Singularity and Lie Theory

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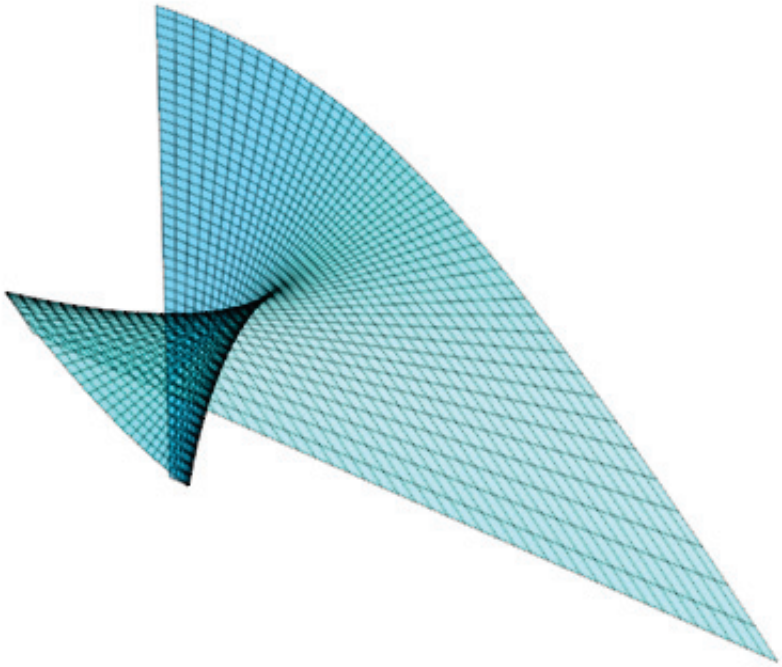
Singularité et Théorie de Lie



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Résumé

Soit Γ un sous-groupe fini de SU_2. F. Klein a publié [Klein84] en 1884 dans lequel il démontre que le quotient $\mathbb{C}^2/\Gamma$ est isomorphe à une surface dans $\mathbb{C}^3$ définie par une équation polynomiale. Cette surface possède une unique singularité isolée et est appelée singularité de Klein, ou singularité simple. Ces singularités ont par la suite été étudiées par de nombreux auteurs. En 1934, P. Du Val a démontré dans [DuVa34] que la préimage du point singulier de la singularité de Klein par sa résolution minimale est une union de droites projectives dont la matrice d'intersection est l'opposée de la matrice de Cartan d'une algèbre de Lie simplement lacée de type A_r, D_r, E_6, E_7 ou E_8. Cette relation entre la théorie de Lie et les singularités simples a depuis été exploitée par, entre autres, E. Brieskorn qui a montré dans [Bries71] que la déformation semiuniverselle d'une singularité simple $\mathbb{C}^2/\Gamma$ peut être obtenue par le quotient adjoint de l'algèbre de Lie de même type que $\mathbb{C}^2/\Gamma$. Il a aussi démontré une conjecture d'A. Grothendieck portant sur une résolution simultanée de ce même quotient adjoint. En 1978, P. Slodowy a donné une description en termes de systèmes de racines des singularités présentes dans les fibres d'une déformation d'une singularité de Klein différentes de la fibre spéciale. Puis en 1998, P. Slodowy et H. Cassens publièrent [CaSlo98] dans lequel ils construisent les déformations semiuniverselles des singularités simples dans un contexte algébro-géométrique en utilisant la théorie des représentations de carquois ainsi que des travaux de P.B. Kronheimer en géométrie différentielle. À la base de cette construction, on trouve la correspondance de McKay, découverte en 1980 par J. McKay ([McK80]), et qui établit un lien entre les représentations irréductibles des sous-groupes finis de SU_2 et les diagrammes de Dynkin étendus de types $\widetilde{A_r}$, $\widetilde{D_r}$, $\widetilde{E_6}$, $\widetilde{E_7}$ et $\widetilde{E_8}$.

En théorie de Lie, la classification des algèbres de Lie simples permet leur séparation en deux catégories: d'un côté les algèbres de Lie simplement lacées de types A_r, D_r, E_6, E_7 et E_8, de l'autre les algèbres de Lie non simplement lacées (ou inhomogènes) de types B_r, C_r, F_4 et G_2. P. Slodowy a étendu la notion de singularité simple aux types inhomogènes en 1978 et a défini des singularités simples de types B_r, C_r, F_4 et G_2. Une singularité simple inhomogène est alors un couple (X_0, Ω) où $X_0 = \mathbb{C}^2/\Gamma$ est une singularité simple homogène et Ω est un groupe de symétrie du diagramme de Dynkin associé à la singularité. Il a ensuite montré que l'on peut trouver un sous-groupe fini Γ' de SU_2 qui contient Γ comme sous-groupe distingué et tel que le quotient $\Gamma'/\Gamma \cong \Omega$ agit sur la singularité $\mathbb{C}^2/\Gamma$. Cette action peut être relevée sur la résolution minimale de $\mathbb{C}^2/\Gamma$ et induit une action sur les diviseurs exceptionnels qui correspond au groupe d'automorphismes du diagramme de Dynkin associé à $\mathbb{C}^2/\Gamma$. P. Slodowy a par

la suite démontré des résultats généralisant ceux obtenus pour les singularités simples homogènes. Il a notamment construit la déformation semiuniverselle d'une singularité de Klein inhomogène à partir du quotient adjoint d'une algèbre de Lie homogène et d'un groupe de symétrie du diagramme de Dynkin associé.
La correspondance de McKay se généralise également aux cas inhomogènes. En effet, il est possible d'établir un lien entre les diagrammes de Dynkin de types B_r, C_r, F_4 et G_2 et certains couples (Γ, Γ') de sous-groupes finis de SU_2 avec $\Gamma \lhd \Gamma'$ et ce de deux manières différentes. La première, dite par induction, consiste à induire des représentations de Γ' à partir des représentations irréductibles de Γ. On peut à partir de là retrouver la matrice de Cartan transposée du diagramme de Dynkin étendu $\widetilde{\Delta^\vee}(\Gamma, \Gamma')$ associé à la singularité simple inhomogène $(\mathbb{C}^2/\Gamma, \Gamma'/\Gamma)$ de type $\Delta(\Gamma, \Gamma')$. La seconde méthode restreint les représentations irréductibles de Γ' en des représentations de Γ et obtient la matrice de Cartan de type $\widetilde{\Delta^\vee}(\Gamma, \Gamma')$, transposée de celle obtenue par induction.

L'objectif de cette thèse est de généraliser la construction de H. Cassens et P. Slodowy d'une déformation semiuniverselle aux singularités de types B_r, C_r, F_4 et G_2. On va commencer par rappeler les définitions et propriétés de base des algèbres de Lie semisimples sur un corps algébriquement clos de caractéristique nulle, et on donnera les résultats de E. Brieskorn et P. Slodowy sur le lien entre le quotient adjoint et les singularités simples. On établira également un lien entre les diagrammes de Dynkin homogènes et inhomogènes à travers un procédé appelé pliage. Dans un second temps, on présentera les définitions et théorèmes nécessaires à la compréhension des résultats de P.B. Kronheimer en géométrie différentielle et symplectique, ainsi que d'un résultat de G. Kempf et L. Ness ([KemNes79]) reliant les quotients hyperkählériens et les quotients GIT. Le troisième chapitre sera l'occasion d'introduire la notion de représentation de carquois ainsi que son lien avec les systèmes de racines d'algèbres de Lie. On donnera également une interprétation du pliage en termes de carquois. Le chapitre 4 sera consacré à la généralisation des travaux de H. Cassens et P. Slodowy. Pour ce faire, on étudiera l'espace des représentations d'un carquois appelé carquois de McKay défini à partir du diagramme de Dynkin associé à $\mathbb{C}^2/\Gamma$, ainsi que d'un groupe de symétrie $\Omega = \Gamma'/\Gamma$ de ce même diagramme de Dynkin en se basant sur la correspondance de McKay inhomogène. L'orientation de ce carquois, présente dans la construction de H. Cassens et P. Slodowy mais jusqu'ici arbitraire, jouera ici un rôle bien plus important. En effet, la déformation semiuniverselle est construite à partir d'une application moment dépendant de l'orientation du carquois. Si l'action du groupe de symétrie Ω est symplectique, alors l'application moment devient naturellement Ω-équivariante. La déformation semiuniverselle devient elle aussi Ω-équivariante et grâce à des travaux de P. Slodowy on obtient la déformation semiuniverselle d'une singularité inhomogène. On déterminera tout d'abord dans quelles conditions l'action de Ω est symplectique ainsi que la compatibilité de cette action avec la symétrie du diagramme de Dynkin. On calculera ensuite explicitement la déformation semiuniverselle des singularités de types A_{2r-1} (r quelconque), C_3, F_4 et G_2. Ces calculs vont nécessiter l'utilisation d'un système de coordonnées particulier sur l'espace de base de la déformation: les coordonnées plates

de K. Saito. Après avoir rappelé leur définition ainsi que la raison pour laquelle on les a sélectionnées, des formules explicites en seront données. La déformation semiuniverselle inhomogène obtenue vérifie que chacune de ces fibres est munie d'une action de Ω. En passant chaque fibre et donc la déformation au quotient, on obtient une nouvelle déformation mais cette fois d'une singularité simple homogène. On verra que le caractère semiuniverselle est perdu par passage au quotient. On terminera par une étude de la régularité des fibres de cette déformation sur certains exemples.

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Introduction

In [Klein84], F. Klein studied the polynomial equations of degree 5. For this he looked at the rotation groups of the five Platonic solids and at the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, which are (up to conjugacy) the cyclic group $\mathcal{C}_n$, the binary dihedral group $\mathcal{D}_n$ and the binary polyhedral groups $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$. They are exactly the finite subgroups of SU_2. F. Klein proved that for such a group Γ, the quotient $\mathbb{C}^2/\Gamma$ is a surface S in $\mathbb{C}^3$ defined by a polynomial equation $R(X,Y,Z) = 0$. The surface has an isolated singularity and is called a Kleinian (or simple) singularity. It was proved that a minimal resolution exists for all surface singularities and thus for all Kleinian singularities, and P. Du Val showed in [DuVa34] that, if $\pi : \widetilde{S} \to S$ is the minimal resolution of the simple singularity S and s is the isolated singular point, then the exceptional divisor $\pi^{-1}(s)$ is a union of projective lines $\mathbb{P}^1(\mathbb{C})$ whose intersection matrix is the opposite of the Cartan matrix of a root system of type A_r, D_r, E_6, E_7 or E_8 that we note $\Delta(\Gamma)$. This result is summarized in the following table:

Γ	R	$\Delta(\Gamma)$
$\mathcal{C}_n$	$X^n + YZ$	A_{n-1}
$\mathcal{D}_n$	$X(Y^2 - X^n) + Z^2$	D_{n+2}
$\mathcal{T}$	$X^4 + Y^3 + Z^2$	E_6
$\mathcal{O}$	$X^3 + XY^3 + Z^2$	E_7
$\mathcal{I}$	$X^5 + Y^3 + Z^2$	E_8

This connection between Lie theory and Kleinian singularities has since been extensively studied. Let Γ be a finite subgroup of SU_2, $\Delta(\Gamma)$ the associated Dynkin diagram and $\mathfrak{g}$ a simple Lie algebra of type $\Delta(\Gamma)$. E. Brieskorn proved in 1971 ([Bries71]) that the semiuniversal deformation of the simple singularity $\mathbb{C}^2/\Gamma$ can be obtained by restricting the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to a particular subspace of $\mathfrak{g}$ called a slice and linked to subregular nilpotent elements of $\mathfrak{g}$. He also proved a conjecture by A. Grothendieck relative to the construction of a simultaneous resolution of the adjoint quotient. P. Slodowy showed in his thesis in 1978 (cf. the enlarged version [Slo80]) how to describe the singularities that appear in the fibers of a deformation of a simple singularity $\mathbb{C}^2/\Gamma$ other than the special fiber containing $\mathbb{C}^2/\Gamma$ in terms of subdiagrams of $\Delta(\Gamma)$. Let us also mention T.A. Springer's resolution of the nilpotent variety of $\mathfrak{g}$ ([Sprin69]).

In [McK80] J. McKay discovered a connection between the finite subgroups of SU_2 and the simply-laced Lie algebras without using a resolution. He exhibited a way of constructing the Cartan matrix of the extended Dynkin diagram $\widetilde{\Delta}(\Gamma)$ from the irreducible representations of Γ. From this correspondence one can construct the representation space $M(\Gamma)$ of a quiver obtained from $\Delta(\Gamma)$ and called a McKay quiver. The dimension vector of the said quiver is composed of the dimensions of the irreducible representations of Γ. It happens that $M(\Gamma)$ can be equipped with a symplectic structure based on the orientation of the McKay quiver. P.B. Kronheimer took advantage of such a structure and constructed in [Kron89] a semiuniversal deformation of $\mathbb{C}^2/\Gamma$ using hyperkähler reduction. Then in [CaSlo98] H. Cassens and P. Slodowy worked on P.B. Kronheimer's results to obtain the semiuniversal deformation of $\mathbb{C}^2/\Gamma$ and its minimal resolution in an algebraic-geometric context.

Dynkin diagrams can be separated in two classes: the simply-laced (or homogeneous) ones, namely A_r, D_r, E_6, E_7 and E_8 and the non simply-laced (or inhomogeneous) ones B_r, C_r, F_4 and G_2. In 1978 P. Slodowy extended the definition of a simple singularity to the inhomogeneous types in the following way: a simple singularity of type B_r, C_r, F_4 or G_2 is a pair (X_0, Ω) where X_0 is a simple singularity of type A_r, D_r, E_6, E_7 or E_8 and Ω is a group of automorphisms of the Dynkin diagram associated to X_0 as indicated in the table below:

type (X_0, Ω)	type X_0	Ω	Γ	Γ'
B_r	A_{2r-1}	$\mathbb{Z}/2\mathbb{Z}$	$\mathcal{C}_{2r}$	$\mathcal{D}_r$
C_r	D_{r+1}	$\mathbb{Z}/2\mathbb{Z}$	$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$
F_4	E_6	$\mathbb{Z}/2\mathbb{Z}$	$\mathcal{T}$	$\mathcal{O}$
G_2	D_4	$\mathfrak{S}_3$	$\mathcal{D}_2$	$\mathcal{O}$

Let Γ be a finite subgroup of SU_2 and $X_0 = \mathbb{C}^2/\Gamma$. Then it is possible to find a finite subgroup Γ' of SU_2 such that $\Gamma \lhd \Gamma'$ and $\Gamma'/\Gamma = \Omega$ acts on X_0. This action can then be lifted to the minimal resolution of the singularity and induces an action on the exceptional divisors that corresponds to a group of automorphisms of the Dynkin diagram of X_0. P. Slodowy generalized the McKay correspondence to the inhomogeneous types. Let (Γ, Γ') be a pair as in the table above and $\Delta(\Gamma, \Gamma')$ the associated inhomogeneous Dynkin diagram. By restricting the irreducible representations of Γ' to Γ, one can construct the extended Cartan matrix of type $\widetilde{\Delta^\vee}(\Gamma, \Gamma')$. The correspondence can also be obtained by inducing representations of Γ' from representations of Γ, in which case we obtain the Cartan matrix of the dual diagram.

In this thesis we aim to generalize the construction by H. Cassens and P. Slodowy to

the inhomogeneous cases. Starting from the representation $M(\Gamma)$ of a McKay quiver and a symmetry group Ω of the Dynkin diagram $\Delta(\Gamma)$ associated to Γ and based on the inhomogeneous McKay correspondence, we will construct the semiuniversal deformations of the simple singularities of types B_r, C_r, F_4 and G_2. In order to do so, we study the representation space of a quiver called McKay quiver, that is defined by using the Dynkin diagram associated to the simple singularity $\mathbb{C}^2/\Gamma$ as well as a symmetry group Ω of the same diagram based on the inhomogeneous McKay correspondence. The action of the group Ω on $M(\Gamma)$ is obtained by lifting its action on the underlying graph. The choice of the orientation of the quiver, which did not play any particular role in the homogeneous case, will now have to be carefully made (Subsection 4.3.3). Indeed the representation space of the McKay quiver can be equipped with a symplectic structure depending on the orientation of the McKay quiver. This symplectic structure induces a moment map from which the semiuniversal deformation is obtained. If the action of Ω is symplectic, then the semiuniversal deformation becomes Ω-equivariant. We will need to determine the conditions in which the action is symplectic as well as their compatibility with the action of Ω on the special fiber of the deformation. In fact we will prove the following theorem (cf. Theorems 4.3.2 and 4.3.4)

Theorem. *1. The action of $\Omega = \Gamma'/\Gamma$ on $M(\Gamma)$ is symplectic when*

(i) For $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$, Ω reverses the orientation of the McKay quiver.

(ii) For the other cases, Ω preserves the orientation of the McKay quiver.

2. For any inhomogeneous singularity $(\mathbb{C}^2/\Gamma, \Omega)$, there exists an action of Ω on $M(\Gamma)$ such that

(i) Ω acts symplectically on $M(\Gamma)$.

(ii) The action of Ω on $M(\Gamma)$ induces the natural action on the singularity $\mathbb{C}^2/\Gamma$.

With the semiuniversal deformation now Ω-equivariant, results by P. Slodowy will lead to the semiuniversal deformation of a simple singularity of inhomogeneous type. This semiuniversal deformation is such that the group Ω induces an action on each of its fibers. In order to analyze the family to which the quotient of the fibers will lead we need to obtain explicit expressions for the semiuniversal deformation. In that sense we will compute explicitly the semiuniversal deformations of the inhomogeneous simple singularities of types A_{2r-1} (r arbitrary), C_3, F_4 and G_2. After quotient, we obtain a new deformation, but this time of the simple homogeneous singularity $\mathbb{C}^2/\Gamma'$. We will see that the semiuniversal nature of the deformation is lost in the process. We will also prove results regarding the regularity of the fibers of the new deformation. For example, when Γ is of type A_3 or D_4 (and Ω is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathfrak{S}_3$), we will see that every single fiber is singular after quotient by Ω (cf. Propositions 4.4.1, 4.4.2 and 4.4.3).

In the first chapter in Part I, we recall the main definitions and properties of semisimple Lie algebras over an algebraically closed field of characteristic zero (Section 1.1). Then a relation between the simply-laced and non-simply laced cases will be exhibited

through a process called folding in Section 1.2, after which known results on homogeneous simple singularities from E. Brieskorn and P. Slodowy will be presented and explained in Section 1.4. Following P. Slodowy's definition of the inhomogeneous simple singularities, we will provide in Subsection 1.4.2 some of his results generalizing those already obtained on the simple homogeneous ones. The chapter will end with an explanation as well as computations of the McKay correspondence (Section 1.5), in both homogeneous and inhomogeneous settings.
The second chapter will be devoted to the symplectic reduction. In particular, we will present a theorem by G. Kempf and L. Ness linking hyperkähler quotients and GIT quotients (Section 2.3) that will be of use later on.
In the third chapter we define quivers, representation spaces as well as basic notions related to them (Section 3.1). We will then present results by T. Tanisaki ([Tani80]) that explain the folding process using quivers representation spaces (Section 3.4), although it will not be used in our construction, but is related nonetheless.

The construction by H. Cassens and P. Slodowy as well as its generalization to the inhomogeneous cases will occupy Part II and the final chapter of this thesis. At first we recall the work of P.B. Kronheimer on which the original construction is based (Section 4.1), present the construction itself (Section 4.2.1), and then study the case A_{2r-1} as an example (Section 4.2.2). The work on a generalization of the construction to the inhomogeneous cases will occupy the rest of this thesis.
We study the orientation of the McKay quiver as well as the action of Ω from Subsection 4.3.1 to Subsection 4.3.4. Then we will compute explicitly the semiuniversal deformations of the inhomogeneous simple singularities of types A_{2r-1} (r arbitrary), C_3, F_4 and G_2. These computations require the use of a particular set of coordinates on the base space of the deformation: K. Saito's flat coordinates. After defining them and explaining why they can be of use for our construction, explicit formulae will be given (Subsection 4.3.5). The computations of the semiuniversal deformations of the simple inhomogeneous singularities will be presented in Subsections 4.3.6 through 4.3.9. The quotient of the semiuniversal deformation by the action of Ω will then be done in Subsections 4.4.2 through 4.4.5. The thesis will be concluded with computations on the regularity of the fibers of this deformation in some special cases.

Part I.

Background

1. Lie Theory

1.1. Definitions and basic properties

In this section we give basic definitions on Lie algebras as well as some elementary properties. Many of the proofs will be omitted. If not stated otherwise, the reader may consult the book [Hum78] by J.E. Humphreys for the proofs.
We fix an algebraically closed field $\Bbbk$ of characteristic 0.

1.1.1. Generalities on Lie algebras

Definition 1.1.1. *A vector space $\mathfrak{L}$ over a field $\Bbbk$, with an operation* $\begin{array}{ccc} \mathfrak{L}\times\mathfrak{L} & \to & \mathfrak{L} \\ (x,y) & \mapsto & [x,y] \end{array}$
called bracket, is called a ***Lie algebra*** *if the three following properties are verified:*

(i) The bracket is bilinear.

(ii) $[x,x]=0$ *for all* $x\in\mathfrak{L}$.

(iii) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ *for all* $x,y,z\in\mathfrak{L}$.

The second axiom is equivalent to the bracket being skew-symmetric and the third one is commonly called **Jacobi's identity**. The notion of subalgebra of a Lie algebra $\mathfrak{L}$ is defined naturally as a subspace of $\mathfrak{L}$ that is a Lie algebra with respect to the restriction of the Lie structure on $\mathfrak{L}$.
In what follows all the Lie algebras will have finite dimension (i.e. the underlying vector space of the Lie algebra is finite dimensional). The study of infinite dimensional Lie algebras requires other tools that we will not present in this thesis. A detailed account on this subject can be found in [Kac90].

Examples 1.1.2. • *If V is a $\Bbbk$-vector space of dimension n, let us write $\mathfrak{gl}(V)=\mathrm{End}(V)$ for the set of endomorphisms of the vector space V. By choosing a base of V, one can identify $\mathfrak{gl}(V)$ with the set of matrices of size $n\times n$. It is a Lie algebra with bracket $[x,y]=xy-yx$ and is called* ***the general linear algebra****. All the Lie algebras in these examples will be equipped with the same bracket. $\mathfrak{sl}(V)$ is the subalgebra of $\mathfrak{gl}(V)$ composed of all the traceless matrices of $\mathfrak{gl}(V)$. It is called* ***the special linear algebra****.*

The structure of $\mathfrak{sl}(V)$ *when* $n = 2$ *will be seen in details later on. Any subalgebra of* $\mathfrak{gl}(V)$ *is called a* ***linear Lie algebra****.*

- *Let* V *be* $\mathbb{k}$*-vector space of dimension* $2n$*. Set* $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ *with* I_n *the identity matrix of size* $n \times n$*. One writes* $\mathfrak{sp}(V) = \{M \in \mathfrak{gl}(V) \mid {}^tAJ + JA = 0\}$*, a space that will be referred to as* ***the symplectic algebra****. One can check that it is indeed a Lie algebra.*

- ***The orthogonal algebra*** *is defined in the same way as the symplectic one. If* $\dim V = 2n$*, set* $J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ *and if* $\dim V = 2n+1$*, set* $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$*. The orthogonal algebra is defined as* $\mathfrak{o}(V) = \{M \in \mathfrak{gl}(V) \mid {}^tAJ + JA = 0\}$*. It is yet again a Lie algebra.*

Usually, when it is clear what vector space is chosen, (V) *is omitted and the dimension of* V *is put as index. The preceding examples are thus* $\mathfrak{gl}_n(\mathbb{k}), \mathfrak{sl}_n(\mathbb{k}), \mathfrak{sp}_{2n}(\mathbb{k}), \mathfrak{o}_{2n}(\mathbb{k})$ *and* $\mathfrak{o}_{2n+1}(\mathbb{k})$*.*

A **morphism** of Lie algebras $\varphi : \mathfrak{L} \to \mathfrak{L}'$ is a linear transformation such that $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{L}$. A **representation** (or **module**) of a Lie algebra $\mathfrak{L}$ is a morphism $\varphi : \mathfrak{L} \to \mathfrak{gl}(V)$ where V is a vector space. When the context is clear, we will write $h.x$ for $\varphi(h)(x)$. If V is a representation of $\mathfrak{L}$ and $W \subset V$ is stable by $\mathfrak{L}$, then W is an $\mathfrak{L}$-sub-representation (or submodule). A representation $\varphi : \mathfrak{L} \to \mathfrak{gl}(V)$ is said to be **irreducible** if the only $\mathfrak{L}$-submodules of V are $\{0\}$ and itself. One important kind of representation is the **adjoint representation**. It is the morphism $\mathrm{ad} : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ such that $\mathrm{ad}(x)(y) = [x,y]$ for all $x, y \in \mathfrak{L}$. One can check explicitly that ad preserves the Lie bracket and is linear. It is then a morphism of Lie algebras. An interesting fact is that $\mathrm{Ker}\ \mathrm{ad} = \mathfrak{z}(\mathfrak{L})$, the center of the Lie algebra. So if $\mathfrak{L}$ is **simple**, i.e. if $\mathfrak{L}$ has no non-trivial ideal for the Lie bracket and $\mathfrak{L}$ is not abelian, then $\mathrm{Ker}\ \mathrm{ad} = \{0\}$ and $\mathfrak{L}$ injects into $\mathfrak{gl}(\mathfrak{L})$. Therefore any simple Lie algebra can be seen as a linear Lie algebra.

It will be shown at the end of this section that the algebras $\mathfrak{sl}_n(\mathbb{k})$ $(n \geq 2)$, $\mathfrak{sp}_{2n}(\mathbb{k})$ $(n \geq 2)$, $\mathfrak{o}_{2n}(\mathbb{k})$ $(n \geq 3)$, and $\mathfrak{o}_{2n+1}(\mathbb{k})$ $(n \geq 2)$ in the preceding examples are in fact simple Lie algebras.

Example 1.1.3. *One can check that* $\mathfrak{sl}_2(\mathbb{C})$ *has a basis consisting of* $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *with the relations* $[e,f] = h$*,* $[h,e] = 2e$ *and* $[h,f] = -2f$*. An infinite dimensional representation of* $\mathfrak{sl}_2(\mathbb{C})$ *noted* $\varphi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}[X,Y])$ *is defined as follows:*

$$\varphi(e) = X\frac{\partial}{\partial Y},\ \varphi(f) = Y\frac{\partial}{\partial X}\ \text{and}\ \varphi(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

It can be seen that φ verifies the requirements for making $\mathbb{C}[X,Y]$ into an $\mathfrak{sl}_2(\mathbb{C})$-module. The action is given by the following formulas:

$$e.X^mY^n = nX^{m+1}Y^{n-1},\ f.X^mY^n = mX^{m-1}Y^{n+1},\ h.X^mY^n = (m-n)X^mY^n.$$

The action preserves the degree of the monomials, so $\mathbb{C}[X,Y]$ splits as a direct sum $\mathbb{C}[X,Y]_0 \oplus \mathbb{C}[X,Y]_1 \oplus \mathbb{C}[X,Y]_2 \oplus \dots$ as an $\mathfrak{sl}_2(\mathbb{C})$-module, where $\mathbb{C}[X,Y]_k$ is the submodule of homogeneous polynomials of degree k. The monomials of degree k form a base for $\mathbb{C}[X,Y]_k$, so $\mathbb{C}[X,Y]_k$ is a $\mathbb{C}$-vector space of dimension $k+1$.
We claim that $\mathbb{C}[X,Y]_k$ is irreducible. Let $x \neq 0$ in an $\mathfrak{sl}_2(\mathbb{C})$-submodule of $\mathbb{C}[X,Y]_k$. Then $x = \sum_{i=0}^{k} a_i X^i Y^{k-i}$ with the a_i's being complex numbers. Let $k_0 = \min\{0 \le i \le k \mid a_i \neq 0\}$. Therefore $\varphi(e)^{k-k_0}(x) = a_{k_0}(k-k_0)!X^k$ and X^k belongs to the submodule. Because $\varphi(f)^j(X^k) = k(k-1)\dots(k-j+1)X^{k-j}Y^j$, it follows that all the elements of the base of $\mathbb{C}[X,Y]_k$ are in the submodule, so the submodule is all of $\mathbb{C}[X,Y]_k$. This proves that $\mathbb{C}[X,Y]_k$ is irreducible, for all $k \in \mathbb{Z}_{\ge 0}$. Since $\dim \mathbb{C}[X,Y]_k = k+1$, all the $\mathbb{C}[X,Y]_k$ are non-isomorphic.

Proposition 1.1.4. *Let V be a $\Bbbk$-vector space of finite dimension and let $x \in \mathrm{End}(V)$.*

(i) There exists $x_s, x_n \in \mathrm{End}(V)$ such that: $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. Furthermore, x_s and x_n are unique.

(ii) There exists two polynomials $p(t)$ and $q(t)$ of one variable such that: $p(0) = q(0) = 0$, $x_s = p(x)$ and $x_n = q(x)$. In particular, x_s and x_n commute with every endomorphism commuting with x.

(iii) If $A \subset B \subset V$ are subspaces such that x maps B into A, then x_s and x_n also map B into A.

The decomposition $x = x_s + x_n$ is called **the Jordan-Chevalley decomposition**. We call x_s and x_n the semisimple and nilpotent parts of x respectively.

Lemma 1.1.5. *Let $x \in \mathrm{End}(V)$ ($\dim V < +\infty$) and $x = x_s + x_n$ its Jordan-Chevalley decomposition. Then $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ is the Jordan-Chevalley decomposition of $\mathrm{ad}\, x$ in $\mathrm{End}(\mathrm{End}(V))$.*

Let $\mathfrak{L}$ be a Lie algebra over $\Bbbk$. Its **derived series** is a sequence of ideals of $\mathfrak{L}$ defined by $\mathfrak{L}^{(0)} = \mathfrak{L}$, $\mathfrak{L}^{(1)} = [\mathfrak{L}^{(0)}, \mathfrak{L}^{(0)}]$, $\mathfrak{L}^{(2)} = [\mathfrak{L}^{(1)}, \mathfrak{L}^{(1)}], \dots$. The Lie algebra $\mathfrak{L}$ is **solvable** if there exists $n \in \mathbb{Z}_{>0}$ such that $\mathfrak{L}^{(n)} = \{0\}$.

Example 1.1.6. *Let $\mathfrak{L}$ be the Lie algebra of upper triangular matrices of size $n \times n$ with coefficients in $\Bbbk$ and the Lie bracket $[A,B] = AB - BA$. It is indeed a Lie algebra.*

One computes that $[\mathfrak{L},\mathfrak{L}]$ *is the set of upper triangular matrices with a zero diagonal. Each time a new bracket is computed the first non-zero diagonal goes up one place above. Therefore,* $\mathfrak{L}^{(n)} = \{0\}$ *and* $\mathfrak{L}$ *is solvable.*

The maximal solvable ideal of a Lie algebra $\mathfrak{L}$ is called the **radical** of $\mathfrak{L}$. If the radical is zero, then $\mathfrak{L}$ is called a **semisimple Lie algebra**. For a semisimple Lie algebra $\mathfrak{L}$, it can be proved that the space Der $\mathfrak{L}$ of derivations of $\mathfrak{L}$ coincides with the space ad $\mathfrak{L}$ and that $\mathfrak{L} \to$ ad $\mathfrak{L}$ is a bijection. So if $x \in \mathfrak{L}$, then there exists $s, n \in \mathfrak{L}$ such that ad x = ad s + ad n is the usual Jordan-Chevalley decomposition of ad x in End($\mathfrak{L}$). This means that $x = s + n$, with commuting s and n, ad s being semisimple and ad n being nilpotent. We call s and n the semisimple and nilpotent part of x, respectively. This decomposition is called the **abstract Jordan-Chevalley decomposition**. It is linked to the usual decomposition thanks to the following theorem and corollary.

Theorem 1.1.7. *Let* $\mathfrak{L} \subset \mathfrak{gl}(V)$ *be a semisimple linear Lie algebra over* $\Bbbk$ *(*V *finite dimensional). Then* $\mathfrak{L}$ *contains the semisimple and nilpotent parts in* $\mathfrak{gl}(V)$ *of all its elements. In particular, the abstract and usual Jordan-Chevalley decompositions coincide.*

Corollary 1.1.8. *Let* $\mathfrak{L}$ *be a semisimple Lie algebra over* $\Bbbk$ *and* $\phi : \mathfrak{L} \to \mathfrak{gl}(V)$ *a finite dimensional representation of* $\mathfrak{L}$*. If* $x = s + n$ *is the abstract Jordan-Chevalley decomposition of* $x \in \mathfrak{L}$*, then* $\phi(x) = \phi(s) + \phi(n)$ *is the usual Jordan-Chevalley decomposition of* $\phi(x)$*.*

Example 1.1.9. *Let* V *be a finite dimensional* $\mathfrak{sl}_2(\mathbb{C})$*-module. As* h *is semisimple,* h *acts diagonally on* V*. Thus one can write* $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$ *where* $V_\lambda = \{x \in V \mid h.x = \lambda x\}$*.* V *is decomposed with respect to the eigenvalues of* h*. If* $\lambda \in \mathbb{C}$ *is such that* $V_\lambda \neq \{0\}$*,* λ *is called a* ***weight*** *of* h *and* V_λ *is a* ***weight space****. A quick calculation shows that if* $v \in V_\lambda$*, then* $e.v \in V_{\lambda+2}$ *and* $f.v \in V_{\lambda-2}$*.*
As V *is of finite dimension, there is only a finite number of* λ *such that* V_λ *is not zero. So there exists* λ *such that* $V_\lambda \neq \{0\}$ *and* $V_{\lambda+2} = \{0\}$*. An element from this* V_λ *is a* ***highest weight vector*** *of weight* λ *(cf. Section 1.1.4).*
Let V *be an irreducible and finite dimensional* $\mathfrak{sl}_2(\mathbb{C})$*-module. Fix* $v_0 \in V_\lambda$ *a highest weight vector. We note* $v_{-1} = 0$ *and* $v_i = \frac{1}{i!} f^i.v_0$ *(*$i \geq 0$*). We then compute that*

$$(i)\ h.v_i = (\lambda - 2i)v_i,$$
$$(ii)\ f.v_i = (i+1)v_{i+1},$$
$$(iii)\ e.v_i = (\lambda - i + 1)v_{i-1}.$$

The v_i*'s turn out to be eigenvectors of* h *for distinct eigenvalues and thus are linearly independent. As* V *is finite dimensional, there exists a smaller* $m \in \mathbb{Z}_{\geq 0}$ *such that* $v_m \neq 0$ *and* $v_{m+1} = 0$*. By taking into account the preceding formulas, one deduces that*

$(v_0, v_1, \ldots, v_m)$ is a base of a non-zero $\mathfrak{sl}_2(\mathbb{C})$-submodule of V. By the irreducibility of V, $(v_0, v_1, \ldots, v_m)$ is a base of V. If in (iii) one sets $i = m+1$, we obtain $0 = (\lambda - m)v_m$. As $v_m \neq 0$, it implies that $\lambda = m \in \mathbb{Z}_{\geq 0}$. We say that m is the highest weight of V. One also notices that all the weight spaces are unidimensional and appear only once in the decomposition of V. The following theorem summarises our conclusions:

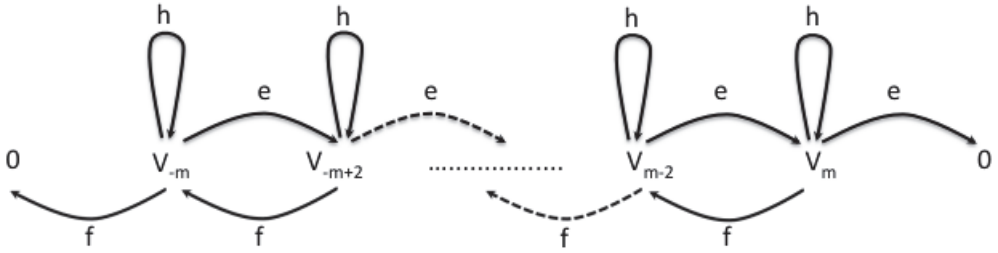
Theorem 1.1.10. *Let V be an irreducible finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module.*

(i) With respect to h, V is the direct sum of weight spaces V_μ where $\mu = m, m-2, \ldots, -m+2, -m$ with $\dim V = m+1$ and $\dim V_\mu = 1$ for all μ. We then write $V = L(m)$.

(ii) V has a unique maximal vector (up to multiplication by a scalar) of weight m.

(iii) The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is explicitly given by the previous formulas. In particular, there exists (up to isomorphism) at most one irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $m+1$ for all $m \geq 0$.

We can illustrate this theorem in the following manner.



Because of the previous example, one sees that for all $n \in \mathbb{Z}_{\geq 0}$, $\mathbb{C}[X,Y]_n \cong L(n)$ as $\mathfrak{sl}_2(\mathbb{C})$-modules.

The next theorem gives a hint on the structure of the representation space for a certain kind of algebra and will be of use later.

Theorem 1.1.11. (Lie). *Let $\mathfrak{L}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$ with V a finite dimensional $\Bbbk$-vector space. Then $\mathfrak{L}$ stabilizes a complete flag in V, i.e. there is a sequence of proper subspaces $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_n = V$ such that $\dim V_i = i$ and V_i is an $\mathfrak{L}$-module for all $0 \leq i \leq n$.*

1.1.2. Structure of semisimple Lie algebras

A semisimple Lie algebra can be described in a unique manner as a sum of simple Lie subalgebras:

Theorem 1.1.12. *Let $\mathfrak{L}$ be a semisimple Lie algebra over $\Bbbk$. Then there exists ideals $L_1, \ldots, L_t$ of $\mathfrak{L}$ which are simple (as Lie algebras), such that $\mathfrak{L} = \mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_t$. Every simple ideal of $\mathfrak{L}$ coincides with one of the $\mathfrak{L}_i$.*

A semisimple Lie algebra also has the property, proved by H. Weyl, that any finite dimensional module is completely reducible.

Theorem 1.1.13. (Weyl). *Let $\mathfrak{L}$ be a semisimple Lie algebra over $\Bbbk$ and V be a finite dimensional $\mathfrak{L}$-module. Then V is completely reducible, i.e. V can be uniquely decomposed as a direct sum of simple $\mathfrak{L}$-modules.*

Let $\mathfrak{L}$ be a non-zero semisimple Lie algebra over $\Bbbk$. If every element of $\mathfrak{L}$ was nilpotent, then by a theorem of F. Engel, $\mathfrak{L}$ would be nilpotent and so solvable. But then $\{0\} = \text{Rad } \mathfrak{L} = \mathfrak{L}$ because $\mathfrak{L}$ is semisimple. It contradicts $\mathfrak{L}$ being non-zero. So $\mathfrak{L}$ contains an element $x \in \mathfrak{L}$ such that its semisimple part x_s is not zero. Hence there exists non-zero subalgebras of $\mathfrak{L}$ composed of semisimple elements. Such a subalgebra is said to be **toral**. It can be proved that a toral subalgebra is abelian. A maximal (for the inclusion) toral subalgebra is called a **Cartan subalgebra**.

Theorem 1.1.14. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Cartan subalgebras of a semisimple Lie algebra $\mathfrak{L}$ over $\Bbbk$. Then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjugated under the group generated by the elements of the form $\exp(\text{ad } x)$ for $x \in \mathfrak{L}$ nilpotent. This implies that all Cartan subalgebras have same dimension, called the **rank** of $\mathfrak{L}$.*

Let $\mathfrak{H}$ denote a Cartan subalgebra of $\mathfrak{L}$. As $\mathfrak{H}$ is abelian, $\text{ad}_{\mathfrak{L}}(\mathfrak{H})$ is a family of commuting semisimple elements. It is therefore possible to simultaneously diagonalize $\text{ad}_{\mathfrak{L}}(\mathfrak{H})$. $\mathfrak{L}$ becomes the direct sum of subspaces $\mathfrak{L}_\alpha = \{x \in \mathfrak{L} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{H}\}$ with $\alpha \in \mathfrak{H}^*$. We write Φ for the set of $\alpha \neq 0$ such that $\mathfrak{L}_\alpha \neq \{0\}$. If $\alpha = 0$, we have $\mathfrak{L}_0 = C_{\mathfrak{L}}(\mathfrak{H})$, the centralizer in $\mathfrak{L}$ of $\mathfrak{H}$. The decomposition is then

$$\mathfrak{L} = C_{\mathfrak{L}}(\mathfrak{H}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha.$$

The α's in this decomposition are called the **roots** of $\mathfrak{L}$ relative to $\mathfrak{H}$ (and they are in finite number), Φ is the **root system** of $\mathfrak{L}$ and the preceding equality is called the **root space decomposition** of $\mathfrak{L}$. Since the Cartan subalgebras are all conjugated, the root system of $\mathfrak{L}$ does not depend on the choice of a Cartan subalgebra $\mathfrak{H}$.

Proposition 1.1.15. *Let $\mathfrak{H}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{L}$ over $\Bbbk$. Then $C_{\mathfrak{L}}(\mathfrak{H}) = \mathfrak{H}$ and the restriction of the Killing form* $\begin{array}{rccc} \kappa : & \mathfrak{L} \times \mathfrak{L} & \to & \Bbbk \\ & (x, y) & \mapsto & \text{Tr}(\text{ad}(x)\text{ad}(y)) \end{array}$ *to $\mathfrak{H}$ is non-degenerate.*

The previous proposition enables us to write $\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha$ and to identify $\mathfrak{H}$ with $\mathfrak{H}^*$ via the Killing form κ. To any element $\phi \in \mathfrak{H}^*$ one associates an element $t_\phi \in \mathfrak{H}$ by $\phi(h) = \kappa(t_\phi, h)$, $\forall h \in \mathfrak{H}$. We define a bilinear form $(.,.)$ on $\mathfrak{H}^*$ by $(\delta, \gamma) = \kappa(t_\delta, t_\gamma)$. Because the root system generates $\mathfrak{H}^*$, one can choose a base $\{\alpha_1, \ldots, \alpha_r\}$ consisting of roots. If $\beta \in \Phi$, then $\beta = \sum_{i=1}^r c_i\alpha_i$, where $c_i \in \Bbbk$. Furthermore, one can show that $c_i \in \mathbb{Q}$ for all i. Therefore the $\mathbb{Q}$-sub-vector space $E_\mathbb{Q}$ of $\mathfrak{H}^*$ generated by the roots has $\mathbb{Q}$-dimension $r = \dim_\Bbbk \mathfrak{H}^*$. Additionally it is known that $(\alpha, \beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$. This implies that the restriction of $(.,.)$ to $E_\mathbb{Q}$ is a non-degenerate form on $E_\mathbb{Q}$. By definition, for $\lambda, \mu \in \mathfrak{H}^*$,

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha\in\Phi} (\alpha, \lambda)(\alpha, \mu).$$

In particular $(\lambda, \lambda) = \sum_{\alpha\in\Phi}(\alpha, \lambda)^2$, so that if $\lambda \in E_\mathbb{Q}$, then (λ, λ) is a sum of squares of rational numbers and is hence positive (unless $\lambda = 0$). Therefore the form $(.,.)$ is positive definite on $E_\mathbb{Q}$.
We extend $E_\mathbb{Q}$ to the field of real numbers by setting $E = \mathbb{R} \otimes_\mathbb{Q} E_\mathbb{Q}$. The bilinear form extends to E and is positive definite, which makes E a Euclidean space. We have that Φ contains a base of E and $\dim_\mathbb{R} E = r$. The properties of E and Φ are summarized in the following theorem:

Theorem 1.1.16. *Let $\mathfrak{L}, \mathfrak{H}, \Phi$ and E be as above. Then:*

(i) *Φ generates E and 0 does not belong to Φ.*

(ii) *If $\alpha \in \Phi$, then $\Bbbk\alpha \bigcap \Phi = \{\pm\alpha\}$.*

(iii) *If $\alpha, \beta \in \Phi$, then $\beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$.*

(iv) *If $\alpha, \beta \in \Phi$, then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.*

Set $\langle\alpha, \beta\rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. If $\alpha \in E$, one defines the reflection $\sigma_\alpha \in \mathrm{GL}(E)$ by $\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\alpha$. If $\alpha \in \Phi$, the preceding theorem shows that σ_α leaves Φ invariant, and one can prove that it preserves the bracket $\langle.,.\rangle$. Let us write W for the subgroup of $\mathrm{GL}(E)$ generated by the σ_α, $\alpha \in \Phi$, and call it the **Weyl group** of $\mathfrak{L}$. As W permutes the elements of Φ, it can be seen as a subgroup of the symmetric group of Φ, and is thus finite.

A subset Π of a root system Φ is called a **base** if:

- Π is a base of E. This implies that $|\Pi| = \mathrm{rank}(\mathfrak{L})$.
- Every $\beta \in \Phi$ can be written $\beta = \sum_{\alpha\in\Pi} k_\alpha\alpha$ with integer coefficients all nonnegative (β is a **positive root**) or all nonpositive (β is a **negative root**).

An element of Π is called **simple**. One can show that a base exists for Φ but is not

unique. We shall see that the classification of simple Lie algebras can be achieved by studying the simple roots of a Lie algebra.

For $\alpha \in \Phi$, set $H_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$. The hyperplanes H_α partition E in a finite number of regions. The connected components of $E \setminus \bigcup_{\alpha\in\Phi} H_\alpha$ are called **Weyl chambers**. The Weyl chambers are in bijection with the bases of Φ, because the walls of a Weyl chamber define the elements of a base. If Π is a base of Φ, the Weyl chamber $\mathfrak{C}(\Pi)$ defined by

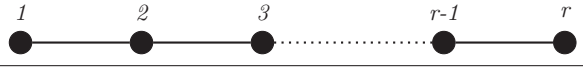
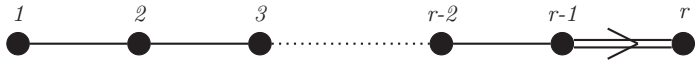
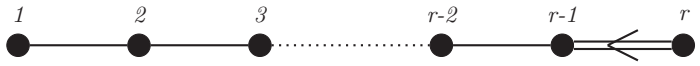
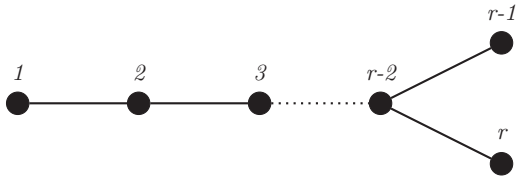
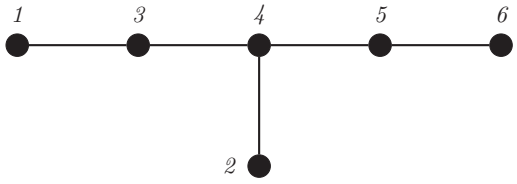
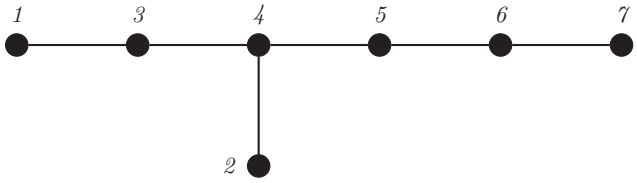
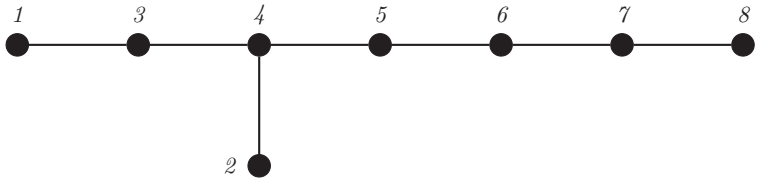
$$\mathfrak{C}(\Pi) = \{\beta \in E \mid (\beta, \alpha) > 0, \quad \forall \alpha \in \Pi\}$$

is called the **fundamental Weyl chamber**. The Weyl chambers are permuted by the Weyl group, and so are the bases of the root system.

Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a base of a root system Φ. The matrix $(\langle \alpha_i, \alpha_j \rangle)_{1\leq i,j\leq r}$ is called the **Cartan matrix** of Φ. The Cartan matrix does not depend on the choice of a base of the root system Φ because the Weyl group permutes the bases and preserves the bracket $\langle ., . \rangle$.
If α, β are two distinct non-proportional roots, it can be shown that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2$ or 3. One defines the **Coxeter graph** of Φ as the graph with r vertices, and such that i-th and the j-th vertices $(i \neq j)$ are linked by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. If all simple roots have the same length (the length of a root is its norm), the Coxeter graph determines the $\langle \alpha_i, \alpha_j \rangle$ because then $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$ and between two distinct vertices there is at most one edge. If simple roots have different lengths, there can be double or triple edges. In that situation, on a multiple edge, one adds an arrow pointing to the shortest root. With this complementary information, one can reconstruct the Cartan matrix from the graph. The graph is then called a **Dynkin diagram** and is noted Δ. The Dynkin diagram determines entirely the root system and their classification is given below.

A root system Φ is **irreducible** if it cannot be partitioned in two subsets such that each root of one subset is orthogonal to each root of the other subset. Φ is irreducible if and only if a base Π cannot be partitioned in the aforementioned manner. This is equivalent to saying that the Dynkin diagram of Φ is connected. In general, a Dynkin diagram is composed of several connected components and each one has a base Π_k. We write $\Pi = \Pi_1 \bigcup \ldots \bigcup \Pi_l$ the corresponding decomposition of the base. We then get a decomposition $\Phi = \Phi_1 \bigcup \ldots \bigcup \Phi_l$ of Φ as a union of sub-root systems and this decomposition is unique. Each of these irreducible sub-root systems is the root system of an irreducible component of the semisimple Lie algebra. Thus a semisimple Lie algebra is simple if and only if its root system is irreducible.

Theorem 1.1.17. *Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$ of rank r with an irreducible root system Φ. Then the Dynkin diagram Δ of Φ is one of the following (each diagram has r vertices):*

Type of Δ	Δ
A_r $(r \geq 1)$	
B_r $(r \geq 2)$	
C_r $(r \geq 3)$	
D_r $(r \geq 4)$	
E_6	
E_7	
E_8	
F_4	
G_2	

Remark 1.1.18. The previous table contains exceptions for small ranks in the A, B, C, D types. This is done to avoid overlaps of isomorphic classes and obtain only non-isomorphic simple Lie algebras. In small ranks, these overlaps are called **exceptional isomorphisms**. They are as follows:

$$A_1 \cong B_1 \cong C_1,$$
$$B_2 \cong C_2,$$
$$A_3 \cong D_3.$$

The classification of simple Lie algebras over the complex number field was first achieved by W. Killing in [Kill88]. The proof was revised by E. Cartan in his Ph.D thesis [Car94] and he added the classification over the real number field but did not make use of simple roots whose existence was unknown at the time. In [Weyl25], H. Weyl introduced the notion of positive root. B.L. Van der Waerden simplified the proof of the classification in [VdW33], and then E.B. Dynkin used the diagrams bearing his name and simplified the proof still further in [Dyn46] and [Dyn47]. Dynkin diagrams are instances of Coxeter graphs ([Cox34]), and E. Witt used these graphs in [Witt41] in the context of complex semisimple Lie algebras.

By means of Theorem 1.1.17, any Dynkin diagram of a semisimple Lie algebra is a disjoint union of some of the previous simple diagrams. One can find the Cartan matrices associated to the diagrams in the table as well as explicit realizations of their root systems in e.g. [Bou68].

Examples 1.1.19. *Let us go back to the previous examples:*

- *The special linear algebra* $\mathfrak{sl}_{r+1}(\mathbb{C})$ *is of type* A_r.
- *The orthogonal algebra* $\mathfrak{o}_{2r+1}(\mathbb{C})$ *is of type* B_r.
- *The symplectic algebra* $\mathfrak{sp}_{2r}(\mathbb{C})$ *is of type* C_r.
- *The orthogonal algebra* $\mathfrak{o}_{2r}(\mathbb{C})$ *is of type* D_r.

The Lie algebras of type A, B, C and D are called **classical** and the others **exceptional**. Contrary to the classical algebras seen in the preceding example, the exceptional Lie algebras do not have such simple realizations. It requires more sophisticated tools like the octonion algebra for G_2 or isometries of projective spaces for F_4 and the types E. The reader can find in [Ada96] more information on the exceptional Lie algebras.

From a simple Lie algebra one can construct an associated Dynkin diagram. The reverse process is possible. One can construct a simple Lie algebra from a Dynkin diagram. Because the Cartan matrix of Φ determines Φ up to isomorphism, one can construct Φ from the knowledge of the Cartan matrix, and thus from the Dynkin diagram (cf. [Hum78]). The following theorem is due to J.P. Serre and describes a simple Lie algebra in terms of generators and relations.

Theorem 1.1.20. (Serre). *Let* Δ *be a Dynkin diagram,* Φ *the corresponding root system and* $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ *a base of* Φ. *Let* $\mathfrak{L}$ *be the Lie algebra generated by* $3r$ *elements* $\{e_i, f_i, h_i \mid 1 \leq i \leq r\}$, *subject to the relations*

$$
\begin{aligned}
&(S1) \quad [h_i, h_j] = 0 \qquad (1 \le i, j \le r).\\
&(S2) \quad [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \text{ if } i \ne j.\\
&(S3) \quad [h_i, e_j] = \langle \alpha_j, \alpha_i \rangle e_j, \quad [h_i, f_j] = -\langle \alpha_j, \alpha_i \rangle f_j.\\
&(S_{ij}^{+}) \quad (\text{ad } e_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(e_j) = 0 \qquad (i \ne j).\\
&(S_{ij}^{-}) \quad (\text{ad } f_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(f_j) = 0 \qquad (i \ne j).
\end{aligned}
$$

Then $\mathfrak{L}$ is a finite dimensional semisimple Lie algebra, with Cartan subalgebra spanned by the h_i, and with corresponding root system Φ. The e_i, f_i, h_i's are called the ***Chevalley generators*** *of $\mathfrak{L}$.*

1.1.3. Weyl group invariants

Let $\mathfrak{L}$ be a semisimple Lie algebra over $\Bbbk$ with root system Φ, $\mathfrak{H}$ a Cartan subalgebra of $\mathfrak{L}$ and W the associated Weyl group. As a set of simple roots of Φ is a base of $\mathfrak{H}^*$ and the Weyl group W permutes the elements of the root system Φ, there is an action of W on $\mathfrak{H}^*$. But $\mathfrak{H}$ and $\mathfrak{H}^*$ are identified by means of the Killing form thus W acts on the Cartan subalgebra $\mathfrak{H}$. It is known that W is finite and generated by reflections. The next theorem was proved by C. Chevalley in [Che55].

Theorem 1.1.21. (Chevalley). *Let $\mathfrak{L}$, $\mathfrak{H}$ and W be as above. The algebra of polynomial invariants $\Bbbk[\mathfrak{H}]^W$ is generated by* $\text{rank}(\mathfrak{L})$ *algebraically independent homogeneous elements.*

We now define several notions associated to the Weyl group which will be of use later on.

Definition 1.1.22. *Let $\mathfrak{L}$ be a Lie algebra over $\Bbbk$, W its Weyl group, $\{\alpha_1, \ldots, \alpha_r\}$ a set of simple roots and $\{s_1, \ldots, s_r\}$ the simple reflections of the Weyl group associated to the simple roots. Then $s = s_1 \ldots s_r$ is called a* ***Coxeter element****.*

Proposition 1.1.23. *All Coxeter elements are conjugated in W.*

Because of the previous proposition, all Coxeter elements have the same order h. It is called the **Coxeter number** of W. As all Coxeter elements are conjugated, they have the same characteristic polynomial and therefore the same eigenvalues. If ζ a primitive h-th root of unity in $\mathbb{C}$, these eigenvalues are of the form ζ^m with $0 \le m < h$. The **exponents** of W are the various m involved, written as

$$m_1 \le \ldots \le m_n.$$

The following theorem gives a practical way to compute the exponents of a Weyl group.

Theorem 1.1.24. *Let $\mathfrak{L}$ be a Lie algebra over $\Bbbk$ of rank r, $\mathfrak{H}$ a Cartan subalgebra and W the Weyl group. We set $f_1, \dots, f_r$ invariant homogeneous polynomials (cf. Theorem 1.1.21) such that $S(\mathfrak{H})^W = \Bbbk[f_1, \dots f_r]$ and $\deg(f_1) \leq \dots \leq \deg(f_r)$. Then the exponents of W are $\deg(f_1) - 1, \dots, \deg(f_r) - 1$ and the Coxeter number is $h = \deg(f_r)$.*

Proof. We can find a proof of this theorem in [Bou68]. □

Examples 1.1.25. *We are going to compute the Coxeter number as well as the exponents for the Lie algebras of types A and D.*

• *Case A_r: $\mathfrak{L} = \mathfrak{sl}_{r+1}(\mathbb{C})$, $\mathfrak{H} = \{\mathrm{diag}(x_1, \dots, x_{r+1}) \mid x_1 + \dots + x_{r+1} = 0\}$ and $W = \mathfrak{S}_{r+1}$.*
We have $S(\mathfrak{H}) = \mathbb{C}[X_1, \dots, X_{r+1}]/(X_1 + \dots + X_{r+1})$ with X_i being the dual of $\epsilon_i = ((\epsilon_i)_{kj})_{1 \leq k,j \leq r+1}$ with $(\epsilon_i)_{kj} = \begin{cases} 1 \text{ if } k = j = i, \\ 0 \text{ otherwise.} \end{cases}$ The family $(\epsilon_i - \epsilon_{i+1})_{1 \leq i \leq r}$ is a base of $\mathfrak{H}$ embedded in $\mathfrak{gl}_{r+1}(\mathbb{C})$.
It is known that $\mathbb{C}[X_1, \dots, X_{r+1}]^W = \mathbb{C}[\sigma_1, \dots, \sigma_{r+1}]$ with σ_i the i-th elementary symmetric polynomial. But in $S(\mathfrak{H})$, $\sigma_1 = 0$ so $S(\mathfrak{H})^W = \mathbb{C}[\sigma_2, \dots, \sigma_{r+1}]$. The degrees of the generators are $2, 3, \dots, r+1$ so the exponents of W are $1, 2, \dots, r$ and the Coxeter number is $h = r + 1$.

One can also compute directly the eigenvalues of the Coxeter element. The Weyl group is $\mathfrak{S}_{r+1}$ which acts as the permutation group of the base $(\epsilon_i)_{1 \leq i \leq r+1}$ (Bourbaki's notations). A base of the root system of type A_r is given by $(\alpha_i = \epsilon_i - \epsilon_{i+1})_{1 \leq i \leq r}$. Let s be the Coxeter element corresponding to the permutation $(123 \cdots r+1)$. Then $s(\alpha_1) = -\sum_{k=1}^{r} \alpha_k$ and $s(\alpha_i) = \alpha_{i-1}$ if $i \geq 2$. One can express s as a matrix in the base $(\alpha_1, \dots, \alpha_r)$ and compute its characteristic polynomial χ_s. By induction one obtains $\chi_s(x) = (-1)^r \sum_{k=0}^{r} x^k$. So the eigenvalues of s are $\exp(\frac{2ik\pi}{r+1})$ for $1 \leq k \leq r$, and we find the same result as before for the exponents.

• *Case D_r: $\mathfrak{L} = \mathfrak{o}_{2r}(\mathbb{C})$, $\mathfrak{H}$ is generated by $h_i = \begin{pmatrix} \Lambda_i & 0 \\ 0 & -\Lambda_i \end{pmatrix}$ with $\Lambda_i = e_{i,i} - e_{i+1,i+1}$, $1 \leq i < r$, and $h_r = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}$ with $\Gamma = e_{r-1,r-1} + e_{r,r}$, and $W = (\mathbb{Z}/2\mathbb{Z})^{r-1} \rtimes \mathfrak{S}_r$. $\mathfrak{S}_r$ permutes the h_i and $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ acts by $h_i \longmapsto (\pm 1)_i h_i$ such that $\prod_i (\pm 1)_i = 1$.*
We have a group homomorphism $\varphi : \begin{array}{ccc} (\mathbb{Z}/2\mathbb{Z})^r & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ (\alpha_1, \dots, \alpha_r) & \longmapsto & \alpha_1 \dots \alpha_r \end{array}$. The action of the $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ part of W on $\mathfrak{H}$ corresponds to the kernel of φ which is of index 2: it is then $(\mathbb{Z}/2\mathbb{Z})^{r-1}$.
Because $S(\mathfrak{H}) = \mathbb{C}[X_1, \dots, X_r]$ and $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ is normal in W, one finds $S(\mathfrak{H})^W = (\mathbb{C}[X_1, \dots X_r]^{(\mathbb{Z}/2\mathbb{Z})^{r-1}})^{\mathfrak{S}_r}$. Let $P \in \mathbb{C}[X_1, \dots X_r]^{(\mathbb{Z}/2\mathbb{Z})^{r-1}}$ be a homogeneous

polynomial. Let X_i be a variable in a monomial factor of P. If X_i is alone, it has to have an even power. Otherwise it must be multiplied by $\prod_{j\neq i} X_j$. Therefore $P \in \mathbb{C}[X_1^2, \dots, X_r^2, X_1 \dots X_r]$.
Let $P \in S(\mathfrak{H})^W$. One sees that $P \in \mathbb{C}[\sigma_1(X^2), \dots, \sigma_{r-1}(X^2), \sigma_r(X)]$. Here $\sigma_i(X^2) := \sigma_i(X_1^2, \dots, X_r^2)$ for all $1 \leq i \leq r-1$. So $S(\mathfrak{H})^W \subset \mathbb{C}[\sigma_1(X^2), \dots, \sigma_{r-1}(X^2), \sigma_r(X)]$ and the other inclusion follows immediately.
We thus have $S(\mathfrak{H})^W = \mathbb{C}[\sigma_1(X^2), \dots, \sigma_{r-1}(X^2), \sigma_r(X)]$. The degrees of the polynomials are $2, 4, \dots, 2r-2, r$, hence the exponents of W are $1, 3, \dots, 2r-3, r-1$ and the Coxeter number is $h = 2r-2$.

The next proposition ([Hum90]) will give a connection between the Coxeter number of a simple Lie algebra and its dimension.

Proposition 1.1.26. *Let $\mathfrak{L}$ be a simple Lie algebra over $\Bbbk$ of rank r and dimension n. Then its Coxeter number h satisfies the following relation*

$$rh = 2N$$

with N the number of positive roots of $\mathfrak{L}$. Because of the root space decomposition of $\mathfrak{L}$, the previous relation implies

$$n = r(h+1).$$

In the next table, we give some numerical values regarding the Coxeter numbers and the dimensions of the simple Lie algebras. These values can be found in [Bou68].

type of Lie algebra $\mathfrak{L}$	rank of $\mathfrak{L}$	exponents of W	Coxeter number h of $\mathfrak{L}$
A_r	r	$1, 2, .., r$	$r+1$
B_r	r	$1, 3, 5, \dots, 2r-1$	$2r$
C_r	r	$1, 3, 5, \dots, 2r-1$	$2r$
D_r	r	$1, 3, 5, \dots, 2r-3, r-1$	$2r-2$
E_6	6	$1, 4, 5, 7, 8, 11$	12
E_7	7	$1, 5, 7, 9, 11, 13, 17$	18
E_8	8	$1, 7, 11, 13, 17, 19, 23, 29$	30
F_4	4	$1, 5, 7, 11$	12
G_2	2	$1, 5$	6

Table 1.1.

Using the formula in Proposition 1.1.26 and the values of the ranks and the Coxeter numbers in Table 1.1, one can compute the dimensions of the simple Lie algebras.

1.1.4. Highest weight modules

If not specified otherwise, the proofs of the results in this subsection can be found in [Hum08].

Let $\mathfrak{L}$ be a finite dimensional Lie algebra over $\Bbbk$. We write $T(\mathfrak{L})$ for the tensor algebra of $\mathfrak{L}$ and I for the two-sided ideal of $T(\mathfrak{L})$ generated by $x \otimes y - y \otimes x - [x, y]$, for all $x, y \in \mathfrak{L}$. We define $U(\mathfrak{L}) = T(\mathfrak{L})/I$ and call it the **universal enveloping algebra** of $\mathfrak{L}$. Let A be an associative $\Bbbk$-algebra with unit. A **filtration** on A is an increasing sequence $\{0\} \subsetneq A_0 \subsetneq A_1 \subsetneq \dots$ such that $A = \bigcup_{i \in \mathbb{Z}_{\geq 0}} A_i$ and $A_i \cdot A_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}_{\geq 0}$. The **associated graded algebra** to the filtration $\{A_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is the direct sum

$$\operatorname{Gr} A = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A_i / A_{i-1}, \quad A_{-1} = \{0\},$$

where the algebra structure on A induces the one on $\operatorname{Gr} A$.
One can define a filtration on $U(\mathfrak{L})$. For $m \in \mathbb{Z}_{\geq 0}$, set $T_m(\mathfrak{L}) = \bigoplus_{k=0}^{m} \mathfrak{L}^{\otimes k}$. Then $\{T_m(\mathfrak{L})\}_{m \in \mathbb{Z}_{\geq 0}}$ is a filtration on $T(\mathfrak{L})$. It induces the one on $U(\mathfrak{L})$ by setting $U_m(\mathfrak{L}) = \pi(T_m(\mathfrak{L}))$ where $\pi : T(\mathfrak{L}) \to U(\mathfrak{L})$ is the canonical projection. This filtration on $U(\mathfrak{L})$ is called the **standard filtration** of $U(\mathfrak{L})$. Because $\operatorname{Gr} T(\mathfrak{L}) = T(\mathfrak{L})$, the canonical projection induces a surjective morphism of algebras $\varphi : T(\mathfrak{L}) \to \operatorname{Gr} U(\mathfrak{L})$. Let J be the ideal of $T(\mathfrak{L})$ generated by $x \otimes y - y \otimes x$, for all $x, y \in \mathfrak{L}$. We have $\pi(x \otimes y - y \otimes x) \in U_2(\mathfrak{L})$ by definition $(x, y \in \mathfrak{L})$. But in fact, $\pi(x \otimes y - y \otimes x) = \pi([x, y]) \in U_1(\mathfrak{L})$. Therefore in $\operatorname{Gr} U(\mathfrak{L})$, one finds $\varphi(x \otimes y - y \otimes x) = 0$ and thus $J \subset \operatorname{Ker} \varphi$. Hence φ induces a surjective morphism of algebras $\omega : S(\mathfrak{L}) \to \operatorname{Gr} U(\mathfrak{L})$. The next theorem is called the Poincaré-Birkhoff-Witt theorem, often shortened as PBW:

Theorem 1.1.27. (PBW). *The morphism $\omega : S(\mathfrak{L}) \to \operatorname{Gr} U(\mathfrak{L})$ is an isomorphism of algebras.*

Corollary 1.1.28. *Let $(x_1, x_2, \dots)$ be an ordered base of $\mathfrak{L}$. Then the elements $x_{i(1)} \dots x_{i(m)} = \pi(x_{i(1)} \otimes \dots \otimes x_{i(m)})$, $m \in \mathbb{Z}_{\geq 0}$, $i(1) \leq \dots \leq i(m)$, along with the unit, form a base of $U(\mathfrak{L})$.*

We fix $\mathfrak{L}$ a finite dimensional semisimple Lie algebra over $\Bbbk$, $\mathfrak{H}$ a Cartan subalgebra of $\mathfrak{L}$, Φ the root system with base $\Pi = \{\alpha_1, \dots, \alpha_r\}$ and $\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha$ the root space decomposition. A **Borel subalgebra** of $\mathfrak{L}$ is a maximal solvable subalgebra of $\mathfrak{L}$. Just like Cartan subalgebras, all Borel subalgebras are conjugated in $\mathfrak{L}$. One can show

that $\mathfrak{H} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{L}_\alpha$, with Φ^+ the set of positive roots relative to the base Π, is a Borel subalgebra. For example, in the case of $\mathfrak{sl}_n(\mathbb{C})$, the diagonal matrices form a Cartan subalgebra and the upper triangular matrices form a Borel subalgebra.

Let V be a finite dimensional $\mathfrak{L}$-module. Because the elements of $\mathfrak{H}$ are semisimple, V can be decomposed as $V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda$ with

$$V_\lambda = \{v \in V \mid h.v = \lambda(h)v, \quad \forall h \in \mathfrak{H}\}.$$

Whenever $V_\lambda \neq \{0\}$ it is called a **weight space** and λ is called a **weight** of V. An $\mathfrak{L}$-module V is called a **highest weight module** of highest weight Λ if there exists $v \in V \setminus \{0\}$, called a **highest weight vector**, such that

- $\mathfrak{L}_\alpha.v = 0, \quad \alpha \in \Phi^+$,
- $h.v = \Lambda(h)v, \quad \forall h \in \mathfrak{H}$,
- v generates V as a $\mathfrak{L}$-module.

Let $\Lambda \in \mathfrak{H}^*$ and $\mathbb{C}_\Lambda := \mathbb{C}v_\Lambda$ be the $\mathfrak{b}_+ = \mathfrak{H} \oplus \mathfrak{n}_+$-module, with $\mathfrak{n}_+ := \bigoplus_{\alpha \in \Phi^+} \mathfrak{L}_\alpha$, defined by

$$\mathfrak{n}_+.v_\Lambda = 0 \qquad \text{and} \qquad h.v_\Lambda = \Lambda(h)v_\Lambda, \text{ for any } h \in \mathfrak{H}.$$

Set

$$M(\Lambda) = U(\mathfrak{L}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\Lambda.$$

$M(\Lambda)$ is a highest weight module of infinite dimension by construction, and because of the PBW theorem (Theorem 1.1.27) it is also $U(\mathfrak{n}_-)$-free, with $\mathfrak{n}_- := \bigoplus_{\alpha \in \Phi^+} \mathfrak{L}_{-\alpha}$. The module $M(\Lambda)$ is called the **Verma module of highest weight** Λ and $1 \otimes v_\Lambda$ is a highest weight vector.

Theorem 1.1.29. *Let $\Lambda \in \mathfrak{H}^*$ and E a highest weight $\mathfrak{L}$-module of highest weight Λ with a highest weight vector v_E. Then there exists a surjection*

$$\begin{array}{ccc} M(\Lambda) & \twoheadrightarrow & E \\ 1 \otimes v_\Lambda & \mapsto & v_E \end{array}.$$

It implies that any highest weight $\mathfrak{L}$-module of highest weight Λ is a quotient of the Verma module $M(\Lambda)$.

It can be proved that if $\lambda \in \mathfrak{H}^*$, then there exists a unique irreducible highest weight $\mathfrak{L}$-module (not necessarily finite dimensional) of highest weight λ. We write $L(\lambda)$ for this module. Furthermore, one can show that any sum of proper submodules of $M(\lambda)$ is a proper submodule of $M(\lambda)$. This implies that there exists a unique maximal proper submodule of $M(\lambda)$ noted $N(\lambda)$. It is a known result that $L(\lambda)$ is the quotient $M(\lambda)/N(\lambda)$.

Any finite dimensional irreducible $\mathfrak{L}$-module is a highest weight module. Indeed, if V is such a module, then the Borel subalgebra $\mathfrak{H} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{L}_\alpha$ has a common eigenvector $v \in V$ of weight λ by Theorem 1.1.11, and v is annihilated by all the $\mathfrak{L}_\alpha$, $\alpha \in \Phi^+$. Thus the submodule generated by v is a highest weight module of weight λ. However V is supposed to be irreducible and hence $V = L(\lambda)$. Furthermore, it can be shown that $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq r$. On the other hand, any $L(\lambda)$ for $\lambda \in \mathfrak{H}^*$ is not necessarily finite dimensional. It requires an additional condition on λ. $\lambda \in \mathfrak{H}^*$ is called a **dominant integral weight** if $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq r$. We have just seen that if $L(\lambda)$ is finite dimensional, then λ is dominant integral. It turns out that the converse is true.

Theorem 1.1.30. *Let $\lambda \in \mathfrak{H}^*$. Then*

$$L(\lambda) \text{ is finite dimensional} \iff \lambda \text{ is a dominant integral weight.}$$

Furthermore, if λ is dominant integral, then the Weyl group W permutes the weights appearing in $L(\lambda)$ and preserves the dimensions of the weight spaces.

The next theorem gives a presentation of the submodule $N(\lambda)$.

Theorem 1.1.31. *Let $\Lambda \in \mathfrak{H}^*$ be a dominant integral weight. The submodule $N(\Lambda)$ of the Verma module $M(\Lambda)$ with a highest weight vector v_Λ is generated by the vectors $f_i^{\Lambda(h_i)+1}.v_\Lambda$.*

Corollary 1.1.32. *Any finite dimensional highest weight module is irreducible.*

Proof. Let V be a finite dimensional highest weight $\mathfrak{L}$-module of highest weight λ and highest weight vector v. By Serre's theorem (Theorem 1.1.20), $\mathfrak{L}$ is generated by the Chevalley generators e_i, f_i, h_i, for all $1 \leq i \leq r$. Let $\mathfrak{s}_i$ $(1 \leq i \leq r)$ be the Lie algebra generated by e_i, f_i and h_i. It is isomorphic to $\mathfrak{sl}_2(\mathbb{k})$. By restriction one can look at V as an $\mathfrak{s}_i$-module, for all $1 \leq i \leq r$. Let V_i be the $\mathfrak{s}_i$-submodule of V generated by v. V_i is finite dimensional because V is. So according to Example 1.1.9, the highest weight of V_i as an $\mathfrak{s}_i$-module is a positive integer and so $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$. This result is true for any $i \in \{1, \ldots, r\}$ and thus λ is a dominant integral weight. Furthermore, it was shown in Example 1.1.9 that $f_i^{\lambda(h_i)+1}.v = 0$, for all $1 \leq i \leq r$. Therefore, according to Theorem 1.1.31, $N(\lambda)$ is contained in the kernel of the morphism $\begin{array}{ccc} M(\lambda) & \to & V \\ x & \mapsto & x.v \end{array}$.

This kernel is a proper submodule of $M(\lambda)$ and $N(\lambda)$ is the unique maximal one, so $N(\lambda)$ is the kernel. By the isomorphism theorem, one finds $V \cong M(\lambda)/N(\lambda) = L(\lambda)$ and V is irreducible. □

Later on, we will compute some irreducible highest weight modules for some particular kind of weights, which we define now. Let $\{\alpha_1, \ldots, \alpha_r\}$ be a base of the root system of a semisimple Lie algebra $\mathfrak{L}$. Let $\{\lambda_1, \ldots, \lambda_r\}$ be the dual set of the simple roots $\{\alpha_1, \ldots, \alpha_r\}$, i.e. $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$, for all $1 \leq i, j \leq r$. The elements of this dual base are called the **fundamental weights** of the root system and they are dominant integral weights.

When λ is a dominant integral weight, the dimension of the associated irreducible highest weight module can be computed by means of a formula proved by H. Weyl.

Theorem 1.1.33. (Weyl). *Let $\lambda \in \mathfrak{H}^*$ be a dominant integral weight. Then*

$$\dim\, L(\lambda) = \prod_{\beta \in \Phi^+} \frac{(\lambda + \rho, \beta)}{(\rho, \beta)}$$

with ρ the half-sum of the positive roots of the root system Φ.

1.1.5. Lie groups, algebraic groups and Lie algebras

We finish this section by a link between Lie groups, algebraic groups and Lie algebras as well as some notations. The references used here are [DuiKol00] and [OV90].

Definition 1.1.34. *A **Lie group** is a group G with a smooth manifold structure over $\mathbb{R}$ or $\mathbb{C}$, and such that the multiplication $m : G \times G \to G$ and the inversion $i : G \to G$ are smooth maps.*

Remark 1.1.35. A Lie group can also be defined over a p-adic field, but being smooth is replaced by being locally analytic (cf. [Schnei11]). However, in this thesis, we will not work over p-adic fields.

Let G be a Lie group. Because of its smooth manifold structure, one can define the **tangent space** to G at a point $x \in G$ by $T_xG = \{\gamma'(0) \mid \gamma : [-\epsilon, \epsilon] \to G$ is a C^1 curve on G and $\gamma(0) = x\}$. A **vector field** on G is a derivation $X : C^\infty(G) \to C^\infty(G)$, i.e. a linear map such that on a product of functions f and g

$$X(fg) = f(Xg) + (Xf)g.$$

Locally one can choose coordinates so that such a derivation is a linear combination of the derivatives with respect to the coordinates

$$X = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i},$$

and with respect to this choice of coordinates, a vector field is given at each point by the n-vector $(a_1, \ldots, a_n)$. If X is a vector field on G and $g \in G$, then the value of X at g is $X_g : C^\infty(G) \to \Bbbk$ defined by $X_g(f) = X(f)(g)$ for any $f \in C^\infty(G)$. The set of all values at a point $g \in G$ of all vector fields on G is the tangent space at g noted T_gG. It can be thought of in terms of derivations acting on germs of functions at g. We have a diffeomorphism of G defined by the left multiplication by an element $g \in G$ and noted L_g. It induces the map $d_hL_g : T_hG \to T_{gh}G$ for any $h \in G$. A vector field X is called **left-invariant** if

$$(d_hL_g)(X_h) = X_{gh},$$

The set ${}^{\text{inv}}\mathfrak{X}(G)$ of all left-invariant vector fields is called the **Lie algebra** of the Lie group G and is isomorphic to T_eG as a G-module. The Lie bracket for this Lie algebra is $[X, Y](f) = X(Y(f)) - Y(X(f))$ for any X, Y left-invariant vector fields and $f \in C^\infty(G)$.

Definition 1.1.36. *An **algebraic group** is an algebraic variety G over $\Bbbk$ which is also a group and such that the maps defining the group structure $\mu : G \times G \to G$ with $\mu(x, y) = xy$ and $\iota : x \mapsto x^{-1}$ are morphisms of varieties.*

The following definition of the tangent space to an algebraic group comes from [Sprin09].

Definition 1.1.37. *Let G be an algebraic group over a field $\Bbbk$. The **tangent space** to G at $x \in G$ is defined by $T_xG = \mathrm{Der}_\Bbbk(\Bbbk[G], \Bbbk_x)$, i.e. the set of $\Bbbk$-derivations from $\Bbbk[G]$ to $\Bbbk_x$, with $\Bbbk_x$ the field $\Bbbk$ seen as $\Bbbk[G]$-module by the morphism $f \to f(x)$. An element $X \in T_xG$ is thus a map $X : \Bbbk[G] \to \Bbbk$ such that $X(fg) = X(f)g(x) + f(x)X(g)$ for any $f, g \in \Bbbk[G]$.*

The tangent space of an algebraic group G can also be expressed using the regular functions on G. If $x \in G$, we write $M_x = \{f \in \Bbbk[G] \mid f(x) = 0\}$. It can be proved that the tangent space to G at x is isomorphic to $(M_x/M_x^2)^*$ (cf. [Sprin09]).

The **Lie algebra** of an algebraic group G is the tangent space at the unit element T_eG and is usually noted $\mathfrak{g}$. We can check that $\mathfrak{g}$ is a Lie algebra in the sense given at the beginning of Section 1.1.1. Indeed, there are two maps

$$^{\mathrm{inv}}\mathfrak{X}(G) \quad \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \quad \mathrm{Der}_{\Bbbk}(\Bbbk[G], \Bbbk_e) = \mathfrak{g}$$

defined by:

- for any $X \in {}^{\mathrm{inv}}\mathfrak{X}(G)$, $\varphi(X)(f) := X(f)(e)$ for e the unit in G and any $f \in \Bbbk[G]$.
- for any $x \in \mathrm{Der}_{\Bbbk}(\Bbbk[G], \Bbbk_e)$, $\psi(x)(f)(g) := x(L_{g^{-1}} f)$ for any $f \in C^\infty(G)$ and $g \in G$.

The maps φ and ψ are both well defined isomorphisms and inverse to each other (cf. [Hum75] Sections 9.1 and 9.2). Thus the Lie bracket on $\mathfrak{g}$ is given by the one for the left-invariant vector fields. From now on we will write $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ for a Cartan subalgebra, a Borel subalgebra and a Lie algebra respectively.

An algebraic group over a field $\Bbbk$ is defined using polynomials. So if $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, an algebraic group is always a Lie group. But the converse is not always true. The easiest example is $D^1 = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, the pointed unit disc without its border in $\mathbb{R}^2$, the group action being the product. The metaplectic group $\mathrm{Mp}_2(\mathbb{R})$, which is the connected double cover of $\mathrm{SL}_2(\mathbb{R})$, as well as the subgroup of $\mathrm{GL}_3(\mathbb{R})$ given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & t & 0 \\ y & 0 & t^\alpha \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}, t \in \mathbb{R}^+ \right\}$$

with α irrational, are non-trivial examples. But there are particular situations in which the converse is true. The next theorem is proved in [OV90].

Theorem 1.1.38. *Let G be a connected, complex, semisimple Lie group. Then G admits a unique structure of complex algebraic group.*

Under these conditions, the classification of simple Lie groups over $\mathbb{C}$ and the classification of simple algebraic groups over $\mathbb{C}$ lead to the same results. In what follows, the base field will be the complex number field, hence we will identify Lie groups and algebraic groups. If G is an algebraic group, then its tangent space as Lie group coincides with its tangent space as algebraic group. In a similar fashion, the two definitions of the Lie algebra of G coincide.

If $g \in G$, we define

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$$

as the derived map at the unit element e of the conjugation $x \mapsto gxg^{-1}$ from G to G. This is the **Adjoint action** of G on $\mathfrak{g}$. Also we define

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

as the differential at e of $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$. It coincides with the adjoint representation of $\mathfrak{g}$ on itself and thus one finds $\mathrm{ad}(X)(Y) = [X, Y]$, $\forall X, Y \in \mathfrak{g}$.

1.2. Folding of Dynkin diagrams

We fix an algebraically closed field $\Bbbk$ of characteristic 0.

1.2.1. Definition of the folding of a Dynkin diagram

Let $\mathfrak{g}$ be a simple Lie algebra of finite dimension over $\Bbbk$ with root system Φ and $x \in \mathfrak{g}$. If $\mathrm{ad}\ x$ is nilpotent of order (say) $n+1$, then $\exp(\mathrm{ad}\ x) = 1 + \mathrm{ad}\ x + \frac{(\mathrm{ad}\ x)^2}{2} + \ldots + \frac{(\mathrm{ad}\ x)^n}{n!} \in \mathrm{Aut}(\mathfrak{g})$ is well defined. Any automorphism of $\mathfrak{g}$ that can be written $\exp(\mathrm{ad}\ x)$ with x nilpotent is called **inner** and the group generated by inner automorphisms is denoted by $\mathrm{Inn}(\mathfrak{g})$. It is a connected and normal subgroup of $\mathrm{Aut}(\mathfrak{g})$, the group of automorphisms of $\mathfrak{g}$. An **outer** automorphism is an element of $\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$. One can find the proof of the next proposition in [FulHar91] (or [Bou68] chapter VI, §1, n°5, Proposition 16).

Proposition 1.2.1. $\mathrm{Out}(\mathfrak{g})$ *is isomorphic to the automorphism group of the Dynkin diagram* Δ *of* $\mathfrak{g}$.

One can verify that the irreducible Dynkin diagrams that have a non-trivial outer automorphism group are those of type A_r ($r \geq 2$), D_r ($r \geq 3$) and E_6. Let σ be an automorphism of one of these Dynkin diagrams. Then σ induces an automorphism of the corresponding Lie algebra. This is achieved by taking a section s of the morphism $\pi : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ and get $s(\sigma)$. It has been seen that every vertex of the Dynkin diagram corresponds to an element of a base Π of Φ. With this identification, σ induces a permutation of Π. From Serre's theorem (Theorem 1.1.20), it is known that the Lie algebra is generated by the Chevalley generators $e_\alpha, f_\alpha, h_\alpha$, with α running through Π. Define $s(\sigma)$ by $s(\sigma)(x_\alpha) = x_{\sigma(\alpha)}$ for any $\alpha \in \Pi$, $x = e, f$ or h. It can be verified that $s(\sigma)$ is an automorphism of $\mathfrak{g}$ and $\pi \circ s(\sigma) = \sigma$. One can see that $s(\sigma)$ preserves the Cartan subalgebra $\mathfrak{h}$ and is thus, by definition, an automorphism of the root system. But $s(\sigma)$ preserves also the Borel subalgebra $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. This implies that $s(\sigma)$ induces a permutation, which we call ρ, of the positive roots through $s(\sigma)(\mathfrak{g}_\alpha) = \mathfrak{g}_{\rho(\alpha)}$. Hence the permutation ρ is induced by an automorphism of the root system Φ and preserves Φ^+, thus it preserves the base Π. As it also preserves the bracket $\langle ., . \rangle$, one sees that the

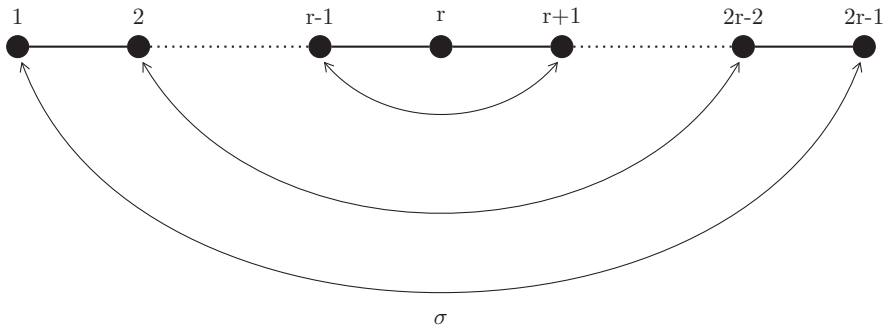
Cartan matrix is unchanged by ρ, and so is the Dynkin diagram. Therefore ρ induces an automorphism of the Dynkin diagram which is equal to σ.

In order to simplify the notations, we will write $\dot{\sigma}$ for the automorphism $s(\sigma)$. By definition of σ, $\dot{\sigma}$ has finite order. Assuming the order is $r \in \mathbb{Z}_{\geq 0}$, $\dot{\sigma}$ induces a gradation on $\mathfrak{g}$ such that $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} \mathfrak{g}_i$ with $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \dot{\sigma}(x) = \omega^i x\}$, ω being a primitive r-th root of unity. Indeed, because $\dot{\sigma}$ is of finite order, it is semisimple. Hence there exists a base of $\mathfrak{g}$ composed of eigenvectors of $\dot{\sigma}$ and the eigenspaces are the summands of the gradation.

In the next subsections we are going to compute explicitly the gradation of the Lie algebra by a case by case analysis of the Dynkin diagrams. Furthermore we will also compute the invariants of the root lattice by the action of σ and the corresponding Dynkin diagram. The **folding of a Dynkin diagram** consists in computing the invariants of the automorphism $\dot{\sigma}$. The reader may consult [Kac90] for more results on Lie algebras automorphisms.

1.2.2. A_{2r-1} and B_r

Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ of type A_{2r-1}, Δ its Dynkin diagram and $\{\alpha_1, \ldots, \alpha_{2r-1}\}$ a base of the root system. We identify the simple roots of $\mathfrak{g}$ with their respective vertices in Δ, so that an automorphism of Δ can be seen as a permutation of the simple roots. We define σ the automorphism of Δ by $\sigma(\alpha_i) = \alpha_{2r-i}$.



a) Folding of the Lie algebra: σ is lifted to an automorphism $\dot{\sigma}$ of $\mathfrak{g}$ defined by $\dot{\sigma}(x_i) = x_{\sigma(i)}$, with $x = e, f$ or h, the Chevalley generators.

$\dot{\sigma}$ is of order 2 so $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We are going to compute $\mathfrak{g}_0$ and $\mathfrak{g}_1$.

$\mathfrak{g}$ is generated by e_i, f_i, h_i, $1 \leq i \leq 2r-1$, with Serre's relations (Theorem 1.1.20). The simple roots are $\alpha_1, \dots \alpha_{2r-1}$ and the positives ones are $\sum_{i \leq k < j} \alpha_k$, $1 \leq i < j \leq 2r$. The root space decomposition gives $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ with $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$.

In the following computations, we are going to use that if $\dot{\sigma}$ is of finite order r and $x \in \mathfrak{g}$, then $y = \sum_{i=0}^{r-1} \zeta^{-i} \dot{\sigma}^i(x)$ with $\zeta^r = 1$ is an eigenvector of $\dot{\sigma}$ for the eigenvalue ζ.

- $\underline{\mathfrak{g}_0}$: set

$$\begin{cases} E_i &= e_i + \dot{\sigma}(e_i),\ 1 \leq i \leq r-1 \\ E_r &= e_r \end{cases}, \begin{cases} F_i &= f_i + \dot{\sigma}(f_i), 1 \leq i \leq r-1 \\ F_r &= f_r \end{cases} \text{ and}$$

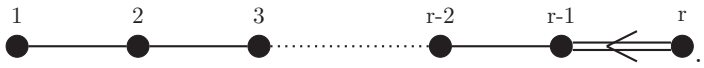
$$\begin{cases} H_i &= h_i + \dot{\sigma}(h_i),\ 1 \leq i \leq r-1, \\ H_r &= h_r. \end{cases}$$

We have the following relations

$$\begin{cases} [E_i, F_i] = \delta_{i,j} H_i, \quad 1 \leq i \leq r, \\ [H_i, H_j] = 0, \quad \forall i, j, \\ [H_i, E_j] = c_{ij} E_j, \\ [H_i, F_j] = -c_{ij} F_j, \\ (\text{ad } E_i)^{1-c_{ij}} E_j = 0 \quad (i \neq j), \\ (\text{ad } F_i)^{1-c_{ij}} F_j = 0 \quad (i \neq j). \end{cases}$$

where the matrix $(c_{i,j})_{1 \leq i,j \leq r}$ is $\begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}$.

According to Serre's theorem (Theorem 1.1.20), the E_i, F_i, H_i are the Chevalley generators of a simple Lie algebra $\mathfrak{a} \subset \mathfrak{g}_0$ whose corresponding Dynkin diagram is



which is of type C_r.

• $\mathfrak{g}_1$: we have $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, so $[\mathfrak{a}, \mathfrak{g}_1] \subset \mathfrak{g}_1$. $\mathfrak{g}_1$ is then a representation of $\mathfrak{a}$. Let $(\lambda_i)_{1 \leq i \leq r}$ be the fundamental weights of $\mathfrak{a}$. Set $\begin{cases} \eta_1 = \alpha_1 + \ldots + \alpha_{2r-2}, \\ \eta_2 = \alpha_2 + \ldots + \alpha_{2r-1}, \end{cases}$ and $v = e_{\eta_1} - e_{\eta_2}$. One can see that $v \in \mathfrak{g}_1$ and $v \neq 0$ because $\eta_1 \neq \eta_2$. Let V be the $\mathfrak{a}$-module generated by v. By direct computations one finds $\begin{cases} [H_i, v] = \lambda_2(H_i)v, \ \forall i, \\ [E_i, v] = 0, \ \forall i. \end{cases}$ Therefore V is a highest weight module of highest weight λ_2 and is finite dimensional. By Corollary 1.1.32, V is irreducible and so $V \cong L(\lambda_2)$.

By Weyl's formula (Theorem 1.1.33), we obtain $\dim V = 2r^2 - r - 1$. Furthermore

$$\begin{array}{ccccc} \mathfrak{g} = & \mathfrak{g}_0 & \oplus & \mathfrak{g}_1 \\ & \cup & & \cup \\ & \mathfrak{a} & & V \end{array}$$

and $\dim \mathfrak{a} + \dim V = r(2r+1) + 2r^2 - r - 1 = 4r^2 - 1 = \dim \mathfrak{g}$, with $\dim \mathfrak{a}$ given in Table 1.1. The simple Lie algebra $\mathfrak{a} \subset \mathfrak{g}_0$ which is of type C_r is therefore equal to $\mathfrak{g}_0$. We conclude that $\mathfrak{g}_0 = \mathfrak{a}$ is the simple Lie algebra of type C_r and $\mathfrak{g}_1 = V$ is an irreducible highest weight $\mathfrak{a}$-module of highest weight λ_2.

b) Folding of the root lattice Q: let $Q = \bigoplus_{i=1}^{2r-1} \mathbb{Z}\alpha_i$ be the root lattice of a root system of type A_{2r-1}. We have defined σ by $\sigma(\alpha_i) = \alpha_{2r-i}$ and extend it to Q by linearity. Let us compute the invariants Q^σ.

$$\begin{aligned} \alpha = \sum_{i=1}^{2r-1} k_i \alpha_i \in Q^\sigma \quad &\Leftrightarrow \quad k_i = k_{2r-i}, \ 1 \leq i \leq r-1, \\ &\Leftrightarrow \quad \alpha = \sum_{i=1}^{r-1} k_i(\alpha_i + \alpha_{2r-i}) + k_r \alpha_r. \end{aligned}$$

Set

$$\begin{cases} \beta_i = \alpha_i + \alpha_{2r-i}, & 1 \leq i \leq r-1, \\ \beta_r = \alpha_r. \end{cases}$$

The β_i's generate the lattice Q^σ. From direct computation, one has

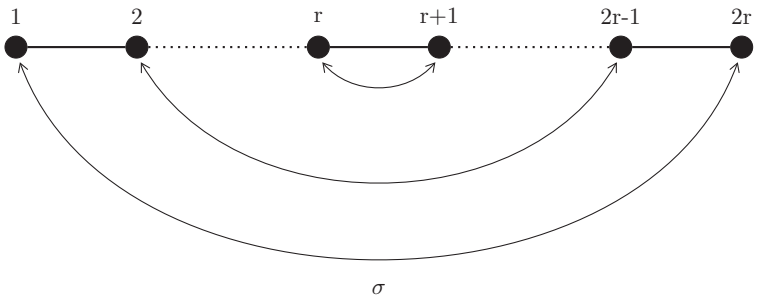
$$(\langle \beta_i, \beta_j \rangle)_{1 \leq i,j \leq r} = \begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The corresponding diagram is

It is a Dynkin diagram of type B_r. We deduce that Q^σ is the root lattice of a Lie algebra of type B_r.

1.2.3. A_{2r} and C_r

Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ of type A_{2r}, Δ its Dynkin diagram and $\{\alpha_1, \ldots, \alpha_{2r}\}$ a base of the root system. We identify the simple roots of $\mathfrak{g}$ with their respective vertices in Δ, so that an automorphism of Δ can be seen as a permutation of the simple roots. We define σ the automorphism of Δ by $\sigma(\alpha_i) = \alpha_{2r+1-i}$.



a) Folding of the Lie algebra: σ is lifted to an automorphism $\dot{\sigma}$ of $\mathfrak{g}$ defined by $\dot{\sigma}(x_i) = x_{\sigma(i)}$, with $x = e, f$ or h, the Chevalley generators.

$\dot{\sigma}$ is of order 2 so $\mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$. We are going to compute $\mathfrak{g}_0$ and $\mathfrak{g}_1$.

$\mathfrak{g}$ is generated by e_i, f_i, h_i, $1 \leq i \leq 2r$, with Serre's relations (Theorem 1.1.20). The simple roots are $\alpha_1, \dots \alpha_{2r}$ and the positives ones are $\sum_{i \leq k < j} \alpha_k$, $1 \leq i < j \leq 2r$. The root space decomposition gives $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ with $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$.

- $\mathfrak{g}_0$: set

$$\begin{cases} E_i = e_i + \dot{\sigma}(e_i), \ 1 \leq i \leq r, \\ F_i = f_i + \dot{\sigma}(f_i), 1 \leq i \leq r, \\ H_i = h_i + \dot{\sigma}(h_i), \ 1 \leq i \leq r. \end{cases}$$

We have the following relations

$$\begin{cases} [E_i, F_i] = \delta_{i,j} H_i, \quad 1 \leq i \leq r, \\ [H_i, H_j] = 0, \quad \forall i, j, \\ [H_i, E_j] = c_{ij} E_j, \\ [H_i, F_j] = -c_{ij} F_j, \\ (\text{ad } E_i)^{1 - c_{ij}} E_j = 0 \quad (i \neq j), \\ (\text{ad } F_i)^{1 - c_{ij}} F_j = 0 \quad (i \neq j). \end{cases}$$

where the matrix $(c_{i,j})_{1\leq i,j\leq r}$ is $\begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$.

According to Serre's theorem (Theorem 1.1.20), the E_i, F_i, H_i are the Chevalley generators of a simple Lie algebra $\mathfrak{a} \subset \mathfrak{g}_0$ whose corresponding Dynkin diagram is

which is of type B_r.

• $\mathfrak{g}_1$: we have $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, so $[\mathfrak{a}, \mathfrak{g}_1] \subset \mathfrak{g}_1$. $\mathfrak{g}_1$ is then a representation of $\mathfrak{a}$. Let $(\lambda_i)_{1\leq i\leq r}$ be the fundamental weights of $\mathfrak{a}$. Set $\theta = \alpha_1 + \ldots + \alpha_{2r}$ and $v = e_\theta$. One can see that $v \in \mathfrak{g}_1$ and $v \neq 0$. Let V be the $\mathfrak{a}$-module generated by v. By direct computations one finds
$\begin{cases} [H_i, v] = 2\lambda_1(H_i)v, \ \forall i, & \text{Therefore } V \text{ is a highest weight module of highest weight } 2\lambda_1 \\ [E_i, v] = 0, \ \forall i. \end{cases}$
and is finite dimensional. By Corollary 1.1.32, V is irreducible and so $V \cong L(2\lambda_1)$.

By Weyl's formula (Theorem 1.1.33), we obtain $\dim V = r(2r+3)$. Furthermore $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\dim \mathfrak{a} + \dim V = r(2r+1) + r(2r+3) = 4r(r+1) = \dim \mathfrak{g}$, with $\mathfrak{a} \cup \mathfrak{g}_0$ and $V \cup \mathfrak{g}_1$ given in Table 1.1. We conclude that $\mathfrak{g}_0 = \mathfrak{a}$ is the simple Lie algebra of type B_r and $\mathfrak{g}_1 = V$ is an irreducible highest weight $\mathfrak{a}$-module of highest weight $2\lambda_1$.

b) Folding of the root lattice Q: let $Q = \bigoplus_{i=1}^{2r} \mathbb{Z}\alpha_i$ be the root lattice of a root system of type A_{2r}. We have defined σ by $\sigma(\alpha_i) = \alpha_{2r+1-i}$ and extend it to Q by linearity. Let us compute the invariants Q^σ.

$$\alpha = \sum_{i=1}^{2r} k_i\alpha_i \in Q^\sigma \quad \Leftrightarrow \quad k_i = k_{2r+1-i},\ 1 \le i \le r,$$
$$\Leftrightarrow \quad \alpha = \sum_{i=1}^{r} k_i(\alpha_i + \alpha_{2r+1-i}).$$

Set

$$\beta_i = \alpha_i + \alpha_{2r+1-i}, \quad 1 \le i \le r.$$

The β_i's generate the lattice Q^σ. From direct computation, one has

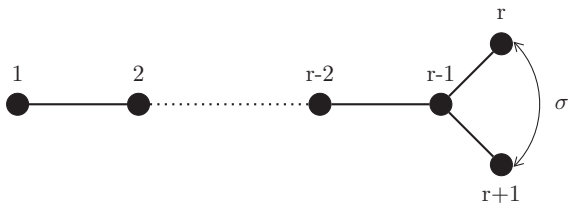
$$(\langle \beta_i, \beta_j \rangle)_{1\le i,j\le r} = \begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

The corresponding diagram is

It is a Dynkin diagram of type C_r. We deduce that Q^σ is the root lattice of a Lie algebra of type C_r.

1.2.4. D_{r+1} and C_r

Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ of type D_{r+1}, Δ its Dynkin diagram and $\{\alpha_1, \ldots, \alpha_{r+1}\}$ a base of the root system. We define σ an automorphism of Δ by $\begin{cases} \sigma(\alpha_i) = \alpha_i \text{ if } i \ne r, r+1, \\ \sigma(\alpha_r) = \alpha_{r+1}, \\ \sigma(\alpha_{r+1}) = \alpha_r. \end{cases}$



a)Folding of the Lie algebra: σ is lifted to an automorphism $\dot{\sigma}$ of $\mathfrak{g}$ defined by $\dot{\sigma}(x_i) = x_{\sigma(i)}$, with $x = e, f$ or h, the Chevalley generators.

As previously one sees that $\mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$.

• $\mathfrak{g}_0$ contains

$$\begin{cases} E_i &= e_i,\ 1 \le i \le r-1, \\ E_r &= e_r + \dot{\sigma}(e_r), \end{cases} \text{ as well as } \begin{cases} F_i &= f_i,\ 1 \le i \le r-1, \\ F_r &= f_r + \dot{\sigma}(f_r), \end{cases}$$

$$\text{and } \begin{cases} H_i &= h_i,\ 1 \le i \le r-1, \\ H_r &= h_r + \dot{\sigma}(h_r). \end{cases}$$

Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}_0$ generated by the E_i, F_i and H_i's. As in the previous subsection, by means of Serre's theorem (Theorem 1.1.20), one sees that the E_i, F_i, H_i's are the Chevalley generators of the simple Lie algebra $\mathfrak{a} \subset \mathfrak{g}_0$ whose Dynkin diagram is

which is of type B_r.

• $\mathfrak{g}_1$: as in the previous subsection, we find that $\mathfrak{g}_1$ is a representation of $\mathfrak{a}$. Let $(\lambda_i)_{1 \le i \le r}$ be the fundamental weights of $\mathfrak{a}$. Let $\begin{cases} \eta_1 &= \alpha_1 + \ldots + \alpha_{r-1} + \alpha_r, \\ \eta_2 &= \alpha_1 + \ldots + \alpha_{r-1} + \alpha_{r+1}, \end{cases}$ and $v = e_{\eta_1} - e_{\eta_2}$. One can see that $v \in \mathfrak{g}_1$ and $v \neq 0$ because $\eta_1 \neq \eta_2$. Let V be the $\mathfrak{a}$-module generated by v. By direct computations we find $\begin{cases} [H_i, v] = \lambda_1(H_i)v,\ \forall i, \\ [E_i, v] = 0,\ \forall i. \end{cases}$ Therefore V is a highest weight module of highest weight λ_1 and is finite dimensional. By Corollary 1.1.32, V is irreducible and so $V \cong L(\lambda_1)$.

By means of Weyl's formula (Theorem 1.1.33), we obtain $\dim V = 2r+1$. Furthermore $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\dim \mathfrak{a} + \dim V = (2r+1)r + 2r + 1 = (2r+1)(r+1) = \dim \mathfrak{g}$, with

$$\begin{array}{ccc} \mathfrak{g} = & \mathfrak{g}_0 & \oplus \quad \mathfrak{g}_1 \\ & \cup & \cup \\ & \mathfrak{a} & V \end{array}$$

$\dim \mathfrak{a}$ given in Table 1.1. We conclude that $\mathfrak{g}_0 = \mathfrak{a}$ is the simple Lie algebra of type B_r and $\mathfrak{g}_1 = V$ is an irreducible highest weight $\mathfrak{a}$-module of highest weight λ_1.

b) Folding of the root lattice Q: let $Q = \bigoplus_{i=1}^{r+1} \mathbb{Z}\alpha_i$ be the root lattice of a root system of type D_{r+1}. We have defined σ by $\begin{cases} \sigma(\alpha_i) = \alpha_i \text{ if } i \neq r, r+1, \\ \sigma(\alpha_r) = \alpha_{r+1}, \\ \sigma(\alpha_{r+1}) = \alpha_r, \end{cases}$ and extend it to Q by linearity. We compute the invariants Q^σ.

$$\begin{aligned} \alpha = \sum_{i=1}^{r+1} k_i\alpha_i \in Q^\sigma \quad &\Leftrightarrow \quad k_r = k_{r+1}, \\ &\Leftrightarrow \quad \alpha = \sum_{i=1}^{r-1} k_i\alpha_i + k_r(\alpha_r + \alpha_{r+1}). \end{aligned}$$

Set

$$\begin{cases} \beta_i = \alpha_i, 1 \leq i \leq r-1, \\ \beta_r = \alpha_r + \alpha_{r+1}. \end{cases}$$

The β_i's generate the lattice Q^σ. From direct computation, one has

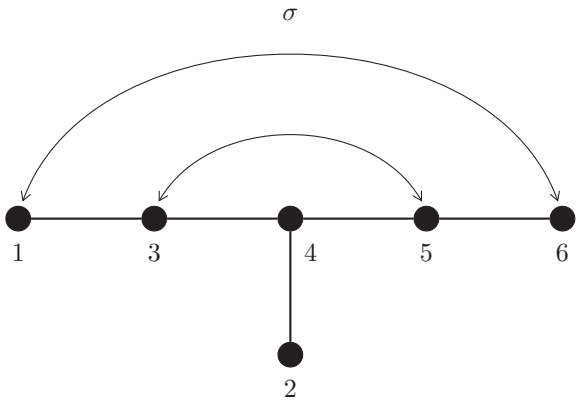
$$(\langle \beta_i, \beta_j \rangle)_{1 \leq i,j \leq r} = \begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

The corresponding diagram is

It is a Dynkin diagram of type C_r. We deduce that Q^σ is the root lattice of a Lie algebra of type C_r.

1.2.5. E_6 and F_4

Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ of type E_6, Δ its Dynkin diagram and $\{\alpha_1, \dots, \alpha_6\}$ a base of the root system. We define σ an automorphism of Δ by
$$\begin{cases} \sigma(\alpha_1) = \alpha_6, \\ \sigma(\alpha_3) = \alpha_5, \\ \sigma(\alpha_5) = \alpha_3, \\ \sigma(\alpha_6) = \alpha_1, \\ \sigma(\alpha_i) = \alpha_i \text{ otherwise.} \end{cases}$$



a) Folding of the Lie algebra: σ is lifted to an automorphism $\dot{\sigma}$ of $\mathfrak{g}$ defined by $\dot{\sigma}(x_i) = x_{\sigma(i)}$, with $x = e, f$ or h, the Chevalley generators.

As previously we prove that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

• $\mathfrak{g}_0$ contains $\begin{cases} E_1 = e_2, \\ E_2 = e_4 \\ E_3 = e_3 + e_5, \\ E_4 = e_1 + e_6, \end{cases}$, $\begin{cases} F_1 = f_2, \\ F_2 = f_4 \\ F_3 = f_3 + f_5, \\ F_4 = f_1 + f_6, \end{cases}$ as well as $\begin{cases} H_1 = h_2, \\ H_2 = h_4 \\ H_3 = h_3 + h_5, \\ H_4 = h_1 + h_6. \end{cases}$

Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}_0$ generated by the E_i, F_i and H_i's. From direct computations and by Serre's theorem (Theorem 1.1.20), one sees that the E_i, F_i, H_i's are the Chevalley generators of the simple Lie algebra $\mathfrak{a} \subset \mathfrak{g}_0$ whose Dynkin diagram is

which is of type F_4.

• $\mathfrak{g}_1$: as in the previous subsection, we find that $\mathfrak{g}_1$ is a representation of $\mathfrak{a}$. Let $(\lambda_i)_{1\leq i\leq 4}$ be the fundamental weights of $\mathfrak{a}$. Let $\begin{cases} \eta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\ \eta_2 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \end{cases}$ and $v = e_{\eta_1} - e_{\eta_2}$. One can see that $v \in \mathfrak{g}_1$ and $v \neq 0$ because $\eta_1 \neq \eta_2$. Let V be the $\mathfrak{a}$-module generated by v. By direct computations we find $\begin{cases} [H_i, v] = \lambda_4(H_i)v, \ \forall i, \\ [E_i, v] = 0, \ \forall i. \end{cases}$ Therefore V is a highest weight module of highest weight λ_4 and is finite dimensional. By Corollary 1.1.32, V is irreducible and so $V \cong L(\lambda_4)$.

By Weyl's formula (Theorem 1.1.33), we obtain $\dim V = 26$. Furthermore

$$\begin{array}{ccccc} \mathfrak{g} = & \mathfrak{g}_0 & \oplus & \mathfrak{g}_1 \\ & \cup & & \cup \\ & \mathfrak{a} & & V \end{array}$$

and $\dim \mathfrak{a} + \dim V = 52 + 26 = 78 = \dim \mathfrak{g}$, with $\dim \mathfrak{a}$ given in Table 1.1.

We conclude that $\mathfrak{g}_0 = \mathfrak{a}$ is the simple Lie algebra of type F_4 and $\mathfrak{g}_1 = V$ is an irreducible highest weight $\mathfrak{a}$-module of highest weight λ_4.

b) Folding of the root lattice Q: let $Q = \bigoplus_{i=1}^{6} \mathbb{Z}\alpha_i$ be the root lattice of a root system of

type E_6. We have defined σ by $\begin{cases} \sigma(\alpha_1) = \alpha_6, \\ \sigma(\alpha_3) = \alpha_5, \\ \sigma(\alpha_5) = \alpha_3, \\ \sigma(\alpha_6) = \alpha_1, \\ \sigma(\alpha_i) = \alpha_i \text{ otherwise,} \end{cases}$ and extend it to Q by linearity.

Let us compute the invariants Q^σ.

$$\begin{aligned} \alpha = \sum_{i=1}^{6} k_i\alpha_i \in Q^\sigma \quad &\Leftrightarrow \quad k_1 = k_6 \text{ and } k_3 = k_5, \\ &\Leftrightarrow \quad \alpha = k_1(\alpha_1 + \alpha_6) + k_3(\alpha_3 + \alpha_5) + k_4\alpha_4 + k_2\alpha_2. \end{aligned}$$

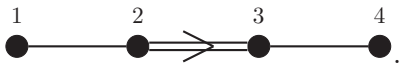
Set

$$\begin{cases} \beta_1 = \alpha_1 + \alpha_6, \\ \beta_2 = \alpha_3 + \alpha_5, \\ \beta_3 = \alpha_4, \\ \beta_4 = \alpha_2. \end{cases}$$

The β_i's generate the lattice Q^σ. From direct computation, one has

$$(\langle \beta_i, \beta_j \rangle)_{1\leq i,j\leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

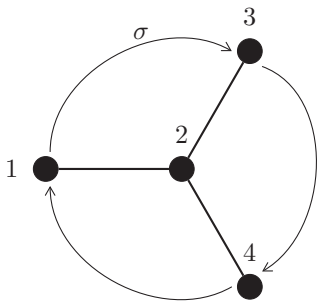
The corresponding diagram is



It is a Dynkin diagram of type F_4. We deduce that Q^σ is the root lattice of a Lie algebra of type F_4.

1.2.6. D_4 and G_2

Let $\mathfrak{g}$ be a Lie algebra of type D_4, Δ its Dynkin diagram and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ a base of the root system. We define σ an automorphism of Δ by $\begin{cases} \sigma(\alpha_1) = \alpha_3, \\ \sigma(\alpha_3) = \alpha_4, \\ \sigma(\alpha_4) = \alpha_1, \\ \sigma(\alpha_2) = \alpha_2. \end{cases}$



a) Folding of the Lie algebra: σ is lifted to an automorphism $\dot{\sigma}$ of $\mathfrak{g}$ defined by $\dot{\sigma}(x_i) = x_{\sigma(i)}$, with $x = e, f$ or h, the Chevalley generators.

As before we can prove that $\mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ because $\dot{\sigma}$ is of order 3.

- $\mathfrak{g}_0$ contains $\begin{cases} E_1 = e_1 + e_3 + e_4, \\ E_2 = e_2, \end{cases}$ $\begin{cases} F_1 = f_1 + f_3 + f_4, \\ F_2 = f_2, \end{cases}$ and $\begin{cases} H_1 = h_1 + h_3 + h_4, \\ H_2 = h_2. \end{cases}$

Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}_0$ generated by the E_i, F_i and H_i's. As in the previous subsection, by means of Serre's theorem (Theorem 1.1.20), one concludes that the E_i, F_i, H_i's are the Chevalley generators of the simple Lie algebra $\mathfrak{a} \subset \mathfrak{g}_0$ whose Dynkin diagram is

which is of type G_2.

• $\underline{\mathfrak{g}_1}$: as in the previous subsection, we find that $\mathfrak{g}_1$ is a representation of $\mathfrak{a}$. Let (λ_1, λ_2) be the fundamental weights of $\mathfrak{a}$. Let $\begin{cases} \eta_1 &= \alpha_1+\alpha_2+\alpha_4 \\ \eta_2 &= \alpha_3+\alpha_2+\alpha_1 \\ \eta_3 &= \alpha_4+\alpha_2+\alpha_3 \end{cases}$ and $v = e_{\eta_1} + j^2 e_{\eta_2} + j e_{\eta_3}$.
One can see that $v \in \mathfrak{g}_1$ and $v \neq 0$ because η_1, η_2 and η_3 are distinct. Let V be the $\mathfrak{a}$-module generated by v. Through direct computations one finds $\begin{cases} [H_i, v] = \lambda_1(H_i)v, \ \forall i, \\ [E_i, v] = 0, \ \forall i. \end{cases}$
Therefore V is a highest weight module of highest weight λ_4 and is finite dimensional. By Corollary 1.1.32, V is irreducible and so $V \cong L(\lambda_1)$.

With Weyl's formula (Theorem 1.1.33), we obtain $\dim V = 7$.

• $\underline{\mathfrak{g}_2}$: we have $[\mathfrak{g}_0, \mathfrak{g}_2] \subset \mathfrak{g}_2$, so $[\mathfrak{a}, \mathfrak{g}_2] \subset \mathfrak{g}_2$. Thus $\mathfrak{g}_2$ is a representation of $\mathfrak{a}$. Set $w = e_{\eta_1} + j e_{\eta_2} + j^2 e_{\eta_3}$. One can see that $w \in \mathfrak{g}_2$ and $w \neq 0$ because η_1, η_2 and η_3 are distinct. Let W be the $\mathfrak{a}$-module generated by w. From direct computations we find $\begin{cases} [H_i, w] = \lambda_1(H_i)w, \ \forall i, \\ [E_i, w] = 0, \ \forall i. \end{cases}$ Therefore W is a highest weight module of highest weight λ_1 and is finite dimensional. By Corollary 1.1.32, W is irreducible and so $W \cong L(\lambda_1)$. Therefore $V \cong W \cong L(\lambda_1)$.

We have $\begin{array}{ccccccc} \mathfrak{g} = & \mathfrak{g}_0 & \oplus & \mathfrak{g}_1 & \oplus & \mathfrak{g}_2 \\ & \cup & & \cup & & \cup \\ & \mathfrak{a} & & V & & U \end{array}$ and $\dim \mathfrak{a} + \dim V + \dim W = 14 + 7 + 7 = 28 = \dim \mathfrak{g}$,
with $\dim \mathfrak{a}$ given in Table 1.1. We conclude that $\mathfrak{g}_0 = \mathfrak{a}$ is the simple Lie algebra of type G_2 and $\mathfrak{g}_1 = V \cong W = \mathfrak{g}_2$ is an irreducible highest weight $\mathfrak{a}$-module of highest weight λ_1.

b) Folding of the root lattice Q: let $Q = \bigoplus_{i=1}^{4} \mathbb{Z}\alpha_i$ be the root lattice of a root system of type D_4. We defined σ by $\begin{cases} \sigma(\alpha_1) = \alpha_3, \\ \sigma(\alpha_3) = \alpha_4, \\ \sigma(\alpha_4) = \alpha_1, \\ \sigma(\alpha_2) = \alpha_2, \end{cases}$ and extend it to Q by linearity. Let us compute the invariants Q^σ.

$$\alpha = \sum_{i=1}^{4} k_i\alpha_i \in Q^\sigma \quad \Leftrightarrow \quad k_1 = k_3 = k_4,$$
$$\Leftrightarrow \quad \alpha = k_1(\alpha_1 + \alpha_3 + \alpha_4) + k_2\alpha_2.$$

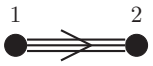
Set

$$\begin{cases} \beta_1 = \alpha_1 + \alpha_3 + \alpha_4, \\ \beta_2 = \alpha_2. \end{cases}$$

The β_i's generate the lattice Q^σ. From direct computation, one has

$$(\langle \beta_i, \beta_j \rangle)_{1\leq i,j\leq 2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

The corresponding diagram is



This is a Dynkin diagram of type G_2. We deduce that Q^σ is the root lattice of a Lie algebra of type G_2.

1.2.7. Conclusions

We summarize the results we obtained in the previous subsections in the following table:

type of $\mathfrak{g}$	A_{2r-1}	A_{2r}	D_{r+1}	E_6	D_4
type of $\mathfrak{g}_0$	C_r	B_r	B_r	F_4	G_2
type of Q^σ	B_r	C_r	C_r	F_4	G_2
order of σ	2	2	2	2	3

Table 1.2.

One notices that, in all five cases, the types of $\mathfrak{g}_0$ and Q^σ are dual to each other. This is due to the fact that the short roots and the long roots are switched when we go from the Lie algebra to the root lattice. For example, with the case A_{2r-1}, $\mathfrak{g}_0$ is of type C_r with simple roots $(\frac{1}{2}(\alpha_1+\alpha_{2r-1}),\dots,\frac{1}{2}(\alpha_{r-1}+\alpha_{r+1}),\alpha_r)$ so we have $r-1$ short roots and one long. Q^σ is of type B_r with base $(\alpha_1+\alpha_{2r-1},\dots,\alpha_{r-1}+\alpha_{r+1},\alpha_r)$ so we have $r-1$ long roots and one short. The Coxeter graphs will then be identical but the Dynkin diagrams will differ by the direction of the arrow on the multiple edge.

1.3. Adjoint quotient

1.3.1. Definition of the affine quotient

In this subsection we follow the presentation of the affine quotient given in [Muk03].

Let $X = \operatorname{Spm} \Bbbk[X]$ be an affine variety on which acts an algebraic group G. We want to define a quotient "X/G " which would characterize the action of G on X. A most natural way is to look at a map of the following form $\varphi : \begin{array}{ccc} X & \to & \mathbb{A}^n \\ x & \mapsto & (f_1(x),\dots,f_n(x)) \end{array}$ with the f_i being G-invariant regular functions. This map is then constant on each G-orbit. We say that G is **linearly reductive** if for any surjective map $f : V \to W$ of representations of G, the induced map $f^G : V^G \to W^G$ on the G-invariants is surjective. It is equivalent to the assertion that every finite dimensional representation of G is completely reducible (cf. [Pro07] Chapter 7, Section 3.1, Lemma). Furthermore, in characteristic 0, a linear algebraic group is linearly reductive if and only if it is reductive (cf. [Pro07] Chapter 7, Section 3.6, Theorem 2). One can show that $\mathrm{SL}_n(\mathbb{C})$, as well as compact Lie groups (which include the exceptional Lie groups), are linearly reductive. D. Hilbert proved the following theorem in [Hilb90]:

Theorem 1.3.1. (Hilbert). *Let G be a linearly reductive algebraic group, which acts on a polynomial ring S, preserving the grading. Then the ring of invariant polynomials S^G is finitely generated.*

Hilbert's theorem states that $\Bbbk[X]^G$ has a finite number of generators. So we take the above f_i's to be a set of generators of the invariants. One wonders if this map can separate the orbits, or what does the image of φ looks like.

Example 1.3.2. *Let us look at the action of the multiplicative group $\mathbb{G}_m$ on the affine plane $X = \mathbb{A}^2$ given by $t.(x,y) = (tx, t^{-1}y)$, $t \in \mathbb{G}_m, (x,y) \in \mathbb{A}^2$. The orbits of this action are of three kinds:*

(i) the origin $(0,0)$,

(ii) every hyperbola $xy = a$ *with* a *not zero*,

(iii) the x *– axis without the origin and the* y *– axis without the origin.*

On the other hand G *acts on the coordinate ring of* X*, which is* $\Bbbk[X] = \Bbbk[x,y]$*. It is easy to see that* $\Bbbk[X]^G = \Bbbk[xy]$ *and so there is a map*

$$\begin{array}{rccc} \varphi: & \mathbb{A}^2 & \to & \Bbbk \\ & (x,y) & \mapsto & xy \end{array} .$$

This map separates the orbits of type (ii), but makes no difference between the types (i) and (iii).

According to the previous example, the invariant functions are not sufficient to separate orbits. This is due to topology. The map φ is polynomial so continuous. This implies that each fiber is closed, so if one G-orbit is not closed, φ cannot separate the orbits. We are going to see that φ does not separate orbits but certain equivalence classes of orbits.

Definition 1.3.3. *Two* G*-orbits* $\mathcal{O}$ *and* $\mathcal{O}' \subset X$ *are said to be* ***closure-equivalent*** *if there exists a sequence of orbits*

$$\mathcal{O} = \mathcal{O}_0,\ \mathcal{O}_1, \ldots,\ \mathcal{O}_{r-1},\ \mathcal{O}_r = \mathcal{O}'$$

such that $\overline{\mathcal{O}_i} \bigcap \overline{\mathcal{O}_{i+1}} \neq \varnothing$*, for all* $i = 0, \ldots, r-1$.

Theorem 1.3.4. (Mumford-Nagata). *Let* G *be a linearly reductive group acting on an affine variety* X *and* $\mathcal{O}$*,* $\mathcal{O}' \subset X$ *two orbits in* X*. The following assertions are equivalent:*

(i) $\overline{\mathcal{O}} \bigcap \overline{\mathcal{O}'} \neq \varnothing$.

(ii) $\mathcal{O}$ *and* $\mathcal{O}'$ *are closure-equivalent.*

(iii) $\mathcal{O}$ *and* $\mathcal{O}'$ *are not separated by the* G*-invariants* $\Bbbk[X]^G$.

According to the theorem, $\varphi(X)$ parametrizes the closure-equivalence classes of G-orbits in X.

Corollary 1.3.5. *Let G be a linearly reductive group acting on an affine variety X. Then the distinct closed G-orbits are separated by φ. Furthermore, each closure-equivalence class contains one unique closed orbit.*

The corollary implies that the image of φ parametrizes the closed G-orbits. Now we can define our quotient variety.

Definition 1.3.6. *We note $X//G$ for the affine variety* $\mathrm{Spm}(\Bbbk[X]^G)$. *The inclusion $\Bbbk[X]^G \subset \Bbbk[X]$ defines a map*

$$\Phi : X \to X//G,$$

which is called the ***affine quotient map****. The variety $X//G$ is called the* ***affine GIT quotient****.*

It is possible to show (cf. [Muk03] Theorem 5.9) that if G is linearly reductive and acts on an affine variety X, then Φ is surjective and there is a bijective correspondence between the points of $X//G$ and the closure-equivalence classes of G-orbits in X.

Assuming that G is a finite group, the affine quotient has an interesting property. Indeed, as G is finite, every orbit $G.x$, $x \in X$, is of finite cardinal and is thus closed. Therefore $X//G$ characterizes all the orbits of G. This turns $X//G$ into a geometric quotient, and as such, is noted X/G. By definition, a **geometric quotient** of a G-variety X by an algebraic group G is a morphism $\pi : X \to Y$ such that:

(i) π is surjective, and its fibers are exactly the G-orbits in X.

(ii) The topology of Y is the quotient topology: a subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open.

(iii) For any open subset $U \subset Y$, the comorphism $\pi^{\#}$ yields an isomorphism $\Bbbk[U] \cong \Bbbk[\pi^{-1}(U)]^G$.

The reader may consult [Mum94] for properties of a geometric quotient. In particular, if X is irreducible, then so is Y, and there is an equality of function fields $\Bbbk(Y) = \Bbbk(X)^G$.

From conditions (i) and (ii), as a set, the geometric quotient is the set of all orbits. However, the geometric quotient and the set theoric quotient are not always the same, because the geometric quotient does not always exists.

Examples 1.3.7. *(1) Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^n$ by scalar multiplication. The orbits are the lines through 0 minus the origin, and the origin. But there is no geometric quotient, since 0 lies in every orbit closure.*

(2) Let $G = \mathbb{G}_m$ act on $\mathbb{A}^2 \setminus \{0\}$ via $t \cdot (x, y) := (tx, t^{-1}y)$, as in Example 1.3.2. Recall that the orbits are the hyperbolae $\{xy = a\}$, with $a \in \Bbbk^*$, the x-axis minus the origin

and the y-axis minus the origin. All the orbits are closed in $\mathbb{A}^2 \setminus \{0\}$*, but there is no geometric quotient. Indeed,* $\Bbbk(X)^G = \Bbbk(xy)$ *and thus, the G-orbits* $(x = 0) \setminus \{0\}$ *and* $(y = 0) \setminus \{0\}$ *are not separated by G-invariant rational functions.*

One may add that if the geometric quotient and the affine GIT quotient both exist, then they are identical.

1.3.2. Adjoint quotient

We return to Lie theory. Let G be a semisimple algebraic group over $\Bbbk$ and $\mathfrak{g}$ its Lie algebra. The adjoint action of G on $\mathfrak{g}$ has already been defined. Let us look at the morphism $\mathfrak{g} \to \mathfrak{g}//G$.
Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and W the corresponding Weyl group. We write $S(\mathfrak{h}) \subset S(\mathfrak{g})$ for the symmetric algebras of $\mathfrak{h}$ and $\mathfrak{g}$ respectively. C. Chevalley proved the following theorem:

Theorem 1.3.8. (Chevalley). *Let G be a semisimple algebraic group over* $\Bbbk$*,* $\mathfrak{g}$ *its Lie algebra,* $\mathfrak{h} \subset \mathfrak{g}$ *a Cartan subalgebra and W the corresponding Weyl group. There is an isomorphism*

$$S(\mathfrak{g})^G \xrightarrow{\cong} S(\mathfrak{h})^W.$$

Proof. A proof due to R. Steinberg can be found in [Hum78]. □

Examples 1.3.9. *In Examples 1.1.25 we computed the exponents for the Lie algebras of types A and D in the complex settings, and for this we obtained a set of generators for the algebra* $S(\mathfrak{h})^W$*.*
For the type A_r*, we found* $S(\mathfrak{h})^W = \mathbb{C}[\sigma_2(X), \dots, \sigma_{r+1}(X)]$ *with* σ_i *the i-th elementary symmetric polynomial. According to Theorem 1.3.8, the algebra* $S(\mathfrak{g})^G$ *of G-invariants is generated by* $\sigma_2(X), \dots, \sigma_{r+1}(X)$*.*
For the type D_r*, we had* $S(\mathfrak{h})^W = \mathbb{C}[\sigma_1(X^2), \dots, \sigma_{r-1}(X^2), \sigma_r(X)] \cong S(\mathfrak{g})^G$*.*
For the cases B_r *et* C_r*, one finds that* $S(\mathfrak{g})^G \cong S(\mathfrak{h})^W = \mathbb{C}[\sigma_1(X^2), \dots, \sigma_r(X^2)]$*.*

In the settings of Chevalley's theorem, $\mathfrak{g}$ is semisimple and therefore the Killing form κ of $\mathfrak{g}$ is non-degenerate. Thus it determines an isomorphism $\Xi : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\Xi(x)(y) = \kappa(x, y), \quad \forall x, y \in \mathfrak{g}.$$

This isomorphism enables us to identify $S(\mathfrak{g})^G$ (respectively $S(\mathfrak{h})^W$) with $\Bbbk[\mathfrak{g}]^G$ (respectively $\Bbbk[\mathfrak{h}]^W$). Looking at the spectrum in Chevalley's theorem, one obtains the isomorphism $\mathfrak{g}//G \cong \mathfrak{h}/W$.

One can write $\mathfrak{h}/W$ because W is a finite group and so the quotient $\mathfrak{h}//W$ is geometric (cf. Subsection 1.3.1). So there is a surjective morphism $\chi : \mathfrak{g} \to \mathfrak{h}/W$ that we call the **adjoint quotient** of $\mathfrak{g}$. This morphism will be described in more details.

The next two lemmas can be found in [ColMcG93].

Lemma 1.3.10. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$, G its corresponding group acting by the Adjoint action, and $\mathfrak{h}$ a Cartan subalgebra. We define a map*

$$\begin{array}{rccc} \widetilde{\mu}: & \mathfrak{h} & \longrightarrow & \{G\text{-orbits of semisimple elements}\}\ . \\ & h & \longmapsto & \mathcal{O}_h = G.h \end{array}$$

Then $\widetilde{\mu}$ is surjective.

Remark 1.3.11. The closed orbits of G in its Lie algebra $\mathfrak{g}$ are exactly the orbits of semisimple elements (cf. [ColMcG93] Theorem 2.3.1).

Lemma 1.3.12. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$, G its corresponding group acting by the Adjoint action, $\mathfrak{h}$ a Cartan subalgebra and W the associated Weyl group. Then we have a natural isomorphism*

$$W \cong N_G(\mathfrak{h})/C_G(\mathfrak{h}),$$

with $N_G(\mathfrak{h}) = \{g \in G \mid g.\mathfrak{h} = \mathfrak{h}\}$ and $C_G(\mathfrak{h}) = \{g \in G \mid g.h = h\ \forall h \in \mathfrak{h}\}$ the normalizer and the centralizer of $\mathfrak{h}$ in G, respectively.

Proposition 1.3.13. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$, $\mathfrak{h}$ a Cartan subalgebra and W the associated Weyl group. Then the set of conjugacy classes of semisimple elements of $\mathfrak{g}$ is in bijection with $\mathfrak{h}/W$.*

Proof. Consider the map

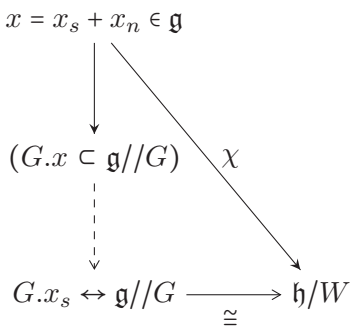
$$\begin{array}{rccc} \mu: & \mathfrak{h}/W & \longrightarrow & \{\mathcal{O}_x \mid x \text{ is semisimple}\} \\ & \overline{h} & \mapsto & \mathcal{O}_h \end{array}$$

with $\overline{h}$ being the image of $h \in \mathfrak{h}$ by the natural projection $\mathfrak{h} \to \mathfrak{h}/W$. Say h and h_1 are two distinct representatives of the conjugacy class $\overline{h}$. Then by Lemma 1.3.12 , h and h_1 are G-conjugated and so $\mathcal{O}_h = \mathcal{O}_{h_1}$. Therefore μ is well defined. Lemma 1.3.10 implies that μ is surjective.
We now prove that μ is injective. Assume $\mu(\overline{h_1}) = \mu(\overline{h_2})$. Then $\mathcal{O}_{h_1} = \mathcal{O}_{h_2}$ and so there exists $g \in G$ such that $g.h_1 = h_2$. One notices that $\mathfrak{h}$ and $g.\mathfrak{h}$ are two Cartan subalgebras containing h_2. Since Cartan subalgebras are abelian, $\mathfrak{h}$ and $g.\mathfrak{h}$ centralize h_2, i.e.

$$\mathfrak{h},\ g.\mathfrak{h} \subset \mathfrak{g}^{h_2}.$$

These are both Cartan subalgebras of $\mathfrak{g}^{h_2}$, which is reductive. Therefore there exists g_2 an element of the adjoint group of $\mathfrak{g}^{h_2}$ such that $g_2.g.\mathfrak{h} = \mathfrak{h}$ and $g_2.g.h_1 = h_2$. This implies that $g_2.g$ is in $N_G(\mathfrak{h})$ and sends h_1 to h_2. By Lemma 1.3.12, there exists $w \in W$ such that $w.h_1 = h_2$. In other words, $\overline{h_1} = \overline{h_2}$ and μ is injective. □

We have the following diagram:



with x_s and x_n the semisimple and nilpotent parts of x and χ the adjoint quotient.

Example 1.3.14. *If $\mathfrak{g} = \mathfrak{sl}_n(\Bbbk)$, $A \in \mathfrak{g}$, we have $\chi(A) = (\mathrm{Tr}(A^2), \ldots, \mathrm{Tr}(A^n)) \in \mathfrak{h}/W \cong \Bbbk^{n-1}$.*

Let $\mathfrak{g}$ be a simple Lie algebra over $\Bbbk$ of rank r, $\mathfrak{h}$ a Cartan subalgebra and W the Weyl group. We have $\Bbbk[\mathfrak{h}]^W \cong \Bbbk[\mathfrak{g}]^G = \Bbbk[\chi_1, \ldots, \chi_r]$ where the χ_i are the homogeneous generators from Chevalley's theorem (Theorem 1.1.21). One can then realize the morphism $\chi : \mathfrak{g} \to \mathfrak{h}/W$ by

$$\begin{array}{rccc} \chi : & \mathfrak{g} & \longrightarrow & \mathfrak{h}/W \cong \Bbbk^r \\ & x & \longmapsto & (\chi_1(x), \ldots, \chi_r(x)) \end{array} .$$

More details on the adjoint quotient can be found in [Hum95].

We conclude this section with a theorem on the structure of the orbits of the Adjoint action.
Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$ with adjoint Lie group G acting via the Adjoint action on $\mathfrak{g}$ and $\mathcal{O}$ an orbit for this action. Let $(.,.)$ be a G-invariant inner product on $\mathfrak{g}$, for instance the opposite of the Killing form. Each element U of the Lie algebra $\mathfrak{g}$ induces a vector field X_U on $\mathcal{O}$ defined by

$$X_U(Z) = [U, Z] = -\mathrm{ad}(Z)(U), \quad \forall Z \in \mathcal{O}.$$

X_U is called the **fundamental vector field** attached to U.
Since G acts transitively on $\mathcal{O}$, at any point $Z \in \mathcal{O}$, the values of the fundamental vector fields X_U generate the whole tangent space $T_Z\mathcal{O}$ of $\mathcal{O}$ at Z. We deduce that $T_Z\mathcal{O}$ is the affine sub-space of $\mathfrak{g}$ at Z associated with the (vector) subspace $\mathrm{Im}(\mathrm{ad}\ Z)$, the image of the endomorphism ad Z in $\mathfrak{g}$.

The next theorem is proved in [Bes87].

Theorem 1.3.15. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\Bbbk$ with adjoint algebraic group G acting via the Adjoint action on $\mathfrak{g}$. Let $\mathcal{O}$ be an orbit in $\mathfrak{g}$. Then $\mathcal{O}$ possesses a symplectic structure (cf. Definition 2.1.1) given by the symplectic form F defined by*

$$F_Z(X, Y) = (Z, [U, V]), \quad \forall X, Y \in T_Z\mathcal{O}, Z \in \mathcal{O},$$

where U and V are any elements of $\mathfrak{g}$ whose image by ad Z *are X and Y respectively (here $T_Z\mathcal{O}$ is viewed as a vector space and not as an affine space). The symplectic form F is called the* ***Kostant-Kirillov-Souriau form****.*

Remark 1.3.16. The Kostant-Kirillov-Souriau form is initially a symplectic form on the co-Adjoint orbits. However, as $\mathfrak{g}$ is semisimple, the Killing form enables us to identify $\mathfrak{g}$ and its dual, thus giving the orbits of the Adjoint action a symplectic structure.

1.3.3. Nilpotent cone

From now on, the base field $\Bbbk$ will always be the complex number field $\mathbb{C}$. Because of Theorem 1.1.38, we can now use indifferently the terms Lie group and algebraic group in our setting of semisimple groups.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra, W the associated Weyl group and $\chi : \mathfrak{g} \to \mathfrak{h}/W$ the adjoint quotient. The next theorem is due to R. Steinberg and proved in [Stein74]:

Theorem 1.3.17. *The adjoint quotient χ is a flat morphism. As a consequence, all fibers have the same dimension* $\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$. *Furthermore, any fiber is a union of a finite number of conjugacy classes.*

We call $\mathcal{N}(\mathfrak{g}) = \chi^{-1}(\chi(0))$ the **nilpotent cone** of $\mathfrak{g}$. It was explained in the previous section that if $x = x_s + x_n \in \mathfrak{g}$ is the Jordan-Chevalley decomposition of an element of $\mathfrak{g}$, then $\chi(x) = \chi(x_s)$. It follows that $x \in \mathcal{N}(\mathfrak{g})$ if and only if $x_s = 0$, which means x nilpotent. $\mathcal{N}(\mathfrak{g})$ is indeed a cone because it is stable by scalar multiplication. It is clear that the algebraic group G associated to $\mathfrak{g}$ acts by the Adjoint action on $\mathcal{N}(\mathfrak{g})$. Because the nilpotent cone is a fiber of the adjoint quotient, we have the following corollary:

Corollary 1.3.18. *Let $\mathfrak{g}$ be a simple Lie algebra with Weyl group W and nilpotent cone $\mathcal{N}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\}$. Let Φ be the set of roots of $\mathfrak{g}$. Then* $\dim \mathcal{N}(\mathfrak{g}) = |\Phi| = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$.

It is also possible to find more information on $\mathcal{N}(\mathfrak{g})$ as it is done in [Sprin66]:

Proposition 1.3.19. *$\mathcal{N}(\mathfrak{g})$ is an irreducible variety as well as a Zariski-closed subset of the affine space modelled on $\mathfrak{g}$.*

An element $x \in \mathfrak{g}$ is called **regular** if $\dim Z_G(x) = \operatorname{rank}(\mathfrak{g})$, where $Z_G(x)$ is the centralizer of x in G under the adjoint action. B. Kostant proved the following in [Kos63]:

Proposition 1.3.20. *Let F be a fiber of the adjoint quotient. Then F contains a unique regular orbit. Furthermore this orbit is dense in F.*

It is proved in [Kos59] that for all $x \in \mathfrak{g}$, the centralizer $\mathfrak{z}_{\mathfrak{g}}(x)$ of x in $\mathfrak{g}$ contains a commutative subalgebra of dimension $\operatorname{rank}(\mathfrak{g})$. So $\dim \mathfrak{z}_{\mathfrak{g}}(x) \geq \operatorname{rank}(\mathfrak{g})$. It is also known that $\exp : \mathfrak{g} \to G$ is a local homeomorphism. We compute $\frac{d}{dt}\big|_{t=0}$ on each side of $\operatorname{ad}(\exp(tX))(x) = x$ with $X \in \mathfrak{g}$, and find $[X, x] = 0$ so $X \in \mathfrak{z}_{\mathfrak{g}}(x)$. One deduces that $T_e(Z_G(x)) = \mathfrak{z}_{\mathfrak{g}}(x)$, implying $\dim Z_G(x) = \dim \mathfrak{z}_{\mathfrak{g}}(x) \geq \operatorname{rank}(\mathfrak{g})$. So an element is regular if and only if its centralizer is of minimal dimension, or equivalently if its orbit is of maximal dimension.
One can explain this in another fashion. Let $\chi : \mathfrak{g} \to \mathfrak{h}/W \cong \mathbb{A}^r$ be the adjoint quotient. Then $\chi = (\chi_1, \ldots, \chi_r)$, the χ_i are algebraically independent and each fiber has dimension $\dim \mathfrak{g} - r$, because of the flatness of χ. Each fiber of χ is G-invariant and is a union of a finite number of G-orbits. Moreover, as the dimension of an orbit is $\dim G - \dim Z_G(x)$, where x is a point in the orbit, we have $\dim Z_G(x) \geq r$ for any $x \in \mathfrak{g}$.

In [Stein74], R. Steinberg defined an element $x \in \mathfrak{g}$ as being **subregular** if $\dim Z_G(x) = \operatorname{rank}(\mathfrak{g}) + 2$. It was stated in Theorem 1.3.15 that all orbits have a symplectic struc-

ture and because of Remark 2.1.2, their dimensions are even. Therefore the orbits of subregular elements are those of maximal size right after the regular ones.

We have the following propositions regarding the orbits in the nilpotent cone.

Proposition 1.3.21. *There exists a unique regular orbit* $\mathcal{O}_{\text{reg}}$ *in* $\mathcal{N}(\mathfrak{g})$. *If* x *is regular nilpotent, then* $\dim \mathcal{O}_x = \dim G - \dim Z_G(x) = \dim \mathfrak{g} - \text{rank}(\mathfrak{g}) = \dim \mathcal{N}(\mathfrak{g})$.

The existence of the regular orbit is due to the fact that $\mathcal{N}(\mathfrak{g})$ is a finite union of orbits and a finite union of subspaces of dimensions strictly inferior than n cannot be a space of dimension n. The unicity is due to Proposition 1.3.20. There is a similar result for the subregular nilpotent elements due to R. Steinberg ([Stein74]):

Proposition 1.3.22. *There exists a unique subregular orbit* $\mathcal{O}_{\text{subreg}}$ *in* $\mathcal{N}(\mathfrak{g})$. *It is open and dense in* $\mathcal{N}(\mathfrak{g}) \setminus \mathcal{O}_{reg}$. *If* x *is subregular nilpotent, then* $\dim \mathcal{O}_x = \dim G - \dim Z_G(x) = \dim \mathfrak{g} - \text{rank}(\mathfrak{g}) - 2 = \dim \mathcal{N}(\mathfrak{g}) - 2$.

1.4. Simple singularities and Dynkin diagrams

1.4.1. Simple singularities of type ADE

1.4.1.1. Definition of simple singularities

F. Klein studied the finite subgroups of $\text{SL}_2(\mathbb{C})$ in terms of transformations of $\mathbb{C}^2$ and classified them in [Klein84]. Later H.S.M. Coxeter and W. Moser gave a presentation for each one in [CoxMos57]. Let Γ be a finite subgroup of $\text{SL}_2(\mathbb{C})$. Then Γ is conjugate to a unique subgroup of the form:

- $\mathcal{C}_n = \langle a \mid a^n = 1 \rangle (n \geq 1)$, the cyclic group of order n,
- $\mathcal{D}_n = \langle a, b, c \mid a^2 = b^2 = c^n = abc \rangle (n \geq 2)$, the binary dihedral group of order $4n$,
- $\mathcal{T} = \langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle$, the binary tetrahedral group of order 24,
- $\mathcal{O} = \langle a, b, c \mid a^2 = b^3 = c^4 = abc \rangle$, the binary octahedral group of order 48,
- $\mathcal{I} = \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle$, the binary icosahedral group of order 120.

The natural action of Γ on $\mathbb{C}^2$ induces an action of Γ on the ring $\mathbb{C}[\mathbb{C}^2]$ of polynomial functions on $\mathbb{C}^2$. The ring $\mathbb{C}[\mathbb{C}^2]^\Gamma$ of Γ-invariant polynomials is then generated by three

elements X, Y, Z satisfying a unique relation $R(X, Y, Z) = 0$. Indeed, such relations were determined by F. Klein in [Klein84].

Theorem 1.4.1. *Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then $\mathbb{C}^2/\Gamma$ injects into $\mathbb{C}^3$ as the zeros of a polynomial $R \in \mathbb{C}[X, Y, Z]$, which is given in the following table:*

Γ	R
$\mathcal{C}_n$	$X^n + YZ$
$\mathcal{D}_n$	$X(Y^2 - X^n) + Z^2$
$\mathcal{T}$	$X^4 + Y^3 + Z^2$
$\mathcal{O}$	$X^3 + XY^3 + Z^2$
$\mathcal{I}$	$X^5 + Y^3 + Z^2$

Proof. Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. As Γ is conjugated to a subgroup of SU_2, without loss of generality, we can assume that Γ is a finite subgroup of SU_2.

- Let $\Gamma = \mathcal{C}_n = \mathbb{Z}/n\mathbb{Z} \cong < \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} >$ where $\omega = \exp(\frac{2i\pi}{n})$. Set $g = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$. $\mathbb{C}^2$ has a base (e_1, e_2) with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So $(\mathbb{C}^2)^*$ has a base (z_1, z_2) with $z_1(x) = x_1$ and $z_2(x) = x_2$, for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2$. $\mathcal{C}_n$ acts on $(\mathbb{C}^2)^*$ by $(\sigma.f)(x) = f(\sigma^{-1}.x)$, for $\sigma \in \mathcal{C}_n, f \in (\mathbb{C}^2)^*$ and $x \in \mathbb{C}^2$. Thus $(g.z_1)(x) = z_1(g^{-1}.x) = \omega^{-1}x_1 = \omega^{-1}z_1(x)$, $\forall\ x \in \mathbb{C}^2$. Hence $g.z_1 = \omega^{-1}z_1$, and in a similar manner, $g.z_2 = \omega z_2$.

The polynomial ring on $\mathbb{C}^2$ is $\mathbb{C}[z_1, z_2]$. Let $k, l \in \mathbb{Z}_{\geq 0}$. The element g acts on the monomial $z_1^k z_2^l$ by $g.(z_1^k z_2^l) = \omega^{l-k} z_1^k z_2^l$. Hence Γ acts diagonally on each monomial, and it follows that $\mathbb{C}[z_1, z_2]^{\mathcal{C}_n} = \mathbb{C}[z_1^n, z_1 z_2, z_2^n]$.

There is a surjection $\varphi : \mathbb{C}[X,Y,Z] \twoheadrightarrow \mathbb{C}[z_1^n, z_1z_2, z_2^n]$.

$$\begin{array}{ccc} X & \longmapsto & z_1z_2 \\ Y & \longmapsto & z_1^n \\ Z & \longmapsto & z_2^n \end{array}$$

We show that $(X^n - YZ) = \mathrm{Ker}(\varphi)$. As the inclusion $\subset$ follows immediately, we show the other inclusion. Let $P \in \mathrm{Ker}(\varphi)$. The division of P by $X^n - YZ$ seen as a polynomial in the X variable gives $P(X,Y,Z) = Q(X,Y,Z)(X^n - YZ) + T(X,Y,Z)$ with $T(X,Y,Z) = a_{n-1}(Y,Z)X^{n-1} + \ldots + a_1(Y,Z)X + a_0(Y,Z)$. By using the relation $T(z_1z_2, z_1^n, z_2^n) = 0$ and then looking at the degrees of z_1 and z_2 modulo n, one sees that $a_0 = \ldots = a_{n-1} = 0$. Hence P is divisible by $X^n - YZ$, which implies that $\mathrm{Ker}(\varphi) \subset (X^n - YZ)$. Therefore $\mathrm{Ker}(\varphi) = (X^n - YZ)$.

It has been proved that $\mathbb{C}[z_1,z_2]^{\mathcal{C}_n} = \mathbb{C}[z_1^n, z_1z_2, z_2^n] \cong \mathbb{C}[X,Y,Z]/(X^n - YZ)$. So $\mathbb{C}^2/\mathcal{C}_n$ injects into $\mathbb{C}^3$ as the solutions of the equation $X^n - YZ = 0$. By a linear transformation, the equation becomes $X^n + YZ = 0$.

Representations of $\mathbb{R}^3 \bigcap (\mathbb{C}^2/\mathcal{C}_n)$ for $n = 2$ and 3.

The cases where $\Gamma = \mathcal{D}_n, \mathcal{T}$ and $\mathcal{O}$ are solved in a similar manner. The realizations of the groups can be found in [Co76] or in Chapter 4 of [Sprin77].

- Let $\Gamma = \mathcal{I} = < \begin{pmatrix} \exp(\frac{2i\pi}{10}) & 0 \\ 0 & \exp(-\frac{2i\pi}{10}) \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix} >$, where $\eta = \exp(\frac{2i\pi}{5})$.

This case cannot be solved easily through direct computations as the previous ones. Another approach is needed. Here are some preliminary definitions.

Let (z_1, z_2) be the coordinate system of $\mathbb{C}^2$ on which $\mathcal{I}$ acts by matrix multiplication.

Definition 1.4.2. *Let G be a group acting on $\mathbb{C}^2$. A homogeneous polynomial $F \in \mathbb{C}[z_1, z_2]$ is a **relative invariant** of G if for all $g \in G$, $g.F = \lambda_g F$ with $\lambda_g \in \mathbb{C}^*$. The map $\chi_F : G \to \mathbb{C}^*, g \to \lambda_g$ is called the **character** of F.*

Definition 1.4.3. *A **Grundform** is a relative invariant Φ whose divisor of zeros is an **exceptional orbit** (i.e. an orbit with a non-trivial stabilizer).*

The Grundformen will form a basis of the invariant polynomials. The following theorem gives the necessary conditions for it to happen (cf. [Dolg07]).

Theorem 1.4.4. *Assume that G is finite and there exist two Grundformen Φ_1 and Φ_2 whose orbits have cardinalities $|G|/e_1$ and $|G|/e_2$ such that the characters satisfy*

$$\chi_{\Phi_1}^{e_2} = \chi_{\Phi_2}^{e_1}.$$

Then every relative invariant is a polynomial in Grundformen.

Our aim is to apply the previous theorem to $\mathcal{I}$.

The group $\mathcal{I}$ acts on $\mathbb{C}[z_1, z_2]$ and preserves the degree. Let $F \in \mathbb{C}[z_1, z_2]$ be a homogeneous polynomial of degree d. Note that if (x, y) is a zero of F, then $F(\lambda x, \lambda y) = \lambda^d F(x, y) = 0$, so (x, y) defines a line of zeros passing through the origin. It defines therefore a point in $\mathbb{P}^1(\mathbb{C})$. If $g \in \mathcal{I}$, then $g.F(x, y) = F(g^{-1}.(x, y)) = 0$ because (x, y) is a zero of F. Hence $\mathcal{I}$ permutes the zeros of F and the lines they define in $\mathbb{P}^1(\mathbb{C})$. Therefore there is an action of $\mathcal{I}$ on $\mathbb{P}^1(\mathbb{C})$.

Let an icosahedron $\mathfrak{I}$ be centered at the origin and included in the unit sphere. Choose two diametrically opposite vertices and set them on the north and south poles of the sphere. Let ξ, μ, ζ denote the coordinates of the unit sphere $\{\xi^2 + \mu^2 + \zeta^2 = 1\}$. The stereographic projection from the north pole to the plane $\{\zeta = 0\}$ of the icosahedron sends a vertex of coordinates (ξ, μ, ζ) to a point (x, y) of the plane $\{\zeta = 0\}$ with

$$x = \frac{\xi}{1 - \zeta}, \quad y = \frac{\mu}{1 - \zeta}.$$

The point (x, y) will be given an affine coordinate $z = x + iy$. Hence the projection sends (ξ, μ, ζ) to $z = \frac{\xi + i\mu}{1 - \zeta}$. The north pole and the south pole are sent to ∞ and 0, respectively.

Let I be the group of rotations of the icosahedron $\mathfrak{I}$. Then $\mathcal{I}$ is the preimage of I under the projection $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. With the stereographic projection previously mentioned, a point on the sphere is given an affine coordinate and can thus be seen as a point in $\mathbb{P}^1(\mathbb{C})$. A point of affix z can be seen as $(z_1 : z_2) \in \mathbb{P}^1(\mathbb{C})$ with $z = \frac{z_1}{z_2}$. The point (z_1, z_2) corresponds to the affine coordinate $z = \frac{z_1}{z_2}$. The action of $\mathcal{I}$ preserves the icosahedron $\mathfrak{I}$ because the action of I does so. If (z_1, z_2) is fixed by $\mathcal{I}$, then $z = \frac{z_1}{z_2} \in \mathbb{P}^1(\mathbb{C}) \cong S^2$ is fixed by I. However, the only points of $\mathfrak{I}$ that are fixed by an element of I are the vertices, the centers of the faces and the midpoints of the edges. Under the action of I, the vertices, the faces and the edges form three orbits. This property is naturally transferred to $\mathcal{I}$. Hence the exceptional orbits are the one of the vertices, the one of the centers of the faces and the one of the midpoints of the edges. More details on the binary icosahedral group as well as the other finite subgroups of SU_2 can be found in [Sprin77]. By applying all elements of $\mathcal{I}$ to the south pole, one obtains the affine coordinates of all the vertices:

$$z = 0,\ \infty,\ \eta^\nu(\eta + \eta^4),\ \eta^\nu(\eta^2 + \eta^3), \text{ for } \nu = 0, 1, 2, 3, 4.$$

Set

$$f(z_1, z_2) = z_1 z_2 \prod_{\nu=0}^{4} (z_1 - \eta^\nu(\eta + \eta^4) z_2) \prod_{\nu=0}^{4} (z_1 - \eta^\nu(\eta^2 + \eta^3) z_2).$$

The zeros of f correspond to the vertices of the icosahedron $\mathfrak{I}$ and thus form an exceptional orbit. Because the zeros are permuted by $\mathcal{I}$, if $g \in \mathcal{I}$, then $g.f$ and f have the same degree and the same roots for all $g \in \mathcal{I}$. Hence there exists $\chi_f : \mathcal{I} \to \mathbb{C}^*$ such that $g.f = \chi_f(g) f$ for any $g \in \mathcal{I}$, implying that f is a Grundform. One can prove that χ_f is in fact trivial.

Set

$$H = \frac{1}{121} \begin{vmatrix} \frac{\partial^2 f}{\partial z_1^2} & \frac{\partial^2 f}{\partial z_1 z_2} \\ \frac{\partial^2 f}{\partial z_1 z_2} & \frac{\partial^2 f}{\partial z_2^2} \end{vmatrix} \quad \text{and} \quad T = \frac{1}{20} \begin{vmatrix} \frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} \\ \frac{\partial H}{\partial z_1} & \frac{\partial H}{\partial z_2} \end{vmatrix}.$$

From explicit computations, one finds $g.H = (\det(g))^2 H$ and $g.T = (\det(g)) T$ for $g \in \mathcal{I}$. But $\mathcal{I}$ is a subgroup of SU_2, hence $\det(g) = 1$ for any $g \in \mathcal{I}$. One deduces that H and T are $\mathcal{I}$-invariants, and thus their zeros are unions of $\mathcal{I}$-orbits.

The reader is reminded that

$$\begin{aligned} f(z_1,z_2) &= z_1z_2(z_1^{10}+11z_1^5z_2^5-z_2^{10}),\\ H(z_1,z_2) &= -(z_1^{20}+z_2^{20})+288(z_1^{15}z_2^5-z_1^5z_2^{15})-494z_1^{10}z_2^{10},\\ T(z_1,z_2) &= (z_1^{30}+z_2^{30})+522(z_1^{25}z_2^5-z_1^5z_2^{25})-10005(z_1^{20}z_2^{10}-z_1^{10}z_2^{20}). \end{aligned}$$

H is an I-invariant of degree 20, which implies that the set of his zeros in $\mathbb{P}^1(\mathbb{C}) \cong S^2$ is a union of I-orbits and is of cardinal ≤ 20. However, the orbits of the group I are of cardinal either 12, 20, 30 or 60 and H is not proportional to a power of f. The only possibility is that the zeros of H are the 20 centers of the faces of $\mathfrak{I}$. Hence they form an exceptional orbit and H is a Grundform. In a similar manner one shows that T is the Grundform corresponding to the midpoints of the edges. All Grundformen have thus been found.

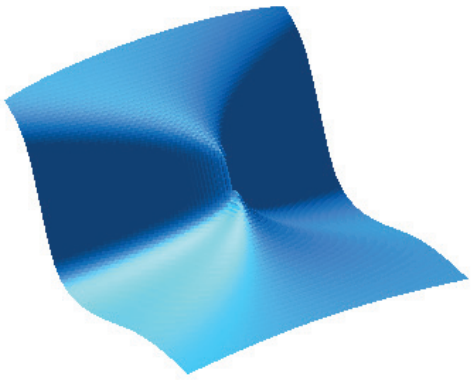
It was shown above that $\chi_H = \chi_T = 1$. Therefore H and T satisfy the condition of Theorem 1.4.4. It follows that the ring of invariants $\mathbb{C}[z_1,z_2]^{\mathcal{I}}$ is generated by the Grundformen f, H and T.

There is a surjection $\varphi:\ \mathbb{C}[X,Y,Z] \twoheadrightarrow \mathbb{C}[f,H,T]$.

$$\begin{aligned} X &\longmapsto f\\ Y &\longmapsto H\\ Z &\longmapsto T \end{aligned}$$

Let $P \in \mathrm{Ker}(\varphi)$. The division of P by $1728X^5 - Y^3 - Z^2$ seen as a polynomial in the Z variable gives $P(X,Y,Z) = Q(X,Y,Z)[1728X^5 - Y^3 - Z^2] + S(X,Y,Z)$ with $S(X,Y,Z) = a_1(X,Y)Z + a_0(X,Y)$. We have $a_1(f,H)T + a_0(f,H) = 0$. The first member of the expression is of homogeneous degree 2 modulo 4 in z_1, z_2, whereas the second is of homogeneous degree 0 modulo 4 in z_1, z_2. Thus $a_1 = a_0 = 0$ and $1728X^5 - Y^3 - Z^2$ divides P. This implies $\mathrm{Ker}(\varphi) \subset (1728X^5 - Y^3 - Z^2)$ and the other inclusion follows immediately, proving $\mathrm{Ker}(\varphi) = (1728X^5 - Y^3 - Z^2)$.

One finally obtains $\mathbb{C}[z_1,z_2]^{\mathcal{I}} \cong \mathbb{C}[X,Y,Z]/(1728X^5 - Y^3 - Z^2)$. So $\mathbb{C}^2/\mathcal{I}$ injects into $\mathbb{C}^3$ as the solutions of the equation $1728X^5 - Y^3 - Z^2 = 0$. By a linear transformation, the equation becomes $X^5 + Y^3 + Z^2 = 0$.



Representation of $\mathbb{R}^3 \bigcap (\mathbb{C}^2/\mathcal{I})$.

□

According to the above theorem, $\mathbb{C}^2/\Gamma$ can be seen as a hypersurface in $\mathbb{C}^3$. Γ acts freely and properly on $\mathbb{C}^2 \setminus \{0\}$ and as $\mathbb{C}^2 \setminus \{0\}$ is smooth, so is $(\mathbb{C}^2/\Gamma) \setminus \{0\}$. Hence $\mathbb{C}^2/\Gamma$ has at most one singularity at the origin. Using the polynomial equation $R = 0$, one can verify that $\mathbb{C}^2/\Gamma$ has a unique singularity at the origin in all cases. The quotient $\mathbb{C}^2/\Gamma$ is called a **simple singularity** or a **Kleinian singularity**.

Remark 1.4.5. In [Durf79], A.H. Durfee gave many different names for these types of singularities like rational double points, quotient singularities or absolutely isolated double points among others. In this thesis, the chosen denomination will be simple or Kleinian singularities.

Singularities are usually defined as germs (see below) of analytic spaces. However we will see that with simple singularities one can work with the algebraic settings given by the polynomials equations of the previous theorem.

Definition 1.4.6. *Let $\mathcal{A}$ be the set of pairs (X, x) consisting of an analytic space X and a point $x \in X$. On the set $\mathcal{A}$ we define the relation $\sim$ as follows:*

$(X, x) \sim (Y, y) \quad \Leftrightarrow$ *there exists a neighborhood $U \subset X$ of x (for the complex topology), a neighborhood $V \subset Y$ of y, and an isomorphism $f : U \cong V$ such that $f(x) = y$.*

The relation $\sim$ is an equivalence relation on $\mathcal{A}$. Let $\mathcal{G}$ denote the quotient $\mathcal{A}/\sim$. An element of $\mathcal{G}$ is called a ***germ of the analytic space*** *and we denote it by a representative (X, x).*

The next theorem is due to M. Artin and allows us to switch from the analytic setting to the algebraic one (cf. [Ishii14] Theorem 4.2.4).

Theorem 1.4.7. (Artin's Algebraization Theorem). *For a germ (X, x) of analytic space, if x is an isolated singularity, there exists an algebraic variety $\mathcal{X}$ over $\mathbb{C}$ and a point p of $\mathcal{X}$ such that*

$$(X, x) = (\mathcal{X}, p).$$

This theorem allows us to see isolated singularities as algebraic varieties. Furthermore, when it comes to simple singularities, the following proposition (proved in [Lamo86] Chapter 4) tells us that one can work globally using the polynomial equations R of Theorem 1.4.1:

Proposition 1.4.8. *Let ${}_3\mathcal{O}$ denote the germs of holomorphic functions $(\mathbb{C}^3, 0) \to \mathbb{C}$ and let $f \in {}_3\mathcal{O}$ be a germ without multiple factors. If $Z(f) = \{x \in \mathbb{C}^3 \mid f(x) = 0\}$ is isomorphic to the simple singularity $\mathbb{C}^2/\Gamma$ defined from a finite subgroup Γ of SU_2 , then there is a biholomorphic germ $\varphi : (\mathbb{C}^3, 0) \cong (\mathbb{C}^3, 0)$ such that $R = f \circ \varphi$ is the polynomial associated to Γ by Theorem 1.4.1.*

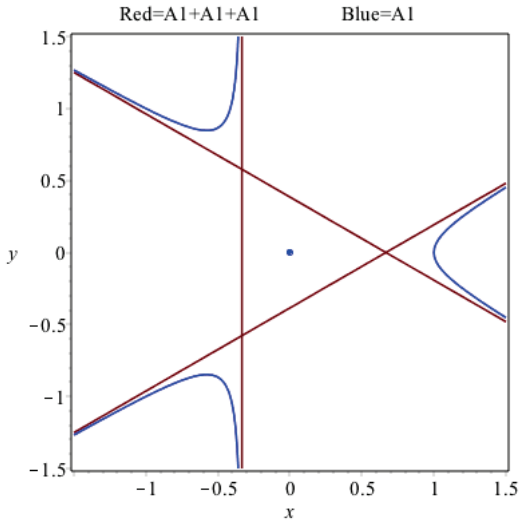
The proof of this proposition is based on the fact that every polynomial of Theorem 1.4.1 is a weighted homogeneous polynomial. The reader may want to consult [Lamo86] for more details on weighted homogeneous functions.

Let $p \in \mathbb{Z}_{>0}$ and choose $T_1, \ldots, T_p$ among A_r $(r \geq 1)$, D_r $(r \geq 4)$, E_6, E_7 and E_8. A **singular configuration** of type $T_1 + \ldots + T_p$ is a complex algebraic surface with p isolated singularities $s_1, \ldots, s_p$, such that, for each $1 \leq i \leq p$, locally around s_i the surface is a simple singularity of type T_i.

Example 1.4.9. *Let f :*

$$\begin{array}{ccc} \mathbb{C}^4 & \to & \mathbb{C}^2 \\ (x, y, z, t) & \mapsto & (z^2 - x^3 + 3xy^2 + t(x^2 + y^2), t) \end{array}$$

. The fiber $f^{-1}(0, 0)$ is a simple singularity of type D_4. Let us look at the other singular fibers. One computes that, except from the origin, the singular fibers are of two types: $f^{-1}(0, t)$ and $f^{-1}(\frac{4}{27}t^3, t)$, $t \neq 0$. The former has a single singularity of type A_1 at the origin, whereas the latter has three singular points, namely $(\frac{2}{3}t, 0, 0)$, $(-\frac{t}{3}, \frac{t}{\sqrt{3}}, 0)$ and $(-\frac{t}{3}, -\frac{t}{\sqrt{3}}, 0)$, each of type A_1. Therefore $f^{-1}(0, t)$ is a simple singularity of type A_1 and $f^{-1}(\frac{4}{27}t^3, t)$ is a singular configuration of type $A_1 + A_1 + A_1$. Below is an illustration in the plane $\{z = 0\}$ of both fibers for $t = 1$:



1.4.1.2. Resolutions of simple singularities

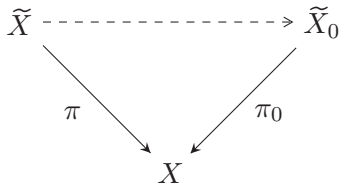
The study of singularities requires an essential tool:

Definition 1.4.10. *Let X be a complex algebraic variety and X^{reg} its set of regular points. A proper morphism $\pi : \widetilde{X} \to X$ is called a* ***resolution*** *of X if:*

- *$\widetilde{X}$ is a smooth complex algebraic variety.*
- *$\pi^{-1}(X^{\mathrm{reg}})$ is dense in $\widetilde{X}$ and the restriction $\pi|_{\pi^{-1}(X^{\mathrm{reg}})} : \pi^{-1}(X^{\mathrm{reg}}) \to X^{\mathrm{reg}}$ is an isomorphism.*

The subvariety $E = \widetilde{X} \setminus \pi^{-1}(X^{\mathrm{reg}})$ is called the **exceptional locus** of π.

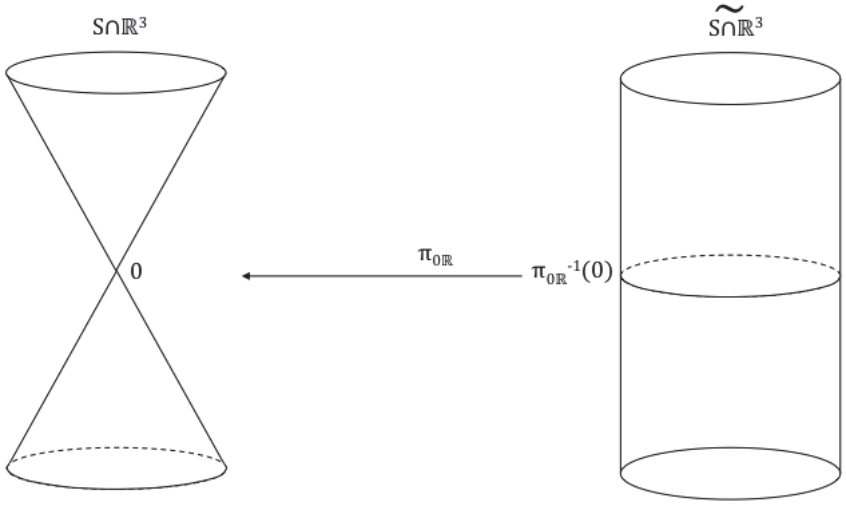
A resolution $\pi_0 : \widetilde{X}_0 \to X$ is said **minimal** if any other resolution $\pi : \widetilde{X} \to X$ can be factorized through π_0:



Minimal resolutions exist for singularities of curves (proved by I. Newton) and surfaces (proved algebraically by O. Zariski in [Zari39]) and are unique up to isomorphism.

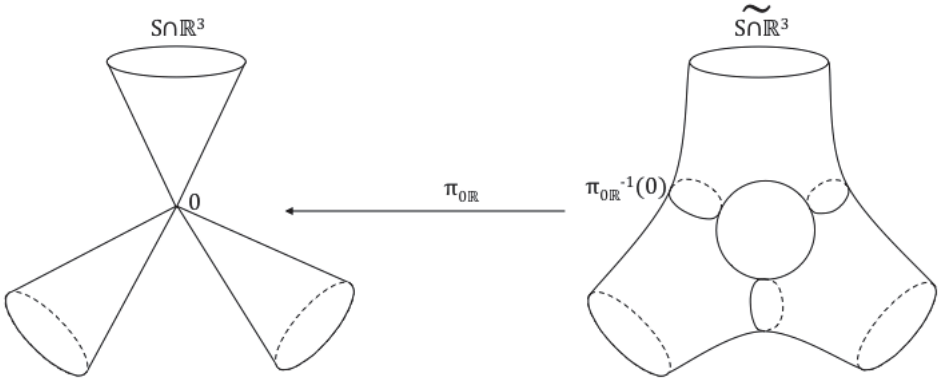
Examples 1.4.11. *In case of the groups $\mathcal{C}_2$ and $\mathcal{D}_2$, the real part $\widetilde{X}_0 \cap \mathbb{R}^3$ of the minimal resolution $\widetilde{X}_0$ gives a quite faithful idea of the complex situation.*

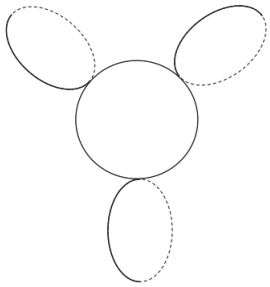
• *We know that $\mathcal{C}_2$ is associated with the surface $S = \{X^2 + YZ = 0\}$. The restriction $\pi_{0\mathbb{R}} := \pi_0|_{\mathbb{R}^3}$ of the minimal resolution π_0 of the singularity $\mathbb{C}^2/\mathcal{C}_2$ can be obtained by blowing up the origin. We obtain*



The circle $\pi_{0\mathbb{R}}^{-1}(0)$ is contracted on the origin.

• *$\mathcal{D}_2$ is associated with the surface $S = \{X(Y^2 - X^2) + Z^2 = 0\}$. Again, $\pi_{0\mathbb{R}}$ is obtained by blowing up the origin several times:*

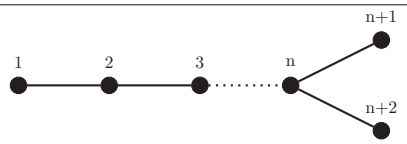
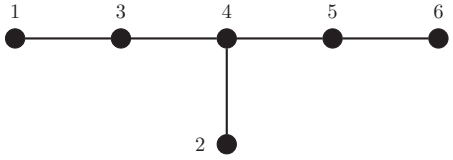
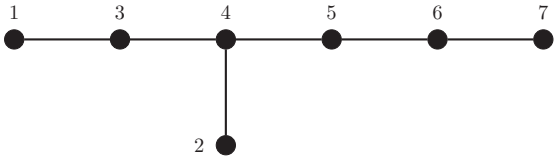
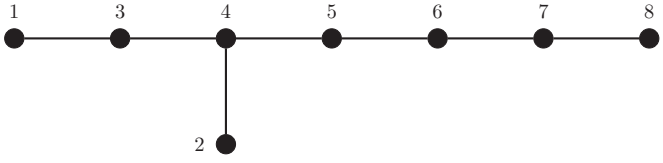


This time, $\pi_{0\mathbb{R}}^{-1}(0)$ *is composed of four circles:*  .

In the case when $S = \mathbb{C}^2/\Gamma$ is a simple singularity, P. Du Val proved in [DuVa34] that if s is the singular point and $\pi_0 : \widetilde{S} \to S$ is the minimal resolution of S, then the preimage of s is a union of projective lines

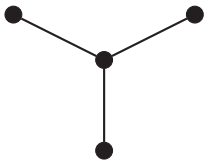
$$\pi_0^{-1}(s) = C_1 \bigcup \ldots \bigcup C_r, \quad C_i \cong \mathbb{P}^1(\mathbb{C}).$$

Each projective line has self-intersection -2 and intersects with others transversally in at most one point. We associate to every line C_i a vertex $\bullet$ in a graph, and two vertices are linked by an edge if the corresponding lines intersect. We obtain the following diagrams $\Delta(\Gamma)$:

Γ	$\Delta(\Gamma)$	Name of $\Delta(\Gamma)$
$\mathcal{C}_n$		A_{n-1}
$\mathcal{D}_n$		D_{n+2}
$\mathcal{T}$		E_6
$\mathcal{O}$		E_7
$\mathcal{I}$		E_8

Examples 1.4.12. *Let us rework the preceding examples with the illustrations of the real resolutions.*

• *We have seen that with* $\mathcal{C}_2$, $\pi_{0\mathbb{R}}^{-1}(0)$ *is a circle, which is* $\mathbb{P}^1(\mathbb{R})$. *The corresponding diagram is then* A_1.

• *In the case of* $\mathcal{D}_2$, *the diagram corresponding to* $\pi_{0\mathbb{R}}^{-1}(0)$ *is*  , *which is called* D_4.

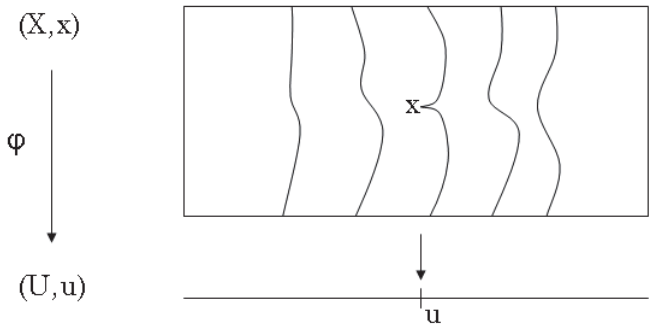
The diagrams $\Delta(\Gamma)$ are the Dynkin diagrams of the homogeneous root systems of the simple Lie algebras. We will see another connection between the simple singularities and the homogeneous Dynkin diagrams in the next section.

Remark 1.4.13. D. Kirby in [Kirby57] has characterized the Kleinian singularities as being the only double points whose minimal resolution can be obtained by successive blowups. Kleinian singularities have many characterisations which have been summarized by A.H. Durfee in [Durf79].

1.4.1.3. Semiuniversal deformation

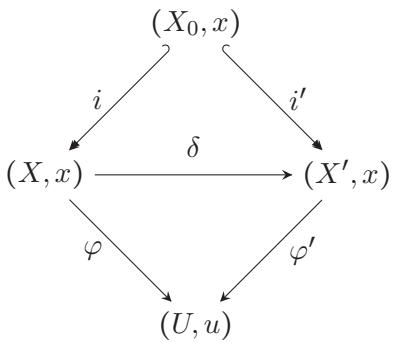
We start this section by defining a deformation of a singularity based on [Ishii14].

Definition 1.4.14. *Let* X_0 *be a complex integral algebraic variety with an isolated singularity* x. *A* ***deformation*** *of* (X_0, x) *is a flat morphism of germs of algebraic varieties* $\varphi : (X, x) \to (U, u)$ *with an isomorphism* $i : (X_0, x) \xrightarrow{\cong} (\varphi^{-1}(u), x)$.

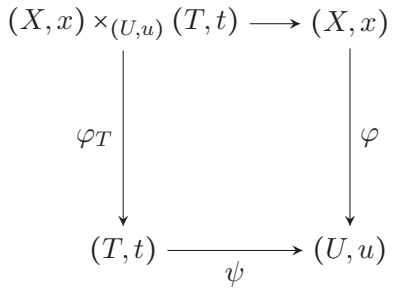


The space (U, u) is called the **base** of the deformation (φ, i).

An **isomorphism** $\delta : (\varphi, i) \to (\varphi', i')$ of two deformations $\varphi : (X, x) \to (U, u)$ and $\varphi' : (X', x) \to (U, u)$ of (X_0, x) is an isomorphism $\delta : (X, x) \to (X', x)$ such that the following diagram commutes:



If $\varphi : (X, x) \to (U, u)$ is a deformation of (X_0, x) and $\psi : (T, t) \to (U, u)$ is any morphism, then the pullback $\varphi_T : (X, x) \times_{(U,u)} (T, t) \to (T, t)$ of φ by ψ is a deformation of (X_0, x) over (T, t) called the **induced deformation** by ψ from φ.



A deformation $\varphi : (X, x) \to (U, u)$ of (X_0, x) is called **semiuniversal** if any other deformation $\varphi' : (X', x) \to (T, t)$ of (X_0, x) is isomorphic to a deformation induced from φ by a base change $\psi : (T, t) \to (U, u)$ whose differential at $t \in T$ is uniquely determined.

More details on deformations of singularities can be found in M. Artin's lecture notes [Artin76].

We are now able to give one of the important theorems of this section.

Theorem 1.4.15. (Brieskorn). *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type A_r, D_r or E_r and x a subregular nilpotent element of $\mathfrak{g}$. Let $S \subset \mathfrak{g}$ be a transversal slice (notion*

defined below) at x to the G-orbit of x in $\mathfrak{g}$. Then the restriction $\chi|_S : (S, x) \to (\mathfrak{h}/W, \overline{0})$ of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ is a semiuniversal deformation of the simple singularity $(\chi|_S^{-1}(\overline{0}), x)$, which is of the same type as $\mathfrak{g}$.

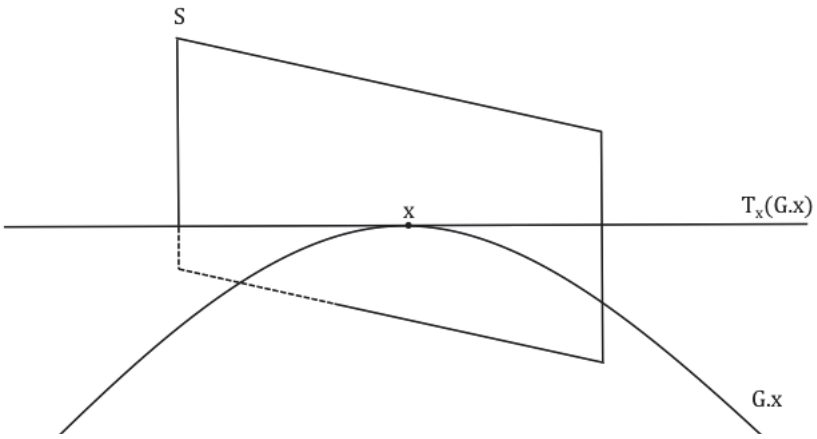
Proof. The theorem was first proved in [Bries71] in the setting of deformations of germs of complex spaces, and then in [Slo80] Section 8.7 in an algebraic setting. □

According to the theorem, $(S \bigcap \mathcal{N}(\mathfrak{g}), x) = (S \bigcap \chi^{-1}(\chi(0)), x)$ is a simple singularity of the same type as $\mathfrak{g}$.

A **transversal slice** S at x to the orbit of x is a locally closed subvariety $S \subset \mathfrak{g}$, $x \in S$, such that the morphism $G \times S \to \mathfrak{g}$ is smooth and dim S = codim $G.x$.

$$(g, s) \mapsto (\text{Ad } g)s$$

In our context, a transversal slice is easily obtained by choosing an affine subspace in $\mathfrak{g}$ complementary to the tangent space of $G.x$ at x and passing through x.



Brieskorn's theorem gives us the possibility to study singularities linked to subregular nilpotent elements of $\mathfrak{g}$. In order to use this theorem, a more precise description of the transversal slice S is necessary. One will be constructed explicitly, but before, here are some definitions and preliminary results (cf. [Huse94] Chapter 4, Section 5).

Definition 1.4.16. *Let G be an algebraic group, $H \subset G$ an algebraic subgroup of G and X a variety on which H acts. The* ***associated bundle*** *$G \times^H X$ over G/H is the quotient of $G \times X$ by the action of H defined by*

$$h.(g, x) = (g.h^{-1}, h.x), \ \forall h \in H, g \in G, x \in X.$$

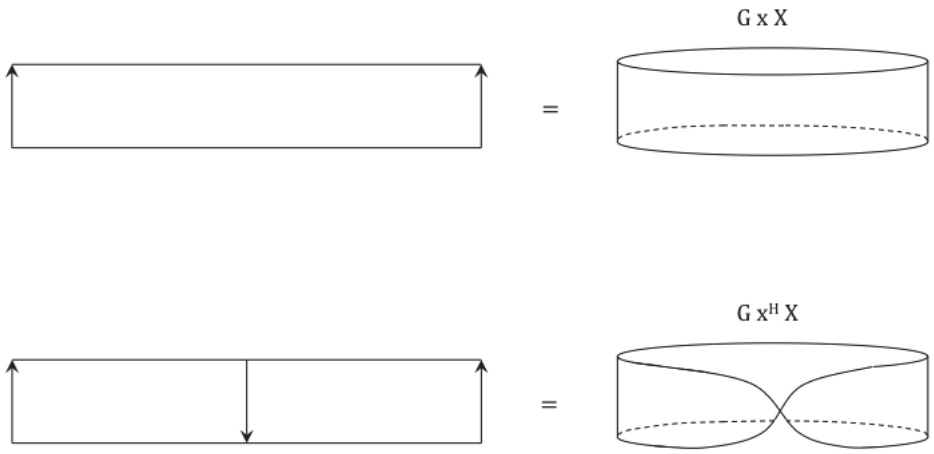
We write $g * x$ for the class of (g, x) in $G \times^H X$. The morphism

$$\begin{array}{ccc} G \times^H X & \to & G/H \\ g * x & \mapsto & gH \end{array}$$

gives a fiber bundle structure of fiber X.

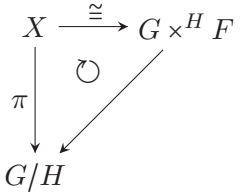
When G and H are linear and X is an affine algebraic variety, the associated bundle $G \times^H X$ exists as a variety.

Example 1.4.17. *Let $G = S^1$ be the unit circle, $H = \{\pm 1\}$ and $X = \mathbb{R}$. The action of H on X is the multiplication. Then the associated bundle $G \times^H X$ is the Möbius strip. The effect of the quotient is illustrated on the following representation:*



The following lemma will be of use later on.

Lemma 1.4.18. *Let X be a G-variety and $\pi : X \to G/H$ a G-equivariant morphism. Then π is G-isomorphic to the bundle $G \times^H F \to G/H$, where F is the fiber $\pi^{-1}(e.H)$. We have the following diagram:*



Corollary 1.3.5 tells us that every fiber of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ contains a unique closed orbit. This orbit is composed of the semisimple elements of the fiber. In fact, the closed orbits are the orbits of semisimple elements, as stated in the next theorem (cf. [Warn72] Proposition 1.3.5.5).

Theorem 1.4.19. (Borel, Harish-Chandra). *Let $\mathfrak{g}$ be a reductive Lie algebra, i.e. the representation* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is semisimple, and G its associated adjoint Lie group. Then $x \in \mathfrak{g}$ is semisimple if and only if its orbit* $\mathrm{Ad}(G)(x)$ *is closed.*

Moreover one can prove (cf. [Stein74] 3.4, Corollary 1) that if $x = x_s + x_n$ is the Jordan decomposition of an element $x \in \mathfrak{g}$, then $\chi(x) = \chi(x_s)$. Let $h \in \mathfrak{h}$ with $\overline{h}$ the class of h in $\mathfrak{h}/W$. One can identify the closed orbit contained in $\chi^{-1}(\overline{h})$ with $G/Z_G(h)$, where $Z_G(h)$ is the stabilizer of h in G. If $x \in \mathfrak{g}$ is such that $\chi(x) = \overline{h}$, then x_s is conjugated to h. So by sending x on its semisimple part, one obtains a G-morphism $\sigma : \chi^{-1}(\overline{h}) \to G/Z_G(h)$. σ is indeed a morphism because for any $x \in \mathfrak{g}$, there exists a polynomial P_x without contant term and depending only on the characteristic polynomial of x such that $P_x(x) = x_s$, and all the elements of $\chi^{-1}(\overline{h})$ have the same characteristic polynomial for any faithful linear representation of $\mathfrak{g}$ (cf. [Slo80] 3.10, proof of the lemma). Because of Lemma 1.4.18, one sees that $\chi^{-1}(\overline{h})$ is isomorphic to the associated bundle $G \times^{Z_G(h)} F$ where F is the fiber of σ over $e.Z_G(h)$. Furthermore $F = \sigma^{-1}(e.Z_G(h)) = \{x = x_s + x_n \mid x_s \in Z_G(h)$ and x_s conjugated to $h\}$. As $G.h \cong G/Z_G(h)$, an element contained in the orbit of h as well as in $Z_G(h)$ can only be h. Hence $F = h + \mathcal{N}(\mathfrak{z}_\mathfrak{g}(h))$, with $\mathcal{N}(\mathfrak{z}_\mathfrak{g}(h))$ being the set of nilpotent elements of $\mathfrak{z}_\mathfrak{g}(h) = \mathrm{Lie}\ Z_G(h)$. One can identify F and $\mathcal{N}(\mathfrak{z}_\mathfrak{g}(h))$, therefore $\chi^{-1}(\overline{h}) \cong G \times^{Z_G(h)} \mathcal{N}(\mathfrak{z}_\mathfrak{g}(h))$.

The following theorem is a key result about nilpotent elements in a Lie algebra.

Theorem 1.4.20. (Jacobson-Morozov). *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $f \in \mathfrak{g}$ be a nilpotent element. Then there exists $e, h \in \mathfrak{g}$ such that (e, f, h) is a $\mathfrak{sl}_2$-triple. This means that the Lie algebra generated by e, f and h is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ equipped with its standard basis.*

Proof. The proof of this theorem is quite long. Only the main steps will be given below. A detailed version can be found in [Kos59].

1. Let us take E a $\mathbb{C}$-vector space of finite dimension and $A, B \in \mathrm{End}_\mathbb{C}(E)$. One shows that if A is nilpotent and $[A, [A, B]] = 0$ then AB is also nilpotent.
2. Let $e \in \mathfrak{g}$ be nilpotent. Then $e \in \mathrm{Im}(\mathrm{ad}\ e)^2$. One deduces that there exists $f \in \mathfrak{g}$ such that $[[f, e], e] = e$. We set $x = [f, e]$.
3. Let m be the smallest positive integer such that $(\mathrm{ad}\ e)^{m+1} = 0$. One proves that $\prod_{p=0}^{m}(\mathrm{ad}\ x - \frac{1}{2}p\mathrm{id})$ is zero on $\mathrm{Ker}(\mathrm{ad}\ e)$. It implies that $\mathrm{ad}\ x + \mathrm{id}$ is invertible on $\mathrm{Ker}(\mathrm{ad}\ e)$.
4. If $[x, f] = -f$ then the subalgebra generated by e, x, f is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. If $[x, f] \neq -f$, one can show that there exists a unique $g \in \mathrm{Ker}(\mathrm{ad}\ e)$ such that $[x, f] + f = [x, g] + g$. It follows that the subalgebra generated by $e, x, f - g$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Thus for all $f \in \mathfrak{g}$ nilpotent, there exists a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ containing f such that $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C})$. □

It is possible to prove that Brieskorn's theorem does not depend on the choice of the transversal slice. It is thus possible to choose one that is convenient. The following construction is due to P. Slodowy.
Let e be a nilpotent element in a semisimple Lie algebra $\mathfrak{g}$. Because of the discussion right before Theorem 1.3.15, it is known that $T_e(G.e) = e + [\mathfrak{g}, e]$. A transversal slice can then be written

$$S = e + \mathfrak{Z},$$

where $\mathfrak{Z}$ is a complementary of $[\mathfrak{g}, e]$ in $\mathfrak{g}$.
According to Jacobson-Morozov theorem (Theorem 1.4.20), there exists f nilpotent and h semisimple, two elements of $\mathfrak{g}$ such that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$ and $< e, f, h > \cong \mathfrak{sl}_2(\mathbb{C})$. The inclusion gives us a morphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$. By composing this morphism with the adjoint representation of $\mathfrak{g}$, one can see $\mathfrak{g}$ as an $\mathfrak{sl}_2(\mathbb{C})$-module. Then Theorem 1.1.13 implies that there is a decomposition

$$\mathfrak{g} = \bigoplus_i E_i$$

of $\mathfrak{g}$ in a direct sum of irreducible submodules E_i, and

$$[\mathfrak{g}, e] = (\text{ad } e)(\mathfrak{g}) = \bigoplus_i (\text{ad } e)(E_i).$$

In each E_i, isomorphic to an irreducible representation V_{n_i} of $\mathfrak{sl}_2(\mathbb{C})$, one chooses as complementary of $(\text{ad } e)(E_i)$ the space of smallest weight of weight $-(n_i - 1)$ for the action of ad h, which is $\mathfrak{z}_{E_i}(f) = \{x \in E_i \mid [f, x] = 0\}$. It follows that $\mathfrak{z}_{\mathfrak{g}}(f) = \oplus_i \mathfrak{z}_{E_i}(f)$ is the complementary of $[\mathfrak{g}, e]$ in $\mathfrak{g}$ and thus

$$S = e + \mathfrak{z}_{\mathfrak{g}}(f)$$

is a transversal slice to the orbit of e at e. We will denote it by S_e and call it the **Slodowy slice** at e (cf. [Slo80(2)]).

Now that we know a transversal slice, explicit computations can be made.

Example 1.4.21. *Let $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ and S_e the Slodowy slice for $e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.*

One can check that $e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $h = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *form an* $\mathfrak{sl}_2$*-triple. We then compute* $\mathfrak{z}_{\mathfrak{sl}_4(\mathbb{C})}(f) = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ z & y & x & w \\ t & 0 & 0 & -3x \end{pmatrix} \middle| \; x, y, z, t, w \in \mathbb{C} \right\}$. *Finally one obtains* $S_e = \left\{ \begin{pmatrix} x & 1 & 0 & 0 \\ y & x & 1 & 0 \\ z & y & x & w \\ t & 0 & 0 & -3x \end{pmatrix} \middle| \; x, y, z, t, w \in \mathbb{C} \right\}$.

Let $A \in S_e$. *Then* $\det(\lambda I - A) = \lambda^4 - p(A)\lambda^2 + q(A)\lambda + r(A)$ *with* $p(A) = 6x^2 + 2y$, $q(A) = 8x^3 - 4xy - z$ *and* $r(A) = 6x^2y - 3xz - tw - 3x^4$. *The adjoint quotient is then*

$$\begin{array}{rccc} \chi|_{S_e} : & S_e & \longrightarrow & \mathfrak{h}/W \\ & A & \longmapsto & (p(A), q(A), r(A)) \end{array} .$$

(i) *Let us compute* $(\chi|_{S_e})^{-1}(a, b, c)$ *for* $a, b, c \in \mathbb{C}$. *It is equivalent to solving the system*

$$\begin{cases} 6x^2 + 2y & = & a, \\ 8x^3 - 4xy - z & = & b, \\ 6x^2y - 3xz - tw - 3x^4 & = & c. \end{cases}$$

By replacing the expressions of y *and* z *depending on* x *in the last equation one gets* $81x^4 - 9ax^2 - 3bx + tw + c = 0$.
So $(\chi|_{S_e})^{-1}(a, b, c)$ *is defined by the equation* $F(x, t, w) = 81x^4 - 9ax^2 - 3bx + tw + c = 0$.

(ii) *We now compute the discriminant* $D \subset \mathfrak{h}/W$ *of* $\chi|_{S_e}$. $(\chi|_{S_e})^{-1}(a, b, c)$ *has a singularity if and only if there exists* x, t, w *such that* $F(x, t, w) = \partial_x F(x, t, w) = \partial_t F(x, t, w) = \partial_w F(x, t, w) = 0$, *which is equivalent to*

$$\begin{cases} F(x,t,w) &= 81x^4 - 9ax^2 - 3bx + tw + c &= 0, \\ \partial_x F(x,t,w) &= 324x^3 - 18ax - 3b &= 0, \\ \partial_t F(x,t,w) &= w &= 0, \\ \partial_w F(x,t,w) &= t &= 0. \end{cases}$$

Set $P(x) = 81x^4 - 9ax^2 - 3bx + c$. *Then* $(\chi|_{S_e})^{-1}(a,b,c)$ *has a singularity if and only if* $P(x) = P'(x) = 0$. *So* P *has a multiple root. The resultant of* P *and* P' *is* $R = 43046721(4a^3b^2 - 27b^4 + 16a^4c - 144ab^2c - 128a^2c^2 + 256c^3)$. *Hence* $(\chi|_{S_e})^{-1}(a,b,c)$ *has a singularity if and only if* $4a^3b^2 - 27b^4 + 16a^4c - 144ab^2c - 128a^2c^2 + 256c^3 = 0$. *Therefore the discriminant is*

$$D = \{(a,b,c) \in \mathbb{C}^3 \mid 4a^3b^2 - 27b^4 + 16a^4c - 144ab^2c - 128a^2c^2 + 256c^3 = 0\}.$$

(iii) *Let* $\pi : \mathfrak{h} \to \mathfrak{h}/W$ *be the natural projection. Let us find* $\pi^{-1}(D)$. *It is known that* $\mathfrak{h} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -x-y-z \end{pmatrix} \middle| \; (x,y,z) \in \mathbb{C}^3 \right\}$. *If* $h \in \mathfrak{h}$ *then* $\pi(h) = (x^2+y^2+z^2+xy+xz+yz, (x+y)(x+z)(y+z), -xyz(x+y+z))$. *Wanting* $(x,y,z) \in \pi^{-1}(D)$ *is equivalent to*

$$(x-y)^2(x-z)^2(y-z)^2(2x+y+z)^2(x+2y+z)^2(x+y+2z)^2 = 0.$$

Thus $\pi^{-1}(D)$ *is the union of six surfaces:* $\{y = x\} \cup \{z = x\} \cup \{z = y\} \cup \{2x + y + z = 0\} \cup \{x + 2y + z = 0\} \cup \{x + y + 2z = 0\}$.

It is known that $\mathfrak{h}$ *is generated by* $h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $h_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. *Moreover* α_1, α_2 *and* α_3 *form a base of the root system of* $\mathfrak{sl}_4(\mathbb{C})$.

The surface $\{x = y\}$ *corresponds to* $\left\{\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -2x-z \end{pmatrix}\right\}$ *and is generated by* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = h_1 + 2h_2 + 2h_3$ *and* $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = h_3$. *The Cartan matrix of* $\mathfrak{sl}_4(\mathbb{C})$ *is* $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ *so* $\alpha_1(h_1 + 2h_2 + 2h_3) = \alpha_1(h_3) = 0$. *As* α_1 *is a linear form on* $\mathfrak{h}$ *which is of dimension 3,* $\mathrm{Ker}(\alpha_1)$ *is of dimension 2. Hence* $\mathrm{Ker}(\alpha_1) = \mathrm{Vect}_{\mathbb{C}}(h_1 + 2h_2 + 2h_3, h_3) = \{x = y\}$.

In a similar fashion one can show that

$$\begin{aligned} \{z = x\} &= \mathrm{Ker}(\alpha_1 + \alpha_2), \\ \{z = y\} &= \mathrm{Ker}(\alpha_2), \\ \{2x + y + z = 0\} &= \mathrm{Ker}(\alpha_1 + \alpha_2 + \alpha_3), \\ \{x + 2y + z = 0\} &= \mathrm{Ker}(\alpha_2 + \alpha_3), \\ \{x + y + 2z = 0\} &= \mathrm{Ker}(\alpha_3). \end{aligned}$$

However $\mathfrak{sl}_4(\mathbb{C})$ *is of dimension 15 and* $\mathfrak{h}$ *of dimension 3. There are six positive roots. As all the roots previously given are clearly positive, one obtains* $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. *Therefore* $\pi^{-1}(D) = \bigcup_{\alpha \in \Phi^+} \mathrm{Ker}(\alpha)$.

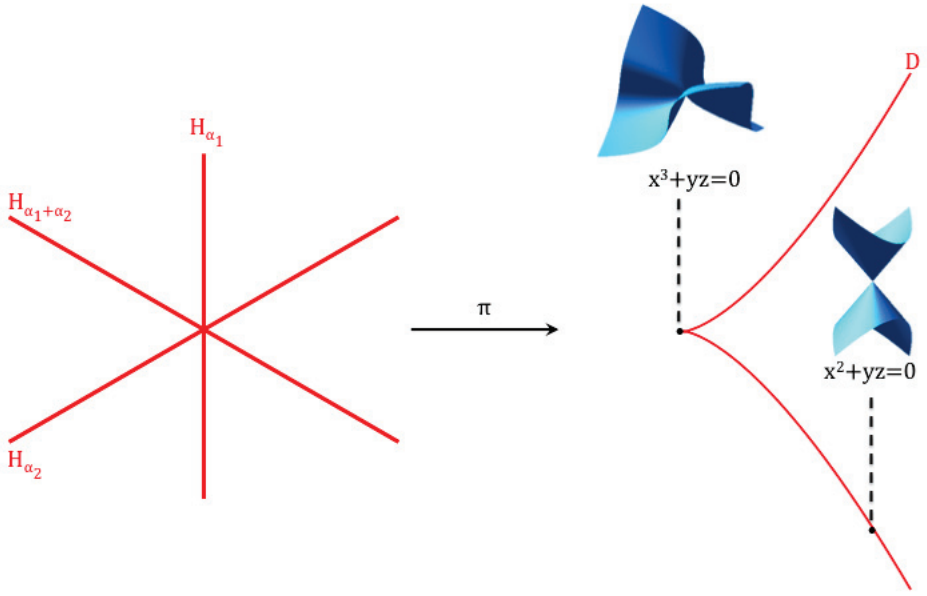
We remind the reader of an interesting fact:

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, W the Weyl group, Φ the root system and S_e a Slodowy slice. We have the following diagram

$$\begin{array}{ccc} & & S_e \\ & & \downarrow \quad \chi|_{S_e} \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W \\ \bigcup\limits_{\alpha \in \Phi^+} H_\alpha & \mapsto & D \end{array}$$

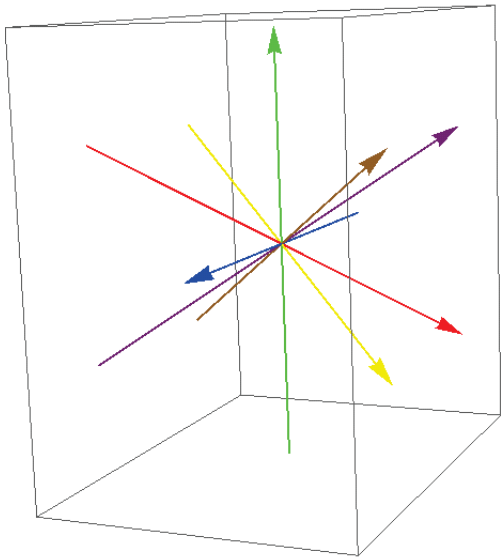
with H_α the kernel of α and D the discriminant of χ.

In the case of A_2, let $p \in \bigcup_{\alpha\in\Phi} H_\alpha$. Then $p \in H_{\alpha_1} \setminus (H_{\alpha_2} \bigcup H_{\alpha_1+\alpha_2})$, or $p \in H_{\alpha_2} \setminus (H_{\alpha_1} \bigcup H_{\alpha_1+\alpha_2})$, or $p \in H_{\alpha_1+\alpha_2} \setminus (H_{\alpha_1} \bigcup H_{\alpha_2})$, or $p \in H_{\alpha_1} \bigcap H_{\alpha_2} \bigcap H_{\alpha_1+\alpha_2}$. It can be showed that if p belongs to only one H_α ($\alpha = \alpha_1, \alpha_2$ or $\alpha_1 + \alpha_2$), then $\chi|_{S_e}^{-1}(\pi(p))$ is a singularity of type A_1. However if $p \in H_{\alpha_1} \bigcap H_{\alpha_2} \bigcap H_{\alpha_1+\alpha_2}$, $\chi|_{S_e}^{-1}(\pi(p))$ is a singularity of type A_2. An illustration of this property is shown below



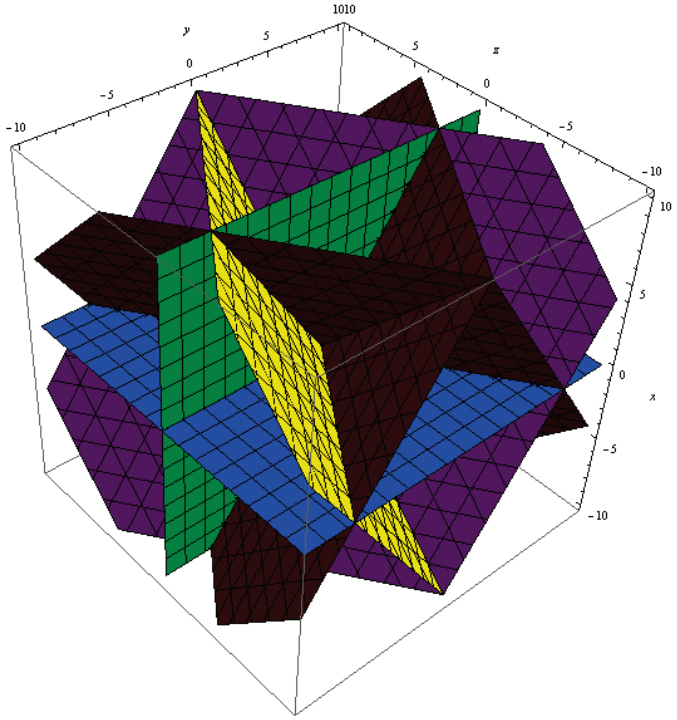
Example 1.4.22. *We go back to our previous example with* $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ *and* S_e *the computed Slodowy slice. Let* $p \in \bigcup_{\alpha\in\Phi^+} H_\alpha$ *and set* $\Phi_p = \{\alpha \in \Phi \mid p \in H_\alpha\}$. *For every* Φ_p, *we are going to compute* $\chi|_{S_e}^{-1}(\pi(p))$.

The Cartan subalgebra is $\mathfrak{h} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -x-y-z \end{pmatrix} \middle| \; (x,y,z) \in \mathbb{C}^3 \right\}$. *The simple roots are* $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$ *and* $\alpha_3 = \epsilon_3 - \epsilon_4$, *with* $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ *an orthonormal base of* $(\mathbb{C}^4)^*$. *One can check that* $\widehat{(\alpha_1, \alpha_2)} = \frac{2\pi}{3}$, $\widehat{(\alpha_2, \alpha_3)} = \frac{2\pi}{3}$ *and* $\widehat{(\alpha_1, \alpha_3)} = \frac{\pi}{2}$. *In order to picture them in* $\mathbb{R}^3$, *we take* $\alpha_1 = (1,0,0)$, $\alpha_2 = (-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2})$ *and* $\alpha_3 = (0,0,1)$. *The following picture with the associated color-coding follows:*



α_1	*Blue*
α_2	*Red*
α_3	*Green*
$\alpha_1 + \alpha_2$	*Yellow*
$\alpha_2 + \alpha_3$	*Purple*
$\alpha_1 + \alpha_2 + \alpha_3$	*Brown*

We can then represent the H_α's, keeping the same colors:



The Φ_p^+ are determined by looking at the intersections. We regroup them depending on their number of elements and arrange them in the next table:

$\lvert\Phi_p^+\rvert$	Φ_p^+	Φ_p
1	$\{\alpha_1\},\{\alpha_2\},\{\alpha_3\},\{\alpha_1+\alpha_2\},\{\alpha_2+\alpha_3\},\{\alpha_1+\alpha_2+\alpha_3\}$	A_1
2	$\{\alpha_1,\alpha_3\},\{\alpha_2,\alpha_1+\alpha_2+\alpha_3\},\{\alpha_1+\alpha_2,\alpha_2+\alpha_3\}$	A_1+A_1
3	$\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\},\{\alpha_1,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\},$ $\{\alpha_2,\alpha_3,\alpha_2+\alpha_3\},\{\alpha_3,\alpha_1+\alpha_2,\alpha_1+\alpha_2+\alpha_3\}$	A_2
6	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$	A_3

One can see that, by adding the negative roots, the Φ_p's are root systems extracted from A_3. They are written in the last column.

We are now going to compute $\chi|_{S_e}^{-1}(\pi(p))$ for $p\in\Phi_p^+$ with $|\Phi_p^+|=1,2,3$ and 6 and check that the singularities obtained are of the same types as the Φ_p.

$\bullet\underline{\Phi_p^+=\{\alpha_1\}}$*:* $p=\begin{pmatrix} x & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & z & 0\\ 0 & 0 & 0 & -2x-z\end{pmatrix}$ *with $x\neq z$ because $p\notin H_{\alpha_2}$ and $z\neq -x$ because $p\notin H_{\alpha_3}$. Subsequently $\pi(p)=(3x^2+2xz+z^2,2x(x+z)^2,-x^2z(z+2x))$. By taking $x=2$ and $z=-1$, one gets $\pi(p)=(9,4,12)$. $\chi|_{S_e}^{-1}(9,4,12)$ is given by the equation $f(x,t,w)=81x^4-81x^2-12x+tw+12=0$. The singular point is $(-\frac{2}{3},0,0)$. In order to see what happens near the singularity, in the equation f we make the substitution $x\to x-\frac{2}{3}$. After an analytic change of variables, we find a singularity of type A_1.*

$\bullet\underline{\Phi_p^+=\{\alpha_1,\alpha_3\}}$*:* $p=\begin{pmatrix} x & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & -x & 0\\ 0 & 0 & 0 & -x\end{pmatrix}$ *with $x\neq 0$. So $\pi(p)=(2x^2,0,x^4)$. By taking $x=1$, one finds $\pi(p)=(2,0,1)$. $\chi|_{S_e}^{-1}(2,0,1)$ is given by the equation $f(x,t,w)=81x^4-18x^2+tw+1=0$. There are two singular points: $(\frac{1}{3},0,0)$ and $(-\frac{1}{3},0,0)$. In order to see what happens near the singularity, in the equation f we make the substitution $x\to x+\frac{1}{3}$. After an analytic change of variables, we find a singularity of type A_1. The same result is obtained for the substitution $x\to x-\frac{1}{3}$. Therefore the fiber is a singular configuration of type A_1+A_1.*

$\bullet\underline{\Phi_p^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}}$: $p = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -x-y-z \end{pmatrix}$ *with* $x = y$, $y = z$ *and* $x = z$. *So* $p = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix}$ *with* $x \neq 0$. *Then* $\pi(p) = (6x^2, 8x^3, -3x^4)$. *By taking* $x = 1$, *one gets* $\pi(p) = (6, 8, -3)$. $\chi|_{S_e}^{-1}(6, 8, -3)$ *is given by the equation* $f(x,t,w) = 81x^4 - 54x^2 - 24x + tw - 3 = 0$. *The singular point is* $(-\frac{1}{3}, 0, 0)$. *In order to see what happens near the singularity, in the equation* f *we make the substitution* $x \to x - \frac{1}{3}$. *After an analytic change of variables, we find an* A_2*-type of singularity.*

$\bullet\underline{|\Phi_p^+| = 6}$: $p = (0,0,0)$. *Then* $\pi(p) = (0,0,0)$ *and* $\chi|_{S_e}^{-1}(0,0,0)$ *is given by the equation* $f(x,t,w) = 81x^4 + tw = 0$, *which is a singularity of type* A_3.

The preceding example is a nice illustration of the following proposition. The proof can be found in [Slo80].

Proposition 1.4.23. *Let* $\chi : X \to U$ *be a deformation of a simple singularity* X_0 *with associated Dynkin diagram* Δ. *Let* $X_\epsilon \neq X_0$ *be a non-special fibre of* χ. *Then there is a proper subdiagram* $\Delta' \subset \Delta$ *and a type-preserving bijection of the components of* Δ' *onto the singular points of* X_ϵ. *This means that each connected component of* Δ' *is sent to a singular point of* X_ϵ, *which is a simple singularity of the corresponding type. Furthermore, if* χ *is semiuniversal, then all subdiagrams of* Δ *are realized as singular configurations in non-special fibers.*

Since the beginning of our study, all the singularities have been isolated. The fact that a singularity is isolated or not can be algebraically characterized (cf. [Loo84]).

Theorem 1.4.24. *Let* A *be the ring of convergent series* $\mathbb{C}\{x_1, \ldots, x_n\}$. *Let* $f \in A$ *such that* $f(0, \ldots, 0) = 0$ *and let* $J(f)$ *be the ideal of* A *generated by* f *and the* $\dfrac{\partial f}{\partial x_i}$, $1 \leq i \leq n$. *Assuming that* $(f^{-1}(0), 0)$ *is a singular point, it is isolated if and only if* $\dim(A/J(f)) < \infty$.

Proof. One can show that the ring A is Noetherian and local with $\mathfrak{m} = (x_1, ..., x_n)$ as its unique maximal ideal. The proof of the theorem is based on the following proposition:

Proposition 1.4.25. *Let $\mathfrak{a} \subset A$ be an ideal of A. The following assertions are equivalent:*

(i) $\dim A/\mathfrak{a} < \infty$.

(ii) $\exists k \in \mathbb{Z}_{>0}$ *such that* $\mathfrak{m}^k \subset \mathfrak{a}$.

(iii) $Z(\mathfrak{a}) := \{(a_1, ..., a_n) \in \mathbb{C}^n \mid g(a_1, ..., a_n) = 0 \ \forall g \in \mathfrak{a}\} = \{0\}$.

Proof. $(ii) \Rightarrow (i)$ Assume there exists $k \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^k \subset \mathfrak{a}$. There is a surjection

$$\begin{array}{ccc} A/\mathfrak{m}^k & \twoheadrightarrow & A/\mathfrak{a} \\ x \bmod \mathfrak{m}^k & \longmapsto & x \bmod \mathfrak{a} \end{array} .$$

It follows that $\dim A/\mathfrak{a} \leq \dim A/\mathfrak{m}^k$. Furthermore the elements of $A/\mathfrak{m}^k$ are polynomials whose degree is at most $k-1$. Thus $A/\mathfrak{m}^k$ has a finite base, which implies $\dim A/\mathfrak{m}^k < \infty$ and $\dim A/\mathfrak{a} < \infty$.

$(ii) \Rightarrow (iii)$ Assume there exists $k \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^k \subset \mathfrak{a}$. If $x \in Z(\mathfrak{a})$, then x is a common zero to all elements in $\mathfrak{m}^k$. But A is an integral domain, implying that x cancels all elements in $\mathfrak{m}$. As the only common zero of the elements of $\mathfrak{m}$ is 0, it follows that $x = 0$ and $Z(\mathfrak{a}) = \{0\}$.

$(i) \Rightarrow (ii)$ Assume that $\dim A/\mathfrak{a} < \infty$. There is a sequence of vector spaces:

$$A/\mathfrak{a} \supset (\mathfrak{m} + \mathfrak{a})/\mathfrak{a} \supset (\mathfrak{m}^2 + \mathfrak{a})/\mathfrak{a} \supset ....$$

According to the hypothesis, there exists $k \in \mathbb{Z}_{>0}$ such that $(\mathfrak{m}^{k+1} + \mathfrak{a})/\mathfrak{a} = (\mathfrak{m}^k + \mathfrak{a})/\mathfrak{a}$, hence $\mathfrak{m}.(\mathfrak{m}^k + \mathfrak{a})/\mathfrak{a} = (\mathfrak{m}^k + \mathfrak{a})/\mathfrak{a}$. Then Nakayama's lemma (cf. [Eis95] Corollary 4.8) implies that $(\mathfrak{m}^k + \mathfrak{a})/\mathfrak{a} = 0$, which means $\mathfrak{m}^k \subset \mathfrak{a}$.

$(iii) \Rightarrow (ii)$ The analytic version of Hilbert's Nullstellenstaz (cf. [GinKhen90] First Part, Chapter 2, Theorem 2.3) tells us that $I(Z(\mathfrak{a})) = \{f \in A \mid f(x) = 0 \ \forall x \in Z(\mathfrak{a})\} = \sqrt{\mathfrak{a}}$. If one assume that $Z(\mathfrak{a}) = \{0\}$, then $\sqrt{\mathfrak{a}} = \{f \in A \mid f(0) = 0\} = \mathfrak{m}$.
Set $k \in \mathbb{Z}_{>0}$. $\mathfrak{m}$ is generated by $x_1, ..., x_n$ so an element in $\mathfrak{m}^k$ can be written

$$\sum \alpha_{\beta_1,...,\beta_n}(x_1, ..., x_n) x_1^{\beta_1} ... x_n^{\beta_n} \text{ with } \beta_1 + ... + \beta_n = k \text{ and } \alpha_{\beta_1,...,\beta_n} \in A.$$

For any partition of k in n elements, there exists one part higher or equal to $\frac{k}{n}$. Indeed, if the partition $(\beta_1, \ldots, \beta_n)$ is such that $\beta_i < \frac{k}{n}$ for any $1 \leq i \leq n$, then $k = \beta_1 + ... + \beta_n < n\frac{k}{n} = k$, which is impossible.
As $\mathfrak{m} = \sqrt{\mathfrak{a}}$, for any $1 \leq i \leq n$, there exists $\alpha_i \in \mathbb{Z}_{>0}$ such that $x_i^{\alpha_i} \in \mathfrak{a}$. Set $k \in \mathbb{Z}_{>0}$ such that $\frac{k}{n} \geq \max(\alpha_1, ..., \alpha_n)$. Then for any partition $(\beta_1, \ldots, \beta_n)$ of k in n elements, there exists $1 \leq i \leq n$ such that $\beta_i \geq \frac{k}{n} \geq \alpha_i$. Thus $x_i^{\beta_i} \in \mathfrak{a}$ and because $\mathfrak{a}$ is an ideal, one finds $\alpha_{\beta_1,...,\beta_n}(x_1, ..., x_n)x_1^{\beta_1}...x_n^{\beta_n} \in \mathfrak{a}$. Hence any element of $\mathfrak{m}^k$ is a sum of elements of $\mathfrak{a}$ and so is in $\mathfrak{a}$. It follows that $\mathfrak{m}^k \subset \mathfrak{a}$. □

The singularity $(f^{-1}(0), 0)$ is isolated if and only if 0 is the only solution of the system

$$\begin{cases} f &= 0, \\ \dfrac{\partial f}{\partial x_i} &= 0, \ 1 \leq i \leq n. \end{cases}$$

Hence $\{0\} = Z(J(f))$. According to Proposition 1.4.25, this is equivalent to $\dim(A/J(f)) < \infty$, which proves the theorem. □

The number $\dim(A/J(f))$ is called the **Tjurina number** of the singularity (cf. [Loo84]).

The next theorem gives us a way of computing the semiuniversal deformation of any simple singularity. It was proved by A. Kas and M. Schlessinger in [KasSchle72].

Theorem 1.4.26. (Kas-Schlessinger). *Let X_0 be a Kleinian singularity defined in $\mathbb{C}^3$ by the polynomial equation $f(X, Y, Z) = 0$. Because of the previous proposition (or by explicit computation), the vector space $V = \mathbb{C}[X, Y, Z]/(f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z})$ is finite dimensional. Let $(b_i(X, Y, Z))_{1 \leq i \leq k}$ be a base of V. Then the semiuniversal deformation of the simple singularity X_0 is given by the map*

$$\begin{array}{rccc} \varphi : & \mathbb{C}^3 \times \mathbb{C}^k & \to & \mathbb{C} \\ & (X, Y, Z, a_1, \ldots, a_k) & \mapsto & f(X, Y, Z) + \sum_{i=1}^{k} a_k b_k(X, Y, Z) \end{array}.$$

1.4.1.4. Springer's resolution

The theorem that will be proved shortly is about the nilpotent variety. Its proof requires some reminders on regular nilpotent elements. The following theorem is proved in [Stein65].

Theorem 1.4.27. *Let x be a nilpotent element of a reductive Lie algebra $\mathfrak{g}$. The following properties are equivalent:*

(i) x *is regular.*

(ii) x *is contained in a unique Borel subalgebra of* $\mathfrak{g}$*.*

(iii) If $\mathfrak{b} = \mathfrak{h} \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ *is the decomposition in root spaces of a Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ *and* $x \in \mathfrak{b}$*, then the projection of* x *on* $\mathfrak{g}_\alpha$ *is not zero for all the simple roots.*

Let B be a Borel subgroup of G, T a maximal torus contained in B, $\mathfrak{b} = \text{Lie } B$ and $\mathfrak{h} = \text{Lie } T$. It is known that $\mathfrak{b}$ decomposes as $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{n}$ the nilradical of $\mathfrak{b}$. As $\mathfrak{n}$ is stable under the Adjoint action of B, one can define the associated bundle $G \times^B \mathfrak{n}$. The next proposition can be found in [Sprin09].

Proposition 1.4.28. *With the same notations as before, the quotient* G/B *is a projective variety called the flag variety for* G*, and is in bijection with the set of Borel subalgebras of* $\mathfrak{g}$*.*

The flag variety contains interesting information and will be studied in the following example.

Example 1.4.29. *Let* $G = \text{SL}_n(\mathbb{C})$*,* B *a Borel subgroup of* G *and* $\mathfrak{g}, \mathfrak{b}$ *the respective Lie algebras. We have the following morphism*

$$G/B \xrightarrow{\varphi} \{\{0\} = V_0 \subsetneq V_1 \subsetneq \ \dots \ \subsetneq V_n = \mathbb{C}^n \mid \dim V_i = i\}$$

sending a Borel subalgebra of $\mathfrak{g}$ *to the flag it stabilizes. Indeed, let* $(e_1, \dots, e_n)$ *be the canonical base of* $\mathbb{C}^n$ *on which* G *acts, and for any* $1 \leq i \leq n$*, let* $V_i = \text{Vect}_{\mathbb{C}}(e_1, \dots, e_i)$*. Then the stabilizer of the flag* $\{V_i\}_{1 \leq i \leq n}$ *is the space of upper triangular matrices in* G *and is a Borel subalgebra. But as all Borel subalgebras are conjugated, any Borel subalgebra* $\mathfrak{b}$ *appears as the stabilizer of a unique flag we call* $\varphi(\mathfrak{b})$*. This defines the morphism* φ*. There are also the following morphisms*

$$\begin{array}{ccccc} & & g * x & \mapsto & \text{Ad}(g)(x) \\ g * x & & G \times^B \mathfrak{n} & \to & \mathfrak{g} \\ \downarrow & & \downarrow & & \\ gB & & G/B & & \end{array}$$

Set $x = \begin{pmatrix} 0 & 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 0 \\ 0 & 0 & 0 & . & . & 0 & 1 \\ 0 & 0 & 0 & . & . & 0 & 0 \end{pmatrix} \in \mathfrak{g}$ *and let us find the flags stabilized by* x.

Let $(v_1, \dots v_n)$ *be the canonical base of* $\mathbb{C}^n$. *Because* x *is nilpotent, its only eigenvalue is* 0. *Set* $\widetilde{v_1} = \alpha_1 v_1 + \alpha_2 v_2$ *an eigenvector.*

• *The aim is to find* $\widetilde{v_2}$ *such that the vector space generated by* $\widetilde{v_1}$ *and* $\widetilde{v_2}$ *is stable by* x *and of dimension 2. This will imply a relation* $x\widetilde{v_2} = \mu\widetilde{v_1} + \lambda\widetilde{v_2}$. *On* $\langle \widetilde{v_1}, \widetilde{v_2} \rangle$ *the matrix of* x *will be* $\begin{pmatrix} 0 & \mu \\ 0 & \lambda \end{pmatrix}$. *But* x *is nilpotent so* $\lambda = 0$ *and so* $x\widetilde{v_2} = \mu\widetilde{v_1}$. *By writing* $\widetilde{v_2} = \sum_{i=1}^{n} \beta_i v_i$ *and applying* x, *one obtains* $\beta_3 v_2 + \ldots + \beta_n v_{n-1} = \mu\alpha_1 v_1 + \mu\alpha_2 v_2$. *Thus* $\beta_3 = \mu\alpha_2$, $\beta_4 = \ldots = \beta_n = 0$ *and* $\mu\alpha_1 = 0$.
If $\alpha_1 \neq 0$, *set* $\widetilde{v_2} = v_2$. *This implies that* $V_1 = \mathbb{C}\widetilde{v_1} \subsetneq V_2 =< v_1, v_2 >$ *are stable by* x.
If $\alpha_1 = 0$, *set* $\widetilde{v_2} = v_3$. *Therefore* $V_1 = \mathbb{C}v_2 \subsetneq V_2 =< v_2, v_3 >$ *are stable by* x.

• *We now aim to construct a vector* $\widetilde{v_3}$ *such that the* $\mathbb{C}$*-vector space* $V_3 = \langle \widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3} \rangle$ *is stable by* x *and of dimension 3. The computation leads to:*
If $\alpha_1 = 0$, $x\widetilde{v_3} = \alpha v_2 + \beta v_3$. *By setting* $\widetilde{v_3} = \beta_1 v_1 + \beta_4 v_4$, *one finds* $V_3 =< v_2, v_3, \beta_1 v_1 + \beta_4 v_4 >$.
If $\alpha_1 \neq 0$, *one obtains* $\widetilde{v_3} = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$ *thus* $V_3 =< v_1, v_2, v_3 >$.

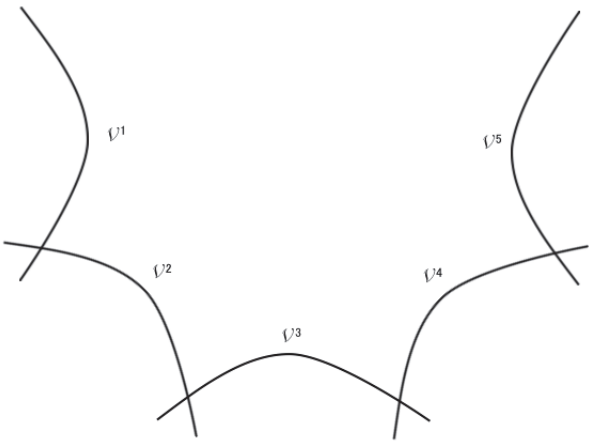
If one keeps on computing the case $\alpha_1 \neq 0$, *therefore* $V_i =< v_1, v_2, \dots, v_i >$, *for all* $1 \leq i \leq n$, *is a* $\mathbb{C}$*-vector space stable by* x.

Let us go back to the case $\alpha_1 = 0$. *One eventually finds that every flag stabilized by* x *is indexed by an* $1 \leq i \leq n-1$ *and an element* $(\lambda : \mu) \in \mathbb{P}^1(\mathbb{C})$ *such that*

$$V_j^{i,(\lambda:\mu)} = \begin{cases} < v_2, \dots, v_{j+1} > & \text{if } j < i, \\ < v_2, \dots, v_i, \lambda v_1 + \mu v_{i+1} > & \text{if } i = j, \\ < v_1, \dots v_j > & \text{if } j > i. \end{cases}$$

The previously mentioned flag will be denoted as $V^{i,(\lambda:\mu)}$ *and the family of flags indexed by* $\mathbb{P}^1(\mathbb{C})$ *as* $\mathcal{V}^i = \{V^{i,(\lambda:\mu)} \mid (\lambda : \mu) \in \mathbb{P}^1(\mathbb{C})\}$.

Let $1 \leq i < k \leq n-1$. *It can be verified that* $\mathcal{V}^i \bigcap \mathcal{V}^k = \begin{cases} \text{one point if } k = i+1 \\ \varnothing \text{ otherwise.} \end{cases}$. *We therefore have the following configuration:*



Assuming that the self-intersection is -2 *for each* $\mathcal{V}^i$, *the intersection matrix of the flag configuration is the opposite of the Cartan matrix of* A_{n-1}, *which is the Cartan matrix of* $\mathrm{SL}_n(\mathbb{C})$.

One can also look at the Borel subalgebras containing x *and notice that they are in bijection with the flags stable by* x.

The next theorem is due to T.A. Springer and can be found in [Sprin69].

Theorem 1.4.30. (Springer). *The morphism* $\psi_0 : \begin{array}{ccc} G \times^B \mathfrak{n} & \to & \mathcal{N}(\mathfrak{g}) \\ g * n & \mapsto & \mathrm{Ad}(g)n \end{array}$ *is a resolution of singularities for the nilpotent variety* $\mathcal{N}(\mathfrak{g})$.

Proof. The associated bundle $G \times^B \mathfrak{n}$ is smooth and has the same dimension as $\mathcal{N}(\mathfrak{g})$. Indeed, $\phi : G \times^B \mathfrak{n} \to G/B$ is a vector bundle and a surjective map. Hence, for any point $p \in G/B$, the dimension of the fiber $\phi^{-1}(p) \cong \mathfrak{n}$ is $\dim(G \times^B \mathfrak{n}) - \dim G/B$. So

$$\begin{aligned} \dim(G \times^B \mathfrak{n}) = \dim G/B + \dim \mathfrak{n} &= \frac{|\Phi|}{2} + \frac{|\Phi|}{2}, \\ &= |\Phi|, \\ &= \dim \mathcal{N}(\mathfrak{g}), \end{aligned}$$

with Φ the root system of the Lie algebra $\mathfrak{g}$, as previously seen. We have the morphism

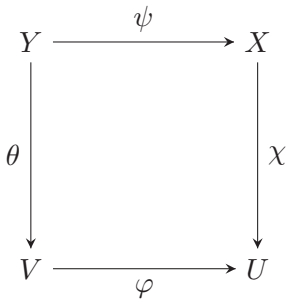
$$\begin{array}{rccc} \alpha: & G \times^B \mathfrak{n} & \to & \{(A,x) \in \mathcal{B} \times \mathcal{N}(\mathfrak{g}) \mid x \in A\} \\ & g * n & \mapsto & (\mathfrak{b}^g, \mathrm{Ad}(g)n) \end{array}$$

with $\mathfrak{b}^g = \mathrm{Ad}(g)\mathfrak{b}$ and $\mathcal{B} = G/B$ the set of Borel subalgebras of $\mathfrak{g}$. Because $\mathcal{N}(\mathfrak{g})$ is G-stable and the action of B on $\mathcal{N}(\mathfrak{g})$ is the restriction of the G-action, one has an isomorphism $\varphi : G \times^B \mathcal{N}(\mathfrak{g}) \xrightarrow{\cong} G/B \times \mathcal{N}(\mathfrak{g})$ given by $\varphi(g * x) = (gB, \mathrm{Ad}(g)(x))$ (cf. [Slo80] 3.7, Lemma 1). Hence the inclusion $G \times^B \mathfrak{n} \hookrightarrow G \times^B \mathcal{N}(\mathfrak{g})$ composed with φ is a closed embedding of $G \times^B \mathfrak{n}$ in $\mathcal{B} \times \mathcal{N}(\mathfrak{g})$ and is exactly the morphism α. We conclude that $G \times^B \mathfrak{n}$ can be embedded onto the closed subvariety $\{(A,x) \in \mathcal{B} \times \mathcal{N}(\mathfrak{g}) \mid x \in A\}$ of $\mathcal{B} \times \mathcal{N}(\mathfrak{g})$. Under this identification, the morphism ψ_0 corresponds to the projection on $\mathcal{N}(\mathfrak{g})$. One deduces that ψ_0 is proper. Moreover, ψ_0 is surjective because every nilpotent element is contained in the nilradical of some Borel subalgebra. As ψ_0 is G-equivariant and the orbit of a regular element is dense in $\mathcal{N}(\mathfrak{g})$, it is enough to prove that the preimage under ψ_0 of any regular element of $\mathcal{N}(\mathfrak{g})$ contains only one point. According to the identification, one sees $\psi_0^{-1}(x)$ with $x \in \mathcal{N}(\mathfrak{g})$ as $\mathcal{B}_x = \{A \in \mathcal{B} \mid x \in A\}$. Then because of Theorem 1.4.27, $\mathcal{B}_x$ is a point if x is regular. The theorem is thus proved.

□

1.4.1.5. Simultaneous resolutions

Definition 1.4.31. *Let $\chi : X \to U$ be a flat morphism of complex algebraic integral varieties. A **simultaneous resolution** of χ is a commutative diagram of morphisms of algebraic varieties*



such that:

- *θ is smooth,*

- *ψ and φ are proper and surjective,*
- *ψ induces $\psi_v : \theta^{-1}(v) \to \chi^{-1}(\varphi(v))$, which is a resolution of singularities for $\chi^{-1}(\varphi(v))$, for all $v \in V$.*

We will speak of simultaneous resolution of χ over φ.

Let $\chi : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a flat morphism whose only singular fiber $\chi^{-1}(0)$ has a unique isolated singularity. The next theorem was obtained by E. Brieskorn in [Bries66] in the setting of holomorphic maps of complex manifolds, and later algebraized by M. Artin in [Artin74] .

Theorem 1.4.32. *There exists a simultaneous resolution of χ if and only if $\chi^{-1}(0)$ has a simple singularity.*

This theorem provides us a means of proving the existence of a simultaneous resolution.

We say that two simultaneous resolutions of χ,

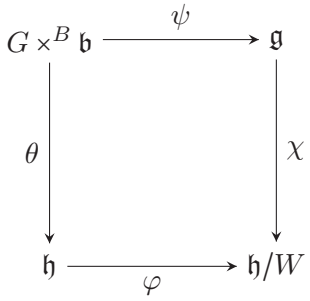
$$\begin{array}{ccc} Y_1 & \xrightarrow{\psi_1} & X \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\chi} \\ V & \xrightarrow[\varphi]{} & U \end{array} \qquad \begin{array}{ccc} Y_2 & \xrightarrow{\psi_2} & X \\ {\scriptstyle\theta_2}\downarrow & & \downarrow{\scriptstyle\chi} \\ V & \xrightarrow[\varphi]{} & U \end{array}$$

are **isomorphic** if there exists an isomorphism $\iota : Y_1 \to Y_2$ such that $\psi_2 \circ \iota = \psi_1$ and $\theta_2 \circ \iota = \theta_1$.

Set χ as in the previous theorem and let $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a base change such that χ has a simultaneous resolution over φ. As $\chi^{-1}(0)$ is a simple singularity one can assign it a Dynkin diagram as well as a Weyl group W. The next result comes from [Bries68] (analytically) and [Artin74] (algebraically).

Theorem 1.4.33. *The number of non-isomorphic simultaneous resolutions of χ over φ is exactly the order of W.*

Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ a Cartan and a Borel subalgebras of $\mathfrak{g}$. The triangular decomposition of $\mathfrak{g}$ gives $\mathfrak{b} = \mathfrak{h} \bigoplus \mathfrak{n}$ with $\mathfrak{n}$ the subalgebra composed of the nilpotent elements of $\mathfrak{b}$. Let W be the Weyl group of $\mathfrak{g}$ associated to $\mathfrak{h}$ as well as G and B the Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{b}$. We have the following diagram:



with χ the adjoint quotient map, φ the canonical projection of $\mathfrak{h}$ on $\mathfrak{h}/W$ and θ and ψ defined by $\psi(g * b) = \mathrm{Ad}(g)b$ and $\theta(g * (h + n)) = h$ for $g \in G, b = h + n \in \mathfrak{b} = \mathfrak{h} \bigoplus \mathfrak{n}$.

The next theorem was conjectured by A. Grothendieck and then proved by E. Brieskorn.

Theorem 1.4.34. (Grothendieck). *The aforementioned diagram is a simultaneous resolution of the adjoint quotient χ.*

Proof. Several points need to be verified:

(1) The diagram is commutative. Indeed, if $g * x \in G \times^B \mathfrak{b}$, then $\chi \circ \psi(g * x) = G.x_s \bigcap \mathfrak{h}$ with x_s the semisimple part of x and $\varphi \circ \theta(g * x) = W.h$ with $x = h + n$ the decomposition of x in $\mathfrak{b}$. But as $W = N(H)/H$, $W.h$ is the set of elements of $\mathfrak{h}$ conjugated to h. On the other hand $G.x_s \bigcap \mathfrak{h}$ is the set of elements of $\mathfrak{h}$ conjugated to x_s. However we know that x_s and h are conjugated so $\chi \circ \psi(g * x) = \varphi \circ \theta(g * x)$.

(2) The action of B on $\mathfrak{h} \cong \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ is trivial because the action of its Lie algebra $\mathfrak{b}$ on $\mathfrak{h}$ is trivial by definition. There is therefore an isomorphism $G \times^B \mathfrak{h} \overset{\cong}{\to} G/B \times \mathfrak{h}$. The morphism θ is smooth because it can be factorized as

$$\begin{array}{ccccccc} G \times^B \mathfrak{b} & \to & G \times^B \mathfrak{h} & \overset{\cong}{\to} & G/B \times \mathfrak{h} & \to & \mathfrak{h} \\ g * (h + n) & \mapsto & g * h & \mapsto & (gB, h) & \mapsto & h \end{array}$$

and each morphism of the factorization is smooth.

(3) The morphism φ is clearly proper and surjective, and so is ψ because every element of $\mathfrak{g}$ is contained in a Borel subalgebra and Borel subalgebras are conjugated.

(4) Let $\overline{h}$ denote the conjugacy class of h in $\mathfrak{h}/W$. We aim to show that $\psi_h : \theta^{-1}(h) \to \chi^{-1}(\overline{h})$ is a resolution of singularities for all $h \in \mathfrak{h}$.

It has been seen before that $\chi^{-1}(\overline{h}) \cong G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h))$. By composing the G-equivariant morphism ψ_h with the projection on $G/Z(h)$, one can see $\theta^{-1}(h)$ as an associated bundle:

$$\begin{array}{ccccccc} \theta^{-1}(h) & \to & \chi^{-1}(\overline{h}) & \cong & G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h)) & \to & G/Z(h) \\ g * (h+n) & \mapsto & \mathrm{ad}(g)(h+n) & \mapsto & g * ((h+n)_n) & \mapsto & gZ(h) \end{array}$$

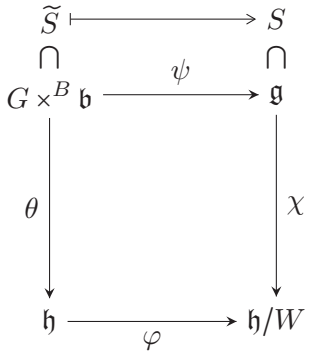
According to Lemma 1.4.18, we need to find the $Z(h)$-stable fiber of this composition above $eZ(h) \in G/Z(h)$. One can see that $h + \mathcal{N}(\mathfrak{z}(h))$ is the set of elements whose semisimple part is h and $G.(h + \mathcal{N}(\mathfrak{z}(h))) \cong G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h))$. The preimage of $eZ(h)$ by the projection on $G/Z(h)$ is $e * \mathcal{N}(\mathfrak{z}(h))$, which corresponds to the set $h + \mathcal{N}(\mathfrak{z}(h))$. The preimage of the composition above $eZ(h)$ is thus the set of elements $g * (h+n) \in G \times^B (h+\mathfrak{n})$ such that the semisimple part of $\mathrm{Ad}(g)(h+n)$ is h. One can replace $g*(h+n)$ by an equivalent $gb^{-1} * \mathrm{Ad}(b)(h+n)$ with a $b \in B$ chosen such that $(\mathrm{Ad}(b)(h+n))_s = h$. As $(\mathrm{Ad}(gb^{-1})(\mathrm{Ad}(b)(h+n)))_s = h$, we get $\mathrm{Ad}(gb^{-1})(\mathrm{Ad}(b)(h+n))_s = h$, which means that $\mathrm{Ad}(gb^{-1})h = h$. We deduce that $gb^{-1} \in Z(h)$. Hence the fiber above $eZ(h)$ is $\{g*(h+n) \in G \times^B \mathfrak{n} \mid g \in Z(h) \text{ and } n \in \mathfrak{n}(h)\}$ with $\mathfrak{n}(h) = \mathfrak{n} \bigcap \mathfrak{z}(h)$. But this is isomorphic to $Z(h) \times^{B(h)} \mathfrak{n}(h)$, where $B(h) = B \bigcap Z(h)$ is a Borel subgroup of $Z(h)$ whose Lie algebra has $\mathfrak{n}(h)$ for nilradical.
Finally, writing $\psi_0(h) : Z(h) \times^{B(h)} \mathfrak{n}(h) \to \mathcal{N}(\mathfrak{z}(h))$ for the restriction of the Springer resolution, one sees that ψ_h is induced by $\psi_0(h)$. Therefore ψ_h is isomorphic to

$$G \times^{Z(h)} \psi_0(h) : G \times^{Z(h)} (Z(h) \times^{B(h)} \mathfrak{n}(h)) \to G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h)).$$

As $\psi_0(h)$ is a resolution by Springer's theorem (Theorem 1.4.30), so is ψ_h. □

Let us consider the diagram of the previous theorem. Let S be a transversal slice to a nilpotent orbit (not necessarily subregular) in $\mathfrak{g}$ and $\widetilde{S} = \psi^{-1}(S)$ the preimage of S by ψ in $G \times^B \mathfrak{b}$.

Corollary 1.4.35. *The restriction of the Grothendieck resolution is a simultaneous resolution.*



Proof. One sees that $\widetilde{S} \cong (G \times^B \mathfrak{b}) \times_{\mathfrak{g}} S$. We know that S is transversal to an orbit at x in $\mathfrak{g}$. As ψ is surjective, let $y \in G \times^B \mathfrak{b}$ such that $\psi(y) = x$. Then $\widetilde{S}$ is a transversal slice to the orbit at y in $G \times^B \mathfrak{b}$. Indeed, one can see that:

- $y \in \widetilde{S}$,
- the morphism $G \times \widetilde{S} \to G \times^B \mathfrak{b}$ is smooth (cf. [Slo80] Section 5.1, Lemma 2),
- codim $\widetilde{S}$ = dim $G.y$. Indeed, because dim $G \times^B \mathfrak{b}$ − dim G = dim G/B + dim $\mathfrak{b}$ − dim G = 0, any fiber of ψ is discrete. The restriction $\psi|_{G.y} : G.y \to G.x$ is surjective and the dimension of any fiber is dim $G.y$ − dim $G.x$ = 0 because it is discrete. So

$$\begin{aligned} \dim \widetilde{S} = \dim G \times^B \mathfrak{b} + \dim S - \dim \mathfrak{g} &= \dim G \times^B \mathfrak{b} - \dim G.x \text{ because } S \text{ is transversal,} \\ &= \dim G \times^B \mathfrak{b} - \dim G.y, \\ &= \operatorname{codim} G.y. \end{aligned}$$

Then it follows that $\widetilde{S}$ is transversal to the G-orbit at y in $G \times^B \mathfrak{b}$.
One concludes that the composition

$$\begin{array}{ccccc} G \times \widetilde{S} & \to & G \times^B \mathfrak{b} & \to & \mathfrak{h} \\ (g, s) & \mapsto & g * s & \mapsto & \theta(g * s) \end{array}$$

is smooth. As it factorizes in

$$G \times \widetilde{S} \xrightarrow{\pi_2} \widetilde{S} \xrightarrow{\theta|_{\widetilde{S}}} \mathfrak{h},$$

then $\widetilde{\theta} = \theta|_{\widetilde{S}}$ is smooth. Therefore $T_s\widetilde{S}$ is well defined for all $s \in \widetilde{S}$ so $\widetilde{S}$ is smooth. One can easily check that the other properties of a simultaneous resolution are verified as well. □

Let S be a transversal slice to a nilpotent subregular orbit of a simple Lie algebra of type A_r, D_r or E_r. Using Theorem 1.4.15 and Corollary 1.4.35, one obtains a simultaneous resolution of the semiuniversal deformation of the corresponding simple singularity.

1.4.2. Simple singularities of type BCFG

The simple singularities have been defined and some of their properties proved. These singularities are linked to the homogeneous (or simply-laced) Dynkin diagrams of simple Lie algebras. However in Section 1.1.2 it was shown that there are other kinds of simple Lie algebras, and as such Dynkin diagrams, which are the inhomogeneous (or non-simply-laced) types B_r ($r \geq 2$), C_r ($r \geq 3$), F_4 and G_2. P. Slodowy gave a definition of simple singularities of inhomogeneous types. This definition requires some preparation.

1.4.2.1. Group action on a simple singularity

Let $\Gamma \subset \Gamma'$ be finite subgroups of SU_2. We would like Γ' to act on $\mathbb{C}^2/\Gamma$. Γ' acts on $\mathbb{C}^2$ and, if $\Gamma \lhd \Gamma'$, this action induces an action of the group Γ'/Γ on $\mathbb{C}^2/\Gamma$. It is thus natural to require that Γ be normal in Γ'. In the following table are all normal subgroups Γ of the finite subgroups Γ' of SU_2 (cf. [Cox91]).

Γ'	Γ	order of Γ
$\mathcal{C}_n$	$\mathcal{C}_r, \forall r/n$	r
$\mathcal{D}_n$	$\mathcal{D}_{\frac{n}{2}}$ if n even	$2n$
	$\mathcal{C}_{\frac{n}{q}}, \forall q/n$ and $\frac{n}{q}$ odd	$\frac{n}{q}$
	$\mathcal{C}_{\frac{2n}{q}}, \forall q/n$	$\frac{2n}{q}$
$\mathcal{T}$	$\mathcal{D}_2$	8
	$\{\pm 1\}$	2
$\mathcal{O}$	$\mathcal{T}$	24
	$\mathcal{D}_2$	8
	$\{\pm 1\}$	2
$\mathcal{I}$	$\{\pm 1\}$	2

The symmetry groups of the simply-laced Dynkin diagrams have been determined in Section 1.2. For type A_{2r-1}, the symmetry group is $\Omega = \mathbb{Z}/2\mathbb{Z}$ and $\Gamma = \mathcal{C}_{2r}$. We are

looking for Γ' in SU_2 such that $\Gamma \triangleleft \Gamma'$ and Γ' acts on $\mathbb{C}^2/\Gamma$. The action of Γ' can be lifted through the minimal resolution of $\mathbb{C}^2/\Gamma$. As the minimal resolution restricts to an isomorphism on a dense open subset and Γ acts trivially on $\mathbb{C}^2/\Gamma$, it follows that Γ acts trivially on the minimal resolution. So the action of Γ' on the whole minimal resolution factors through an action of Γ'/Γ. As the origin is a fixed point of Γ' in $\mathbb{C}^2/\Gamma$, Γ'/Γ acts on the exceptional locus, which is composed of a union of projective lines whose intersection pattern corresponds to the Dynkin diagram A_{2r-1}. Therefore we want Γ'/Γ to act on the exceptional locus the same way Ω acts on the Dynkin diagram. This can be achieved by requiring $\Gamma'/\Gamma \cong \Omega$ and $\Gamma \triangleleft \Gamma'$.

• Type A_{2r-1}: $\Gamma = \mathcal{C}_{2r}$ and $\Omega = \mathbb{Z}/2\mathbb{Z}$. Based on the previous table, the possibilities are $\Gamma' = \mathcal{C}_{4r}$ or $\mathcal{D}_r$.
The singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X^{2r} - YZ = 0\}$ with $X = z_1 z_2$, $Y = z_1^{2r}$, $Z = z_2^{2r}$, where (z_1, z_2) is the dual of the canonical base of $\mathbb{C}^2$ (cf. Theorem 1.4.1).

- $\mathcal{C}_{4r}$ is generated by $g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ with $\zeta = \exp(\frac{2i\pi}{4r})$. The action is $g.z_1 = \zeta^{-1} z_1$ and $g.z_2 = \zeta z_2$. Then

$$\begin{cases} g.X &= X, \\ g.Y &= -Y, \\ g.Z &= -Z. \end{cases}$$

- $\mathcal{D}_r$ is generated by g^2 and $h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. g^2 fixes $\mathbb{C}^2/\Gamma$ and $h.z_1 = -iz_2$, $h.z_2 = -iz_1$. Then

$$\begin{cases} h.X &= -X, \\ h.Y &= (-1)^r Z, \\ h.Z &= (-1)^r Y. \end{cases} .$$

In order to determine which group is the appropriate one, it is necessary to extend the action of Γ' to the minimal resolution of $\mathbb{C}^2/\Gamma$ and check that Γ' permutes the components of the exceptional locus the same way Ω acts on the Dynkin diagram. In the next subsection, it will be explicitly proved that $\Gamma' = \mathcal{D}_r$.

• Type A_{2r}: $\Gamma = \mathcal{C}_{2r+1}$ and $\Omega = \mathbb{Z}/2\mathbb{Z}$. Based on the previous table, the only possibility

is $\Gamma' = \mathcal{C}_{4r+2}$.
The singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X^{2r+1} - YZ = 0\}$ with $X = z_1 z_2$, $Y = z_1^{2r+1}$, $Z = z_2^{2r+1}$, where (z_1, z_2) is the dual of the canonical base of $\mathbb{C}^2$ (cf. Theorem 1.4.1).

$\mathcal{C}_{4r+2}$ is generated by $g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ with $\zeta = \exp(\frac{2i\pi}{4r+2})$. The action is $g.z_1 = \zeta^{-1} z_1$ and $g.z_2 = \zeta z_2$. Then

$$\begin{cases} g.X &= X, \\ g.Y &= -Y, \\ g.Z &= -Z. \end{cases}$$

It is necessary to extend the action of Γ' to the minimal resolution of $\mathbb{C}^2/\Gamma$ and check that Γ' permutes the components of the exceptional locus the same way Ω acts on the Dynkin diagram. In the next subsection, it will be shown that is does not. Therefore a group Γ' satisfying our conditions does not exist.

• Type D_{r+1}: $\Gamma = \mathcal{D}_{r-1}$ and $\Omega = \mathbb{Z}/2\mathbb{Z}$. The only possibility is $\Gamma' = \mathcal{D}_{2(r-1)}$.
If r is odd, the singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X(Y^2 - X^{r-1}) + Z^2 = 0\}$ with $X = 4^{\frac{1}{r}}(z_1 z_2)^2$, $Y = 4^{-\frac{1}{2r}}(z_1^{2(r-1)} + z_2^{2(r-1)})$, $Z = iz_1 z_2(z_1^{2(r-1)} - z_2^{2(r-1)})$, where (z_1, z_2) is the dual of the canonical base of $\mathbb{C}^2$ (cf. Theorem 1.4.1).
If r is even, the singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X(Y^2 + X^{r-1}) + Z^2 = 0\}$ with $X = 4^{\frac{1}{r}}(z_1 z_2)^2$, $Y = 4^{-\frac{1}{2r}}(z_1^{2(r-1)} - z_2^{2(r-1)})$, $Z = iz_1 z_2(z_1^{2(r-1)} + z_2^{2(r-1)})$, where (z_1, z_2) is defined similarly as when r is odd.

$\mathcal{D}_{2(r-1)}$ is generated by $g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ with $\xi = \exp(\frac{2i\pi}{4(r-1)})$ and $h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then

$$\begin{cases} g.X &= X, \\ g.Y &= -Y, \\ g.Z &= -Z, \end{cases} \qquad \text{and} \qquad \begin{cases} h.X &= X, \\ h.Y &= Y, \\ h.Z &= Z. \end{cases}$$

• Type E_6: $\Gamma = \mathcal{T}$ and $\Omega = \mathbb{Z}/2\mathbb{Z}$. According to the previous table, $\Gamma' = \mathcal{O}$.
The singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X^4 + Y^3 + Z^2 = 0\}$ with $X = 108^{\frac{1}{4}} z_1 z_2(z_1^4 - z_2^4)$, $Y = \exp(\frac{i\pi}{3})(z_1^8 + z_2^8 + 14(z_1 z_2)^4)$, $Z = (z_1^4 + z_2^4)^3 - 36(z_1 z_2)^4(z_1^4 + z_2^4)$, with (z_1, z_2) the dual of the canonical base of $\mathbb{C}^2$ (cf. Theorem 1.4.1).

$\mathcal{O} =< g, \mathcal{T} >$ with $g = \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^5 \end{pmatrix}$ and $\epsilon = \exp(\frac{i\pi}{4})$. It is know that $\mathcal{T}$ fixes $\mathbb{C}^2/\Gamma$ and

$$\begin{cases} g.X &= -X, \\ g.Y &= Y, \\ g.Z &= -Z. \end{cases}$$

• Type D_4: $\Gamma = \mathcal{D}_2$ and $\Omega = \mathfrak{S}_3$. The only possibility is $\Gamma' = \mathcal{O}$.
The singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X(Y^2 - X^2) + Z^2 = 0\}$ with $X = 4^{\frac{1}{3}}(z_1 z_2)^2$, $Y = 4^{-\frac{1}{6}}(z_1^4 + z_2^4)$, $Z = iz_1z_2(z_1^4 - z_2^4)$, with (z_1, z_2) the dual of the canonical base of $\mathbb{C}^2$ (cf. Theorem 1.4.1).

$\mathcal{O} =< \mathcal{D}_2, g, h >$ with $g = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix}$ and $h = \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^5 \end{pmatrix}$ with $\epsilon = \exp(\frac{i\pi}{4})$. It is known that $\mathcal{D}_2$ fixes $\mathbb{C}^2/\Gamma$, and

$$\begin{cases} g.X &= \frac{1}{2}(Y - X), \\ g.Y &= -\frac{1}{2}(Y + 3X), \\ g.Z &= Z, \end{cases} \quad \text{and} \quad \begin{cases} h.X &= X, \\ h.Y &= -Y, \\ h.Z &= -Z. \end{cases}$$

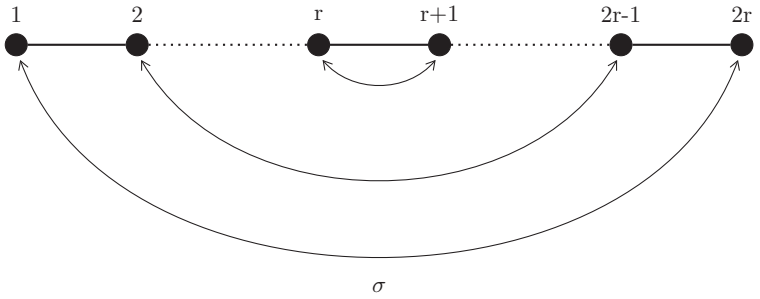
We summarize the results we obtained in the following table:

Homogeneous	Inhomogeneous	Γ	Γ'	Ω
A_{2r-1}	B_r	$\mathcal{C}_{2r}$	$\mathcal{D}_r$	$\mathbb{Z}/2\mathbb{Z}$
A_{2r}	C_r	$\mathcal{C}_{2r+1}$	$\times$	$\mathbb{Z}/2\mathbb{Z}$
D_{r+1}	C_r	$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	$\mathbb{Z}/2\mathbb{Z}$
E_6	F_4	$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}/2\mathbb{Z}$
D_4	G_2	$\mathcal{D}_2$	$\mathcal{T}$ / $\mathcal{O}$	$\mathbb{Z}/3\mathbb{Z}$ / $\mathfrak{S}_3$

Table 1.3.

The "inhomogeneous" column corresponds to the foldings of the root lattices whose types compose the first column. They were computed in Section 1.2. We add that if we chose Ω to be $\mathbb{Z}/3\mathbb{Z}$ in the case $\Gamma = \mathcal{D}_2$, it can be proved that $\Gamma' = \mathcal{T}$.

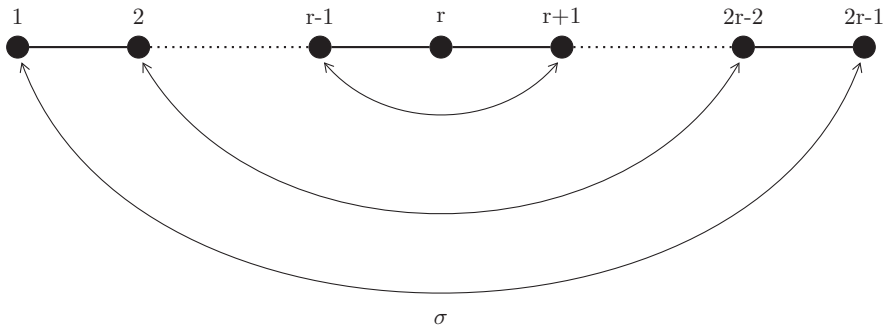
Remark 1.4.36. One may wonder why the Dynkin diagram A_{2r} was disregarded. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ of type A_{2r}, Δ its Dynkin diagram and $\{\alpha_1, \ldots, \alpha_{2r}\}$ a base of the root system. We identify the simple roots of $\mathfrak{g}$ with their respective vertices in Δ, so that an automorphism of Δ can be seen as a permutation of the simple roots. We define σ the automorphism of Δ by $\sigma(\alpha_i) = \alpha_{2r+1-i}$.



The folding of the Lie algebra $\mathfrak{g}$ gives a Lie algebra $\mathfrak{g}_0$ of type B_r and the folding of the root lattice gives a root lattice Q^σ of type C_r. This case should a priori be included in Table 1.3. However, it will be shown in Section 1.4.2.2 that for any Γ' such that $\Gamma'/\Gamma \cong \Omega = \mathbb{Z}/2\mathbb{Z}$, Γ'/Γ fails to permute the components of the exceptional locus of the minimal resolution of A_{2r} like the symmetry group Ω of the Dynkin diagram permutes its vertices.

1.4.2.2. Computation with A_{2r-1} and B_r

Let σ be the automorphism of the Dynkin diagram of type A_{2r-1} defined, as in Section 1.2.2, by



The group corresponding to A_{2r-1} is $\Gamma = \mathcal{C}_{2r}$. Set $\Gamma' = \mathcal{D}_r$. Let us check that Γ' permutes the components of the exceptional locus of A_{2r-1} the same way σ permutes the vertices of the Dynkin diagram.

The singularity of type A_{2r-1} is $X_{2r} = \{(x,y,z) \in \mathbb{C}^3 \mid x^{2r} - yz = 0\}$ and its minimal resolution is given by (cf. [Lamo86])

$$\begin{array}{rccc} \psi : & \widetilde{X}_{2r} & \to & X_{2r} \\ & ((x,y,z), \{(a_i : b_i)\}_{1 \le i \le 2r-1}) & \mapsto & (x,y,z) \end{array}$$

with $\widetilde{X}_{2r}$ the surface in $\mathbb{C}^3 \times (\mathbb{P}^1)^{2r-1}$ defined by $\begin{cases} a_{2r-1}x = b_{2r-1}y, \\ a_i b_{i+1} x = a_{i+1} b_i, \quad 1 \le i \le 2r-2, \\ a_1 z = b_1 x. \end{cases}$

It is known that $\mathcal{D}_r = \langle g, h \rangle$ with $g = \begin{pmatrix} \exp(\frac{2i\pi}{2r}) & 0 \\ 0 & \exp(-\frac{2i\pi}{2r}) \end{pmatrix}$ and $h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. It was shown in Subsection 1.4.2.1 that $\mathcal{D}_r$ acts on X_{2r} by $\begin{cases} g.(x,y,z) = (x,y,z), \\ h.(x,y,z) = (-x, (-1)^r z, (-1)^r y). \end{cases}$

Let $\alpha = ((x,y,z), (a_1 : b_1), \ldots, (a_{2r-1} : b_{2r-1})) \in \widetilde{X}_{2r}$. Our goal is to define $g.\alpha$ and $h.\alpha$ and make ψ a Γ'-equivariant morphism.

By definition $\Gamma = \langle g \rangle$ fixes every point in X_{2r} so $g.\alpha = ((x,y,z), (a'_1 : b'_1), \ldots, (a'_{2r-1} :$

$b'_{2r-1})) \in \widetilde{X}_{2r}$ thus $\begin{cases} a'_{2r-1}x = b'_{2r-1}y, \\ a'_i b'_{i+1}x = a'_{i+1}b'_i,\ 1 \le i \le 2r-2, \\ a'_1 z = b'_1 x, \end{cases}$ and $\begin{cases} a_{2r-1}x = b_{2r-1}y, \\ a_i b_{i+1}x = a_{i+1}b_i,\ 1 \le i \le 2r-2, \\ a_1 z = b_1 x. \end{cases}$

One deduces from these systems that $(a'_i : b'_i) = (a_i : b_i)$ for all i, hence $g.\alpha = \alpha$, $\forall \alpha \in \widetilde{X}_{2r}$.

Furthermore, $h.\alpha \in \widetilde{X}_{2r}$ with $\begin{cases} a'_{2r-1}(-x) &= b'_{2r-1}(-1)^r z, \\ a'_i b'_{i+1}(-x) &= a'_{i+1}b'_i, \qquad 1 \le i \le 2r-2, \\ a'_1(-1)^r y &= b'_1(-x). \end{cases}$. But $\alpha \in \widetilde{X}_{2r}$ so $\dfrac{x}{y} = \dfrac{b_{2r-1}}{a_{2r-1}}$ and $\dfrac{x}{z} = \dfrac{a_1}{b_1}$. One deduces that $\dfrac{a'_1(-1)^{r+1}}{b'_1} = \dfrac{b_{2r-1}}{a_{2r-1}}$ and $\dfrac{b'_{2r-1}(-1)^{r+1}}{a'_{2r-1}} = \dfrac{a_1}{b_1}$. Then $\dfrac{a'_1}{b'_1} = (-1)^{r+1}\dfrac{b_{2r-1}}{a_{2r-1}}$ and $\dfrac{a'_{2r-1}}{b'_{2r-1}} = (-1)^{r+1}\dfrac{b_1}{a_1}$. Thus $\dfrac{a'_{2r-2}}{b'_{2r-2}} = \dfrac{1}{-x}\dfrac{a'_{2r-1}}{b'_{2r-1}} = (-1)^{r+2}\dfrac{b_1}{a_1 x} = (-1)^{r+2}\dfrac{b_2}{a_2}$. The computation finally gives $\dfrac{a'_i}{b'_i} = (-1)^{r+2r-i}\dfrac{b_{2r-i}}{a_{2r-i}}$, $1 \le i \le 2r-1$. Therefore the action of h is given by $h.\alpha = ((-x, (-1)^r z, (-1)^r y), \{((-1)^{r+2r-i} b_{2r-i} : a_{2r-i})\}_{1 \le i \le 2r-1}) \in \widetilde{X}_{2r}$.

Let $E = \psi^{-1}(0)$ be the exceptional locus. One sees that $E = \bigcup_{i=1}^{2r-1} E_i$ with $E_i = \{(0,0,0), (1:0), \ldots, (1:0), \underbrace{(a_i : b_i)}_{\text{i-th position}}, (0:1), \ldots, (0:1)\} \subset \widetilde{X}_{2r}$. The action of g on the components of E is $g.E_i = E_i$ for any $1 \le i \le 2r-1$, and $h.E_i = \{(0,0,0), (1:0), \ldots, (1:0), \underbrace{((-1)^{r+2r-(2r-i)} b_i : a_i)}_{\text{(2r-i)-th position}}, (0:1), \ldots, (0:1)\}$. Thus $h.E_i = E_{2r-i}$ for any $1 \le i \le 2r-1$.

Finally Γ' permutes the components of the exceptional locus the same way Γ'/Γ permutes the vertices of the Dynkin diagram.

We go back to the case A_{2r} for $\Gamma = \mathcal{C}_{2r+1}$ and explain why it differs from the others. For A_{2r} one cannot find a finite subgroup Γ' of SU_2 such that $\Gamma \lhd \Gamma'$ and Γ' permutes the components of the exceptional locus of the minimal resolution of $\mathbb{C}^2/\Gamma$ the same way the symmetry group of the Dynkin diagram permutes the vertices of the Dynkin diagram. Indeed, let us take $\Gamma = \mathcal{C}_{2r+1}$. The singularity $\mathbb{C}^2/\Gamma$ is of type A_{2r}. The symmetry group of the Dynkin diagram of type A_{2r} is $\Omega = \mathbb{Z}/2\mathbb{Z}$ and permutes the vertices i and $2r+1-i$, $1 \le i \le 2r$. Based on the table in Section 1.4.2.1, if one wants Γ' such that $\Gamma \lhd \Gamma'$ and $\Gamma'/\Gamma = \Omega$, then the only possibility is $\Gamma' = \mathcal{C}_{4r+2}$.
The singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X^{2r+1} - YZ = 0\}$ with $X = z_1 z_2$, $Y = z_1^{2r+1}$, $Z = z_2^{2r+1}$, where (z_1, z_2) is the dual of the canonical base of $\mathbb{C}^2$ (cf. Theorem 1.4.1).

$\mathcal{C}_{4r+2}$ is generated by $g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ with $\zeta = \exp(\frac{2i\pi}{4r+2})$. The action is $g.z_1 = \zeta^{-1} z_1$ and $g.z_2 = \zeta z_2$. Then

$$\begin{cases} g.X &= X, \\ g.Y &= -Y, \\ g.Z &= -Z. \end{cases}$$

If one lifts this action to the minimal resolution of A_{2r} like we did previously for A_{2r-1}, then Γ' stabilizes every component of the exceptional locus. However the action of Ω on the Dynkin diagram is not the identity, so Γ'/Γ acts differently on the exceptional locus than Ω acts on the Dynkin diagram of type A_{2r}. Therefore a finite subgroup Γ' of SU_2 that fits our requirements does not exist.

1.4.2.3. Definition and deformations of BCFG singularities

Let us begin with the definition the inhomogeneous Kleinian singularities given by P. Slodowy in [Slo80].

Definition 1.4.37. *A **simple singularity** of type B_r ($r \geq 2$), C_r ($r \geq 3$), F_4 or G_2 is a pair (X_0, Ω) of a simple singularity (in the former sense) and a group Ω of automorphisms of X_0 according to the following list:*

Type of (X_0, Ω)	*Type of* X_0	Γ	Γ'	Ω
$B_r,\ r \geq 2$	A_{2r-1}	$\mathcal{C}_{2r}$	$\mathcal{D}_r$	$\mathbb{Z}/2\mathbb{Z}$
$C_r,\ r \geq 3$	D_{r+1}	$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	$\mathcal{D}_2$	$\mathcal{O}$	$\mathfrak{S}_3$

A simple singularity of inhomogeneous type is then a simple homogeneous singularity with a symmetry of the Dynkin diagram. One notices from the Section 1.2 that the type of (X_0, Ω) is the same as the type of the folding of the root lattice of type X_0.

Remark 1.4.38. One notices that the case where X_0 is of type A_{2r} does not appear in the preceding table. Although the Dynkin diagram of type A_{2r} has a non-trivial

symmetry group, it was proved in Subsection 1.4.2.2 that the action of this symmetry group fails to lift to the exceptional locus of the minimal resolution of X_0.

The notion of symmetry has been added to simple singularities, therefore it is necessary to include this symmetry in the definition of deformations of singularities of type B_r, C_r, F_4 and G_2. P. Slodowy gave the following definition.

Definition 1.4.39. *A **deformation** of a simple singularity (X_0, Ω) is a deformation $\chi : X \to U$ of X_0 together with a trivial action of Ω on U and with an action of Ω on X which induces the given one on X_0 and such that χ is Ω-invariant.*

An **isomorphism** of two deformations of (X_0, Ω) is an isomorphism in the previous sense which is in addition Ω-equivariant. A **semiuniversal deformation** is then defined in an analogous way as done in Subsection 1.4.1.3.

In [Slo80], P. Slodowy constructs a semiuniversal deformation of (X_0, Ω) in the case of X_0 a hypersurface with an isolated singularity and Ω a reductive group. More precisely, he shows the existence of a semiuniversal deformation $\chi : X \to U$ of X_0 in the original sense with the following additional property: there are Ω-actions on X and U such that χ is Ω-equivariant and the restriction of Ω on X_0 is the given one. Furthermore, if $Y \to T$ is any Ω-equivariant deformation of X_0, it can be induced from χ by a Ω-equivariant morphism. From here, a semiuniversal deformation χ_Ω of (X_0, Ω) is obtained by the deformation induced from χ by the base change $U^\Omega \to U$, with U^Ω being the fixed point set of Ω in U.

$$\begin{array}{ccc} X \times_U U^\Omega & \longrightarrow & X \\ \downarrow{\scriptstyle \chi_\Omega} & & \downarrow{\scriptstyle \chi} \\ U^\Omega & \longrightarrow & U \end{array}$$

1.4.2.4. Inhomogeneous singularities from Lie algebras of type B_r, C_r, F_4 and G_2

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with adjoint group G and let $e \in \mathfrak{g}$ be a nilpotent element. It is known that there exist $f, h \in \mathfrak{g}$ such that (e, f, h) is an $\mathfrak{sl}_2$-triple. In [Slo80] Section 7.5, P. Slodowy defines $C(e) := Z_G(e) \cap Z_G(h)$ and calls it the **reductive centralizer** of e with respect to h.

Lemma 1.4.40. *The reductive centralizer of e centralizes the $\mathfrak{sl}_2$-Lie subalgebra of $\mathfrak{g}$ generated by e, f and h. As a consequence, the Slodowy slice $S_e = e + \mathfrak{z}_{\mathfrak{g}}(f)$ is $C(e)$-stable.*

Proof. Let $c \in C(e)$. Then $(ce, ch, cf) = (e, h, cf)$ is an $\mathfrak{sl}_2$-triple for e. Based on the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, the map ad $x : \mathfrak{g}(-2) \to \mathfrak{g}(0)$ is injective and $h = [e, f] = [e, cf]$. It follows that $cf = f$ and the $\mathfrak{sl}_2$-Lie subalgebra of $\mathfrak{g}$ is stable. It implies naturally that $S_e = e + \mathfrak{z}_{\mathfrak{g}}(f)$ is $C(e)$-stable. □

Remark 1.4.41. In fact, the reductive centralizer $C(e)$ is exactly the centralizer of the $\mathfrak{sl}_2$-Lie subalgebra.

Proposition 1.4.42. *If $e \in \mathfrak{g}$ is regular nilpotent then $C(e) = \{1\}$, and in particular $Z_G(e)$ is connected. If e is subregular nilpotent, then according to type we have*

Type of $\mathfrak{g}$	$A_r,\ r > 1$	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$C(e)$	$\mathbb{G}_m$	$\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathfrak{S}_3$

In the semidirect product $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{G}_m$ by $x \mapsto x^{-1}$.

Proof. The case e regular was done in [SprinStein70]. The rest was done by various authors with case by case analysis. All the references can be found in [Slo80] Section 7.5. □

In Subsection 1.4.1.3 we stated a theorem due to E. Brieskorn (Theorem 1.4.15) which gives the semiuniversal deformation of a Kleinian singularity of type A_r, D_r, or E_r from the adjoint quotient of a simple Lie algebra of the same type. This theorem can be extended to the inhomogeneous case.

Theorem 1.4.43. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type B_r, C_r, F_4 or G_2 and e a subregular nilpotent element of $\mathfrak{g}$. Then there exists a finite subgroup Ξ of the reductive centralizer $C(e)$ of e and a Ξ-stable transversal slice S at e to the G-orbit of e such that the (Ξ-invariant) restriction of the adjoint quotient map $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to S realizes a semiuniversal deformation of a simple singularity of the same type as $\mathfrak{g}$.*

The proof can be found in [Slo80].

As a special case, the theorem states that the intersection $X_0 = S \bigcap \mathcal{N}(\mathfrak{g})$ of S with the nilpotent variety of $\mathfrak{g}$ plus the induced action of Ξ on X_0 is a simple singularity (X_0, Ξ) of the same type B_r, C_r, F_4 or G_2 as $\mathfrak{g}$.

It was mentioned in Proposition 1.4.23 that for a simple singularity whose Dynkin diagram Δ is of type A_r, D_r, or E_r, singularities whose types correspond to subdiagrams of Δ can be found in the fibers of the semiuniversal deformation near the singular point. There is a similar result for the inhomogeneous singularities.

If Δ is an inhomogeneous Dynkin diagram, it was shown in Section 1.2 that it can be obtained from a simply-laced Dynkin diagram ${}_h\Delta$ as a quotient by the action of a group Ξ of diagram isometries. The preimages of this quotient map in ${}_h\Delta$ of subdiagrams of Δ are the Ξ-stable subdiagrams of ${}_h\Delta$. The following corollary is also proved in [Slo80].

Corollary 1.4.44. *Let $\chi : X \to U$ be a deformation of a simple singularity (X_0, Ξ) of inhomogeneous type Δ, and let $X_\epsilon \neq X_0$ be a non-special fiber. Then there is an Ξ-stable subdiagram ${}_h\Delta'$ of ${}_h\Delta$ and a Ξ-equivariant type-preserving bijection of the connected components of ${}_h\Delta'$ onto the singular points of X_ϵ.*

Example 1.4.45. *For the Lie algebra G_2, one obtains a semiuniversal deformation of type $(D_4, \mathfrak{S}_3)$. Consider the projection of ${}_h\Delta$ onto Δ:*

There are only two non-trivial subdiagrams of Δ:

(1) Omit the red vertex  *and it gives :*  $= A_1 + A_1 + A_1$ *plus a permutation by $\mathfrak{S}_3$.*

(2) Omit the blue vertex  *and it gives :*  $= A_1$ *which is invariant under $\mathfrak{S}_3$.*

1.4.2.5. A stable Slodowy slice for A_{2r-1}, D_r and E_6

Let Δ be a Dynkin diagram of type A_{2r-1}, D_r, or E_6 and let $\mathfrak{g}$ be a simple Lie algebra of type Δ with adjoint simple group G. Take (e, f, h) an $\mathfrak{sl}_2$-triple with e a nilpotent element of $\mathfrak{g}$. In [Slo80] Section 7.6, $CA(e) := \{\sigma \in \mathrm{Aut}(\mathfrak{g}) \mid \sigma(e) = e \text{ and } \sigma(h) = h\}$ is called the **outer reductive centralizer** of e with respect to h. The group $CA(e)$ can be described with the following proposition:

Proposition 1.4.46. *If $e \in \mathfrak{g}$ is regular nilpotent, then $CA(e)$ is isomorphic to $\mathrm{Aut}(\Delta)$. If $e \in \mathfrak{g}$ is subregular nilpotent, then $CA(e)$ has the following form depending on the type of Δ:*

Type of Δ	A_{2r-1}	D_4	$D_r,\ r>4$	E_6
$CA(e)$	$E(\mathbb{Z}/2\mathbb{Z},\mathbb{G}_m)$	$\mathfrak{S}_3$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Here $E(\mathbb{Z}/2\mathbb{Z},\mathbb{G}_m)$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{G}_m$ with non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{G}_m$. For r odd this extension is the semi-direct product, for r even it is the unique non-trivial extension.

Proof. Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be an automorphism of $\mathfrak{g}$. Then $(\sigma(e),\sigma(f),\sigma(h))$ is an $\mathfrak{sl}_2$-triple for $\sigma(e)$, which is again regular resp. subregular. By the uniqueness of the regular resp. subregular G-orbit, there exists $g \in G$ such that $g.\sigma(e,f,h) = (e,f,h)$. This implies that $g.\sigma \in CA(e)$. From the surjectivity of $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\Delta)$, it follows that the composition $CA(e) \to \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\Delta)$ is surjective and its kernel is $C(e) = CA(e) \bigcap G$. For e regular as well as for $\Delta = D_r$ or E_6 and e subregular, the kernel is trivial and thus $CA(e) \cong \mathrm{Aut}(\Delta)$. For $\Delta = A_{2r-1}$ and e subregular we have $C(e) = \mathbb{G}_m$ and $\mathrm{Aut}(\Delta) = \mathbb{Z}/2\mathbb{Z}$. Thus $CA(e)$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{G}_m$. Let $s \in CA(e) \setminus C(e)$. To prove the remaining statements, we need to show that s induces the non-trivial automorphism of $C(e) = \mathbb{G}_m$ and that $s^2 = 1$, respectively $s^2 = -1$, for r odd, respectively even. We verify this on an example.

Let e be the subregular element of $\mathfrak{sl}_{2r}(\mathbb{C})$ given by the matrix

$$\begin{pmatrix} 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & . & 0 \end{pmatrix}$$

and let $\sigma \in \mathrm{Aut}(\mathfrak{sl}_{2r}(\mathbb{C}))$ be the outer automorphism $\sigma(m) = {}^t(-m)$. Let

$$g = \begin{pmatrix} 0 & . & . & . & . & 1 & 0 \\ 0 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 1 & . & . & . & . \\ 0 & -1 & 0 & . & . & 0 & . \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 1 \end{pmatrix} \in \mathrm{GL}_{2r}(\mathbb{C})$$

and $\Phi := (\mathrm{Inn}\, g) \circ \sigma \in \mathrm{Aut}(\mathfrak{sl}_{2r}(\mathbb{C}))$. A computation gives $\Phi(e) = e$ and $\Phi^2 = 1$. With respect to a suitable choice of an $\mathfrak{sl}_2$-triple (e, f, h), the Lie algebra $\mathfrak{c}(e)$ of $C(e)$ is

$$\mathfrak{c}(e) = \left\{ m(t) = \begin{pmatrix} t & 0 & . & . & . & 0 \\ 0 & t & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & t & 0 \\ 0 & 0 & 0 & . & . & -(2r-1)t \end{pmatrix} \middle| \; t \in \mathbb{C} \right\}.$$

Then Φ transforms $\mathfrak{c}(e)$ into itself by sending $m(t)$ to $m(-t)$. The lemma is hence proved in this special case. □

Take (e, f, h) an $\mathfrak{sl}_2$-triple with e a subregular nilpotent element of $\mathfrak{g}$ and $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$ a Slodowy slice at e. With the same arguments as for the proof of Lemma 1.4.40, $CA(e)$ stabilizes the slice S. Let $\delta : S \to \mathfrak{h}/W$ denote the restriction of the adjoint quotient of $\mathfrak{g}$ to the Slodowy slice. The group $\mathrm{Aut}(\Delta) \cong \mathrm{Aut}(\mathfrak{g})/G$ acts naturally on $\mathfrak{g}//G \cong \mathfrak{h}/W$ and can be considered as a subgroup of $CA(e)$.

Corollary 1.4.47. $\mathrm{Aut}(\Delta)$ *is a finite subgroup of* $CA(e)$ *and stabilizes the Slodowy slice* S.

1.4.2.6. Inhomogeneous singularities from Lie algebras of type A_r, D_r and E_6

Like in the previous section, Δ is a Dynkin diagram of type A_{2r-1}, D_r, or E_6, $\mathfrak{g}$ is a simple Lie algebra of type Δ with adjoint simple group G, $e \in \mathfrak{g}$ a subregular nilpotent

element of $\mathfrak{g}$, (e, f, h) an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ and $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$ a Slodowy slice at e. Because of Corollary 1.4.47, $\mathrm{Aut}(\Delta)$ acts on S and on $\mathfrak{h}/W$ and makes δ equivariant. As a result, there is an action of $\mathrm{Aut}(\Delta)$ on the special fiber $X = \delta^{-1}(0)$. Now let Δ_0 be the unique inhomogeneous Dynkin diagram such that ${}_h\Delta_0 = \Delta$ and $AS(\Delta_0) = \mathrm{Aut}(\Delta)$ with $AS(\Delta_0)$ being the associated symmetry group of Δ_0 defined by

$$AS(\Delta_0) = \begin{cases} \mathfrak{S}_3 \text{ if } \Delta_0 = G_2, \\ \mathbb{Z}/2\mathbb{Z} \text{ otherwise.} \end{cases}$$

The following two theorems are proved in Section 8.8 of [Slo80]:

Theorem 1.4.48. *$(X, AS(\Delta_0))$ is a simple singularity of type Δ_0.*

Let G_0 denote the simple adjoint group of type Δ_0 with Lie algebra $\mathfrak{g}_0$. Let (e_0, f_0, h_0) be an $\mathfrak{sl}_2$-triple with e_0 a subregular nilpotent element of $\mathfrak{g}_0$ and $S_0 = e_0 + \mathfrak{z}_{\mathfrak{g}_0}(f_0)$. Let $\delta_0 : S_0 \to \mathfrak{h}_0/W_0$ denote the restriction to S_0 of the adjoint quotient map of $\mathfrak{g}_0$.

Theorem 1.4.49. *The $AS(\Delta_0)$-equivariant deformation $\delta : S \to \mathfrak{h}/W$ of X is $AS(\Delta_0)$-semiuniversal, and the restriction $\delta^{AS(\Delta_0)}$ of δ over the fixed point space $(\mathfrak{h}/W)^{AS(\Delta_0)}$ is isomorphic to δ_0.*

Remark 1.4.50. The theorem above allows an identification of $\mathfrak{h}_0/W_0$ with $(\mathfrak{h}/W)^{AS(\Delta_0)}$. However another identification is possible. The group $AS(\Delta_0)$ acts on the Dynkin diagram Δ and its action can be naturally extended to $\mathfrak{h}$. Set $\mathfrak{h}_1 = \mathfrak{h}^{AS(\Delta_0)}$ and $W_1 = \{w \in W \mid w\gamma = \gamma w, \forall \gamma \in AS(\Delta_0)\}$. Then the natural map $\mathfrak{h}_1 \to (\mathfrak{h}/W)^{AS(\Delta_0)}$ induces a $\mathbb{G}_m$-equivariant morphism $\mathfrak{h}_1/W_1 \to (\mathfrak{h}/W)^{AS(\Delta_0)}$. The $\mathbb{G}_m$-weights of this morphism are the same on $\mathfrak{h}_1/W_1$ and $(\mathfrak{h}/W)^{AS(\Delta_0)}$, and are strictly positive. Hence using [Slo80] Section 8.1, Lemma 3, one finds that it is an isomorphism.

1.5. McKay correspondence

1.5.1. Homogeneous correspondence

In 1980, J. McKay noticed in [McK80] a link between the irreducible representations of the finite subgroups of SU_2 and the extended Dynkin diagrams of types A_r, D_r and E_r.

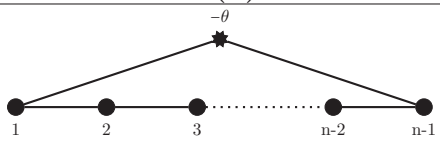
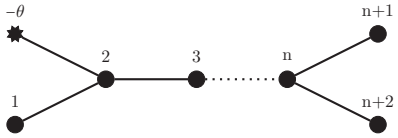
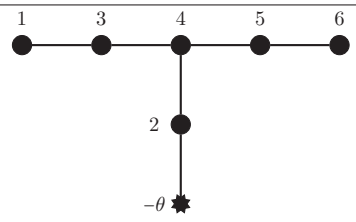
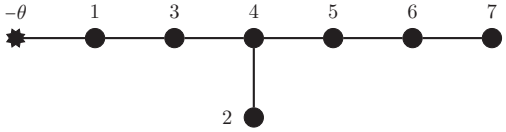
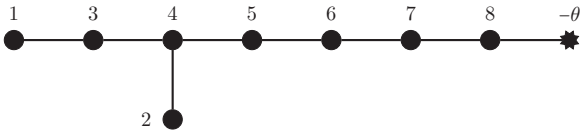
1.5.1.1. Statement of the correspondence

Let Γ be a finite subgroup of SU_2. As such Γ acts naturally on $V_{\mathrm{nat}} := \mathbb{C}^2$. For every irreducible representation V_i, $0 \le i \le r$, of Γ (V_0 being the trivial representation), one can define the representation $V_{\mathrm{nat}} \otimes V_i$. Using representation theory of finite groups (cf. [Ser78]), these tensor products decompose as:

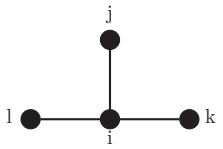
$$V_{\mathrm{nat}} \otimes V_i = \bigoplus_{j=0}^{r} m_{ij} V_j, \quad 0 \le i \le r,$$

with $m_{ij} \in \mathbb{Z}_{\ge 0}$, for all $0 \le i, j \le r$. m_{ij} is the multiplicity of V_j in $V_{\mathrm{nat}} \otimes V_i$. Set $M = (m_{ij})_{0 \le i,j \le r}$ the corresponding matrix and I the $(r+1) \times (r+1)$ identity matrix. J. McKay observed the following:

McKay correspondence: The matrix $2I - M$ is the Cartan matrix of the extended Dynkin diagram $\widetilde{\Delta}(\Gamma)$ associated to Γ. We write $\Delta(\Gamma)$ for the Dynkin diagram associated to Γ through the resolution of $\mathbb{C}^2/\Gamma$. The extended Dynkin diagram $\widetilde{\Delta}(\Gamma)$ is obtained by adding $-\theta$ to $\Delta(\Gamma)$, with θ being the highest root of the root system of $\Delta(\Gamma)$. The obtained diagrams are listed in the table below:

Γ	$\widetilde{\Delta}(\Gamma)$	Name of $\Delta(\Gamma)$
$\mathcal{C}_n$		A_{n-1}
$\mathcal{D}_n$		D_{n+2}
$\mathcal{T}$		E_6
$\mathcal{O}$		E_7
$\mathcal{I}$		E_8

McKay correspondence can be interpreted directly from the diagram $\widetilde{\Delta}(\Gamma)$. To each vertex of $\widetilde{\Delta}(\Gamma)$ one associates an irreducible representation of Γ, the trivial one V_0 corresponding to the vertex of the highest root. Recall that the tensor product $V_{\text{nat}} \otimes V_i$ decomposes as a direct sum of irreducible representations. Draw an edge between the vertices V_i and V_j if V_j appears in the decomposition of $V_{\text{nat}} \otimes V_i$. The process is illustrated with this example:



$$V_{\text{nat}} \otimes V_i = V_l \oplus V_k \oplus V_j \text{ and } m_{il} = m_{ik} = m_{ij} = 1.$$

One can also interpret McKay's result using the eigenvectors and eigenvalues of the Cartan matrix of $\widetilde{\Delta}(\Gamma)$. Let $\chi_{\text{nat}} : \Gamma \to \mathbb{C}$ (respectively χ_i) the character

$$\gamma \mapsto \text{Tr}(\rho_{\text{nat}}(\gamma))$$

associated to the representation $\rho_{\text{nat}} : \Gamma \to \text{GL}(V_{\text{nat}})$ of Γ on V_{nat} (respectively ρ_i on V_i). Let $\gamma_0 = 1, \gamma_1, \ldots, \gamma_r$ be representatives of the conjugacy classes of Γ. As characters are central functions, they are defined by their values on γ_j, $0 \leq j \leq r$. By putting all these values in a table one finds the character table of Γ:

	γ_0	γ_1	$\cdots$	γ_k	$\cdots$	γ_r
χ_0	1	1	$\cdots$	1	$\cdots$	1
χ_1	d_1	$\chi_1(\gamma_1)$				$\chi_1(\gamma_r)$
$\vdots$	$\vdots$					
χ_j	d_j			$\chi_j(\gamma_k)$		$\chi_j(\gamma_r)$
$\vdots$	$\vdots$					
χ_r	d_r			$\chi_r(\gamma_k)$		$\chi_r(\gamma_r)$

The formula $V_{\text{nat}} \otimes V_i = \bigoplus_{j=0}^{r} m_{ij} V_j$ gives us $\chi_{\text{nat}}(\gamma_k)\chi_i(\gamma_k) = \sum_{j=0}^{r} m_{ij}\chi_j(\gamma_k)$ for all $0 \leq$

$i, k \leq r$, which can be written $M \begin{pmatrix} \chi_0(\gamma_k) \\ \vdots \\ \chi_r(\gamma_k) \end{pmatrix} = \chi_{\text{nat}}(\gamma_k) \begin{pmatrix} \chi_0(\gamma_k) \\ \vdots \\ \chi_r(\gamma_k) \end{pmatrix}$. This can be interpreted as:

> The k-th column of the character table is an eigenvector of M (respectively $2I - M$) of eigenvalue $\chi_{\text{nat}}(\gamma_k)$ (respectively $2 - \chi_{\text{nat}}(\gamma_k)$).

In particular, as V_{nat} is of dimension 2, for $k = 0$, one finds that the column $(d_0, \ldots, d_r)$ is annihilated by $2I - M$, with $d_i = \dim V_i$, for $0 \leq i \leq r$. This is due to the fact that this column corresponds to the coefficients in the decomposition of the highest root θ in terms of the simple roots $\{\alpha_1, \ldots, \alpha_r\}$ of the root system associated to $\Delta(\Gamma)$. Indeed $2I - M = (\langle \alpha_i, \alpha_j \rangle)_{i,j} = ((\alpha_i, \alpha_j))_{i,j}$ because we work on A_r, D_r or E_r hence all the roots have the same length. Therefore $(2I - M) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_r \end{pmatrix} = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{r} d_j(\alpha_i, \alpha_j) = 0, \forall i.$

$$\Leftrightarrow \quad (\alpha_i, \sum_{j=0}^{r} d_j \alpha_j) = 0, \forall i.$$

$$\Leftrightarrow \quad \sum_{j=0}^{r} d_j \alpha_j = 0.$$

$$\Leftrightarrow \quad \theta = -\alpha_0 = \sum_{j=1}^{r} d_j \alpha_j.$$

This correspondence was obtained by J. McKay through explicit computation. R. Steinberg has since proved the result in a more abstract way in [Stein85].

1.5.1.2. Example

Let $\Gamma = \mathcal{I}$, $\Delta(\Gamma) = E_8$. As Γ is the preimage of $\mathfrak{A}_5$ by the morphism $f : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$, we first need to compute the character table of $\mathfrak{A}_5$.

$\mathfrak{A}_5$ is the rotation group of the dodecahedron, which is composed of 12 faces, 30 edges and 20 vertices. Each vertex of the dodecahedron has a stabilizer containing 3 elements: id, a rotation of angle $\frac{2\pi}{3}$ and a rotation of angle $-\frac{2\pi}{3}$. There are 1 element of order 1 and 2 elements of order 3. Furthermore, two opposite vertices have the same stabilizer.

Hence there are $10 \times 2 = 20$ elements of order 3.
In a similar fashion, each face has a stabilizer composed of 5 elements: id and rotations of angles $\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$. We have 1 element of order 1 and 4 elements of order 5. Furthermore, two opposite faces have the same stabilizer. We thus have $6 \times 4 = 24$ elements of order 5.
Each edge has a stabilizer composed of 2 elements: id and a rotation of angle π. We have 1 element of order 1 and 1 element of order 2. But two opposite edges have the same stabilizer so we have $15 \times 1 = 15$ elements of order 2.
This far, $\mathfrak{A}_5$ contains $1+20+24+15 = 60 = |\mathfrak{A}_5|$ elements. Hence we have all the elements of the group.

It is an obvious fact that two elements in the same orbit will have conjugated stabilizers. As the subgroups of order 2 are stabilizers of elements in the same orbit (the action of $\mathfrak{A}_5$ on the edges is transitive), these groups are conjugated. The same goes for the subgroups of order 3 (vertices) and of order 5 (faces).

Let us find the conjugacy classes of $\mathfrak{A}_5$.

1. As the subgroups of order 2 are conjugated, we deduce that all the elements of order 2 form one unique conjugacy class of order 15.

2. Let x be the rotation of angle $\frac{2\pi}{3}$ and of vertex v. x is conjugated to any rotation of angle $\frac{2\pi}{3}$ of another vertex. Let x' be the rotation of angle $\frac{2\pi}{3}$ and of vertex v', the opposite of v. x' is conjugated to x and $x' = x^2$. Thus all elements of order 3 are conjugated. They form a unique conjugacy class of order 20.

3. In a similar manner, rotations of angle $\frac{2\pi}{5}$ and $-\frac{2\pi}{5}$ form a conjugacy class of order 12.

4. Furthermore, rotations of angle $\frac{4\pi}{5}$ and $-\frac{4\pi}{5}$ form a conjugacy class of order 12. These last two classes cannot be conjugated because that would mean there exists a class of order $24 \nmid 60 = |\mathfrak{A}_5|$.

The class equation of $\mathfrak{A}_5$ is thus $60 = 1+15+20+12+12$. There are 5 conjugacy classes. One can check that $60 = 1^2 + 3^2 + 3^2 + 4^2 + 5^2$.

The conjugacy classes are:

Class	Order
$c_1 = \{1\}$	1
$c_2 = \{x \text{ rotation of } \pi\}$	15
$c_3 = \{y \text{ rotation of } \frac{2\pi}{3}, -\frac{2\pi}{3}\}$	20
$c_4 = \{z \text{ rotation of } \frac{2\pi}{5}, -\frac{2\pi}{5}\}$	12
$c_5 = \{z^2 \text{ rotation of } \frac{4\pi}{5}, -\frac{4\pi}{5}\}$	12

Let ρ_0 be the trivial representation and ρ_1 the representation of dimension 3 as a subgroup of $\mathrm{SO}_3(\mathbb{R})$. Denote ρ_2, ρ_3, ρ_4 the other irreducible representations and χ_i, $0 \leq i \leq 4$, the character of ρ_i.

	1	x	y	z	z^2
χ_0	1	1	1	1	1
χ_1	3	-1	0	α	β

with $\begin{cases} \alpha = 1 + 2\cos\frac{2\pi}{5} = \frac{1+\sqrt{5}}{2}, \\ \beta = 1 + 2\cos\frac{4\pi}{5} = \frac{1-\sqrt{5}}{2}. \end{cases}$

$\mathfrak{A}_5$ acts on the set $\{1,2,3,4,5\}$. Hence there exists a representation ρ' of dimension 5. One can see from the dimensions that there is a decomposition $\rho' = \rho_0 + \rho_3$. ρ' realizes x as an even element of order 2. It is a product of transpositions and thus fixes one element. So $\chi'(x) = 1$ and $\chi_3(x) = 0$. In a similar manner, one finds the values of χ_3 on the other conjugacy classes and obtains

	1	x	y	z	z^2
χ_3	4	0	1	-1	-1

$\mathfrak{A}_5$ permutes the six axes linking the opposite faces of the dodecahedron, so there is a representation ρ'' of dimension 6. It follows that $\rho'' = \rho_0 + \rho_4$. $\rho''(x)$ fixes two diagonals, implying $\chi''(x) = 2$ and $\chi_4(x) = 1$. The other values of χ_4 are obtained in a similar fashion. Eventually one finds

	1	x	y	z	z^2
χ_4	5	1	-1	0	0

$\mathfrak{A}_5$ is a normal subgroup of $\mathfrak{S}_5$ so we can define the conjugation σ by an element in $\mathfrak{S}_5 \setminus \mathfrak{A}_5$. σ is an automorphism of $\mathfrak{A}_5$ and σ permutes c_4 and c_5. Thus $\rho_1 \circ \sigma$ is an irreducible representation of dimension 3 but is not ρ_1. It has then to be ρ_2.
The action of σ permutes z and z^2. Hence $\begin{cases} \chi_2(z) = \chi_1(z^2) = \beta, \\ \chi_2(z^2) = \chi_1(z) = \alpha, \\ \chi_2(x) = \chi_1(x) = -1, \\ \chi_2(y) = \chi_1(y) = 0. \end{cases}$

The character table of $\mathfrak{A}_5$ is now entirely determined:

	1	x	y	z	z^2
χ_0	1	1	1	1	1
χ_1	3	-1	0	α	β
χ_2	3	-1	0	β	α
χ_3	4	0	1	-1	-1
χ_4	5	1	-1	0	0

Because the morphism f is a double cover of $\mathrm{SO}_3(\mathbb{R})$, the preimage of a conjugacy class from $\mathfrak{A}_5$ by the morphism f is one conjugacy class in Γ with order doubled or two classes of same order. 1 lifts to two classes: 1 and -1. c_3, c_4 and c_5 each transform in two classes and c_2 turns into one class twice its original size. There are 60 new elements divided as follows: $60 = 1+15+20+12+12$. There are 4 additional classes hence $60 = 2^2+2^2+4^2+6^2$. Therefore Γ has 9 conjugacy classes.

Γ is generated by $g_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^4-\eta & \eta^2-\eta^3 \\ \eta^2-\eta^3 & \eta-\eta^4 \end{pmatrix}$ and $g_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^2-\eta^4 & \eta^4-1 \\ 1-\eta & \eta^3-\eta \end{pmatrix}$, with $\eta = e^{\frac{2i\pi}{5}}$, and its conjugacy classes are:

Number of elements	1	1	30	20	20	12	12	12	12
Order of the elements	1	2	4	6	3	10	5	5	10
Name of the class	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
Representative	1	-1	$g_1^3g_2^3$	$g_1^2g_2^2$	$-g_1^2g_2^2$	g_1g_2	$(g_1g_2)^4$	$(g_1g_2)^2$	$(g_1g_2)^3$
Representative in $\mathfrak{A}_5$	1		x	y		z		z^2	

We remind the reader that a rotation of angle θ around the axis O_z can be written $\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$ in SU_2.

Let $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4$ denote the lifts of the characters of $\mathfrak{A}_5$ and ρ_8 the representation of Γ as a subgroup of SU_2. The corresponding characters are noted ψ_i.

1. One finds $\psi_8 = [2, -2, 0, 1, -1, \alpha, -\alpha, -\beta, \beta]$ and $(\psi_8, \psi_8) = 1$ implying that ψ_8 is irreducible.

2. One can check that $\psi_8\psi_2 = [6, -6, 0, 0, 0, \alpha\beta, -\alpha\beta, -\alpha\beta, \alpha\beta]$ thus $\rho_8 \otimes \rho_2$ is irreducible of dimension 6. Let us call it ρ_6.

3. A computation gives $\psi_8\psi_1 = [6, -6, 0, 0, 0, \alpha^2, -\alpha^2, -\beta^2, \beta^2]$. The only possible decompositions are then $\rho_8 \otimes \rho_1 = \rho_5 + \rho_7$ or $\rho_8 \otimes \rho_1 = \rho_7 + \rho_8$. If it is $\rho_7 + \rho_8$, then $\psi_7 = [4, -4, 0, -1, 1, 1, -1, -1, 1]$ which is irreducible. Hence we found ρ_7.

4. ψ_5 is obtained using the orthogonality of the columns of the character table.

The character table of Γ is:

	1	-1	$g_1^3g_2^3$	$g_1^2g_2^2$	$-g_1^2g_2^2$	g_1g_2	$(g_1g_2)^4$	$(g_1g_2)^2$	$(g_1g_2)^3$
ψ_0	1	1	1	1	1	1	1	1	1
ψ_1	3	3	-1	0	0	α	α	β	β
ψ_2	3	3	-1	0	0	β	β	α	α
ψ_3	4	4	0	1	1	-1	-1	-1	-1
ψ_4	5	5	1	-1	-1	0	0	0	0
ψ_5	2	-2	0	1	-1	β	$-\beta$	$-\alpha$	α
ψ_6	6	-6	0	0	0	-1	1	1	-1
ψ_7	4	-4	0	-1	1	1	-1	-1	1
ψ_8	2	-2	0	1	-1	α	$-\alpha$	$-\beta$	β

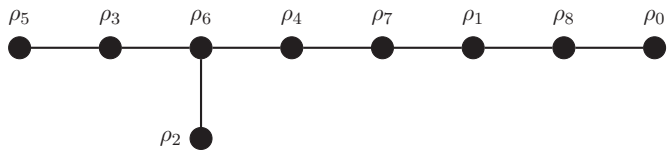
Let us look at the tensor products $\rho_8 \otimes \rho_i$, $0 \leq i \leq 8$.

- $\rho_8 \otimes \rho_0 = \rho_8$ because ρ_0 is trivial.

- $\psi_8\psi_1 = [6, -6, 0, 0, 0, \alpha^2, -\alpha^2, -\beta^2, \beta^2]$. By inversing the character table, we get $\rho_8 \otimes \rho_1 = \rho_7 + \rho_8$. With the same method it follows that

$$\begin{cases} \rho_8 \otimes \rho_2 = \rho_6, \\ \rho_8 \otimes \rho_3 = \rho_5 + \rho_6, \\ \rho_8 \otimes \rho_4 = \rho_6 + \rho_7, \\ \rho_8 \otimes \rho_5 = \rho_3, \\ \rho_8 \otimes \rho_6 = \rho_2 + \rho_3 + \rho_4, \\ \rho_8 \otimes \rho_7 = \rho_1 + \rho_4, \\ \rho_8 \otimes \rho_8 = \rho_0 + \rho_1. \end{cases}$$

The corresponding diagram is



which is indeed the Dynkin diagram of $\widetilde{E}_8$.

1.5.2. Inhomogeneous correspondence by restriction

1.5.2.1. Statement of the correspondence

In Subsection 1.4.2.1 we associated some finite subgroups $\Gamma \triangleleft \Gamma'$ of SU_2 to inhomogeneous Dynkin diagrams in the following way:

Inhomogeneous $\Delta(\Gamma,\Gamma')$	Homogeneous $\Delta(\Gamma)$	Γ	Γ'	Ω
$B_r,\ r \geq 2$	A_{2r-1}	$\mathcal{C}_{2r}$	$\mathcal{D}_r$	$\mathbb{Z}/2\mathbb{Z}$
$C_r,\ r \geq 3$	D_{r+1}	$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	$\mathcal{D}_2$	$\mathcal{O}$	$\mathfrak{S}_3$

(In the case of D_4 the group $\mathcal{O}$ can be replaced with the smaller group $\mathcal{T}$ and what follows remains valid. The difference will be $\Omega = \mathbb{Z}/3\mathbb{Z}$)

This association can be explained using group-theoretic arguments. The next theorem is proved in [Slo80] Appendix III.

Theorem 1.5.1. *Let $\Gamma \triangleleft \Gamma'$ be a pair of finite subgroups of SU_2 as in the previous table. By restriction, the irreducible representations of Γ' may be regarded as representations of Γ. Let $S_1, \ldots, S_r$ denote the equivalence classes (with respect to Γ) of these representations and let N be the natural representation of Γ as a subgroup of SU_2, which can be seen as the restriction of the natural representation of Γ'. It follows that the tensor product $N \otimes S_i$ decomposes as:*

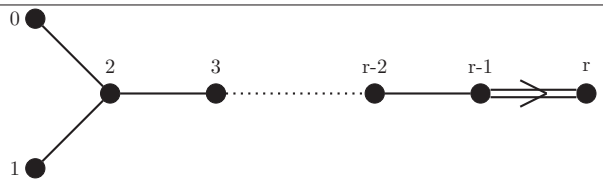
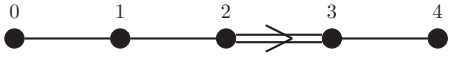
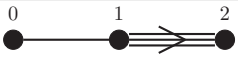
$$N \otimes S_i = \bigoplus_{j=1}^{r} b_{ji} S_j, \quad 1 \leq i \leq r,$$

which defines an $r \times r$ matrix $B = (b_{ij})_{1 \leq i,j \leq r}$. One verifies explicitly that the matrix

$$C = 2I - B$$

is the Cartan matrix of the extended Dynkin diagram $\widetilde{\Delta^\vee}(\Gamma,\Gamma')$ of the dual of $\Delta(\Gamma,\Gamma')$.

The diagrams obtained in this correspondence are given in the table below:

Γ	Γ'	Type of $\Delta(\Gamma,\Gamma')$	$\widetilde{\Delta^\vee}(\Gamma,\Gamma')$
$\mathcal{C}_{2r}$	$\mathcal{D}_r$	B_r	
$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	C_r	
$\mathcal{T}$	$\mathcal{O}$	F_4	
$\mathcal{D}_2$	$\mathcal{T}$	G_2	

In the following subsection the case $\Delta(\Gamma,\Gamma') = G_2$ will be explicitly computed.

1.5.2.2. Computations for $\Delta(\Gamma,\Gamma') = G_2$

Set $\Gamma = \mathcal{D}_2$, $\Gamma' = \mathcal{T}$, $\Omega = \mathbb{Z}/3\mathbb{Z}$ and $\Delta(\Gamma,\Gamma') = G_2$.

$\Gamma =< a, b >$ with $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\Gamma' =< \Gamma, c >$ with $c = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1} \\ -\epsilon & \epsilon \end{pmatrix}$ and $\epsilon = \exp(\frac{i\pi}{4})$.

The irreducible characters of $\mathcal{T}$ are

	1	-1	b	c	$-c$	c^2	$-c^2$
χ_0	1	1	1	1	1	1	1
χ_1	1	1	1	ω	ω	ω^2	ω^2
χ_2	1	1	1	ω^2	ω^2	ω	ω
χ_3	2	-2	0	1	-1	-1	1
χ_4	2	-2	0	ω	$-\omega$	$-\omega^2$	ω^2
χ_5	2	-2	0	ω^2	$-\omega^2$	$-\omega$	ω
χ_6	3	3	-1	0	0	0	0

with $\omega = \exp(\frac{2i\pi}{3})$. For any $0 \leq i \leq 6$, one denotes by $\widetilde{\chi}_i$ the restriction of χ_i to $\mathcal{D}_2$. It leads to

	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(b)$
$\widetilde{\chi}_0$	1	1	1
$\widetilde{\chi}_1$	1	1	1
$\widetilde{\chi}_2$	1	1	1
$\widetilde{\chi}_3$	2	-2	0
$\widetilde{\chi}_4$	2	-2	0
$\widetilde{\chi}_5$	2	-2	0
$\widetilde{\chi}_6$	3	3	-1

Set $\begin{cases} \psi_0 = \widetilde{\chi}_0 = \widetilde{\chi}_1 = \widetilde{\chi}_2 \\ \psi_1 = \widetilde{\chi}_3 = \widetilde{\chi}_4 = \widetilde{\chi}_5 \\ \psi_2 = \widetilde{\chi}_6 \end{cases}$. The ψ_i are the equivalence classes of the restrictions of the irreducible characters of $\mathcal{T}$ to $\mathcal{D}_2$. The restricted character table is thus

	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(b)$
ψ_0	1	1	1
ψ_1	2	−2	0
ψ_2	3	3	−1

Let ψ_{nat} be the restriction to $\mathcal{D}_2$ of χ_{nat}, the character of the representation of $\mathcal{T}$ as a subgroup of SU_2. The character table of ψ_{nat} is

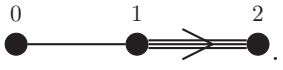
	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(b)$
ψ_{nat}	2	−2	0

Therefore $\begin{cases} \psi_{\text{nat}}\psi_0 = [2,-2,0] \\ \psi_{\text{nat}}\psi_1 = [4,4,0] \\ \psi_{\text{nat}}\psi_2 = [6,-6,0] \end{cases}$. By inversing the character table one obtains

$\psi_{\text{nat}}\psi_0$	=	ψ_1
$\psi_{\text{nat}}\psi_1$	=	$\psi_0 + \psi_2$
$\psi_{\text{nat}}\psi_2$	=	$3\psi_1$

.

The matrix $C = 2I - B$ introduced in Subsection 1.5.2.1 is then $C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$, and the corresponding diagram is



.

It is an extended Dynkin diagram of type $\widetilde{G_2} = \widetilde{G_2^\vee} = \widetilde{\Delta^\vee}(\Gamma, \Gamma')$.

If we take $\Gamma' = \mathcal{O}$ and $\Omega = \mathfrak{S}_3$, the restriction of the irreducible characters of Γ' to $\mathcal{D}_2$ are

	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(b)$
$\widetilde{\chi}_0$	1	1	1
$\widetilde{\chi}_1$	1	1	1
$\widetilde{\chi}_2$	2	−2	0
$\widetilde{\chi}_3$	2	−2	0
$\widetilde{\chi}_4$	2	2	2
$\widetilde{\chi}_5$	3	3	−1
$\widetilde{\chi}_6$	3	3	−1
$\widetilde{\chi}_7$	4	−4	0

Set $\begin{cases} \psi_0 = \widetilde{\chi}_0 \\ \psi_1 = \widetilde{\chi}_2 \\ \psi_2 = \widetilde{\chi}_5 \end{cases}$. The ψ_i are the equivalence classes of the restrictions of the irreducible characters of $\mathcal{O}$ to $\mathcal{D}_2$. The restricted character table is then

	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(b)$
ψ_0	1	1	1
ψ_1	2	−2	0
ψ_2	3	3	−1

It is the exact same table as in the case $\Gamma' = \mathcal{T}$. As the restriction of χ_{nat} of $\mathcal{O}$ to $\mathcal{D}_2$ is the same as the restriction of χ_{nat} of $\mathcal{T}$ to $\mathcal{D}_2$, one obtains the same result as before: the matrix C is the Cartan matrix of $\widetilde{\Delta^\vee}(\Gamma, \Gamma')$.

1.5.3. Inhomogeneous correspondence by induction

1.5.3.1. Statement of the correspondence

In the Subsection 1.5.2.1 a connection between pairs $\Gamma \triangleleft \Gamma'$ of finite subgroups SU_2 and inhomogeneous Dynkin diagrams using a restriction approach on the representations of Γ' was presented. A similar argument can be made using induction. The following result appears in [Slo80] Appendix III.

Theorem 1.5.2. *Let $\Gamma \triangleleft \Gamma'$ be a pair of finite subgroups of SU_2 as in the table in Subsection 1.5.2.1. By induction, one can study the Γ'-equivalence classes $Q_1, \ldots, Q_l$ of the representations of Γ' induced from the irreducible representations of Γ. Then $l = r$, r being the number of Γ-equivalence classes of the restrictions to Γ of the irreducible representations of Γ'. Furthermore, with a convenient ordering of the Q_i, there is the following decomposition formula:*

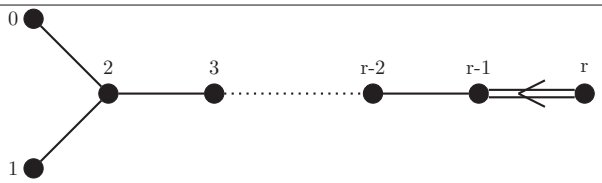
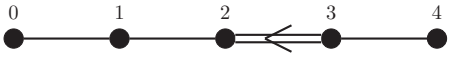
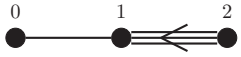
$$N \otimes Q_i = \bigoplus_{j=1}^{r} b_{ij} Q_j,\ 1 \leq i \leq r,$$

with N the natural representation of Γ' as a subgroup of SU_2. This decomposition leads to an $r \times r$ matrix ${}^tB = (b_{ji})_{1 \leq i,j \leq r}$, which is the transpose of the matrix found in Subsection 1.5.2.1. Therefore the matrix

$$C = 2I - {}^tB,$$

is the Cartan matrix of $(\widetilde{\Delta^\vee}(\Gamma, \Gamma'))^\vee$, the dual of the type obtained with the restriction argument.

The diagrams obtained in this correspondence are given in the table below:

Γ	Γ'	Type of $\Delta(\Gamma,\Gamma')$	$(\widetilde{\Delta^\vee}(\Gamma,\Gamma'))^\vee$
$\mathcal{C}_{2r}$	$\mathcal{D}_r$	B_r	
$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	C_r	
$\mathcal{T}$	$\mathcal{O}$	F_4	
$\mathcal{D}_2$	$\mathcal{T}$	G_2	

In the following subsection the case $\Delta(\Gamma,\Gamma') = G_2$ will be explicitly computed.

1.5.3.2. Computations for type $\Delta(\Gamma,\Gamma') = G_2$

Set $\Gamma = \mathcal{D}_2$, $\Gamma' = \mathcal{T}$, $\Omega = \mathbb{Z}/3\mathbb{Z}$ and $\Delta(\Gamma,\Gamma') = G_2$.

$\Gamma' =< r, b, c >$ and $\Gamma =< r, s >$ with $r = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $c = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1} \\ -\epsilon & \epsilon \end{pmatrix}$ with $\epsilon = \exp(\frac{i\pi}{4})$ and $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The natural representation of $\mathcal{T}$ as a subgroup of SU_2 is denoted N.

Set

$$\begin{cases} (\psi_i)_i \text{ the irreducible characters of } \mathcal{T}, \\ (\chi_j)_j \text{ the irreducible characters of } \mathcal{D}_2, \\ (\psi_i^{\downarrow})_i \text{ the restrictions of the } (\psi_i)_i \text{ to } \mathcal{D}_2, \\ (\chi_j^{\uparrow})_j \text{ the inductions of the } (\chi_j)_j \text{ to } \mathcal{T}, \end{cases}$$

and $(.,.)_{\mathcal{T}}$ (respectively $(.,.)_{\mathcal{D}_2}$) the scalar product of characters on $\mathcal{T}$ (respectively $\mathcal{D}_2$). Frobenius reciprocity formula (cf. [Ser78]) states

$$\chi_j^{\uparrow} = \sum_i (\psi_i, \chi_j^{\uparrow})_{\mathcal{T}} \psi_i = \sum_i (\psi_i^{\downarrow}, \chi_j)_{\mathcal{D}_2} \psi_i.$$

The computations of the $(\psi_i^{\downarrow})_i$ will lead us to the decomposition in irreducible components of the induced representations.

The irreducible characters of $\mathcal{T}$ are given by

	$Cl_{\mathcal{T}}(1)$	$Cl_{\mathcal{T}}(-1)$	$Cl_{\mathcal{T}}(b)$	$Cl_{\mathcal{T}}(c)$	$Cl_{\mathcal{T}}(c^2)$	$Cl_{\mathcal{T}}(-c)$	$Cl_{\mathcal{T}}(-c^2)$
ψ_0	1	1	1	1	1	1	1
ψ_1	1	1	1	j	j^2	j	j^2
ψ_2	1	1	1	j^2	j	j^2	j
ψ_3	2	-2	0	1	-1	-1	1
ψ_4	2	-2	0	j	$-j^2$	$-j$	j^2
ψ_5	2	-2	0	j^2	$-j$	$-j^2$	j
ψ_6	3	3	-1	0	0	0	0

with $j = \exp(\frac{2i\pi}{3})$.

It is known that $\mathcal{D}_2$ is the quaternion group and is generated by I, J, K with $I = r, J = s, K = rs$. The conjugacy classes of $\mathcal{D}_2$ are $\{1\}, \{-1\}, \{I, -I\}, \{J, -J\}, \{K, -K\}$.

If one looks at the restrictions of the conjugacy classes of $\mathcal{T}$ to $\mathcal{D}_2$, then the classes $Cl_{\mathcal{T}}(c)$, $Cl_{\mathcal{T}}(c^2)$, $Cl_{\mathcal{T}}(-c)$ and $Cl_{\mathcal{T}}(-c^2)$ disappear and $Cl_{\mathcal{T}}(b)$ splits into $\{I, -I\}, \{J, -J\}$, $\{K, -K\}$. The character table of the $\psi_i^{\downarrow}$ is then

	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(I)$	$Cl_{\mathcal{D}_2}(J)$	$Cl_{\mathcal{D}_2}(K)$
$\psi_0^{\downarrow} = \psi_1^{\downarrow} = \psi_2^{\downarrow}$	1	1	1	1	1
$\psi_3^{\downarrow} = \psi_4^{\downarrow} = \psi_5^{\downarrow}$	2	-2	0	0	0
$\psi_6^{\downarrow}$	3	3	-1	-1	-1

The character table of $\mathcal{D}_2$ is

	$Cl_{\mathcal{D}_2}(1)$	$Cl_{\mathcal{D}_2}(-1)$	$Cl_{\mathcal{D}_2}(I)$	$Cl_{\mathcal{D}_2}(J)$	$Cl_{\mathcal{D}_2}(K)$
χ_0	1	1	1	1	1
χ_1	1	1	1	−1	−1
χ_2	1	1	−1	−1	1
χ_3	1	1	−1	1	−1
χ_4	2	−2	0	0	0

The scalar product on the character space of $\mathcal{D}_2$ is defined by $(\varphi, \chi)_{\mathcal{D}_2} = \frac{1}{|\mathcal{D}_2|} \sum_{t \in \mathcal{D}_2} \varphi(t)\overline{\chi(t)}$ and the values of $(\psi_i^{\downarrow}, \chi_j)_{\mathcal{D}_2}$ are put in the following table:

	$\psi_0^{\downarrow} = \psi_1^{\downarrow} = \psi_2^{\downarrow}$	$\psi_3^{\downarrow} = \psi_4^{\downarrow} = \psi_5^{\downarrow}$	$\psi_6^{\downarrow}$
χ_0	1	0	0
χ_1	0	0	1
χ_2	0	0	1
χ_3	0	0	1
χ_4	0	1	0

Therefore $\begin{cases} \chi_0^{\uparrow} \cong \psi_0 + \psi_1 + \psi_2, \\ \chi_1^{\uparrow} = \chi_2^{\uparrow} = \chi_3^{\uparrow} \cong \psi_6, \\ \chi_4^{\uparrow} \cong \psi_3 + \psi_4 + \psi_5, \end{cases}$, so $\begin{cases} \widehat{X}_0 \cong Y_0 + Y_1 + Y_2, \\ \widehat{X}_1 = \widehat{X}_2 = \widehat{X}_3 \cong Y_6, \\ \widehat{X}_4 \cong Y_3 + Y_4 + Y_5, \end{cases}$ where the Y_i are the irreducible representations of $\mathcal{T}$ and the X_j are the irreducible representations of $\mathcal{D}_2$.

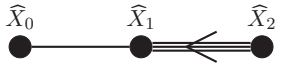
One can verify that $Y_3 = N$ and the character table of $\mathcal{T}$ gives $\begin{cases} Y_0 \otimes N \cong Y_3, \\ Y_1 \otimes N \cong Y_4, \\ Y_2 \otimes N \cong Y_5, \\ Y_3 \otimes N \cong Y_0 \oplus Y_6, \\ Y_4 \otimes N \cong Y_1 \oplus Y_6, \\ Y_5 \otimes N \cong Y_2 \oplus Y_6, \\ Y_6 \otimes N \cong Y_3 \oplus Y_4 \oplus Y_5, \end{cases}$

which implies $\begin{cases} \widehat{X}_0 \otimes N \cong (Y_0 + Y_1 + Y_2) \otimes N \cong Y_3 + Y_4 + Y_5 \cong \widehat{X}_4, \\ \widehat{X}_1 \otimes N \cong Y_6 \otimes N \cong Y_3 + Y_4 + Y_5 \cong \widehat{X}_4, \\ \widehat{X}_2 \otimes N \cong Y_6 \otimes N \cong Y_3 + Y_4 + Y_5 \cong \widehat{X}_4, \\ \widehat{X}_3 \otimes N \cong Y_6 \otimes N \cong Y_3 + Y_4 + Y_5 \cong \widehat{X}_4, \\ \widehat{X}_4 \otimes N \cong (Y_3 + Y_4 + Y_5) \otimes N \cong Y_0 + Y_1 + Y_2 + 3Y_6 \cong \widehat{X}_0 + 3\widehat{X}_1. \end{cases}$

The non-equivalent induced representations are $\widehat{X}_0, \widehat{X}_1$ and $\widehat{X}_4$. We reorder the indices via the permutation $1 \to 2,\ 4 \to 1,\ 0 \to 0$. The system then becomes

$$\begin{cases} \widehat{X}_0 \otimes N \cong \widehat{X}_1, \\ \widehat{X}_1 \otimes N \cong \widehat{X}_0 + 3\widehat{X}_2, \\ \widehat{X}_2 \otimes N \cong \widehat{X}_1. \end{cases}$$

The matrix introduced in Subsection 1.5.3.1 is then $C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$ whose diagram is



It is the extended Dynkin diagram of type $(\widetilde{G}_2)^\vee = (\widetilde{G_2^\vee})^\vee = (\widetilde{\Delta^\vee}(\Gamma, \Gamma'))^\vee$.

2. Symplectic geometry and moment map

2.1. Reminders on symplectic geometry and group actions

Let X be a smooth manifold.

Definition 2.1.1. *X is a **symplectic manifold** if it is equipped with a symplectic form ω, i.e. a non-degenerate, closed differential 2-form. This translates as:*

- $\forall x \in X,\ \omega_x : T_xX \times T_xX \to \mathbb{C}$ *is skew-symmetric and bilinear,*
- $d\omega = 0$,
- $\forall x \in X,$ *if there exists $u \in T_xX$ such that $\omega_x(u, v) = 0$ for all $v \in T_xX$, then $u = 0$.*

The symplectic manifold will be referred to as (X, ω).

Remark 2.1.2. The non-degeneracy of the symplectic form implies that any symplectic manifold is even-dimensional.

Let (X, ω) be a symplectic manifold and let $f : Y \to X$ be a morphism of manifolds. The **pullback** of ω by f, noted $f^*\omega$, is the 2-form on Y defined by

$$(f^*\omega)_y(v_1, v_2) = \omega_{f(y)}(d_y f(v_1), d_y f(v_2)), \quad \forall y \in Y, \forall v_1, v_2 \in T_yY.$$

Example 2.1.3. *Equip $\mathbb{R}^{2n}$ with a coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The form $\omega_{\mathbb{R}^{2n}}$ defined by*

$$\omega_{\mathbb{R}^{2n}} = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is a symplectic form and thus $\mathbb{R}^{2n}$ is a symplectic variety. The reader is reminded that the dx_i, dy_i's form the dual base of $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i})_{1 \le i \le n}$, base of the tangent space. One can also see $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ and thus as a complex variety. An elementary computation shows that the form becomes

$$\omega_{\mathbb{C}^n} = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z_j}.$$

A theorem due to G. Darboux states that any symplectic form is locally of the form of $\omega_{\mathbb{R}^{2n}}$.

It was stated in Remark 2.1.2 that a symplectic manifold is necessary even-dimensional. However the converse is not true, i.e. any variety of even dimension does not necessarily have a symplectic structure. An example is given below.

Example 2.1.4. *We are going to show that the sphere* S^{2n} *is not a symplectic manifold if* $n > 1$. *Let* (X, ω) *be a compact symplectic variety without boundary and of dimension* $2n$. *The 2-form* ω *is closed by definition and it defines a non-zero element in* $H^2_{dR}(X)$, *the second de Rahm cohomology group of* X. *Indeed, assume that there exists a 1-form* α *on* X *such that* $\omega = d\alpha$. *Then*

$$\int_X \omega^n = \int_X d(\alpha \wedge \omega^{n-1}) = \int_{\partial X} \alpha \wedge \omega^{n-1} = 0$$

using Stokes' theorem and the fact that $\partial X = \varnothing$. *But this is impossible because* ω^n *is a volume form and therefore not zero (it follows from the non-degeneracy of* ω*). Hence* ω *is not exact and its class* $[\omega]$ *is not zero in* $H^2_{dR}(X)$. *This implies that* $H^2_{dR}(X) \neq \{0\}$. *However a computation of the de Rahm cohomology of the sphere gives* $H^2_{dR}(S^{2n}) = \{0\}$ *if* $n > 1$. *Therefore, if* $n > 1$, *the sphere* S^{2n} *cannot be a symplectic manifold.*

Definition 2.1.5. *Let* (X, ω) *be a symplectic variety and* G *a Lie group. An action of* G *on* X *is defined by a smooth map* $\Phi : G \times X \to X$ *such that* $\Phi(g.h, x) = \Phi(g, \Phi(h, x))$ *and* $\Phi(e, x) = x$ *for* $g, h \in G, x \in X$ *and* e *the unit in* G. *Let* $\Phi_g(x) := \Phi(g, x)$. *The action of* G *is* ***symplectic*** *if* ω *is invariant by the pullback of the action, i.e.*

$$\omega(d\Phi_g(.), d\Phi_g(.)) = \omega(.,.), \quad \forall g \in G.$$

Definition 2.1.6. *A* ***vector field*** *on* X *is a section of the tangent bundle* $TX \to X$. *If a Lie group* G *acts on* X, *any* $\xi \in \mathfrak{g}$ *(the Lie algebra of* G*) induces a vector field* V_ξ *on* X *called* ***infinitesimal action*** *defined by*

$$V_\xi(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi).x$$

where $\exp : \mathfrak{g} \to G$ *is the exponential map of* G *defined using one-parameter subgroups of* G *(cf. [Helga01]).*

Remarks 2.1.7. 1. The exponential map $\exp : \mathfrak{g} \to G$ corresponds to the classical matrix exponential when G is a linear algebraic group.

2. When X is a Lie group acting on itself by left multiplication, this definition of a vector field on X coincides with the one given in Section 1.1.5.

The **Adjoint action** Ad of G on $\mathfrak{g}$ is defined as follows (as in Section 1.1.5):

$$\mathrm{Ad}_g(\xi) = \left.\frac{d}{dt}\right|_{t=0} g.\exp(t\xi).g^{-1}, \qquad \text{for } \xi \in \mathfrak{g}, g \in G.$$

For $g \in G$, the **coAdjoint action** $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ is given by

$$< \mathrm{Ad}^*_g(l), \xi >=< l, \mathrm{Ad}_{g^{-1}}(\xi) >, \quad \forall l \in \mathfrak{g}^*.$$

where $< ., . >$ is the dual pairing.

All necessary elements have been given in order to define a crucial notion for later reasonings: the moment map.

Definition 2.1.8. *Let (X, ω) be a symplectic variety with a symplectic G-action. A **moment map** is a morphism $\mu : X \to \mathfrak{g}^*$ such that*

(i) $d\mu_\xi = \omega(V_\xi, .)$,

(ii) $\mu(g.x) = \mathrm{Ad}^*_g(\mu(x)), \forall x \in X, g \in G$ *(equivariance of μ),*

where $\mu_\xi : \begin{array}{ccc} X & \to & \mathbb{C} \\ x & \mapsto & < \mu(x), \xi > \end{array}$ *for all $\xi \in \mathfrak{g}$.*

One notices that $\mu_\xi = i_\xi \circ \mu$ with $i_\xi(l) :=< l, \xi >$ for all $l \in \mathfrak{g}^*$, and that i_ξ is linear. Therefore $d_x\mu_\xi = d_x(i_\xi \circ \mu) = i_\xi \circ d_x\mu$. Hence for any $v_x \in T_xX$, one obtains

$$\begin{aligned} d_x\mu_\xi(v_x) &= i_\xi \circ d_x\mu(v_x), \qquad \text{according to the (i) condition.} \\ &= < d_x\mu(v_x), \xi >, \\ &= \omega_x(V_\xi(x), v_x), \end{aligned}$$

Example 2.1.9. *Let M be the vector space $\mathbb{C}^n$ equipped with the Hermitian product H given by*

$$H(z, \zeta) = \sum_{i=1}^{n} z_i \overline{\zeta_i},$$

for any $z, \zeta \in \mathbb{C}^n$. *Set* $\omega = \mathrm{Im}(H)$. *Then* ω *is a symplectic form on* $\mathbb{C}^n$. *By identifying every tangent space of* M *to* $\mathbb{C}^n$, *one identifies* ω *with a symplectic form on* M. *A system of real coordinates* $(x_i, y_i)_{1 \leq i \leq n}$ *on* M *is given by*

$$z_k = x_k + iy_k,$$

and the symplectic form can be written $\omega = \sum_{k=1}^{n} dx_k \wedge dy_k$.
Let $K := \mathrm{U}_n(\mathbb{C})$ *act on* M *naturally. By definition,* K *preserves* H *which means* $H(u.v_1, u.v_2) = H(v_1, v_2)$ *for any* $v_1, v_2 \in M, u \in K$. *The Lie algebra of* K *will be referred to as* $\mathfrak{k}$. *One computes*

$$\mathfrak{k} = \mathfrak{u}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid {}^t\overline{X} = -X\}.$$

Let us verify that the following map

$$\begin{array}{rccc} \mu: & M & \to & \mathfrak{k}^* \\ & m & \mapsto & \mu(m) \end{array}$$

with $\mu_X(m) := < \mu(m), X > = -\frac{i}{2} H(X.m, m)$, $\forall X \in \mathfrak{k}$, *is a moment map.*
The map is equivariant because K *preserves* H: *set* $g \in K, m \in M$ *and* $X \in \mathfrak{k}$. *Then* $< \mu(g.m), X > = -\frac{i}{2} H(X.g.m, g.m) = -\frac{i}{2} H(g^{-1} X g.m, m) = < \mu(m), \mathrm{Ad}_{g^{-1}} X >$
$= < \mathrm{Ad}_g^* \mu(m), X >$ *and therefore* $\mu(g.m) = \mathrm{Ad}_g^* \mu(m)$.
Set $X \in \mathfrak{k}, m \in M$ *and* $\xi \in T_m M \cong \mathbb{C}^n$. *By definition,*

$$d_m(\mu_X)(\xi) = -\frac{i}{2}(H(X\xi, m) + H(Xm, \xi)).$$

Furthermore it is known that if $X \in \mathfrak{k}$, *then its adjoint for* H *is* ${}^t\overline{X} = -X$. *Hence*

$$\begin{array}{rcl} H(X\xi, m) & = & -H(\xi, Xm), \\ & = & -\overline{H(Xm, \xi)}. \end{array}$$

It follows that

$$\begin{array}{rcl} -\frac{i}{2}(-\overline{H(Xm, \xi)} + H(Xm, \xi)) & = & -\frac{i}{2}(2i\omega(Xm, \xi)), \\ & = & \omega(Xm, \xi). \end{array}$$

Therefore $d_m(\mu_X)(\xi) = \omega(Xm, \xi)$*, which is the condition (i) in the definition of a moment map.*

The next example gives a hindsight on why a moment map is thus called.

Example 2.1.10. *The tangent bundle* $T^*\mathbb{R}^n$ *of* $\mathbb{R}^n$ *is identified with* $\mathbb{R}^n \times \mathbb{R}^n$ *through the inner product. The group* $\mathrm{SO}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid {}^tA = A^{-1}\}$ *acts on* $T^*\mathbb{R}$ *by*

$$A.(p,q) = (Ap, Aq) \text{ for any } (p,q) \in \mathbb{R}^n \times \mathbb{R}^n.$$

This action preserves the symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$*. The Lie algebra of* $\mathrm{SO}_n(\mathbb{R})$ *is* $\mathfrak{so}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid {}^tA = -A\}$*. A moment map*

$$\mu : T^*\mathbb{R}^n \to \mathfrak{so}_n^*$$

for the action of $\mathrm{SO}_n(\mathbb{R})$ *on* $T^*\mathbb{R}^n$ *is then given by*

$$\mu_A(p,q) := < \mu(p,q), A >= \langle p, Aq \rangle$$

with $\langle .,. \rangle$ *the inner product in* $\mathbb{R}^n$*,* $A \in \mathfrak{so}_n$ *and* $(p,q) \in T^*\mathbb{R}^n$*.*
When $n = 3$ *there is a bijective correspondence*

$$\begin{array}{ccc} \mathfrak{so}_3 & \to & \mathbb{R}^3 \\ A & \mapsto & \alpha \end{array}$$

given by

$$Av = \alpha \times v$$

for $v \in \mathbb{R}^3$ *(*$\times$ *is the cross product). This correspondence identifies* μ_A *with the angular momentum about the axis directed by* α*.*

The next lemma gives a condition on the action of a group Ω on a symplectic manifold (M, ω) equipped with an action of a Lie group G so that the associated moment map $\mu : M \to \mathfrak{g}^*$ is Ω-equivariant.

Lemma 2.1.11. *Let $\mu : M \to \mathfrak{g}^*$ be a moment map on a symplectic manifold (M, ω) with an action of a semisimple Lie group G. Assume a group Ω acts on M by symplectomorphisms and that Ω is a subgroup of the outer automorphism group of G. Furthermore assume that the action of G lifts to an action of $G \rtimes \Omega$. Then μ is Ω-equivariant.*

Proof. The action of Ω on G induces an action on the Lie algebra $\mathfrak{g}$ as well as on its dual $\mathfrak{g}^*$. If $\varpi \in \Omega$, let $\sigma_\varpi : M \to M$ and $\sigma_\varpi : \mathfrak{g}^* \to \mathfrak{g}^*$ denote the action of ϖ on M and $\mathfrak{g}^*$. The $\mathfrak{g}^* \times \mathfrak{g}$ pairing is denoted by $\langle ., . \rangle$. By definition of the moment map, it is known that for any $x \in M, v \in T_x M$ and $\xi \in \mathfrak{g}$,

$$\langle d_x \mu(v), \xi \rangle = \omega_x(V_\xi(x), v).$$

Hence

$$\begin{aligned} \langle d_{\sigma_\varpi(x)} \mu(d\sigma_\varpi(v)), \xi \rangle &= \omega_{\sigma_\varpi(x)}(V_\xi(\sigma_\varpi(x)), d\sigma_\varpi(v)), \\ &= \omega_x((d\sigma_\varpi)^{-1} V_\xi(\sigma_\varpi(x)), v) \text{ because } \sigma_\varpi \text{ preserves the symplectic form.} \end{aligned}$$

$$\begin{aligned} \text{But } (d\sigma_\varpi)^{-1} V_\xi(\sigma_\varpi(x)) = (d\sigma_\varpi)^{-1} \left.\tfrac{d}{dt}\right|_{t=0} \exp(t\xi).(\sigma_\varpi(x)) &= \left.\tfrac{d}{dt}\right|_{t=0} \sigma_{\varpi^{-1}} \exp(t\xi).(\sigma_\varpi(x)), \\ &= \left.\tfrac{d}{dt}\right|_{t=0} \exp(t\sigma_{\varpi^{-1}}(\xi)).(x), \\ &= V_{\sigma_{\varpi^{-1}}(\xi)}(x). \end{aligned}$$

So

$$\begin{aligned} \langle d_{\sigma_\varpi(x)} \mu(d\sigma_\varpi(v)), \xi \rangle &= \omega_x(V_{\sigma_{\varpi^{-1}}(\xi)}(x), v), \\ &= \langle d_x \mu(v), \sigma_{\varpi^{-1}}(\xi) \rangle, \\ &= \langle \sigma_\varpi \circ d_x \mu(v), \xi \rangle \text{ by the definition of the action on the dual space.} \end{aligned}$$

Thus $d_{\sigma_\varpi(x)} \mu \circ d\sigma_\varpi = \sigma_\varpi \circ d_x \mu$. It follows that $\mu(\varpi.x) = \varpi.\mu(x) + f_\varpi$, with f_ϖ an element of $\mathfrak{g}^*$ (we replaced σ_ϖ by $\varpi.$ in order to lighten the notation). Let us prove that $f_\varpi = 0$. For any $g \in G$, we have

$$
\begin{aligned}
f_\varpi &= \mu(\varpi.(g.x)) - \varpi.(\mu(g.x)),\\
&= \mu(\varpi.(g.x)) - \varpi.(g.\mu(x)) \text{ because } \mu \text{ is } G\text{-equivariant},\\
&= \mu((\varpi.g).(\varpi.x)) - \varpi.(g.\mu(x)) \text{ because as the action lifts to the semi-direct product}\\
&\quad \text{one finds that for any } x \in M \text{ and } (g,\varpi) \in G \rtimes \Omega,\ \varpi.(g.x) = (\varpi.g).(\varpi.x),\\
&= (\varpi.g).\mu(\varpi.x) - \varpi.(g.\mu(x)) \text{ because } \mu \text{ is } G\text{-equivariant},\\
&= (\varpi.g).(\varpi.\mu(x) + f_\varpi) - \varpi.(g.\mu(x)),\\
&= (\varpi.g).(\varpi.\mu(x)) + (\varpi.g).f_\varpi - \varpi.(g.\mu(x)),\\
&= \varpi.(g.\mu(x)) + (\varpi.g).f_\varpi - \varpi.(g.\mu(x)),\\
&= (\varpi.g).f_\varpi.
\end{aligned}
$$

Hence for any $g \in G$, we have $g.f_\varpi = f_\varpi$, implying that f_ϖ is a G-invariant element of $\mathfrak{g}^*$. But as G is semisimple, Theorems 1.1.21, 1.1.24 and 1.3.8 state that the ring $S(\mathfrak{g}^*)^G$ is generated by $r = \mathrm{rank}(G)$ homogeneous polynomials $f_1, \ldots, f_r$ such that the exponents $m_1, \ldots, m_r$ of $\mathfrak{g}$ verify $m_i = \deg f_i - 1$ for all i. But as $m_i \geq 1$ for all i (cf. [Hum90] Section 3.16, Lemma), it follows that the non-constant G-invariant elements of $S(\mathfrak{g}^*)$ are of degree at least 2. As f_ϖ is of degree 1, $f_\varpi = 0$. We thus found $\mu(\varpi.x) = \varpi.\mu(x)$ for any $x \in M$, $\varpi \in \Omega$, and so μ is Ω-equivariant. □

2.2. Symplectic reduction

In this section we study the Marsden-Weinstein quotient, which is the quotient of a symplectic manifold by a symplectic action.

Let (X, ω) be a symplectic manifold and G a compact Lie group acting symplectically on X. Let $\mu : X \to \mathfrak{g}^*$ be a moment map for this action. If $l \in \mathfrak{g}^*$, let G_l be the isotropy subgroup of G relative to l for the Ad^*-action defined by

$$G_l = \{g \in G \mid \mathrm{Ad}^*_g(l) = l\}.$$

The group G_l acts on $\mu^{-1}(l)$ because if $x \in \mu^{-1}(l)$ and $g \in G_l$ then $\mu(g.x) = \mathrm{Ad}^*_g(\mu(x)) = \mathrm{Ad}^*_g(l) = l$. The orbit space

$$X_l := \mu^{-1}(l)/G_l$$

is called **symplectic reduction**. In what follows it will be shown that, under suitable

assumptions, X_l is in fact a symplectic manifold. Before we can state this result, some notions remain to be defined.

Let $f : X \to Y$ be a morphism between two manifolds X and Y. A point $y \in Y$ is said to be a **regular value** of f if for any $x \in f^{-1}(y)$, the derivative $d_x f : T_x X \to T_{f(x)} Y$ is surjective.
An action of a Lie group G on X is **free** if for any $x \in X$, its stabilizer $G_x = \{g \in G \mid g.x = x\}$ is trivial. Furthermore, the action of G is **proper** if the map

$$\begin{array}{ccc} G \times X & \to & X \times X \\ (g, x) & \mapsto & (g.x, x) \end{array}$$

is proper, i.e. the preimage of any compact is compact. One can show that the action of a compact group is always proper (cf. [Bou71] Chapter III, §4, Proposition 2).

Lemma 2.2.1. *Let M be a smooth manifold and G a Lie group acting smoothly, freely and properly on M. Then the orbit space M/G has a natural structure of manifold such that the quotient map $\pi : M \to M/G$ is smooth. Furthermore, for any $m \in M$, there is an isomorphism*

$$T_{\pi(m)}(M/G) \cong (T_m M)/T_m(G.m).$$

Proof. The first assertion comes from the quotient manifold theorem which can be found in [Lee13] Theorem 21.10. According to the quotient manifold theorem, π is a smooth submersion and thus, for any $m \in M$, $d_m\pi : T_m M \to T_{\pi(m)}(M/G)$ is surjective. One can prove ([Lee13] Proposition 5.38) that $\operatorname{Ker} d_m\pi = T_m(G.m)$. It follows from the isomorphism theorem that $(T_m M)/T_m(G.m) = (T_m M)/(\operatorname{Ker} d_m\pi) \cong \operatorname{Im} d_m\pi = T_{\pi(m)}(M/G)$. □

The next theorem is proved in [MaWein74].

Theorem 2.2.2. (Marsden-Weinstein). *Let (X, ω) be a symplectic manifold, G a Lie group acting symplectically on X and $\mu : X \to \mathfrak{g}^*$ a moment map. Assume that $l \in \mathfrak{g}^*$ is a regular value of μ and that the isotropy group G_l under the Ad^* action on $\mathfrak{g}^*$ acts freely and properly on $\mu^{-1}(l)$. Then the following assertions are verified:*

(i) The symplectic reduction $X_l = \mu^{-1}(l)/G_l$ is a smooth manifold of dimension $\dim X - \dim G - \dim G_l$.

(ii) There exists a unique symplectic form ω^{red} on X_l such that $\pi_l^* \omega^{\mathrm{red}} = i_l^* \omega$ where $i_l : \mu^{-1}(l) \hookrightarrow X$ is the inclusion and $\pi_l : \mu^{-1}(l) \to \mu^{-1}(l)/G_l$ is the quotient map.

The proof of this theorem will require the following lemma:

Lemma 2.2.3. *Let $x \in \mu^{-1}(l)$. Then*

(i) $T_x(G_l.x) = T_x(G.x) \cap T_x(\mu^{-1}(l))$, and

(ii) $T_x(\mu^{-1}(l))$ *is the* ω*-orthogonal complement of* $T_x(G.x)$.

Proof. (i) Viewing the tangent space to Y in y as $T_yY = \{\gamma'(0) \mid \gamma :]-\epsilon, \epsilon[\to Y$ is C^1 and $\gamma(0) = y\}$, one sees

$$T_x(G.x) = \{V_\xi(x) \mid \xi \in \mathfrak{g}\},$$
$$T_x(G_l.x) = \{V_\xi(x) \mid \xi \in \mathfrak{g}_l\},$$

where $\mathfrak{g}_l$ is the Lie algebra of G_l. Hence the desired equality is equivalent to

$$V_\xi(x) \in T_x(\mu^{-1}(l)) \Longleftrightarrow \xi \in \mathfrak{g}_l.$$

It is known that the moment map μ is Ad^*-equivariant so $d_x\mu(V_\xi(x)) = V_\xi^*(\mu(x))$ where V_ξ^* is the vector field on $\mathfrak{g}^*$ induced by ξ. Indeed, let $\varphi_g : M \to M$ denote the action of $g \in G$ on M. Then the equivariance of μ implies $\mu \circ \varphi_{\exp(t\xi)} = \mathrm{Ad}^*_{\exp(t\xi)} \circ \mu$ for any $t \in \mathbb{R}$. By differentiating with respect to t at $t = 0$, one obtains the desired equality (cf. [AbMar78] Proposition 4.1.28). Hence

$$V_\xi(x) \in T_x(\mu^{-1}(l)) = \mathrm{Ker}\ d_x\mu \Longleftrightarrow V_\xi^*(l) = 0.$$

Then $V_\xi^*(l) = 0$ $\Leftrightarrow < V_\xi^*(l), x >= 0$ for all $x \in X$,
$\Leftrightarrow < \frac{d}{dt}\big|_{t=0} \mathrm{Ad}^*_{\exp(-t\xi)}(l), x >= 0$ for all $x \in X$,
$\Leftrightarrow \frac{d}{dt}\big|_{t=0} < l, \mathrm{Ad}_{\exp(t\xi)}x >= 0$ for all $x \in X$,
$\Leftrightarrow < l, [\xi, x] >= 0$ for all $x \in X$,
$\Leftrightarrow \mathrm{ad}^*_\xi(l) = 0$ because $< ., . >$ is non-degenerate,
$\Leftrightarrow \xi \in \mathfrak{g}_l$.

(ii) By definition it is known that

$$\omega_x(V_\xi(x), v_x) =< d_x\mu(v_x), \xi >.$$

$$\begin{aligned}
\text{Therefore}\quad v_x \in T_x(\mu^{-1}(l)) = \text{Ker}\ (d_x\mu) &\iff d_x\mu(v_x) = 0,\\
&\iff < d_x\mu(v_x), \xi >= 0,\ \forall \xi \in \mathfrak{g},\\
&\iff \omega_x(V_\xi(x), v_x) = 0,\ \forall \xi \in \mathfrak{g},\\
&\iff v_x \in (T_x(G.x))^\omega,
\end{aligned}$$

according to the equality verified by $T_x(G.x)$ at the beginning of this proof. □

Proof. (Marsden-Weinstein Theorem)
(i) The preimage theorem (cf. [Lee13] Corollary 5.14) implies that $\mu^{-1}(l)$ is a smooth manifold. As the action of G_l is free and proper, Lemma 2.2.1 states that X_l is a smooth manifold. Furthermore, as $l \in \mathfrak{g}^*$ is regular, the map $d_x\mu$ is surjective for any $x \in \mu^{-1}(l)$ hence $\dim\ \mu^{-1}(l) = \dim\ \text{Ker}(d_x\mu) = \dim\ T_xX - \dim\ \text{Im}(d_x\mu) = \dim\ X - \dim\ \mathfrak{g} = \dim\ X - \dim\ G$. Finally $\dim\ X_l = \dim\ \mu^{-1}(l) - \dim\ G_l = \dim\ X - \dim\ G - \dim\ G_l$.
(ii) For $v_x \in T_x(\mu^{-1}(l))$ let $[v_x]$ denote the equivalence class of v_x in $T_x(\mu^{-1}(l))/T_x(G_l.x) \cong T_x(\mu^{-1}(l)/G_l)$. The relation we aim to prove is

$$\pi_l^*\omega^{\text{red}} = i_l^*\omega \iff \omega^{\text{red}}([v],[w]) = \omega(v,w), \quad \forall v, w \in T_x(\mu^{-1}(l)).$$

Because of Lemma 2.2.3 (ii), ω^{red} is a well defined 2-form. Additionally, π_l is a surjective submersion thus admits locally a smooth section σ_l at every point of its image. Therefore, by definition of ω^{red}, one verifies that $\omega^{\text{red}} = \sigma_l^*\omega$. Hence ω^{red} is a composition of smooth maps and is thus smooth.
Let us show that ω^{red} is closed. By definition ω is symplectic and so $d(\pi_l^*\omega^{\text{red}}) = d(i_l^*\omega) = i_l^*(d\omega) = 0$. Hence $\pi_l^*(d\omega^{\text{red}}) = 0$. As π_l is a submersion, $d\pi_l$ is surjective and one concludes that $d\omega^{\text{red}} = 0$.
For the non-degeneracy of ω^{red}, assume that there exists $v \in T_x(\mu^{-1}(l))$ such that $\omega^{\text{red}}([v],[w]) = 0$ for any $w \in T_x(\mu^{-1}(l))$. It implies that $\omega(v,w) = 0$ for any $w \in T_x(\mu^{-1}(l))$ and thus $v \in (T_x(\mu^{-1}(l)))^\omega = T_x(G.x)$ according to Lemma 2.2.3. So $[v] = 0$ and ω^{red} is non-degenerate. □

Example 2.2.4. *Let us rework the example of $\mathbb{C}^n$ equipped with a Hermitian form H. The group $\mathrm{U}_n(\mathbb{C})$ acts naturally on $\mathbb{C}^n$ and it has been seen before that there is a moment map $\mu : \mathbb{C}^n \to \mathfrak{u}_n(\mathbb{C})^*$ given by*

$$\mu_X(v) = -\tfrac{i}{2}H(Xv, v),$$

for $v \in \mathbb{C}^n$ and $X \in \mathfrak{u}_n(\mathbb{C})$.
Consider the circle $S^1 = \{z\mathrm{Id}_n \in \mathrm{U}_n(\mathbb{C}) \mid z \in \mathbb{C} \text{ and } |z| = 1\}$. As it is a subgroup of $\mathrm{U}_n(\mathbb{C})$, it can be shown that its action on $\mathbb{C}^n$ leads to a moment map which is the compose of the moment map μ with the transpose of the inclusion $\mathrm{Lie}(S^1) = 2i\pi\mathbb{R} \hookrightarrow \mathfrak{u}_n(\mathbb{C})$. Define $1^* \in Lie(S^1)^*$ such that

$$< 1^*, 2i\pi >= 1.$$

It follows that

$$\begin{aligned}\mu^{-1}(1^*) &= \{v \in \mathbb{C}^n \mid \mu_{2i\pi}(v) = 1\},\\ &= \{v \in \mathbb{C}^n \mid \pi H(v,v) = 1\},\\ &= \{v \in \mathbb{C}^n \mid \|v\|^2 = \tfrac{1}{\pi}\}.\end{aligned}$$

Hence $\mu^{-1}(1^*)$ is a sphere of radius $\sqrt{\frac{1}{\pi}}$ in $\mathbb{C}^n$, and is thus homeomorphic to S^{2n-1}. If $\lambda \in S^1$ fixes $v \in S^{2n-1}$, it means that $\lambda = e^{2i\pi k}$ with $k \in \mathbb{Z}$. One deduces that the stabilizer of v is discrete (in fact, finite) and so 1^* is a regular value of μ. One applies Theorem 2.2.2 and obtains that $\mu^{-1}(1^*)/S^1 \cong S^{2n-1}/S^1 \cong \mathbb{P}(\mathbb{C}^n)$ is a symplectic manifold. The symplectic form obtained by reduction on $\mathbb{P}(\mathbb{C}^n)$ is called the ***Fubini-Study form****. One notices that for $n = 2$, the result is $S^3/S^1 \cong \mathbb{P}^1$, which is the Hopf fibration.*

It was shown that a symplectic reduction has a symplectic structure. In the next section, a particular symplectic reduction will be investigated, namely the one coming from $\mu^{-1}(0)$.

2.3. Kempf-Ness theorem

The aim of this section is to present a result of G. Kempf and L. Ness that will prove the existence of an isomorphism between a symplectic reduction and a GIT quotient. Some preliminary definitions are necessary.

2.3.1. Kähler quotients

Definition 2.3.1. *Let M be a smooth manifold.*

1. *The* ***Lie bracket*** *$[X,Y]$ of two vector fields X and Y on M is defined as the vector field on M such that*
$$[X,Y](f) = X(Y(f)) - Y(X(f)) \text{ for any } f \in C^\infty(M).$$
2. *An* ***almost complex structure*** *J on M is a linear map $J : TM \to TM$ on the tangent bundle of M such that*
$$J^2 = -\mathrm{id}.$$

Locally, this implies that for any $p \in M$ there is an endomorphism $J_p : T_pM \to T_pM$ which satisfies $J_p^2 = -\mathrm{id}|_{T_pM}$ and which depends smoothly on $p \in M$.

3. *Assume M is endowed with an almost complex structure. A complex vector field Z on M is **holomorphic** if $J(Z) = iZ$.*
4. *Assume M is equipped with an almost complex structure J. The almost complex structure is **integrable** if the Lie bracket of any two holomorphic vector fields is again a holomorphic vector field.*

We can now define the main object of this part.

Definition 2.3.2. *A **Kähler manifold** X is a symplectic manifold (X, ω) with an integrable almost complex structure J which is compatible with the complex structure in the sense that the bilinear form $\omega(., J.)$ is symmetric and positive definite.*

More information on Kähler manifolds can be found in [McD99].

Let (X, ω, J) be a compact Kähler manifold and G a reductive complex Lie group acting symplectically on X. Because G is reductive, it can be seen as the complexification of $G_{\mathbb{R}}$, a maximal compact subgroup. Assume that the action of $G_{\mathbb{R}}$ on X preserves the symplectic form ω (one can eventually replace ω by $\int_{G_{\mathbb{R}}} \omega$) and assume there exists $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$ a moment map associated to the action of $G_{\mathbb{R}}$ on X. $\mathfrak{g}_{\mathbb{R}}^*$ is a vector space on which a $G_{\mathbb{R}}$-invariant inner product and thus a norm $\|.\|$ is fixed. Set

$$\begin{array}{rccc} f: & X & \to & \mathbb{R} \\ & x & \mapsto & \|\mu(x)\|^2 \end{array} .$$

For any $x \in X$, the path of steepest descent of f from x is the trajectory $\{x_t,\ t \geq 0\}$ of $-\mathrm{grad}(f)$ and let

$$\zeta(x) = \{y \in X \mid \text{every neighbourhood of } y \in X \text{ contains points } x_t \text{ for } t \text{ arbitrarily large}\}$$

be the set of limit points of the trajectory starting from x. Let X_{min} denote the subspace of X composed of the $x \in X$ such that $\zeta(x) \bigcap \mu^{-1}(0) \neq \varnothing$. X_{min} will play a preponderant role in this section.

Lemma 2.3.3. $G = G_{\mathbb{R}} \exp(i\mathfrak{g}_{\mathbb{R}})$.

Proof. The proof is based on the polar decomposition of G and on the universal complexification of $G_{\mathbb{R}}$. The details can be found in [HiNe12]. □

Remark 2.3.4. The **universal complexification** $G_{\mathbb{C}}$ of a real Lie group $G_{\mathbb{R}}$ is a pair $(\eta_{G_{\mathbb{R}}}, G_{\mathbb{C}})$ of a complex Lie group $G_{\mathbb{C}}$ and a morphism $\eta_{G_{\mathbb{R}}} : G_{\mathbb{R}} \to G_{\mathbb{C}}$ such that, for every homomorphism $\alpha : G_{\mathbb{R}} \to H$ to a complex Lie group H, there exists a unique holomorphic homomorphism $\alpha_{\mathbb{C}} : G_{\mathbb{C}} \to H$ with $\alpha_{\mathbb{C}} \circ \eta_{G_{\mathbb{R}}} = \alpha$. This is called the universal property of $G_{\mathbb{C}}$.

Lemma 2.3.5. *If $x \in \mu^{-1}(0)$ then $G.x \bigcap \mu^{-1}(0) = G_{\mathbb{R}}.x$.*

Proof. Let $g \in G$ such that $g.x \in \mu^{-1}(0)$. We aim to show that there exists $k \in G_{\mathbb{R}}$ such that $g.x = k.x$. Because μ is $G_{\mathbb{R}}$-equivariant, it follows that $G_{\mathbb{R}}$ acts on $\mu^{-1}(0)$. By Lemma 2.3.3 one can assume that $g = \exp(iz)$ with $z \in \mathfrak{g}_{\mathbb{R}}$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) =< \mu(\exp(izt).x), z >$. Then $h(0) = h(1) = 0$ because x and $\exp(iz).x = g.x$ both lie in $\mu^{-1}(0)$. As h is a non-zero continuous function, it follows that there exists $t \in [0,1]$ such that

$$0 = h'(t) =< d_y\mu(iV_z(y)), z >= -\omega_y(iV_z(y), V_z(y)) = -\langle V_z(y), V_z(y)\rangle$$

where $y = \exp(izt).x$ and $\langle ., . \rangle := \omega(i., .)$ is the metric induced by the Kähler structure. Hence $V_z(y) = 0$ so that $\exp(iz\mathbb{R})$ fixes y and also x. Therefore $g.x = \exp(iz).x = x \in G_{\mathbb{R}}x$ and the proof is complete. □

The proof of the following lemma can be found in [Kir84].

Lemma 2.3.6. *Set x and y in $\mu^{-1}(0)$ such that $x \notin G_{\mathbb{R}}.y$. Then there exists disjoint G-stable neighbourhoods of x and y in X.*

The theorem we were aiming for can now be proved. It was first proved in [KemNes79].

Theorem 2.3.7. (Kempf-Ness). *Let X be a compact Kähler manifold on which acts G, the complexification of a maximal compact subgroup $G_{\mathbb{R}}$. Assume that $G_{\mathbb{R}}$ preserves the Kähler form ω (in this context, the symplectic form is taken to be the Kähler form) and write $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$ for a moment map associated to this action. Suppose that for any $x \in \mu^{-1}(0)$, the stabiliser $(G_{\mathbb{R}})_x$ is finite. Then $X_{min} = G.\mu^{-1}(0)$ and there is a homeomorphism $\mu^{-1}(0)/G_{\mathbb{R}} \longrightarrow X_{min}/G$ between both orbit spaces.*

Proof. The minimal value that the function $\|\mu\|^2$ can take is 0 implying that any point of $\mu^{-1}(0)$ is in X_{min}. Furthermore, it is shown in [Kir84] that X_{min} is G-stable. It follows that $G.\mu^{-1}(0) \subset X_{min}$. Conversely if $x \in X_{min}$ then there exists $y \in \mu^{-1}(0)$ lying in the closure of the path of steepest descent for $\|\mu\|^2$ from x. Following [Kir84], this path is included in $G.x$, hence $y \in \overline{G.x}$. One deduces that $G.y \subset \overline{G.x}$ and so either $y \in G.x$ or $\dim G.y < \dim G.x$. However $(G_{\mathbb{R}})_y$ is supposed to be finite, and as $G = G_{\mathbb{R}} \exp(i\mathfrak{g}_{\mathbb{R}})$, it

follows that $\dim G_y = 0$, because G_y is the complexification of $(G_{\mathbb{R}})_y$ (cf. [Sze14] Lemma 5.24). Therefore $\dim G.y = \dim G \geq \dim G.x$. One sees that $y \in G.x$ and so $x \in G.\mu^{-1}(0)$. This proves the first assertion.
There is an injection $\mu^{-1}(0) \hookrightarrow X_{min}$ as well as a surjection $X_{min} \to X_{min}/G$. The composition of both these maps and the quotient of $\mu^{-1}(0)$ by $G_{\mathbb{R}}$ gives a map $\mu^{-1}(0)/G_{\mathbb{R}} \longrightarrow X_{min}/G$. Additionally, according to the first assertion and the structure of G, the map is surjective. Lemma 2.3.5 implies that this map is injective and Lemma 2.3.6 that the space X_{min}/G is Hausdorff. Thus the map is a continuous bijection from a compact space to a Hausdorff space and therefore it is a homeomorphism. □

The preceding theorem establishes a link between Kähler quotients and algebraic quotients and will help to show the existence of a link between the symplectic reduction $\mu^{-1}(0)/G_{\mathbb{R}}$ and the GIT quotient $X//G$, assuming that X has an algebraic structure. We first need to define the notion of stability for the manifold X.

Until the end of this subsection the Kähler manifold X is a non-singular complex projective variety in $\mathbb{P}^n$ and G is a connected reductive complex group acting linearly on X, i.e. via a morphism $G \to \mathrm{GL}_{n+1}(\mathbb{C})$.

Definition 2.3.8. *Let $X \subset \mathbb{P}^n$ be a complex non-singular projective variety equipped with a G-action (i.e. G acts on $\mathbb{P}^n$ and stabilizes X) and x a point in X. x is **semi-stable** if there exists a polynomial function $F \in \mathbb{C}[X_0, \dots, X_n]$ on X that is homogeneous, non-constant, G-invariant and such that $F(x) \neq 0$. x is **stable** if it is semi-stable, i.e. there exists F a homogeneous G-invariant polynomial such that $F(x) \neq 0$, all orbits of G contained in*

$$X_F := \{y \in X \mid F(y) \neq 0\}$$

are closed in X_F and G_x is finite.
Let $X^s \subset X^{ss}$ denote respectively the sets of stable and semi-stable elements of X.

The following theorem can be found in various references, like [Kir84] or [Mum94].

Theorem 2.3.9. *Set $X \subset \mathbb{P}^n$ a complex non-singular projective variety and G a complex reductive algebraic group acting on X via a morphism $\varphi : G \to \mathrm{GL}_{n+1}(\mathbb{C})$. Assume that G has a maximal compact subgroup $G_{\mathbb{R}}$ such that $\varphi(G_{\mathbb{R}}) \subset \mathrm{U}_{n+1}(\mathbb{C})$. Then the set X^{ss} coincides with the set X_{min} for the function $\|\mu\|^2$ where $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$ is a moment map for the action of $G_{\mathbb{R}}$ and $\| \cdot \|$ is the norm associated to any $G_{\mathbb{R}}$-invariant inner product on $\mathfrak{g}_{\mathbb{R}}$.*

Let us place ourselves in the context of the previous theorem. The inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ induces a rational map $X \to X//G = \mathrm{Proj}\ \mathbb{C}[X]^G$. Its domain of definition is X^{ss}

and it gives a G-invariant surjective morphism $\psi : X^{ss} \to X//G$.
Assume now that G_x is finite for any $x \in X$ semi-stable. Then if $x \in X^{ss}$, there exists $F \in \mathbb{C}[X]^G$ such that $F(x) \neq 0$. As all elements in X_F are semi-stable, the stabilizer of each one of these elements is finite and so its orbit is of same dimension as G. This implies that all orbits in X_F are closed. Indeed, it is known that $\overline{G.x} \setminus G.x$ is a union of orbits of dimension strictly smaller that dim $G.x$ (cf. Proposition 3.3.3). As all orbits have the same dimension, $\overline{G.x} \setminus G.x = \varnothing$ and $G.x$ is closed. Therefore x is stable and one finds $X^{ss} = X^s$. However according to [Mum94] Theorem 1.10, every fiber of ψ that meets $X^s = X^{ss}$ is a unique G-orbit. It follows that there exists a continuous bijection $\widetilde{\psi} : X^{ss}/G \to X//G$. Because X^{ss}/G and $X//G$ are compact and Hausdorff, one deduces that $\widetilde{\psi}$ is a homeomorphism. By means of Theorem 2.3.9 and Theorem 2.3.7, one obtains the following proposition:

Proposition 2.3.10. *With the hypotheses of the last paragraph, there exists a homeomorphism*

$$\mu^{-1}(0)/G_{\mathbb{R}} \longrightarrow X//G$$

between the symplectic reduction and the GIT quotient.

Remark 2.3.11. A theorem due to W.L. Chow states that an analytic subspace of complex projective space that is closed (in the ordinary topological sense) is an algebraic subvariety. Thus if we suppose our manifold X to be analytic, non-singular and closed in some $\mathbb{P}^n$, then it is algebraic and non-singular and Proposition 2.3.10 remains valid.

2.3.2. Hyperkähler quotients

The last proposition of the previous subsection can be extended to another class of manifold, namely hyperkähler manifolds.

Definition 2.3.12. *Let M be a Riemanniann manifold with a metric g. M is a **hyperkähler manifold** if M is equipped with three complex structures I, J, K such that*

- *g is Kähler for I, J and K.*
- *I, J and K verify the quaternionic relations, i.e. $I^2 = J^2 = K^2 = IJK = -1$.*

More information on hyperkähler manifolds can be found in [Dan99].

Let M be a simply-connected hyperkähler manifold on which a compact Lie group $G_{\mathbb{R}}$ acts freely. Assume that the action of $G_{\mathbb{R}}$ preserves the symplectic forms induced be the complex structures I, J and K. Because M is simply-connected, there exist

three moment maps $\mu_I, \mu_J, \mu_K : M \to \mathfrak{g}_{\mathbb{R}}^*$. One can regroup these maps to construct a hyperkähler moment map

$$\begin{array}{rccl} \mu_{\mathbb{H}} : & M & \to & \mathfrak{g}_{\mathbb{R}}^* \otimes \mathbb{H}_0 \\ & x & \mapsto & \mu_I(x) \otimes I + \mu_J(x) \otimes J + \mu_K(x) \otimes K. \end{array}$$

with $\mathbb{H}_0 = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ the set of pure imaginaries.

Proposition 2.3.13. *The quotient $\mu_{\mathbb{H}}^{-1}(0)/G_{\mathbb{R}}$ is a hyperkähler manifold.*

This proposition is proved in [Hi87]. It can also be extended to a whole family of quotients. Let $Z = \mathfrak{z}(\mathfrak{g}_{\mathbb{R}})^*$ denote the dual of the center of $\mathfrak{g}_{\mathbb{R}}$.

Theorem 2.3.14. *Assume $G_{\mathbb{R}}$ acts freely on $m \in \mu_{\mathbb{H}}^{-1}(\zeta)$, with $\zeta = (\zeta_I, \zeta_J, \zeta_K) \in Z \otimes \mathbb{H}_0$. Then the quotient $\mu_{\mathbb{H}}^{-1}(\zeta)/G_{\mathbb{R}}$ is non-singular in $\pi(m)$ and is a hyperkähler manifold of dimension* $\dim M - 4\dim G_{\mathbb{R}}$*, with $\pi : \mu_{\mathbb{H}}^{-1}(\zeta) \to \mu_{\mathbb{H}}^{-1}(\zeta)/G_{\mathbb{R}}$ the canonical projection.*

Proof. cf. [Hi87] Sections 3.B, 3.C, 3.D. □

Assume that M has a complex vector space structure and let N be an affine algebraic subvariety of M stable by the action of G, the complexification of $G_{\mathbb{R}}$. The next theorem links hyperkähler quotient and GIT quotient.

Theorem 2.3.15. *The composition $\mu_I^{-1}(0) \bigcap N \hookrightarrow N$ with the affine algebraic quotient $\pi : N \to N//G$ induces a homeomorphism*

$$(\mu_I^{-1}(0) \bigcap N)/G_{\mathbb{R}} \xrightarrow{\cong} N//G.$$

Proof. cf. [RS90] Theorem 7.7. □

One can also construct the following moment map:

$$\begin{array}{rccl} \mu_{\mathbb{C}} : & M & \to & \mathfrak{g}^* \otimes \mathbb{C} \\ & x & \mapsto & \mu_J(x) + i\mu_K(x). \end{array}$$

with $\mathfrak{g}$ the Lie algebra of the complexification G of $G_{\mathbb{R}}$. It is shown in [Hi87] that $\mu_{\mathbb{C}}$ is a moment map for the action of G on M equipped with the form $\omega_{\mathbb{C}} = \omega_J + i\omega_K$.

Corollary 2.3.16. *By setting $N = \mu_{\mathbb{C}}^{-1}(Z \otimes \mathbb{C})$ in Theorem 2.3.15, there is a homeomorphism*

$$(\mu_I^{-1}(0) \bigcap \mu_{\mathbb{C}}^{-1}(Z \otimes \mathbb{C}))/G_{\mathbb{R}} \overset{\cong}{\to} \mu_{\mathbb{C}}^{-1}(Z \otimes \mathbb{C})//G.$$

Hyperkähler manifolds and simple singularities are linked. Indeed, using Hilbert schemes of points one can prove that the minimal resolutions of the simple singularities have a hyperkähler structure (cf. [Naka99]). Furthermore, in the next chapter, a construction by P.B. Kronheimer will be presented in which he constructs the semiuniversal deformations of the simple singularities with aid of hyperkähler quotients.

3. Quivers

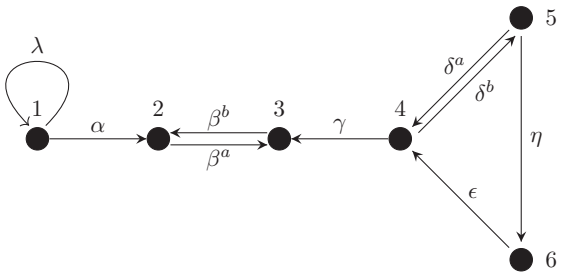
The first three sections of this chapter are largely based on the lecture notes by M. Brion [Brion00] and the ones by W. Crawley-Boevey [Crawley].

3.1. Definitions and basic properties

Let $\Bbbk$ be an algebraically closed field.

Definition 3.1.1. *A **quiver** is a quadruplet $Q = (Q_0, Q_1, s, t)$ with Q_0 a set of vertices, Q_1 a set of arrows between the vertices and $s, t : Q_1 \to Q_0$ two maps that give to each arrow a source and a target respectively. We shall denote the vertices by letters $i, j, \ldots$. The arrow α such that $s(\alpha) = i$ and $t(\alpha) = j$ will be written $\alpha : i \to j$. There can be several arrows between two vertices. An **edge loop** in the quiver Q is an edge α with $s(\alpha) = t(\alpha)$. An **oriented cycle** in Q is a succession of edges $\alpha_1, \ldots, \alpha_k$ such that $t(\alpha_1) = s(\alpha_2), t(\alpha_2) = s(\alpha_3), \ldots, t(\alpha_k) = s(\alpha_1)$. Edges loops and oriented cycles are allowed as well.*

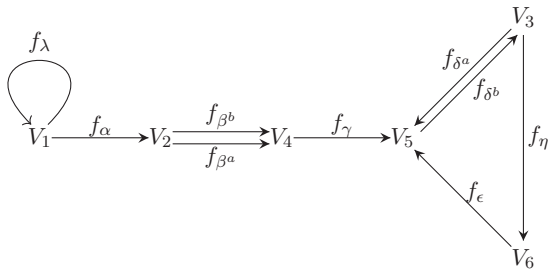
Example 3.1.2. *The reader will find below an example of a quiver:*



All through this chapter it will be assume that for a quiver Q, the sets of vertices Q_0 and edges Q_1 are finite.

Definition 3.1.3. *A **representation** $V = ((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$ of a quiver Q is given by a family of $\Bbbk$-vector spaces V_i indexed by the vertices $i \in Q_0$ as well as a family of linear maps $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ indexed by the arrows $\alpha \in Q_1$.*

Example 3.1.4. *A representation of the quiver in Example 3.1.2 will have the form*



with the V_i's being vector spaces and the f_*'s linear maps.

Definition 3.1.5. *Let $M = ((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ and $N = ((W_i)_{i\in Q_0}, (g_\alpha)_{\alpha\in Q_1})$ be two representations of a quiver Q. A **morphism** $u : M \to N$ between two quiver representations is a collection of maps $u_i : V_i \to W_i$ indexed by $i \in Q_0$ such that the following diagram commutes:*

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{f_\alpha} & V_{t(\alpha)} \\ \downarrow{\scriptstyle u_{s(\alpha)}} & & \downarrow{\scriptstyle u_{t(\alpha)}} \\ W_{s(\alpha)} & \xrightarrow[g_\alpha]{} & W_{t(\alpha)} \end{array}$$

for all $\alpha \in Q_1$.

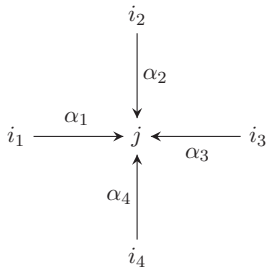
The composition of two morphisms between quiver representations is defined naturally and thus the category $\mathrm{Rep}(Q)$ of the representations of the quiver Q can be defined. One can verify that this category is abelian.

Definition 3.1.6. *The **dimension vector** of a representation $M = ((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ of a quiver Q is*

$$\underline{\dim}\ M := (\dim_{\Bbbk} V_i)_{i\in Q_0}.$$

The next example shows that the classification of the representations of a given quiver can be linked to the classification of other objects.

Example 3.1.7. *The quiver with $r+1$ vertices $i_1, \dots, i_r, j$, equipped with arrows $\alpha_1, \dots, \alpha_r$ whose sources are the $i_1, \dots, i_r$ and whose target is j is commonly noted as S_r. Below is an example of S_4 (it is also known as $\widetilde{D}_4$):*



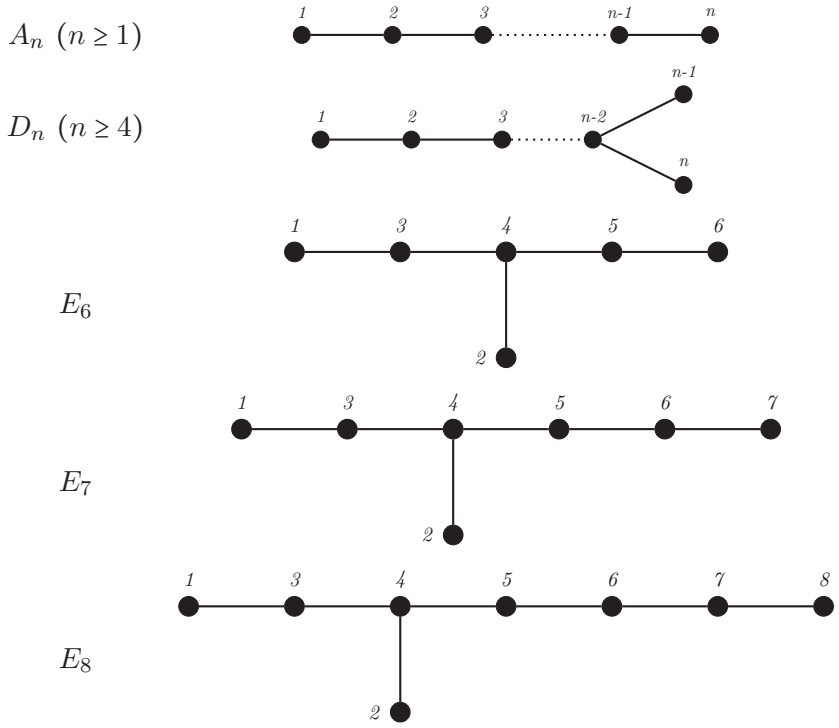
Hence a representation M of S_r is given by $\Bbbk$-vector spaces $V_1, \dots, V_r, W$ whose dimensions are $m_1, \dots, m_r, n$,respectively, and linear maps $f_{\alpha_j} : V_j \to W$ for $1 \leq j \leq r$. By sending M to the images of the f_{α_j}'s, one obtains a bijection between the isomorphism classes of representations of dimension vector $(m_1, \dots, m_r, n)$ and the orbits of the general linear group $\mathrm{GL}_n(\Bbbk)$ acting on r-uplets $(E_1, \dots, E_r)$ of subspaces of $\Bbbk^n$ such that $\dim E_i \leq m_i$ for any i, the action being $g.(E_1, \dots, E_r) = (g.E_1, \dots, g.E_r)$ for any $g \in \mathrm{GL}_n(\Bbbk)$. Therefore there is an equivalence between the classification of the representations of the quiver S_r for a given dimension vector and the classification of r-uplets of subspaces of prescribed dimensions of a given $\Bbbk$-vector space.

Definition 3.1.8. *A quiver Q is of **finite orbit type** if Q has only finitely many isomorphism classes of representations of any prescribed dimension vector.*

Example 3.1.9. *One may check that the quivers S_1, S_2 and S_3 of Example 3.1.7 are of finite orbit type, but $S_4 = \widetilde{D}_4$ is not.*

The next important theorem is due to P. Gabriel who proved it in [Gab72].

Theorem 3.1.10. (Gabriel). *A quiver is of finite orbit type if and only if every connected component of its underlying graph is a simply-laced Dynkin diagram, i.e. it is one of the following:*



3.2. The path algebra

Let $\Bbbk$ be an algebraically closed field and $Q = (Q_0, Q_1, s, t)$ a quiver. For a representation $((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ of Q, one defines the $\Bbbk$-vector space

$$V := \bigoplus_{i\in Q_0} V_i$$

as well as two families of linear maps:

- The projections

$$\begin{array}{rccc} f_i : & V & \to & V \\ & (v_1, \ldots, v_i, \ldots, v_r) & \mapsto & (0, \ldots, 0, v_i, 0, \ldots, 0) \end{array} \quad , \quad i \in Q_0.$$

- The maps

$$\begin{array}{rccccc} f_\alpha : & V & \to & V & , & \alpha \in Q_1. \\ & (v_1, \ldots, v_{s(\alpha)}, \ldots, v_r) & \mapsto & (0, \ldots, 0, \underbrace{f_\alpha(v_{s(\alpha)})}_{t(\alpha)\text{-th position}}, 0, \ldots, 0) & & \end{array}$$

One notices that these maps verify the relations

$$f_i^2 = f_i,\ f_i f_j = 0\ (i \neq j),\ f_{t(\alpha)} f_\alpha = f_\alpha f_{s(\alpha)} = f_\alpha,$$

and all other compositions are zero. This leads us to the next definition.

Definition 3.2.1. *The **path algebra** $\Bbbk Q$ of Q is the $\Bbbk$-associative algebra defined by the generators e_i, $i \in Q_0$, and α, $\alpha \in Q_1$, satisfying the following identities:*

$$e_i^2 = e_i,\ e_i e_j = 0\ (i \neq j),\ e_{t(\alpha)} \alpha = \alpha e_{s(\alpha)} = \alpha,$$

One notices that the e_i's are idempotent and pairwise orthogonal in $\Bbbk Q$. An important relation is $\sum_{i \in Q_0} e_i = 1$.
The $\Bbbk$-vector space V defined above can be seen as a left-$\Bbbk Q$-module. Indeed, by defining injections $\iota_i : \begin{array}{ccc} V_i & \to & V \\ v_i & \mapsto & (0, \cdots, 0, v_i, 0, \cdots, 0) \end{array}$ and projections $\pi_i : \begin{array}{ccc} V & \to & V_i \\ (v_1, \cdots, v_{|Q_0|}) & \mapsto & v_i \end{array}$
naturally, the action of $\Bbbk Q$ on V is given by:

$$\alpha_{i_1} \ldots \alpha_{i_n}.v = \iota_{t(\alpha_{i_1})} f_{\alpha_{i_1}} \ldots f_{\alpha_{i_n}} \pi_{s(\alpha_{i_n})}(v) \text{ and } e_i.v = \iota_i \pi_i(v),$$

for any path $\alpha_{i_1} \ldots \alpha_{i_n}$ of $\Bbbk Q$, $v \in V$ and $i \in Q_0$.

The next proposition establishes a link between the representations of a quiver Q and the $\Bbbk Q$-modules.

Proposition 3.2.2. *The category of representations of a quiver Q is equivalent to the category of $\Bbbk Q$-left-modules.*

Proof. We have previously seen how, from a representation M of a quiver Q one can construct a $\Bbbk Q$-module V. Let us do the converse. Let V be a $\Bbbk Q$-module. Define $V_i = e_i.V$ for any $i \in Q_0$. As $\sum_{i \in Q_0} e_i = 1$, one finds $V = \bigoplus_{i \in Q_0} V_i$. Furthermore, for any arrow $\alpha : i \to j$, there is a map $f_\alpha : V_i \to V_j$ given by the multiplication by α, which has its image in V_j because $\alpha = e_j \alpha$. One can verify that both constructions extend to functors

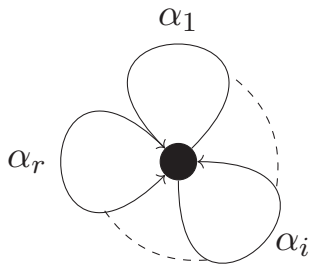
between the category of representations of Q and the category of $\Bbbk Q$-left-modules, and each one is the inverse of the other. The details of the computations can be found in [AuRe97] Chapter III, Theorem 1.5. □

Remark 3.2.3. One notices that as a $\Bbbk$-vector space, $\Bbbk Q$ is finite dimensional if and only if Q does not contain any edge loops nor oriented cycles.

We will now define the notion of a quiver with relations.

Definition 3.2.4. *A **relation** on a quiver Q is a subspace of $\Bbbk Q$ generated by linear combinations of paths with same source and same target, and with a length of at least 2. A **quiver with relations** is a pair (Q, I) with Q a quiver and I an ideal of $\Bbbk Q$ generated by relations. The quotient $\Bbbk Q/I$ is called the path algebra of (Q, I).*

Example 3.2.5. *Consider the quiver Q with one vertex and n edge loops $\alpha_1, \ldots, \alpha_n$:*



Its path algebra is $\Bbbk Q = \Bbbk\langle X_1, \ldots, X_n\rangle$, the ring over $\Bbbk$ generated by n independent elements. $\Bbbk Q$ is clearly infinite dimensional due to the edge loops. For example, if one wants the order in which the edge loops are browsed not to matter, one defines I the ideal of $\Bbbk Q$ generated by the relations $X_iX_j - X_jX_i$, for any $i \neq j$. The path algebra then becomes $\Bbbk\langle X_1, \ldots, X_1\rangle/(X_iX_j - X_jX_i, 1 \leq i, j \leq n) = \Bbbk[X_1, \ldots, X_n]$ the polynomial algebra in n variables.

Remark 3.2.6. We mention that the representations of a finite dimensional algebra can be described in terms of quivers. Starting from an algebra A, one can define a quiver with relations (Q, I) such that the category of representations $\mathrm{Rep}(A)$ is equivalent to the category $\mathrm{Rep}(Q, I)$. More details on this procedure can be found in the fourth chapter of [Ben98].

Let Q be a quiver without oriented cycles. We are going to classify the **simple representations** of Q, i.e. those which do not contain non-trivial subrepresentations. First let $S(i)$ $(i \in Q_0)$ be the representation of Q given by

$$S(i)_i = \Bbbk,\ S(i)_j = \{0\}\ (j \in Q_0, j \neq i),\ f_\alpha = 0\ (\alpha \in Q_1).$$

$S(i)$ is clearly simple and its dimension vector is $\epsilon_i = (0, \ldots, 0, \underbrace{1}_{\text{i-th position}}, 0, \ldots, 0)$.

Proposition 3.2.7. *Let Q be a quiver without oriented cycles nor edge loops. Any simple representation of Q is isomorphic to some $S(i)$ with $i \in Q_0$. Furthermore, any finite dimensional semisimple representation, i.e. a direct sum of its simple submodules, is completely determined by its dimension vector.*

Proof. Let M be a non-trivial simple $\Bbbk Q$-module. One sees that $M \neq \Bbbk Q_{r\geq 1} M$ with $\Bbbk Q_{r\geq 1}$ the ideal of $\Bbbk Q$ generated by the non-trivial arrows of Q. Indeed, if $M = \Bbbk Q_{r\geq 1} M$, then $M = (\Bbbk Q_{r\geq 1})^n M$ for any $n \geq 1$. But $(\Bbbk Q_{r\geq 1})^n$ is a $\Bbbk$-vector space generated by the paths whose length is at least n. As Q does not contain any oriented cycle nor edge loop, it follows that there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(\Bbbk Q_{r\geq 1})^n = \{0\}$, which implies $M = \{0\}$ and contradicting the non-triviality of M. Because M is simple and $\Bbbk Q_{r\geq 1} M$ is a proper submodule of M, one deduces that $\Bbbk Q_{r\geq 1} M = \{0\}$. Hence M can be seen as a module over

$$\Bbbk Q / \Bbbk Q_{r\geq 1} \cong \bigoplus_{i \in Q_0} \Bbbk e_i \cong \prod_{i \in Q_0} \Bbbk.$$

Therefore any subspace $e_i M$ is a $\Bbbk Q$-submodule de M. Thus $e_i M = \{0\}$ or M for any $i \in Q_0$. It is known that $\sum_{i \in Q_0} e_i = 1$, implying that $e_i M = \{0\}$ cannot happen for all $i \in Q_0$. Consequently there exists $i \in Q_0$ such that $e_i M = M$. Assume that there exists $j \neq i$ such that $e_j M = M$. Then $\{0\} = e_i e_j M = e_i M = M$ because $e_i e_j = 0$. It follows that there exists a unique $i \in Q_0$ such that $e_i M = M$ and then $M = S(i)$.
Let M be a semisimple $\Bbbk Q$-module of finite dimension. According to the first statement of the proposition,

$$M = \bigoplus_{i \in Q_0} m_i S(i)$$

with $m_i \in \mathbb{Z}_{\geq 0}$ for any $i \in Q_0$. Thus

$$\underline{\dim}\ M = \sum_{i \in Q_0} m_i\ \underline{\dim}\ S(i) = \sum_{i \in Q_0} m_i \epsilon_i$$

with ϵ_i being the i-th basis vector of $\mathbb{Z}_{\geq 0}^{Q_0}$. □

We finish this section by introducing an object which will give us a numerical criterion for the classification of quivers.

Definition 3.2.8. *The **Euler form** of a quiver Q is the bilinear form on $\mathbb{R}^{Q_0}$ denoted by $\langle .,. \rangle_Q$ and defined by*

$$\langle \underline{m}, \underline{n} \rangle_Q = \sum_{i \in Q_0} m_i n_i - \sum_{\alpha \in Q_1} m_{s(\alpha)} n_{t(\alpha)}$$

for any $\underline{m} = (m_i)_{i \in Q_0}, \underline{n} = (n_i)_{i \in Q_0} \in \mathbb{R}^{Q_0}$.
*The quadratic form associated to the Euler form is denoted by q_Q and is called the **Tits form**. Explicitly,*

$$q_Q(\underline{n}) = \langle \underline{n}, \underline{n} \rangle_Q = \sum_{i \in Q_0} n_i^2 - \sum_{\alpha \in Q_1} n_{s(\alpha)} n_{t(\alpha)}.$$

for any $\underline{n} = (n_i)_{i \in Q_0} \in \mathbb{R}^{Q_0}$.

Using a result called the standard resolution (cf. [Crawley]), one can prove the next formula:

$$q_Q(\underline{\dim}\ M) = \dim \operatorname{End}_Q(M) - \dim \operatorname{Ext}^1_Q(M, M).$$

Therefore the Tits form gives us information on self-extensions of a finite dimensional representation of a quiver. But it is not its only use. Some of its properties give conditions on whether or not a quiver is of finite orbit type. The next theorem can be found in [Ben98].

Theorem 3.2.9. *Let Q be a quiver. The following assertions are equivalent:*

(i) The Tits form q_Q is positive definite.

(ii) $q_Q(\underline{n}) \geq 1$ for any $\underline{n} \in \mathbb{Z}_{\geq 0}^{Q_0} \setminus \{0\}$.

(iii) The underlying non-oriented graph of Q is a union of simply-laced Dynkin diagrams.

This theorem enables us to reformulate Gabriel's theorem in the following manner:

$$Q \text{ is of finite orbit type} \iff q_Q \text{ is positive definite.}$$

3.3. Orbits and indecomposable representations

In this section we define and study a structure of module on the representation space. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\underline{n} = (n_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$ a dimension vector.

Definition 3.3.1. *The **representation space** of a quiver Q with dimension vector $\underline{n} = (n_i)_{i \in Q_0}$ is*

$$\mathrm{Rep}(Q, \underline{n}) := \bigoplus_{\alpha : i \to j} \mathrm{Hom}(\Bbbk^{n_i}, \Bbbk^{n_j}).$$

It is a $\Bbbk$-vector space of dimension $\sum_{\alpha : i \to j} n_i n_j$.

Let $\underline{n}$ be a dimension vector of a quiver Q. The group

$$\mathrm{GL}_{\underline{n}}(\Bbbk) = \prod_{i \in Q_0} \mathrm{GL}_{n_i}(\Bbbk)$$

acts naturally on each $\mathrm{Hom}(\Bbbk^{n_i}, \Bbbk^{n_j})$ by

$$(g_i)_{i \in Q_0} . x_\alpha := g_j x_\alpha g_i^{-1},$$

and hence acts on $\mathrm{Rep}(Q, \underline{n})$ by preserving the decomposition.

Any point $x \in \mathrm{Rep}(Q, \underline{n})$ defines a representation M_x of Q. Furthermore, two representations M_x and M_y are isomorphic if and only if x and y are in the same $\mathrm{GL}_{\underline{n}}(\Bbbk)$-orbit. The next lemma follows naturally.

Lemma 3.3.2. *The map which associates to any $x \in \mathrm{Rep}(Q, \underline{n})$ a representation M_x of Q defines a bijection between the set of $\mathrm{GL}_{\underline{n}}(\Bbbk)$-orbits in $\mathrm{Rep}(Q, \underline{n})$ and the isomorphism classes of representations of Q with dimension vector $\underline{n}$. Additionally, the isotropy group of x,*

$$\mathrm{GL}_{\underline{n}}(\Bbbk)_x = \{g \in \mathrm{GL}_{\underline{n}}(\Bbbk) \mid g.x = x\}$$

is isomorphic to the group $\mathrm{Aut}_Q(M_x)$ of automorphisms of M_x.

For a quiver Q let us write $\mathcal{O}_M$ for the orbit in $\mathrm{Rep}(Q,\underline{n})$ associated to the representation M. Then $\mathcal{O}_M = \mathrm{GL}_{\underline{n}}(\Bbbk).x$ where x is any element of $\mathrm{Rep}(Q,\underline{n})$ such that $M_x \cong M$. In particular one may notice that a quiver is of finite orbit type if and only if the number of $\mathrm{GL}_{\underline{n}}(\Bbbk)$-orbits in $\mathrm{Rep}(Q,\underline{n})$ is finite for any dimension vector $\underline{n}$.

The following proposition is stated in a setting broader than that of quivers but can be easily adapted.

Proposition 3.3.3. *Let X be a variety with an algebraic action of an algebraic group G and x a point in X. The following assertions are true:*

(i) The isotropy group G_x is closed in G.

(ii) The orbit $G.x$ is a non-singular subvariety of X and locally closed in X. The connected components of $G.x$ all have the same dimension: $\dim G - \dim G_x$.

(iii) The closure $\overline{G.x}$ of the orbit is the union of $G.x$ and of orbits of strictly smaller dimensions. An orbit with the smallest dimension is then closed.

(iv) The variety G is connected if and only if it is irreducible. If so the orbit $G.x$ as well as its closure $\overline{G.x}$ are irreducible.

Proof. (i) Consider the map $\varphi_x : \begin{array}{ccc} G & \to & X \\ g & \mapsto & g.x \end{array}$. This map is continuous and every fiber is a set gG_x, for some $g \in G$. As all points are closed, it follows that gG_x is closed and so is G_x.
(ii) The orbit $G.x$ is the image of the morphism φ_x and is thus a constructible subset of X (cf. Corollary 14.7 in [Eis95]). Hence $G.x$ contains an open subset U dense in its closure (cf. [Bo91] page 2). One deduces that $G.x = \bigcup_{g\in G} gU$ implying that $G.x$ is open in its closure. In a similar manner, $G.x$ is non-singular and all its connected components have the same dimension.
(iii) According to (ii), $\overline{G.x} \setminus G.x$ is of dimension strictly inferior to dim $G.x$. As it is G-stable, one deduces that it is a union of orbits of dimensions strictly inferior to dim $G.x$. Let us denote by $\mathcal{O}_{min}$ an orbit with the smallest dimension in $\overline{G.x}$. It follows that $\overline{\mathcal{O}_{min}} \setminus \mathcal{O}_{min}$ is empty and thus $\mathcal{O}_{min}$ is closed.
(iv) The group G can be seen as the orbit of the action of G on itself. According to (ii), G is non-singular. Because it is an algebraic group, G is connected if and only if it is irreducible. Hence if G is irreducible, $G.x$ is the image of G by the morphism φ_x and so $G.x$ and $\overline{G.x}$ are irreducible. □

Proposition 3.3.4. *Let $x = (x_\alpha)_{\alpha:i\to j} \in \mathrm{Rep}(Q,\underline{n})$ and M be the corresponding representation.*

(i) The following sequence is exact:

$$0 \to \mathrm{End}_Q(M) \to \mathrm{End}(\underline{n}) \xrightarrow{c_x} \mathrm{Rep}(Q,\underline{n}) \to \mathrm{Ext}^1_Q(M,M) \to 0$$

with $\mathrm{End}(\underline{n}) = \prod_{i\in Q_0} \mathrm{End}(\Bbbk^{n_i})$ *and* $c_x((f_i)_{i\in Q_0}) = (f_j x_\alpha - x_\alpha f_i)_{\alpha:i\to j}$.

(ii) The map c_x can be identified with the differential at the identity of the map

$$\begin{array}{rccc} \varphi_x : & \mathrm{GL}(\underline{n}) & \to & \mathrm{Rep}(Q,\underline{n}) \\ & g & \mapsto & g.x \end{array}$$

(iii) The image of c_x is the tangent space $T_x(\mathrm{GL}(\underline{n}).x)$ seen as a subspace of $T_x(\mathrm{Rep}(Q,\underline{n})) \cong \mathrm{Rep}(Q,\underline{n})$.

Proof. cf. [Brion00] Theorem 2.2.3. □

The next proposition gives us some details on the structure of the orbits of $\mathrm{GL}(\underline{n})$ in $\mathrm{Rep}(Q,\underline{n})$. Its proof can be found in [Brion00] Theorem 2.3.1.

Proposition 3.3.5. *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finite dimensional representations of a quiver Q. Then $\overline{\mathcal{O}_M}$ contains $\mathcal{O}_{M'\oplus M''}$. Furthermore, the exact sequence splits if and only if $\mathcal{O}_M = \mathcal{O}_{M'\oplus M''}$.

We can now give a more precise version of Gabriel's theorem (cf. [Gab72]).

Definition 3.3.6. *A representation M of a quiver Q is a* ***Schur representation*** *if* $\mathrm{End}_Q(M) = \Bbbk \mathrm{id}_M$.

Theorem 3.3.7. *Let Q be a quiver with $|Q_0| = r$. Assume that the Tits form q_Q of Q is positive definite. Then:*

1. *Any* ***indecomposable*** *representation M (i.e. M cannot be decomposed as a direct sum of non-zero submodules) is a Schur representation and has no non-zero self-extension.*
2. *The dimension vectors of the indecomposable representations are exactly the $\underline{n} \in \mathbb{Z}^r_{\geq 0}$ such that $q_Q(\underline{n}) = 1$.*
3. *Any indecomposable representation is uniquely determined (up to isomorphism) by its dimension vector.*

4. *If the underlying graph of Q is connected, then it is a Dynkin diagram Δ of type A_r, D_r or E_r. Let $\{\alpha_1, \ldots, \alpha_r\}$ and $L_Q = \sum_{i=1}^{r} \mathbb{Z}\alpha_i$ be a base of the root system of type Δ and the associated root lattice. Identify the vertices of the Dynkin diagram with their respective roots. Then the map*

$$\begin{array}{rcl} \mathrm{Rep}(Q) & \to & L_Q \\ M = (M_i)_{1 \le i \le r} & \mapsto & \sum_{i=1}^{r} \dim(M_{\alpha_i})\alpha_i \end{array}$$

gives a bijection from the set of all isomorphism classes of indecomposable representations onto the set of all positive roots of the root system of type Δ.

All quivers are not of finite orbit type. Let Q be a quiver. If there exists a full embedding of the category of representations of the quiver

into the category of representations of Q, Q is called **wild**. If Q is neither wild nor of finite orbit type then it is **tame**.
In [Gab75], P. Gabriel presented the different types of quivers and stated that the representations of a wild quiver are not classifiable. Indeed, let Q be the previous quiver with one vertex and two edge loops. The path algebra of Q is the free algebra $\Bbbk\langle X, Y\rangle$ generated by two indeterminates X and Y. One can show that for any $\Bbbk$-algebra of finite type A there is a fully faithful functor $F : \mathrm{Modf}_A \to \mathrm{Modf}_{\Bbbk\langle X,Y\rangle}$ from the category of finite dimensional A-modules to the category of finite dimensional $\Bbbk\langle X, Y\rangle$-modules. If A is generated by n elements $a_1, ..., a_n$ and $M \in \mathrm{Modf}_A$, then $F(M) = M^{n+2}$ and X and Y act on $F(M)$ by

$$\begin{pmatrix} 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 \\ 0 & 0 & 0 & . & 0 & 1 \\ 0 & 0 & 0 & . & . & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & . & . & . & . & 0 \\ 1 & 0 & . & . & . & 0 \\ a_1 & 1 & . & . & . & . \\ 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 0 & a_n & 1 & 0 \end{pmatrix}.$$

A complete classification of the finite dimensional representations of the quiver Q would imply a complete classification of all finite dimensional modules for all finite type $\Bbbk$-algebras, which is not practical.

When the underlying graph of the quiver Q is not a Dynkin diagram, the root system (in the sense of Section 1.1.2) of the Lie algebra $\mathfrak{g}_Q$ (defined below) whose Coxeter graph is Q is different than what has been previously described. Let Q be a quiver whose underlying graph is connected and simply-laced without edge loops. The Cartan matrix

$$C_Q = 2I - A$$

with A the adjacent matrix of Q:

$$A_{i,j} = \text{number of edges between vertices } i \text{ and } j \in Q_0.$$

This matrix can be used to define a **Kac-Moody Lie algebra** $\mathfrak{g}_Q$ (cf. [Kiri16]). In particular it defines a root lattice and a Weyl group.

The root lattice L_Q of Q is defined by

$$L_Q = \sum_{i \in Q_0} \mathbb{Z}\alpha_i \cong \mathbb{Z}^{Q_0},$$

where the α_i's are the **simple roots**. α_i can be seen as the vector dimension of a representation $S(i)$ of Q with 1 at the i-th vertex and 0 elsewhere. The Cartan matrix defines an integral symmetric bilinear form on L_Q, called the **symmetrized Euler form**:

$$(\alpha_i, \alpha_j) = (C_Q)_{i,j} = 2\delta_{ij} - A_{i,j}.$$

For every $i \in Q_0$, the corresponding **simple reflection** s_i is defined as

$$\begin{array}{rccc} s_i : & L_Q & \to & L_Q \\ & \alpha & \mapsto & \alpha - (\alpha, \alpha_i)\alpha_i \end{array} .$$

The subgroup of $\mathrm{GL}(L_Q \otimes_{\mathbb{Z}} \mathbb{R})$ generated by the simple reflections is the **Weyl group** W_Q of $\mathfrak{g}_Q$.

By looking at the adjoint action of $\mathfrak{g}_Q$ on itself, one finds the root system Φ_Q of $\mathfrak{g}_Q$. In the special case where Q is a Dynkin diagram or an extended Dynkin diagram, the root system is given by

$$\Phi_Q = \{\alpha \in L_Q \setminus \{0\} \mid (\alpha, \alpha) \leq 2\}.$$

Like in Section 1.1.2, one can show that $\Phi_Q = \Phi_Q^+ \sqcup \Phi_Q^-$ with $\Phi_Q^+ := (\sum_{i \in Q_0} \mathbb{Z}_{\geq 0}\alpha_i) \cap \Phi_Q$ being the set of positive roots of Φ_Q and $\Phi_Q^- := (\sum_{i \in Q_0} \mathbb{Z}_{\leq 0}\alpha_i) \cap \Phi_Q$ the set of negative roots of Φ_Q.

A root $\alpha \in \Phi_Q$ is called **real** if there exists $w \in W_Q$ and $i \in Q_0$ such that $\alpha = w.\alpha_i$. All the other roots are called **imaginary**, so that

$$\Phi_Q = \Phi_Q^{re} \sqcup \Phi_Q^{im}.$$

When the underlying graph of Q is a Dynkin diagram or an extended Dynkin diagram, the next proposition (cf. [Kac90]) characterizes Φ_Q^{re} and Φ_Q^{im} more precisely.

Proposition 3.3.8. *1. Let Q be a quiver such that its underlying graph is a Dynkin diagram. Then $\Phi_Q = \Phi_Q^{re}$.*

2. Let Q be a quiver such that its underlying graph is an extended Dynkin diagram $\widetilde{\Delta}$. Let $\mathring{\Phi}_Q$ and $\mathring{\Pi}_Q$ denote the root system and a basis of type Δ. Let $\alpha_0 \in \Phi_Q$ be the simple root associated to the extended vertex of Q. Then $\Pi_Q = \mathring{\Pi}_Q \cup \{\alpha_0\}$ is a base of Φ_Q,

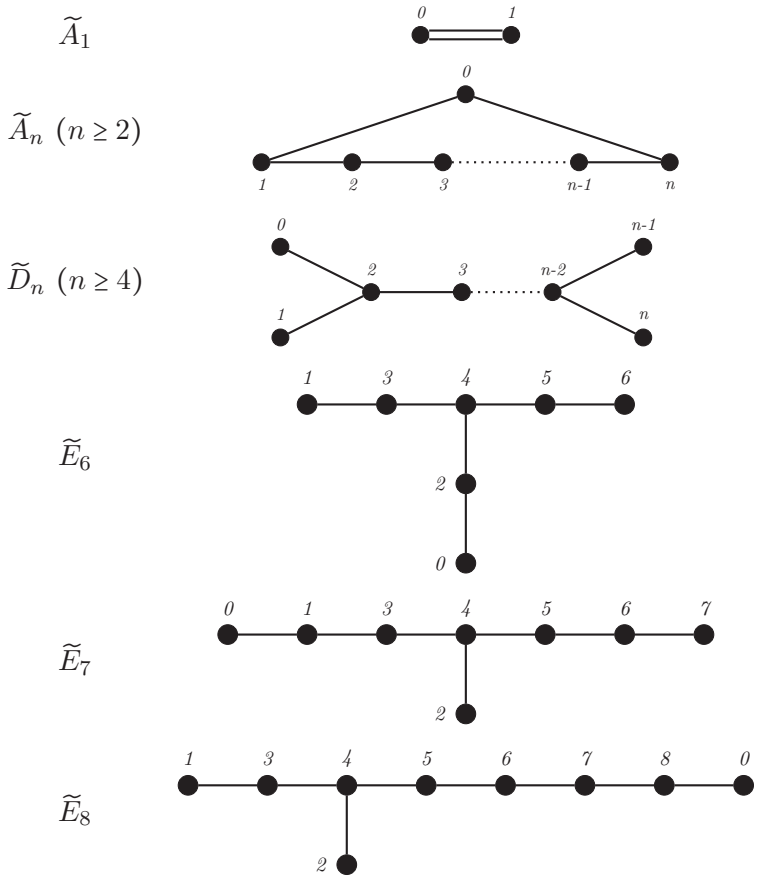
$$\Phi_Q^{im} = \mathbb{Z}^*\delta \text{ and } \Phi_Q^{re} = \{\alpha + n\delta \mid \alpha \in \mathring{\Phi}_Q,\ n \in \mathbb{Z}\}$$

with $\delta = \alpha_0 + \theta$ and θ being the highest root of $\mathring{\Phi}_Q$ with respect to $\mathring{\Pi}_Q$.

Remark 3.3.9. Let Q be a quiver such that its underlying graph is a simply-laced extended Dynkin diagram $\widetilde{\Delta}$. The dimension vector of the imaginary root δ is the vector $(d_0, \ldots, d_r)$ from Subsection 1.5.1.1 as well as the dimension vector of the McKay quiver which will be used in Section 4.2.

The tame quivers were classified by L.A. Nazarova [Nazar73] and by P. Donovan and M.R. Freislich [DonFrei73].

Theorem 3.3.10. (Donovan-Freislich-Nazarova). *A finite connected quiver Q is tame if and only if its underlying graph is $\widetilde{A}_1$ or a simply-laced extended Dynkin diagram, i.e. it is one of the following:*



Assume Q is tame of type $\widetilde{\Delta}$. Let $\{\alpha_0, \ldots, \alpha_r\}$ and $L_Q = \sum_{i=0}^{r} \mathbb{Z}\alpha_i$ be a base of the root system of type $\widetilde{\Delta}$ and the associated root lattice. Identify the vertices of the extended Dynkin diagram with their respective roots. Then the map

$$\begin{array}{rccc} f: & \mathrm{Rep}(Q) & \to & L_Q \\ & M = (M_i)_{0 \leq i \leq r} & \mapsto & \sum_{i=0}^{r} \dim(M_{\alpha_i})\alpha_i \end{array}$$

gives a surjection from the set of all isomorphism classes of indecomposable representations onto the set of all positive roots of the root system of type $\widetilde{\Delta}$. Furthermore,

1. *If α is a positive real root then $f^{-1}(\alpha)$ is a singleton.*
2. *If α is a positive imaginary root, then there exists a finite subset $D = \{p_1, ..., p_k\} \subset \mathbb{P}^1$ and a collection of positive integers $l_p > 1$, $p \in D$, such that the set of isomorphism classes of indecomposable representations of Q of dimension α is in bijection with*

$$(\mathbb{P}^1 \setminus D) \cup \bigcup_{p \in D} \mathbb{Z}/l_p\mathbb{Z}.$$

Finally, in [Kac80] and [Kac83] V. Kac generalized the previous results to an arbitrary connected quiver without edge loops.

Theorem 3.3.11. (Kac). *Let Q be a connected quiver without edge loops. Let $\alpha = \sum_{i \in Q_0} n_i \alpha_i \in \mathbb{Z}^{Q_0}$ be positive, i.e. $n_i \geq 0$ for all $i \in Q_0$, $\alpha \neq 0$. Then:*

1. *An indecomposable representation M of Q of dimension α exists if and only if α is a positive root.*
2. *If α is a positive real root, then the indecomposable representation M_α with $\underline{\dim}\ M_\alpha = \alpha$ is unique up to isomorphism.*
3. *If α is a positive imaginary root, then there are infinitely many non-isomorphic indecomposable representations of dimension α. The set of isomorphism classes of indecomposable representations of dimension α can be parametrized by a finite union of algebraic varieties $Z_1, ..., Z_n$ such that $\max\{\dim\ Z_i\}_i = 1 - \frac{(\alpha,\alpha)}{2}$.*

In Subsection 1.1.2, a root system Φ was defined as a finite subset of a $\Bbbk$-vector space E coming from a semisimple Lie algebra. It was then equipped with a non-degenerate bilinear form $(.,.)$ coming from the Killing form and if $\alpha, \beta \in \Phi$, we defined $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. The definition of $\langle .,. \rangle$ can be extended to equip affine root systems corresponding to infinite-dimensional Kac-Moody algebras (cf. [Kac90]).

Definition 3.3.12. *A root system Φ (in the sense axioms (R1), (R2) and (R3) of [Hum75] Appendix) is **crystallographic** if for any $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle \in \mathbb{Z}$.*

It was said in Theorem 1.1.16 that finite root systems are crystallographic. It can be shown that extended root systems are crystallographic as well.

The notion of folding of a root system via automorphism was presented in Section 1.2 for the finite root systems. However the notion can be generalized to a larger class of root systems. The next proposition comes from [Stem08].

Proposition 3.3.13. *Any crystallographic root system may be realized as a folding of a simply-laced root system by some diagram automorphism.*

3.4. Folding of root systems and quivers

In the following subsection we present a construction by T. Tanisaki of some kind of folding of the representation space of a quiver whose underlying graph is a simply-laced Dynkin diagram. In the subsequent subsection observations on the different types of folding will be given.

3.4.1. Foldings and quivers representations

It has been stated in the previous part that a connected quiver Q has only finitely many isomorphism classes of finite dimensional indecomposable representations if and only if Q is a Dynkin diagram Δ of type A_r, D_r or E_r. If Q is of such type, then there is a bijection from the set of all isomorphism classes of indecomposable representations onto the set of all positive roots of a root system of type Δ given by the function $\underline{\dim}$ (cf. Theorems 3.1.10 and 3.3.7). This result has been extended by T. Tanisaki in [Tani80] to the non-simply-laced Dynkin diagrams using an approach based on foldings of root systems. In what follows we present his results.

Let $Q = (Q_0, Q_1, s, t)$ be a simply-laced quiver. An **automorphism** of Q is a permutation σ of the set $Q_0 = \{\alpha_i\}_{i\in I}$ such that a pair $(\sigma(\alpha_i), \sigma(\alpha_j))$ forms an edge of Q if and only if the pair (α_i, α_j) is an edge of Q. The set of automorphisms of Q is a group $\mathrm{Aut}(Q)$ given by

$$\mathrm{Aut}(Q) = \{\sigma = (\sigma_0, \sigma_1) \in \mathfrak{S}_0^{|Q_0|} \times \mathfrak{S}_1^{|Q_1|} \mid s(\sigma_1(\alpha)) = \sigma_0(s(\alpha)),\ t(\sigma_1(\alpha)) = \sigma_0(t(\alpha)) \text{ for all } \alpha \in Q_1\}.$$

For any $\sigma \in \mathrm{Aut}(Q)$, one defines the functor $F^\sigma : \mathrm{Rep}(Q) \to \mathrm{Rep}(Q)$ in the following way:

- For any $(V_i, f_\alpha)_{i\in Q_0, \alpha\in Q_1}$, then $F^\sigma((V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1}) = (W_i, g_\alpha)_{i\in Q_0,\alpha\in Q_1}$ with $W_i = V_{\sigma_0^{-1}(i)}$ for any $i \in Q_0$, and $g_\alpha = f_{\sigma_1^{-1}(\alpha)}$ for any $\alpha \in Q_1$.
- For any $\varphi : (V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1} \to (W_i, g_\alpha)_{i\in Q_0,\alpha\in Q_1}$, then $F^\sigma(\varphi) : F^\sigma(V_i, f_\alpha) \to F^\sigma(W_i, g_\alpha)$ is given by $(F^\sigma(\varphi))_i = \varphi_{\sigma_0^{-1}(i)}$ for all $i \in Q_0$.

Definition 3.4.1. *Let G be a subgroup of $\mathrm{Aut}(Q)$. Define the category $\mathrm{Rep}(Q)^G$, which is a full subcategory of $\mathrm{Rep}(Q)$, as follows: if $(V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}(Q)$, then $(V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1} \in \mathrm{Rep}(Q)^G$ if for any $g \in G$, $F^g((V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1})$ is isomorphic to $(V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1}$ in the category $\mathrm{Rep}(Q)$.*

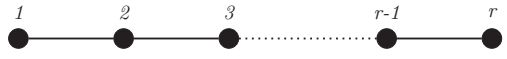
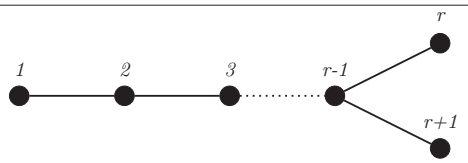
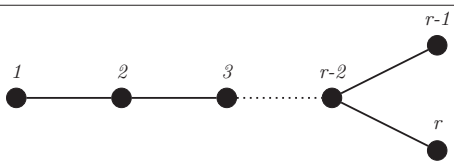
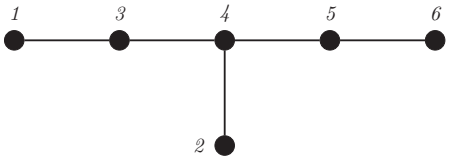
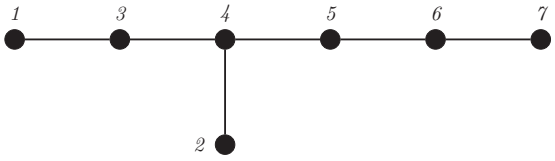
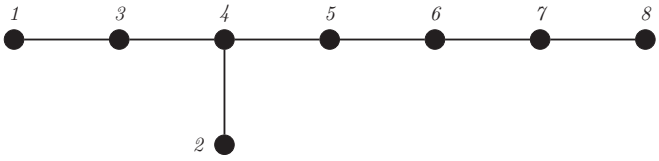
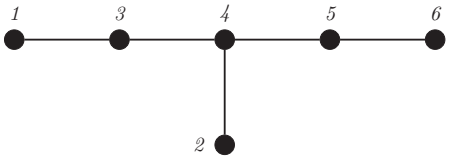
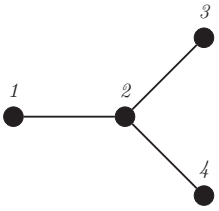
One can see $\mathrm{Rep}(Q)^G$ as the invariants up to isomorphism of $\mathrm{Rep}(Q)$ by the action of G through the functor F.

We are now able to state Tanisaki's theorem (cf. [Tani80]).

Theorem 3.4.2. (Tanisaki). *Let Q be a connected quiver and G a subgroup of $\mathrm{Aut}(Q)$ which preserves the orientation of Q. Then the following assertions are verified:*

(i) In the category $\mathrm{Rep}(Q)^G$, the theorem of Krull-Remak-Schmidt (which states the unicity of the decomposition of a module into a direct sum of indecomposable submodules) is valid.

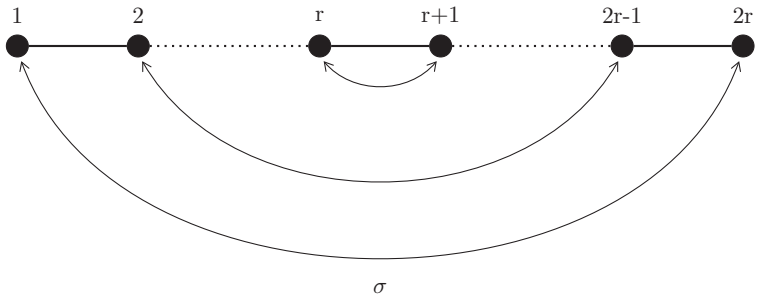
(ii) There are only finitely many isomorphism classes of indecomposable objects in $\mathrm{Rep}(Q)^G$ *if and only if* Q *and* G *are in the following table:*

$\Delta(Q,G)$	*Graph of* Q	G
A_r $(r \geq 1)$		$\{1\}$
B_r $(r \geq 2)$		$\{1,\tau\}$ *with* $\tau(\alpha_i) = \alpha_{2r-i}$
C_r $(r \geq 3)$		$\{1,\tau\}$ *with* $\tau(\alpha_r) = \alpha_{r+1}$, $\tau(\alpha_{r+1}) = \alpha_r$, $\tau(\alpha_i) = \alpha_i$ *if* $i \neq r, r+1$.
D_r $(r \geq 4)$		$\{1\}$
E_6		$\{1\}$
E_7		$\{1\}$
E_8		$\{1\}$
F_4		$\{1,\tau\}$ *with* $\tau(\alpha_1) = \alpha_6, \tau(\alpha_2) = \alpha_2,$ $\tau(\alpha_3) = \alpha_5, \tau(\alpha_4) = \alpha_4,$ $\tau(\alpha_5) = \alpha_3, \tau(\alpha_6) = \alpha_1.$
G_2		*G acts transitively on* $\{\alpha_1, \alpha_3, \alpha_4\}$ *and fixes* α_2.

(iii) Let $(Q, G, \Delta(Q,G))$ be a triple as in the previous table. Then there exists a natural one-to-one correspondence between the set of all isomorphism classes of indecomposable objects of $\mathrm{Rep}(Q)^G$ and the set of positive roots of the root system of type $\Delta(Q,G)$.

Remarks 3.4.3. 1. One notices that $\Delta(Q,G)$ is the type obtained by folding the root system associated with the graph of Q by the action of G.

2. The reader might wonder why T. Tanisaki did not work on the Dynkin diagram A_{2r} with the $\mathbb{Z}/2\mathbb{Z} =< \sigma >$ symmetry given by $\sigma(\alpha_i) = \alpha_{2r+1-i}$:



Theorem 3.4.2 is based upon choosing an orientation of the quiver Q and a subgroup G of $\mathrm{Aut}(Q)$ which preserves the orientation. However the non-trivial symmetry group of A_{2r} is $\mathbb{Z}/2\mathbb{Z}$ and the action of σ would send the arrow $r \to r+1$ to $r+1 \to r$, thus reversing the orientation of this edge. It is the same if we choose to orient the edge by $r+1 \to r$. Therefore there cannot be a subgroup G of $\mathrm{Aut}(Q)$ which preserves the orientation and so Tanisaki's result is not valid anymore.

3.4.2. Observations on foldings

Throughout the several previous sections we have obtained connections between the simply-laced Dynkin diagrams and the non-simply-laced ones. In this subsection the different results will be compared.

In Section 1.2, we realized foldings on:

1. simply-laced Lie algebras.
2. root systems of simply-laced Lie algebras.

These foldings give results dual to each others.

Type of $\mathfrak{g}$	A_{2r-1}	D_{r+1}	E_6	D_4
Ω	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathfrak{S}_3$
Type of $\mathfrak{g}_0$	C_r	B_r	F_4	G_2
Type of Q^σ	B_r	C_r	F_4	G_2

Table 3.1. – Foldings of root systems and Lie algebras

Remark 3.4.4. The case of type A_{2r} had to be left out because of the absence of a group Γ' acting on the singularity with the required properties (cf. Remark 1.4.38).

P. Slodowy defined the inhomogeneous simple singularities based on the symmetries of the Dynkin diagrams in [Slo80] and [Slo80(2)]:

Γ	Γ'	$\Omega = \Gamma'/\Gamma$	Type of X_0	Type of (X_0, Ω)
$\mathcal{C}_{2r}$	$\mathcal{D}_r$	$\mathbb{Z}/2\mathbb{Z}$	A_{2r-1}	$B_r,\ r \geq 2$
$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	$\mathbb{Z}/2\mathbb{Z}$	D_{r+1}	$C_r,\ r \geq 3$
$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}/2\mathbb{Z}$	E_6	F_4
$\mathcal{D}_2$	$\mathcal{O}$	$\mathfrak{S}_3$	D_4	G_2

Remark 3.4.5. When (X_0, Ω) is of type G_2, the possibility $\Gamma' = \mathcal{T}$ was disregarded because the quotient $\mathcal{T}/\mathcal{D}_2$ leads to the same invariants as $\mathcal{O}/\mathcal{D}_2$.

He also computed two variants of the McKay correspondence: one by induction and one by restriction. The results are as follows:

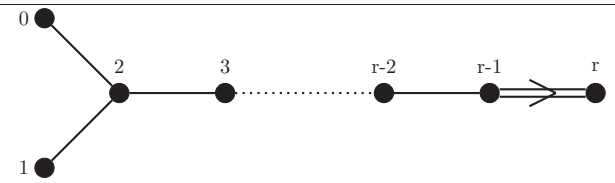
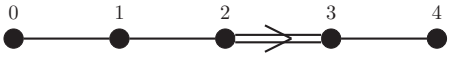
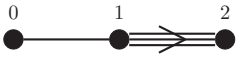
Γ	Γ'	Type of $\Delta(\Gamma)$	Type of $\Delta(\Gamma,\Gamma')$	$\widetilde{\Delta^\vee}(\Gamma,\Gamma')$
$\mathcal{C}_{2r}$	$\mathcal{D}_r$	A_{2r-1}	B_r	
$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	D_{r+1}	C_r	
$\mathcal{T}$	$\mathcal{O}$	E_6	F_4	
$\mathcal{D}_2$	$\mathcal{O}$	D_4	G_2	

Table 3.2. – McKay's correspondence by restriction

and

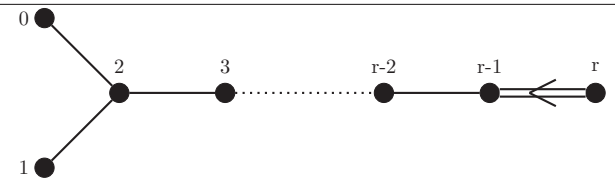
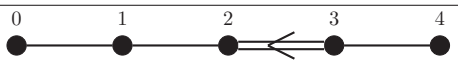
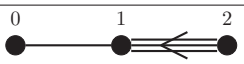
Γ	Γ'	Type of $\Delta(\Gamma)$	Type of $\Delta(\Gamma,\Gamma')$	$(\widetilde{\Delta^\vee}(\Gamma,\Gamma'))^\vee$
$\mathcal{C}_{2r}$	$\mathcal{D}_r$	A_{2r-1}	B_r	
$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	D_{r+1}	C_r	
$\mathcal{T}$	$\mathcal{O}$	E_6	F_4	
$\mathcal{D}_2$	$\mathcal{O}$	D_4	G_2	

Table 3.3. – McKay's correspondence by induction

Finally the foldings by T. Tanisaki using quiver representations gave:

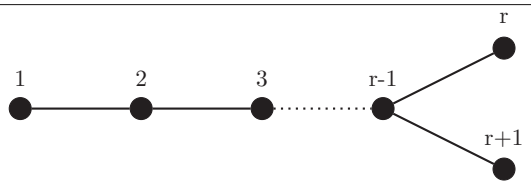
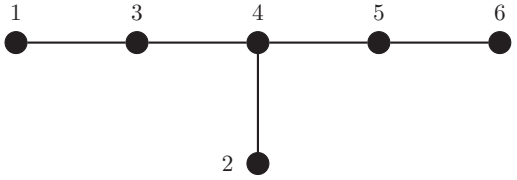
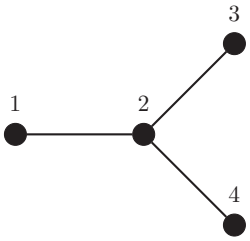
$\Delta(Q)$	$\Delta(Q,G)$	Graph of Q	G
A_{2r-1}	B_r $(r \geq 2)$		$\mathbb{Z}/2\mathbb{Z}$
D_{r+1} $(r \geq 3)$	C_r $(r \geq 3)$		$\mathbb{Z}/2\mathbb{Z}$
E_6	F_4		$\mathbb{Z}/2\mathbb{Z}$
D_4	G_2		$\mathfrak{S}_3$

Table 3.4. – Foldings using quiver representations

Remark 3.4.6. In the case of (D_4, G_2) (cf. Remark 3.4.5), the group $\mathfrak{S}_3$ can be replaced by the smaller group $\mathbb{Z}/3\mathbb{Z}$ and the results of Theorem 3.4.2 remain valid.

We notice that the way P. Slodowy defined the inhomogeneous singularities is in adequacy with the foldings of the root systems as well as the foldings using quiver representations. Furthermore, the McKay correspondences by induction and restriction establish a similar link between the homogeneous and the inhomogeneous Dynkin diagrams, however less direct because of the extension of the Dynkin diagrams as well as their duals.

The connections between the different foldings are summarized in the following diagrams:

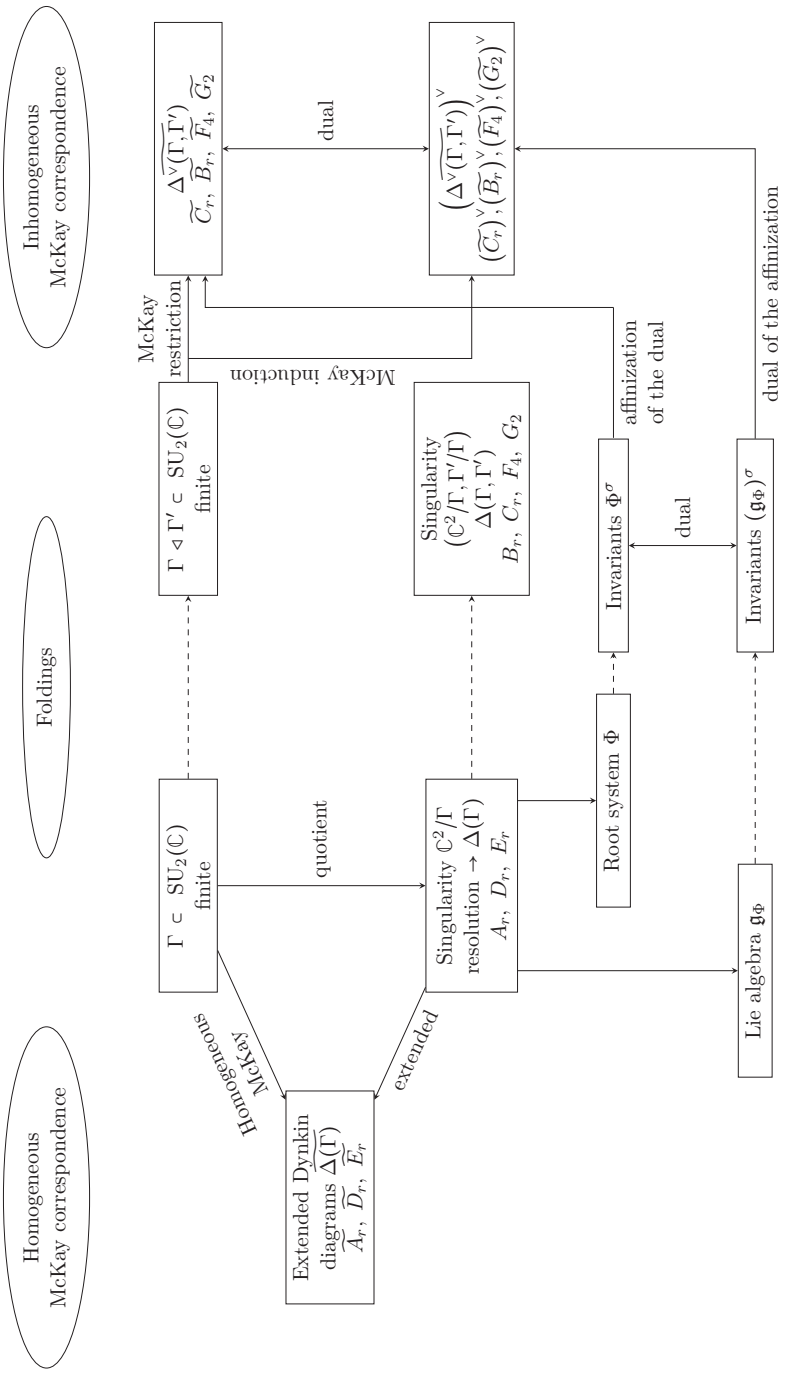


Figure 3.1. – Foldings and McKay correspondences

In the next chapter we are going to present a construction by H. Cassens and P. Slodowy of the semiuniversal deformations of the simple singularities using quiver rep-

resentations and symplectic geometry. Their method will then be generalized to obtain the semiuniversal deformations of the inhomogeneous simple singularities. The process is based on the affinization of the Dynkin diagram associated to the singularity through its minimal resolution, as illustrated below:

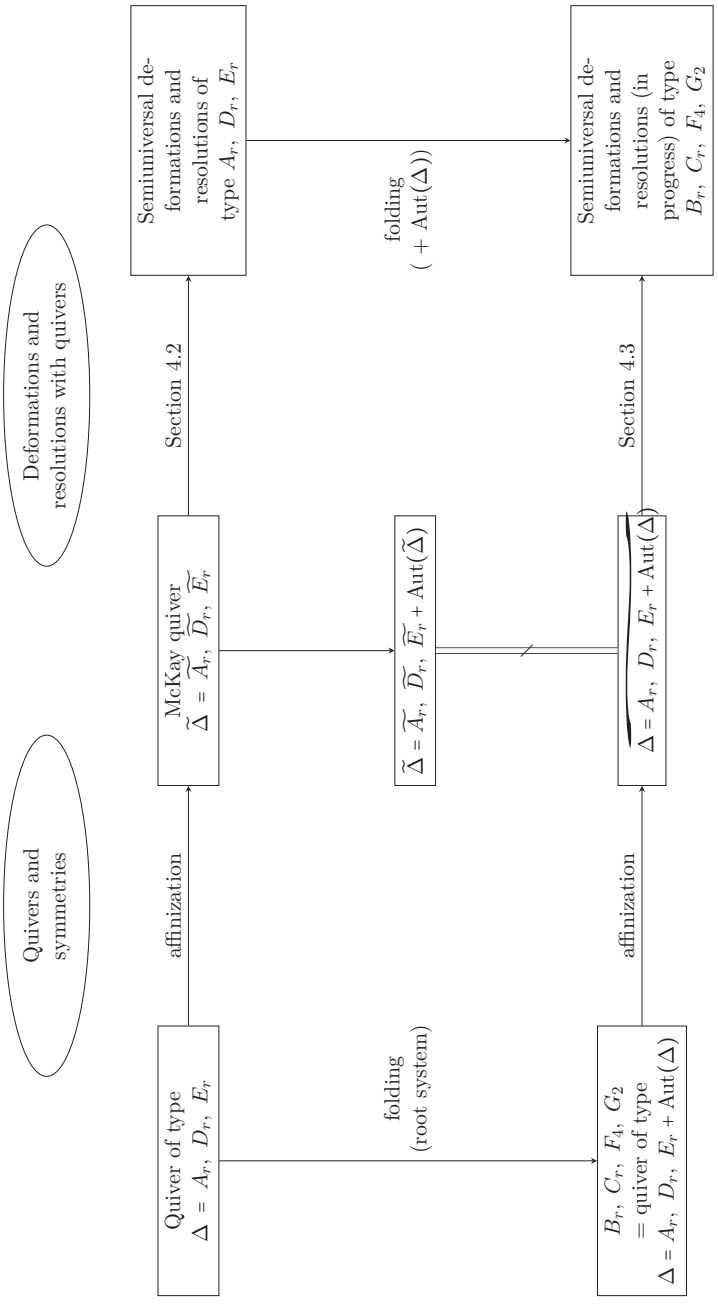


Figure 3.2. – Quivers, symmetries and simple singularities

Part II.

Quiver varieties and simple singularities

4. Deformations and quiver representations

Throughout this chapter the base field will be the complex number field $\mathbb{C}$.

4.1. Kronheimer's construction

P.B. Kronheimer gave in [Kron89] a new approach regarding the deformations and the resolutions of Kleinian singularities. His work is based on finite subgroups of SU_2 and hyperkähler quotients. Some of his ideas will be recalled in this section.

4.1.1. Definition of a hyperkähler manifold $M(\Gamma)$

Let Γ be a finite subgroup of SU_2, N its natural representation as a matrix group and R its regular representation. Set

$$M = \mathrm{End}(R) \otimes N$$

where $\mathrm{End}(R) = \mathrm{Hom}_{\mathbb{C}}(R, R)$. There is a linear action of Γ on M with the action on $\mathrm{End}(R)$ being the conjugation and the action on N the natural one. Let us focus on the space

$$M(\Gamma) := (\mathrm{End}(R) \otimes N)^{\Gamma}$$

of Γ-invariants of M.

By providing N with an orthonormal base, as $\dim_{\mathbb{C}} N = 2$, any element of M can be seen as a pair (α, β) with α, β elements of $\mathrm{End}(R)$.

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ be the field of quaternions. It is known that I, J, K satisfy the relations

$$I^2 = J^2 = K^2 = IJK = -1.$$

Fix a Γ-invariant Hermitian form on R and, if $a \in \mathrm{End}(R)$, let a^* denote the adjoint of a relative to the Hermitian form.
Let us give M an $\mathbb{H}$-module structure by defining the action of $\mathbb{H}$ with the following formulae:

$$\begin{aligned} I.(\alpha, \beta) &= (i\alpha, i\beta), \\ J.(\alpha, \beta) &= (-\beta^*, \alpha^*), \\ K.(\alpha, \beta) &= (-i\beta^*, i\alpha^*). \end{aligned}$$

Hence I, J and K can be seen as elements of $\mathrm{End}(TM)$, where every tangent space of M is identified with M.

Let $\mathrm{U}(R)$ be the subgroup of unitary transformations of $\mathrm{End}(R)$ with respect to the Γ-invariant Hermitian form on R, and set

$$\mathrm{U}(\Gamma) = \{g \in \mathrm{U}(R) \mid g\gamma = \gamma g \ \ \forall \gamma \in \Gamma\}$$

the subgroup of $\mathrm{U}(R)$ composed of elements commuting with the action of Γ. $\mathrm{U}(\Gamma)$ acts naturally on $M(\Gamma)$ by conjugation:

$$u.(\alpha, \beta) = (u\alpha u^{-1}, u\beta u^{-1}), \quad \forall u \in \mathrm{U}(\Gamma), \forall (\alpha, \beta) \in M(\Gamma).$$

It is a classical result of finite group theory that R can be decomposed as a sum of the irreducible representations of Γ. Let $R_0, R_1, \ldots, R_r$ denote the irreducible representations of Γ with R_0 being the trivial one and set $d_i = \dim R_i$. Then

$$R = \bigoplus_{i=0}^{r} R_i \otimes \mathbb{C}^{d_i}.$$

One can also decompose $\mathrm{U}(\Gamma)$. The elements of $\mathrm{End}(R)$ commuting with Γ are $\mathrm{Hom}_\Gamma(R, R) = \mathrm{Hom}_\Gamma(\bigoplus_{i=0}^r R_i \otimes \mathbb{C}^{d_i}, \bigoplus_{i=0}^r R_i \otimes \mathbb{C}^{d_i}) = \bigoplus_{i,j} \mathrm{Hom}_\Gamma(R_i, R_j) \otimes \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$. However Schur's lemma implies that the elements of $\mathrm{Hom}_\Gamma(R_i, R_j)$ are scalars if $i = j$ or are invertible or zero if $i \neq j$. But by definition $R_i \ncong R_j$, therefore $\mathrm{Hom}_\Gamma(R_i, R_j) = \{0\}$ if $i \neq j$ and $\mathrm{Hom}_\Gamma(R_i, R_i) \cong \mathbb{C}$. It follows that $\mathrm{Hom}_\Gamma(R, R) = \bigoplus_{i=0}^r \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_i})$ and looking at the unitary elements one finds

$$\mathrm{U}(\Gamma) \cong \prod_{i=0}^{r} \mathrm{U}_{d_i}(\mathbb{C}).$$

Hence any element of $\mathrm{U}(\Gamma)$ can be seen as an $(r+1)$-uplet $(u_0, u_1, \ldots, u_r)$ with $u_i \in \mathrm{U}_{d_i}(\mathbb{C})$ for any $0 \leq i \leq r$. Furthermore, one sees that

$$\begin{aligned} M(\Gamma) &= (\mathrm{End}(R) \otimes N)^{\Gamma} = (R^* \otimes R \otimes N)^{\Gamma}, \\ &= \mathrm{Hom}_{\Gamma}(R, R \otimes N) = \mathrm{Hom}_{\Gamma}(\bigoplus_{i=0}^{r} R_i \otimes \mathbb{C}^{d_i}, \bigoplus_{j=0}^{r} R_j \otimes N \otimes \mathbb{C}^{d_j}), \\ &= \bigoplus_{i,j=0}^{r} \mathrm{Hom}_{\Gamma}(R_i, R_j \otimes N) \otimes \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}). \end{aligned}$$

Any element of $M(\Gamma)$ can be written as $\sum_{i,j=0}^{r} r_{ij} \otimes f_{ij}$ with $r_{ij} \in \mathrm{Hom}_{\Gamma}(R_i, R_j \otimes N)$ and $f_{ij} \in \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$. The action of $\mathrm{U}(\Gamma)$ on $M(\Gamma)$ is given by

$$(u_0, u_1, \ldots, u_r).(\sum_{i,j=0}^{r} r_{ij} \otimes f_{ij}) = \sum_{i,j=0}^{r} r_{ij} \otimes u_j f_{ij} u_i^{-1}.$$

It can be verified that the action of $\mathrm{U}(\Gamma)$ on $M(\Gamma)$ preserves the quaternionic structure, i.e. the induced action of $\mathrm{U}(\Gamma)$ on $\mathrm{End}(TM(\Gamma))$ preserves the real vector space generated by I, J and K. One can thus obtain three real $\mathrm{U}(\Gamma)$-invariant symplectic structures on $M(\Gamma)$. More specifically, one can choose a $\mathrm{U}(\Gamma)$-invariant scalar product $\langle ., . \rangle$ on $M(\Gamma)$ such that the quaternionic operators I, J and K are anti-self-adjoint with respect to $\langle ., . \rangle$. One finds three real symplectic forms given by

$$\begin{aligned} \omega_I(v, w) &= \langle v, Iw \rangle, \\ \omega_J(v, w) &= \langle v, Jw \rangle, \\ \omega_K(v, w) &= \langle v, Kw \rangle, \end{aligned}$$

for any $v, w \in TM(\Gamma)$. As $M(\Gamma)$ is a vector space, for any $x \in M(\Gamma)$, one has $T_x M(\Gamma) \cong M(\Gamma)$. Hence the symplectic forms I, J and K can be seen as elements of $\mathrm{End}(M(\Gamma))$.

One also notices that the group of scalars $\mathbb{T} = \{(\lambda \mathrm{Id}_{d_0}, \ldots, \lambda \mathrm{Id}_{d_r}) \in \prod_{i=0}^{r} \mathrm{U}_{d_i}(\mathbb{C}) \mid \lambda \in \mathrm{U}_1(\mathbb{C}) \cong S^1\} \subset \mathrm{U}(\Gamma)$ acts trivially on $M(\Gamma)$ and so $\mathrm{U}(\Gamma)/\mathbb{T}$ acts on $M(\Gamma)$. Let $\mathfrak{su}(\Gamma)$ denote the Lie algebra of $\mathrm{U}(\Gamma)/\mathbb{T}$. Three moment maps $\mu_I, \mu_J, \mu_K : M(\Gamma) \to \mathfrak{su}(\Gamma)^*$ can be obtained, each relative to one symplectic structure. They are given by

$$\mu_I(\alpha,\beta) = \tfrac{i}{2}([\alpha,\alpha^*] + [\beta,\beta^*]),$$
$$\mu_J(\alpha,\beta) = \tfrac{1}{2}([\alpha,\beta] + [\alpha^*,\beta^*]),$$
$$\mu_K(\alpha,\beta) = \tfrac{i}{2}(-[\alpha,\beta] + [\alpha^*,\beta^*]).$$

With these maps one defines two new moment maps $\mu_{\mathbb{H}}$ and $\mu_{\mathbb{C}}$ on $M(\Gamma)$ by

$$\begin{array}{rccl} \mu_{\mathbb{H}}: & M(\Gamma) & \to & \mathfrak{su}(\Gamma)^* \otimes \mathbb{H}_0 \\ & (\alpha,\beta) & \mapsto & \mu_I(\alpha,\beta)\otimes I + \mu_J(\alpha,\beta)\otimes J + \mu_K(\alpha,\beta)\otimes K \\ & & \text{and} & \\ \mu_{\mathbb{C}} := \mu_J + i\mu_K: & M(\Gamma) & \to & \mathfrak{su}(\Gamma)^* \otimes \mathbb{C} \\ & (\alpha,\beta) & \mapsto & [\alpha,\beta] \end{array}$$

with $\mathbb{H}_0 = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ the set of pure imaginaries.

4.1.2. Hyperkähler quotients and simple singularities

Let $\mathfrak{u}(\Gamma)$ be the Lie algebra of $\mathrm{U}(\Gamma)$ and define the following form

$$\begin{array}{rccc} <.,.>: & \mathfrak{u}(\Gamma)\times\mathfrak{u}(\Gamma) & \to & \mathbb{C} \\ & (A,B) & \mapsto & \mathrm{Tr}(AB^*) = -\mathrm{Tr}(AB) \end{array}.$$

It is bilinear, symmetric, non-degenerate and invariant under the adjoint action of $\mathrm{U}(\Gamma)$. Set $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}) = \{(\lambda\mathrm{Id}_{d_0},\dots,\lambda\mathrm{Id}_{d_r}) \in \prod_{i=1}^r \mathfrak{u}_{d_i}(\mathbb{C})) \mid \lambda \in \mathfrak{u}_1(\mathbb{C})\}$ and $\mathfrak{t}^\perp = \{u \in \mathfrak{u}(\Gamma) \mid < u,t >= 0, \forall t \in \mathfrak{t}\} = \{u \in \mathfrak{u}(\Gamma) \mid \mathrm{Tr}(u) = 0\}$. The restriction of the bilinear form $<.,.>$ to $\mathfrak{t}^\perp \times \mathfrak{u}(\Gamma)$, induces the bilinear form $<.,.>: \mathfrak{t}^\perp \times \mathfrak{u}(\Gamma)/\mathfrak{t} \to \mathbb{C}$, which is symmetric and non-degenerate. Therefore

$$\begin{array}{ccc} \mathfrak{t}^\perp & \to & (\mathfrak{u}(\Gamma)/\mathfrak{t})^* \\ u & \mapsto & <u,.> \end{array}$$

is an isomorphism. But it is known that $\mathfrak{t}^\perp = \mathfrak{su}(\Gamma)$, hence $\mathfrak{su}(\Gamma) \cong (\mathfrak{u}(\Gamma)/\mathfrak{t})^*$. This isomorphism turns $(\mathfrak{u}(\Gamma)/\mathfrak{t})^*$ into a Lie algebra. Let $\mathfrak{c}$ be the center of $(\mathfrak{u}(\Gamma)/\mathfrak{t})^*$. Then the isomorphism gives

$$\mathfrak{c} \cong \{(z_0\mathrm{Id}_{d_0}, \dots, z_r\mathrm{Id}_{d_r}) \in \mathfrak{su}(\Gamma) \mid z_i \in \mathfrak{u}_1(\mathbb{C}) \text{ and } \sum_{i=0}^{r} z_i d_i = 0\}.$$

Let $\pi_j : R \to R_j \otimes \mathbb{C}^{d_j}$ denote the canonical projection, $0 \leq j \leq r$. The $i\pi_j$ (here i is the square root of -1, not an index) generate the center of $\mathfrak{u}(\Gamma)$. Set

$$\begin{array}{rccc} \rho : & \mathfrak{z}(\mathfrak{u}(\Gamma)) & \to & \mathfrak{h}^*_{\mathbb{R}} \\ & i\pi_j & \mapsto & d_j\alpha_j \end{array}$$

with $(\alpha_j)_{1\leq j\leq r}$ a set of simple roots of the Lie algebra $\mathfrak{g}$ of type $\Delta(\Gamma)$ relative to a Cartan subalgebra $\mathfrak{h}$, $\mathfrak{h}^*_{\mathbb{R}} = \{\sum_{i=1}^r a_i\alpha_i \mid a_i \in \mathbb{R}, \forall i\}$ the $\mathbb{R}$-span of the roots of $\mathfrak{g}$, and $\alpha_0 = -\sum_{j=1}^r d_j\alpha_j$. The root system of $\mathfrak{g}$ is denoted by Φ. ρ is surjective and its kernel is $\mathfrak{t}$ thus ρ induces an isomorphism $\begin{array}{rccc} \overline{\rho} : & \mathfrak{z}(\mathfrak{u}(\Gamma))/\mathfrak{t} & \overset{\cong}{\to} & \mathfrak{h}^*_{\mathbb{R}} \\ & i\pi_j + \mathfrak{t} & \mapsto & d_j\alpha_j \end{array}$. However, we know that $\mathfrak{c} = \mathfrak{z}((\mathfrak{u}(\Gamma)/\mathfrak{t})^*) = (\mathfrak{z}(\mathfrak{u}(\Gamma))/\mathfrak{t})^* = \mathfrak{z}(\mathfrak{su}(\Gamma))$, therefore the transpose of $\overline{\rho}$ gives an isomorphism

$$\begin{array}{rccl} \tau^{-1} : & \mathfrak{h}_{\mathbb{R}} & \overset{\cong}{\to} & \mathfrak{c} \\ & h & \mapsto & z \text{ with } z(\pi_j) = h \circ \rho(\pi_j) = d_j\alpha_j(h) \end{array}$$

with $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{R},\ \forall \alpha \in \Phi\} = \{\sum_{i=1}^r a_i\alpha_i^\vee \mid a_i \in \mathbb{R}, \forall i\}$. By definition, $\alpha_i^\vee$ is the only element of $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_i}]$ such that $\alpha_i(\alpha_i^\vee) = 2$ and is called the coroot associated to α_i. As $\pi_j \in \mathfrak{u}(\Gamma)/\mathfrak{t}$ and $z \in \mathfrak{su}(\Gamma) \cong (\mathfrak{u}(\Gamma)/\mathfrak{t})^*$, it implies that $z(\pi_j) =< z, \pi_j >= -\mathrm{Tr}(\pi_j z) = -d_j z_j$.

Finally we obtain an isomorphism

$$\begin{array}{rccl} \tau : & \mathfrak{c} & \overset{\cong}{\to} & \mathfrak{h}_{\mathbb{R}} \\ & (z_0\mathrm{Id}_{d_0}, \dots, z_r\mathrm{Id}_{d_r}) & \mapsto & h \text{ with } \alpha_j(h) = -z_j, \quad \forall\ 0 \leq j \leq r \end{array} .$$

There is a natural identification of $\mathfrak{c} \otimes \mathbb{C} = \mathfrak{c} \otimes J + \mathfrak{c} \otimes K$ with the Cartan subalgebra $\mathfrak{h}$ of type $\Delta(\Gamma)$.

If α is a root, set $D_\alpha = \mathrm{Ker}(\alpha \circ \tau) \subset \mathfrak{c}$. If $\xi \in \mathfrak{c} \setminus \bigcup_\alpha D_\alpha$, ξ is called **generic**. It is known that $\mathfrak{c} = \mathfrak{su}(\Gamma)^{\mathrm{U}(\Gamma)}$. Hence $\mathrm{U}(\Gamma)$ acts on every fiber $\mu_{\mathbb{H}}^{-1}(\xi)$, with $\xi = \xi_1 I + \xi_2 J + \xi_3 K \in \mathfrak{c} \otimes \mathbb{H}_0$. Let X_ξ denote the quotient $\mu_{\mathbb{H}}^{-1}(\xi)/\mathrm{U}(\Gamma)$. It was explained in Section 2.2 that, upon some hypotheses, X_ξ can have a symplectic structure. This result can be strengthened. First let us define the **good set** $(\mathfrak{c} \otimes_{\mathbb{R}} \mathbb{H}_0)^\circ$ (denomination due to P.B. Kronheimer) by

$$(\mathfrak{c} \otimes_{\mathbb{R}} \mathbb{H}_0)^\circ = (\mathfrak{c} \otimes_{\mathbb{R}} \mathbb{H}_0) \setminus \bigcup_\alpha (D_\alpha \otimes_{\mathbb{R}} \mathbb{H}_0).$$

where α runs through the roots of the root system of type $\Delta(\Gamma)$. The next proposition is proved in [Kron89].

Proposition 4.1.1. *If $\xi \in (\mathfrak{c} \otimes_{\mathbb{R}} \mathbb{H}_0)^\circ$, then X_ξ is a non-singular hyperkähler manifold of dimension 4 over $\mathbb{R}$.*

The main theorem of this section can now be stated. Its proof may be found in [Kron89].

Theorem 4.1.2. (Kronheimer). *We have*

$$\mu_{\mathbb{H}}^{-1}(0)/\mathrm{U}(\Gamma) \cong \mathbb{C}^2/\Gamma.$$

Furthermore, the complex r-parameter family

$$\mu_{\mathbb{H}}^{-1}(\mathfrak{c} \otimes \mathbb{C})/\mathrm{U}(\Gamma) \longrightarrow \mathfrak{c} \otimes \mathbb{C}$$

is the pullback (via $\mathfrak{h} \to \mathfrak{h}/W$ with W the Weyl group) of the semiuniversal deformation of $\mathbb{C}^2/\Gamma$.
For $\xi_1 \in \mathfrak{c}$ generic, one obtains a simultaneous resolution of the previous family by the following diagram:

$$\begin{array}{ccc} \mu_{\mathbb{H}}^{-1}(\xi_1 I + \mathfrak{c} \otimes \mathbb{C})/\mathrm{U}(\Gamma) & \longrightarrow & \mu_{\mathbb{H}}^{-1}(\mathfrak{c} \otimes \mathbb{C})/\mathrm{U}(\Gamma) \\ \downarrow & \circlearrowleft & \downarrow \\ \xi_1 I + \mathfrak{c} \otimes \mathbb{C} & \xrightarrow{\cong} & \mathfrak{c} \otimes \mathbb{C} \end{array}$$

Finally, all resolutions obtained in this way are minimal.

We conclude this section by mentioning that P.B. Kronheimer gave in [Kron89] an interpretation of $M(\Gamma)$ in terms of quivers. This interpretation will be presented in the next section and is the base of a construction by H. Cassens and P. Slodowy of the semiuniversal deformations of Kleinian singularities using geometric invariant theory and quiver theory.

4.2. Deformations of homogeneous simple singularities

In [CaSlo98] H. Cassens and P. Slodowy gave a construction of the semiuniversal deformations of the simple singularities based on quiver theory and P.B. Kronheimer's work as well as H. Cassens' Ph.D thesis [Cas95]. This construction is presented in the following subsection.

4.2.1. Presentation of the construction

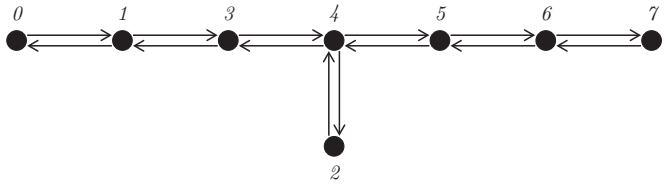
Let Γ be a finite subgroup of SU_2 and $\Delta(\Gamma)$ the associated Dynkin diagram (cf. Subsection 1.4.1.2). We write $R_0, \ldots, R_r$ for the irreducible representations of Γ, R_0 being the trivial one, and N for the natural representation obtained through the inclusion $\Gamma \subset \mathrm{SU}_2$. The regular representation is $R = \bigoplus_{i=0}^{r} d_i R_i$ with $d_i = \dim R_i$. The space $M(\Gamma)$ of the previous section can be interpreted in terms of quivers in the following way:

$$\begin{aligned}
M(\Gamma) &= (\mathrm{End}(R) \otimes N)^{\Gamma} = (R^* \otimes R \otimes N)^{\Gamma}, \\
&= \mathrm{Hom}_{\Gamma}(R, R \otimes N) = \mathrm{Hom}_{\Gamma}(\bigoplus_{i=0}^{r} R_i \otimes \mathbb{C}^{d_i}, \bigoplus_{j=0}^{r} R_j \otimes N \otimes \mathbb{C}^{d_j}), \\
&= \bigoplus_{i,j=0}^{r} \mathrm{Hom}_{\Gamma}(R_i, R_j \otimes N) \otimes \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}), \\
&= \bigoplus_{\overset{i}{\bullet}\!\!-\!\!\!-\!\!\overset{j}{\bullet}} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) \text{ by McKay's correspondence,} \\
&= \mathrm{Rep}(Q, \underline{d}),
\end{aligned}$$

for a quiver Q whose vertices are the vertices of the extended Dynkin diagram $\widetilde{\Delta}(\Gamma)$, with two arrows (one in each direction) for any edge in $\widetilde{\Delta}(\Gamma)$, and whose dimension vector is $\underline{d} = (d_0, d_1, \ldots, d_r)$. It is called a **McKay quiver**. For every arrow $a : i \to j$ of Q, the opposite arrow $j \to i$ is denoted $\bar{a}$.

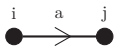
Remark 4.2.1. The dimension vector $\underline{d}$ associated to the McKay quiver Q based on $\widetilde{\Delta}(\Gamma)$ is the same as the minimal imaginary root of the extended root system of type $\widetilde{\Delta}(\Gamma)$ (cf. [Kac90]).

Example 4.2.2. *Take $\Gamma = \mathcal{O}$. Then $\Delta(\Gamma) = E_7$ and the quiver Q is*

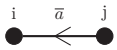


with dimension vector $\underline{d} = \{1, 2, 2, 3, 4, 3, 2, 1\}$.

The group $\prod_{i=0}^{r} \mathrm{GL}_{d_i}(\mathbb{C})$ acts on $M(\Gamma)$ by simultaneous conjugation and $\mathbb{C}^* = \{(z\mathrm{Id}_{d_0}, \ldots, z\mathrm{Id}_{d_r}) \mid z \in \mathbb{C}^*\}$ acts trivially. Therefore an action of $G(\Gamma) = (\prod_{i=0}^{r} \mathrm{GL}_{d_i}(\mathbb{C}))/\mathbb{C}^*$ on $M(\Gamma)$ is induced. We are going to define a $G(\Gamma)$-invariant symplectic form on $M(\Gamma)$. An orientation of Q is a function $\epsilon : Q_1 \to \mathbb{C}^*$, such that $\epsilon(\overline{a}) = -\epsilon(a) = -1$ for every arrow



belonging to a fixed orientation of the edges of Q and its opposite arrow



For every pair $\varphi = (\varphi_a, a \in Q_1), \psi = (\psi_a, a \in Q_1)$ of elements of $M(\Gamma)$, set

$$\langle \varphi, \psi \rangle = \sum_{a \in Q_1} \epsilon(a) \mathrm{Tr}(\varphi_a \psi_{\overline{a}}).$$

Then one can verify that $\langle ., . \rangle$ is a non-degenerate $G(\Gamma)$-invariant symplectic form on $M(\Gamma)$ and induces a moment map

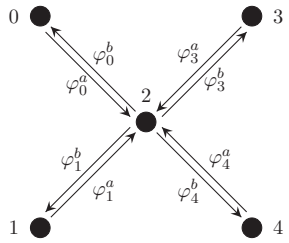
$$\mu_{CS} : M(\Gamma) \to (\mathrm{Lie}\ G(\Gamma))^* \subset \bigoplus_{i=0}^{r} M_{d_i}(\mathbb{C})$$

given by

$$\mu_{CS}(\varphi) = (\ldots, \underbrace{\sum_{\text{j a i}} \epsilon(a) \varphi_a \varphi_{\overline{a}}}_{i\text{-th entry}}, \ldots).$$

Here Lie $G(\Gamma)$ is identified with its dual $(\mathrm{Lie}\ G(\Gamma))^*$ using the bilinear form $< ., . >$ defined at the beginning of Subsection 4.1.2.

Example 4.2.3. *For $\Gamma = \mathcal{D}_2$, the associated Dynkin diagram is $\Delta(\Gamma) = D_4$. A representation of the quiver Q is denoted by $\varphi = (\varphi_i^j)$ with $i = 0, 1, 3$ or 4 and $j = a$ or b. The complete notation can be found on the following diagram*



For the orientation, take the φ^a's to be positive. The moment map is then

$$\mu_{CS}(\varphi) = (-\varphi_0^b\varphi_0^a, -\varphi_1^b\varphi_1^a, \varphi_0^a\varphi_0^b + \varphi_1^a\varphi_1^b + \varphi_3^a\varphi_3^b + \varphi_4^a\varphi_4^b, -\varphi_3^b\varphi_3^a, -\varphi_4^b\varphi_4^a).$$

It can be verified through explicit computations that μ_{CS} is equal to the map $\mu_{\mathbb{C}}$ defined in the previous section.

Let Z be the dual of the center of Lie $G(\Gamma)$. As previously seen, by means of the bilinear form from Subsection 4.1.2, Z is identified with

$$\mathfrak{c} \otimes \mathbb{C} = \{(\mu_0 \mathrm{Id}_0, \mu_1 \mathrm{Id}_1, \ldots, \mu_r \mathrm{Id}_r) \in \prod_{i=0}^{r} M_{d_i}(\mathbb{C}) \mid \mu_i \in \mathbb{C}, \sum_{i=0}^{r} d_i\mu_i = 0\}.$$

As the moment map is $G(\Gamma)$-equivariant, for all $z \in Z$, $G(\Gamma)$ acts on the fiber $\mu_{CS}^{-1}(z)$ and we can form the quotient $\mu_{CS}^{-1}(z)//G(\Gamma)$. Moreover Kempf-Ness' Theorem (Theorem 2.3.7) and Proposition 2.3.10 can be adapted to work on a symplectic vector space instead of a compact symplectic manifold (cf. [Kiri16] Sections 9.9 and 9.10). It follows that there exists an isomorphism

$$\mu_{CS}^{-1}(z)//G(\Gamma) = \mu_{\mathbb{C}}^{-1}(z)//G(\Gamma) \cong \mu_{\mathbb{H}}^{-1}(0, \zeta_2, \zeta_3)/\mathrm{U}(\Gamma)$$

with $z = \xi_2 + i\xi_3 \in \mathfrak{c} \otimes \mathbb{C} = Z$. Furthermore, because of Theorem 4.1.2, one obtains that

$$\mu_{\mathbb{H}}^{-1}(Z)/\mathrm{U}(\Gamma) \longrightarrow Z$$

is the pullback of the semiuniversal deformation of the Kleinian singularity $\mathbb{C}^2/\Gamma$. We hence have a construction of the pullback of the semiuniversal deformation of $\mathbb{C}^2/\Gamma$ purely in terms of invariant theory:

$$\begin{array}{ccc} \mu_{CS}^{-1}(Z)//G(\Gamma) & \xrightarrow{\cong} & \mathcal{X} \times_{\mathfrak{h}/W} \mathfrak{h} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\cong} & \mathfrak{h} \end{array}$$

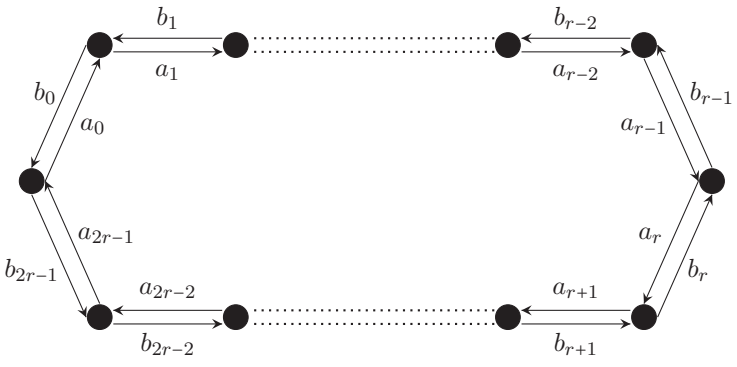
Remark 4.2.4. H. Cassens and P. Slodowy also used the space $M(\Gamma)$ in [CaSlo98] to work on resolutions of the Kleinian singularities. Indeed, by linearizing the quotient $\mu_{CS}^{-1}(Z)//G(\Gamma)$ with a well suited character $\chi : G(\Gamma) \to \mathbb{C}^*$ (cf. [CaSlo98] Section 7), they obtain the following diagram:

$$\begin{array}{ccc} \mu_{CS}^{-1}(Z)//^{\chi}G(\Gamma) & \longrightarrow & \mu_{CS}^{-1}(Z)//G(\Gamma) \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\cong} & \mathfrak{c} \otimes \mathbb{C} \end{array}$$

which is a simultaneous minimal resolution of the simple singularity $\mathbb{C}^2/\Gamma$.

4.2.2. Example: $\widetilde{A}_{2r-1}$

Start with the McKay quiver of type $\widetilde{A}_{2r-1}$ with dimension vector $(1, \ldots, 1)$:



.

The representation space of this quiver is $M(\Gamma) = \{(a_0, \dots a_{2r-1}, b_0, \dots, b_{2r-1}) \in \mathbb{C}^{2r} \oplus \mathbb{C}^{2r}\}$. The orientation of the quiver is fixed by $\epsilon(a_i) = -\epsilon(b_i) = 1, \quad 0 \leq i \leq 2r-1$. Moreover $G(\Gamma) = (\mathbb{C}^*)^{2r}/\mathbb{C}^* \cong (\mathbb{C}^*)^{2r-1}$ and the action of any $g \in G(\Gamma)$ on an element of $M(\Gamma)$ is done by conjugation:

$$g.(a_0, \dots a_{2r-1}, b_0, \dots, b_{2r-1}) = (g_1 g_0^{-1} a_0, \dots g_0 g_{2r-1}^{-1} a_{2r-1}, g_0 g_1^{-1} b_0, \dots, g_{2r-1} g_0^{-1} b_{2r-1}).$$

The fundamental invariants are then $\begin{cases} z_i = a_i b_i, \ 0 \leq i \leq 2r-1, \\ x = a_0 \dots a_{2r-1}, \\ y = b_0 \dots b_{2r-1}, \end{cases}$ with the relation

$$z_0 z_1 \dots z_{2r-1} = xy.$$

Therefore $M(\Gamma)//G(\Gamma)$ is the hypersurface

$$\{(z_0, z_1, \dots, z_{2r-1}, x, y) \in \mathbb{C}^{2r+2} \mid z_0 z_1 \dots z_{2r-1} = xy\}.$$

The moment map for the action of $G(\Gamma)$ on $M(\Gamma)$ is

$$\begin{array}{rccc} \mu : & M(\Gamma) & \to & \prod_{i=0}^{2r-1} \mathbb{C} \\ & \varphi & \mapsto & (\dots, \sum_{\bullet_j \xrightarrow{a} \bullet_i} \epsilon(a) \varphi_a \varphi_{\overline{a}}, \dots) \end{array} .$$

By quotient of the previous map one gets

$$\begin{array}{ccc} M(\Gamma)//G(\Gamma) & \xrightarrow{\overline{\mu}} & Z = \{(\mu_0, \dots, \mu_{2r-1}) \in \prod_{i=0}^{2r-1} \mathbb{C} \mid \sum_{i=0}^{2r-1} \mu_i = 0\} \\ (z_0, \dots, z_{2r-1}, x, y) & \mapsto & (z_{2r-1} - z_0, z_0 - z_1, \dots, z_{2r-2} - z_{2r-1}) \end{array} .$$

We change the variables by setting $\begin{cases} z = \dfrac{1}{2r} \sum_{i=0}^{2r-1} z_i, \\ \lambda_i = z - z_i, \ 0 \leq i \leq 2r-1. \end{cases}$ Furthermore, Z is identified with the Cartan subalgebra $\mathfrak{h} = \{\mathrm{diag}(\lambda_0, \dots, \lambda_{2r-1}) \mid \sum_{i=0}^{2r-1} \lambda_i = 0\} \subset \mathfrak{sl}_{2r}(\mathbb{C})$ through the isomorphism $\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\cong} & Z \\ (\lambda_0, \dots, \lambda_{2r-1}) & \mapsto & (\lambda_1 - \lambda_0, \dots, \lambda_0 - \lambda_{2r-1}) \end{array}$.

The computations in this example are fairly simple because $\mu(M(\Gamma)) = Z$. For the types D_r and E_6 it is not the case anymore.

In the new coordinate system, one finds

$$\begin{array}{rcl} \{\prod_{i=0}^{2r-1}(z-\lambda_i) = xy\} = M(\Gamma)//G(\Gamma) & \stackrel{\widetilde{\alpha}}{\to} & \mathfrak{h} \\ (\lambda_0,\ldots,\lambda_{2r-1},x,y,z) & \mapsto & (\lambda_0,\ldots,\lambda_{2r-1}) \end{array}$$

which is the pullback of the semiuniversal deformation of the hypersurface $X_0 = \widetilde{\alpha}^{-1}(0) = \{(x,y,z) \in \mathbb{C}^3 \mid z^{2r} = xy\}$, i.e. a simple singularity of type A_{2r-1}.

The following diagram

$$\begin{array}{ccc} X \times_{\mathfrak{h}/W} \mathfrak{h} & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle \widetilde{\alpha}} & \circlearrowleft & \downarrow{\scriptstyle \alpha} \\ \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W \end{array}$$

is the pullback of the semiuniversal deformation $X \stackrel{\alpha}{\to} \mathfrak{h}/W$ of X_0. W is the Weyl group of type A_{2r-1}, i.e. $\mathfrak{S}_{2r}$.

It is known that $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_0,\ldots,x_{2r-1}]/(\sum_{i=0}^{2r-1} x_i)$ and $\mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[\mathfrak{h}]^W$, $= \mathbb{C}[\epsilon_2(x),\ldots,\epsilon_{2r}(x)]$, with ϵ_i the i-th elementary symmetric polynomial with $2r$ variables. There is a natural injection $\mathbb{C}[\mathfrak{h}/W] \hookrightarrow \mathbb{C}[\mathfrak{h}]$ which induces

$$\begin{array}{rccc} \pi: & \mathfrak{h} & \twoheadrightarrow & \mathfrak{h}/W \\ & (\lambda_0,\ldots,\lambda_{2r-1}) & \mapsto & (\epsilon_2(\lambda),\ldots,\epsilon_{2r}(\lambda)) \end{array}.$$

A polynomial expansion gives $\prod_{i=0}^{2r-1}(z-\lambda_i) = z^{2r} + \sum_{i=2}^{2r}(-1)^i \epsilon_i(\lambda) z^{2r-i}$. Set

$$X = \{(x,y,z,t_2,\ldots,t_{2r}) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^{2r} + \sum_{i=2}^{2r}(-1)^i t_i z^{2r-i} = xy\}.$$

There are two natural maps

$$\begin{array}{rccc} \alpha: & X & \to & \mathfrak{h}/W \\ & (x,y,z,t_2,\dots,t_{2r}) & \mapsto & (t_2,\dots,t_{2r}) \end{array} \quad \text{and}$$

$$\begin{array}{rccc} \pi: & \mathfrak{h} & \twoheadrightarrow & \mathfrak{h}/W \\ & (\lambda_0,\dots,\lambda_{2r-1}) & \mapsto & (\epsilon_2(\lambda),\dots,\epsilon_{2r}(\lambda)) \end{array} .$$

The pullback of these maps gives

$$\begin{aligned} & X \times_{\mathfrak{h}/W} \mathfrak{h} \\ = \ & \{((x,y,z,t_2,\dots,t_{2r}),(\lambda_0,\dots,\lambda_{2r-1})) \in X \times \mathfrak{h} \mid \alpha(x,y,z,t_2,\dots,t_{2r}) = \pi(\lambda_0,\dots,\lambda_{2r-1})\}, \\ = \ & \{((x,y,z,t_2,\dots,t_{2r}),(\lambda_0,\dots,\lambda_{2r-1})) \in X \times \mathfrak{h} \mid (t_2,\dots,t_{2r}) = (\epsilon_2(\lambda),\dots,\epsilon_{2r}(\lambda))\}, \\ = \ & \{(x,y,z,\lambda_0,\dots,\lambda_{2r-1}) \in \mathbb{C}^3 \times \mathfrak{h} \mid z^{2r} + \sum_{i=2}^{2r} (-1)^i \epsilon_i(\lambda) z^{2r-i} = xy\}, \\ = \ & \{(x,y,z,\lambda_0,\dots,\lambda_{2r-1}) \in \mathbb{C}^3 \times \mathfrak{h} \mid \prod_{i=0}^{2r-1} (z-\lambda_i) = xy\}, \\ = \ & M(\Gamma)//G(\Gamma), \end{aligned}$$

and $\alpha^{-1}(0) = \{(x,y,z) \in \mathbb{C}^3 \mid z^{2r} = xy\}$. It is a simple singularity of type A_{2r-1} and α is its semiuniversal deformation.

4.3. Deformations of inhomogeneous simple singularities

The aim of this section is to extend the results of H. Cassens and P. Slodowy to the inhomogeneous simple singularities. This means that we aim to construct the semiuniversal deformations of the simple singularities of type B_r, C_r, F_4 and G_2 using McKay quivers. First we will lay out the objectives of the task. Then computations will be made in order to ensure that the objectives are validated. The semiuniversal deformations will then be explicitly constructed. However, as the semiuniversal deformation of a simple inhomogeneous singularity is a Ω-equivariant morphism for Ω a symmetry group of the simply-laced Dynkin diagram associated to the singularity, the fibers above the fixed points are all acted upon. We will then compute the quotients of these fibers and look at some of their properties.

4.3.1. Objectives

This section aims to extend the construction of Section 4.2 to the inhomogeneous simple singularities of type B_r, C_r, F_4 and G_2.

Let us start with a Dynkin diagram $\Delta(\Gamma)$ of type A_{2r-1}, D_{r+1} or E_6 with Γ being the associated finite group of SU_2. The notations and results of Section 4.2 give the following diagram:

$$\begin{array}{ccc}
\mu_{CS}^{-1}(Z) \subset M(\Gamma) & & \\
\big\downarrow & & \\
\mu_{CS}^{-1}(Z)//G(\Gamma) = X \times_{\mathfrak{h}/W} \mathfrak{h} & \xrightarrow{\psi} & X \\
\big\downarrow \widetilde{\alpha} & \circlearrowleft & \big\downarrow \alpha \\
Z \cong \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W
\end{array}$$

with α the semiuniversal deformation of the singularity $\mathbb{C}^2/\Gamma$ of type $\Delta(\Gamma)$. Let Γ' be the finite subgroup of SU_2 such that there exists an inhomogeneous simple singularity of type $\Delta(\Gamma, \Gamma')$ (cf. Definition 1.4.37 and the first table in Subsection 1.5.2.1). Then $\Omega = \Gamma'/\Gamma$ acts on the singularity $X_0 = \alpha^{-1}(0)$. Our aim is to define natural actions of Ω on X and $\mathfrak{h}/W$ such that α becomes Ω-equivariant. The next theorem is a direct consequence of Theorem 1.4.49.

Theorem 4.3.1. *With the previously defined notations, the restriction $\alpha|_{\alpha^{-1}((\mathfrak{h}/W)^\Omega)}$ over the fixed points $(\mathfrak{h}/W)^\Omega$ of a semiuniversal deformation of the singularity $\mathbb{C}^2/\Gamma$ is a semiuniversal deformation of the inhomogeneous singularity of type $\Delta(\Gamma, \Gamma')$.*

A natural way to accomplish this, is to render $\widetilde{\alpha}$ Ω-equivariant. By Lemma 2.1.11, it suffices to analyze when the action of Ω on $M(\Gamma)$ is symplectic.

We need to define actions of Ω on every object in the preceding diagram such that:

1. The restriction of the action on X to the singularity X_0 coincides with the natural action computed with matrices in Subsection 1.4.2.1.
2. The action on $M(\Gamma)$ is symplectic.

3. The action on $G(\Gamma)$ is induced from the one on $M(\Gamma)$.

4. The action on $\mathrm{Lie}(G(\Gamma))$ stabilizes Z. Hence the isomorphism $\tau : Z \to \mathfrak{h}$ (cf. Subsection 4.1.2) becomes Ω-equivariant, with the action on $\mathfrak{h}$ coming from the action on the Dynkin diagram $\Delta(\Gamma)$.

In Subsection 4.3.2, we will compute the conditions required on the action of Ω so that it restricts to the natural action on the singularity $\mathbb{C}^2/\Gamma$ seen in Subsection 1.4.2.1. In order for Ω to act symplectically on $M(\Gamma)$, it has to preserve the symplectic form $\langle .,. \rangle$. However, by definition, this form depends on an orientation of the McKay quiver based on $\Delta(\Gamma)$. The orientation will then have to be chosen accordingly, which will be done in Subsection 4.3.3. The questions regarding the action of Ω on $G(\Gamma)$ as well as the Ω-equivariance of $\tau : Z \to \mathfrak{h}$ will be handled in Subsection 4.3.4.

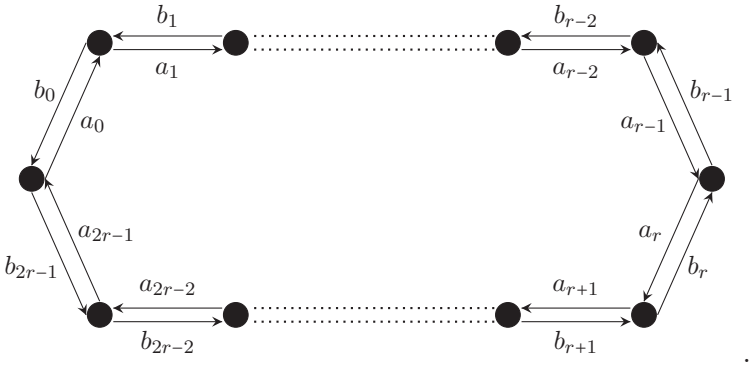
Because of Theorem 4.3.1, $\alpha|_{\alpha^{-1}((\mathfrak{h}/W)^\Omega)}$ is an Ω-invariant morphism, and as such the group Ω acts on every fiber of $\alpha|_{\alpha^{-1}((\mathfrak{h}/W)^\Omega)}$. In order to analyse the quotients of the fibers, an explicit description of the semiuniversal deformation is required. We will introduce in Subsection 4.3.5 a system of coordinates on the base space $\mathfrak{h}/W$ of the semiuniversal deformation called flat coordinates. It will simplify the computations because with this system the action of Ω on $\mathfrak{h}/W$ becomes linear. The remaining subsections will be devoted to the explicit computations of the semiuniversal deformations of simple singularities of inhomogeneous types. The study of the quotients of the fibers will be conducted in Section 4.4.

4.3.2. Action of Γ'/Γ on $\mathbb{C}^2/\Gamma$

The action of $\Omega = \Gamma'/\Gamma$ on $M(\Gamma)$ induces an action on $\mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma)$ and it is known that $\mathbb{C}^2/\Gamma \subset \mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma)$. However it has been seen in Subsection 1.4.2.1 that Γ'/Γ acts on the singularity $\mathbb{C}^2/\Gamma$ in a natural way. We therefore want to impose that the action of Γ'/Γ on $M(\Gamma)$ that is to be defined shall induce the right action on the singularity $\mathbb{C}^2/\Gamma$.

4.3.2.1. Computations for $\mathbb{C}^2/\Gamma$ of type $\widetilde{A}_{2r-1}$

The McKay quiver when $\Delta(\Gamma) = A_{2r-1}$ is



It is known that $\Gamma = \mathcal{C}_{2n}$, $\Gamma' = \mathcal{D}_n$ and $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$. The action of Ω on the Dynkin diagram exchanges the vertices i and $2r-i$, $1 \leq i \leq 2r-1$. We extend this action to the extended Dynkin diagram of type $\widetilde{\Delta}(\Gamma)$ by making Ω fix the additional vertex labelled 0. The action of Ω on $M(\Gamma)$ arises naturally from its action on the extended Dynkin diagram. Hence the action of Ω on $M(\Gamma)$ needs to verify

$$\sigma.(a_0, \ldots, a_{2r-1}, b_0, \ldots, b_{2r-1}) = (\lambda_{2r-1}b_{2r-1}, \ldots, \lambda_0 b_0, \delta_{2r-1}a_{2r-1}, \ldots, \delta_0 a_0).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

The action of $\mathbb{Z}/2\mathbb{Z}$ needs to be consistent with the action on the singularity computed with matrices. Recall that the singularity $\mathbb{C}^2/\Gamma$ is defined by $\{Z^{2n} - XY = 0\}$ with $Z = z_1 z_2, X = z_1^{2r}, Y = z_2^{2r}$, (cf. Theorem 1.4.1 for the notation), and that $\begin{cases} \sigma.X = (-1)^r Y, \\ \sigma.Y = (-1)^r X, \\ \sigma.Z = -Z. \end{cases}$

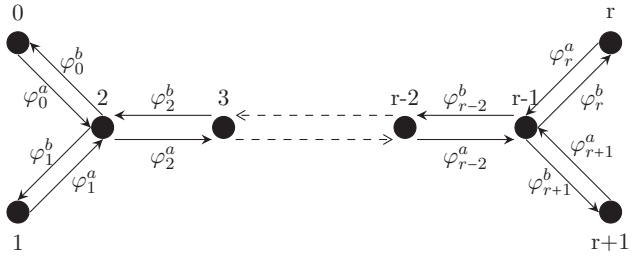
After computation of the special fiber of the moment map, we find that $\mathbb{C}^2/\Gamma \cong \{z^{2r} - xy = 0\}$ with $\begin{cases} x = \prod_{i=0}^{2r-1} a_i, \\ y = \prod_{i=0}^{2r-1} b_i, \\ z = \frac{1}{2r} \sum_{i=0}^{2r-1} a_i b_i, \end{cases}$ which implies

$$\begin{cases} \sigma.x = (\prod_{i=0}^{2r-1} \lambda_i) y, \\ \sigma.y = (\prod_{i=0}^{2r-1} \delta_i) x, \\ \sigma.z = \frac{1}{2r} \sum_{i=0}^{2r-1} \lambda_i \delta_i a_i b_i. \end{cases}$$

For the action to satisfy our conditions, it needs to verify $\begin{cases} \lambda_i\delta_i = -1, \quad 0 \leq i \leq 2r-1, \\ \prod_{i=0}^{2r-1}\lambda_i = \prod_{i=0}^{2r-1}\delta_i = (-1)^r. \end{cases}$

4.3.2.2. Computations for $\mathbb{C}^2/\Gamma$ of type $\widetilde{D}_{r+1}$

The McKay quiver when $\Delta(\Gamma) = D_{r+1}$ is



It is known that $\Gamma = \mathcal{D}_{r-1}$, $\Gamma' = \mathcal{D}_{2(r-1)}$ and $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$. The action of Ω on the Dynkin diagram exchanges the vertices r and $r+1$ and fixes the others. We extend this action to the extended Dynkin diagram of type $\widetilde{\Delta}(\Gamma)$ by making Ω fix the additional vertex labelled 0. The action of Ω on $M(\Gamma)$ arises naturally from its action on the extended Dynkin diagram. Hence the action of Ω on $M(\Gamma)$ needs to verify

$$\sigma.(\varphi_0^a, \varphi_0^b, \ldots, \varphi_{r-1}^a, \varphi_{r-1}^b, \varphi_r^a, \varphi_r^b, \varphi_{r+1}^a, \varphi_{r+1}^b) = (\varphi_0^a, \varphi_0^b, \ldots, \varphi_{r-1}^a, \varphi_{r-1}^b, \lambda_{r+1}\varphi_{r+1}^a, \delta_{r+1}\varphi_{r+1}^b, \lambda_r\varphi_r^a, \delta_r\varphi_r^b).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

Assume r odd. The action of σ has to induce the correct action on the singularity $\mathbb{C}^2/\Gamma$. From Theorem 1.4.1 we know that the singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X(Y^2 - X^{r-1}) + Z^2 = 0\}$ with $X = 4^{\frac{1}{r}}(z_1z_2)^2$, $Y = 4^{-\frac{1}{2r}}(z_1^{2(r-1)} + z_2^{2(r-1)})$, $Z = iz_1z_2(z_1^{2(r-1)} - z_2^{2(r-1)})$. Furthermore, Γ' is generated by $g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ with $\xi = \exp(\frac{2i\pi}{4(r-1)})$ and $h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. h fixes X, Y and Z and

$$\begin{cases} g.X &= X, \\ g.Y &= -Y, \\ g.Z &= -Z. \end{cases}$$

After computation of the special fiber of the moment map, we get that $\mathbb{C}^2/\Gamma \cong \{x^r + y^2x + z^2 = 0\}$ with $\begin{cases} x = 4^{-\frac{1}{r}} p_{r,r+1}, \\ y = i4^{\frac{1}{2r}}(q_{0,\dots,r} - \frac{1}{2}p_{r,r+1}^{\frac{r-1}{2}}), \\ z = s_{0,\dots,r,r+1}, \end{cases}$ and

$\begin{cases} p_{r,r+1} = \mathrm{Tr}(\varphi_r^b \varphi_{r+1}^a \varphi_{r+1}^b \varphi_r^a), \\ q_{0,\dots,r} = \mathrm{Tr}(\varphi_0^b \dots \varphi_{r-2}^b \varphi_r^a \varphi_r^b \varphi_{r-2}^a \dots \varphi_0^a), \\ s_{0,\dots,r,r+1} = \mathrm{Tr}(\varphi_0^b \dots \varphi_{r-2}^b \varphi_{r+1}^a \varphi_{r+1}^b \varphi_r^a \varphi_r^b \varphi_{r-2}^a \dots \varphi_0^a), \end{cases}$ which implies

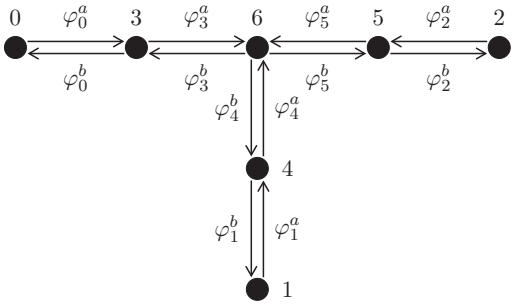
$$\begin{cases} \sigma.x = \lambda_r \delta_r \lambda_{r+1} \delta_{r+1} x, \\ \sigma.y = i4^{\frac{1}{2r}}((-\lambda_{r+1}\delta_{r+1}) q_{0,\dots,r} + (\lambda_{r+1}\delta_{r+1} - \frac{1}{2}\lambda_r \delta_r \lambda_{r+1}\delta_{r+1}) p_{r,r+1}^{\frac{r-1}{2}}), \\ \sigma.z = -\lambda_r \delta_r \lambda_{r+1} \delta_{r+1} z. \end{cases}$$

In order for the action to satisfy our conditions, we need $\lambda_r \delta_r = \lambda_{r+1}\delta_{r+1} = 1$.

The conditions when r is even are also $\lambda_r \delta_r = \lambda_{r+1}\delta_{r+1} = 1$.

4.3.2.3. Computations for $\mathbb{C}^2/\Gamma$ of type $\widetilde{E}_6$

The McKay quiver when $\Delta(\Gamma) = E_6$ is



It is known that $\Gamma = \mathcal{T}$, $\Gamma' = \mathcal{O}$ and $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$. The action of Ω on the Dynkin diagram permutes the vertices $1 \leftrightarrow 2$, $4 \leftrightarrow 5$ and fixes the others. We extend this action to the extended Dynkin diagram of type $\widetilde{\Delta}(\Gamma)$ by making Ω fix the additional vertex labelled 0. The action of Ω on $M(\Gamma)$ arises naturally from its action on the extended Dynkin diagram. Hence the action of Ω on $M(\Gamma)$ needs to verify

$$\sigma.(\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \varphi_2^a, \varphi_2^b, \varphi_3^a, \varphi_3^b, \varphi_4^a, \varphi_4^b, \varphi_5^a, \varphi_5^b) = (\varphi_0^a, \varphi_0^b, \lambda_2\varphi_2^a, \delta_2\varphi_2^b, \lambda_1\varphi_1^a, \delta_1\varphi_1^b, \varphi_3^a, \varphi_3^b, \lambda_5\varphi_5^a, \delta_5\varphi_5^b, \lambda_4\varphi_4^a, \delta_4\varphi_4^b).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

The action of $\mathbb{Z}/2\mathbb{Z}$ needs to be consistent with the action on the singularity computed with matrices. It is known that the singularity $\mathbb{C}^2/\Gamma$ is defined by $\{X^4+Y^3+Z^2=0\}$ with $X = 108^{\frac{1}{4}} z_1z_2(z_1^4-z_2^4)$, $Y = \exp(\frac{i\pi}{3})(z_1^8+z_2^8+14(z_1z_2)^4)$, $Z = (z_1^4+z_2^4)^3-36(z_1z_2)^4(z_1^4+z_2^4)$, (cf. Theorem 1.4.1 for the notation), and that

$$\begin{cases} \sigma.X &= -X, \\ \sigma.Y &= Y, \\ \sigma.Z &= -Z. \end{cases}$$

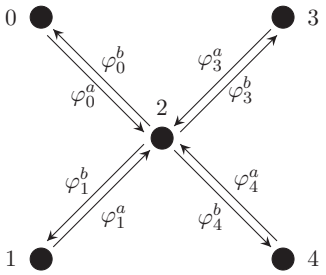
We adopt the following notation: $\Phi(i_1i_2\dots i_k) \coloneqq \mathrm{Tr}(\varphi^a_{i_1}\varphi^b_{i_1}\varphi^a_{i_2}\varphi^b_{i_2}\dots\varphi^a_{i_k}\varphi^b_{i_k})$. After computation of the special fiber of the moment map, we find that $\mathbb{C}^2/\Gamma \cong \{x^4+y^3+z^2=0\}$ with $\begin{cases} x = \frac{\exp(\frac{i\pi}{4})}{\sqrt{2}}\Phi(4^25) \\ y = \Phi(4^25^2), \\ z = \Phi(3^24^25^2) + \frac{1}{2}\Phi(4^25)^2, \end{cases}$ which implies

$$\begin{cases} \sigma.x = -(\lambda_5\delta_5)^2\lambda_4\delta_4 x, \\ \sigma.y = (\lambda_4\delta_4\lambda_5\delta_5)^2 y, \\ \sigma.z = -(\lambda_4\delta_4\lambda_5\delta_5)^2\Phi(3^24^25^2) + (-(\lambda_4\delta_4\lambda_5\delta_5)^2 + \frac{1}{2}(\lambda_4\delta_4(\lambda_5\delta_5)^2)^2)\Phi(4^25)^2. \end{cases}$$

For the action to satisfy our conditions, it needs to verify $\lambda_4\delta_4 = 1$ and $\lambda_5\delta_5 = \pm 1$.

4.3.2.4. Computations for $\mathbb{C}^2/\Gamma$ of type $\widetilde{D}_4$ and $\Gamma'/\Gamma = \mathfrak{S}_3$

The McKay quiver of type $\widetilde{D}_4$ is



We have $\Gamma = \mathcal{D}_2 = \langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle$ and $\Gamma' = \mathcal{O} = \langle a = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, c = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1} \\ -\epsilon & \epsilon \end{pmatrix} \rangle$ with $\epsilon = \exp(\frac{i\pi}{4})$. Then $\Gamma'/\Gamma = \{\overline{1}, \overline{c}, \overline{c}^2, \overline{a}, \overline{ac}, \overline{ac}^2\} = \langle \overline{c}, \overline{a} \rangle = \mathfrak{S}_3$ with $\overline{c}^3 = 1$ and $\overline{a}^2 = 1$, the symbol $^-$ meaning the class modulo $\mathcal{D}_2$.

The action of Ω on the Dynkin diagram permutes the vertices with $\overline{c}$ acting as (143) and $\overline{a}$ as (34). We extend this action to the extended Dynkin diagram of type $\widetilde{\Delta}(\Gamma)$ by making Ω fix the additional vertex labelled 0. The action of Ω on $M(\Gamma)$ arises naturally from its action on the extended Dynkin diagram. Hence the action of Ω on $M(\Gamma)$ needs to verify

$$\begin{array}{l} \overline{c}.(\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \varphi_3^a, \varphi_3^b, \varphi_4^a, \varphi_4^b) = (\varphi_0^a, \varphi_0^b, \lambda_3\varphi_3^a, \delta_3\varphi_3^b, \lambda_4\varphi_4^a, \delta_4\varphi_4^b, \lambda_1\varphi_1^a, \delta_1\varphi_1^b), \\ \overline{a}.(\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \varphi_3^a, \varphi_3^b, \varphi_4^a, \varphi_4^b) = (\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \alpha_4\varphi_4^a, \beta_4\varphi_4^b, \alpha_3\varphi_3^a, \beta_3\varphi_3^b). \end{array} .$$

with $\lambda_i, \delta_i, \alpha_i, \beta_i \in \mathbb{C}$.

It was proved in Theorem 1.4.1 that $\mathbb{C}^2/\Gamma \cong \{X^3 + Y^2X + Z^2 = 0\}$ with $\begin{cases} X = (z_1z_2)^2, \\ Y = \frac{i}{2}(z_1^4 + z_2^4), \\ Z = \frac{-1}{2}z_1z_2(z_1^4 - z_2^4), \end{cases}$. We can then compute

$$\begin{cases} \overline{c}.X = \frac{-X+iY}{2}, \\ \overline{c}.Y = \frac{3iX-Y}{2}, \\ \overline{c}.Z = Z, \end{cases} \quad \text{and} \quad \begin{cases} \overline{a}.X = X, \\ \overline{a}.Y = -Y, \\ \overline{a}.Z = -Z. \end{cases}$$

After computation of the special fiber of the moment map, we get that $\mathbb{C}^2/\Gamma \cong \{x^3 + y^2x + z^2 = 0\}$ with $\begin{cases} x = 2^{-\frac{2}{3}} \exp(\frac{2i\pi}{3})p_{34}, \\ y = 2^{\frac{1}{3}} \exp(\frac{i\pi}{6})(p_{03} + \frac{1}{2}p_{34}), \\ z = q_{034}, \end{cases}$ and $\begin{cases} p_{03} = \mathrm{Tr}(\varphi_0^b\varphi_3^a\varphi_3^b\varphi_0^a), \\ p_{34} = \mathrm{Tr}(\varphi_3^b\varphi_4^a\varphi_4^b\varphi_3^a), \\ q_{034} = \mathrm{Tr}(\varphi_0^b\varphi_4^a\varphi_4^b\varphi_3^a\varphi_3^b\varphi_0^a), \end{cases}$ which implies

$$\begin{cases} \overline{c}.x = \lambda_1\delta_1\lambda_4\delta_4\frac{-x+iy}{3}, \\ \overline{c}.y = 2^{\frac{1}{3}} \exp(\frac{i\pi}{6})(-\lambda_4\delta_4p_{34} + \lambda_4\delta_4(\frac{\lambda_1\delta_1}{2} - 1)p_{03}), \\ \overline{c}.z = z, \end{cases} \quad \text{and}$$

$$\begin{cases} \overline{a}.x = \alpha_3\beta_3\alpha_4\beta_4 x, \\ \overline{a}.y = 2^{\frac{1}{3}}\exp(\frac{i\pi}{6})(p_{03}(-\alpha_4\beta_4) + p_{34}(\frac{1}{2}\alpha_3\beta_3\alpha_4\beta_4 - \alpha_4\beta_4)), \\ \overline{a}.z = -\alpha_3\beta_3\alpha_4\beta_4 z. \end{cases}$$

In order for the action of Γ'/Γ on $M(\Gamma)$ to induce the natural action of $\mathfrak{S}_3$ on $\mathbb{C}^2/\Gamma$ we need $\begin{cases} \lambda_1\delta_1 = \lambda_3\delta_3 = \lambda_4\delta_4 = 1, \\ \alpha_3\beta_3 = \alpha_4\beta_4 = -1. \end{cases}$

If $\Gamma'/\Gamma = \mathbb{Z}/3\mathbb{Z}$, the results are similar.

4.3.2.5. Conclusions

In the last four subsections, conditions have been found in order to ensure that the restriction of the action of Ω on $M(\Gamma)$ to the singularity $\mathbb{C}^2/\Gamma$ corresponds to the action we computed using matrices in Theorem 1.4.1. The results are summarized in the following table:

Type of singularity	Conditions on the action
$B_r = (A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$	$\begin{cases} \lambda_i\delta_i = -1, \quad 0 \le i \le 2r-1, \\ \prod_{i=0}^{2r-1}\lambda_i = \prod_{i=0}^{2r-1}\delta_i = (-1)^r. \end{cases}$
$C_r = (D_{r+1}, \mathbb{Z}/2\mathbb{Z})$	$\lambda_r\delta_r = \lambda_{r+1}\delta_{r+1} = 1$
$F_4 = (E_6, \mathbb{Z}/2\mathbb{Z})$	$\lambda_4\delta_4 = 1$ and $\lambda_5\delta_5 = \pm 1$
$G_2 = (D_4, \mathfrak{S}_3)$	$\begin{cases} \lambda_1\delta_1 = \lambda_3\delta_3 = \lambda_4\delta_4 = 1, \\ \alpha_3\beta_3 = \alpha_4\beta_4 = -1. \end{cases}$

4.3.3. Choice of an orientation

Let $M(\Gamma)$ be the representation space of the McKay quiver Q defined in Subsection 4.2.1. By definition, we have $M(\Gamma) = \bigoplus_{a\in Q_1}\mathrm{Hom}(V_{s(a)}, V_{t(a)})$. It is known that between two connected vertices there are two arrows, one in each direction. For every such pair of arrows, let us choose one. All the selected arrows form a subset Q_1^+ of Q_1. Hence one can write

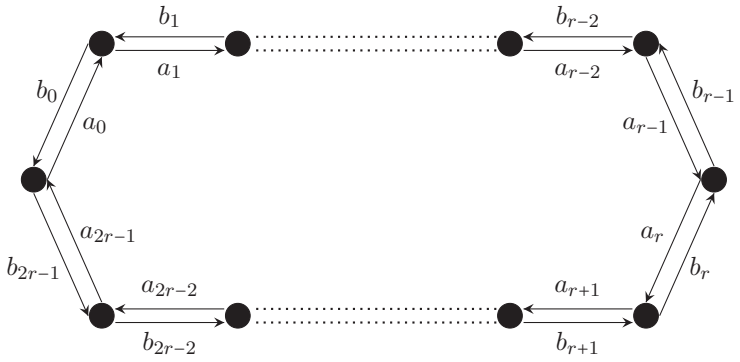
$$\begin{aligned} M(\Gamma) &= \oplus_{a\in Q_1^+}(\mathrm{Hom}(V_{s(a)}, V_{t(a)}) \oplus \mathrm{Hom}(V_{s(\bar{a})}, V_{t(\bar{a})})), \\ &= \oplus_{a\in Q_1^+} \mathrm{Hom}(V_{s(a)}, V_{t(a)}) \oplus \oplus_{a\in Q_1^+} \mathrm{Hom}(V_{t(a)}, V_{s(a)}), \\ &= \oplus_{a\in Q_1^+} \mathrm{Hom}(V_{s(a)}, V_{t(a)}) \oplus \oplus_{a\in Q_1^+} \mathrm{Hom}(V_{s(a)}, V_{t(a)})^*, \\ &= \oplus_{a\in Q_1^+} \mathrm{Hom}(V_{s(a)}, V_{t(a)}) \oplus (\oplus_{a\in Q_1^+} \mathrm{Hom}(V_{s(a)}, V_{t(a)}))^*. \end{aligned}$$

Therefore $M(\Gamma)$ can be seen as the cotangent space on the vector space $\oplus_{a\in Q_1^+} \mathrm{Hom}(V_{s(a)}, V_{t(a)})$. The choice of Q_1^+, which is the equivalent to the choice of an orientation of Q, determines the space on which the cotangent fiber is constructed. Conversely, the choice of the base space of the cotangent fiber will define a subset Q_1^+ of Q_1 and so an orientation of Q.

Theorem 4.3.2. *The action of $\Omega = \Gamma'/\Gamma$ on $M(\Gamma)$ is symplectic when:*
- *for $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$, Ω reverses the orientation of the McKay quiver.*
- *for the other cases, Ω preserves the orientation of the McKay quiver.*

4.3.3.1. Proof of Theorem 4.3.2 for the type $(\widetilde{A}_{2r-1}, \mathbb{Z}/2\mathbb{Z})$

The McKay quiver of type $\widetilde{A}_{2r-1}$ is



It was showed in Subsection 4.3.2 that the action of $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$ needs to verify

$$\sigma.(a_0, \ldots, a_{2r-1}, b_0, \ldots, b_{2r-1}) = (\lambda_{2r-1}b_{2r-1}, \ldots, \lambda_0 b_0, \delta_{2r-1}a_{2r-1}, \ldots, \delta_0 a_0).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

Let $\varphi = (a_0, \ldots, a_{2r-1}, b_0, \ldots, b_{2r-1})$ and $\psi = (c_0, \ldots, c_{2r-1}, d_0, \ldots, d_{2r-1})$ be two elements of $M(\Gamma)$. We are going to compute $\langle \varphi, \psi \rangle$ and $\langle \sigma.\varphi, \sigma.\psi \rangle$ with $\langle ., . \rangle$ the symplectic

form defined in Subsection 4.2.1. $M(\Gamma)$ is a vector space so it is identified with its tangent space. Furthermore, σ acts linearly thus $\sigma = d\sigma$. Then

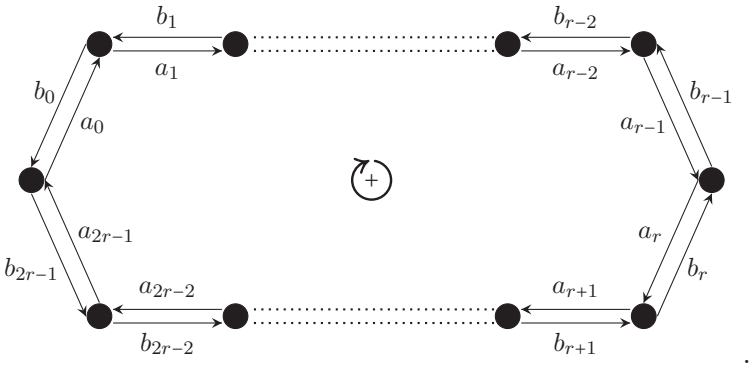
$$\langle \varphi, \psi \rangle = \sum_{i=0}^{2r-1} (\epsilon(a_i)(a_i d_i) + \epsilon(b_i)(b_i c_i)) = \sum_{i=0}^{2r-1} \epsilon(a_i)(a_i d_i - b_i c_i)$$

and

$$\langle \sigma.\varphi, \sigma.\psi \rangle = \sum_{i=0}^{2r-1} \epsilon(b_{2r-1-i})\lambda_i \delta_i (a_i d_i - b_i c_i).$$

For Ω to act symplectically it requires that $\epsilon(a_i) = \epsilon(b_{2r-1-i})\lambda_i\delta_i \Leftrightarrow \lambda_i\delta_i = \epsilon(a_i)\epsilon(b_{2r-1-i})$ for all i. However, for σ to induce the right action on $\mathbb{C}^2/\Gamma$, it is necessary that $\lambda_i\delta_i = -1$ for all i (cf. Subsection 4.3.2). Then we get $\epsilon(a_i)\epsilon(b_{2r-1-i}) = -1$. But the arrow a_i is sent to the arrow b_{2r-1-i} by σ, thus the equality $\epsilon(a_i)\epsilon(b_{2r-1-i}) = -1$ means that σ reverses the orientation of Q.

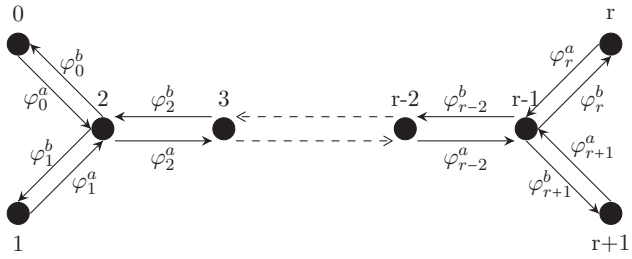
From now on the McKay quiver of type $\widetilde{A}_{2r-1}$ will be equipped with the clockwise orientation:



Then all a_i's are positive and all b_i's are negative.

4.3.3.2. Proof of Theorem 4.3.2 for the type $(\widetilde{D}_{r+1}, \mathbb{Z}/2\mathbb{Z})$

The McKay quiver of type $\widetilde{D}_{r+1}$ is



It was showed in Subsection 4.3.2 that the action of $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$ needs to verify

$$\sigma.(\varphi_0^a, \varphi_0^b, \ldots, \varphi_{r-1}^a, \varphi_{r-1}^b, \varphi_r^a, \varphi_r^b, \varphi_{r+1}^a, \varphi_{r+1}^b) = (\varphi_0^a, \varphi_0^b, \ldots, \varphi_{r-1}^a, \varphi_{r-1}^b, \lambda_{r+1}\varphi_{r+1}^a, \delta_{r+1}\varphi_{r+1}^b, \lambda_r\varphi_r^a, \delta_r\varphi_r^b).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

Let $\varphi = (\varphi_0^a, \varphi_0^b, \ldots, \varphi_r^a, \varphi_r^b, \varphi_{r+1}^a, \varphi_{r+1}^b)$ and $\psi = (\psi_0^a, \psi_0^b, \ldots, \psi_r^a, \psi_r^b, \psi_{r+1}^a, \psi_{r+1}^b)$ be two elements of $M(\Gamma)$. We are going to compute $\langle \varphi, \psi \rangle$ and $\langle \sigma.\varphi, \sigma.\psi \rangle$ with $\langle ., . \rangle$ the symplectic form defined in Subsection 4.2.1. $M(\Gamma)$ is a vector space so it is identified with its tangent space. Furthermore, σ acts linearly thus $\sigma = d\sigma$. Then

$$\begin{aligned}\langle \varphi, \psi \rangle &= \sum_{i=0}^{r-2}(\epsilon(\varphi_i^a)\mathrm{Tr}(\varphi_i^a\psi_i^b) + \epsilon(\varphi_i^b)\mathrm{Tr}(\varphi_i^b\psi_i^a)) + \sum_{i=r}^{r+1}(\epsilon(\varphi_i^a)\mathrm{Tr}(\varphi_i^a\psi_i^b) + \epsilon(\varphi_i^b)\mathrm{Tr}(\varphi_i^b\psi_i^a)),\\ &= \sum_{i=0}^{r-2}\epsilon(\varphi_i^a)(\mathrm{Tr}(\varphi_i^a\psi_i^b) - \mathrm{Tr}(\varphi_i^b\psi_i^a)) + \sum_{i=r}^{r+1}\epsilon(\varphi_i^a)(\mathrm{Tr}(\varphi_i^a\psi_i^b) - \mathrm{Tr}(\varphi_i^b\psi_i^a)),\end{aligned}$$

and

$$\begin{aligned}\langle \sigma.\varphi, \sigma.\psi \rangle = \sum_{i=0}^{r-2}\epsilon(\varphi_i^a)(\mathrm{Tr}(\varphi_i^a\psi_i^b) - \mathrm{Tr}(\varphi_i^b\psi_i^a)) + \epsilon(\varphi_r^a)\lambda_{r+1}\delta_{r+1}(\mathrm{Tr}(\varphi_{r+1}^a\psi_{r+1}^b) -\\ \mathrm{Tr}(\varphi_{r+1}^b\psi_{r+1}^a)) + \epsilon(\varphi_{r+1}^a)\lambda_r\delta_r(\mathrm{Tr}(\varphi_r^a\psi_r^b) - \mathrm{Tr}(\varphi_r^b\psi_r^a)).\end{aligned}$$

For Ω to act symplectically it requires that $\epsilon(\varphi_r^a)\lambda_{r+1}\delta_{r+1} = \epsilon(\varphi_{r+1}^a)$ and $\epsilon(\varphi_{r+1}^a)\lambda_r\delta_r = \epsilon(\varphi_r^a)$, which is equivalent to the system

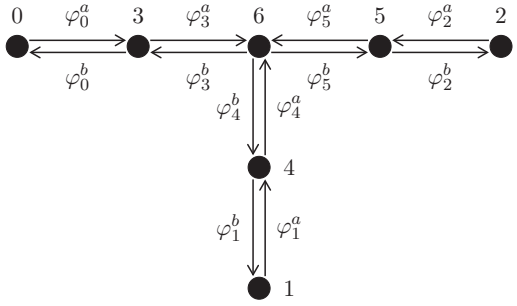
$$\begin{cases}\epsilon(\varphi_r^a)\epsilon(\varphi_{r+1}^a) = \lambda_{r+1}\delta_{r+1},\\ \epsilon(\varphi_{r+1}^a)\epsilon(\varphi_r^a) = \lambda_r\delta_r.\end{cases}$$

However, for σ to induce the right action on $\mathbb{C}^2/\Gamma$, it is necessary that $\lambda_r\delta_r = \lambda_{r+1}\delta_{r+1} = 1$ (cf. Subsection 4.3.2). Then we get $\epsilon(\varphi_r^a)\epsilon(\varphi_{r+1}^a) = 1$. But the arrow φ_r^a is sent to the arrow φ_{r+1}^a by σ, thus the equality $\epsilon(\varphi_r^a)\epsilon(\varphi_{r+1}^a) = 1$ means that σ preserves the orientation of Q.

From now on, the orientation of the McKay quiver of type $\widetilde{D}_{r+1}$ will be all a_i's are positive and all b_i's are negative.

4.3.3.3. Proof of Theorem 4.3.2 for the type $(\widetilde{E}_6, \mathbb{Z}/2\mathbb{Z})$

The McKay quiver of type $\widetilde{E}_6$ is



It was showed in Subsection 4.3.2 that the action of $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$ needs to verify

$$\sigma.(\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \varphi_2^a, \varphi_2^b, \varphi_3^a, \varphi_3^b, \varphi_4^a, \varphi_4^b, \varphi_5^a, \varphi_5^b) =$$
$$(\varphi_0^a, \varphi_0^b, \lambda_2\varphi_2^a, \delta_2\varphi_2^b, \lambda_1\varphi_1^a, \delta_1\varphi_1^b, \varphi_3^a, \varphi_3^b, \lambda_5\varphi_5^a, \delta_5\varphi_5^b, \lambda_4\varphi_4^a, \delta_4\varphi_4^b).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

Let $\varphi = (\varphi_0^a, \varphi_0^b, \ldots, \varphi_5^a, \varphi_5^b)$ and $\psi = (\psi_0^a, \psi_0^b, \ldots, \psi_5^a, \psi_5^b)$ be two elements of $M(\Gamma)$. We are going to compute $\langle\varphi, \psi\rangle$ and $\langle\sigma.\varphi, \sigma.\psi\rangle$ with $\langle ., . \rangle$ the symplectic form defined in Subsection 4.2.1. $M(\Gamma)$ is a vector space so it is identified with its tangent space. Furthermore, σ acts linearly thus $\sigma = d\sigma$. Then

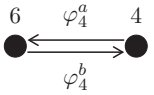
$$\langle\varphi, \psi\rangle = \sum_{i=0}^{5} \epsilon(\varphi_i^a)(\mathrm{Tr}(\varphi_i^a\psi_i^b) - \mathrm{Tr}(\varphi_i^b\psi_i^a))$$

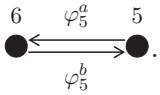
and

$$\begin{aligned}\langle \sigma.\varphi, \sigma.\psi\rangle = \quad & \epsilon(\varphi_0^a)(\mathrm{Tr}(\varphi_0^a\psi_0^b) - \mathrm{Tr}(\varphi_0^b\psi_0^a)) + \epsilon(\varphi_1^a)\lambda_2\delta_2(\mathrm{Tr}(\varphi_2^a\psi_2^b) - \mathrm{Tr}(\varphi_2^b\psi_2^a)) \\ & +\epsilon(\varphi_2^a)\lambda_1\delta_1(\mathrm{Tr}(\varphi_1^a\psi_1^b) - \mathrm{Tr}(\varphi_1^b\psi_1^a)) + \epsilon(\varphi_3^a)(\mathrm{Tr}(\varphi_3^a\psi_3^b) - \mathrm{Tr}(\varphi_3^b\psi_3^a)) \\ & +\epsilon(\varphi_4^a)\lambda_5\delta_5(\mathrm{Tr}(\varphi_5^a\psi_5^b) - \mathrm{Tr}(\varphi_5^b\psi_5^a)) + \epsilon(\varphi_5^a)\lambda_4\delta_4(\mathrm{Tr}(\varphi_4^a\psi_4^b) - \mathrm{Tr}(\varphi_4^b\psi_4^a)).\end{aligned}$$

For Ω to act symplectically it requires that

$$\begin{cases} \epsilon(\varphi_1^a)\epsilon(\varphi_2^a) = \lambda_1\delta_1 = \lambda_2\delta_2, \\ \epsilon(\varphi_4^a)\epsilon(\varphi_5^a) = \lambda_4\delta_4 = \lambda_5\delta_5. \end{cases}$$

However, for σ to induce the right action on $\mathbb{C}^2/\Gamma$, it is necessary that $\lambda_4\delta_4 = 1$ (cf. Subsection 4.3.2). Then we get $\lambda_5\delta_5 = \epsilon(\varphi_4^a)\epsilon(\varphi_5^a) = 1$. But the arrow φ_4^a is sent to the arrow φ_5^a by σ, thus the equality means that σ preserves the orientation between 

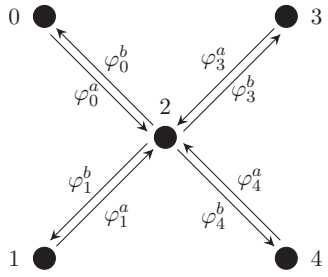
and 

. One notices that there are no conditions on the orientation of the arrows indexed by 1 and 2.

From now on, the orientation of the McKay quiver of type $\widetilde{E}_6$ will be all a_i's are positive and all b_i's are negative.

4.3.3.4. Proof of Theorem 4.3.2 for the type $(\widetilde{D}_4, \mathfrak{S}_3)$

The McKay quiver of type $\widetilde{D}_4$ is



It was showed in Subsection 4.3.2 that the action of $\Gamma'/\Gamma = \mathbb{Z}/3\mathbb{Z} =< \sigma >$ needs to verify

$$\sigma.(\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \varphi_3^a, \varphi_3^b, \varphi_4^a, \varphi_4^b) = (\varphi_0^a, \varphi_0^b, \lambda_3\varphi_3^a, \delta_3\varphi_3^b, \lambda_4\varphi_4^a, \delta_4\varphi_4^b, \lambda_1\varphi_1^a, \delta_1\varphi_1^b).$$

with $\lambda_i, \delta_i \in \mathbb{C}$.

Let $\varphi = (\varphi_0^a, \varphi_0^b, \dots, \varphi_4^a, \varphi_4^b)$ and $\psi = (\psi_0^a, \psi_0^b, \dots, \psi_4^a, \psi_4^b)$ be two elements of $M(\Gamma)$. We are going to compute $\langle \varphi, \psi \rangle$ and $\langle \sigma.\varphi, \sigma.\psi \rangle$ with $\langle ., . \rangle$ the symplectic form defined in Subsection 4.2.1. $M(\Gamma)$ is a vector space so it is identified with its tangent space. Furthermore, σ acts linearly thus $\sigma = d\sigma$. Then

$$\langle \varphi, \psi \rangle = \sum_{i=0,1,3,4} (\epsilon(\varphi_i^a)\mathrm{Tr}(\varphi_i^a\psi_i^b) + \epsilon(\varphi_i^b)\mathrm{Tr}(\varphi_i^b\psi_i^a)) = \sum_{i=0,1,3,4} \epsilon(\varphi_i^a)(\mathrm{Tr}(\varphi_i^a\psi_i^b) - \mathrm{Tr}(\varphi_i^b\psi_i^a))$$

and

$$\begin{gathered}\langle \sigma.\varphi, \sigma.\psi \rangle = \epsilon(\varphi_0^a)(\mathrm{Tr}(\varphi_0^a\psi_0^b) - \mathrm{Tr}(\varphi_0^b\psi_0^a)) + \epsilon(\varphi_1^a)\lambda_3\delta_3(\mathrm{Tr}(\varphi_3^a\psi_3^b) - \mathrm{Tr}(\varphi_3^b\psi_3^a)) + \\ \epsilon(\varphi_3^a)\lambda_4\delta_4(\mathrm{Tr}(\varphi_4^a\psi_4^b) - \mathrm{Tr}(\varphi_4^b\psi_4^a)) + \epsilon(\varphi_4^a)\lambda_1\delta_1(\mathrm{Tr}(\varphi_1^a\psi_1^b) - \mathrm{Tr}(\varphi_1^b\psi_1^a)).\end{gathered}$$

For Ω to act symplectically it requires that

$$\begin{cases} \epsilon(\varphi_4^a)\lambda_1\delta_1 = \epsilon(\varphi_1^a), \\ \epsilon(\varphi_1^a)\lambda_3\delta_3 = \epsilon(\varphi_3^a), \\ \epsilon(\varphi_3^a)\lambda_4\delta_4 = \epsilon(\varphi_4^a). \end{cases} \Leftrightarrow \begin{cases} \lambda_1\delta_1 = \epsilon(\varphi_1^a)\epsilon(\varphi_4^a), \\ \lambda_3\delta_3 = \epsilon(\varphi_1^a)\epsilon(\varphi_3^a), \\ \lambda_4\delta_4 = \epsilon(\varphi_3^a)\epsilon(\varphi_4^a). \end{cases}$$

However, for σ to induce the right action on $\mathbb{C}^2/\Gamma$, it is necessary that $\lambda_1\delta_1 = \lambda_3\delta_3 = \lambda_4\delta_4 = 1$ (cf. Subsection 4.3.2). Then we get $\epsilon(\varphi_1^a)\epsilon(\varphi_4^a) = \epsilon(\varphi_1^a)\epsilon(\varphi_3^a) = \epsilon(\varphi_3^a)\epsilon(\varphi_4^a) = 1$. But σ permutes circularly the triple $(\varphi_1^a, \varphi_3^a, \varphi_4^a)$, thus σ preserves the orientation of Q.

Remark 4.3.3. One can replace $\mathbb{Z}/3\mathbb{Z}$ by $\mathfrak{S}_3$ and the result remains the same.

From now on, the orientation of the McKay quiver of type $\widetilde{D}_4$ will be all a_i's are positive and all b_i's are negative.

4.3.3.5. Conclusions

During the last four subsections, conditions have been found in order to ensure that the action of Ω on $M(\Gamma)$ is symplectic. The results are summarized in the following table:

Type of singularity	Conditions on the action
$B_r = (A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$	$\epsilon(a_i)\epsilon(b_{2r-1-i}) = \lambda_i\delta_i,\ 0 \leq i \leq 2r-1$
$C_r = (D_{r+1}, \mathbb{Z}/2\mathbb{Z})$	$\epsilon(\varphi_r^a)\epsilon(\varphi_{r+1}^a) = \lambda_r\delta_r = \lambda_{r+1}\delta_{r+1}$
$F_4 = (E_6, \mathbb{Z}/2\mathbb{Z})$	$\begin{cases} \epsilon(\varphi_1^a)\epsilon(\varphi_2^a) = \lambda_1\delta_1 = \lambda_2\delta_2, \\ \epsilon(\varphi_4^a)\epsilon(\varphi_5^a) = \lambda_4\delta_4 = \lambda_5\delta_5. \end{cases}$
$G_2 = (D_4, \mathfrak{S}_3)$	$\begin{cases} \lambda_1\delta_1 = \epsilon(\varphi_1^a)\epsilon(\varphi_4^a), \\ \lambda_3\delta_3 = \epsilon(\varphi_1^a)\epsilon(\varphi_3^a), \\ \lambda_4\delta_4 = \epsilon(\varphi_3^a)\epsilon(\varphi_4^a). \end{cases}$

One can see that by setting constants λ_i, δ_i such that they verify the conditions in Subsection 4.3.2.5, and an orientation like in Theorem 4.3.2, the action of Ω on $M(\Gamma)$ is both symplectic and induces the natural action on the singularity $\mathbb{C}^2/\Gamma$. It follows that we proved the following result:

Theorem 4.3.4. *For any McKay quiver built on a Dynkin diagram of type A_{2r-1}, D_{r+1} or E_6, there exists an action of $\Omega = \Gamma'/\Gamma$ on $M(\Gamma)$ that is both symplectic and induces the natural action on the singularity $\mathbb{C}^2/\Gamma$.*

4.3.4. Compatibility of the action of Γ'/Γ on $G(\Gamma)$

In the preceding section we saw how to define a natural action of $\Omega = \Gamma'/\Gamma$ on $M(\Gamma)$ such that the action is symplectic and induces the right action on the singularity $\mathbb{C}^2/\Gamma$. We now want to define an action of Γ'/Γ on $G(\Gamma)$ such that $\gamma.(g.\varphi) = (\gamma.g).(\gamma.\varphi)$ for all $\gamma \in \Gamma'/\Gamma$, $g \in G(\Gamma)$ and $\varphi \in M(\Gamma)$. Furthermore the morphism $\tau : Z \to \mathfrak{h}$ has to be Γ'/Γ-equivariant. The action will be found by a case by case analysis.

4.3.4.1. Case $\Delta(\Gamma) = A_{2r-1}$

The orientation of the quiver is all a_i's are positive. Let $g = (g_0, \ldots, g_{2r-1})$ be a representative of an element in $G(\Gamma)$ and let $\varphi = (a_0, \ldots, a_{2r-1}, b_0, \ldots, b_{2r-1}) \in M(\Gamma)$. We have $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$. Set

$$\sigma.\varphi = (-b_{2r-1}, \ldots, -b_r, b_{r-1}, \ldots, b_0, a_{2r-1}, \ldots, a_r, -a_{r-1}, \ldots, -a_0).$$

It can be verified that this action abides by the conditions of the previous section and is then symplectic, induces the natural action on the singularity $\mathbb{C}^2/\Gamma$ and verifies $\sigma^2 = \mathrm{Id}$.

Let Ω act on $G(\Gamma)$ by

$$\sigma.g = (g_0, g_{2r-1}, \dots, g_1), \text{ for any } g \in G(\Gamma).$$

One may check explicitly that the desired relation $\sigma.(g.\varphi) = (\sigma.g).(\sigma.\varphi)$ is satisfied.

Proposition 4.3.5. *The action of Ω on $G(\Gamma)$ induces the correct action on the Cartan subalgebra $\mathfrak{h}$ of type A_{2r-1}.*

Proof. By definition $G(\Gamma) = \prod_{i=0}^{2r-1} \mathrm{GL}_{d_i}(\mathbb{C})/\mathbb{C}^*$ with $d_i = 1$ for all i. Γ'/Γ acts on $G(\Gamma)$ by permuting the i-th and $(2r-i)$-th coordinates for all $i \geq 1$ and fixing the 0-th coordinate. As the action is linear, it stays the same on $\mathfrak{g}(\Gamma) = \mathrm{Lie}(G(\Gamma)) = \oplus_{i=0}^{2r-1} \mathfrak{gl}_{d_i}(\mathbb{C})/\mathbb{C}$. Z is defined as the dual of the center of $\mathfrak{g}(\Gamma)$. Let $z \in Z$. Then there exists $z_0, \dots, z_{2r-1} \in \mathbb{C}$ such that $\sum_{i=0}^{2r-1} d_i z_i = 0$ and $z = (z_0 \mathrm{Id}_{d_0}, \dots, z_{2r-1}\mathrm{Id}_{d_{2r-1}})$. Then the action of Γ'/Γ on Z is given by

$$\sigma.z = (z_0, z_{2r-1}, \dots, z_1).$$

Thus Γ'/Γ acts on Z by permuting the coordinates i and $2r-i$.
Let $(\alpha_1^\vee, \dots, \alpha_{2r-1}^\vee)$ be a base of a Cartan subalgebra $\mathfrak{h}$ of the Lie algebra of type A_{2r-1}, with $(\alpha_1, \dots, \alpha_{2r-1})$ being a base of the root system of type A_{2r-1}. Γ'/Γ acts naturally on $\mathfrak{h}$ by permuting the i-th and $(2r-i)$-th coordinates. The isomorphism $\tau : Z \xrightarrow{\cong} \mathfrak{h}$ was computed in Section 4.1.2 and is given by

$$\begin{array}{rccc} \tau : & Z & \to & \mathfrak{h} \\ & z & \mapsto & h \end{array}$$

with $\alpha_i(h) = -z_i$, for all $1 \leq i \leq 2r-1$. Using the fundamental root system and its coroot system, it can be verified that $\tau(\sigma.z) = \sigma.\tau(z)$ for all $z \in Z$. Hence τ is Γ'/Γ-equivariant. □

4.3.4.2. Case $\Delta(\Gamma) = D_{r+1}$

Let $g = (g_0, \dots, g_{r+1})$ be a representative of an element in $G(\Gamma)$ and let $\varphi = (\varphi_i^a, \varphi_i^b)_{\substack{0 \leq i \leq r+1 \\ i \neq r-1}} \in M(\Gamma)$. By definition $\Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$. Set

$$\sigma.\varphi = (\varphi_0^a, \varphi_0^b, \varphi_1^a, \varphi_1^b, \dots, \varphi_{r+1}^a, \varphi_{r+1}^b, \varphi_r^a, \varphi_r^b).$$

It can be checked that this action abides by the conditions of the previous section (an analysis has to be made depending on the parity of r) and is then symplectic and induces the natural action on the singularity $\mathbb{C}^2/\Gamma$.

Let Ω act on $G(\Gamma)$ by

$$\sigma.g = (g_0, g_1, \ldots, g_{r-1}, g_{r+1}, g_r),$$

Then the desired relation $\sigma.(g.\varphi) = (\sigma.g).(\sigma.\varphi)$ is verified.

Proposition 4.3.6. *The action of Ω on $G(\Gamma)$ induces the correct action on the Cartan subalgebra $\mathfrak{h}$ of type D_{r+1}.*

Proof. $G(\Gamma) = \prod_{i=0}^{r+1} \mathrm{GL}_{d_i}(\mathbb{C})/\mathbb{C}^*$ with $d_i = 1$ for $i = 0, 1, r, r+1$, and $d_i = 2$ otherwise. Γ'/Γ acts on $G(\Gamma)$ by permuting the r-th and $(r+1)$-th coordinates. As the action is linear, it stays the same on $\mathfrak{g}(\Gamma) = \mathrm{Lie}(G(\Gamma)) = \oplus_{i=0}^{r+1} \mathfrak{gl}_{d_i}(\mathbb{C})/\mathbb{C}$. Z is defined as the dual of the center of $\mathfrak{g}(\Gamma)$. Let $z \in Z$. Then there exists $z_0, \ldots, z_{r+1} \in \mathbb{C}$ such that $\sum_{i=0}^{r+1} d_i z_i = 0$ and $z = (z_0 \mathrm{Id}_{d_0}, \ldots, z_{r+1}\mathrm{Id}_{d_{r+1}})$. The action of Γ'/Γ on Z is given by

$$\sigma.z = (z_0, z_1, \ldots, z_{r-1}, z_{r+1}, z_r).$$

Hence Γ'/Γ acts on Z by permuting the coordinates r and $r+1$.
Let $(\alpha_1^\vee, \ldots, \alpha_{r+1}^\vee)$ be a base of a Cartan subalgebra $\mathfrak{h}$ of the Lie algebra of type D_{r+1}, with $(\alpha_1, \ldots, \alpha_{r+1})$ being a base of the root system of type D_{r+1}. Γ'/Γ acts naturally on $\mathfrak{h}$ by permuting the r-th and $(r+1)$-th coordinates. The isomorphism $\tau : Z \xrightarrow{\cong} \mathfrak{h}$ is

$$\begin{array}{rccc} \tau : & Z & \to & \mathfrak{h} \\ & z & \mapsto & h \end{array}$$

with $\alpha_i(h) = -z_i$, for all $1 \leq i \leq r+1$. Using the fundamental root system and its coroot system, one may verify that $\tau(\sigma.z) = \sigma.\tau(z)$ for all $z \in Z$. Therefore τ is Γ'/Γ-equivariant. □

The cases for E_6 as well as $(D_4, \mathfrak{S}_3)$ are dealt with in a similar fashion.

4.3.5. Flat coordinates

The field on which we work is the complex number field $\mathbb{C}$.

4.3.5.1. Definition of flat coordinates

Flat coordinates were defined by K. Saito in [Saito79] and [Saito83] in the context of flat structures of tangent bundles of parameter spaces of universal unfoldings. This setting goes beyond the scope of this thesis, so the definition will be given only for the simple singularities.

Let $\mathfrak{h}$ be a Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$ of rank r with simply-laced Dynkin diagram, and W the associated Weyl group. According to Chevalley's Theorem (Theorem 1.1.21), the ring $S(\mathfrak{h}^*)^W$ of W-invariant polynomials on $\mathfrak{h}$ is generated by r algebraically independent homogeneous polynomials $P_1, \dots, P_r$ of degrees $m_1 + 1 = 2 < \dots < m_r + 1 = h$, h being the Coxeter number of $\mathfrak{g}$. Let Δ be the product of all the linear functions defining reflection hyperplanes of reflections in W (Δ^2 is called the discriminant of $\mathfrak{h}$). Then Δ is a fundamental anti-invariant of W and is a homogeneous element in $S(\mathfrak{h})^{\frac{rh}{2}}$. The polynomial Δ^2 is W-invariant and can thus be written as $\Delta^2 = a_0 P_r^r + a_1 P_r^{r-1} + \dots + a_r$, with a_i a polynomial in $P_1, \dots, P_{r-1}$ of degree hi. Let s be a Coxeter element (cf. Definition 1.1.22). The eigenvalues of s are h-th roots of unity. Let $\xi \in \mathfrak{h}$ be an eigenvector of s associated to the eigenvalue $\exp(\frac{2i\pi}{h})$. Because $\deg P_i < h$ for $1 \leq i < r$, one has $P_i(\xi) = 0$ for $1 \leq i < r$. Therefore $a_i(\xi) = 0$ for $1 \leq i \leq r$ and $\Delta(\xi)^2 = a_0 P_r(\xi)^r$. One can show (cf. Proposition 3, §6, V in [Bou68]) that $\Delta(\xi) \neq 0$, hence $a_0 \neq 0$.

Let κ be the Killing form on $\mathfrak{h}$. It induces an inner product

$$I(dP_i, dP_j) = \sum_{m,n} \frac{\partial P_i}{\partial x_m} \frac{\partial P_j}{\partial x_n} \kappa(x_m, x_n)$$

on the cotangent vectors dP_i ($i = 1, \dots, r$) on $\mathfrak{h}/W$. The following proposition can be found in [SekYan79].

Proposition 4.3.7. *There exists a constant c such that*

$$\det(I(dP_i, dP_j)_{1 \leq i,j \leq r}) = c\Delta^2.$$

The proposition implies that the inner product degenerates along the discriminant. Set $D := \frac{\partial}{\partial P_r}$. Because the degree of P_r is maximal among the generators of $S(\mathfrak{h}^*)^W$, D is unique up to scalar multiplication. We define a new inner product $J(dP_i, dP_j) = DI(dP_i, dP_j)$ on $\mathfrak{h}/W$. The next theorem comes from [SYS80].

Theorem 4.3.8. *The following assertions are verified:*

1. *J is a non-degenerate bilinear form and* $\det(J(dP_i, dP_j)_{1 \leq i,j \leq r}) = a_0$.
2. *The inner product J does not depend on the coordinates $P_1, \dots, P_r$.*

3. *There exists affine linear coordinates, say $Q_1, \dots, Q_r$, on $\mathfrak{h}/W$, such that, for any $1 \leq i, j \leq r$, $J(dQ_i, dQ_j)$ is a constant. These coordinates are called* ***flat coordinates****.*

The next proposition is proved in [Yano80] and explains why we take interest in flat coordinates.

Proposition 4.3.9. *Let $\mathfrak{h}$ be a Cartan subalgebra of a simply-laced simple Lie algebra $\mathfrak{g}$, and W the associated Weyl group. Then the action of any automorphism of the Dynkin diagram of $\mathfrak{g}$ on $\mathfrak{h}/W$ is linear relative to flat coordinates.*

In the next subsections flat coordinates for $\mathfrak{g}$ of type A_{2r-1}, D_{r+1} and E_6 will be explicitly given for later purposes. For more details on flat coordinates, the reader may consult [Saito93] and [Saito01].

4.3.5.2. Flat coordinates for A_{2r-1}

Set $\mathfrak{g}$ a simple Lie algebra of type A_{2r-1}, $\mathfrak{h}$ a Cartan subalgebra and W the associated Weyl group. Let $\xi_1, \dots, \xi_{2r}$ be an orthonormal basis of a Euclidian space $V \cong \mathbb{R}^{2r}$. $\mathfrak{h}$ is isomorphic to the subspace of V defined by $\{\sum_{i=1}^{2r} x_i \xi_i \in V \mid \sum_{i=1}^{2r} x_i = 0\}$. The set of roots of A_{2r-1} consists of $\xi_i - \xi_j$ $(1 \leq i, j \leq 2r$ and $i \neq j)$. The Weyl group of type A_{2r-1} is $\mathfrak{S}_{2r}$ and permutes the ξ_i. Hence the ring of W-invariant polynomials is generated by $\epsilon_2, \epsilon_3, \dots, \epsilon_{2r}$ with $\epsilon_i(\xi)$ the i-th elementary symmetric polynomial in $2r$ variables $\xi = (\xi_1, \dots, \xi_{2r})$. Set $I = \{2, 3, \dots, 2r\}$.

According to [SYS80], flat coordinates for A_{2r-1} are $\psi_2, \psi_3, \dots, \psi_{2r}$ with

$$\psi_i = \sum_{d \geq 1} \frac{(-1)^{d-1}(\frac{1}{h}(h-i+1), d-1))}{d!} X_i^d$$

where

$$X_i^d = \sum_{\substack{i_1 + \ldots + i_d = i \\ i_j \in I}} \epsilon_{i_1} \dots \epsilon_{i_d}.$$

and $(a, n) = a(a+1)\dots(a+n-1)$. For any $i \in I$, ψ_i is a homogeneous polynomial of degree i.

4.3.5.3. Flat coordinates for D_{r+1}

Set $\mathfrak{g}$ a simple Lie algebra of type D_{r+1}, $\mathfrak{h}$ a Cartan subalgebra and W the associated Weyl group. Let $\xi_1, \ldots, \xi_{r+1}$ be an orthonormal basis of a Euclidian space $V \cong \mathbb{R}^{r+1}$. The set of roots of D_{r+1} consists of $\pm\xi_i \pm \xi_j$ $(1 \leq i < j \leq r+1)$. The Weyl group of type D_{r+1} is the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^r \rtimes \mathfrak{S}_{r+1}$. $\mathfrak{S}_{r+1}$ permutes the ξ_i and $(\mathbb{Z}/2\mathbb{Z})^r$ acts by $\xi_i \mapsto (\pm 1)_i \xi_i$ such that $\prod_i (\pm 1)_i = 1$. Hence the ring of W-invariant polynomials is generated by $x_2, x_4, \ldots, x_{2r}, \psi$ with $x_{2i} = \epsilon_i(\xi^2)$ with $\xi = (\xi_1, \ldots, \xi_{r+1})$, ϵ_i the i-th elementary symmetric polynomial in $r+1$ variables, and $\psi = \prod_{i=1}^{r+1} \xi_i$. Set $I = \{2, 4, \ldots, 2r\}$.

According to [SYS80], flat coordinates for D_{r+1} are $\psi_2, \psi_4, \ldots, \psi_{2r}, \psi$ with

$$\psi_i = \sum_{d \geq 1} \frac{(-1)^{d-1}(\frac{1}{h}(h-i+1), d-1))}{d!} X_i^d$$

where

$$X_i^d = \sum_{\substack{i_1 + \ldots + i_d = i \\ i_j \in I}} x_{i_1} \ldots x_{i_d}.$$

and $(a, n) = a(a+1)\ldots(a+n-1)$ (the formulas are similar to the case A_{2r-1}, except for X_i^d who is now a function of the x_i's rather than the ϵ_i's). For any $i \in I$, ψ_i is a homogeneous polynomial of degree i.

According to the preceding formulae, flat coordinates for D_4 are ψ_2, ψ_4, ψ_6 and ψ with

$$\begin{cases} \psi_2 = x_2 = \sum_{i=1}^{4} \xi_i^2, \\ \psi_4 = x_4 - \frac{1}{4}x_2^2 = \sum_{1 \leq i < j \leq 4} \xi_i^2 \xi_j^2 - \frac{1}{4}(\sum_{i=1}^{4} \xi_i^2)^2, \\ \psi_6 = x_6 - \frac{1}{6}x_2 x_4 + \frac{7}{216}x_2^3 = \sum_{1 \leq i < j < k \leq 4} \xi_i^2 \xi_j^2 \xi_k^2 - \frac{1}{6}(\sum_{i=1}^{4} \xi_i^2)(\sum_{1 \leq i < j \leq 4} \xi_i^2 \xi_j^2) + \frac{7}{216}(\sum_{i=1}^{4} \xi_i^2)^3, \\ \psi = \xi_1 \xi_2 \xi_3 \xi_4. \end{cases}$$

They will be used in Subsections 4.3.7 and 4.3.8.

4.3.5.4. Flat coordinates for E_6

The computation of flat coordinates for type E_6 requires a more delicate approach than the previous cases. In [Frame51], J.S. Frame studied the Weyl group W of type E_6 as the group of automorphisms of the 27 lines on a non-singular cubic surface. He described the 27 lines as 27 complex triples:

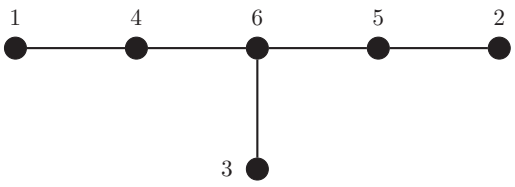
$$\begin{cases} (0, \omega^\lambda, \omega^\mu) \text{ with } \lambda, \mu = 1, 2, 3, \\ (-\omega^\mu, 0, \omega^\lambda) \text{ with } \lambda, \mu = 1, 2, 3, \\ (\omega^\lambda, -\omega^\mu, 0) \text{ with } \lambda, \mu = 1, 2, 3, \end{cases}$$

with $\omega = \exp(\frac{2i\pi}{3})$. By taking $x_1, y_1, x_2, y_2, x_3, y_3$ the real and imaginary parts of the complex triples, Frame identified the 27 lines with the 27 vertices of a polyhedron in $\mathbb{R}^6$ and W as its group of symmetries. The polyhedron has 36 hyperplanes of symmetries, each given by its normal vector which is one of the following:

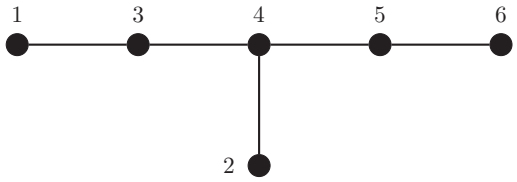
$$\begin{cases} D_{\kappa,\lambda,\mu} = \frac{1}{\sqrt{3}}(c_\kappa, s_\kappa, c_\lambda, s_\lambda, c_\mu, s_\mu)^T \text{ with } \kappa, \lambda, \mu = 1, 2 \text{ or } 3, \\ D_{\kappa,0,0} = (-s_\kappa, c_\kappa, 0, 0, 0, 0)^T \text{ with } \kappa = 1, 2 \text{ or } 3, \\ D_{0,\lambda,0} = (0, 0, -s_\lambda, c_\lambda, 0, 0)^T \text{ with } \lambda = 1, 2 \text{ or } 3, \\ D_{0,0,\mu} = (0, 0, 0, 0, -s_\mu, c_\mu)^T \text{ with } \mu = 1, 2 \text{ or } 3, \end{cases}$$

with $c_a = \cos(\frac{2a\pi}{3})$ and $s_a = \sin(\frac{2a\pi}{3})$. Let $s_k = \mathrm{Id} - 2D_k D_k^T$, with k a triple, denote the reflection in $\mathbb{R}^6$ of hyperplane whose normal vector is D_k. Frame proved that W is generated by $s_{1,0,0}, s_{0,1,0}, s_{0,0,1}, s_{3,0,0}, s_{0,0,3}$ and $s_{3,3,3}$.

Let $\alpha_1, \ldots, \alpha_6$ be a basis of the root system of type E_6, and index the Dynkin diagram in the following way (it is the indexation that will be used for the computation of the semiuniversal deformation):



Remark 4.3.10. The indexation used here is not the standard one used in reference books like [Bou68], which is



If s_{α_i} denote the simple reflection in W associated to α_i, then $s_{\alpha_i}(\alpha_j) = \alpha_j - c_{ji}\alpha_i$, with $(c_{ij})_{1\leq i\leq 6} = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 2 \end{pmatrix}$ the Cartan matrix of E_6. With the Cartan matrix, one can identify the s_{α_i}'s with their corresponding s_k, and as such identify the α_i's with the D_k's. The correspondence is as follows:

$$\alpha_1 \leftrightarrow \sqrt{2}D_{3,0,0} = \sqrt{2}(0,1,0,0,0,0)^T,$$

$$\alpha_2 \leftrightarrow \sqrt{2}D_{0,0,3} = \sqrt{2}(0,0,0,0,0,1)^T,$$

$$\alpha_3 \leftrightarrow \sqrt{2}D_{0,1,0} = \sqrt{2}(0,0,-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2},0,0)^T,$$

$$\alpha_4 \leftrightarrow \sqrt{2}D_{1,0,0} = \sqrt{2}(-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2},0,0,0,0)^T,$$

$$\alpha_5 \leftrightarrow \sqrt{2}D_{0,0,1} = \sqrt{2}(0,0,0,0,-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2})^T,$$

$$\alpha_6 \leftrightarrow \sqrt{2}D_{3,3,3} = \sqrt{\tfrac{2}{3}}(1,0,1,0,1,0)^T,$$

so $c_{ii} = 2 = \|\alpha_i\|^2$, for any $1 \leq i \leq 6$.

Set $p_i = x_i^2 + y_i^2$, $q_i = \frac{1}{3}x_i^3 - x_i y_i^2$ for $1 \leq i \leq 3$, and define two differential operators Θ and Δ by

$$\Theta = \textstyle\sum_{i=1}^{3}(3q_i(p_j - p_k) - 2p_i(q_j - q_k))\frac{\partial}{\partial p_i} + (\frac{1}{2}p_i^2(p_j - p_k) - 3q_i(q_j - q_k))\frac{\partial}{\partial q_i}$$

$$\text{for } (ijk) = (123), (231), (312),$$

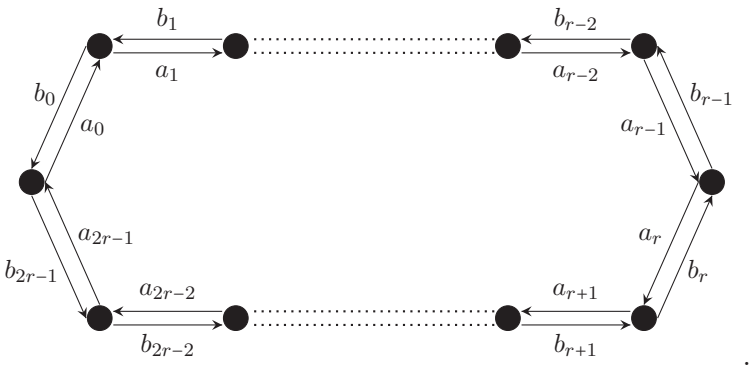
$$\Delta = \textstyle\sum_{i=1}^{3} 4\frac{\partial}{\partial p_i}p_i\frac{\partial}{\partial p_i} + 12q_i\frac{\partial^2}{\partial p_i \partial q_i} + p_i^2\frac{\partial^2}{\partial q_i^2}.$$

Set $A = p_1 + p_2 + p_3$, $B = \frac{1}{5}\Theta A$, $H = \Theta B$, $C = \frac{1}{16}\Delta H$, $J = \frac{1}{9}(\Theta C - 3A^2B)$, $K = \frac{2}{3}\Theta J$. Then it is proved in [SYS80] that flat coordinates of type E_6 are given by

$$\begin{cases} \psi_2 = A, \\ \psi_5 = B, \\ \psi_6 = C - \frac{1}{8}A^3, \\ \psi_8 = H - \frac{1}{4}AC + \frac{5}{192}A^4, \\ \psi_9 = J, \\ \psi_{12} = K - \frac{1}{8}A^2H - \frac{1}{8}C^2 + \frac{5}{96}A^3C - AB^2 - \frac{1}{256}A^6. \end{cases}$$

4.3.6. Semiuniversal deformation of type $B_r = (A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$

We have $\Gamma = \mathcal{C}_{2r}$ and $\Gamma' = \mathcal{D}_r$. The McKay quiver $\widetilde{A}_{2r-1}$ is



The orientation has been chosen as $\epsilon(a_i) = -\epsilon(b_i) = 1$, for any $0 \le i \le 2r-1$. $M(\Gamma)$ is the space of representation of this quiver with dimension vector $(1, \dots, 1)$:

$$\begin{array}{ll} M(\Gamma) & = \mathbb{C}^{2r} \oplus \mathbb{C}^{2r} \\ & \{(a_0, \dots, a_{2r-1}, b_0, \dots, b_{2r-1}) \mid a_i, b_i \in \mathbb{C}\} \end{array}.$$

Furthermore $G(\Gamma) = (\mathbb{C}^*)^{2r}/\mathbb{C}^* \cong \mathbb{C}^{2r-1}$ and the action of $g \in G(\Gamma)$ on $M(\Gamma)$ is done by conjugation:

$$g.(a_0,\dots a_{2r-1},b_0,\dots,b_{2r-1}) = (g_1g_0^{-1}a_0,\dots g_0g_{2r-1}^{-1}a_{2r-1},g_0g_1^{-1}b_0,\dots,g_{2r-1}g_0^{-1}b_{2r-1}).$$

It was computed in Subsection 4.2.2 that the procedure by H. Cassens and P. Slodowy leads to

$$\begin{array}{rcl}\{\prod_{i=0}^{2r-1}(z-\lambda_i)=xy\}=M(\Gamma)//G(\Gamma) & \xrightarrow{\widetilde{\alpha}} & \mathfrak{h} \\ (\lambda_0,\dots,\lambda_{2r-1},x,y,z) & \mapsto & (\lambda_0,\dots,\lambda_{2r-1})\end{array},$$

which, according to Section 4.2.1, is the pullback of the semiuniversal deformation of $X_0=\widetilde{\alpha}^{-1}(0)=\{(x,y,z)\in\mathbb{C}^3 \mid z^{2r}=xy\}$, a singularity of type A_{2r-1}.

The following diagram:

$$\begin{array}{ccc} M(\Gamma)//G(\Gamma)\cong X_\Gamma\times_{\mathfrak{h}/W}\mathfrak{h} & \xrightarrow{\Psi} & X_\Gamma \\ \downarrow{\scriptstyle\widetilde{\alpha}} & \circlearrowleft & \downarrow{\scriptstyle\alpha} \\ \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W \end{array}$$

is the pullback of the semiuniversal deformation $X_\Gamma\xrightarrow{\alpha}\mathfrak{h}/W$ of X_0.

It is known that $\mathbb{C}[\mathfrak{h}]=\mathbb{C}[x_0,\dots,x_{2r-1}]/(x_0+\dots+x_{2r-1})$ and $\mathbb{C}[\mathfrak{h}/W]=\mathbb{C}[\mathfrak{h}]^W=\mathbb{C}[\psi_2(x),\dots,\psi_{2r}(x)]$ with the ψ_i's being the flat coordinates from Subsection 4.3.5.2. Because W is finite and $\mathbb{C}[\mathfrak{h}/W]\hookrightarrow\mathbb{C}[\mathfrak{h}]$, there is a surjection

$$\begin{array}{rccc}\pi: & \mathfrak{h} & \twoheadrightarrow & \mathfrak{h}/W \\ & (\lambda_0,\dots,\lambda_{2r-1}) & \mapsto & (\psi_2(\lambda),\dots,\psi_{2r}(\lambda))\end{array}.$$

If $\lambda\in\mathfrak{h}$, then one computes $\prod_{i=0}^{2r-1}(z-\lambda_i)=z^{2r}+\sum_{i=2}^{2r}(-1)^i\epsilon_i(\lambda)z^{2r-i}$ with ϵ_i being the i-th elementary symmetric polynomial in $2r$ variables. It is proved in [SYS80] that

$$\epsilon_i=\sum_{d\geq 1}\frac{(2r-i+1,d-1)}{d!(2r)^{d-1}}Y_i^d$$

where

$$Y_i^d = \sum_{\substack{i_1+\ldots+i_d=i \\ i_j\in\{2,\ldots,2r\}}} \psi_{i_1}\ldots\psi_{i_d}.$$

and $(a,n) = a(a+1)\ldots(a+n-1)$. Hence for any $i \in \{2,\ldots,2r\}$, there exists a polynomial f_i such that $\epsilon_i = f_i(\psi)$, with $\psi = (\psi_2,\ldots,\psi_{2r})$. Set

$$X_\Gamma = \{(x,y,z,t_2,\ldots,t_{2r}) \in \mathbb{C}^3\times\mathfrak{h}/W \mid z^{2r} + \sum_{i=2}^{2r}(-1)^i f_i(t_2,\ldots,t_{2r})z^{2r-i} = xy\}.$$

There are two maps

$$\begin{array}{rccc} \alpha: & X_\Gamma & \to & \mathfrak{h}/W \\ & (x,y,z,t_2,\ldots,t_{2r}) & \mapsto & (t_2,\ldots,t_{2r}) \end{array} \quad \text{and}$$
$$\begin{array}{rccc} \pi: & \mathfrak{h} & \twoheadrightarrow & \mathfrak{h}/W \\ & (\lambda_0,\ldots,\lambda_{2r-1}) & \mapsto & (\psi_2(\lambda),\ldots,\psi_{2r}(\lambda)) \end{array}.$$

Then the pullback of these maps (or fibered product of these two maps) is

$$\begin{aligned} & X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h} \\ = \ & \{((x,y,z,t_2,\ldots,t_{2r}),(\lambda_0,\ldots,\lambda_{2r-1})) \in X\times\mathfrak{h} \mid \alpha(x,y,z,t_2,\ldots,t_{2r}) = \pi(\lambda_0,\ldots,\lambda_{2r-1})\}, \\ = \ & \{((x,y,z,t_2,\ldots,t_{2r}),(\lambda_0,\ldots,\lambda_{2r-1})) \in X\times\mathfrak{h} \mid (t_2,\ldots,t_{2r}) = (\psi_2(\lambda),\ldots,\psi_{2r}(\lambda))\}, \\ = \ & \{(x,y,z,\lambda_0,\ldots,\lambda_{2r-1}) \in \mathbb{C}^3\times\mathfrak{h} \mid \sum_{i=0}^{2r}(-1)^i f_i(\psi_2(\lambda),\ldots,\psi_{2r}(\lambda))z^{2r-i} = xy\}, \\ = \ & \{(x,y,z,\lambda_0,\ldots,\lambda_{2r-1}) \in \mathbb{C}^3\times\mathfrak{h} \mid \prod_{i=0}^{2r-1}(z-\lambda_i) = xy\}, \\ = \ & M(\Gamma)//G(\Gamma), \end{aligned}$$

and $X_{\Gamma,0} := \alpha^{-1}(0) = \{(x,y,z) \in \mathbb{C}^3 \mid z^{2r} = xy\}$. It is indeed an A_{2r-1}-type singularity.

We now need to verify that all the maps in the pullback diagram are Ω-equivariant, with $\Omega := \Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$.

One sees that

$$\begin{array}{rccc} \Psi: & X_\Gamma\times_{\mathfrak{h}/W}\mathfrak{h} & \to & X_\Gamma \\ & (x,y,z,\lambda_0,\ldots,\lambda_{2r-1}) & \mapsto & (x,y,z,\psi_2(\lambda),\ldots,\psi_{2r}(\lambda)) \end{array}.$$

Ψ is surjective if and only if for all $(x,y,z,t_2,\ldots,t_{2r}) \in \mathbb{C}^3\times\mathfrak{h}/W$ with $z^{2r}+\sum_{i=2}^{2r}(-1)^i f_i(t_2,\ldots,t_{2r})z^{2r-i} = xy$, there exists $\lambda\in\mathfrak{h}$ such that $\psi_2(\lambda) = t_2,\ldots,\psi_{2r}(\lambda) =$

t_{2r}. It is a system of $2r-1$ polynomial equations in $2r$ variables with the relation $\sum_{i=0}^{2r-1}\lambda_i = 0$ because $\mathfrak{g} = \mathfrak{sl}_{2r}(\mathbb{C})$. As the field $\mathbb{C}$ is algebraically closed and the ψ_i's are algebraically independent, there exists a solution. Hence Ψ is surjective.

• Let $(x,y,z,\lambda_0,\dots,\lambda_{2r-1}) \in M(\Gamma)//G(\Gamma)$. With the action of Ω on $M(\Gamma)$ from Subsection 4.3.4.1 and the change of variables we did in Subsection 4.2.2, we obtain

$$\sigma.(x,y,z,\lambda_0,\dots,\lambda_{2r-1}) = ((-1)^r y,(-1)^r x,-z,-\lambda_{2r-1},\dots,-\lambda_0) \in X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h}.$$

Therefore $\widetilde{\alpha}(\sigma.(x,y,z,\lambda_0,\dots,\lambda_{2r-1})) = (-\lambda_{2r-1},\dots,-\lambda_0) \begin{array}{l} =: \sigma.(\lambda_0,\dots,\lambda_{2r-1}), \\ = \sigma.\widetilde{\alpha}(\lambda_0,\dots,\lambda_{2r-1}), \end{array}$
and $\widetilde{\alpha}$ is Ω-equivariant.

• If $(x,y,z,\lambda_0,\dots,\lambda_{2r-1}) \in M(\Gamma)//G(\Gamma)$, then

$$\begin{array}{ll} \Psi(\sigma.(x,y,z,\lambda_0,\dots,\lambda_{2r-1})) & = \Psi(((-1)^r y,(-1)^r x,-z,-\lambda_{2r-1},\dots,-\lambda_0)), \\ & = ((-1)^r y,(-1)^r x,-z,\psi_2(-\lambda),\dots,\psi_{2r}(-\lambda)), \\ & =: \sigma.(x,y,z,\psi_2(\lambda),\dots,\psi_{2r}(\lambda)), \\ & = \sigma.\Psi(x,y,z,\lambda_0,\dots,\lambda_{2r-1})). \end{array}$$

Hence the action of Ω on X_Γ makes Ψ an Ω-equivariant map.

• If $(\lambda_0,\dots,\lambda_{2r-1}) \in \mathfrak{h}$, then $\begin{array}{ll} \pi(\sigma.(\lambda_0,\dots,\lambda_{2r-1})) & = \pi(-\lambda_{2r-1},\dots,-\lambda_0), \\ & = (\psi_2(-\lambda),\dots,\psi_{2r}(-\lambda)), \\ & = (\psi_2(\lambda),-\psi_3(\lambda),\dots,\psi_{2r}(\lambda)), \\ & := \sigma.(\psi_2(\lambda),\psi_3(\lambda),\dots,\psi_{2r}(\lambda)), \\ & = \sigma.\pi(\lambda_0,\dots,\lambda_{2r-1}). \end{array}$
Hence the action of Ω on $\mathfrak{h}/W$ makes π an Ω-equivariant map.

We have just defined the action of Ω on X_Γ. It needs to induce the natural action on the special fiber $X_{\Gamma,0} = \alpha^{-1}(0) \subset X_\Gamma$, i.e. the one obtained in Subsection 1.4.2.1. $X_{\Gamma,0}$ is defined by $\{z^{2r} = xy\}$ and Ω acts on $X_{\Gamma,0}$ by $\left\{ \begin{array}{l} \sigma.x = (-1)^r y, \\ \sigma.y = (-1)^r x, \\ \sigma.z = -z. \end{array} \right.$ The action of Ω on $X_\Gamma = \{(x,y,z,t_2,\dots,t_{2r}) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^{2r} + \sum_{i=2}^{2r}(-1)^i f_i(t_2,\dots,t_{2r}) z^{2r-i} = xy\}$ is

$$\sigma.(x,y,z,t_2,\dots,t_{2r}) = ((-1)^r y,(-1)^r x,-z,t_2,-t_3,t_4,-t_5,\dots,t_{2r}).$$

Then on $X_{\Gamma,0}$, $t_i = 0$ for $i \geq 2$ and $\sigma.(x,y,z,0,..,0) = ((-1)^r y,(-1)^r x,-z,0,\dots,0)$, which is the expected action on $\mathbb{C}^2/\Gamma$.

• Let us check the equivariance of α. For any $(x, y, z, t_2, \dots, t_{2r}) \in X_\Gamma$,

$$\begin{aligned}\alpha(\sigma.(x, y, z, t_2, \dots, t_{2r})) &= \alpha((-1)^r y, (-1)^r x, -z, t_2, -t_3, t_4, -t_5, \dots, t_{2r}),\\ &= (t_2, -t_3, t_4, -t_5, \dots, t_{2r}),\end{aligned}$$

and $\sigma.\alpha(x, y, z, t_2, \dots, t_{2r}) = \sigma.(t_2, \dots, t_{2r}) = (t_2, -t_3, t_4, -t_5, \dots, t_{2r})$. Hence α is Ω-equivariant.

Finally the diagram

$$\begin{array}{ccc} X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h} & \xrightarrow{\Psi} & X_\Gamma \\ \downarrow \widetilde{\alpha} & \circlearrowright & \downarrow \alpha \\ \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W \end{array}$$

is commutative and Ω-equivariant.

Let us place ourselves above $(\mathfrak{h}/W)^\Omega$. If $(x, y, z, t_2, \dots, t_{2r}) \in \alpha^{-1}((\mathfrak{h}/W)^\Omega)$, then $t_{2i+1} = 0$ for any $1 \leq i \leq r-1$. Furthermore $\alpha(\sigma.(x, y, z, t_2, 0, \dots, t_{2r})) = \sigma.(t_2, 0, \dots, t_{2r}) = \alpha(x, y, z, t_2, 0, \dots, t_{2r})$.
Set $X_{\Gamma,\Omega} = \{(x, y, z, t_2, 0, \dots, t_{2r}) \in X\}$. Then the restriction $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ of α is Ω-invariant and

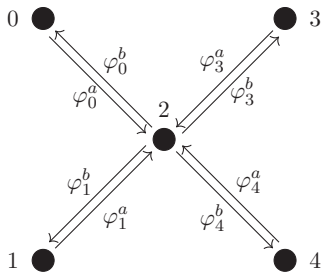
$$\begin{aligned}(\alpha^\Omega)^{-1}(0) &= \alpha^{-1}(0) \bigcap X_{\Gamma,\Omega} \text{ because } 0 \in (\mathfrak{h}/W)^\Omega,\\ &= X_{\Gamma,0} \bigcap X_{\Gamma,\Omega},\\ &= X_{\Gamma,0} \text{ because } X_{\Gamma,0} \subset X_{\Gamma,\Omega}.\end{aligned}$$

Therefore α^Ω is a semiuniversal deformation of type $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z}) = B_r$, result we expected from Theorem 4.3.1.

Remark 4.3.11. As the elementary symmetric polynomials ϵ_i appear naturally in the expression of $M(\Gamma)//G(\Gamma)$, instead of the flat coordinates we could have chosen $(\epsilon_2, \dots, \epsilon_{2r})$ as a system of coordinates for $\mathfrak{h}/W$.

4.3.7. Semiuniversal deformation of type $C_3 = (D_4, \mathbb{Z}/2\mathbb{Z})$

Set $\Gamma = \mathcal{D}_2$ and $\Gamma' = \mathcal{D}_4$. The McKay quiver of type $\widetilde{D_4}$ is

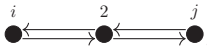


The dimension vector of this quiver is $(1,1,2,1,1)$, which corresponds to the coordinates of the highest root of the root system D_4. The orientation is all a's are positive and all b's are negative.
We have $Z = \{(\mu_0, \mu_1, \mu_2 \mathrm{Id}_2, \mu_3, \mu_4) \mid \mu_0 + \mu_1 + 2\mu_2 + \mu_3 + \mu_4 = 0\}$.

In [LeBryPro90], it is proved that the ring of invariants of the representation space of a quiver by the product of general linear groups associated with vertices is generated by the traces of the oriented cycles of the said quiver.

We have $\mu_{\mathbb{C}}(\varphi) = (-\varphi_0^b\varphi_0^a, -\varphi_1^b\varphi_1^a, \varphi_0^a\varphi_0^b + \varphi_1^a\varphi_1^b + \varphi_3^a\varphi_3^b + \varphi_4^a\varphi_4^b, -\varphi_3^b\varphi_3^a, -\varphi_4^b\varphi_4^a)$ and compute $\mu_{\mathbb{C}}^{-1}(\mu_0, \mu_1, \mu_2 \mathrm{I}_2, \mu_3, \mu_4)//G(\Gamma)$.

- Cycles of degree 2: $\mathrm{Tr}(\varphi_i^a\varphi_i^b) = \varphi_i^b\varphi_i^a = -\mu_i,\ i = 0,1,3,4.$

- Cycles of degree 4: p_{ij} = 

. For example, $p_{01} = \mathrm{Tr}(\varphi_0^b\varphi_1^a\varphi_1^b\varphi_0^a) = \mathrm{Tr}(\varphi_0^a\varphi_0^b\varphi_1^a\varphi_1^b)$.
We have $p_{ii} + p_{ij} + p_{ik} + p_{il} = \mathrm{Tr}(\varphi_i^a\varphi_i^b(\varphi_i^a\varphi_i^b + \varphi_j^a\varphi_j^b + \varphi_k^a\varphi_k^b + \varphi_l^a\varphi_l^b))$ with $\{i,j,k,l\} = \{0,1,3,4\}$,
$= \mathrm{Tr}(\mu_2\varphi_i^a\varphi_i^b)$,
$= -\mu_2\mu_i$.

So $\begin{cases} p_{01} + p_{03} + p_{04} = -\mu_0(\mu_0 + \mu_2), \\ p_{01} + p_{13} + p_{14} = -\mu_1(\mu_1 + \mu_2), \\ p_{03} + p_{13} + p_{34} = -\mu_3(\mu_3 + \mu_2), \\ p_{04} + p_{14} + p_{34} = -\mu_4(\mu_4 + \mu_2). \end{cases}$

There are 4 linear equations with 6 unknowns. The matrix of the system is of rank 4. There are then 2 independent elements and we choose p_{03} and p_{34}.

- Cycles of degree 6: They are $q_{ijk} = \mathrm{Tr}(\varphi_k^a\varphi_k^b\varphi_j^a\varphi_j^b\varphi_i^a\varphi_i^b) = q_{jki} = q_{kij}$.

We have $q_{ijk} + q_{ijl} + q_{iji} + q_{ijj} = \mathrm{Tr}((\varphi_k^a\varphi_k^b + \varphi_l^a\varphi_l^b + \varphi_i^a\varphi_i^b + \varphi_j^a\varphi_j^b)\varphi_j^a\varphi_j^b\varphi_i^a\varphi_i^b)$ with $\{i,j,k,l\} = \{0,1,3,4\}$, $= \mu_2\mathrm{Tr}(\varphi_j^a\varphi_j^b\varphi_i^a\varphi_i^b)$, $= \mu_2 p_{ij}$.

Furthermore $q_{iji} = -\mu_i p_{ij}$ and $q_{ijj} = -\mu_j p_{ij}$. So $q_{ijk} + q_{ijl} = (\mu_i + \mu_j + \mu_2)p_{ij}$.
Finally
$$\begin{cases} q_{013} + q_{014} = (\mu_0 + \mu_1 + \mu_2)p_{01} = q_{103} + q_{104}, \\ q_{103} + q_{034} = (\mu_0 + \mu_3 + \mu_2)p_{03} = q_{013} + q_{304}, \\ q_{104} + q_{304} = (\mu_0 + \mu_4 + \mu_2)p_{04} = q_{014} + q_{034}, \\ q_{134} + q_{013} = (\mu_1 + \mu_3 + \mu_2)p_{13} = q_{314} + q_{103}, \\ q_{014} + q_{314} = (\mu_1 + \mu_4 + \mu_2)p_{14} = q_{104} + q_{134}, \\ q_{134} + q_{034} = (\mu_3 + \mu_4 + \mu_2)p_{34} = q_{314} + q_{304}. \end{cases}$$
There are 12 linear equations with 8 unknowns. The matrix of the system is of rank 7. Therefore there are $8 - 7 = 1$ independent element and we choose q_{034}.

• Cycles of degree 8: They are $z_{ijkl} = \mathrm{Tr}(\varphi_l^a\varphi_l^b\varphi_k^a\varphi_k^b\varphi_j^a\varphi_j^b\varphi_i^a\varphi_i^b)$.
We have $z_{ijkl} + z_{ijkk} + z_{ijkj} + z_{ijki} = \mathrm{Tr}((\varphi_l^a\varphi_l^b + \varphi_k^a\varphi_k^b + \varphi_j^a\varphi_j^b + \varphi_i^a\varphi_i^b)\varphi_k^a\varphi_k^b\varphi_j^a\varphi_j^b\varphi_i^a\varphi_i^b)$, $\{i,j,k,l\} = \{0,1,3,4\}$, $= \mu_2\mathrm{Tr}(\varphi_k^a\varphi_k^b\varphi_j^a\varphi_j^b\varphi_i^a\varphi_i^b)$, $= \mu_2 q_{ijk}$.

We compute that $z_{ijkk} = -\mu_k q_{ijk}$, $z_{ijki} = -\mu_i q_{ijk}$ and $z_{ijkj} = p_{jk}p_{ij}$.
So $z_{ijkl} = (\mu_i + \mu_k + \mu_2)q_{ijk} - p_{ij}p_{jk}$.

Thus all degree 8 cycles are determined by elements with inferior degrees.

• Cycles of degree ≥ 10: Starting from degree 10, for any cycle, there is at least one repetition in the indices. Hence any element of degree $k \geq 10$ breaks down as the product of two elements with degrees strictly smaller than k.

Eventually there are 3 linearly independent elements: p_{03}, p_{34} and q_{034}. By [LeBryPro90] they generate the ring of invariants.

Because the space we want to exhibit is the pullback of the semiuniversal deformation of a simple singularity, and that such a deformation is defined by a single equation (cf. [KasSchle72]), a unique relation has to link the generators (or one can compute the dimension of the quotient variety and see that it is a hypersurface). Let us find this relation.

$$\begin{aligned} & q_{034}^2 \\ = \; & q_{034}q_{034}, \\ = \; & q_{034}((\mu_0 + \mu_3 + \mu_2)p_{03} - (\mu_0 + \mu_1 + \mu_2)p_{01} + (\mu_0 + \mu_4 + \mu_2)p_{04} - q_{304}), \\ = \; & q_{034}((\mu_0 + \mu_3 + \mu_2)p_{03} - (\mu_0 + \mu_1 + \mu_2)p_{01} + (\mu_0 + \mu_4 + \mu_2)p_{04}) - q_{034}q_{304}, \\ = \; & q_{034}((\mu_0 + \mu_3 + \mu_2)p_{03} - (\mu_0 + \mu_1 + \mu_2)p_{01} + (\mu_0 + \mu_4 + \mu_2)p_{04}) - p_{03}p_{04}p_{34}. \end{aligned}$$

We know that $2(p_{01}-p_{34}) = -\mu_0(\mu_0+\mu_2)-\mu_1(\mu_1+\mu_2)+\mu_3(\mu_3+\mu_2)+\mu_4(\mu_4+\mu_2)$ and $p_{01}+p_{03}+p_{04} = -\mu_0(\mu_0+\mu_2)$ thus p_{01} and p_{04} can be expressed using p_{03} and p_{34}. By reinjecting these expressions in the preceding equation, one finds

$$q_{034}^2 = q_{034}(p_{03}(\mu_3-\mu_4)+p_{34}(\mu_3-\mu_0)+C)+p_{03}p_{34}(p_{03}+p_{34})+p_{03}p_{34}E,$$

with $C = \mu_3(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3)$ and $E = \frac{1}{2}(\mu_0(\mu_0+\mu_2)-\mu_1(\mu_1+\mu_2)+\mu_3(\mu_3+\mu_2)+\mu_4(\mu_4+\mu_2))$.

Set $q'_{034} = q_{034}-\frac{1}{2}(p_{03}(\mu_3-\mu_4)+p_{34}(\mu_3-\mu_0)+C)$. The equation becomes

$$q'^2_{034} = p_{03}p_{34}(p_{03}+p_{34})+p_{03}p_{34}(D+E)+Ap_{03}^2+Bp_{34}^2+Fp_{03}+Gp_{34}+H,$$

with

$$\begin{cases} A = \frac{1}{4}(\mu_3-\mu_4)^2, \\ B = \frac{1}{4}(\mu_3-\mu_0)^2, \\ D = \frac{1}{2}(\mu_3-\mu_4)(\mu_3-\mu_0), \\ F = \frac{1}{2}(\mu_3-\mu_4)C, \\ G = \frac{1}{2}(\mu_3-\mu_0)C, \\ H = \frac{1}{4}C^2. \end{cases}$$

Set $\begin{cases} p'_{03} = p_{03}+B \\ p'_{34} = p_{34}+A \end{cases}$. The equation now becomes

$$q'^2_{034} = p'_{03}p'_{34}(p'_{03}+p'_{34})+\mathcal{A}p'_{03}p'_{34}+\mathcal{B}p'_{03}+\mathcal{C}p'_{34}+\mathcal{D},$$

with

$$\begin{cases} \mathcal{A} = -\mu_1\mu_2-\mu_2\mu_3-\mu_2\mu_4-\mu_2^2-\frac{1}{2}\mu_1\mu_4-\frac{1}{2}\mu_1\mu_3-\frac{1}{2}\mu_3\mu_4-\frac{1}{2}\mu_1^2-\frac{1}{2}\mu_3^2-\frac{1}{2}\mu_4^2, \\ \mathcal{B} = \frac{1}{16}(\mu_3-\mu_4)(\mu_3+\mu_4)(2\mu_2+\mu_3+\mu_4)(2\mu_1+2\mu_2+\mu_3+\mu_4), \\ \mathcal{C} = \frac{1}{16}(\mu_1-\mu_4)(\mu_1+\mu_4)(2\mu_2+\mu_1+\mu_4)(2\mu_3+2\mu_2+\mu_1+\mu_4), \\ \mathcal{D} = -\frac{1}{32}(2\mu_1\mu_2+\mu_1\mu_3+\mu_1\mu_4+2\mu_2^2+2\mu_2\mu_3+2\mu_2\mu_4+\mu_3\mu_4+\mu_4^2)(\mu_1\mu_3+\mu_1\mu_4 \\ \qquad +2\mu_2\mu_4+\mu_3\mu_4+\mu_4^2)(\mu_1\mu_3-\mu_1\mu_4-2\mu_2\mu_4-\mu_3\mu_4-\mu_4^2). \end{cases}$$

The equation is indeed of type D_4 as predicted by Theorem 1.4.26. However we want to look at the fibers over $\mathfrak{h}/W$, so the coefficients of the equation need to be invariant by the Weyl group W. The coefficients $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ have to be expressible using Weyl group invariants.

From Subsection 4.1.2, it is known that $\tau : \begin{array}{ccc} Z & \xrightarrow{\cong} & \mathfrak{h} \\ (\mu_0\mathrm{Id}_{d_0},\dots,\mu_4\mathrm{Id}_{d_4}) & \mapsto & h \end{array}$ such that $\tau(z) = h$ with $\alpha_i(h) = -\mu_i, \forall 0 \leq i \leq 4$. Hence an element $\mu \in Z$ can be identified with $\sum_{i=1}^4 -\mu_i\Lambda_i^\vee$, with $(\Lambda_i^\vee)_i$ being the fundamental coweights, i.e. the dual base of the simple roots $(\alpha_i)_i$.
The Weyl group W of type D_4 is generated by the $r_{\alpha_j^\vee}$'s with

$$\begin{aligned} r_{\alpha_j^\vee}(\Lambda_i^\vee) &= \Lambda_i^\vee - \frac{2(\Lambda_i^\vee,\alpha_j^\vee)}{(\alpha_j^\vee,\alpha_j^\vee)}\alpha_j^\vee, \\ &= \begin{cases} \Lambda_i^\vee \text{ if } i \neq j, \\ \Lambda_i^\vee - \alpha_i^\vee \text{ if } i = j. \end{cases} \end{aligned}$$

It is known that $\alpha_i^\vee = \sum_{j=1}^4 c_{ij}\Lambda_j^\vee$ with $c = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ the Cartan matrix of type D_4. One sees that $\begin{cases} \alpha_1^\vee = 2\Lambda_1^\vee - \Lambda_2^\vee, \\ \alpha_2^\vee = -\Lambda_1^\vee + 2\Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee, \\ \alpha_3^\vee = -\Lambda_2^\vee + 2\Lambda_3^\vee, \\ \alpha_4^\vee = -\Lambda_2^\vee + 2\Lambda_4^\vee. \end{cases}$
Hence $r_{\alpha_1^\vee}(\mu) = r_{\alpha_1^\vee}(\sum_{i=1}^4 -\mu_i\Lambda_i^\vee) = \mu_1\Lambda_1^\vee - (\mu_1+\mu_2)\Lambda_2^\vee - \mu_3\Lambda_3^\vee - \mu_4\Lambda_4^\vee$, which means $r_{\alpha_1^\vee} : \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \mapsto \begin{pmatrix} -\mu_1 \\ \mu_1+\mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}$. In a similar fashion we compute $r_{\alpha_2^\vee} : \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \mapsto \begin{pmatrix} \mu_1+\mu_2 \\ -\mu_2 \\ \mu_2+\mu_3 \\ \mu_2+\mu_4 \end{pmatrix}$, $r_{\alpha_3^\vee} : \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \mapsto \begin{pmatrix} \mu_1 \\ \mu_2+\mu_3 \\ -\mu_3 \\ \mu_4 \end{pmatrix}$ and $r_{\alpha_4^\vee} : \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \mapsto \begin{pmatrix} \mu_1 \\ \mu_2+\mu_4 \\ \mu_3 \\ -\mu_4 \end{pmatrix}$.
It can be verified that the coefficients $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$ are invariant by these transformations and thus are invariant by W.

The base we chose to use for $\mathfrak{h}/W$ is the one with flat coordinates. Therefore the coefficients $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$ have to be expressed using ψ_2,ψ_4,ψ_6 and ψ from Subsection 4.3.5.3.
It is known (cf. [Bou68]) that the simple roots of the root system of type D_4 are

$\begin{cases} \alpha_1 = e_1 - e_2, \\ \alpha_2 = e_2 - e_3, \\ \alpha_3 = e_3 - e_4, \\ \alpha_4 = e_3 + e_4. \end{cases}$ with $(e_i)_{1\leq i\leq 4}$ the canonical base of $\mathfrak{h}^*$. Let $(\overline{e_i})_{1\leq i\leq 4}$ be its dual base. Again from [Bou68] one finds $\begin{cases} \Lambda_1^\vee = \overline{e_1}, \\ \Lambda_2^\vee = \overline{e_1} + \overline{e_2}, \\ \Lambda_3^\vee = \frac{1}{2}(\overline{e_1} + \overline{e_2} + \overline{e_3} - \overline{e_4}), \\ \Lambda_4^\vee = \frac{1}{2}(\overline{e_1} + \overline{e_2} + \overline{e_3} + \overline{e_4}). \end{cases}$ We have a correspondence $\mu \leftrightarrow \sum_{i=1}^4 -\mu_i\Lambda_i^\vee = \sum_{i=1}^4 \xi_i\overline{e_i}$. It follows that $\begin{cases} \xi_1 = -\mu_1 - \mu_2 - \frac{1}{2}(\mu_3 + \mu_4), \\ \xi_2 = -\mu_2 - \frac{1}{2}(\mu_3 + \mu_4), \\ \xi_3 = -\frac{1}{2}(\mu_3 + \mu_4), \\ \xi_4 = -\frac{1}{2}(\mu_4 - \mu_3). \end{cases}$ and so $\begin{cases} \mu_1 = -\xi_1 + \xi_2, \\ \mu_2 = -\xi_2 + \xi_3, \\ \mu_3 = -\xi_3 + \xi_4, \\ \mu_4 = -\xi_3 - \xi_4. \end{cases}$ We can now replace the μ_i in the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ and we obtain

$$\begin{cases} \mathcal{A} = -\frac{1}{2}\epsilon_1(\xi^2), \\ \mathcal{B} = -\epsilon_4(\xi), \\ \mathcal{C} = \frac{1}{16}\epsilon_1(\xi^2)^2 - \frac{1}{2}\epsilon_4(\xi) - \frac{1}{4}\epsilon_2(\xi^2), \\ \mathcal{D} = \frac{1}{4}(\epsilon_3(\xi^2) + \epsilon_4(\xi)\epsilon_1(\xi^2)), \end{cases}$$

with ϵ_i the i-th elementary symmetric polynomial, $\xi = (\xi_1, \ldots, \xi_4)$ and $\xi^2 = (\xi_1^2, \ldots, \xi_4^2)$. Using the expressions from Subsection 4.3.5.3 one finds

$$\begin{cases} \mathcal{A} = -\frac{1}{2}\psi_2, \\ \mathcal{B} = -\psi, \\ \mathcal{C} = -\frac{1}{2}(\psi + \frac{1}{2}\psi_4), \\ \mathcal{D} = \frac{1}{4}(\psi_6 + \frac{1}{6}\psi_2\psi_4 + \psi\psi_2 + \frac{1}{108}\psi_2^3), \end{cases}$$

Set $x = p'_{34}$, $y = p'_{03}$ and $z = q'_{034}$. The equation of the deformation becomes:

$$z^2 = xy(x+y) - \frac{1}{2}\psi_2 xy - \psi y - \frac{1}{2}(\psi + \frac{1}{2}\psi_4)x + \frac{1}{4}(\psi_6 + \frac{1}{6}\psi_2\psi_4 + \psi\psi_2 + \frac{1}{108}\psi_2^3).$$

The coefficients are indeed invariants of $W = \mathfrak{S}_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$. According to Subsection 4.2.1, it is the equation of the pullback of the semiuniversal deformation of type D_4.

Set $X_\Gamma = \{(x,y,z,t_2,t_4,t_6,t) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^2 = xy(x+y) - \frac{1}{2}t_2xy - ty - \frac{1}{2}(t+\frac{1}{2}t_4)x + \frac{1}{4}(t_6 + \frac{1}{6}t_2t_4 + tt_2 + \frac{1}{108}t_2^3)\}$.

We have $\pi : \mathfrak{h} \to \mathfrak{h}/W$ and
$(\xi_1,\xi_2,\xi_3,\xi_4) \mapsto (\psi_2(\xi),\psi_4(\xi),\psi_6(\xi),\psi(\xi))$

$\alpha : X_\Gamma \to \mathfrak{h}/W$.
$(x,y,z,t_2,t_4,t_6,t) \mapsto (t_2,t_4,t_6,t)$

Then $X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h}$

$= \{((x,y,z,t_2,t_4,t_6,t),(\xi_1,\xi_2,\xi_3,\xi_4)) \in X \times \mathfrak{h} \mid \alpha(x,y,z,t_i,t) = \pi(\xi_1,\xi_2,\xi_3,\xi_4)\}$,

$= \{(x,y,z,t_i,t,\xi_j) \mid t_2 = \psi_2(\xi), t_4 = \psi_4(\xi), t_6 = \psi_6(\xi), t = \psi(\xi)$ and $z^2 = xy(x+y) - \frac{1}{2}t_2xy - ty - \frac{1}{2}(t+\frac{1}{2}t_4)x + \frac{1}{4}(t_6 + \frac{1}{6}t_2t_4 + tt_2 + \frac{1}{108}t_2^3)\}$,

$= \{(x,y,z,\xi_j) \mid z^2 = xy(x+y) - \frac{1}{2}\psi_2(\xi)xy - \psi(\xi)y - \frac{1}{2}(\psi(\xi) + \frac{1}{2}\psi_4(\xi))x + \frac{1}{4}(\psi_6(\xi) + \frac{1}{6}\psi_2(\xi)\psi_4(\xi) + \psi(\xi)\psi_2(\xi) + \frac{1}{108}\psi_2(\xi)^3)\}$,

$= \mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma)$.

Finally we obtain

$$\begin{array}{ccc} X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h} & \xrightarrow{\Psi} & X_\Gamma \\ \downarrow \widetilde{\alpha} & \circlearrowleft & \downarrow \alpha \\ \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W \end{array}$$

with α being the semiuniversal deformation of $X_{\Gamma,0} := \alpha^{-1}(0) = \{(x,y,z) \in \mathbb{C}^3 \mid z^2 = xy(x+y)\}$, which is a D_4 singularity.

Previously we defined the action of $\Omega = \Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$ on $\mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma)$ as being the natural $3 \leftrightarrow 4$ permutation. Ω also acts naturally on $\mathfrak{h}$ by the permutation $\alpha_3^\vee \leftrightarrow \alpha_4^\vee$, which corresponds to $\overline{e_4} \leftrightarrow -\overline{e_4}$. By setting $\sigma.(t_2,t_4,t_6,t) = (t_2,t_4,t_6,-t)$ for the action of Ω on $\mathfrak{h}/W$, we find that π becomes Ω-equivariant.

We look at what happens above $(\mathfrak{h}/W)^\Omega$, i.e. the elements of $\mathfrak{h}/W$ that are Ω-invariant. One finds $\alpha^{-1}((\mathfrak{h}/W)^\Omega) = \{(x,y,z,t_2,t_4,t_6,0) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^2 = xy(x+y) - \frac{1}{2}t_2xy - \frac{1}{4}t_4x + \frac{1}{4}(t_6 + \frac{1}{6}t_2t_4 + \frac{1}{108}t_2^3)\}$. The action of Ω on $X_{\Gamma,\Omega} := \alpha^{-1}((\mathfrak{h}/W)^\Omega)$ is induced by the $\mu_3 \leftrightarrow \mu_4$

permutation. It was proved that $\left\{\begin{array}{lcl} x & = & p_{34}+\frac{1}{4}(\mu_3-\mu_4)^2, \\ y & = & p_{03}+\frac{1}{4}(\mu_3-\mu_0)^2, \\ z & = & q_{034}-\frac{1}{2}(p_{03}(\mu_3-\mu_4)+p_{34}(\mu_3-\mu_0) \\ & & +\mu_3(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3)), \end{array}\right.$
and $\mu_3 \overset{\sigma}{\leftrightarrow} \mu_4$.

Because $p_{04}=-p_{03}-p_{01}-\mu_0(\mu_0+\mu_2)$ and $2(p_{01}-p_{34})=-\mu_0(\mu_0+\mu_2)-\mu_1(\mu_1+\mu_2)+\mu_3(\mu_3+\mu_2)+\mu_4(\mu_4+\mu_2)$, we obtain the following relations:

$$\left\{\begin{array}{lcl} \sigma.x & = & x, \\ \sigma.y & = & -x-y+\frac{1}{2}t_2, \\ \sigma.z & = & -z. \end{array}\right.$$

Above the origin, the system is $\left\{\begin{array}{lcccl} \sigma.x & = & x, & & \\ \sigma.y & = & p_{04} & = & -x-y, \\ \sigma.z & = & q_{043} & = & -z. \end{array}\right.$ It is known that the equation of the D_4-singularity is $X(Y^2-X^2)+Z^2=0$ and the natural Ω-action (cf. Subsection 1.4.2.1) is $\left\{\begin{array}{lcl} \sigma.X & = & X, \\ \sigma.Y & = & -Y, \\ \sigma.Z & = & -Z. \end{array}\right.$
With our computations from the quiver we obtained the equation $z^2=xy(x+y)$. Set $\left\{\begin{array}{lcl} X & = & -4^{-1/3}x, \\ Y & = & -4^{1/6}(y+\frac{x}{2}). \end{array}\right.$ Then $X(Y^2-X^2)+z^2=0$ and $\left\{\begin{array}{lcl} \sigma.X & = & X, \\ \sigma.Y & = & -Y, \\ \sigma.z & = & -z. \end{array}\right.$ The action on the singularity is thus the correct one.

Finally our pullback diagram is Ω-equivariant with the natural action on the singularity, so the restriction $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ of α is Ω-invariant and

$$\begin{aligned} (\alpha^\Omega)^{-1}(0) &= \alpha^{-1}(0) \bigcap X_{\Gamma,\Omega} \text{ because } 0 \in (\mathfrak{h}/W)^\Omega, \\ &= X_{\Gamma,0} \bigcap X_{\Gamma,\Omega}, \\ &= X_{\Gamma,0} \text{ because } X_{\Gamma,0} \subset X_{\Gamma,\Omega}. \end{aligned}$$

Therefore α^Ω is a semiuniversal deformation of type $(D_4, \mathbb{Z}/2\mathbb{Z}) = C_3$ as expected from Theorem 4.3.1.

4.3.8. Semiuniversal deformation of type $G_2 = (D_4, \mathfrak{S}_3)$

Set $\Gamma = \mathcal{D}_2$ and $\Gamma' = \mathcal{O}$. The McKay quiver of type $\widetilde{D_4}$ is given in the previous subsection and we keep the same notations.

The pullback diagram

$$\begin{array}{ccc} X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h} & \xrightarrow{\Psi} & X_\Gamma \\ \downarrow \widetilde{\alpha} & \circlearrowright & \downarrow \alpha \\ \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W \end{array}$$

of the semiuniversal deformation α of the singularity $\mathbb{C}^2/\Gamma$ of type D_4 was computed in the previous subsection. We now check the action of $\Omega = \Gamma'/\Gamma = \mathfrak{S}_3 = \langle \sigma, \rho \rangle$ with $\sigma = (34)$ and $\rho = (134)$. Ω induces a natural action on $\mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma)$ and permutes of the vertices $1, 3$ and 4.
There is also a natural action of Ω on $\mathfrak{h}$ given by

$$\alpha_1^\vee \longrightarrow \alpha_3^\vee \longrightarrow \alpha_4^\vee \quad \text{and} \quad \alpha_3^\vee \overset{\sigma}{\longleftrightarrow} \alpha_4^\vee \ .$$

From the previous case $(D_4, \mathbb{Z}/2\mathbb{Z})$, it is known that σ corresponds to the permutation $\overline{e_4} \leftrightarrow -\overline{e_4}$. We determine the action of ρ on the base $(\overline{e_i})_{1 \le i \le 4}$ (cf. Subsection 4.3.7) of $\mathfrak{h}$.

We have $\begin{cases} \alpha_1^\vee = \overline{e_1} - \overline{e_2}, \\ \alpha_2^\vee = \overline{e_2} - \overline{e_3}, \\ \alpha_3^\vee = \overline{e_3} - \overline{e_4}, \\ \alpha_4^\vee = \overline{e_3} + \overline{e_4}, \end{cases}$ and ρ is represented by a 4×4 matrix P in the base $(\overline{e_i})_{1 \le i \le 4}$ the canonical orthonormal base of $\mathfrak{h}$. Because of how ρ permutes the $\alpha_i^\vee$'s, it follows that $P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$. This implies that the action of ρ on the $\overline{e_i}$'s is given by

$$\begin{cases} \overline{e_1} \mapsto \frac{1}{2}(\overline{e_1} + \overline{e_2} + \overline{e_3} - \overline{e_4}), \\ \overline{e_2} \mapsto \frac{1}{2}(\overline{e_1} + \overline{e_2} - \overline{e_3} + \overline{e_4}), \\ \overline{e_3} \mapsto \frac{1}{2}(\overline{e_1} - \overline{e_2} + \overline{e_3} + \overline{e_4}), \\ \overline{e_4} \mapsto \frac{1}{2}(\overline{e_1} - \overline{e_2} - \overline{e_3} - \overline{e_4}). \end{cases}$$

We already know from Subsection 4.3.5.3 that

$$\begin{array}{rccc} \pi: & \mathfrak{h} & \to & \mathfrak{h}/W \\ & (\xi_1,\xi_2,\xi_3,\xi_4) & \mapsto & (\psi_2(\xi),\psi_4(\xi),\psi_6(\xi),\psi(\xi)) \end{array} \quad \text{and thus}$$

$$\begin{aligned} \pi(\rho.(\xi_1,\xi_2,\xi_3,\xi_4)) = \pi(P\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{pmatrix}) &= \pi\begin{pmatrix} \frac{1}{2}(\xi_1+\xi_2+\xi_3+\xi_4)\\ \frac{1}{2}(\xi_1+\xi_2-\xi_3-\xi_4)\\ \frac{1}{2}(\xi_1-\xi_2+\xi_3-\xi_4)\\ \frac{1}{2}(-\xi_1+\xi_2+\xi_3-\xi_4)\end{pmatrix}, \\ &= \pi(Y_1,Y_2,Y_3,Y_4), \\ &= (\psi_2(Y),\psi_4(Y),\psi_6(Y),\psi(Y)), \\ &= \begin{pmatrix} \psi_2(\xi)\\ -\frac{1}{2}\psi_4(\xi)-3\psi(\xi)\\ \psi_6(\xi)\\ \frac{1}{4}\psi_4(\xi)-\frac{1}{2}\psi(\xi)\end{pmatrix}. \end{aligned}$$

We define the action of ρ on $\mathfrak{h}/W$ by $\rho.(t_2,t_4,t_6,t) = \begin{pmatrix} t_2\\ -\frac{1}{2}t_4-3t\\ t_6\\ \frac{1}{4}t_4-\frac{1}{2}t\end{pmatrix}$ and π becomes $\mathbb{Z}/3\mathbb{Z}$-equivariant. In the preceding section it was defined $\sigma.(t_2,t_4,t_6,t) = (t_2,t_4,t_6,-t)$. Hence π is Ω-equivariant and the action of Ω on $\mathfrak{h}/W$ is linear.

Let us work above $(\mathfrak{h}/W)^\Omega$, which implies $\begin{cases} t_4 = -\frac{1}{2}t_4-3t, & \text{and } t=-t. \text{ Hence} \\ t = \frac{1}{4}t_4-\frac{1}{2}t, \end{cases}$ $t_4 = t = 0$. It implies that $\alpha^{-1}((\mathfrak{h}/W)^\Omega) = \{(x,y,z,t_2,0,t_6,0) \in \mathbb{C}^3\times\mathfrak{h}/W \mid z^2 = xy(x+y)-\frac{1}{2}t_2xy+\frac{1}{4}(t_6+\frac{1}{108}t_2^3)\}$.
Ω induces a natural action on $X_{\Gamma,\Omega} := \alpha^{-1}((\mathfrak{h}/W)^\Omega)$ via the permutations of $1,3$ and 4. It was proved in the previous section that $\begin{cases} x &= p_{34}+\frac{1}{4}(\mu_3-\mu_4)^2, \\ y &= p_{03}+\frac{1}{4}(\mu_3-\mu_0)^2, \\ z &= q_{034}-\frac{1}{2}(p_{03}(\mu_3-\mu_4)+p_{34}(\mu_3-\mu_0) \\ & \quad +\mu_3(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3)). \end{cases}$
The action of Ω is given by $\begin{cases} \sigma.\mu_3=\mu_4 \\ \sigma.\mu_4=\mu_3 \end{cases}$ and $\begin{cases} \rho.\mu_1=\mu_3 \\ \rho.\mu_3=\mu_4 \\ \rho.\mu_4=\mu_1 \end{cases}$. Hence we have

$$\begin{cases} \sigma.x &= x, \\ \sigma.y &= -x-y+\frac{1}{2}t_2, \\ \sigma.z &= -z, \end{cases} \quad \text{and} \quad \begin{cases} \rho.x &= y, \\ \rho.y &= -x-y+\frac{1}{2}t_2, \\ \rho.z &= z. \end{cases}$$

We have previously seen that the action of σ on the singularity is the correct one. Let

us check what happens with ρ.

Above the origin, $\begin{cases} \rho.x &= p_{14} &= p_{03} &= y \\ \rho.y &= p_{04} &= -p_{03}-p_{01} &= -x-y, \\ \rho.z &= q_{041} &= q_{034} &= z. \end{cases}$ It is known that the equation of the D_4-singularity is $X(Y^2-X^2)+Z^2=0$ and the natural action is

$$\begin{cases} \rho.X &= \frac{1}{2}(Y-X), \\ \rho.Y &= -\frac{1}{2}(Y+3X), \\ \rho.Z &= Z. \end{cases}$$

In our case we have $z^2 = xy(x+y)$. Set $\begin{cases} X &= -\frac{1}{\sqrt[3]{4}}x, \\ Y &= -\sqrt[6]{4}(y+\frac{1}{2}x). \end{cases}$ Then $X(Y^2-X^2)+z^2=0$ and $\begin{cases} \rho.X &= \frac{1}{2}(Y-X), \\ \rho.Y &= -\frac{1}{2}(Y+3X), \\ \rho.z &= z. \end{cases}$ The action on the singularity is the expected one (cf. Subsection 1.4.2.1).

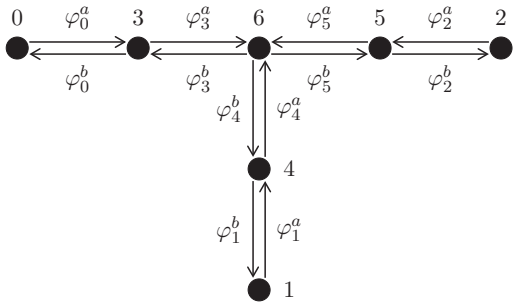
Finally the pullback diagram is Ω-equivariant with the natural action on the singularity, hence the restriction $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ of α is Ω-invariant and

$$\begin{aligned} (\alpha^\Omega)^{-1}(0) &= \alpha^{-1}(0) \bigcap X_{\Gamma,\Omega} \text{ because } 0 \in (\mathfrak{h}/W)^\Omega, \\ &= X_{\Gamma,0} \bigcap X_{\Gamma,\Omega}, \\ &= X_{\Gamma,0} \text{ because } X_{\Gamma,0} \subset X_{\Gamma,\Omega}. \end{aligned}$$

Therefore α^Ω is a semiuniversal deformation of an inhomogeneous singularity of type $(D_4, \mathfrak{S}_3) = G_2$, as predicted by Theorem 4.3.1.

4.3.9. Semiuniversal deformation of type $F_4 = (E_6, \mathbb{Z}/2\mathbb{Z})$

Set $\Gamma = \mathcal{T}$ and $\Gamma' = \mathcal{O}$. The McKay quiver of type $\widetilde{E_6}$ is



The dimension vector of this quiver is $(1,1,1,2,2,2,3)$ and corresponds to the coordinates of the highest root of the root system of type E_6. The orientation is that all a's are positive and all b's are negative.
By definition $Z = \{(\mu_0, \mu_1, \mu_2, \mu_3 \mathrm{Id}_2, \mu_4 \mathrm{Id}_2, \mu_5 \mathrm{Id}_2, \mu_6 \mathrm{Id}_3) \mid \mu_0 + \mu_1 + \mu_2 + 2(\mu_3 + \mu_4 + \mu_5) + 3\mu_6 = 0\}$.

In [LeBryPro90], it is proved that the ring of invariants of the representation space of a quiver by the product of general linear groups associated with vertices is generated by the traces of the oriented cycles of the said quiver.

We have $\mu_{\mathbb{C}}(\varphi) = (-\varphi_0^b\varphi_0^a, -\varphi_1^b\varphi_1^a, -\varphi_2^b\varphi_2^a, \varphi_0^a\varphi_0^b - \varphi_3^b\varphi_3^a, \varphi_1^a\varphi_1^b - \varphi_4^b\varphi_4^a, \varphi_2^a\varphi_2^b - \varphi_5^b\varphi_5^a, \varphi_3^a\varphi_3^b + \varphi_4^a\varphi_4^b + \varphi_5^a\varphi_5^b)$ and compute $\mu_{\mathbb{C}}^{-1}(\mu_0, \mu_1, \mu_2, \mu_3 \mathrm{Id}_2, \mu_4 \mathrm{Id}_2, \mu_5 \mathrm{Id}_2, \mu_6 \mathrm{Id}_3)//G(\Gamma)$.

Like we did for the McKay quiver of type $\widetilde{D_4}$, we compute the traces of the oriented cycles of the quiver and find those which are linearly independent. Set

$$\Phi(i_1 i_2 \dots i_k) = \mathrm{Tr}((\varphi_{i_1}^a \varphi_{i_1}^b)(\varphi_{i_2}^a \varphi_{i_2}^b) \dots (\varphi_{i_k}^a \varphi_{i_k}^b))$$

for $i_1, i_2, \dots, i_k \in \{3,4,5\}$. We find that there are only three linearly independent elements: $X := \Phi(4^2 5)$, $Y := \Phi(4^2 5^2)$ and $Z := \Phi(3^2 4^2 5^2)$. These are therefore the generators of the ring of invariants. Because this ring is the coordinate ring of the pullback of the semiuniversal deformation of a simple singularity, and that such a deformation is defined by a single equation (cf. [KasSchle72]), a unique relation has to link the generators. We then compute the relation between the generators X, Y, Z. It leads to the following equation:

$$\begin{aligned} X^2Z + Y^3 + Z^2 \;=\; & a_{X^3}X^3 + a_{X^2Y}X^2Y + a_{XY^2}XY^2 + a_{X^2}X^2 + a_{XY}XY + a_{XZ}XZ \\ & + a_{Y^2}Y^2 + a_{YZ}YZ + a_X X + a_Y Y + a_Z Z + a_0, \end{aligned}$$

with $a_0, a_X, a_Y, a_Z, a_{XZ}, a_{YZ}, a_{XY}, a_{X^2}, a_{X^3}, a_{Y^2}, a_{X^2Y}$ and a_{XY^2} homogeneous polynomials in $\mu_1, \dots, \mu_6$. They are given in the Appendix 4.4.5.

In order for the equation of the deformation to have the form predicted by Theorem 1.4.26, it requires a change of variables: replace (X, Y, Z) by $(X+\alpha, Y+\beta X+\psi, Z+\delta X+\epsilon Y+kX^2+\mu)$ with

$$\begin{cases} k = -\frac{1}{2}, \\ \beta = \frac{1}{3}a_{XY^2}, \\ \epsilon = \frac{1}{2}a_{YZ}, \\ \alpha = -a_{X^3} - \frac{1}{3}a_{X^2Y}a_{XY^2} - \frac{2}{27}a_{XY^2}^3 + \frac{1}{2}a_{XZ} + \frac{1}{6}a_{YZ}a_{XY^2}, \\ \psi = -a_{XY^2}a_{X^3} - \frac{1}{3}a_{XY^2}^2a_{X^2Y} - \frac{2}{27}a_{XY^2}^4 + \frac{1}{2}a_{XY^2}a_{XZ} \\ \qquad +\frac{1}{6}a_{XY^2}^2a_{YZ} + a_{Y^2} + \frac{1}{4}a_{YZ}^2, \\ \delta = \frac{1}{2}(a_{XZ} - 2\alpha + \frac{1}{3}a_{YZ}a_{XY^2}), \\ \mu = \frac{1}{2}(a_{XZ}\alpha + a_{YZ}\psi - \alpha^2 + a_Z). \end{cases}$$

It follows that the equation becomes

$$-\tfrac{1}{4}X^4 + Y^3 + Z^2 + A_{X^2Y}X^2Y + A_{X^2}X^2 + A_{XY}XY + A_XX + A_YY + A_0 = 0$$

with

$$A_{XY} = \tfrac{4}{81}a_{XY^2}^5 - a_{X^3}a_{YZ} + \tfrac{2}{3}a_{X^2Y}^2a_{XY^2} + 2a_{X^2Y}a_{X^3} + \tfrac{10}{27}a_{X^2Y}a_{XY^2}^3 - a_{X^2Y}a_{XZ} - \tfrac{2}{3}a_{XY^2}a_{Y^2} + \tfrac{2}{3}a_{XY^2}^2a_{X^3} - \tfrac{1}{3}a_{XY^2}^2a_{XZ} - \tfrac{5}{27}a_{XY^2}^3a_{YZ} - \tfrac{2}{3}a_{X^2Y}a_{YZ}a_{XY^2} - a_{XY}.$$

$$A_{X^2Y} = -\tfrac{1}{3}a_{XY^2}^2 + \tfrac{1}{2}a_{YZ} - a_{X^2Y}.$$

$$A_Y = -\tfrac{1}{9}a_{X^2Y}^3a_{XY^2}^2 - \tfrac{1}{6}a_{X^2Y}a_{XZ}a_{YZ}a_{XY^2} + \tfrac{2}{3}a_{X^2Y}a_{YZ}a_{XY^2}a_{X^3} - \tfrac{1}{4}a_{X^2Y}a_{XZ}^2 - a_{X^2Y}a_{X^3}^2 - \tfrac{1}{2}a_{YZ}a_Z + \tfrac{2}{27}a_{XY}a_{XY^2}^3 - \tfrac{1}{2}a_{XY}a_{XZ} - \tfrac{1}{8}a_{XZ}^2a_{YZ} + \tfrac{1}{2}a_{X^3}^2a_{YZ} + a_{XY}a_{X^3} - \tfrac{1}{72}a_{XY^2}^2a_{YZ}^3 - \tfrac{1}{6}a_{Y^2}a_{YZ}^2 + \tfrac{8}{729}a_{XY^2}^6a_{YZ} + \tfrac{4}{81}a_{XY^2}^4a_{Y^2} - \tfrac{1}{108}a_{XY^2}^4a_{YZ}^2 - \tfrac{1}{12}a_{XY^2}^2a_{XZ}^2 - \tfrac{1}{3}a_{XY^2}^2a_{X^3}^2 - \tfrac{4}{81}a_{XY^2}^5a_{X^3} - \tfrac{7}{81}a_{XY^2}^4a_{X^2Y}^2 - \tfrac{16}{729}a_{XY^2}^6a_{X^2Y} + \tfrac{2}{81}a_{XY^2}^5a_{XZ} + a_{X^2Y}a_{XZ}a_{X^3} - \tfrac{2}{3}a_{X^3}a_{X^2Y}^2a_{XY^2} + \tfrac{1}{3}a_{XZ}a_{X^2Y}^2a_{XY^2} + \tfrac{1}{6}a_{YZ}a_{XY^2}^2a_{X^2Y}^2 - \tfrac{1}{12}a_{XY^2}a_{XZ}a_{YZ}^2 + \tfrac{1}{3}a_{XY}a_{X^2Y}a_{XY^2} - \tfrac{1}{6}a_{XY}a_{YZ}a_{XY^2} + \tfrac{2}{9}a_{XY^2}^2a_{X^2Y}a_{Y^2} + \tfrac{7}{81}a_{XY^2}^4a_{X^2Y}a_{YZ} - \tfrac{1}{18}a_{XY^2}^3a_{XZ}a_{YZ} - \tfrac{1}{3}a_{XY^2}a_{XZ}a_{Y^2} - \tfrac{1}{9}a_{XY^2}^2a_{YZ}a_{Y^2} + \tfrac{5}{27}a_{XY^2}^3a_{X^3}a_{YZ} - \tfrac{10}{27}a_{XY^2}^3a_{X^3}a_{X^2Y} + \tfrac{5}{27}a_{XY^2}^3a_{X^2Y}a_{XZ} + \tfrac{1}{3}a_{XY^2}^2a_{X^3}a_{XZ} - \tfrac{1}{36}a_{XY^2}^2a_{X^2Y}a_{YZ}^2 + \tfrac{2}{3}a_{XY^2}a_{X^3}a_{Y^2} - \tfrac{1}{48}a_{YZ}^4 - \tfrac{4}{2187}a_{XY^2}^8 - \tfrac{1}{3}a_{Y^2}^2 - a_Y.$$

$$A_X = -\tfrac{2}{9}a_{X^2Y}a_{XY^2}a_{YZ}a_{Y^2} + 2a_{X^3}a_{XZ}a_{X^2Y}a_{XY^2} - \tfrac{2}{3}a_{X^3}a_{XZ}a_{YZ}a_{XY^2} + \tfrac{2}{3}a_{X^3}a_{YZ}a_{X^2Y}^2a_{XY^2}a_{X^2Y} - -\tfrac{2}{9}a_{X^2Y}a_{XY^2}^2a_{XZ}a_{YZ} - \tfrac{1}{3}a_{X^3}a_{YZ}a_{Y^2} - \tfrac{1}{18}a_{X^2Y}a_{XY^2}a_{YZ}^3 - \tfrac{1}{18}a_{YZ}^2a_{XY^2}a_{Y^2} - \tfrac{1}{3}a_{X^2Y}a_{XY^2}a_Z - \tfrac{1}{3}a_{X^2}a_{YZ}a_{XY^2} - \tfrac{8}{81}a_{XY^2}^3a_{YZ}a_{Y^2} - a_X + \tfrac{2}{3}a_{X^2}a_{X^2Y}a_{XY^2} + \tfrac{2}{9}a_{XY}a_{XY^2}^2a_{X^2Y} - 2a_{X^3}^3 - \tfrac{16}{19683}a_{XY^2}^9 - \tfrac{2}{27}a_{XY^2}^3a_Z - \tfrac{1}{12}a_{X^3}a_{YZ}^3 - a_{X^3}a_Z + \tfrac{8}{243}a_{XY^2}^5a_{Y^2} - \tfrac{1}{3}a_Ya_{XY^2} - \tfrac{1}{12}a_{XY}a_{YZ}^2 + \tfrac{4}{81}a_{XY}a_{XY^2}^4 - \tfrac{1}{3}a_{XY}a_{Y^2} + 2a_{X^2}a_{X^3} + \tfrac{4}{27}a_{X^2}a_{XY^2}^3 - a_{X^2}a_{XZ} - \tfrac{1}{36}a_{XZ}a_{XY^2}^2a_{YZ}^2 - \tfrac{4}{81}a_{XY^2}^4a_{XZ}a_{YZ} + \tfrac{1}{9}a_{X^2Y}^2a_{XY^2}^3a_{YZ} + \tfrac{1}{3}a_{X^2Y}^2a_{XY^2}^2a_{XZ} - \tfrac{1}{162}a_{X^2Y}a_{XY^2}^3a_{YZ}^2 - \tfrac{1}{3}a_{X^2Y}a_{XY^2}a_{XZ}^2 + \tfrac{4}{81}a_{X^2Y}a_{XY^2}^5a_{YZ} - 2a_{X^3}^2a_{X^2Y}a_{XY^2} + \tfrac{4}{27}a_{X^2Y}a_{XY^2}^4a_{XZ} + a_{YZ}a_{XY^2}a_{X^3}^2$$

$$-\tfrac{8}{27}a_{X^3}a_{X^2Y}a_{XY^2}^4-\tfrac{2}{3}a_{X^3}a_{X^2Y}^2a_{XY^2}^2-\tfrac{1}{18}a_{X^3}a_{XY^2}^2a_{YZ}^2+\tfrac{4}{27}a_{X^3}a_{YZ}a_{XY^2}^4+\tfrac{4}{9}a_{X^3}a_{XZ}a_{XY^2}^3$$
$$-\tfrac{2}{9}a_{XY^2}a_{Y^2}^2-\tfrac{5}{324}a_{YZ}^3a_{XY^2}^3+\tfrac{4}{729}a_{XY^2}^7a_{YZ}-\tfrac{1}{243}a_{XY^2}^5a_{YZ}^2-\tfrac{8}{729}a_{X^2Y}a_{XY^2}^7+\tfrac{4}{243}a_{XY^2}^6a_{XZ}$$
$$-\tfrac{2}{27}a_{XY^2}^3a_{XZ}^2-\tfrac{8}{243}a_{X^3}a_{XY^2}^6-\tfrac{2}{27}a_{X^2Y}^3a_{XY^2}^3-\tfrac{4}{81}a_{X^2Y}^2a_{XY^2}^5+3a_{XZ}a_{X^3}^2-a_{X^3}a_{XZ}^2-\tfrac{4}{9}a_{X^3}^2a_{XY^2}^3$$
$$-\tfrac{1}{12}a_{X^2Y}a_{XZ}a_{YZ}^2+\tfrac{4}{9}a_{XY^2}^2a_{X^3}a_{Y^2}-\tfrac{1}{3}a_{X^2Y}a_{XZ}a_{Y^2}-\tfrac{2}{9}a_{XY^2}^2a_{XZ}a_{Y^2}+\tfrac{16}{81}a_{X^2Y}a_{XY^2}^3a_{Y^2}$$
$$+\tfrac{1}{18}a_{X^2Y}^2a_{XY^2}a_{YZ}^2+\tfrac{2}{3}a_{X^2Y}a_{X^3}a_{Y^2}+\tfrac{1}{6}a_{X^2Y}a_{X^3}a_{YZ}^2-\tfrac{1}{9}a_{XY}a_{XY^2}^2a_{YZ}+\tfrac{2}{9}a_{X^2Y}^2a_{XY^2}a_{Y^2}$$
$$+\tfrac{2}{3}a_{XY}a_{XY^2}a_{X^3}-\tfrac{1}{3}a_{XY}a_{XY^2}a_{XZ}.$$

$$A_{X^2}=\tfrac{2}{27}a_{X^2Y}a_{XY^2}^4-\tfrac{1}{3}a_{X^2Y}a_{Y^2}-\tfrac{1}{12}a_{X^2Y}a_{YZ}^2+\tfrac{1}{6}a_{X^2Y}^2a_{XY^2}^2-\tfrac{1}{9}a_{XZ}a_{XY^2}^3-\tfrac{1}{27}a_{YZ}a_{XY^2}^4$$
$$-\tfrac{2}{9}a_{XY^2}^2a_{Y^2}-\tfrac{1}{72}a_{XY^2}^2a_{YZ}^2+\tfrac{1}{6}a_{YZ}a_{Y^2}+\tfrac{1}{12}a_{XZ}a_{YZ}a_{XY^2}+a_{X^3}a_{X^2Y}a_{XY^2}-\tfrac{1}{2}a_{YZ}a_{XY^2}a_{X^3}$$
$$-\tfrac{1}{2}a_{XZ}a_{X^2Y}a_{XY^2}-\tfrac{1}{6}a_{YZ}a_{XY^2}^2a_{X^2Y}-\tfrac{1}{3}a_{XY}a_{XY^2}-a_{X2}+\tfrac{3}{2}a_{X^3}^2+\tfrac{2}{243}a_{XY^2}^6+\tfrac{1}{8}a_{XZ}^2+\tfrac{1}{24}a_{YZ}^3$$
$$+\tfrac{1}{2}a_Z+\tfrac{2}{9}a_{X^3}a_{XY^2}^3-\tfrac{3}{2}a_{X^3}a_{XZ}.$$

$$A_0=\tfrac{2}{243}a_{X^2Y}^3a_{XY^2}^6-\tfrac{1}{8}a_{X^3}a_{XZ}^2a_{YZ}a_{XY^2}-\tfrac{1}{72}a_{X^3}a_{XZ}a_{XY^2}^2a_{YZ}^2+\tfrac{7}{108}a_{XZ}a_{X^2Y}^2a_{XY^2}^3a_{YZ}$$
$$+\tfrac{1}{9}a_{XZ}a_{X^2Y}^2a_{XY^2}a_{Y^2}+\tfrac{1}{36}a_{XZ}a_{X^2Y}^2a_{XY^2}a_{YZ}^2+\tfrac{1}{18}a_{YZ}a_{XY^2}^2a_{X^2Y}^2a_{Y^2}-\tfrac{2}{9}a_{X^3}a_{X^2Y}^2a_{XY^2}a_{Y^2}$$
$$-\tfrac{1}{6}a_{YZ}a_{Y^2}a_Z+\tfrac{1}{3}a_Xa_{X^2Y}a_{XY^2}-\tfrac{1}{6}a_Xa_{YZ}a_{XY^2}+\tfrac{1}{3}a_Ya_{XY^2}a_{X^3}+\tfrac{1}{9}a_Ya_{XY^2}^2a_{X^2Y}-\tfrac{1}{6}a_Ya_{XY^2}a_{XZ}$$
$$-\tfrac{1}{18}a_Ya_{XY^2}^2a_{YZ}-\tfrac{1}{3}a_{XY}a_{XY^2}a_{X^3}^2-\tfrac{4}{81}a_{XY}a_{X^3}a_{XY^2}^4+\tfrac{1}{3}a_{XY}a_{X^3}a_{Y^2}-\tfrac{1}{4}a_Z^2+\tfrac{1}{18}a_{X^2Y}^2a_{XY^2}^2a_Z$$
$$+\tfrac{2}{27}a_{X^3}a_{XY^2}^3a_Z-\tfrac{1}{72}a_{XY^2}^2a_{YZ}^2a_Z+\tfrac{1}{6}a_{X^3}^2a_{YZ}a_{Y^2}-\tfrac{1}{24}a_{XZ}^2a_{YZ}a_{Y^2}-\tfrac{4}{81}a_{X^2}a_{X^2Y}a_{XY^2}^4$$
$$-\tfrac{1}{9}a_{X^2}a_{X^2Y}^2a_{XY^2}^2-\tfrac{4}{27}a_{X^2}a_{X^3}a_{XY^2}^3-\tfrac{1}{36}a_{X^2}a_{XY^2}^2a_{YZ}^2+\tfrac{7}{162}a_{X^2Y}a_{XZ}^2a_{XY^2}^4-\tfrac{2}{243}a_{X^2Y}a_{XZ}a_{XY^2}^7$$
$$+\tfrac{2}{27}a_{X^2}a_{XZ}a_{XY^2}^3-\tfrac{1}{12}a_{X^2Y}a_{X^3}^2a_{YZ}^2-\tfrac{1}{3}a_{X^2Y}a_{X^3}^2a_{Y^2}-\tfrac{1}{432}a_{X^2Y}a_{XY^2}^2a_{YZ}^4-\tfrac{1}{72}a_{YZ}^3a_{XY^2}^2a_{Y^2}$$
$$-\tfrac{1}{864}a_{YZ}^5a_{XY^2}^2-\tfrac{1}{108}a_{XY^2}^3a_{XZ}^3+\tfrac{2}{729}a_{XY^2}^6a_Z-\tfrac{1}{96}a_{XZ}^2a_{YZ}^3-\tfrac{1}{8}a_{XZ}^2a_Z+\tfrac{1}{9}a_{X^3}a_{X^2Y}^3a_{XY^2}^3$$
$$+\tfrac{7}{648}a_{X^3}a_{YZ}^3a_{XY^2}^3-\tfrac{1}{18}a_{XZ}a_{X^2Y}^3a_{XY^2}^3+\tfrac{11}{1944}a_{X^2Y}a_{YZ}^3a_{XY^2}^4+\tfrac{13}{4374}a_{X^2Y}a_{YZ}^2a_{XY^2}^6$$
$$-\tfrac{2}{729}a_{X^2Y}a_{YZ}a_{XY^2}^8-\tfrac{1}{48}a_{X^2Y}a_{XZ}^2a_{YZ}^2-\tfrac{1}{12}a_{X^2Y}a_{XZ}^2a_{Y^2}+\tfrac{1}{108}a_{X^2Y}^4a_{XY^2}^4-\tfrac{1}{9}a_{X^3}^2a_{YZ}a_{XY^2}^4$$
$$-\tfrac{1}{3}a_{X^3}^2a_{XZ}a_{XY^2}^3+\tfrac{2}{81}a_{X^2}a_{YZ}a_{XY^2}^4+a_{X^2}a_{XZ}a_{X^3}+a_{X^3}^3a_{X^2Y}a_{XY^2}-\tfrac{1}{72}a_{XY}a_{YZ}^3a_{XY^2}$$
$$-\tfrac{1}{24}a_{XY}a_{XZ}a_{YZ}^2-\tfrac{1}{24}a_{X^2Y}a_{XZ}^3a_{XY^2}+\tfrac{7}{8}a_{X^3}^2a_{XZ}^2-\tfrac{3}{2}a_{XZ}a_{X^3}^3+\tfrac{8}{19683}a_{X^2Y}a_{XY^2}^{10}+\tfrac{2}{729}a_{X^2Y}^2a_{XY^2}^8+$$
$$\tfrac{3}{4}a_{X^3}^4-\tfrac{1}{6}a_{XY}a_{XZ}a_{Y^2}-\tfrac{1}{324}a_{XY}a_{XY^2}^3a_{YZ}^2-\tfrac{1}{12}a_{XY}a_{XY^2}a_{XZ}^2+\tfrac{7}{54}a_{X^3}a_{XY^2}^3a_{XZ}^2+\tfrac{2}{81}a_{XY}a_{XY^2}^3a_{Y^2}$$
$$-\tfrac{4}{2187}a_{XY}a_{XY^2}^7-\tfrac{4}{729}a_{X^2}a_{XY^2}^6-\tfrac{1}{2}a_{YZ}a_{XY^2}a_{X^3}^3+\tfrac{1}{24}a_{X^3}^2a_{YZ}^3+\tfrac{1}{2}a_{X^3}^2a_Z+\tfrac{2}{243}a_{XY}a_{XY^2}^5a_{YZ}-$$
$$\tfrac{4}{243}a_{XY}a_{X^2Y}a_{XY^2}^5+\tfrac{2}{81}a_{XY}a_{XY^2}^4a_{XZ}+\tfrac{1}{12}a_{XY}a_{X^3}a_{YZ}^2-\tfrac{1}{27}a_{XY}a_{X^2Y}^2a_{XY^2}^3-\tfrac{1}{54}a_{YZ}a_{XY^2}^4a_{X^2Y}^3-$$
$$\tfrac{1}{108}a_{X^2Y}^3a_{XY^2}^2a_{YZ}^2-\tfrac{1}{72}a_{Y^2}a_{YZ}^4-\tfrac{1}{18}a_{Y^2}^2a_{YZ}^2-\tfrac{1}{12}a_Ya_{YZ}^2-\tfrac{1}{3}a_Ya_{Y^2}+\tfrac{11}{324}a_{X^3}a_{X^2Y}a_{XY^2}^3a_{YZ}^2+$$
$$\tfrac{7}{81}a_{X^3}a_{XY^2}^4a_{XZ}a_{YZ}-\tfrac{2}{27}a_{X^3}a_{X^2Y}a_{XY^2}^5a_{YZ}+\tfrac{7}{12}a_{X^3}a_{X^2Y}a_{XY^2}a_{XZ}^2-\tfrac{1}{6}a_{X^3}a_{X^2Y}^2a_{XY^2}^3a_{YZ}-$$
$$\tfrac{2}{9}a_{X^3}a_{X^2Y}a_{XY^2}^4a_{XZ}-\tfrac{1}{2}a_{X^3}a_{X^2Y}^2a_{XY^2}^2a_{XZ}+\tfrac{7}{12}a_{X^3}^2a_{XZ}a_{YZ}a_{XY^2}-\tfrac{1}{2}a_{X^3}^2a_{YZ}a_{XY^2}^2a_{X^2Y}-$$
$$\tfrac{1}{18}a_{XY}a_{YZ}a_{XY^2}a_{Y^2}-\tfrac{3}{2}a_{X^3}^2a_{XZ}a_{X^2Y}a_{XY^2}+\tfrac{2}{81}a_Ya_{XY^2}^4-\tfrac{1}{2}a_Xa_{XZ}+\tfrac{2}{27}a_Xa_{XY^2}^3-\tfrac{1}{24}a_{YZ}^3a_Z+$$
$$a_Xa_{X^3}+\tfrac{4}{243}a_{XY^2}^4a_{Y^2}^2+\tfrac{1}{36}a_{XY}a_{X^2Y}a_{XY^2}a_{YZ}^2-\tfrac{1}{18}a_{XY}a_{XZ}a_{XY^2}^2a_{YZ}+\tfrac{1}{9}a_{XY}a_{X^2Y}a_{XY^2}^2a_{XZ}+$$
$$\tfrac{1}{27}a_{XY}a_{X^2Y}a_{XY^2}^3a_{YZ}+\tfrac{1}{9}a_{XY}a_{X^2Y}a_{XY^2}a_{Y^2}+\tfrac{1}{3}a_{XY}a_{X^3}a_{XY^2}a_{XZ}+\tfrac{1}{9}a_{XY}a_{X^3}a_{XY^2}^2a_{YZ}+$$
$$\tfrac{1}{3}a_{X^2}a_{YZ}a_{XY^2}a_{X^3}+\tfrac{1}{9}a_{X^2}a_{YZ}a_{XY^2}^2a_{X^2Y}-\tfrac{2}{3}a_{X^2}a_{X^3}a_{X^2Y}a_{XY^2}-\tfrac{2}{9}a_{XY}a_{X^3}a_{XY^2}^2a_{X^2Y}-$$
$$\tfrac{1}{27}a_{XZ}a_{XY^2}^3a_{YZ}a_{Y^2}-\tfrac{1}{18}a_{XY^2}a_{XZ}a_{Y^2}a_{YZ}^2+\tfrac{1}{18}a_{XY^2}a_{X^3}a_{Y^2}a_{YZ}^2-\tfrac{1}{12}a_{XZ}a_{YZ}a_{XY^2}a_Z$$
$$+\tfrac{1}{3}a_{X^3}a_{X^2Y}a_{XY^2}a_Z+\tfrac{1}{3}a_{X^2}a_{XZ}a_{X^2Y}a_{XY^2}-\tfrac{1}{6}a_{X^2}a_{XZ}a_{YZ}a_{XY^2}+\tfrac{1}{648}a_{X^2Y}a_{XZ}a_{YZ}^2a_{XY^2}^3-$$
$$\tfrac{1}{27}a_{XZ}a_{X^2Y}^2a_{XY^2}^5-\tfrac{1}{8}a_{X^3}a_{XZ}^3+\tfrac{4}{177147}a_{XY^2}^{12}+\tfrac{8}{6561}a_{X^3}a_{XY^2}^9-\tfrac{1}{64}a_{XZ}^4-\tfrac{1}{18}a_{XY^2}^2a_{XZ}^2a_{Y^2}+$$
$$\tfrac{2}{81}a_{X^3}^2a_{XY^2}^6-\tfrac{1}{486}a_{XY^2}^4a_{Y^2}a_{YZ}^2+\tfrac{14}{2187}a_{XY^2}^6a_{YZ}a_{Y^2}-\tfrac{1}{972}a_{XY^2}^5a_{XZ}a_{YZ}^2-\tfrac{1}{864}a_{YZ}^6$$
$$-\tfrac{2}{81}a_{X^3}a_{XY^2}^6a_{XZ}-\tfrac{1}{108}a_{XY^2}^4a_{XZ}^2a_{YZ}-\tfrac{5}{288}a_{XY^2}^2a_{XZ}^2a_{YZ}^2-\tfrac{1}{18}a_{X^3}a_{X^2Y}^2a_{XY^2}a_{YZ}^2$$
$$+\tfrac{5}{486}a_{X^3}a_{XY^2}^5a_{YZ}^2-\tfrac{2}{243}a_{X^3}a_{XY^2}^7a_{YZ}-\tfrac{1}{81}a_{YZ}a_{XY^2}^6a_{X^2Y}^2-\tfrac{1}{24}a_{X^2Y}a_{XZ}^2a_{YZ}a_{XY^2}^2$$
$$+\tfrac{7}{243}a_{X^2Y}a_{XZ}a_{YZ}a_{XY^2}^5-\tfrac{16}{81}a_{X^2Y}a_{X^3}a_{XY^2}^3a_{Y^2}+\tfrac{1}{108}a_{X^2Y}a_{XY^2}^2a_{YZ}^2a_{Y^2}$$
$$+\tfrac{10}{243}a_{X^2Y}a_{YZ}a_{XY^2}^4a_{Y^2}+\tfrac{2}{9}a_{X^3}^2a_{X^2Y}a_{XY^2}^4-\tfrac{10}{243}a_{X^2Y}^2a_{XY^2}^4a_{Y^2}+\tfrac{4}{243}a_{XY^2}^5a_{XZ}a_{Y^2}$$
$$+\tfrac{7}{2187}a_{XY^2}^7a_{XZ}a_{YZ}+\tfrac{4}{243}a_{X^3}a_{X^2Y}a_{XY^2}^7+\tfrac{2}{27}a_{X^3}a_{X^2Y}^2a_{XY^2}^5+\tfrac{2}{9}a_{X^3}^3a_{XY^2}^3-\tfrac{1}{144}a_{XY^2}a_{XZ}a_{YZ}^4+$$

$\frac{2}{9}a_{XY^2}a_{X^3}a_{Y^2}^2 - \frac{2}{27}a_{Y^2}^3 - \frac{1}{9}a_{XY^2}a_{XZ}a_{Y^2}^2 - \frac{28}{2187}a_{X^2Y}a_{XY^2}^6a_{Y^2} - \frac{1}{4}a_{X^2}a_{XZ}^2 - a_{X^2}a_{X^3}^2$
$+ \frac{2}{27}a_{X^2Y}a_{XY^2}^2a_{Y^2}^2 - \frac{5}{5184}a_{XY^2}^4a_{YZ}^4 - \frac{4}{6561}a_{XY^2}^9a_{XZ} - \frac{8}{243}a_{X^3}a_{XY^2}^5a_{Y^2} - \frac{4}{19683}a_{XY^2}^{10}a_{YZ} -$
$\frac{8}{6561}a_{XY^2}^8a_{Y^2} + \frac{5}{13122}a_{XY^2}^8a_{YZ}^2 + \frac{8}{81}a_{X^3}a_{XY^2}^3a_{YZ}a_{Y^2} + \frac{2}{9}a_{X^3}a_{XY^2}^2a_{XZ}a_{Y^2}$
$+ \frac{8}{81}a_{X^2Y}a_{XZ}a_{XY^2}^3a_{Y^2} + \frac{1}{12}a_{X^2Y}a_{XZ}a_{X^3}a_{YZ}^2 + \frac{1}{3}a_{X^2Y}a_{XZ}a_{X^3}a_{Y^2} + \frac{1}{18}a_{X^2Y}a_{YZ}^3a_{XY^2}a_{X^3} -$
$\frac{1}{72}a_{X^2Y}a_{XZ}a_{YZ}^3a_{XY^2} - a_0 + \frac{7}{1458}a_{XY^2}^6a_{XZ}^2 + \frac{7}{72}a_{XZ}^2a_{X^2Y}^2a_{XY^2}^2 + \frac{1}{72}a_{YZ}^3a_{XY^2}^2a_{X^2Y}^2$
$+ \frac{7}{1944}a_{YZ}^2a_{XY^2}^4a_{X^2Y}^2 - \frac{2}{9}a_{XY^2}^2a_{X^3}^2a_{Y^2} - \frac{1}{48}a_{XZ}^3a_{YZ}a_{XY^2} + \frac{2}{81}a_{X^2Y}a_{XY^2}^4a_Z - \frac{1}{27}a_{YZ}a_{XY^2}^2a_{Y^2}^2 -$
$\frac{1}{144}a_{XZ}a_{XY^2}^3a_{YZ}^3 + \frac{1}{2}a_{X^3}^2a_{X^2Y}^2a_{XY^2}^2 + \frac{5}{72}a_{X^3}^2a_{XY^2}^2a_{YZ}^2 - \frac{1}{27}a_{X^2Y}^3a_{XY^2}^2a_{Y^2} + \frac{5}{8748}a_{XY^2}^6a_{YZ}^3 +$
$\frac{2}{9}a_{X^2Y}a_{YZ}a_{XY^2}a_{X^3}a_{Y^2} - \frac{1}{18}a_{X^2Y}a_{XZ}a_{YZ}a_{XY^2}a_{Y^2} + \frac{7}{18}a_{X^3}a_{X^2Y}a_{XY^2}^2a_{XZ}a_{YZ}.$

According to Subsection 4.1.2, there is an isomorphism

$$\begin{array}{rccc} \tau: & Z & \xrightarrow{\cong} & \mathfrak{h} \\ & (\mu_0,\mu_1,\mu_2,\mu_3\mathrm{Id}_2,\mu_4\mathrm{Id}_2,\mu_5\mathrm{Id}_2,\mu_6\mathrm{Id}_3) & \mapsto & h \text{ with } \alpha_i(h) = -\mu_i, \forall 0 \leq i \leq 6 \end{array}$$

with $\mathfrak{h}$ a Cartan subalgebra of the simple Lie algebra of type E_6. Hence an element $\mu \in Z$ can be identified with $\sum_{i=1}^6 -\mu_i\Lambda_i^\vee$, with $(\Lambda_i^\vee)_{1\leq i\leq 6}$ being the fundamental coweights, i.e. the dual base of the simple roots $(\alpha_i)_{1\leq i\leq 6}$ of the root system of type E_6.
It is known that the Weyl group W is generated by the $r_{\alpha_j^\vee}$'s, with

$$\begin{array}{rcl} r_{\alpha_j^\vee}(\Lambda_i^\vee) & = & \Lambda_i^\vee - \dfrac{2(\Lambda_i^\vee,\alpha_j^\vee)}{(\alpha_j^\vee,\alpha_j^\vee)}\alpha_j^\vee, \\ & = & \begin{cases} \Lambda_i^\vee \text{ if } i \neq j, \\ \Lambda_i^\vee - \alpha_i^\vee \text{ if } i = j, \end{cases} \end{array}$$

for $1 \leq j \leq 6$. It is also known that $\alpha_i^\vee = \sum_{j=1}^6 c_{ij}\Lambda_j^\vee$ with $c = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 2 \end{pmatrix}$
the Cartan matrix of type E_6 according to our labelling of the vertices. Therefore one can write

$$\begin{cases} \alpha_1^\vee = 2\Lambda_1^\vee - \Lambda_4^\vee, \\ \alpha_2^\vee = 2\Lambda_2^\vee - \Lambda_5^\vee, \\ \alpha_3^\vee = 2\Lambda_3^\vee - \Lambda_6^\vee, \\ \alpha_4^\vee = -\Lambda_1^\vee + 2\Lambda_4^\vee - \Lambda_6^\vee, \\ \alpha_5^\vee = -\Lambda_2^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee, \\ \alpha_6^\vee = -\Lambda_3^\vee - \Lambda_4^\vee - \Lambda_5^\vee + 2\Lambda_6^\vee. \end{cases}$$

A computation gives us the action of the generators of W on $\mathfrak{h}$: $r_{\alpha_1^\vee} : \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\end{pmatrix} \mapsto \begin{pmatrix}-\mu_1\\ \mu_2\\ \mu_3\\ \mu_1+\mu_4\\ \mu_5\\ \mu_6\end{pmatrix}$, $r_{\alpha_2^\vee} : \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\end{pmatrix} \mapsto \begin{pmatrix}\mu_1\\ -\mu_2\\ \mu_3\\ \mu_4\\ \mu_2+\mu_5\\ \mu_6\end{pmatrix}$, $r_{\alpha_3^\vee} : \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\end{pmatrix} \mapsto \begin{pmatrix}\mu_1\\ \mu_2\\ -\mu_3\\ \mu_4\\ \mu_5\\ \mu_3+\mu_6\end{pmatrix}$, $r_{\alpha_4^\vee} : \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\end{pmatrix} \mapsto \begin{pmatrix}\mu_1+\mu_4\\ \mu_2\\ \mu_3\\ -\mu_4\\ \mu_5\\ \mu_4+\mu_6\end{pmatrix}$, $r_{\alpha_5^\vee} : \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\end{pmatrix} \mapsto \begin{pmatrix}\mu_1\\ \mu_2+\mu_5\\ \mu_3\\ \mu_4\\ -\mu_5\\ \mu_5+\mu_6\end{pmatrix}$ and $r_{\alpha_6^\vee} : \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\end{pmatrix} \mapsto \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3+\mu_6\\ \mu_4+\mu_6\\ \mu_5+\mu_6\\ -\mu_6\end{pmatrix}$.

Using a computer software (in our case Maple 2015) one can explicitly verify that the coefficients $A_0, A_X, A_Y, A_{X^2}, A_{XY}, A_{X^2Y}$ are invariant by the $r_{\alpha_j^\vee}$'s and are thus W-invariant. They can then be expressed using the flat coordinates defined in Subsection 4.3.5.4. As the coefficients and the flat coordinates are homogeneous, the relations have to homogeneous as well.

According to our indexation of the vertices of the Dynkin diagram, the expressions of the fundamental weights, i.e. the duals of the coweights, are given below (cf. [Bou68]):

$$\begin{cases} \Lambda_1 = \frac{1}{3}(4\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 6\alpha_6), \\ \Lambda_2 = \frac{1}{3}(2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6), \\ \Lambda_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 3\alpha_6, \\ \Lambda_4 = \frac{1}{3}(5\alpha_1 + 4\alpha_2 + 6\alpha_3 + 10\alpha_4 + 8\alpha_5 + 12\alpha_6), \\ \Lambda_5 = \frac{1}{3}(4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 8\alpha_4 + 10\alpha_5 + 12\alpha_6), \\ \Lambda_6 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6. \end{cases}$$

Therefore it follows that

$$\begin{cases} \Lambda_1 = (-\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{2}}{2}, 0, 0)^T, \\ \Lambda_2 = (0, 0, \frac{\sqrt{6}}{6}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2})^T, \\ \Lambda_3 = (0, 0, 0, -\sqrt{2}, 0, 0)^T, \\ \Lambda_4 = (-\frac{\sqrt{6}}{3}, 0, \frac{\sqrt{6}}{3}, -\sqrt{2}, 0, 0)^T, \\ \Lambda_5 = (0, 0, \frac{\sqrt{6}}{3}, -\sqrt{2}, -\frac{\sqrt{6}}{3}, 0)^T, \\ \Lambda_6 = (0, 0, \frac{\sqrt{6}}{2}, -\frac{3}{\sqrt{2}}, 0, 0)^T. \end{cases}$$

So if one writes $(x_1, y_1, \dots, y_3)^T = \sum_{i=1}^{6} \lambda_i \Lambda_i$, one finds

$$\begin{cases} x_1 = -\frac{\sqrt{6}}{6}\lambda_1 - \frac{\sqrt{6}}{3}\lambda_4, \\ y_1 = \frac{\sqrt{2}}{2}\lambda_1, \\ x_2 = \frac{\sqrt{6}}{6}\lambda_1 + \frac{\sqrt{6}}{6}\lambda_2 + \frac{\sqrt{6}}{3}\lambda_4 + \frac{\sqrt{6}}{3}\lambda_5 + \frac{\sqrt{6}}{2}\lambda_6, \\ y_2 = -\frac{\sqrt{2}}{2}\lambda_1 - \frac{\sqrt{2}}{2}\lambda_2 - \sqrt{2}\lambda_3 - \sqrt{2}\lambda_4 - \sqrt{2}\lambda_5 - \frac{3}{\sqrt{2}}\lambda_6, \\ x_3 = -\frac{\sqrt{6}}{6}\lambda_2 - \frac{\sqrt{6}}{3}\lambda_5, \\ y_3 = \frac{\sqrt{2}}{2}\lambda_2. \end{cases}$$

Using the isomorphism τ, it follows that $\mu_i = -\lambda_i$ for any $1 \leq i \leq 6$, and thus the coefficients of the equation can be expressed with the flat coordinates:

$$\begin{cases} A_0 = \frac{1}{576}(\psi_{12}(\mu) - \frac{1}{8}\psi_8(\mu)\psi_2(\mu)^2 - \frac{1}{8}\psi_6(\mu)^2 + \frac{1}{96}\psi_6(\mu)\psi_2(\mu)^3 - \psi_5(\mu)^2\psi_2(\mu)), \\ A_X = \frac{\sqrt{6}}{144}(-\psi_9(\mu) + \frac{1}{4}\psi_5(\mu)\psi_2(\mu)^2), \\ A_Y = \frac{1}{48}(-\psi_8(\mu) + \frac{1}{4}\psi_6(\mu)\psi_2(\mu) - \frac{1}{192}\psi_2(\mu)^4), \\ A_{X^2} = \frac{1}{48}(\psi_6(\mu) - \frac{1}{8}\psi_2(\mu)^3), \\ A_{XY} = \frac{1}{2\sqrt{6}}\psi_5(\mu), \\ A_{X^2Y} = -\frac{1}{4}\psi_2(\mu). \end{cases}$$

Hence

$$
\begin{aligned}
& \mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma) \\
= \; & \{(X,Y,Z,\mu_1,\dots,\mu_6) \mid -\tfrac{1}{4}X^4+Y^3+Z^2-\tfrac{1}{4}\psi_2(\mu)X^2Y+\tfrac{1}{2\sqrt{6}}\psi_5(\mu)XY \\
& +\tfrac{1}{48}(\psi_6(\mu)-\tfrac{1}{8}\psi_2(\mu)^3)X^2+\tfrac{1}{48}(-\psi_8(\mu)+\tfrac{1}{4}\psi_6(\mu)\psi_2(\mu)-\tfrac{1}{192}\psi_2(\mu)^4)Y \\
& +\tfrac{\sqrt{6}}{144}(-\psi_9(\mu)+\tfrac{1}{4}\psi_5(\mu)\psi_2(\mu)^2)X \\
& +\tfrac{1}{576}(\psi_{12}(\mu)-\tfrac{1}{8}\psi_8(\mu)\psi_2(\mu)^2-\tfrac{1}{8}\psi_6(\mu)^2+\tfrac{1}{96}\psi_6(\mu)\psi_2(\mu)^3-\psi_5(\mu)^2\psi_2(\mu))=0\}.
\end{aligned}
$$

Set

$$
\begin{aligned}
X_\Gamma \;=\; & \{(X,Y,Z,t_2,t_5,t_6,t_8,t_9,t_{12}) \in \mathbb{C}^3\times\mathfrak{h}/W \mid -\tfrac{1}{4}X^4+Y^3+Z^2-\tfrac{1}{4}t_2X^2Y+\tfrac{1}{2\sqrt{6}}t_5XY \\
& +\tfrac{1}{48}(t_6-\tfrac{1}{8}t_2^3)X^2+\tfrac{1}{48}(-t_8+\tfrac{1}{4}t_6t_2-\tfrac{1}{192}t_2^4)Y+\tfrac{\sqrt{6}}{144}(-t_9+\tfrac{1}{4}t_5t_2^2)X \\
& +\tfrac{1}{576}(t_{12}-\tfrac{1}{8}t_8t_2^2-\tfrac{1}{8}t_6^2+\tfrac{1}{96}t_6t_2^3-t_5^2t_2)=0\}.
\end{aligned}
$$

We have $\pi:\ \begin{array}{ccc}\mathfrak{h} & \to & \mathfrak{h}/W \\ (\mu_1,\dots,\mu_6) & \mapsto & (\psi_2(\mu),\psi_5(\mu),\psi_6(\mu),\psi_8(\mu),\psi_9(\mu),\psi_{12}(\mu))\end{array}$ (as $Z\cong\mathfrak{h}$ and $(\mu_1,\dots,\mu_6)$ are coordinates on Z, we use the same coordinates on $\mathfrak{h}$) and

$\alpha:\ \begin{array}{ccc} X_\Gamma & \to & \mathfrak{h}/W \\ (X,Y,Z,t_2,t_5,t_6,t_8,t_9,t_{12}) & \mapsto & (t_2,t_5,t_6,t_8,t_9,t_{12})\end{array}$.

Then

$$
\begin{aligned}
& X_\Gamma\times_{\mathfrak{h}/W}\mathfrak{h} \\
= \; & \{((X,Y,Z,t_2,t_5,t_6,t_8,t_9,t_{12}),(\mu_1,\dots,\mu_6))\in X\times\mathfrak{h} \mid \alpha(X,Y,Z,t_i)=\pi(\mu_1,\dots,\mu_6)\}, \\
= \; & \{(X,Y,Z,t_i,\mu_j) \mid t_2=\psi_2(\mu),t_5=\psi_5(\mu),t_6=\psi_6(\mu),t_8=\psi_8(\mu),t_9=\psi_9(\mu),t_{12}=\psi_{12}(\mu), \\
& \text{and } -\tfrac{1}{4}X^4+Y^3+Z^2-\tfrac{1}{4}t_2X^2Y+\tfrac{1}{2\sqrt{6}}t_5XY+\tfrac{1}{48}(t_6-\tfrac{1}{8}t_2^3)X^2+\tfrac{1}{48}(-t_8+\tfrac{1}{4}t_6t_2-\tfrac{1}{192}t_2^4)Y \\
& +\tfrac{\sqrt{6}}{144}(-t_9+\tfrac{1}{4}t_5t_2^2)X+\tfrac{1}{576}(t_{12}-\tfrac{1}{8}t_8t_2^2-\tfrac{1}{8}t_6^2+\tfrac{1}{96}t_6t_2^3-t_5^2t_2)=0\}, \\
= \; & \mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma).
\end{aligned}
$$

Finally we obtain

$$
\begin{array}{ccc}
X_\Gamma\times_{\mathfrak{h}/W}\mathfrak{h} & \xrightarrow{\Psi} & X_\Gamma \\
\big\downarrow{\scriptstyle\widetilde{\alpha}} & \circlearrowright & \big\downarrow{\scriptstyle\alpha} \\
\mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W
\end{array}
$$

with α being a semiuniversal deformation of $X_{\Gamma,0} := \alpha^{-1}(0) = \{(X,Y,Z) \in \mathbb{C}^3 \mid -\frac{1}{4}X^4 + Y^3 + Z^2 = 0\}$, which is an E_6 singularity.

Finally our pullback diagram is Ω-equivariant with the natural action on the singularity, so the restriction $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ of α is Ω-invariant and

$$\begin{aligned}(\alpha^\Omega)^{-1}(0) &= \alpha^{-1}(0) \bigcap X_{\Gamma,\Omega} \text{ because } 0 \in (\mathfrak{h}/W)^\Omega,\\ &= X_{\Gamma,0} \bigcap X_{\Gamma,\Omega},\\ &= X_{\Gamma,0} \text{ because } X_{\Gamma,0} \subset X_{\Gamma,\Omega}.\end{aligned}$$

Therefore α^Ω is a semiuniversal deformation of type $(E_6, \mathbb{Z}/2\mathbb{Z}) = F_4$ as expected from Theorem 4.3.1.

4.4. Quotient of the inhomogeneous singularity $(\mathbb{C}^2/\Gamma, \Omega)$ by Ω

4.4.1. Objectives

It was shown in the previous section that the restriction $\alpha^\Omega := \alpha|_{\alpha^{-1}((\mathfrak{h}/W)^\Omega)} : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ over the fixed points $(\mathfrak{h}/W)^\Omega$ of a semiuniversal deformation of the singularity $\mathbb{C}^2/\Gamma$ is a semiuniversal deformation of the inhomogeneous singularity of type $\Delta(\Gamma,\Gamma')$ and is thus Ω-invariant. Hence Ω acts on each fiber of α^Ω and the fibers can be quotiented. It is known that $(\alpha^\Omega)^{-1}(\overline{0}) = X_0 = \mathbb{C}^2/\Gamma$. Hence the fiber above the origin of the quotient map is also a Kleinian singularity. Indeed, $(\alpha^\Omega)^{-1}(\overline{0})/\Omega = X_0/\Omega \cong (\mathbb{C}^2/\Gamma)/(\Gamma'/\Gamma) \cong \mathbb{C}^2/\Gamma'$. As Γ' is a finite subgroup of SU_2, $\mathbb{C}^2/\Gamma'$ is a Kleinian singularity. Therefore the family given by the quotient map $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ is a deformation of the simple singularity $\mathbb{C}^2/\Gamma'$.

With Proposition 1.4.23, P. Slodowy described the singularities in the fibers around the special fiber $\mathbb{C}^2/\Gamma$ in terms of subdiagrams of the Dynkin diagram of type $\Delta(\Gamma)$ when $\mathbb{C}^2/\Gamma$ is of homogeneous type. Furthermore, he gave a relation mentioned in Remark 1.4.50 between $(\mathfrak{h}/W)^\Omega$ and a quotient $\mathfrak{h}_0/W_0$. This raises two questions:

1. how is the map $\overline{\alpha^\Omega}$ related with the semiuniversal deformation of type $\mathbb{C}^2/\Gamma'$, and how to describe the base space $(\mathfrak{h}/W)^\Omega$ using a Cartan subalgebra and a Weyl group of type $\Delta(\Gamma')$?
2. can we describe the singularities in the neighbouring fibers of $\mathbb{C}^2/\Gamma'$ in terms of sub-root systems of the root system of type $\Delta(\Gamma')$, like P. Slodowy did for the homogeneous case?

The content of Section 4.4 is only a first step in the search for answers to the previous questions. In order to investigate the nature and the regularity of the fibers of the quotient of a semiuniversal deformation of a simple singularity of inhomogeneous type, we use an explicit realization of the map $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ given in the previous section. The quotient map $\overline{\alpha^\Omega}$ is computed for the cases $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$, $(D_4, \mathbb{Z}/2\mathbb{Z})$, $(D_4, \mathfrak{S}_3)$ and $(E_6, \mathbb{Z}/2\mathbb{Z})$. In particular, when the original singularity $\mathbb{C}^2/\Gamma$ is of type A_3 or D_4, one notices that every fiber of the map $\overline{\alpha^\Omega}$ is singular.

4.4.2. Case $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$

We have $\Gamma = \mathcal{C}_{2r}$ and $\Gamma' = \mathcal{D}_r$. In Subsection 4.3.6, a semiuniversal deformation of type $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$ was computed. It is the projection $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ with $X_{\Gamma,\Omega} = \{(x, y, z, t_2, 0, t_4, 0, \ldots, t_{2r}) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^{2r} + \sum_{i=1}^{r} f_{2i}(t_2, \ldots, t_{2r}) z^{2(r-i)} = xy\}$ with the same notations as in Subsection 4.3.6. The morphism α^Ω is Ω-invariant and thus Ω acts on every fiber of α^Ω. The purpose of this subsection is to compute the quotient morphism $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$, and study its nature.

4.4.2.1. The morphism $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$

The action of $\Omega = \Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$ on $X_{\Gamma,\Omega}$ is $\sigma.(x, y, z, t_2, 0, t_4, \ldots, 0, t_{2r}) = ((-1)^r y, (-1)^r x, -z, t_2, 0, t_4, \ldots, 0, t_{2r})$. Let us compute the Ω-invariant polynomial ring of the coordinate ring of $X_{\Gamma,\Omega}$.

Set $p \in \mathbb{C}[X_{\Gamma,\Omega}]$. Then $p = p(x, y, z, t_2, t_4, .., t_{2r})$ and $\sigma.p = p((-1)^r y, (-1)^r x, -z, t_2, t_4, \ldots, t_{2r})$. As the t_{2k}, $1 \leq k \leq r$, are invariant by the action of Ω on p, they can be treated as constants and will be omitted in order to simplify the notations.
We can write $p = p_0(x, y) + p_1(x, y)z + p_2(x, y)z^2 + \ldots + p_n(x, y)z^n = \sum_{k=0}^{n} p_k(x, y)z^k \in \mathbb{C}[x, y][z]$ with the p_i's being polynomials in x and y. Let us assume that p is Ω-invariant. Then

$$\begin{aligned} \sigma.p &= p_0((-1)^r y, (-1)^r x) - p_1((-1)^r y, (-1)^r x)z + \ldots + (-1)^n p_n((-1)^r y, (-1)^r x)z^n, \\ &= p, \\ &= p_0(x, y) + p_1(x, y)z + p_2(x, y)z^2 + \ldots + p_n(x, y)z^n. \end{aligned}$$

By identification, one obtains $\begin{cases} p_0((-1)^r y,(-1)^r x) = p_0(x,y), \\ p_1((-1)^r y,(-1)^r x) = -p_1(x,y), \\ p_2((-1)^r y,(-1)^r x) = p_2(x,y), \\ \vdots \\ p_n((-1)^r y,(-1)^r x) = (-1)^n p_n(x,y). \end{cases}$

Assume r to be even. Then the system of equations becomes

$$\begin{cases} p_0(y,x) = p_0(x,y), \\ p_1(y,x) = -p_1(x,y), \\ p_2(y,x) = p_2(x,y), \\ \vdots \\ p_n(y,x) = (-1)^n p_n(x,y). \end{cases}$$

It follows that p_i is symmetric in (x,y) if i is even and skew-symmetric otherwise. The polynomial p can be rewritten in the following form

$$p = \underbrace{p_0(x,y)}_{\text{symmetric}} + z^2 \underbrace{p_2(x,y)}_{\text{symmetric}} + \ldots + z \underbrace{p_1(x,y)}_{\text{skew-symmetric}} + z^3 \underbrace{p_3(x,y)}_{\text{skew-symmetric}} + \ldots$$

However it is known that a skew-symmetric polynomial in two variables can be written $(x-y)\mathrm{sym}(x,y)$ where $\mathrm{sym}(x,y)$ is some symmetric polynomial. Hence p can be expressed as

$$p = \mathrm{sym}_0(x,y) + z^2\mathrm{sym}_2(x,y) + \ldots + z(x-y)\mathrm{sym}_1(x,y) + z^2 z(x-y)\mathrm{sym}_3(x,y) + \ldots$$

Therefore any Ω-invariant polynomial is generated by $z^2, z(x-y)$ and symmetric polynomials in (x,y). It is also known that the latters are generated by $(x+y)$ and xy. The converse is trivial and we have proved

$$\mathbb{C}[X_{\Gamma,\Omega}]^{\Omega} = \\ \mathbb{C}[z^2, z(x-y), x+y, xy, t_2, \ldots, t_{2r}]/(z^{2r} + f_2(t_2,\ldots,t_{2r})z^{2r-2} + \ldots f_{2r}(t_2,\ldots,t_{2r}) = xy).$$

Set $X = x+y, Y = xy, Z = z^2$ and $W = iz(x-y)$. The relation between these variables is $X^2Z + W^2 = 4ZY$. This implies that the coordinate ring of our quotient space is given by

$$\mathbb{C}[X_{\Gamma,\Omega}]^{\Omega} = \mathbb{C}[X, Z, W, t_2, \ldots, t_{2r}]/(Z(X^2 - 4Z^r) + W^2 - 4f_2(t_2, \ldots, t_{2r})Z^r - 4f_4(t_2, \ldots, t_{2r})Z^{r-1} - \ldots - 4f_{2r}(t_2, \ldots, t_{2r})Z = 0).$$

and thus

$$\begin{aligned} X_{\Gamma,\Omega}/\Omega \quad = \quad & \{(X, Z, W, t_2, \ldots, t_{2r}) \in \mathbb{C}^3 \times (\mathfrak{h}/W)^{\Omega} \mid Z(X^2 - 4Z^r) + W^2 \\ & -4f_2(t_2, \ldots, t_{2r})Z^r - 4f_4(t_2, \ldots, t_{2r})Z^{r-1} - \ldots - 4f_{2r}(t_2, \ldots, t_{2r})Z = 0\}. \end{aligned}$$

The quotient of the map α^{Ω} is then $\overline{\alpha^{\Omega}}$: $\begin{array}{ccc} X_{\Gamma,\Omega}/\Omega & \to & (\mathfrak{h}/W)^{\Omega} \\ (X, Z, W, t_2, \ldots, t_{2r}) & \mapsto & (0, t_2, 0, \ldots 0, t_{2r}) \end{array}$,

which is a deformation of $(\overline{\alpha^{\Omega}})^{-1}(0) = \{(X, Z, W) \in \mathbb{C}^3 \mid Z(X^2 - 4Z^r) + W^2 = 0\}$. It is a D_{r+2} singularity. However, the deformation $\overline{\alpha^{\Omega}}$ is not semiuniversal. Indeed, according to Theorem 1.4.15, the base space of a semiuniversal deformation of type D_{r+2} is $\mathfrak{h}_{D_{r+2}}/W_{D_{r+2}}$, which is of dimension $r+2$. But our base space is $(\mathfrak{h}/W)^{\Omega}$ and of dimension $r < r + 2$. Hence the deformation cannot be semiuniversal.

We now assume that r is odd. Then for $p = \sum_{k=0}^{n} p_k(x, y)z^k \in \mathbb{C}[x, y][z]$ Ω-invariant, we have

$$\begin{cases} p_0(-y, -x) = p_0(x, y), \\ p_1(-y, -x) = -p_1(x, y), \\ \vdots \\ p_n(-y, -x) = (-1)^n p_n(x, y). \end{cases}$$

$p_0(-y, -x) = p_0(x, y)$, so $p_0(y, x) = p_0(-x, -y)$. We write $p_0(x, y) = p_0(x, y)_{even} + p_0(x, y)_{odd}$ the decomposition of p_0 in a sum of monomial factors of even degrees and a sum of monomial factors of odd degrees. Then $p_0(y, x) = p_0(y, x)_{even} + p_0(y, x)_{odd} = p_0(-x, -y) = p_0(x, y)_{even} - p_0(x, y)_{odd}$. Therefore the even part is symmetric and the odd part is skew-symmetric. So $p_0(x, y) =$ symmetric of even degree + skew-symmetric of odd degree. All p_i, with i even, can be expressed in a similar fashion.

We have $p_1(x, y) = p_1(x, y)_{even} + p_1(x, y)_{odd}$ the decomposition of p_1. As $p_1(-y, -x) = -p_1(x, y)$, we get that $p_1(x, y) =$ skew-symmetric of even degree + symmetric of odd degree. All p_i, with i odd, have a similar decomposition.

Then if p is Ω-invariant, we have $p(x, y) = \text{sym}(x, y)_{even} + \text{skew-sym}(x, y)_{odd}$ $+ z(\text{skew-sym}(x, y)_{even} + \text{sym}(x, y)_{odd}) + z^2(\text{sym}(x, y)_{even} + \text{skew-sym}(x, y)_{odd}) + \ldots$. But

we know that skew-sym$(x,y)_{odd}$ = $(x-y)$sym$(x,y)_{even}$ and skew-sym$(x,y)_{even}$ = $(x-y)$sym$(x,y)_{odd}$. Then $p(x,y)$ = sym$(x,y)_{even}$+$(x-y)$sym$(x,y)_{even}$+$z((x-y)$sym$(x,y)_{odd}$+ sym$(x,y)_{odd}$) + z^2(sym$(x,y)_{even}$ + $(x-y)$sym$(x,y)_{even}$) + ...
We deduce that p is generated $x-y, xy, z^2, z(x+y)$ and the t_{2k}, $1 \leq k \leq r$. Hence

$$\begin{aligned}\mathbb{C}[X_{\Gamma,\Omega}]^{\Omega} &= \mathbb{C}[x-y, xy, z^2, z(x+y), t_2, \ldots, t_{2r}]/(z^{2r} + f_2(t_2, \ldots, t_{2r})z^{2r-2} \\ &\quad + \ldots + f_{2r}(t_2, \ldots, t_{2r}) = xy).\end{aligned}$$

Set $Z = z^2, X = x-y, Y = xy$ and $W = iz(x+y)$. We have the relation $W^2 + ZX^2 = -4YZ$. As we did when r was even, we show that

$$\begin{gathered}\mathbb{C}[X_{\Gamma,\Omega}]^{\Omega} = \mathbb{C}[X, Z, W, t_i]/(X^2Z + W^2 = \\ -4Z(Z^r + f_2(t_2, \ldots, t_{2r})Z^{r-1} + \ldots + f_{2r}(t_2, \ldots, t_{2r}))).\end{gathered}$$

We deduce that

$$\begin{aligned}X_{\Gamma,\Omega}/\Omega &= \{(X, Z, W, t_2, \ldots, t_{2r}) \in \mathbb{C}^3 \times (\mathfrak{h}/W)^{\Omega} \mid Z(X^2 + 4Z^r) + W^2 \\ &\quad + 4f_2(t_2, \ldots, t_{2r})Z^r + 4f_4(t_2, \ldots, t_{2r})Z^{r-1} \ldots + 4f_{2r}(t_2, \ldots, t_{2r})Z = 0\}.\end{aligned}$$

The quotient $\overline{\alpha^{\Omega}}$ is then $\overline{\alpha^{\Omega}}: \begin{array}{ccc} X_{\Gamma,\Omega}/\Omega & \to & (\mathfrak{h}/W)^{\Omega} \\ (X, Z, W, t_2, \ldots, t_{2r}) & \mapsto & (0, t_2, 0, \ldots 0, t_{2r}) \end{array}$, which is a deformation of $\overline{\alpha^{\Omega}}^{-1}(0) = \{(X, Z, W) \in \mathbb{C}^3 \mid Z(X^2 + 4Z^r) + W^2 = 0\}$. It is a D_{r+2} singularity. With the same argument as when r is even, the deformation cannot be semiuniversal.

4.4.2.2. Example: regularity of the fiber of $\overline{\alpha^{\Omega}}$ when $r = 2$

We look at the regularity of $X_{\Gamma,\Omega}$ and $X_{\Gamma,\Omega}/\Omega$ on an example. Set $r = 2$. Then

$$\begin{aligned}X_{\Gamma,\Omega} &= \{(x, y, z, t_2, t_4) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^4 + f_2(t_2, t_4)z^2 + f_4(t_2, t_4) = xy\}, \\ X_{\Gamma,\Omega}/\Omega &= \{(X, Z, W, t_2, t_4) \in \mathbb{C}^3 \times (\mathfrak{h}/W)^{\Omega} \mid Z(X^2 - 4Z^2) + W^2 - 4f_2(t_2, t_4)Z^2 \\ &\quad -4f_4(t_2, t_4)Z = 0\}.\end{aligned}$$

For a fiber in $X_{\Gamma,\Omega}$ other than $(\alpha^\Omega)^{-1}(0)$ to be singular, it requires $16f_4(t_2,t_4)(f_2(t_2,t_4)^2 - 4f_4(t_2,t_4))^2 = 0 \Leftrightarrow f_4(t_2,t_4) = 0$ or $f_2(t_2,t_4)^2 = 4f_4(t_2,t_4)$.

- If $f_4(t_2,t_4) = t_4 + \frac{1}{8}t_2^2 = 0$ then $(t_2,t_4) = (t_2, -\frac{1}{8}t_2^2)$ and $(\alpha^\Omega)^{-1}(t_2, -\frac{1}{8}t_2^2)$ is singular at the origin.
- If $f_2(t_2,t_4)^2 = t_2^2 = 4f_4(t_2,t_4) = 4(t_4 + \frac{1}{8}t_2^2)$, then $(t_2,t_4) = (t_2, \frac{1}{8}t_2^2)$ and $(\alpha^\Omega)^{-1}(t_2, \frac{1}{8}t_2^2)$ is singular.

In $X_{\Gamma,\Omega}/\Omega$, for any fiber $(\overline{\alpha^\Omega})^{-1}(t_2,t_4)$, the point $(X,W,Z) = (\pm 2\sqrt{f_4(t_2,t_4)}, 0, 0)$ is a singular point. Hence all fibers of $\overline{\alpha^\Omega}$ are singular. We have proved the next proposition.

Proposition 4.4.1. *Every fiber of the deformation $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ of type D_4 is singular.*

4.4.3. Case $(D_4, \mathbb{Z}/2\mathbb{Z})$

We have $\Gamma = \mathcal{D}_2$ and $\Gamma' = \mathcal{D}_4$ and $\Omega = \Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$. In Subsection 4.3.7, a semiuniversal deformation of type $(D_4, \mathbb{Z}/2\mathbb{Z})$ was computed. It is the projection $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ with $X_{\Gamma,\Omega} = \{(x,y,z,t_2,t_4,t_6,0) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^2 = xy(x+y) - \frac{1}{2}t_2xy - \frac{1}{4}t_4x + \frac{1}{4}(t_6 + \frac{1}{6}t_2t_4 + \frac{1}{108}t_2^3)\}$ with the same notations as in Subsection 4.3.7. The morphism α^Ω is Ω-invariant and thus Ω acts on every fiber of α^Ω. The purpose of this subsection is to compute the quotient morphism $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$, and study its nature.

4.4.3.1. The morphism $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$

We obtained in Subsection 4.3.7 the following relations:

$$\begin{cases} \sigma.x &= x, \\ \sigma.y &= -x - y + \frac{1}{2}t_2, \\ \sigma.z &= -z, \\ \sigma.t_i &= t_i,\ i = 2,4,6. \end{cases}$$

The relations imply that $\mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z \oplus \mathbb{C}t_2$ is a $\mathbb{C}$-vector space on which Ω acts linearly. The eigenvalues of σ are -1 and 1. Set $y' = \frac{1}{2}x + y - \frac{1}{4}t_2$ and then $\sigma.y' = -y'$. With this change of variable, one has $X_{\Gamma,\Omega} = \{(x,y,z,t_2,t_4,t_6,0) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^2 = xy(x+y) \frac{1}{2}t_2xy - \frac{1}{4}t_4x + \frac{1}{4}(t_6 + \frac{1}{6}t_2t_4 + \frac{1}{108}t_2^3)\}$,
$= \{(x,y',z,t_2,t_4,t_6,0) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^2 = \frac{1}{4}t_6 + \frac{1}{24}t_2t_4 + \frac{1}{432}t_2^3 + (-\frac{1}{16}t_2^2 - \frac{1}{4}t_4)x - \frac{1}{4}x^3 + \frac{1}{4}t_2x^2 + x(y')^2\}$.

and

$$\left\{\begin{array}{lcl}\sigma.x & = & x,\\ \sigma.y' & = & -y',\\ \sigma.z & = & -z.\\ \sigma.t_i & = & t_i,\ i=2,4,6.\end{array}\right.$$

Let $p \in \mathbb{C}[X_{\Gamma,\Omega}]^{\Omega}$. As x and the t_i's are not altered by σ, we omit them from the notation. We have $\sigma.p(y',z) = p(-y',-z) = p(y',z)$. It is then clear that p is generated by y'^2, z^2 and $y'z$. So $\mathbb{C}[x,y,z,t_2,t_4,t_6]^{\Omega} = \mathbb{C}[x,y',z,t_2,t_4,t_6]^{\Omega} = \mathbb{C}[x,y'^2,z^2,y'z,t_2,t_4,t_6]$.
Set $X = x, Y = y'^2, Z = z^2$ and $W = y'z$. The equations between the invariants are

$$\left\{\begin{array}{lcl} Z & = & \frac{1}{4}t_6 + \frac{1}{24}t_2t_4 + \frac{1}{432}t_2^3 + (-\frac{1}{16}t_2^2 - \frac{1}{4}t_4)X - \frac{1}{4}X^3 + \frac{1}{4}t_2X^2 + XY,\\ YZ & = & W^2,\end{array}\right.$$

hence $W^2 = (\frac{1}{4}t_6 + \frac{1}{24}t_2t_4 + \frac{1}{432}t_2^3)Y + (-\frac{1}{16}t_2^2 - \frac{1}{4}t_4)XY - \frac{1}{4}X^3Y + \frac{1}{4}t_2X^2Y + XY^2$. We finally obtain

$$\begin{array}{rcl}\mathbb{C}[X_{\Gamma,\Omega}]^{\Omega} & = & \mathbb{C}[X,Y,W,t_2,t_4,t_6]/((\frac{1}{4}t_6 + \frac{1}{24}t_2t_4 + \frac{1}{432}t_2^3)Y + (-\frac{1}{16}t_2^2 - \frac{1}{4}t_4)XY\\ & & -\frac{1}{4}X^3Y + \frac{1}{4}t_2X^2Y + XY^2 - W^2),\end{array}$$

with X of degree 4, Y of degree 4 and W of degree 6. The t_i are of degree 2,4 and 6, respectively.

With the following substitution: $Y \to Y + \frac{1}{8}X^2 - \frac{1}{8}Xt_2 + \frac{1}{32}t_2^2 + \frac{1}{8}t_4$, the equation becomes

$$-\tfrac{1}{64}X^5 + XY^2 - W^2 + A_{X^4}X^4 + A_{X^3}X^3 + A_{X^2}X^2 + A_XX + A_YY + A_0 = 0.$$

with

$$\left\{\begin{array}{l} A_{X^4} = \frac{t_2}{32},\\ A_{X^3} = -\frac{3}{128}t_2^2 - \frac{1}{32}t_4,\\ A_{X^2} = \frac{7}{192}t_2t_4 + \frac{1}{32}t_6 + \frac{7}{864}t_2^3,\\ A_X = -\frac{1}{32}t_6t_2 - \frac{5}{384}t_2^2t_4 - \frac{35}{27648}t_2^4 - \frac{1}{64}t_4^2,\\ A_Y = \frac{1}{4}t_6 + \frac{1}{24}t_2t_4 + \frac{1}{432}t_2^3,\\ A_0 = \frac{1}{128}t_6t_2^2 + \frac{1}{32}t_6t_4 + \frac{11}{6912}t_2^3t_4 + \frac{1}{192}t_2t_4^2 + \frac{1}{13824}t_2^5.\end{array}\right.$$

Hence a fiber of $\overline{\alpha^{\Omega}}$ in $X_{\Gamma,\Omega}/\Omega$ is defined by the preceding equation. We notice that it is a subfamily of the semiuniversal deformation of a D_6-singularity. It follows that the projection

$$\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$$

is a deformation of type D_6. However it is not semiuniversal. Using Theorem 1.4.15, one sees that the dimension of the base space of a semiuniversal deformation of a simple singularity of type D_6 is of dimension 6. But we have $\dim(\mathfrak{h}/W)^\Omega = 3$. So $\overline{\alpha^\Omega}$ is not semiuniversal.

4.4.3.2. The discriminant of $\overline{\alpha^\Omega}$

Let us determine the discriminant of $\overline{\alpha^\Omega}$, i.e. the elements $(t_2, t_4, t_6) \in (\mathfrak{h}/W)^\Omega$ such that the fiber $(\overline{\alpha^\Omega})^{-1}(t_2, t_4, t_6)$ is singular.
The fiber $(\overline{\alpha^\Omega})^{-1}(t_2, t_4, t_6)$ is defined by the equation $f(X, Y, Z) = -\frac{1}{64}X^5 + XY^2 - W^2 + A_{X^4}X^4 + A_{X^3}X^3 + A_{X^2}X^2 + A_X X + A_Y Y + A_0 = 0$, with A_i's depending on the t_k's and whose expressions we can find above. The fiber is singular if and only if the following system has a solution:

$$\begin{cases} f(X,Y,Z) &= 0, \\ \frac{\partial f}{\partial X}(X,Y,Z) &= 0, \\ \frac{\partial f}{\partial Y}(X,Y,Z) &= 0, \\ \frac{\partial f}{\partial Z}(X,Y,Z) &= 0. \end{cases}$$

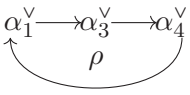
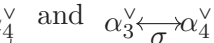
This system leads to the following equations:

$$\begin{cases} W = 0, \\ Y^2 = \frac{5}{64}X^4 - 4A_{X^4}X^3 - 3A_{X^3}X^2 - 2A_{X^2}X - A_X, \\ -\frac{3}{32}X^5 + 5A_{X^4}X^4 + 4A_{X^3}X^3 + 3A_{X^2}X^2 + 2A_X X + A_0 = 0. \end{cases}$$

The polynomial equation of degree 5 in X always has a solution X_s because the base field is $\mathbb{C}$ which is algebraically closed. It follows that the second equation has two solutions: $\pm Y_s$, depending on X_s. Hence for any $(t_2, t_4, t_6) \in (\mathfrak{h}/W)^\Omega$, there exists $(X_s, \pm Y_s, 0) \in (\overline{\alpha^\Omega})^{-1}(t_2, t_4, t_6)$ that is a singularity. We obtained the following result:

Proposition 4.4.2. *Every fiber of the deformation $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ of type D_6 is singular.*

4.4.4. Case $(D_4, \mathfrak{S}_3)$

We have seen that the action of $\Omega = \mathfrak{S}_3 = \langle \sigma, \rho \rangle$ on $\mu_{\mathbb{C}}^{-1}(Z)//G(\Gamma)$ comes from the permutations of the vertices $1, 3$ and 4.
We also have a natural action on $\mathfrak{h}$ given by  and  . We computed in Subsection 4.3.8 that $X_{\Gamma,\Omega} = \{(x, y, z, t_2, 0, t_6, 0) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid z^2 = xy(x+y) - \frac{1}{2}t_2xy + \frac{1}{4}(t_6 + \frac{1}{108}t_2^3)\}$.
Ω induces a natural action on $X_{\Gamma,\Omega}$ via the permutations $\sigma = (34)$ and $\rho = (134)$.

4.4.4.1. The morphism $\overline{\alpha^{\Omega}} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^{\Omega}$

In Subsection 4.3.8 we obtained

$$\begin{cases} \sigma.x &= x, \\ \sigma.y &= -x - y + \frac{1}{2}t_2, \\ \sigma.z &= -z, \\ \sigma.t_i &= t_i,\ i = 2, 6. \end{cases} \quad \text{and} \quad \begin{cases} \rho.x &= y, \\ \rho.y &= -x - y + \frac{1}{2}t_2, \\ \rho.z &= z, \\ \rho.t_i &= t_i,\ i = 2, 6. \end{cases}$$

Hence $\mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}t_2$ is a $\mathbb{C}$-vector space one which ρ acts linearly. The eigenvalues of ρ are $1, \omega = \exp(\frac{2i\pi}{3})$ and ω^2. Set

$$\begin{cases} X = (-3 - i\sqrt{3})x + (-3 + i\sqrt{3})y + t_2, \\ Y = (-3 + i\sqrt{3})x + (-3 - i\sqrt{3})y + t_2. \end{cases}$$

Then

$$\begin{cases} \rho.X = \omega X, \\ \rho.Y = \omega^2 Y, \end{cases}$$

and ρ fixes z, t_2 and t_6.

Set $p \in \mathbb{C}[(\alpha^{\Omega})^{-1}(t_2, t_6)]^{\mathbb{Z}/3\mathbb{Z}} = (\mathbb{C}[x, y, z, t_2, t_6]/(-z^2 + xy(x+y) - \frac{1}{2}t_2xy + \frac{1}{4}(t_6 + \frac{1}{108}t_2^3))^{\mathbb{Z}/3\mathbb{Z}} = (\mathbb{C}[X, Y, z, t_2, t_6]/(-z^2 - \frac{1}{216}Y^3 - \frac{1}{216}X^3 - \frac{1}{432}t_2^3 + \frac{1}{72}XYt_2 + \frac{1}{4}t_6))^{\mathbb{Z}/3\mathbb{Z}}$. It follows that $\rho.p(X, Y) = p(\omega X, \omega^2 Y) = p(X, Y)$. It is clear that p is generated by X^3, Y^3 and XY.

$$\begin{aligned}\text{Hence } \mathbb{C}[(\alpha^{\Omega})^{-1}(t_2,t_6)]^{\mathbb{Z}/3\mathbb{Z}} &= \mathbb{C}[X^3,Y^3,XY,z,t_2,t_6]/(-z^2-\tfrac{1}{216}Y^3-\tfrac{1}{216}X^3-\tfrac{1}{432}t_2^3\\ &\quad +\tfrac{1}{72}XYt_2+\tfrac{1}{4}t_6)),\\ &= \mathbb{C}[\mathcal{X},\mathcal{Y},\mathcal{W},z,t_2,t_6]/(-z^2-\tfrac{1}{216}(\mathcal{X}+\mathcal{Y}+\tfrac{1}{2}t_2^3)+\tfrac{1}{72}\mathcal{W}t_2\\ &\quad +\tfrac{1}{4}t_6,\mathcal{X}\mathcal{Y}-\mathcal{W}^3).\end{aligned}$$

Now let us look at the action of $\mathbb{Z}/2\mathbb{Z} =< \sigma >$ on $\mathcal{X},\mathcal{Y},\mathcal{W},z,t_2,t_6$. We compute that

$$\begin{cases}\sigma.\mathcal{X}=\mathcal{Y},\\ \sigma.\mathcal{Y}=\mathcal{X},\\ \sigma.\mathcal{W}=\mathcal{W},\\ \sigma.z=-z,\\ \sigma.t_i=t_i, i=2,6.\end{cases}$$

Set $\begin{cases}\mathfrak{X}=\mathcal{X}+\mathcal{Y},\\ \mathfrak{Y}=\mathcal{X}-\mathcal{Y}.\end{cases}$ Then $\mathbb{C}[(\alpha^{\Omega})^{-1}(t_2,t_3)]^{\mathbb{Z}/3\mathbb{Z}} = \mathbb{C}[\mathfrak{X},\mathfrak{Y},\mathcal{W},z,t_2,t_6]/(-z^2-\frac{1}{216}\mathfrak{X}-\frac{1}{432}t_2^3 +\frac{1}{72}\mathcal{W}t_2+\frac{1}{4}t_6,\frac{1}{4}(\mathfrak{X}^2-\mathfrak{Y}^2)-\mathcal{W}^3)$.

The action of $\mathbb{Z}/2\mathbb{Z}$ is given by

$$\begin{cases}\sigma.\mathfrak{X}=\mathfrak{X},\\ \sigma.\mathfrak{Y}=-\mathfrak{Y},\\ \sigma.\mathcal{W}=\mathcal{W},\\ \sigma.z=-z\\ \sigma.t_i=t_i,\ i=2,6.\end{cases}$$

$$\begin{aligned}\text{Therefore } ((\mathbb{C}[(\alpha^{\Omega})^{-1}(t_2,t_6)])^{\mathbb{Z}/3\mathbb{Z}})^{\mathbb{Z}/2\mathbb{Z}} &= \mathbb{C}[\alpha^{-1}(t_2,t_6)]^{\mathfrak{S}_3},\\ &= \mathbb{C}[\mathfrak{X},\mathfrak{Y}^2,z^2,\mathfrak{Y}z,\mathcal{W},t_2,t_6]/(-z^2-\tfrac{1}{216}\mathfrak{X}-\tfrac{1}{432}t_2^3\\ &\quad +\tfrac{1}{72}\mathcal{W}t_2+\tfrac{1}{4}t_6,\tfrac{1}{4}(\mathfrak{X}^2-\mathfrak{Y}^2)-\mathcal{W}^3).\end{aligned}$$

We have $\mathfrak{X} = 216(-z^2-\frac{1}{432}t_2^3+\frac{1}{72}\mathcal{W}t_2+\frac{1}{4}t_6)$, thus $\mathcal{W}^3 = -\frac{1}{4}\mathfrak{Y}^2+\frac{t_2^6}{16}-\frac{3}{4}\mathcal{W}t_2^4+54t_2^3z^2+\frac{9}{4}\mathcal{W}^2t_2^2-324\mathcal{W}t_2z^2-\frac{27}{2}t_2^3t_6+11664z^4+81\mathcal{W}t_2t_6-5832t_6z^2+729t_6^2$.

We then have $\mathbb{C}[(\alpha^{\Omega})^{-1}(t_2,t_6)]^{\mathfrak{S}_3} = \mathbb{C}[\mathfrak{Y}^2,z^2,\mathfrak{Y}z,\mathcal{W},t_2,t_6]/(P(\mathcal{W},\mathcal{Y},z^2))$ with

$$\begin{aligned}P(\mathcal{W},\mathcal{Y},z^2) &= -\mathcal{W}^3-\tfrac{1}{4}\mathfrak{Y}^2+\tfrac{t_2^6}{16}-\tfrac{3}{4}\mathcal{W}t_2^4+54t_2^3z^2+\tfrac{9}{4}\mathcal{W}^2t_2^2-324\mathcal{W}t_2z^2-\tfrac{27}{2}t_2^3t_6\\ &\quad +11664z^4+81\mathcal{W}t_2t_6-5832t_6z^2+729t_6^2.\end{aligned}$$

Set $\begin{cases}X=\mathfrak{Y}^2,\\ Y=\mathfrak{Y}z,\\ Z=z^2.\end{cases}$ Then $\mathbb{C}(\alpha^{\Omega})^{-1}(t_2,t_6)]^{\mathfrak{S}_3} = \mathbb{C}[X,Y,Z,\mathcal{W},t_2,t_6]/(Q(X,Y,Z),R(X,Y,Z))$ with

$$\begin{array}{rcl} Q(X,Y,Z) & = & -\mathcal{W}^3 - \frac{1}{4}X + \frac{t_2^6}{16} - \frac{3}{4}\mathcal{W}t_2^4 + 54t_2^3Z + \frac{9}{4}\mathcal{W}^2t_2^2 - 324\mathcal{W}t_2Z - \frac{27}{2}t_2^3t_6 + 11664Z^2 \\ & & +81\mathcal{W}t_2t_6 - 5832t_6Z + 729t_6^2, \\ R(X,Y,Z) & = & XZ - Y^2. \end{array}$$

We deduce that $\mathbb{C}[(\alpha^\Omega)^{-1}(t_2,t_6)]^{\mathfrak{S}_3} = \mathbb{C}[Z,Y,\mathcal{W},t_2,t_6]/(S(Z,Y,\mathcal{W}))$, with

$$S(Z,Y,\mathcal{W}) = \mathcal{W}^3Z + \frac{1}{4}Y^2 - 11664Z^3 - (a_1Z + a_2Z^2 + b\mathcal{W}Z + \frac{9}{4}t_2^2\mathcal{W}^2Z - 324t_2\mathcal{W}Z^2)$$

and $\left\{ \begin{array}{l} a_1 = \frac{t_2^6}{16} - \frac{27}{2}t_2^3t_6 + 729t_6^2, \\ a_2 = 54t_2^3 - 5832t_6, \\ b = 81t_2t_6 - \frac{3}{4}t_2^4. \end{array} \right.$

After renaming the variables $X := \mathcal{W}$, $Y := Z$, and $Z := Y$, one finds $\mathbb{C}[(\alpha^\Omega)^{-1}(t_2,t_6)]^{\mathfrak{S}_3} = \mathbb{C}[X,Y,Z,t_2,t_6]/(T(X,Y,Z))$, with

$$T(X,Y,Z) = X^3Y - 11664Y^3 + \frac{1}{4}Z^2 - (a_1Y + a_2Y^2 + bXY + \frac{9}{4}t_2^2X^2Y - 324t_2XY^2)$$

and $\left\{ \begin{array}{l} a_1 = \frac{t_2^6}{16} - \frac{27}{2}t_2^3t_6 + 729t_6^2, \\ a_2 = 54t_2^3 - 5832t_6, \\ b = 81t_2t_6 - \frac{3}{4}t_2^4. \end{array} \right.$ Looking at the degrees of the generators, one finds that

$$\left\{ \begin{array}{l} X \text{ is of degree } 4, \\ Y \text{ is of degree } 6, \\ Z \text{ is of degree } 9, \\ t_2 \text{ is of degree } 2, \\ t_6 \text{ is of degree } 6. \end{array} \right.$$

In order for our equation to have the form predicted by Theorem 1.4.26, we do the following substitutions: $\left\{ \begin{array}{l} X \to -(11664)^{1/9}X + \frac{3}{4}t_2^2, \\ Y \to -(11664)^{-1/3}Y, \\ Z \to 2Z. \end{array} \right.$

The equation is finally

$$\begin{array}{c} X^3Y + Y^3 + Z^2 - \frac{1}{36}t_2 11664^{\frac{4}{9}}XY^2 + 11664^{\frac{7}{9}}(-\frac{1}{6912}t_2^4 - \frac{1}{11664}b)XY + 11664^{\frac{1}{3}}(-\frac{1}{11664}a_2 + \\ \frac{1}{48}t_2^3)Y^2 + 11664^{\frac{2}{3}}(\frac{1}{13824}t_2^6 + \frac{1}{11664}a_1 + \frac{1}{15552}bt_2^2)Y = 0. \end{array}$$

Hence a fiber of $\overline{\alpha^\Omega}$ in $X_{\Gamma,\Omega}/\Omega$ is defined by the preceding equation. We notice that it is a subfamily of the semiuniversal deformation of an E_7-singularity. It follows that the projection

$$\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$$

is a deformation of type E_7. However it is not semiuniversal. Using Theorem 1.4.15, one sees that the dimension of the base space of a semiuniversal deformation of a simple singularity of type E_7 is 7. But we have $\dim(\mathfrak{h}/W)^\Omega = 2$. So $\overline{\alpha^\Omega}$ is not semiuniversal.

4.4.4.2. The discriminant of $\overline{\alpha^\Omega}$

Let us determine the discriminant of $\overline{\alpha^\Omega}$, i.e. the elements $(t_2, t_6) \in (\mathfrak{h}/W)^\Omega$ such that the fiber $(\overline{\alpha^\Omega})^{-1}(t_2, t_6)$ is singular.
The fiber $(\overline{\alpha^\Omega})^{-1}(t_2, t_6)$ is defined as the zero locus of the function

$$\begin{aligned} f(X,Y,Z) = \; & X^3Y + Y^3 + Z^2 - \tfrac{1}{36}t_2 11664^{\frac{4}{9}} XY^2 + 11664^{\frac{7}{9}}(-\tfrac{1}{6912}t_2^4 - \tfrac{1}{11664}b)XY \\ & + 11664^{\frac{1}{3}}(-\tfrac{1}{11664}a_2 + \tfrac{1}{48}t_2^3)Y^2 + 11664^{\frac{2}{3}}(\tfrac{1}{13824}t_2^6 + \tfrac{1}{11664}a_1 + \tfrac{1}{15552}bt_2^2)Y. \end{aligned}$$

The fiber is singular if and only if the following system has a solution:

$$\begin{cases} f(X,Y,Z) & = 0, \\ \dfrac{\partial f}{\partial X}(X,Y,Z) & = 0, \\ \dfrac{\partial f}{\partial Y}(X,Y,Z) & = 0, \\ \dfrac{\partial f}{\partial Z}(X,Y,Z) & = 0. \end{cases}$$

If $Y \neq 0$, this system leads to the following 2 equations:

$$\begin{cases} P(X) = (1458^{\frac{5}{9}}8Xt_2^2 + 117t_2^4 1458^{\frac{1}{9}} + 1458^{\frac{1}{9}}7776t_2t_6 - 7776X^2)(t_2^2 1458^{\frac{5}{9}} - 216X)^2 = 0, \\ Q(X) = (t_2^2 1458^{\frac{5}{9}} + 648X)(t_2^2 1458^{\frac{5}{9}} - 216X)^3 = 0. \end{cases}$$

We notice that P and Q have a common factor, namely $(t_2^2 1458^{\frac{5}{9}} - 216X)$. But this factor is a degree 1 polynomial and thus always has a solution. It follows that there exists X_s such that $P(X_s) = Q(X_s) = 0$.

If $Y = 0$, this system leads to the following equation:

$$R(X) = X^3 + 11664^{\frac{7}{9}}(-\tfrac{5}{62208}t_2^4 - \tfrac{1}{144}t_2t_6)X + 11664^{\frac{2}{3}}(\tfrac{11}{373248}t_2^6 + \tfrac{7}{1728}t_2^3t_6 + (1/16)t_6^2) = 0.$$

The polynomial R is of degree 3 and because the base field is $\mathbb{C}$, which is algebraically closed, $R(X) = 0$ always has a solution.

Finally for any $(t_2, t_6) \in (\mathfrak{h}/W)^\Omega$, the fiber $(\overline{\alpha^\Omega})^{-1}(t_2, t_6)$ has a singularity. We obtained the following result:

Proposition 4.4.3. *Every fiber of the deformation $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ of type E_7 is singular.*

4.4.5. Case $(E_6, \mathbb{Z}/2\mathbb{Z})$

We have $\Gamma = \mathcal{T}$ and $\Gamma' = \mathcal{O}$. In Subsection 4.3.9, a semiuniversal deformation of type $(E_6, \mathbb{Z}/2\mathbb{Z})$ was computed. It is the projection $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ with

$$\begin{aligned} X_{\Gamma,\Omega} \;=\; & \{(X, Y, Z, t_2, 0, t_6, t_8, 0, t_{12}) \in \mathbb{C}^3 \times \mathfrak{h}/W \mid -\tfrac{1}{4}X^4 + Y^3 + Z^2 - \tfrac{1}{4}t_2X^2Y \\ & + \tfrac{1}{48}(t_6 - \tfrac{1}{8}t_2^3)X^2 + \tfrac{1}{48}(-t_8 + \tfrac{1}{4}t_6t_2 - \tfrac{1}{192}t_2^4)Y + \tfrac{1}{576}(t_{12} - \tfrac{1}{8}t_8t_2^2 - \tfrac{1}{8}t_6^2 + \tfrac{1}{96}t_6t_2^3) = 0\}. \end{aligned}$$

with the same notations as in Subsection 4.3.9. The morphism α^Ω is Ω-invariant and thus Ω acts on every fiber of α^Ω. The purpose of this subsection is to compute the quotient morphism $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$, and study its nature.

The action of $\Omega = \Gamma'/\Gamma = \mathbb{Z}/2\mathbb{Z} =< \sigma >$ on $X_{\Gamma,\Omega}$ is

$$\begin{cases} \sigma.X = -X, \\ \sigma.Y = Y, \\ \sigma.Z = -Z, \\ \sigma.t_i = t_i, \;\; i = 2, 6, 8, 12. \end{cases}$$

It follows that

$$\begin{aligned} & \mathbb{C}[(\alpha^\Omega)^{-1}(t_2, t_6, t_8, t_{12})]^\Omega \\ = \;& \mathbb{C}[X^2, Z^2, XZ, Y, t_2, t_6, t_8, t_{12}]/(-\tfrac{1}{4}X^4 + Y^3 + Z^2 - \tfrac{1}{4}t_2X^2Y + \tfrac{1}{48}(t_6 - \tfrac{1}{8}t_2^3)X^2 \\ & + \tfrac{1}{48}(-t_8 + \tfrac{1}{4}t_6t_2 - \tfrac{1}{192}t_2^4)Y + \tfrac{1}{576}(t_{12} - \tfrac{1}{8}t_8t_2^2 - \tfrac{1}{8}t_6^2 + \tfrac{1}{96}t_6t_2^3)), \\ = \;& \mathbb{C}[\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z}, t_2, t_6, t_8, t_{12}]/(-\tfrac{1}{4}\mathcal{X}^2 + \mathcal{Y}^3 + \mathcal{Z} - \tfrac{1}{4}t_2\mathcal{X}\mathcal{Y} + \tfrac{1}{48}(t_6 - \tfrac{1}{8}t_2^3)\mathcal{X} \\ & + \tfrac{1}{48}(-t_8 + \tfrac{1}{4}t_6t_2 - \tfrac{1}{192}t_2^4)\mathcal{Y} + \tfrac{1}{576}(t_{12} - \tfrac{1}{8}t_8t_2^2 - \tfrac{1}{8}t_6^2 + \tfrac{1}{96}t_6t_2^3), \mathcal{X}\mathcal{Z} - \mathcal{W}^2). \end{aligned}$$

We deduce that $\mathbb{C}[(\alpha^\Omega)^{-1}(t_2, t_6, t_8, t_{12})]^\Omega = \mathbb{C}[\mathcal{X}, \mathcal{Y}, \mathcal{W}, t_2, t_6, t_8, t_{12}]/(R(\mathcal{X}, \mathcal{Y}, \mathcal{W}))$, with

$$\begin{aligned} R(\mathcal{X}, \mathcal{Y}, \mathcal{W}) = \; & -\tfrac{1}{4}\mathcal{X}^3 + \mathcal{X}\mathcal{Y}^3 - \tfrac{1}{4}t_2\mathcal{X}^2\mathcal{Y} + \tfrac{1}{48}(t_6 - \tfrac{1}{8}t_2^3)\mathcal{X}^2 + \tfrac{1}{48}(-t_8 + \tfrac{1}{4}t_6t_2 - \tfrac{1}{192}t_2^4)\mathcal{X}\mathcal{Y} \\ & + \tfrac{1}{576}(t_{12} - \tfrac{1}{8}t_8t_2^2 - \tfrac{1}{8}t_6^2 + \tfrac{1}{96}t_6t_2^3)\mathcal{X} + \mathcal{W}^2. \end{aligned}$$

After renaming the variables $X := \mathcal{X}$, $Y := \mathcal{Y}$ and $Z := \mathcal{W}$, the equation is finally

$$\begin{aligned} & -\tfrac{1}{4}X^3 + XY^3 + Z^2 - \tfrac{1}{4}t_2X^2Y + \tfrac{1}{48}(t_6 - \tfrac{1}{8}t_2^3)X^2 + \tfrac{1}{48}(-t_8 + \tfrac{1}{4}t_6t_2 - \tfrac{1}{192}t_2^4)XY \\ & + \tfrac{1}{576}(t_{12} - \tfrac{1}{8}t_8t_2^2 - \tfrac{1}{8}t_6^2 + \tfrac{1}{96}t_6t_2^3)X = 0. \end{aligned}$$

Hence a fiber of $\overline{\alpha^\Omega}$ in $X_{\Gamma,\Omega}/\Omega$ is defined by the preceding equation. We notice that it is a subfamily of the semiuniversal deformation of an E_7-singularity. It follows that the projection

$$\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$$

is a deformation of type E_7. However it is not semiuniversal. Using Theorem 1.4.15, one sees that the dimension of the base space of a semiuniversal deformation of a simple singularity of type E_7 is 7. But we have $\dim(\mathfrak{h}/W)^\Omega = 4$. So $\overline{\alpha^\Omega}$ is not semiuniversal.

Appendix

After computation of the relation between the generators of the coordinate ring of the invariants of the representation space of the McKay quiver of type $\widetilde{E}_6$ (cf. Subsection 4.3.9), we obtain the following equation

$$\begin{aligned} Y^3 + X^2Z + Z^2 \quad = \quad & a_{X^3}X^3 + a_{XY^2}XY^2 + a_{X^2Y}X^2Y + a_{X^2}X^2 + a_{Y^2}Y^2 + a_{XY}XY \\ & + a_{XZ}XZ + a_{YZ}YZ + a_X X + a_Y Y + a_Z Z + a_0. \end{aligned}$$

with $X = \Phi(4^2 5)$ of degree 3, $Y = \Phi(4^2 5^2)$ of degree 4 and $Z = \Phi(3^2 4^2 5^2)$ of degree 6. The coefficients are homogeneous polynomials in $\mu_1, \ldots, \mu_6$ and are given below.

$a_{X^3} = -2\mu_1\mu_2\mu_4 + 2\mu_1\mu_2\mu_5 - 4\mu_1\mu_4\mu_5 - 2\mu_1\mu_4\mu_6 + 2\mu_1\mu_5^2 - 2\mu_2\mu_4^2 + 4\mu_2\mu_4\mu_5 + 2\mu_2\mu_5\mu_6 - 4\mu_4^2\mu_5 - 2\mu_4^2\mu_6 + 4\mu_4\mu_5^2 + 2\mu_5^2\mu_6.$

$$\begin{aligned} a_X = {} & 4\mu_1^6\mu_2^2\mu_4 - 2\mu_1^6\mu_2^2\mu_5 + 6\mu_1^6\mu_2\mu_3\mu_4 - 2\mu_1^6\mu_2\mu_3\mu_5 + 4\mu_1^6\mu_2\mu_4^2 + 14\mu_1^6\mu_2\mu_4\mu_5 \\ & + 12\mu_1^6\mu_2\mu_4\mu_6 - 6\mu_1^6\mu_2\mu_5^2 - 4\mu_1^6\mu_2\mu_5\mu_6 + 2\mu_1^6\mu_3^2\mu_4 + 2\mu_1^6\mu_3\mu_4^2 + 12\mu_1^6\mu_3\mu_4\mu_5 + 8\mu_1^6\mu_3\mu_4\mu_6 \\ & - 2\mu_1^6\mu_3\mu_5^2 + 8\mu_1^6\mu_4^2\mu_5 + 4\mu_1^6\mu_4^2\mu_6 + 14\mu_1^6\mu_4\mu_5^2 + 24\mu_1^6\mu_4\mu_5\mu_6 + 8\mu_1^6\mu_4\mu_6^2 - 4\mu_1^6\mu_5^3 \\ & - 4\mu_1^6\mu_5^2\mu_6 - 2\mu_1^5\mu_2^3\mu_4 + 2\mu_1^5\mu_2^3\mu_5 + 2\mu_1^5\mu_2^2\mu_3\mu_4 + \mu_1^5\mu_2^2\mu_3\mu_5 + 28\mu_1^5\mu_2^2\mu_4^2 - 21\mu_1^5\mu_2^2\mu_4\mu_5 \\ & + 6\mu_1^5\mu_2^2\mu_4\mu_6 + 10\mu_1^5\mu_2^2\mu_5^2 + 2\mu_1^5\mu_2^2\mu_5\mu_6 + 8\mu_1^5\mu_2\mu_3^2\mu_4 - \mu_1^5\mu_2\mu_3^2\mu_5 + 54\mu_1^5\mu_2\mu_3\mu_4^2 \\ & - 4\mu_1^5\mu_2\mu_3\mu_4\mu_5 + 32\mu_1^5\mu_2\mu_3\mu_4\mu_6 + 3\mu_1^5\mu_2\mu_3\mu_5^2 - 4\mu_1^5\mu_2\mu_3\mu_5\mu_6 + 30\mu_1^5\mu_2\mu_4^3 + 101\mu_1^5\mu_2\mu_4^2\mu_5 \\ & + 100\mu_1^5\mu_2\mu_4^2\mu_6 - 51\mu_1^5\mu_2\mu_4\mu_5^2 + 32\mu_1^5\mu_2\mu_4\mu_6^2 + 16\mu_1^5\mu_2\mu_5^3 + 6\mu_1^5\mu_2\mu_5^2\mu_6 - 4\mu_1^5\mu_2\mu_5\mu_6^2 + \\ & 4\mu_1^5\mu_3^3\mu_4 + 24\mu_1^5\mu_3^2\mu_4^2 + 16\mu_1^5\mu_3^2\mu_4\mu_5 + 22\mu_1^5\mu_3^2\mu_4\mu_6 - \mu_1^5\mu_3^2\mu_5^2 + 20\mu_1^5\mu_3\mu_4^3 + 108\mu_1^5\mu_3\mu_4^2\mu_5 \\ & + 84\mu_1^5\mu_3\mu_4^2\mu_6 - 4\mu_1^5\mu_3\mu_4\mu_5^2 + 64\mu_1^5\mu_3\mu_4\mu_5\mu_6 + 40\mu_1^5\mu_3\mu_4\mu_6^2 + 2\mu_1^5\mu_3\mu_5^3 - 4\mu_1^5\mu_3\mu_5^2\mu_6 \\ & + 60\mu_1^5\mu_4^3\mu_5 + 30\mu_1^5\mu_4^3\mu_6 + 101\mu_1^5\mu_4^2\mu_5^2 + 200\mu_1^5\mu_4^2\mu_5\mu_6 + 72\mu_1^5\mu_4^2\mu_6^2 - 34\mu_1^5\mu_4\mu_5^3 + \\ & 64\mu_1^5\mu_4\mu_5\mu_6^2 + 24\mu_1^5\mu_4\mu_6^3 + 8\mu_1^5\mu_5^4 + 4\mu_1^5\mu_5^3\mu_6 - 4\mu_1^5\mu_5^2\mu_6^2 - 3\mu_1^4\mu_2^4\mu_4 - \mu_1^4\mu_2^4\mu_5 - 11\mu_1^4\mu_2^3\mu_3\mu_4 \\ & + 2\mu_1^4\mu_2^3\mu_3\mu_5 - 26\mu_1^4\mu_2^3\mu_4^2 - 7\mu_1^4\mu_2^3\mu_4\mu_5 - 26\mu_1^4\mu_2^3\mu_4\mu_6 - 6\mu_1^4\mu_2^3\mu_5^2 + 2\mu_1^4\mu_2^3\mu_5\mu_6 - 10\mu_1^4\mu_2^2\mu_3^2\mu_4 \\ & + 5\mu_1^4\mu_2^2\mu_3^2\mu_5 - 6\mu_1^4\mu_2^2\mu_3\mu_4^2 - 44\mu_1^4\mu_2^2\mu_3\mu_4\mu_5 - 43\mu_1^4\mu_2^2\mu_3\mu_4\mu_6 + 11\mu_1^4\mu_2^2\mu_3\mu_5^2 + 15\mu_1^4\mu_2^2\mu_3\mu_5\mu_6 \\ & + 72\mu_1^4\mu_2^2\mu_4^3 - 148\mu_1^4\mu_2^2\mu_4^2\mu_5 - 18\mu_1^4\mu_2^2\mu_4^2\mu_6 + 12\mu_1^4\mu_2^2\mu_4\mu_5^2 - 106\mu_1^4\mu_2^2\mu_4\mu_5\mu_6 - 51\mu_1^4\mu_2^2\mu_4\mu_6^2 \\ & - 14\mu_1^4\mu_2^2\mu_5^3 + 12\mu_1^4\mu_2^2\mu_5^2\mu_6 + 13\mu_1^4\mu_2^2\mu_5\mu_6^2 + 2\mu_1^4\mu_2\mu_3^3\mu_5 + 58\mu_1^4\mu_2\mu_3^2\mu_4^2 - 36\mu_1^4\mu_2\mu_3^2\mu_4\mu_5 \\ & + 2\mu_1^4\mu_2\mu_3^2\mu_4\mu_6 + 15\mu_1^4\mu_2\mu_3^2\mu_5^2 + 9\mu_1^4\mu_2\mu_3^2\mu_5\mu_6 + 204\mu_1^4\mu_2\mu_3\mu_4^3 - 28\mu_1^4\mu_2\mu_3\mu_4^2\mu_5 \\ & + 204\mu_1^4\mu_2\mu_3\mu_4^2\mu_6 - 66\mu_1^4\mu_2\mu_3\mu_4\mu_5^2 - 149\mu_1^4\mu_2\mu_3\mu_4\mu_5\mu_6 - 2\mu_1^4\mu_2\mu_3\mu_4\mu_6^2 + 18\mu_1^4\mu_2\mu_3\mu_5^3 + \\ & 45\mu_1^4\mu_2\mu_3\mu_5^2\mu_6 + 15\mu_1^4\mu_2\mu_3\mu_5\mu_6^2 + 94\mu_1^4\mu_2\mu_4^4 + 282\mu_1^4\mu_2\mu_4^3\mu_5 + 336\mu_1^4\mu_2\mu_4^3\mu_6 \\ & - 288\mu_1^4\mu_2\mu_4^2\mu_5^2 - 72\mu_1^4\mu_2\mu_4^2\mu_5\mu_6 + 164\mu_1^4\mu_2\mu_4^2\mu_6^2 + 38\mu_1^4\mu_2\mu_4\mu_5^3 - 162\mu_1^4\mu_2\mu_4\mu_5^2\mu_6 \end{aligned}$$

$$
\begin{aligned}
&- 173\mu_1^4\mu_2\mu_4\mu_5\mu_6^2 - 12\mu_1^4\mu_2\mu_4\mu_6^3 - 15\mu_1^4\mu_2\mu_5^4 + 20\mu_1^4\mu_2\mu_5^3\mu_6 + 39\mu_1^4\mu_2\mu_5^2\mu_6^2 + 10\mu_1^4\mu_2\mu_5\mu_6^3 \\
&+ 2\mu_1^4\mu_3^4\mu_4 + 36\mu_1^4\mu_3^3\mu_4^2 + 16\mu_1^4\mu_3^3\mu_4\mu_6 + 2\mu_1^4\mu_3^3\mu_5^2 + 116\mu_1^4\mu_3^2\mu_4^3 + 116\mu_1^4\mu_3^2\mu_4^2\mu_5 + \\
&188\mu_1^4\mu_3^2\mu_4^2\mu_6 - 36\mu_1^4\mu_3^2\mu_4\mu_5^2 + 4\mu_1^4\mu_3^2\mu_4\mu_5\mu_6 + 44\mu_1^4\mu_3^2\mu_4\mu_6^2 + 10\mu_1^4\mu_3^2\mu_5^3 + 9\mu_1^4\mu_3^2\mu_5^2\mu_6 + \\
&82\mu_1^4\mu_3\mu_4^4 + 408\mu_1^4\mu_3\mu_4^3\mu_5 + 368\mu_1^4\mu_3\mu_4^3\mu_6 - 28\mu_1^4\mu_3\mu_4^2\mu_5^2 + 408\mu_1^4\mu_3\mu_4^2\mu_5\mu_6 + 310\mu_1^4\mu_3\mu_4^2\mu_6^2 \\
&- 44\mu_1^4\mu_3\mu_4\mu_5^3 - 149\mu_1^4\mu_3\mu_4\mu_5^2\mu_6 - 4\mu_1^4\mu_3\mu_4\mu_5\mu_6^2 + 48\mu_1^4\mu_3\mu_4\mu_6^3 + 9\mu_1^4\mu_3\mu_5^4 + 30\mu_1^4\mu_3\mu_5^3\mu_6 \\
&+ 15\mu_1^4\mu_3\mu_5^2\mu_6^2 + 188\mu_1^4\mu_4^4\mu_5 + 94\mu_1^4\mu_4^4\mu_6 + 282\mu_1^4\mu_4^3\mu_5^2 + 672\mu_1^4\mu_4^3\mu_5\mu_6 + 264\mu_1^4\mu_4^3\mu_6^2 \\
&- 192\mu_1^4\mu_4^2\mu_5^3 - 72\mu_1^4\mu_4^2\mu_5^2\mu_6 + 328\mu_1^4\mu_4^2\mu_5\mu_6^2 + 156\mu_1^4\mu_4^2\mu_6^3 + 19\mu_1^4\mu_4\mu_5^4 - 108\mu_1^4\mu_4\mu_5^3\mu_6 - \\
&173\mu_1^4\mu_4\mu_5^2\mu_6^2 - 24\mu_1^4\mu_4\mu_5\mu_6^3 + 16\mu_1^4\mu_4\mu_6^4 - 6\mu_1^4\mu_5^5 + 10\mu_1^4\mu_5^4\mu_6 + 26\mu_1^4\mu_5^3\mu_6^2 + 10\mu_1^4\mu_5^2\mu_6^3 - \\
&2\mu_1^3\mu_2^5\mu_4 - 10\mu_1^3\mu_2^4\mu_3\mu_4 - \mu_1^3\mu_2^4\mu_3\mu_5 - 24\mu_1^3\mu_2^4\mu_4^2 - 25\mu_1^3\mu_2^4\mu_4\mu_5 - 24\mu_1^3\mu_2^4\mu_4\mu_6 - 2\mu_1^3\mu_2^4\mu_5\mu_6 \\
&- 18\mu_1^3\mu_2^3\mu_3^2\mu_4 - 88\mu_1^3\mu_2^3\mu_3\mu_4^2 - 62\mu_1^3\mu_2^3\mu_3\mu_4\mu_5 - 86\mu_1^3\mu_2^3\mu_3\mu_4\mu_6 - 6\mu_1^3\mu_2^3\mu_3\mu_5^2 - 112\mu_1^3\mu_2^3\mu_4^3 - \\
&130\mu_1^3\mu_2^3\mu_4^2\mu_5 - 212\mu_1^3\mu_2^3\mu_4^2\mu_6 - 104\mu_1^3\mu_2^3\mu_4\mu_5^2 - 156\mu_1^3\mu_2^3\mu_4\mu_5\mu_6 - 100\mu_1^3\mu_2^3\mu_4\mu_6^2 - 12\mu_1^3\mu_2^3\mu_5^2\mu_6 \\
&- 2\mu_1^3\mu_2^3\mu_5\mu_6^2 - 12\mu_1^3\mu_2^2\mu_3^3\mu_4 + 2\mu_1^3\mu_2^2\mu_3^3\mu_5 - 80\mu_1^3\mu_2^2\mu_3^2\mu_4^2 - 64\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_5 - 92\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_6 \\
&+ \mu_1^3\mu_2^2\mu_3^2\mu_5^2 + 12\mu_1^3\mu_2^2\mu_3^2\mu_5\mu_6 - 80\mu_1^3\mu_2^2\mu_3\mu_4^3 - 394\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_5 - 372\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_6 \\
&- 140\mu_1^3\mu_2^2\mu_3\mu_4\mu_5^2 - 352\mu_1^3\mu_2^2\mu_3\mu_4\mu_5\mu_6 - 222\mu_1^3\mu_2^2\mu_3\mu_4\mu_6^2 - 14\mu_1^3\mu_2^2\mu_3\mu_5^3 + 4\mu_1^3\mu_2^2\mu_3\mu_5^2\mu_6 + \\
&22\mu_1^3\mu_2^2\mu_3\mu_5\mu_6^2 + 80\mu_1^3\mu_2^2\mu_4^4 - 554\mu_1^3\mu_2^2\mu_4^3\mu_5 - 240\mu_1^3\mu_2^2\mu_4^3\mu_6 - 261\mu_1^3\mu_2^2\mu_4^2\mu_5^2 - 960\mu_1^3\mu_2^2\mu_4^2\mu_5\mu_6 \\
&- 464\mu_1^3\mu_2^2\mu_4^2\mu_6^2 - 206\mu_1^3\mu_2^2\mu_4\mu_5^3 - 376\mu_1^3\mu_2^2\mu_4\mu_5^2\mu_6 - 424\mu_1^3\mu_2^2\mu_4\mu_5\mu_6^2 - 174\mu_1^3\mu_2^2\mu_4\mu_6^3 \\
&- 28\mu_1^3\mu_2^2\mu_5^3\mu_6 - 6\mu_1^3\mu_2^2\mu_5^2\mu_6^2 + 12\mu_1^3\mu_2^2\mu_5\mu_6^3 - 2\mu_1^3\mu_2\mu_3^4\mu_4 + \mu_1^3\mu_2\mu_3^4\mu_5 - 28\mu_1^3\mu_2\mu_3^3\mu_4\mu_5 - \\
&24\mu_1^3\mu_2\mu_3^3\mu_4\mu_6 + 6\mu_1^3\mu_2\mu_3^3\mu_5^2 + 8\mu_1^3\mu_2\mu_3^3\mu_5\mu_6 + 172\mu_1^3\mu_2\mu_3^2\mu_4^3 - 254\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_5 \\
&- 18\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_6 - 84\mu_1^3\mu_2\mu_3^2\mu_4\mu_5^2 - 256\mu_1^3\mu_2\mu_3^2\mu_4\mu_5\mu_6 - 100\mu_1^3\mu_2\mu_3^2\mu_4\mu_6^2 + 2\mu_1^3\mu_2\mu_3^2\mu_5^3 + \\
&36\mu_1^3\mu_2\mu_3^2\mu_5^2\mu_6 + 24\mu_1^3\mu_2\mu_3^2\mu_5\mu_6^2 + 420\mu_1^3\mu_2\mu_3\mu_4^4 - 208\mu_1^3\mu_2\mu_3\mu_4^3\mu_5 + 520\mu_1^3\mu_2\mu_3\mu_4^3\mu_6 - \\
&654\mu_1^3\mu_2\mu_3\mu_4^2\mu_5^2 - 1176\mu_1^3\mu_2\mu_3\mu_4^2\mu_5\mu_6 - 146\mu_1^3\mu_2\mu_3\mu_4^2\mu_6^2 - 156\mu_1^3\mu_2\mu_3\mu_4\mu_5^3 \\
&- 540\mu_1^3\mu_2\mu_3\mu_4\mu_5^2\mu_6 - 656\mu_1^3\mu_2\mu_3\mu_4\mu_5\mu_6^2 - 176\mu_1^3\mu_2\mu_3\mu_4\mu_6^3 - 15\mu_1^3\mu_2\mu_3\mu_5^4 + 8\mu_1^3\mu_2\mu_3\mu_5^3\mu_6 \\
&+ 66\mu_1^3\mu_2\mu_3\mu_5^2\mu_6^2 + 32\mu_1^3\mu_2\mu_3\mu_5\mu_6^3 + 162\mu_1^3\mu_2\mu_4^5 + 385\mu_1^3\mu_2\mu_4^4\mu_5 + 600\mu_1^3\mu_2\mu_4^4\mu_6 \\
&- 990\mu_1^3\mu_2\mu_4^3\mu_5^2 - 684\mu_1^3\mu_2\mu_4^3\mu_5\mu_6 + 280\mu_1^3\mu_2\mu_4^3\mu_6^2 - 262\mu_1^3\mu_2\mu_4^2\mu_5^3 - 1608\mu_1^3\mu_2\mu_4^2\mu_5^2\mu_6 - \\
&1476\mu_1^3\mu_2\mu_4^2\mu_5\mu_6^2 - 220\mu_1^3\mu_2\mu_4^2\mu_6^3 - 205\mu_1^3\mu_2\mu_4\mu_5^4 - 440\mu_1^3\mu_2\mu_4\mu_5^3\mu_6 - 672\mu_1^3\mu_2\mu_4\mu_5^2\mu_6^2 - \\
&520\mu_1^3\mu_2\mu_4\mu_5\mu_6^3 - 112\mu_1^3\mu_2\mu_4\mu_6^4 - 30\mu_1^3\mu_2\mu_5^4\mu_6 - 8\mu_1^3\mu_2\mu_5^3\mu_6^2 + 36\mu_1^3\mu_2\mu_5^2\mu_6^3 + 16\mu_1^3\mu_2\mu_5\mu_6^4 + \\
&14\mu_1^3\mu_3^4\mu_4^2 - 4\mu_1^3\mu_3^4\mu_4\mu_5 + 2\mu_1^3\mu_3^4\mu_4\mu_6 + \mu_1^3\mu_3^4\mu_5^2 + 128\mu_1^3\mu_3^3\mu_4^3 + 112\mu_1^3\mu_3^3\mu_4^2\mu_6 - 28\mu_1^3\mu_3^3\mu_4\mu_5^2 \\
&- 48\mu_1^3\mu_3^3\mu_4\mu_5\mu_6 + 8\mu_1^3\mu_3^3\mu_4\mu_6^2 + 4\mu_1^3\mu_3^3\mu_5^3 + 8\mu_1^3\mu_3^3\mu_5^2\mu_6 + 290\mu_1^3\mu_3^2\mu_4^4 + 344\mu_1^3\mu_3^2\mu_4^3\mu_5 + \\
&644\mu_1^3\mu_3^2\mu_4^3\mu_6 - 254\mu_1^3\mu_3^2\mu_4^2\mu_5^2 - 36\mu_1^3\mu_3^2\mu_4^2\mu_5\mu_6 + 288\mu_1^3\mu_3^2\mu_4^2\mu_6^2 - 56\mu_1^3\mu_3^2\mu_4\mu_5^3 \\
&- 256\mu_1^3\mu_3^2\mu_4\mu_5^2\mu_6 - 200\mu_1^3\mu_3^2\mu_4\mu_5\mu_6^2 + 4\mu_1^3\mu_3^2\mu_4\mu_6^3 + \mu_1^3\mu_3^2\mu_5^4 + 24\mu_1^3\mu_3^2\mu_5^3\mu_6 + 24\mu_1^3\mu_3^2\mu_5^2\mu_6^2 \\
&+ 176\mu_1^3\mu_3\mu_4^5 + 840\mu_1^3\mu_3\mu_4^4\mu_5 + 860\mu_1^3\mu_3\mu_4^4\mu_6 - 208\mu_1^3\mu_3\mu_4^3\mu_5^2 + 1040\mu_1^3\mu_3\mu_4^3\mu_5\mu_6 + \\
&980\mu_1^3\mu_3\mu_4^3\mu_6^2 - 436\mu_1^3\mu_3\mu_4^2\mu_5^3 - 1176\mu_1^3\mu_3\mu_4^2\mu_5^2\mu_6 - 292\mu_1^3\mu_3\mu_4^2\mu_5\mu_6^2 + 268\mu_1^3\mu_3\mu_4^2\mu_6^3 \\
&- 78\mu_1^3\mu_3\mu_4\mu_5^4 - 360\mu_1^3\mu_3\mu_4\mu_5^3\mu_6 - 656\mu_1^3\mu_3\mu_4\mu_5^2\mu_6^2 - 352\mu_1^3\mu_3\mu_4\mu_5\mu_6^3 - 16\mu_1^3\mu_3\mu_4\mu_6^4 - 6\mu_1^3\mu_3\mu_5^5 \\
&+ 4\mu_1^3\mu_3\mu_5^4\mu_6 + 44\mu_1^3\mu_3\mu_5^3\mu_6^2 + 32\mu_1^3\mu_3\mu_5^2\mu_6^3 + 324\mu_1^3\mu_4^5\mu_5 + 162\mu_1^3\mu_4^5\mu_6 + 385\mu_1^3\mu_4^4\mu_5^2 + \\
&1200\mu_1^3\mu_4^4\mu_5\mu_6 + 520\mu_1^3\mu_4^4\mu_6^2 - 660\mu_1^3\mu_4^3\mu_5^3 - 684\mu_1^3\mu_4^3\mu_5^2\mu_6 + 560\mu_1^3\mu_4^3\mu_5\mu_6^2 + 408\mu_1^3\mu_4^3\mu_6^3 \\
&- 131\mu_1^3\mu_4^2\mu_5^4 - 1072\mu_1^3\mu_4^2\mu_5^3\mu_6 - 1476\mu_1^3\mu_4^2\mu_5^2\mu_6^2 - 440\mu_1^3\mu_4^2\mu_5\mu_6^3 + 56\mu_1^3\mu_4^2\mu_6^4 - 82\mu_1^3\mu_4\mu_5^5 \\
&- 220\mu_1^3\mu_4\mu_5^4\mu_6 - 448\mu_1^3\mu_4\mu_5^3\mu_6^2 - 520\mu_1^3\mu_4\mu_5^2\mu_6^3 - 224\mu_1^3\mu_4\mu_5\mu_6^4 - 16\mu_1^3\mu_4\mu_6^5 - 12\mu_1^3\mu_5^5\mu_6 - \\
&4\mu_1^3\mu_5^4\mu_6^2 + 24\mu_1^3\mu_5^3\mu_6^3 + 16\mu_1^3\mu_5^2\mu_6^4 - 3\mu_1^2\mu_2^5\mu_3\mu_4 - 10\mu_1^2\mu_2^5\mu_4^2 - \mu_1^2\mu_2^5\mu_4\mu_5 - 6\mu_1^2\mu_2^5\mu_4\mu_6 - \\
&9\mu_1^2\mu_2^4\mu_3^2\mu_4 - 51\mu_1^2\mu_2^4\mu_3\mu_4^2 - 34\mu_1^2\mu_2^4\mu_3\mu_4\mu_5 - 41\mu_1^2\mu_2^4\mu_3\mu_4\mu_6 - \mu_1^2\mu_2^4\mu_3\mu_5\mu_6 - 66\mu_1^2\mu_2^4\mu_4^3 - \\
&108\mu_1^2\mu_2^4\mu_4^2\mu_5 - 122\mu_1^2\mu_2^4\mu_4^2\mu_6 - 8\mu_1^2\mu_2^4\mu_4\mu_5^2 - 66\mu_1^2\mu_2^4\mu_4\mu_5\mu_6 - 47\mu_1^2\mu_2^4\mu_4\mu_6^2 - \mu_1^2\mu_2^4\mu_5\mu_6^2 - \\
&10\mu_1^2\mu_2^3\mu_3^3\mu_4 - 91\mu_1^2\mu_2^3\mu_3^2\mu_4^2 - 68\mu_1^2\mu_2^3\mu_3^2\mu_4\mu_5 - 76\mu_1^2\mu_2^3\mu_3^2\mu_4\mu_6 - 242\mu_1^2\mu_2^3\mu_3\mu_4^3 \\
&- 348\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_5 - 442\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_6 - 142\mu_1^2\mu_2^3\mu_3\mu_4\mu_5^2 - 302\mu_1^2\mu_2^3\mu_3\mu_4\mu_5\mu_6 - 182\mu_1^2\mu_2^3\mu_3\mu_4\mu_6^2 \\
&- 6\mu_1^2\mu_2^3\mu_3\mu_5^2\mu_6 - 2\mu_1^2\mu_2^3\mu_3\mu_5\mu_6^2 - 216\mu_1^2\mu_2^3\mu_4^4 - 410\mu_1^2\mu_2^3\mu_4^3\mu_5 - 588\mu_1^2\mu_2^3\mu_4^3\mu_6 - 425\mu_1^2\mu_2^3\mu_4^2\mu_5^2
\end{aligned}
$$

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\begin{aligned}
&- 832\mu_1^2\mu_2^3\mu_4^2\mu_5\mu_6 - 520\mu_1^2\mu_2^3\mu_4^2\mu_6^2 - 26\mu_1^2\mu_2^3\mu_4\mu_5^3 - 268\mu_1^2\mu_2^3\mu_4\mu_5^2\mu_6 - 338\mu_1^2\mu_2^3\mu_4\mu_5\mu_6^2 - \\
&140\mu_1^2\mu_2^3\mu_4\mu_6^3 - 6\mu_1^2\mu_2^3\mu_5^2\mu_6^2 - 2\mu_1^2\mu_2^3\mu_5\mu_6^3 - 4\mu_1^2\mu_2^4\mu_3^2\mu_4 - 60\mu_1^2\mu_2^4\mu_3^3\mu_4^2 - 44\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_5 - \\
&50\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_6 + 2\mu_1^2\mu_2^4\mu_3^2\mu_5\mu_6 - 220\mu_1^2\mu_2^4\mu_3^2\mu_4^3 - 404\mu_1^2\mu_2^4\mu_3^2\mu_4^2\mu_5 - 469\mu_1^2\mu_2^4\mu_3^2\mu_4^2\mu_6 \\
&- 188\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_5^2 - 353\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_5\mu_6 - 203\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_6^2 + \mu_1^2\mu_2^4\mu_3^2\mu_5^2\mu_6 + 7\mu_1^2\mu_2^4\mu_3^2\mu_5\mu_6^2 - \\
&200\mu_1^2\mu_2^4\mu_3\mu_4^4 - 1116\mu_1^2\mu_2^4\mu_3\mu_4^3\mu_5 - 1058\mu_1^2\mu_2^4\mu_3\mu_4^3\mu_6 - 890\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_5^2 - 2059\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_5\mu_6 \\
&- 1153\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_6^2 - 288\mu_1^2\mu_2^4\mu_3\mu_4\mu_5^3 - 827\mu_1^2\mu_2^4\mu_3\mu_4\mu_5^2\mu_6 - 863\mu_1^2\mu_2^4\mu_3\mu_4\mu_5\mu_6^2 \\
&- 336\mu_1^2\mu_2^4\mu_3\mu_4\mu_6^3 - 14\mu_1^2\mu_2^4\mu_3\mu_5^3\mu_6 - 7\mu_1^2\mu_2^4\mu_3\mu_5^2\mu_6^2 + 8\mu_1^2\mu_2^4\mu_3\mu_5\mu_6^3 + 32\mu_1^2\mu_2^4\mu_4^5 \\
&- 1027\mu_1^2\mu_2^4\mu_4^4\mu_5 - 600\mu_1^2\mu_2^4\mu_4^4\mu_6 - 964\mu_1^2\mu_2^4\mu_4^3\mu_5^2 - 2740\mu_1^2\mu_2^4\mu_4^3\mu_5\mu_6 - 1346\mu_1^2\mu_2^4\mu_4^3\mu_6^2 - \\
&820\mu_1^2\mu_2^4\mu_4^2\mu_5^3 - 2140\mu_1^2\mu_2^4\mu_4^2\mu_5^2\mu_6 - 2439\mu_1^2\mu_2^4\mu_4^2\mu_5\mu_6^2 - 920\mu_1^2\mu_2^4\mu_4^2\mu_6^3 - 43\mu_1^2\mu_2^4\mu_4\mu_5^4 \\
&- 532\mu_1^2\mu_2^4\mu_4\mu_5^3\mu_6 - 915\mu_1^2\mu_2^4\mu_4\mu_5^2\mu_6^2 - 666\mu_1^2\mu_2^4\mu_4\mu_5\mu_6^3 - 195\mu_1^2\mu_2^4\mu_4\mu_6^4 - 14\mu_1^2\mu_2^4\mu_5^3\mu_6^2 - \\
&8\mu_1^2\mu_2^4\mu_5^2\mu_6^3 + 3\mu_1^2\mu_2^4\mu_5\mu_6^4 - 10\mu_1^2\mu_2\mu_3^4\mu_4^2 - 8\mu_1^2\mu_2\mu_3^4\mu_4\mu_5 - 8\mu_1^2\mu_2\mu_3^4\mu_4\mu_6 + \mu_1^2\mu_2\mu_3^4\mu_5\mu_6 \\
&- 172\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_5 - 120\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_6 - 72\mu_1^2\mu_2\mu_3^3\mu_4\mu_5^2 - 138\mu_1^2\mu_2\mu_3^3\mu_4\mu_5\mu_6 - 64\mu_1^2\mu_2\mu_3^3\mu_4\mu_6^2 \\
&+ 6\mu_1^2\mu_2\mu_3^3\mu_5^2\mu_6 + 6\mu_1^2\mu_2\mu_3^3\mu_5\mu_6^2 + 266\mu_1^2\mu_2\mu_3^2\mu_4^4 - 676\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_5 - 92\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_6 - \\
&666\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5^2 - 1453\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5\mu_6 - 516\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_6^2 - 240\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^3 \\
&- 603\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^2\mu_6 - 613\mu_1^2\mu_2\mu_3^2\mu_4\mu_5\mu_6^2 - 192\mu_1^2\mu_2\mu_3^2\mu_4\mu_6^3 + 2\mu_1^2\mu_2\mu_3^2\mu_5^3\mu_6 + 21\mu_1^2\mu_2\mu_3^2\mu_5^2\mu_6^2 \\
&+ 14\mu_1^2\mu_2\mu_3^2\mu_5\mu_6^3 + 504\mu_1^2\mu_2\mu_3\mu_4^5 - 504\mu_1^2\mu_2\mu_3\mu_4^4\mu_5 + 700\mu_1^2\mu_2\mu_3\mu_4^4\mu_6 - 1896\mu_1^2\mu_2\mu_3\mu_4^3\mu_5^2 \\
&- 3294\mu_1^2\mu_2\mu_3\mu_4^3\mu_5\mu_6 - 564\mu_1^2\mu_2\mu_3\mu_4^3\mu_6^2 - 1084\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^3 - 3525\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^2\mu_6 \\
&- 3671\mu_1^2\mu_2\mu_3\mu_4^2\mu_5\mu_6^2 - 936\mu_1^2\mu_2\mu_3\mu_4^2\mu_6^3 - 290\mu_1^2\mu_2\mu_3\mu_4\mu_5^4 - 1050\mu_1^2\mu_2\mu_3\mu_4\mu_5^3\mu_6 \\
&- 1497\mu_1^2\mu_2\mu_3\mu_4\mu_5^2\mu_6^2 - 1046\mu_1^2\mu_2\mu_3\mu_4\mu_5\mu_6^3 - 252\mu_1^2\mu_2\mu_3\mu_4\mu_6^4 - 15\mu_1^2\mu_2\mu_3\mu_5^4\mu_6 \\
&- 10\mu_1^2\mu_2\mu_3\mu_5^3\mu_6^2 + 24\mu_1^2\mu_2\mu_3\mu_5^2\mu_6^3 + 15\mu_1^2\mu_2\mu_3\mu_5\mu_6^4 + 166\mu_1^2\mu_2\mu_6^6 + 280\mu_1^2\mu_2\mu_4^5\mu_5 + \\
&632\mu_1^2\mu_2\mu_4^5\mu_6 - 1785\mu_1^2\mu_2\mu_4^4\mu_5^2 - 1692\mu_1^2\mu_2\mu_4^4\mu_5\mu_6 + 180\mu_1^2\mu_2\mu_4^4\mu_6^2 - 1108\mu_1^2\mu_2\mu_4^3\mu_5^3 \\
&- 4692\mu_1^2\mu_2\mu_4^3\mu_5^2\mu_6 - 4254\mu_1^2\mu_2\mu_4^3\mu_5\mu_6^2 - 760\mu_1^2\mu_2\mu_4^3\mu_6^3 - 805\mu_1^2\mu_2\mu_4^2\mu_5^4 - 2616\mu_1^2\mu_2\mu_4^2\mu_5^3\mu_6 \\
&- 4197\mu_1^2\mu_2\mu_4^2\mu_5^2\mu_6^2 - 2942\mu_1^2\mu_2\mu_4^2\mu_5\mu_6^3 - 608\mu_1^2\mu_2\mu_4^2\mu_6^4 - 36\mu_1^2\mu_2\mu_4\mu_5^5 - 530\mu_1^2\mu_2\mu_4\mu_5^4\mu_6 - \\
&1154\mu_1^2\mu_2\mu_4\mu_5^3\mu_6^2 - 1158\mu_1^2\mu_2\mu_4\mu_5^2\mu_6^3 - 611\mu_1^2\mu_2\mu_4\mu_5\mu_6^4 - 120\mu_1^2\mu_2\mu_4\mu_6^5 - 15\mu_1^2\mu_2\mu_5^4\mu_6^2 - \\
&12\mu_1^2\mu_2\mu_5^3\mu_6^3 + 9\mu_1^2\mu_2\mu_5^2\mu_6^4 + 6\mu_1^2\mu_2\mu_5\mu_6^5 + 36\mu_1^2\mu_3^4\mu_4^3 - 20\mu_1^2\mu_3^4\mu_4^2\mu_5 + 10\mu_1^2\mu_3^4\mu_4^2\mu_6 - \\
&8\mu_1^2\mu_3^4\mu_4\mu_5^2 - 16\mu_1^2\mu_3^4\mu_4\mu_5\mu_6 - 2\mu_1^2\mu_3^4\mu_4\mu_6^2 + \mu_1^2\mu_3^4\mu_5^2\mu_6 + 224\mu_1^2\mu_3^3\mu_4^4 + 288\mu_1^2\mu_3^3\mu_4^3\mu_6 - \\
&172\mu_1^2\mu_3^3\mu_4^2\mu_5^2 - 240\mu_1^2\mu_3^3\mu_4^2\mu_5\mu_6 + 40\mu_1^2\mu_3^3\mu_4^2\mu_6^2 - 48\mu_1^2\mu_3^3\mu_4\mu_5^3 \\
&- 138\mu_1^2\mu_3^3\mu_4\mu_5^2\mu_6 - 128\mu_1^2\mu_3^3\mu_4\mu_5\mu_6^2 - 16\mu_1^2\mu_3^3\mu_4\mu_6^3 + 4\mu_1^2\mu_3^3\mu_5^3\mu_6 + 6\mu_1^2\mu_3^3\mu_5^2\mu_6^2 + 396\mu_1^2\mu_3^2\mu_4^5 \\
&+ 532\mu_1^2\mu_3^2\mu_4^4\mu_5 + 1102\mu_1^2\mu_3^2\mu_4^4\mu_6 - 676\mu_1^2\mu_3^2\mu_4^3\mu_5^2 - 184\mu_1^2\mu_3^2\mu_4^3\mu_5\mu_6 + 712\mu_1^2\mu_3^2\mu_4^3\mu_6^2 - \\
&444\mu_1^2\mu_3^2\mu_4^2\mu_5^3 - 1453\mu_1^2\mu_3^2\mu_4^2\mu_5^2\mu_6 - 1032\mu_1^2\mu_3^2\mu_4^2\mu_5\mu_6^2 + 12\mu_1^2\mu_3^2\mu_4^2\mu_6^3 - 120\mu_1^2\mu_3^2\mu_4\mu_5^4 \\
&- 402\mu_1^2\mu_3^2\mu_4\mu_5^3\mu_6 - 613\mu_1^2\mu_3^2\mu_4\mu_5^2\mu_6^2 - 384\mu_1^2\mu_3^2\mu_4\mu_5\mu_6^3 - 46\mu_1^2\mu_3^2\mu_4\mu_6^4 + \mu_1^2\mu_3^2\mu_5^4\mu_6 \\
&+ 14\mu_1^2\mu_3^2\mu_5^3\mu_6^2 + 14\mu_1^2\mu_3^2\mu_5^2\mu_6^3 + 208\mu_1^2\mu_3\mu_4^6 + 1008\mu_1^2\mu_3\mu_4^5\mu_5 + 1128\mu_1^2\mu_3\mu_4^5\mu_6 \\
&- 504\mu_1^2\mu_3\mu_4^4\mu_5^2 + 1400\mu_1^2\mu_3\mu_4^4\mu_5\mu_6 + 1590\mu_1^2\mu_3\mu_4^4\mu_6^2 - 1264\mu_1^2\mu_3\mu_4^3\mu_5^3 - 3294\mu_1^2\mu_3\mu_4^3\mu_5^2\mu_6 \\
&- 1128\mu_1^2\mu_3\mu_4^3\mu_5\mu_6^2 + 592\mu_1^2\mu_3\mu_4^3\mu_6^3 - 542\mu_1^2\mu_3\mu_4^2\mu_5^4 - 2350\mu_1^2\mu_3\mu_4^2\mu_5^3\mu_6 - 3671\mu_1^2\mu_3\mu_4^2\mu_5^2\mu_6^2 \\
&- 1872\mu_1^2\mu_3\mu_4^2\mu_5\mu_6^3 - 104\mu_1^2\mu_3\mu_4^2\mu_6^4 - 116\mu_1^2\mu_3\mu_4\mu_5^5 - 525\mu_1^2\mu_3\mu_4\mu_5^4\mu_6 - 998\mu_1^2\mu_3\mu_4\mu_5^3\mu_6^2 - \\
&1046\mu_1^2\mu_3\mu_4\mu_5^2\mu_6^3 - 504\mu_1^2\mu_3\mu_4\mu_5\mu_6^4 - 56\mu_1^2\mu_3\mu_4\mu_6^5 - 6\mu_1^2\mu_3\mu_5^5\mu_6 - 5\mu_1^2\mu_3\mu_5^4\mu_6^2 + 16\mu_1^2\mu_3\mu_5^3\mu_6^3 \\
&+ 15\mu_1^2\mu_3\mu_5^2\mu_6^4 + 332\mu_1^2\mu_4^6\mu_5 + 166\mu_1^2\mu_4^6\mu_6 + 280\mu_1^2\mu_4^5\mu_5^2 + 1264\mu_1^2\mu_4^5\mu_5\mu_6 + 600\mu_1^2\mu_4^5\mu_6^2 - \\
&1190\mu_1^2\mu_4^4\mu_5^3 - 1692\mu_1^2\mu_4^4\mu_5^2\mu_6 + 360\mu_1^2\mu_4^4\mu_5\mu_6^2 + 564\mu_1^2\mu_4^4\mu_6^3 - 554\mu_1^2\mu_4^3\mu_5^4 - 3128\mu_1^2\mu_4^3\mu_5^3\mu_6 - \\
&4254\mu_1^2\mu_4^3\mu_5^2\mu_6^2 - 1520\mu_1^2\mu_4^3\mu_5\mu_6^3 + 64\mu_1^2\mu_4^3\mu_6^4 - 322\mu_1^2\mu_4^2\mu_5^5 - 1308\mu_1^2\mu_4^2\mu_5^4\mu_6 - 2798\mu_1^2\mu_4^2\mu_5^3\mu_6^2 \\
&- 2942\mu_1^2\mu_4^2\mu_5^2\mu_6^3 - 1216\mu_1^2\mu_4^2\mu_5\mu_6^4 - 96\mu_1^2\mu_4^2\mu_6^5 - 12\mu_1^2\mu_4\mu_5^6 - 212\mu_1^2\mu_4\mu_5^5\mu_6 - 577\mu_1^2\mu_4\mu_5^4\mu_6^2 - \\
&772\mu_1^2\mu_4\mu_5^3\mu_6^3 - 611\mu_1^2\mu_4\mu_5^2\mu_6^4 - 240\mu_1^2\mu_4\mu_5\mu_6^5 - 24\mu_1^2\mu_4\mu_6^6 - 6\mu_1^2\mu_5^5\mu_6^2 - 6\mu_1^2\mu_5^4\mu_6^3 + 6\mu_1^2\mu_5^3\mu_6^4 + \\
&6\mu_1^2\mu_5^2\mu_6^5 - \mu_1\mu_2^5\mu_3^2\mu_4 - 9\mu_1\mu_2^5\mu_3\mu_4^2 - 4\mu_1\mu_2^5\mu_3\mu_4\mu_6 - 16\mu_1\mu_2^5\mu_4^3 - 3\mu_1\mu_2^5\mu_4^2\mu_5 - 18\mu_1\mu_2^5\mu_4^2\mu_6 \\
&+ \mu_1\mu_2^5\mu_4\mu_5^2 - 4\mu_1\mu_2^5\mu_4\mu_6^2 - 2\mu_1\mu_2^4\mu_3^3\mu_4 - 27\mu_1\mu_2^4\mu_3^2\mu_4^2 - 10\mu_1\mu_2^4\mu_3^2\mu_4\mu_5 - 15\mu_1\mu_2^4\mu_3^2\mu_4\mu_6
\end{aligned}
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\begin{aligned}
&- 82\mu_1\mu_2^4\mu_3\mu_4^3 - 96\mu_1\mu_2^4\mu_3\mu_4^2\mu_5 - 123\mu_1\mu_2^4\mu_3\mu_4^2\mu_6 - 2\mu_1\mu_2^4\mu_3\mu_4\mu_5^2 - 41\mu_1\mu_2^4\mu_3\mu_4\mu_5\mu_6 - \\
&35\mu_1\mu_2^4\mu_3\mu_4\mu_6^2 - 75\mu_1\mu_2^4\mu_4^4 - 166\mu_1\mu_2^4\mu_4^3\mu_5 - 196\mu_1\mu_2^4\mu_4^3\mu_6 - 24\mu_1\mu_2^4\mu_4^2\mu_5^2 - 186\mu_1\mu_2^4\mu_4^2\mu_5\mu_6 \\
&- 141\mu_1\mu_2^4\mu_4^2\mu_6^2 + 6\mu_1\mu_2^4\mu_4\mu_5^3 + 2\mu_1\mu_2^4\mu_4\mu_5^2\mu_6 - 37\mu_1\mu_2^4\mu_4\mu_5\mu_6^2 - 26\mu_1\mu_2^4\mu_4\mu_6^3 - \mu_1\mu_2^3\mu_3^4\mu_4 \\
&- 30\mu_1\mu_2^3\mu_3^3\mu_4^2 - 16\mu_1\mu_2^3\mu_3^3\mu_4\mu_5 - 16\mu_1\mu_2^3\mu_3^3\mu_4\mu_6 - 146\mu_1\mu_2^3\mu_3^2\mu_4^3 - 204\mu_1\mu_2^3\mu_3^2\mu_4^2\mu_5 \\
&- 228\mu_1\mu_2^3\mu_3^2\mu_4^2\mu_6 - 42\mu_1\mu_2^3\mu_3^2\mu_4\mu_5^2 - 116\mu_1\mu_2^3\mu_3^2\mu_4\mu_5\mu_6 - 68\mu_1\mu_2^3\mu_3^2\mu_4\mu_6^2 - 275\mu_1\mu_2^3\mu_3\mu_4^4 - \\
&572\mu_1\mu_2^3\mu_3\mu_4^3\mu_5 - 712\mu_1\mu_2^3\mu_3\mu_4^3\mu_6 - 390\mu_1\mu_2^3\mu_3\mu_4^2\mu_5^2 - 906\mu_1\mu_2^3\mu_3\mu_4^2\mu_5\mu_6 - 546\mu_1\mu_2^3\mu_3\mu_4^2\mu_6^2 \\
&- 12\mu_1\mu_2^3\mu_3\mu_4\mu_5^3 - 172\mu_1\mu_2^3\mu_3\mu_4\mu_5^2\mu_6 - 264\mu_1\mu_2^3\mu_3\mu_4\mu_5\mu_6^2 - 112\mu_1\mu_2^3\mu_3\mu_4\mu_6^3 - 192\mu_1\mu_2^3\mu_4^5 - \\
&485\mu_1\mu_2^3\mu_4^4\mu_5 - 670\mu_1\mu_2^3\mu_4^4\mu_6 - 642\mu_1\mu_2^3\mu_4^3\mu_5^2 - 1352\mu_1\mu_2^3\mu_4^3\mu_5\mu_6 - 840\mu_1\mu_2^3\mu_4^3\mu_6^2 \\
&- 78\mu_1\mu_2^3\mu_4^2\mu_5^3 - 732\mu_1\mu_2^3\mu_4^2\mu_5^2\mu_6 - 1002\mu_1\mu_2^3\mu_4^2\mu_5\mu_6^2 - 420\mu_1\mu_2^3\mu_4^2\mu_6^3 + 15\mu_1\mu_2^3\mu_4\mu_5^4 \\
&+ 8\mu_1\mu_2^3\mu_4\mu_5^3\mu_6 - 140\mu_1\mu_2^3\mu_4\mu_5^2\mu_6^2 - 184\mu_1\mu_2^3\mu_4\mu_5\mu_6^3 - 64\mu_1\mu_2^3\mu_4\mu_6^4 - 12\mu_1\mu_2^2\mu_3^4\mu_4^2 \\
&- 6\mu_1\mu_2^2\mu_3^4\mu_4\mu_5 - 5\mu_1\mu_2^2\mu_3^4\mu_4\mu_6 - 96\mu_1\mu_2^2\mu_3^3\mu_4^3 - 144\mu_1\mu_2^2\mu_3^3\mu_4^2\mu_5 - 150\mu_1\mu_2^2\mu_3^3\mu_4^2\mu_6 \\
&- 48\mu_1\mu_2^2\mu_3^3\mu_4\mu_5^2 - 86\mu_1\mu_2^2\mu_3^3\mu_4\mu_5\mu_6 - 42\mu_1\mu_2^2\mu_3^3\mu_4\mu_6^2 - 250\mu_1\mu_2^2\mu_3^2\mu_4^4 - 680\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_5 - \\
&754\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_6 - 570\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5^2 - 1131\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6 - 609\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_6^2 - 88\mu_1\mu_2^2\mu_3^2\mu_4\mu_5^3 \\
&- 339\mu_1\mu_2^2\mu_3^2\mu_4\mu_5^2\mu_6 - 359\mu_1\mu_2^2\mu_3^2\mu_4\mu_5\mu_6^2 - 126\mu_1\mu_2^2\mu_3^2\mu_4\mu_6^3 - 192\mu_1\mu_2^2\mu_3\mu_4^5 \\
&- 1280\mu_1\mu_2^2\mu_3\mu_4^4\mu_5 - 1215\mu_1\mu_2^2\mu_3\mu_4^4\mu_6 - 1500\mu_1\mu_2^2\mu_3\mu_4^3\mu_5^2 - 3414\mu_1\mu_2^2\mu_3\mu_4^3\mu_5\mu_6 \\
&- 1862\mu_1\mu_2^2\mu_3\mu_4^3\mu_6^2 - 780\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^3 - 2505\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6 - 2721\mu_1\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^2 \\
&- 1008\mu_1\mu_2^2\mu_3\mu_4^2\mu_6^3 - 26\mu_1\mu_2^2\mu_3\mu_4\mu_5^4 - 358\mu_1\mu_2^2\mu_3\mu_4\mu_5^3\mu_6 - 757\mu_1\mu_2^2\mu_3\mu_4\mu_5^2\mu_6^2 \\
&- 582\mu_1\mu_2^2\mu_3\mu_4\mu_5\mu_6^3 - 161\mu_1\mu_2^2\mu_3\mu_4\mu_6^4 - 900\mu_1\mu_2^2\mu_4^5\mu_5 - 576\mu_1\mu_2^2\mu_4^5\mu_6 - 1185\mu_1\mu_2^2\mu_4^4\mu_5^2 - \\
&3150\mu_1\mu_2^2\mu_4^4\mu_5\mu_6 - 1555\mu_1\mu_2^2\mu_4^4\mu_6^2 - 1228\mu_1\mu_2^2\mu_4^3\mu_5^3 - 3528\mu_1\mu_2^2\mu_4^3\mu_5^2\mu_6 - 4030\mu_1\mu_2^2\mu_4^3\mu_5\mu_6^2 \\
&- 1492\mu_1\mu_2^2\mu_4^3\mu_6^3 - 129\mu_1\mu_2^2\mu_4^2\mu_5^4 - 1428\mu_1\mu_2^2\mu_4^2\mu_5^3\mu_6 - 2709\mu_1\mu_2^2\mu_4^2\mu_5^2\mu_6^2 - 2070\mu_1\mu_2^2\mu_4^2\mu_5\mu_6^3 \\
&- 585\mu_1\mu_2^2\mu_4^2\mu_6^4 + 20\mu_1\mu_2^2\mu_4\mu_5^5 + 14\mu_1\mu_2^2\mu_4\mu_5^4\mu_6 - 270\mu_1\mu_2^2\mu_4\mu_5^3\mu_6^2 - 506\mu_1\mu_2^2\mu_4\mu_5^2\mu_6^3 - \\
&325\mu_1\mu_2^2\mu_4\mu_5\mu_6^4 - 74\mu_1\mu_2^2\mu_4\mu_6^5 - 16\mu_1\mu_2\mu_3^4\mu_4^3 - 30\mu_1\mu_2\mu_3^4\mu_4^2\mu_5 - 24\mu_1\mu_2\mu_3^4\mu_4^2\mu_6 \\
&- 12\mu_1\mu_2\mu_3^4\mu_4\mu_5^2 - 15\mu_1\mu_2\mu_3^4\mu_4\mu_5\mu_6 - 6\mu_1\mu_2\mu_3^4\mu_4\mu_6^2 - 288\mu_1\mu_2\mu_3^3\mu_4^3\mu_5 - 192\mu_1\mu_2\mu_3^3\mu_4^3\mu_6 - \\
&252\mu_1\mu_2\mu_3^3\mu_4^2\mu_5^2 - 462\mu_1\mu_2\mu_3^3\mu_4^2\mu_5\mu_6 - 192\mu_1\mu_2\mu_3^3\mu_4^2\mu_6^2 - 64\mu_1\mu_2\mu_3^3\mu_4\mu_5^3 - 162\mu_1\mu_2\mu_3^3\mu_4\mu_5^2\mu_6 \\
&- 138\mu_1\mu_2\mu_3^3\mu_4\mu_5\mu_6^2 - 40\mu_1\mu_2\mu_3^3\mu_4\mu_6^3 + 216\mu_1\mu_2\mu_3^2\mu_4^5 - 760\mu_1\mu_2\mu_3^2\mu_4^4\mu_5 - 120\mu_1\mu_2\mu_3^2\mu_4^4\mu_6 \\
&- 1164\mu_1\mu_2\mu_3^2\mu_4^3\mu_5^2 - 2394\mu_1\mu_2\mu_3^2\mu_4^3\mu_5\mu_6 - 832\mu_1\mu_2\mu_3^2\mu_4^3\mu_6^2 - 732\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^3 \\
&- 2025\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6 - 1983\mu_1\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^2 - 576\mu_1\mu_2\mu_3^2\mu_4^2\mu_6^3 - 90\mu_1\mu_2\mu_3^2\mu_4\mu_5^4 \\
&- 446\mu_1\mu_2\mu_3^2\mu_4\mu_5^3\mu_6 - 669\mu_1\mu_2\mu_3^2\mu_4\mu_5^2\mu_6^2 - 426\mu_1\mu_2\mu_3^2\mu_4\mu_5\mu_6^3 - 98\mu_1\mu_2\mu_3^2\mu_4\mu_6^4 \\
&+ 336\mu_1\mu_2\mu_3\mu_4^6 - 480\mu_1\mu_2\mu_3\mu_4^5\mu_5 + 528\mu_1\mu_2\mu_3\mu_4^5\mu_6 - 2190\mu_1\mu_2\mu_3\mu_4^4\mu_5^2 \\
&- 3765\mu_1\mu_2\mu_3\mu_4^4\mu_5\mu_6 - 700\mu_1\mu_2\mu_3\mu_4^4\mu_6^2 - 1856\mu_1\mu_2\mu_3\mu_4^3\mu_5^3 - 5970\mu_1\mu_2\mu_3\mu_4^3\mu_5^2\mu_6 \\
&- 6030\mu_1\mu_2\mu_3\mu_4^3\mu_5\mu_6^2 - 1520\mu_1\mu_2\mu_3\mu_4^3\mu_6^3 - 780\mu_1\mu_2\mu_3\mu_4^2\mu_5^4 - 3198\mu_1\mu_2\mu_3\mu_4^2\mu_5^3\mu_6 \\
&- 4887\mu_1\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^2 - 3330\mu_1\mu_2\mu_3\mu_4^2\mu_5\mu_6^3 - 756\mu_1\mu_2\mu_3\mu_4^2\mu_6^4 - 24\mu_1\mu_2\mu_3\mu_4\mu_5^5 \\
&- 365\mu_1\mu_2\mu_3\mu_4\mu_5^4\mu_6 - 986\mu_1\mu_2\mu_3\mu_4\mu_5^3\mu_6^2 - 1074\mu_1\mu_2\mu_3\mu_4\mu_5^2\mu_6^3 - 545\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6^4 \\
&- 104\mu_1\mu_2\mu_3\mu_4\mu_6^5 + 96\mu_1\mu_2\mu_4^7 + 126\mu_1\mu_2\mu_4^6\mu_5 + 392\mu_1\mu_2\mu_4^6\mu_6 - 1548\mu_1\mu_2\mu_4^5\mu_5^2 \\
&- 1620\mu_1\mu_2\mu_4^5\mu_5\mu_6 + 48\mu_1\mu_2\mu_4^5\mu_6^2 - 1400\mu_1\mu_2\mu_4^4\mu_5^3 - 5430\mu_1\mu_2\mu_4^4\mu_5^2\mu_6 - 4905\mu_1\mu_2\mu_4^4\mu_5\mu_6^2 \\
&- 920\mu_1\mu_2\mu_4^4\mu_6^3 - 1200\mu_1\mu_2\mu_4^3\mu_5^4 - 4352\mu_1\mu_2\mu_4^3\mu_5^3\mu_6 - 7050\mu_1\mu_2\mu_4^3\mu_5^2\mu_6^2 - 4844\mu_1\mu_2\mu_4^3\mu_5\mu_6^3 \\
&- 992\mu_1\mu_2\mu_4^3\mu_6^4 - 108\mu_1\mu_2\mu_4^2\mu_5^5 - 1410\mu_1\mu_2\mu_4^2\mu_5^4\mu_6 - 3414\mu_1\mu_2\mu_4^2\mu_5^3\mu_6^2 - 3690\mu_1\mu_2\mu_4^2\mu_5^2\mu_6^3 \\
&- 1929\mu_1\mu_2\mu_4^2\mu_5\mu_6^4 - 360\mu_1\mu_2\mu_4^2\mu_6^5 + 14\mu_1\mu_2\mu_4\mu_5^6 + 12\mu_1\mu_2\mu_4\mu_5^5\mu_6 - 265\mu_1\mu_2\mu_4\mu_5^4\mu_6^2 - \\
&644\mu_1\mu_2\mu_4\mu_5^3\mu_6^3 - 591\mu_1\mu_2\mu_4\mu_5^2\mu_6^4 - 250\mu_1\mu_2\mu_4\mu_5\mu_6^5 - 40\mu_1\mu_2\mu_4\mu_6^6 + 40\mu_1\mu_3^4\mu_4^4 \\
&- 32\mu_1\mu_3^4\mu_4^3\mu_5 + 16\mu_1\mu_3^4\mu_4^3\mu_6 - 30\mu_1\mu_3^4\mu_4^2\mu_5^2 - 48\mu_1\mu_3^4\mu_4^2\mu_5\mu_6 - 6\mu_1\mu_3^4\mu_4^2\mu_6^2 - 8\mu_1\mu_3^4\mu_4\mu_5^3 - \\
&15\mu_1\mu_3^4\mu_4\mu_5^2\mu_6 - 12\mu_1\mu_3^4\mu_4\mu_5\mu_6^2 - 2\mu_1\mu_3^4\mu_4\mu_6^3 + 192\mu_1\mu_3^3\mu_4^5 + 320\mu_1\mu_3^3\mu_4^4\mu_6 - 288\mu_1\mu_3^3\mu_4^3\mu_5^2 \\
&- 384\mu_1\mu_3^3\mu_4^3\mu_5\mu_6 + 64\mu_1\mu_3^3\mu_4^3\mu_6^2 - 168\mu_1\mu_3^3\mu_4^2\mu_5^3 - 462\mu_1\mu_3^3\mu_4^2\mu_5^2\mu_6 - 384\mu_1\mu_3^3\mu_4^2\mu_5\mu_6^2 - \\
&48\mu_1\mu_3^3\mu_4^2\mu_6^3 - 32\mu_1\mu_3^3\mu_4\mu_5^4 - 108\mu_1\mu_3^3\mu_4\mu_5^3\mu_6 - 138\mu_1\mu_3^3\mu_4\mu_5^2\mu_6^2 - 80\mu_1\mu_3^3\mu_4\mu_5\mu_6^3
\end{aligned}
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\begin{aligned}
&- 12\mu_1\mu_3^3\mu_4\mu_6^4 + 280\mu_1\mu_3^2\mu_4^6 + 432\mu_1\mu_3^2\mu_4^5\mu_5 + 936\mu_1\mu_3^2\mu_4^5\mu_6 - 760\mu_1\mu_3^2\mu_4^4\mu_5^2 \\
&- 240\mu_1\mu_3^2\mu_4^4\mu_5\mu_6 + 780\mu_1\mu_3^2\mu_4^4\mu_6^2 - 776\mu_1\mu_3^2\mu_4^3\mu_5^3 - 2394\mu_1\mu_3^2\mu_4^3\mu_5^2\mu_6 - 1664\mu_1\mu_3^2\mu_4^3\mu_5\mu_6^2 \\
&+ 16\mu_1\mu_3^2\mu_4^3\mu_6^3 - 366\mu_1\mu_3^2\mu_4^2\mu_5^4 - 1350\mu_1\mu_3^2\mu_4^2\mu_5^3\mu_6 - 1983\mu_1\mu_3^2\mu_4^2\mu_5^2\mu_6^2 - 1152\mu_1\mu_3^2\mu_4^2\mu_5\mu_6^3 \\
&- 138\mu_1\mu_3^2\mu_4^2\mu_6^4 - 36\mu_1\mu_3^2\mu_4\mu_5^5 - 223\mu_1\mu_3^2\mu_4\mu_5^4\mu_6 - 446\mu_1\mu_3^2\mu_4\mu_5^3\mu_6^2 - 426\mu_1\mu_3^2\mu_4\mu_5^2\mu_6^3 - \\
&196\mu_1\mu_3^2\mu_4\mu_5\mu_6^4 - 26\mu_1\mu_3^2\mu_4\mu_6^5 + 128\mu_1\mu_3\mu_4^7 + 672\mu_1\mu_3\mu_4^6\mu_5 + 784\mu_1\mu_3\mu_4^6\mu_6 - 480\mu_1\mu_3\mu_4^5\mu_5^2 \\
&+ 1056\mu_1\mu_3\mu_4^5\mu_5\mu_6 + 1320\mu_1\mu_3\mu_4^5\mu_6^2 - 1460\mu_1\mu_3\mu_4^4\mu_5^3 - 3765\mu_1\mu_3\mu_4^4\mu_5^2\mu_6 - 1400\mu_1\mu_3\mu_4^4\mu_5\mu_6^2 \\
&+ 620\mu_1\mu_3\mu_4^4\mu_6^3 - 928\mu_1\mu_3\mu_4^3\mu_5^4 - 3980\mu_1\mu_3\mu_4^3\mu_5^3\mu_6 - 6030\mu_1\mu_3\mu_4^3\mu_5^2\mu_6^2 - 3040\mu_1\mu_3\mu_4^3\mu_5\mu_6^3 \\
&- 176\mu_1\mu_3\mu_4^3\mu_6^4 - 312\mu_1\mu_3\mu_4^2\mu_5^5 - 1599\mu_1\mu_3\mu_4^2\mu_5^4\mu_6 - 3258\mu_1\mu_3\mu_4^2\mu_5^3\mu_6^2 - 3330\mu_1\mu_3\mu_4^2\mu_5^2\mu_6^3 \\
&- 1512\mu_1\mu_3\mu_4^2\mu_5\mu_6^4 - 168\mu_1\mu_3\mu_4^2\mu_6^5 - 8\mu_1\mu_3\mu_4\mu_5^6 - 146\mu_1\mu_3\mu_4\mu_5^5\mu_6 - 493\mu_1\mu_3\mu_4\mu_5^4\mu_6^2 - \\
&716\mu_1\mu_3\mu_4\mu_5^3\mu_6^3 - 545\mu_1\mu_3\mu_4\mu_5^2\mu_6^4 - 208\mu_1\mu_3\mu_4\mu_5\mu_6^5 - 24\mu_1\mu_3\mu_4\mu_6^6 + 192\mu_1\mu_4^7\mu_5 + 96\mu_1\mu_4^7\mu_6 \\
&+ 126\mu_1\mu_4^6\mu_5^2 + 784\mu_1\mu_4^6\mu_5\mu_6 + 392\mu_1\mu_4^6\mu_6^2 - 1032\mu_1\mu_4^5\mu_5^3 - 1620\mu_1\mu_4^5\mu_5^2\mu_6 + 96\mu_1\mu_4^5\mu_5\mu_6^2 \\
&+ 432\mu_1\mu_4^5\mu_6^3 - 700\mu_1\mu_4^4\mu_5^4 - 3620\mu_1\mu_4^4\mu_5^3\mu_6 - 4905\mu_1\mu_4^4\mu_5^2\mu_6^2 - 1840\mu_1\mu_4^4\mu_5\mu_6^3 + 40\mu_1\mu_4^4\mu_6^4 - \\
&480\mu_1\mu_4^3\mu_5^5 - 2176\mu_1\mu_4^3\mu_5^4\mu_6 - 4700\mu_1\mu_4^3\mu_5^3\mu_6^2 - 4844\mu_1\mu_4^3\mu_5^2\mu_6^3 - 1984\mu_1\mu_4^3\mu_5\mu_6^4 - 160\mu_1\mu_4^3\mu_6^5 \\
&- 36\mu_1\mu_4^2\mu_5^6 - 564\mu_1\mu_4^2\mu_5^5\mu_6 - 1707\mu_1\mu_4^2\mu_5^4\mu_6^2 - 2460\mu_1\mu_4^2\mu_5^3\mu_6^3 - 1929\mu_1\mu_4^2\mu_5^2\mu_6^4 - 720\mu_1\mu_4^2\mu_5\mu_6^5 \\
&- 72\mu_1\mu_4^2\mu_6^6 + 4\mu_1\mu_4\mu_5^7 + 4\mu_1\mu_4\mu_5^6\mu_6 - 106\mu_1\mu_4\mu_5^5\mu_6^2 - 322\mu_1\mu_4\mu_5^4\mu_6^3 - 394\mu_1\mu_4\mu_5^3\mu_6^4 - \\
&250\mu_1\mu_4\mu_5^2\mu_6^5 - 80\mu_1\mu_4\mu_5\mu_6^6 - 8\mu_1\mu_4\mu_6^7 - \mu_2^5\mu_3^2\mu_4^2 - 6\mu_2^5\mu_3\mu_4^3 - 4\mu_2^5\mu_3\mu_4^2\mu_6 - 8\mu_2^5\mu_4^4 - 2\mu_2^5\mu_4^3\mu_5 \\
&- 12\mu_2^5\mu_4^3\mu_6 + \mu_2^5\mu_4^2\mu_5^2 - 4\mu_2^5\mu_4^2\mu_6^2 - 2\mu_2^4\mu_3^3\mu_4^2 - 18\mu_2^4\mu_3^2\mu_4^3 - 10\mu_2^4\mu_3^2\mu_4^2\mu_5 - 15\mu_2^4\mu_3^2\mu_4^2\mu_6 \\
&- 41\mu_2^4\mu_3\mu_4^4 - 64\mu_2^4\mu_3\mu_4^3\mu_5 - 82\mu_2^4\mu_3\mu_4^3\mu_6 - 2\mu_2^4\mu_3\mu_4^2\mu_5^2 - 41\mu_2^4\mu_3\mu_4^2\mu_5\mu_6 - 35\mu_2^4\mu_3\mu_4^2\mu_6^2 - \\
&30\mu_2^4\mu_4^5 - 83\mu_2^4\mu_4^4\mu_5 - 98\mu_2^4\mu_4^4\mu_6 - 16\mu_2^4\mu_4^3\mu_5^2 - 124\mu_2^4\mu_4^3\mu_5\mu_6 - 94\mu_2^4\mu_4^3\mu_6^2 + 6\mu_2^4\mu_4^2\mu_5^3 + \\
&2\mu_2^4\mu_4^2\mu_5^2\mu_6 - 37\mu_2^4\mu_4^2\mu_5\mu_6^2 - 26\mu_2^4\mu_4^2\mu_6^3 - \mu_2^3\mu_3^4\mu_4^2 - 20\mu_2^3\mu_3^3\mu_4^3 - 16\mu_2^3\mu_3^3\mu_4^2\mu_5 - 16\mu_2^3\mu_3^3\mu_4^2\mu_6 - \\
&73\mu_2^3\mu_3^2\mu_4^4 - 136\mu_2^3\mu_3^2\mu_4^3\mu_5 - 152\mu_2^3\mu_3^2\mu_4^3\mu_6 - 42\mu_2^3\mu_3^2\mu_4^2\mu_5^2 - 116\mu_2^3\mu_3^2\mu_4^2\mu_5\mu_6 - 68\mu_2^3\mu_3^2\mu_4^2\mu_6^2 - \\
&110\mu_2^3\mu_3\mu_4^5 - 286\mu_2^3\mu_3\mu_4^4\mu_5 - 356\mu_2^3\mu_3\mu_4^4\mu_6 - 260\mu_2^3\mu_3\mu_4^3\mu_5^2 - 604\mu_2^3\mu_3\mu_4^3\mu_5\mu_6 - 364\mu_2^3\mu_3\mu_4^3\mu_6^2 \\
&- 12\mu_2^3\mu_3\mu_4^2\mu_5^3 - 172\mu_2^3\mu_3\mu_4^2\mu_5^2\mu_6 - 264\mu_2^3\mu_3\mu_4^2\mu_5\mu_6^2 - 112\mu_2^3\mu_3\mu_4^2\mu_6^3 - 64\mu_2^3\mu_4^6 - 194\mu_2^3\mu_4^5\mu_5 - \\
&268\mu_2^3\mu_4^5\mu_6 - 321\mu_2^3\mu_4^4\mu_5^2 - 676\mu_2^3\mu_4^4\mu_5\mu_6 - 420\mu_2^3\mu_4^4\mu_6^2 - 52\mu_2^3\mu_4^3\mu_5^3 - 488\mu_2^3\mu_4^3\mu_5^2\mu_6 \\
&- 668\mu_2^3\mu_4^3\mu_5\mu_6^2 - 280\mu_2^3\mu_4^3\mu_6^3 + 15\mu_2^2\mu_4^2\mu_5^4 + 8\mu_2^3\mu_4^2\mu_5^3\mu_6 - 140\mu_2^3\mu_4^2\mu_5^2\mu_6^2 - 184\mu_2^3\mu_4^2\mu_5\mu_6^3 - \\
&64\mu_2^3\mu_4^2\mu_6^4 - 8\mu_2^2\mu_3^4\mu_4^3 - 6\mu_2^2\mu_3^4\mu_4^2\mu_5 - 5\mu_2^2\mu_3^4\mu_4^2\mu_6 - 48\mu_2^2\mu_3^3\mu_4^4 - 96\mu_2^2\mu_3^3\mu_4^3\mu_5 - 100\mu_2^2\mu_3^3\mu_4^3\mu_6 \\
&- 48\mu_2^2\mu_3^3\mu_4^2\mu_5^2 - 86\mu_2^2\mu_3^3\mu_4^2\mu_5\mu_6 - 42\mu_2^2\mu_3^3\mu_4^2\mu_6^2 - 100\mu_2^2\mu_3^2\mu_4^5 - 340\mu_2^2\mu_3^2\mu_4^4\mu_5 - 377\mu_2^2\mu_3^2\mu_4^4\mu_6 \\
&- 380\mu_2^2\mu_3^2\mu_4^3\mu_5^2 - 754\mu_2^2\mu_3^2\mu_4^3\mu_5\mu_6 - 406\mu_2^2\mu_3^2\mu_4^3\mu_6^2 - 88\mu_2^2\mu_3^2\mu_4^2\mu_5^3 - 339\mu_2^2\mu_3^2\mu_4^2\mu_5^2\mu_6 \\
&- 359\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6^2 - 126\mu_2^2\mu_3^2\mu_4^2\mu_6^3 - 64\mu_2^2\mu_3\mu_4^6 - 512\mu_2^2\mu_3\mu_4^5\mu_5 - 486\mu_2^2\mu_3\mu_4^5\mu_6 - 750\mu_2^2\mu_3\mu_4^4\mu_5^2 \\
&- 1707\mu_2^2\mu_3\mu_4^4\mu_5\mu_6 - 931\mu_2^2\mu_3\mu_4^4\mu_6^2 - 520\mu_2^2\mu_3\mu_4^3\mu_5^3 - 1670\mu_2^2\mu_3\mu_4^3\mu_5^2\mu_6 - 1814\mu_2^2\mu_3\mu_4^3\mu_5\mu_6^2 \\
&- 672\mu_2^2\mu_3\mu_4^3\mu_6^3 - 26\mu_2^2\mu_3\mu_4^2\mu_5^4 - 358\mu_2^2\mu_3\mu_4^2\mu_5^3\mu_6 - 757\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6^2 - 582\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^3 - \\
&161\mu_2^2\mu_3\mu_4^2\mu_6^4 - 300\mu_2^2\mu_4^6\mu_5 - 192\mu_2^2\mu_4^6\mu_6 - 474\mu_2^2\mu_4^5\mu_5^2 - 1260\mu_2^2\mu_4^5\mu_5\mu_6 - 622\mu_2^2\mu_4^5\mu_6^2 - \\
&614\mu_2^2\mu_4^4\mu_5^3 - 1764\mu_2^2\mu_4^4\mu_5^2\mu_6 - 2015\mu_2^2\mu_4^4\mu_5\mu_6^2 - 746\mu_2^2\mu_4^4\mu_6^3 - 86\mu_2^2\mu_4^3\mu_5^4 - 952\mu_2^2\mu_4^3\mu_5^3\mu_6 - \\
&1806\mu_2^2\mu_4^3\mu_5^2\mu_6^2 - 1380\mu_2^2\mu_4^3\mu_5\mu_6^3 - 390\mu_2^2\mu_4^3\mu_6^4 + 20\mu_2^2\mu_4^2\mu_5^5 + 14\mu_2^2\mu_4^2\mu_5^4\mu_6 - 270\mu_2^2\mu_4^2\mu_5^3\mu_6^2 - \\
&506\mu_2^2\mu_4^2\mu_5^2\mu_6^3 - 325\mu_2^2\mu_4^2\mu_5\mu_6^4 - 74\mu_2^2\mu_4^2\mu_6^5 - 8\mu_2\mu_3^4\mu_4^4 - 20\mu_2\mu_3^4\mu_4^3\mu_5 - 16\mu_2\mu_3^4\mu_4^3\mu_6 \\
&- 12\mu_2\mu_3^4\mu_4^2\mu_5^2 - 15\mu_2\mu_3^4\mu_4^2\mu_5\mu_6 - 6\mu_2\mu_3^4\mu_4^2\mu_6^2 - 144\mu_2\mu_3^3\mu_4^4\mu_5 - 96\mu_2\mu_3^3\mu_4^4\mu_6 - 168\mu_2\mu_3^3\mu_4^3\mu_5^2 \\
&- 308\mu_2\mu_3^3\mu_4^3\mu_5\mu_6 - 128\mu_2\mu_3^3\mu_4^3\mu_6^2 - 64\mu_2\mu_3^3\mu_4^2\mu_5^3 - 162\mu_2\mu_3^3\mu_4^2\mu_5^2\mu_6 - 138\mu_2\mu_3^3\mu_4^2\mu_5\mu_6^2 - \\
&40\mu_2\mu_3^3\mu_4^2\mu_6^3 + 72\mu_2\mu_3^2\mu_4^6 - 304\mu_2\mu_3^2\mu_4^5\mu_5 - 48\mu_2\mu_3^2\mu_4^5\mu_6 - 582\mu_2\mu_3^2\mu_4^4\mu_5^2 - 1197\mu_2\mu_3^2\mu_4^4\mu_5\mu_6 \\
&- 416\mu_2\mu_3^2\mu_4^4\mu_6^2 - 488\mu_2\mu_3^2\mu_4^3\mu_5^3 - 1350\mu_2\mu_3^2\mu_4^3\mu_5^2\mu_6 - 1322\mu_2\mu_3^2\mu_4^3\mu_5\mu_6^2 - 384\mu_2\mu_3^2\mu_4^3\mu_6^3 - \\
&90\mu_2\mu_3^2\mu_4^2\mu_5^4 - 446\mu_2\mu_3^2\mu_4^2\mu_5^3\mu_6 - 669\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6^2 - 426\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^3 - 98\mu_2\mu_3^2\mu_4^2\mu_6^4 \\
&+ 96\mu_2\mu_3\mu_4^7 - 160\mu_2\mu_3\mu_4^6\mu_5 + 176\mu_2\mu_3\mu_4^6\mu_6 - 876\mu_2\mu_3\mu_4^5\mu_5^2 - 1506\mu_2\mu_3\mu_4^5\mu_5\mu_6 \\
&- 280\mu_2\mu_3\mu_4^5\mu_6^2 - 928\mu_2\mu_3\mu_4^4\mu_5^3 - 2985\mu_2\mu_3\mu_4^4\mu_5^2\mu_6 - 3015\mu_2\mu_3\mu_4^4\mu_5\mu_6^2 - 760\mu_2\mu_3\mu_4^4\mu_6^3 - \\
&520\mu_2\mu_3\mu_4^3\mu_5^4 - 2132\mu_2\mu_3\mu_4^3\mu_5^3\mu_6 - 3258\mu_2\mu_3\mu_4^3\mu_5^2\mu_6^2 - 2220\mu_2\mu_3\mu_4^3\mu_5\mu_6^3 - 504\mu_2\mu_3\mu_4^3\mu_6^4 -
\end{aligned}
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$24\mu_2\mu_3\mu_4^2\mu_5^5$ - $365\mu_2\mu_3\mu_4^2\mu_5^4\mu_6$ - $986\mu_2\mu_3\mu_4^2\mu_5^3\mu_6^2$ - $1074\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^3$ - $545\mu_2\mu_3\mu_4^2\mu_5\mu_6^4$ - $104\mu_2\mu_3\mu_4^2\mu_6^5 + 24\mu_2\mu_4^8 + 36\mu_2\mu_4^7\mu_5 + 112\mu_2\mu_4^7\mu_6$ - $516\mu_2\mu_4^6\mu_5^2$
- $540\mu_2\mu_4^6\mu_5\mu_6 + 16\mu_2\mu_4^6\mu_6^2$ - $560\mu_2\mu_4^5\mu_5^3$ - $2172\mu_2\mu_4^5\mu_5^2\mu_6$ - $1962\mu_2\mu_4^5\mu_5\mu_6^2$ - $368\mu_2\mu_4^5\mu_6^3$ - $600\mu_2\mu_4^4\mu_5^4$ - $2176\mu_2\mu_4^4\mu_5^3\mu_6$ - $3525\mu_2\mu_4^4\mu_5^2\mu_6^2$ - $2422\mu_2\mu_4^4\mu_5\mu_6^3$ - $496\mu_2\mu_4^4\mu_6^4$ - $72\mu_2\mu_4^3\mu_5^5$ - $940\mu_2\mu_4^3\mu_5^4\mu_6$ - $2276\mu_2\mu_4^3\mu_5^3\mu_6^2$ - $2460\mu_2\mu_4^3\mu_5^2\mu_6^3$ - $1286\mu_2\mu_4^3\mu_5\mu_6^4$ - $240\mu_2\mu_4^3\mu_6^5 + 14\mu_2\mu_4^2\mu_5^6$
+ $12\mu_2\mu_4^2\mu_5^5\mu_6$ - $265\mu_2\mu_4^2\mu_5^4\mu_6^2$ - $644\mu_2\mu_4^2\mu_5^3\mu_6^3$ - $591\mu_2\mu_4^2\mu_5^2\mu_6^4$ - $250\mu_2\mu_4^2\mu_5\mu_6^5$ - $40\mu_2\mu_4^2\mu_6^6$
+ $16\mu_3^4\mu_5^5$ - $16\mu_3^4\mu_4^4\mu_5 + 8\mu_3^4\mu_4^4\mu_6$ - $20\mu_3^4\mu_4^3\mu_5^2$ - $32\mu_3^4\mu_4^3\mu_5\mu_6$ - $4\mu_3^4\mu_4^3\mu_6^2$ - $8\mu_3^4\mu_4^2\mu_5^3$ - $15\mu_3^4\mu_4^2\mu_5^2\mu_6$ - $12\mu_3^4\mu_4^2\mu_5\mu_6^2$ - $2\mu_3^4\mu_4^2\mu_6^3 + 64\mu_3^3\mu_4^6$ + $128\mu_3^3\mu_4^5\mu_6$ - $144\mu_3^3\mu_4^4\mu_5^2$ - $192\mu_3^3\mu_4^4\mu_5\mu_6$
+ $32\mu_3^3\mu_4^4\mu_6^2$ - $112\mu_3^3\mu_4^3\mu_5^3$ - $308\mu_3^3\mu_4^3\mu_5^2\mu_6$ - $256\mu_3^3\mu_4^3\mu_5\mu_6^2$ - $32\mu_3^3\mu_4^3\mu_6^3$ - $32\mu_3^3\mu_4^2\mu_5^4$
- $108\mu_3^3\mu_4^2\mu_5^3\mu_6$ - $138\mu_3^3\mu_4^2\mu_5^2\mu_6^2$ - $80\mu_3^3\mu_4^2\mu_5\mu_6^3$ - $12\mu_3^3\mu_4^2\mu_6^4 + 80\mu_3^2\mu_4^7$ + $144\mu_3^2\mu_4^6\mu_5$ +
$312\mu_3^2\mu_4^6\mu_6$ - $304\mu_3^2\mu_4^5\mu_5^2$ - $96\mu_3^2\mu_4^5\mu_5\mu_6 + 312\mu_3^2\mu_4^5\mu_6^2$ - $388\mu_3^2\mu_4^4\mu_5^3$ - $1197\mu_3^2\mu_4^4\mu_5^2\mu_6$
- $832\mu_3^2\mu_4^4\mu_5\mu_6^2 + 8\mu_3^2\mu_4^4\mu_6^3$ - $244\mu_3^2\mu_4^3\mu_5^4$ - $900\mu_3^2\mu_4^3\mu_5^3\mu_6$ - $1322\mu_3^2\mu_4^3\mu_5^2\mu_6^2$ - $768\mu_3^2\mu_4^3\mu_5\mu_6^3$
- $92\mu_3^2\mu_4^3\mu_6^4$ - $36\mu_3^2\mu_4^2\mu_5^5$ - $223\mu_3^2\mu_4^2\mu_5^4\mu_6$ - $446\mu_3^2\mu_4^2\mu_5^3\mu_6^2$ - $426\mu_3^2\mu_4^2\mu_5^2\mu_6^3$ - $196\mu_3^2\mu_4^2\mu_5\mu_6^4$ -
$26\mu_3^2\mu_4^2\mu_6^5 + 32\mu_3\mu_4^8 + 192\mu_3\mu_4^7\mu_5 + 224\mu_3\mu_4^7\mu_6$ - $160\mu_3\mu_4^6\mu_5^2 + 352\mu_3\mu_4^6\mu_5\mu_6 + 440\mu_3\mu_4^6\mu_6^2$
- $584\mu_3\mu_4^5\mu_5^3$ - $1506\mu_3\mu_4^5\mu_5^2\mu_6$ - $560\mu_3\mu_4^5\mu_5\mu_6^2 + 248\mu_3\mu_4^5\mu_6^3$ - $464\mu_3\mu_4^4\mu_5^4$ - $1990\mu_3\mu_4^4\mu_5^3\mu_6$ -
$3015\mu_3\mu_4^4\mu_5^2\mu_6^2$ - $1520\mu_3\mu_4^4\mu_5\mu_6^3$ - $88\mu_3\mu_4^4\mu_6^4$ - $208\mu_3\mu_4^3\mu_5^5$ - $1066\mu_3\mu_4^3\mu_5^4\mu_6$ - $2172\mu_3\mu_4^3\mu_5^3\mu_6^2$
- $2220\mu_3\mu_4^3\mu_5^2\mu_6^3$ - $1008\mu_3\mu_4^3\mu_5\mu_6^4$ - $112\mu_3\mu_4^3\mu_6^5$ - $8\mu_3\mu_4^2\mu_5^6$ - $146\mu_3\mu_4^2\mu_5^5\mu_6$ - $493\mu_3\mu_4^2\mu_5^4\mu_6^2$ -
$716\mu_3\mu_4^2\mu_5^3\mu_6^3$ - $545\mu_3\mu_4^2\mu_5^2\mu_6^4$ - $208\mu_3\mu_4^2\mu_5\mu_6^5$ - $24\mu_3\mu_4^2\mu_6^6 + 48\mu_4^8\mu_5 + 24\mu_4^8\mu_6 + 36\mu_4^7\mu_5^2$
+ $224\mu_4^7\mu_5\mu_6 + 112\mu_4^7\mu_6^2$ - $344\mu_4^6\mu_5^3$ - $540\mu_4^6\mu_5^2\mu_6 + 32\mu_4^6\mu_5\mu_6^2 + 144\mu_4^6\mu_6^3$ - $280\mu_4^5\mu_5^4$ -
$1448\mu_4^5\mu_5^3\mu_6$ - $1962\mu_4^5\mu_5^2\mu_6^2$ - $736\mu_4^5\mu_5\mu_6^3 + 16\mu_4^5\mu_6^4$ - $240\mu_4^4\mu_5^5$ - $1088\mu_4^4\mu_5^4\mu_6$ - $2350\mu_4^4\mu_5^3\mu_6^2$
- $2422\mu_4^4\mu_5^2\mu_6^3$ - $992\mu_4^4\mu_5\mu_6^4$ - $80\mu_4^4\mu_6^5$ - $24\mu_4^3\mu_5^6$ - $376\mu_4^3\mu_5^5\mu_6$ - $1138\mu_4^3\mu_5^4\mu_6^2$ - $1640\mu_4^3\mu_5^3\mu_6^3$
- $1286\mu_4^3\mu_5^2\mu_6^4$ - $480\mu_4^3\mu_5\mu_6^5$ - $48\mu_4^3\mu_6^6 + 4\mu_4^2\mu_5^7 + 4\mu_4^2\mu_5^6\mu_6$ - $106\mu_4^2\mu_5^5\mu_6^2$ - $322\mu_4^2\mu_5^4\mu_6^3$ -
$394\mu_4^2\mu_5^3\mu_6^4$ - $250\mu_4^2\mu_5^2\mu_6^5$ - $80\mu_4^2\mu_5\mu_6^6$ - $8\mu_4^2\mu_6^7$.

$a_Y = -2\mu_1^6\mu_2^2$ - $3\mu_1^6\mu_2\mu_3$ - $\mu_1^6\mu_2\mu_4$ - $8\mu_1^6\mu_2\mu_5$ - $6\mu_1^6\mu_2\mu_6$ - $\mu_1^6\mu_3^2$ - $6\mu_1^6\mu_3\mu_5$ - $4\mu_1^6\mu_3\mu_6$ +
$\mu_1^6\mu_4^2$ - $2\mu_1^6\mu_4\mu_5$ - $8\mu_1^6\mu_5^2$ - $12\mu_1^6\mu_5\mu_6$ - $4\mu_1^6\mu_6^2$ - $2\mu_1^5\mu_2^3$ - $6\mu_1^5\mu_2^2\mu_3$ - $18\mu_1^5\mu_2^2\mu_4$ - $10\mu_1^5\mu_2^2\mu_5$ -
$14\mu_1^5\mu_2^2\mu_6$ - $6\mu_1^5\mu_2\mu_3^2$ - $30\mu_1^5\mu_2\mu_3\mu_4$ - $22\mu_1^5\mu_2\mu_3\mu_5$ - $26\mu_1^5\mu_2\mu_3\mu_6$ - $8\mu_1^5\mu_2\mu_4^2$ - $70\mu_1^5\mu_2\mu_4\mu_5$
- $54\mu_1^5\mu_2\mu_4\mu_6$ - $18\mu_1^5\mu_2\mu_5^2$ - $52\mu_1^5\mu_2\mu_5\mu_6$ - $28\mu_1^5\mu_2\mu_6^2$ - $2\mu_1^5\mu_3^3$ - $12\mu_1^5\mu_3^2\mu_4$ - $12\mu_1^5\mu_3^2\mu_5$ -
$12\mu_1^5\mu_3^2\mu_6$ - $2\mu_1^5\mu_3\mu_4^2$ - $60\mu_1^5\mu_3\mu_4\mu_5$ - $40\mu_1^5\mu_3\mu_4\mu_6$ - $22\mu_1^5\mu_3\mu_5^2$ - $52\mu_1^5\mu_3\mu_5\mu_6$ - $24\mu_1^5\mu_3\mu_6^2$ +
$8\mu_1^5\mu_4^3$ - $16\mu_1^5\mu_4^2\mu_5 + 4\mu_1^5\mu_4^2\mu_6$ - $70\mu_1^5\mu_4\mu_5^2$ - $108\mu_1^5\mu_4\mu_5\mu_6$ - $32\mu_1^5\mu_4\mu_6^2$ - $12\mu_1^5\mu_5^3$ - $52\mu_1^5\mu_5^2\mu_6$
- $56\mu_1^5\mu_5\mu_6^2$ - $16\mu_1^5\mu_6^3$ - $\mu_1^4\mu_2^4$ - $4\mu_1^4\mu_2^3\mu_3$ - $10\mu_1^4\mu_2^3\mu_4$ - $8\mu_1^4\mu_2^3\mu_5$ - $10\mu_1^4\mu_2^3\mu_6$ - $6\mu_1^4\mu_2^2\mu_3^2$
- $43\mu_1^4\mu_2^2\mu_3\mu_4$ - $22\mu_1^4\mu_2^2\mu_3\mu_5$ - $30\mu_1^4\mu_2^2\mu_3\mu_6$ - $63\mu_1^4\mu_2^2\mu_4^2$ - $53\mu_1^4\mu_2^2\mu_4\mu_5$ - $86\mu_1^4\mu_2^2\mu_4\mu_6$ -
$24\mu_1^4\mu_2^2\mu_5^2$ - $56\mu_1^4\mu_2^2\mu_5\mu_6$ - $35\mu_1^4\mu_2^2\mu_6^2$ - $4\mu_1^4\mu_2\mu_3^3$ - $53\mu_1^4\mu_2\mu_3^2\mu_4$ - $22\mu_1^4\mu_2\mu_3^2\mu_5$ - $30\mu_1^4\mu_2\mu_3^2\mu_6$
- $129\mu_1^4\mu_2\mu_3\mu_4^2$ - $164\mu_1^4\mu_2\mu_3\mu_4\mu_5$ - $204\mu_1^4\mu_2\mu_3\mu_4\mu_6$ - $42\mu_1^4\mu_2\mu_3\mu_5^2$ - $112\mu_1^4\mu_2\mu_3\mu_5\mu_6$ -
$69\mu_1^4\mu_2\mu_3\mu_6^2$ - $26\mu_1^4\mu_2\mu_4^3$ - $246\mu_1^4\mu_2\mu_4^2\mu_5$ - $194\mu_1^4\mu_2\mu_4^2\mu_6$ - $99\mu_1^4\mu_2\mu_4\mu_5^2$ - $328\mu_1^4\mu_2\mu_4\mu_5\mu_6$ -
$185\mu_1^4\mu_2\mu_4\mu_6^2$ - $32\mu_1^4\mu_2\mu_5^3$ - $108\mu_1^4\mu_2\mu_5^2\mu_6$ - $132\mu_1^4\mu_2\mu_5\mu_6^2$ - $50\mu_1^4\mu_2\mu_6^3$ - $\mu_1^4\mu_3^4$ - $20\mu_1^4\mu_3^3\mu_4$ -
$8\mu_1^4\mu_3^3\mu_5$ - $10\mu_1^4\mu_3^3\mu_6$ - $62\mu_1^4\mu_3^2\mu_4^2$ - $106\mu_1^4\mu_3^2\mu_4\mu_5$ - $111\mu_1^4\mu_3^2\mu_4\mu_6$ - $22\mu_1^4\mu_3^2\mu_5^2$ - $60\mu_1^4\mu_3^2\mu_5\mu_6$
- $34\mu_1^4\mu_3^2\mu_6^2$ - $16\mu_1^4\mu_3\mu_4^3$ - $258\mu_1^4\mu_3\mu_4^2\mu_5$ - $177\mu_1^4\mu_3\mu_4^2\mu_6$ - $164\mu_1^4\mu_3\mu_4\mu_5^2$ - $408\mu_1^4\mu_3\mu_4\mu_5\mu_6$ -
$193\mu_1^4\mu_3\mu_4\mu_6^2$ - $28\mu_1^4\mu_3\mu_5^3$ - $112\mu_1^4\mu_3\mu_5^2\mu_6$ - $138\mu_1^4\mu_3\mu_5\mu_6^2$ - $48\mu_1^4\mu_3\mu_6^3 + 27\mu_1^4\mu_4^4$ - $52\mu_1^4\mu_4^3\mu_5$
+ $30\mu_1^4\mu_4^3\mu_6$ - $246\mu_1^4\mu_4^2\mu_5^2$ - $388\mu_1^4\mu_4^2\mu_5\mu_6$ - $95\mu_1^4\mu_4^2\mu_6^2$ - $66\mu_1^4\mu_4\mu_5^3$ - $328\mu_1^4\mu_4\mu_5^2\mu_6$
- $370\mu_1^4\mu_4\mu_5\mu_6^2$ - $102\mu_1^4\mu_4\mu_6^3$ - $16\mu_1^4\mu_5^4$ - $72\mu_1^4\mu_5^3\mu_6$ - $132\mu_1^4\mu_5^2\mu_6^2$ - $100\mu_1^4\mu_5\mu_6^3$ - $24\mu_1^4\mu_6^4$
- $\mu_1^3\mu_2^4\mu_3$ - $\mu_1^3\mu_2^4\mu_4$ - $2\mu_1^3\mu_2^4\mu_6$ - $2\mu_1^3\mu_2^3\mu_3^2$ - $14\mu_1^3\mu_2^3\mu_3\mu_4$ - $8\mu_1^3\mu_2^3\mu_3\mu_5$ - $12\mu_1^3\mu_2^3\mu_3\mu_6$ -

$$10\mu_1^3\mu_2^3\mu_4^2 - 12\mu_1^3\mu_2^3\mu_4\mu_5 - 22\mu_1^3\mu_2^3\mu_4\mu_6 - 16\mu_1^3\mu_2^3\mu_5\mu_6 - 14\mu_1^3\mu_2^3\mu_6^2 - 2\mu_1^3\mu_2^2\mu_3^3 - 34\mu_1^3\mu_2^2\mu_3^2\mu_4 - 12\mu_1^3\mu_2^2\mu_3^2\mu_5 - 18\mu_1^3\mu_2^2\mu_3^2\mu_6 - 122\mu_1^3\mu_2^2\mu_3\mu_4^2 - 88\mu_1^3\mu_2^2\mu_3\mu_4\mu_5 - 142\mu_1^3\mu_2^2\mu_3\mu_4\mu_6 - 24\mu_1^3\mu_2^2\mu_3\mu_5^2 - 70\mu_1^3\mu_2^2\mu_3\mu_5\mu_6 - 47\mu_1^3\mu_2^2\mu_3\mu_6^2 - 110\mu_1^3\mu_2^2\mu_4^3 - 68\mu_1^3\mu_2^2\mu_4^2\mu_5 - 178\mu_1^3\mu_2^2\mu_4^2\mu_6 - 44\mu_1^3\mu_2^2\mu_4\mu_5^2 - 140\mu_1^3\mu_2^2\mu_4\mu_5\mu_6 - 117\mu_1^3\mu_2^2\mu_4\mu_6^2 - 48\mu_1^3\mu_2^2\mu_5^2\mu_6 - 82\mu_1^3\mu_2^2\mu_5\mu_6^2 - 36\mu_1^3\mu_2^2\mu_6^3 - \mu_1^3\mu_2\mu_3^4 - 30\mu_1^3\mu_2\mu_3^3\mu_4 - 8\mu_1^3\mu_2\mu_3^3\mu_5 - 12\mu_1^3\mu_2\mu_3^3\mu_6 - 192\mu_1^3\mu_2\mu_3^2\mu_4^2 - 134\mu_1^3\mu_2\mu_3^2\mu_4\mu_5 - 200\mu_1^3\mu_2\mu_3^2\mu_4\mu_6 - 24\mu_1^3\mu_2\mu_3^2\mu_5^2 - 70\mu_1^3\mu_2\mu_3^2\mu_5\mu_6 - 47\mu_1^3\mu_2\mu_3^2\mu_6^2 - 306\mu_1^3\mu_2\mu_3\mu_4^3 - 494\mu_1^3\mu_2\mu_3\mu_4^2\mu_5 - 656\mu_1^3\mu_2\mu_3\mu_4^2\mu_6 - 180\mu_1^3\mu_2\mu_3\mu_4\mu_5^2 - 564\mu_1^3\mu_2\mu_3\mu_4\mu_5\mu_6 - 388\mu_1^3\mu_2\mu_3\mu_4\mu_6^2 - 32\mu_1^3\mu_2\mu_3\mu_5^3 - 138\mu_1^3\mu_2\mu_3\mu_5^2\mu_6 - 182\mu_1^3\mu_2\mu_3\mu_5\mu_6^2 - 74\mu_1^3\mu_2\mu_3\mu_6^3 - 45\mu_1^3\mu_2\mu_4^4 - 448\mu_1^3\mu_2\mu_4^3\mu_5 - 360\mu_1^3\mu_2\mu_4^3\mu_6 - 144\mu_1^3\mu_2\mu_4^2\mu_5^2 - 730\mu_1^3\mu_2\mu_4^2\mu_5\mu_6 - 455\mu_1^3\mu_2\mu_4^2\mu_6^2 - 64\mu_1^3\mu_2\mu_4\mu_5^3 - 288\mu_1^3\mu_2\mu_4\mu_5^2\mu_6 - 470\mu_1^3\mu_2\mu_4\mu_5\mu_6^2 - 210\mu_1^3\mu_2\mu_4\mu_6^3 - 64\mu_1^3\mu_2\mu_5^3\mu_6 - 162\mu_1^3\mu_2\mu_5^2\mu_6^2 - 140\mu_1^3\mu_2\mu_5\mu_6^3 - 40\mu_1^3\mu_2\mu_6^4 - 8\mu_1^3\mu_3^4\mu_4 - 2\mu_1^3\mu_3^4\mu_5 - 2\mu_1^3\mu_3^4\mu_6 - 80\mu_1^3\mu_3^3\mu_4^2 - 60\mu_1^3\mu_3^3\mu_4\mu_5 - 78\mu_1^3\mu_3^3\mu_4\mu_6 - 8\mu_1^3\mu_3^3\mu_5^2 - 24\mu_1^3\mu_3^3\mu_5\mu_6 - 14\mu_1^3\mu_3^3\mu_6^2 - 172\mu_1^3\mu_3^2\mu_4^3 - 384\mu_1^3\mu_3^2\mu_4^2\mu_5 - 420\mu_1^3\mu_3^2\mu_4^2\mu_6 - 134\mu_1^3\mu_3^2\mu_4\mu_5^2 - 400\mu_1^3\mu_3^2\mu_4\mu_5\mu_6 - 242\mu_1^3\mu_3^2\mu_4\mu_6^2 - 16\mu_1^3\mu_3^2\mu_5^3 - 70\mu_1^3\mu_3^2\mu_5^2\mu_6 - 94\mu_1^3\mu_3^2\mu_5\mu_6^2 - 36\mu_1^3\mu_3^2\mu_6^3 - 50\mu_1^3\mu_3\mu_4^4 - 612\mu_1^3\mu_3\mu_4^3\mu_5 - 438\mu_1^3\mu_3\mu_4^3\mu_6 - 494\mu_1^3\mu_3\mu_4^2\mu_5^2 - 1312\mu_1^3\mu_3\mu_4^2\mu_5\mu_6 - 642\mu_1^3\mu_3\mu_4^2\mu_6^2 - 120\mu_1^3\mu_3\mu_4\mu_5^3 - 564\mu_1^3\mu_3\mu_4\mu_5^2\mu_6 - 776\mu_1^3\mu_3\mu_4\mu_5\mu_6^2 - 290\mu_1^3\mu_3\mu_4\mu_6^3 - 16\mu_1^3\mu_3\mu_5^4 - 92\mu_1^3\mu_3\mu_5^3\mu_6 - 182\mu_1^3\mu_3\mu_5^2\mu_6^2 - 148\mu_1^3\mu_3\mu_5\mu_6^3 - 40\mu_1^3\mu_3\mu_6^4 + 50\mu_1^3\mu_4^5 - 90\mu_1^3\mu_4^4\mu_5 + 90\mu_1^3\mu_4^4\mu_6 - 448\mu_1^3\mu_4^3\mu_5^2 - 720\mu_1^3\mu_4^3\mu_5\mu_6 - 122\mu_1^3\mu_4^3\mu_6^2 - 96\mu_1^3\mu_4^2\mu_5^3 - 730\mu_1^3\mu_4^2\mu_5^2\mu_6 - 910\mu_1^3\mu_4^2\mu_5\mu_6^2 - 238\mu_1^3\mu_4^2\mu_6^3 - 32\mu_1^3\mu_4\mu_5^4 - 192\mu_1^3\mu_4\mu_5^3\mu_6 - 470\mu_1^3\mu_4\mu_5^2\mu_6^2 - 420\mu_1^3\mu_4\mu_5\mu_6^3 - 108\mu_1^3\mu_4\mu_6^4 - 32\mu_1^3\mu_5^4\mu_6 - 108\mu_1^3\mu_5^3\mu_6^2 - 140\mu_1^3\mu_5^2\mu_6^3 - 80\mu_1^3\mu_5\mu_6^4 - 16\mu_1^3\mu_6^5 + 2\mu_1^2\mu_2^5\mu_4 + 3\mu_1^2\mu_2^4\mu_3\mu_4 - \mu_1^2\mu_2^4\mu_3\mu_6 + 11\mu_1^2\mu_2^4\mu_4^2 + 20\mu_1^2\mu_2^4\mu_4\mu_5 + 14\mu_1^2\mu_2^4\mu_4\mu_6 - \mu_1^2\mu_2^4\mu_6^2 - 2\mu_1^2\mu_2^3\mu_3^2\mu_4 - 2\mu_1^2\mu_2^3\mu_3^2\mu_6 - 6\mu_1^2\mu_2^3\mu_3\mu_4^2 + 14\mu_1^2\mu_2^3\mu_3\mu_4\mu_5 - 2\mu_1^2\mu_2^3\mu_3\mu_4\mu_6 - 8\mu_1^2\mu_2^3\mu_3\mu_5\mu_6 - 8\mu_1^2\mu_2^3\mu_3\mu_6^2 + 20\mu_1^2\mu_2^3\mu_4^3 + 68\mu_1^2\mu_2^3\mu_4^2\mu_5 + 50\mu_1^2\mu_2^3\mu_4^2\mu_6 + 76\mu_1^2\mu_2^3\mu_4\mu_5^2 + 96\mu_1^2\mu_2^3\mu_4\mu_5\mu_6 + 26\mu_1^2\mu_2^3\mu_4\mu_6^2 - 8\mu_1^2\mu_2^3\mu_5\mu_6^2 - 6\mu_1^2\mu_2^3\mu_6^3 - 8\mu_1^2\mu_2^2\mu_3^3\mu_4 - 2\mu_1^2\mu_2^2\mu_3^3\mu_6 - 74\mu_1^2\mu_2^2\mu_3^2\mu_4^2 - 24\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_5 - 56\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_6 - 12\mu_1^2\mu_2^2\mu_3^2\mu_5\mu_6 - 12\mu_1^2\mu_2^2\mu_3^2\mu_6^2 - 178\mu_1^2\mu_2^2\mu_3\mu_4^3 - 90\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_5 - 232\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_6 + 20\mu_1^2\mu_2^2\mu_3\mu_4\mu_5^2 - 62\mu_1^2\mu_2^2\mu_3\mu_4\mu_5\mu_6 - 94\mu_1^2\mu_2^2\mu_3\mu_4\mu_6^2 - 24\mu_1^2\mu_2^2\mu_3\mu_5^2\mu_6 - 48\mu_1^2\mu_2^2\mu_3\mu_5\mu_6^2 - 23\mu_1^2\mu_2^2\mu_3\mu_6^3 - 105\mu_1^2\mu_2^2\mu_4^4 + 58\mu_1^2\mu_2^2\mu_4^3\mu_5 - 132\mu_1^2\mu_2^2\mu_4^3\mu_6 + 156\mu_1^2\mu_2^2\mu_4^2\mu_5^2 + 186\mu_1^2\mu_2^2\mu_4^2\mu_5\mu_6 - 29\mu_1^2\mu_2^2\mu_4^2\mu_6^2 + 144\mu_1^2\mu_2^2\mu_4\mu_5^3 + 248\mu_1^2\mu_2^2\mu_4\mu_5^2\mu_6 + 103\mu_1^2\mu_2^2\mu_4\mu_5\mu_6^2 - 10\mu_1^2\mu_2^2\mu_4\mu_6^3 - 24\mu_1^2\mu_2^2\mu_5^2\mu_6^2 - 36\mu_1^2\mu_2^2\mu_5\mu_6^3 - 13\mu_1^2\mu_2^2\mu_6^4 - 5\mu_1^2\mu_2\mu_3^4\mu_4 - \mu_1^2\mu_2\mu_3^4\mu_6 - 86\mu_1^2\mu_2\mu_3^3\mu_4^2 - 36\mu_1^2\mu_2\mu_3^3\mu_4\mu_5 - 60\mu_1^2\mu_2\mu_3^3\mu_4\mu_6 - 8\mu_1^2\mu_2\mu_3^3\mu_5\mu_6 - 8\mu_1^2\mu_2\mu_3^3\mu_6^2 - 358\mu_1^2\mu_2\mu_3^2\mu_4^3 - 320\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5 - 522\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_6 - 60\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^2 - 248\mu_1^2\mu_2\mu_3^2\mu_4\mu_5\mu_6 - 202\mu_1^2\mu_2\mu_3^2\mu_4\mu_6^2 - 24\mu_1^2\mu_2\mu_3^2\mu_5^2\mu_6 - 48\mu_1^2\mu_2\mu_3^2\mu_5\mu_6^2 - 23\mu_1^2\mu_2\mu_3^2\mu_6^3 - 423\mu_1^2\mu_2\mu_3\mu_4^4 - 776\mu_1^2\mu_2\mu_3\mu_4^3\mu_5 - 1104\mu_1^2\mu_2\mu_3\mu_4^3\mu_6 - 234\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^2 - 1042\mu_1^2\mu_2\mu_3\mu_4^2\mu_5\mu_6 - 844\mu_1^2\mu_2\mu_3\mu_4^2\mu_6^2 + 12\mu_1^2\mu_2\mu_3\mu_4\mu_5^3 - 174\mu_1^2\mu_2\mu_3\mu_4\mu_5^2\mu_6 - 428\mu_1^2\mu_2\mu_3\mu_4\mu_5\mu_6^2 - 236\mu_1^2\mu_2\mu_3\mu_4\mu_6^3 - 32\mu_1^2\mu_2\mu_3\mu_5^3\mu_6 - 96\mu_1^2\mu_2\mu_3\mu_5^2\mu_6^2 - 92\mu_1^2\mu_2\mu_3\mu_5\mu_6^3 - 28\mu_1^2\mu_2\mu_3\mu_6^4 - 46\mu_1^2\mu_2\mu_4^5 - 464\mu_1^2\mu_2\mu_4^4\mu_5 - 380\mu_1^2\mu_2\mu_4^4\mu_6 + 54\mu_1^2\mu_2\mu_4^3\mu_5^2 - 680\mu_1^2\mu_2\mu_4^3\mu_5\mu_6 - 530\mu_1^2\mu_2\mu_4^3\mu_6^2 + 176\mu_1^2\mu_2\mu_4^2\mu_5^3 + 258\mu_1^2\mu_2\mu_4^2\mu_5^2\mu_6 - 262\mu_1^2\mu_2\mu_4^2\mu_5\mu_6^2 - 258\mu_1^2\mu_2\mu_4^2\mu_6^3 + 140\mu_1^2\mu_2\mu_4\mu_5^4 + 304\mu_1^2\mu_2\mu_4\mu_5^3\mu_6 + 153\mu_1^2\mu_2\mu_4\mu_5^2\mu_6^2 - 84\mu_1^2\mu_2\mu_4\mu_5\mu_6^3 - 64\mu_1^2\mu_2\mu_4\mu_6^4 - 32\mu_1^2\mu_2\mu_5^3\mu_6^2 - 72\mu_1^2\mu_2\mu_5^2\mu_6^3 - 52\mu_1^2\mu_2\mu_5\mu_6^4 - 12\mu_1^2\mu_2\mu_6^5 - 24\mu_1^2\mu_3^4\mu_4^2 - 10\mu_1^2\mu_3^4\mu_4\mu_5 - 11\mu_1^2\mu_3^4\mu_4\mu_6 - 2\mu_1^2\mu_3^4\mu_5\mu_6 - \mu_1^2\mu_3^4\mu_6^2 - 160\mu_1^2\mu_3^3\mu_4^3 - 172\mu_1^2\mu_3^3\mu_4^2\mu_5 - 230\mu_1^2\mu_3^3\mu_4^2\mu_6 - 36\mu_1^2\mu_3^3\mu_4\mu_5^2 - 120\mu_1^2\mu_3^3\mu_4\mu_5\mu_6 - 76\mu_1^2\mu_3^3\mu_4\mu_6^2 - 8\mu_1^2\mu_3^3\mu_5^2\mu_6 - 16\mu_1^2\mu_3^3\mu_5\mu_6^2 - 6\mu_1^2\mu_3^3\mu_6^3 - 266\mu_1^2\mu_3^2\mu_4^4 - 716\mu_1^2\mu_3^2\mu_4^3\mu_5 - 810\mu_1^2\mu_3^2\mu_4^3\mu_6 - 320\mu_1^2\mu_3^2\mu_4^2\mu_5^2 - 1044\mu_1^2\mu_3^2\mu_4^2\mu_5\mu_6 - 668\mu_1^2\mu_3^2\mu_4^2\mu_6^2 - 40\mu_1^2\mu_3^2\mu_4\mu_5^3 - 248\mu_1^2\mu_3^2\mu_4\mu_5^2\mu_6 - 404\mu_1^2\mu_3^2\mu_4\mu_5\mu_6^2 - 176\mu_1^2\mu_3^2\mu_4\mu_6^3 - 16\mu_1^2\mu_3^2\mu_5^3\mu_6 - 48\mu_1^2\mu_3^2\mu_5^2\mu_6^2 - 46\mu_1^2\mu_3^2\mu_5\mu_6^3 - 13\mu_1^2\mu_3^2\mu_6^4 - 76\mu_1^2\mu_3\mu_4^5 - 846\mu_1^2\mu_3\mu_4^4\mu_5 - 629\mu_1^2\mu_3\mu_4^4\mu_6 - 776\mu_1^2\mu_3\mu_4^3\mu_5^2 - 2208\mu_1^2\mu_3\mu_4^3\mu_5\mu_6 - 1118\mu_1^2\mu_3\mu_4^3\mu_6^2 - 156\mu_1^2\mu_3\mu_4^2\mu_5^3 - 1042\mu_1^2\mu_3\mu_4^2\mu_5^2\mu_6 - 1688\mu_1^2\mu_3\mu_4^2\mu_5\mu_6^2$$

$$
\begin{aligned}
&- 688\mu_1^2\mu_3\mu_4^2\mu_6^3 + 6\mu_1^2\mu_3\mu_4\mu_5^4 - 116\mu_1^2\mu_3\mu_4\mu_5^3\mu_6 - 428\mu_1^2\mu_3\mu_4\mu_5^2\mu_6^2 - 472\mu_1^2\mu_3\mu_4\mu_5\mu_6^3 - \\
&155\mu_1^2\mu_3\mu_4\mu_6^4 - 16\mu_1^2\mu_3\mu_5^4\mu_6 - 64\mu_1^2\mu_3\mu_5^3\mu_6^2 - 92\mu_1^2\mu_3\mu_5^2\mu_6^3 - 56\mu_1^2\mu_3\mu_5\mu_6^4 - 12\mu_1^2\mu_3\mu_6^5 + \\
&54\mu_1^2\mu_4^6 - 92\mu_1^2\mu_4^5\mu_5 + 134\mu_1^2\mu_4^5\mu_6 - 464\mu_1^2\mu_4^4\mu_5^2 - 760\mu_1^2\mu_4^4\mu_5\mu_6 - 51\mu_1^2\mu_4^4\mu_6^2 + 36\mu_1^2\mu_4^3\mu_5^3 - \\
&680\mu_1^2\mu_4^3\mu_5^2\mu_6 - 1060\mu_1^2\mu_4^3\mu_5\mu_6^2 - 252\mu_1^2\mu_4^3\mu_6^3 + 88\mu_1^2\mu_4^2\mu_5^4 + 172\mu_1^2\mu_4^2\mu_5^3\mu_6 - 262\mu_1^2\mu_4^2\mu_5^2\mu_6^2 \\
&- 516\mu_1^2\mu_4^2\mu_5\mu_6^3 - 160\mu_1^2\mu_4^2\mu_6^4 + 56\mu_1^2\mu_4\mu_5^5 + 152\mu_1^2\mu_4\mu_5^4\mu_6 + 102\mu_1^2\mu_4\mu_5^3\mu_6^2 - 84\mu_1^2\mu_4\mu_5^2\mu_6^3 \\
&- 128\mu_1^2\mu_4\mu_5\mu_6^4 - 38\mu_1^2\mu_4\mu_6^5 - 16\mu_1^2\mu_5^4\mu_6^2 - 48\mu_1^2\mu_5^3\mu_6^3 - 52\mu_1^2\mu_5^2\mu_6^4 - 24\mu_1^2\mu_5\mu_6^5 - 4\mu_1^2\mu_6^6 + \\
&2\mu_1\mu_2^5\mu_3\mu_4 + 6\mu_1\mu_2^5\mu_4^2 + 2\mu_1\mu_2^5\mu_4\mu_5 + 4\mu_1\mu_2^5\mu_4\mu_6 + 3\mu_1\mu_2^4\mu_3^2\mu_4 + 15\mu_1\mu_2^4\mu_3\mu_4^2 \\
&+ 18\mu_1\mu_2^4\mu_3\mu_4\mu_5 + 16\mu_1\mu_2^4\mu_3\mu_4\mu_6 + 24\mu_1\mu_2^4\mu_4^3 + 60\mu_1\mu_2^4\mu_4^2\mu_5 + 54\mu_1\mu_2^4\mu_4^2\mu_6 + 15\mu_1\mu_2^4\mu_4\mu_5^2 \\
&+ 44\mu_1\mu_2^4\mu_4\mu_5\mu_6 + 24\mu_1\mu_2^4\mu_4\mu_6^2 + 2\mu_1\mu_2^3\mu_3^3\mu_4 + 6\mu_1\mu_2^3\mu_3^2\mu_4^2 + 20\mu_1\mu_2^3\mu_3^2\mu_4\mu_5 \\
&+ 14\mu_1\mu_2^3\mu_3^2\mu_4\mu_6 + 16\mu_1\mu_2^3\mu_3\mu_4^3 + 90\mu_1\mu_2^3\mu_3\mu_4^2\mu_5 + 66\mu_1\mu_2^3\mu_3\mu_4^2\mu_6 + 64\mu_1\mu_2^3\mu_3\mu_4\mu_5^2 \\
&+ 108\mu_1\mu_2^3\mu_3\mu_4\mu_5\mu_6 + 42\mu_1\mu_2^3\mu_3\mu_4\mu_6^2 + 40\mu_1\mu_2^3\mu_4^4 + 160\mu_1\mu_2^3\mu_4^3\mu_5 + 144\mu_1\mu_2^3\mu_4^3\mu_6 + \\
&228\mu_1\mu_2^3\mu_4^2\mu_5^2 + 384\mu_1\mu_2^3\mu_4^2\mu_5\mu_6 + 162\mu_1\mu_2^3\mu_4^2\mu_6^2 + 46\mu_1\mu_2^3\mu_4\mu_5^3 + 178\mu_1\mu_2^3\mu_4\mu_5^2\mu_6 + \\
&178\mu_1\mu_2^3\mu_4\mu_5\mu_6^2 + 52\mu_1\mu_2^3\mu_4\mu_6^3 + \mu_1\mu_2^2\mu_3^4\mu_4 - 12\mu_1\mu_2^2\mu_3^3\mu_4^2 + 8\mu_1\mu_2^2\mu_3^3\mu_4\mu_5 \\
&+ 2\mu_1\mu_2^2\mu_3^3\mu_4\mu_6 - 80\mu_1\mu_2^2\mu_3^2\mu_4^3 - 60\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_6 + 50\mu_1\mu_2^2\mu_3^2\mu_4\mu_5^2 + 58\mu_1\mu_2^2\mu_3^2\mu_4\mu_5\mu_6 \\
&+ 6\mu_1\mu_2^2\mu_3^2\mu_4\mu_6^2 - 145\mu_1\mu_2^2\mu_3\mu_4^4 - 4\mu_1\mu_2^2\mu_3\mu_4^3\mu_5 - 180\mu_1\mu_2^2\mu_3\mu_4^3\mu_6 + 204\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^2 + \\
&234\mu_1\mu_2^2\mu_3\mu_4^2\mu_5\mu_6 + 116\mu_1\mu_2^2\mu_3\mu_4\mu_5^3 + 278\mu_1\mu_2^2\mu_3\mu_4\mu_5^2\mu_6 + 194\mu_1\mu_2^2\mu_3\mu_4\mu_5\mu_6^2 \\
&+ 34\mu_1\mu_2^2\mu_3\mu_4\mu_6^3 - 60\mu_1\mu_2^2\mu_4^5 + 155\mu_1\mu_2^2\mu_4^4\mu_5 - 20\mu_1\mu_2^2\mu_4^4\mu_6 + 400\mu_1\mu_2^2\mu_4^3\mu_5^2 \\
&+ 652\mu_1\mu_2^2\mu_4^3\mu_5\mu_6 + 176\mu_1\mu_2^2\mu_4^3\mu_6^2 + 432\mu_1\mu_2^2\mu_4^2\mu_5^3 + 1032\mu_1\mu_2^2\mu_4^2\mu_5^2\mu_6 + 801\mu_1\mu_2^2\mu_4^2\mu_5\mu_6^2 \\
&+ 186\mu_1\mu_2^2\mu_4^2\mu_6^3 + 73\mu_1\mu_2^2\mu_4\mu_5^4 + 352\mu_1\mu_2^2\mu_4\mu_5^3\mu_6 + 498\mu_1\mu_2^2\mu_4\mu_5^2\mu_6^2 + 270\mu_1\mu_2^2\mu_4\mu_5\mu_6^3 + \\
&48\mu_1\mu_2^2\mu_4\mu_6^4 - 9\mu_1\mu_2\mu_3^4\mu_4^2 + 2\mu_1\mu_2\mu_3^4\mu_4\mu_5 - 2\mu_1\mu_2\mu_3^4\mu_4\mu_6 - 112\mu_1\mu_2\mu_3^3\mu_4^3 - 60\mu_1\mu_2\mu_3^3\mu_4^2\mu_5 \\
&- 108\mu_1\mu_2\mu_3^3\mu_4^2\mu_6 + 12\mu_1\mu_2\mu_3^3\mu_4\mu_5^2 - 4\mu_1\mu_2\mu_3^3\mu_4\mu_5\mu_6 - 18\mu_1\mu_2\mu_3^3\mu_4\mu_6^2 - 345\mu_1\mu_2\mu_3^2\mu_4^4 - \\
&372\mu_1\mu_2\mu_3^2\mu_4^3\mu_5 - 644\mu_1\mu_2\mu_3^2\mu_4^3\mu_6 - 36\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^2 - 324\mu_1\mu_2\mu_3^2\mu_4^2\mu_5\mu_6 - 324\mu_1\mu_2\mu_3^2\mu_4^2\mu_6^2 \\
&+ 60\mu_1\mu_2\mu_3^2\mu_4\mu_5^3 + 90\mu_1\mu_2\mu_3^2\mu_4\mu_5^2\mu_6 - 10\mu_1\mu_2\mu_3^2\mu_4\mu_5\mu_6^2 - 38\mu_1\mu_2\mu_3^2\mu_4\mu_6^3 - 324\mu_1\mu_2\mu_3\mu_4^5 - \\
&670\mu_1\mu_2\mu_3\mu_4^4\mu_5 - 1000\mu_1\mu_2\mu_3\mu_4^4\mu_6 - 108\mu_1\mu_2\mu_3\mu_4^3\mu_5^2 - 956\mu_1\mu_2\mu_3\mu_4^3\mu_5\mu_6 - 912\mu_1\mu_2\mu_3\mu_4^3\mu_6^2 \\
&+ 228\mu_1\mu_2\mu_3\mu_4^2\mu_5^3 + 306\mu_1\mu_2\mu_3\mu_4^2\mu_5^2\mu_6 - 192\mu_1\mu_2\mu_3\mu_4^2\mu_5\mu_6^2 - 264\mu_1\mu_2\mu_3\mu_4^2\mu_6^3 \\
&+ 110\mu_1\mu_2\mu_3\mu_4\mu_5^4 + 340\mu_1\mu_2\mu_3\mu_4\mu_5^3\mu_6 + 330\mu_1\mu_2\mu_3\mu_4\mu_5^2\mu_6^2 + 88\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6^3 \\
&- 12\mu_1\mu_2\mu_3\mu_4\mu_6^4 - 28\mu_1\mu_2\mu_6^6 - 288\mu_1\mu_2\mu_4^5\mu_5 - 240\mu_1\mu_2\mu_4^5\mu_6 + 225\mu_1\mu_2\mu_4^4\mu_5^2 - 290\mu_1\mu_2\mu_4^4\mu_5\mu_6 \\
&- 340\mu_1\mu_2\mu_4^4\mu_6^2 + 480\mu_1\mu_2\mu_4^3\mu_5^3 + 1092\mu_1\mu_2\mu_4^3\mu_5^2\mu_6 + 416\mu_1\mu_2\mu_4^3\mu_5\mu_6^2 - 96\mu_1\mu_2\mu_4^3\mu_6^3 + \\
&420\mu_1\mu_2\mu_4^2\mu_5^4 + 1296\mu_1\mu_2\mu_4^2\mu_5^3\mu_6 + 1431\mu_1\mu_2\mu_4^2\mu_5^2\mu_6^2 + 588\mu_1\mu_2\mu_4^2\mu_5\mu_6^3 + 48\mu_1\mu_2\mu_4^2\mu_6^4 + \\
&60\mu_1\mu_2\mu_4\mu_5^5 + 350\mu_1\mu_2\mu_4\mu_5^4\mu_6 + 640\mu_1\mu_2\mu_4\mu_5^3\mu_6^2 + 498\mu_1\mu_2\mu_4\mu_5^2\mu_6^3 + 164\mu_1\mu_2\mu_4\mu_5\mu_6^4 \\
&+ 16\mu_1\mu_2\mu_4\mu_6^5 - 32\mu_1\mu_3^4\mu_4^3 - 18\mu_1\mu_3^4\mu_4^2\mu_5 - 21\mu_1\mu_3^4\mu_4^2\mu_6 + 2\mu_1\mu_3^4\mu_4\mu_5^2 - 4\mu_1\mu_3^4\mu_4\mu_5\mu_6 - \\
&3\mu_1\mu_3^4\mu_4\mu_6^2 - 160\mu_1\mu_3^3\mu_4^4 - 224\mu_1\mu_3^3\mu_4^3\mu_5 - 304\mu_1\mu_3^3\mu_4^3\mu_6 - 60\mu_1\mu_3^3\mu_4^2\mu_5^2 - 216\mu_1\mu_3^3\mu_4^2\mu_5\mu_6 - \\
&144\mu_1\mu_3^3\mu_4^2\mu_6^2 + 8\mu_1\mu_3^3\mu_4\mu_5^3 - 4\mu_1\mu_3^3\mu_4\mu_5^2\mu_6 - 36\mu_1\mu_3^3\mu_4\mu_5\mu_6^2 - 18\mu_1\mu_3^3\mu_4\mu_6^3 - 216\mu_1\mu_3^2\mu_4^5 - \\
&690\mu_1\mu_3^2\mu_4^4\mu_5 - 795\mu_1\mu_3^2\mu_4^4\mu_6 - 372\mu_1\mu_3^2\mu_4^3\mu_5^2 - 1288\mu_1\mu_3^2\mu_4^3\mu_5\mu_6 - 852\mu_1\mu_3^2\mu_4^3\mu_6^2 - 24\mu_1\mu_3^2\mu_4^2\mu_5^3 \\
&- 324\mu_1\mu_3^2\mu_4^2\mu_5^2\mu_6 - 648\mu_1\mu_3^2\mu_4^2\mu_5\mu_6^2 - 312\mu_1\mu_3^2\mu_4^2\mu_6^3 + 30\mu_1\mu_3^2\mu_4\mu_5^4 + 60\mu_1\mu_3^2\mu_4\mu_5^3\mu_6 - \\
&10\mu_1\mu_3^2\mu_4\mu_5^2\mu_6^2 - 76\mu_1\mu_3^2\mu_4\mu_5\mu_6^3 - 33\mu_1\mu_3^2\mu_4\mu_6^4 - 56\mu_1\mu_3\mu_4^6 - 648\mu_1\mu_3\mu_4^5\mu_5 - 492\mu_1\mu_3\mu_4^5\mu_6 \\
&- 670\mu_1\mu_3\mu_4^4\mu_5^2 - 2000\mu_1\mu_3\mu_4^4\mu_5\mu_6 - 1035\mu_1\mu_3\mu_4^4\mu_6^2 - 72\mu_1\mu_3\mu_4^3\mu_5^3 - 956\mu_1\mu_3\mu_4^3\mu_5^2\mu_6 - \\
&1824\mu_1\mu_3\mu_4^3\mu_5\mu_6^2 - 796\mu_1\mu_3\mu_4^3\mu_6^3 + 114\mu_1\mu_3\mu_4^2\mu_5^4 + 204\mu_1\mu_3\mu_4^2\mu_5^3\mu_6 - 192\mu_1\mu_3\mu_4^2\mu_5^2\mu_6^2 \\
&- 528\mu_1\mu_3\mu_4^2\mu_5\mu_6^3 - 225\mu_1\mu_3\mu_4^2\mu_6^4 + 44\mu_1\mu_3\mu_4\mu_5^5 + 170\mu_1\mu_3\mu_4\mu_5^4\mu_6 + 220\mu_1\mu_3\mu_4\mu_5^3\mu_6^2 \\
&+ 88\mu_1\mu_3\mu_4\mu_5^2\mu_6^3 - 24\mu_1\mu_3\mu_4\mu_5\mu_6^4 - 18\mu_1\mu_3\mu_4\mu_6^5 + 32\mu_1\mu_4^7 - 56\mu_1\mu_4^6\mu_5 + 98\mu_1\mu_4^6\mu_6 - \\
&288\mu_1\mu_4^5\mu_5^2 - 480\mu_1\mu_4^5\mu_5\mu_6 + 12\mu_1\mu_4^5\mu_6^2 + 150\mu_1\mu_4^4\mu_5^3 - 290\mu_1\mu_4^4\mu_5^2\mu_6 - 680\mu_1\mu_4^4\mu_5\mu_6^2 - \\
&140\mu_1\mu_4^4\mu_6^3 + 240\mu_1\mu_4^3\mu_5^4 + 728\mu_1\mu_4^3\mu_5^3\mu_6 + 416\mu_1\mu_4^3\mu_5^2\mu_6^2 - 192\mu_1\mu_4^3\mu_5\mu_6^3 - 104\mu_1\mu_4^3\mu_6^4 + \\
&168\mu_1\mu_4^2\mu_5^5 + 648\mu_1\mu_4^2\mu_5^4\mu_6 + 954\mu_1\mu_4^2\mu_5^3\mu_6^2 + 588\mu_1\mu_4^2\mu_5^2\mu_6^3 + 96\mu_1\mu_4^2\mu_5\mu_6^4 - 18\mu_1\mu_4^2\mu_6^5 +
\end{aligned}
$$

$20\mu_1\mu_4\mu_5^6 + 140\mu_1\mu_4\mu_5^5\mu_6 + 320\mu_1\mu_4\mu_5^4\mu_6^2 + 332\mu_1\mu_4\mu_5^3\mu_6^3 + 164\mu_1\mu_4\mu_5^2\mu_6^4 + 32\mu_1\mu_4\mu_5\mu_6^5$
$+ 2\mu_2^5\mu_3\mu_4^2 + 4\mu_2^5\mu_4^3 + 2\mu_2^5\mu_4^2\mu_5 + 4\mu_2^5\mu_4^2\mu_6 + 3\mu_2^4\mu_3^2\mu_4^2 + 10\mu_2^4\mu_3\mu_4^3 + 18\mu_2^4\mu_3\mu_4^2\mu_5 +$
$16\mu_2^4\mu_3\mu_4^2\mu_6 + 12\mu_2^4\mu_4^4 + 40\mu_2^4\mu_4^3\mu_5 + 36\mu_2^4\mu_4^3\mu_6 + 15\mu_2^4\mu_4^2\mu_5^2 + 44\mu_2^4\mu_4^2\mu_5\mu_6 + 24\mu_2^4\mu_4^2\mu_6^2$
$+ 2\mu_2^3\mu_3^3\mu_4^2 + 4\mu_2^3\mu_3^2\mu_4^3 + 20\mu_2^3\mu_3^2\mu_4^2\mu_5 + 14\mu_2^3\mu_3^2\mu_4^2\mu_6 + 8\mu_2^3\mu_3\mu_4^4 + 60\mu_2^3\mu_3\mu_4^3\mu_5 +$
$44\mu_2^3\mu_3\mu_4^3\mu_6 + 64\mu_2^3\mu_3\mu_4^2\mu_5^2 + 108\mu_2^3\mu_3\mu_4^2\mu_5\mu_6 + 42\mu_2^3\mu_3\mu_4^2\mu_6^2 + 16\mu_2^3\mu_4^5 + 80\mu_2^3\mu_4^4\mu_5 +$
$72\mu_2^3\mu_4^4\mu_6 + 152\mu_2^3\mu_4^3\mu_5^2 + 256\mu_2^3\mu_4^3\mu_5\mu_6 + 108\mu_2^3\mu_4^3\mu_6^2 + 46\mu_2^3\mu_4^2\mu_5^3 + 178\mu_2^3\mu_4^2\mu_5^2\mu_6 +$
$178\mu_2^3\mu_4^2\mu_5\mu_6^2 + 52\mu_2^3\mu_4^2\mu_6^3 + \mu_2^2\mu_3^4\mu_4^2 - 8\mu_2^2\mu_3^3\mu_4^3 + 8\mu_2^2\mu_3^3\mu_4^2\mu_5 + 2\mu_2^2\mu_3^3\mu_4^2\mu_6 - 40\mu_2^2\mu_3^2\mu_4^4$
$- 40\mu_2^2\mu_3^2\mu_4^3\mu_6 + 50\mu_2^2\mu_3^2\mu_4^2\mu_5^2 + 58\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6 + 6\mu_2^2\mu_3^2\mu_4^2\mu_6^2 - 58\mu_2^2\mu_3\mu_4^5 - 2\mu_2^2\mu_3\mu_4^4\mu_5$
$- 90\mu_2^2\mu_3\mu_4^4\mu_6 + 136\mu_2^2\mu_3\mu_4^3\mu_5^2 + 156\mu_2^2\mu_3\mu_4^3\mu_5\mu_6 + 116\mu_2^2\mu_3\mu_4^2\mu_5^3 + 278\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6 +$
$194\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^2 + 34\mu_2^2\mu_3\mu_4^2\mu_6^3 - 20\mu_2^2\mu_4^6 + 62\mu_2^2\mu_4^5\mu_5 - 8\mu_2^2\mu_4^5\mu_6 + 200\mu_2^2\mu_4^4\mu_5^2$
$+ 326\mu_2^2\mu_4^4\mu_5\mu_6 + 88\mu_2^2\mu_4^4\mu_6^2 + 288\mu_2^2\mu_4^3\mu_5^3 + 688\mu_2^2\mu_4^3\mu_5^2\mu_6 + 534\mu_2^2\mu_4^3\mu_5\mu_6^2 + 124\mu_2^2\mu_4^3\mu_6^3$
$+ 73\mu_2^2\mu_4^2\mu_5^4 + 352\mu_2^2\mu_4^2\mu_5^3\mu_6 + 498\mu_2^2\mu_4^2\mu_5^2\mu_6^2 + 270\mu_2^2\mu_4^2\mu_5\mu_6^3 + 48\mu_2^2\mu_4^2\mu_6^4 - 6\mu_2\mu_3^4\mu_4^3$
$+ 2\mu_2\mu_3^4\mu_4^2\mu_5 - 2\mu_2\mu_3^4\mu_4^2\mu_6 - 56\mu_2\mu_3^3\mu_4^4 - 40\mu_2\mu_3^3\mu_4^3\mu_5 - 72\mu_2\mu_3^3\mu_4^3\mu_6 + 12\mu_2\mu_3^3\mu_4^2\mu_5^2 -$
$4\mu_2\mu_3^3\mu_4^2\mu_5\mu_6 - 18\mu_2\mu_3^3\mu_4^2\mu_6^2 - 138\mu_2\mu_3^2\mu_4^5 - 186\mu_2\mu_3^2\mu_4^4\mu_5 - 322\mu_2\mu_3^2\mu_4^4\mu_6 - 24\mu_2\mu_3^2\mu_4^3\mu_5^2$
$- 216\mu_2\mu_3^2\mu_4^3\mu_5\mu_6 - 216\mu_2\mu_3^2\mu_4^3\mu_6^2 + 60\mu_2\mu_3^2\mu_4^2\mu_5^3 + 90\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6 - 10\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^2$
$- 38\mu_2\mu_3^2\mu_4^2\mu_6^3 - 108\mu_2\mu_3\mu_4^6 - 268\mu_2\mu_3\mu_4^5\mu_5 - 400\mu_2\mu_3\mu_4^5\mu_6 - 54\mu_2\mu_3\mu_4^4\mu_5^2 - 478\mu_2\mu_3\mu_4^4\mu_5\mu_6$
$- 456\mu_2\mu_3\mu_4^4\mu_6^2 + 152\mu_2\mu_3\mu_4^3\mu_5^3 + 204\mu_2\mu_3\mu_4^3\mu_5^2\mu_6 - 128\mu_2\mu_3\mu_4^3\mu_5\mu_6^2 - 176\mu_2\mu_3\mu_4^3\mu_6^3 +$
$110\mu_2\mu_3\mu_4^2\mu_5^4 + 340\mu_2\mu_3\mu_4^2\mu_5^3\mu_6 + 330\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^2 + 88\mu_2\mu_3\mu_4^2\mu_5\mu_6^3 - 12\mu_2\mu_3\mu_4^2\mu_6^4 -$
$8\mu_2\mu_4^7 - 96\mu_2\mu_4^6\mu_5 - 80\mu_2\mu_4^6\mu_6 + 90\mu_2\mu_4^5\mu_5^2 - 116\mu_2\mu_4^5\mu_5\mu_6 - 136\mu_2\mu_4^5\mu_6^2 + 240\mu_2\mu_4^4\mu_5^3 +$
$546\mu_2\mu_4^4\mu_5^2\mu_6 + 208\mu_2\mu_4^4\mu_5\mu_6^2 - 48\mu_2\mu_4^4\mu_6^3 + 280\mu_2\mu_4^3\mu_5^4 + 864\mu_2\mu_4^3\mu_5^3\mu_6 + 954\mu_2\mu_4^3\mu_5^2\mu_6^2$
$+ 392\mu_2\mu_4^3\mu_5\mu_6^3 + 32\mu_2\mu_4^3\mu_6^4 + 60\mu_2\mu_4^2\mu_5^5 + 350\mu_2\mu_4^2\mu_5^4\mu_6 + 640\mu_2\mu_4^2\mu_5^3\mu_6^2 + 498\mu_2\mu_4^2\mu_5^2\mu_6^3$
$+ 164\mu_2\mu_4^2\mu_5\mu_6^4 + 16\mu_2\mu_4^2\mu_6^5 - 16\mu_3^4\mu_4^4 - 12\mu_3^4\mu_4^3\mu_5 - 14\mu_3^4\mu_4^3\mu_6 + 2\mu_3^4\mu_4^2\mu_5^2 - 4\mu_3^4\mu_4^2\mu_5\mu_6$
$- 3\mu_3^4\mu_4^2\mu_6^2 - 64\mu_3^3\mu_4^5 - 112\mu_3^3\mu_4^4\mu_5 - 152\mu_3^3\mu_4^4\mu_6 - 40\mu_3^3\mu_4^3\mu_5^2 - 144\mu_3^3\mu_4^3\mu_5\mu_6 - 96\mu_3^3\mu_4^3\mu_6^2$
$+ 8\mu_3^3\mu_4^2\mu_5^3 - 4\mu_3^3\mu_4^2\mu_5^2\mu_6 - 36\mu_3^3\mu_4^2\mu_5\mu_6^2 - 18\mu_3^3\mu_4^2\mu_6^3 - 72\mu_3^2\mu_4^6 - 276\mu_3^2\mu_4^5\mu_5 - 318\mu_3^2\mu_4^5\mu_6$
$- 186\mu_3^2\mu_4^4\mu_5^2 - 644\mu_3^2\mu_4^4\mu_5\mu_6 - 426\mu_3^2\mu_4^4\mu_6^2 - 16\mu_3^2\mu_4^3\mu_5^3 - 216\mu_3^2\mu_4^3\mu_5^2\mu_6 - 432\mu_3^2\mu_4^3\mu_5\mu_6^2 -$
$208\mu_3^2\mu_4^3\mu_6^3 + 30\mu_3^2\mu_4^2\mu_5^4 + 60\mu_3^2\mu_4^2\mu_5^3\mu_6 - 10\mu_3^2\mu_4^2\mu_5^2\mu_6^2 - 76\mu_3^2\mu_4^2\mu_5\mu_6^3 - 33\mu_3^2\mu_4^2\mu_6^4 - 16\mu_3\mu_4^7 -$
$216\mu_3\mu_4^6\mu_5 - 164\mu_3\mu_4^6\mu_6 - 268\mu_3\mu_4^5\mu_5^2 - 800\mu_3\mu_4^5\mu_5\mu_6 - 414\mu_3\mu_4^5\mu_6^2 - 36\mu_3\mu_4^4\mu_5^3 - 478\mu_3\mu_4^4\mu_5^2\mu_6$
$- 912\mu_3\mu_4^4\mu_5\mu_6^2 - 398\mu_3\mu_4^4\mu_6^3 + 76\mu_3\mu_4^3\mu_5^4 + 136\mu_3\mu_4^3\mu_5^3\mu_6 - 128\mu_3\mu_4^3\mu_5^2\mu_6^2 - 352\mu_3\mu_4^3\mu_5\mu_6^3$
$- 150\mu_3\mu_4^3\mu_6^4 + 44\mu_3\mu_4^2\mu_5^5 + 170\mu_3\mu_4^2\mu_5^4\mu_6 + 220\mu_3\mu_4^2\mu_5^3\mu_6^2 + 88\mu_3\mu_4^2\mu_5^2\mu_6^3 - 24\mu_3\mu_4^2\mu_5\mu_6^4 -$
$18\mu_3\mu_4^2\mu_6^5 + 8\mu_4^8 - 16\mu_4^7\mu_5 + 28\mu_4^7\mu_6 - 96\mu_4^6\mu_5^2 - 160\mu_4^6\mu_5\mu_6 + 4\mu_4^6\mu_6^2 + 60\mu_4^5\mu_5^3 - 116\mu_4^5\mu_5^2\mu_6$
$- 272\mu_4^5\mu_5\mu_6^2 - 56\mu_4^5\mu_6^3 + 120\mu_4^4\mu_5^4 + 364\mu_4^4\mu_5^3\mu_6 + 208\mu_4^4\mu_5^2\mu_6^2 - 96\mu_4^4\mu_5\mu_6^3 - 52\mu_4^4\mu_6^4 +$
$112\mu_4^3\mu_5^5 + 432\mu_4^3\mu_5^4\mu_6 + 636\mu_4^3\mu_5^3\mu_6^2 + 392\mu_4^3\mu_5^2\mu_6^3 + 64\mu_4^3\mu_5\mu_6^4 - 12\mu_4^3\mu_6^5 + 20\mu_4^2\mu_5^6 +$
$140\mu_4^2\mu_5^5\mu_6 + 320\mu_4^2\mu_5^4\mu_6^2 + 332\mu_4^2\mu_5^3\mu_6^3 + 164\mu_4^2\mu_5^2\mu_6^4 + 32\mu_4^2\mu_5\mu_6^5.$

$a_{X^2} = 2\mu_1^4\mu_2\mu_4 - \mu_1^4\mu_2\mu_5 + \mu_1^4\mu_3\mu_4 + 4\mu_1^4\mu_4\mu_5 + 2\mu_1^4\mu_4\mu_6 - \mu_1^4\mu_5^2 - 6\mu_1^3\mu_2^2\mu_4 +$
$4\mu_1^3\mu_2^2\mu_5 - 4\mu_1^3\mu_2\mu_3\mu_4 + 3\mu_1^3\mu_2\mu_3\mu_5 + 6\mu_1^3\mu_2\mu_4^2 - 23\mu_1^3\mu_2\mu_4\mu_5 - 10\mu_1^3\mu_2\mu_4\mu_6 + 12\mu_1^3\mu_2\mu_5^2$
$+ 6\mu_1^3\mu_2\mu_5\mu_6 + \mu_1^3\mu_3^2\mu_4 + 7\mu_1^3\mu_3\mu_4^2 - 8\mu_1^3\mu_3\mu_4\mu_5 + 3\mu_1^3\mu_3\mu_5^2 + 12\mu_1^3\mu_4^2\mu_5 + 6\mu_1^3\mu_4^2\mu_6$
$- 23\mu_1^3\mu_4\mu_5^2 - 20\mu_1^3\mu_4\mu_5\mu_6 - 4\mu_1^3\mu_4\mu_6^2 + 8\mu_1^3\mu_5^3 + 6\mu_1^3\mu_5^2\mu_6 - 2\mu_1^2\mu_2^2\mu_4 - 8\mu_1^2\mu_2^2\mu_3\mu_4 +$
$4\mu_1^2\mu_2^2\mu_3\mu_5 - 30\mu_1^2\mu_2^2\mu_4^2 + 7\mu_1^2\mu_2^2\mu_4\mu_5 - 22\mu_1^2\mu_2^2\mu_4\mu_6 + 8\mu_1^2\mu_2^2\mu_5\mu_6 - 5\mu_1^2\mu_2\mu_3^2\mu_4 + 3\mu_1^2\mu_2\mu_3^2\mu_5$
$- 20\mu_1^2\mu_2\mu_3\mu_4^2 - 13\mu_1^2\mu_2\mu_3\mu_4\mu_5 - 26\mu_1^2\mu_2\mu_3\mu_4\mu_6 + 12\mu_1^2\mu_2\mu_3\mu_5^2 + 13\mu_1^2\mu_2\mu_3\mu_5\mu_6 + 4\mu_1^2\mu_2\mu_4^3$
$- 97\mu_1^2\mu_2\mu_4^2\mu_5 - 54\mu_1^2\mu_2\mu_4^2\mu_6 + 33\mu_1^2\mu_2\mu_4\mu_5^2 - 40\mu_1^2\mu_2\mu_4\mu_5\mu_6 - 34\mu_1^2\mu_2\mu_4\mu_6^2 + 24\mu_1^2\mu_2\mu_5^2\mu_6$
$+ 15\mu_1^2\mu_2\mu_5\mu_6^2 + 5\mu_1^2\mu_3^2\mu_4^2 - 10\mu_1^2\mu_3^2\mu_4\mu_5 - 3\mu_1^2\mu_3^2\mu_4\mu_6 + 3\mu_1^2\mu_3^2\mu_5^2 + 18\mu_1^2\mu_3\mu_4^3 - 40\mu_1^2\mu_3\mu_4^2\mu_5$

$$- 13\mu_1^2\mu_3\mu_4\mu_5^2 - 52\mu_1^2\mu_3\mu_4\mu_5\mu_6 - 13\mu_1^2\mu_3\mu_4\mu_6^2 + 8\mu_1^2\mu_3\mu_5^3 + 13\mu_1^2\mu_3\mu_5^2\mu_6 + 8\mu_1^2\mu_4^3\mu_5 + 4\mu_1^2\mu_4^3\mu_6 - 97\mu_1^2\mu_4^2\mu_5^2 - 108\mu_1^2\mu_4^2\mu_5\mu_6 - 24\mu_1^2\mu_4^2\mu_6^2 + 22\mu_1^2\mu_4\mu_5^3 - 40\mu_1^2\mu_4\mu_5^2\mu_6 - 68\mu_1^2\mu_4\mu_5\mu_6^2 - 14\mu_1^2\mu_4\mu_6^3 + 16\mu_1^2\mu_5^3\mu_6 + 15\mu_1^2\mu_5^2\mu_6^2 - \mu_1\mu_2^4\mu_4 + 2\mu_1\mu_2^4\mu_5 - 3\mu_1\mu_2^3\mu_3\mu_4 + 2\mu_1\mu_2^3\mu_3\mu_5 - 6\mu_1\mu_2^3\mu_4^2 - 5\mu_1\mu_2^3\mu_4\mu_5 - 6\mu_1\mu_2^3\mu_4\mu_6 + 12\mu_1\mu_2^3\mu_5^2 + 8\mu_1\mu_2^3\mu_5\mu_6 - 3\mu_1\mu_2^2\mu_3^2\mu_4 + \mu_1\mu_2^2\mu_3^2\mu_5 - 24\mu_1\mu_2^2\mu_3\mu_4^2 - \mu_1\mu_2^2\mu_3\mu_4\mu_5 - 17\mu_1\mu_2^2\mu_3\mu_4\mu_6 + 10\mu_1\mu_2^2\mu_3\mu_5^2 + 10\mu_1\mu_2^2\mu_3\mu_5\mu_6 - 48\mu_1\mu_2^2\mu_4^3 - 3\mu_1\mu_2^2\mu_4^2\mu_5 - 66\mu_1\mu_2^2\mu_4^2\mu_6 - 7\mu_1\mu_2^2\mu_4\mu_5^2 + 2\mu_1\mu_2^2\mu_4\mu_5\mu_6 - 21\mu_1\mu_2^2\mu_4\mu_6^2 + 28\mu_1\mu_2^2\mu_5^3 + 36\mu_1\mu_2^2\mu_5^2\mu_6 + 14\mu_1\mu_2^2\mu_5\mu_6^2 - 15\mu_1\mu_2\mu_3^2\mu_4^2 + \mu_1\mu_2\mu_3^2\mu_4\mu_5 - 8\mu_1\mu_2\mu_3^2\mu_4\mu_6 + 3\mu_1\mu_2\mu_3^2\mu_5^2 + 4\mu_1\mu_2\mu_3^2\mu_5\mu_6 - 32\mu_1\mu_2\mu_3\mu_4^3 - 57\mu_1\mu_2\mu_3\mu_4^2\mu_5 - 78\mu_1\mu_2\mu_3\mu_4^2\mu_6 + 15\mu_1\mu_2\mu_3\mu_4\mu_5^2 - 13\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6 - 28\mu_1\mu_2\mu_3\mu_4\mu_6^2 + 16\mu_1\mu_2\mu_3\mu_5^3 + 30\mu_1\mu_2\mu_3\mu_5^2\mu_6 + 14\mu_1\mu_2\mu_3\mu_5\mu_6^2 - 148\mu_1\mu_2\mu_4^3\mu_5 - 88\mu_1\mu_2\mu_4^3\mu_6 + 27\mu_1\mu_2\mu_4^2\mu_5^2 - 156\mu_1\mu_2\mu_4^2\mu_5\mu_6 - 102\mu_1\mu_2\mu_4^2\mu_6^2 - 4\mu_1\mu_2\mu_4\mu_5^3 + 42\mu_1\mu_2\mu_4\mu_5^2\mu_6 - 13\mu_1\mu_2\mu_4\mu_5\mu_6^2 - 24\mu_1\mu_2\mu_4\mu_6^3 + 30\mu_1\mu_2\mu_5^4 + 56\mu_1\mu_2\mu_5^3\mu_6 + 42\mu_1\mu_2\mu_5^2\mu_6^2 + 12\mu_1\mu_2\mu_5\mu_6^3 + 8\mu_1\mu_3^2\mu_4^3 - 30\mu_1\mu_3^2\mu_4^2\mu_5 - 9\mu_1\mu_3^2\mu_4^2\mu_6 + \mu_1\mu_3^2\mu_4\mu_5^2 - 16\mu_1\mu_3^2\mu_4\mu_5\mu_6 - 4\mu_1\mu_3^2\mu_4\mu_6^2 + 2\mu_1\mu_3^2\mu_5^3 + 4\mu_1\mu_3^2\mu_5^2\mu_6 + 20\mu_1\mu_3\mu_4^4 - 64\mu_1\mu_3\mu_4^3\mu_5 - 57\mu_1\mu_3\mu_4^2\mu_5^2 - 156\mu_1\mu_3\mu_4^2\mu_5\mu_6 - 39\mu_1\mu_3\mu_4^2\mu_6^2 + 10\mu_1\mu_3\mu_4\mu_5^3 - 13\mu_1\mu_3\mu_4\mu_5^2\mu_6 - 56\mu_1\mu_3\mu_4\mu_5\mu_6^2 - 12\mu_1\mu_3\mu_4\mu_6^3 + 8\mu_1\mu_3\mu_5^4 + 20\mu_1\mu_3\mu_5^3\mu_6 + 14\mu_1\mu_3\mu_5^2\mu_6^2 - 148\mu_1\mu_4^3\mu_5^2 - 176\mu_1\mu_4^3\mu_5\mu_6 - 40\mu_1\mu_4^3\mu_6^2 + 18\mu_1\mu_4^2\mu_5^3 - 156\mu_1\mu_4^2\mu_5^2\mu_6 - 204\mu_1\mu_4^2\mu_5\mu_6^2 - 42\mu_1\mu_4^2\mu_6^3 - 2\mu_1\mu_4\mu_5^4 + 28\mu_1\mu_4\mu_5^3\mu_6 - 13\mu_1\mu_4\mu_5^2\mu_6^2 - 48\mu_1\mu_4\mu_5\mu_6^3 - 8\mu_1\mu_4\mu_6^4 + 12\mu_1\mu_5^5 + 28\mu_1\mu_5^4\mu_6 + 28\mu_1\mu_5^3\mu_6^2 + 12\mu_1\mu_5^2\mu_6^3 + \mu_2^4\mu_3\mu_5 - \mu_2^4\mu_4^2 + 4\mu_2^4\mu_4\mu_5 + 2\mu_2^4\mu_5\mu_6 + \mu_2^3\mu_3^2\mu_5 - 3\mu_2^3\mu_3\mu_4^2 + 4\mu_2^3\mu_3\mu_4\mu_5 + 7\mu_2^3\mu_3\mu_5^2 + 6\mu_2^3\mu_3\mu_5\mu_6 - 4\mu_2^3\mu_4^3 - 5\mu_2^3\mu_4^2\mu_5 - 6\mu_2^3\mu_4^2\mu_6 + 24\mu_2^3\mu_4\mu_5^2 + 16\mu_2^3\mu_4\mu_5\mu_6 + 12\mu_2^3\mu_5^2\mu_6 + 8\mu_2^3\mu_5\mu_6^2 - 3\mu_2^2\mu_3^2\mu_4^2 + 2\mu_2^2\mu_3^2\mu_4\mu_5 + 5\mu_2^2\mu_3^2\mu_5^2 + 3\mu_2^2\mu_3^2\mu_5\mu_6 - 16\mu_2^2\mu_3\mu_4^3 - \mu_2^2\mu_3\mu_4^2\mu_5 - 17\mu_2^2\mu_3\mu_4^2\mu_6 + 20\mu_2^2\mu_3\mu_4\mu_5^2 + 20\mu_2^2\mu_3\mu_4\mu_5\mu_6 + 18\mu_2^2\mu_3\mu_5^3 + 30\mu_2^2\mu_3\mu_5^2\mu_6 + 11\mu_2^2\mu_3\mu_5\mu_6^2 - 24\mu_2^2\mu_4^4 - 2\mu_2^2\mu_4^3\mu_5 - 44\mu_2^2\mu_4^3\mu_6 - 7\mu_2^2\mu_4^2\mu_5^2 + 2\mu_2^2\mu_4^2\mu_5\mu_6 - 21\mu_2^2\mu_4^2\mu_6^2 + 56\mu_2^2\mu_4\mu_5^3 + 72\mu_2^2\mu_4\mu_5^2\mu_6 + 28\mu_2^2\mu_4\mu_5\mu_6^2 + 28\mu_2^2\mu_5^3\mu_6 + 36\mu_2^2\mu_5^2\mu_6^2 + 10\mu_2^2\mu_5\mu_6^3 - 10\mu_2\mu_3^2\mu_4^3 + \mu_2\mu_3^2\mu_4^2\mu_5 - 8\mu_2\mu_3^2\mu_4^2\mu_6 + 6\mu_2\mu_3^2\mu_4\mu_5^2 + 8\mu_2\mu_3^2\mu_4\mu_5\mu_6 + 8\mu_2\mu_3^2\mu_5^3 + 9\mu_2\mu_3^2\mu_5^2\mu_6 + 2\mu_2\mu_3^2\mu_5\mu_6^2 - 16\mu_2\mu_3\mu_4^4 - 38\mu_2\mu_3\mu_4^3\mu_5 - 52\mu_2\mu_3\mu_4^3\mu_6 + 15\mu_2\mu_3\mu_4^2\mu_5^2 - 13\mu_2\mu_3\mu_4^2\mu_5\mu_6 - 28\mu_2\mu_3\mu_4^2\mu_6^2 + 32\mu_2\mu_3\mu_4\mu_5^3 + 60\mu_2\mu_3\mu_4\mu_5^2\mu_6 + 28\mu_2\mu_3\mu_4\mu_5\mu_6^2 + 20\mu_2\mu_3\mu_5^4 + 48\mu_2\mu_3\mu_5^3\mu_6 + 33\mu_2\mu_3\mu_5^2\mu_6^2 + 6\mu_2\mu_3\mu_5\mu_6^3 - 74\mu_2\mu_4^4\mu_5 - 44\mu_2\mu_4^4\mu_6 + 18\mu_2\mu_4^3\mu_5^2 - 104\mu_2\mu_4^3\mu_5\mu_6 - 68\mu_2\mu_4^3\mu_6^2 - 4\mu_2\mu_4^2\mu_5^3 + 42\mu_2\mu_4^2\mu_5^2\mu_6 - 13\mu_2\mu_4^2\mu_5\mu_6^2 - 24\mu_2\mu_4^2\mu_6^3 + 60\mu_2\mu_4\mu_5^4 + 112\mu_2\mu_4\mu_5^3\mu_6 + 84\mu_2\mu_4\mu_5^2\mu_6^2 + 24\mu_2\mu_4\mu_5\mu_6^3 + 30\mu_2\mu_5^4\mu_6 + 56\mu_2\mu_5^3\mu_6^2 + 30\mu_2\mu_5^2\mu_6^3 + 4\mu_2\mu_5\mu_6^4 + 4\mu_3^2\mu_4^4 - 20\mu_3^2\mu_4^3\mu_5 - 6\mu_3^2\mu_4^3\mu_6 + \mu_3^2\mu_4^2\mu_5^2 - 16\mu_3^2\mu_4^2\mu_5\mu_6 - 4\mu_3^2\mu_4^2\mu_6^2 + 4\mu_3^2\mu_4\mu_5^3 + 8\mu_3^2\mu_4\mu_5^2\mu_6 + 4\mu_3^2\mu_5^4 + 6\mu_3^2\mu_5^3\mu_6 + 2\mu_3^2\mu_5^2\mu_6^2 + 8\mu_3\mu_4^5 - 32\mu_3\mu_4^4\mu_5 - 38\mu_3\mu_4^3\mu_5^2 - 104\mu_3\mu_4^3\mu_5\mu_6 - 26\mu_3\mu_4^3\mu_6^2 + 10\mu_3\mu_4^2\mu_5^3 - 13\mu_3\mu_4^2\mu_5^2\mu_6 - 56\mu_3\mu_4^2\mu_5\mu_6^2 - 12\mu_3\mu_4^2\mu_6^3 + 16\mu_3\mu_4\mu_5^4 + 40\mu_3\mu_4\mu_5^3\mu_6 + 28\mu_3\mu_4\mu_5^2\mu_6^2 + 8\mu_3\mu_5^5 + 24\mu_3\mu_5^4\mu_6 + 22\mu_3\mu_5^3\mu_6^2 + 6\mu_3\mu_5^2\mu_6^3 - 74\mu_4^4\mu_5^2 - 88\mu_4^4\mu_5\mu_6 - 20\mu_4^4\mu_6^2 + 12\mu_4^3\mu_5^3 - 104\mu_4^3\mu_5^2\mu_6 - 136\mu_4^3\mu_5\mu_6^2 - 28\mu_4^3\mu_6^3 - 2\mu_4^2\mu_5^4 + 28\mu_4^2\mu_5^3\mu_6 - 13\mu_4^2\mu_5^2\mu_6^2 - 48\mu_4^2\mu_5\mu_6^3 - 8\mu_4^2\mu_6^4 + 24\mu_4\mu_5^5 + 56\mu_4\mu_5^4\mu_6 + 56\mu_4\mu_5^3\mu_6^2 + 24\mu_4\mu_5^2\mu_6^3 + 12\mu_5^5\mu_6 + 28\mu_5^4\mu_6^2 + 20\mu_5^3\mu_6^3 + 4\mu_5^2\mu_6^4.$$

$$a_Z = \mu_1^4\mu_2^2 + \mu_1^4\mu_2\mu_3 - \mu_1^4\mu_2\mu_4 + 4\mu_1^4\mu_2\mu_5 + 2\mu_1^4\mu_2\mu_6 - 2\mu_1^4\mu_3\mu_4 + 2\mu_1^4\mu_3\mu_5 - 2\mu_1^4\mu_4^2 - 2\mu_1^4\mu_4\mu_5 - 4\mu_1^4\mu_4\mu_6 + 4\mu_1^4\mu_5^2 + 4\mu_1^4\mu_5\mu_6 + \mu_1^3\mu_2^2\mu_3 + 3\mu_1^3\mu_2^2\mu_4 + 2\mu_1^3\mu_2^2\mu_6 + \mu_1^3\mu_2\mu_3^2 + 2\mu_1^3\mu_2\mu_3\mu_4 + 4\mu_1^3\mu_2\mu_3\mu_5 + 4\mu_1^3\mu_2\mu_3\mu_6 - 9\mu_1^3\mu_2\mu_4^2 + 12\mu_1^3\mu_2\mu_4\mu_5 + 8\mu_1^3\mu_2\mu_5\mu_6 + 4\mu_1^3\mu_2\mu_6^2 - 2\mu_1^3\mu_3^2\mu_4 + 2\mu_1^3\mu_3^2\mu_5 - 14\mu_1^3\mu_3\mu_4^2 + 4\mu_1^3\mu_3\mu_4\mu_5 - 12\mu_1^3\mu_3\mu_4\mu_6 + 4\mu_1^3\mu_3\mu_5^2 + 8\mu_1^3\mu_3\mu_5\mu_6 - 12\mu_1^3\mu_4^3 - 18\mu_1^3\mu_4^2\mu_5 - 32\mu_1^3\mu_4^2\mu_6 + 12\mu_1^3\mu_4\mu_5^2 - 16\mu_1^3\mu_4\mu_6^2 + 8\mu_1^3\mu_5^2\mu_6 + 8\mu_1^3\mu_5\mu_6^2 - 2\mu_1^2\mu_2^3\mu_4 + \mu_1^2\mu_2^2\mu_3\mu_6 - \mu_1^2\mu_2^2\mu_4^2 - 11\mu_1^2\mu_2^2\mu_4\mu_5 - 4\mu_1^2\mu_2^2\mu_4\mu_6 + \mu_1^2\mu_2^2\mu_6^2 + 2\mu_1^2\mu_2\mu_3^2\mu_4 + \mu_1^2\mu_2\mu_3^2\mu_6$$

$$- 6\mu_1^2\mu_2\mu_3\mu_4^2 + 2\mu_1^2\mu_2\mu_3\mu_4\mu_5 + 4\mu_1^2\mu_2\mu_3\mu_5\mu_6 + 3\mu_1^2\mu_2\mu_3\mu_6^2 - 26\mu_1^2\mu_2\mu_4^3 - 2\mu_1^2\mu_2\mu_4^2\mu_5 - 32\mu_1^2\mu_2\mu_4^2\mu_6 - 21\mu_1^2\mu_2\mu_4\mu_5^2 - 12\mu_1^2\mu_2\mu_4\mu_5\mu_6 - 8\mu_1^2\mu_2\mu_4\mu_6^2 + 4\mu_1^2\mu_2\mu_5\mu_6^2 + 2\mu_1^2\mu_2\mu_6^3 - 10\mu_1^2\mu_3^2\mu_4^2 + 4\mu_1^2\mu_3^2\mu_4\mu_5 - 6\mu_1^2\mu_3^2\mu_4\mu_6 + 2\mu_1^2\mu_3^2\mu_5\mu_6 - 36\mu_1^2\mu_3\mu_4^3 - 12\mu_1^2\mu_3\mu_4^2\mu_5 - 60\mu_1^2\mu_3\mu_4^2\mu_6 + 2\mu_1^2\mu_3\mu_4\mu_5^2 - 22\mu_1^2\mu_3\mu_4\mu_6^2 + 4\mu_1^2\mu_3\mu_5^2\mu_6 + 6\mu_1^2\mu_3\mu_5\mu_6^2 - 26\mu_1^2\mu_4^4 - 52\mu_1^2\mu_4^3\mu_5 - 88\mu_1^2\mu_4^3\mu_6 - 2\mu_1^2\mu_4^2\mu_5^2 - 64\mu_1^2\mu_4^2\mu_5\mu_6 - 80\mu_1^2\mu_4^2\mu_6^2 - 14\mu_1^2\mu_4\mu_5^3 - 12\mu_1^2\mu_4\mu_5^2\mu_6 - 16\mu_1^2\mu_4\mu_5\mu_6^2 - 20\mu_1^2\mu_4\mu_6^3 + 4\mu_1^2\mu_5^2\mu_6^2 + 4\mu_1^2\mu_5\mu_6^3 - 2\mu_1\mu_2^3\mu_3\mu_4 - 6\mu_1\mu_2^3\mu_4^2 - 2\mu_1\mu_2^3\mu_4\mu_5 - 4\mu_1\mu_2^3\mu_4\mu_6 - 2\mu_1\mu_2^2\mu_3^2\mu_4 - 6\mu_1\mu_2^2\mu_3\mu_4^2 - 10\mu_1\mu_2^2\mu_3\mu_4\mu_5 - 8\mu_1\mu_2^2\mu_3\mu_4\mu_6 - 8\mu_1\mu_2^2\mu_4^3 - 33\mu_1\mu_2^2\mu_4^2\mu_5 - 24\mu_1\mu_2^2\mu_4^2\mu_6 - 8\mu_1\mu_2^2\mu_4\mu_5^2 - 24\mu_1\mu_2^2\mu_4\mu_5\mu_6 - 12\mu_1\mu_2^2\mu_4\mu_6^2 - 6\mu_1\mu_2\mu_3^2\mu_4\mu_5 - 2\mu_1\mu_2\mu_3^2\mu_4\mu_6 - 16\mu_1\mu_2\mu_3\mu_4^3 - 18\mu_1\mu_2\mu_3\mu_4^2\mu_5 - 24\mu_1\mu_2\mu_3\mu_4^2\mu_6 - 18\mu_1\mu_2\mu_3\mu_4\mu_5^2 - 26\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6 - 10\mu_1\mu_2\mu_3\mu_4\mu_6^2 - 30\mu_1\mu_2\mu_4^4 - 28\mu_1\mu_2\mu_4^3\mu_5 - 64\mu_1\mu_2\mu_4^3\mu_6 - 63\mu_1\mu_2\mu_4^2\mu_5^2 - 84\mu_1\mu_2\mu_4^2\mu_5\mu_6 - 48\mu_1\mu_2\mu_4^2\mu_6^2 - 12\mu_1\mu_2\mu_4\mu_5^3 - 48\mu_1\mu_2\mu_4\mu_5^2\mu_6 - 42\mu_1\mu_2\mu_4\mu_5\mu_6^2 - 12\mu_1\mu_2\mu_4\mu_6^3 - 16\mu_1\mu_3^2\mu_4^3 - 18\mu_1\mu_3^2\mu_4^2\mu_6 - 6\mu_1\mu_3^2\mu_4\mu_5^2 - 4\mu_1\mu_3^2\mu_4\mu_5\mu_6 - 4\mu_1\mu_3^2\mu_4\mu_6^2 - 40\mu_1\mu_3\mu_4^4 - 32\mu_1\mu_3\mu_4^3\mu_5 - 96\mu_1\mu_3\mu_4^3\mu_6 - 18\mu_1\mu_3\mu_4^2\mu_5^2 - 48\mu_1\mu_3\mu_4^2\mu_5\mu_6 - 66\mu_1\mu_3\mu_4^2\mu_6^2 - 12\mu_1\mu_3\mu_4\mu_5^3 - 26\mu_1\mu_3\mu_4\mu_5^2\mu_6 - 20\mu_1\mu_3\mu_4\mu_5\mu_6^2 - 12\mu_1\mu_3\mu_4\mu_6^3 - 24\mu_1\mu_4^5 - 60\mu_1\mu_4^4\mu_5 - 100\mu_1\mu_4^4\mu_6 - 28\mu_1\mu_4^3\mu_5^2 - 128\mu_1\mu_4^3\mu_5\mu_6 - 128\mu_1\mu_4^3\mu_6^2 - 42\mu_1\mu_4^2\mu_5^3 - 84\mu_1\mu_4^2\mu_5^2\mu_6 - 96\mu_1\mu_4^2\mu_5\mu_6^2 - 60\mu_1\mu_4^2\mu_6^3 - 6\mu_1\mu_4\mu_5^4 - 32\mu_1\mu_4\mu_5^3\mu_6 - 42\mu_1\mu_4\mu_5^2\mu_6^2 - 24\mu_1\mu_4\mu_5\mu_6^3 - 8\mu_1\mu_4\mu_6^4 - 2\mu_2^3\mu_3\mu_4^2 - 4\mu_2^3\mu_4^3 - 2\mu_2^3\mu_4^2\mu_5 - 4\mu_2^3\mu_4^2\mu_6 - 2\mu_2^2\mu_3^2\mu_4^2 - 4\mu_2^2\mu_3\mu_4^3 - 10\mu_2^2\mu_3\mu_4^2\mu_5 - 8\mu_2^2\mu_3\mu_4^2\mu_6 - 4\mu_2^2\mu_4^4 - 22\mu_2^2\mu_4^3\mu_5 - 16\mu_2^2\mu_4^3\mu_6 - 8\mu_2^2\mu_4^2\mu_5^2 - 24\mu_2^2\mu_4^2\mu_5\mu_6 - 12\mu_2^2\mu_4^2\mu_6^2 - 6\mu_2\mu_3^2\mu_4^2\mu_5 - 2\mu_2\mu_3^2\mu_4^2\mu_6 - 8\mu_2\mu_3\mu_4^4 - 12\mu_2\mu_3\mu_4^3\mu_5 - 16\mu_2\mu_3\mu_4^3\mu_6 - 18\mu_2\mu_3\mu_4^2\mu_5^2 - 26\mu_2\mu_3\mu_4^2\mu_5\mu_6 - 10\mu_2\mu_3\mu_4^2\mu_6^2 - 12\mu_2\mu_4^5 - 14\mu_2\mu_4^4\mu_5 - 32\mu_2\mu_4^4\mu_6 - 42\mu_2\mu_4^3\mu_5^2 - 56\mu_2\mu_4^3\mu_5\mu_6 - 32\mu_2\mu_4^3\mu_6^2 - 12\mu_2\mu_4^2\mu_5^3 - 48\mu_2\mu_4^2\mu_5^2\mu_6 - 42\mu_2\mu_4^2\mu_5\mu_6^2 - 12\mu_2\mu_4^2\mu_6^3 - 8\mu_3^2\mu_4^4 - 12\mu_3^2\mu_4^3\mu_6 - 6\mu_3^2\mu_4^2\mu_5^2 - 4\mu_3^2\mu_4^2\mu_5\mu_6 - 4\mu_3^2\mu_4^2\mu_6^2 - 16\mu_3\mu_4^5 - 16\mu_3\mu_4^4\mu_5 - 48\mu_3\mu_4^4\mu_6 - 12\mu_3\mu_4^3\mu_5^2 - 32\mu_3\mu_4^3\mu_5\mu_6 - 44\mu_3\mu_4^3\mu_6^2 - 12\mu_3\mu_4^2\mu_5^3 - 26\mu_3\mu_4^2\mu_5^2\mu_6 - 20\mu_3\mu_4^2\mu_5\mu_6^2 - 12\mu_3\mu_4^2\mu_6^3 - 8\mu_4^6 - 24\mu_4^5\mu_5 - 40\mu_4^5\mu_6 - 14\mu_4^4\mu_5^2 - 64\mu_4^4\mu_5\mu_6 - 64\mu_4^4\mu_6^2 - 28\mu_4^3\mu_5^3 - 56\mu_4^3\mu_5^2\mu_6 - 64\mu_4^3\mu_5\mu_6^2 - 40\mu_4^3\mu_6^3 - 6\mu_4^2\mu_5^4 - 32\mu_4^2\mu_5^3\mu_6 - 42\mu_4^2\mu_5^2\mu_6^2 - 24\mu_4^2\mu_5\mu_6^3 - 8\mu_4^2\mu_6^4.$$

$$a_{Y^2} = - \mu_1^4 + \mu_1^3\mu_3 - 5\mu_1^3\mu_4 - 2\mu_1^3\mu_6 - 2\mu_1^2\mu_2^2 + \mu_1^2\mu_2\mu_3 - 8\mu_1^2\mu_2\mu_5 - 6\mu_1^2\mu_2\mu_6 + \mu_1^2\mu_3^2 + 7\mu_1^2\mu_3\mu_4 + 2\mu_1^2\mu_3\mu_5 + 3\mu_1^2\mu_3\mu_6 - 9\mu_1^2\mu_4^2 - 8\mu_1^2\mu_4\mu_6 - 8\mu_1^2\mu_5^2 - 12\mu_1^2\mu_5\mu_6 - 5\mu_1^2\mu_6^2 - 2\mu_1\mu_2^3 - \mu_1\mu_2^2\mu_3 - 8\mu_1\mu_2^2\mu_4 - 10\mu_1\mu_2^2\mu_5 - 10\mu_1\mu_2^2\mu_6 + 5\mu_1\mu_2\mu_3\mu_4 - 3\mu_1\mu_2\mu_3\mu_5 - 2\mu_1\mu_2\mu_3\mu_6 - 27\mu_1\mu_2\mu_4\mu_5 - 20\mu_1\mu_2\mu_4\mu_6 - 18\mu_1\mu_2\mu_5^2 - 36\mu_1\mu_2\mu_5\mu_6 - 16\mu_1\mu_2\mu_6^2 + 5\mu_1\mu_3^2\mu_4 + 15\mu_1\mu_3\mu_4^2 + 10\mu_1\mu_3\mu_4\mu_5 + 15\mu_1\mu_3\mu_4\mu_6 - 3\mu_1\mu_3\mu_5^2 - 4\mu_1\mu_3\mu_5\mu_6 - \mu_1\mu_3\mu_6^2 - 8\mu_1\mu_4^3 - 12\mu_1\mu_4^2\mu_6 - 27\mu_1\mu_4\mu_5^2 - 40\mu_1\mu_4\mu_5\mu_6 - 16\mu_1\mu_4\mu_6^2 - 12\mu_1\mu_5^3 - 36\mu_1\mu_5^2\mu_6 - 32\mu_1\mu_5\mu_6^2 - 8\mu_1\mu_6^3 - \mu_2^4 - \mu_2^3\mu_3 - 4\mu_2^3\mu_4 - 7\mu_2^3\mu_5 - 6\mu_2^3\mu_6 - \mu_2^2\mu_3^2 - 2\mu_2^2\mu_3\mu_4 - 5\mu_2^2\mu_3\mu_5 - 5\mu_2^2\mu_3\mu_6 - 8\mu_2^2\mu_4^2 - 20\mu_2^2\mu_4\mu_5 - 20\mu_2^2\mu_4\mu_6 - 19\mu_2^2\mu_5^2 - 32\mu_2^2\mu_5\mu_6 - 13\mu_2^2\mu_6^2 - 3\mu_2\mu_3^2\mu_5 - 2\mu_2\mu_3^2\mu_6 + 5\mu_2\mu_3\mu_4^2 - 6\mu_2\mu_3\mu_4\mu_5 - 4\mu_2\mu_3\mu_4\mu_6 - 9\mu_2\mu_3\mu_5^2 - 17\mu_2\mu_3\mu_5\mu_6 - 7\mu_2\mu_3\mu_6^2 - 27\mu_2\mu_4^2\mu_5 - 20\mu_2\mu_4^2\mu_6 - 36\mu_2\mu_4\mu_5^2 - 72\mu_2\mu_4\mu_5\mu_6 - 32\mu_2\mu_4\mu_6^2 - 24\mu_2\mu_5^3 - 60\mu_2\mu_5^2\mu_6 - 48\mu_2\mu_5\mu_6^2 - 12\mu_2\mu_6^3 + 5\mu_3^2\mu_4^2 - 3\mu_3^2\mu_5^2 - 4\mu_3^2\mu_5\mu_6 - \mu_3^2\mu_6^2 + 10\mu_3\mu_4^3 + 10\mu_3\mu_4^2\mu_5 + 15\mu_3\mu_4^2\mu_6 - 6\mu_3\mu_4\mu_5^2 - 8\mu_3\mu_4\mu_5\mu_6 - 2\mu_3\mu_4\mu_6^2 - 6\mu_3\mu_5^3 - 17\mu_3\mu_5^2\mu_6 - 14\mu_3\mu_5\mu_6^2 - 3\mu_3\mu_6^3 - 4\mu_4^4 - 8\mu_4^3\mu_6 - 27\mu_4^2\mu_5^2 - 40\mu_4^2\mu_5\mu_6 - 16\mu_4^2\mu_6^2 - 24\mu_4\mu_5^3 - 72\mu_4\mu_5^2\mu_6 - 64\mu_4\mu_5\mu_6^2 - 16\mu_4\mu_6^3 - 12\mu_5^4 - 40\mu_5^3\mu_6 - 48\mu_5^2\mu_6^2 - 24\mu_5\mu_6^3 - 4\mu_6^4.$$

$$a_{X^2Y} = 4\mu_1\mu_2 + \mu_1\mu_3 - \mu_1\mu_4 + 8\mu_1\mu_5 + 4\mu_1\mu_6 + \mu_2\mu_3 + 8\mu_2\mu_4 - \mu_2\mu_5 + 4\mu_2\mu_6 + \mu_3^2 + 2\mu_3\mu_4 + 2\mu_3\mu_5 + 3\mu_3\mu_6 - \mu_4^2 + 16\mu_4\mu_5 + 8\mu_4\mu_6 - \mu_5^2 + 8\mu_5\mu_6 + 4\mu_6^2.$$

$a_{XY^2} = 2\mu_1 - 2\mu_2 + 4\mu_4 - 4\mu_5.$

$a_{XY} = -2\mu_1^4\mu_2 - \mu_1^4\mu_3 + \mu_1^4\mu_4 - 4\mu_1^4\mu_5 - 2\mu_1^4\mu_6 + 4\mu_1^3\mu_2^2 + 4\mu_1^3\mu_2\mu_3 - 8\mu_1^3\mu_2\mu_4 + 16\mu_1^3\mu_2\mu_5 + 8\mu_1^3\mu_2\mu_6 - 8\mu_1^3\mu_3\mu_4 + 8\mu_1^3\mu_3\mu_5 + 2\mu_1^3\mu_3\mu_6 + 4\mu_1^3\mu_4^2 - 16\mu_1^3\mu_4\mu_5 - 6\mu_1^3\mu_4\mu_6 + 16\mu_1^3\mu_5^2 + 16\mu_1^3\mu_5\mu_6 + 4\mu_1^3\mu_6^2 + 4\mu_1^2\mu_2^3 + 10\mu_1^2\mu_2^2\mu_3 + 28\mu_1^2\mu_2^2\mu_4 + 18\mu_1^2\mu_2^2\mu_5 + 24\mu_1^2\mu_2^2\mu_6 + 8\mu_1^2\mu_2\mu_3^2 + 24\mu_1^2\mu_2\mu_3\mu_4 + 38\mu_1^2\mu_2\mu_3\mu_5 + 36\mu_1^2\mu_2\mu_3\mu_6 - 8\mu_1^2\mu_2\mu_4^2 + 102\mu_1^2\mu_2\mu_4\mu_5 + 62\mu_1^2\mu_2\mu_4\mu_6 + 30\mu_1^2\mu_2\mu_5^2 + 84\mu_1^2\mu_2\mu_5\mu_6 + 42\mu_1^2\mu_2\mu_6^2 + 2\mu_1^2\mu_3^3 + 16\mu_1^2\mu_3^2\mu_5 + 12\mu_1^2\mu_3^2\mu_6 - 24\mu_1^2\mu_3\mu_4^2 + 48\mu_1^2\mu_3\mu_4\mu_5 + 12\mu_1^2\mu_3\mu_4\mu_6 + 38\mu_1^2\mu_3\mu_5^2 + 72\mu_1^2\mu_3\mu_5\mu_6 + 27\mu_1^2\mu_3\mu_6^2 + 6\mu_1^2\mu_4^3 - 16\mu_1^2\mu_4^2\mu_5 + 102\mu_1^2\mu_4\mu_5^2 + 124\mu_1^2\mu_4\mu_5\mu_6 + 37\mu_1^2\mu_4\mu_6^2 + 20\mu_1^2\mu_5^3 + 84\mu_1^2\mu_5^2\mu_6 + 84\mu_1^2\mu_5\mu_6^2 + 22\mu_1^2\mu_6^3 + 2\mu_1\mu_2^4 + 6\mu_1\mu_2^3\mu_3 + 16\mu_1\mu_2^3\mu_4 + 14\mu_1\mu_2^3\mu_5 + 16\mu_1\mu_2^3\mu_6 + 8\mu_1\mu_2^2\mu_3^2 + 40\mu_1\mu_2^2\mu_3\mu_4 + 34\mu_1\mu_2^2\mu_3\mu_5 + 38\mu_1\mu_2^2\mu_3\mu_6 + 60\mu_1\mu_2^2\mu_4^2 + 80\mu_1\mu_2^2\mu_4\mu_5 + 104\mu_1\mu_2^2\mu_4\mu_6 + 38\mu_1\mu_2^2\mu_5^2 + 82\mu_1\mu_2^2\mu_5\mu_6 + 46\mu_1\mu_2^2\mu_6^2 + 4\mu_1\mu_2\mu_3^3 + 32\mu_1\mu_2\mu_3^2\mu_4 + 30\mu_1\mu_2\mu_3^2\mu_5 + 30\mu_1\mu_2\mu_3^2\mu_6 + 48\mu_1\mu_2\mu_3\mu_4^2 + 152\mu_1\mu_2\mu_3\mu_4\mu_5 + 144\mu_1\mu_2\mu_3\mu_4\mu_6 + 66\mu_1\mu_2\mu_3\mu_5^2 + 144\mu_1\mu_2\mu_3\mu_5\mu_6 + 72\mu_1\mu_2\mu_3\mu_6^2 + 210\mu_1\mu_2\mu_4^2\mu_5 + 138\mu_1\mu_2\mu_4^2\mu_6 + 144\mu_1\mu_2\mu_4\mu_5^2 + 368\mu_1\mu_2\mu_4\mu_5\mu_6 + 180\mu_1\mu_2\mu_4\mu_6^2 + 48\mu_1\mu_2\mu_5^3 + 150\mu_1\mu_2\mu_5^2\mu_6 + 164\mu_1\mu_2\mu_5\mu_6^2 + 56\mu_1\mu_2\mu_6^3 + \mu_1\mu_3^4 + 8\mu_1\mu_3^3\mu_4 + 8\mu_1\mu_3^3\mu_5 + 8\mu_1\mu_3^3\mu_6 + 64\mu_1\mu_3^2\mu_4\mu_5 + 48\mu_1\mu_3^2\mu_4\mu_6 + 30\mu_1\mu_3^2\mu_5^2 + 60\mu_1\mu_3^2\mu_5\mu_6 + 27\mu_1\mu_3^2\mu_6^2 - 32\mu_1\mu_3\mu_4^3 + 96\mu_1\mu_3\mu_4^2\mu_5 + 24\mu_1\mu_3\mu_4^2\mu_6 + 152\mu_1\mu_3\mu_4\mu_5^2 + 288\mu_1\mu_3\mu_4\mu_5\mu_6 + 108\mu_1\mu_3\mu_4\mu_6^2 + 44\mu_1\mu_3\mu_5^3 + 144\mu_1\mu_3\mu_5^2\mu_6 + 144\mu_1\mu_3\mu_5\mu_6^2 + 42\mu_1\mu_3\mu_6^3 + 5\mu_1\mu_4^4 + 12\mu_1\mu_4^3\mu_6 + 210\mu_1\mu_4^2\mu_5^2 + 276\mu_1\mu_4^2\mu_5\mu_6 + 87\mu_1\mu_4^2\mu_6^2 + 96\mu_1\mu_4\mu_5^3 + 368\mu_1\mu_4\mu_5^2\mu_6 + 360\mu_1\mu_4\mu_5\mu_6^2 + 94\mu_1\mu_4\mu_6^3 + 24\mu_1\mu_5^4 + 100\mu_1\mu_5^3\mu_6 + 164\mu_1\mu_5^2\mu_6^2 + 112\mu_1\mu_5\mu_6^3 + 24\mu_1\mu_6^4 + \mu_2^4\mu_3 + 4\mu_2^4\mu_4 - \mu_2^4\mu_5 + 2\mu_2^4\mu_6 + 2\mu_2^3\mu_3^2 + 12\mu_2^3\mu_3\mu_4 + 8\mu_2^3\mu_3\mu_5 + 10\mu_2^3\mu_3\mu_6 + 16\mu_2^3\mu_4^2 + 28\mu_2^3\mu_4\mu_5 + 32\mu_2^3\mu_4\mu_6 - 6\mu_2^3\mu_5^2 + 10\mu_2^3\mu_5\mu_6 + 12\mu_2^3\mu_6^2 + 2\mu_2^2\mu_3^3 + 16\mu_2^2\mu_3^2\mu_4 + 12\mu_2^2\mu_3^2\mu_5 + 14\mu_2^2\mu_3^2\mu_6 + 40\mu_2^2\mu_3\mu_4^2 + 68\mu_2^2\mu_3\mu_4\mu_5 + 76\mu_2^2\mu_3\mu_4\mu_6 + 24\mu_2^2\mu_3\mu_5^2 + 58\mu_2^2\mu_3\mu_5\mu_6 + 33\mu_2^2\mu_3\mu_6^2 + 40\mu_2^2\mu_4^3 + 80\mu_2^2\mu_4^2\mu_5 + 104\mu_2^2\mu_4^2\mu_6 + 76\mu_2^2\mu_4\mu_5^2 + 164\mu_2^2\mu_4\mu_5\mu_6 + 92\mu_2^2\mu_4\mu_6^2 - 14\mu_2^2\mu_5^3 + 20\mu_2^2\mu_5^2\mu_6 + 59\mu_2^2\mu_5\mu_6^2 + 26\mu_2^2\mu_6^3 + \mu_2\mu_3^4 + 8\mu_2\mu_3^3\mu_4 + 8\mu_2\mu_3^3\mu_5 + 8\mu_2\mu_3^3\mu_6 + 32\mu_2\mu_3^2\mu_4^2 + 60\mu_2\mu_3^2\mu_4\mu_5 + 60\mu_2\mu_3^2\mu_4\mu_6 + 24\mu_2\mu_3^2\mu_5^2 + 54\mu_2\mu_3^2\mu_5\mu_6 + 27\mu_2\mu_3^2\mu_6^2 + 32\mu_2\mu_3\mu_4^3 + 152\mu_2\mu_3\mu_4^2\mu_5 + 144\mu_2\mu_3\mu_4^2\mu_6 + 132\mu_2\mu_3\mu_4\mu_5^2 + 288\mu_2\mu_3\mu_4\mu_5\mu_6 + 144\mu_2\mu_3\mu_4\mu_6^2 + 32\mu_2\mu_3\mu_5^3 + 114\mu_2\mu_3\mu_5^2\mu_6 + 126\mu_2\mu_3\mu_5\mu_6^2 + 42\mu_2\mu_3\mu_6^3 + 140\mu_2\mu_4^3\mu_5 + 92\mu_2\mu_4^3\mu_6 + 144\mu_2\mu_4^2\mu_5^2 + 368\mu_2\mu_4^2\mu_5\mu_6 + 180\mu_2\mu_4^2\mu_6^2 + 96\mu_2\mu_4\mu_5^3 + 300\mu_2\mu_4\mu_5^2\mu_6 + 328\mu_2\mu_4\mu_5\mu_6^2 + 112\mu_2\mu_4\mu_6^3 - 15\mu_2\mu_5^4 + 20\mu_2\mu_5^3\mu_6 + 105\mu_2\mu_5^2\mu_6^2 + 94\mu_2\mu_5\mu_6^3 + 24\mu_2\mu_6^4 + 2\mu_3^4\mu_4 + 2\mu_3^4\mu_5 + \mu_3^4\mu_6 + 8\mu_3^3\mu_4^2 + 16\mu_3^3\mu_4\mu_5 + 16\mu_3^3\mu_4\mu_6 + 8\mu_3^3\mu_5^2 + 16\mu_3^3\mu_5\mu_6 + 6\mu_3^3\mu_6^2 + 64\mu_3^2\mu_4^2\mu_5 + 48\mu_3^2\mu_4^2\mu_6 + 60\mu_3^2\mu_4\mu_5^2 + 120\mu_3^2\mu_4\mu_5\mu_6 + 54\mu_3^2\mu_4\mu_6^2 + 16\mu_3^2\mu_5^3 + 54\mu_3^2\mu_5^2\mu_6 + 54\mu_3^2\mu_5\mu_6^2 + 15\mu_3^2\mu_6^3 - 16\mu_3\mu_4^4 + 64\mu_3\mu_4^3\mu_5 + 16\mu_3\mu_4^3\mu_6 + 152\mu_3\mu_4^2\mu_5^2 + 288\mu_3\mu_4^2\mu_5\mu_6 + 108\mu_3\mu_4^2\mu_6^2 + 88\mu_3\mu_4\mu_5^3 + 288\mu_3\mu_4\mu_5^2\mu_6 + 288\mu_3\mu_4\mu_5\mu_6^2 + 84\mu_3\mu_4\mu_6^3 + 16\mu_3\mu_5^4 + 76\mu_3\mu_5^3\mu_6 + 126\mu_3\mu_5^2\mu_6^2 + 84\mu_3\mu_5\mu_6^3 + 18\mu_3\mu_6^4 + 2\mu_4^5 + 6\mu_4^4\mu_6 + 140\mu_4^3\mu_5^2 + 184\mu_4^3\mu_5\mu_6 + 58\mu_4^3\mu_6^2 + 96\mu_4^2\mu_5^3 + 368\mu_4^2\mu_5^2\mu_6 + 360\mu_4^2\mu_5\mu_6^2 + 94\mu_4^2\mu_6^3 + 48\mu_4\mu_5^4 + 200\mu_4\mu_5^3\mu_6 + 328\mu_4\mu_5^2\mu_6^2 + 224\mu_4\mu_5\mu_6^3 + 48\mu_4\mu_6^4 - 6\mu_5^5 + 10\mu_5^4\mu_6 + 70\mu_5^3\mu_6^2 + 94\mu_5^2\mu_6^3 + 48\mu_5\mu_6^4 + 8\mu_6^5.$

$a_{XZ} = -\mu_1^2\mu_3 - 3\mu_1^2\mu_4 - 2\mu_1^2\mu_6 - 2\mu_1\mu_2^2 - 2\mu_1\mu_2\mu_3 - 2\mu_1\mu_2\mu_4 - 6\mu_1\mu_2\mu_5 - 4\mu_1\mu_2\mu_6 - \mu_1\mu_3^2 - 4\mu_1\mu_3\mu_4 - 4\mu_1\mu_3\mu_5 - 4\mu_1\mu_3\mu_6 - 9\mu_1\mu_4^2 - 4\mu_1\mu_4\mu_5 - 12\mu_1\mu_4\mu_6 - 6\mu_1\mu_5^2 - 8\mu_1\mu_5\mu_6 - 4\mu_1\mu_6^2 -$

$\mu_2^2\mu_3$ - $4\mu_2^2\mu_4$ + $\mu_2^2\mu_5$ - $2\mu_2^2\mu_6$ - $\mu_2\mu_3^2$ - $4\mu_2\mu_3\mu_4$ - $4\mu_2\mu_3\mu_5$ - $4\mu_2\mu_3\mu_6$ - $2\mu_2\mu_4^2$ - $12\mu_2\mu_4\mu_5$ - $8\mu_2\mu_4\mu_6$ + $3\mu_2\mu_5^2$ - $4\mu_2\mu_5\mu_6$ - $4\mu_2\mu_6^2$ - $2\mu_3^2\mu_4$ - $2\mu_3^2\mu_5$ - $\mu_3^2\mu_6$ - $4\mu_3\mu_4^2$ - $8\mu_3\mu_4\mu_5$ - $8\mu_3\mu_4\mu_6$ - $4\mu_3\mu_5^2$ - $8\mu_3\mu_5\mu_6$ - $3\mu_3\mu_6^2$ - $6\mu_4^3$ - $4\mu_4^2\mu_5$ - $12\mu_4^2\mu_6$ - $12\mu_4\mu_5^2$ - $16\mu_4\mu_5\mu_6$ - $8\mu_4\mu_6^2$ + $2\mu_5^3$ - $4\mu_5^2\mu_6$ - $8\mu_5\mu_6^2$ - $2\mu_6^3$.

a_{YZ} = $2\mu_1^2$ + $2\mu_1\mu_2$ + $\mu_1\mu_3$ + $7\mu_1\mu_4$ + $4\mu_1\mu_5$ + $6\mu_1\mu_6$ + $2\mu_2^2$ + $\mu_2\mu_3$ + $4\mu_2\mu_4$ + $7\mu_2\mu_5$ + $6\mu_2\mu_6$ + μ_3^2 + $2\mu_3\mu_4$ + $2\mu_3\mu_5$ + $3\mu_3\mu_6$ + $7\mu_4^2$ + $8\mu_4\mu_5$ + $12\mu_4\mu_6$ + $7\mu_5^2$ + $12\mu_5\mu_6$ + $5\mu_6^2$.

a_0 = $2\mu_1^8\mu_2^3\mu_4$ - $\mu_1^8\mu_2^3\mu_5$ + $5\mu_1^8\mu_2^2\mu_3\mu_4$ - $2\mu_1^8\mu_2^2\mu_3\mu_5$ + $4\mu_1^8\mu_2^2\mu_4^2$ + $10\mu_1^8\mu_2^2\mu_4\mu_5$
+ $10\mu_1^8\mu_2^2\mu_4\mu_6$ - $5\mu_1^8\mu_2^2\mu_5^2$ - $4\mu_1^8\mu_2^2\mu_5\mu_6$ + $4\mu_1^8\mu_2\mu_3^2\mu_4$ - $\mu_1^8\mu_2\mu_3^2\mu_5$ + $6\mu_1^8\mu_2\mu_3\mu_4^2$
+ $18\mu_1^8\mu_2\mu_3\mu_4\mu_5$ + $16\mu_1^8\mu_2\mu_3\mu_4\mu_6$ - $6\mu_1^8\mu_2\mu_3\mu_5^2$ - $4\mu_1^8\mu_2\mu_3\mu_5\mu_6$ + $2\mu_1^8\mu_2\mu_4^3$ + $15\mu_1^8\mu_2\mu_4^2\mu_5$
+ $12\mu_1^8\mu_2\mu_4^2\mu_6$ + $18\mu_1^8\mu_2\mu_4\mu_5^2$ + $36\mu_1^8\mu_2\mu_4\mu_5\mu_6$ + $16\mu_1^8\mu_2\mu_4\mu_6^2$ - $8\mu_1^8\mu_2\mu_5^3$ - $12\mu_1^8\mu_2\mu_5^2\mu_6$
- $4\mu_1^8\mu_2\mu_5\mu_6^2$ + $\mu_1^8\mu_3^3\mu_4$ + $2\mu_1^8\mu_3^2\mu_4^2$ + $8\mu_1^8\mu_3^2\mu_4\mu_5$ + $6\mu_1^8\mu_3^2\mu_4\mu_6$ - $\mu_1^8\mu_3^2\mu_5^2$ + $\mu_1^8\mu_3\mu_4^3$ +
$12\mu_1^8\mu_3\mu_4^2\mu_5$ + $8\mu_1^8\mu_3\mu_4^2\mu_6$ + $18\mu_1^8\mu_3\mu_4\mu_5^2$ + $32\mu_1^8\mu_3\mu_4\mu_5\mu_6$ + $12\mu_1^8\mu_3\mu_4\mu_6^2$ - $4\mu_1^8\mu_3\mu_5^3$ -
$4\mu_1^8\mu_3\mu_5^2\mu_6$ + $4\mu_1^8\mu_4^3\mu_5$ + $2\mu_1^8\mu_4^3\mu_6$ + $15\mu_1^8\mu_4^2\mu_5^2$ + $24\mu_1^8\mu_4^2\mu_5\mu_6$ + $8\mu_1^8\mu_4^2\mu_6^2$ + $12\mu_1^8\mu_4\mu_5^3$
+ $36\mu_1^8\mu_4\mu_5^2\mu_6$ + $32\mu_1^8\mu_4\mu_5\mu_6^2$ + $8\mu_1^8\mu_4\mu_6^3$ - $4\mu_1^8\mu_5^4$ - $8\mu_1^8\mu_5^3\mu_6$ - $4\mu_1^8\mu_5^2\mu_6^2$ + $2\mu_1^7\mu_2^4\mu_4$
+ $10\mu_1^7\mu_2^3\mu_3\mu_4$ - $2\mu_1^7\mu_2^3\mu_3\mu_5$ + $26\mu_1^7\mu_2^3\mu_4^2$ + $6\mu_1^7\mu_2^3\mu_4\mu_5$ + $22\mu_1^7\mu_2^3\mu_4\mu_6$ - $4\mu_1^7\mu_2^3\mu_5\mu_6$ +
$17\mu_1^7\mu_2^2\mu_3^2\mu_4$ - $4\mu_1^7\mu_2^2\mu_3^2\mu_5$ + $73\mu_1^7\mu_2^2\mu_3\mu_4^2$ + $36\mu_1^7\mu_2^2\mu_3\mu_4\mu_5$ + $72\mu_1^7\mu_2^2\mu_3\mu_4\mu_6$ - $10\mu_1^7\mu_2^2\mu_3\mu_5^2$
- $16\mu_1^7\mu_2^2\mu_3\mu_5\mu_6$ + $46\mu_1^7\mu_2^2\mu_4^3$ + $136\mu_1^7\mu_2^2\mu_4^2\mu_5$ + $146\mu_1^7\mu_2^2\mu_4^2\mu_6$ - $2\mu_1^7\mu_2^2\mu_4\mu_5^2$ + $84\mu_1^7\mu_2^2\mu_4\mu_5\mu_6$
+ $76\mu_1^7\mu_2^2\mu_4\mu_6^2$ - $20\mu_1^7\mu_2^2\mu_5^2\mu_6$ - $16\mu_1^7\mu_2^2\mu_5\mu_6^2$ + $12\mu_1^7\mu_2\mu_3^3\mu_4$ - $2\mu_1^7\mu_2\mu_3^3\mu_5$ + $66\mu_1^7\mu_2\mu_3^2\mu_4^2$ +
$54\mu_1^7\mu_2\mu_3^2\mu_4\mu_5$ + $74\mu_1^7\mu_2\mu_3^2\mu_4\mu_6$ - $12\mu_1^7\mu_2\mu_3^2\mu_5^2$ - $12\mu_1^7\mu_2\mu_3^2\mu_5\mu_6$ + $76\mu_1^7\mu_2\mu_3\mu_4^3$
+ $270\mu_1^7\mu_2\mu_3\mu_4^2\mu_5$ + $260\mu_1^7\mu_2\mu_3\mu_4^2\mu_6$ + $48\mu_1^7\mu_2\mu_3\mu_4\mu_5^2$ + $232\mu_1^7\mu_2\mu_3\mu_4\mu_5\mu_6$
+ $152\mu_1^7\mu_2\mu_3\mu_4\mu_6^2$ - $16\mu_1^7\mu_2\mu_3\mu_5^3$ - $48\mu_1^7\mu_2\mu_3\mu_5^2\mu_6$ - $24\mu_1^7\mu_2\mu_3\mu_5\mu_6^2$ + $22\mu_1^7\mu_2\mu_4^4$
+ $174\mu_1^7\mu_2\mu_4^3\mu_5$ + $146\mu_1^7\mu_2\mu_4^3\mu_6$ + $252\mu_1^7\mu_2\mu_4^2\mu_5^2$ + $540\mu_1^7\mu_2\mu_4^2\mu_5\mu_6$ + $256\mu_1^7\mu_2\mu_4^2\mu_6^2$ -
$16\mu_1^7\mu_2\mu_4\mu_5^3$ + $120\mu_1^7\mu_2\mu_4\mu_5^2\mu_6$ + $248\mu_1^7\mu_2\mu_4\mu_5\mu_6^2$ + $104\mu_1^7\mu_2\mu_4\mu_6^3$ - $32\mu_1^7\mu_2\mu_5^3\mu_6$
- $48\mu_1^7\mu_2\mu_5^2\mu_6^2$ - $16\mu_1^7\mu_2\mu_5\mu_6^3$ + $3\mu_1^7\mu_3^4\mu_4$ + $19\mu_1^7\mu_3^3\mu_4^2$ + $24\mu_1^7\mu_3^3\mu_4\mu_5$ + $24\mu_1^7\mu_3^3\mu_4\mu_6$ -
$2\mu_1^7\mu_3^3\mu_5^2$ + $29\mu_1^7\mu_3^2\mu_4^3$ + $132\mu_1^7\mu_3^2\mu_4^2\mu_5$ + $110\mu_1^7\mu_3^2\mu_4^2\mu_6$ + $54\mu_1^7\mu_3^2\mu_4\mu_5^2$ + $148\mu_1^7\mu_3^2\mu_4\mu_5\mu_6$
+ $72\mu_1^7\mu_3^2\mu_4\mu_6^2$ - $8\mu_1^7\mu_3^2\mu_5^3$ - $12\mu_1^7\mu_3^2\mu_5^2\mu_6$ + $13\mu_1^7\mu_3\mu_4^4$ + $152\mu_1^7\mu_3\mu_4^3\mu_5$ + $108\mu_1^7\mu_3\mu_4^3\mu_6$
+ $270\mu_1^7\mu_3\mu_4^2\mu_5^2$ + $520\mu_1^7\mu_3\mu_4^2\mu_5\mu_6$ + $212\mu_1^7\mu_3\mu_4^2\mu_6^2$ + $32\mu_1^7\mu_3\mu_4\mu_5^3$ + $232\mu_1^7\mu_3\mu_4\mu_5^2\mu_6$ +
$304\mu_1^7\mu_3\mu_4\mu_5\mu_6^2$ + $96\mu_1^7\mu_3\mu_4\mu_6^3$ - $8\mu_1^7\mu_3\mu_5^4$ - $32\mu_1^7\mu_3\mu_5^3\mu_6$ - $24\mu_1^7\mu_3\mu_5^2\mu_6^2$ + $44\mu_1^7\mu_4^4\mu_5$ +
$22\mu_1^7\mu_4^4\mu_6$ + $174\mu_1^7\mu_4^3\mu_5^2$ + $292\mu_1^7\mu_4^3\mu_5\mu_6$ + $100\mu_1^7\mu_4^3\mu_6^2$ + $168\mu_1^7\mu_4^2\mu_5^3$ + $540\mu_1^7\mu_4^2\mu_5^2\mu_6$ +
$512\mu_1^7\mu_4^2\mu_5\mu_6^2$ + $136\mu_1^7\mu_4^2\mu_6^3$ - $8\mu_1^7\mu_4\mu_5^4$ + $80\mu_1^7\mu_4\mu_5^3\mu_6$ + $248\mu_1^7\mu_4\mu_5^2\mu_6^2$ + $208\mu_1^7\mu_4\mu_5\mu_6^3$
+ $48\mu_1^7\mu_4\mu_6^4$ - $16\mu_1^7\mu_5^4\mu_6$ - $32\mu_1^7\mu_5^3\mu_6^2$ - $16\mu_1^7\mu_5^2\mu_6^3$ + $\mu_1^6\mu_2^5\mu_4$ + $7\mu_1^6\mu_2^4\mu_3\mu_4$ + $19\mu_1^6\mu_2^4\mu_4^2$ +
$12\mu_1^6\mu_2^4\mu_4\mu_5$ + $16\mu_1^6\mu_2^4\mu_4\mu_6$ + $18\mu_1^6\mu_2^3\mu_3^2\mu_4$ - $\mu_1^6\mu_2^3\mu_3^2\mu_5$ + $107\mu_1^6\mu_2^3\mu_3\mu_4^2$ + $42\mu_1^6\mu_2^3\mu_3\mu_4\mu_5$ +
$83\mu_1^6\mu_2^3\mu_3\mu_4\mu_6$ - $6\mu_1^6\mu_2^3\mu_3\mu_5\mu_6$ + $142\mu_1^6\mu_2^3\mu_4^3$ + $115\mu_1^6\mu_2^3\mu_4^2\mu_5$ + $228\mu_1^6\mu_2^3\mu_4^2\mu_6$ + $52\mu_1^6\mu_2^3\mu_4\mu_5^2$
+ $100\mu_1^6\mu_2^3\mu_4\mu_5\mu_6$ + $91\mu_1^6\mu_2^3\mu_4\mu_6^2$ - $6\mu_1^6\mu_2^3\mu_5\mu_6^2$ + $22\mu_1^6\mu_2^2\mu_3^3\mu_4$ - $2\mu_1^6\mu_2^2\mu_3^3\mu_5$ + $199\mu_1^6\mu_2^2\mu_3^2\mu_4^2$
+ $73\mu_1^6\mu_2^2\mu_3^2\mu_4\mu_5$ + $152\mu_1^6\mu_2^2\mu_3^2\mu_4\mu_6$ - $5\mu_1^6\mu_2^2\mu_3^2\mu_5^2$ - $16\mu_1^6\mu_2^2\mu_3^2\mu_5\mu_6$ + $450\mu_1^6\mu_2^2\mu_3\mu_4^3$ +
$533\mu_1^6\mu_2^2\mu_3\mu_4^2\mu_5$ + $825\mu_1^6\mu_2^2\mu_3\mu_4^2\mu_6$ + $96\mu_1^6\mu_2^2\mu_3\mu_4\mu_5^2$ + $356\mu_1^6\mu_2^2\mu_3\mu_4\mu_5\mu_6$ + $334\mu_1^6\mu_2^2\mu_3\mu_4\mu_6^2$
- $30\mu_1^6\mu_2^2\mu_3\mu_5^2\mu_6$ - $36\mu_1^6\mu_2^2\mu_3\mu_5\mu_6^2$ + $230\mu_1^6\mu_2^2\mu_4^4$ + $776\mu_1^6\mu_2^2\mu_4^3\mu_5$ + $882\mu_1^6\mu_2^2\mu_4^3\mu_6$
+ $269\mu_1^6\mu_2^2\mu_4^2\mu_5^2$ + $1150\mu_1^6\mu_2^2\mu_4^2\mu_5\mu_6$ + $845\mu_1^6\mu_2^2\mu_4^2\mu_6^2$ + $108\mu_1^6\mu_2^2\mu_4\mu_5^3$ + $240\mu_1^6\mu_2^2\mu_4\mu_5^2\mu_6$ +

$404\mu_1^6\mu_2^2\mu_4\mu_5\mu_6^2 + 236\mu_1^6\mu_2^2\mu_4\mu_6^3 - 30\mu_1^6\mu_2^2\mu_5^2\mu_6^2 - 24\mu_1^6\mu_2^2\mu_5\mu_6^3 + 13\mu_1^6\mu_2\mu_3^4\mu_4 - \mu_1^6\mu_2\mu_3^4\mu_5$
$+ 152\mu_1^6\mu_2\mu_3^3\mu_4^2 + 69\mu_1^6\mu_2\mu_3^3\mu_4\mu_5 + 118\mu_1^6\mu_2\mu_3^3\mu_4\mu_6 - 6\mu_1^6\mu_2\mu_3^3\mu_5^2 - 10\mu_1^6\mu_2\mu_3^3\mu_5\mu_6 +$
$455\mu_1^6\mu_2\mu_3^2\mu_4^3 + 724\mu_1^6\mu_2\mu_3^2\mu_4^2\mu_5 + 924\mu_1^6\mu_2\mu_3^2\mu_4^2\mu_6 + 111\mu_1^6\mu_2\mu_3^2\mu_4\mu_5^2 + 492\mu_1^6\mu_2\mu_3^2\mu_4\mu_5\mu_6$
$+ 386\mu_1^6\mu_2\mu_3^2\mu_4\mu_6^2 - 8\mu_1^6\mu_2\mu_3^2\mu_5^3 - 48\mu_1^6\mu_2\mu_3^2\mu_5^2\mu_6 - 34\mu_1^6\mu_2\mu_3^2\mu_5\mu_6^2 + 422\mu_1^6\mu_2\mu_3\mu_4^4$
$+ 1709\mu_1^6\mu_2\mu_3\mu_4^3\mu_5 + 1750\mu_1^6\mu_2\mu_3\mu_4^3\mu_6 + 957\mu_1^6\mu_2\mu_3\mu_4^2\mu_5^2 + 3012\mu_1^6\mu_2\mu_3\mu_4^2\mu_5\mu_6$
$+ 1854\mu_1^6\mu_2\mu_3\mu_4^2\mu_6^2 + 108\mu_1^6\mu_2\mu_3\mu_4\mu_5^3 + 570\mu_1^6\mu_2\mu_3\mu_4\mu_5^2\mu_6 + 1102\mu_1^6\mu_2\mu_3\mu_4\mu_5\mu_6^2$
$+ 544\mu_1^6\mu_2\mu_3\mu_4\mu_6^3 - 48\mu_1^6\mu_2\mu_3\mu_5^3\mu_6 - 108\mu_1^6\mu_2\mu_3\mu_5^2\mu_6^2 - 48\mu_1^6\mu_2\mu_3\mu_5\mu_6^3 + 106\mu_1^6\mu_2\mu_4^5 +$
$883\mu_1^6\mu_2\mu_4^4\mu_5 + 774\mu_1^6\mu_2\mu_4^4\mu_6 + 1476\mu_1^6\mu_2\mu_4^3\mu_5^2 + 3346\mu_1^6\mu_2\mu_4^3\mu_5\mu_6 + 1670\mu_1^6\mu_2\mu_4^3\mu_6^2 +$
$308\mu_1^6\mu_2\mu_4^2\mu_5^3 + 2082\mu_1^6\mu_2\mu_4^2\mu_5^2\mu_6 + 3092\mu_1^6\mu_2\mu_4^2\mu_5\mu_6^2 + 1228\mu_1^6\mu_2\mu_4^2\mu_6^3 + 110\mu_1^6\mu_2\mu_4\mu_5^4$
$+ 280\mu_1^6\mu_2\mu_4\mu_5^3\mu_6 + 666\mu_1^6\mu_2\mu_4\mu_5^2\mu_6^2 + 788\mu_1^6\mu_2\mu_4\mu_5\mu_6^3 + 280\mu_1^6\mu_2\mu_4\mu_6^4 - 48\mu_1^6\mu_2\mu_5^3\mu_6^2$
$- 72\mu_1^6\mu_2\mu_5^2\mu_6^3 - 24\mu_1^6\mu_2\mu_5\mu_6^4 + 3\mu_1^6\mu_3^5\mu_4 + 41\mu_1^6\mu_3^4\mu_4^2 + 26\mu_1^6\mu_3^4\mu_4\mu_5 + 33\mu_1^6\mu_3^4\mu_4\mu_6 -$
$\mu_1^6\mu_3^4\mu_5^2 + 146\mu_1^6\mu_3^3\mu_4^3 + 304\mu_1^6\mu_3^3\mu_4^2\mu_5 + 324\mu_1^6\mu_3^3\mu_4^2\mu_6 + 69\mu_1^6\mu_3^3\mu_4\mu_5^2 + 236\mu_1^6\mu_3^3\mu_4\mu_5\mu_6$
$+ 141\mu_1^6\mu_3^3\mu_4\mu_6^2 - 4\mu_1^6\mu_3^3\mu_5^3 - 10\mu_1^6\mu_3^3\mu_5^2\mu_6 + 181\mu_1^6\mu_3^2\mu_4^4 + 910\mu_1^6\mu_3^2\mu_4^3\mu_5 + 821\mu_1^6\mu_3^2\mu_4^3\mu_6$
$+ 724\mu_1^6\mu_3^2\mu_4^2\mu_5^2 + 1848\mu_1^6\mu_3^2\mu_4^2\mu_5\mu_6 + 952\mu_1^6\mu_3^2\mu_4^2\mu_6^2 + 74\mu_1^6\mu_3^2\mu_4\mu_5^3 + 492\mu_1^6\mu_3^2\mu_4\mu_5^2\mu_6 +$
$772\mu_1^6\mu_3^2\mu_4\mu_5\mu_6^2 + 294\mu_1^6\mu_3^2\mu_4\mu_6^3 - 4\mu_1^6\mu_3^2\mu_5^4 - 32\mu_1^6\mu_3^2\mu_5^3\mu_6 - 34\mu_1^6\mu_3^2\mu_5^2\mu_6^2 + 73\mu_1^6\mu_3\mu_4^5 +$
$844\mu_1^6\mu_3\mu_4^4\mu_5 + 636\mu_1^6\mu_3\mu_4^4\mu_6 + 1709\mu_1^6\mu_3\mu_4^3\mu_5^2 + 3500\mu_1^6\mu_3\mu_4^3\mu_5\mu_6 + 1523\mu_1^6\mu_3\mu_4^3\mu_6^2 +$
$638\mu_1^6\mu_3\mu_4^2\mu_5^3 + 3012\mu_1^6\mu_3\mu_4^2\mu_5^2\mu_6 + 3708\mu_1^6\mu_3\mu_4^2\mu_5\mu_6^2 + 1232\mu_1^6\mu_3\mu_4^2\mu_6^3 + 54\mu_1^6\mu_3\mu_4\mu_5^4 +$
$380\mu_1^6\mu_3\mu_4\mu_5^3\mu_6 + 1102\mu_1^6\mu_3\mu_4\mu_5^2\mu_6^2 + 1088\mu_1^6\mu_3\mu_4\mu_5\mu_6^3 + 300\mu_1^6\mu_3\mu_4\mu_6^4 - 24\mu_1^6\mu_3\mu_5^4\mu_6 -$
$72\mu_1^6\mu_3\mu_5^3\mu_6^2 - 48\mu_1^6\mu_3\mu_5^2\mu_6^3 + 212\mu_1^6\mu_4^5\mu_5 + 106\mu_1^6\mu_4^5\mu_6 + 883\mu_1^6\mu_4^4\mu_5^2 + 1548\mu_1^6\mu_4^4\mu_5\mu_6 +$
$544\mu_1^6\mu_4^4\mu_6^2 + 984\mu_1^6\mu_4^3\mu_5^3 + 3346\mu_1^6\mu_4^3\mu_5^2\mu_6 + 3340\mu_1^6\mu_4^3\mu_5\mu_6^2 + 930\mu_1^6\mu_4^3\mu_6^3 + 154\mu_1^6\mu_4^2\mu_5^4$
$+ 1388\mu_1^6\mu_4^2\mu_5^3\mu_6 + 3092\mu_1^6\mu_4^2\mu_5^2\mu_6^2 + 2456\mu_1^6\mu_4^2\mu_5\mu_6^3 + 592\mu_1^6\mu_4^2\mu_6^4 + 44\mu_1^6\mu_4\mu_5^5$
$+ 140\mu_1^6\mu_4\mu_5^4\mu_6 + 444\mu_1^6\mu_4\mu_5^3\mu_6^2 + 788\mu_1^6\mu_4\mu_5^2\mu_6^3 + 560\mu_1^6\mu_4\mu_5\mu_6^4 + 120\mu_1^6\mu_4\mu_6^5 - 24\mu_1^6\mu_5^4\mu_6^2$
$- 48\mu_1^6\mu_5^3\mu_6^3 - 24\mu_1^6\mu_5^2\mu_6^4 + 2\mu_1^5\mu_2^5\mu_3\mu_4 + 6\mu_1^5\mu_2^5\mu_4^2 + \mu_1^5\mu_2^5\mu_4\mu_5 + 4\mu_1^5\mu_2^5\mu_4\mu_6 + 8\mu_1^5\mu_2^4\mu_3^2\mu_4$
$+ 53\mu_1^5\mu_2^4\mu_3\mu_4^2 + 25\mu_1^5\mu_2^4\mu_3\mu_4\mu_5 + 38\mu_1^5\mu_2^4\mu_3\mu_4\mu_6 + 72\mu_1^5\mu_2^4\mu_4^3 + 79\mu_1^5\mu_2^4\mu_4^2\mu_5$
$+ 114\mu_1^5\mu_2^4\mu_4^2\mu_6 + 10\mu_1^5\mu_2^4\mu_4\mu_5^2 + 52\mu_1^5\mu_2^4\mu_4\mu_5\mu_6 + 42\mu_1^5\mu_2^4\mu_4\mu_6^2 + 14\mu_1^5\mu_2^3\mu_3^3\mu_4$
$+ 159\mu_1^5\mu_2^3\mu_3^2\mu_4^2 + 66\mu_1^5\mu_2^3\mu_3^2\mu_4\mu_5 + 106\mu_1^5\mu_2^3\mu_3^2\mu_4\mu_6 - 2\mu_1^5\mu_2^3\mu_3^2\mu_5\mu_6 + 482\mu_1^5\mu_2^3\mu_3\mu_4^3 +$
$414\mu_1^5\mu_2^3\mu_3\mu_4^2\mu_5 + 706\mu_1^5\mu_2^3\mu_3\mu_4^2\mu_6 + 116\mu_1^5\mu_2^3\mu_3\mu_4\mu_5^2 + 299\mu_1^5\mu_2^3\mu_3\mu_4\mu_5\mu_6$
$+ 246\mu_1^5\mu_2^3\mu_3\mu_4\mu_6^2 - 6\mu_1^5\mu_2^3\mu_3\mu_5\mu_6^2 + 430\mu_1^5\mu_2^3\mu_4^4 + 532\mu_1^5\mu_2^3\mu_4^3\mu_5 + 980\mu_1^5\mu_2^3\mu_4^3\mu_6$
$+ 366\mu_1^5\mu_2^3\mu_4^2\mu_5^2 + 892\mu_1^5\mu_2^3\mu_4^2\mu_5\mu_6 + 736\mu_1^5\mu_2^3\mu_4^2\mu_6^2 + 38\mu_1^5\mu_2^3\mu_4\mu_5^3 + 246\mu_1^5\mu_2^3\mu_4\mu_5^2\mu_6 +$
$333\mu_1^5\mu_2^3\mu_4\mu_5\mu_6^2 + 178\mu_1^5\mu_2^3\mu_4\mu_6^3 - 4\mu_1^5\mu_2^3\mu_5\mu_6^3 + 13\mu_1^5\mu_2^2\mu_3^4\mu_4 + 218\mu_1^5\mu_2^2\mu_3^3\mu_4^2$
$+ 77\mu_1^5\mu_2^2\mu_3^3\mu_4\mu_5 + 136\mu_1^5\mu_2^2\mu_3^3\mu_4\mu_6 - 4\mu_1^5\mu_2^2\mu_3^3\mu_5\mu_6 + 997\mu_1^5\mu_2^2\mu_3^2\mu_4^3 + 868\mu_1^5\mu_2^2\mu_3^2\mu_4^2\mu_5 +$
$1459\mu_1^5\mu_2^2\mu_3^2\mu_4^2\mu_6 + 202\mu_1^5\mu_2^2\mu_3^2\mu_4\mu_5^2 + 561\mu_1^5\mu_2^2\mu_3^2\mu_4\mu_5\mu_6 + 486\mu_1^5\mu_2^2\mu_3^2\mu_4\mu_6^2$
$- 10\mu_1^5\mu_2^2\mu_3^2\mu_5^2\mu_6 - 20\mu_1^5\mu_2^2\mu_3^2\mu_5\mu_6^2 + 1554\mu_1^5\mu_2^2\mu_3\mu_4^4 + 2695\mu_1^5\mu_2^2\mu_3\mu_4^3\mu_5 + 4020\mu_1^5\mu_2^2\mu_3\mu_4^3\mu_6$
$+ 1214\mu_1^5\mu_2^2\mu_3\mu_4^2\mu_5^2 + 3875\mu_1^5\mu_2^2\mu_3\mu_4^2\mu_5\mu_6 + 3093\mu_1^5\mu_2^2\mu_3\mu_4^2\mu_6^2 + 254\mu_1^5\mu_2^2\mu_3\mu_4\mu_5^3$
$+ 887\mu_1^5\mu_2^2\mu_3\mu_4\mu_5^2\mu_6 + 1293\mu_1^5\mu_2^2\mu_3\mu_4\mu_5\mu_6^2 + 722\mu_1^5\mu_2^2\mu_3\mu_4\mu_6^3 - 30\mu_1^5\mu_2^2\mu_3\mu_5^2\mu_6^2$
$- 32\mu_1^5\mu_2^2\mu_3\mu_5\mu_6^3 + 658\mu_1^5\mu_2^2\mu_4^5 + 2462\mu_1^5\mu_2^2\mu_4^4\mu_5 + 2946\mu_1^5\mu_2^2\mu_4^4\mu_6 + 1500\mu_1^5\mu_2^2\mu_4^3\mu_5^2 +$
$5488\mu_1^5\mu_2^2\mu_4^3\mu_5\mu_6 + 3956\mu_1^5\mu_2^2\mu_4^3\mu_6^2 + 794\mu_1^5\mu_2^2\mu_4^2\mu_5^3 + 2620\mu_1^5\mu_2^2\mu_4^2\mu_5^2\mu_6 + 4057\mu_1^5\mu_2^2\mu_4^2\mu_5\mu_6^2$
$+ 2092\mu_1^5\mu_2^2\mu_4^2\mu_6^3 + 69\mu_1^5\mu_2^2\mu_4\mu_5^4 + 544\mu_1^5\mu_2^2\mu_4\mu_5^3\mu_6 + 993\mu_1^5\mu_2^2\mu_4\mu_5^2\mu_6^2 + 946\mu_1^5\mu_2^2\mu_4\mu_5\mu_6^3$
$+ 380\mu_1^5\mu_2^2\mu_4\mu_6^4 - 20\mu_1^5\mu_2^2\mu_5^2\mu_6^3 - 16\mu_1^5\mu_2^2\mu_5\mu_6^4 + 6\mu_1^5\mu_2\mu_3^5\mu_4 + 139\mu_1^5\mu_2\mu_3^4\mu_4^2 + 47\mu_1^5\mu_2\mu_3^4\mu_4\mu_5$
$+ 82\mu_1^5\mu_2\mu_3^4\mu_4\mu_6 - 2\mu_1^5\mu_2\mu_3^4\mu_5\mu_6 + 824\mu_1^5\mu_2\mu_3^3\mu_4^3 + 815\mu_1^5\mu_2\mu_3^3\mu_4^2\mu_5 + 1234\mu_1^5\mu_2\mu_3^3\mu_4^2\mu_6 +$
$147\mu_1^5\mu_2\mu_3^3\mu_4\mu_5^2 + 486\mu_1^5\mu_2\mu_3^3\mu_4\mu_5\mu_6 + 400\mu_1^5\mu_2\mu_3^3\mu_4\mu_6^2 - 12\mu_1^5\mu_2\mu_3^3\mu_5^2\mu_6 - 14\mu_1^5\mu_2\mu_3^3\mu_5\mu_6^2$
$+ 1741\mu_1^5\mu_2\mu_3^2\mu_4^4 + 3833\mu_1^5\mu_2\mu_3^2\mu_4^3\mu_5 + 4924\mu_1^5\mu_2\mu_3^2\mu_4^3\mu_6 + 1650\mu_1^5\mu_2\mu_3^2\mu_4^2\mu_5^2$
$+ 5478\mu_1^5\mu_2\mu_3^2\mu_4^2\mu_5\mu_6 + 3930\mu_1^5\mu_2\mu_3^2\mu_4^2\mu_6^2 + 272\mu_1^5\mu_2\mu_3^2\mu_4\mu_5^3 + 1047\mu_1^5\mu_2\mu_3^2\mu_4\mu_5^2\mu_6 +$

$$1734\mu_1^5\mu_2\mu_3^2\mu_4\mu_5\mu_6^2 + 908\mu_1^5\mu_2\mu_3^2\mu_4\mu_6^3 - 16\mu_1^5\mu_2\mu_3^2\mu_5^3\mu_6 - 60\mu_1^5\mu_2\mu_3^2\mu_5^2\mu_6^2 - 36\mu_1^5\mu_2\mu_3^2\mu_5\mu_6^3$$
$$+ 1344\mu_1^5\mu_2\mu_3\mu_4^5 + 6057\mu_1^5\mu_2\mu_3\mu_4^4\mu_5 + 6522\mu_1^5\mu_2\mu_3\mu_4^4\mu_6 + 5193\mu_1^5\mu_2\mu_3\mu_4^3\mu_5^2$$
$$+ 15464\mu_1^5\mu_2\mu_3\mu_4^3\mu_5\mu_6 + 9620\mu_1^5\mu_2\mu_3\mu_4^3\mu_6^2 + 1600\mu_1^5\mu_2\mu_3\mu_4^2\mu_5^3 + 7389\mu_1^5\mu_2\mu_3\mu_4^2\mu_5^2\mu_6$$
$$+ 11644\mu_1^5\mu_2\mu_3\mu_4^2\mu_5\mu_6^2 + 5364\mu_1^5\mu_2\mu_3\mu_4^2\mu_6^3 + 265\mu_1^5\mu_2\mu_3\mu_4\mu_5^4 + 1176\mu_1^5\mu_2\mu_3\mu_4\mu_5^3\mu_6 +$$
$$2403\mu_1^5\mu_2\mu_3\mu_4\mu_5^2\mu_6^2 + 2584\mu_1^5\mu_2\mu_3\mu_4\mu_5\mu_6^3 + 976\mu_1^5\mu_2\mu_3\mu_4\mu_6^4 - 48\mu_1^5\mu_2\mu_3\mu_5^3\mu_6^2$$
$$- 96\mu_1^5\mu_2\mu_3\mu_5^2\mu_6^3 - 40\mu_1^5\mu_2\mu_3\mu_5\mu_6^4 + 294\mu_1^5\mu_2\mu_4^6 + 2576\mu_1^5\mu_2\mu_4^5\mu_5 + 2350\mu_1^5\mu_2\mu_4^5\mu_6 +$$
$$4806\mu_1^5\mu_2\mu_4^4\mu_5^2 + 11470\mu_1^5\mu_2\mu_4^4\mu_5\mu_6 + 5990\mu_1^5\mu_2\mu_4^4\mu_6^2 + 1936\mu_1^5\mu_2\mu_4^3\mu_5^3 + 10584\mu_1^5\mu_2\mu_4^3\mu_5^2\mu_6$$
$$+ 15216\mu_1^5\mu_2\mu_4^3\mu_5\mu_6^2 + 6136\mu_1^5\mu_2\mu_4^3\mu_6^3 + 825\mu_1^5\mu_2\mu_4^2\mu_5^4 + 3456\mu_1^5\mu_2\mu_4^2\mu_5^3\mu_6$$
$$+ 7755\mu_1^5\mu_2\mu_4^2\mu_5^2\mu_6^2 + 7888\mu_1^5\mu_2\mu_4^2\mu_5\mu_6^3 + 2656\mu_1^5\mu_2\mu_4^2\mu_6^4 + 60\mu_1^5\mu_2\mu_4\mu_5^5 + 570\mu_1^5\mu_2\mu_4\mu_5^4\mu_6$$
$$+ 1320\mu_1^5\mu_2\mu_4\mu_5^3\mu_6^2 + 1770\mu_1^5\mu_2\mu_4\mu_5^2\mu_6^3 + 1368\mu_1^5\mu_2\mu_4\mu_5\mu_6^4 + 400\mu_1^5\mu_2\mu_4\mu_6^5 - 32\mu_1^5\mu_2\mu_5^3\mu_6^3$$
$$- 48\mu_1^5\mu_2\mu_5^2\mu_6^4 - 16\mu_1^5\mu_2\mu_5\mu_6^5 + \mu_1^5\mu_3^6\mu_4 + 33\mu_1^5\mu_3^5\mu_4^2 + 12\mu_1^5\mu_3^5\mu_4\mu_5 + 18\mu_1^5\mu_3^5\mu_4\mu_6 +$$
$$236\mu_1^5\mu_3^4\mu_4^3 + 278\mu_1^5\mu_3^4\mu_4^2\mu_5 + 363\mu_1^5\mu_3^4\mu_4^2\mu_6 + 47\mu_1^5\mu_3^4\mu_4\mu_5^2 + 164\mu_1^5\mu_3^4\mu_4\mu_5\mu_6$$
$$+ 114\mu_1^5\mu_3^4\mu_4\mu_6^2 - 2\mu_1^5\mu_3^4\mu_5^2\mu_6 + 608\mu_1^5\mu_3^3\mu_4^4 + 1648\mu_1^5\mu_3^3\mu_4^3\mu_5 + 1848\mu_1^5\mu_3^3\mu_4^3\mu_6$$
$$+ 815\mu_1^5\mu_3^3\mu_4^2\mu_5^2 + 2468\mu_1^5\mu_3^3\mu_4^2\mu_5\mu_6 + 1531\mu_1^5\mu_3^3\mu_4^2\mu_6^2 + 98\mu_1^5\mu_3^3\mu_4\mu_5^3 + 486\mu_1^5\mu_3^3\mu_4\mu_5^2\mu_6 +$$
$$800\mu_1^5\mu_3^3\mu_4\mu_5\mu_6^2 + 352\mu_1^5\mu_3^3\mu_4\mu_6^3 - 8\mu_1^5\mu_3^3\mu_5^3\mu_6 - 14\mu_1^5\mu_3^3\mu_5^2\mu_6^2 + 635\mu_1^5\mu_3^2\mu_4^5 + 3482\mu_1^5\mu_3^2\mu_4^4\mu_5$$
$$+ 3341\mu_1^5\mu_3^2\mu_4^4\mu_6 + 3833\mu_1^5\mu_3^2\mu_4^3\mu_5^2 + 9848\mu_1^5\mu_3^2\mu_4^3\mu_5\mu_6 + 5328\mu_1^5\mu_3^2\mu_4^3\mu_6^2 + 1100\mu_1^5\mu_3^2\mu_4^2\mu_5^3$$
$$+ 5478\mu_1^5\mu_3^2\mu_4^2\mu_5^2\mu_6 + 7860\mu_1^5\mu_3^2\mu_4^2\mu_5\mu_6^2 + 3122\mu_1^5\mu_3^2\mu_4^2\mu_6^3 + 136\mu_1^5\mu_3^2\mu_4\mu_5^4 + 698\mu_1^5\mu_3^2\mu_4\mu_5^3\mu_6$$
$$+ 1734\mu_1^5\mu_3^2\mu_4\mu_5^2\mu_6^2 + 1816\mu_1^5\mu_3^2\mu_4\mu_5\mu_6^3 + 576\mu_1^5\mu_3^2\mu_4\mu_6^4 - 8\mu_1^5\mu_3^2\mu_5^4\mu_6 - 40\mu_1^5\mu_3^2\mu_5^3\mu_6^2 -$$
$$36\mu_1^5\mu_3^2\mu_5^2\mu_6^3 + 231\mu_1^5\mu_3\mu_4^6 + 2688\mu_1^5\mu_3\mu_4^5\mu_5 + 2132\mu_1^5\mu_3\mu_4^5\mu_6 + 6057\mu_1^5\mu_3\mu_4^4\mu_5^2$$
$$+ 13044\mu_1^5\mu_3\mu_4^4\mu_5\mu_6 + 5987\mu_1^5\mu_3\mu_4^4\mu_6^2 + 3462\mu_1^5\mu_3\mu_4^3\mu_5^3 + 15464\mu_1^5\mu_3\mu_4^3\mu_5^2\mu_6$$
$$+ 19240\mu_1^5\mu_3\mu_4^3\mu_5\mu_6^2 + 6696\mu_1^5\mu_3\mu_4^3\mu_6^3 + 800\mu_1^5\mu_3\mu_4^2\mu_5^4 + 4926\mu_1^5\mu_3\mu_4^2\mu_5^3\mu_6$$
$$+ 11644\mu_1^5\mu_3\mu_4^2\mu_5^2\mu_6^2 + 10728\mu_1^5\mu_3\mu_4^2\mu_5\mu_6^3 + 3092\mu_1^5\mu_3\mu_4^2\mu_6^4 + 106\mu_1^5\mu_3\mu_4\mu_5^5$$
$$+ 588\mu_1^5\mu_3\mu_4\mu_5^4\mu_6 + 1602\mu_1^5\mu_3\mu_4\mu_5^3\mu_6^2 + 2584\mu_1^5\mu_3\mu_4\mu_5^2\mu_6^3 + 1952\mu_1^5\mu_3\mu_4\mu_5\mu_6^4$$
$$+ 480\mu_1^5\mu_3\mu_4\mu_6^5 - 24\mu_1^5\mu_3\mu_5^4\mu_6^2 - 64\mu_1^5\mu_3\mu_5^3\mu_6^3 - 40\mu_1^5\mu_3\mu_5^2\mu_6^4 + 588\mu_1^5\mu_4^6\mu_5 + 294\mu_1^5\mu_4^6\mu_6$$
$$+ 2576\mu_1^5\mu_4^5\mu_5^2 + 4700\mu_1^5\mu_4^5\mu_5\mu_6 + 1692\mu_1^5\mu_4^5\mu_6^2 + 3204\mu_1^5\mu_4^4\mu_5^3 + 11470\mu_1^5\mu_4^4\mu_5^2\mu_6 +$$
$$11980\mu_1^5\mu_4^4\mu_5\mu_6^2 + 3474\mu_1^5\mu_4^4\mu_6^3 + 968\mu_1^5\mu_4^3\mu_5^4 + 7056\mu_1^5\mu_4^3\mu_5^3\mu_6 + 15216\mu_1^5\mu_4^3\mu_5^2\mu_6^2$$
$$+ 12272\mu_1^5\mu_4^3\mu_5\mu_6^3 + 3088\mu_1^5\mu_4^3\mu_6^4 + 330\mu_1^5\mu_4^2\mu_5^5 + 1728\mu_1^5\mu_4^2\mu_5^4\mu_6 + 5170\mu_1^5\mu_4^2\mu_5^3\mu_6^2 +$$
$$7888\mu_1^5\mu_4^2\mu_5^2\mu_6^3 + 5312\mu_1^5\mu_4^2\mu_5\mu_6^4 + 1192\mu_1^5\mu_4^2\mu_6^5 + 20\mu_1^5\mu_4\mu_5^6 + 228\mu_1^5\mu_4\mu_5^5\mu_6$$
$$+ 660\mu_1^5\mu_4\mu_5^4\mu_6^2 + 1180\mu_1^5\mu_4\mu_5^3\mu_6^3 + 1368\mu_1^5\mu_4\mu_5^2\mu_6^4 + 800\mu_1^5\mu_4\mu_5\mu_6^5 + 160\mu_1^5\mu_4\mu_6^6$$
$$- 16\mu_1^5\mu_5^4\mu_6^3 - 32\mu_1^5\mu_5^3\mu_6^4 - 16\mu_1^5\mu_5^2\mu_6^5 - \mu_1^4\mu_2^6\mu_4^2 + \mu_1^4\mu_2^5\mu_3^2\mu_4 + 7\mu_1^4\mu_2^5\mu_3\mu_4^2 + \mu_1^4\mu_2^5\mu_3\mu_4\mu_5 +$$
$$5\mu_1^4\mu_2^5\mu_3\mu_4\mu_6 + 10\mu_1^4\mu_2^5\mu_4^3 - 7\mu_1^4\mu_2^5\mu_4^2\mu_5 + 10\mu_1^4\mu_2^5\mu_4^2\mu_6 + 2\mu_1^4\mu_2^5\mu_4\mu_5\mu_6 + 5\mu_1^4\mu_2^5\mu_4\mu_6^2 +$$
$$3\mu_1^4\mu_2^4\mu_3^3\mu_4 + 46\mu_1^4\mu_2^4\mu_3^2\mu_4^2 + 14\mu_1^4\mu_2^4\mu_3^2\mu_4\mu_5 + 26\mu_1^4\mu_2^4\mu_3^2\mu_4\mu_6 + 157\mu_1^4\mu_2^4\mu_3\mu_4^3$$
$$+ 115\mu_1^4\mu_2^4\mu_3\mu_4^2\mu_5 + 204\mu_1^4\mu_2^4\mu_3\mu_4^2\mu_6 + 12\mu_1^4\mu_2^4\mu_3\mu_4\mu_5^2 + 71\mu_1^4\mu_2^4\mu_3\mu_4\mu_5\mu_6$$
$$+ 64\mu_1^4\mu_2^4\mu_3\mu_4\mu_6^2 + 139\mu_1^4\mu_2^4\mu_4^4 + 178\mu_1^4\mu_2^4\mu_3^3\mu_5 + 302\mu_1^4\mu_2^4\mu_3^3\mu_6 - \mu_1^4\mu_2^4\mu_4^2\mu_5^2$$
$$+ 196\mu_1^4\mu_2^4\mu_4^2\mu_5\mu_6 + 202\mu_1^4\mu_2^4\mu_4^2\mu_6^2 + \mu_1^4\mu_2^4\mu_4\mu_5^3 + 24\mu_1^4\mu_2^4\mu_4\mu_5^2\mu_6 + 73\mu_1^4\mu_2^4\mu_4\mu_5\mu_6^2 +$$
$$46\mu_1^4\mu_2^4\mu_4\mu_6^3 + 4\mu_1^4\mu_2^3\mu_3^4\mu_4 + 102\mu_1^4\mu_2^3\mu_3^3\mu_4^2 + 33\mu_1^4\mu_2^3\mu_3^3\mu_4\mu_5 + 52\mu_1^4\mu_2^3\mu_3^3\mu_4\mu_6$$
$$+ 587\mu_1^4\mu_2^3\mu_3^2\mu_4^3 + 451\mu_1^4\mu_2^3\mu_3^2\mu_4^2\mu_5 + 735\mu_1^4\mu_2^3\mu_3^2\mu_4^2\mu_6 + 75\mu_1^4\mu_2^3\mu_3^2\mu_4\mu_5^2$$
$$+ 255\mu_1^4\mu_2^3\mu_3^2\mu_4\mu_5\mu_6 + 207\mu_1^4\mu_2^3\mu_3^2\mu_4\mu_6^2 - \mu_1^4\mu_2^3\mu_3^2\mu_5\mu_6^2 + 1201\mu_1^4\mu_2^3\mu_3\mu_4^4 + 1433\mu_1^4\mu_2^3\mu_3\mu_4^3\mu_5$$
$$+ 2482\mu_1^4\mu_2^3\mu_3\mu_4^3\mu_6 + 622\mu_1^4\mu_2^3\mu_3\mu_4^2\mu_5^2 + 1932\mu_1^4\mu_2^3\mu_3\mu_4^2\mu_5\mu_6 + 1611\mu_1^4\mu_2^3\mu_3\mu_4^2\mu_6^2$$
$$+ 51\mu_1^4\mu_2^3\mu_3\mu_4\mu_5^3 + 371\mu_1^4\mu_2^3\mu_3\mu_4\mu_5^2\mu_6 + 601\mu_1^4\mu_2^3\mu_3\mu_4\mu_5\mu_6^2 + 324\mu_1^4\mu_2^3\mu_3\mu_4\mu_6^3$$
$$- 2\mu_1^4\mu_2^3\mu_3\mu_5\mu_6^3 + 802\mu_1^4\mu_2^3\mu_4^5 + 1203\mu_1^4\mu_2^3\mu_4^4\mu_5 + 2294\mu_1^4\mu_2^3\mu_4^4\mu_6 + 960\mu_1^4\mu_2^3\mu_4^3\mu_5^2$$
$$+ 2766\mu_1^4\mu_2^3\mu_4^3\mu_5\mu_6 + 2410\mu_1^4\mu_2^3\mu_4^3\mu_6^2 + 84\mu_1^4\mu_2^3\mu_4^2\mu_5^3 + 1130\mu_1^4\mu_2^3\mu_4^2\mu_5^2\mu_6$$
$$+ 1922\mu_1^4\mu_2^3\mu_4^2\mu_5\mu_6^2 + 1080\mu_1^4\mu_2^3\mu_4^2\mu_6^3 + 5\mu_1^4\mu_2^3\mu_4\mu_5^4 + 102\mu_1^4\mu_2^3\mu_4\mu_5^3\mu_6$$

$$+ 385\mu_1^4\mu_2^3\mu_4\mu_5^2\mu_6^2 + 430\mu_1^4\mu_2^3\mu_4\mu_5\mu_6^3 + 173\mu_1^4\mu_2^3\mu_4\mu_6^4 - \mu_1^4\mu_2^3\mu_5\mu_6^4 + 3\mu_1^4\mu_2^2\mu_3^5\mu_4$$
$$+ 109\mu_1^4\mu_2^2\mu_3^4\mu_4^2 + 33\mu_1^4\mu_2^2\mu_3^4\mu_4\mu_5 + 52\mu_1^4\mu_2^2\mu_3^4\mu_4\mu_6 + 908\mu_1^4\mu_2^2\mu_3^3\mu_4^3 + 665\mu_1^4\mu_2^2\mu_3^3\mu_4^2\mu_5 +$$
$$1096\mu_1^4\mu_2^2\mu_3^3\mu_4^2\mu_6 + 121\mu_1^4\mu_2^2\mu_3^3\mu_4\mu_5^2 + 357\mu_1^4\mu_2^2\mu_3^3\mu_4\mu_5\mu_6 + 290\mu_1^4\mu_2^2\mu_3^3\mu_4\mu_6^2 - 2\mu_1^4\mu_2^2\mu_3^3\mu_5\mu_6^2$$
$$+ 2789\mu_1^4\mu_2^2\mu_3^2\mu_4^4 + 3608\mu_1^4\mu_2^2\mu_3^2\mu_4^3\mu_5 + 5887\mu_1^4\mu_2^2\mu_3^2\mu_4^3\mu_6 + 1533\mu_1^4\mu_2^2\mu_3^2\mu_4^2\mu_5^2$$
$$+ 4630\mu_1^4\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6 + 3757\mu_1^4\mu_2^2\mu_3^2\mu_4^2\mu_6^2 + 180\mu_1^4\mu_2^2\mu_3^2\mu_4\mu_5^3 + 879\mu_1^4\mu_2^2\mu_3^2\mu_4\mu_5^2\mu_6$$
$$+ 1324\mu_1^4\mu_2^2\mu_3^2\mu_4\mu_5\mu_6^2 + 712\mu_1^4\mu_2^2\mu_3^2\mu_4\mu_6^3 - 5\mu_1^4\mu_2^2\mu_3^2\mu_5^2\mu_6^2 - 8\mu_1^4\mu_2^2\mu_3^2\mu_5\mu_6^3 + 3332\mu_1^4\mu_2^2\mu_3\mu_4^5$$
$$+ 7268\mu_1^4\mu_2^2\mu_3\mu_4^4\mu_5 + 10911\mu_1^4\mu_2^2\mu_3\mu_4^4\mu_6 + 4665\mu_1^4\mu_2^2\mu_3\mu_4^3\mu_5^2 + 15190\mu_1^4\mu_2^2\mu_3\mu_4^3\mu_5\mu_6 +$$
$$12009\mu_1^4\mu_2^2\mu_3\mu_4^3\mu_6^2 + 1478\mu_1^4\mu_2^2\mu_3\mu_4^2\mu_5^3 + 6454\mu_1^4\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6 + 10010\mu_1^4\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^2 +$$
$$5326\mu_1^4\mu_2^2\mu_3\mu_4^2\mu_6^3 + 98\mu_1^4\mu_2^2\mu_3\mu_4\mu_5^4 + 874\mu_1^4\mu_2^2\mu_3\mu_4\mu_5^3\mu_6 + 2017\mu_1^4\mu_2^2\mu_3\mu_4\mu_5^2\mu_6^2$$
$$+ 2018\mu_1^4\mu_2^2\mu_3\mu_4\mu_5\mu_6^3 + 798\mu_1^4\mu_2^2\mu_3\mu_4\mu_6^4 - 10\mu_1^4\mu_2^2\mu_3\mu_5^3\mu_6^3 - 10\mu_1^4\mu_2^2\mu_3\mu_5\mu_6^4 + 1190\mu_1^4\mu_2^2\mu_4^6$$
$$+ 4832\mu_1^4\mu_2^2\mu_4^5\mu_5 + 6070\mu_1^4\mu_2^2\mu_4^5\mu_6 + 3793\mu_1^4\mu_2^2\mu_4^4\mu_5^2 + 13902\mu_1^4\mu_2^2\mu_4^4\mu_5\mu_6 + 10255\mu_1^4\mu_2^2\mu_4^4\mu_6^2$$
$$+ 2260\mu_1^4\mu_2^2\mu_4^3\mu_5^3 + 9026\mu_1^4\mu_2^2\mu_4^3\mu_5^2\mu_6 + 14778\mu_1^4\mu_2^2\mu_4^3\mu_5\mu_6^2 + 7718\mu_1^4\mu_2^2\mu_4^3\mu_6^3 + 222\mu_1^4\mu_2^2\mu_4^2\mu_5^4$$
$$+ 2760\mu_1^4\mu_2^2\mu_4^2\mu_5^3\mu_6 + 6440\mu_1^4\mu_2^2\mu_4^2\mu_5^2\mu_6^2 + 6716\mu_1^4\mu_2^2\mu_4^2\mu_5\mu_6^3 + 2649\mu_1^4\mu_2^2\mu_4^2\mu_6^4 + 9\mu_1^4\mu_2^2\mu_4\mu_5^5$$
$$+ 196\mu_1^4\mu_2^2\mu_4\mu_5^4\mu_6 + 910\mu_1^4\mu_2^2\mu_4\mu_5^3\mu_6^2 + 1438\mu_1^4\mu_2^2\mu_4\mu_5^2\mu_6^3 + 1075\mu_1^4\mu_2^2\mu_4\mu_5\mu_6^4$$
$$+ 332\mu_1^4\mu_2^2\mu_4\mu_6^5 - 5\mu_1^4\mu_2^2\mu_5^2\mu_6^4 - 4\mu_1^4\mu_2^2\mu_5\mu_6^5 + \mu_1^4\mu_2\mu_3^6\mu_4 + 54\mu_1^4\mu_2\mu_3^5\mu_4^2 + 15\mu_1^4\mu_2\mu_3^5\mu_4\mu_5 +$$
$$24\mu_1^4\mu_2\mu_3^5\mu_4\mu_6 + 618\mu_1^4\mu_2\mu_3^4\mu_4^3 + 445\mu_1^4\mu_2\mu_3^4\mu_4^2\mu_5 + 718\mu_1^4\mu_2\mu_3^4\mu_4^2\mu_6 + 75\mu_1^4\mu_2\mu_3^4\mu_4\mu_5^2 +$$
$$223\mu_1^4\mu_2\mu_3^4\mu_4\mu_5\mu_6 + 180\mu_1^4\mu_2\mu_3^4\mu_4\mu_6^2 - \mu_1^4\mu_2\mu_3^4\mu_5\mu_6^2 + 2476\mu_1^4\mu_2\mu_3^3\mu_4^4 + 3618\mu_1^4\mu_2\mu_3^3\mu_4^3\mu_5$$
$$+ 5388\mu_1^4\mu_2\mu_3^3\mu_4^3\mu_6 + 1383\mu_1^4\mu_2\mu_3^3\mu_4^2\mu_5^2 + 4398\mu_1^4\mu_2\mu_3^3\mu_4^2\mu_5\mu_6 + 3404\mu_1^4\mu_2\mu_3^3\mu_4^2\mu_6^2$$
$$+ 176\mu_1^4\mu_2\mu_3^3\mu_4\mu_5^3 + 759\mu_1^4\mu_2\mu_3^3\mu_4\mu_5^2\mu_6 + 1176\mu_1^4\mu_2\mu_3^3\mu_4\mu_5\mu_6^2 + 620\mu_1^4\mu_2\mu_3^3\mu_4\mu_6^3$$
$$- 6\mu_1^4\mu_2\mu_3^3\mu_5^2\mu_6^2 - 6\mu_1^4\mu_2\mu_3^3\mu_5\mu_6^3 + 4083\mu_1^4\mu_2\mu_3^2\mu_4^5 + 11113\mu_1^4\mu_2\mu_3^2\mu_4^4\mu_5 + 14554\mu_1^4\mu_2\mu_3^2\mu_4^4\mu_6$$
$$+ 7302\mu_1^4\mu_2\mu_3^2\mu_4^3\mu_5^2 + 23425\mu_1^4\mu_2\mu_3^2\mu_4^3\mu_5\mu_6 + 16756\mu_1^4\mu_2\mu_3^2\mu_4^3\mu_6^2 + 2164\mu_1^4\mu_2\mu_3^2\mu_4^2\mu_5^3 +$$
$$9480\mu_1^4\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6 + 14959\mu_1^4\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^2 + 7484\mu_1^4\mu_2\mu_3^2\mu_4^2\mu_6^3 + 195\mu_1^4\mu_2\mu_3^2\mu_4\mu_5^4$$
$$+ 1248\mu_1^4\mu_2\mu_3^2\mu_4\mu_5^3\mu_6 + 2730\mu_1^4\mu_2\mu_3^2\mu_4\mu_5^2\mu_6^2 + 2822\mu_1^4\mu_2\mu_3^2\mu_4\mu_5\mu_6^3 + 1088\mu_1^4\mu_2\mu_3^2\mu_4\mu_6^4$$
$$- 8\mu_1^4\mu_2\mu_3^2\mu_5^3\mu_6^2 - 24\mu_1^4\mu_2\mu_3^2\mu_5^2\mu_6^3 - 13\mu_1^4\mu_2\mu_3^2\mu_5\mu_6^4 + 2688\mu_1^4\mu_2\mu_3\mu_4^6 + 13305\mu_1^4\mu_2\mu_3\mu_4^5\mu_5$$
$$+ 14926\mu_1^4\mu_2\mu_3\mu_4^5\mu_6 + 14598\mu_1^4\mu_2\mu_3\mu_4^4\mu_5^2 + 43500\mu_1^4\mu_2\mu_3\mu_4^4\mu_5\mu_6 + 27710\mu_1^4\mu_2\mu_3\mu_4^4\mu_6^2 +$$
$$6464\mu_1^4\mu_2\mu_3\mu_4^3\mu_5^3 + 30678\mu_1^4\mu_2\mu_3\mu_4^3\mu_5^2\mu_6 + 47787\mu_1^4\mu_2\mu_3\mu_4^3\mu_5\mu_6^2 + 22184\mu_1^4\mu_2\mu_3\mu_4^3\mu_6^3 +$$
$$1595\mu_1^4\mu_2\mu_3\mu_4^2\mu_5^4 + 9044\mu_1^4\mu_2\mu_3\mu_4^2\mu_5^3\mu_6 + 20364\mu_1^4\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^2 + 21150\mu_1^4\mu_2\mu_3\mu_4^2\mu_5\mu_6^3 +$$
$$7756\mu_1^4\mu_2\mu_3\mu_4^2\mu_6^4 + 87\mu_1^4\mu_2\mu_3\mu_4\mu_5^5 + 940\mu_1^4\mu_2\mu_3\mu_4\mu_5^4\mu_6 + 2832\mu_1^4\mu_2\mu_3\mu_4\mu_5^3\mu_6^2$$
$$+ 4110\mu_1^4\mu_2\mu_3\mu_4\mu_5^2\mu_6^3 + 3137\mu_1^4\mu_2\mu_3\mu_4\mu_5\mu_6^4 + 944\mu_1^4\mu_2\mu_3\mu_4\mu_6^5 - 16\mu_1^4\mu_2\mu_3\mu_5^3\mu_6^3$$
$$- 30\mu_1^4\mu_2\mu_3\mu_5^2\mu_6^4 - 12\mu_1^4\mu_2\mu_3\mu_5\mu_6^5 + 516\mu_1^4\mu_2\mu_4^7 + 4760\mu_1^4\mu_2\mu_4^6\mu_5 + 4498\mu_1^4\mu_2\mu_4^6\mu_6 +$$
$$9684\mu_1^4\mu_2\mu_4^5\mu_5^2 + 24258\mu_1^4\mu_2\mu_4^5\mu_5\mu_6 + 13190\mu_1^4\mu_2\mu_4^5\mu_6^2 + 5180\mu_1^4\mu_2\mu_4^4\mu_5^3$$
$$+ 27942\mu_1^4\mu_2\mu_4^4\mu_5^2\mu_6 + 40932\mu_1^4\mu_2\mu_4^4\mu_5\mu_6^2 + 17012\mu_1^4\mu_2\mu_4^4\mu_6^3 + 2430\mu_1^4\mu_2\mu_4^3\mu_5^4$$
$$+ 12520\mu_1^4\mu_2\mu_4^3\mu_5^3\mu_6 + 29874\mu_1^4\mu_2\mu_4^3\mu_5^2\mu_6^2 + 30758\mu_1^4\mu_2\mu_4^3\mu_5\mu_6^3 + 10560\mu_1^4\mu_2\mu_4^3\mu_6^4$$
$$+ 216\mu_1^4\mu_2\mu_4^2\mu_5^5 + 3010\mu_1^4\mu_2\mu_4^2\mu_5^4\mu_6 + 9036\mu_1^4\mu_2\mu_4^2\mu_5^3\mu_6^2 + 13668\mu_1^4\mu_2\mu_4^2\mu_5^2\mu_6^3$$
$$+ 10533\mu_1^4\mu_2\mu_4^2\mu_5\mu_6^4 + 3048\mu_1^4\mu_2\mu_4^2\mu_6^5 + 7\mu_1^4\mu_2\mu_4\mu_5^6 + 174\mu_1^4\mu_2\mu_4\mu_5^5\mu_6 + 980\mu_1^4\mu_2\mu_4\mu_5^4\mu_6^2$$
$$+ 2016\mu_1^4\mu_2\mu_4\mu_5^3\mu_6^3 + 2187\mu_1^4\mu_2\mu_4\mu_5^2\mu_6^4 + 1306\mu_1^4\mu_2\mu_4\mu_5\mu_6^5 + 320\mu_1^4\mu_2\mu_4\mu_6^6 - 8\mu_1^4\mu_2\mu_5^3\mu_6^4$$
$$- 12\mu_1^4\mu_2\mu_5^2\mu_6^5 - 4\mu_1^4\mu_2\mu_5\mu_6^6 + 9\mu_1^4\mu_3^6\mu_4^2 + 2\mu_1^4\mu_3^6\mu_4\mu_5 + 3\mu_1^4\mu_3^6\mu_4\mu_6 + 150\mu_1^4\mu_3^5\mu_4^3 +$$
$$108\mu_1^4\mu_3^5\mu_4^2\mu_5 + 162\mu_1^4\mu_3^5\mu_4^2\mu_6 + 15\mu_1^4\mu_3^5\mu_4\mu_5^2 + 48\mu_1^4\mu_3^5\mu_4\mu_5\mu_6 + 36\mu_1^4\mu_3^5\mu_4\mu_6^2 + 742\mu_1^4\mu_3^4\mu_4^4$$
$$+ 1236\mu_1^4\mu_3^4\mu_4^3\mu_5 + 1650\mu_1^4\mu_3^4\mu_4^3\mu_6 + 445\mu_1^4\mu_3^4\mu_4^2\mu_5^2 + 1436\mu_1^4\mu_3^4\mu_4^2\mu_5\mu_6 + 1016\mu_1^4\mu_3^4\mu_4^2\mu_6^2$$
$$+ 50\mu_1^4\mu_3^4\mu_4\mu_5^3 + 223\mu_1^4\mu_3^4\mu_4\mu_5^2\mu_6 + 360\mu_1^4\mu_3^4\mu_4\mu_5\mu_6^2 + 174\mu_1^4\mu_3^4\mu_4\mu_6^3 - \mu_1^4\mu_3^4\mu_5^2\mu_6^2$$
$$+ 1520\mu_1^4\mu_3^3\mu_4^5 + 4952\mu_1^4\mu_3^3\mu_4^4\mu_5 + 5772\mu_1^4\mu_3^3\mu_4^4\mu_6 + 3618\mu_1^4\mu_3^3\mu_4^3\mu_5^2 + 10776\mu_1^4\mu_3^3\mu_4^3\mu_5\mu_6 +$$
$$6890\mu_1^4\mu_3^3\mu_4^3\mu_6^2 + 922\mu_1^4\mu_3^3\mu_4^2\mu_5^3 + 4398\mu_1^4\mu_3^3\mu_4^2\mu_5^2\mu_6 + 6808\mu_1^4\mu_3^3\mu_4^2\mu_5\mu_6^2 + 3084\mu_1^4\mu_3^3\mu_4^2\mu_6^3$$
$$+ 88\mu_1^4\mu_3^3\mu_4\mu_5^4 + 506\mu_1^4\mu_3^3\mu_4\mu_5^3\mu_6 + 1176\mu_1^4\mu_3^3\mu_4\mu_5^2\mu_6^2 + 1240\mu_1^4\mu_3^3\mu_4\mu_5\mu_6^3 + 435\mu_1^4\mu_3^3\mu_4\mu_6^4$$

$$
\begin{aligned}
&- 4\mu_1^4\mu_3^3\mu_5^3\mu_6^2 - 6\mu_1^4\mu_3^3\mu_5^2\mu_6^3 + 1369\mu_1^4\mu_3^2\mu_4^6 + 8166\mu_1^4\mu_3^2\mu_4^5\mu_5 + 8209\mu_1^4\mu_3^2\mu_4^5\mu_6 \\
&+ 11113\mu_1^4\mu_3^2\mu_4^4\mu_5^2 + 29108\mu_1^4\mu_3^2\mu_4^4\mu_5\mu_6 + 16384\mu_1^4\mu_3^2\mu_4^4\mu_6^2 + 4868\mu_1^4\mu_3^2\mu_4^3\mu_5^3 \\
&+ 23425\mu_1^4\mu_3^2\mu_4^3\mu_5^2\mu_6 + 33512\mu_1^4\mu_3^2\mu_4^3\mu_5\mu_6^2 + 13796\mu_1^4\mu_3^2\mu_4^3\mu_6^3 + 1082\mu_1^4\mu_3^2\mu_4^2\mu_5^4 \\
&+ 6320\mu_1^4\mu_3^2\mu_4^2\mu_5^3\mu_6 + 14959\mu_1^4\mu_3^2\mu_4^2\mu_5^2\mu_6^2 + 14968\mu_1^4\mu_3^2\mu_4^2\mu_5\mu_6^3 + 4930\mu_1^4\mu_3^2\mu_4^2\mu_6^4 \\
&+ 78\mu_1^4\mu_3^2\mu_4\mu_5^5 + 624\mu_1^4\mu_3^2\mu_4\mu_5^4\mu_6 + 1820\mu_1^4\mu_3^2\mu_4\mu_5^3\mu_6^2 + 2822\mu_1^4\mu_3^2\mu_4\mu_5^2\mu_6^3 \\
&+ 2176\mu_1^4\mu_3^2\mu_4\mu_5\mu_6^4 + 594\mu_1^4\mu_3^2\mu_4\mu_6^5 - 4\mu_1^4\mu_3^2\mu_5^4\mu_6^2 - 16\mu_1^4\mu_3^2\mu_5^3\mu_6^3 - 13\mu_1^4\mu_3^2\mu_5^2\mu_6^4 \\
&+ 450\mu_1^4\mu_3\mu_4^7 + 5376\mu_1^4\mu_3\mu_4^6\mu_5 + 4444\mu_1^4\mu_3\mu_4^6\mu_6 + 13305\mu_1^4\mu_3\mu_4^5\mu_5^2 + 29852\mu_1^4\mu_3\mu_4^5\mu_5\mu_6 \\
&+ 14299\mu_1^4\mu_3\mu_4^5\mu_6^2 + 9732\mu_1^4\mu_3\mu_4^4\mu_5^3 + 43500\mu_1^4\mu_3\mu_4^4\mu_5^2\mu_6 + 55420\mu_1^4\mu_3\mu_4^4\mu_5\mu_6^2 \\
&+ 20072\mu_1^4\mu_3\mu_4^4\mu_6^3 + 3232\mu_1^4\mu_3\mu_4^3\mu_5^4 + 20452\mu_1^4\mu_3\mu_4^3\mu_5^3\mu_6 + 47787\mu_1^4\mu_3\mu_4^3\mu_5^2\mu_6^2 \\
&+ 44368\mu_1^4\mu_3\mu_4^3\mu_5\mu_6^3 + 13312\mu_1^4\mu_3\mu_4^3\mu_6^4 + 638\mu_1^4\mu_3\mu_4^2\mu_5^5 + 4522\mu_1^4\mu_3\mu_4^2\mu_5^4\mu_6 \\
&+ 13576\mu_1^4\mu_3\mu_4^2\mu_5^3\mu_6^2 + 21150\mu_1^4\mu_3\mu_4^2\mu_5^2\mu_6^3 + 15512\mu_1^4\mu_3\mu_4^2\mu_5\mu_6^4 + 3992\mu_1^4\mu_3\mu_4^2\mu_6^5 \\
&+ 29\mu_1^4\mu_3\mu_4\mu_5^6 + 376\mu_1^4\mu_3\mu_4\mu_5^5\mu_6 + 1416\mu_1^4\mu_3\mu_4\mu_5^4\mu_6^2 + 2740\mu_1^4\mu_3\mu_4\mu_5^3\mu_6^3 \\
&+ 3137\mu_1^4\mu_3\mu_4\mu_5^2\mu_6^4 + 1888\mu_1^4\mu_3\mu_4\mu_5\mu_6^5 + 420\mu_1^4\mu_3\mu_4\mu_6^6 - 8\mu_1^4\mu_3\mu_5^4\mu_6^3 - 20\mu_1^4\mu_3\mu_5^3\mu_6^4 - \\
&12\mu_1^4\mu_3\mu_5^2\mu_6^5 + 1032\mu_1^4\mu_4^7\mu_5 + 516\mu_1^4\mu_4^7\mu_6 + 4760\mu_1^4\mu_4^6\mu_5^2 + 8996\mu_1^4\mu_4^6\mu_5\mu_6 + 3308\mu_1^4\mu_4^6\mu_6^2 \\
&+ 6456\mu_1^4\mu_4^5\mu_5^3 + 24258\mu_1^4\mu_4^5\mu_5^2\mu_6 + 26380\mu_1^4\mu_4^5\mu_5\mu_6^2 + 7922\mu_1^4\mu_4^5\mu_6^3 + 2590\mu_1^4\mu_4^4\mu_5^4 + \\
&18628\mu_1^4\mu_4^4\mu_5^3\mu_6 + 40932\mu_1^4\mu_4^4\mu_5^2\mu_6^2 + 34024\mu_1^4\mu_4^4\mu_5\mu_6^3 + 8912\mu_1^4\mu_4^4\mu_6^4 + 972\mu_1^4\mu_4^3\mu_5^5 + \\
&6260\mu_1^4\mu_4^3\mu_5^4\mu_6 + 19916\mu_1^4\mu_4^3\mu_5^3\mu_6^2 + 30758\mu_1^4\mu_4^3\mu_5^2\mu_6^3 + 21120\mu_1^4\mu_4^3\mu_5\mu_6^4 + 4960\mu_1^4\mu_4^3\mu_6^5 \\
&+ 72\mu_1^4\mu_4^2\mu_5^6 + 1204\mu_1^4\mu_4^2\mu_5^5\mu_6 + 4518\mu_1^4\mu_4^2\mu_5^4\mu_6^2 + 9112\mu_1^4\mu_4^2\mu_5^3\mu_6^3 + 10533\mu_1^4\mu_4^2\mu_5^2\mu_6^4 + \\
&6096\mu_1^4\mu_4^2\mu_5\mu_6^5 + 1288\mu_1^4\mu_4^2\mu_6^6 + 2\mu_1^4\mu_4\mu_5^7 + 58\mu_1^4\mu_4\mu_5^6\mu_6 + 392\mu_1^4\mu_4\mu_5^5\mu_6^2 + 1008\mu_1^4\mu_4\mu_5^4\mu_6^3 \\
&+ 1458\mu_1^4\mu_4\mu_5^3\mu_6^4 + 1306\mu_1^4\mu_4\mu_5^2\mu_6^5 + 640\mu_1^4\mu_4\mu_5\mu_6^6 + 120\mu_1^4\mu_4\mu_6^7 - 4\mu_1^4\mu_5^4\mu_6^4 - 8\mu_1^4\mu_5^3\mu_6^5 - \\
&4\mu_1^4\mu_5^2\mu_6^6 - 2\mu_1^3\mu_2^6\mu_3\mu_4^2 - 6\mu_1^3\mu_2^6\mu_4^3 - 2\mu_1^3\mu_2^6\mu_4^2\mu_5 - 4\mu_1^3\mu_2^6\mu_4^2\mu_6 - \mu_1^3\mu_2^5\mu_3^2\mu_4^2 + \mu_1^3\mu_2^5\mu_3^2\mu_4\mu_6 - \\
&2\mu_1^3\mu_2^5\mu_3\mu_4^3 - 20\mu_1^3\mu_2^5\mu_3\mu_4^2\mu_5 - 5\mu_1^3\mu_2^5\mu_3\mu_4^2\mu_6 + \mu_1^3\mu_2^5\mu_3\mu_4\mu_5\mu_6 + 3\mu_1^3\mu_2^5\mu_3\mu_4\mu_6^2 - 5\mu_1^3\mu_2^5\mu_4^4 \\
&- 64\mu_1^3\mu_2^5\mu_4^3\mu_5 - 28\mu_1^3\mu_2^5\mu_4^3\mu_6 - 19\mu_1^3\mu_2^5\mu_4^2\mu_5^2 - 48\mu_1^3\mu_2^5\mu_4^2\mu_5\mu_6 - 13\mu_1^3\mu_2^5\mu_4^2\mu_6^2 + \mu_1^3\mu_2^5\mu_4\mu_5\mu_6^2 + \\
&2\mu_1^3\mu_2^5\mu_4\mu_6^3 + 11\mu_1^3\mu_2^4\mu_3^3\mu_4^2 + \mu_1^3\mu_2^4\mu_3^3\mu_4\mu_5 + 4\mu_1^3\mu_2^4\mu_3^3\mu_4\mu_6 + 100\mu_1^3\mu_2^4\mu_3^2\mu_4^3 + 25\mu_1^3\mu_2^4\mu_3^2\mu_4^2\mu_5 \\
&+ 90\mu_1^3\mu_2^4\mu_3^2\mu_4^2\mu_6 + 2\mu_1^3\mu_2^4\mu_3^2\mu_4\mu_5^2 + 20\mu_1^3\mu_2^4\mu_3^2\mu_4\mu_5\mu_6 + 22\mu_1^3\mu_2^4\mu_3^2\mu_4\mu_6^2 + 235\mu_1^3\mu_2^4\mu_3\mu_4^4 + \\
&141\mu_1^3\mu_2^4\mu_3\mu_4^3\mu_5 + 388\mu_1^3\mu_2^4\mu_3\mu_4^3\mu_6 - 55\mu_1^3\mu_2^4\mu_3\mu_4^2\mu_5^2 + 106\mu_1^3\mu_2^4\mu_3\mu_4^2\mu_5\mu_6 + 196\mu_1^3\mu_2^4\mu_3\mu_4^2\mu_6^2 \\
&+ \mu_1^3\mu_2^4\mu_3\mu_4\mu_5^3 + 18\mu_1^3\mu_2^4\mu_3\mu_4\mu_5^2\mu_6 + 56\mu_1^3\mu_2^4\mu_3\mu_4\mu_5\mu_6^2 + 38\mu_1^3\mu_2^4\mu_3\mu_4\mu_6^3 + 146\mu_1^3\mu_2^4\mu_4^5 + \\
&125\mu_1^3\mu_2^4\mu_4^4\mu_5 + 360\mu_1^3\mu_2^4\mu_4^4\mu_6 - 226\mu_1^3\mu_2^4\mu_4^3\mu_5^2 + 34\mu_1^3\mu_2^4\mu_4^3\mu_5\mu_6 + 288\mu_1^3\mu_2^4\mu_4^3\mu_6^2 - 69\mu_1^3\mu_2^4\mu_4^2\mu_5^3 \\
&- 176\mu_1^3\mu_2^4\mu_4^2\mu_5^2\mu_6 + 18\mu_1^3\mu_2^4\mu_4^2\mu_5\mu_6^2 + 106\mu_1^3\mu_2^4\mu_4^2\mu_6^3 + 2\mu_1^3\mu_2^4\mu_4\mu_5^3\mu_6 + 18\mu_1^3\mu_2^4\mu_4\mu_5^2\mu_6^2 + \\
&38\mu_1^3\mu_2^4\mu_4\mu_5\mu_6^3 + 20\mu_1^3\mu_2^4\mu_4\mu_6^4 + 24\mu_1^3\mu_2^3\mu_3^4\mu_4^2 + 3\mu_1^3\mu_2^3\mu_3^4\mu_4\mu_5 + 7\mu_1^3\mu_2^3\mu_3^4\mu_4\mu_6 \\
&+ 304\mu_1^3\mu_2^3\mu_3^3\mu_4^3 + 163\mu_1^3\mu_2^3\mu_3^3\mu_4^2\mu_5 + 288\mu_1^3\mu_2^3\mu_3^3\mu_4^2\mu_6 + 11\mu_1^3\mu_2^3\mu_3^3\mu_4\mu_5^2 \\
&+ 62\mu_1^3\mu_2^3\mu_3^3\mu_4\mu_5\mu_6 + 56\mu_1^3\mu_2^3\mu_3^3\mu_4\mu_6^2 + 1180\mu_1^3\mu_2^3\mu_3^2\mu_4^4 + 1191\mu_1^3\mu_2^3\mu_3^2\mu_4^3\mu_5 \\
&+ 2078\mu_1^3\mu_2^3\mu_3^2\mu_4^3\mu_6 + 258\mu_1^3\mu_2^3\mu_3^2\mu_4^2\mu_5^2 + 1183\mu_1^3\mu_2^3\mu_3^2\mu_4^2\mu_5\mu_6 + 1069\mu_1^3\mu_2^3\mu_3^2\mu_4^2\mu_6^2 \\
&+ 13\mu_1^3\mu_2^3\mu_3^2\mu_4\mu_5^3 + 135\mu_1^3\mu_2^3\mu_3^2\mu_4\mu_5^2\mu_6 + 275\mu_1^3\mu_2^3\mu_3^2\mu_4\mu_5\mu_6^2 + 158\mu_1^3\mu_2^3\mu_3^2\mu_4\mu_6^3 \\
&+ 1824\mu_1^3\mu_2^3\mu_3\mu_4^5 + 2565\mu_1^3\mu_2^3\mu_3\mu_4^4\mu_5 + 4725\mu_1^3\mu_2^3\mu_3\mu_4^4\mu_6 + 1187\mu_1^3\mu_2^3\mu_3\mu_4^3\mu_5^2 \\
&+ 4638\mu_1^3\mu_2^3\mu_3\mu_4^3\mu_5\mu_6 + 4254\mu_1^3\mu_2^3\mu_3\mu_4^3\mu_6^2 - 11\mu_1^3\mu_2^3\mu_3\mu_4^2\mu_5^3 + 1099\mu_1^3\mu_2^3\mu_3\mu_4^2\mu_5^2\mu_6 \\
&+ 2507\mu_1^3\mu_2^3\mu_3\mu_4^2\mu_5\mu_6^2 + 1540\mu_1^3\mu_2^3\mu_3\mu_4^2\mu_6^3 + 5\mu_1^3\mu_2^3\mu_3\mu_4\mu_5^4 + 90\mu_1^3\mu_2^3\mu_3\mu_4\mu_5^3\mu_6 \\
&+ 353\mu_1^3\mu_2^3\mu_3\mu_4\mu_5^2\mu_6^2 + 444\mu_1^3\mu_2^3\mu_3\mu_4\mu_5\mu_6^3 + 187\mu_1^3\mu_2^3\mu_3\mu_4\mu_6^4 + 966\mu_1^3\mu_2^3\mu_4^6 \\
&+ 1550\mu_1^3\mu_2^3\mu_4^5\mu_5 + 3240\mu_1^3\mu_2^3\mu_4^5\mu_6 + 1130\mu_1^3\mu_2^3\mu_4^4\mu_5^2 + 4230\mu_1^3\mu_2^3\mu_4^4\mu_5\mu_6 + 4195\mu_1^3\mu_2^3\mu_4^4\mu_6^2 \\
&- 328\mu_1^3\mu_2^3\mu_4^3\mu_5^3 + 1386\mu_1^3\mu_2^3\mu_4^3\mu_5^2\mu_6 + 3834\mu_1^3\mu_2^3\mu_4^3\mu_5\mu_6^2 + 2564\mu_1^3\mu_2^3\mu_4^3\mu_6^3 - 130\mu_1^3\mu_2^3\mu_4^2\mu_5^4 - \\
&242\mu_1^3\mu_2^3\mu_4^2\mu_5^3\mu_6 + 735\mu_1^3\mu_2^3\mu_4^2\mu_5^2\mu_6^2 + 1478\mu_1^3\mu_2^3\mu_4^2\mu_5\mu_6^3 + 731\mu_1^3\mu_2^3\mu_4^2\mu_6^4 + 10\mu_1^3\mu_2^3\mu_4\mu_5^4\mu_6 \\
&+ 90\mu_1^3\mu_2^3\mu_4\mu_5^3\mu_6^2 + 240\mu_1^3\mu_2^3\mu_4\mu_5^2\mu_6^3 + 231\mu_1^3\mu_2^3\mu_4\mu_5\mu_6^4 + 78\mu_1^3\mu_2^3\mu_4\mu_6^5 + 21\mu_1^3\mu_2^2\mu_3^2\mu_4^2 \\
&+ 3\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_5 + 6\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_6 + 368\mu_1^3\mu_2^2\mu_3^4\mu_4^3 + 211\mu_1^3\mu_2^2\mu_3^4\mu_4^2\mu_5 + 344\mu_1^3\mu_2^2\mu_3^4\mu_4^2\mu_6 +
\end{aligned}
$$

$$17\mu_1^3\mu_2^2\mu_3^4\mu_4\mu_5^2 + 75\mu_1^3\mu_2^2\mu_3^4\mu_4\mu_5\mu_6 + 64\mu_1^3\mu_2^2\mu_3^4\mu_4\mu_6^2 + 2040\mu_1^3\mu_2^2\mu_3^3\mu_4^4 + 2168\mu_1^3\mu_2^2\mu_3^3\mu_4^3\mu_5$$
$$+ 3584\mu_1^3\mu_2^2\mu_3^3\mu_4^3\mu_6 + 667\mu_1^3\mu_2^2\mu_3^3\mu_4^2\mu_5^2 + 2221\mu_1^3\mu_2^2\mu_3^3\mu_4^2\mu_5\mu_6 + 1828\mu_1^3\mu_2^2\mu_3^3\mu_4^2\mu_6^2$$
$$+ 34\mu_1^3\mu_2^2\mu_3^3\mu_4\mu_5^3 + 262\mu_1^3\mu_2^2\mu_3^3\mu_4\mu_5^2\mu_6 + 467\mu_1^3\mu_2^2\mu_3^3\mu_4\mu_5\mu_6^2 + 256\mu_1^3\mu_2^2\mu_3^3\mu_4\mu_6^3$$
$$+ 4731\mu_1^3\mu_2^2\mu_3^2\mu_4^5 + 7855\mu_1^3\mu_2^2\mu_3^2\mu_4^4\mu_5 + 12870\mu_1^3\mu_2^2\mu_3^2\mu_4^4\mu_6 + 4439\mu_1^3\mu_2^2\mu_3^2\mu_4^3\mu_5^2$$
$$+ 14419\mu_1^3\mu_2^2\mu_3^2\mu_4^3\mu_5\mu_6 + 11850\mu_1^3\mu_2^2\mu_3^2\mu_4^3\mu_6^2 + 762\mu_1^3\mu_2^2\mu_3^2\mu_4^2\mu_5^3 + 4631\mu_1^3\mu_2^2\mu_3^2\mu_4^2\mu_5^2\mu_6 +$$
$$7809\mu_1^3\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6^2 + 4272\mu_1^3\mu_2^2\mu_3^2\mu_4^2\mu_6^3 + 29\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_5^4 + 360\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_5^3\mu_6$$
$$+ 1071\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_5^2\mu_6^2 + 1202\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_5\mu_6^3 + 484\mu_1^3\mu_2^2\mu_3^2\mu_4\mu_6^4 + 4592\mu_1^3\mu_2^2\mu_3\mu_4^6$$
$$+ 11775\mu_1^3\mu_2^2\mu_3\mu_4^5\mu_5 + 18006\mu_1^3\mu_2^2\mu_3\mu_4^5\mu_6 + 9175\mu_1^3\mu_2^2\mu_3\mu_4^4\mu_5^2 + 31405\mu_1^3\mu_2^2\mu_3\mu_4^4\mu_5\mu_6 +$$
$$25340\mu_1^3\mu_2^2\mu_3\mu_4^4\mu_6^2 + 3298\mu_1^3\mu_2^2\mu_3\mu_4^3\mu_5^3 + 17330\mu_1^3\mu_2^2\mu_3\mu_4^3\mu_5^2\mu_6 + 29329\mu_1^3\mu_2^2\mu_3\mu_4^3\mu_5\mu_6^2 +$$
$$16072\mu_1^3\mu_2^2\mu_3\mu_4^3\mu_6^3 + 142\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_5^4 + 3236\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_5^3\mu_6 + 9703\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6^2$$
$$+ 11050\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^3 + 4528\mu_1^3\mu_2^2\mu_3\mu_4^2\mu_6^4 + 9\mu_1^3\mu_2^2\mu_3\mu_4\mu_5^5 + 185\mu_1^3\mu_2^2\mu_3\mu_4\mu_5^4\mu_6$$
$$+ 912\mu_1^3\mu_2^2\mu_3\mu_4\mu_5^3\mu_6^2 + 1676\mu_1^3\mu_2^2\mu_3\mu_4\mu_5^2\mu_6^3 + 1372\mu_1^3\mu_2^2\mu_3\mu_4\mu_5\mu_6^4 + 434\mu_1^3\mu_2^2\mu_3\mu_4\mu_6^5 +$$
$$1408\mu_1^3\mu_2^2\mu_4^7 + 6132\mu_1^3\mu_2^2\mu_4^6\mu_5 + 8050\mu_1^3\mu_2^2\mu_4^6\mu_6 + 5466\mu_1^3\mu_2^2\mu_4^5\mu_5^2 + 21140\mu_1^3\mu_2^2\mu_4^5\mu_5\mu_6 +$$
$$16182\mu_1^3\mu_2^2\mu_4^5\mu_6^2 + 3170\mu_1^3\mu_2^2\mu_4^4\mu_5^3 + 15630\mu_1^3\mu_2^2\mu_4^4\mu_5^2\mu_6 + 28295\mu_1^3\mu_2^2\mu_4^4\mu_5\mu_6^2$$
$$+ 15470\mu_1^3\mu_2^2\mu_4^4\mu_6^3 - 174\mu_1^3\mu_2^2\mu_4^3\mu_5^4 + 4644\mu_1^3\mu_2^2\mu_4^3\mu_5^3\mu_6 + 14934\mu_1^3\mu_2^2\mu_4^3\mu_5^2\mu_6^2$$
$$+ 17966\mu_1^3\mu_2^2\mu_4^3\mu_5\mu_6^3 + 7534\mu_1^3\mu_2^2\mu_4^3\mu_6^4 - 143\mu_1^3\mu_2^2\mu_4^2\mu_5^5 - 86\mu_1^3\mu_2^2\mu_4^2\mu_5^4\mu_6$$
$$+ 2500\mu_1^3\mu_2^2\mu_4^2\mu_5^3\mu_6^2 + 5902\mu_1^3\mu_2^2\mu_4^2\mu_5^2\mu_6^3 + 5302\mu_1^3\mu_2^2\mu_4^2\mu_5\mu_6^4 + 1760\mu_1^3\mu_2^2\mu_4^2\mu_6^5$$
$$+ 18\mu_1^3\mu_2^2\mu_4\mu_5^5\mu_6 + 185\mu_1^3\mu_2^2\mu_4\mu_5^4\mu_6^2 + 620\mu_1^3\mu_2^2\mu_4\mu_5^3\mu_6^3 + 867\mu_1^3\mu_2^2\mu_4\mu_5^2\mu_6^4$$
$$+ 568\mu_1^3\mu_2^2\mu_4\mu_5\mu_6^5 + 148\mu_1^3\mu_2^2\mu_4\mu_6^6 + 7\mu_1^3\mu_2\mu_3^6\mu_4^2 + \mu_1^3\mu_2\mu_3^6\mu_4\mu_5 + 2\mu_1^3\mu_2\mu_3^6\mu_4\mu_6$$
$$+ 192\mu_1^3\mu_2\mu_3^5\mu_4^3 + 105\mu_1^3\mu_2\mu_3^5\mu_4^2\mu_5 + 168\mu_1^3\mu_2\mu_3^5\mu_4^2\mu_6 + 9\mu_1^3\mu_2\mu_3^5\mu_4\mu_5^2 + 36\mu_1^3\mu_2\mu_3^5\mu_4\mu_5\mu_6$$
$$+ 30\mu_1^3\mu_2\mu_3^5\mu_4\mu_6^2 + 1460\mu_1^3\mu_2\mu_3^4\mu_4^4 + 1572\mu_1^3\mu_2\mu_3^4\mu_4^3\mu_5 + 2504\mu_1^3\mu_2\mu_3^4\mu_4^3\mu_6$$
$$+ 489\mu_1^3\mu_2\mu_3^4\mu_4^2\mu_5^2 + 1551\mu_1^3\mu_2\mu_3^4\mu_4^2\mu_5\mu_6 + 1232\mu_1^3\mu_2\mu_3^4\mu_4^2\mu_6^2 + 28\mu_1^3\mu_2\mu_3^4\mu_4\mu_5^3$$
$$+ 183\mu_1^3\mu_2\mu_3^4\mu_4\mu_5^2\mu_6 + 311\mu_1^3\mu_2\mu_3^4\mu_4\mu_5\mu_6^2 + 166\mu_1^3\mu_2\mu_3^4\mu_4\mu_6^3 + 4448\mu_1^3\mu_2\mu_3^3\mu_4^5$$
$$+ 8420\mu_1^3\mu_2\mu_3^3\mu_4^4\mu_5 + 12560\mu_1^3\mu_2\mu_3^3\mu_4^4\mu_6 + 4680\mu_1^3\mu_2\mu_3^3\mu_4^3\mu_5^2 + 15056\mu_1^3\mu_2\mu_3^3\mu_4^3\mu_5\mu_6 +$$
$$11624\mu_1^3\mu_2\mu_3^3\mu_4^3\mu_6^2 + 1008\mu_1^3\mu_2\mu_3^3\mu_4^2\mu_5^3 + 4935\mu_1^3\mu_2\mu_3^3\mu_4^2\mu_5^2\mu_6 + 7914\mu_1^3\mu_2\mu_3^3\mu_4^2\mu_5\mu_6^2 +$$
$$4124\mu_1^3\mu_2\mu_3^3\mu_4^2\mu_6^3 + 40\mu_1^3\mu_2\mu_3^3\mu_4\mu_5^4 + 400\mu_1^3\mu_2\mu_3^3\mu_4\mu_5^3\mu_6 + 1065\mu_1^3\mu_2\mu_3^3\mu_4\mu_5^2\mu_6^2$$
$$+ 1154\mu_1^3\mu_2\mu_3^3\mu_4\mu_5\mu_6^3 + 452\mu_1^3\mu_2\mu_3^3\mu_4\mu_6^4 + 6041\mu_1^3\mu_2\mu_3^2\mu_4^6 + 19335\mu_1^3\mu_2\mu_3^2\mu_4^5\mu_5$$
$$+ 25800\mu_1^3\mu_2\mu_3^2\mu_4^5\mu_6 + 16485\mu_1^3\mu_2\mu_3^2\mu_4^4\mu_5^2 + 53015\mu_1^3\mu_2\mu_3^2\mu_4^4\mu_5\mu_6 + 38340\mu_1^3\mu_2\mu_3^2\mu_4^4\mu_6^2 +$$
$$6496\mu_1^3\mu_2\mu_3^2\mu_4^3\mu_5^3 + 30789\mu_1^3\mu_2\mu_3^2\mu_4^3\mu_5^2\mu_6 + 49434\mu_1^3\mu_2\mu_3^2\mu_4^3\mu_5\mu_6^2 + 24928\mu_1^3\mu_2\mu_3^2\mu_4^3\mu_6^3 +$$
$$885\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_5^4 + 6896\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_5^3\mu_6 + 17013\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6^2 + 18174\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^3 +$$
$$7004\mu_1^3\mu_2\mu_3^2\mu_4^2\mu_6^4 + 27\mu_1^3\mu_2\mu_3^2\mu_4\mu_5^5 + 405\mu_1^3\mu_2\mu_3^2\mu_4\mu_5^4\mu_6 + 1592\mu_1^3\mu_2\mu_3^2\mu_4\mu_5^3\mu_6^2$$
$$+ 2658\mu_1^3\mu_2\mu_3^2\mu_4\mu_5^2\mu_6^3 + 2111\mu_1^3\mu_2\mu_3^2\mu_4\mu_5\mu_6^4 + 650\mu_1^3\mu_2\mu_3^2\mu_4\mu_6^5 + 3456\mu_1^3\mu_2\mu_3\mu_4^7$$
$$+ 18711\mu_1^3\mu_2\mu_3\mu_4^6\mu_5 + 21658\mu_1^3\mu_2\mu_3\mu_4^6\mu_6 + 24381\mu_1^3\mu_2\mu_3\mu_4^5\mu_5^2 + 73744\mu_1^3\mu_2\mu_3\mu_4^5\mu_5\mu_6 +$$
$$48088\mu_1^3\mu_2\mu_3\mu_4^5\mu_6^2 + 13220\mu_1^3\mu_2\mu_3\mu_4^4\mu_5^3 + 65865\mu_1^3\mu_2\mu_3\mu_4^4\mu_5^2\mu_6 + 104525\mu_1^3\mu_2\mu_3\mu_4^4\mu_5\mu_6^2$$
$$+ 49500\mu_1^3\mu_2\mu_3\mu_4^4\mu_6^3 + 3760\mu_1^3\mu_2\mu_3\mu_4^3\mu_5^4 + 25384\mu_1^3\mu_2\mu_3\mu_4^3\mu_5^3\mu_6 + 62463\mu_1^3\mu_2\mu_3\mu_4^3\mu_5^2\mu_6^2$$
$$+ 66992\mu_1^3\mu_2\mu_3\mu_4^3\mu_5\mu_6^3 + 25064\mu_1^3\mu_2\mu_3\mu_4^3\mu_6^4 + 177\mu_1^3\mu_2\mu_3\mu_4^2\mu_5^5 + 3755\mu_1^3\mu_2\mu_3\mu_4^2\mu_5^4\mu_6 +$$
$$14392\mu_1^3\mu_2\mu_3\mu_4^2\mu_5^3\mu_6^2 + 23910\mu_1^3\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^3 + 19155\mu_1^3\mu_2\mu_3\mu_4^2\mu_5\mu_6^4 + 5856\mu_1^3\mu_2\mu_3\mu_4^2\mu_6^5$$
$$+ 7\mu_1^3\mu_2\mu_3\mu_4\mu_5^6 + 168\mu_1^3\mu_2\mu_3\mu_4\mu_5^5\mu_6 + 1015\mu_1^3\mu_2\mu_3\mu_4\mu_5^4\mu_6^2 + 2464\mu_1^3\mu_2\mu_3\mu_4\mu_5^3\mu_6^3$$
$$+ 2994\mu_1^3\mu_2\mu_3\mu_4\mu_5^2\mu_6^4 + 1868\mu_1^3\mu_2\mu_3\mu_4\mu_5\mu_6^5 + 472\mu_1^3\mu_2\mu_3\mu_4\mu_6^6 + 588\mu_1^3\mu_2\mu_4^8$$
$$+ 5744\mu_1^3\mu_2\mu_4^7\mu_5 + 5584\mu_1^3\mu_2\mu_4^7\mu_6 + 12600\mu_1^3\mu_2\mu_4^6\mu_5^2 + 32942\mu_1^3\mu_2\mu_4^6\mu_5\mu_6$$
$$+ 18522\mu_1^3\mu_2\mu_4^6\mu_6^2 + 7832\mu_1^3\mu_2\mu_4^5\mu_5^3 + 43980\mu_1^3\mu_2\mu_4^5\mu_5^2\mu_6 + 66600\mu_1^3\mu_2\mu_4^5\mu_5\mu_6^2$$
$$+ 28560\mu_1^3\mu_2\mu_4^5\mu_6^3 + 3625\mu_1^3\mu_2\mu_4^4\mu_5^4 + 22800\mu_1^3\mu_2\mu_4^4\mu_5^3\mu_6 + 59715\mu_1^3\mu_2\mu_4^4\mu_5^2\mu_6^2$$
$$+ 64170\mu_1^3\mu_2\mu_4^4\mu_5\mu_6^3 + 22760\mu_1^3\mu_2\mu_4^4\mu_6^4 - 12\mu_1^3\mu_2\mu_4^3\mu_5^5 + 5580\mu_1^3\mu_2\mu_4^3\mu_5^4\mu_6$$

$$+ 22200\mu_1^3\mu_2\mu_4^3\mu_5^3\mu_6^2 + 38514\mu_1^3\mu_2\mu_4^3\mu_5^2\mu_6^3 + 31574\mu_1^3\mu_2\mu_4^3\mu_5\mu_6^4 + 9488\mu_1^3\mu_2\mu_4^3\mu_6^5$$
$$- 91\mu_1^3\mu_2\mu_4^2\mu_5^6 + 42\mu_1^3\mu_2\mu_4^2\mu_5^5\mu_6 + 3015\mu_1^3\mu_2\mu_4^2\mu_5^4\mu_6^2 + 8848\mu_1^3\mu_2\mu_4^2\mu_5^3\mu_6^3$$
$$+ 11520\mu_1^3\mu_2\mu_4^2\mu_5^2\mu_6^4 + 7470\mu_1^3\mu_2\mu_4^2\mu_5\mu_6^5 + 1904\mu_1^3\mu_2\mu_4^2\mu_6^6 + 14\mu_1^3\mu_2\mu_4\mu_5^6\mu_6$$
$$+ 168\mu_1^3\mu_2\mu_4\mu_5^5\mu_6^2 + 690\mu_1^3\mu_2\mu_4\mu_5^4\mu_6^3 + 1272\mu_1^3\mu_2\mu_4\mu_5^3\mu_6^4 + 1236\mu_1^3\mu_2\mu_4\mu_5^2\mu_6^5$$
$$+ 636\mu_1^3\mu_2\mu_4\mu_5\mu_6^6 + 136\mu_1^3\mu_2\mu_4\mu_6^7 + 32\mu_1^3\mu_3^6\mu_4^3 + 14\mu_1^3\mu_3^6\mu_4^2\mu_5 + 21\mu_1^3\mu_3^6\mu_4^2\mu_6 + \mu_1^3\mu_3^6\mu_4\mu_5^2$$
$$+ 4\mu_1^3\mu_3^6\mu_4\mu_5\mu_6 + 3\mu_1^3\mu_3^6\mu_4\mu_6^2 + 360\mu_1^3\mu_3^5\mu_4^4 + 384\mu_1^3\mu_3^5\mu_4^3\mu_5 + 576\mu_1^3\mu_3^5\mu_4^3\mu_6$$
$$+ 105\mu_1^3\mu_3^5\mu_4^2\mu_5^2 + 336\mu_1^3\mu_3^5\mu_4^2\mu_5\mu_6 + 252\mu_1^3\mu_3^5\mu_4^2\mu_6^2 + 6\mu_1^3\mu_3^5\mu_4\mu_5^3 + 36\mu_1^3\mu_3^5\mu_4\mu_5^2\mu_6 +$$
$$60\mu_1^3\mu_3^5\mu_4\mu_5\mu_6^2 + 30\mu_1^3\mu_3^5\mu_4\mu_6^3 + 1376\mu_1^3\mu_3^4\mu_4^5 + 2920\mu_1^3\mu_3^4\mu_4^4\mu_5 + 3960\mu_1^3\mu_3^4\mu_4^4\mu_6$$
$$+ 1572\mu_1^3\mu_3^4\mu_4^3\mu_5^2 + 5008\mu_1^3\mu_3^4\mu_4^3\mu_5\mu_6 + 3588\mu_1^3\mu_3^4\mu_4^3\mu_6^2 + 326\mu_1^3\mu_3^4\mu_4^2\mu_5^3 + 1551\mu_1^3\mu_3^4\mu_4^2\mu_5^2\mu_6$$
$$+ 2464\mu_1^3\mu_3^4\mu_4^2\mu_5\mu_6^2 + 1206\mu_1^3\mu_3^4\mu_4^2\mu_6^3 + 14\mu_1^3\mu_3^4\mu_4\mu_5^4 + 122\mu_1^3\mu_3^4\mu_4\mu_5^3\mu_6 + 311\mu_1^3\mu_3^4\mu_4\mu_5^2\mu_6^2$$
$$+ 332\mu_1^3\mu_3^4\mu_4\mu_5\mu_6^3 + 123\mu_1^3\mu_3^4\mu_4\mu_6^4 + 2352\mu_1^3\mu_3^3\mu_4^6 + 8896\mu_1^3\mu_3^3\mu_4^5\mu_5 + 10656\mu_1^3\mu_3^3\mu_4^5\mu_6 +$$
$$8420\mu_1^3\mu_3^3\mu_4^4\mu_5^2 + 25120\mu_1^3\mu_3^3\mu_4^4\mu_5\mu_6 + 16420\mu_1^3\mu_3^3\mu_4^4\mu_6^2 + 3120\mu_1^3\mu_3^3\mu_4^3\mu_5^3$$
$$+ 15056\mu_1^3\mu_3^3\mu_4^3\mu_5^2\mu_6 + 23248\mu_1^3\mu_3^3\mu_4^3\mu_5\mu_6^2 + 10760\mu_1^3\mu_3^3\mu_4^3\mu_6^3 + 504\mu_1^3\mu_3^3\mu_4^2\mu_5^4$$
$$+ 3290\mu_1^3\mu_3^3\mu_4^2\mu_5^3\mu_6 + 7914\mu_1^3\mu_3^3\mu_4^2\mu_5^2\mu_6^2 + 8248\mu_1^3\mu_3^3\mu_4^2\mu_5\mu_6^3 + 2965\mu_1^3\mu_3^3\mu_4^2\mu_6^4 + 16\mu_1^3\mu_3^3\mu_4\mu_5^5$$
$$+ 200\mu_1^3\mu_3^3\mu_4\mu_5^4\mu_6 + 710\mu_1^3\mu_3^3\mu_4\mu_5^3\mu_6^2 + 1154\mu_1^3\mu_3^3\mu_4\mu_5^2\mu_6^3 + 904\mu_1^3\mu_3^3\mu_4\mu_5\mu_6^4 + 264\mu_1^3\mu_3^3\mu_4\mu_6^5$$
$$+ 1856\mu_1^3\mu_3^2\mu_4^7 + 12082\mu_1^3\mu_3^2\mu_4^6\mu_5 + 12551\mu_1^3\mu_3^2\mu_4^6\mu_6 + 19335\mu_1^3\mu_3^2\mu_4^5\mu_5^2 + 51600\mu_1^3\mu_3^2\mu_4^5\mu_5\mu_6$$
$$+ 29904\mu_1^3\mu_3^2\mu_4^5\mu_6^2 + 10990\mu_1^3\mu_3^2\mu_4^4\mu_5^3 + 53015\mu_1^3\mu_3^2\mu_4^4\mu_5^2\mu_6 + 76680\mu_1^3\mu_3^2\mu_4^4\mu_5\mu_6^2$$
$$+ 32440\mu_1^3\mu_3^2\mu_4^4\mu_6^3 + 3248\mu_1^3\mu_3^2\mu_4^3\mu_5^4 + 20526\mu_1^3\mu_3^2\mu_4^3\mu_5^3\mu_6 + 49434\mu_1^3\mu_3^2\mu_4^3\mu_5^2\mu_6^2$$
$$+ 49856\mu_1^3\mu_3^2\mu_4^3\mu_5\mu_6^3 + 16908\mu_1^3\mu_3^2\mu_4^3\mu_6^4 + 354\mu_1^3\mu_3^2\mu_4^2\mu_5^5 + 3448\mu_1^3\mu_3^2\mu_4^2\mu_5^4\mu_6$$
$$+ 11342\mu_1^3\mu_3^2\mu_4^2\mu_5^3\mu_6^2 + 18174\mu_1^3\mu_3^2\mu_4^2\mu_5^2\mu_6^3 + 14008\mu_1^3\mu_3^2\mu_4^2\mu_5\mu_6^4 + 3962\mu_1^3\mu_3^2\mu_4^2\mu_6^5$$
$$+ 9\mu_1^3\mu_3^2\mu_4\mu_5^6 + 162\mu_1^3\mu_3^2\mu_4\mu_5^5\mu_6 + 796\mu_1^3\mu_3^2\mu_4\mu_5^4\mu_6^2 + 1772\mu_1^3\mu_3^2\mu_4\mu_5^3\mu_6^3$$
$$+ 2111\mu_1^3\mu_3^2\mu_4\mu_5^2\mu_6^4 + 1300\mu_1^3\mu_3^2\mu_4\mu_5\mu_6^5 + 312\mu_1^3\mu_3^2\mu_4\mu_6^6 + 552\mu_1^3\mu_3\mu_4^8 + 6912\mu_1^3\mu_3\mu_4^7\mu_5$$
$$+ 5888\mu_1^3\mu_3\mu_4^7\mu_6 + 18711\mu_1^3\mu_3\mu_4^6\mu_5^2 + 43316\mu_1^3\mu_3\mu_4^6\mu_5\mu_6 + 21413\mu_1^3\mu_3\mu_4^6\mu_6^2$$
$$+ 16254\mu_1^3\mu_3\mu_4^5\mu_5^3 + 73744\mu_1^3\mu_3\mu_4^5\mu_5^2\mu_6 + 96176\mu_1^3\mu_3\mu_4^5\mu_5\mu_6^2 + 35928\mu_1^3\mu_3\mu_4^5\mu_6^3$$
$$+ 6610\mu_1^3\mu_3\mu_4^4\mu_5^4 + 43910\mu_1^3\mu_3\mu_4^4\mu_5^3\mu_6 + 104525\mu_1^3\mu_3\mu_4^4\mu_5^2\mu_6^2 + 99000\mu_1^3\mu_3\mu_4^4\mu_5\mu_6^3$$
$$+ 30680\mu_1^3\mu_3\mu_4^4\mu_6^4 + 1504\mu_1^3\mu_3\mu_4^3\mu_5^5 + 12692\mu_1^3\mu_3\mu_4^3\mu_5^4\mu_6 + 41642\mu_1^3\mu_3\mu_4^3\mu_5^3\mu_6^2$$
$$+ 66992\mu_1^3\mu_3\mu_4^3\mu_5^2\mu_6^3 + 50128\mu_1^3\mu_3\mu_4^3\mu_5\mu_6^4 + 13392\mu_1^3\mu_3\mu_4^3\mu_6^5 + 59\mu_1^3\mu_3\mu_4^2\mu_5^6$$
$$+ 1502\mu_1^3\mu_3\mu_4^2\mu_5^5\mu_6 + 7196\mu_1^3\mu_3\mu_4^2\mu_5^4\mu_6^2 + 15940\mu_1^3\mu_3\mu_4^2\mu_5^3\mu_6^3 + 19155\mu_1^3\mu_3\mu_4^2\mu_5^2\mu_6^4$$
$$+ 11712\mu_1^3\mu_3\mu_4^2\mu_5\mu_6^5 + 2732\mu_1^3\mu_3\mu_4^2\mu_6^6 + 2\mu_1^3\mu_3\mu_4\mu_5^7 + 56\mu_1^3\mu_3\mu_4\mu_5^6\mu_6$$
$$+ 406\mu_1^3\mu_3\mu_4\mu_5^5\mu_6^2 + 1232\mu_1^3\mu_3\mu_4\mu_5^4\mu_6^3 + 1996\mu_1^3\mu_3\mu_4\mu_5^3\mu_6^4 + 1868\mu_1^3\mu_3\mu_4\mu_5^2\mu_6^5$$
$$+ 944\mu_1^3\mu_3\mu_4\mu_5\mu_6^6 + 192\mu_1^3\mu_3\mu_4\mu_6^7 + 1176\mu_1^3\mu_4^8\mu_5 + 588\mu_1^3\mu_4^8\mu_6 + 5744\mu_1^3\mu_4^7\mu_5^2$$
$$+ 11168\mu_1^3\mu_4^7\mu_5\mu_6 + 4176\mu_1^3\mu_4^7\mu_6^2 + 8400\mu_1^3\mu_4^6\mu_5^3 + 32942\mu_1^3\mu_4^6\mu_5^2\mu_6 + 37044\mu_1^3\mu_4^6\mu_5\mu_6^2$$
$$+ 11438\mu_1^3\mu_4^6\mu_6^3 + 3916\mu_1^3\mu_4^5\mu_5^4 + 29320\mu_1^3\mu_4^5\mu_5^3\mu_6 + 66600\mu_1^3\mu_4^5\mu_5^2\mu_6^2 + 57120\mu_1^3\mu_4^5\mu_5\mu_6^3$$
$$+ 15472\mu_1^3\mu_4^5\mu_6^4 + 1450\mu_1^3\mu_4^4\mu_5^5 + 11400\mu_1^3\mu_4^4\mu_5^4\mu_6 + 39810\mu_1^3\mu_4^4\mu_5^3\mu_6^2 + 64170\mu_1^3\mu_4^4\mu_5^2\mu_6^3$$
$$+ 45520\mu_1^3\mu_4^4\mu_5\mu_6^4 + 11120\mu_1^3\mu_4^4\mu_6^5 - 4\mu_1^3\mu_4^3\mu_5^6 + 2232\mu_1^3\mu_4^3\mu_5^5\mu_6 + 11100\mu_1^3\mu_4^3\mu_5^4\mu_6^2 +$$
$$25676\mu_1^3\mu_4^3\mu_5^3\mu_6^3 + 31574\mu_1^3\mu_4^3\mu_5^2\mu_6^4 + 18976\mu_1^3\mu_4^3\mu_5\mu_6^5 + 4208\mu_1^3\mu_4^3\mu_6^6 - 26\mu_1^3\mu_4^2\mu_5^7$$
$$+ 14\mu_1^3\mu_4^2\mu_5^6\mu_6 + 1206\mu_1^3\mu_4^2\mu_5^5\mu_6^2 + 4424\mu_1^3\mu_4^2\mu_5^4\mu_6^3 + 7680\mu_1^3\mu_4^2\mu_5^3\mu_6^4 + 7470\mu_1^3\mu_4^2\mu_5^2\mu_6^5 +$$
$$3808\mu_1^3\mu_4^2\mu_5\mu_6^6 + 760\mu_1^3\mu_4^2\mu_6^7 + 4\mu_1^3\mu_4\mu_5^7\mu_6 + 56\mu_1^3\mu_4\mu_5^6\mu_6^2 + 276\mu_1^3\mu_4\mu_5^5\mu_6^3 + 636\mu_1^3\mu_4\mu_5^4\mu_6^4$$
$$+ 824\mu_1^3\mu_4\mu_5^3\mu_6^5 + 636\mu_1^3\mu_4\mu_5^2\mu_6^6 + 272\mu_1^3\mu_4\mu_5\mu_6^7 + 48\mu_1^3\mu_4\mu_6^8 - \mu_1^2\mu_2^6\mu_3^2\mu_4^2 - 8\mu_1^2\mu_2^6\mu_3\mu_4^3$$
$$- 2\mu_1^2\mu_2^6\mu_3\mu_4^2\mu_5 - 4\mu_1^2\mu_2^6\mu_3\mu_4^2\mu_6 - 13\mu_1^2\mu_2^6\mu_4^4 - 8\mu_1^2\mu_2^6\mu_4^3\mu_5 - 16\mu_1^2\mu_2^6\mu_4^3\mu_6 - \mu_1^2\mu_2^6\mu_4^2\mu_5^2 -$$
$$4\mu_1^2\mu_2^6\mu_4^2\mu_5\mu_6 - 4\mu_1^2\mu_2^6\mu_4^2\mu_6^2 - 2\mu_1^2\mu_2^5\mu_3^3\mu_4^2 - 14\mu_1^2\mu_2^5\mu_3^2\mu_4^3 - 12\mu_1^2\mu_2^5\mu_3^2\mu_4^2\mu_5 - 11\mu_1^2\mu_2^5\mu_3^2\mu_4^2\mu_6$$
$$- 31\mu_1^2\mu_2^5\mu_3\mu_4^4 - 90\mu_1^2\mu_2^5\mu_3\mu_4^3\mu_5 - 70\mu_1^2\mu_2^5\mu_3\mu_4^3\mu_6 - 18\mu_1^2\mu_2^5\mu_3\mu_4^2\mu_5^2 - 51\mu_1^2\mu_2^5\mu_3\mu_4^2\mu_5\mu_6 -$$
$$25\mu_1^2\mu_2^5\mu_3\mu_4^2\mu_6^2 - 30\mu_1^2\mu_2^5\mu_4^5 - 152\mu_1^2\mu_2^5\mu_4^4\mu_5 - 114\mu_1^2\mu_2^5\mu_4^4\mu_6 - 76\mu_1^2\mu_2^5\mu_4^3\mu_5^2 - 212\mu_1^2\mu_2^5\mu_4^3\mu_5\mu_6$$

$$
\begin{aligned}
&- 102\mu_1^2\mu_2^5\mu_4^3\mu_6^2 - 8\mu_1^2\mu_2^5\mu_4^2\mu_5^3 - 40\mu_1^2\mu_2^5\mu_4^2\mu_5^2\mu_6 - 59\mu_1^2\mu_2^5\mu_4^2\mu_5\mu_6^2 - 22\mu_1^2\mu_2^5\mu_4^2\mu_6^3 - \mu_1^2\mu_2^4\mu_3^4\mu_4^2 \\
&+ 14\mu_1^2\mu_2^4\mu_3^3\mu_4^3 - 11\mu_1^2\mu_2^4\mu_3^3\mu_4^2\mu_5 + 2\mu_1^2\mu_2^4\mu_3^3\mu_4^2\mu_6 + \mu_1^2\mu_2^4\mu_3^3\mu_4\mu_5\mu_6 + \mu_1^2\mu_2^4\mu_3^3\mu_4\mu_6^2 \\
&+ 110\mu_1^2\mu_2^4\mu_3^2\mu_4^4 - 40\mu_1^2\mu_2^4\mu_3^2\mu_4^3\mu_5 + 100\mu_1^2\mu_2^4\mu_3^2\mu_4^3\mu_6 - 46\mu_1^2\mu_2^4\mu_3^2\mu_4^2\mu_5^2 - 43\mu_1^2\mu_2^4\mu_3^2\mu_4^2\mu_5\mu_6 \\
&+ 24\mu_1^2\mu_2^4\mu_3^2\mu_4^2\mu_6^2 + 2\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_5^2\mu_6 + 6\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_5\mu_6^2 + 4\mu_1^2\mu_2^4\mu_3^2\mu_4\mu_6^3 + 199\mu_1^2\mu_2^4\mu_3\mu_4^5 - \\
&27\mu_1^2\mu_2^4\mu_3\mu_4^4\mu_5 + 334\mu_1^2\mu_2^4\mu_3\mu_4^4\mu_6 - 340\mu_1^2\mu_2^4\mu_3\mu_4^3\mu_5^2 - 286\mu_1^2\mu_2^4\mu_3\mu_4^3\mu_5\mu_6 + 144\mu_1^2\mu_2^4\mu_3\mu_4^3\mu_6^2 \\
&- 63\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_5^3 - 199\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_5^2\mu_6 - 105\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_5\mu_6^2 + 30\mu_1^2\mu_2^4\mu_3\mu_4^2\mu_6^3 + \mu_1^2\mu_2^4\mu_3\mu_4\mu_5^3\mu_6 \\
&+ 6\mu_1^2\mu_2^4\mu_3\mu_4\mu_5^2\mu_6^2 + 10\mu_1^2\mu_2^4\mu_3\mu_4\mu_5\mu_6^3 + 5\mu_1^2\mu_2^4\mu_3\mu_4\mu_6^4 + 86\mu_1^2\mu_2^4\mu_4^6 - 76\mu_1^2\mu_2^4\mu_4^5\mu_5 + \\
&186\mu_1^2\mu_2^4\mu_4^5\mu_6 - 623\mu_1^2\mu_2^4\mu_4^4\mu_5^2 - 618\mu_1^2\mu_2^4\mu_4^4\mu_5\mu_6 + 64\mu_1^2\mu_2^4\mu_4^4\mu_6^2 - 286\mu_1^2\mu_2^4\mu_4^3\mu_5^3 \\
&- 944\mu_1^2\mu_2^4\mu_4^3\mu_5^2\mu_6 - 658\mu_1^2\mu_2^4\mu_4^3\mu_5\mu_6^2 - 36\mu_1^2\mu_2^4\mu_4^3\mu_6^3 - 27\mu_1^2\mu_2^4\mu_4^2\mu_5^4 - 154\mu_1^2\mu_2^4\mu_4^2\mu_5^3\mu_6 \\
&- 267\mu_1^2\mu_2^4\mu_4^2\mu_5^2\mu_6^2 - 136\mu_1^2\mu_2^4\mu_4^2\mu_5\mu_6^3 - 4\mu_1^2\mu_2^4\mu_4^2\mu_6^4 + \mu_1^2\mu_2^4\mu_4\mu_5^3\mu_6^2 + 4\mu_1^2\mu_2^4\mu_4\mu_5^2\mu_6^3 \\
&+ 5\mu_1^2\mu_2^4\mu_4\mu_5\mu_6^4 + 2\mu_1^2\mu_2^4\mu_4\mu_6^5 + 56\mu_1^2\mu_2^3\mu_3^4\mu_4^3 + 9\mu_1^2\mu_2^3\mu_3^4\mu_4^2\mu_5 + 29\mu_1^2\mu_2^3\mu_3^4\mu_4^2\mu_6 \\
&+ 3\mu_1^2\mu_2^3\mu_3^4\mu_4\mu_5\mu_6 + 3\mu_1^2\mu_2^3\mu_3^4\mu_4\mu_6^2 + 472\mu_1^2\mu_2^3\mu_3^3\mu_4^4 + 322\mu_1^2\mu_2^3\mu_3^3\mu_4^3\mu_5 + 632\mu_1^2\mu_2^3\mu_3^3\mu_4^3\mu_6 \\
&+ 3\mu_1^2\mu_2^3\mu_3^3\mu_4^2\mu_5^2 + 192\mu_1^2\mu_2^3\mu_3^3\mu_4^2\mu_5\mu_6 + 218\mu_1^2\mu_2^3\mu_3^3\mu_4^2\mu_6^2 + 11\mu_1^2\mu_2^3\mu_3^3\mu_4\mu_5^2\mu_6 \\
&+ 29\mu_1^2\mu_2^3\mu_3^3\mu_4\mu_5\mu_6^2 + 18\mu_1^2\mu_2^3\mu_3^3\mu_4\mu_6^3 + 1389\mu_1^2\mu_2^3\mu_3^2\mu_4^5 + 1648\mu_1^2\mu_2^3\mu_3^2\mu_4^4\mu_5 \\
&+ 3089\mu_1^2\mu_2^3\mu_3^2\mu_4^4\mu_6 + 282\mu_1^2\mu_2^3\mu_3^2\mu_4^3\mu_5^2 + 2142\mu_1^2\mu_2^3\mu_3^2\mu_4^3\mu_5\mu_6 + 2206\mu_1^2\mu_2^3\mu_3^2\mu_4^3\mu_6^2 \\
&- 71\mu_1^2\mu_2^3\mu_3^2\mu_4^2\mu_5^3 + 165\mu_1^2\mu_2^3\mu_3^2\mu_4^2\mu_5^2\mu_6 + 789\mu_1^2\mu_2^3\mu_3^2\mu_4^2\mu_5\mu_6^2 + 570\mu_1^2\mu_2^3\mu_3^2\mu_4^2\mu_6^3 \\
&+ 13\mu_1^2\mu_2^3\mu_3^2\mu_4\mu_5^3\mu_6 + 60\mu_1^2\mu_2^3\mu_3^2\mu_4\mu_5^2\mu_6^2 + 86\mu_1^2\mu_2^3\mu_3^2\mu_4\mu_5\mu_6^3 + 39\mu_1^2\mu_2^3\mu_3^2\mu_4\mu_6^4 \\
&+ 1728\mu_1^2\mu_2^3\mu_3\mu_4^6 + 2685\mu_1^2\mu_2^3\mu_3\mu_4^5\mu_5 + 5306\mu_1^2\mu_2^3\mu_3\mu_4^5\mu_6 + 1031\mu_1^2\mu_2^3\mu_3\mu_4^4\mu_5^2 \\
&+ 5749\mu_1^2\mu_2^3\mu_3\mu_4^4\mu_5\mu_6 + 5937\mu_1^2\mu_2^3\mu_3\mu_4^4\mu_6^2 - 554\mu_1^2\mu_2^3\mu_3\mu_4^3\mu_5^3 + 686\mu_1^2\mu_2^3\mu_3\mu_4^3\mu_5^2\mu_6 \\
&+ 3898\mu_1^2\mu_2^3\mu_3\mu_4^3\mu_5\mu_6^2 + 2920\mu_1^2\mu_2^3\mu_3\mu_4^3\mu_6^3 - 115\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_5^4 - 308\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_5^3\mu_6 \\
&+ 307\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_5^2\mu_6^2 + 1080\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_5\mu_6^3 + 607\mu_1^2\mu_2^3\mu_3\mu_4^2\mu_6^4 + 5\mu_1^2\mu_2^3\mu_3\mu_4\mu_5^4\mu_6 \\
&+ 39\mu_1^2\mu_2^3\mu_3\mu_4\mu_5^3\mu_6^2 + 98\mu_1^2\mu_2^3\mu_3\mu_4\mu_5^2\mu_6^3 + 100\mu_1^2\mu_2^3\mu_3\mu_4\mu_5\mu_6^4 + 36\mu_1^2\mu_2^3\mu_3\mu_4\mu_6^5 \\
&+ 756\mu_1^2\mu_2^3\mu_4^7 + 1226\mu_1^2\mu_2^3\mu_4^6\mu_5 + 2872\mu_1^2\mu_2^3\mu_4^6\mu_6 + 564\mu_1^2\mu_2^3\mu_4^5\mu_5^2 + 3638\mu_1^2\mu_2^3\mu_4^5\mu_5\mu_6 + \\
&4310\mu_1^2\mu_2^3\mu_4^5\mu_6^2 - 1214\mu_1^2\mu_2^3\mu_4^4\mu_5^3 - 262\mu_1^2\mu_2^3\mu_4^4\mu_5^2\mu_6 + 3557\mu_1^2\mu_2^3\mu_4^4\mu_5\mu_6^2 + 3182\mu_1^2\mu_2^3\mu_4^4\mu_6^3 - \\
&570\mu_1^2\mu_2^3\mu_4^3\mu_5^4 - 1988\mu_1^2\mu_2^3\mu_4^3\mu_5^3\mu_6 - 910\mu_1^2\mu_2^3\mu_4^3\mu_5^2\mu_6^2 + 1532\mu_1^2\mu_2^3\mu_4^3\mu_5\mu_6^3 + 1194\mu_1^2\mu_2^3\mu_4^3\mu_6^4 \\
&- 50\mu_1^2\mu_2^3\mu_4^2\mu_5^5 - 310\mu_1^2\mu_2^3\mu_4^2\mu_5^4\mu_6 - 540\mu_1^2\mu_2^3\mu_4^2\mu_5^3\mu_6^2 - 88\mu_1^2\mu_2^3\mu_4^2\mu_5^2\mu_6^3 + 387\mu_1^2\mu_2^3\mu_4^2\mu_5\mu_6^4 \\
&+ 214\mu_1^2\mu_2^3\mu_4^2\mu_6^5 + 5\mu_1^2\mu_2^3\mu_4\mu_5^4\mu_6^2 + 26\mu_1^2\mu_2^3\mu_4\mu_5^3\mu_6^3 + 49\mu_1^2\mu_2^3\mu_4\mu_5^2\mu_6^4 + 40\mu_1^2\mu_2^3\mu_4\mu_5\mu_6^5 \\
&+ 12\mu_1^2\mu_2^3\mu_4\mu_6^6 + 54\mu_1^2\mu_2^2\mu_3^5\mu_4^3 + 15\mu_1^2\mu_2^2\mu_3^5\mu_4^2\mu_5 + 30\mu_1^2\mu_2^2\mu_3^5\mu_4^2\mu_6 + 3\mu_1^2\mu_2^2\mu_3^5\mu_4\mu_5\mu_6 + \\
&3\mu_1^2\mu_2^2\mu_3^5\mu_4\mu_6^2 + 624\mu_1^2\mu_2^2\mu_3^4\mu_4^4 + 514\mu_1^2\mu_2^2\mu_3^4\mu_4^3\mu_5 + 856\mu_1^2\mu_2^2\mu_3^4\mu_4^3\mu_6 + 71\mu_1^2\mu_2^2\mu_3^4\mu_4^2\mu_5^2 + \\
&346\mu_1^2\mu_2^2\mu_3^4\mu_4^2\mu_5\mu_6 + 307\mu_1^2\mu_2^2\mu_3^4\mu_4^2\mu_6^2 + 17\mu_1^2\mu_2^2\mu_3^4\mu_4\mu_5^2\mu_6 + 42\mu_1^2\mu_2^2\mu_3^4\mu_4\mu_5\mu_6^2 \\
&+ 25\mu_1^2\mu_2^2\mu_3^4\mu_4\mu_6^3 + 2608\mu_1^2\mu_2^2\mu_3^3\mu_4^5 + 3564\mu_1^2\mu_2^2\mu_3^3\mu_4^4\mu_5 + 5952\mu_1^2\mu_2^2\mu_3^3\mu_4^4\mu_6 \\
&+ 1458\mu_1^2\mu_2^2\mu_3^3\mu_4^3\mu_5^2 + 5234\mu_1^2\mu_2^2\mu_3^3\mu_4^3\mu_5\mu_6 + 4412\mu_1^2\mu_2^2\mu_3^3\mu_4^3\mu_6^2 + 82\mu_1^2\mu_2^2\mu_3^3\mu_4^2\mu_5^3 \\
&+ 1008\mu_1^2\mu_2^2\mu_3^3\mu_4^2\mu_5^2\mu_6 + 2047\mu_1^2\mu_2^2\mu_3^3\mu_4^2\mu_5\mu_6^2 + 1178\mu_1^2\mu_2^2\mu_3^3\mu_4^2\mu_6^3 + 34\mu_1^2\mu_2^2\mu_3^3\mu_4\mu_5^3\mu_6 + \\
&141\mu_1^2\mu_2^2\mu_3^3\mu_4\mu_5^2\mu_6^2 + 187\mu_1^2\mu_2^2\mu_3^3\mu_4\mu_5\mu_6^3 + 80\mu_1^2\mu_2^2\mu_3^3\mu_4\mu_6^4 + 4881\mu_1^2\mu_2^2\mu_3^2\mu_4^6 \\
&+ 9764\mu_1^2\mu_2^2\mu_3^2\mu_4^5\mu_5 + 16145\mu_1^2\mu_2^2\mu_3^2\mu_4^5\mu_6 + 6662\mu_1^2\mu_2^2\mu_3^2\mu_4^4\mu_5^2 + 22912\mu_1^2\mu_2^2\mu_3^2\mu_4^4\mu_5\mu_6 + \\
&19195\mu_1^2\mu_2^2\mu_3^2\mu_4^4\mu_6^2 + 1248\mu_1^2\mu_2^2\mu_3^2\mu_4^3\mu_5^3 + 9534\mu_1^2\mu_2^2\mu_3^2\mu_4^3\mu_5^2\mu_6 + 17596\mu_1^2\mu_2^2\mu_3^2\mu_4^3\mu_5\mu_6^2 + \\
&9968\mu_1^2\mu_2^2\mu_3^2\mu_4^3\mu_6^3 - 43\mu_1^2\mu_2^2\mu_3^2\mu_4^2\mu_5^4 + 870\mu_1^2\mu_2^2\mu_3^2\mu_4^2\mu_5^3\mu_6 + 3805\mu_1^2\mu_2^2\mu_3^2\mu_4^2\mu_5^2\mu_6^2 \\
&+ 4956\mu_1^2\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6^3 + 2127\mu_1^2\mu_2^2\mu_3^2\mu_4^2\mu_6^4 + 29\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_5^4\mu_6 + 180\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_5^3\mu_6^2 + \\
&394\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_5^2\mu_6^3 + 366\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_5\mu_6^4 + 123\mu_1^2\mu_2^2\mu_3^2\mu_4\mu_6^5 + 4000\mu_1^2\mu_2^2\mu_3\mu_4^7 \\
&+ 11709\mu_1^2\mu_2^2\mu_3\mu_4^6\mu_5 + 18210\mu_1^2\mu_2^2\mu_3\mu_4^6\mu_6 + 10453\mu_1^2\mu_2^2\mu_3\mu_4^5\mu_5^2 + 37552\mu_1^2\mu_2^2\mu_3\mu_4^5\mu_5\mu_6 \\
&+ 30975\mu_1^2\mu_2^2\mu_3\mu_4^5\mu_6^2 + 3774\mu_1^2\mu_2^2\mu_3\mu_4^4\mu_5^3 + 24155\mu_1^2\mu_2^2\mu_3\mu_4^4\mu_5^2\mu_6 + 44402\mu_1^2\mu_2^2\mu_3\mu_4^4\mu_5\mu_6^2 \\
&+ 25196\mu_1^2\mu_2^2\mu_3\mu_4^4\mu_6^3 - 412\mu_1^2\mu_2^2\mu_3\mu_4^3\mu_5^4 + 4204\mu_1^2\mu_2^2\mu_3\mu_4^3\mu_5^3\mu_6 + 18042\mu_1^2\mu_2^2\mu_3\mu_4^3\mu_5^2\mu_6^2 + \\
&23380\mu_1^2\mu_2^2\mu_3\mu_4^3\mu_5\mu_6^3 + 10132\mu_1^2\mu_2^2\mu_3\mu_4^3\mu_6^4 - 123\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_5^5 - 184\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_5^4\mu_6
\end{aligned}
$$

$$+ 1778\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_5^3\mu_6^2 + 5252\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6^3 + 5211\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^4 + 1806\mu_1^2\mu_2^2\mu_3\mu_4^2\mu_6^5 +$$
$$9\mu_1^2\mu_2^2\mu_3\mu_4\mu_5^5\mu_6 + 87\mu_1^2\mu_2^2\mu_3\mu_4\mu_5^4\mu_6^2 + 292\mu_1^2\mu_2^2\mu_3\mu_4\mu_5^3\mu_6^3 + 450\mu_1^2\mu_2^2\mu_3\mu_4\mu_5^2\mu_6^4$$
$$+ 327\mu_1^2\mu_2^2\mu_3\mu_4\mu_5\mu_6^5 + 91\mu_1^2\mu_2^2\mu_3\mu_4\mu_6^6 + 1072\mu_1^2\mu_2^2\mu_4^8 + 5016\mu_1^2\mu_2^2\mu_4^7\mu_5 + 6812\mu_1^2\mu_2^2\mu_4^7\mu_6$$
$$+ 4846\mu_1^2\mu_2^2\mu_4^6\mu_5^2 + 19964\mu_1^2\mu_2^2\mu_4^6\mu_5\mu_6 + 15810\mu_1^2\mu_2^2\mu_4^6\mu_6^2 + 2416\mu_1^2\mu_2^2\mu_4^5\mu_5^3$$
$$+ 15674\mu_1^2\mu_2^2\mu_4^5\mu_5^2\mu_6 + 31688\mu_1^2\mu_2^2\mu_4^5\mu_5\mu_6^2 + 18158\mu_1^2\mu_2^2\mu_4^5\mu_6^3 - 1287\mu_1^2\mu_2^2\mu_4^4\mu_5^4$$
$$+ 2852\mu_1^2\mu_2^2\mu_4^4\mu_5^3\mu_6 + 17567\mu_1^2\mu_2^2\mu_4^4\mu_5^2\mu_6^2 + 25048\mu_1^2\mu_2^2\mu_4^4\mu_5\mu_6^3 + 11257\mu_1^2\mu_2^2\mu_4^4\mu_6^4$$
$$- 662\mu_1^2\mu_2^2\mu_4^3\mu_5^5 - 2304\mu_1^2\mu_2^2\mu_4^3\mu_5^4\mu_6 + 900\mu_1^2\mu_2^2\mu_4^3\mu_5^3\mu_6^2 + 8828\mu_1^2\mu_2^2\mu_4^3\mu_5^2\mu_6^3$$
$$+ 10138\mu_1^2\mu_2^2\mu_4^3\mu_5\mu_6^4 + 3720\mu_1^2\mu_2^2\mu_4^3\mu_6^5 - 54\mu_1^2\mu_2^2\mu_4^2\mu_5^6 - 362\mu_1^2\mu_2^2\mu_4^2\mu_5^5\mu_6 - 580\mu_1^2\mu_2^2\mu_4^2\mu_5^4\mu_6^2 +$$
$$568\mu_1^2\mu_2^2\mu_4^2\mu_5^3\mu_6^3 + 2167\mu_1^2\mu_2^2\mu_4^2\mu_5^2\mu_6^4 + 1918\mu_1^2\mu_2^2\mu_4^2\mu_5\mu_6^5 + 576\mu_1^2\mu_2^2\mu_4^2\mu_6^6 + 9\mu_1^2\mu_2^2\mu_4\mu_5^5\mu_6^2$$
$$+ 58\mu_1^2\mu_2^2\mu_4\mu_5^4\mu_6^3 + 146\mu_1^2\mu_2^2\mu_4\mu_5^3\mu_6^4 + 180\mu_1^2\mu_2^2\mu_4\mu_5^2\mu_6^5 + 109\mu_1^2\mu_2^2\mu_4\mu_5\mu_6^6 + 26\mu_1^2\mu_2^2\mu_4\mu_6^7$$
$$+ 18\mu_1^2\mu_2\mu_3^6\mu_4^3 + 5\mu_1^2\mu_2\mu_3^6\mu_4^2\mu_5 + 10\mu_1^2\mu_2\mu_3^6\mu_4^2\mu_6 + \mu_1^2\mu_2\mu_3^6\mu_4\mu_5\mu_6 + \mu_1^2\mu_2\mu_3^6\mu_4\mu_6^2 +$$
$$336\mu_1^2\mu_2\mu_3^5\mu_4^4 + 270\mu_1^2\mu_2\mu_3^5\mu_4^3\mu_5 + 432\mu_1^2\mu_2\mu_3^5\mu_4^3\mu_6 + 45\mu_1^2\mu_2\mu_3^5\mu_4^2\mu_5^2 + 180\mu_1^2\mu_2\mu_3^5\mu_4^2\mu_5\mu_6$$
$$+ 150\mu_1^2\mu_2\mu_3^5\mu_4^2\mu_6^2 + 9\mu_1^2\mu_2\mu_3^5\mu_4\mu_5^2\mu_6 + 21\mu_1^2\mu_2\mu_3^5\mu_4\mu_5\mu_6^2 + 12\mu_1^2\mu_2\mu_3^5\mu_4\mu_6^3$$
$$+ 1928\mu_1^2\mu_2\mu_3^4\mu_4^5 + 2726\mu_1^2\mu_2\mu_3^4\mu_4^4\mu_5 + 4332\mu_1^2\mu_2\mu_3^4\mu_4^4\mu_6 + 1206\mu_1^2\mu_2\mu_3^4\mu_4^3\mu_5^2$$
$$+ 3934\mu_1^2\mu_2\mu_3^4\mu_4^3\mu_5\mu_6 + 3128\mu_1^2\mu_2\mu_3^4\mu_4^3\mu_6^2 + 124\mu_1^2\mu_2\mu_3^4\mu_4^2\mu_5^3 + 864\mu_1^2\mu_2\mu_3^4\mu_4^2\mu_5^2\mu_6$$
$$+ 1524\mu_1^2\mu_2\mu_3^4\mu_4^2\mu_5\mu_6^2 + 818\mu_1^2\mu_2\mu_3^4\mu_4^2\mu_6^3 + 28\mu_1^2\mu_2\mu_3^4\mu_4\mu_5^3\mu_6 + 108\mu_1^2\mu_2\mu_3^4\mu_4\mu_5^2\mu_6^2$$
$$+ 135\mu_1^2\mu_2\mu_3^4\mu_4\mu_5\mu_6^3 + 55\mu_1^2\mu_2\mu_3^4\mu_4\mu_6^4 + 4768\mu_1^2\mu_2\mu_3^3\mu_4^6 + 10984\mu_1^2\mu_2\mu_3^3\mu_4^5\mu_5$$
$$+ 16448\mu_1^2\mu_2\mu_3^3\mu_4^5\mu_6 + 7860\mu_1^2\mu_2\mu_3^3\mu_4^4\mu_5^2 + 25608\mu_1^2\mu_2\mu_3^3\mu_4^4\mu_5\mu_6 + 19852\mu_1^2\mu_2\mu_3^3\mu_4^4\mu_6^2 +$$
$$2272\mu_1^2\mu_2\mu_3^3\mu_4^3\mu_5^3 + 11910\mu_1^2\mu_2\mu_3^3\mu_4^3\mu_5^2\mu_6 + 19616\mu_1^2\mu_2\mu_3^3\mu_4^3\mu_5\mu_6^2 + 10296\mu_1^2\mu_2\mu_3^3\mu_4^3\mu_6^3 +$$
$$120\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_5^4 + 1632\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_5^3\mu_6 + 4833\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_5^2\mu_6^2 + 5488\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_5\mu_6^3 +$$
$$2180\mu_1^2\mu_2\mu_3^3\mu_4^2\mu_6^4 + 40\mu_1^2\mu_2\mu_3^3\mu_4\mu_5^4\mu_6 + 224\mu_1^2\mu_2\mu_3^3\mu_4\mu_5^3\mu_6^2 + 453\mu_1^2\mu_2\mu_3^3\mu_4\mu_5^2\mu_6^3$$
$$+ 395\mu_1^2\mu_2\mu_3^3\mu_4\mu_5\mu_6^4 + 126\mu_1^2\mu_2\mu_3^3\mu_4\mu_6^5 + 5518\mu_1^2\mu_2\mu_3^2\mu_4^7 + 20277\mu_1^2\mu_2\mu_3^2\mu_4^6\mu_5$$
$$+ 27416\mu_1^2\mu_2\mu_3^2\mu_4^6\mu_6 + 20958\mu_1^2\mu_2\mu_3^2\mu_4^5\mu_5^2 + 67917\mu_1^2\mu_2\mu_3^2\mu_4^5\mu_5\mu_6 + 49608\mu_1^2\mu_2\mu_3^2\mu_4^5\mu_6^2 +$$
$$10028\mu_1^2\mu_2\mu_3^2\mu_4^4\mu_5^3 + 50202\mu_1^2\mu_2\mu_3^2\mu_4^4\mu_5^2\mu_6 + 82177\mu_1^2\mu_2\mu_3^2\mu_4^4\mu_5\mu_6^2 + 41904\mu_1^2\mu_2\mu_3^2\mu_4^4\mu_6^3 +$$
$$1590\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_5^4 + 14784\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_5^3\mu_6 + 39552\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_5^2\mu_6^2 + 43756\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_5\mu_6^3$$
$$+ 17136\mu_1^2\mu_2\mu_3^2\mu_4^3\mu_6^4 - 9\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5^5 + 1140\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5^4\mu_6 + 6032\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5^3\mu_6^2$$
$$+ 11448\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6^3 + 9660\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^4 + 3054\mu_1^2\mu_2\mu_3^2\mu_4^2\mu_6^5 + 27\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^5\mu_6$$
$$+ 210\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^4\mu_6^2 + 616\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^3\mu_6^3 + 864\mu_1^2\mu_2\mu_3^2\mu_4\mu_5^2\mu_6^4 + 585\mu_1^2\mu_2\mu_3^2\mu_4\mu_5\mu_6^5 +$$
$$154\mu_1^2\mu_2\mu_3^2\mu_4\mu_6^6 + 2784\mu_1^2\mu_2\mu_3\mu_4^8 + 16530\mu_1^2\mu_2\mu_3\mu_4^7\mu_5 + 19532\mu_1^2\mu_2\mu_3\mu_4^7\mu_6$$
$$+ 24759\mu_1^2\mu_2\mu_3\mu_4^6\mu_5^2 + 75952\mu_1^2\mu_2\mu_3\mu_4^6\mu_5\mu_6 + 50376\mu_1^2\mu_2\mu_3\mu_4^6\mu_6^2 + 15536\mu_1^2\mu_2\mu_3\mu_4^5\mu_5^3 +$$
$$80820\mu_1^2\mu_2\mu_3\mu_4^5\mu_5^2\mu_6 + 130787\mu_1^2\mu_2\mu_3\mu_4^5\mu_5\mu_6^2 + 62976\mu_1^2\mu_2\mu_3\mu_4^5\mu_6^3 + 4630\mu_1^2\mu_2\mu_3\mu_4^4\mu_5^4 +$$
$$36812\mu_1^2\mu_2\mu_3\mu_4^4\mu_5^3\mu_6 + 97584\mu_1^2\mu_2\mu_3\mu_4^4\mu_5^2\mu_6^2 + 108146\mu_1^2\mu_2\mu_3\mu_4^4\mu_5\mu_6^3 + 41292\mu_1^2\mu_2\mu_3\mu_4^4\mu_6^4$$
$$- 162\mu_1^2\mu_2\mu_3\mu_4^3\mu_5^5 + 5620\mu_1^2\mu_2\mu_3\mu_4^3\mu_5^4\mu_6 + 28288\mu_1^2\mu_2\mu_3\mu_4^3\mu_5^3\mu_6^2 + 52620\mu_1^2\mu_2\mu_3\mu_4^3\mu_5^2\mu_6^3 +$$
$$44450\mu_1^2\mu_2\mu_3\mu_4^3\mu_5\mu_6^4 + 13984\mu_1^2\mu_2\mu_3\mu_4^3\mu_6^5 - 77\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^6 - 36\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^5\mu_6$$
$$+ 2360\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^4\mu_6^2 + 8344\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^3\mu_6^3 + 11991\mu_1^2\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^4 + 8148\mu_1^2\mu_2\mu_3\mu_4^2\mu_5\mu_6^5$$
$$+ 2152\mu_1^2\mu_2\mu_3\mu_4^2\mu_6^6 + 7\mu_1^2\mu_2\mu_3\mu_4\mu_5^6\mu_6 + 81\mu_1^2\mu_2\mu_3\mu_4\mu_5^5\mu_6^2 + 340\mu_1^2\mu_2\mu_3\mu_4\mu_5^4\mu_6^3$$
$$+ 700\mu_1^2\mu_2\mu_3\mu_4\mu_5^3\mu_6^4 + 765\mu_1^2\mu_2\mu_3\mu_4\mu_5^2\mu_6^5 + 427\mu_1^2\mu_2\mu_3\mu_4\mu_5\mu_6^6 + 96\mu_1^2\mu_2\mu_3\mu_4\mu_6^7$$
$$+ 424\mu_1^2\mu_2\mu_4^9 + 4436\mu_1^2\mu_2\mu_4^8\mu_5 + 4396\mu_1^2\mu_2\mu_4^8\mu_6 + 10512\mu_1^2\mu_2\mu_4^7\mu_5^2 + 28372\mu_1^2\mu_2\mu_4^7\mu_5\mu_6$$
$$+ 16332\mu_1^2\mu_2\mu_4^7\mu_6^2 + 7240\mu_1^2\mu_2\mu_4^6\mu_5^3 + 42660\mu_1^2\mu_2\mu_4^6\mu_5^2\mu_6 + 66552\mu_1^2\mu_2\mu_4^6\mu_5\mu_6^2$$
$$+ 29232\mu_1^2\mu_2\mu_4^6\mu_6^3 + 3060\mu_1^2\mu_2\mu_4^5\mu_5^4 + 24072\mu_1^2\mu_2\mu_4^5\mu_5^3\mu_6 + 69204\mu_1^2\mu_2\mu_4^5\mu_5^2\mu_6^2$$
$$+ 77510\mu_1^2\mu_2\mu_4^5\mu_5\mu_6^3 + 28256\mu_1^2\mu_2\mu_4^5\mu_6^4 - 816\mu_1^2\mu_2\mu_4^4\mu_5^5 + 4540\mu_1^2\mu_2\mu_4^4\mu_5^4\mu_6$$
$$+ 28020\mu_1^2\mu_2\mu_4^4\mu_5^3\mu_6^2 + 56052\mu_1^2\mu_2\mu_4^4\mu_5^2\mu_6^3 + 48897\mu_1^2\mu_2\mu_4^4\mu_5\mu_6^4 + 15224\mu_1^2\mu_2\mu_4^4\mu_6^5$$
$$- 434\mu_1^2\mu_2\mu_4^3\mu_5^6 - 1572\mu_1^2\mu_2\mu_4^3\mu_5^5\mu_6 + 2260\mu_1^2\mu_2\mu_4^3\mu_5^4\mu_6^2 + 14592\mu_1^2\mu_2\mu_4^3\mu_5^3\mu_6^3$$

$+ 23250\mu_1^2\mu_2\mu_4^3\mu_5^2\mu_6^4 + 16500\mu_1^2\mu_2\mu_4^3\mu_5\mu_6^5 + 4416\mu_1^2\mu_2\mu_4^3\mu_6^6 - 32\mu_1^2\mu_2\mu_4^2\mu_5^7 - 238\mu_1^2\mu_2\mu_4^2\mu_5^6\mu_6$
$- 372\mu_1^2\mu_2\mu_4^2\mu_5^5\mu_6^2 + 940\mu_1^2\mu_2\mu_4^2\mu_5^4\mu_6^3 + 3560\mu_1^2\mu_2\mu_4^2\mu_5^3\mu_6^4 + 4470\mu_1^2\mu_2\mu_4^2\mu_5^2\mu_6^5$
$+ 2616\mu_1^2\mu_2\mu_4^2\mu_5\mu_6^6 + 600\mu_1^2\mu_2\mu_4^2\mu_6^7 + 7\mu_1^2\mu_2\mu_4\mu_5^6\mu_6^2 + 54\mu_1^2\mu_2\mu_4\mu_5^5\mu_6^3 + 170\mu_1^2\mu_2\mu_4\mu_5^4\mu_6^4$
$+ 280\mu_1^2\mu_2\mu_4\mu_5^3\mu_6^5 + 255\mu_1^2\mu_2\mu_4\mu_5^2\mu_6^6 + 122\mu_1^2\mu_2\mu_4\mu_5\mu_6^7 + 24\mu_1^2\mu_2\mu_4\mu_6^8 + 56\mu_1^2\mu_3^6\mu_4^4 +$
$36\mu_1^2\mu_3^6\mu_4^3\mu_5 + 54\mu_1^2\mu_3^6\mu_4^3\mu_6 + 5\mu_1^2\mu_3^6\mu_4^2\mu_5^2 + 20\mu_1^2\mu_3^6\mu_4^2\mu_5\mu_6 + 15\mu_1^2\mu_3^6\mu_4^2\mu_6^2 + \mu_1^2\mu_3^6\mu_4\mu_5^2\mu_6$
$+ 2\mu_1^2\mu_3^6\mu_4\mu_5\mu_6^2 + \mu_1^2\mu_3^6\mu_4\mu_6^3 + 480\mu_1^2\mu_3^5\mu_4^5 + 672\mu_1^2\mu_3^5\mu_4^4\mu_5 + 1008\mu_1^2\mu_3^5\mu_4^4\mu_6$
$+ 270\mu_1^2\mu_3^5\mu_4^3\mu_5^2 + 864\mu_1^2\mu_3^5\mu_4^3\mu_5\mu_6 + 648\mu_1^2\mu_3^5\mu_4^3\mu_6^2 + 30\mu_1^2\mu_3^5\mu_4^2\mu_5^3 + 180\mu_1^2\mu_3^5\mu_4^2\mu_5^2\mu_6$
$+ 300\mu_1^2\mu_3^5\mu_4^2\mu_5\mu_6^2 + 150\mu_1^2\mu_3^5\mu_4^2\mu_6^3 + 6\mu_1^2\mu_3^5\mu_4\mu_5^3\mu_6 + 21\mu_1^2\mu_3^5\mu_4\mu_5^2\mu_6^2 + 24\mu_1^2\mu_3^5\mu_4\mu_5\mu_6^3$
$+ 9\mu_1^2\mu_3^5\mu_4\mu_6^4 + 1504\mu_1^2\mu_3^4\mu_4^6 + 3856\mu_1^2\mu_3^4\mu_4^5\mu_5 + 5280\mu_1^2\mu_3^4\mu_4^5\mu_6 + 2726\mu_1^2\mu_3^4\mu_4^4\mu_5^2 +$
$8664\mu_1^2\mu_3^4\mu_4^4\mu_5\mu_6 + 6254\mu_1^2\mu_3^4\mu_4^4\mu_6^2 + 804\mu_1^2\mu_3^4\mu_4^3\mu_5^3 + 3934\mu_1^2\mu_3^4\mu_4^3\mu_5^2\mu_6 + 6256\mu_1^2\mu_3^4\mu_4^3\mu_5\mu_6^2$
$+ 3084\mu_1^2\mu_3^4\mu_4^3\mu_6^3 + 62\mu_1^2\mu_3^4\mu_4^2\mu_5^4 + 576\mu_1^2\mu_3^4\mu_4^2\mu_5^3\mu_6 + 1524\mu_1^2\mu_3^4\mu_4^2\mu_5^2\mu_6^2 + 1636\mu_1^2\mu_3^4\mu_4^2\mu_5\mu_6^3$
$+ 611\mu_1^2\mu_3^4\mu_4^2\mu_6^4 + 14\mu_1^2\mu_3^4\mu_4\mu_5^4\mu_6 + 72\mu_1^2\mu_3^4\mu_4\mu_5^3\mu_6^2 + 135\mu_1^2\mu_3^4\mu_4\mu_5^2\mu_6^3 + 110\mu_1^2\mu_3^4\mu_4\mu_5\mu_6^4$
$+ 33\mu_1^2\mu_3^4\mu_4\mu_6^5 + 2208\mu_1^2\mu_3^3\mu_4^7 + 9536\mu_1^2\mu_3^3\mu_4^6\mu_5 + 11616\mu_1^2\mu_3^3\mu_4^6\mu_6 + 10984\mu_1^2\mu_3^3\mu_4^5\mu_5^2 +$
$32896\mu_1^2\mu_3^3\mu_4^5\mu_5\mu_6 + 21800\mu_1^2\mu_3^3\mu_4^5\mu_6^2 + 5240\mu_1^2\mu_3^3\mu_4^4\mu_5^3 + 25608\mu_1^2\mu_3^3\mu_4^4\mu_5^2\mu_6$
$+ 39704\mu_1^2\mu_3^3\mu_4^4\mu_5\mu_6^2 + 18620\mu_1^2\mu_3^3\mu_4^4\mu_6^3 + 1136\mu_1^2\mu_3^3\mu_4^3\mu_5^4 + 7940\mu_1^2\mu_3^3\mu_4^3\mu_5^3\mu_6$
$+ 19616\mu_1^2\mu_3^3\mu_4^3\mu_5^2\mu_6^2 + 20592\mu_1^2\mu_3^3\mu_4^3\mu_5\mu_6^3 + 7510\mu_1^2\mu_3^3\mu_4^3\mu_6^4 + 48\mu_1^2\mu_3^3\mu_4^2\mu_5^5$
$+ 816\mu_1^2\mu_3^3\mu_4^2\mu_5^4\mu_6 + 3222\mu_1^2\mu_3^3\mu_4^2\mu_5^3\mu_6^2 + 5488\mu_1^2\mu_3^3\mu_4^2\mu_5^2\mu_6^3 + 4360\mu_1^2\mu_3^3\mu_4^2\mu_5\mu_6^4$
$+ 1296\mu_1^2\mu_3^3\mu_4^2\mu_6^5 + 16\mu_1^2\mu_3^3\mu_4\mu_5^5\mu_6 + 112\mu_1^2\mu_3^3\mu_4\mu_5^4\mu_6^2 + 302\mu_1^2\mu_3^3\mu_4\mu_5^3\mu_6^3 + 395\mu_1^2\mu_3^3\mu_4\mu_5^2\mu_6^4$
$+ 252\mu_1^2\mu_3^3\mu_4\mu_5\mu_6^5 + 63\mu_1^2\mu_3^3\mu_4\mu_6^6 + 1544\mu_1^2\mu_3^2\mu_4^8 + 11036\mu_1^2\mu_3^2\mu_4^7\mu_5 + 11698\mu_1^2\mu_3^2\mu_4^7\mu_6 +$
$20277\mu_1^2\mu_3^2\mu_4^6\mu_5^2 + 54832\mu_1^2\mu_3^2\mu_4^6\mu_5\mu_6 + 32368\mu_1^2\mu_3^2\mu_4^6\mu_6^2 + 13972\mu_1^2\mu_3^2\mu_4^5\mu_5^3$
$+ 67917\mu_1^2\mu_3^2\mu_4^5\mu_5^2\mu_6 + 99216\mu_1^2\mu_3^2\mu_4^5\mu_5\mu_6^2 + 42712\mu_1^2\mu_3^2\mu_4^5\mu_6^3 + 5014\mu_1^2\mu_3^2\mu_4^4\mu_5^4$
$+ 33468\mu_1^2\mu_3^2\mu_4^4\mu_5^3\mu_6 + 82177\mu_1^2\mu_3^2\mu_4^4\mu_5^2\mu_6^2 + 83808\mu_1^2\mu_3^2\mu_4^4\mu_5\mu_6^3 + 28954\mu_1^2\mu_3^2\mu_4^4\mu_6^4$
$+ 636\mu_1^2\mu_3^2\mu_4^3\mu_5^5 + 7392\mu_1^2\mu_3^2\mu_4^3\mu_5^4\mu_6 + 26368\mu_1^2\mu_3^2\mu_4^3\mu_5^3\mu_6^2 + 43756\mu_1^2\mu_3^2\mu_4^3\mu_5^2\mu_6^3$
$+ 34272\mu_1^2\mu_3^2\mu_4^3\mu_5\mu_6^4 + 9908\mu_1^2\mu_3^2\mu_4^3\mu_6^5 - 3\mu_1^2\mu_3^2\mu_4^2\mu_5^6 + 456\mu_1^2\mu_3^2\mu_4^2\mu_5^5\mu_6$
$+ 3016\mu_1^2\mu_3^2\mu_4^2\mu_5^4\mu_6^2 + 7632\mu_1^2\mu_3^2\mu_4^2\mu_5^3\mu_6^3 + 9660\mu_1^2\mu_3^2\mu_4^2\mu_5^2\mu_6^4 + 6108\mu_1^2\mu_3^2\mu_4^2\mu_5\mu_6^5$
$+ 1508\mu_1^2\mu_3^2\mu_4^2\mu_6^6 + 9\mu_1^2\mu_3^2\mu_4\mu_5^6\mu_6 + 84\mu_1^2\mu_3^2\mu_4\mu_5^5\mu_6^2 + 308\mu_1^2\mu_3^2\mu_4\mu_5^4\mu_6^3 + 576\mu_1^2\mu_3^2\mu_4\mu_5^3\mu_6^4$
$+ 585\mu_1^2\mu_3^2\mu_4\mu_5^2\mu_6^5 + 308\mu_1^2\mu_3^2\mu_4\mu_5\mu_6^6 + 66\mu_1^2\mu_3^2\mu_4\mu_6^7 + 416\mu_1^2\mu_3\mu_4^9 + 5568\mu_1^2\mu_3\mu_4^8\mu_5 +$
$4832\mu_1^2\mu_3\mu_4^8\mu_6 + 16530\mu_1^2\mu_3\mu_4^7\mu_5^2 + 39064\mu_1^2\mu_3\mu_4^7\mu_5\mu_6 + 19702\mu_1^2\mu_3\mu_4^7\mu_6^2$
$+ 16506\mu_1^2\mu_3\mu_4^6\mu_5^3 + 75952\mu_1^2\mu_3\mu_4^6\mu_5^2\mu_6 + 100752\mu_1^2\mu_3\mu_4^6\mu_5\mu_6^2 + 38408\mu_1^2\mu_3\mu_4^6\mu_6^3$
$+ 7768\mu_1^2\mu_3\mu_4^5\mu_5^4 + 53880\mu_1^2\mu_3\mu_4^5\mu_5^3\mu_6 + 130787\mu_1^2\mu_3\mu_4^5\mu_5^2\mu_6^2 + 125952\mu_1^2\mu_3\mu_4^5\mu_5\mu_6^3 +$
$39880\mu_1^2\mu_3\mu_4^5\mu_6^4 + 1852\mu_1^2\mu_3\mu_4^4\mu_5^5 + 18406\mu_1^2\mu_3\mu_4^4\mu_5^4\mu_6 + 65056\mu_1^2\mu_3\mu_4^4\mu_5^3\mu_6^2$
$+ 108146\mu_1^2\mu_3\mu_4^4\mu_5^2\mu_6^3 + 82584\mu_1^2\mu_3\mu_4^4\mu_5\mu_6^4 + 22616\mu_1^2\mu_3\mu_4^4\mu_6^5 - 54\mu_1^2\mu_3\mu_4^3\mu_5^6$
$+ 2248\mu_1^2\mu_3\mu_4^3\mu_5^5\mu_6 + 14144\mu_1^2\mu_3\mu_4^3\mu_5^4\mu_6^2 + 35080\mu_1^2\mu_3\mu_4^3\mu_5^3\mu_6^3 + 44450\mu_1^2\mu_3\mu_4^3\mu_5^2\mu_6^4 +$
$27968\mu_1^2\mu_3\mu_4^3\mu_5\mu_6^5 + 6728\mu_1^2\mu_3\mu_4^3\mu_6^6 - 22\mu_1^2\mu_3\mu_4^2\mu_5^7 - 12\mu_1^2\mu_3\mu_4^2\mu_5^6\mu_6 + 944\mu_1^2\mu_3\mu_4^2\mu_5^5\mu_6^2 +$
$4172\mu_1^2\mu_3\mu_4^2\mu_5^4\mu_6^3 + 7994\mu_1^2\mu_3\mu_4^2\mu_5^3\mu_6^4 + 8148\mu_1^2\mu_3\mu_4^2\mu_5^2\mu_6^5 + 4304\mu_1^2\mu_3\mu_4^2\mu_5\mu_6^6$
$+ 912\mu_1^2\mu_3\mu_4^2\mu_6^7 + 2\mu_1^2\mu_3\mu_4\mu_5^7\mu_6 + 27\mu_1^2\mu_3\mu_4\mu_5^6\mu_6^2 + 136\mu_1^2\mu_3\mu_4\mu_5^5\mu_6^3 + 350\mu_1^2\mu_3\mu_4\mu_5^4\mu_6^4$
$+ 510\mu_1^2\mu_3\mu_4\mu_5^3\mu_6^5 + 427\mu_1^2\mu_3\mu_4\mu_5^2\mu_6^6 + 192\mu_1^2\mu_3\mu_4\mu_5\mu_6^7 + 36\mu_1^2\mu_3\mu_4\mu_6^8 + 848\mu_1^2\mu_4^9\mu_5 +$
$424\mu_1^2\mu_4^9\mu_6 + 4436\mu_1^2\mu_4^8\mu_5^2 + 8792\mu_1^2\mu_4^8\mu_5\mu_6 + 3324\mu_1^2\mu_4^8\mu_6^2 + 7008\mu_1^2\mu_4^7\mu_5^3$
$+ 28372\mu_1^2\mu_4^7\mu_5^2\mu_6 + 32664\mu_1^2\mu_4^7\mu_5\mu_6^2 + 10276\mu_1^2\mu_4^7\mu_6^3 + 3620\mu_1^2\mu_4^6\mu_5^4 + 28440\mu_1^2\mu_4^6\mu_5^3\mu_6$
$+ 66552\mu_1^2\mu_4^6\mu_5^2\mu_6^2 + 58464\mu_1^2\mu_4^6\mu_5\mu_6^3 + 16208\mu_1^2\mu_4^6\mu_6^4 + 1224\mu_1^2\mu_4^5\mu_5^5 + 12036\mu_1^2\mu_4^5\mu_5^4\mu_6$
$+ 46136\mu_1^2\mu_4^5\mu_5^3\mu_6^2 + 77510\mu_1^2\mu_4^5\mu_5^2\mu_6^3 + 56512\mu_1^2\mu_4^5\mu_5\mu_6^4 + 14192\mu_1^2\mu_4^5\mu_6^5 - 272\mu_1^2\mu_4^4\mu_5^6 +$
$1816\mu_1^2\mu_4^4\mu_5^5\mu_6 + 14010\mu_1^2\mu_4^4\mu_5^4\mu_6^2 + 37368\mu_1^2\mu_4^4\mu_5^3\mu_6^3 + 48897\mu_1^2\mu_4^4\mu_5^2\mu_6^4 + 30448\mu_1^2\mu_4^4\mu_5\mu_6^5$
$+ 6984\mu_1^2\mu_4^4\mu_6^6 - 124\mu_1^2\mu_4^3\mu_5^7 - 524\mu_1^2\mu_4^3\mu_5^6\mu_6 + 904\mu_1^2\mu_4^3\mu_5^5\mu_6^2 + 7296\mu_1^2\mu_4^3\mu_5^4\mu_6^3$

$$+ 15500\mu_1^2\mu_4^3\mu_5^3\mu_6^4 + 16500\mu_1^2\mu_4^3\mu_5^2\mu_6^5 + 8832\mu_1^2\mu_4^3\mu_5\mu_6^6 + 1840\mu_1^2\mu_4^3\mu_6^7 - 8\mu_1^2\mu_4^2\mu_5^8$$
$$- 68\mu_1^2\mu_4^2\mu_5^7\mu_6 - 124\mu_1^2\mu_4^2\mu_5^6\mu_6^2 + 376\mu_1^2\mu_4^2\mu_5^5\mu_6^3 + 1780\mu_1^2\mu_4^2\mu_5^4\mu_6^4 + 2980\mu_1^2\mu_4^2\mu_5^3\mu_6^5$$
$$+ 2616\mu_1^2\mu_4^2\mu_5^2\mu_6^6 + 1200\mu_1^2\mu_4^2\mu_5\mu_6^7 + 224\mu_1^2\mu_4^2\mu_6^8 + 2\mu_1^2\mu_4\mu_5^7\mu_6^2 + 18\mu_1^2\mu_4\mu_5^6\mu_6^3$$
$$+ 68\mu_1^2\mu_4\mu_5^5\mu_6^4 + 140\mu_1^2\mu_4\mu_5^4\mu_6^5 + 170\mu_1^2\mu_4\mu_5^3\mu_6^6 + 122\mu_1^2\mu_4\mu_5^2\mu_6^7 + 48\mu_1^2\mu_4\mu_5\mu_6^8 + 8\mu_1^2\mu_4\mu_6^9$$
$$- 2\mu_1\mu_2^6\mu_3^2\mu_4^3 - 10\mu_1\mu_2^6\mu_3\mu_4^4 - 4\mu_1\mu_2^6\mu_3\mu_4^3\mu_5 - 8\mu_1\mu_2^6\mu_3\mu_4^3\mu_6 - 12\mu_1\mu_2^6\mu_4^5 - 10\mu_1\mu_2^6\mu_4^4\mu_5$$
$$- 20\mu_1\mu_2^6\mu_4^4\mu_6 - 2\mu_1\mu_2^6\mu_4^3\mu_5^2 - 8\mu_1\mu_2^6\mu_4^3\mu_5\mu_6 - 8\mu_1\mu_2^6\mu_4^3\mu_6^2 - 4\mu_1\mu_2^5\mu_3^3\mu_4^3 - 20\mu_1\mu_2^5\mu_3^2\mu_4^4 -$$
$$24\mu_1\mu_2^5\mu_3^2\mu_4^3\mu_5 - 24\mu_1\mu_2^5\mu_3^2\mu_4^3\mu_6 - 36\mu_1\mu_2^5\mu_3\mu_4^5 - 115\mu_1\mu_2^5\mu_3\mu_4^4\mu_5 - 100\mu_1\mu_2^5\mu_3\mu_4^4\mu_6$$
$$- 36\mu_1\mu_2^5\mu_3\mu_4^3\mu_5^2 - 104\mu_1\mu_2^5\mu_3\mu_4^3\mu_5\mu_6 - 56\mu_1\mu_2^5\mu_3\mu_4^3\mu_6^2 - 28\mu_1\mu_2^5\mu_4^6 - 144\mu_1\mu_2^5\mu_4^5\mu_5$$
$$- 120\mu_1\mu_2^5\mu_4^5\mu_6 - 95\mu_1\mu_2^5\mu_4^4\mu_5^2 - 270\mu_1\mu_2^5\mu_4^4\mu_5\mu_6 - 140\mu_1\mu_2^5\mu_4^4\mu_6^2 - 16\mu_1\mu_2^5\mu_4^3\mu_5^3$$
$$- 80\mu_1\mu_2^5\mu_4^3\mu_5^2\mu_6 - 120\mu_1\mu_2^5\mu_4^3\mu_5\mu_6^2 - 48\mu_1\mu_2^5\mu_4^3\mu_6^3 - 2\mu_1\mu_2^4\mu_3^4\mu_4^3 + 10\mu_1\mu_2^4\mu_3^3\mu_4^4$$
$$- 24\mu_1\mu_2^4\mu_3^3\mu_4^3\mu_5 - 4\mu_1\mu_2^4\mu_3^3\mu_4^3\mu_6 + 3\mu_1\mu_2^4\mu_3^3\mu_4^2\mu_5\mu_6 + 3\mu_1\mu_2^4\mu_3^3\mu_4^2\mu_6^2 + 72\mu_1\mu_2^4\mu_3^2\mu_4^5$$
$$- 85\mu_1\mu_2^4\mu_3^2\mu_4^4\mu_5 + 60\mu_1\mu_2^4\mu_3^2\mu_4^4\mu_6 - 96\mu_1\mu_2^4\mu_3^2\mu_4^3\mu_5^2 - 126\mu_1\mu_2^4\mu_3^2\mu_4^3\mu_5\mu_6 + 4\mu_1\mu_2^4\mu_3^2\mu_4^3\mu_6^2 +$$
$$6\mu_1\mu_2^4\mu_3^2\mu_4^2\mu_5^2\mu_6 + 18\mu_1\mu_2^4\mu_3^2\mu_4^2\mu_5\mu_6^2 + 12\mu_1\mu_2^4\mu_3^2\mu_4^2\mu_6^3 + 105\mu_1\mu_2^4\mu_3\mu_6^4 - 117\mu_1\mu_2^4\mu_3\mu_4^5\mu_5 +$$
$$168\mu_1\mu_2^4\mu_3\mu_4^5\mu_6 - 455\mu_1\mu_2^4\mu_3\mu_4^4\mu_5^2 - 535\mu_1\mu_2^4\mu_3\mu_4^4\mu_5\mu_6 + 20\mu_1\mu_2^4\mu_3\mu_4^4\mu_6^2 - 128\mu_1\mu_2^4\mu_3\mu_4^3\mu_5^3$$
$$- 434\mu_1\mu_2^4\mu_3\mu_4^3\mu_5^2\mu_6 - 322\mu_1\mu_2^4\mu_3\mu_4^3\mu_5\mu_6^2 - 16\mu_1\mu_2^4\mu_3\mu_4^3\mu_6^3 + 3\mu_1\mu_2^4\mu_3\mu_4^2\mu_5^3\mu_6$$
$$+ 18\mu_1\mu_2^4\mu_3\mu_4^2\mu_5^2\mu_6^2 + 30\mu_1\mu_2^4\mu_3\mu_4^2\mu_5\mu_6^3 + 15\mu_1\mu_2^4\mu_3\mu_4^2\mu_6^4 + 32\mu_1\mu_2^4\mu_4^7 - 126\mu_1\mu_2^4\mu_4^6\mu_5 +$$
$$42\mu_1\mu_2^4\mu_4^6\mu_6 - 612\mu_1\mu_2^4\mu_4^5\mu_5^2 - 762\mu_1\mu_2^4\mu_4^5\mu_5\mu_6 - 96\mu_1\mu_2^4\mu_4^5\mu_6^2 - 360\mu_1\mu_2^4\mu_4^4\mu_5^3$$
$$- 1240\mu_1\mu_2^4\mu_4^4\mu_5^2\mu_6 - 1005\mu_1\mu_2^4\mu_4^4\mu_5\mu_6^2 - 160\mu_1\mu_2^4\mu_4^4\mu_6^3 - 54\mu_1\mu_2^4\mu_4^3\mu_5^4 - 312\mu_1\mu_2^4\mu_4^3\mu_5^3\mu_6$$
$$- 570\mu_1\mu_2^4\mu_4^3\mu_5^2\mu_6^2 - 348\mu_1\mu_2^4\mu_4^3\mu_5\mu_6^3 - 48\mu_1\mu_2^4\mu_4^3\mu_6^4 + 3\mu_1\mu_2^4\mu_4^2\mu_5^3\mu_6^2 + 12\mu_1\mu_2^4\mu_4^2\mu_5^2\mu_6^3 +$$
$$15\mu_1\mu_2^4\mu_4^2\mu_5\mu_6^4 + 6\mu_1\mu_2^4\mu_4^2\mu_6^5 + 60\mu_1\mu_2^3\mu_3^4\mu_4^4 + 12\mu_1\mu_2^3\mu_3^4\mu_4^3\mu_5 + 44\mu_1\mu_2^3\mu_3^4\mu_4^3\mu_6$$
$$+ 9\mu_1\mu_2^3\mu_3^4\mu_4^2\mu_5\mu_6 + 9\mu_1\mu_2^3\mu_3^4\mu_4^2\mu_6^2 + 384\mu_1\mu_2^3\mu_3^3\mu_4^5 + 320\mu_1\mu_2^3\mu_3^3\mu_4^4\mu_5 + 660\mu_1\mu_2^3\mu_3^3\mu_4^4\mu_6$$
$$- 16\mu_1\mu_2^3\mu_3^3\mu_4^3\mu_5^2 + 260\mu_1\mu_2^3\mu_3^3\mu_4^3\mu_5\mu_6 + 324\mu_1\mu_2^3\mu_3^3\mu_4^3\mu_6^2 + 33\mu_1\mu_2^3\mu_3^3\mu_4^2\mu_5^2\mu_6$$
$$+ 87\mu_1\mu_2^3\mu_3^3\mu_4^2\mu_5\mu_6^2 + 54\mu_1\mu_2^3\mu_3^3\mu_4^2\mu_6^3 + 917\mu_1\mu_2^3\mu_3^2\mu_4^6 + 1263\mu_1\mu_2^3\mu_3^2\mu_4^5\mu_5$$
$$+ 2460\mu_1\mu_2^3\mu_3^2\mu_4^5\mu_6 + 165\mu_1\mu_2^3\mu_3^2\mu_4^4\mu_5^2 + 2030\mu_1\mu_2^3\mu_3^2\mu_4^4\mu_5\mu_6 + 2240\mu_1\mu_2^3\mu_3^2\mu_4^4\mu_6^2$$
$$- 168\mu_1\mu_2^3\mu_3^2\mu_4^3\mu_5^3 + 60\mu_1\mu_2^3\mu_3^2\mu_4^3\mu_5^2\mu_6 + 1028\mu_1\mu_2^3\mu_3^2\mu_4^3\mu_5\mu_6^2 + 824\mu_1\mu_2^3\mu_3^2\mu_4^3\mu_6^3$$
$$+ 39\mu_1\mu_2^3\mu_3^2\mu_4^2\mu_5^3\mu_6 + 180\mu_1\mu_2^3\mu_3^2\mu_4^2\mu_5^2\mu_6^2 + 258\mu_1\mu_2^3\mu_3^2\mu_4^2\mu_5\mu_6^3 + 117\mu_1\mu_2^3\mu_3^2\mu_4^2\mu_6^4$$
$$+ 960\mu_1\mu_2^3\mu_3\mu_4^7 + 1659\mu_1\mu_2^3\mu_3\mu_4^6\mu_5 + 3416\mu_1\mu_2^3\mu_3\mu_4^6\mu_6 + 525\mu_1\mu_2^3\mu_3\mu_4^5\mu_5^2$$
$$+ 4116\mu_1\mu_2^3\mu_3\mu_4^5\mu_5\mu_6 + 4572\mu_1\mu_2^3\mu_3\mu_4^5\mu_6^2 - 820\mu_1\mu_2^3\mu_3\mu_4^4\mu_5^3 - 70\mu_1\mu_2^3\mu_3\mu_4^4\mu_5^2\mu_6$$
$$+ 3340\mu_1\mu_2^3\mu_3\mu_4^4\mu_5\mu_6^2 + 2840\mu_1\mu_2^3\mu_3\mu_4^4\mu_6^3 - 240\mu_1\mu_2^3\mu_3\mu_4^3\mu_5^4 - 796\mu_1\mu_2^3\mu_3\mu_4^3\mu_5^3\mu_6$$
$$- 92\mu_1\mu_2^3\mu_3\mu_4^3\mu_5^2\mu_6^2 + 1272\mu_1\mu_2^3\mu_3\mu_4^3\mu_5\mu_6^3 + 840\mu_1\mu_2^3\mu_3\mu_4^3\mu_6^4 + 15\mu_1\mu_2^3\mu_3\mu_4^2\mu_5^4\mu_6$$
$$+ 117\mu_1\mu_2^3\mu_3\mu_4^2\mu_5^3\mu_6^2 + 294\mu_1\mu_2^3\mu_3\mu_4^2\mu_5^2\mu_6^3 + 300\mu_1\mu_2^3\mu_3\mu_4^2\mu_5\mu_6^4 + 108\mu_1\mu_2^3\mu_3\mu_4^2\mu_6^5$$
$$+ 360\mu_1\mu_2^3\mu_4^8 + 608\mu_1\mu_2^3\mu_4^7\mu_5 + 1536\mu_1\mu_2^3\mu_4^7\mu_6 + 112\mu_1\mu_2^3\mu_4^6\mu_5^2 + 1946\mu_1\mu_2^3\mu_4^6\mu_5\mu_6 +$$
$$2632\mu_1\mu_2^3\mu_4^6\mu_6^2 - 1260\mu_1\mu_2^3\mu_4^5\mu_5^3 - 1146\mu_1\mu_2^3\mu_4^5\mu_5^2\mu_6 + 1968\mu_1\mu_2^3\mu_4^5\mu_5\mu_6^2 + 2280\mu_1\mu_2^3\mu_4^5\mu_6^3 -$$
$$725\mu_1\mu_2^3\mu_4^4\mu_5^4 - 2740\mu_1\mu_2^3\mu_4^4\mu_5^3\mu_6 - 2100\mu_1\mu_2^3\mu_4^4\mu_5^2\mu_6^2 + 820\mu_1\mu_2^3\mu_4^4\mu_5\mu_6^3 + 1060\mu_1\mu_2^3\mu_4^4\mu_6^4$$
$$- 100\mu_1\mu_2^3\mu_4^3\mu_5^5 - 640\mu_1\mu_2^3\mu_4^3\mu_5^4\mu_6 - 1260\mu_1\mu_2^3\mu_4^3\mu_5^3\mu_6^2 - 656\mu_1\mu_2^3\mu_4^3\mu_5^2\mu_6^3 + 312\mu_1\mu_2^3\mu_4^3\mu_5\mu_6^4$$
$$+ 272\mu_1\mu_2^3\mu_4^3\mu_6^5 + 15\mu_1\mu_2^3\mu_4^2\mu_5^4\mu_6^2 + 78\mu_1\mu_2^3\mu_4^2\mu_5^3\mu_6^3 + 147\mu_1\mu_2^3\mu_4^2\mu_5^2\mu_6^4 + 120\mu_1\mu_2^3\mu_4^2\mu_5\mu_6^5$$
$$+ 36\mu_1\mu_2^3\mu_4^2\mu_6^6 + 60\mu_1\mu_2^2\mu_3^5\mu_4^4 + 24\mu_1\mu_2^2\mu_3^5\mu_4^3\mu_5 + 48\mu_1\mu_2^2\mu_3^5\mu_4^3\mu_6 + 9\mu_1\mu_2^2\mu_3^5\mu_4^2\mu_5\mu_6 +$$
$$9\mu_1\mu_2^2\mu_3^5\mu_4^2\mu_6^2 + 528\mu_1\mu_2^2\mu_3^4\mu_4^5 + 560\mu_1\mu_2^2\mu_3^4\mu_4^4\mu_5 + 940\mu_1\mu_2^2\mu_3^4\mu_4^4\mu_6 + 108\mu_1\mu_2^2\mu_3^4\mu_4^3\mu_5^2 +$$
$$542\mu_1\mu_2^2\mu_3^4\mu_4^3\mu_5\mu_6 + 486\mu_1\mu_2^2\mu_3^4\mu_4^3\mu_6^2 + 51\mu_1\mu_2^2\mu_3^4\mu_4^2\mu_5^2\mu_6 + 126\mu_1\mu_2^2\mu_3^4\mu_4^2\mu_5\mu_6^2$$
$$+ 75\mu_1\mu_2^2\mu_3^4\mu_4^2\mu_6^3 + 1792\mu_1\mu_2^2\mu_3^3\mu_4^6 + 2976\mu_1\mu_2^2\mu_3^3\mu_4^5\mu_5 + 4992\mu_1\mu_2^2\mu_3^3\mu_4^5\mu_6$$
$$+ 1520\mu_1\mu_2^2\mu_3^3\mu_4^4\mu_5^2 + 5630\mu_1\mu_2^2\mu_3^3\mu_4^4\mu_5\mu_6 + 4790\mu_1\mu_2^2\mu_3^3\mu_4^4\mu_6^2 + 96\mu_1\mu_2^2\mu_3^3\mu_4^3\mu_5^3$$
$$+ 1492\mu_1\mu_2^2\mu_3^3\mu_4^3\mu_5^2\mu_6 + 3160\mu_1\mu_2^2\mu_3^3\mu_4^3\mu_5\mu_6^2 + 1844\mu_1\mu_2^2\mu_3^3\mu_4^3\mu_6^3 + 102\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_5^3\mu_6 +$$
$$423\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5^2\mu_6^2 + 561\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6^3 + 240\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_6^4 + 2832\mu_1\mu_2^2\mu_3^2\mu_4^7$$

$$+ 6622\mu_1\mu_2^2\mu_3^2\mu_4^6\mu_5 + 10997\mu_1\mu_2^2\mu_3^2\mu_4^6\mu_6 + 5331\mu_1\mu_2^2\mu_3^2\mu_4^5\mu_5^2 + 18843\mu_1\mu_2^2\mu_3^2\mu_4^5\mu_5\mu_6 +$$
$$15924\mu_1\mu_2^2\mu_3^2\mu_4^5\mu_6^2 + 1110\mu_1\mu_2^2\mu_3^2\mu_4^4\mu_5^3 + 9670\mu_1\mu_2^2\mu_3^2\mu_4^4\mu_5^2\mu_6 + 18585\mu_1\mu_2^2\mu_3^2\mu_4^4\mu_5\mu_6^2 +$$
$$10680\mu_1\mu_2^2\mu_3^2\mu_4^4\mu_6^3 - 144\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_5^4 + 1020\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_5^3\mu_6 + 5468\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_5^2\mu_6^2$$
$$+ 7508\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_5\mu_6^3 + 3286\mu_1\mu_2^2\mu_3^2\mu_4^3\mu_6^4 + 87\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5^4\mu_6 + 540\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5^3\mu_6^2 +$$
$$1182\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5^2\mu_6^3 + 1098\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_5\mu_6^4 + 369\mu_1\mu_2^2\mu_3^2\mu_4^2\mu_6^5 + 2016\mu_1\mu_2^2\mu_3\mu_4^8$$
$$+ 6672\mu_1\mu_2^2\mu_3\mu_4^7\mu_5 + 10464\mu_1\mu_2^2\mu_3\mu_4^7\mu_6 + 6783\mu_1\mu_2^2\mu_3\mu_4^6\mu_5^2 + 24990\mu_1\mu_2^2\mu_3\mu_4^6\mu_5\mu_6 +$$
$$20839\mu_1\mu_2^2\mu_3\mu_4^6\mu_6^2 + 2550\mu_1\mu_2^2\mu_3\mu_4^5\mu_5^3 + 18588\mu_1\mu_2^2\mu_3\mu_4^5\mu_5^2\mu_6 + 35685\mu_1\mu_2^2\mu_3\mu_4^5\mu_5\mu_6^2 +$$
$$20592\mu_1\mu_2^2\mu_3\mu_4^5\mu_6^3 - 760\mu_1\mu_2^2\mu_3\mu_4^4\mu_5^4 + 3070\mu_1\mu_2^2\mu_3\mu_4^4\mu_5^3\mu_6 + 17360\mu_1\mu_2^2\mu_3\mu_4^4\mu_5^2\mu_6^2$$
$$+ 24020\mu_1\mu_2^2\mu_3\mu_4^4\mu_5\mu_6^3 + 10670\mu_1\mu_2^2\mu_3\mu_4^4\mu_6^4 - 264\mu_1\mu_2^2\mu_3\mu_4^3\mu_5^5 - 738\mu_1\mu_2^2\mu_3\mu_4^3\mu_5^4\mu_6$$
$$+ 1732\mu_1\mu_2^2\mu_3\mu_4^3\mu_5^3\mu_6^2 + 7152\mu_1\mu_2^2\mu_3\mu_4^3\mu_5^2\mu_6^3 + 7678\mu_1\mu_2^2\mu_3\mu_4^3\mu_5\mu_6^4 + 2744\mu_1\mu_2^2\mu_3\mu_4^3\mu_6^5$$
$$+ 27\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^5\mu_6 + 261\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^4\mu_6^2 + 876\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^3\mu_6^3 + 1350\mu_1\mu_2^2\mu_3\mu_4^2\mu_5^2\mu_6^4 +$$
$$981\mu_1\mu_2^2\mu_3\mu_4^2\mu_5\mu_6^5 + 273\mu_1\mu_2^2\mu_3\mu_4^2\mu_6^6 + 480\mu_1\mu_2^2\mu_4^9 + 2448\mu_1\mu_2^2\mu_4^8\mu_5 + 3384\mu_1\mu_2^2\mu_4^8\mu_6 +$$
$$2592\mu_1\mu_2^2\mu_4^7\mu_5^2 + 11072\mu_1\mu_2^2\mu_4^7\mu_5\mu_6 + 8928\mu_1\mu_2^2\mu_4^7\mu_6^2 + 1148\mu_1\mu_2^2\mu_4^6\mu_5^3 + 9422\mu_1\mu_2^2\mu_4^6\mu_5^2\mu_6$$
$$+ 20370\mu_1\mu_2^2\mu_4^6\mu_5\mu_6^2 + 11970\mu_1\mu_2^2\mu_4^6\mu_6^3 - 1440\mu_1\mu_2^2\mu_4^5\mu_5^4 + 636\mu_1\mu_2^2\mu_4^5\mu_5^3\mu_6$$
$$+ 12120\mu_1\mu_2^2\mu_4^5\mu_5^2\mu_6^2 + 19278\mu_1\mu_2^2\mu_4^5\mu_5\mu_6^3 + 8988\mu_1\mu_2^2\mu_4^5\mu_6^4 - 850\mu_1\mu_2^2\mu_4^4\mu_5^5$$
$$- 3370\mu_1\mu_2^2\mu_4^4\mu_5^4\mu_6 - 1150\mu_1\mu_2^2\mu_4^4\mu_5^3\mu_6^2 + 7340\mu_1\mu_2^2\mu_4^4\mu_5^2\mu_6^3 + 9905\mu_1\mu_2^2\mu_4^4\mu_5\mu_6^4$$
$$+ 3820\mu_1\mu_2^2\mu_4^4\mu_6^5 - 108\mu_1\mu_2^2\mu_4^3\mu_5^6 - 760\mu_1\mu_2^2\mu_4^3\mu_5^5\mu_6 - 1530\mu_1\mu_2^2\mu_4^3\mu_5^4\mu_6^2 - 104\mu_1\mu_2^2\mu_4^3\mu_5^3\mu_6^3 +$$
$$2600\mu_1\mu_2^2\mu_4^3\mu_5^2\mu_6^4 + 2700\mu_1\mu_2^2\mu_4^3\mu_5\mu_6^5 + 856\mu_1\mu_2^2\mu_4^3\mu_6^6 + 27\mu_1\mu_2^2\mu_4^2\mu_5^5\mu_6^2 + 174\mu_1\mu_2^2\mu_4^2\mu_5^4\mu_6^3$$
$$+ 438\mu_1\mu_2^2\mu_4^2\mu_5^3\mu_6^4 + 540\mu_1\mu_2^2\mu_4^2\mu_5^2\mu_6^5 + 327\mu_1\mu_2^2\mu_4^2\mu_5\mu_6^6 + 78\mu_1\mu_2^2\mu_4^2\mu_6^7 + 20\mu_1\mu_2\mu_3^6\mu_4^4 +$$
$$8\mu_1\mu_2\mu_3^6\mu_4^3\mu_5 + 16\mu_1\mu_2\mu_3^6\mu_4^3\mu_6 + 3\mu_1\mu_2\mu_3^6\mu_4^2\mu_5\mu_6 + 3\mu_1\mu_2\mu_3^6\mu_4^2\mu_6^2 + 288\mu_1\mu_2\mu_3^5\mu_4^5 +$$
$$300\mu_1\mu_2\mu_3^5\mu_4^4\mu_5 + 480\mu_1\mu_2\mu_3^5\mu_4^4\mu_6 + 72\mu_1\mu_2\mu_3^5\mu_4^3\mu_5^2 + 288\mu_1\mu_2\mu_3^5\mu_4^3\mu_5\mu_6$$
$$+ 240\mu_1\mu_2\mu_3^5\mu_4^3\mu_6^2 + 27\mu_1\mu_2\mu_3^5\mu_4^2\mu_5^2\mu_6 + 63\mu_1\mu_2\mu_3^5\mu_4^2\mu_5\mu_6^2 + 36\mu_1\mu_2\mu_3^5\mu_4^2\mu_6^3$$
$$+ 1344\mu_1\mu_2\mu_3^4\mu_4^6 + 2328\mu_1\mu_2\mu_3^4\mu_4^5\mu_5 + 3696\mu_1\mu_2\mu_3^4\mu_4^5\mu_6 + 1320\mu_1\mu_2\mu_3^4\mu_4^4\mu_5^2$$
$$+ 4350\mu_1\mu_2\mu_3^4\mu_4^4\mu_5\mu_6 + 3460\mu_1\mu_2\mu_3^4\mu_4^4\mu_6^2 + 192\mu_1\mu_2\mu_3^4\mu_4^3\mu_5^3 + 1362\mu_1\mu_2\mu_3^4\mu_4^3\mu_5^2\mu_6 +$$
$$2426\mu_1\mu_2\mu_3^4\mu_4^3\mu_5\mu_6^2 + 1304\mu_1\mu_2\mu_3^4\mu_4^3\mu_6^3 + 84\mu_1\mu_2\mu_3^4\mu_4^2\mu_5^3\mu_6 + 324\mu_1\mu_2\mu_3^4\mu_4^2\mu_5^2\mu_6^2$$
$$+ 405\mu_1\mu_2\mu_3^4\mu_4^2\mu_5\mu_6^3 + 165\mu_1\mu_2\mu_3^4\mu_4^2\mu_6^4 + 2816\mu_1\mu_2\mu_3^3\mu_4^7 + 7616\mu_1\mu_2\mu_3^3\mu_4^6\mu_5$$
$$+ 11424\mu_1\mu_2\mu_3^3\mu_4^6\mu_6 + 6624\mu_1\mu_2\mu_3^3\mu_4^5\mu_5^2 + 21696\mu_1\mu_2\mu_3^3\mu_4^5\mu_5\mu_6 + 16848\mu_1\mu_2\mu_3^3\mu_4^5\mu_6^2 +$$
$$2400\mu_1\mu_2\mu_3^3\mu_4^4\mu_5^3 + 12930\mu_1\mu_2\mu_3^3\mu_4^4\mu_5^2\mu_6 + 21510\mu_1\mu_2\mu_3^3\mu_4^4\mu_5\mu_6^2 + 11320\mu_1\mu_2\mu_3^3\mu_4^4\mu_6^3 +$$
$$160\mu_1\mu_2\mu_3^3\mu_4^3\mu_5^4 + 2464\mu_1\mu_2\mu_3^3\mu_4^3\mu_5^3\mu_6 + 7536\mu_1\mu_2\mu_3^3\mu_4^3\mu_5^2\mu_6^2 + 8668\mu_1\mu_2\mu_3^3\mu_4^3\mu_5\mu_6^3 +$$
$$3456\mu_1\mu_2\mu_3^3\mu_4^3\mu_6^4 + 120\mu_1\mu_2\mu_3^3\mu_4^2\mu_5^4\mu_6 + 672\mu_1\mu_2\mu_3^3\mu_4^2\mu_5^3\mu_6^2 + 1359\mu_1\mu_2\mu_3^3\mu_4^2\mu_5^2\mu_6^3$$
$$+ 1185\mu_1\mu_2\mu_3^3\mu_4^2\mu_5\mu_6^4 + 378\mu_1\mu_2\mu_3^3\mu_4^2\mu_6^5 + 2844\mu_1\mu_2\mu_3^2\mu_4^8 + 11856\mu_1\mu_2\mu_3^2\mu_4^7\mu_5$$
$$+ 16128\mu_1\mu_2\mu_3^2\mu_4^7\mu_6 + 14364\mu_1\mu_2\mu_3^2\mu_4^6\mu_5^2 + 46711\mu_1\mu_2\mu_3^2\mu_4^6\mu_5\mu_6 + 34272\mu_1\mu_2\mu_3^2\mu_4^6\mu_6^2 +$$
$$8136\mu_1\mu_2\mu_3^2\mu_4^5\mu_5^3 + 41769\mu_1\mu_2\mu_3^2\mu_4^5\mu_5^2\mu_6 + 68952\mu_1\mu_2\mu_3^2\mu_4^5\mu_5\mu_6^2 + 35328\mu_1\mu_2\mu_3^2\mu_4^5\mu_6^3 +$$
$$1500\mu_1\mu_2\mu_3^2\mu_4^4\mu_5^4 + 15280\mu_1\mu_2\mu_3^2\mu_4^4\mu_5^3\mu_6 + 42315\mu_1\mu_2\mu_3^2\mu_4^4\mu_5^2\mu_6^2 + 47460\mu_1\mu_2\mu_3^2\mu_4^4\mu_5\mu_6^3$$
$$+ 18700\mu_1\mu_2\mu_3^2\mu_4^4\mu_6^4 - 72\mu_1\mu_2\mu_3^2\mu_4^3\mu_5^5 + 1470\mu_1\mu_2\mu_3^2\mu_4^3\mu_5^4\mu_6 + 8880\mu_1\mu_2\mu_3^2\mu_4^3\mu_5^3\mu_6^2$$
$$+ 17580\mu_1\mu_2\mu_3^2\mu_4^3\mu_5^2\mu_6^3 + 15098\mu_1\mu_2\mu_3^2\mu_4^3\mu_5\mu_6^4 + 4808\mu_1\mu_2\mu_3^2\mu_4^3\mu_6^5 + 81\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^5\mu_6 +$$
$$630\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^4\mu_6^2 + 1848\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^3\mu_6^3 + 2592\mu_1\mu_2\mu_3^2\mu_4^2\mu_5^2\mu_6^4 + 1755\mu_1\mu_2\mu_3^2\mu_4^2\mu_5\mu_6^5 +$$
$$462\mu_1\mu_2\mu_3^2\mu_4^2\mu_6^6 + 1280\mu_1\mu_2\mu_3\mu_4^9 + 8388\mu_1\mu_2\mu_3\mu_4^8\mu_5 + 10008\mu_1\mu_2\mu_3\mu_4^8\mu_6$$
$$+ 14256\mu_1\mu_2\mu_3\mu_4^7\mu_5^2 + 44032\mu_1\mu_2\mu_3\mu_4^7\mu_5\mu_6 + 29440\mu_1\mu_2\mu_3\mu_4^7\mu_6^2 + 10248\mu_1\mu_2\mu_3\mu_4^6\mu_5^3 +$$
$$54474\mu_1\mu_2\mu_3\mu_4^6\mu_5^2\mu_6 + 88977\mu_1\mu_2\mu_3\mu_4^6\mu_5\mu_6^2 + 43176\mu_1\mu_2\mu_3\mu_4^6\mu_6^3 + 3300\mu_1\mu_2\mu_3\mu_4^5\mu_5^4 +$$
$$28944\mu_1\mu_2\mu_3\mu_4^5\mu_5^3\mu_6 + 79623\mu_1\mu_2\mu_3\mu_4^5\mu_5^2\mu_6^2 + 89580\mu_1\mu_2\mu_3\mu_4^5\mu_5\mu_6^3 + 34512\mu_1\mu_2\mu_3\mu_4^5\mu_6^4$$
$$- 420\mu_1\mu_2\mu_3\mu_4^4\mu_5^5 + 4675\mu_1\mu_2\mu_3\mu_4^4\mu_5^4\mu_6 + 28040\mu_1\mu_2\mu_3\mu_4^4\mu_5^3\mu_6^2 + 55020\mu_1\mu_2\mu_3\mu_4^4\mu_5^2\mu_6^3 +$$
$$47520\mu_1\mu_2\mu_3\mu_4^4\mu_5\mu_6^4 + 15120\mu_1\mu_2\mu_3\mu_4^4\mu_6^5 - 168\mu_1\mu_2\mu_3\mu_4^3\mu_5^6 - 408\mu_1\mu_2\mu_3\mu_4^3\mu_5^5\mu_6$$

$$+ 2690\mu_1\mu_2\mu_3\mu_4^3\mu_5^4\mu_6^2 + 11760\mu_1\mu_2\mu_3\mu_4^3\mu_5^3\mu_6^3 + 17994\mu_1\mu_2\mu_3\mu_4^3\mu_5^2\mu_6^4$$
$$+ 12560\mu_1\mu_2\mu_3\mu_4^3\mu_5\mu_6^5 + 3360\mu_1\mu_2\mu_3\mu_4^3\mu_6^6 + 21\mu_1\mu_2\mu_3\mu_4^2\mu_5^6\mu_6 + 243\mu_1\mu_2\mu_3\mu_4^2\mu_5^5\mu_6^2 +$$
$$1020\mu_1\mu_2\mu_3\mu_4^2\mu_5^4\mu_6^3 + 2100\mu_1\mu_2\mu_3\mu_4^2\mu_5^3\mu_6^4 + 2295\mu_1\mu_2\mu_3\mu_4^2\mu_5^2\mu_6^5 + 1281\mu_1\mu_2\mu_3\mu_4^2\mu_5\mu_6^6 +$$
$$288\mu_1\mu_2\mu_3\mu_4^2\mu_6^7 + 176\mu_1\mu_2\mu_4^{10} + 2000\mu_1\mu_2\mu_4^9\mu_5 + 2000\mu_1\mu_2\mu_4^9\mu_6 + 5184\mu_1\mu_2\mu_4^8\mu_5^2 +$$
$$14220\mu_1\mu_2\mu_4^8\mu_5\mu_6 + 8280\mu_1\mu_2\mu_4^8\mu_6^2 + 3968\mu_1\mu_2\mu_4^7\mu_5^3 + 24000\mu_1\mu_2\mu_4^7\mu_5^2\mu_6$$
$$+ 38016\mu_1\mu_2\mu_4^7\mu_5\mu_6^2 + 16896\mu_1\mu_2\mu_4^7\mu_6^3 + 1610\mu_1\mu_2\mu_4^6\mu_5^4 + 14952\mu_1\mu_2\mu_4^6\mu_5^3\mu_6$$
$$+ 45318\mu_1\mu_2\mu_4^6\mu_5^2\mu_6^2 + 51842\mu_1\mu_2\mu_4^6\mu_5\mu_6^3 + 19152\mu_1\mu_2\mu_4^6\mu_6^4 - 972\mu_1\mu_2\mu_4^5\mu_5^5$$
$$+ 2100\mu_1\mu_2\mu_4^5\mu_5^4\mu_6 + 20304\mu_1\mu_2\mu_4^5\mu_5^3\mu_6^2 + 44154\mu_1\mu_2\mu_4^5\mu_5^2\mu_6^3 + 39732\mu_1\mu_2\mu_4^5\mu_5\mu_6^4 +$$
$$12576\mu_1\mu_2\mu_4^5\mu_6^5 - 560\mu_1\mu_2\mu_4^4\mu_5^6 - 2400\mu_1\mu_2\mu_4^4\mu_5^5\mu_6 + 375\mu_1\mu_2\mu_4^4\mu_5^4\mu_6^2 + 13040\mu_1\mu_2\mu_4^4\mu_5^3\mu_6^3$$
$$+ 23355\mu_1\mu_2\mu_4^4\mu_5^2\mu_6^4 + 17280\mu_1\mu_2\mu_4^4\mu_5\mu_6^5 + 4720\mu_1\mu_2\mu_4^4\mu_6^6 - 64\mu_1\mu_2\mu_4^3\mu_5^7 - 504\mu_1\mu_2\mu_4^3\mu_5^6\mu_6$$
$$- 1080\mu_1\mu_2\mu_4^3\mu_5^5\mu_6^2 + 500\mu_1\mu_2\mu_4^3\mu_5^4\mu_6^3 + 4576\mu_1\mu_2\mu_4^3\mu_5^3\mu_6^4 + 6468\mu_1\mu_2\mu_4^3\mu_5^2\mu_6^5$$
$$+ 3960\mu_1\mu_2\mu_4^3\mu_5\mu_6^6 + 928\mu_1\mu_2\mu_4^3\mu_6^7 + 21\mu_1\mu_2\mu_4^2\mu_5^6\mu_6^2 + 162\mu_1\mu_2\mu_4^2\mu_5^5\mu_6^3 + 510\mu_1\mu_2\mu_4^2\mu_5^4\mu_6^4$$
$$+ 840\mu_1\mu_2\mu_4^2\mu_5^3\mu_6^5 + 765\mu_1\mu_2\mu_4^2\mu_5^2\mu_6^6 + 366\mu_1\mu_2\mu_4^2\mu_5\mu_6^7 + 72\mu_1\mu_2\mu_4^2\mu_6^8 + 48\mu_1\mu_3^6\mu_4^5 +$$
$$40\mu_1\mu_3^6\mu_4^4\mu_5 + 60\mu_1\mu_3^6\mu_4^4\mu_6 + 8\mu_1\mu_3^6\mu_4^3\mu_5^2 + 32\mu_1\mu_3^6\mu_4^3\mu_5\mu_6 + 24\mu_1\mu_3^6\mu_4^3\mu_6^2 + 3\mu_1\mu_3^6\mu_4^2\mu_5^2\mu_6$$
$$+ 6\mu_1\mu_3^6\mu_4^2\mu_5\mu_6^2 + 3\mu_1\mu_3^6\mu_4^2\mu_6^3 + 336\mu_1\mu_3^5\mu_4^6 + 576\mu_1\mu_3^5\mu_4^5\mu_5 + 864\mu_1\mu_3^5\mu_4^5\mu_6$$
$$+ 300\mu_1\mu_3^5\mu_4^4\mu_5^2 + 960\mu_1\mu_3^5\mu_4^4\mu_5\mu_6 + 720\mu_1\mu_3^5\mu_4^4\mu_6^2 + 48\mu_1\mu_3^5\mu_4^3\mu_5^3 + 288\mu_1\mu_3^5\mu_4^3\mu_5^2\mu_6 +$$
$$480\mu_1\mu_3^5\mu_4^3\mu_5\mu_6^2 + 240\mu_1\mu_3^5\mu_4^3\mu_6^3 + 18\mu_1\mu_3^5\mu_4^2\mu_5^3\mu_6 + 63\mu_1\mu_3^5\mu_4^2\mu_5^2\mu_6^2 + 72\mu_1\mu_3^5\mu_4^2\mu_5\mu_6^3$$
$$+ 27\mu_1\mu_3^5\mu_4^2\mu_6^4 + 896\mu_1\mu_3^4\mu_4^7 + 2688\mu_1\mu_3^4\mu_4^6\mu_5 + 3696\mu_1\mu_3^4\mu_4^6\mu_6 + 2328\mu_1\mu_3^4\mu_4^5\mu_5^2 +$$
$$7392\mu_1\mu_3^4\mu_4^5\mu_5\mu_6 + 5352\mu_1\mu_3^4\mu_4^5\mu_6^2 + 880\mu_1\mu_3^4\mu_4^4\mu_5^3 + 4350\mu_1\mu_3^4\mu_4^4\mu_5^2\mu_6 + 6920\mu_1\mu_3^4\mu_4^4\mu_5\mu_6^2$$
$$+ 3420\mu_1\mu_3^4\mu_4^4\mu_6^3 + 96\mu_1\mu_3^4\mu_4^3\mu_5^4 + 908\mu_1\mu_3^4\mu_4^3\mu_5^3\mu_6 + 2426\mu_1\mu_3^4\mu_4^3\mu_5^2\mu_6^2 + 2608\mu_1\mu_3^4\mu_4^3\mu_5\mu_6^3$$
$$+ 976\mu_1\mu_3^4\mu_4^3\mu_6^4 + 42\mu_1\mu_3^4\mu_4^2\mu_5^4\mu_6 + 216\mu_1\mu_3^4\mu_4^2\mu_5^3\mu_6^2 + 405\mu_1\mu_3^4\mu_4^2\mu_5^2\mu_6^3 + 330\mu_1\mu_3^4\mu_4^2\mu_5\mu_6^4$$
$$+ 99\mu_1\mu_3^4\mu_4^2\mu_5^5 + 1152\mu_1\mu_3^3\mu_4^8 + 5632\mu_1\mu_3^3\mu_4^7\mu_5 + 6912\mu_1\mu_3^3\mu_4^7\mu_6 + 7616\mu_1\mu_3^3\mu_4^6\mu_5^2 +$$
$$22848\mu_1\mu_3^3\mu_4^6\mu_5\mu_6 + 15232\mu_1\mu_3^3\mu_4^6\mu_6^2 + 4416\mu_1\mu_3^3\mu_4^5\mu_5^3 + 21696\mu_1\mu_3^3\mu_4^5\mu_5^2\mu_6$$
$$+ 33696\mu_1\mu_3^3\mu_4^5\mu_5\mu_6^2 + 15888\mu_1\mu_3^3\mu_4^5\mu_6^3 + 1200\mu_1\mu_3^3\mu_4^4\mu_5^4 + 8620\mu_1\mu_3^3\mu_4^4\mu_5^3\mu_6$$
$$+ 21510\mu_1\mu_3^3\mu_4^4\mu_5^2\mu_6^2 + 22640\mu_1\mu_3^3\mu_4^4\mu_5\mu_6^3 + 8300\mu_1\mu_3^3\mu_4^4\mu_6^4 + 64\mu_1\mu_3^3\mu_4^3\mu_5^5$$
$$+ 1232\mu_1\mu_3^3\mu_4^3\mu_5^4\mu_6 + 5024\mu_1\mu_3^3\mu_4^3\mu_5^3\mu_6^2 + 8668\mu_1\mu_3^3\mu_4^3\mu_5^2\mu_6^3 + 6912\mu_1\mu_3^3\mu_4^3\mu_5\mu_6^4$$
$$+ 2064\mu_1\mu_3^3\mu_4^3\mu_6^5 + 48\mu_1\mu_3^3\mu_4^2\mu_5^5\mu_6 + 336\mu_1\mu_3^3\mu_4^2\mu_5^4\mu_6^2 + 906\mu_1\mu_3^3\mu_4^2\mu_5^3\mu_6^3$$
$$+ 1185\mu_1\mu_3^3\mu_4^2\mu_5^2\mu_6^4 + 756\mu_1\mu_3^3\mu_4^2\mu_5\mu_6^5 + 189\mu_1\mu_3^3\mu_4^2\mu_6^6 + 720\mu_1\mu_3^2\mu_4^9 + 5688\mu_1\mu_3^2\mu_4^8\mu_5 +$$
$$6084\mu_1\mu_3^2\mu_4^8\mu_6 + 11856\mu_1\mu_3^2\mu_4^7\mu_5^2 + 32256\mu_1\mu_3^2\mu_4^7\mu_5\mu_6 + 19200\mu_1\mu_3^2\mu_4^7\mu_6^2 + 9576\mu_1\mu_3^2\mu_4^6\mu_5^3$$
$$+ 46711\mu_1\mu_3^2\mu_4^6\mu_5^2\mu_6 + 68544\mu_1\mu_3^2\mu_4^6\mu_5\mu_6^2 + 29736\mu_1\mu_3^2\mu_4^6\mu_6^3 + 4068\mu_1\mu_3^2\mu_4^5\mu_5^4$$
$$+ 27846\mu_1\mu_3^2\mu_4^5\mu_5^3\mu_6 + 68952\mu_1\mu_3^2\mu_4^5\mu_5^2\mu_6^2 + 70656\mu_1\mu_3^2\mu_4^5\mu_5\mu_6^3 + 24600\mu_1\mu_3^2\mu_4^5\mu_6^4$$
$$+ 600\mu_1\mu_3^2\mu_4^4\mu_5^5 + 7640\mu_1\mu_3^2\mu_4^4\mu_5^4\mu_6 + 28210\mu_1\mu_3^2\mu_4^4\mu_5^3\mu_6^2 + 47460\mu_1\mu_3^2\mu_4^4\mu_5^2\mu_6^3$$
$$+ 37400\mu_1\mu_3^2\mu_4^4\mu_5\mu_6^4 + 10900\mu_1\mu_3^2\mu_4^4\mu_6^5 - 24\mu_1\mu_3^2\mu_4^3\mu_5^6 + 588\mu_1\mu_3^2\mu_4^3\mu_5^5\mu_6$$
$$+ 4440\mu_1\mu_3^2\mu_4^3\mu_5^4\mu_6^2 + 11720\mu_1\mu_3^2\mu_4^3\mu_5^3\mu_6^3 + 15098\mu_1\mu_3^2\mu_4^3\mu_5^2\mu_6^4 + 9616\mu_1\mu_3^2\mu_4^3\mu_5\mu_6^5$$
$$+ 2392\mu_1\mu_3^2\mu_4^3\mu_6^6 + 27\mu_1\mu_3^2\mu_4^2\mu_5^6\mu_6 + 252\mu_1\mu_3^2\mu_4^2\mu_5^5\mu_6^2 + 924\mu_1\mu_3^2\mu_4^2\mu_5^4\mu_6^3$$
$$+ 1728\mu_1\mu_3^2\mu_4^2\mu_5^3\mu_6^4 + 1755\mu_1\mu_3^2\mu_4^2\mu_5^2\mu_6^5 + 924\mu_1\mu_3^2\mu_4^2\mu_5\mu_6^6 + 198\mu_1\mu_3^2\mu_4^2\mu_6^7 + 176\mu_1\mu_3\mu_4^{10}$$
$$+ 2560\mu_1\mu_3\mu_4^9\mu_5 + 2240\mu_1\mu_3\mu_4^9\mu_6 + 8388\mu_1\mu_3\mu_4^8\mu_5^2 + 20016\mu_1\mu_3\mu_4^8\mu_5\mu_6$$
$$+ 10188\mu_1\mu_3\mu_4^8\mu_6^2 + 9504\mu_1\mu_3\mu_4^7\mu_5^3 + 44032\mu_1\mu_3\mu_4^7\mu_5^2\mu_6 + 58880\mu_1\mu_3\mu_4^7\mu_5\mu_6^2$$
$$+ 22656\mu_1\mu_3\mu_4^7\mu_6^3 + 5124\mu_1\mu_3\mu_4^6\mu_5^4 + 36316\mu_1\mu_3\mu_4^6\mu_5^3\mu_6 + 88977\mu_1\mu_3\mu_4^6\mu_5^2\mu_6^2$$
$$+ 86352\mu_1\mu_3\mu_4^6\mu_5\mu_6^3 + 27608\mu_1\mu_3\mu_4^6\mu_6^4 + 1320\mu_1\mu_3\mu_4^5\mu_5^5 + 14472\mu_1\mu_3\mu_4^5\mu_5^4\mu_6$$
$$+ 53082\mu_1\mu_3\mu_4^5\mu_5^3\mu_6^2 + 89580\mu_1\mu_3\mu_4^5\mu_5^2\mu_6^3 + 69024\mu_1\mu_3\mu_4^5\mu_5\mu_6^4 + 19104\mu_1\mu_3\mu_4^5\mu_6^5$$
$$- 140\mu_1\mu_3\mu_4^4\mu_5^6 + 1870\mu_1\mu_3\mu_4^4\mu_5^5\mu_6 + 14020\mu_1\mu_3\mu_4^4\mu_5^4\mu_6^2 + 36680\mu_1\mu_3\mu_4^4\mu_5^3\mu_6^3$$
$$+ 47520\mu_1\mu_3\mu_4^4\mu_5^2\mu_6^4 + 30240\mu_1\mu_3\mu_4^4\mu_5\mu_6^5 + 7360\mu_1\mu_3\mu_4^4\mu_6^6 - 48\mu_1\mu_3\mu_4^3\mu_5^7 - 136\mu_1\mu_3\mu_4^3\mu_5^6\mu_6$$

$$
\begin{aligned}
&+ 1076\mu_1\mu_3\mu_4^3\mu_5^5\mu_6^2 + 5880\mu_1\mu_3\mu_4^3\mu_5^4\mu_6^3 + 11996\mu_1\mu_3\mu_4^3\mu_5^3\mu_6^4 + 12560\mu_1\mu_3\mu_4^3\mu_5^2\mu_6^5 \\
&+ 6720\mu_1\mu_3\mu_4^3\mu_5\mu_6^6 + 1440\mu_1\mu_3\mu_4^3\mu_6^7 + 6\mu_1\mu_3\mu_4^2\mu_5^7\mu_6 + 81\mu_1\mu_3\mu_4^2\mu_5^6\mu_6^2 + 408\mu_1\mu_3\mu_4^2\mu_5^5\mu_6^3 \\
&+ 1050\mu_1\mu_3\mu_4^2\mu_5^4\mu_6^4 + 1530\mu_1\mu_3\mu_4^2\mu_5^3\mu_6^5 + 1281\mu_1\mu_3\mu_4^2\mu_5^2\mu_6^6 + 576\mu_1\mu_3\mu_4^2\mu_5\mu_6^7 \\
&+ 108\mu_1\mu_3\mu_4^2\mu_6^8 + 352\mu_1\mu_4^{10}\mu_5 + 176\mu_1\mu_4^{10}\mu_6 + 2000\mu_1\mu_4^9\mu_5^2 + 4000\mu_1\mu_4^9\mu_5\mu_6 \\
&+ 1520\mu_1\mu_4^9\mu_6^2 + 3456\mu_1\mu_4^8\mu_5^3 + 14220\mu_1\mu_4^8\mu_5^2\mu_6 + 16560\mu_1\mu_4^8\mu_5\mu_6^2 + 5256\mu_1\mu_4^8\mu_6^3 + \\
&1984\mu_1\mu_4^7\mu_5^4 + 16000\mu_1\mu_4^7\mu_5^3\mu_6 + 38016\mu_1\mu_4^7\mu_5^2\mu_6^2 + 33792\mu_1\mu_4^7\mu_5\mu_6^3 + 9472\mu_1\mu_4^7\mu_6^4 + \\
&644\mu_1\mu_4^6\mu_5^5 + 7476\mu_1\mu_4^6\mu_5^4\mu_6 + 30212\mu_1\mu_4^6\mu_5^3\mu_6^2 + 51842\mu_1\mu_4^6\mu_5^2\mu_6^3 + 38304\mu_1\mu_4^6\mu_5\mu_6^4 + \\
&9744\mu_1\mu_4^6\mu_6^5 - 324\mu_1\mu_4^5\mu_6^5 + 840\mu_1\mu_4^5\mu_5^5\mu_6 + 10152\mu_1\mu_4^5\mu_5^4\mu_6^2 + 29436\mu_1\mu_4^5\mu_5^3\mu_6^3 \\
&+ 39732\mu_1\mu_4^5\mu_5^2\mu_6^4 + 25152\mu_1\mu_4^5\mu_5\mu_6^5 + 5856\mu_1\mu_4^5\mu_6^6 - 160\mu_1\mu_4^4\mu_5^7 - 800\mu_1\mu_4^4\mu_5^6\mu_6 \\
&+ 150\mu_1\mu_4^4\mu_5^5\mu_6^2 + 6520\mu_1\mu_4^4\mu_5^4\mu_6^3 + 15570\mu_1\mu_4^4\mu_5^3\mu_6^4 + 17280\mu_1\mu_4^4\mu_5^2\mu_6^5 + 9440\mu_1\mu_4^4\mu_5\mu_6^6 \\
&+ 2000\mu_1\mu_4^4\mu_6^7 - 16\mu_1\mu_4^3\mu_5^8 - 144\mu_1\mu_4^3\mu_5^7\mu_6 - 360\mu_1\mu_4^3\mu_5^6\mu_6^2 + 200\mu_1\mu_4^3\mu_5^5\mu_6^3 + 2288\mu_1\mu_4^3\mu_5^4\mu_6^4 \\
&+ 4312\mu_1\mu_4^3\mu_5^3\mu_6^5 + 3960\mu_1\mu_4^3\mu_5^2\mu_6^6 + 1856\mu_1\mu_4^3\mu_5\mu_6^7 + 352\mu_1\mu_4^3\mu_6^8 + 6\mu_1\mu_4^2\mu_5^7\mu_6^2 \\
&+ 54\mu_1\mu_4^2\mu_5^6\mu_6^3 + 204\mu_1\mu_4^2\mu_5^5\mu_6^4 + 420\mu_1\mu_4^2\mu_5^4\mu_6^5 + 510\mu_1\mu_4^2\mu_5^3\mu_6^6 + 366\mu_1\mu_4^2\mu_5^2\mu_6^7 \\
&+ 144\mu_1\mu_4^2\mu_5\mu_6^8 + 24\mu_1\mu_4^2\mu_6^9 - \mu_2^6\mu_3^2\mu_4^4 - 4\mu_2^6\mu_3\mu_4^5 - 2\mu_2^6\mu_3\mu_4^4\mu_5 - 4\mu_2^6\mu_3\mu_4^4\mu_6 - 4\mu_2^6\mu_4^6 - \\
&4\mu_2^6\mu_4^5\mu_5 - 8\mu_2^6\mu_4^5\mu_6 - \mu_2^6\mu_4^4\mu_5^2 - 4\mu_2^6\mu_4^4\mu_5\mu_6 - 4\mu_2^6\mu_4^4\mu_6^2 - 2\mu_2^5\mu_3^3\mu_4^4 - 8\mu_2^5\mu_3^2\mu_4^5 \\
&- 12\mu_2^5\mu_3^2\mu_4^4\mu_5 - 12\mu_2^5\mu_3^2\mu_4^4\mu_6 - 12\mu_2^5\mu_3\mu_4^6 - 46\mu_2^5\mu_3\mu_4^5\mu_5 - 40\mu_2^5\mu_3\mu_4^5\mu_6 - 18\mu_2^5\mu_3\mu_4^4\mu_5^2 - \\
&52\mu_2^5\mu_3\mu_4^4\mu_5\mu_6 - 28\mu_2^5\mu_3\mu_4^4\mu_6^2 - 8\mu_2^5\mu_4^7 - 48\mu_2^5\mu_4^6\mu_5 - 40\mu_2^5\mu_4^6\mu_6 - 38\mu_2^5\mu_4^5\mu_5^2 - 108\mu_2^5\mu_4^5\mu_5\mu_6 \\
&- 56\mu_2^5\mu_4^5\mu_6^2 - 8\mu_2^5\mu_4^4\mu_5^3 - 40\mu_2^5\mu_4^4\mu_5^2\mu_6 - 60\mu_2^5\mu_4^4\mu_5\mu_6^2 - 24\mu_2^5\mu_4^4\mu_6^3 - \mu_2^4\mu_3^4\mu_4^4 + 4\mu_2^4\mu_3^3\mu_4^5 - \\
&12\mu_2^4\mu_3^3\mu_4^4\mu_5 - 2\mu_2^4\mu_3^3\mu_4^4\mu_6 + 2\mu_2^4\mu_3^3\mu_4^3\mu_5\mu_6 + 2\mu_2^4\mu_3^3\mu_4^3\mu_6^2 + 24\mu_2^4\mu_3^2\mu_4^6 - 34\mu_2^4\mu_3^2\mu_4^5\mu_5 + \\
&24\mu_2^4\mu_3^2\mu_4^5\mu_6 - 48\mu_2^4\mu_3^2\mu_4^4\mu_5^2 - 63\mu_2^4\mu_3^2\mu_4^4\mu_5\mu_6 + 2\mu_2^4\mu_3^2\mu_4^4\mu_6^2 + 4\mu_2^4\mu_3^2\mu_4^3\mu_5^2\mu_6 + 12\mu_2^4\mu_3^2\mu_4^3\mu_5\mu_6^2 \\
&+ 8\mu_2^4\mu_3^2\mu_4^3\mu_6^3 + 30\mu_2^4\mu_3\mu_4^7 - 39\mu_2^4\mu_3\mu_4^6\mu_5 + 56\mu_2^4\mu_3\mu_4^6\mu_6 - 182\mu_2^4\mu_3\mu_4^5\mu_5^2 - 214\mu_2^4\mu_3\mu_4^5\mu_5\mu_6 + \\
&8\mu_2^4\mu_3\mu_4^5\mu_6^2 - 64\mu_2^4\mu_3\mu_4^4\mu_5^3 - 217\mu_2^4\mu_3\mu_4^4\mu_5^2\mu_6 - 161\mu_2^4\mu_3\mu_4^4\mu_5\mu_6^2 - 8\mu_2^4\mu_3\mu_4^4\mu_6^3 + 2\mu_2^4\mu_3\mu_4^3\mu_5^3\mu_6 \\
&+ 12\mu_2^4\mu_3\mu_4^3\mu_5^2\mu_6^2 + 20\mu_2^4\mu_3\mu_4^3\mu_5\mu_6^3 + 10\mu_2^4\mu_3\mu_4^3\mu_6^4 + 8\mu_2^4\mu_4^8 - 36\mu_2^4\mu_4^7\mu_5 + 12\mu_2^4\mu_4^7\mu_6 \\
&- 204\mu_2^4\mu_4^6\mu_5^2 - 254\mu_2^4\mu_4^6\mu_5\mu_6 - 32\mu_2^4\mu_4^6\mu_6^2 - 144\mu_2^4\mu_4^5\mu_5^3 - 496\mu_2^4\mu_4^5\mu_5^2\mu_6 - 402\mu_2^4\mu_4^5\mu_5\mu_6^2 \\
&- 64\mu_2^4\mu_4^5\mu_6^3 - 27\mu_2^4\mu_4^4\mu_5^4 - 156\mu_2^4\mu_4^4\mu_5^3\mu_6 - 285\mu_2^4\mu_4^4\mu_5^2\mu_6^2 - 174\mu_2^4\mu_4^4\mu_5\mu_6^3 - 24\mu_2^4\mu_4^4\mu_6^4 + \\
&2\mu_2^4\mu_4^3\mu_5^3\mu_6^2 + 8\mu_2^4\mu_4^3\mu_5^2\mu_6^3 + 10\mu_2^4\mu_4^3\mu_5\mu_6^4 + 4\mu_2^4\mu_4^3\mu_6^5 + 24\mu_2^3\mu_3^4\mu_4^5 + 6\mu_2^3\mu_3^4\mu_4^4\mu_5 \\
&+ 22\mu_2^3\mu_3^4\mu_4^4\mu_6 + 6\mu_2^3\mu_3^4\mu_4^3\mu_5\mu_6 + 6\mu_2^3\mu_3^4\mu_4^3\mu_6^2 + 128\mu_2^3\mu_3^3\mu_4^6 + 128\mu_2^3\mu_3^3\mu_4^5\mu_5 + 264\mu_2^3\mu_3^3\mu_4^5\mu_6 \\
&- 8\mu_2^3\mu_3^3\mu_4^4\mu_5^2 + 130\mu_2^3\mu_3^3\mu_4^4\mu_5\mu_6 + 162\mu_2^3\mu_3^3\mu_4^4\mu_6^2 + 22\mu_2^3\mu_3^3\mu_4^3\mu_5^2\mu_6 + 58\mu_2^3\mu_3^3\mu_4^3\mu_5\mu_6^2 + \\
&36\mu_2^3\mu_3^3\mu_4^3\mu_6^3 + 262\mu_2^3\mu_3^2\mu_4^7 + 421\mu_2^3\mu_3^2\mu_4^6\mu_5 + 820\mu_2^3\mu_3^2\mu_4^6\mu_6 + 66\mu_2^3\mu_3^2\mu_4^5\mu_5^2 \\
&+ 812\mu_2^3\mu_3^2\mu_4^5\mu_5\mu_6 + 896\mu_2^3\mu_3^2\mu_4^5\mu_6^2 - 84\mu_2^3\mu_3^2\mu_4^4\mu_5^3 + 30\mu_2^3\mu_3^2\mu_4^4\mu_5^2\mu_6 + 514\mu_2^3\mu_3^2\mu_4^4\mu_5\mu_6^2 \\
&+ 412\mu_2^3\mu_3^2\mu_4^4\mu_6^3 + 26\mu_2^3\mu_3^2\mu_4^3\mu_5^3\mu_6 + 120\mu_2^3\mu_3^2\mu_4^3\mu_5^2\mu_6^2 + 172\mu_2^3\mu_3^2\mu_4^3\mu_5\mu_6^3 + 78\mu_2^3\mu_3^2\mu_4^3\mu_6^4 \\
&+ 240\mu_2^3\mu_3\mu_4^8 + 474\mu_2^3\mu_3\mu_4^7\mu_5 + 976\mu_2^3\mu_3\mu_4^7\mu_6 + 175\mu_2^3\mu_3\mu_4^6\mu_5^2 + 1372\mu_2^3\mu_3\mu_4^6\mu_5\mu_6 + \\
&1524\mu_2^3\mu_3\mu_4^6\mu_6^2 - 328\mu_2^3\mu_3\mu_4^5\mu_5^3 - 28\mu_2^3\mu_3\mu_4^5\mu_5^2\mu_6 + 1336\mu_2^3\mu_3\mu_4^5\mu_5\mu_6^2 + 1136\mu_2^3\mu_3\mu_4^5\mu_6^3 - \\
&120\mu_2^3\mu_3\mu_4^4\mu_5^4 - 398\mu_2^3\mu_3\mu_4^4\mu_5^3\mu_6 - 46\mu_2^3\mu_3\mu_4^4\mu_5^2\mu_6^2 + 636\mu_2^3\mu_3\mu_4^4\mu_5\mu_6^3 + 420\mu_2^3\mu_3\mu_4^4\mu_6^4 + \\
&10\mu_2^3\mu_3\mu_4^3\mu_5^4\mu_6 + 78\mu_2^3\mu_3\mu_4^3\mu_5^3\mu_6^2 + 196\mu_2^3\mu_3\mu_4^3\mu_5^2\mu_6^3 + 200\mu_2^3\mu_3\mu_4^3\mu_5\mu_6^4 + 72\mu_2^3\mu_3\mu_4^3\mu_6^5 + \\
&80\mu_2^3\mu_4^9 + 152\mu_2^3\mu_4^8\mu_5 + 384\mu_2^3\mu_4^8\mu_6 + 32\mu_2^3\mu_4^7\mu_5^2 + 556\mu_2^3\mu_4^7\mu_5\mu_6 + 752\mu_2^3\mu_4^7\mu_6^2 - 420\mu_2^3\mu_4^6\mu_5^3 \\
&- 382\mu_2^3\mu_4^6\mu_5^2\mu_6 + 656\mu_2^3\mu_4^6\mu_5\mu_6^2 + 760\mu_2^3\mu_4^6\mu_6^3 - 290\mu_2^3\mu_4^5\mu_5^4 - 1096\mu_2^3\mu_4^5\mu_5^3\mu_6 - 840\mu_2^3\mu_4^5\mu_5^2\mu_6^2 \\
&+ 328\mu_2^3\mu_4^5\mu_5\mu_6^3 + 424\mu_2^3\mu_4^5\mu_6^4 - 50\mu_2^3\mu_4^4\mu_5^5 - 320\mu_2^3\mu_4^4\mu_5^4\mu_6 - 630\mu_2^3\mu_4^4\mu_5^3\mu_6^2 - 328\mu_2^3\mu_4^4\mu_5^2\mu_6^3 + \\
&156\mu_2^3\mu_4^4\mu_5\mu_6^4 + 136\mu_2^3\mu_4^4\mu_6^5 + 10\mu_2^3\mu_4^3\mu_5^4\mu_6^2 + 52\mu_2^3\mu_4^3\mu_5^3\mu_6^3 + 98\mu_2^3\mu_4^3\mu_5^2\mu_6^4 + 80\mu_2^3\mu_4^3\mu_5\mu_6^5 \\
&+ 24\mu_2^3\mu_4^3\mu_6^6 + 24\mu_2^2\mu_3^5\mu_4^5 + 12\mu_2^2\mu_3^5\mu_4^4\mu_5 + 24\mu_2^2\mu_3^5\mu_4^4\mu_6 + 6\mu_2^2\mu_3^5\mu_4^3\mu_5\mu_6 + 6\mu_2^2\mu_3^5\mu_4^3\mu_6^2 + \\
&176\mu_2^2\mu_3^4\mu_4^6 + 224\mu_2^2\mu_3^4\mu_4^5\mu_5 + 376\mu_2^2\mu_3^4\mu_4^5\mu_6 + 54\mu_2^2\mu_3^4\mu_4^4\mu_5^2 + 271\mu_2^2\mu_3^4\mu_4^4\mu_5\mu_6 \\
&+ 243\mu_2^2\mu_3^4\mu_4^4\mu_6^2 + 34\mu_2^2\mu_3^4\mu_4^3\mu_5^2\mu_6 + 84\mu_2^2\mu_3^4\mu_4^3\mu_5\mu_6^2 + 50\mu_2^2\mu_3^4\mu_4^3\mu_6^3 + 512\mu_2^2\mu_3^3\mu_4^7 \\
&+ 992\mu_2^2\mu_3^3\mu_4^6\mu_5 + 1664\mu_2^2\mu_3^3\mu_4^6\mu_6 + 608\mu_2^2\mu_3^3\mu_4^5\mu_5^2 + 2252\mu_2^2\mu_3^3\mu_4^5\mu_5\mu_6 + 1916\mu_2^2\mu_3^3\mu_4^5\mu_6^2
\end{aligned}
$$

$$+ 48\mu_2^2\mu_3^3\mu_4^4\mu_5^3 + 746\mu_2^2\mu_3^3\mu_4^4\mu_5^2\mu_6 + 1580\mu_2^2\mu_3^3\mu_4^4\mu_5\mu_6^2 + 922\mu_2^2\mu_3^3\mu_4^4\mu_6^3 + 68\mu_2^2\mu_3^3\mu_4^3\mu_5^3\mu_6$$
$$+ 282\mu_2^2\mu_3^3\mu_4^3\mu_5^2\mu_6^2 + 374\mu_2^2\mu_3^3\mu_4^3\mu_5\mu_6^3 + 160\mu_2^2\mu_3^3\mu_4^3\mu_6^4 + 708\mu_2^2\mu_3^2\mu_4^8 + 1892\mu_2^2\mu_3^2\mu_4^7\mu_5 +$$
$$3142\mu_2^2\mu_3^2\mu_4^7\mu_6 + 1777\mu_2^2\mu_3^2\mu_4^6\mu_5^2 + 6281\mu_2^2\mu_3^2\mu_4^6\mu_5\mu_6 + 5308\mu_2^2\mu_3^2\mu_4^6\mu_6^2 + 444\mu_2^2\mu_3^2\mu_4^5\mu_5^3 +$$
$$3868\mu_2^2\mu_3^2\mu_4^5\mu_5^2\mu_6 + 7434\mu_2^2\mu_3^2\mu_4^5\mu_5\mu_6^2 + 4272\mu_2^2\mu_3^2\mu_4^5\mu_6^3 - 72\mu_2^2\mu_3^2\mu_4^4\mu_5^4 + 510\mu_2^2\mu_3^2\mu_4^4\mu_5^3\mu_6 +$$
$$2734\mu_2^2\mu_3^2\mu_4^4\mu_5^2\mu_6^2 + 3754\mu_2^2\mu_3^2\mu_4^4\mu_5\mu_6^3 + 1643\mu_2^2\mu_3^2\mu_4^4\mu_6^4 + 58\mu_2^2\mu_3^2\mu_4^3\mu_5^4\mu_6 + 360\mu_2^2\mu_3^2\mu_4^3\mu_5^3\mu_6^2$$
$$+ 788\mu_2^2\mu_3^2\mu_4^3\mu_5^2\mu_6^3 + 732\mu_2^2\mu_3^2\mu_4^3\mu_5\mu_6^4 + 246\mu_2^2\mu_3^2\mu_4^3\mu_6^5 + 448\mu_2^2\mu_3\mu_4^9 + 1668\mu_2^2\mu_3\mu_4^8\mu_5 +$$
$$2616\mu_2^2\mu_3\mu_4^8\mu_6 + 1938\mu_2^2\mu_3\mu_4^7\mu_5^2 + 7140\mu_2^2\mu_3\mu_4^7\mu_5\mu_6 + 5954\mu_2^2\mu_3\mu_4^7\mu_6^2 + 850\mu_2^2\mu_3\mu_4^6\mu_5^3 +$$
$$6196\mu_2^2\mu_3\mu_4^6\mu_5^2\mu_6 + 11895\mu_2^2\mu_3\mu_4^6\mu_5\mu_6^2 + 6864\mu_2^2\mu_3\mu_4^6\mu_6^3 - 304\mu_2^2\mu_3\mu_4^5\mu_5^4 + 1228\mu_2^2\mu_3\mu_4^5\mu_5^3\mu_6$$
$$+ 6944\mu_2^2\mu_3\mu_4^5\mu_5^2\mu_6^2 + 9608\mu_2^2\mu_3\mu_4^5\mu_5\mu_6^3 + 4268\mu_2^2\mu_3\mu_4^5\mu_6^4 - 132\mu_2^2\mu_3\mu_4^4\mu_5^5 - 369\mu_2^2\mu_3\mu_4^4\mu_5^4\mu_6$$
$$+ 866\mu_2^2\mu_3\mu_4^4\mu_5^3\mu_6^2 + 3576\mu_2^2\mu_3\mu_4^4\mu_5^2\mu_6^3 + 3839\mu_2^2\mu_3\mu_4^4\mu_5\mu_6^4 + 1372\mu_2^2\mu_3\mu_4^4\mu_6^5$$
$$+ 18\mu_2^2\mu_3\mu_4^3\mu_5^5\mu_6 + 174\mu_2^2\mu_3\mu_4^3\mu_5^4\mu_6^2 + 584\mu_2^2\mu_3\mu_4^3\mu_5^3\mu_6^3 + 900\mu_2^2\mu_3\mu_4^3\mu_5^2\mu_6^4 + 654\mu_2^2\mu_3\mu_4^3\mu_5\mu_6^5$$
$$+ 182\mu_2^2\mu_3\mu_4^3\mu_6^6 + 96\mu_2^2\mu_4^{10} + 544\mu_2^2\mu_4^9\mu_5 + 752\mu_2^2\mu_4^9\mu_6 + 648\mu_2^2\mu_4^8\mu_5^2 + 2768\mu_2^2\mu_4^8\mu_5\mu_6 +$$
$$2232\mu_2^2\mu_4^8\mu_6^2 + 328\mu_2^2\mu_4^7\mu_5^3 + 2692\mu_2^2\mu_4^7\mu_5^2\mu_6 + 5820\mu_2^2\mu_4^7\mu_5\mu_6^2 + 3420\mu_2^2\mu_4^7\mu_6^3 - 480\mu_2^2\mu_4^6\mu_5^4$$
$$+ 212\mu_2^2\mu_4^6\mu_5^3\mu_6 + 4040\mu_2^2\mu_4^6\mu_5^2\mu_6^2 + 6426\mu_2^2\mu_4^6\mu_5\mu_6^3 + 2996\mu_2^2\mu_4^6\mu_6^4 - 340\mu_2^2\mu_4^5\mu_5^5$$
$$- 1348\mu_2^2\mu_4^5\mu_5^4\mu_6 - 460\mu_2^2\mu_4^5\mu_5^3\mu_6^2 + 2936\mu_2^2\mu_4^5\mu_5^2\mu_6^3 + 3962\mu_2^2\mu_4^5\mu_5\mu_6^4 + 1528\mu_2^2\mu_4^5\mu_6^5$$
$$- 54\mu_2^2\mu_4^4\mu_5^6 - 380\mu_2^2\mu_4^4\mu_5^5\mu_6 - 765\mu_2^2\mu_4^4\mu_5^4\mu_6^2 - 52\mu_2^2\mu_4^4\mu_5^3\mu_6^3 + 1300\mu_2^2\mu_4^4\mu_5^2\mu_6^4 + 1350\mu_2^2\mu_4^4\mu_5\mu_6^5$$
$$+ 428\mu_2^2\mu_4^4\mu_6^6 + 18\mu_2^2\mu_4^3\mu_5^5\mu_6^2 + 116\mu_2^2\mu_4^3\mu_5^4\mu_6^3 + 292\mu_2^2\mu_4^3\mu_5^3\mu_6^4 + 360\mu_2^2\mu_4^3\mu_5^2\mu_6^5$$
$$+ 218\mu_2^2\mu_4^3\mu_5\mu_6^6 + 52\mu_2^2\mu_4^3\mu_6^7 + 8\mu_2\mu_3^6\mu_4^5 + 4\mu_2\mu_3^6\mu_4^4\mu_5 + 8\mu_2\mu_3^6\mu_4^4\mu_6 + 2\mu_2\mu_3^6\mu_4^3\mu_5\mu_6 +$$
$$2\mu_2\mu_3^6\mu_4^3\mu_6^2 + 96\mu_2\mu_3^5\mu_4^6 + 120\mu_2\mu_3^5\mu_4^5\mu_5 + 192\mu_2\mu_3^5\mu_4^5\mu_6 + 36\mu_2\mu_3^5\mu_4^4\mu_5^2 + 144\mu_2\mu_3^5\mu_4^4\mu_5\mu_6$$
$$+ 120\mu_2\mu_3^5\mu_4^4\mu_6^2 + 18\mu_2\mu_3^5\mu_4^3\mu_5^2\mu_6 + 42\mu_2\mu_3^5\mu_4^3\mu_5\mu_6^2 + 24\mu_2\mu_3^5\mu_4^3\mu_6^3 + 384\mu_2\mu_3^4\mu_4^7$$
$$+ 776\mu_2\mu_3^4\mu_4^6\mu_5 + 1232\mu_2\mu_3^4\mu_4^6\mu_6 + 528\mu_2\mu_3^4\mu_4^5\mu_5^2 + 1740\mu_2\mu_3^4\mu_4^5\mu_5\mu_6 + 1384\mu_2\mu_3^4\mu_4^5\mu_6^2$$
$$+ 96\mu_2\mu_3^4\mu_4^4\mu_5^3 + 681\mu_2\mu_3^4\mu_4^4\mu_5^2\mu_6 + 1213\mu_2\mu_3^4\mu_4^4\mu_5\mu_6^2 + 652\mu_2\mu_3^4\mu_4^4\mu_6^3 + 56\mu_2\mu_3^4\mu_4^3\mu_5^3\mu_6$$
$$+ 216\mu_2\mu_3^4\mu_4^3\mu_5^2\mu_6^2 + 270\mu_2\mu_3^4\mu_4^3\mu_5\mu_6^3 + 110\mu_2\mu_3^4\mu_4^3\mu_6^4 + 704\mu_2\mu_3^3\mu_4^8 + 2176\mu_2\mu_3^3\mu_4^7\mu_5 +$$
$$3264\mu_2\mu_3^3\mu_4^7\mu_6 + 2208\mu_2\mu_3^3\mu_4^6\mu_5^2 + 7232\mu_2\mu_3^3\mu_4^6\mu_5\mu_6 + 5616\mu_2\mu_3^3\mu_4^6\mu_6^2 + 960\mu_2\mu_3^3\mu_4^5\mu_5^3 +$$
$$5172\mu_2\mu_3^3\mu_4^5\mu_5^2\mu_6 + 8604\mu_2\mu_3^3\mu_4^5\mu_5\mu_6^2 + 4528\mu_2\mu_3^3\mu_4^5\mu_6^3 + 80\mu_2\mu_3^3\mu_4^4\mu_5^4 + 1232\mu_2\mu_3^3\mu_4^4\mu_5^3\mu_6$$
$$+ 3768\mu_2\mu_3^3\mu_4^4\mu_5^2\mu_6^2 + 4334\mu_2\mu_3^3\mu_4^4\mu_5\mu_6^3 + 1728\mu_2\mu_3^3\mu_4^4\mu_6^4 + 80\mu_2\mu_3^3\mu_4^3\mu_5^4\mu_6$$
$$+ 448\mu_2\mu_3^3\mu_4^3\mu_5^3\mu_6^2 + 906\mu_2\mu_3^3\mu_4^3\mu_5^2\mu_6^3 + 790\mu_2\mu_3^3\mu_4^3\mu_5\mu_6^4 + 252\mu_2\mu_3^3\mu_4^3\mu_6^5 + 632\mu_2\mu_3^2\mu_4^9 +$$
$$2964\mu_2\mu_3^2\mu_4^8\mu_5 + 4032\mu_2\mu_3^2\mu_4^8\mu_6 + 4104\mu_2\mu_3^2\mu_4^7\mu_5^2 + 13346\mu_2\mu_3^2\mu_4^7\mu_5\mu_6 + 9792\mu_2\mu_3^2\mu_4^7\mu_6^2$$
$$+ 2712\mu_2\mu_3^2\mu_4^6\mu_5^3 + 13923\mu_2\mu_3^2\mu_4^6\mu_5^2\mu_6 + 22984\mu_2\mu_3^2\mu_4^6\mu_5\mu_6^2 + 11776\mu_2\mu_3^2\mu_4^6\mu_6^3$$
$$+ 600\mu_2\mu_3^2\mu_4^5\mu_5^4 + 6112\mu_2\mu_3^2\mu_4^5\mu_5^3\mu_6 + 16926\mu_2\mu_3^2\mu_4^5\mu_5^2\mu_6^2 + 18984\mu_2\mu_3^2\mu_4^5\mu_5\mu_6^3$$
$$+ 7480\mu_2\mu_3^2\mu_4^5\mu_6^4 - 36\mu_2\mu_3^2\mu_4^4\mu_5^5 + 735\mu_2\mu_3^2\mu_4^4\mu_5^4\mu_6 + 4440\mu_2\mu_3^2\mu_4^4\mu_5^3\mu_6^2 + 8790\mu_2\mu_3^2\mu_4^4\mu_5^2\mu_6^3$$
$$+ 7549\mu_2\mu_3^2\mu_4^4\mu_5\mu_6^4 + 2404\mu_2\mu_3^2\mu_4^4\mu_6^5 + 54\mu_2\mu_3^2\mu_4^3\mu_5^5\mu_6 + 420\mu_2\mu_3^2\mu_4^3\mu_5^4\mu_6^2$$
$$+ 1232\mu_2\mu_3^2\mu_4^3\mu_5^3\mu_6^3 + 1728\mu_2\mu_3^2\mu_4^3\mu_5^2\mu_6^4 + 1170\mu_2\mu_3^2\mu_4^3\mu_5\mu_6^5 + 308\mu_2\mu_3^2\mu_4^3\mu_6^6 + 256\mu_2\mu_3\mu_4^{10}$$
$$+ 1864\mu_2\mu_3\mu_4^9\mu_5 + 2224\mu_2\mu_3\mu_4^9\mu_6 + 3564\mu_2\mu_3\mu_4^8\mu_5^2 + 11008\mu_2\mu_3\mu_4^8\mu_5\mu_6 + 7360\mu_2\mu_3\mu_4^8\mu_6^2$$
$$+ 2928\mu_2\mu_3\mu_4^7\mu_5^3 + 15564\mu_2\mu_3\mu_4^7\mu_5^2\mu_6 + 25422\mu_2\mu_3\mu_4^7\mu_5\mu_6^2 + 12336\mu_2\mu_3\mu_4^7\mu_6^3$$
$$+ 1100\mu_2\mu_3\mu_4^6\mu_5^4 + 9648\mu_2\mu_3\mu_4^6\mu_5^3\mu_6 + 26541\mu_2\mu_3\mu_4^6\mu_5^2\mu_6^2 + 29860\mu_2\mu_3\mu_4^6\mu_5\mu_6^3$$
$$+ 11504\mu_2\mu_3\mu_4^6\mu_6^4 - 168\mu_2\mu_3\mu_4^5\mu_5^5 + 1870\mu_2\mu_3\mu_4^5\mu_5^4\mu_6 + 11216\mu_2\mu_3\mu_4^5\mu_5^3\mu_6^2$$
$$+ 22008\mu_2\mu_3\mu_4^5\mu_5^2\mu_6^3 + 19008\mu_2\mu_3\mu_4^5\mu_5\mu_6^4 + 6048\mu_2\mu_3\mu_4^5\mu_6^5 - 84\mu_2\mu_3\mu_4^4\mu_5^6 - 204\mu_2\mu_3\mu_4^4\mu_5^5\mu_6$$
$$+ 1345\mu_2\mu_3\mu_4^4\mu_5^4\mu_6^2 + 5880\mu_2\mu_3\mu_4^4\mu_5^3\mu_6^3 + 8997\mu_2\mu_3\mu_4^4\mu_5^2\mu_6^4 + 6280\mu_2\mu_3\mu_4^4\mu_5\mu_6^5$$
$$+ 1680\mu_2\mu_3\mu_4^4\mu_6^6 + 14\mu_2\mu_3\mu_4^3\mu_5^6\mu_6 + 162\mu_2\mu_3\mu_4^3\mu_5^5\mu_6^2 + 680\mu_2\mu_3\mu_4^3\mu_5^4\mu_6^3$$
$$+ 1400\mu_2\mu_3\mu_4^3\mu_5^3\mu_6^4 + 1530\mu_2\mu_3\mu_4^3\mu_5^2\mu_6^5 + 854\mu_2\mu_3\mu_4^3\mu_5\mu_6^6 + 192\mu_2\mu_3\mu_4^3\mu_6^7 + 32\mu_2\mu_4^{11} +$$
$$400\mu_2\mu_4^{10}\mu_5 + 400\mu_2\mu_4^{10}\mu_6 + 1152\mu_2\mu_4^9\mu_5^2 + 3160\mu_2\mu_4^9\mu_5\mu_6 + 1840\mu_2\mu_4^9\mu_6^2 + 992\mu_2\mu_4^8\mu_5^3$$
$$+ 6000\mu_2\mu_4^8\mu_5^2\mu_6 + 9504\mu_2\mu_4^8\mu_5\mu_6^2 + 4224\mu_2\mu_4^8\mu_6^3 + 460\mu_2\mu_4^7\mu_5^4 + 4272\mu_2\mu_4^7\mu_5^3\mu_6 +$$

$$12948\mu_2\mu_4^7\mu_5^2\mu_6^2 + 14812\mu_2\mu_4^7\mu_5\mu_6^3 + 5472\mu_2\mu_4^7\mu_6^4 - 324\mu_2\mu_4^6\mu_5^5 + 700\mu_2\mu_4^6\mu_5^4\mu_6$$
$$+ 6768\mu_2\mu_4^6\mu_5^3\mu_6^2 + 14718\mu_2\mu_4^6\mu_5^2\mu_6^3 + 13244\mu_2\mu_4^6\mu_5\mu_6^4 + 4192\mu_2\mu_4^6\mu_6^5 - 224\mu_2\mu_4^5\mu_5^6 -$$
$$960\mu_2\mu_4^5\mu_5^5\mu_6 + 150\mu_2\mu_4^5\mu_5^4\mu_6^2 + 5216\mu_2\mu_4^5\mu_5^3\mu_6^3 + 9342\mu_2\mu_4^5\mu_5^2\mu_6^4 + 6912\mu_2\mu_4^5\mu_5\mu_6^5 +$$
$$1888\mu_2\mu_4^5\mu_6^6 - 32\mu_2\mu_4^4\mu_5^7 - 252\mu_2\mu_4^4\mu_5^6\mu_6 - 540\mu_2\mu_4^4\mu_5^5\mu_6^2 + 250\mu_2\mu_4^4\mu_5^4\mu_6^3 + 2288\mu_2\mu_4^4\mu_5^3\mu_6^4$$
$$+ 3234\mu_2\mu_4^4\mu_5^2\mu_6^5 + 1980\mu_2\mu_4^4\mu_5\mu_6^6 + 464\mu_2\mu_4^4\mu_6^7 + 14\mu_2\mu_4^3\mu_5^6\mu_6^2 + 108\mu_2\mu_4^3\mu_5^5\mu_6^3$$
$$+ 340\mu_2\mu_4^3\mu_5^4\mu_6^4 + 560\mu_2\mu_4^3\mu_5^3\mu_6^5 + 510\mu_2\mu_4^3\mu_5^2\mu_6^6 + 244\mu_2\mu_4^3\mu_5\mu_6^7 + 48\mu_2\mu_4^3\mu_6^8 + 16\mu_3^6\mu_4^6 +$$
$$16\mu_3^6\mu_4^5\mu_5 + 24\mu_3^6\mu_4^5\mu_6 + 4\mu_3^6\mu_4^4\mu_5^2 + 16\mu_3^6\mu_4^4\mu_5\mu_6 + 12\mu_3^6\mu_4^4\mu_6^2 + 2\mu_3^6\mu_4^3\mu_5^2\mu_6 + 4\mu_3^6\mu_4^3\mu_5\mu_6^2$$
$$+ 2\mu_3^6\mu_4^3\mu_6^3 + 96\mu_3^5\mu_4^7 + 192\mu_3^5\mu_4^6\mu_5 + 288\mu_3^5\mu_4^6\mu_6 + 120\mu_3^5\mu_4^5\mu_5^2 + 384\mu_3^5\mu_4^5\mu_5\mu_6$$
$$+ 288\mu_3^5\mu_4^5\mu_6^2 + 24\mu_3^5\mu_4^4\mu_5^3 + 144\mu_3^5\mu_4^4\mu_5^2\mu_6 + 240\mu_3^5\mu_4^4\mu_5\mu_6^2 + 120\mu_3^5\mu_4^4\mu_6^3 + 12\mu_3^5\mu_4^3\mu_5^3\mu_6$$
$$+ 42\mu_3^5\mu_4^3\mu_5^2\mu_6^2 + 48\mu_3^5\mu_4^3\mu_5\mu_6^3 + 18\mu_3^5\mu_4^3\mu_6^4 + 224\mu_3^4\mu_4^8 + 768\mu_3^4\mu_4^7\mu_5 + 1056\mu_3^4\mu_4^7\mu_6 +$$
$$776\mu_3^4\mu_4^6\mu_5^2 + 2464\mu_3^4\mu_4^6\mu_5\mu_6 + 1784\mu_3^4\mu_4^6\mu_6^2 + 352\mu_3^4\mu_4^5\mu_5^3 + 1740\mu_3^4\mu_4^5\mu_5^2\mu_6$$
$$+ 2768\mu_3^4\mu_4^5\mu_5\mu_6^2 + 1368\mu_3^4\mu_4^5\mu_6^3 + 48\mu_3^4\mu_4^4\mu_5^4 + 454\mu_3^4\mu_4^4\mu_5^3\mu_6 + 1213\mu_3^4\mu_4^4\mu_5^2\mu_6^2$$
$$+ 1304\mu_3^4\mu_4^4\mu_5\mu_6^3 + 488\mu_3^4\mu_4^4\mu_6^4 + 28\mu_3^4\mu_4^3\mu_5^4\mu_6 + 144\mu_3^4\mu_4^3\mu_5^3\mu_6^2 + 270\mu_3^4\mu_4^3\mu_5^2\mu_6^3$$
$$+ 220\mu_3^4\mu_4^3\mu_5\mu_6^4 + 66\mu_3^4\mu_4^3\mu_6^5 + 256\mu_3^3\mu_4^9 + 1408\mu_3^3\mu_4^8\mu_5 + 1728\mu_3^3\mu_4^8\mu_6 + 2176\mu_3^3\mu_4^7\mu_5^2$$
$$+ 6528\mu_3^3\mu_4^7\mu_5\mu_6 + 4352\mu_3^3\mu_4^7\mu_6^2 + 1472\mu_3^3\mu_4^6\mu_5^3 + 7232\mu_3^3\mu_4^6\mu_5^2\mu_6 + 11232\mu_3^3\mu_4^6\mu_5\mu_6^2 +$$
$$5296\mu_3^3\mu_4^6\mu_6^3 + 480\mu_3^3\mu_4^5\mu_5^4 + 3448\mu_3^3\mu_4^5\mu_5^3\mu_6 + 8604\mu_3^3\mu_4^5\mu_5^2\mu_6^2 + 9056\mu_3^3\mu_4^5\mu_5\mu_6^3$$
$$+ 3320\mu_3^3\mu_4^5\mu_6^4 + 32\mu_3^3\mu_4^4\mu_5^5 + 616\mu_3^3\mu_4^4\mu_5^4\mu_6 + 2512\mu_3^3\mu_4^4\mu_5^3\mu_6^2 + 4334\mu_3^3\mu_4^4\mu_5^2\mu_6^3$$
$$+ 3456\mu_3^3\mu_4^4\mu_5\mu_6^4 + 1032\mu_3^3\mu_4^4\mu_6^5 + 32\mu_3^3\mu_4^3\mu_5^5\mu_6 + 224\mu_3^3\mu_4^3\mu_5^4\mu_6^2 + 604\mu_3^3\mu_4^3\mu_5^3\mu_6^3$$
$$+ 790\mu_3^3\mu_4^3\mu_5^2\mu_6^4 + 504\mu_3^3\mu_4^3\mu_5\mu_6^5 + 126\mu_3^3\mu_4^3\mu_6^6 + 144\mu_3^2\mu_4^{10} + 1264\mu_3^2\mu_4^9\mu_5 + 1352\mu_3^2\mu_4^9\mu_6$$
$$+ 2964\mu_3^2\mu_4^8\mu_5^2 + 8064\mu_3^2\mu_4^8\mu_5\mu_6 + 4800\mu_3^2\mu_4^8\mu_6^2 + 2736\mu_3^2\mu_4^7\mu_5^3 + 13346\mu_3^2\mu_4^7\mu_5^2\mu_6 +$$
$$19584\mu_3^2\mu_4^7\mu_5\mu_6^2 + 8496\mu_3^2\mu_4^7\mu_6^3 + 1356\mu_3^2\mu_4^6\mu_5^4 + 9282\mu_3^2\mu_4^6\mu_5^3\mu_6 + 22984\mu_3^2\mu_4^6\mu_5^2\mu_6^2 +$$
$$23552\mu_3^2\mu_4^6\mu_5\mu_6^3 + 8200\mu_3^2\mu_4^6\mu_6^4 + 240\mu_3^2\mu_4^5\mu_5^5 + 3056\mu_3^2\mu_4^5\mu_5^4\mu_6 + 11284\mu_3^2\mu_4^5\mu_5^3\mu_6^2$$
$$+ 18984\mu_3^2\mu_4^5\mu_5^2\mu_6^3 + 14960\mu_3^2\mu_4^5\mu_5\mu_6^4 + 4360\mu_3^2\mu_4^5\mu_6^5 - 12\mu_3^2\mu_4^4\mu_5^6 + 294\mu_3^2\mu_4^4\mu_5^5\mu_6$$
$$+ 2220\mu_3^2\mu_4^4\mu_5^4\mu_6^2 + 5860\mu_3^2\mu_4^4\mu_5^3\mu_6^3 + 7549\mu_3^2\mu_4^4\mu_5^2\mu_6^4 + 4808\mu_3^2\mu_4^4\mu_5\mu_6^5 + 1196\mu_3^2\mu_4^4\mu_6^6$$
$$+ 18\mu_3^2\mu_4^3\mu_5^6\mu_6 + 168\mu_3^2\mu_4^3\mu_5^5\mu_6^2 + 616\mu_3^2\mu_4^3\mu_5^4\mu_6^3 + 1152\mu_3^2\mu_4^3\mu_5^3\mu_6^4 + 1170\mu_3^2\mu_4^3\mu_5^2\mu_6^5 +$$
$$616\mu_3^2\mu_4^3\mu_5\mu_6^6 + 132\mu_3^2\mu_4^3\mu_6^7 + 32\mu_3\mu_4^{11} + 512\mu_3\mu_4^{10}\mu_5 + 448\mu_3\mu_4^{10}\mu_6 + 1864\mu_3\mu_4^9\mu_5^2 +$$
$$4448\mu_3\mu_4^9\mu_5\mu_6 + 2264\mu_3\mu_4^9\mu_6^2 + 2376\mu_3\mu_4^8\mu_5^3 + 11008\mu_3\mu_4^8\mu_5^2\mu_6 + 14720\mu_3\mu_4^8\mu_5\mu_6^2 +$$
$$5664\mu_3\mu_4^8\mu_6^3 + 1464\mu_3\mu_4^7\mu_5^4 + 10376\mu_3\mu_4^7\mu_5^3\mu_6 + 25422\mu_3\mu_4^7\mu_5^2\mu_6^2 + 24672\mu_3\mu_4^7\mu_5\mu_6^3 +$$
$$7888\mu_3\mu_4^7\mu_6^4 + 440\mu_3\mu_4^6\mu_5^5 + 4824\mu_3\mu_4^6\mu_5^4\mu_6 + 17694\mu_3\mu_4^6\mu_5^3\mu_6^2 + 29860\mu_3\mu_4^6\mu_5^2\mu_6^3$$
$$+ 23008\mu_3\mu_4^6\mu_5\mu_6^4 + 6368\mu_3\mu_4^6\mu_6^5 - 56\mu_3\mu_4^5\mu_5^6 + 748\mu_3\mu_4^5\mu_5^5\mu_6 + 5608\mu_3\mu_4^5\mu_5^4\mu_6^2$$
$$+ 14672\mu_3\mu_4^5\mu_5^3\mu_6^3 + 19008\mu_3\mu_4^5\mu_5^2\mu_6^4 + 12096\mu_3\mu_4^5\mu_5\mu_6^5 + 2944\mu_3\mu_4^5\mu_6^6 - 24\mu_3\mu_4^4\mu_5^7 -$$
$$68\mu_3\mu_4^4\mu_5^6\mu_6 + 538\mu_3\mu_4^4\mu_5^5\mu_6^2 + 2940\mu_3\mu_4^4\mu_5^4\mu_6^3 + 5998\mu_3\mu_4^4\mu_5^3\mu_6^4 + 6280\mu_3\mu_4^4\mu_5^2\mu_6^5$$
$$+ 3360\mu_3\mu_4^4\mu_5\mu_6^6 + 720\mu_3\mu_4^4\mu_6^7 + 4\mu_3\mu_4^3\mu_5^7\mu_6 + 54\mu_3\mu_4^3\mu_5^6\mu_6^2 + 272\mu_3\mu_4^3\mu_5^5\mu_6^3$$
$$+ 700\mu_3\mu_4^3\mu_5^4\mu_6^4 + 1020\mu_3\mu_4^3\mu_5^3\mu_6^5 + 854\mu_3\mu_4^3\mu_5^2\mu_6^6 + 384\mu_3\mu_4^3\mu_5\mu_6^7 + 72\mu_3\mu_4^3\mu_6^8 + 64\mu_4^{11}\mu_5$$
$$+ 32\mu_4^{11}\mu_6 + 400\mu_4^{10}\mu_5^2 + 800\mu_4^{10}\mu_5\mu_6 + 304\mu_4^{10}\mu_6^2 + 768\mu_4^9\mu_5^3 + 3160\mu_4^9\mu_5^2\mu_6$$
$$+ 3680\mu_4^9\mu_5\mu_6^2 + 1168\mu_4^9\mu_6^3 + 496\mu_4^8\mu_5^4 + 4000\mu_4^8\mu_5^3\mu_6 + 9504\mu_4^8\mu_5^2\mu_6^2 + 8448\mu_4^8\mu_5\mu_6^3$$
$$+ 2368\mu_4^8\mu_6^4 + 184\mu_4^7\mu_5^5 + 2136\mu_4^7\mu_5^4\mu_6 + 8632\mu_4^7\mu_5^3\mu_6^2 + 14812\mu_4^7\mu_5^2\mu_6^3 + 10944\mu_4^7\mu_5\mu_6^4$$
$$+ 2784\mu_4^7\mu_5^5 - 108\mu_4^6\mu_5^6 + 280\mu_4^6\mu_5^5\mu_6 + 3384\mu_4^6\mu_5^4\mu_6^2 + 9812\mu_4^6\mu_5^3\mu_6^3 + 13244\mu_4^6\mu_5^2\mu_6^4 +$$
$$8384\mu_4^6\mu_5\mu_6^5 + 1952\mu_4^6\mu_6^6 - 64\mu_4^5\mu_5^7 - 320\mu_4^5\mu_5^6\mu_6 + 60\mu_4^5\mu_5^5\mu_6^2 + 2608\mu_4^5\mu_5^4\mu_6^3 + 6228\mu_4^5\mu_5^3\mu_6^4$$
$$+ 6912\mu_4^5\mu_5^2\mu_6^5 + 3776\mu_4^5\mu_5\mu_6^6 + 800\mu_4^5\mu_6^7 - 8\mu_4^4\mu_5^8 - 72\mu_4^4\mu_5^7\mu_6 - 180\mu_4^4\mu_5^6\mu_6^2 + 100\mu_4^4\mu_5^5\mu_6^3$$
$$+ 1144\mu_4^4\mu_5^4\mu_6^4 + 2156\mu_4^4\mu_5^3\mu_6^5 + 1980\mu_4^4\mu_5^2\mu_6^6 + 928\mu_4^4\mu_5\mu_6^7 + 176\mu_4^4\mu_6^8 + 4\mu_4^3\mu_5^7\mu_6^2 +$$
$$36\mu_4^3\mu_5^6\mu_6^3 + 136\mu_4^3\mu_5^5\mu_6^4 + 280\mu_4^3\mu_5^4\mu_6^5 + 340\mu_4^3\mu_5^3\mu_6^6 + 244\mu_4^3\mu_5^2\mu_6^7 + 96\mu_4^3\mu_5\mu_6^8 + 16\mu_4^3\mu_6^9.$$

Whereas the case A_{2r-1} was easily done by hand, the case E_6 requires the help of a computer software. In order to find these coefficients, we used the program Maple 2015.

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Singularité et Théorie de Lie

Résumé : Soit Γ un sous-groupe fini de $\mathrm{SU}_2(\mathbb{C})$. Alors le quotient $\mathbb{C}^2/\Gamma$ peut être plongé dans $\mathbb{C}^3$ sous la forme d'une surface munie d'une singularité isolée. Le quotient $\mathbb{C}^2/\Gamma$ est appelé singularité de Klein, d'après F. Klein qui fut le premier à les décrire en 1884. À travers leurs résolutions minimales, ces singularités ont un lien étroit avec les diagrammes de Dynkin simplement lacés de types A_r, D_r et E_r. Dans les années 1970, E. Brieskorn et P. Slodowy ont tiré profit de cette connection pour décrire les résolutions et les déformations de ces singularités à l'aide de la théorie de Lie. En 1998 P. Slodowy et H. Cassens ont construit les déformations semiuniverselles des $\mathbb{C}^2/\Gamma$ à l'aide de la théorie des carquois ainsi que des travaux de P.B. Kronheimer en géométrie symplectique datant de 1989.
En théorie de Lie, la classification des algèbres de Lie simples divise ces dernières en deux classes : les algèbres de Lie de types A_r, D_r et E_r qui sont simplement lacées, et celles de types B_r, C_r, F_4 et G_2 appelées non-homogènes. A l'aide d'un second sous-groupe fini Γ' de $\mathrm{SU}_2(\mathbb{C})$ tel que $\Gamma \lhd \Gamma'$, P. Slodowy a étendu en 1978 la notion de singularité de Klein aux algèbres de Lie non-homogènes en ajoutant à $\mathbb{C}^2/\Gamma$ le groupe d'automorphismes $\Omega = \Gamma'/\Gamma$ du diagramme de Dynkin associé à la singularité.
L'objectif de cette thèse est de généraliser la construction de H. Cassens et P. Slodowy à ces singularités de types B_r, C_r, F_4 et G_2. Il en résultera des constructions explicites des déformations semiuniverselles de types inhomogènes sur les fibres desquelles le groupe Ω agit. Le passage au quotient d'une telle application révèle alors une déformation d'une singularité de type $\mathbb{C}^2/\Gamma'$.

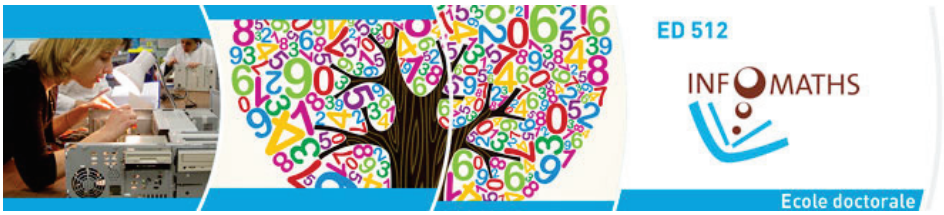
Mots clés : Systèmes de racines ; pliage ; singularité simple ; réduction symplectique ; carquois ; déformations de singularités.

Singularity and Lie Theory

Abstract : Let Γ be a finite subgroup of $\mathrm{SU}_2(\mathbb{C})$. Then the quotient $\mathbb{C}^2/\Gamma$ can be embedded in $\mathbb{C}^3$ as a surface with an isolated singularity. The quotient $\mathbb{C}^2/\Gamma$ is called a Kleinian singularity, after F. Klein who studied them first in 1884. Through their minimal resolutions, these singularities have a deep connection with simply-laced Dynkin diagrams of types A_r, D_r and E_r. In the 1970's E. Brieskorn and P. Slodowy took advantage of this connection to describe the resolutions and deformations of these singularities in terms of Lie theory. In 1998 P. Slodowy and H. Cassens constructed the semiuniversal deformations of the Kleinian singularities using quiver theory and work from 1989 by P.B. Kronheimer on symplectic geometry.
In Lie theory, the classification of simple Lie algebras allows for a separation in two classes : those simply-laced of types A_r, D_r and E_r, and those of types B_r, C_r, F_4 and G_2 called inhomogeneous. With the use of a second finite subgroup Γ' of $\mathrm{SU}_2(\mathbb{C})$ such that $\Gamma \lhd \Gamma'$, P. Slodowy extended in 1978 the definition of a Kleinian singularity to the inhomogeneous types by adding to $\mathbb{C}^2/\Gamma$ the group of automorphisms $\Omega = \Gamma'/\Gamma$ of the Dynkin diagram associated to the singularity.
The purpose of this thesis is to generalize H. Cassens' and P. Slodowy's construction to the singularities of types B_r, C_r, F_4 and G_2. It will lead to explicit semiuniversal deformations of inhomogeneous types on the fibers of which the group Ω acts. By quotienting such a map we obtain a deformation of a singularity $\mathbb{C}^2/\Gamma'$.

Keywords : Root systems ; folding ; simple singularity ; symplectic reduction ; quiver ; deformations of singularities.

Image en couverture : La queue d'aronde, ou le discrimant de la déformation semiuniverselle d'une singularité simple de type A_4.



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