers of

$$\min_{v \in L^2(D, \mathbb{R}^N)} \mathcal{J}^{**} + \int_D v(x) \cdot g(x) dx. \quad (1.3.6)$$

On the other hand, minimizers of (1.3.6) have to be convex combinations of minimizers of the problem (1.3.5).

Anyhow, the function \mathcal{J}^{**} cannot be always made explicit. For this reason, the authors use a notion of "local" convex envelope, which can be written roughly in a particular form and can be easier implemented. For more details, see [23].

1.3.3 Γ -convergence

The notion of Γ -convergence was introduced in a paper by E. De Giorgi and T. Franzoni in 1975 [27]. It has been largely used in the calculus of variations in particular in homogenization theory, phase transitions, image processing, and material science. This type of convergence, together with compactness (equicoercivity), guarantees us the convergence (up to subsequences) of minimizers of the limiting functional. This converging property is not ensured by pointwise convergence, which is quite different from the Γ -convergence. The Γ -convergence assures also the convergence of the minimum energy of \bar{F}_ε to that F . For this reasons, this type of convergence is said to be a variational convergence.

The fundamental theorem of Γ -convergence, is summarized by the implication

$$\Gamma - \text{convergence} + \text{equi-equicoercivity} \Rightarrow \text{convergence of minimum problems.}$$

One of the first illustration of this concept was the work of Modica, Mortola [41]. In the pioneering work by Modica and Mortola [40] (see also [1]), it is proved that the functional

$$F_{MM}^\varepsilon(u) = \begin{cases} \varepsilon \int_D |\nabla u|^2 + \frac{1}{\varepsilon} \int_D W(u) & \text{if } u \in H^1(D), \\ +\infty & \text{otherwise.} \end{cases}$$

where W is a double well potential, converges as $\varepsilon \rightarrow 0$ in a sense to be precised, to the perimeter functional

$$F_{MM}(u) = \begin{cases} c \text{Per}_D(\{u = 1\}) & \text{if } u \in BV(D, \{0, 1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

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where $c = 2 \int_0^1 \sqrt{W(x)} dx$.

The original motivation for Modica and Mortola was a mathematical justification of convergence for some two phase problem based upon a model by Cahn and Hilliard [25, 41].

Later, this procedure gave rise to a method to perform a numerical approximation of a wide class of variational problems. Indeed, in order to minimize numerically the geometrical functional given by the perimeter, one may find it convenient to minimize the more regular functionals F_{MM}^ε , which are of elliptic type. This idea was used by many authors in the last two decades, with quite satisfactory results [4, 15, 19, 45].

In [11] and [10], Ambrosio and Tortorelli proposed two elliptic approximations to the weak formulation of the Mumford and Shah problem. The approach they presented in [11] is simpler than the approximation proposed in [10]. Indeed in [11], the strategy of Ambrosio and Tortorelli uses the particular potential $W(u) = \frac{1}{4}(1-u)^2$ and develops the approximation as follows.

Let $X = L^2(D)^2$ and let us define

$$F_{AT}^\varepsilon(u, v) = \int_D (u - g)^2 dx + a \int_D v^2 |\nabla u|^2 dx + b \int_D \left(\varepsilon |\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon} \right) dx$$

if $(u, v) \in W^{1,2}(D)^2$, $0 \leq v \leq 1$, and $F_{AT}^\varepsilon(u, v) = +\infty$ otherwise.

Let us denote $SBV(D)$ as the subset of $BV(D)$ whose functions are such that their gradient measure have no Cantor part in the Lebesgue decomposition.

The authors also define the limiting Mumford-Shah functional,

$$F_{MS}(u, v) = \begin{cases} \int_D (u - g)^2 dx + a \int_D |\nabla u|^2 dx + b \mathcal{H}^{d-1}(S_u) & \text{if } u \in SBV(D), v \equiv 1, \\ +\infty & \text{otherwise.} \end{cases}$$

It can be shown that the functional F_{AT}^ε Γ -converges to F_{MS} as ε tends towards 0 (in a decreasing way) in $L^2(D)$. Moreover, F_{AT}^ε admits a minimizer $(u_\varepsilon, v_\varepsilon)$ such that, up to subsequences, u_ε converges to some $u \in SBV(D)$, which is a minimizer of $F_{MS}(u, 1)$, and such that $\inf F_{AT}^\varepsilon(u_\varepsilon, v_\varepsilon)$ converges to $F_{MS}(u, 1)$.

In [2], Alberti and Bellettini studied the asymptotic behaviour as $\varepsilon \rightarrow 0$, of the nonlocal models for phase transition described by the scaled free energy

$$F_{AB}^\varepsilon(u) := \frac{1}{4\varepsilon} \int_{D \times D} J_\varepsilon(x' - x) |u(x') - u(x)|^2 dx' dx + \frac{1}{\varepsilon} \int_D W(u(x)) dx, \quad (1.3.7)$$

where u is a scalar density function, W is a double-well potential which vanishes at ± 1 , J is a nonnegative interaction potential and $J_\varepsilon(h) := \varepsilon^{-N} J(h/\varepsilon)$. They proved that the functionals F_{AB}^ε converge in a variational sense to the anisotropic surface energy

$$F_{AB}(u) := \int_{S_u} \sigma(\nu_u),$$

where u is allowed to take the values ± 1 only, ν_u is the normal to the interface S_u between the phases $\{u = +1\}$ and $\{u = -1\}$, and σ is the surface tension.

In [35], Esedoğlu and Otto introduced an approximation of the weighted surface area functional in dimension d

$$E_{OE}(\Omega_1, \dots, \Omega_N) = \sum_{i \neq j}^N \alpha_{ij} \text{Area}(\partial\Omega_i \cap \partial\Omega_j). \quad (1.3.8)$$

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In the spirit of (1.3.7), the idea to approximate the surface area of the boundary $(\partial\Omega_i \cap \partial\Omega_j)$ in (1.3.8) by the term:

$$\text{Area}(\partial\Omega_i \cap \partial\Omega_j) \approx \frac{1}{\varepsilon} \int \chi_{\Omega_i} G^\varepsilon * \chi_{\Omega_j} dx,$$

where

$$G^\varepsilon(x) = \frac{1}{(4\pi\varepsilon^2)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\varepsilon^2}}.$$

Therefore the approximate energy of (1.3.8) becomes:

$$E_{OE}^\varepsilon(\Omega_1, \dots, \Omega_N) = \frac{1}{\varepsilon} \sum_{i \neq j}^N \alpha_{ij} \int \chi_{\Omega_i} G^\varepsilon * \chi_{\Omega_j} dx.$$

In [12, 13], the authors use different techniques for several reasons: indeed, the functional of Modica-Mortola does not accept the characteristic functions, and moreover the derivative with respect to u involves $-\Delta u$, which can yield a high number of iterations in optimization processes with fine grids (CFL condition). A gradient-free perimeter approximation is used with the following functional:

$$\begin{aligned} \tilde{F}_\varepsilon : L^\infty(D, [0, 1]) &\rightarrow \mathbb{R} \\ u &\mapsto \frac{1}{\varepsilon} \int_D (L_\varepsilon u)(1 - u) dx, \end{aligned} \quad (1.3.9)$$

where $L_\varepsilon u$ is the (weak) solution of the boundary value problem with unknown $v \in H^1(D)$:

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } D, \\ \partial_n v_\varepsilon = 0 & \text{on } \partial D. \end{cases} \quad (1.3.10)$$

They prove that the functionals \tilde{F}_ε Γ -converge, when $\varepsilon \rightarrow 0$, to the functional

$$\tilde{F}(u) = \begin{cases} \frac{1}{2} \text{Per}_D(\{u = 1\}) & \text{if } u \in BV(D, \{0, 1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

strongly in $L^1(D)$.

In [12, 13], the authors also prove equicoercivity. In addition, the Legendre-Fenchel transformation provides

$$\tilde{F}_\varepsilon(u) = \inf_{v \in H^1(D)} \left\{ \varepsilon \|\nabla v\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v\|_{L^2(D)}^2 + \int_D u(1 - 2L_\varepsilon u) dx \right) \right\},$$

which fits well with the use of alternating algorithms. The functional \tilde{F}_ε is used in [12] for multiphase optimal partitioning with uniform surface tensions, i.e., $\alpha_{ij} = \alpha$. In this framework, the authors prove convergence results and develop optimization algorithms for image classification and deblurring.

1.4 The present work

The aim of this thesis is to extend the results and algorithms of [12] to interface energies with general surface tensions α_{ij} . We define the functional $G_\varepsilon : L^\infty(D, [0, 1]) \times L^\infty(D, [0, 1]) \rightarrow \mathbb{R}$ by

$$G_\varepsilon(u_i, u_j) = \frac{1}{\varepsilon} \int_D (L_\varepsilon u_i) u_j dx \quad \forall i \neq j,$$

where $L_\varepsilon u_i$ is the (weak) solution of (1.3.10). We also define the functional

$$G(u_i, u_j) = \begin{cases} \frac{1}{2} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) & \text{if } u_i, u_j \in BV(D, \{0, 1\}), u_i = \chi_{\Omega_i}, u_j = \chi_{\Omega_j}, \\ +\infty & \text{otherwise.} \end{cases}$$

where Ω_i, Ω_j are two subsets of finite perimeter of D .

We address the following theoretical issues:

- pointwise convergence of G_ε to G ,
- lower semicontinuity of

$$\frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D)$$

under the condition (1.1.3) and relaxation,

- Γ -convergence, i.e.,

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i, u_j) \xrightarrow{\Gamma} \frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} G(u_i, u_j)$$

in appropriate space,

- equicoercivity of

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon).$$

The numerical issues are

- efficient numerical solution of (1.3.10),
- design of optimization algorithms to solve

$$\min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i, u_j) \right\}, \quad (1.4.1)$$

with $\tilde{\mathcal{E}}_N$ defined by

$$\tilde{\mathcal{E}}_N = \left\{ (u_1, \dots, u_N) \in L^\infty(D, [0, 1])^N, \sum_{i=1}^N u_i = 1 \right\}.$$

Various applications are studied, in particular we look at binary and multilabel minimal partition problems, including supervised or automatic image classification and deblurring.

1.5 Thesis outline

The subsequent chapters of the thesis are organised as follows:

Chapter 2

In chapter 2 we introduce and present all the mathematical tools needed to develop the theory of our procedure: in section 1 we recall some basic functional analysis, definitions and results.

In section 2 we introduce some basic geometry measure theory, while in section 3 we recall the definition of Γ -convergence as well as some properties of it.

The Legendre-Fenchel transform is introduced in section 4, where also the Fenchel-Moreau-Rockafellar theorem is stated.

Section 5 recalls the Lax-Milgram theorem and classical notions on homogeneous elliptic Neumann.

The operator L_ε which plays a central role, is introduced with some properties in section 6.

Chapter 3

Chapter 3 is devoted to the regularization approach: the classical Modica-Mortola theorem is recalled, as well as the gradient-free perimeter approximation. Our approach is based on an approximation of interface energy in which the functionals G_ε converges pointwise to G .

We provide then numerical validation for such a regularization approach (see Section 3.3.2), as well as some numerical examples. The notions of lower semicontinuity and equicoercivity are discussed. Theorem 3.11 states important results concerning the equicoercivity of functionals.

Chapter 4

In chapter 4 we focus on the surface tensions α_{ij} . We show that the interface energy can be rewritten as a linear combination of perimeters and we are able to compute the coefficients. Lemmas 4.1, 4.2 and 4.3 are technical results which allow us to obtain new coefficients that lead to better properties. Moreover, an algorithm which searches for a conical combination provides us new coefficients. Theorem 4.4 guarantees the existence of positive coefficients assuming that $N = 3, 4$ and the surface tensions α_{ij} satisfy the condition (1.1.3). Theorem 4.5 ensure us that if the new coefficients are nonnegative, then the surface tensions α_{ij} satisfy the condition (1.1.3) for $N = 3, 4, 5$.

The positiveness of the coefficients implies Γ -convergence, as shown in Theorem 4.8. The sign of the coefficients lead us to choose a strategy among several optimization algorithms, such as

- primal algorithm, with a variant that incorporates a volume constraint,
- primal dual algorithm, which relies on Theorem 4.9,
- saddle point algorithm.

We show that the saddle point and primal algorithms are in fact identical under some assumptions. In addition, some examples are described, using the primal algorithm with a volume constraint.

Finally, we show an example where the surface tensions α_{ij} does not satisfy the condition (1.1.3).

Chapter 5

Chapter 5 concerns an algorithm based on Legendre-Fenchel duality, which depends on the conditional negative definiteness of the matrix Q of the coefficients (α_{ij}) . In Theorem 5.4 we provide also a sufficient condition for the conditional negative definiteness of the coefficient matrix Q if $N = 3, 4$. The converse of this theorem is not true, that is to say the necessary condition does not hold, as shown by Remark 5.3. Moreover, in Remark 5.4 an example with $N \geq 5$, for which the theorem is not true, is presented. Then we present the dual formulation of the interface energy by Legendre-Fenchel transformation, both in the continuous framework and in the discrete one. Assuming that Q is conditionally negative definite, Theorem 5.7 and Corollaries 5.8, 5.9 allow us to obtain an alternating algorithm to solve the minimal partition problem. Other numerical examples are provided. We compare the primal algorithm and the algorithm based on Legendre-Fenchel duality.

Chapter 6

In this chapter we begin by introducing the two technical Lemmas 6.2 and 6.3. These results give us the framework on which we can apply the projected gradient algorithm. We focus on the applications of this algorithm to

- image deblurring of greyscale image,
- image deblurring of colour image,
- medical imaging by Radon transform.

In the end, we present some numerical examples.

Chapter 2

Mathematical tools

In the proof of our results we will need different theoretical tools, which are recalled in this chapter.

2.1 Basic functional analysis

All the results of this section, except for Lemma 2.24, Corollary 2.25 and Theorem 2.27, are detailed in [3].

2.1.1 Some results from integration

Let D be an open set of \mathbb{R}^N equipped with the Lebesgue measure. We define by $L^2(D)$ the space of measurable functions which are square integrable in D . Under the scalar product

$$\langle f, g \rangle_{L^2} = \int_D f(x)g(x)dx,$$

$L^2(D)$ is a Hilbert space. We denote the corresponding norm by

$$\|f\|_{L^2(D)} = \left(\int_D |f(x)|^2 dx \right)^{1/2}.$$

We denote by $C_c^\infty(D)$ (or $\mathcal{D}(D)$) the space of functions of class C^∞ with compact support in D .

Theorem 2.1. *The space $C_c^\infty(D)$ is dense in $L^2(D)$, that is, for all $f \in L^2(D)$ there exists a sequence $f_n \in C_c^\infty(D)$ such that*

$$\lim_{n \rightarrow +\infty} \|f - f_n\|_{L^2(D)} = 0.$$

Corollary 2.2. *Let us take $f \in L^2(D)$. If for every function $\phi \in C_c^\infty(D)$, we have*

$$\int_D f(x)\phi(x)dx = 0,$$

then $f(x) = 0$ almost everywhere in D .

2.1.2 Weak differentiation

We will define the concept of the weak derivative in $L^2(D)$.

Definition 2.3. Let v be a function of $L^2(D)$. We say that v is differentiable in the weak sense in $L^2(D)$ if there exist functions $w_i \in L^2(D)$, for $i \in \{1, \dots, N\}$, such that, for every function $\phi \in C_c^\infty(D)$, we have

$$\int_D v(x) \frac{\partial \phi}{\partial x_i}(x) dx = - \int_D w_i(x) \phi(x) dx.$$

Each w_i is called the i th weak partial derivative of v and is written from now on as $\frac{\partial v}{\partial x_i}$.

Lemma 2.4. Let v be a function of $L^2(D)$. If there exists a constant $C > 0$ such that, for every function $\phi \in C_c^\infty(D)$ and for all indices $i \in \{1, \dots, N\}$, we have

$$\left| \int_D v(x) \frac{\partial \phi}{\partial x_i}(x) dx \right| \leq C \|\phi\|_{L^2(D)},$$

then v is differentiable in the weak sense.

Definition 2.5. Let σ be a function from D into \mathbb{R}^N with all components belonging to $L^2(D)$ (we say $\sigma \in L^2(D)^N$). We say that σ has divergence in the weak sense in $L^2(D)$ if there exists a function $w \in L^2(D)$ such that, for every function $\phi \in C_c^\infty(D)$, we have

$$\int_D \sigma(x) \cdot \nabla \phi(x) dx = - \int_D w(x) \phi(x) dx.$$

The function w is called the weak divergence of σ and from now on will be denoted as $\operatorname{div} \sigma$.

Lemma 2.6. Let σ be a function of $L^2(D)^N$. If there exists a constant $C > 0$ such that, for every function $\phi \in C_c^\infty(D)$, we have

$$\left| \int_D \sigma(x) \cdot \nabla \phi(x) dx \right| \leq C \|\phi\|_{L^2(D)},$$

then σ has a divergence in the weak sense.

2.1.3 The space $H^1(D)$

Definition 2.7. Let D be an open set of \mathbb{R}^N . The Sobolev space $H^1(D)$ is defined by

$$H^1(D) = \left\{ v \in L^2(D) \text{ such that } \forall i \in \{1, \dots, N\} \frac{\partial v}{\partial x_i} \in L^2(D) \right\},$$

where $\frac{\partial v}{\partial x_i}$ is the weak partial derivative of v in the sense of definition 2.3.

Proposition 2.8. Equipped with the scalar product

$$\langle u, v \rangle_{H^1} = \int_D (u(x)v(x) + \nabla u(x) \cdot \nabla v(x)) dx$$

and with the norm

$$\|u\|_{H^1(D)} = \left(\int_D (|u(x)|^2 + |\nabla u(x)|^2) dx \right)^{1/2}$$

the Sobolev space $H^1(D)$ is a Hilbert space.

2.1. Basic functional analysis

Theorem 2.9 (density). *If D is a regular open bounded set of class C^1 , or if $D = \mathbb{R}_+^N$, or even if $D = \mathbb{R}^N$, then $C_c^\infty(\overline{D})$ is dense in $H^1(D)$.*

Theorem 2.10 (trace). *Let D be an open bounded regular set of class C^1 , or $D = \mathbb{R}_+^N$. We define the trace mapping γ_0*

$$\begin{aligned} H^1(D) \cap C(\overline{D}) &\rightarrow L^2(\partial D) \cap C(\overline{\partial D}) \\ v &\mapsto \gamma_0(v) = v|_{\partial D}. \end{aligned}$$

This mapping γ_0 is extended by continuity to a continuous linear mapping of $H^1(D)$ into $L^2(\partial D)$, again called γ_0 . In particular, there exists a constant $C > 0$ such that, for every function $v \in H^1(D)$, we have

$$\|\gamma_0(v)\|_{L^2(\partial D)} \leq C\|v\|_{H^1(D)}.$$

2.1.4 The space $H^{1/2}(\partial D)$

Definition 2.11. *Let D be an open bounded regular set of class C^1 , or $D = \mathbb{R}_+^N$. The space $H^{1/2}$ is defined by*

$$H^{1/2}(\partial D) = \gamma_0(H^1(D)).$$

Definition 2.12. *Equipped with the norm*

$$\|v\|_{H^{1/2}(\partial D)} = \inf \{ \|v\|_{H^1(D)} \text{ such that } \gamma_0(\phi) = v \},$$

$H^{1/2}(\partial D)$ is a Banach space (and even a Hilbert space). We then define $H^{-1/2}(\partial D)$ as the dual of $H^{1/2}(\partial D)$.

2.1.5 The space $H^{\text{div}}(D)$

Definition 2.13. *The space H^{div} is defined by*

$$H^{\text{div}}(D) = \{ \sigma \in L^2(D)^N \text{ such that } \text{div} \sigma \in L^2(D) \},$$

where $\text{div} \sigma$ is the weak divergence of σ in the sense of the definition 2.5.

Proposition 2.14. *Equipped with the scalar product*

$$\langle \sigma, \tau \rangle_{H^{\text{div}}(D)} = \langle \sigma, \tau \rangle_{L^2} + \langle \text{div} \sigma, \text{div} \tau \rangle_{L^2}, \quad (2.1.1)$$

and with the norm $\|\sigma\|_{H^{\text{div}}} = \sqrt{\langle \sigma, \sigma \rangle_{H^{\text{div}}}}$, the space $H^{\text{div}}(D)$ is a Hilbert space.

Theorem 2.15. *Let D be a Lipschitz open set in \mathbb{R}^N . Then the set of vector functions belonging to $\mathcal{D}(\overline{D})$ is dense in $H^{\text{div}}(D)$.*

Theorem 2.16 (divergence formula). *Let D be an open bounded regular set of class C^1 . We define the 'normal trace' mapping γ_n*

$$\begin{aligned} H^{\text{div}} \cap C(\overline{D}) &\rightarrow H^{-1/2}(\partial D) \cap C(\overline{\partial D}) \\ \sigma = (\sigma_i)_{1 \leq i \leq N} &\mapsto \gamma_n(\sigma) = (\sigma \cdot n)|_{\partial D}, \end{aligned}$$

where $n = (n_i)_{1 \leq i \leq N}$ is the outward unit normal to ∂D . This mapping γ_n is extended by continuity to a continuous linear mapping from H^{div} into $H^{-1/2}(\partial D)$. Further, if $\sigma \in H^{\text{div}}$ and $\phi \in H^1(D)$, we have

$$\int_D \text{div} \sigma \phi dx + \int_D \sigma \cdot \nabla \phi dx = \langle \gamma_n(\sigma), \gamma_0(\phi) \rangle_{H^{-1/2}, H^{1/2}(\partial D)}.$$

Definition 2.17. $H_0^{\text{div}}(D)$ is the closure of $\mathcal{D}(D)$ in $H^{\text{div}}(D)$.

Theorem 2.18. $H_0^{\text{div}}(D) = \ker \gamma_n = \{\sigma \in H^{\text{div}}(D), \gamma_n(\sigma) = 0\}$.

2.1.6 Linear operators

Throughout this subsection, E and F denote two Banach spaces.

Definition 2.19. Let $T : E \rightarrow F$ be a linear map. We say that T is **bounded (or continuous)** if there is a constant $C \geq 0$ such that

$$\|Tu\| \leq C\|u\| \quad \forall u \in E.$$

The norm of a bounded operator is defined by

$$\|T\|_{(E,F)} = \sup_{u \neq 0} \frac{\|Tu\|}{\|u\|}.$$

Definition 2.20. The space of **continuous linear** operators from E into F , denoted by $\mathcal{L}(E, F)$ is equipped with the norm

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} \|Tx\|.$$

As usual, one writes $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

Definition 2.21. Let $B_E = \{x \in E : \|x\| \leq 1\}$. A bounded operator $T \in \mathcal{L}(E, F)$ is said to be **compact** if $T(B_E)$ has compact closure in F (in the strong topology).

Definition 2.22. The dual space of E , denoted by $E' := \mathcal{L}(E, \mathbb{R})$, is the space of all continuous linear functionals on E . If $M \subset E$ is a linear subspace we set

$$M^\perp = \{f \in E'; \langle f, x \rangle_{(E',E)} = 0 \quad \forall x \in M\}.$$

If $N \subset E'$ is a linear subspace we set

$$N^\perp = \{x \in E; \langle f, x \rangle_{(E',E)} = 0 \quad \forall f \in N\}.$$

Definition 2.23. Let $T \in \mathcal{L}(E, F)$. We define the **adjoint** operator $T^* \in \mathcal{L}(F', E')$ by

$$\langle T^*u, v \rangle_{(E',E)} = \langle u, Tv \rangle_{(F',F)} \quad \forall u \in F', \forall v \in E.$$

Lemma 2.24. If E is reflexive and $T \in \mathcal{L}(E, F)$ is injective, then $\text{Im}T^*$ is dense in E' .

Proof. According to Remark 6 and Corollary 2.18 in [20], we have

$$\overline{\text{Im}T^*} = (\text{Im}T^*)^{\perp\perp} = (\ker T)^\perp = \{0\}^\perp = E'.$$

□

Corollary 2.25. Assume that E is a reflexive Banach space and let $E \subset F$ with continuous canonical injection. Then the canonical embedding from F' into E' has dense range.

2.2. Basic geometric measure theory

Proof. Let $I : E \rightarrow F$ be the canonical injection from E into F and $I^* : F' \rightarrow E'$. Then we have

$$\forall \varphi \in F', \forall x \in E \quad \langle I^*(\varphi), x \rangle_{E', E} = \langle \varphi, I(x) \rangle_{F', F} = \langle \varphi, x \rangle_{F', F},$$

that is to say, $I^*(\varphi) = \varphi|_E$.

Applying Lemma 2.24, we deduce that $\text{Im} I^*$ is dense in E' . □

Definition 2.26. Let H be a Hilbert space identified with its dual space H' . A bounded operator $T \in \mathcal{L}(H)$ is said to be self-adjoint if $T = T^*$, i.e.,

$$\langle Tu, v \rangle_H = \langle u, Tv \rangle_H \quad \forall u, v \in H.$$

Theorem 2.27 (Rellich). If D is an open bounded regular set of class C^1 , then for every bounded sequence of $H^1(D)$ we can extract a convergent subsequence in $L^2(D)$ (we say that the canonical injection of $H^1(D)$ into $L^2(D)$ is compact).

Remark 2.1. The above result can be generalized to Lipschitz domains (see section 1.3 in [39] on this subject).

2.2 Basic geometric measure theory

Throughout this section, D denotes an open subset of \mathbb{R}^N .

2.2.1 BV space

Definition 2.28. A function $u \in L^1(D)$ whose partial derivatives in the sense of distributions are measures with finite total mass in D is called a **function of bounded variation**. The vector space of functions of bounded variation in D is denoted by $BV(D)$. Thus $u \in BV(D)$ if and only if $u \in L^1(D)$ and there are Radon measures μ_1, \dots, μ_N with finite total mass in D such that

$$\int_D u \frac{\partial \varphi}{\partial x_i} dx = - \int_D \varphi d\mu_i \quad \forall \varphi \in C_c^1(D), \quad i = 1, \dots, N.$$

If $u \in BV(D)$, the total variation of the measure Du is

$$\|Du\| = \sup \left\{ \int_D u \text{div} \phi dx : \phi \in C_c^1(D, \mathbb{R}^N), |\phi(x)| \leq 1 \text{ for } x \in D \right\} < \infty.$$

The space $BV(D)$, endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + \|Du\|,$$

is a Banach space.

We also use $|Du|(D)$ to denote the total variation $\|Du\|$. For a complete introduction to the structure of BV functions in any dimension we refer to [9].

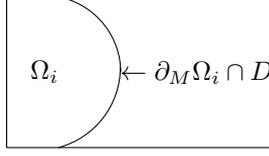


Figure 2.1: The measure theoretical (or essential) boundary.

2.2.2 Sets of finite perimeter

We consider a N -dimensional Euclidean space \mathbb{R}^N , with $N \geq 2$. The Lebesgue measure of a Lebesgue measurable set $E \subseteq \mathbb{R}^N$ will be denoted by $|E|$.

Definition 2.29. A Borel set $E \subset \mathbb{R}^N$ has finite perimeter in a open set D if $\chi_{E|_D} \in BV(D)$. The perimeter of E in D in that case is

$$P(E, D) = |D\chi_E|(D) = \sup \left\{ \int_E \operatorname{div} g dx : g \in C_0^\infty(D, \mathbb{R}^d), |\phi(x)| \leq 1 \right\}.$$

E is a set of finite perimeter if $\chi_E \in BV(\mathbb{R}^N)$.

Definition 2.30 (Reduced boundary). We say a point $x_0 \in \partial E$ belongs to the reduced boundary $\partial^* E$ if

$$\lim_{r \rightarrow 0} \frac{\int_{B_r} D\chi_E}{\int_{B_r} |D\chi_E|} = \nu(x_0)$$

for some unit vector $\nu(x_0) \in S^{N-1}$. The function $\nu_E : \partial^* E \rightarrow S^{N-1}$ is called the generalised inner normal to E .

Definition 2.31 (Points of density t). For every $t \in [0, 1]$ and every \mathcal{L}^N -measurable set $E \subseteq \mathbb{R}^N$, define

$$E^t = \{x \in \mathbb{R}^N : D(E, x) = t\},$$

where $D(E, x)$ is the density of E at x defined by

$$D(E, x) := \lim_{r \rightarrow 0} \frac{|E \cap B(x, r)|}{|B(x, r)|}.$$

Then E^t is the set of all points where E has density t .

We establish some elementary facts about the essential boundaries of arbitrary subsets of \mathbb{R}^N .

Definition 2.32 (The measure theoretical (or essential) boundary). Let E be an \mathcal{L}^N -measurable set in \mathbb{R}^N . The essential boundary $\partial_M E$ of E is the set

$$\partial_M E = \mathbb{R}^N \setminus (E^0 \cup E^1)$$

(see Figure 2.1 for an illustration).

We refer to [9] and [17] for the following theorems, respectively.

Theorem 2.33 (Federer). *If $E \subseteq D$ is a set of finite perimeter, then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial_M E, \quad \mathcal{H}^{N-1}(D \setminus (E^0 \cup \partial^* E \cup E^1)) = 0,$$

such that $\partial^ E$ is the reduced boundary of E . In particular, E has density either 0 or $\frac{1}{2}$ or 1 at \mathcal{H}^{N-1} -a.e. $x \in D$, and \mathcal{H}^{N-1} -a.e. $x \in \partial_M E \cap D$ belongs to $\partial^* E$.*

Theorem 2.34 (De Giorgi's rectifiability theorem). *Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in D . Then $\partial_M E$ is rectifiable; i.e., there exists a countable family (Γ_i) of graphs of C^1 functions of $(N-1)$ variables such that $\mathcal{H}^{N-1}(\partial_M E \setminus \cup_{i=1}^{\infty} \Gamma_i) = 0$. Moreover the perimeter of E in $D' \subset D$ is given by*

$$P(E, D') = \mathcal{H}^{N-1}(\partial_M E \cap D').$$

2.3 Γ -convergence

The notion of Γ -convergence, introduced by de Giorgi, is a suitable tool for the study of the convergence of variational problems. We recall the definition and some main properties of Γ -convergence from [14] and [18].

Definition 2.35. *Let (X, d) be a metrizable space, or more generally a first countable topological space, $(F_n)_{n \in \mathbb{N}}$ a sequence of extended real-valued functions $F_n : X \rightarrow \mathbb{R} \cup \{+\infty\}$, and $F : X \rightarrow \mathbb{R} \cup \{+\infty\}$. The sequence $(F_n)_{n \in \mathbb{N}}$ (sequentially) Γ -converges to F at $x \in X$ iff both the following assertions hold:*

(i) (**liminf inequality**) *for all sequences $(x_n)_{n \in \mathbb{N}}$ converging to x in X , one has*

$$F(x) \leq \liminf_{n \rightarrow +\infty} F_n(x_n), \quad (2.3.1)$$

(ii) (**limsup inequality**) *there exists a sequence $(y_n)_{n \in \mathbb{N}}$ converging to x in X such that*

$$F(x) \geq \limsup_{n \rightarrow +\infty} F_n(y_n), \quad (2.3.2)$$

Or,

(ii)' (**existence of a recovery sequence**) *there exists a sequence $(y_n)_{n \in \mathbb{N}}$ converging to x in X such that*

$$F(x) = \lim_{n \rightarrow +\infty} F_n(y_n). \quad (2.3.3)$$

Here there are some main properties of the Γ -convergence. We recall the following from [14].

Theorem 2.36. *Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of functions $F_n : X \rightarrow \mathbb{R} \cup \{+\infty\}$ which Γ -converges to some function $F : X \rightarrow \mathbb{R} \cup \{+\infty\}$. Then the following assertions hold:*

(i) *Let $x_n \in X$ be such that $F_n(x_n) \leq \inf \{F_n(x) : x \in X\} + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$ when $n \rightarrow +\infty$. Assume that $\{x_n, n \in \mathbb{N}\}$ is relatively compact; then every cluster point \bar{x} of $\{x_n : n \in \mathbb{N}\}$ is a minimizer of F and*

$$\lim_{n \rightarrow +\infty} \inf \{F_n(x) : x \in X\} = F(\bar{x}).$$

2.4. Legendre-Fenchel transform

(ii) If $G : X \rightarrow \mathbb{R}$ is continuous, then $(F_n + G)_{n \in \mathbb{N}}$ Γ -converges to $F + G$.

Definition 2.37 (Equicoercivity). We say that the sequence (f_ε) is equicoercive (on X), if for every $t \in \mathbb{R}$ there exists a closed countably compact subset K_t of X such that $\{f_\varepsilon \leq t\} \subset K_t$ for every $\varepsilon \in \mathbb{N}$.

For more precise details about Γ -convergence or equicoercivity, we refer the reader to [14] and [18].

2.4 Legendre-Fenchel transform

In this section, $(V, \|\cdot\|_V)$ is a general normed linear space with topological dual V' .

Definition 2.38. The (effective) **domain** of a function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is the set

$$\text{dom} f = \{x \in V : f(x) < +\infty\}.$$

The function f is said to be **proper** if $\text{dom} f \neq \emptyset$.

Definition 2.39 (Closed convex function). A convex function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is called **closed (or lower semicontinuous)** if its epigraph is a closed set.

Definition 2.40. Let V be a normed linear space and let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. The Legendre-Fenchel conjugate of f is the function

$$f^* : V' \rightarrow \mathbb{R} \cup \{+\infty\}$$

defined by

$$f^*(v^*) = \sup_{v \in V} \{\langle v^*, v \rangle_{(V', V)} - f(v)\}.$$

Theorem 2.41 (Fenchel-Moreau-Rockafellar). Assume that V is a reflexive Banach space. Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ be an extended real valued function. Then

$$f^{**} = \text{cl}(\text{conv} f),$$

where $\text{cl}(f)$ is the closure of the function f and $\text{conv} f$ is its convex envelope.

For a proof we refer to [16].

Corollary 2.42. Let V be a normed space and let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a closed convex proper function. Then

$$f = f^{**}.$$

i.e., f is equal to its biconjugate. Equivalently,

$$\forall v \in V \quad f(v) = \sup_{v^* \in V^*} \{\langle v^*, v \rangle_{(V', V)} - f(v^*)\}.$$

2.5 Elliptic boundary value problems

2.5.1 The Lax-Milgram theorem

The Lax-Milgram theorem is a very simple and efficient tool for solving linear elliptic partial differential equations. We refer the following theorem from [14].

Theorem 2.43 (Lax-Milgram). *Let V be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ the associated norm.*

Let $a : V \times V \rightarrow \mathbb{R}$ be a bilinear form which satisfies (i) and (ii):

(i) a is continuous, that is, there exists a constant $C \in \mathbb{R}^+$ such that

$$\forall u, v \in H \quad |a(u, v)| \leq \|u\| \cdot \|v\|;$$

(ii) a is coercive, that is, there exists a constant $\alpha > 0$ such that

$$\forall v \in H \quad a(v, v) \geq \alpha \|v\|^2.$$

Then for any $L \in V'$ (L is a linear continuous form on V) there exists a unique $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V.$$

2.5.2 Homogeneous Neumann problem

Let $D \subset \mathbb{R}^N$ be a bounded domain of class C^1 . We look for a function $u : \bar{D} \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u + u = f & \text{in } D, \\ \partial_n u = 0 & \text{on } \partial D, \end{cases} \quad (2.5.1)$$

where f is given on D ; ∂_n denotes the outward normal derivative of u , i.e., $\partial_n u = \nabla u \cdot \mathbf{n}$, where \mathbf{n} is the unit normal vector to ∂D , pointing outward. The boundary condition $\partial_n u = 0$ on ∂D is called the (homogeneous) *Neumann condition*.

Definition 2.44. *A classical solution of (2.5.1) is a function $u \in C^2(\bar{D})$ satisfying (2.5.1). A weak solution of (2.5.1) is a function $u \in H^1(D)$ satisfying*

$$\int_D \nabla u \cdot \nabla v + \int_D uv = \int_D fv \quad \forall v \in H^1(D). \quad (2.5.2)$$

We can apply the Lax-Milgram theorem 2.43, which proves the existence and uniqueness of the solution of the variational formulation (2.5.2).

2.5.3 The maximum principle

We refer the following proposition from [20].

Proposition 2.45 (maximum principle for the Neumann problem). *Let $f \in L^2(D)$ and $u \in H^1(D)$ be such that*

$$\int_D \nabla u \cdot \nabla \varphi + \int_D u\varphi = \int_D f\varphi \quad \forall \varphi \in H^1(D).$$

Then we have, for a.e. $x \in D$,

$$\inf_D f \leq u(x) \leq \sup_D f.$$

2.6 The operator L_ε

Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. We look for a function $v : \bar{D} \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} -\varepsilon^2 \Delta v + v &= u & \text{in } D, \\ \partial_n v &= 0 & \text{on } \partial D, \end{cases} \quad (2.6.1)$$

where u is given on D .

We define the operator $P_\varepsilon \in \mathcal{L}(H^1(D), (H^1(D))')$ by

$$\langle P_\varepsilon u, v \rangle_{(H^1(D))', H^1(D)} = \int_D \varepsilon^2 \nabla u \cdot \nabla v + uv \quad \forall u, v \in H^1(D), \quad (2.6.2)$$

which is called the *variational formulation* of problem (2.6.1). Applying Lax-Milgram's theorem (Theorem 2.43), we deduce that P_ε is invertible. We then define $L_\varepsilon = P_\varepsilon^{-1} \in \mathcal{L}((H^1(D))', H^1(D))$.

Now, we present below some properties of the operator L_ε .

- (1) The operator L_ε is an isomorphism from $(H^1(D))'$ into $H^1(D)$ defined by

$$\begin{aligned} L_\varepsilon : (H^1(D))' &\rightarrow H^1(D) \\ u &\mapsto \hat{u} = L_\varepsilon u, \end{aligned}$$

with

$$\int_D (\varepsilon^2 \nabla \hat{u} \cdot \nabla \varphi + \hat{u} \varphi) dx = \langle u, \varphi \rangle_{(H^1(D))', H^1(D)} \quad \forall \varphi \in H^1(D). \quad (2.6.3)$$

- (2) If $u, v \in (H^1(D))'$, $\hat{u} = L_\varepsilon u$ and $\hat{v} = L_\varepsilon v$, then we have

$$\langle u, L_\varepsilon v \rangle_{(H^1(D))', H^1(D)} = \int_D (\varepsilon^2 \nabla \hat{u} \cdot \nabla \hat{v} + \hat{u} \hat{v}) dx = \langle v, L_\varepsilon u \rangle_{(H^1(D))', H^1(D)}. \quad (2.6.4)$$

- (3) If $u, v \in H^1(D)$, $\hat{u} = L_\varepsilon u$ and $\hat{v} = L_\varepsilon v$, then we have

$$\int_D (\varepsilon^2 \nabla \hat{u} \cdot \nabla v + \hat{u} v) dx = \int_D uv dx = \int_D (\varepsilon^2 \nabla u \cdot \nabla \hat{v} + u \hat{v}) dx. \quad (2.6.5)$$

- (4) By restriction of the operator L_ε , we can define $L_\varepsilon \in \mathcal{L}(L^2)$, which is self-adjoint (this follows from (2.6.4)) and compact (this follows from Rellich theorem, Theorem 2.27).

- (5) If $u, v \in L^2(D)$, $\hat{u} = L_\varepsilon u$ and choosing $\varphi = \hat{u}$ in (2.6.3) we obtain

$$\|\hat{u}\|_{L^2}^2 \leq \int_D (\varepsilon^2 |\nabla \hat{u}|^2 + \hat{u}^2) dx = \int_D u \hat{u} dx.$$

We deduce by using the Cauchy-Schwarz inequality

$$\|\hat{u}\|_{L^2}^2 \leq \int_D u \hat{u} dx \leq \|u\|_{L^2} \|\hat{u}\|_{L^2},$$

which implies

$$\|L_\varepsilon\|_{\mathcal{L}(L^2)} \leq 1. \quad (2.6.6)$$

Chapter 3

Approximation of interface energies

In this chapter, we present the regularization approach developed for the problem in all details. Our strategy will use in approximation of interface energy.

3.1 The Modica Mortola functional

A classical Γ -convergence result is the Modica Mortola theorem. The result stated below is due to Modica and Mortola [41], and it provides an approximation of the perimeter using Γ -convergence.

Theorem 3.1 (Modica-Mortola). *Let D be a bounded open set and let $W : \mathbb{R} \rightarrow [0, \infty)$ be a continuous function such that $W(z) = 0$ if and only if $z \in \{0, 1\}$. Denote $c = 2 \int_0^1 \sqrt{W(s)} ds$. We define $F_{MM}^\varepsilon, F_{MM} : L^1(D) \rightarrow [0, +\infty]$ by*

$$F_{MM}^\varepsilon(u) = \begin{cases} \varepsilon \int_D |\nabla u|^2 + \frac{1}{\varepsilon} \int_D W(u) & \text{if } u \in H^1(D), \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$F_{MM}(u) = \begin{cases} c \text{Per}_D(\{u = 1\}) & \text{if } u \in BV(D, \{0, 1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

Then F_{MM}^ε Γ -converge to F_{MM} in the $L^1(D)$ topology.

3.2 A gradient-free perimeter approximation

Let us begin with some definitions and notation. Let D be an open rectangle of \mathbb{R}^2 . We define the set

$$\mathcal{E} = L^\infty(D, \{0, 1\})$$

of characteristic functions in D and the functional $F : \mathcal{E} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$F(u) = \begin{cases} \frac{1}{2} |Du|(D) & \text{if } u \in BV(D, \{0, 1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

3.2. A gradient-free perimeter approximation

We recall that $|Du|(D) = \mathcal{H}^1(\partial_M \Omega \cap D)$ when u is a characteristic function of $\Omega \subset D$. We define also

$$\tilde{\mathcal{E}} = L^\infty(D, [0, 1]),$$

the convex hull of \mathcal{E} , and the functional $\tilde{F} : \tilde{\mathcal{E}} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\tilde{F}(u) = \begin{cases} F(u) & \text{if } u \in \mathcal{E} \\ +\infty & \text{otherwise.} \end{cases}$$

It is shown in [13] that a suitable approximation of \tilde{F} is provided by the functionals \tilde{F}_ε defined as

$$\tilde{F}_\varepsilon(u) = \inf_{v \in H^1(D)} \left\{ \varepsilon \|\nabla v\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v\|_{L^2(D)}^2 + \langle u, 1 - 2v \rangle \right) \right\}. \quad (3.2.1)$$

In everything that follows we denote

$$\langle u, v \rangle = \int_D u(x)v(x)dx$$

for every pair of functions u, v having suitable regularity. Now, we recall below some theoretical tools needed to prove our results (Proposition 3.2 proved in [9], Theorem 3.3 and Theorem 3.4 proved in [12]).

The following proposition establishes the solution to the minimization problem (3.2.1).

Proposition 3.2. *Let $u \in L^2(D)$ be given and $v_\varepsilon \in H^1(D)$ be the (weak) solution of*

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } D, \\ \partial_n v_\varepsilon = 0 & \text{on } \partial D. \end{cases} \quad (3.2.2)$$

Then we have

$$\tilde{F}_\varepsilon(u) = \frac{1}{\varepsilon} \langle 1 - v_\varepsilon, u \rangle.$$

The following theorem establishes the Γ -convergence of the approximating functionals.

Theorem 3.3. *For any $u \in L^\infty(D, [0, 1])$, define*

$$\tilde{F}_\varepsilon(u) = \frac{1}{\varepsilon} \langle L_\varepsilon u, 1 - u \rangle = \frac{1}{\varepsilon} \langle 1 - L_\varepsilon u, u \rangle. \quad (3.2.3)$$

When $\varepsilon \rightarrow 0$, the functionals \tilde{F}_ε Γ -converge in $\tilde{\mathcal{E}}$ endowed with the strong topology of $L^1(D)$ to the functional

$$\tilde{F}(u) = \begin{cases} \frac{1}{2} |Du|(D) & \text{if } u \in BV(D, \{0, 1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

The following theorem provides the pointwise convergence of $\tilde{F}_\varepsilon(u)$.

Theorem 3.4. *For all $u \in \tilde{\mathcal{E}}$ it holds*

$$\lim_{\varepsilon \rightarrow 0} \tilde{F}_\varepsilon(u) = \tilde{F}(u). \quad (3.2.4)$$

3.3 Approximation of interface energy: pointwise convergence

In the previous section, we obtained an approximation of the perimeter $\mathcal{H}^1(\partial_M \Omega \cap D)$. Given two subsets Ω_i, Ω_j of D , we look for an approximation of the interface energy $\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D)$.

3.3.1 Mathematical results

As a particular case of Proposition 1 in [8], let us consider the following situation: take ($d=2$).

Proposition 3.5. *Let Ω_i, Ω_j be sets of finite perimeter. Then*

$$\mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j)) = \mathcal{H}^1(\partial_M \Omega_i) + \mathcal{H}^1(\partial_M \Omega_j) - 2\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j)$$

whenever $|\Omega_i \cap \Omega_j| = 0$.

The following lemma is a variant of Proposition 3.5. The proof is adapted from [8].

Lemma 3.6. *Let Ω_i, Ω_j be subsets of finite perimeter of D . Then*

$$\mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j) \cap D) = \mathcal{H}^1(\partial_M \Omega_i \cap D) + \mathcal{H}^1(\partial_M \Omega_j \cap D) - 2\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D)$$

whenever $|\Omega_i \cap \Omega_j| = 0$.

Proof. From the proof of Proposition 3.5 in [8], we have

$$\begin{aligned} \partial_M(\Omega_i \cup \Omega_j) \cup \partial_M(\Omega_i \cap \Omega_j) &\subset \partial_M \Omega_i \cup \partial_M \Omega_j, \\ \partial_M(\Omega_i \cup \Omega_j) \cap \partial_M(\Omega_i \cap \Omega_j) &\subset \partial_M \Omega_i \cap \partial_M \Omega_j. \end{aligned}$$

It follows that

$$\begin{aligned} \partial_M(\Omega_i \cup \Omega_j) \cup \partial_M(\Omega_i \cap \Omega_j) \cap D &\subset (\partial_M \Omega_i \cup \partial_M \Omega_j) \cap D, \\ \partial_M(\Omega_i \cup \Omega_j) \cap \partial_M(\Omega_i \cap \Omega_j) \cap D &\subset (\partial_M \Omega_i \cap \partial_M \Omega_j) \cap D. \end{aligned}$$

Here, we refer to Section 2.2.2 for the point of density 1/2. If $|\Omega_i \cap \Omega_j| = 0$, we denote by L the \mathcal{H}^1 -negligible set

$$\left((\partial_M \Omega_i \cap D) \setminus (\Omega_i^{\frac{1}{2}} \cap \Omega) \right) \cap \left((\partial_M \Omega_j \cap D) \setminus (\Omega_j^{\frac{1}{2}} \cap D) \right)$$

and notice that

$$\begin{aligned} \partial_M(\Omega_i \cup \Omega_j) \cap D \setminus L &\subset (\partial_M \Omega_i \cap D) \Delta (\partial_M \Omega_j \cap D), \\ (\partial_M \Omega_i \cap D) \Delta (\partial_M \Omega_j \cap D) &\subset \partial_M(\Omega_i \cup \Omega_j) \cap D. \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j) \cap D) &= \mathcal{H}^1((\partial_M \Omega_i \cap D) \Delta (\partial_M \Omega_j \cap D)) \\ &= \mathcal{H}^1(\partial_M \Omega_i \cap D) + \mathcal{H}^1(\partial_M \Omega_j \cap D) - 2\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D). \end{aligned}$$

□

We now arrive at the desired pointwise convergence result.

3.3. Approximation of interface energy: pointwise convergence

Theorem 3.7. *Let Ω_i, Ω_j be two subsets of finite perimeter of D . If $u_i = \chi_{\Omega_i}$ and $u_j = \chi_{\Omega_j}$, then*

$$\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) = \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon} \langle L_\varepsilon u_i, u_j \rangle,$$

whenever $|\Omega_i \cap \Omega_j| = 0$.

Proof. By Lemma 3.6, we have

$$\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) = \frac{1}{2} [\mathcal{H}^1(\partial_M \Omega_i \cap D) + \mathcal{H}^1(\partial_M \Omega_j \cap D) - \mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j) \cap D)].$$

According to Theorem 3.4 we get

$$\begin{aligned} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) &= \lim_{\varepsilon \rightarrow 0} [\tilde{F}_\varepsilon(u_i) + \tilde{F}_\varepsilon(u_j) - \tilde{F}_\varepsilon(u_i + u_j)] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon} \langle 1 - L_\varepsilon u_i, u_i \rangle + \frac{1}{\varepsilon} \langle 1 - L_\varepsilon u_j, u_j \rangle - \frac{1}{\varepsilon} \langle 1 - L_\varepsilon(u_i + u_j), u_i + u_j \rangle \right] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon} \langle L_\varepsilon u_i, u_j \rangle. \end{aligned}$$

□

From now on, for simplicity, we denote

$$G_\varepsilon(u_i, u_j) = \frac{1}{\varepsilon} \langle L_\varepsilon u_i, u_j \rangle. \quad (3.3.1)$$

3.3.2 Algorithm

We present an algorithm to check the pointwise convergence (see Algorithm 1).

<p>Data: Given a partition $\Omega_1, \dots, \Omega_N$.</p> <p>1 set $u_i = \chi_{\Omega_i} \forall i = 1, \dots, N$;</p> <p>2 set $\varepsilon_{min}, \varepsilon_{max}$;</p> <p>3 set $\varepsilon = \varepsilon_{max}$;</p> <p>4 repeat</p> <p>5 solve</p> $\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } D \\ \partial_n v_\varepsilon = 0 & \text{on } \partial D; \end{cases}$ <p style="padding-left: 20px;">compute $G_\varepsilon(u_i, u_j) = \frac{1}{\varepsilon} \langle L_\varepsilon u_i, u_j \rangle$;</p> <p>6 update $\varepsilon \leftarrow \frac{\varepsilon}{2}$;</p> <p>7 until $\varepsilon \geq \varepsilon_{min}$;</p>
--

Algorithm 1: Check the pointwise convergence of the function $G_\varepsilon(u_i, u_j)$.

Remark 3.1. The parameter ε has the dimension of a length. In fact, in view of (3.2.2), it is a characteristic width of the diffuse interface represented by the *slow* variable v . Thus we start with a characteristic size of D , namely $\varepsilon_0 = \varepsilon_{max} = \max(m, n)$. Then we divide ε by two at each

3.4. Lower semicontinuity

iteration of an outer loop, that is, we choose $\varepsilon_i = \varepsilon_{max}/2^i$. In order to approximate (3.2.2) properly, ε must not be taken significantly smaller than the grid size. In fact, when we use the FEM, the discrete Maximum principle is not guaranteed if ε is smaller than the mesh size. Thus we stop the algorithm as soon as $\varepsilon_i \leq \varepsilon_{min} = 1$. In fact, numerical tests show that almost no more evolution occurs when ε goes below this value (for more details see the appendix).

3.3.3 Numerical examples

We present four examples to illustrate the pointwise convergence of the functional G_ε in Figures 3.1 and 3.2. The values of the function G_ε are computed in two cases namely the finite element method (FEM) and the finite difference method (FDM), and compared with the exact value. We notice that when we use the FDM, the value of G_ε is less than the exact value, for the FEM it is above.

3.4 Lower semicontinuity

Lower semicontinuity is very important in variational problems, because together with coercivity it ensures the existence of minimizers.

Theorem 3.8. *Let X be a metric space and $f : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$. f be lower semicontinuous (or just lsc for short) if and only if*

$$u_\varepsilon \rightarrow u \Rightarrow f(u) \leq \liminf_{\varepsilon \rightarrow 0} f(u^\varepsilon).$$

The following important result is found in [7].

Theorem 3.9. *The condition (1.1.3) is necessary and sufficient for the function*

$$\mathcal{I} : (\Omega_1, \dots, \Omega_N) \mapsto \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap \Omega)$$

to be lower semicontinuous (w.r.t. convergence in measure).

3.5 Equicoercivity

We refer to the following theorem from [12].

Theorem 3.10 (Equicoercivity). *Let u^ε be a sequence of functions of $\tilde{\mathcal{E}}$ such that $\sup_\varepsilon \tilde{F}_\varepsilon(u^\varepsilon) < +\infty$. There exists $u \in \mathcal{E}$ such that $u^\varepsilon \rightarrow u$ strongly in $L^1(\Omega)$ for a subsequence.*

For simplicity, we denote

$$\mathcal{I}_\varepsilon(u_1^\varepsilon, \dots, u_N^\varepsilon) = \sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon).$$

We now show the equicoercivity of the functionals \mathcal{I}_ε .

Theorem 3.11. *Let $\alpha_{ij} > 0$ and let $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ be a sequence of functions of $\tilde{\mathcal{E}}$ such that $\sup_\varepsilon \mathcal{I}_\varepsilon(u_1^\varepsilon, \dots, u_N^\varepsilon) < +\infty$ for any i, j . For all i , there exists $u_i \in \mathcal{E}$ such that $u_i^\varepsilon \rightarrow u_i$ strongly in $L^1(D)$ for a subsequence.*

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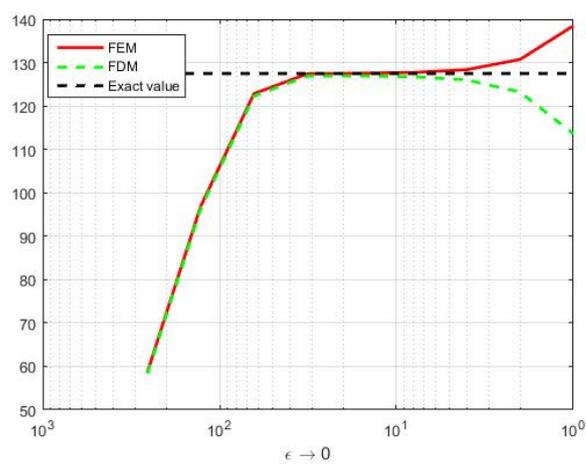
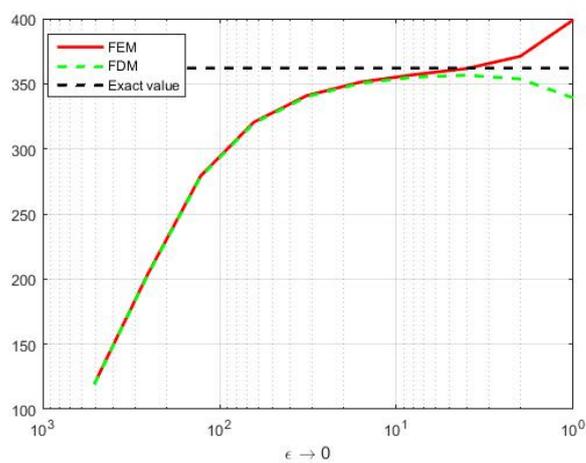
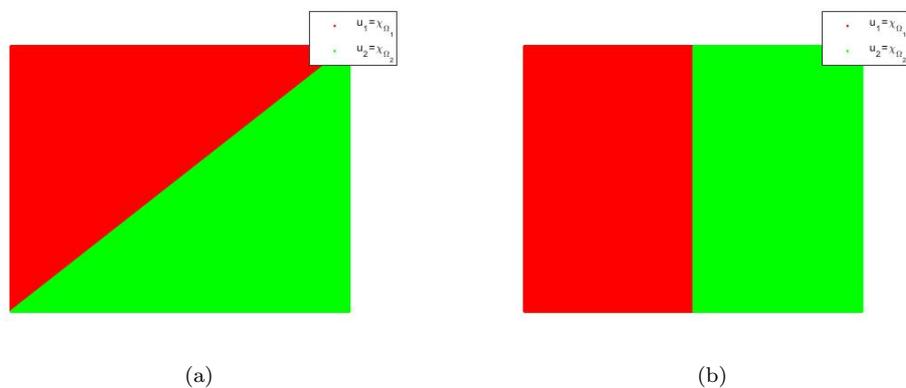
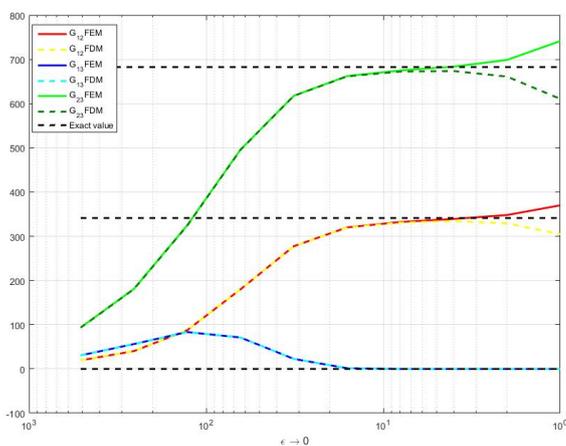
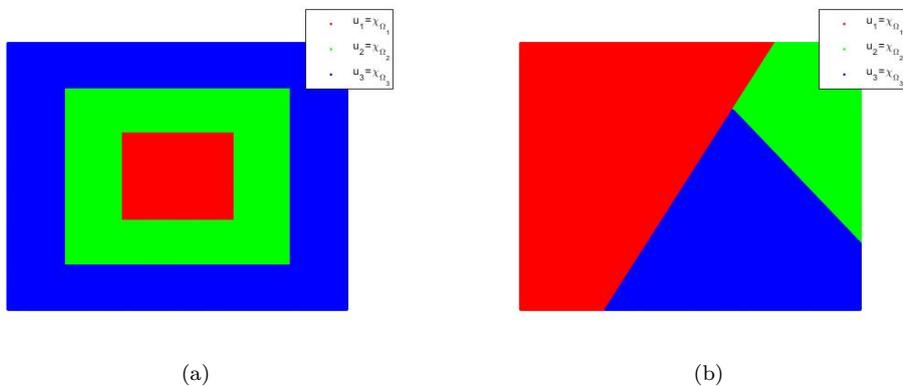
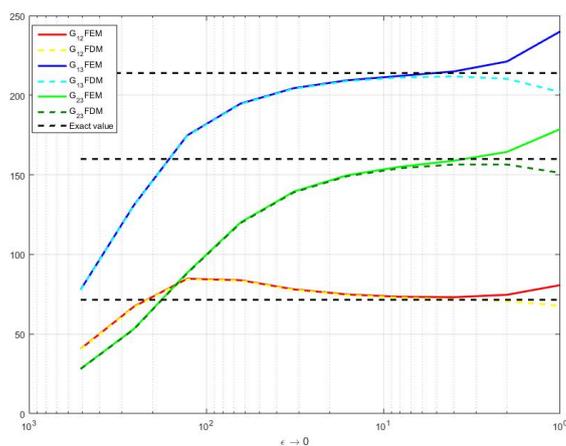


Figure 3.1: (a) Example 1: given partition, (b) Example 2: given partition and (c), (d), convergence history of $G_\epsilon(u_1, u_2)$ for example 1, 2, respectively with the FEM, the FDM and the exact values.

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(c)



(d)

Figure 3.2: (a) Example 3: given partition, (b) Example 4: given partition and (c), (d), convergence history of $G_\epsilon(u_i, u_j)$ for examples 3, 4, respectively with the FEM, the FDM and the exact values.

3.5. Equicoercivity

Proof. By taking $\alpha_{ij} \geq \bar{\alpha} > 0$ and from (3.3.1), we obtain

$$\begin{aligned}
\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) &= \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i^\varepsilon, u_j^\varepsilon \rangle \\
&\geq \frac{\bar{\alpha}}{\varepsilon} \sum_{1 \leq i < j \leq N} \langle L_\varepsilon u_i^\varepsilon, u_j^\varepsilon \rangle \\
&= \frac{\bar{\alpha}}{2\varepsilon} \sum_{\substack{i,j=1 \\ i \neq j}}^N \langle L_\varepsilon u_i^\varepsilon, u_j^\varepsilon \rangle \\
&= \frac{\bar{\alpha}}{2\varepsilon} \sum_{i=1}^N \langle L_\varepsilon u_i^\varepsilon, \sum_{\substack{j=1 \\ j \neq i}}^N u_j^\varepsilon \rangle.
\end{aligned}$$

By using the fact that $\sum_{j \neq i}^N u_j^\varepsilon = 1 - u_i^\varepsilon$ and Theorem 3.2.3, we deduce that

$$\begin{aligned}
\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) &\geq \frac{\bar{\alpha}}{2} \sum_{i=1}^N \frac{1}{\varepsilon} \langle L_\varepsilon u_i^\varepsilon, 1 - u_i^\varepsilon \rangle \\
&= \frac{\bar{\alpha}}{2} \sum_{i=1}^N \tilde{F}_\varepsilon(u_i^\varepsilon).
\end{aligned}$$

It is easy to see that if

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) \leq C, \quad C > 0,$$

then

$$\tilde{F}_\varepsilon(u_i^\varepsilon) \leq \frac{2C}{\bar{\alpha}} \quad \forall i.$$

From Theorem 3.10, we get

$$u_i^\varepsilon \xrightarrow{L^1} u_i,$$

for some $u_i \in \mathcal{E}$. □

We will study the Γ -convergence in chapter 4.

Chapter 4

Formulation of the interface energy as a linear combination of perimeters

In this chapter, we will rewrite the surface tensions α_{ij} in the form of linear combination that allows us to rewrite the interface energy as a combination of perimeters and we will explain how compute the coefficients. If all the coefficients are positive we will prove Γ -convergence in Theorem 4.8. We will provide several optimization algorithms depending on the sign of the coefficients. In the end, we will present some numerical illustrations.

4.1 Reformulation in terms of perimeters

For technical simplicity we restrict ourselves to the cases where $N \leq 5$.

4.1.1 Algebraic properties of interface energies

Lemma 4.1. *Let $\Omega_1, \dots, \Omega_N$ be subsets of finite perimeter of D such that $D = \cup_{i=1}^N \Omega_i$. Let $\mathbb{L}_{ij} = \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D)$, $\mathbb{P}_i = \mathcal{H}^1(\partial_M \Omega_i \cap D)$ and $\mathbb{P}_{ij} = \mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j) \cap D)$. Then*

$$\mathbb{P}_i = \sum_{j \neq i} \mathbb{L}_{ij} \text{ and } \mathbb{P}_{ij} = \sum_{k \neq i, j} \mathbb{L}_{ik} + \mathbb{L}_{kj},$$

whenever $|\Omega_i \cap \Omega_j| = 0$ for $i \neq j$.

Proof. By the definition of the essential boundary, we have

$$\partial_M \Omega_i = \partial_M (\mathbb{R}^2 \setminus \Omega_i) = \partial_M \left(\bigcup_{j \neq i} \Omega_j \cup (\mathbb{R}^2 \setminus D) \right).$$

It follows from the proof of Proposition 3.5 in [8] that

$$\partial_M \Omega_i \cap D \subset \partial_M \left(\bigcup_{j \neq i} \Omega_j \cup (\mathbb{R}^2 \setminus D) \right) \cap D \subset \left(\bigcup_{j \neq i} \partial_M \Omega_j \right) \cap D,$$

which implies that

$$\partial_M \Omega_i \cap D \subset \bigcup_{j \neq i} (\partial_M \Omega_j \cap \partial_M \Omega_i \cap D),$$

4.1. Reformulation in terms of perimeters

and thus

$$\partial_M \Omega_i \cap D = \bigcup_{j \neq i} (\partial_M \Omega_j \cap \partial_M \Omega_i \cap D). \quad (4.1.1)$$

Now, we need to prove that if $i \neq j, i \neq k, j \neq k$, then

$$\mathcal{H}^1((\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) \cap (\partial_M \Omega_i \cap \partial_M \Omega_k \cap D)) = \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap \partial_M \Omega_k \cap D) = 0. \quad (4.1.2)$$

By the definition of points of density $\frac{1}{2}$ for $\Omega_i, i = 1, \dots, N$, we underline the fact that if $i \neq j, i \neq k, j \neq k$, then

$$\Omega_i^{\frac{1}{2}} \cap \Omega_j^{\frac{1}{2}} \cap \Omega_k^{\frac{1}{2}} \cap D = \emptyset.$$

As a consequence, it follows that

$$\mathcal{H}^1(\Omega_i^{\frac{1}{2}} \cap \Omega_j^{\frac{1}{2}} \cap \Omega_k^{\frac{1}{2}} \cap D) = 0.$$

According to Theorem 2.33, we have that

$$\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap \partial_M \Omega_k \cap D) = 0. \quad (4.1.3)$$

We deduce from (4.1.1), (4.1.2) and (4.1.3) that

$$\mathbb{P}_i = \sum_{j \neq i} \mathbb{L}_{ij}.$$

From this fact and according to Lemma 3.6, we can obtain that

$$\begin{aligned} \mathbb{P}_{ij} &= \mathbb{P}_i + \mathbb{P}_j - 2\mathbb{L}_{ij} \\ &= \sum_{k \neq i} \mathbb{L}_{ik} + \sum_{k \neq j} \mathbb{L}_{kj} - 2\mathbb{L}_{ij} \\ &= \sum_{k \neq i, j} \mathbb{L}_{ik} + \mathbb{L}_{kj}. \end{aligned}$$

□

4.1.2 Algebraic properties of approximate interface energies

We now prove the approximate counterpart of Lemma 4.1.

Lemma 4.2. *Let $\Omega_i, \dots, \Omega_N$ be subsets of finite perimeter of D such that $D = \bigcup_{i=1}^N \Omega_i$. Let $u_i = \chi_{\Omega_i}$ for all i , $\mathbb{L}_{ij}^\varepsilon = \langle L_\varepsilon u_i, u_j \rangle$, $\mathbb{P}_i^\varepsilon = \frac{1}{\varepsilon} \langle 1 - L_\varepsilon u_i, u_i \rangle$ and $\mathbb{P}_{ij}^\varepsilon = \frac{1}{\varepsilon} \langle 1 - L_\varepsilon (u_i + u_j), u_i + u_j \rangle$. Then*

$$\mathbb{P}_i^\varepsilon = \frac{1}{\varepsilon} \sum_{j \neq i} \mathbb{L}_{ij}^\varepsilon \text{ and } \mathbb{P}_{ij}^\varepsilon = \frac{1}{\varepsilon} \sum_{k \neq i, j} \mathbb{L}_{ik}^\varepsilon + \mathbb{L}_{kj}^\varepsilon,$$

whenever $|\Omega_i \cap \Omega_j| = 0$ for $i \neq j$.

4.1. Reformulation in terms of perimeters

Proof. First, from Theorem 3.3 we have

$$\mathbb{P}_i^\varepsilon = \frac{1}{\varepsilon} \langle 1 - L_\varepsilon u_i, u_i \rangle = \frac{1}{\varepsilon} \langle L_\varepsilon u_i, 1 - u_i \rangle.$$

Using the fact that $1 - u_i = \sum_{j \neq i} u_j$, we obtain

$$\mathbb{P}_i^\varepsilon = \frac{1}{\varepsilon} \langle L_\varepsilon u_i, \sum_{j \neq i} u_j \rangle = \frac{1}{\varepsilon} \sum_{j \neq i} \langle L_\varepsilon u_i, u_j \rangle = \frac{1}{\varepsilon} \sum_{j \neq i} \mathbb{L}_{ij}^\varepsilon.$$

Second, from Theorem 3.3 we have

$$\mathbb{P}_{ij}^\varepsilon = \frac{1}{\varepsilon} \langle 1 - L_\varepsilon(u_i + u_j), u_i + u_j \rangle = \frac{1}{\varepsilon} \langle L_\varepsilon(u_i + u_j), 1 - (u_i + u_j) \rangle.$$

Using again the fact that $1 - (u_i + u_j) = \sum_{k \neq i, j} u_k$, we obtain

$$\begin{aligned} \mathbb{P}_{ij}^\varepsilon &= \frac{1}{\varepsilon} \langle L_\varepsilon(u_i + u_j), \sum_{k \neq i, j} u_k \rangle \\ &= \frac{1}{\varepsilon} \sum_{k \neq i, j} \langle L_\varepsilon(u_i + u_j), u_k \rangle \\ &= \frac{1}{\varepsilon} \sum_{k \neq i, j} \langle L_\varepsilon u_i, u_k \rangle + \langle L_\varepsilon u_j, u_k \rangle \\ &= \frac{1}{\varepsilon} \sum_{k \neq i, j} \mathbb{L}_{ik}^\varepsilon + \mathbb{L}_{kj}^\varepsilon. \end{aligned}$$

□

Remark 4.1. The algebraic properties given in Lemmas 4.1 and 4.2 are identical. This will allow to obtain similar reformulations for the interface energy and its approximation.

4.1.3 Matrix representation of algebraic properties

For $N \leq 5$, we define the column vector \mathbb{L} made of the values (\mathbb{L}_{ij}) in a chosen order. Similarly define the column vector α of the surface tensions (α_{ij}) , \mathbb{P} the vector of the values \mathbb{P}_i and \mathbb{P}_{ij} . We will denote \mathbb{P}_i and \mathbb{P}_{ij} by \mathbb{P}_S , $S \subset \{1, \dots, N\}$, β_i and β_{ij} by β_S . According to lemma 4.1, we can define the matrix $\mathbb{M} = (m_{ij}) \in \mathbb{R}^{(N+C_2^N) \times C_2^N}$ by

$$\mathbb{P} = \mathbb{M}\mathbb{L}.$$

Note that $m_{ij} \in \{0, 1\}$.

To find the relationship between the values of the α_{ij} and β_S , we start with

$$\beta \cdot \mathbb{P} = \beta \cdot \mathbb{M}\mathbb{L} = \mathbb{M}^\top \beta \cdot \mathbb{L}.$$

Hence one has

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} = \sum_{S \subset \{1, \dots, N\}} \beta_S \mathbb{P}_S \quad (4.1.4)$$

whenever the columns of coefficients satisfy the linear system

$$\mathbb{M}^\top \beta = \alpha. \quad (4.1.5)$$

This system is square invertible if $N = 3$ and underdetermined if $N > 3$. The property of complementation in [6] allows us to select the values of S (see Table 4.1).

4.1. Reformulation in terms of perimeters

N=3	{1}, {2} and {3}
N=4	{1}, {2}, {3}, {4}, {12}, {13} and {14}.
N=5	{1}, {2}, {3}, {4}, {5}, {12}, {13}, {14}, {15}, {23}, {24}, {25}, {34}, {35} and {45}.

Table 4.1: The values of S .

Example 4.1. For $N=5$, let us consider $\mathbb{L} = (\mathbb{L}_{12}, \mathbb{L}_{13}, \mathbb{L}_{14}, \mathbb{L}_{15}, \mathbb{L}_{23}, \mathbb{L}_{24}, \mathbb{L}_{25}, \mathbb{L}_{34}, \mathbb{L}_{35}, \mathbb{L}_{45})^\top$ and $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \mathbb{P}_5, \mathbb{P}_{12}, \mathbb{P}_{13}, \mathbb{P}_{14}, \mathbb{P}_{15}, \mathbb{P}_{23}, \mathbb{P}_{24}, \mathbb{P}_{25}, \mathbb{P}_{34}, \mathbb{P}_{35}, \mathbb{P}_{45})^\top$. By lemma 4.1 we have

$$\mathbb{P} = \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \\ \mathbb{P}_4 \\ \mathbb{P}_5 \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{14} \\ \mathbb{P}_{15} \\ \mathbb{P}_{23} \\ \mathbb{P}_{24} \\ \mathbb{P}_{25} \\ \mathbb{P}_{34} \\ \mathbb{P}_{35} \\ \mathbb{P}_{45} \end{pmatrix} = \begin{pmatrix} \mathbb{L}_{12} + \mathbb{L}_{13} + \mathbb{L}_{14} + \mathbb{L}_{15} \\ \mathbb{L}_{12} + \mathbb{L}_{23} + \mathbb{L}_{24} + \mathbb{L}_{25} \\ \mathbb{L}_{13} + \mathbb{L}_{23} + \mathbb{L}_{34} + \mathbb{L}_{25} \\ \mathbb{L}_{14} + \mathbb{L}_{24} + \mathbb{L}_{34} + \mathbb{L}_{45} \\ \mathbb{L}_{15} + \mathbb{L}_{25} + \mathbb{L}_{35} + \mathbb{L}_{45} \\ \mathbb{L}_{13} + \mathbb{L}_{23} + \mathbb{L}_{14} + \mathbb{L}_{24} + \mathbb{L}_{15} + \mathbb{L}_{25} \\ \mathbb{L}_{12} + \mathbb{L}_{23} + \mathbb{L}_{14} + \mathbb{L}_{34} + \mathbb{L}_{15} + \mathbb{L}_{35} \\ \mathbb{L}_{12} + \mathbb{L}_{24} + \mathbb{L}_{13} + \mathbb{L}_{34} + \mathbb{L}_{15} + \mathbb{L}_{45} \\ \mathbb{L}_{12} + \mathbb{L}_{24} + \mathbb{L}_{13} + \mathbb{L}_{34} + \mathbb{L}_{15} + \mathbb{L}_{45} \\ \mathbb{L}_{12} + \mathbb{L}_{13} + \mathbb{L}_{14} + \mathbb{L}_{34} + \mathbb{L}_{25} + \mathbb{L}_{35} \\ \mathbb{L}_{12} + \mathbb{L}_{14} + \mathbb{L}_{23} + \mathbb{L}_{34} + \mathbb{L}_{25} + \mathbb{L}_{35} \\ \mathbb{L}_{12} + \mathbb{L}_{15} + \mathbb{L}_{23} + \mathbb{L}_{35} + \mathbb{L}_{24} + \mathbb{L}_{45} \\ \mathbb{L}_{13} + \mathbb{L}_{14} + \mathbb{L}_{23} + \mathbb{L}_{24} + \mathbb{L}_{35} + \mathbb{L}_{45} \\ \mathbb{L}_{13} + \mathbb{L}_{15} + \mathbb{L}_{23} + \mathbb{L}_{25} + \mathbb{L}_{34} + \mathbb{L}_{45} \\ \mathbb{L}_{14} + \mathbb{L}_{15} + \mathbb{L}_{24} + \mathbb{L}_{25} + \mathbb{L}_{34} + \mathbb{L}_{35} \end{pmatrix}.$$

This can be written as

$$\mathbb{P} = \begin{pmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \\ \mathbb{P}_4 \\ \mathbb{P}_5 \\ \mathbb{P}_{12} \\ \mathbb{P}_{13} \\ \mathbb{P}_{14} \\ \mathbb{P}_{15} \\ \mathbb{P}_{23} \\ \mathbb{P}_{24} \\ \mathbb{P}_{25} \\ \mathbb{P}_{34} \\ \mathbb{P}_{35} \\ \mathbb{P}_{45} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{L}_{12} \\ \mathbb{L}_{13} \\ \mathbb{L}_{14} \\ \mathbb{L}_{15} \\ \mathbb{L}_{23} \\ \mathbb{L}_{24} \\ \mathbb{L}_{25} \\ \mathbb{L}_{34} \\ \mathbb{L}_{35} \\ \mathbb{L}_{45} \end{pmatrix} = \mathbb{M}\mathbb{L}.$$

We noticed that \mathbb{M}^\top is surjective by Lemma 3.6:

$$\mathbb{P}_{ij} = \mathbb{P}_i + \mathbb{P}_j - 2\mathbb{L}_{ij}, \text{ i.e.,}$$

$$\mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j) \cap D) = \mathcal{H}^1(\partial_M \Omega_i \cap D) + \mathcal{H}^1(\partial_M \Omega_j \cap D) - 2\mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D).$$

The issue is to find positive β .

4.1.4 Existence of conical combination

There are multiple ways to find the values of β . In particular, for $N = 3, 4, 5$ we have the following result.

Lemma 4.3. *Let $N=3,4,5$. Let $\beta_{ij} = (-\alpha_{ij} + \alpha_{iN} + \alpha_{jN})/2$ and $\beta_i = \sum_{1 \leq i < j < N} -\beta_{ij} + \alpha_{iN}$. Then*

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} = \sum_{1 \leq i < j < N} \beta_{ij} \mathbb{P}_{ij} + \sum_{1 \leq i < N} \beta_i \mathbb{P}_i.$$

Proof. We have

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} = \sum_{1 \leq i < j < N} \alpha_{ij} \mathbb{L}_{ij} + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{L}_{iN}.$$

It follows from Lemma 4.1 that

$$\begin{aligned} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} &= \sum_{1 \leq i < j < N} \alpha_{ij} \mathbb{L}_{ij} + \sum_{1 \leq i < N} \alpha_{iN} \left(\mathbb{P}_i - \sum_{k \neq i, N} \mathbb{L}_{ik} \right) \\ &= \sum_{1 \leq i < j < N} \alpha_{ij} \mathbb{L}_{ij} + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i - \sum_{1 \leq i < k < N} \alpha_{iN} \mathbb{L}_{ik} - \sum_{1 \leq k < i < N} \alpha_{iN} \mathbb{L}_{ik} \\ &= \sum_{1 \leq i < j < N} \alpha_{ij} \mathbb{L}_{ij} + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i - \sum_{1 \leq i < k < N} \alpha_{iN} \mathbb{L}_{ik} - \sum_{1 \leq i < k < N} \alpha_{kN} \mathbb{L}_{ik} \\ &= \sum_{1 \leq i < j < N} (\alpha_{ij} - \alpha_{iN} - \alpha_{jN}) \mathbb{L}_{ij} + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i. \end{aligned}$$

By using Lemma 3.6, we thus get

$$\begin{aligned} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} &= \sum_{1 \leq i < j < N} \left(\frac{\alpha_{ij} - \alpha_{iN} - \alpha_{jN}}{2} \right) [\mathbb{P}_i + \mathbb{P}_j - \mathbb{P}_{ij}] + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i \\ &= \sum_{1 \leq i < j < N} \beta_{ij} (\mathbb{P}_{ij} - \mathbb{P}_i - \mathbb{P}_j) + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i \\ &= \sum_{1 \leq i < j < N} \beta_{ij} \mathbb{P}_{ij} + \sum_{1 \leq i < j < N} \beta_{ij} (-\mathbb{P}_i - \mathbb{P}_j) + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i, \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} &= \sum_{1 \leq i < j < N} \beta_{ij} \mathbb{P}_{ij} - \sum_{i \neq j} \beta_{ij} \mathbb{P}_i + \sum_{1 \leq i < N} \alpha_{iN} \mathbb{P}_i \\ &= \sum_{1 \leq i < j < N} \beta_{ij} \mathbb{P}_{ij} + \sum_{1 \leq i < N} \left(-\sum_{i \neq j} \beta_{ij} + \alpha_{iN} \right) \mathbb{P}_i, \end{aligned}$$

and thus

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij} = \sum_{1 \leq i < j < N} \beta_{ij} \mathbb{P}_{ij} + \sum_{1 \leq i < N} \beta_i \mathbb{P}_i.$$

□

4.1. Reformulation in terms of perimeters

Now, we define

$$B_N = \left\{ (\alpha_{ij}) \in S_N : \exists (\beta_S) \text{ such that } \sum_{i<j} \alpha_{ij} \mathbb{L}_{ij} = \sum_{S \subset \{1, \dots, N\}} \beta_S \mathbb{P}_S \right\},$$

and

$$B_N^+ = \left\{ (\alpha_{ij}) \in B_N : \exists (\beta_S) \geq 0 \text{ such that } \sum_{i<j} \alpha_{ij} \mathbb{L}_{ij} = \sum_{S \subset \{1, \dots, N\}} \beta_S \mathbb{P}_S \right\}.$$

The following theorem allows us to find the nonnegative β_S for all S when the surface tensions α_{ij} satisfy the triangle inequality.

Theorem 4.4. *If $N=3,4$, then $T_N \subseteq B_N^+$.*

Proof. We distinguish the two cases:

- Case 1: $N = 3$. Using Lemma 4.3 for $N = 3$, we have

$$\beta_1 = \frac{-\alpha_{23} + \alpha_{12} + \alpha_{13}}{2}, \quad \beta_2 = \frac{-\alpha_{13} + \alpha_{12} + \alpha_{23}}{2}, \quad \beta_{12} = \frac{-\alpha_{12} + \alpha_{13} + \alpha_{23}}{2}.$$

If $(\alpha_{ij}) \in T_3$, then β_1, β_2 , and β_{12} are nonnegative, which implies that $T_3 \subseteq B_3^+$.

- Case 2: $N = 4$. Using once more Lemma 4.3 for $N = 4$, we have

$$\begin{aligned} \beta_{12} &= \frac{-\alpha_{12} + \alpha_{14} + \alpha_{24}}{2}, \quad \beta_{13} = \frac{-\alpha_{13} + \alpha_{14} + \alpha_{34}}{2}, \quad \beta_{23} = \frac{-\alpha_{23} + \alpha_{24} + \alpha_{34}}{2}, \\ \beta_1 &= \alpha_{12} + \alpha_{13} - \alpha_{24} - \alpha_{34}, \quad \beta_2 = \alpha_{12} + \alpha_{23} - \alpha_{14} - \alpha_{34} \text{ and } \beta_3 = \alpha_{13} + \alpha_{23} - \alpha_{14} - \alpha_{24}. \end{aligned}$$

It is easy to see that if $(\alpha_{ij}) \in T_4$, then β_{12}, β_{13} , and β_{23} are nonnegative.

Now, We want to prove that if $(\alpha_{ij}) \in T_4$, then β_1, β_2 , and β_3 are nonnegative. Let us define for $i = 1, \dots, 4$, $S_i = \sum_{i \neq j}^4 \alpha_{ij}$. Up to reordering S_i , we assume that $S_4 \leq S_3 \leq S_2 \leq S_1$. Then we have

$$\begin{aligned} \begin{cases} S_4 \leq S_1 \\ S_4 \leq S_2 \\ S_4 \leq S_3 \end{cases} &\Rightarrow \begin{cases} \alpha_{14} + \alpha_{24} + \alpha_{34} \leq \alpha_{12} + \alpha_{13} + \alpha_{14} \\ \alpha_{14} + \alpha_{24} + \alpha_{34} \leq \alpha_{12} + \alpha_{23} + \alpha_{24} \\ \alpha_{14} + \alpha_{24} + \alpha_{34} \leq \alpha_{13} + \alpha_{23} + \alpha_{34} \end{cases} \Rightarrow \begin{cases} \alpha_{24} + \alpha_{34} \leq \alpha_{12} + \alpha_{13} \\ \alpha_{14} + \alpha_{34} \leq \alpha_{12} + \alpha_{23} \\ \alpha_{14} + \alpha_{24} \leq \alpha_{13} + \alpha_{23} \end{cases} \\ &\Rightarrow \begin{cases} \beta_1 \geq 0 \\ \beta_2 \geq 0 \\ \beta_3 \geq 0 \end{cases}. \end{aligned}$$

Thus we indeed have obtained $T_4 \subseteq B_4^+$. □

Theorem 4.5. *If $N=3,4,5$, then $B_N^+ \subseteq T_N$.*

Proof. We distinguish the two cases:

4.1. Reformulation in terms of perimeters

- Case 1: $N = 3$. From (4.1.4) we have

$$\begin{aligned} \sum_{i=1}^3 \beta_i \mathbb{P}_i &= \beta_1(\mathbb{L}_{12} + \mathbb{L}_{13}) + \beta_2(\mathbb{L}_{12} + \mathbb{L}_{23}) + \beta_3(\mathbb{L}_{13} + \mathbb{L}_{23}) \\ &= (\beta_1 + \beta_2)\mathbb{L}_{12} + (\beta_1 + \beta_3)\mathbb{L}_{13} + (\beta_2 + \beta_3)\mathbb{L}_{23}. \end{aligned} \quad (4.1.6)$$

Now, we identify

$$\alpha_{ij} = \beta_i + \beta_j. \quad (4.1.7)$$

If the values of β are nonnegative, then the surface tensions α_{ij} are nonnegative.

Now, we want to prove that the surface tensions α_{ij} satisfy the triangle inequality. From (4.1.7) we have

$$\begin{cases} \alpha_{12} = \beta_1 + \beta_2 \\ \alpha_{13} = \beta_1 + \beta_3 \\ \alpha_{23} = \beta_2 + \beta_3 \end{cases} \Rightarrow \begin{cases} \alpha_{12} \leq \alpha_{13} + \alpha_{23} \\ \alpha_{13} \leq \alpha_{12} + \alpha_{23} \\ \alpha_{23} \leq \alpha_{12} + \alpha_{13}, \end{cases}$$

which implies that $B_3^+ \subseteq T_3$.

- Case 2: $N = 4, 5$. We have

$$\begin{aligned} \sum_{i=1}^N \beta_i \mathbb{P}_i + \sum_{i < j} \beta_{ij} \mathbb{P}_{ij} &= \sum_{i=1}^N \beta_i \left(\sum_{j \neq i} \mathbb{L}_{ij} \right) + \sum_{i < j} \beta_{ij} \left(\sum_{k \neq i, j} \mathbb{L}_{ik} + \mathbb{L}_{kj} \right) \\ &= \sum_{i < j} (\beta_i + \beta_j) \mathbb{L}_{ij} + \sum_{i < j} \beta_{ij} \sum_{k \neq i, j} \mathbb{L}_{ik} + \sum_{i < j} \beta_{ij} \sum_{k \neq i, j} \mathbb{L}_{kj} \\ &= \sum_{i < j} (\beta_i + \beta_j) \mathbb{L}_{ij} + \sum_{i \neq j} \beta_{ij} \sum_{k \neq i, j} \mathbb{L}_{ik} \\ &= \sum_{i < j} (\beta_i + \beta_j) \mathbb{L}_{ij} + \sum_{i \neq k} \left(\sum_{j \neq i} \beta_{ij} \right) \mathbb{L}_{ik} \\ &= \sum_{i < j} (\beta_i + \beta_j) \mathbb{L}_{ij} + \sum_{i \neq j} \left(\sum_{k \neq i} \beta_{ik} \right) \mathbb{L}_{ik} \\ &= \sum_{i < j} (\beta_i + \beta_j) \mathbb{L}_{ij} + \sum_{i < j} \left(\sum_{k \neq i, j} \beta_{ik} + \beta_{kj} \right) \mathbb{L}_{ij} \\ &= \sum_{i < j} \left(\beta_i + \beta_j + \sum_{k \neq i, j} \beta_{ik} + \beta_{kj} \right) \mathbb{L}_{ij}. \end{aligned}$$

Now, we identify

$$\alpha_{ij} = \beta_i + \beta_j + \sum_{k \neq i, j} \beta_{ik} + \beta_{kj}. \quad (4.1.8)$$

If the values of β are nonnegative, then the surface tensions α_{ij} are nonnegative.

4.1. Reformulation in terms of perimeters

Now, we want to prove that the surface tensions α_{ij} satisfy the triangle inequality. We start with

$$\begin{aligned}
-\alpha_{ij} + \alpha_{ik} + \alpha_{kj} &= -\beta_i - \beta_j - \sum_{h \neq i, j} (\beta_{ih} + \beta_{hj}) + \beta_i + \beta_k + \sum_{h \neq i, k} (\beta_{ih} + \beta_{hk}) \\
&\quad + \beta_k + \beta_j + \sum_{h \neq k, j} (\beta_{kh} + \beta_{hj}) \\
&= 2\beta_k - \sum_{h \neq i, j, k} (\beta_{ih} + \beta_{hj}) - \beta_{ik} - \beta_{kj} + \sum_{h \neq i, j, k} (\beta_{ih} + \beta_{hk}) + \beta_{ij} \\
&\quad + \beta_{jk} + \sum_{h \neq i, j, k} (\beta_{kh} + \beta_{hj}) + \beta_{ki} + \beta_{ij} \\
&= 2\beta_k + 2 \sum_{h \neq i, j, k} \beta_{hk} + 2\beta_{ij}.
\end{aligned}$$

Since the values of β are positive, we obtain

$$-\alpha_{ij} + \alpha_{ik} + \alpha_{kj} \geq 0,$$

which implies that $B_N^+ \subseteq T_N$.

□

Now, we want to find positive values of β_S in order to use the primal variational formulation of (3.2.1) for the perimeters. Let us begin with the definition of conical combination and Carathéodory's theorem.

Definition 4.6. *Given a finite number of vectors v_1, v_2, \dots, v_p in a real vector space, a conical combination, of these vectors is a vector of the form*

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_p v_p,$$

where the real numbers λ_i satisfy $\lambda_i \geq 0$.

Theorem 4.7 (Carathéodory). *In a vector space of dimension n , all conical combination of m vectors ($m > n$), can be written as conical combination of n of these vectors.*

We refer to [50] for more details.

Remark 4.2. By the above theorem and the linear system in (4.1.5), when $(\alpha_{ij}) \in B_N^+$, α can be written as a conical combination of C_2^N columns of the matrix \mathbb{M}^Γ .

4.2. Γ -convergence with non negative coefficients

4.1.5 Algorithm to search for a conical combination

Theorem 4.7 lead us to Algorithm 2 in order to search for nonnegative values of β .

Data: Given $\alpha = (\alpha_{ij}) \in \mathbb{R}^{C_2^N \times 1}$, $\mathbb{M} \in \mathbb{R}^{(C_2^N + N) \times C_2^N}$.

1 repeat

2 | Loop on the set of square invertible submatrix $\Lambda \in \mathbb{R}^{C_2^N \times C_2^N}$ of \mathbb{M} ;

3 | Compute $\beta = \Lambda^{-1}\alpha$;

4 until $\beta \geq 0$;

Algorithm 2: Find positive values of β .

At each step we have to solve a linear system of form:

$$\begin{pmatrix} C_2^N \times C_2^N & \dots \\ \Lambda & \dots \\ & \dots \\ & \dots \end{pmatrix} \begin{pmatrix} C_2^N \\ \beta \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha \end{pmatrix}.$$

(4.1.9)

Remark 4.3. If $N=5$, then we have counterexample of Theorem 4.4. Consider the matrix:

$$(\alpha_{ij}) = \begin{pmatrix} 0 & 1 & 3 & 2 & 2 \\ 1 & 0 & 3 & 2 & 2 \\ 3 & 3 & 0 & 3 & 3 \\ 2 & 2 & 3 & 0 & 2 \\ 2 & 2 & 3 & 2 & 0 \end{pmatrix}.$$

It is clear to see that the matrix (α_{ij}) satisfies the triangle inequality, but for all possible values of β found from (4.1.9) at least one coefficient is negative.

4.2 Γ -convergence with non negative coefficients

We now state the main result of this chapter. Let us recall the definition of the following set

$$\tilde{\mathcal{E}}_N = \left\{ (u_1, \dots, u_N) \in \tilde{\mathcal{E}}^N, \sum_{i=1}^N u_i = 1 \right\}.$$

Theorem 4.8. Let D be an open rectangle of \mathbb{R}^2 . If $(\alpha_{ij}) \in B_N^+$, then

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i, u_j) \xrightarrow{\Gamma} \begin{cases} \frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathcal{H}^1(\partial_M \Omega_i \cap \partial_M \Omega_j \cap D) & \text{if } u_i \in BV(D, \{0, 1\}) \forall i, u_i = \chi_{\Omega_i}, \\ +\infty & \text{otherwise,} \end{cases}$$

in $\tilde{\mathcal{E}}_N$, strongly in $L^1(D)^N$.

4.3. Primal algorithm

Proof. For simplicity of notation, we assume that $N \leq 5$. First, we will prove the lim inf inequality. Let $(u_i^\varepsilon) \in \tilde{\mathcal{E}}_N$ such that (u_i^ε) converges to u_i in L^1 . From (3.3.1), (4.1.4) and Remark 4.1 we have

$$\sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) = \frac{1}{\varepsilon} \sum_{i=1}^N \beta_i \langle 1 - L_\varepsilon u_i^\varepsilon, u_i^\varepsilon \rangle + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \beta_{ij} \langle 1 - L_\varepsilon(u_i^\varepsilon + u_j^\varepsilon), u_i^\varepsilon + u_j^\varepsilon \rangle. \quad (4.2.1)$$

Passing to lim inf as $\varepsilon \rightarrow 0$ in (4.2.1), we get

$$\liminf_{\varepsilon \rightarrow 0} \sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) = \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} \sum_{i=1}^N \beta_i \langle 1 - L_\varepsilon u_i^\varepsilon, u_i^\varepsilon \rangle + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \beta_{ij} \langle 1 - L_\varepsilon(u_i^\varepsilon + u_j^\varepsilon), u_i^\varepsilon + u_j^\varepsilon \rangle \right),$$

which implies that

$$\liminf_{\varepsilon \rightarrow 0} \sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) \geq \sum_{i=1}^N \beta_i \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \langle 1 - L_\varepsilon u_i^\varepsilon, u_i^\varepsilon \rangle + \sum_{1 \leq i < j \leq N} \beta_{ij} \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \langle 1 - L_\varepsilon(u_i^\varepsilon + u_j^\varepsilon), u_i^\varepsilon + u_j^\varepsilon \rangle.$$

We deduce from Theorem 3.3 and (4.1.4) that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \sum_{1 \leq i < j \leq N} \alpha_{ij} G_\varepsilon(u_i^\varepsilon, u_j^\varepsilon) &\geq \frac{1}{2} \sum_{i=1}^N \beta_i \mathcal{H}^1(\partial_M \Omega_i \cap \Omega) + \frac{1}{2} \sum_{1 \leq i < j \leq N} \beta_{ij} \mathcal{H}^1(\partial_M(\Omega_i \cup \Omega_j) \cap \Omega) \\ &= \frac{1}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij} \mathbb{L}_{ij}. \end{aligned}$$

Second, as we have the pointwise limit (Theorem 3.7), the lim sup inequality holds for the constant recovery sequence. \square

4.3 Primal algorithm

If the surface tension matrix satisfies $(\alpha_{ij}) \in \beta_N^+$, then we can use the primal variational formulation (3.2.1) for the interface energy. For ε fixed the approximate problem (1.4.1) becomes

$$\mathcal{I}_\varepsilon^P = \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle u_i, g_i \rangle + \sum_{i=1}^N \beta_i \tilde{F}_\varepsilon(u_i) + \sum_{1 \leq i < j \leq N} \beta_{ij} \tilde{F}_\varepsilon(u_i + u_j) \right\}. \quad (4.3.1)$$

Recall that \tilde{F}_ε is defined by (1.3.5). Then from (3.2.1) we have

$$\begin{aligned} \mathcal{I}_\varepsilon^P &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle u_i, g_i \rangle + \sum_{i=1}^N \beta_i \inf_{v_i \in H^1(D)} \left\{ \varepsilon \|\nabla v_i\|_{L^2(D)}^2 + \frac{1}{\varepsilon} (\|v_i\|_{L^2(D)}^2 + \langle u_i, 1 - 2v_i \rangle) \right\} \right. \\ &\quad \left. + \sum_{1 \leq i < j \leq N} \beta_{ij} \inf_{w_{ij} \in H^1(D)} \left\{ \varepsilon \|\nabla w_{ij}\|_{L^2(D)}^2 + \frac{1}{\varepsilon} (\|w_{ij}\|_{L^2(D)}^2 + \langle u_i + u_j, 1 - 2w_{ij} \rangle) \right\} \right\}, \end{aligned} \quad (4.3.2)$$

4.3. Primal algorithm

with $w_{ij} = v_i + v_j$ for $i \neq j$, which implies that

$$\begin{aligned} \mathcal{I}_\varepsilon^P = & \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \inf_{v_i \in H^1(D)} \inf_{w_{ij} \in H^1(D)} & (4.3.3) \\ & \left\{ \sum_{i=1}^N \langle u_i, g_i \rangle + \sum_{i=1}^N \beta_i \left[\varepsilon \|\nabla v_i\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v_i\|_{L^2(D)}^2 + \langle u_i, 1 - 2v_i \rangle \right) \right] \right. \\ & \left. + \sum_{1 \leq i < j \leq N} \beta_{ij} \left[\varepsilon \|\nabla w_{ij}\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|w_{ij}\|_{L^2(D)}^2 + \langle u_i + u_j, 1 - 2w_{ij} \rangle \right) \right] \right\}. \end{aligned} \quad (4.3.4)$$

The simple structure of this problem with respect to each variable (u_1, \dots, u_N) , (v_1, \dots, v_N) and $(w_{12}, \dots, w_{(N-1)N})$ leads us to use an alternating minimization algorithm. The superscript k is used to designate variables computed at iteration k . The iteration k , $k \geq 1$, consists in the three steps described below.

- The minimization with respect to (v_1, \dots, v_N) is straightforward. It consists in solving the boundary value problems

$$\begin{cases} -\varepsilon^2 \Delta v_i^k + v_i^k & = u_i^{k-1} & \text{in } D, \\ \partial_n v_i^k & = 0 & \text{on } \partial D. \end{cases} \quad (4.3.5)$$

- The minimization with respect to $(w_{12}, \dots, w_{(N-1)N})$ is straightforward. It consists in solving the boundary value problems

$$\begin{cases} -\varepsilon^2 \Delta w_{ij}^k + w_{ij}^k & = (u_i + u_j)^{k-1} & \text{in } D, \\ \partial_n w_{ij}^k & = 0 & \text{on } \partial D. \end{cases}$$

In fact $w_{ij}^k = v_i^k + v_j^k$ for $i \neq j$.

- The minimization with respect to (u_1, \dots, u_N) is a linear programming problem in a convex set. Therefore a minimizer can always be found among the extreme points of $\tilde{\mathcal{E}}$. More precisely here, we have to minimize at each point $x \in D$ the linear function $\sum_{i=1}^N \phi_i(\mathbf{s}_i)$ with

$$\phi_i(\mathbf{s}) = \mathbf{s} g_i(x) + \frac{\beta_i \mathbf{s}}{\varepsilon} (1 - 2v_i^k(x)) + \frac{\mathbf{s}}{\varepsilon} \sum_{\substack{1 < j \leq N \\ j \neq i}} \beta_{ij} (1 - 2w_{ij}^k(x))$$

over $\mathbf{s}_i \in [0, 1]$ and $\sum_{i=1}^N \mathbf{s}_i = 1$. Set

$$\xi_i^k(x) = g_i(x) + \frac{\beta_i}{\varepsilon} (1 - 2v_i^k(x)) + \frac{1}{\varepsilon} \sum_{\substack{1 < j \leq N \\ j \neq i}} \beta_{ij} (1 - 2w_{ij}^k(x)).$$

At each point $x \in D$ we find an index $i(x)$ such that

$$\xi_{i(x)}^k = \min \{ \xi_1^k(x), \dots, \xi_N^k(x) \}.$$

We then set

$$u_i^k(x) = \begin{cases} 1 & \text{if } i = i(x), \\ 0 & \text{otherwise.} \end{cases}$$

This algorithm ensures a decrease of the objective function at each iteration. Moreover, each cluster point (in the weak-* topology for u_i and the H^1 norm topology for v_i) is a stationary point. Of course, as the coupled problem in (u_i, v_i) is not convex, local minimizers are theoretically not excluded. An outstanding feature of this algorithm is that u_i^k is always a characteristic function during the iterations.

4.4 Extension: volume constraints

In this section we apply the previous algorithm with volume constraints. Given $m_1, \dots, m_N \in \mathbb{R}^+$ such that $\sum_{i=1}^N m_i = |D|$. We define the set

$$\bar{\mathcal{E}}_N = \left\{ (u_1, \dots, u_N) \in \tilde{\mathcal{E}}^N, \sum_{i=1}^N u_i = 1, \int_D u_i = m_i \quad \forall i \right\}.$$

The problem (4.3.1) with volume constraints and $g_i = 0$ for all i becomes

$$\mathcal{I}_\varepsilon^P = \min_{(u_1, \dots, u_N) \in \bar{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \beta_i \tilde{F}_\varepsilon(u_i) + \sum_{1 \leq i < j \leq N} \beta_{ij} \tilde{F}_\varepsilon(u_i + u_j) \right\}.$$

After transformation as in (4.3.2) we obtain

$$\begin{aligned} \mathcal{I}_\varepsilon^P = & \min_{(u_1, \dots, u_N) \in \bar{\mathcal{E}}_N} \inf_{v_i \in H^1(D)} \inf_{w_{ij} \in H^1(D)} \left\{ \sum_{i=1}^N \beta_i \left[\varepsilon \|\nabla v_i\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v_i\|_{L^2(D)}^2 + \langle u_i, 1 - 2v_i \rangle \right) \right] \right. \\ & \left. + \sum_{1 \leq i < j \leq N} \beta_{ij} \left[\varepsilon \|\nabla w_{ij}\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|w_{ij}\|_{L^2(D)}^2 + \langle u_i + u_j, 1 - 2w_{ij} \rangle \right) \right] \right\}. \end{aligned}$$

Now, we use again an alternating minimization algorithm with respect to (u_1, \dots, u_N) , (v_1, \dots, v_N) and $(w_{12}, \dots, w_{(N-1)N})$.

- The minimization with respect to (v_1, \dots, v_N) consists in solving the boundary value problem

$$\begin{cases} -\varepsilon^2 \Delta v_i^k + v_i^k & = u_i^{k-1} & \text{in } D, \\ \partial_n v_i^k & = 0 & \text{on } \partial D. \end{cases}$$

- The minimization with respect to $(w_{12}, \dots, w_{(N-1)N})$ consists in setting

$$w_{ij}^k = v_i^k + v_j^k \quad \forall i, j.$$

- The minimization with respect to (u_1, \dots, u_N) amounts to solving a linear programming problem.

4.5 Primal dual formulation

The following theorem will allow us to deduce the primal dual algorithm.

4.5. Primal dual formulation

Theorem 4.9. *Consider the problems*

$$(\mathcal{P}) \quad \sup_{v \in H^1(D)} 2\langle u, v \rangle - \varepsilon^2 \|\nabla v\|^2 - \|v\|^2, \quad (4.5.1)$$

$$(\mathcal{P}^*) \quad \inf_{q \in H_0^{\text{div}}(D)} \|u + \varepsilon \operatorname{div} q\|^2 + \|q\|^2. \quad (4.5.2)$$

We have that

1. (\mathcal{P}) is strictly concave and (\mathcal{P}^*) is strictly convex.
2. $\operatorname{sol}(\mathcal{P}) = \{v_\varepsilon\}$ and $\operatorname{sol}(\mathcal{P}^*) = \{\varepsilon \nabla v_\varepsilon\}$ with $v_\varepsilon = L_\varepsilon u$.
3. $\operatorname{val}(\mathcal{P}) = \operatorname{val}(\mathcal{P}^*) = \langle v_\varepsilon, u \rangle$.

Proof. 1. It is clear that (\mathcal{P}) is strictly concave and (\mathcal{P}^*) is strictly convex.

2. Let us start with the existence and uniqueness of a solution of problems (\mathcal{P}) and (\mathcal{P}^*) . Since (\mathcal{P}) strictly concave, continuous, coercive and (\mathcal{P}^*) strictly convex, continuous, coercive, then (\mathcal{P}) and (\mathcal{P}^*) admit one and only one solution. The Euler-Lagrange equations of the minimization problem (\mathcal{P}) are

$$\varepsilon^2 \langle \nabla v_\varepsilon, \nabla \varphi \rangle + \langle v_\varepsilon, \varphi \rangle = \langle u, \varphi \rangle \quad \forall \varphi \in H^1(D).$$

This is the weak formulation of (3.2.2). Therefore $\operatorname{sol}(\mathcal{P}) = \{v_\varepsilon\}$ and

$$\varepsilon^2 \|\nabla v_\varepsilon\|_{L^2(D)}^2 + \|v_\varepsilon\|_{L^2(D)}^2 = \langle v_\varepsilon, u \rangle. \quad (4.5.3)$$

We deduce from (4.5.1) and (4.5.3) that

$$\operatorname{val}(\mathcal{P}) = 2\langle v_\varepsilon, u \rangle - \langle v_\varepsilon, u \rangle = \langle v_\varepsilon, u \rangle. \quad (4.5.4)$$

The Euler-Lagrange equations of the minimization problem (\mathcal{P}^*) are

$$\begin{aligned} 2\langle u + \varepsilon \operatorname{div} q_\varepsilon, \varepsilon \operatorname{div} \varphi \rangle + 2\langle q_\varepsilon, \varphi \rangle &= 0 \quad \forall \varphi \in H_0^{\text{div}}(D) \\ \Leftrightarrow \varepsilon^2 \langle \operatorname{div} q_\varepsilon, \operatorname{div} \varphi \rangle + \langle q_\varepsilon, \varphi \rangle &= -\varepsilon \langle u, \operatorname{div} \varphi \rangle \quad \forall \varphi \in H_0^{\text{div}}(D). \end{aligned} \quad (4.5.5)$$

Let $v_\varepsilon = L_\varepsilon u = \operatorname{sol}(\mathcal{P}) \in H^1(D)$, and denote by $\tilde{q}_\varepsilon = \varepsilon \nabla v_\varepsilon \in L^2(D)$. We have

$$\forall \varphi \in \mathcal{D}(D), \quad \langle \varepsilon \operatorname{div} \tilde{q}_\varepsilon, \varphi \rangle = -\varepsilon \langle \tilde{q}_\varepsilon, \nabla \varphi \rangle = -\varepsilon^2 \langle \nabla v_\varepsilon, \nabla \varphi \rangle = \langle v_\varepsilon - u, \varphi \rangle, \quad (4.5.6)$$

which implies that

$$\operatorname{div} \tilde{q}_\varepsilon = \frac{1}{\varepsilon} (v_\varepsilon - u) \in L^2(D). \quad (4.5.7)$$

From Theorem 2.16 and (4.5.6) we have

$$\forall \varphi \in H^1(D), \quad \int_{\partial D} (\tilde{q}_\varepsilon \cdot n) \varphi = \langle \tilde{q}_\varepsilon, \nabla \varphi \rangle + \langle \operatorname{div} \tilde{q}_\varepsilon, \varphi \rangle = 0.$$

Hence $\tilde{q}_\varepsilon \in H_0^{\text{div}}(D)$.

4.6. Primal dual algorithm

Now, we come back to (4.5.5) to prove that $\tilde{q}_\varepsilon = \text{sol}(\mathcal{P}^*)$. Let $\varphi \in H_0^{\text{div}}(D)$, then

$$\begin{aligned} \langle u + \varepsilon \text{div } \tilde{q}_\varepsilon, \varepsilon \text{div } \varphi \rangle + \langle \tilde{q}_\varepsilon, \varphi \rangle &= \langle u + (v_\varepsilon - u), \varepsilon \text{div } \varphi \rangle + \langle \varepsilon \nabla v_\varepsilon, \varphi \rangle \\ &= \varepsilon \langle v_\varepsilon, \text{div } \varphi \rangle + \varepsilon \langle \nabla v_\varepsilon, \varphi \rangle. \end{aligned}$$

Using Theorem 2.16, we get

$$\langle u + \varepsilon \text{div } \tilde{q}_\varepsilon, \varepsilon \text{div } \varphi \rangle + \langle \tilde{q}_\varepsilon, \varphi \rangle = \varepsilon \int_{\partial D} (\varphi \cdot n) v_\varepsilon = 0. \quad (4.5.8)$$

Therefore \tilde{q}_ε is solution of (\mathcal{P}^*) . Moreover, by uniqueness we have

$$\text{sol}(\mathcal{P}^*) = \{\tilde{q}_\varepsilon\} = \{q_\varepsilon\}. \quad (4.5.9)$$

3. From (4.5.2) and (4.5.9) we have

$$\text{val}(\mathcal{P}^*) = \|u + \varepsilon \text{div } q_\varepsilon\|^2 + \|q_\varepsilon\|^2.$$

In particular, from (4.5.8) we obtain

$$\text{val}(\mathcal{P}^*) = \langle u + \varepsilon \text{div } q_\varepsilon, u \rangle + \underbrace{\langle u + \varepsilon \text{div } q_\varepsilon, \varepsilon \text{div } q_\varepsilon \rangle}_{=-\|q_\varepsilon\|^2} + \|q_\varepsilon\|^2.$$

It follows from (4.5.7) and (4.5.4) that

$$\begin{aligned} \text{val}(\mathcal{P}^*) &= \langle u + \varepsilon \text{div } q_\varepsilon, u \rangle \\ &= \langle u + (v_\varepsilon - u), u \rangle \\ &= \langle v_\varepsilon, u \rangle \\ &= \text{val}(\mathcal{P}). \end{aligned}$$

□

4.6 Primal dual algorithm

We now consider the problem (4.3.1) for $N = 5$, where one value of β_S is negative, let it β_1 . Then we have

$$\mathcal{I}_\varepsilon^{PD} = \min_{(u_1, \dots, u_5) \in \tilde{\mathcal{E}}_5} \left\{ \sum_{i=1}^5 \langle u_i, g_i \rangle + \beta_1 \tilde{F}_\varepsilon(u_1) + \sum_{i=2}^5 \beta_i \tilde{F}_\varepsilon(u_i) + \sum_{1 \leq i < j \leq 5} \beta_{ij} \tilde{F}_\varepsilon(u_i + u_j) \right\}.$$

It follows from (3.2.3) that

$$\begin{aligned} \mathcal{I}_\varepsilon^{PD} = \min_{(u_1, \dots, u_5) \in \tilde{\mathcal{E}}_5} \left\{ \sum_{i=1}^5 \langle u_i, g_i \rangle - \frac{|\beta_1|}{\varepsilon} \langle 1 - L_\varepsilon u_1, u_1 \rangle + \frac{1}{\varepsilon} \sum_{i=2}^5 \beta_i \langle 1 - L_\varepsilon u_i, u_i \rangle \right. \\ \left. + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq 5} \beta_{ij} \langle 1 - L_\varepsilon (u_i + u_j), u_i + u_j \rangle \right\}. \end{aligned}$$

4.6. Primal dual algorithm

According to the values of β_S , we can use the dual expression (4.5.2) for β_1 and otherwise the primal expression (4.5.1) for the positive β_S , i.e.

$$\begin{aligned} \mathcal{I}_\varepsilon^{PD} = \min_{(u_1, \dots, u_5) \in \tilde{\mathcal{E}}_5} & \left\{ \sum_{i=1}^5 \langle u_i, g_i \rangle - \frac{|\beta_1|}{\varepsilon} \langle 1, u_1 \rangle + \frac{|\beta_1|}{\varepsilon} \inf_{q \in H_0^{\text{div}}(D)} \left\{ \|q\|_{L^2(D)}^2 + \|u_1 + \varepsilon \text{div } q\|_{L^2(D)}^2 \right\} \right. \\ & \left. + \frac{1}{\varepsilon} \sum_{i=2}^5 \beta_i \inf_{v_i \in H^1(D)} \left\{ \varepsilon \|\nabla v_i\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v_i\|_{L^2(D)}^2 + \langle u_i, 1 - 2v_i \rangle \right) \right\} \right. \\ & \left. + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq 5} \beta_{ij} \inf_{w_{ij} \in H^1(D)} \left\{ \varepsilon \|\nabla w_{ij}\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|w_{ij}\|_{L^2(D)}^2 + \langle u_i + u_j, 1 - 2w_{ij} \rangle \right) \right\} \right\}. \end{aligned}$$

At ε fixed and we use again an alternating minimization algorithm with respect to (u_1, \dots, u_5) , (v_1, \dots, v_5) , (w_{12}, \dots, w_{45}) and q .

- The minimization with respect to (v_1, \dots, v_5) consists in solving the 5 boundary value problems

$$\begin{cases} -\varepsilon^2 \Delta v_i^k + v_i^k &= u_i^{k-1} & \text{in } D, \\ \partial_n v_i^k &= 0 & \text{on } \partial D. \end{cases}$$

- The minimization with respect to (w_{12}, \dots, w_{45}) consists in solving the C_2^N boundary value problems

$$\begin{cases} -\varepsilon^2 \Delta w_{ij}^k + w_{ij}^k &= (u_i + u_j)^{k-1} & \text{in } D, \\ \partial_n w_{ij}^k &= 0 & \text{on } \partial D. \end{cases}$$

In fact $w_{ij}^k = v_i^k + v_j^k$ for $i \neq j$.

- The minimization with respect to q yields the Euler-Lagrange equation

$$\begin{aligned} 2\langle q_\varepsilon, h \rangle + 2\langle u_1 + \varepsilon \text{div } q_\varepsilon, \varepsilon \text{div } h \rangle &= 0, \quad \forall h \in H_0^{\text{div}}(D) \\ \langle q_\varepsilon, h \rangle - \varepsilon \langle \nabla(u_1 + \varepsilon \text{div } q_\varepsilon), h \rangle &= 0, \quad \forall h \in H_0^{\text{div}}(D) \end{aligned} \quad (4.6.1)$$

Let $q_\varepsilon = \varepsilon \nabla v_\varepsilon^1$. Then a change of q_ε in the above equation entails

$$\begin{aligned} \langle \varepsilon \nabla v_\varepsilon, h \rangle - \varepsilon \langle \nabla(u_1 + \varepsilon \text{div } \varepsilon \nabla v_\varepsilon), h \rangle &= 0, \quad \forall h \in H_0^{\text{div}}(D) \\ \Leftrightarrow \langle \varepsilon \nabla \underbrace{(v_\varepsilon - u_1 - \varepsilon^2 \Delta v_\varepsilon)}_0, h \rangle &= 0, \quad \forall h \in H_0^{\text{div}}(D) \end{aligned} \quad (4.6.2)$$

We deduce from (4.6.2) that q_ε is solution of (4.6.1).

- The minimization with respect to (u_1, \dots, u_5) consists in solving

$$\min_{(u_1, \dots, u_5) \in \tilde{\mathcal{E}}_5} \left\{ \frac{|\beta_1|}{\varepsilon} \|u_1 + \varepsilon \text{div } q_\varepsilon\|^2 + \sum_{i=1}^5 \langle \xi_i, u_i \rangle \right\} =: \mathcal{J}_\varepsilon^{PD}$$

4.6. Primal dual algorithm

with

$$\xi_1 = g_1(x) - \frac{|\beta_1|}{\varepsilon} + \frac{1}{\varepsilon} \sum_{1 < j \leq 5} \beta_{1j}(1 - 2w_{1j}(x)),$$

and

$$\xi_i = g_i(x) + \frac{\beta_i}{\varepsilon}(1 - 2v_i(x)) + \frac{1}{\varepsilon} \sum_{\substack{1 < j \leq 5 \\ j \neq i}} \beta_{ij}(1 - 2w_{ij}(x)) \text{ for } i = 2, \dots, 5.$$

Since $u_5 = 1 - \sum_{i=1}^4 u_i$, then we have

$$\mathcal{J}_\varepsilon^{PD} = \langle \xi_5, 1 \rangle + \min_{\substack{\sum_{i=1}^4 u_i \leq 1 \\ u_i \geq 0}} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \sum_{i=1}^4 \langle \xi_i - \xi_5, u_i \rangle \right\}. \quad (4.6.3)$$

In this situation we have two kinds of problems:

1. the minimization with respect to (u_2, \dots, u_4) is a linear programming problem,
2. the minimization with respect to u_1 is a quadratic programming problem.

It follows that

$$\mathcal{J}_\varepsilon^{PD} = \langle \xi_5, 1 \rangle + \min_{0 \leq u_1 \leq 1} \min_{\substack{\sum_{i=2}^4 u_i \leq 1 - u_1 \\ u_i \geq 0}} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \xi_1 - \xi_5, u_1 \rangle + \sum_{i=2}^4 \langle \xi_i - \xi_5, u_i \rangle \right\}.$$

This provides

$$\mathcal{J}_\varepsilon^{PD} = \langle \xi_5, 1 \rangle + \min_{0 \leq u_1 \leq 1} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \xi_1 - \xi_5, u_1 \rangle + \min_{\substack{\sum_{i=2}^4 u_i \leq 1 - u_1 \\ u_i \geq 0}} \sum_{i=2}^4 \langle \xi_i - \xi_5, u_i \rangle \right\}.$$

We define Θ_i for $i=2,3,4$ by

$$\Theta_i = \begin{cases} \frac{u_i}{1 - u_1} & \text{if } u_1 \neq 1, \\ 0 & \text{if } u_1 = 1, \end{cases}$$

which implies that $u_i = (1 - u_1)\Theta_i \forall i$. Let $\tilde{\xi}_i = \xi_i - \xi_5$, then we get

$$\mathcal{J}_\varepsilon^{PD} = \langle \xi_5, 1 \rangle + \min_{0 \leq u_1 \leq 1} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \tilde{\xi}_1, u_1 \rangle + \min_{\substack{\sum_{i=2}^4 \Theta_i \leq 1 \\ \Theta_i \geq 0}} \sum_{i=2}^4 \langle \tilde{\xi}_i(1 - u_1), \Theta_i \rangle \right\}.$$

The minimization with respect to Θ_i is a linear programming problem, it yields

$$\begin{aligned} \mathcal{J}_\varepsilon^{PD} = \langle \xi_5, 1 \rangle + \min_{0 \leq u_1 \leq 1} & \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \tilde{\xi}_1, u_1 \rangle \right. \\ & \left. + \int_D \min(\tilde{\xi}_2(1 - u_1), \tilde{\xi}_3(1 - u_1), \tilde{\xi}_3(1 - u_1), 0) \right\}. \end{aligned}$$

4.7. Saddle point formulation and algorithm

This implies that

$$\mathcal{J}_\varepsilon^{PD} = \langle \xi_5, 1 \rangle + \min_{0 \leq u_1 \leq 1} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \tilde{\xi}_1, u_1 \rangle + \langle \min(\tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_3, 0), 1 - u_1 \rangle \right\}.$$

We denote ξ by

$$\xi = \min(\tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_3, 0),$$

which implies that

$$\begin{aligned} \mathcal{J}_\varepsilon^{PD} &= \langle \xi_5, 1 \rangle + \langle \xi, 1 \rangle + \min_{0 \leq u_1 \leq 1} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \tilde{\xi}_1 - \xi, u_1 \rangle \right\} \\ &= \langle \xi_5 + \xi, 1 \rangle + \min_{0 \leq u_1 \leq 1} \left\{ \frac{1}{\varepsilon} |\beta_1| \|u_1 + \varepsilon \operatorname{div} q_\varepsilon\|^2 + \langle \tilde{\xi}_1 - \xi, u_1 \rangle \right\} \\ &= \langle \xi_5 + \xi, 1 \rangle + \min_{0 \leq u_1 \leq 1} \left\{ \frac{1}{\varepsilon} |\beta_1| \left\| u_1 + \left[\varepsilon \operatorname{div} q_\varepsilon + \frac{\varepsilon}{2|\beta_1|} (\tilde{\xi}_1 - \xi) \right] \right\|^2 \right. \\ &\quad \left. - \frac{\varepsilon}{4|\beta_1|} \left\| \frac{2}{\varepsilon} |\beta_1| \varepsilon \operatorname{div} q_\varepsilon + (\tilde{\xi}_1 - \xi) \right\|^2 + \frac{1}{\varepsilon} |\beta_1| \|\varepsilon \operatorname{div} q_\varepsilon\|^2 \right\}. \end{aligned}$$

The minimization with respect to u_1 is a quadratic problem, whose solution is

$$u_1^* = P_{[0,1]} \left(\frac{\varepsilon}{2|\beta_1|} (\xi - \tilde{\xi}_1) - \varepsilon \operatorname{div} q_\varepsilon \right),$$

with

$$P_{[0,1]}(\varphi) = \max(0, \min(1, \varphi)).$$

4.7 Saddle point formulation and algorithm

4.7.1 Saddle point formulation

Let us start by defining the functional

$$\tilde{G}_\varepsilon(u_1, u_2) = \inf_{v_1 \in H^1(D)} \sup_{v_2 \in H^1(D)} \left[-\varepsilon \langle \nabla v_1, \nabla v_2 \rangle + \frac{1}{\varepsilon} (\langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle - \langle v_1, v_2 \rangle) \right]. \quad (4.7.1)$$

Definition 4.10. Let $L : X \times Y \rightarrow \bar{\mathbb{R}}$ be a bivariate function where X and Y are arbitrary spaces. A point $(x, y) \in X \times Y$ is called a saddle point of L if

$$\max_{y \in Y} L(\bar{x}, y) = L(\bar{x}, \bar{y}) = \min_{x \in X} L(x, \bar{y}).$$

Equivalently, (\bar{x}, \bar{y}) is a saddle point of L if

$$L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y}).$$

Another way to say this is

$$\begin{cases} (a) \ x \text{ is a solution of the minimization problem } \inf_{x \in X} L(x, \bar{y}), \\ (b) \ y \text{ is a solution of the maximization problem } \max_{y \in Y} L(\bar{x}, y). \end{cases}$$

4.7. Saddle point formulation and algorithm

Let us now define the function $L : H^1(D) \times H^1(D) \rightarrow \mathbb{R}$ as

$$L(v_1, v_2) = -\varepsilon \langle \nabla v_1, \nabla v_2 \rangle + \frac{1}{\varepsilon} (\langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle - \langle v_1, v_2 \rangle).$$

Lemma 4.11. *Let $u_1, u_2 \in L^2(D)$ be given and $v_1^\varepsilon, v_2^\varepsilon \in H^1(D)$ be the (weak) solution of*

$$\begin{cases} -\varepsilon^2 \Delta v_i^\varepsilon + v_i^\varepsilon &= u_i \text{ in } D, \quad i = 1, 2, \\ \partial_n v_i^\varepsilon &= 0 \text{ on } \partial D. \end{cases} \quad (4.7.2)$$

Then L admits as unique saddle point $(v_1^\varepsilon, v_2^\varepsilon)$ and

$$L(v_1^\varepsilon, v_2^\varepsilon) = \frac{1}{\varepsilon} \langle u_1, v_2^\varepsilon \rangle = \frac{1}{\varepsilon} \langle u_2, v_1^\varepsilon \rangle.$$

Proof. We first seek a stationary point of the functional (4.7.1). The Euler-Lagrange equations of (4.7.1) with respect to v_1, v_2 are respectively

$$\begin{aligned} -\varepsilon \langle \nabla \varphi, \nabla v_2^\varepsilon \rangle + \frac{1}{\varepsilon} (\langle u_2, \varphi \rangle - \langle \varphi, v_2^\varepsilon \rangle) &= 0 \quad \forall \varphi \in H^1(D) \\ -\varepsilon \langle \nabla v_1^\varepsilon, \nabla \psi \rangle + \frac{1}{\varepsilon} (\langle u_1, \psi \rangle - \langle v_1^\varepsilon, \psi \rangle) &= 0 \quad \forall \psi \in H^1(D). \end{aligned}$$

We recognize the weak formulation of (4.7.2) for $i = 1, 2$. After simplification we have

$$\varepsilon^2 \langle \nabla v_2^\varepsilon, \nabla \varphi \rangle + \langle v_2^\varepsilon - u_2, \varphi \rangle = 0 \quad \forall \varphi \in H^1(D), \quad (4.7.3)$$

$$\varepsilon^2 \langle \nabla v_1^\varepsilon, \nabla \psi \rangle + \langle v_1^\varepsilon - u_1, \psi \rangle = 0 \quad \forall \psi \in H^1(D). \quad (4.7.4)$$

By the Lax-Milgram theorem the equations (4.7.3) and (4.7.4) admit unique solutions. We infer the existence of a unique stationary point $(v_1^\varepsilon, v_2^\varepsilon)$ of L .

Now, we need to prove that the stationary point is a saddle point. Note that

$$L(v_1^\varepsilon, v_2^\varepsilon) = -\varepsilon \langle \nabla v_1^\varepsilon, \nabla v_2^\varepsilon \rangle + \frac{1}{\varepsilon} (\langle u_1, v_2^\varepsilon \rangle + \langle u_2, v_1^\varepsilon \rangle - \langle v_1^\varepsilon, v_2^\varepsilon \rangle)$$

and if $v_2 \in H^1(D)$, then

$$L(v_1^\varepsilon, v_2) = -\varepsilon \langle \nabla v_1^\varepsilon, \nabla v_2 \rangle + \frac{1}{\varepsilon} (\langle u_1, v_2 \rangle + \langle u_2, v_1^\varepsilon \rangle - \langle v_1^\varepsilon, v_2 \rangle).$$

Now, we calculate

$$\begin{aligned} L(v_1^\varepsilon, v_2^\varepsilon) - L(v_1^\varepsilon, v_2) &= -\varepsilon \langle \nabla v_1^\varepsilon, \nabla v_2^\varepsilon \rangle + \frac{1}{\varepsilon} (\langle u_1, v_2^\varepsilon \rangle + \langle u_2, v_1^\varepsilon \rangle - \langle v_1^\varepsilon, v_2^\varepsilon \rangle) \\ &\quad + \varepsilon \langle \nabla v_1^\varepsilon, \nabla v_2 \rangle - \frac{1}{\varepsilon} (\langle u_1, v_2 \rangle + \langle u_2, v_1^\varepsilon \rangle - \langle v_1^\varepsilon, v_2 \rangle) \\ &= \varepsilon \langle \nabla v_1^\varepsilon, \nabla (v_2 - v_2^\varepsilon) \rangle + \frac{1}{\varepsilon} (\langle u_1, (v_2^\varepsilon - v_2) \rangle - \langle v_1^\varepsilon, (v_2 - v_2^\varepsilon) \rangle) \\ &= \frac{1}{\varepsilon} [\varepsilon^2 \langle \nabla v_1^\varepsilon, \nabla (v_2 - v_2^\varepsilon) \rangle - \langle u_1, v_2 - v_2^\varepsilon \rangle + \langle v_1^\varepsilon, v_2 - v_2^\varepsilon \rangle]. \end{aligned}$$

From (4.7.4) with $\psi = v_2 - v_2^\varepsilon$ we get

$$L(v_1^\varepsilon, v_2^\varepsilon) = L(v_1^\varepsilon, v_2). \quad (4.7.5)$$

4.7. Saddle point formulation and algorithm

Similarly, from (4.7.3) we prove that

$$L(v_1^\varepsilon, v_2^\varepsilon) - L(v_1, v_2^\varepsilon) = 0. \quad (4.7.6)$$

The equations (4.7.5) and (4.7.6) satisfy the definition of saddle point, more specifically

$$L(v_1^\varepsilon, v_2) = L(v_1^\varepsilon, v_2^\varepsilon) = L(v_1, v_2^\varepsilon) \quad \forall v_1, v_2 \in H^1(D).$$

Next, from (4.7.1) and (4.7.3) with $\varphi = v_1^\varepsilon$ we obtain that

$$\begin{aligned} L(v_1^\varepsilon, v_2^\varepsilon) &= -\varepsilon \left(\frac{\langle u_2, v_1^\varepsilon \rangle - \langle v_1^\varepsilon, v_2^\varepsilon \rangle}{\varepsilon^2} \right) + \frac{1}{\varepsilon} (\langle u_1, v_2^\varepsilon \rangle + \langle u_2, v_1^\varepsilon \rangle - \langle v_1^\varepsilon, v_2^\varepsilon \rangle) \\ &= \frac{1}{\varepsilon} \langle u_1, v_2^\varepsilon \rangle. \end{aligned}$$

Similarly, from (4.7.1) and (4.7.4) with $\psi = v_2^\varepsilon$ we can prove that

$$L(v_1^\varepsilon, v_2^\varepsilon) = \frac{1}{\varepsilon} \langle u_2, v_1^\varepsilon \rangle.$$

□

4.7.2 Saddle point algorithm

In this subsection, we introduce another algorithm for the minimal partition problem. We write the minimal partition problem (1.4.1) as

$$\mathcal{I}_\varepsilon^{SD} = \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \sum_{1 \leq i < j \leq N} \alpha_{ij} \tilde{G}_\varepsilon(u_i, u_j) \right\} \quad (4.7.7)$$

with

$$\tilde{G}_\varepsilon(u_i, u_j) = \inf_{v_{ij} \in H^1(D)} \sup_{v_{ji} \in H^1(D)} \left[-\varepsilon \langle \nabla v_{ij}, \nabla v_{ji} \rangle + \frac{1}{\varepsilon} (\langle u_i, v_{ji} \rangle + \langle u_j, v_{ij} \rangle - \langle v_{ij}, v_{ji} \rangle) \right],$$

which implies that

$$\begin{aligned} \mathcal{I}_\varepsilon^{SD} &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \inf_{v_{ij} \in H^1(D)} \sup_{v_{ji} \in H^1(D)} \\ &\quad \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \sum_{1 \leq i < j \leq N} \alpha_{ij} \left[-\varepsilon \langle \nabla v_{ij}, \nabla v_{ji} \rangle + \frac{1}{\varepsilon} (\langle u_i, v_{ji} \rangle + \langle u_j, v_{ij} \rangle - \langle v_{ij}, v_{ji} \rangle) \right] \right\}. \end{aligned} \quad (4.7.8)$$

We use an alternating algorithm with respect to (u_1, \dots, u_N) , $(v_{12}, \dots, v_{(N-1)N})$ and $(v_{21}, \dots, v_{N(N-1)})$. The superscript k is again used to designate these vectors at iteration k .

- The stationary with respect to (v_{ij}, v_{ji}) consists in solving simultaneously

$$\begin{cases} -\varepsilon^2 \Delta v_{ij}^k + v_{ij}^k &= u_i^{k-1} & \text{in } D, \\ \partial_n v_{ij}^k &= 0 & \text{on } \partial D, \end{cases}$$

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and

$$\begin{cases} -\varepsilon^2 \Delta v_{ji}^k + v_{ji}^k &= u_j^{k-1} & \text{in } D, \\ \partial_n v_{ji}^k &= 0 & \text{on } \partial D. \end{cases}$$

In fact $v_{ij}^\varepsilon = v_i^\varepsilon$ and $v_{ji}^\varepsilon = v_j^\varepsilon$.

- The minimization with respect to (u_1, \dots, u_N) is a linear programming problem. Minimizers can be found by exploring the vertices of the polyhedron $\tilde{\mathcal{E}}_N$, that is, \mathcal{E}_N . The practical procedure is the following. Set

$$\xi_i^k = g_i + \frac{1}{\varepsilon} \sum_{j \neq i} \alpha_{ij} v_j^k.$$

At each point $x \in D$ we find an index $i(x)$ such that

$$\xi_{i(x)}^k = \min \{ \xi_1^k(x), \dots, \xi_N^k(x) \}.$$

We then set

$$u_i^k(x) = \begin{cases} 1 & \text{if } i = i(x), \\ 0 & \text{otherwise.} \end{cases}$$

Of course, there is no guarantee that the cost is decreased at each iteration. This issue is addressed in the next section.

4.8 Comparison between saddle point and primal algorithms

We focus on the relationships between the saddle point algorithm and the primal algorithm. We assume that in the primal algorithm the coefficients β_i, β_{ij} are chosen according to lemma 4.3. Let us start with the saddle point algorithm. From (4.7.7) and (4.7.8) we have the following

1. Saddle point algorithm

We have the minimal partition problem (4.7.8)

$$\mathcal{I}_\varepsilon^{SD} = \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \inf_{v_{ij} \in H^1(D)} \sup_{v_{ji} \in H^1(D)} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \sum_{1 \leq i < j \leq N} \alpha_{ij} \left[-\varepsilon \langle \nabla v_{ij}, \nabla v_{ji} \rangle + \frac{1}{\varepsilon} (\langle u_i, v_{ji} \rangle + \langle u_j, v_{ij} \rangle - \langle v_{ij}, v_{ji} \rangle) \right] \right\}. \quad (4.8.1)$$

Let us begin with the initial partition (u_1^0, \dots, u_N^0) . Take $k = 1$ (the first iteration), then the stationary with respect to (v_{ij}, v_{ji}) consists in solving simultaneously

$$\begin{cases} -\varepsilon^2 \Delta v_{ij}^1 + v_{ij}^1 &= u_i^0 & \text{in } D, \\ \partial_n v_{ij}^1 &= 0 & \text{on } \partial D, \end{cases} \quad (4.8.2)$$

and

$$\begin{cases} -\varepsilon^2 \Delta v_{ji}^1 + v_{ji}^1 &= u_j^0 & \text{in } D, \\ \partial_n v_{ji}^1 &= 0 & \text{on } \partial D. \end{cases} \quad (4.8.3)$$

4.8. Comparison between saddle point and primal algorithms

We deduce from (4.8.1), (4.8.2) and (4.8.3) that the second step of the first iteration consists in solving

$$\min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \left\langle g_i + \frac{1}{\varepsilon} \sum_{j \neq i} \alpha_{ij} v_j^1, u_i \right\rangle \right\} =: \mathcal{J}_\varepsilon^{SD^1}.$$

It follows from (4.1.8) that

$$\mathcal{J}_\varepsilon^{SD^1} = \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \left\langle g_i + \frac{1}{\varepsilon} \sum_{j \neq i} \left(\beta_i + \beta_j + \sum_{k \neq i, j} \beta_{ik} + \beta_{kj} \right) v_j^1, u_i \right\rangle \right\}. \quad (4.8.4)$$

2. Primal algorithm

We have the minimal partition problem (4.3.3)

$$\mathcal{I}_\varepsilon^P = \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \inf_{v_i \in H^1(D)} \inf_{w_{ij} \in H^1(D)} \quad (4.8.5)$$

$$\left\{ \sum_{i=1}^N \langle u_i, g_i \rangle + \sum_{i=1}^N \beta_i \left[\varepsilon \|\nabla v_i\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|v_i\|_{L^2(D)}^2 + \langle u_i, 1 - 2v_i \rangle \right) \right] \right. \\ \left. + \sum_{1 \leq i < j \leq N} \beta_{ij} \left[\varepsilon \|\nabla w_{ij}\|_{L^2(D)}^2 + \frac{1}{\varepsilon} \left(\|w_{ij}\|_{L^2(D)}^2 + \langle u_i + u_j, 1 - 2w_{ij} \rangle \right) \right] \right\}. \quad (4.8.6)$$

Let us begin with the initial partition (u_1^0, \dots, u_N^0) . Take $k = 1$, then the minimization of (4.8.5) with respect to (v_1, \dots, v_N) is straightforward. It consists in solving the boundary value problem

$$\begin{cases} -\varepsilon^2 \Delta v_i^1 + v_i^1 &= u_i^0 & \text{in } D, \\ \partial_n v_i^1 &= 0 & \text{on } \partial D. \end{cases} \quad (4.8.7)$$

The minimization of (4.8.5) with respect to $(w_{12}, \dots, w_{(N-1)N})$ is straightforward. It consists in solving the boundary value problem

$$\begin{cases} -\varepsilon^2 \Delta w_{ij}^1 + w_{ij}^1 &= (u_i + u_j)^0 & \text{in } D, \\ \partial_n w_{ij}^1 &= 0 & \text{on } \partial D, \end{cases} \quad (4.8.8)$$

with $w_{ij}^1 = v_i^1 + v_j^1$.

We deduce from (4.8.5), (4.8.7) and (4.8.8) that the second step of the first iteration consists in solving

$$\min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \left\langle g_i + \frac{\beta_i}{\varepsilon} (1 - 2v_i^1) + \frac{1}{\varepsilon} \sum_{j \neq i} \beta_{ij} (1 - 2(v_i^1 + v_j^1)), u_i \right\rangle \right\} =: \mathcal{J}_\varepsilon^{P^1}.$$

3. Comparison

Since $v_i^1 = 1 - \sum_{j \neq i} v_j^1$, then we get

$$\begin{aligned}
 \mathcal{J}_\varepsilon^{P^1} &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \left\langle g_i - \frac{\beta_i}{\varepsilon} + \frac{2\beta_i}{\varepsilon} \sum_{j \neq i} v_j^1 + \frac{1}{\varepsilon} \sum_{j \neq i} \left(-\beta_{ij} + 2\beta_{ij} \sum_{k \neq i} v_k^1 - 2\beta_{ij} v_j^1 \right), u_i \right\rangle \right\} \\
 &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \left\langle g_i - \frac{\beta_i}{\varepsilon} + \frac{2\beta_i}{\varepsilon} \sum_{j \neq i} v_j^1 - \frac{1}{\varepsilon} \sum_{j \neq i} \beta_{ij} + \frac{1}{\varepsilon} \sum_{j \neq i} \left(2\beta_{ij} \sum_{k \neq i} v_k^1 - 2\beta_{ij} v_j^1 \right), u_i \right\rangle \right\} \\
 &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle - \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \langle 1, u_i \rangle - 2\beta_i \sum_{j \neq i} \langle v_j^1, u_i \rangle \right) \right. \\
 &\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[\left\langle \sum_{j \neq i} \left(2\beta_{ij} \sum_{k \neq i} v_k^1 - 2\beta_{ij} v_j^1 \right), u_i \right\rangle - \sum_{k \neq i} \beta_{ik} \langle 1, u_i \rangle \right] \right\}.
 \end{aligned}$$

Using the fact that $\sum_{j=1}^N v_j^1 = 1$, we get

$$\begin{aligned}
 \mathcal{J}_\varepsilon^{P^1} &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle - \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \left\langle \sum_{j=1}^N v_j^1, u_i \right\rangle - 2\beta_i \sum_{j \neq i} \langle v_j^1, u_i \rangle \right) \right. \\
 &\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[2 \sum_{j \neq i} \beta_{ij} \sum_{k \neq i} \langle v_k^1, u_i \rangle - 2 \sum_{j \neq i} \beta_{ij} \langle v_j^1, u_i \rangle - \sum_{k \neq i} \beta_{ik} \left\langle \sum_{j=1}^N v_j^1, u_i \right\rangle \right] \right\} \\
 &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle - \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \sum_{j \neq i} \langle v_j^1, u_i \rangle + \beta_i \langle v_i^1, u_i \rangle - 2\beta_i \sum_{j \neq i} \langle v_j^1, u_i \rangle \right) \right. \\
 &\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[2 \sum_{k \neq i} \beta_{ik} \sum_{j \neq i} \langle v_j^1, u_i \rangle - 2 \sum_{j \neq i} \beta_{ij} \langle v_j^1, u_i \rangle - \sum_{j=1}^N \left(\sum_{k \neq i, j} \beta_{ik} + \beta_{ij} \right) \langle v_j^1, u_i \rangle \right] \right\} \\
 &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \sum_{j \neq i} \langle v_j^1, u_i \rangle - \beta_i \langle v_i^1, u_i \rangle \right) + \frac{1}{\varepsilon} \sum_{i=1}^N \left[2 \left(\sum_{j \neq i} \sum_{k \neq i, j} \beta_{ik} \langle v_j^1, u_i \rangle \right) \right. \right. \\
 &\quad \left. \left. + \sum_{j \neq i} \beta_{ij} \langle v_j^1, u_i \rangle \right) - 2 \sum_{j \neq i} \beta_{ij} \langle v_j^1, u_i \rangle - \sum_{j \neq i} \sum_{k \neq i, j} \beta_{ik} \langle v_j^1, u_i \rangle - \sum_{k \neq i} \beta_{ik} \langle v_i^1, u_i \rangle - \sum_{j=1}^N \beta_{ij} \langle v_j^1, u_i \rangle \right] \right\}.
 \end{aligned}$$

Using the fact that $u_i = 1 - \sum_{j=1}^N u_j$, we get

$$\begin{aligned}
 \mathcal{J}_\varepsilon^{P^1} &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \sum_{j \neq i} \langle v_j^1, u_i \rangle - \beta_i \langle v_i^1, 1 \rangle + \beta_i \sum_{j \neq i} \langle v_i^1, u_j \rangle \right) \right. \\
 &\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[\sum_{j \neq i} \sum_{k \neq i, j} \beta_{ik} \langle v_j^1, u_i \rangle - \sum_{k \neq i} \beta_{ik} \langle v_i^1, u_i \rangle - \sum_{j=1}^N \beta_{ij} \langle v_j^1, u_i \rangle \right] \right\}.
 \end{aligned}$$

4.9. Numerical results

Using again the fact that $u_i = 1 - \sum_{j=1}^N u_j$, we get

$$\begin{aligned}
\mathcal{J}_\varepsilon^{P^1} &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \sum_{j \neq i}^N \langle v_j^1, u_i \rangle - \beta_i \langle v_i^1, 1 \rangle + \sum_{j=1}^N \beta_i \langle v_i^1, u_j \rangle - \beta_i \langle v_i^1, u_i \rangle \right) \right. \\
&\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[\sum_{j \neq i} \sum_{k \neq i, j} \beta_{ik} \langle v_j^1, u_i \rangle - \sum_{k \neq i} \beta_{ik} \langle v_i^1, 1 \rangle + \sum_{k \neq i} \beta_{ik} \sum_{j \neq i} \langle v_i^1, u_j \rangle - \sum_{j=1}^N \beta_{ij} \langle v_j^1, u_i \rangle \right] \right\} \\
&= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \sum_{j \neq i}^N \langle v_j^1, u_i \rangle - \beta_i \langle v_i^1, 1 \rangle + \sum_{j=1}^N \beta_j \langle v_j^1, u_i \rangle - \beta_i \langle v_i^1, u_i \rangle \right) \right. \\
&\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[\sum_{j \neq i} \sum_{k \neq i, j} \beta_{ik} \langle v_j^1, u_i \rangle + \sum_{j \neq i} \sum_{k \neq j} \beta_{jk} \langle v_j^1, u_j \rangle - \sum_{k \neq i} \beta_{ik} \langle v_i^1, 1 \rangle - \sum_{j=1}^N \beta_{ij} \langle v_j^1, u_i \rangle \right] \right\} \\
&= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i \sum_{j \neq i}^N \langle v_j^1, u_i \rangle - \beta_i \langle v_i^1, 1 \rangle + \sum_{j \neq i}^N \beta_j \langle v_j^1, u_i \rangle + \beta_i \langle v_i^1, u_i \rangle - \beta_i \langle v_i^1, u_i \rangle \right) \right. \\
&\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[\sum_{j \neq i} \sum_{k \neq i, j} \beta_{ik} \langle v_j^1, u_i \rangle + \sum_{j \neq i} \sum_{k \neq i, j} \beta_{jk} \langle v_j^1, u_i \rangle - \sum_{k \neq i} \beta_{ik} \langle v_i^1, 1 \rangle \right] \right\} \\
&= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{i=1}^N \sum_{j \neq i}^N (\beta_i + \beta_j) \langle v_j^1, u_i \rangle - \frac{1}{\varepsilon} \sum_{i=1}^N \beta_i \langle v_i^1, 1 \rangle \right. \\
&\quad \left. + \frac{1}{\varepsilon} \sum_{i=1}^N \left[\sum_{j \neq i} \sum_{k \neq i, j} (\beta_{ik} + \beta_{jk}) \langle v_j^1, u_i \rangle - \sum_{k \neq i} \beta_{ik} \langle v_i^1, 1 \rangle \right] \right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{J}_\varepsilon^{P^1} &= \min_{(u_1, \dots, u_N) \in \tilde{\mathcal{E}}_N} \left\{ \sum_{i=1}^N \left\langle g_i + \frac{1}{\varepsilon} \sum_{j \neq i} \left(\beta_i + \beta_j + \sum_{k \neq i, j} \beta_{ik} + \beta_{kj} \right) v_j^1, u_i \right\rangle \right\} \\
&\quad - \frac{1}{\varepsilon} \sum_{i=1}^N \left(\beta_i + \sum_{k \neq i} \beta_{ik} \right) \langle v_i^1, 1 \rangle.
\end{aligned} \tag{4.8.9}$$

We deduce that the minimizers of the problems (4.8.4) and (4.8.9) are identical.

4.9 Numerical results

Let E_0, E_1, \dots, E_N be a given partition of D . We define $g_i, i = 1, \dots, N$, by

$$g_i = \sum_{\substack{0 \leq j \leq N \\ j \neq i}} \chi_{E_j} = 1 - \chi_{E_i}.$$

This means that, in the set E_i , $i \geq 1$, the label i is favored, whereas in the set E_0 there is no preference, or, said otherwise, no information on which label to choose. For the following example we use the finite element method.

We provide an example for the primal algorithm to show the evolution of the partition with respect to ε for $\alpha_{ij} = 1$, which is presented in Figure 4.1. The partition is as shown in Figure 4.1a. The set E_0 is the black disc, while each E_i , $i = 1, 2, 3$, is assigned to a specific color, namely red, green and blue, respectively.

In Figure 4.2 we can observe an analogous example for the saddle point algorithm where the surface tensions α_{ij} do not satisfy the condition (1.1.3). Finally, the example corresponding to Figure 4.3 is characterized by a volume constraint imposed for each phase for the primal algorithm. We note that Figure 4.3f is coherent with Figure 2.5 in [15] for $N = 3$, where in [15] the Modica-Mortola functional is used.

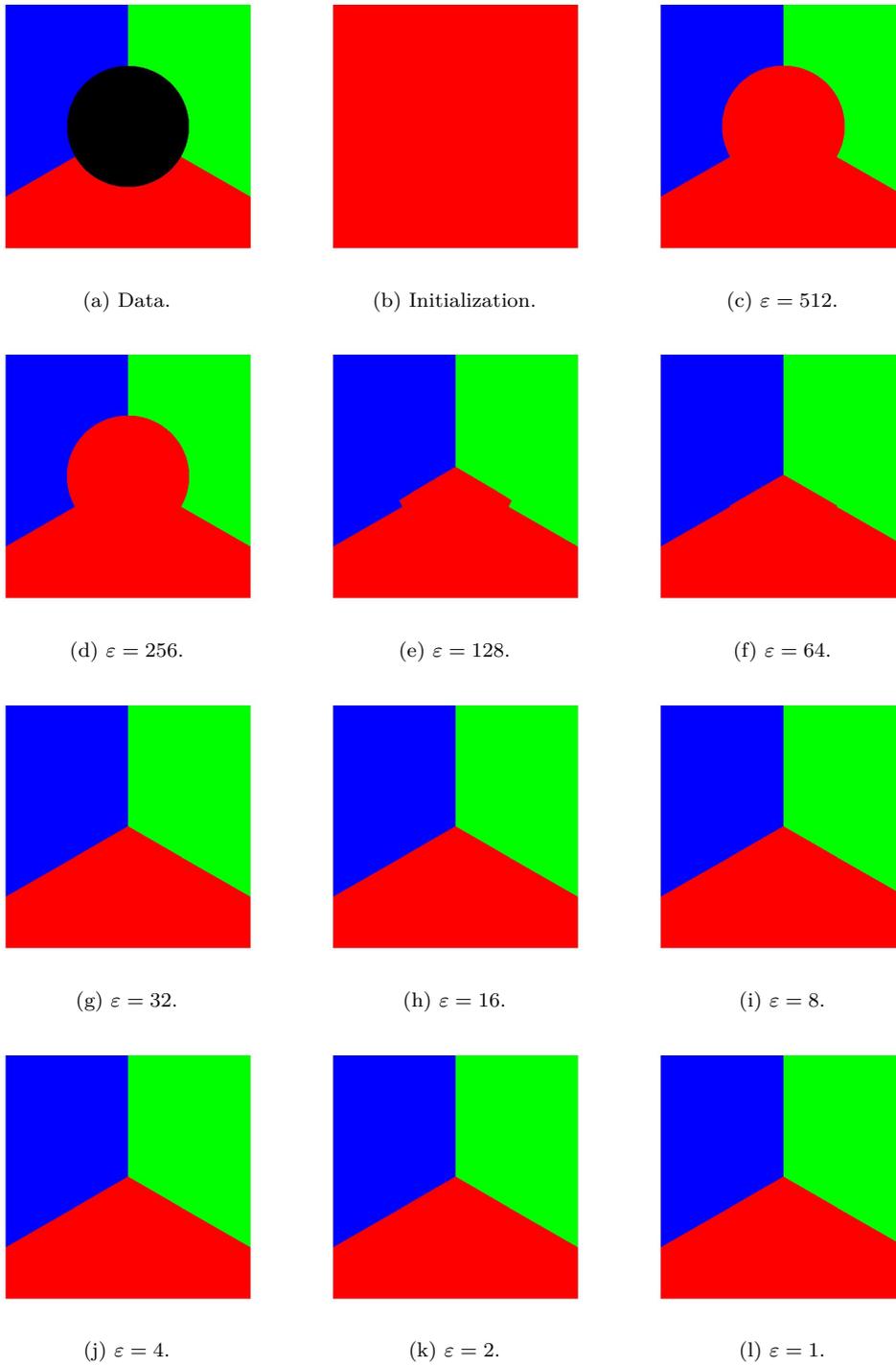
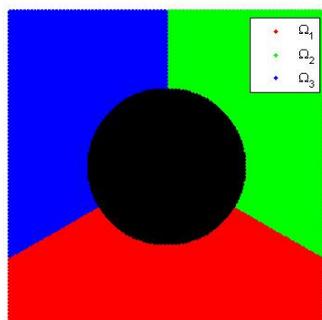
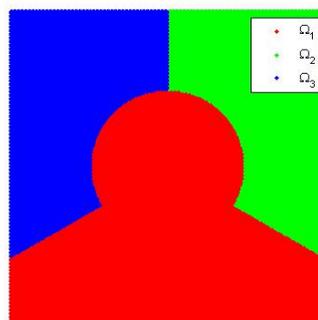


Figure 4.1: Evolution of the partition for the primal algorithm with respect ε for $\alpha_{ij} = 1$.

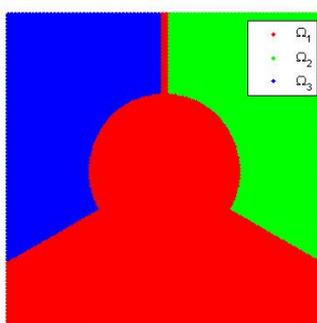
4.9. Numerical results



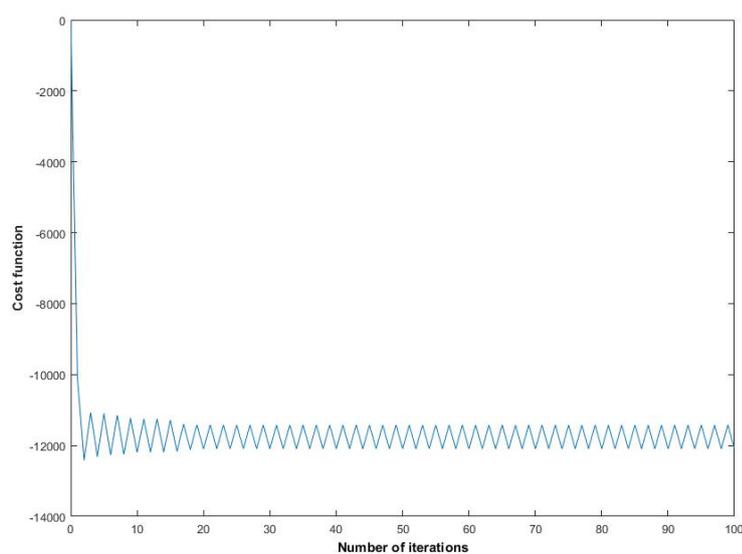
(a) Data.



(b) Initialization.

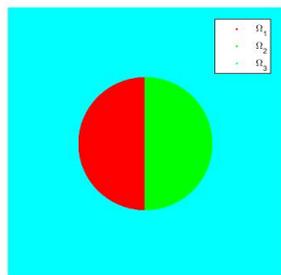


(c) Our solution.

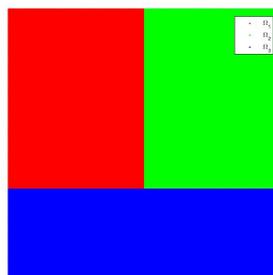


(d) The cost function.

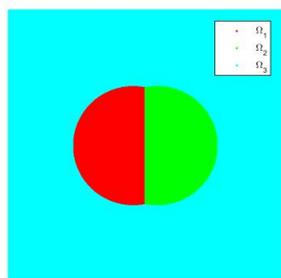
Figure 4.2: Our solution of a triple point problem with $\alpha_{23} = 100$ and $\alpha_{ij} = 1$ otherwise. Saddle point algorithm.



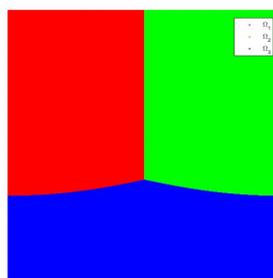
(a) Initialization.



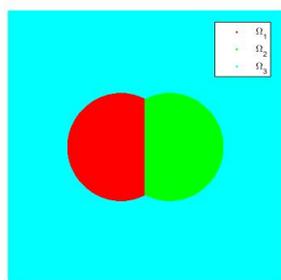
(b) Initialization.



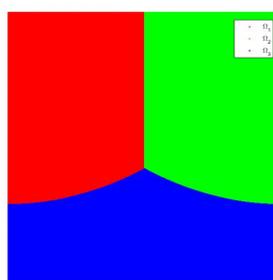
(c) $\alpha_{12} = 1/2$ and $\alpha_{ij} = 1$ otherwise.



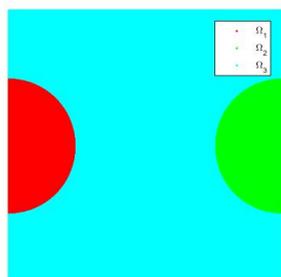
(d) $\alpha_{12} = 1/2$ and $\alpha_{ij} = 1$ otherwise.



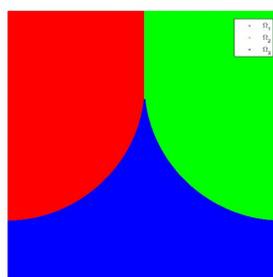
(e) $\alpha_{ij} = 1$.



(f) $\alpha_{ij} = 1$.



(g) $\alpha_{12} = 2$ and $\alpha_{ij} = 1$ otherwise.



(h) $\alpha_{12} = 2$ and $\alpha_{ij} = 1$ otherwise.

Figure 4.3: Minimal partitions with volume constraints for the primal algorithm.

Chapter 5

An algorithm based on Legendre-Fenchel duality

In this chapter we present an algorithm based on the Legendre-Fenchel duality. It requires the conditional negative definiteness of the matrix Q of the coefficients (α_{ij}) .

5.1 Concavity of the approximate interface energy

5.1.1 Condition for concavity: conditional negative definiteness

Let us begin with defining the $(N - 1) \times (N - 1)$ submatrix of $Q = (\alpha_{ij})$ by

$$\tilde{Q} = (\alpha_{ij})_{1 \leq i, j \leq N-1}$$

and defining the column vector $V = (V_i)$ by

$$V = (\alpha_{iN})_{1 \leq i \leq N-1}. \quad (5.1.1)$$

We also define

$$\bar{Q} = \tilde{Q} - \mathbb{1}V^\top - V\mathbb{1}^\top, \quad (5.1.2)$$

where $\mathbb{1} = (1, \dots, 1)^\top$.

Definition 5.1. A real symmetric $N \times N$ matrix $Q = (\alpha_{ij})$ is said to be conditionally negative semidefinite if $\sum_{i,j=1}^N \alpha_{ij} \xi_i \xi_j \leq 0$ for all $\xi = (\xi_1, \dots, \xi_N)^\top \in \mathbb{R}^N$ such that $\sum_{i=1}^N \xi_i = 0$. We denote $Q \preceq 0$.

Definition 5.2. A real symmetric $(N-1) \times (N-1)$ matrix $\bar{Q} = (\bar{\alpha}_{ij})$ is said to be negative semidefinite if $\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \tilde{\xi}_i \tilde{\xi}_j \leq 0$ for all $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_{N-1})^\top \in \mathbb{R}^{N-1}$. We denote $\bar{Q} \leq 0$.

The following result describes the relationship between conditionally negative semidefinite and negative semidefinite matrices.

Lemma 5.3. Let $Q \in S_N$. Then $Q \preceq 0$ if and only if $\bar{Q} \leq 0$.

5.1. Concavity of the approximate interface energy

Proof. Let $\xi = (\xi_1, \dots, \xi_N)^\top \in \mathbb{R}^N$ with $\sum_{i=1}^N \xi_i = 0$ and $\tilde{\xi} = (\xi_i)_{1 \leq i \leq N-1}$. We have

$$\begin{aligned} Q\xi \cdot \xi &= \left(\frac{\tilde{Q}}{V^\top} \middle| \frac{V}{0} \right) \left(\frac{\tilde{\xi}}{-\mathbf{1} \cdot \tilde{\xi}} \right) \cdot \left(\frac{\tilde{\xi}}{-\mathbf{1} \cdot \tilde{\xi}} \right) \\ &= \tilde{Q}\tilde{\xi} \cdot \tilde{\xi} - 2(\mathbf{1} \cdot \tilde{\xi})(V \cdot \tilde{\xi}) \\ &= \tilde{\xi}^\top \tilde{Q}\tilde{\xi} - 2\tilde{\xi}^\top \mathbf{1} V^\top \tilde{\xi} \\ &= \tilde{\xi}^\top (\tilde{Q} - \mathbf{1} V^\top - V \mathbf{1}^\top) \tilde{\xi}. \end{aligned}$$

From (5.1.2), we obtain

$$Q\xi \cdot \xi = \bar{Q}\tilde{\xi} \cdot \tilde{\xi}.$$

It is clear that $Q \preceq 0$ if and only if $\bar{Q} \leq 0$. □

Now, we will provide two remarks needed to prove our results.

Remark 5.1. In the case $\sum_{i=1}^N \xi_i = 1$ where $\xi = (\xi_1, \dots, \xi_N)^\top \in \mathbb{R}^N$ and $Q \in T_N$, we have the following equations:

$$\begin{aligned} Q\xi \cdot \xi &= \left(\frac{\tilde{Q}}{V^\top} \middle| \frac{V}{0} \right) \left(\frac{\tilde{\xi}}{1 - \mathbf{1} \cdot \tilde{\xi}} \right) \cdot \left(\frac{\tilde{\xi}}{1 - \mathbf{1} \cdot \tilde{\xi}} \right) \\ &= \tilde{Q}\tilde{\xi} \cdot \tilde{\xi} + 2(1 - (\mathbf{1} \cdot \tilde{\xi}))(V \cdot \tilde{\xi}) \\ &= \tilde{Q}\tilde{\xi} \cdot \tilde{\xi} - 2(\mathbf{1} \cdot \tilde{\xi})(V \cdot \tilde{\xi}) + 2V \cdot \tilde{\xi} \\ &= \tilde{\xi}^\top \tilde{Q}\tilde{\xi} - 2\tilde{\xi}^\top \mathbf{1} V^\top \tilde{\xi} + 2V \cdot \tilde{\xi} \\ &= \tilde{\xi}^\top (\tilde{Q} - \mathbf{1} V^\top - V \mathbf{1}^\top) \tilde{\xi} + 2V \cdot \tilde{\xi}. \end{aligned}$$

From (5.1.2), we get

$$Q\xi \cdot \xi = \bar{Q}\tilde{\xi} \cdot \tilde{\xi} + 2V \cdot \tilde{\xi}.$$

Remark 5.2. Let $\xi = (\xi_1, \dots, \xi_N)^\top$, $g = (g_1, \dots, g_N)^\top$, $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^N$, and $\sum_{i=1}^N \xi_i = 1$. We have the following equations:

$$\begin{aligned} g \cdot \xi &= \left(\frac{\tilde{g}}{g_N} \right) \cdot \left(\frac{\tilde{\xi}}{1 - \mathbf{1} \cdot \tilde{\xi}} \right) \\ &= \tilde{g} \cdot \tilde{\xi} + g_N(1 - \mathbf{1} \cdot \tilde{\xi}) \\ &= (\tilde{g} - g_N \mathbf{1}) \cdot \tilde{\xi} + g_N. \end{aligned}$$

For simplicity, we denote

$$\bar{g} = \tilde{g} - g_N \mathbf{1}.$$

It follows that

$$g \cdot \xi = \bar{g} \cdot \tilde{\xi} + g_N.$$

5.1.2 Sufficient condition for conditional negative semidefiniteness

Theorem 5.4. *If $N=3,4$ and if $Q \in T_N$, then $Q \preceq 0$.*

Proof. We proceed similarly to Theorem 4.4. For clarity we provide the details.

Case 1: $N = 3$. Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ with $\xi_1 + \xi_2 + \xi_3 = 0$. Then we have

$$\begin{aligned}
 \sum_{\substack{i,j=1 \\ i < j}}^3 \alpha_{ij} \xi_i \xi_j &= \alpha_{12} \xi_1 \xi_2 + \alpha_{13} \xi_1 (-\xi_1 - \xi_2) + \alpha_{23} \xi_2 (-\xi_1 - \xi_2) \\
 &= (\alpha_{12} - \alpha_{13} - \alpha_{23}) \xi_1 \xi_2 - \alpha_{13} \xi_1^2 - \alpha_{23} \xi_2^2 \\
 &= \left(\frac{\alpha_{12} - \alpha_{13} - \alpha_{23}}{2} \right) [(\xi_1 + \xi_2)^2 - \xi_1^2 - \xi_2^2] - \alpha_{13} \xi_1^2 - \alpha_{23} \xi_2^2 \\
 &= \left(\frac{\alpha_{12} - \alpha_{13} - \alpha_{23}}{2} \right) (\xi_1 + \xi_2)^2 + \left(\frac{\alpha_{23} - \alpha_{12} - \alpha_{13}}{2} \right) \xi_1^2 + \left(\frac{\alpha_{23} - \alpha_{12} - \alpha_{13}}{2} \right) \xi_2^2.
 \end{aligned}$$

Since $Q \in T_3$, then $(\alpha_{12} - \alpha_{13} - \alpha_{23})$, $(\alpha_{23} - \alpha_{12} - \alpha_{13})$ and $(\alpha_{23} - \alpha_{12} - \alpha_{13})$ are nonpositive, which implies that

$$\sum_{\substack{i,j=1 \\ i < j}}^3 \alpha_{ij} \xi_i \xi_j \leq 0.$$

Case 2: $N = 4$. Let $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{R}$ with $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$. Then we have

$$\begin{aligned}
 \sum_{\substack{i,j=1 \\ i < j}}^4 \alpha_{ij} \xi_i \xi_j &= \alpha_{12} \xi_1 \xi_2 + \alpha_{13} \xi_1 \xi_3 + \alpha_{14} \xi_1 (-\xi_1 - \xi_2 - \xi_3) + \alpha_{23} \xi_2 \xi_3 \\
 &\quad + \alpha_{24} \xi_2 (-\xi_1 - \xi_2 - \xi_3) + \alpha_{34} \xi_3 (-\xi_1 - \xi_2 - \xi_3) \\
 &= (\alpha_{12} - \alpha_{14} - \alpha_{24}) \xi_1 \xi_2 + (\alpha_{13} - \alpha_{14} - \alpha_{24}) \xi_1 \xi_3 \\
 &\quad + (\alpha_{23} - \alpha_{24} - \alpha_{34}) \xi_2 \xi_3 - \alpha_{14} \xi_1^2 - \alpha_{24} \xi_2^2 - \alpha_{34} \xi_3^2 \\
 &= \left(\frac{\alpha_{12} - \alpha_{14} - \alpha_{24}}{2} \right) [(\xi_1 + \xi_2)^2 - \xi_1^2 - \xi_2^2] + \left(\frac{\alpha_{13} - \alpha_{14} - \alpha_{34}}{2} \right) [(\xi_1 + \xi_3)^2 - \xi_1^2 - \xi_3^2] \\
 &\quad + \left(\frac{\alpha_{23} - \alpha_{24} - \alpha_{34}}{2} \right) [(\xi_2 + \xi_3)^2 - \xi_2^2 - \xi_3^2] - \alpha_{14} \xi_1^2 - \alpha_{24} \xi_2^2 - \alpha_{34} \xi_3^2 \\
 &= \left(\frac{\alpha_{12} - \alpha_{14} - \alpha_{24}}{2} \right) (\xi_1 + \xi_2)^2 + \left(\frac{\alpha_{13} - \alpha_{14} - \alpha_{34}}{2} \right) (\xi_1 + \xi_3)^2 \\
 &\quad + \left(\frac{\alpha_{23} - \alpha_{24} - \alpha_{34}}{2} \right) (\xi_2 + \xi_3)^2 + \left(\frac{\alpha_{24} + \alpha_{34} - \alpha_{12} - \alpha_{13}}{2} \right) \xi_1^2 \\
 &\quad + \left(\frac{\alpha_{14} + \alpha_{34} - \alpha_{12} - \alpha_{23}}{2} \right) \xi_2^2 + \left(\frac{\alpha_{14} + \alpha_{24} - \alpha_{13} - \alpha_{23}}{2} \right) \xi_3^2.
 \end{aligned}$$

Since $Q \in T_4$, then $(\alpha_{12} - \alpha_{14} - \alpha_{24})$, $(\alpha_{23} - \alpha_{24} - \alpha_{34})$ and $(\alpha_{23} - \alpha_{24} - \alpha_{34})$ are nonpositive.

The proof for the values $(\alpha_{24} + \alpha_{34} - \alpha_{12} - \alpha_{13})$, $(\alpha_{14} + \alpha_{34} - \alpha_{12} - \alpha_{23})$ and $(\alpha_{14} + \alpha_{24} - \alpha_{13} - \alpha_{23})$ is essentially as the proof of Theorem 4.4, except replacing nonnegative values with nonpositive values.

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Hence $\sum_{\substack{i,j=1 \\ i < j}}^4 \alpha_{ij} \xi_i \xi_j \leq 0$. □

Remark 5.3. The converse of Theorem 5.4 is false. For $N = 3$ we have the following counterexample. Consider the matrix

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}.$$

From (5.1.2), we have

$$\bar{Q} = \begin{pmatrix} -2 & -3 \\ -3 & -6 \end{pmatrix}.$$

It is easy to see that $\det(\bar{Q}) = 12$ and $\text{trace}(\bar{Q}) = -8$, which implies that $Q \not\leq 0$, but

$$\alpha_{23} > \alpha_{12} + \alpha_{13}.$$

Remark 5.4. We have a counterexample to Theorem 5.4 for $N = 5$. Consider the matrix

$$Q = \begin{pmatrix} 0 & 2 & 3 & 2 & 1 \\ 2 & 0 & 2 & 3 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 \end{pmatrix}.$$

It is clear to see that the matrix Q satisfies the triangle inequality, but \bar{Q} is not negative semidefinite, because the eigenvalues are

$$\text{Sp}(\bar{Q}) = \{-6.2886, -3.4300, -2.5700, \mathbf{0.2886}\}.$$

5.2 Dual formulation of the interface energy by Legendre-Fenchel transform

5.2.1 Dual formulation in the continuous framework

From now on, we denote $\tilde{Q} = \bar{Q}|_{\text{Im}\bar{Q}}$. Let us first of all give two technical lemmas.

Lemma 5.5. *The operator $\tilde{Q} : \text{Im}\bar{Q} \rightarrow \text{Im}\bar{Q}$ is bijective.*

Proof. For all $x \in \ker \tilde{Q}$, we have

$$\begin{cases} x \in \text{Im}\bar{Q}, \\ \tilde{Q}(x) = 0. \end{cases}$$

Since \bar{Q} is symmetric, we obtain

$$\begin{cases} x \in (\ker \tilde{Q})^\perp, \\ x \in \ker \tilde{Q}, \end{cases}$$

which implies that $x = 0$, and thus \tilde{Q} is bijective. □

5.2. Dual formulation of the interface energy by Legendre-Fenchel transform

To simplify the computations, we denote for all $u, v \in H^1(D, \mathbb{R}^N)$

$$\langle u, v \rangle_{H_\varepsilon^1} = \int_D (\varepsilon^2 \nabla u \cdot \nabla v + uv) dx, \quad (5.2.1)$$

and for all $u, v \in H^1(D, \mathbb{R}^{N-1})$

$$[u, v] = \sum_{i=1}^{N-1} \langle u_i, v_i \rangle_{H_\varepsilon^1}. \quad (5.2.2)$$

From now on we will always assume that Q^\dagger is the inverse of \tilde{Q} and we recall that some properties of the operator L_ε are given in Section 2.6.

Lemma 5.6. *Let $\tau \in H^1(D, \text{Im}\tilde{Q})$. Then $[Q^\dagger \tau, \tau] \leq 0$.*

Proof. We have

$$[Q^\dagger \tau, \tau] = \sum_{i=1}^{N-1} \int_D (\varepsilon^2 \nabla (Q^\dagger \tau)_i \cdot \nabla \tau_i + (Q^\dagger \tau)_i \tau_i) dx.$$

Let $\xi = Q^\dagger \tau \in H^1(D, \text{Im}\tilde{Q})$, then we get

$$[Q^\dagger \tau, \tau] = \sum_{i=1}^{N-1} \int_D (\varepsilon^2 \nabla (Q^\dagger(\tilde{Q}\xi))_i \cdot \nabla (\tilde{Q}\xi)_i + (Q^\dagger(\tilde{Q}\xi))_i (\tilde{Q}\xi)_i) dx,$$

which implies that

$$\begin{aligned} [Q^\dagger \tau, \tau] &= \sum_{i=1}^{N-1} \int_D (\varepsilon^2 \nabla (\tilde{Q}\xi)_i \cdot \nabla \xi_i + (\tilde{Q}\xi)_i \xi_i) dx \\ &= \int_D \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} (\varepsilon^2 \nabla \xi_i \cdot \nabla \xi_j + \xi_i \xi_j) dx \\ &= \int_D \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} [(\varepsilon \partial_x \xi_i)(\varepsilon \partial_x \xi_j) + (\varepsilon \partial_y \xi_i)(\varepsilon \partial_y \xi_j) + \xi_i \xi_j] dx. \end{aligned}$$

We deduce from the definition of negative semidefiniteness of \tilde{Q} that

$$[Q^\dagger \tau, \tau] \leq 0.$$

□

Now, we define the function $\mathcal{G}_\varepsilon : L^2(D, \mathbb{R}^N) \rightarrow \mathbb{R}$ as

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \sum_{i,j=1}^N \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle \quad \forall u \in (D, \mathbb{R}^N). \quad (5.2.3)$$

Since $L_\varepsilon \in \mathcal{L}(L^2)$ is self-adjoint and compact, we get

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \sum_{i,j=1}^N \alpha_{ij} \left\langle L_\varepsilon^{\frac{1}{2}} u_i, L_\varepsilon^{\frac{1}{2}} u_j \right\rangle.$$

5.2. Dual formulation of the interface energy by Legendre-Fenchel transform

Now, we use the fact that

$$\sum_{i=1}^N L_\varepsilon^{\frac{1}{2}} u_i = L_\varepsilon^{\frac{1}{2}} \sum_{i=1}^N u_i = L_\varepsilon^{\frac{1}{2}} \mathbf{1} = \mathbf{1},$$

because $\mathbf{1}$ is an eigenfunction of L_ε for the eigenvalue 1. From the previous results and according to Remark 5.1, we get

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon^{\frac{1}{2}} u_i, L_\varepsilon^{\frac{1}{2}} u_j \rangle + 2 \sum_{i=1}^{N-1} \langle V_i, L_\varepsilon^{\frac{1}{2}} u_i \rangle \right].$$

Using again the fact L_ε is self-adjoint, we get

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon u_i, u_j \rangle + 2 \sum_{i=1}^{N-1} \langle L_\varepsilon^{\frac{1}{2}} V_i, u_i \rangle \right].$$

As V_i is constant, then

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon u_i, u_j \rangle + 2 \sum_{i=1}^{N-1} \langle V_i, u_i \rangle \right].$$

Now, we define the function $\bar{\mathcal{G}}_\varepsilon(\tilde{u}) : L^2(D, \mathbb{R}^{N-1}) \rightarrow \mathbb{R}$ by

$$\bar{\mathcal{G}}_\varepsilon(\tilde{u}) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon \tilde{u}_i, \tilde{u}_j \rangle + 2 \sum_{i=1}^{N-1} \langle V_i, \tilde{u}_i \rangle \right], \quad \forall \tilde{u} \in L^2(D, \mathbb{R}^{N-1}). \quad (5.2.4)$$

It is clear that if $\bar{Q} = (\bar{\alpha}_{ij}) \leq 0$, then $\bar{\mathcal{G}}_\varepsilon$ is concave.

Now, we can use the Legendre-Fenchel transform for $\bar{\mathcal{G}}_\varepsilon$ to get the following theorem.

Theorem 5.7. *Let $Q \preceq 0$, $w \in L^2(D, \mathbb{R}^{N-1})$ and $\bar{w} = w - \frac{1}{\varepsilon} V$ with V defined by (5.1.1). Then*

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \begin{cases} \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] & \text{if } \bar{w}(x) \in \text{Im} \bar{Q} \text{ a.e. } x \in D, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. The sign-reversed definition of the Legendre-Fenchel transform of $\bar{\mathcal{G}}_\varepsilon$ is

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \langle w_i, \tilde{u}_i \rangle - \bar{\mathcal{G}}_\varepsilon(\tilde{u}) \quad \forall w \in L^2(D, \mathbb{R}^{N-1}) \right\}.$$

From (5.2.4), we have

$$\begin{aligned} \bar{\mathcal{G}}_\varepsilon^*(w) &= \inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \langle w_i, \tilde{u}_i \rangle - \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon \tilde{u}_i, \tilde{u}_j \rangle + 2 \sum_{i=1}^{N-1} \langle V_i, \tilde{u}_i \rangle \right] \right\} \\ &= \inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \left\langle w_i - \frac{1}{\varepsilon} V_i, \tilde{u}_i \right\rangle - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon \tilde{u}_i, \tilde{u}_j \rangle \right\}, \end{aligned}$$

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which implies that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \langle \bar{w}_i, \tilde{u}_i \rangle - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle L_\varepsilon \tilde{u}_i, \tilde{u}_j \rangle \right\}. \quad (5.2.5)$$

Since L_ε is self-adjoint, we obtain

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \langle \bar{w}_i, \tilde{u}_i \rangle - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \tilde{u}_i, L_\varepsilon \tilde{u}_j \rangle \right\}. \quad (5.2.6)$$

Here we consider two cases: either $\bar{w}_i \in H^1(D)$ or $\bar{w}_i \notin H^1(D)$.

- Case 1: If $\bar{w}_i \in H^1(D)$ for all i , then we will show that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in H^1(D)'} \left\{ \sum_{i=1}^{N-1} \langle \tilde{u}_i, \bar{w}_i \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \tilde{u}_i, L_\varepsilon \tilde{u}_j \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} \right\}.$$

For simplicity, we denote for all $\tilde{u} \in H^1(D, \mathbb{R}^{N-1})'$

$$\mathcal{F}(\tilde{u}) = \sum_{i=1}^{N-1} \langle \tilde{u}_i, \bar{w}_i \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \tilde{u}_i, L_\varepsilon \tilde{u}_j \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})}.$$

On the one hand, it is clear to see that

$$\inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \mathcal{F}(\tilde{u}) \geq \inf_{\tilde{u} \in H^1(D, \mathbb{R}^{N-1})'} \mathcal{F}(\tilde{u}). \quad (5.2.7)$$

On the other hand, according to Corollary 2.25 and since \mathcal{F} is continuous on $H^1(D, \mathbb{R}^{N-1})'$. We have

$$\inf_{\tilde{u} \in L^2(D, \mathbb{R}^{N-1})} \mathcal{F}(\tilde{u}) = \inf_{\tilde{u} \in H^1(D, \mathbb{R}^{N-1})'} \mathcal{F}(\tilde{u}).$$

In this case, the problem (5.2.6) becomes

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in H^1(D, \mathbb{R}^{N-1})'} \left\{ \sum_{i=1}^{N-1} \langle \tilde{u}_i, \bar{w}_i \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \tilde{u}_i, L_\varepsilon \tilde{u}_j \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} \right\},$$

As L_ε is bijective from $H^1(D, \mathbb{R}^{N-1})'$ into $H^1(D, \mathbb{R}^{N-1})$, then there exists $h_i \in H^1(D, \mathbb{R}^{N-1})'$ such that $\bar{w}_i = L_\varepsilon h_i$. It follows that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in H^1(D, \mathbb{R}^{N-1})'} \left\{ \sum_{i=1}^{N-1} \langle \tilde{u}_i, L_\varepsilon h_i \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \tilde{u}_i, L_\varepsilon \tilde{u}_j \rangle_{H^1(D, \mathbb{R}^{N-1})', H^1(D, \mathbb{R}^{N-1})} \right\},$$

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From (2.6.4) and the fact that L_ε is an isomorphism from $H^1(D, \mathbb{R}^{N-1})'$ into $H^1(D, \mathbb{R}^{N-1})$, we obtain

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\hat{u} \in H^1(D, \mathbb{R}^{N-1})'} \left\{ \sum_{i=1}^{N-1} \int_D (\varepsilon^2 \nabla \hat{u}_i \cdot \nabla \bar{w}_i + \hat{u}_i \bar{w}_i) dx - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \int_D (\varepsilon^2 \nabla \hat{u}_i \cdot \nabla \hat{u}_j + \hat{u}_i \hat{u}_j) dx \right\}.$$

Moreover, from (5.2.1) we have

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\hat{u} \in H^1(D, \mathbb{R}^{N-1})'} \left\{ \sum_{i=1}^{N-1} \langle \bar{w}_i, \hat{u}_i \rangle_{H_\varepsilon^1} - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \hat{u}_i, \hat{u}_j \rangle_{H_\varepsilon^1} \right\},$$

we rewrite as

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\psi \in H^1(D, \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \langle \bar{w}_i, \psi_i \rangle_{H_\varepsilon^1} - \frac{1}{2\varepsilon} \sum_{j=1}^{N-1} \langle (\bar{Q}\psi)_j, \psi_j \rangle_{H_\varepsilon^1} \right\}.$$

From (5.2.2), we obtain

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\psi \in H^1(D, \mathbb{R}^{N-1})} \left\{ [\bar{w}, \psi] - \frac{1}{2\varepsilon} [\bar{Q}\psi, \psi] \right\}. \quad (5.2.8)$$

Here, there are two cases depending on whether $\bar{w} \in \text{Im} \bar{Q}$ or not:

- Case 1-1: $\bar{w}(x) \in \text{Im} \bar{Q}$ a.e. $x \in D$. By using Gauss reduction for (5.2.8) and Lemma 5.5, we obtain

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\psi \in H^1(D, \mathbb{R}^{N-1})} \left\{ \left[-\frac{\varepsilon}{2} Q^\dagger \left(\frac{1}{\varepsilon} \bar{Q}\psi - \bar{w} \right), \frac{1}{\varepsilon} \bar{Q}\psi - \bar{w} \right] + \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}. \quad (5.2.9)$$

Using Lemma 5.6, we conclude that a solution of (5.2.9) is $\psi^* \in H^1(D, \mathbb{R}^{N-1})$ such that

$$\frac{1}{\varepsilon} \bar{Q}\psi^* - \bar{w} = 0 \quad (\text{i.e. } \psi^* = \varepsilon Q^\dagger \bar{w}).$$

Equation (5.2.9) becomes now

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}]. \quad (5.2.10)$$

- Case 1-2: There exists $\mathcal{W} \subset D$ such that $\bar{w}(x) \notin \text{Im} \bar{Q}$ for all $x \in \mathcal{W}$ and $|\mathcal{W}| > 0$. It is straightforward that

$$\bar{w}(x) \notin (\text{Ker} \bar{Q})^\perp \quad \forall x \in \mathcal{W} \quad (5.2.11)$$

since the matrix \bar{Q} is symmetric. Take in this case

$$\psi^1 = -P_{\text{Ker} \bar{Q}} \bar{w} \in H^1(D, \text{Ker} \bar{Q}), \quad (5.2.12)$$

where $P_{\text{Ker} \bar{Q}} \bar{w}$ is the orthogonal projection of \bar{w} onto $\text{Ker} \bar{Q}$.

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Now, we will show that

$$[\bar{w}, \psi^k] - \frac{1}{2\varepsilon} [\bar{Q}\psi^k, \psi^k] \xrightarrow{k \rightarrow \infty} -\infty, \quad (5.2.13)$$

where $\psi^k = -k\psi^1$, $k > 0$.

First, let q_1, \dots, q_M be an orthonormal basis of $\text{Ker}\bar{Q}$. Then we have

$$P_{\text{Ker}\bar{Q}}\bar{w} = \sum_{i=1}^M (q_i \cdot \bar{w}) q_i,$$

which implies that

$$\partial_j P_{\text{Ker}\bar{Q}}\bar{w} = \sum_{i=1}^M (q_i \cdot \partial_j \bar{w}) q_i.$$

It follows that

$$P_{\text{Ker}\bar{Q}}\bar{w} \cdot \bar{w} = \sum_{i=1}^M (q_i \cdot \bar{w})(q_i \cdot \bar{w}) = \sum_{i=1}^M (q_i \cdot \bar{w})^2 \geq 0. \quad (5.2.14)$$

Moreover,

$$\begin{aligned} \nabla P_{\text{Ker}\bar{Q}}\bar{w} \cdot \nabla \bar{w} &= \sum_{j=1}^2 \partial_j P_{\text{Ker}\bar{Q}}\bar{w} \cdot \partial_j \bar{w} \\ &= \sum_{j=1}^2 \sum_{i=1}^M (q_i \cdot \partial_j \bar{w})(q_i \cdot \partial_j \bar{w}) \geq 0. \end{aligned} \quad (5.2.15)$$

We deduce from (5.2.11), (5.2.12), (5.2.14) and (5.2.15) that the first term in (5.2.13) satisfies

$$[\bar{w}, \psi^k] = -k \sum_{i=1}^{N-1} \int_D (\varepsilon^2 \nabla \bar{w}_i \cdot \nabla \psi_i^1 + \bar{w}_i \psi_i^1) dx < 0.$$

Hence,

$$[\bar{w}, \psi^k] \xrightarrow{k \rightarrow \infty} -\infty. \quad (5.2.16)$$

Now, we take the second term in (5.2.13)

$$[\bar{Q}\psi^k, \psi^k] = k^2 \sum_{i=1}^{N-1} \int_D (\varepsilon^2 \nabla (\bar{Q}\psi^1)_i \cdot \nabla \psi_i^1 + (\bar{Q}\psi^1)_i \psi_i^1) dx.$$

By the definition of ψ^1 in (5.2.12), it is clear to see that

$$\begin{aligned} [\bar{Q}\psi^k, \psi^k] &= k^2 \sum_{i=1}^{N-1} \int_D [(\varepsilon^2 \nabla (\bar{Q}P_{\text{Ker}\bar{Q}}\bar{w})_i \cdot \nabla (P_{\text{Ker}\bar{Q}}\bar{w})_i + (\bar{Q}P_{\text{Ker}\bar{Q}}\bar{w})_i (P_{\text{Ker}\bar{Q}}\bar{w})_i)] dx \\ &= 0. \end{aligned}$$

We deduce from (5.2.8), (5.2.16), and (5.2.17) that

$$\bar{G}_\varepsilon^*(w) = -\infty. \quad (5.2.17)$$

5.2. Dual formulation of the interface energy by Legendre-Fenchel transform

- Case 2: There exists i_0 such that $\bar{w}_{i_0} \notin H^1(D)$, then there exists a sequence $(\eta^k) \in \mathcal{D}(D)$ such that

$$\int_D \bar{w}_{i_0} \operatorname{div} \eta^k \rightarrow -\infty, \|\eta^k\|_{L^2(D)} \leq 1. \quad (5.2.18)$$

Here, we take

$$\tilde{u}_i^k = \begin{cases} \operatorname{div} \eta^k & \text{if } i = i_0, \\ 0 & \text{otherwise,} \end{cases}$$

and by continuity of L_ε we have

$$\|L_\varepsilon \tilde{u}_i^k\|_{H^1(D)} \leq C \|\tilde{u}_i^k\|_{H^1(D)'} \leq C \|\eta^k\|_{L^2(D)} \leq C,$$

where C is a positive constant independent of k . From this, we infer that

$$\left| \int_D L_\varepsilon \tilde{u}_i^k \tilde{u}_j^k \right| = \left| \int_D \tilde{u}_j^k L_\varepsilon \tilde{u}_i^k \right| \leq \|\tilde{u}_j^k\|_{H^1(D)'} \|L_\varepsilon \tilde{u}_i^k\|_{H^1(D)} \leq C. \quad (5.2.19)$$

Moreover, the second term in (5.2.5) is bounded. From (5.2.18) and (5.2.19), we infer that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = -\infty. \quad (5.2.20)$$

□

Now, we take the concave biconjugate of $\bar{\mathcal{G}}_\varepsilon(\tilde{u})$ is defined by

$$\bar{\mathcal{G}}_\varepsilon^{**}(\tilde{u}) = \inf_{\bar{w} \in L^2(D, \mathbb{R}^{N-1})} \langle \bar{w}, \tilde{u} \rangle - \bar{\mathcal{G}}_\varepsilon^*(w).$$

From (5.2.10), (5.2.1) and (5.2.20) we have

$$\bar{\mathcal{G}}_\varepsilon^{**}(\tilde{u}) = \inf_{\bar{w} \in H^1(D, \operatorname{Im} \bar{Q})} \left\{ \langle \bar{w}, \tilde{u} \rangle - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}. \quad (5.2.21)$$

Corollary 5.8. *If $Q \preceq 0$ and $\tilde{u} \in L^2(D, \mathbb{R}^{N-1})$, then*

$$\bar{\mathcal{G}}_\varepsilon(\tilde{u}) = \inf_{\bar{w} \in H^1(D, \operatorname{Im} \bar{Q})} \left\{ \langle \bar{w}, \tilde{u} \rangle - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}.$$

Proof. Since $\bar{\mathcal{G}} : L^2(D, \mathbb{R}^{N-1}) \rightarrow \mathbb{R}$ is a closed concave proper function, then by using the Corollary of Fenchel-Moreau-Rockafellar theorem, Corollary 2.42, we have

$$\bar{\mathcal{G}}_\varepsilon(\tilde{u}) = \bar{\mathcal{G}}_\varepsilon^{**}(\tilde{u}).$$

From (5.2.21) we have

$$\bar{\mathcal{G}}_\varepsilon(\tilde{u}) = \inf_{\bar{w} \in H^1(D, \operatorname{Im} \bar{Q})} \left\{ \langle \bar{w}, \tilde{u} \rangle - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}.$$

□

We have the following corollary.

Corollary 5.9. *If $Q \preceq 0$ and $u \in \tilde{\mathcal{E}}_N$, then*

$$\mathcal{G}_\varepsilon(u) = \inf_{\bar{w} \in H^1(D, \operatorname{Im} \bar{Q})} \left\{ \langle \bar{w}, \tilde{u} \rangle - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\},$$

where $\tilde{u} = (u_1, \dots, u_{N-1})$.

5.2.2 Dual formulation in the discrete framework

We define the function $\mathcal{G}_\varepsilon : \mathbb{R}^{p \times N} \rightarrow \mathbb{R}$ as

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \sum_{i,j=1}^N \alpha_{ij} M(L_\varepsilon u_i) \cdot u_j \quad \forall u \in \mathbb{R}^{p \times N},$$

where p is the number of pixels, the dot (\cdot) is the standard dot product of \mathbb{R}^N , $L_\varepsilon = (\varepsilon^2 K + M)^{-1} M$ such that M is the global mass matrix and K the global stiffness matrix. We have

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \alpha_{ij} M(L_\varepsilon u_i) \cdot u_j + \sum_{i=1}^{N-1} \alpha_{iN} M(L_\varepsilon u_i) \cdot u_N + \sum_{j=1}^{N-1} \alpha_{jN} M(L_\varepsilon u_N) \cdot u_j \right].$$

Since L_ε is symmetric and $u_N = \mathbb{1} - \sum_{i=1}^{N-1} u_i$, then we get

$$\begin{aligned} \mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} & \left[\sum_{i,j=1}^{N-1} \alpha_{ij} M(L_\varepsilon u_i) \cdot u_j + \sum_{i=1}^{N-1} \alpha_{iN} M(L_\varepsilon u_i) \cdot \left(\mathbb{1} - \sum_{j=1}^{N-1} u_j \right) \right. \\ & \left. + \sum_{j=1}^{N-1} \alpha_{jN} M(L_\varepsilon u_j) \cdot \left(\mathbb{1} - \sum_{i=1}^{N-1} u_i \right) \right], \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} & \left[\sum_{i,j=1}^{N-1} \alpha_{ij} M(L_\varepsilon u_i) \cdot u_j + \sum_{i,j=1}^{N-1} \alpha_{iN} M(L_\varepsilon u_i) \cdot u_j \right. \\ & \left. + \sum_{i,j=1}^{N-1} \alpha_{jN} M(L_\varepsilon u_i) \cdot u_j + 2 \sum_{i=1}^{N-1} \alpha_{iN} M(L_\varepsilon u_i) \cdot \mathbb{1} \right] \\ = \frac{1}{2\varepsilon} & \left[\sum_{i,j=1}^{N-1} (\alpha_{ij} - \alpha_{iN} - \alpha_{jN}) M(L_\varepsilon u_i) \cdot u_j + 2 \sum_{i=1}^{N-1} \alpha_{iN} M(L_\varepsilon \mathbb{1}) \cdot u_i \right]. \end{aligned}$$

For simplicity we denote $\bar{\alpha}_{ij} = \alpha_{ij} - \alpha_{iN} - \alpha_{jN}$. It follows that

$$\mathcal{G}_\varepsilon(u) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} M(L_\varepsilon u_i) \cdot u_j + 2 \sum_{i=1}^{N-1} \alpha_{iN} M \mathbb{1} \cdot u_i \right].$$

Now, we define the function $\bar{\mathcal{G}}_\varepsilon(\tilde{u}) : \mathbb{R}^{p \times (N-1)} \rightarrow \mathbb{R}$ by

$$\bar{\mathcal{G}}_\varepsilon(\tilde{u}) = \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} M(L_\varepsilon \tilde{u}_i) \cdot \tilde{u}_j + 2 \sum_{i=1}^{N-1} \alpha_{iN} M \mathbb{1} \cdot \tilde{u}_i \right], \quad \forall \tilde{u} \in \mathbb{R}^{p \times (N-1)}. \quad (5.2.22)$$

It is clear that if $\bar{Q} = (\bar{\alpha}_{ij}) \leq 0$, then $\bar{\mathcal{G}}$ is concave.

Now, we can use the sign-reversed definition of the Legendre-Fenchel transform to get

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\tilde{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} M w_i \cdot \tilde{u}_i - \bar{\mathcal{G}}_\varepsilon(\tilde{u}) \right\}.$$

5.2. Dual formulation of the interface energy by Legendre-Fenchel transform

From (5.2.22), we obtain

$$\begin{aligned}\bar{\mathcal{G}}_\varepsilon^*(w) &= \inf_{\tilde{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} M w_i \cdot \tilde{u}_i - \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} M(L_\varepsilon \tilde{u}_i) \cdot \tilde{u}_j + 2 \sum_{i=1}^{N-1} \alpha_{iN} M \mathbb{1} \cdot \tilde{u}_i \right] \right\} \\ &= \inf_{\tilde{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} M \left(w_i - \frac{\alpha_{iN}}{\varepsilon} \mathbb{1} \right) \cdot \tilde{u}_i - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} M(L_\varepsilon \tilde{u}_i) \cdot \tilde{u}_j \right\}.\end{aligned}$$

For simplify we denote

$$\bar{w}_i = w_i - \frac{\alpha_{iN}}{\varepsilon} \mathbb{1}.$$

It follows that

$$\mathcal{G}_\varepsilon^*(w) = \inf_{\tilde{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} M \bar{w}_i \cdot \tilde{u}_i - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} M(L_\varepsilon \tilde{u}_i) \cdot \tilde{u}_j \right\}.$$

As M is symmetric, we obtain

$$\mathcal{G}_\varepsilon^*(w) = \inf_{\tilde{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} M \tilde{u}_i \cdot \bar{w}_i - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} (L_\varepsilon \tilde{u}_i) \cdot M \tilde{u}_j \right\}. \quad (5.2.23)$$

By the definition of $\hat{u}_i := L_\varepsilon \tilde{u}_i$, we have

$$M \tilde{u}_i = (\varepsilon^2 K + M) \hat{u}_i. \quad (5.2.24)$$

It follows that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\hat{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} [\varepsilon^2 K \hat{u}_i \cdot \bar{w}_i + M \hat{u}_i \cdot \bar{w}_i] - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} [\varepsilon^2 K \hat{u}_i \cdot \hat{u}_j + M \hat{u}_i \cdot \hat{u}_j] \right\}.$$

To simplify the computations, we denote

$$\langle u_i, v_i \rangle_\varepsilon = \varepsilon^2 K u_i \cdot v_j + M u_i \cdot v_i.$$

It follows that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\hat{u} \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} \langle \hat{u}_i, \bar{w} \rangle_\varepsilon - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \hat{u}_i, \hat{u}_j \rangle_\varepsilon \right\}.$$

As L_ε is invertible, we obtain

$$\begin{aligned}\bar{\mathcal{G}}_\varepsilon^*(w) &= \inf_{\psi \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} \langle \psi_i, \bar{w} \rangle_\varepsilon - \frac{1}{2\varepsilon} \sum_{i,j=1}^{N-1} \bar{\alpha}_{ij} \langle \psi_i, \psi_j \rangle_\varepsilon \right\} \\ &= \inf_{\psi \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} \langle \psi_i, \bar{w} \rangle_\varepsilon - \frac{1}{2\varepsilon} \sum_{i=1}^{N-1} \left\langle \sum_{j=1}^{N-1} \bar{\alpha}_{ij} \psi_i, \psi_j \right\rangle_\varepsilon \right\} \\ &= \inf_{\psi \in \mathbb{R}^{p \times (N-1)}} \left\{ \sum_{i=1}^{N-1} \langle \psi_i, \bar{w} \rangle_\varepsilon - \frac{1}{2\varepsilon} \sum_{i=1}^{N-1} \langle (\bar{Q}\psi)_j, \psi_j \rangle_\varepsilon \right\}.\end{aligned}$$

5.2. Dual formulation of the interface energy by Legendre-Fenchel transform

For simplicity, we denote

$$[u, v]_\varepsilon = \sum_{i=1}^{N-1} \langle u, v \rangle_\varepsilon.$$

It follows that

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\psi \in \mathbb{R}^{p \times (N-1)}} \left\{ [\bar{w}, \psi]_\varepsilon - \frac{1}{2\varepsilon} [\bar{Q}\psi, \psi]_\varepsilon \right\}. \quad (5.2.25)$$

Here, there are two subcases depending on whether $\bar{w} \in \text{Im}\bar{Q}$ or not:

- Case 1-1: $\bar{w}_i \in \text{Im}\bar{Q}$ for all i . By using Gauss reduction for (5.2.25) and Lemma 5.5, we obtain

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \inf_{\psi \in \mathbb{R}^{p \times (N-1)}} \left\{ \left[-\frac{\varepsilon}{2} Q^\dagger \left(\frac{1}{\varepsilon} \tilde{Q}\psi - \bar{w} \right), \frac{1}{\varepsilon} \tilde{Q}\psi - \bar{w} \right] + \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}]_\varepsilon \right\}. \quad (5.2.26)$$

Using the discrete counterpart of Lemma (5.6), we conclude that the solution of (5.2.26) is $\psi^* \in \mathbb{R}^{p \times (N-1)}$ such that

$$\frac{1}{\varepsilon} \tilde{Q}\psi^* - \bar{w} = 0 \quad (\text{i.e. } \psi^* = \varepsilon Q^\dagger \bar{w}).$$

Equation (5.2.26) becomes now

$$\bar{\mathcal{G}}_\varepsilon^*(w) = \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}]_\varepsilon. \quad (5.2.27)$$

- Case 1-2: There exists i such that $\bar{w}_i \notin \text{Im}\bar{Q}$. Take in this case

$$\psi^1 = -P_{\text{Ker}\bar{Q}} \bar{w} \in \mathbb{R}^{p \times (N-1)},$$

where $P_{\text{Ker}\bar{Q}} \bar{w}$ is the row-wise orthogonal projection of \bar{w} on $\text{Ker}\bar{Q}$. By using the same technique as in 5.7, we get

$$\bar{\mathcal{G}}_\varepsilon^*(w) = -\infty. \quad (5.2.28)$$

Now, we take the concave biconjugate of $\bar{\mathcal{G}}_\varepsilon(\tilde{u})$ is defined by

$$\bar{\mathcal{G}}_\varepsilon^{**}(\tilde{u}) = \inf_{\bar{w} \in \mathbb{R}^{p \times (N-1)}} \{ M\bar{w} \cdot \tilde{u} - \bar{\mathcal{G}}_\varepsilon^*(\bar{w}) \}.$$

From (5.2.27) and (5.2.28), we obtain

$$\bar{\mathcal{G}}_\varepsilon^{**}(\tilde{u}) = \inf_{\bar{w} \in (\text{Im}\bar{Q})^p} \left\{ M\bar{w} \cdot \tilde{u} - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}]_\varepsilon \right\}.$$

Using the discrete counterpart of Corollary 5.9, we have

$$\bar{\mathcal{G}}_\varepsilon(\tilde{u}) = \bar{\mathcal{G}}_\varepsilon^{**}(\tilde{u}) = \inf_{\bar{w} \in (\text{Im}\bar{Q})^p} \left\{ M\bar{w} \cdot \tilde{u} - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}]_\varepsilon \right\}.$$

The above result allows us to get

$$\mathcal{G}_\varepsilon(u) = \inf_{\bar{w} \in (\text{Im}\bar{Q})^p} \left\{ M\bar{w} \cdot \tilde{u} - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}]_\varepsilon \right\},$$

where $\tilde{u} = (u_1, \dots, u_{N-1})$.

We shall now describe the algorithm.

5.3 Algorithm

We have the minimal partition problem (1.4.1)

$$\mathcal{I}_\varepsilon^C = \min_{(u_1, \dots, u_N) \in \mathcal{E}_N} \left\{ \sum_{i=1}^N \langle g_i, u_i \rangle + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle \right\}. \quad (5.3.1)$$

We deduce from Corollary 5.9, Remark 5.2 and (5.3.1) that

$$\mathcal{I}_\varepsilon^C = \min_{\substack{\sum_{i=1}^{N-1} \tilde{u}_i \leq 1 \\ \tilde{u}_i \geq 0}} \inf_{\bar{w} \in H^1(D, \text{Im} \tilde{Q})} \left\{ \langle \bar{g} + \bar{w}, \tilde{u} \rangle + g_N - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}. \quad (5.3.2)$$

For this, we use an alternating minimization algorithm with respect to the two $(N-1)$ -tuples of variables $(\tilde{u}_1, \dots, \tilde{u}_{N-1})$ and $(\bar{w}_1, \dots, \bar{w}_{N-1})$.

On the one hand, we have the minimization with respect to \bar{w} :

$$\inf_{\bar{w} \in H^1(D, \text{Im} \tilde{Q})} \left\{ \langle \bar{w}, \tilde{u} \rangle - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}.$$

We deduce from (2.6.5) that this is equal to

$$\inf_{\bar{w} \in H^1(D, \text{Im} \tilde{Q})} \left\{ [\bar{w}, L_\varepsilon \tilde{u}] - \frac{\varepsilon}{2} [Q^\dagger \bar{w}, \bar{w}] \right\}.$$

By using the same technique as in (5.2.9), we get the unique minimizer

$$\bar{w} = \frac{1}{\varepsilon} \tilde{Q} L_\varepsilon \tilde{u}. \quad (5.3.3)$$

On the other hand, we can extend the minimization with respect to the $(N-1)$ -tuple of variables $(\tilde{u}_1, \dots, \tilde{u}_{N-1})$ to the N -tuple of variables (u_1, \dots, u_N) . We obtain

$$\min_{\substack{\sum_{i=1}^{N-1} \tilde{u}_i \leq 1 \\ \tilde{u}_i \geq 0}} \langle \zeta, \tilde{u} \rangle = \min_{\substack{\sum_{i=1}^N u_i = 1 \\ u_i \geq 0}} \langle (\zeta, 0), u \rangle, \quad (5.3.4)$$

where $\zeta = \bar{g} + \bar{w}$.

Now, we arrive at the following alternating algorithm.

1. The equation (5.3.3) allows us to find the minimization with respect to $(\bar{w}_1, \dots, \bar{w}_{N-1})$. It consists in solving the boundary value problems

$$\begin{cases} -\varepsilon^2 \Delta v_i^k + v_i^k &= \tilde{u}_i^{k-1} & \text{in } D, \\ \partial_n v_i^k &= 0 & \text{on } \partial D, \end{cases} \quad (5.3.5)$$

and setting $\bar{w}^k = \frac{1}{\varepsilon} \tilde{Q} v^k$.

2. Thanks to (5.3.4), we realize the minimization with respect to (u_1, \dots, u_N) . Set

$$\zeta_i^k = \bar{w}_i^k + \bar{g}_i^k.$$

At each point $x \in D$, we find an index $j(x)$ such that

$$\zeta_{j(x)}^k = \min \{ \zeta_1^k(x), \dots, \zeta_{N-1}^k(x), 0 \}.$$

We then set

$$u_i^k(x) = \begin{cases} 1 & \text{if } j = j(x), \\ 0 & \text{otherwise.} \end{cases}$$

5.4. Numerical examples

(α_{ij})	1, 1, 0.5	1, 1, 1	1, 1, 2
(β_S)	0.75, 0.25, 0.25	0.5, 0.5, 0.5	0, 1, 1
Eigenvalues of \bar{Q}	-2.2071, -0.7929	-3, -1	-5.2361, -0.7639
Primal algorithm			
Figure	5.1d	5.1f	5.1h
Elapsed time	37 s	32 s	51 s
Cost function	-2.0873e+05	-2.0867e+05	-2.0857e+05
Legendre conjugate algorithm			
Figure	5.1d	5.1f	5.1h
Elapsed time	37 s	31 s	50 s
Cost function	-2.0873e+05	-2.0867e+05	-2.0857e+05

Table 5.1: Triple point: comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) .

5.4 Numerical examples

In the following examples we use the finite difference method (FDM) with the fast Fourier transformation (FFT) (see Appendix A).

In Figure 5.1 and Table 5.1, we compare the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) on the triple point problem.

Figure 5.2 and Table 5.2 concern the comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) on the quadruple point problem (case 1). We also compare the primal algorithm and the Legendre conjugate algorithm on the quadruple point problem (case 2) in Figure 5.3 and Table 5.3, for the same values of (α_{ij}) as in case 1.

The primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) on the Steiner tree problem with 4 phases are discussed in Figure 5.4 and Table 5.4.

In Figure 5.5 and Table 5.5, we compare the primal dual algorithm and the Legendre conjugate algorithm for the same values of (α_{ij}) on the Steiner tree problem with 5 phases.

Figure 5.6 provides an example for the Legendre conjugate algorithm for different values of (α_{ij}) on the Steiner tree problem with 5 phases.

We note that the Figures 5.6c and 5.4d are coherent with the Figures 2.6 and 2.7, respectively, in [4], where in [4] the Modica-Mortola functional is used.

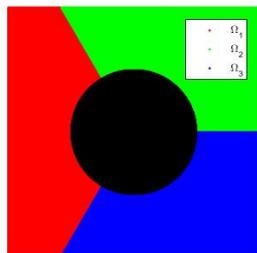
5.5 Image classification

5.5.1 Classification of greyscale images

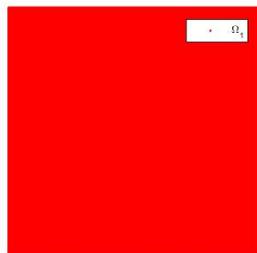
Let $f \in L^\infty(D, [0, 1])$ be a given image with N grey levels $c_1, \dots, c_N \in [0, 1]$. We consider the piecewise constant image

$$w = \sum_{i=1}^N u_i c_i$$

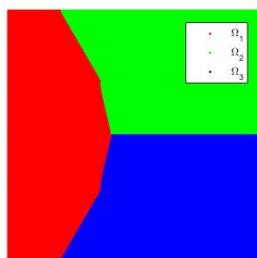
5.5. Image classification



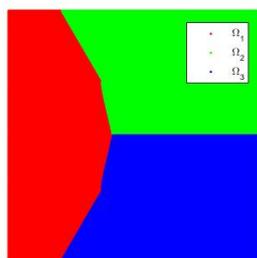
(a) Data.



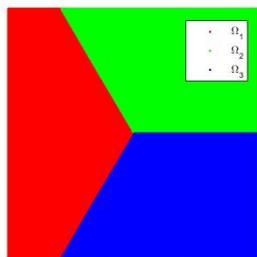
(b) Initialization.



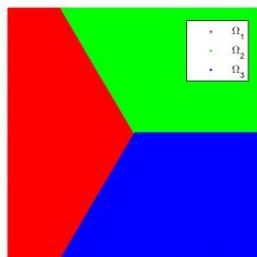
(c) Primal algorithm for $\alpha_{23}=1/2$ and $\alpha_{ij}=1$ otherwise.



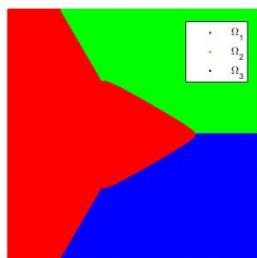
(d) Legendre conjugate algorithm for $\alpha_{23}=1/2$ and $\alpha_{ij}=1$ otherwise.



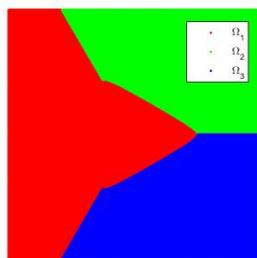
(e) Primal algorithm for $\alpha_{ij} = 1$.



(f) Legendre conjugate algorithm for $\alpha_{ij} = 1$.

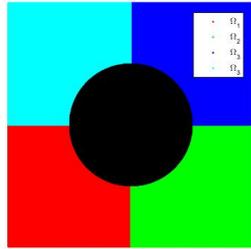


(g) Primal algorithm for $\alpha_{23}=2$ and $\alpha_{ij}=1$ otherwise.

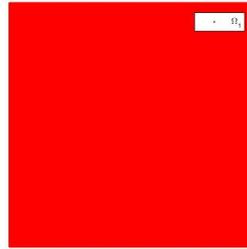


(h) Legendre conjugate algorithm for $\alpha_{23}=2$ and $\alpha_{ij}=1$ otherwise.

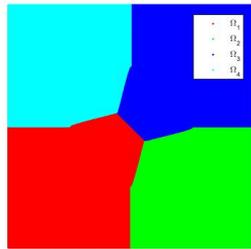
Figure 5.1: Triple point: comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) (see Table 5.1 for more details).



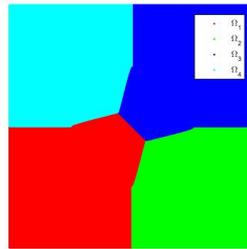
(a) Data.



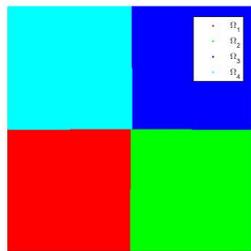
(b) Initialization.



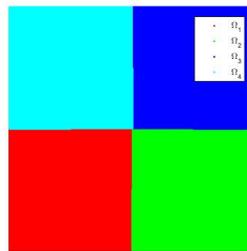
(c) Primal algorithm for $\alpha_{ij} = 1$.



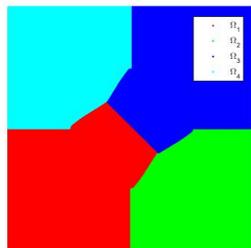
(d) Legendre conjugate algorithm for $\alpha_{ij} = 1$.



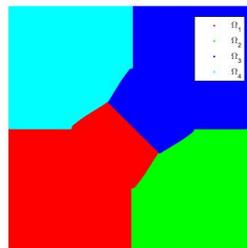
(e) Primal algorithm for $\alpha_{12} = \alpha_{34} = 1/2$, $\alpha_{13} = \alpha_{24} = 3/2$ and $\alpha_{14} = \alpha_{23} = 1$.



(f) Legendre conjugate algorithm for $\alpha_{12} = \alpha_{34} = 1/2$, $\alpha_{13} = \alpha_{24} = 3/2$ and $\alpha_{14} = \alpha_{23} = 1$.

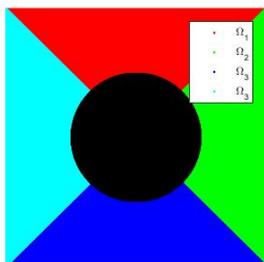


(g) Primal algorithm for $\alpha_{13} = 1/2$ and $\alpha_{ij} = 1$ otherwise.

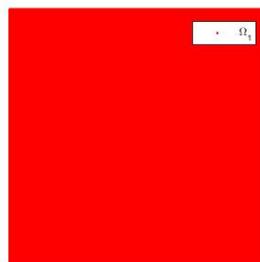


(h) Legendre conjugate algorithm $\alpha_{13} = 1/2$ and $\alpha_{ij} = 1$ otherwise.

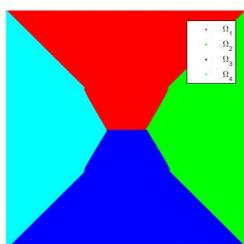
Figure 5.2: Quadruple point (case 1): comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) (see Table 5.2 for more details).



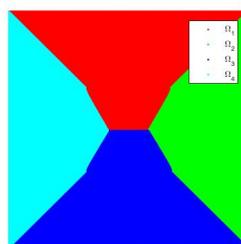
(a) Data.



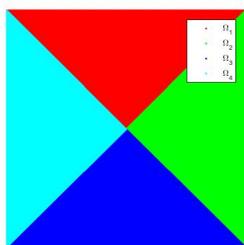
(b) Initialization.



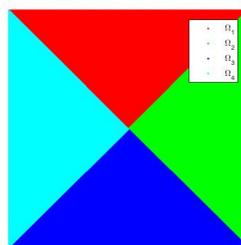
(c) Primal algorithm for $\alpha_{ij} = 1$.



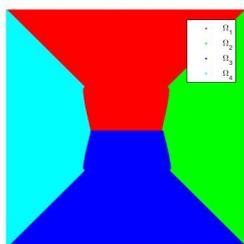
(d) Legendre conjugate algorithm for $\alpha_{ij} = 1$.



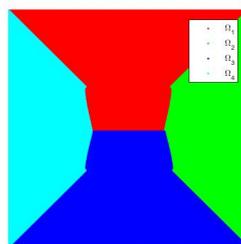
(e) Primal algorithm for $\alpha_{12} = \alpha_{34} = 1/2$, $\alpha_{13} = \alpha_{24} = 3/2$ and $\alpha_{14} = \alpha_{23} = 1$.



(f) Legendre conjugate algorithm for $\alpha_{12} = \alpha_{34} = 1/2$, $\alpha_{13} = \alpha_{24} = 3/2$ and $\alpha_{14} = \alpha_{23} = 1$.



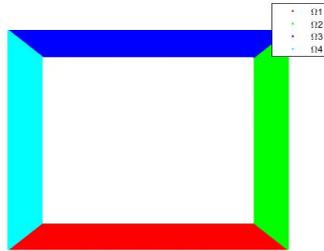
(g) Primal algorithm $\alpha_{13} = 1/2$ and $\alpha_{ij} = 1$ otherwise.



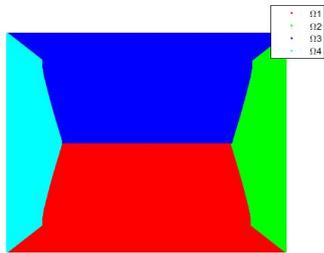
(h) Legendre conjugate algorithm $\alpha_{13} = 1/2$ and $\alpha_{ij} = 1$ otherwise.

Figure 5.3: Quadruple point (case 2): comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) (see Table 5.3 for more details).

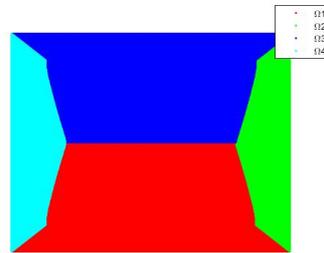
5.5. Image classification



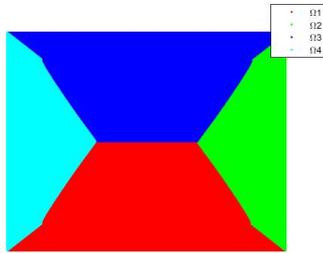
(a) Data.



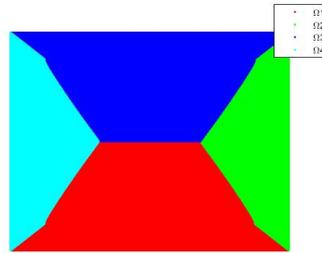
(b) Primal algorithm for $\alpha_{13}=1/2$ and $\alpha_{ij}=1$ otherwise.



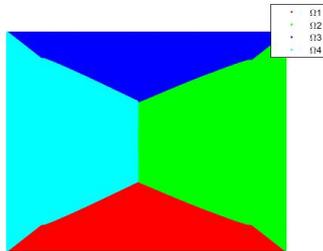
(c) Legendre conjugate algorithm for $\alpha_{13}=1/2$ and $\alpha_{ij}=1$ otherwise.



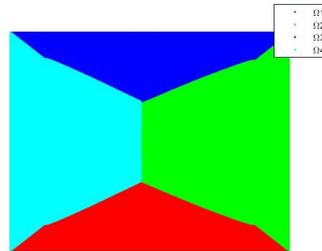
(d) Primal algorithm for $\alpha_{ij}=1$.



(e) Legendre conjugate algorithm for $\alpha_{ij}=1$.

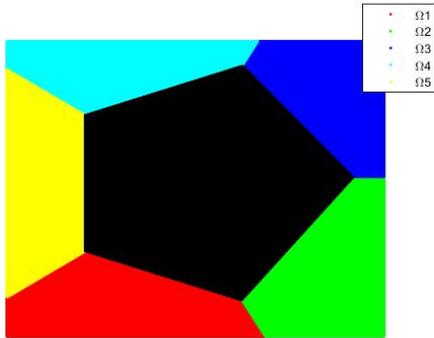


(f) Primal algorithm for $\alpha_{13}=2$ and $\alpha_{ij}=1$ otherwise.

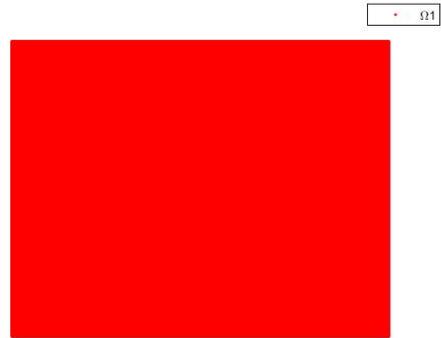


(g) Legendre conjugate algorithm for $\alpha_{13}=2$ and $\alpha_{ij}=1$ otherwise.

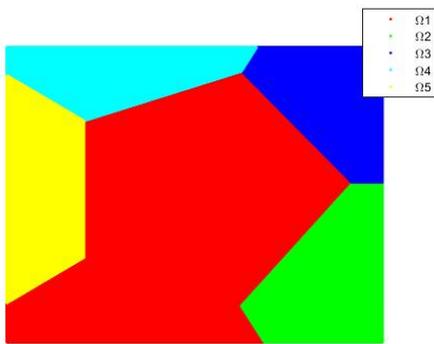
Figure 5.4: Steiner tree problem: comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) with initialization: $u_1 = u_2 = u_3 = u_4 = 1/4$ (see Table 5.4 for more details).



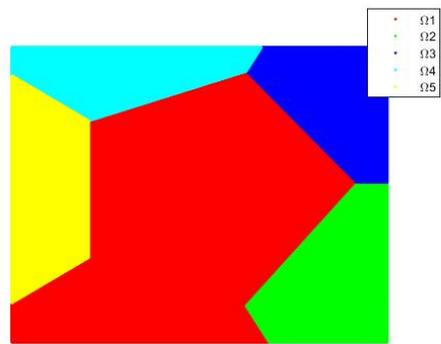
(a) Data



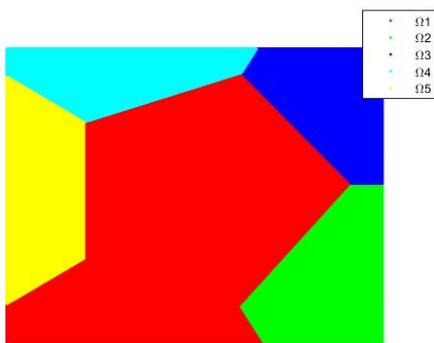
(b) Initialization



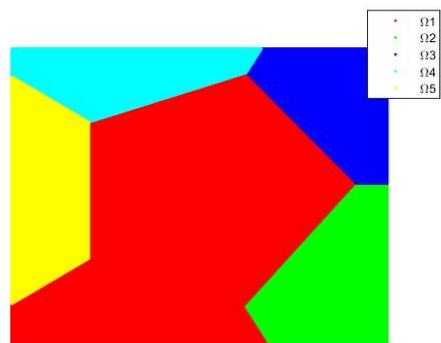
(c) Primal dual algorithm for β_S (case 1).



(d) Primal dual algorithm for β_S (case 2).

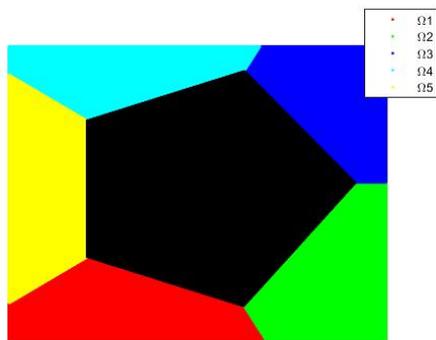


(e) Primal dual algorithm for β_S (case 3).

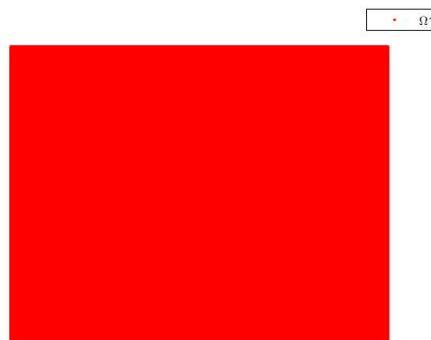


(f) Legendre conjugate algorithm.

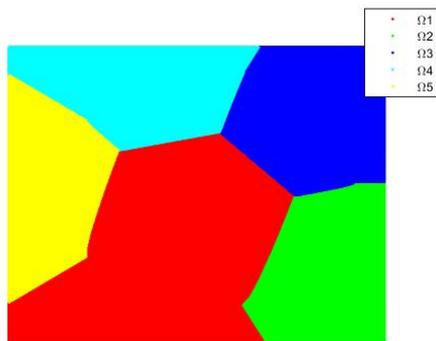
Figure 5.5: Steiner tree problem: comparison between the primal dual algorithm and the Legendre conjugate algorithm for the same values of (α_{ij}) (see Table 5.5 for more details).



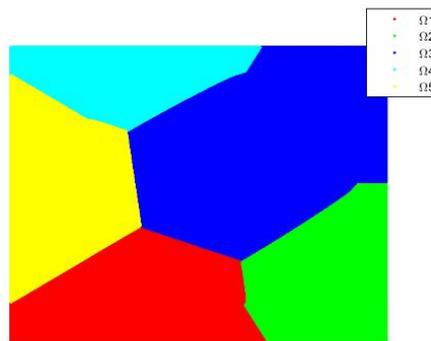
(a) Data



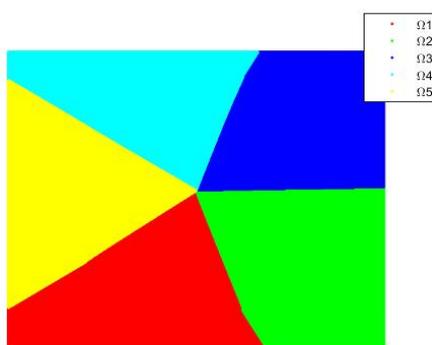
(b) Initialization



(c) $\alpha_{ij} = 1$.



(d) $\alpha_{13} = 2$ and $\alpha_{ij} = 1$ otherwise.



(e) $\alpha_{12} = \alpha_{15} = \alpha_{23} = \alpha_{34} = \alpha_{45} = 1$ and $\alpha_{ij} = 2$ otherwise.

Figure 5.6: Steiner tree problem: Legendre conjugate algorithm for different values of (α_{ij}) (see Table 5.6 for more details).

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(α_{ij})	1, 1, 1, 1, 1, 1	0.5, 1.5, 1, 1, 1.5, 0.5	1, 0.5, 1, 1, 1, 1
(β_S)	0.5, 0.5, 0.5, 0.5, 0, 0, 0	0, 0, 0, 0, 1, 0.5	0.25, 0.5, 0.25, 0.5, 0, 0.25, 0
Eigenvalues of \tilde{Q}	-4,-1,-1	-4.7321,-1.2679, 0	-4.3508, -1.1492, -0.5
Primal algorithm			
Figure	5.2d	5.2f	5.2h
Elapsed time	40 s	10 s	117 s
Cost function	-2.0860e+05	-2.0872e+05	-2.0862e+05
Legendre conjugate algorithm			
Figure	5.2d	5.2f	5.2h
Elapsed time	35 s	8 s	104 s
Cost function	-2.0860e+05	-2.0872e+05	-2.0862e+05

Table 5.2: Quadruple point example 1: comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) .

(α_{ij})	1, 1, 1, 1, 1, 1	0.5, 1.5, 1, 1, 1.5, 0.5	1, 0.5, 1, 1, 1, 1
(β_S)	0.5, 0.5, 0.5, 0.5, 0, 0, 0	0, 0, 0, 0, 1, 0.5	0.25, 0.5, 0.25, 0.5, 0, 0.25, 0
Eigenvalues of \tilde{Q}	-4, -1, -1	-4.7321, -1.2679, 0	-4.3508, -1.1492, -0.5
Primal algorithm			
Figure	5.3d	5.3f	5.3h
Elapsed time	39 s	14 s	66 s
Cost function	670.6850	508.9342	643.7179
Legendre conjugate algorithm			
Figure	5.3d	5.3f	5.3h
Elapsed time	34 S	12 S	58 S
Cost function	670.6850	508.9342	643.7179

Table 5.3: Quadruple point example 2: comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) .

where each u_i is the characteristic function of a subset of Ω_i of D such that $(\Omega_1, \dots, \Omega_N)$ forms a partition of D . We have for any L^p norm on D :

$$\|w - f\|_{L^p(D)}^p = \left\| \sum_{i=1}^N u_i c_i - f \right\|_{L^p(D)}^p = \left\| \sum_{i=1}^N u_i (c_i - f) \right\|_{L^p(D)}^p = \sum_{i=1}^N \int_D u_i |c_i - f|^p dx.$$

The difference between the piecewise constant and original images is measured by:

$$\|w - f\|_{L^p(D)}^p = \sum_{i=1}^N \langle u_i, g_i \rangle, \quad g_i = |c_i - f|^p.$$

When the levels c_i are fixed, we can directly apply the three algorithms (primal, primal dual and Legendre conjugate algorithms) mentioned in this chapter.

For the update of levels, it is often desirable to determine the grey levels within the classes automatically. Thus, we include a third step in the alternating minimization algorithm, consisting

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(α_{ij})	1, 0.5, 1, 1, 1, 1	1, 1, 1, 1, 1, 1	1, 2, 1, 1, 1, 1
(β_S)	0.25, 0.5, 0.25, 0.5, 0, 0.25, 0	0.5, 0.5, 0.5, 0.5, 0, 0, 0	0.5, 0, 0.5, 0, 0.5, 0, 0.5
Eigenvalues of Q	-4.3508, -1.1492, -0.5	-4, -1, -1	-3.4142, -2, -0.5858
Primal algorithm			
Figure	5.4b	5.4d	5.4f
Elapsed time	71 s	77 s	178 s
Cost function	-1.1128e+05	-1.1122e+05	-1.1122e+05
Legendre conjugate algorithm			
Figure	5.4c	5.4e	5.4g
Elapsed time	68 s	72 s	167 s
Cost function	-1.1128e+05	-1.1122e+05	-1.1122e+05

Table 5.4: Steiner tree problem: comparison between the primal algorithm and the Legendre conjugate algorithm for different values of (α_{ij}) .

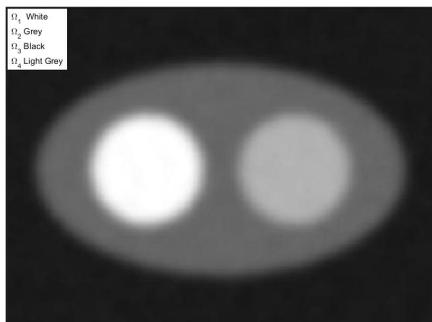
(α_{ij})	1, 1, 1, 1, 1, 2, 3, 2, 2, 3	Figure	Elapsed time	Cost function
Primal dual algorithm				
(β_S) case1	-0.5, 0.5, 0, 1, 1, 0, 0, 0, 0.5, 0.5, 0, 0, 0, 0.5, 0	5.5c	5 s	-1.6064e+05
(β_S) case2	-1, 0, 0, 0.5, 1, 0.5, 0, 0.5, 0.5, 0.5, 0, 0, 0, 0	5.5d	5 s	-1.6064e+05
(β_S) case3	-1, 0.5, 0, 0.5, 0.5, 0, 0, 0.5, 1, 0, 0, 0, 0, 0.5, 0.5	5.5e	5 s	-1.6064e+05
Legendre conjugate algorithm				
Eigenvalues of Q	-14.5432, -2.3656, -0.9360, -0.1553	5.5f	47 s	-1.6065e+05

Table 5.5: Steiner tree problem: comparison between the primal dual algorithm and the Legendre conjugate algorithm for the same values of (α_{ij}) .

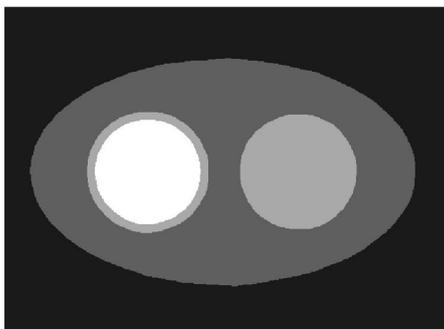
(α_{ij})	1, 1, 1, 1, 1, 1, 1, 1, 1, 1	1, 1, 2, 1, 1, 1, 1, 1, 1, 1	1, 2, 2, 1, 1, 2, 2, 1, 2, 1
Eigenvalues of Q	-5, -1, -1, -1	-4.5616, -2, -1, -0.4384	-8.4051, -2.618, -0.5949, -0.382
Figure	5.6c	5.6d	5.6e
Elapsed time	187 s	207 s	141 s
Cost function	-1.6080e+05	-1.6080e+05	-1.6077e+05

Table 5.6: Steiner tree problem: Legendre conjugate algorithm for different values of (α_{ij}) .

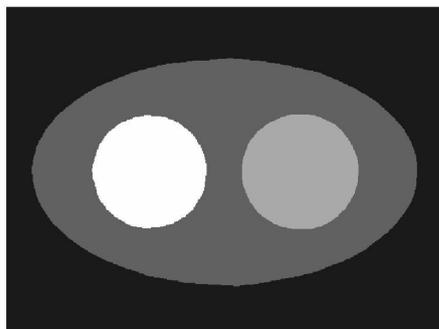
5.5. Image classification



(a) Data.



(b) Unsupervised solution for $\alpha_{ij}=1$.



(c) Unsupervised solution for $\alpha_{12}=0.1$ and $\alpha_{ij}=1$ otherwise.

Figure 5.7: Classification of greyscale images with L^2 norm, 4 phases and initialization: $u_1=1$ and $u_i=0$ otherwise. Primal algorithm.

of solving

$$\min_{(c_1, \dots, c_N) \in [0,1]^N} \sum_{i=1}^N \langle u_i, g_i \rangle = \sum_{i=1}^N \int_D u_i |c_i - f|^p dx.$$

This problem is separable in its variables c_1, \dots, c_N , and each c_i must satisfy

$$c_i \in \operatorname{argmin}_{c \in [0,1]} \int_D u_i |c - f|^p dx.$$

Note that, since $0 \leq f \leq 1$, the constraint $0 \leq c \leq 1$ can be removed. For $p = 2$, this is a standard problem which results in computing the arithmetic mean

$$c_i = \frac{\int_D u_i f dx}{\int_D u_i dx}.$$

In Figure 5.7, an example of unsupervised greyscale image classification with L^2 norm is shown.

5.5.2 Classification of color images

The original image f is represented by the three channels $(f_1, f_2, f_3) \in L^\infty(D, [0, 1])^3$ representing the intensity of red, green and blue, respectively. Each phase Ω_i is associated to a color $(c_{i1}, c_{i2}, c_{i3}) \in [0, 1]^3$ in the same RGB system. The reconstructed image

$$w_j = \sum_{i=1}^N u_i c_{ij}$$

where u_i is the characteristic function of Ω_i . We have for each channel

$$\|w_j - f_j\|_{L^p(D)}^p = \left\| \sum_{i=1}^N u_i c_{ij} - f_j \right\|_{L^p(D)}^p = \left\| \sum_{i=1}^N u_i (c_{ij} - f_j) \right\|_{L^p(D)}^p = \sum_{i=1}^N \int_D u_i |c_{ij} - f_j|^p.$$

The difference between the segmented and original images is measured by

$$\sum_{j=1}^3 \|w_j - f_j\|_{L^p(D)}^p = \sum_{i=1}^N \langle u_i, g_i \rangle, \quad g_i = \sum_{j=1}^3 |c_{ij} - f_j|^p.$$

We then apply the same algorithms as in this chapter. Note that the geometrical variable $u = (u_1, \dots, u_N)$ as well as the auxiliary variable $v = (v_1, \dots, v_N)$ remain N -dimensional vectors, and the update of levels is separable in the channels.

In Figure 5.8, we compare between supervised and unsupervised solution for color image classification for different values of α_{ij} with L^2 norm.

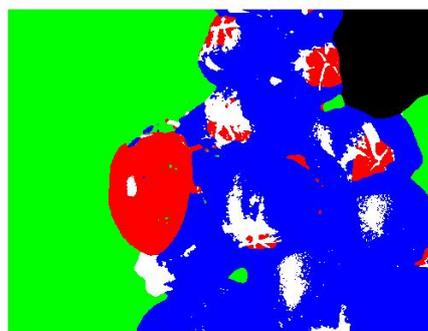
5.5. Image classification



(a) Original image.



(b) Supervised solution for $\alpha_{ij} = 1$.



(c) Supervised solution for $\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 20$ and $\alpha_{ij} = 1$ otherwise.



(d) Unsupervised solution for $\alpha_{ij} = 1$.



(e) Unsupervised solution for $\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 20$ and $\alpha_{ij} = 1$ otherwise.

Figure 5.8: Color image classification with L^2 norm, 5 phases and initialization: $u_1=1$ and $u_i=0$ otherwise.

Chapter 6

Projected gradient algorithm and applications

In this chapter, we present an algorithm based on the projected gradient and some applications.

6.1 Projected gradient algorithm

Let us start with the following lemma from [12].

Lemma 6.1. *Let X be a Hilbert space and $\Phi : X \rightarrow \mathbb{R}$ be a differentiable function such that $\nabla\Phi$ is λ -Lipschitz. We have for all $u \in X$:*

$$\Phi(u) = \inf_{\tilde{u} \in X} \left\{ \Phi(\tilde{u}) + \langle \nabla\Phi(\tilde{u}), u - \tilde{u} \rangle + \frac{\lambda}{2} \|u - \tilde{u}\|^2 \right\}. \quad (6.1.1)$$

We want to minimize the right hand side of (6.1.1) using an alternating algorithm. The minimization with respect to u is a projection problem as shown below.

Lemma 6.2. *Let X be a Hilbert space. Let \mathcal{K} be a convex closed nonempty set of X , $P_{\mathcal{K}}$ be the orthogonal projection onto \mathcal{K} and $\Phi : X \rightarrow \mathbb{R}$ be a differentiable function such that $\nabla\Phi$ is λ -Lipschitz. Let*

$$\psi(u, \tilde{u}) = \Phi(\tilde{u}) + \langle \nabla\Phi(\tilde{u}), u - \tilde{u} \rangle + \frac{\lambda}{2} \|u - \tilde{u}\|^2. \quad (6.1.2)$$

We have for all $u \in X$:

$$\inf_{u \in \mathcal{K}} \psi(u, \tilde{u}) = \psi \left(P_{\mathcal{K}} \left(\tilde{u} - \frac{1}{\lambda} \nabla\Phi(\tilde{u}) \right) \right). \quad (6.1.3)$$

Proof. By using Gauss reduction, we can rewrite (6.1.2) in the form

$$\psi(u, \tilde{u}) = \left\| \sqrt{\frac{\lambda}{2}}(u - \tilde{u}) + \frac{1}{\sqrt{2\lambda}} \nabla\Phi(\tilde{u}) \right\|_{L^2}^2 - \frac{1}{2\lambda} \|\nabla\Phi(\tilde{u})\|_{L^2}^2 + \Phi(\tilde{u}),$$

which implies that

$$\psi(u, \tilde{u}) = \frac{\lambda}{2} \left\| u - \left(\tilde{u} - \frac{1}{\lambda} \nabla\Phi(\tilde{u}) \right) \right\|_{L^2}^2 - \frac{1}{2\lambda} \|\nabla\Phi(\tilde{u})\|_{L^2}^2 + \Phi(\tilde{u}). \quad (6.1.4)$$

6.1. Projected gradient algorithm

The minimization with respect to u is

$$\inf_{u \in \mathcal{K}} \psi(u, \tilde{u}) = \psi \left(P_{\mathcal{K}} \left(\tilde{u} - \frac{1}{\lambda} \nabla \Phi(\tilde{u}) \right) \right). \quad (6.1.5)$$

□

Now, we define the standard simplex set

$$\mathcal{K} = \left\{ \xi = (\xi_1, \dots, \xi_N)^\top \in \mathbb{R}^N : 0 \leq \xi_i \leq 1, \ i = 1, \dots, N \text{ and } \sum_{i=1}^N \xi_i = 1 \right\}. \quad (6.1.6)$$

The minimal partition problem (1.4.1) reads

$$\min_{u \in L^\infty(D, \mathcal{K})} \Phi(u), \quad (6.1.7)$$

with $\Phi : L^2(D, \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\Phi(u) = \sum_{i=1}^N \langle u_i, g_i \rangle + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle \quad \forall u \in L^2(D, \mathbb{R}^N). \quad (6.1.8)$$

We have

$$\nabla \Phi(u) = \begin{pmatrix} g_1 + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon u_i \\ \vdots \\ g_N + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon u_i \end{pmatrix}. \quad (6.1.9)$$

We also define the constant C by

$$C = \max \left\{ \sum_{i \neq 1} \alpha_{1i}^2, \dots, \sum_{i \neq N} \alpha_{iN}^2 \right\}. \quad (6.1.10)$$

To determine the lipschitz constant λ of $\nabla \Phi$, we will use the following observation.

Lemma 6.3. *The gradient $\nabla \Phi$ is λ -lipschitz if*

$$\lambda \geq \frac{\sqrt{(N-1)C}}{\varepsilon}.$$

Proof. We have

$$\|\nabla \Phi(u) - \nabla \Phi(\tilde{u})\|_{L^2}^2 = \left\| \begin{pmatrix} g_1 + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon u_i \\ \vdots \\ g_N + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon u_i \end{pmatrix} - \begin{pmatrix} g_1 + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon \tilde{u}_i \\ \vdots \\ g_N + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon \tilde{u}_i \end{pmatrix} \right\|_{L^2}^2.$$

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It follows that

$$\begin{aligned} \|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 &= \left\| \begin{pmatrix} g_1 + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon(u_i - \tilde{u}_i) \\ \vdots \\ g_N + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon(u_i - \tilde{u}_i) \end{pmatrix} \right\|_{L^2}^2 \\ &= \left\| \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon(u_i - \tilde{u}_i) \right\|_{L^2}^2 + \cdots + \left\| \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon(u_i - \tilde{u}_i) \right\|_{L^2}^2, \end{aligned}$$

which implies that

$$\|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 \leq \|L_\varepsilon\|_{\mathcal{L}(L^2)}^2 \left(\left\| \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} (u_i - \tilde{u}_i) \right\|_{L^2}^2 + \cdots + \left\| \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} (u_i - \tilde{u}_i) \right\|_{L^2}^2 \right).$$

From (2.6.6) we infer that

$$\begin{aligned} \|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 &\leq \left\| \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} (u_i - \tilde{u}_i) \right\|_{L^2}^2 + \cdots + \left\| \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} (u_i - \tilde{u}_i) \right\|_{L^2}^2 \\ &\leq \left(\frac{N-1}{\varepsilon} \right)^2 \left\| \sum_{i \neq 1} \frac{\alpha_{1i}}{N-1} (u_i - \tilde{u}_i) \right\|_{L^2}^2 + \cdots + \left(\frac{N-1}{\varepsilon} \right)^2 \left\| \sum_{i \neq N} \frac{\alpha_{iN}}{N-1} (u_i - \tilde{u}_i) \right\|_{L^2}^2. \end{aligned}$$

By convexity of $\|\cdot\|_{L^2}^2$, we get

$$\begin{aligned} \|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 &\leq \left(\frac{N-1}{\varepsilon^2} \right) \sum_{i \neq 1} \alpha_{1i}^2 \|u_i - \tilde{u}_i\|_{L^2}^2 + \cdots + \left(\frac{N-1}{\varepsilon^2} \right) \sum_{i \neq N} \alpha_{iN}^2 \|u_i - \tilde{u}_i\|_{L^2}^2 \\ &\leq \left(\frac{N-1}{\varepsilon^2} \sum_{i \neq 1} \alpha_{1i}^2 \right) \|u_1 - \tilde{u}_1\|_{L^2}^2 + \cdots + \left(\frac{N-1}{\varepsilon^2} \sum_{i \neq N} \alpha_{iN}^2 \right) \|u_N - \tilde{u}_N\|_{L^2}^2. \end{aligned}$$

Using (6.1.10) and taking the square root of both sides, we get

$$\|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2} \leq \left(\frac{\sqrt{(N-1)C}}{\varepsilon} \right) \|u - \tilde{u}\|_{L^2}.$$

Consequently, we obtain

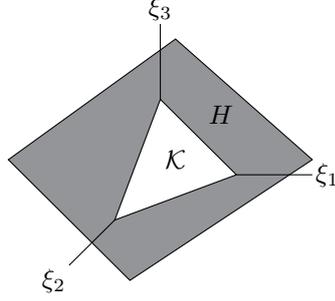
$$\lambda \geq \frac{\sqrt{(N-1)C}}{\varepsilon}.$$

□

6.1.1 Projection onto a simplex

In order to use Lemma 6.2 we need to project onto the convex set \mathcal{K} . We describe the procedure for \mathcal{K} being defined by (6.1.6) in the case $N=3$, i.e.,

$$\mathcal{K} = \left\{ \xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3 : 0 \leq \xi_i \leq 1, \ i = 1, 2, 3 \text{ and } \sum_{i=1}^3 \xi_i = 1 \right\}.$$


 Figure 6.1: The standard simplex in \mathbb{R}^3 and a plane H .

Let

$$H = \left\{ \xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3 : \sum_{i=1}^3 \xi_i = 1 \right\}, \text{ (see Figure 6.1 for illustration).}$$

Let $\xi \in \mathbb{R}^3$ and define $y = P_H(\xi)$, where $P_H(\xi)$ is the orthogonal projection of ξ onto the plane H .

Here, there are two cases depending on whether $y_i \geq 0 \forall i$ or not.

- If $y_i \geq 0 \forall i$, then $y \in \mathcal{K}$, which implies $P_{\mathcal{K}}(\xi) = P_H(\xi) = y$.
- If there exists i_0 such that $y_{i_0} < 0$, then $P_{\mathcal{K}}(\xi) = P_{S_{i_0}}(y)$ with

$$S_{i_0} = \{\xi \in \mathcal{K} : \xi_{i_0} = 0\}.$$

We define the line $L_{i_0} = \{\xi \in H : \xi_{i_0} = 0\}$ and $\bar{y} = P_{L_{i_0}}(y)$. Now, we have two subcases:

- If $\bar{y}_i \geq 0 \forall i$, then $\bar{y} \in S_{i_0}$, which implies $P_{S_{i_0}}(y) = P_{L_{i_0}}(y)$.
- If there exists i_{00} such that $\bar{y}_{i_{00}} < 0$, then $P_{S_{i_0}}(y) = P_{S_{i_{00}}}(\bar{y}) = S_{i_{00}}$ with the vertex

$$S_{i_{00}} = \{\xi \in L_{i_0} : \xi_{i_{00}} = 0\},$$

(i.e. project onto a vertex).

6.1.2 Algorithm

Plugging (6.1.1) into (6.1.7), we obtain that the subproblem at ε fixed consists in solving the following two-level minimization problem

$$\mathcal{I}_\varepsilon^{PG} = \min_{u \in L^\infty(D, \mathcal{K})} \inf_{\tilde{u} \in L^2(D)} \left\{ \Phi(\tilde{u}) + \langle \nabla \Phi(\tilde{u}), u - \tilde{u} \rangle + \frac{\lambda}{2} \|u - \tilde{u}\|^2 \right\},$$

with Φ given by (6.1.8). The simple structure of this problem with respect to each variable u and \tilde{u} allows us to use an alternating minimization algorithm. The two steps are described below.

- According to Lemma 6.1, we obtain the minimization with respect to \tilde{u} is simply achieved by setting $\tilde{u}^k = u^{k-1}$.
- Applying Lemma 6.2 we conclude that the minimization with respect to u is given by

$$u^k(x) = P_{\mathcal{K}} \left(\tilde{u}^k(x) - \frac{1}{\lambda} \nabla \Phi(\tilde{u}^k(x)) \right).$$

6.2 Image deblurring of greyscale image

We will apply a variant of our minimal partition model for multilabel image deconvolution. The blurring kernel is represented by a linear and continuous operator $\mathcal{A} : L^2(D) \rightarrow L^2(D)$ such that $\mathcal{A}1 = 1$. The given blurred (greyscale) image is $f \in L^\infty(D, [0, 1])$, and the reconstructed image is $w = c_1 u_1 + \dots + c_N u_N$, with $u \in \mathcal{E}$ and $c_i \geq 0$ for all i . We have

$$\mathcal{A}w = \sum_{i=1}^N c_i \mathcal{A}u_i.$$

We have for L^2 norm on D :

$$\|\mathcal{A}w - f\|_{L^2}^2 = \left\| \sum_{i=1}^N c_i \mathcal{A}u_i - f \right\|_{L^2}^2.$$

The deblurring problem of greyscale image reads

$$\min_{u \in L^\infty(D, \mathcal{K})} \Phi(u),$$

with $\Phi : L^2(D, \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\Phi(u) = \left\| \sum_{i=1}^N c_i \mathcal{A}u_i - f \right\|_{L^2}^2 + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle \quad \forall u \in L^2(D, \mathbb{R}^N).$$

We have

$$\nabla \Phi(u) = \begin{pmatrix} 2c_1 \mathcal{A}^* \left(\sum_{i=1}^N c_i \mathcal{A}u_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon u_i \\ \vdots \\ 2c_N \mathcal{A}^* \left(\sum_{i=1}^N c_i \mathcal{A}u_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon u_i \end{pmatrix}, \quad (6.2.1)$$

with \mathcal{A}^* the adjoint operator of \mathcal{A} .

In our model the blurring kernel \mathcal{A} is assumed to be known, which occurs in some practical applications, like when the blur is generated by an optical device.

To determinate the lipschitz constant λ of $\nabla \Phi$, we define the constant \bar{C} by

$$\bar{C} = \max \{c_1, \dots, c_N\}, \quad (6.2.2)$$

and will use the following observation.

Lemma 6.4. *The gradient $\nabla \Phi$ is λ -lipschitz if*

$$\lambda \geq \left(8N^2 \bar{C}^4 \|\mathcal{A}^* \mathcal{A}\|^2 + \frac{2(N-1)C}{\varepsilon^2} \right)^{\frac{1}{2}}$$

with C, \bar{C} defined by (6.1.10) and (6.2.2) respectively.

Proof. We have

$$\|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 = \left\| \begin{pmatrix} 2c_1\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}u_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i}L_\varepsilon u_i \\ \vdots \\ 2c_N\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}u_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN}L_\varepsilon u_i \end{pmatrix} - \begin{pmatrix} 2c_1\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}\tilde{u}_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i}L_\varepsilon \tilde{u}_i \\ \vdots \\ 2c_N\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}\tilde{u}_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN}L_\varepsilon \tilde{u}_i \end{pmatrix} \right\|_{L^2}^2.$$

It follows that

$$\begin{aligned} \|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 &= \left\| \begin{pmatrix} 2c_1\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}(u_i - \tilde{u}_i) \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i}L_\varepsilon(u_i - \tilde{u}_i) \\ \vdots \\ 2c_N\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}(u_i - \tilde{u}_i) \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN}L_\varepsilon(u_i - \tilde{u}_i) \end{pmatrix} \right\|_{L^2}^2 \\ &= \left\| 2c_1\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}(u_i - \tilde{u}_i) \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i}L_\varepsilon(u_i - \tilde{u}_i) \right\|_{L^2}^2 \\ &\quad + \cdots + \left\| 2c_N\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}(u_i - \tilde{u}_i) \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN}L_\varepsilon(u_i - \tilde{u}_i) \right\|_{L^2}^2. \end{aligned}$$

By using $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$, we get

$$\begin{aligned} \|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2(D)}^2 &\leq 2 \left\| 2c_1\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}(u_i - \tilde{u}_i) \right) \right\|_{L^2}^2 + 2 \left\| \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i}L_\varepsilon(u_i - \tilde{u}_i) \right\|_{L^2}^2 \\ &\quad + \cdots + 2 \left\| 2c_N\mathcal{A}^* \left(\sum_{i=1}^N c_i\mathcal{A}(u_i - \tilde{u}_i) \right) \right\|_{L^2}^2 + 2 \left\| \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN}L_\varepsilon(u_i - \tilde{u}_i) \right\|_{L^2}^2. \end{aligned}$$

Using the same technique in proof of Lemma 6.3 and from (6.2.2), we get

$$\begin{aligned} \|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2}^2 &\leq 8N^2\bar{C}^4\|\mathcal{A}^*\mathcal{A}\|^2\|u - \tilde{u}\|_{L^2}^2 + 2 \left(\frac{(N-1)C}{\varepsilon^2} \right) \|u - \tilde{u}\|_{L^2}^2 \\ &\leq \left(8N^2\bar{C}^4\|\mathcal{A}^*\mathcal{A}\|^2 + \frac{2(N-1)C}{\varepsilon^2} \right) \|u - \tilde{u}\|_{L^2}^2, \end{aligned}$$

which implies that

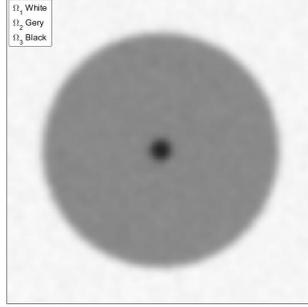
$$\|\nabla\Phi(u) - \nabla\Phi(\tilde{u})\|_{L^2} \leq \left(8N^2\bar{C}^4\|\mathcal{A}^*\mathcal{A}\|^2 + \frac{2(N-1)C}{\varepsilon^2} \right)^{\frac{1}{2}} \|u - \tilde{u}\|_{L^2}.$$

Consequently, we obtain

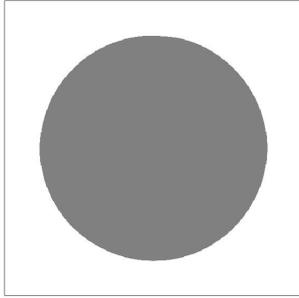
$$\lambda \geq \left(8N^2\bar{C}^4\|\mathcal{A}^*\mathcal{A}\|^2 + \frac{2(N-1)C}{\varepsilon^2} \right)^{\frac{1}{2}}.$$

□

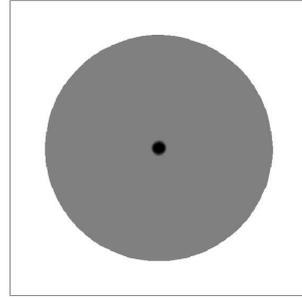
6.3. Image deblurring of colour image



(a) Damaged image with blur and noise effects.



(b) Reconstructed image for $\alpha_{ij} = 10$.



(c) Reconstructed image for $\alpha_{23} = 1$ and $\alpha_{ij} = 10$ otherwise.

Figure 6.2: Image deblurring of greyscale image with initialization: $u_2=1$ and $u_i=0$ otherwise (example 1).

Algorithm

For ε fixed we have to solve the approximate problem

$$\mathcal{I}_\varepsilon^{PG} = \min_{u \in L^\infty(D, \mathcal{K})} \inf_{\tilde{u} \in L^2(D)} \left\{ \Phi(\tilde{u}) + \langle \nabla \Phi(\tilde{u}), u - \tilde{u} \rangle + \frac{\lambda}{2} \|u - \tilde{u}\|^2 \right\},$$

with

$$\Phi(u) = \left\| \sum_{i=1}^N c_i \mathcal{A} u_i - f \right\|_{L^2}^2 + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle.$$

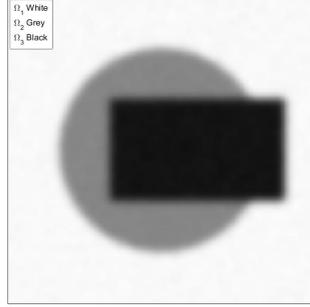
We can directly apply the algorithm of Section 6.1.

We present in Figures 6.2 and 6.3 two examples of image deblurring of greyscale image with the finite difference method (FDM).

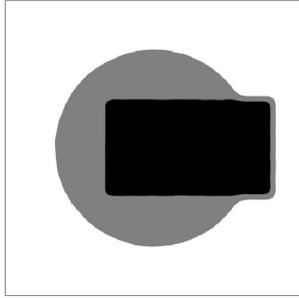
6.3 Image deblurring of colour image

The blurred image f is represented by the three channels $(f_1, f_2, f_3) \in L^\infty(D, [0, 1])^3$ representing the intensity of red, green and blue, respectively. Each phase Ω_i is associated to a color $(c_{i1}, c_{i2}, c_{i3}) \in$

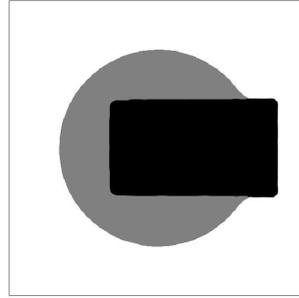
6.3. Image deblurring of colour image



(a) Damaged image with blur and noise effects.



(b) Reconstructed image for $\alpha_{ij} = 1$.



(c) Reconstructed image for $\alpha_{13}=1$ and $\alpha_{ij}=2$ otherwise.

Figure 6.3: Image deblurring of greyscale image with initialization: $u_2=1$ and $u_i=0$ otherwise (example 2).

$[0, 1]^3$ in the same RGB system. The reconstructed image $\omega = (\omega_1, \omega_2, \omega_3)$ is given by

$$w_j = \sum_{i=1}^N c_{ij} u_i,$$

where u_i is the characteristic function of a subset Ω_i of D such that $(\Omega_1, \dots, \Omega_N)$ forms a partition of D . We have for each channel and L^2 norm on D :

$$\|\mathcal{A}w_j - f_j\|_{L^2}^2 = \left\| \sum_{i=1}^N c_{ij} \mathcal{A}u_i - f_j \right\|_{L^2}^2,$$

where \mathcal{A} is a linear and continuous operator defined by

$$\mathcal{A} : L^2(D) \rightarrow L^2(D) \text{ such that } \mathcal{A}1 = 1.$$

Using the fact that $\sum_{i=1}^N \mathcal{A}u_i = 1$, we get

$$\|\mathcal{A}w_j - f_j\|_{L^2}^2 = \left\| \sum_{i=1}^n c_{ij} \mathcal{A}u_i - \sum_{i=1}^N \mathcal{A}u_i f_j \right\|_{L^2}^2.$$

It follows that

$$\|\mathcal{A}w_j - f_j\|_{L^2}^2 = \sum_{j=1}^3 \left\| \sum_{i=1}^N \mathcal{A}u_i(c_{ij} - f_j) \right\|_{L^2}^2. \quad (6.3.1)$$

The deblurring problem of colour image reads

$$\min_{u \in L^\infty(D, \mathcal{K})} \Phi(u),$$

with $\Phi : L^2(D, \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\Phi(u) = \sum_{j=1}^3 \left\| \sum_{i=1}^N \mathcal{A}u_i(c_{ij} - f_j) \right\|_{L^2}^2 + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle \quad \forall u \in L^2(D, \mathbb{R}^N).$$

We have

$$\nabla \Phi(u) = \begin{pmatrix} 2\mathcal{A}^* \left(\sum_{j=1}^3 \sum_{i=1}^N (c_{1j} - f_j) \mathcal{A}u_i(c_{ij} - f_j) \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon u_i \\ \vdots \\ 2\mathcal{A}^* \left(\sum_{j=1}^3 \sum_{i=1}^N (c_{Nj} - f_j) \mathcal{A}u_i(c_{ij} - f_j) \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon u_i \end{pmatrix} \quad (6.3.2)$$

with \mathcal{A}^* the adjoint operator of \mathcal{A} .

Algorithm

For ε fixed we have to solve the approximate problem

$$\mathcal{I}_\varepsilon^{PG} = \min_{u \in L^\infty(D, \mathcal{K})} \inf_{\tilde{u} \in L^2(D)} \left\{ \Phi(\tilde{u}) + \langle \nabla \Phi(\tilde{u}), u - \tilde{u} \rangle + \frac{\lambda}{2} \|u - \tilde{u}\|^2 \right\},$$

with

$$\Phi(u) = \sum_{j=1}^3 \left\| \sum_{i=1}^N \mathcal{A}u_i(c_{ij} - f_j) \right\|_{L^2}^2 + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle.$$

We can directly apply the algorithm of Section 6.1.

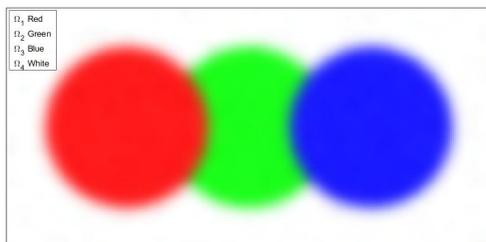
We present in Figure 6.4 an example of image deblurring of colour image with the finite difference method (FDM).

6.4 Medical imaging

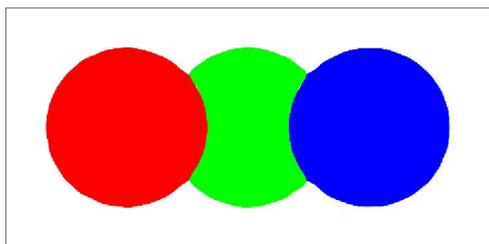
The (2-dimensional) Radon transform \mathcal{R} maps a function $w : \mathbb{R}^2 \rightarrow \mathbb{R}$ into the set of its integrals over the lines of \mathbb{R}^2 . More specifically, if $\theta \in S^1$ and $s \in \mathbb{R}$, then

$$\mathcal{R}w(\theta, s) = \int_{x \cdot \theta = s} w(x) dx = \int_{\theta^\perp} w(s\theta + y) dy$$

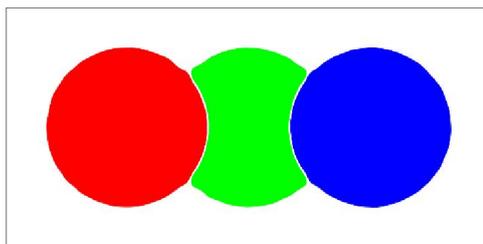
is the integral of $w \in \mathcal{S}(\mathbb{R}^2)$, the Schwartz space, over the line perpendicular to θ with (signed) distance s from the origin. The Radon transform extends by continuity to functions $w \in L^2(D)$ with D bounded. We refer to [43] for more details.



(a) Damaged image with blur and noise effects.



(b) Reconstructed image for $\alpha_{ij} = 1$.



(c) Reconstructed image for $\alpha_{12} = \alpha_{23}=2$ and $\alpha_{ij}=1$ otherwise.

Figure 6.4: Image deblurring of colour image.

The measurement is $f \in L^2(S^1 \times \mathbb{R})$, and the reconstructed image is $w = \sum_{i=1}^N c_i u_i$, with $u \in \mathcal{E}$ and $c_i \geq 0$ for all i . We have

$$\|\mathcal{R}w - f\|_{L^2}^2 = \left\| \sum_{i=1}^N c_i \mathcal{R}u_i - f \right\|_{L^2}^2.$$

The imaging problem reads

$$\min_{u \in L^\infty(D, \mathcal{K})} \Phi(u),$$

with $\Phi : L^2(D, \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\Phi(u) = \left\| \sum_{i=1}^N c_i \mathcal{R}u_i - f \right\|_{L^2}^2 + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle \quad \forall u \in L^2(D, \mathbb{R}^N).$$

We have

$$\nabla\Phi(u) = \begin{pmatrix} 2c_1\mathcal{R}^* \left(\sum_{i=1}^N c_i\mathcal{R}u_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq 1} \alpha_{1i} L_\varepsilon u_i \\ \vdots \\ 2c_N\mathcal{R}^* \left(\sum_{i=1}^N c_i\mathcal{R}u_i - f \right) + \frac{1}{\varepsilon} \sum_{i \neq N} \alpha_{iN} L_\varepsilon u_i \end{pmatrix}, \quad (6.4.1)$$

with \mathcal{R}^* the adjoint operator of \mathcal{R} .

To determinate the lipschitz constant λ , we will use the following observation.

Lemma 6.5. *The gradient $\nabla\Phi$ is λ -lipschitz if*

$$\lambda \geq \left(8N^2\bar{C}^4 \|\mathcal{R}^*\mathcal{R}\|^2 + \frac{C(N-1)}{\varepsilon^2} \right)^{\frac{1}{2}}$$

with C, \bar{C} defined by (6.1.10) and (6.2.2) respectively.

Proof. The proof is the same computation as in the proof of Lemma 6.4. □

Algorithm

For ε fixed we have to solve the approximate problem

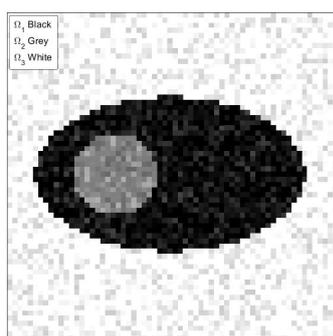
$$\mathcal{I}_\varepsilon^{PG} = \min_{u \in L^\infty(D, \mathcal{K})} \inf_{\tilde{u} \in L^2(D)} \left\{ \Phi(\tilde{u}) + \langle \nabla\Phi(\tilde{u}), u - \tilde{u} \rangle + \frac{\lambda}{2} \|u - \tilde{u}\|^2 \right\},$$

with

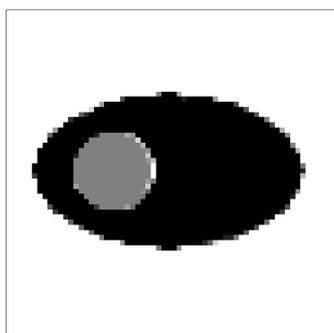
$$\Phi(u) = \left\| \sum_{i=1}^N c_i \mathcal{R}u_i - f \right\|_{L^2}^2 + \frac{1}{\varepsilon} \sum_{1 \leq i < j \leq N} \alpha_{ij} \langle L_\varepsilon u_i, u_j \rangle.$$

We can directly apply the algorithm of Section 6.1.

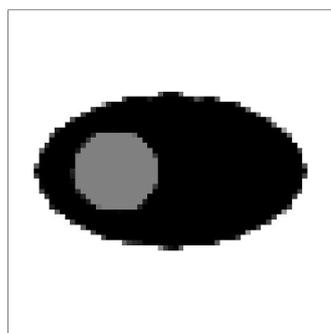
We present in Figure 6.5 an example of reconstruction with the finite difference method (FDM).



(a) Source.



(b) Reconstructed image for $\alpha_{ij} = 280$.



(c) Reconstructed image for $\alpha_{12}=250$ and $\alpha_{ij}=280$ otherwise.

Figure 6.5: Medical imaging with initialization: $u_2=1$ and $u_i=0$ otherwise.

Appendix A

FEM, FDM and FFT

For solving the boundary value problem (2.5.1) we first use finite elements on a Cartesian mesh with $Q1$ shape functions. The mesh nodes coincide with the image pixels, and without any loss of generality, the mesh size is fixed to 1. The discrete counterparts of the variables u and v are therefore vectors of \mathbb{R}^p where p is the number of pixels.

The linear system (5.2.24) is solved in an efficient way with the help of the fast Fourier transform (FFT), according to the following procedure. First, symmetries of the image u are performed in both axial directions, in such a way that we consider a domain \hat{D} with double width and height. Then periodicity conditions are assumed on the boundary of \hat{D} (see Figure A.1 for the symmetrization of the mesh), which is a convenient way of implementing Neumann boundary conditions. In this framework, the matrix products Kv and Mv represent bidimensional discrete convolutions, which are easily transferred to the Fourier domain. The Fourier transform of v is thus obtained, and v itself is retrieved by inverse FFT.

We also use the standard finite difference method (FDM) with five-point stencil for solving the boundary value problem (2.5.1). The periodicity conditions are assumed on the boundary of \hat{D} (see Figure A.2 for the symmetrization of the mesh).

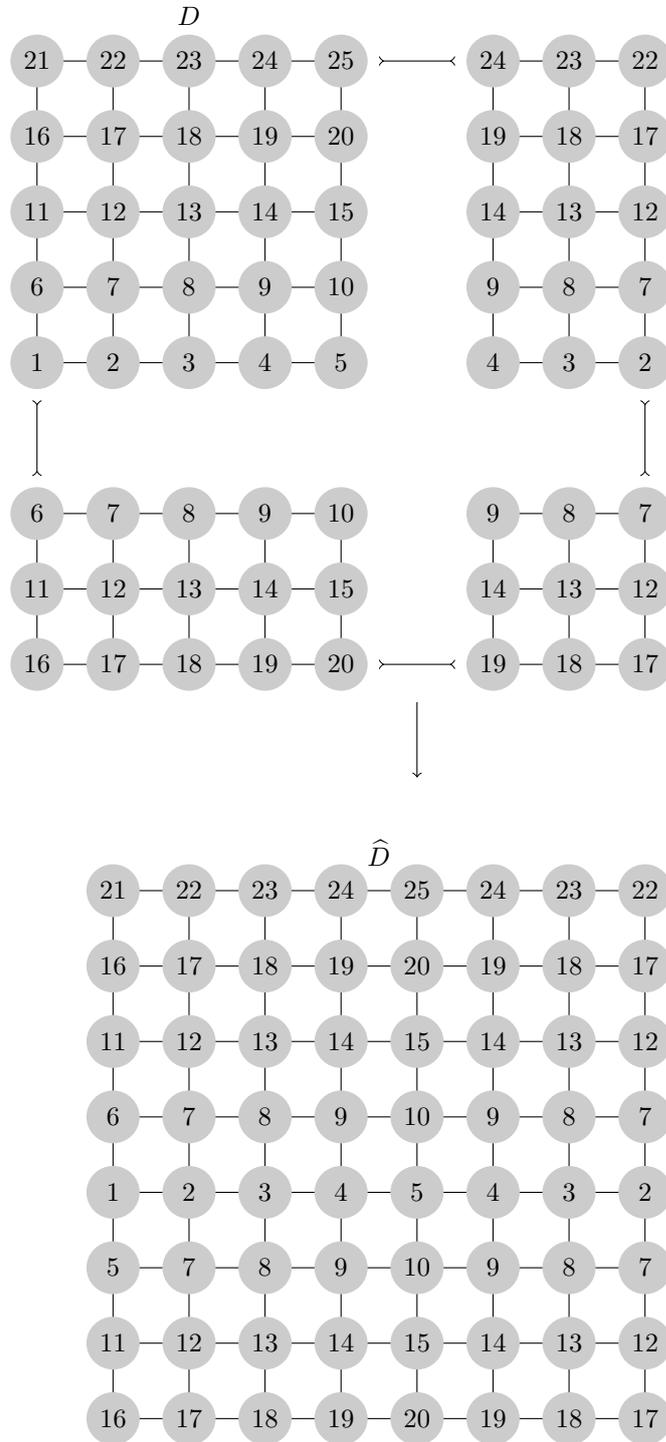


Figure A.1: Construction of \hat{D} in the FEM by mesh symmetrization (global node numbering in D).

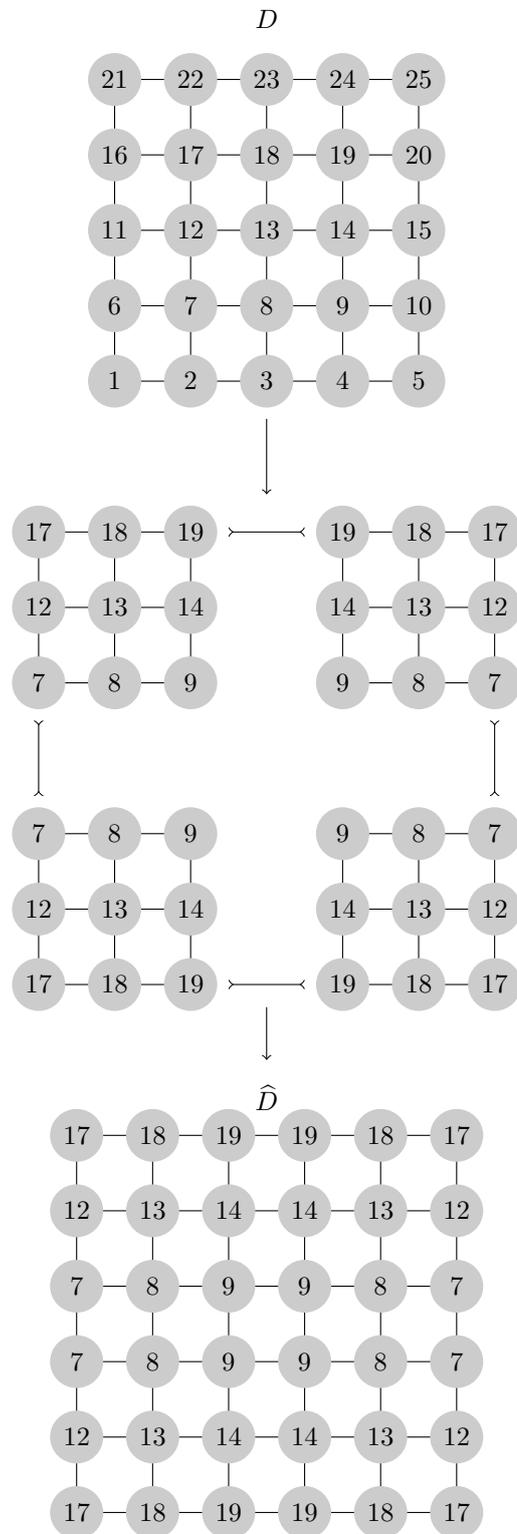


Figure A.2: Construction of \hat{D} in the FDM by mesh symmetrization (global node numbering in D).

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Abstract

Minimal partition problems consist in finding a partition of a domain into a given number of components in order to minimize a geometric criterion. In applicative fields such as image processing or continuum mechanics, it is standard to incorporate in this objective an interface energy that accounts for the lengths of the interfaces between components. The present work is focused on the theoretical and numerical treatment of minimal partition problems with interface energies. The considered approach is based on a Γ -convergence approximation and duality techniques.

Résumé

Les problèmes de partition minimale consistent à déterminer une partition d'un domaine en un nombre donné de composantes de manière à minimiser un critère géométrique. Dans les champs d'application tels que le traitement d'images et la mécanique des milieux continus, il est courant d'incorporer dans cet objectif une énergie d'interface qui prend en compte les longueurs des interfaces entre composantes. Ce travail est focalisé sur le traitement théorique et numérique de problèmes de partition minimale avec énergie d'interface. L'approche considérée est basée sur une approximation par Γ -convergence et des techniques de dualité.