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UNIVERSITÉ FRANÇOIS -RABELAIS DE TOURS

École Doctorale Mathématiques, Informatique, Physique Théorique et Ingénierie des Systèmes

Laboratoire de Mathématiques et Physique Théorique

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Etude qualitative de modèles dispersifs

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A ma Mère, à mon Père...

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Résumé

Dans cette thèse nous nous intéressons aux propriétés qualitatives des solutions de quelques équations d'ondes en milieux dispersifs ou dispersifs-dissipatifs. Dans le premier chapitre nous étudions l'explosion de solutions dans le régime log-log et l'existence globale pour le problème de Cauchy associé à l'équation de Schrödinger L^2 -critique amortie.

Dans le deuxième chapitre, nous considérons l'équation de Schrödinger L^2 -critique avec un amortissement non linéaire. Selon la puissance du terme d'amortissement, nous montrons l'existence globale ou l'explosion en régime log-log.

Dans le troisième chapitre, nous étudions le problème de Cauchy pour l'équation de Kadomtsev-Petviashvili-Burgers-I (KPBI) en deux dimensions. Nous montrons que ce problème est localement et globalement bien posé dans $H^s(\mathbb{R}^2)$, $s > -\frac{1}{2}$.

Dans le dernier chapitre, nous considérons l'équation d'Ostrovsky sur le cercle, et nous construisons une mesure invariante par le flot associée à l'énergie de cette équation.

Mots clés : Equations dispersives et dissipatives, existence locale et globale, espaces de Bourgain, explosion, mesures invariantes.

RÉSUMÉ

Abstract

This thesis deals with the qualitative properties of solutions to some wave equations in dispersive or dispersive-dissipative media. In the first part, we study the blowup in the log-log regime and global existence of solutions to the Cauchy problem for the L^2 -critical damped nonlinear Schrödinger equations. In the second part, we consider the Cauchy problem for the L^2 -critical nonlinear Schrödinger equation with a nonlinear damping. According to the power of the damping term, we prove the global existence or the existence of finite time blowup dynamics with the log-log blowup law. In the third part, we study the Cauchy problem for the Kadomtsev-Petviashvili-Burgers-I (KPBI) equations in two dimensions. We show that this problem is locally and globally well posed in $H^s(\mathbb{R}^2)$, $s > -\frac{1}{2}$.

In the last part, we consider the Ostrovsky equation on the circle, we construct an invariant measure under the flow associated with the energy of the equation.

Keywords : Dispersive and dissipative equation, local and global existence, Bourgain's spaces, invariant measures.

ABSTRACT

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Introduction

De nombreux milieux sont dispersifs. Ce terme signifie que les ondes traversant ce milieu évoluent à des vitesses différentes selon la fréquence imposée par la source. Ce phénomène induit un étalement, sans perte d'énergie, de la solution qui peut être vue comme une superposition d'ondes de différentes longueurs se propageant à des vitesses différentes. Cependant, ces ondes sont souvent soumises à des effets dissipatifs plus ou moins importants, particulièrement pour des ondes longues de petite amplitude et par conséquent, une solution perd de l'énergie au cours du temps. Ces phénomènes relatifs à la propagation d'ondes dans des milieux non-linéaires dispersifs-dissipatifs peuvent être modélisés par des équations aux dérivées partielles d'évolution. L'étude mathématique de ces équations a fait l'objet de recherches intensives ces vingt dernières années et ceci a conduit à l'introduction de nouveaux outils afin de mieux comprendre le comportement local et global de leurs solutions. Ce groupe d'équations comprend entre autre, les équations de Schrödinger et de Korteweg-de-Vries qui en sont les exemples types.

D'un point de vue historique, les équations de type Schrödinger ont fait l'objet de nombreuses recherches dans les années 80 et le problème local pour (NLS) était bien appréhendé dès le début des années 90, avec entre autres les travaux de Ginibre - Vélo [GV85] et de Cazenave - Weissler [CW90]. Ceci a permis à Raphaël et Merle dans les années 2000, d'étudier la dynamique explosive des solutions de ces équations. En ce qui concerne les équations de type KdV, il a fallu attendre les travaux de Bourgain [Bou93b] puis de Kenig, Ponce et Vega [KPV96] pour voir se développer de nouveaux outils permettant d'obtenir une théorie locale optimale pour ces équations. Récemment, Molinet et Ribaud (cf. [MR01], [MR02b], [MR02c]) ont montré que l'on pouvait adapter de façon très satisfaisante la méthode introduite par Bourgain pour traiter les équations à la fois dispersives et dissipatives. Ils ont ainsi obtenu un résultat précis dans l'échelle des espaces de Sobolev pour le problème de Cauchy associé à l'équation de Korteweg-de Vries-Burgers (KdV-B).

L'objet de cette thèse est l'étude mathématique de modèles dispersifs et dissipatifs-dispersifs non linéaires. Les buts étant d'étudier des solutions explosives et l'existence globale pour de telles équations, d'améliorer les résultats existants et d'obtenir ainsi des résultats précis sur les problèmes de Cauchy locaux et/ou des résultats d'existence globale dans des espaces naturels. Enfin, on construira une mesure invariante pour une équation purement dispersive.

L'équation de Schrödinger non linéaire L^2 -critique est donnée en dimension d par :

$$iu_t + \Delta u + |u|^{\frac{4}{d}}u = 0, \tag{NLS}$$

En dimension $d = 2$, cette équation modélise la dynamique de l'enveloppe d'une onde monochromatique dans un milieu non linéaire faiblement dispersif. Le phénomène décrit par la dynamique explosive dans l'espace d'énergie H^1 est alors la focalisation ponctuelle du faisceau laser en un

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point de l'espace.

Dans les années 70, Ott et Sudan [OS70] ont montré que dans certains milieux les collisions de certaines particules imposent d'ajouter un terme d'amortissement à l'équation qui devient:

$$iu_t + \Delta u + |u|^{\frac{4}{d}}u + iau = 0, \quad (\text{NLSa})$$

où $a > 0$ est le coefficient d'amortissement.

Une généralisation de l'équation (NLSa), lorsque l'amortissement est non linéaire, dans ce cas l'équation est donnée par

$$iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia|u|^p u = 0. \quad (\text{NLSap})$$

L'origine du terme d'amortissement non-linéaire est l'absorption multiphotonique (voir [FM01]). Par exemple, dans le cas des solides le nombre p correspond au nombre de photons nécessaires pour faire une transition de l'état stable à un état excité ou au continuum.

D'autre part, l'équation de Korteweg-de Vries-Burgers:

$$\partial_t u + u_{xxx} - u_{xx} + uu_x = 0, \quad (\text{KdV-B})$$

apparaît dans la littérature en tant que version dissipative de l'équation de Korteweg-de Vries:

$$\partial_t u + u_{xxx} + uu_x = 0. \quad (\text{KdV})$$

Dans quelques situations typiques, en raison des effets de viscosité, il est impossible de négliger des effets dissipatifs, et ceci peut mener à l'équation de KdV-Burgers (cf. [OS70]). Elle est donc un modèle pour la propagation d'ondes dans un milieu non-linéaire à la fois dispersif et dissipatif.

Récemment, il a été montré par Leblond [Leb02] que les équations de Kadomtsev-Petviashvili-Burgers:

$$(\partial_t u + u_{xxx} - u_{xx} + uu_x)_x + \epsilon u_{yy} = 0 \quad \epsilon = \pm 1, \quad (\text{KPB})$$

modélisent, dans certains régimes, la propagation d'ondes électromagnétiques dans un milieu ferromagnétique saturé. On peut donc considérer ces équations comme des modèles pour la propagation des ondes bidimensionnelles prenant en compte des effets d'amortissement. Notons que, comme nous sommes intéressés par une propagation presque uni-directionnelle, le terme dissipatif agit seulement dans la direction principale de la propagation dans KPB. Ces équations sont aussi des versions dissipatives des équations de Kadomtsev-Petviashvili:

$$(\partial_t u + u_{xxx} + uu_x)_x + \epsilon u_{yy} = 0 \quad \epsilon = \pm 1. \quad (\text{KP})$$

Dans le contexte des vagues, les équations de (KP) sont des modèles universels pour des ondes dispersives non-linéaires presque unidirectionnelles avec des effets transversaux faibles. Le signe $\epsilon = +1$ correspond à l'équation de (KP-II), tandis que le signe $\epsilon = -1$ correspond à l'équation de (KP-I). L'équation de KP-II modélise de longues vagues avec des effets faibles de tension superficielle, par contre l'équation de KP-I modélise l'écoulement en présence des effets de tension superficielle forts. Ces équations sont des extensions bidimensionnelles de l'équation de Korteweg-de Vries (KdV).

En perturbant l'équation de Korteweg Vries (KdV) avec un terme non-local, on pourra obtenir l'équation d'Ostrovsky:

$$\partial_t u - u_{xxx} + \partial_x^{-1} u + uu_x = 0. \quad (\text{Ost})$$

Cette équation pourra être considérée comme un modèle faiblement non linéaire des ondes longues, pour décrire la propagation des ondes de surface dans l'océan.

Cette thèse se compose de quatre chapitres qui font l'objet de mes travaux dans [Dar12a], [Dar12c], [Dar12b] et [Dar13]. Le premier chapitre est relatif à l'équation de Schrödinger non linéaire L^2 -critique amortie (NLSa), plus précisément à l'étude des solutions explosives et l'existence globale pour cette équation.

Dans un second chapitre, nous avons généralisé les résultats obtenus pour (NLSa), en étudiant le cas d'un amortissement non linéaire.

Un troisième chapitre est consacré à l'équation de Kadomtsev-Petviashvili-Burgers (KPB-I), plus précisément à l'étude précise en dimensions deux d'espace du caractère bien posé local et global pour cette équation.

Le dernier chapitre porte sur l'étude du caractère bien posé et la construction des mesures invariants pour l'équation d'Ostrovsky (Ost) sur le cercle.

Nous présentons maintenant un résumé détaillé de chacun des quatre chapitres:

Explosion pour l'équation de Schrödinger non linéaire L^2 critique amortie

Dans ce premier chapitre, nous étudions l'explosion de solutions et l'existence globale pour le problème de Cauchy de l'équation de Schrödinger L^2 -critique amortie:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}} u + iau = 0, (t, x) \in [0, \infty[\times \mathbb{R}^d, d = 1, 2, 3, 4. \\ u(0) = u_0 \end{cases} \quad (1)$$

avec une donnée initiale $u(0) = u_0 \in H^1(\mathbb{R}^d)$ et $a > 0$ est le coefficient d'amortissement. L'équation (1) apparaît dans divers domaines de l'optique non linéaire, la physique des plasmas et dans la mécanique des fluides.

Il est connu que le problème de Cauchy pour (1) est localement bien posé dans $H^1(\mathbb{R}^d)$ (voir Kato [Kat87] et aussi Cazenave [Caz03]) : Pour tout $u_0 \in H^1(\mathbb{R}^d)$, il existe $T \in (0, \infty]$ et une unique solution $u(t)$ de (1) avec $u(0) = u_0$ telle que $u \in C([0, T), H^1(\mathbb{R}^d))$. De plus, T est le temps maximal d'existence de la solution $u(t)$ dans le sens où si $T < \infty$, alors $\lim_{t \rightarrow T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$. Tsutsumi [Tsu84] puis Ohta et Todorova [OT09] ont étudié le cas surcritique ($|u|^p u$ avec $p > \frac{4}{d}$) et ont montré que l'explosion en temps fini peut être obtenue en utilisant la méthode du viriel. Cependant, cette méthode ne semble pas s'appliquer dans le cas critique. Par conséquent, même si des simulations numériques suggèrent l'existence de solutions qui explosent en temps fini (voir Fibich [FM01]), il n'existe aucune preuve mathématique pour l'explosion dans le cas critique.

Notons que pour $a = 0$ (1) devient l'équation de Schrödinger non linéaire L^2 -critique :

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}} u = 0 \\ u(0) = u_0 \end{cases} \quad u_0 \in H^1(\mathbb{R}^d) \quad (2)$$

Cette équation (2) admet un certain nombre de symétries dans l'espace d'énergie H^1 . En particulier, si $u(t, x)$ est une solution de (2), alors $\forall \lambda_0 \in \mathbb{R}, \lambda_0^{\frac{d}{2}} u(\lambda_0 x, \lambda_0^2 t)$ est aussi une solution de NLS.

Notons que la norme L^2 est invariante par scaling et donc L^2 est l'espace critique associé à cette symétrie.

L'évolution (2) admet les lois de conservation dans l'espace d'énergie H^1 suivants:

Norme L^2 : $\|u(t, x)\|_{L^2} = \|u(0, x)\|_{L^2} = \|u_0(x)\|_{L^2}$.

L'énergie : $E(u(t, x)) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{d}{4+2d}\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = E(u_0)$.

Moment cinétique : $P(u(t)) = \text{Im}(\int \nabla u \bar{u}(t, x)) = P(u_0)$.

Des solutions spéciales jouent un rôle fondamental pour la description de la dynamique de (2), ce sont les ondes solitaires de la forme $u(t, x) = \exp(it)Q(x)$, où Q résout l'équation suivante:

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (3)$$

L'équation (3) est une équation elliptique non linéaire, qui possède une solution unique positive (aux translations près) (voir [BL83], [Lio84], [Kwo89]).

Pour $u_0 \in H^1$, un critère pour l'existence globale a été trouvé par Weinstein [Wei83]: a) Pour $\|u_0\|_{L^2} < \|Q\|_{L^2}$, la solution de (2) est globale dans H^1 . Cela découle de la conservation de l'énergie et de la norme L^2 et l'inégalité de Gagliardo-Nirenberg:

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right).$$

b) Il existe des solutions explosives issues de données initiales $u_0 \in H^1$ avec $\|u_0\|_{L^2} = \|Q\|_{L^2}$.

Cela découle de la symétrie pseudo-conforme appliquée aux ondes solitaires.

Dans une série d'articles [MR02a, MR05], Merle et Raphaël ont étudié l'explosion de l'équation de Schrödinger L^2 -critique (2) et ont prouvé l'existence d'un régime explosif correspond à la loi log-log:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \quad (4)$$

Ce régime a l'avantage d'être stable par des perturbations de l'équation.

Les travaux de Merle-Raphaël [MR02a, MR05] montrent en particulier, le résultat suivant:

Soit u_0 une donnée initiale dans $H^1(\mathbb{R}^d)$ de masse sur-critique:

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0 \quad (5)$$

avec une énergie négative $E(u_0) < 0$, alors la solution correspondante de (2) explose en temps fini avec la vitesse log-log. Dans la preuve de leur résultat, ils utilisent entre autre les quantités conservées. Dans le cas de (1), il n'existe plus de quantités conservées. Cependant, il est facile de montrer que si u est une solution de (1) alors:

$$\|u(t)\|_{L^2} = \exp(-at)\|u_0\|_{L^2}, t \in [0, T), \quad (6)$$

$$\frac{d}{dt}E(u(t)) = -a(\|\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}) \quad (7)$$

et

$$|P(u(t))| = \exp(-2at)|P(u_0)|, t \in [0, T). \quad (8)$$

Maintenant, nous sommes en mesure d'énoncer les résultats principaux de ce chapitre.

Dans le théorème suivant nous montrons que, si $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ la solution de (1) est globale dans H^1 , et qu'il existe des solutions de masse surcritique qui explosent en régime log-log. Plus précisément, nous avons le théorème suivant:

Theorem 0.1. *Etant donnée u_0 dans $H^1(\mathbb{R}^d)$ avec $d = 1, 2, 3, 4$:*

1. *Si $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ alors la solution de (1) est globale dans H^1 .*
2. *Il existe $\delta_0 > 0$ telle que $\forall \alpha > 0$ et $\forall \delta \in]0, \delta_0[$, il existe $u_0 \in H^1$ avec $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, telle que la solution de (1) explose en temps fini dans le régime log-log.*

Principe de la preuve.

Explosion en temps fini pour (NLSa)

Pour montrer l'existence des solutions explosives, nous nous mettons dans le régime décrit par Merle et Raphaël. Nous restreignons désormais notre discussion au cas de données de petite masse surcritique:

$$u_0 \in B_{\alpha^*} = \{u_0 \in H^1; \int Q^2 < \int |u_0|^2 < \int Q^2 + \alpha\}$$

avec α petit.

L'idée est de décomposer notre solution de la façon suivante:

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} (Q_{b(t)} + \varepsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)} \quad (9)$$

où $(x(t), \gamma(t)) \in \mathbb{R}^d \times \mathbb{R}$ et $\|\varepsilon\|_{H^1} \leq \delta(\alpha)$, $\delta(\alpha) \rightarrow 0$ quand $\alpha \rightarrow 0$, $\lambda(t) \sim \frac{1}{\|\nabla u\|_{L^2}}$, $b = -\lambda \lambda_t$ et $Q_{b(t)}$ est un raffinement de Q . Après avoir décomposé la solution u , le régime log-log correspond au contrôle asymptotique suivant:

$$b_s \sim C e^{-\frac{c}{b}}, -\frac{\lambda_s}{\lambda} \sim b \quad (10)$$

et

$$\int |\nabla \varepsilon|^2 \lesssim e^{-\frac{c}{b}}, \quad (11)$$

où nous introduisons le changement d'échelle suivant $\frac{ds}{dt} = \frac{1}{\lambda^2}$.

En fait, (11) est une conséquence de l'estimation préliminaire:

$$\int |\nabla \varepsilon|^2 \lesssim e^{-\frac{c}{b}} + \lambda^2 E(t). \quad (12)$$

On constate alors que, dans le régime log-log, l'intégration des lois (10), donne

$$\lambda \sim e^{-e^{\frac{c}{b}}} \ll e^{-\frac{c}{b}}, b(t) \rightarrow 0, t \rightarrow T. \quad (13)$$

Par conséquent, le terme impliquant la conservation de l'Hamiltonien est asymptotiquement négligeable par rapport à l'ordre dominant le terme $e^{-\frac{c}{b}}$ qui entraîne l'estimation (12) de b . C'était

l'observation centrale faite par Planchon et Raphaël dans [PR07]. En fait, n'importe quelle croissance algébrique de l'énergie inférieure à $\frac{1}{\lambda^2}$ serait suffisante. Dans ce chapitre, nous montrons que dans le régime log-log, la croissance de l'énergie est estimée par:

$$E(u(t)) \lesssim (\log(\lambda(t)))^2. \quad (14)$$

Nous déduisons de (12) que:

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{\epsilon}{b}}. \quad (15)$$

Une caractéristique importante de cette estimation au niveau H^1 est qu'elle repose sur un calcul de flux dans L^2 . Cela permet de récupérer les lois asymptotiques pour les paramètres géométriques (10) et de boucler les estimations de bootstrap du régime log-log.

Existence Globale en temps pour (NLSa)

Pour montrer l'existence globale en temps, nous allons montrer que si la solution explose en temps fini T , alors un phénomène de concentration dans L^2 aura lieu. Plus précisément, pour n'importe quelle fonction $w(t)$ qui vérifie $w(t) \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \rightarrow \infty$ lorsque $t \rightarrow T$, il existe $x(t) \in \mathbb{R}^d$ telle que:

$$\limsup_{t \rightarrow T} \|u(t)\|_{L^2(|x-x(t)| < w(t))} \geq \|Q\|_{L^2(\mathbb{R}^d)}.$$

Ce qui nous donne l'existence globale, pour des données initiales de masse inférieure ou égale à celle de Q .

Sur l'équation de Schrödinger non linéaire L^2 critique avec un amortissement non linéaire

Dans ce chapitre, nous étudions l'explosion et l'existence globale de solutions pour l'équation de Schrödinger NLS (focalisante) avec un terme d'amortissement non linéaire:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia|u|^p u = 0, (t, x) \in [0, \infty[\times \mathbb{R}^d, d = 1, 2, 3, 4. \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (16)$$

avec une donnée initiale $u(0) = u_0 \in H^1(\mathbb{R}^d)$ où $a > 0$ est toujours le coefficient d'amortissement et $p \geq 1$. Notons que si on change $+|u|^{\frac{4}{d}}u$ en $-|u|^{\frac{4}{d}}u$, (16) devient défocalisante.

L'équation (16) entre dans divers domaines de l'optique non linéaire, la physique des plasmas et de la mécanique des fluides. Fibich [FM01] a noté que, dans le contexte optique non linéaire, l'origine du terme d'amortissement non-linéaire est l'absorption multiphotonique. Par exemple, dans le cas des solides le nombre p correspond au nombre de photons nécessaire pour faire la transition entre la bande de valence et la bande de conduction. Un comportement similaire peut se produire avec des atomes libres, dans ce cas p correspond au nombre de photons nécessaires pour faire une transition de l'état fondamental à l'état excité ou au continuum.

Le problème de Cauchy associé à (16) a été étudié par Kato [Kat87] et Cazenave [Caz03] et il est connu que si $p < \frac{4}{d-2}$, alors le problème est localement bien posé dans $H^1(\mathbb{R}^d)$: Pour $u_0 \in H^1(\mathbb{R}^d)$, il existe $T \in (0, \infty]$ et une unique solution $u(t)$ de (16) avec $u(0) = u_0$ telle que $u \in C([0, T]; H^1(\mathbb{R}^d))$. De plus, T est le temps maximal d'existence de la solution $u(t)$ dans le sens où si $T < \infty$ alors $\lim_{t \rightarrow T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$.

INTRODUCTION

Dans le premier chapitre nous avons étudié le cas d'amortissement linéaire ($p = 0$). Nous avons montré l'existence globale dans H^1 pour $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ et l'existence de solutions explosant en temps fini avec la vitesse log-log.

Des observations numériques suggèrent que ce phénomène d'explosion persiste dans le cas d'amortissement non linéaire pour $p < \frac{4}{d}$ (voir [FM01] et [PSS05]). Passot et Sulem [PSS05] ont prouvé que la solution est globale dans $H^1(\mathbb{R}^2)$ pour $p > 2$. Par conséquent, il est intéressant maintenant d'étudier selon la valeur de p , l'explosion ou l'existence globale de solution.

Résultats principaux

Nous montrons l'existence des solutions explosives dans le régime log-log dans le cas $p < \frac{4}{d}$ et nous établissons l'existence globale de solutions dès que $\frac{4}{d} \leq p < \frac{4}{d-2}$ où $\frac{4}{d-2}$ est l'exposant H^1 -critique.

Après nous montrerons que pour $p < \frac{4}{d}$, il n'existe pas de solution de donnée initiale de masse inférieure ou égale à celle de Q qui explose en régime log-log.

Plus précisément, nous avons le théorème suivant:

Theorem 0.2. *Soit u_0 dans $H^1(\mathbb{R}^d)$ avec $d = 1, 2, 3, 4$:*

1. *Si $\frac{4}{d-2} > p \geq \frac{4}{d}$, la solution de (16) est globale dans H^1 .*
2. *Pour $p = 1$ et $d = 1, 2, 3$ ou $p = 2$ et $d = 1$, la solution est globale dans H^1 , pour $u_0 \in H^1$ telle que $\|u_0\|_{L^2} < \alpha$ avec $\alpha > 0$ suffisamment petit.*
3. *Soit $p < \frac{4}{d}$, alors il existe $\delta_0 > 0$ telle que $\forall a > 0$ et $\forall \delta \in]0, \delta_0[$, il existe $u_0 \in H^1$ avec $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, telle que la solution de (16) explose en temps fini dans le régime log-log.*
4. *Soit $1 \leq p < \frac{4}{d}$, alors il n'existe pas de donnée initiale u_0 avec $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ tel que la solution u de (16) explose en temps fini avec la vitesse suivante:*

$$\frac{1}{(T-t)^{\beta-\epsilon}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^{\beta+\epsilon}},$$

quels que soient $\beta \in]0, \frac{2}{pd}[$ et $0 < \epsilon < \frac{2-\beta pd}{8+pd}$.

Dans le cas $p = \frac{4}{d}$, nous avons le théorème suivant:

Theorem 0.3. *Pour $p = \frac{4}{d}$, le problème de Cauchy pour (16) est globalement bien posé dans $H^s(\mathbb{R}^d)$, $s \geq 0$. De plus, il existe un unique u_+ dans L^2 tel que*

$$\|u(\cdot, t) - e^{it\Delta} u_+\|_{L^2} \longrightarrow 0, \quad t \longrightarrow +\infty, \quad (17)$$

où $e^{it\Delta}$ est la solution du problème linéaire.

Principe de la preuve.

Pour montrer la première partie du Théorème 0.2, nous prouvons une estimation a priori sur la norme H^1 de u pour tout $t \geq 0$.

Pour montrer la deuxième partie du Théorème 0.2, nous utilisons une inégalité de Gagliardo-Nirenberg généralisée pour montrer que l'énergie est décroissante. La décroissance de l'énergie et l'inégalité de Gagliardo-Nirenberg suivante

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right).$$

nous donnent alors directement l'existence globale de la solution.

Comme dans le cas d'un amortissement linéaire, afin d'établir l'explosion, nous avons juste besoin de montrer que dans le régime log-log la norme L^2 est non croissante, et que la croissance de l'énergie (resp le moment cinétique) est inférieure à $\frac{1}{\lambda^2}$ (resp $\frac{1}{\lambda}$) où $\lambda \sim \frac{1}{\|\nabla u\|_{L^2}}$.

Pour cela, nous montrons que dans le régime log-log, les croissances de l'énergie et du moment cinétique sont bornées par:

$$E(u(t)) \lesssim \log(\lambda(t)) \lambda(t)^{-\frac{pd}{2}}, \quad P(u(t)) \lesssim \log(\lambda(t)) \lambda(t)^{1-\frac{pd}{4}}.$$

Finalement le Théorème 0.3 se déduit d'une estimation a priori sur la norme de Strichartz critique: $\|u\|_{L^{\frac{4}{d}+2}([0,T]; L^{\frac{4}{d}+2}(\mathbb{R}^d))}$.

Sur le caractère bien posé pour l'équation de Kadomtsev-Petviashvili-Burgers I

Nous considérons dans ce chapitre le problème de Cauchy pour l'équation dispersive-dissipative de Kadomtsev-Petviashvili-Burgers-I (KPBI) en deux dimension :

$$\begin{cases} (\partial_t u + u_{xxx} - u_{xx} + uu_x)_x - u_{yy} = 0, \\ u(0, x, y) = \phi(x, y). \end{cases} \quad (18)$$

Bourgain a développé une nouvelle méthode, précisée ensuite par Ginibre dans [Gin95], pour l'étude du problème de Cauchy liée aux équations non linéaires dispersives (cf. [Bou93c], [Bou93a], [Bou93b]). Cette méthode est basée essentiellement sur l'introduction d'un espace qui mesure la localisation de la transformée de Fourier des fonctions proches d'une parabole liée au symbole associé à la partie linéaire de l'équation considérée. Un tel espace est connu dans la littérature sous la désignation d'espace de Bourgain.

En introduisant un espace Bourgain associée à KPI usuel (lié uniquement à la partie dispersive du symbole linéaire dans l'équation KPBI), Molinet-Ribaud [MR02b] ont prouvé l'existence locale pour le problème de Cauchy associé à l'équation KPBI pour des données initiales dans $H^{s_1, s_2}(\mathbb{R}^2)$, $s_1 > 0$ et $s_2 \geq 0$. Ces solutions sont globales en temps lorsque les données initiales ont une énergie finie. Rappelons que l'énergie de l'équation KP-I, est donnée par:

$$E(\phi) = \frac{1}{2} \int (\phi_x)^2 + \frac{1}{2} \int (\partial_x^{-1} \phi_y)^2 - \frac{1}{6} \int \phi^3.$$

Kojok [Koj07] a prouvé l'existence locale et globale pour (18) pour des petites données initiales dans $L^2(\mathbb{R}^2)$. Dans ce chapitre, nous améliorons les résultats obtenus par Molinet-Ribaud et Kojok, en prouvant l'existence locale et globale pour KPBI pour des données initiales dans $H^{s,0}$ lorsque $s > -\frac{1}{2}$. Notons que cela prouve, en particulier, l'existence globale pour des données initiales arbitraires dans $L^2(\mathbb{R}^2)$ et évite d'imposer des contraintes sur les faibles fréquences des données initiales pour avoir une énergie finie.

Le nouvel ingrédient est une estimation trilinéaire pour KPI prouvé dans [IKT08]. En se référant à [MR02c], nous introduisons un espace Bourgain associé à KPBI. Cet espace est en fait l'intersection de l'espace introduit dans [Bou93c] et d'un espace de Sobolev lié à l'effet dissipatif. L'avantage de cet espace est qu'il contient à la fois les parties dissipative et dispersive du symbole linéaire de (18). A la fin de ce chapitre, nous prouvons aussi que notre résultat est optimal dans le sens où l'application $\phi \mapsto u$ de H^{s_1, s_2} dans $C([0, T]; H^{s_1, s_2})$ (si elle existe) ne peut jamais être régulière pour $s_1 < -\frac{1}{2}$ et $s_2 = 0$. Ce point de vue a été introduit par Bourgain. Notant que (18) n'admet pas un changement d'échelle laissant invariant l'équation, par contre $H^{-1/2, 0}$ est critique pour le changement d'échelle des équations de KP.

Résultats principaux

Dans le but d'expliquer les résultats principaux de ce chapitre, nous commençons par donner quelques notations essentielles. Nous définissons la transformation de Fourier pour une fonction $f = f(x, y)$ notée par $\hat{\cdot}$ de $\mathcal{S}'(\mathbb{R}^2)$ dans $\mathcal{S}'(\mathbb{R}^2)$ par :

$$\hat{f}(v) := \mathcal{F}_{x,y}(f)(v) = \int_{\mathbb{R}^2} e^{-i\langle(x,y),v\rangle} f(x, y) dx dy, \quad \forall f \in \mathcal{S}'(\mathbb{R}^2),$$

et pour une fonction $f = f(t, x, y)$ par:

$$\hat{f}(\tau, v) := \mathcal{F}_{t,x,y}(f)(\tau, v) = \int_{\mathbb{R}^3} e^{-i\langle(t,x,y),\tau,v\rangle} f(t, x, y) dt dx dy$$

et les espaces de Sobolev correspondant à l'espace (resp. espace-temps) H^{s_1, s_2} (resp H^{b, s_1, s_2}) par les normes suivantes :

$$H^{s_1, s_2}(\mathbb{R}^2) =: \{u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{H^{s_1, s_2}(\mathbb{R}^2)} < +\infty\}, \quad (19)$$

$$H^{b, s_1, s_2}(\mathbb{R}^2) =: \{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{H^{b, s_1, s_2}(\mathbb{R}^3)} < +\infty\} \quad (20)$$

où,

$$\|u\|_{H^{s_1, s_2}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(v)|^2 dv, \quad (21)$$

$$\|u\|_{H^{b, s_1, s_2}}^2 = \int_{\mathbb{R}^2} \langle \tau \rangle^b \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\mathcal{F}_{t,x,y}(u)(\tau, v)|^2 dv d\tau, \quad (22)$$

et $v = (\xi, \eta)$. Soit $U(\cdot)$ le groupe unitaire dans H^{s_1, s_2} , $s_1, s_2 \in \mathbb{R}$, définissant l'évolution libre de l'équation (KP-I), qui est donnée par

$$U(t) = \exp(itP(D_x, D_y)), \quad (23)$$

où $P(D_x, D_y)$ est le multiplicateur de Fourier de symbole $P(\xi, \eta) = \xi^3 - \eta^2/\xi$. Par la transformation de Fourier, la partie linéaire de l'équation (18) peut s'écrire comme :

$$i(\tau - \xi^3 - \eta^2/\xi) + \xi^2 =: i(\tau - P(\eta, \xi)) + \xi^2. \quad (24)$$

INTRODUCTION

Nous avons besoin de localiser notre solution, et l'idée de Bourgain était de considérer cette localisation en définissant l'espace $X^{b,s}$ muni de la norme:

$$\|u\|_{X^{b,s_1,s_2}} = \|\langle i(\tau - P(\eta, \xi)) + \xi^2 \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \tilde{u}(\tau, \xi, \eta)\|_{L^2(\mathbb{R}^3)}. \quad (25)$$

Nous aurons besoin aussi de définir la décomposition de Littlewood-Paley. Soit $\eta \in C_0(\mathbb{R})$ telle que $\eta \geq 0$, $\text{supp } \eta \subset [-2, 2]$, $\eta = 1$ sur $[-1, 1]$. On définit ensuite $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$.

Les variables N, L sont en dyadique, i.e. ces variables sont de la forme $N = 2^j$, $j \in \mathbb{Z}$ et $L = 2^l$, $l \in \mathbb{N}$. On fixe $\varphi_N(\xi) = \varphi(\frac{\xi}{N})$ et l'opérateur P_N défini par $\mathcal{F}_x(P_N u) = \varphi_N \mathcal{F}_x(u)$. On introduit $\psi_L(\tau, \zeta) = \varphi_L(\tau - P(\zeta))$ et pour $u \in \mathcal{S}(\mathbb{R}^2)$,

$$\mathcal{F}_x(P_N u(t))(\xi) = \varphi_N(\xi) \mathcal{F}_x(u)(t, \xi), \quad \mathcal{F}(Q_L u)(\tau, \xi, \eta) = \psi_L(\tau, \zeta) \mathcal{F}(u)(\tau, \xi); L > 1$$

et $\mathcal{F}(Q_1 u)(\tau, \xi, \eta) = \eta(\tau - P(\zeta)) \mathcal{F}(u)(\tau, \xi)$. Grosso modo, l'opérateur P_N localise dans l'anneau $\{|\xi| \sim N\}$ et Q_L localise dans la région $\{\langle \tau - P(\zeta) \rangle \sim L\}$. On note $P_N u$ par u_N , $Q_L u$ par u_L et $P_N(Q_L u)$ par $u_{N,L}$.

Pour $T \geq 0$, nous considérons les espaces de restriction de Bourgain X_T^{b,s_1,s_2} muni de la norme

$$\|u\|_{X_T^{b,s_1,s_2}} = \inf_{w \in X^{b,s_1,s_2}} \{ \|w\|_{X^{b,s_1,s_2}}, w(t) = u(t) \text{ sur } [0, T] \}.$$

Nous utilisons également l'espace de Lebesgue $L_{t,x}^{p,q}$ munie de la norme

$$\|u\|_{L_{t,x}^{q,r}} = \left\| \|u\|_{L_x^r} \right\|_{L_t^q},$$

et nous utiliserons la notation $L_{t,x}^2$ pour $L_{t,x}^{2,2}$.

Nous désignons par $W(\cdot)$ le semi groupe associé à l'évolution libre de l'équation (KPB),

$$\mathcal{F}_x(W(t)\phi) = \exp(itP(\xi, \eta) - |\xi|^2 t) \hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \geq 0. \quad (26)$$

Nous prolongeons W en un opérateur linéaire défini sur tout l'axe réel \mathbb{R} de la façon suivante:

$$\mathcal{F}_x(W(t)\phi) = \exp(itP(\xi, \eta) - |\xi|^2 |t|) \hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \in \mathbb{R}. \quad (27)$$

Finalement, la formule de représentation intégrale de Duhamel permet de réécrire l'équation (18) comme

$$u(t) = W(t)\phi - \frac{1}{2} \int_0^t W(t-t') \partial_x(u^2(t')) dt', \quad t \geq 0. \quad (28)$$

Dans le but de montrer notre résultat d'existence, nous appliquerons un argument de point fixe à l'extension de (28), qui est défini sur tout l'axe réel par:

$$u(t) = \psi_T(t) [W(t)\phi - L(\partial_x(\psi_T^2 u^2))(x, t)] =: M(u(t)), \quad (29)$$

où $t \in \mathbb{R}$, ψ_T indique une fonction plateau comme suit:

$$\psi \in C_0^\infty(\mathbb{R}), \quad \text{supp } \psi \subset [-2, 2], \quad \psi = 1 \text{ sur } [-1, 1], \quad (30)$$

$\psi_T(\cdot) = \psi(\cdot/T)$, et

$$L(f)(x, t) = W(t) \int e^{ix\xi} \frac{e^{it\tau} - e^{-|t|\xi^2}}{it + \xi^2} \mathcal{F}(W(-t)f)(\xi, \tau) d\xi d\tau. \quad (31)$$

On peut voir facilement que

$$\chi_{\mathbb{R}_+}(t)\psi(t)L(f)(x, t) = \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t - \tau)f(\tau)d\tau. \quad (32)$$

En effet, en prenant $w = W(\cdot)f$, la quantité à droite de (32) pourra être réécrite comme

$$W(t)\left(\chi_{\mathbb{R}_+}(t)\psi(t) \int e^{ix\xi} \frac{e^{it\tau} - e^{-t|\xi|^2}}{i\tau + \xi^2} \hat{w}(\xi, \tau') d\xi d\tau'\right).$$

Dans [MR02c], les auteurs ont effectué le processus d'itération dans l'espace $X^{s,b}$ muni de la norme:

$$\|u\|_{X^{b,s_1,s_2}} = \|\langle i(\tau - P(\nu)) + \xi^2 \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{u}(\tau, \nu)\|_{L^2(\mathbb{R}^3)},$$

qui tient compte de l'effet mixte dispersif et dissipatif l'équation. Nous allons plutôt travailler dans sa version Besov $X^{s,b,q}$ (avec $q = 1$) munie de la norme

$$\|u\|_{X^{b,s,0,q}} = \left(\sum_N \left[\sum_L \langle L + N^2 \rangle^{bq} \langle N \rangle^{sq} \|P_N Q_L u\|_{L^2_{x,y,t}}^q \right]^{\frac{2}{q}} \right)^{\frac{1}{2}}.$$

Remark 0.4. *Il est clair maintenant que si u résout (29) alors u est une solution de (28) sur $[0, T]$, $T < 1$. Donc il suffit de résoudre (29) pour un temps petit.*

Dans ce cadre nous pouvons maintenant énoncer nos principaux résultats:

Theorem 0.5. *Soient $s_1 > -1/2$ et $\phi \in H^{s_1,0}$. Alors il existe un temps $T = T(\|\phi\|_{H^{-\frac{1}{2},0}}) > 0$ et une solution unique u de (18) dans*

$$Y_T = X_T^{1/2,s_1,0,1}. \quad (33)$$

L'application $\phi \mapsto u$ est C^∞ d'une boule de $H^{s_1,0}$ de rayon $R > 0$ dans $Y_{T(R)}$.

De plus, dans le cas de données initiales à valeurs réelles, la solution est globale en temps et appartient à $C(\mathbb{R}_+; H^{s_1,0}) \cap C(\mathbb{R}_+^; H^{\infty,0})$. \square*

Remark 0.6. *En procédant comme dans [MR01], il est facile de voir que le théorème (0.5) reste vrai en remplaçant $H^{s_1,0}$, $X_T^{1/2,s_1,0,1}$ et $H^{\infty,0}$ respectivement par H^{s_1,s_2} , $X_T^{1/2,s_1,s_2,1}$ et H^{∞,s_2} , avec $s_2 \geq 0$.*

Le deuxième résultat concerne le caractère mal posé de l'équation (18) qui assure que notre résultat d'existence est optimal dans un certain sens.

Theorem 0.7. *Etant donnée $s < -1/2$, il n'existe aucun temps $T > 0$ tel que l'équation (18) admette une unique solution dans $C([0, T], H^{s,0})$ pour toute donnée initiale dans une boule de $H^{s,0}(\mathbb{R}^2)$ centrée à l'origine et telle que l'application*

$$\phi \mapsto u \quad (34)$$

soit C^2 -différentiable à l'origine de $H^{s,0}$ dans $C([0, T], H^{s,0})$. \square

Principe de la preuve.

Caractère bien posé

On résout d'abord l'équation intégrale (29) dans un intervalle du temps $[0, T]$ T assez petit, par une méthode de point fixe consistant à montrer que l'opérateur M défini dans (29) est une contraction dans une boule de $X_T^{1/2, s_1, s_2}$ pour $s_1 > -1/2$, $s_2 \geq 0$. La méthode donne en même temps l'existence et l'unicité de la solution dans Y_T et la continuité de l'application $\phi \mapsto u$ de H^{s_1, s_2} dans Y_T . Si cette situation est réalisée, on dira que le problème (18) est localement bien posé dans H^{s_1, s_2} . Le principe de la preuve du résultat d'existence locale tient en deux étapes:

Première étape: Dans le but d'appliquer un argument standard de point fixe, une approche classique sera d'analyser premièrement les deux termes: terme libre et terme de force de l'équation intégrale (29). Une première étape consiste donc à montrer, à l'aide de l'analyse de Fourier, que l'application $\phi \mapsto \psi(t)W(t)\phi$ est bornée de H^{s_1, s_2} dans $X^{1/2, s_1, s_2}$ pour $s_1, s_2 \in \mathbb{R}$ et que l'opérateur L est borné de l'espace auxiliaire $X^{-1/2, s_1, s_2}$ dans $X^{1/2, s_1, s_2}$ pour $s_1, s_2 \in \mathbb{R}$. De plus, pour obtenir des informations sur la régularité de la solution, on montre que l'opérateur linéaire L admet des effets régularisants.

Deuxième étape: La préparation des estimations linéaires permet de concentrer toute la difficulté sur les estimations non-linéaires. Une deuxième étape nécessite de montrer des estimations bilinéaires assurant que l'application $(u, v) \mapsto \partial_x(uv)$ est bornée et lipschitzienne de $X^{1/2, s_1, s_2} \times X^{1/2, s_1, s_2}$ dans $X^{-1/2, s_1, s_2}$. Dans le but d'établir de telles estimations nous avons besoin de l'estimation dispersive prouvée pour KPI par Ionescu-Kenig-Tataru et de quelques estimations de Strichartz. Ce type d'estimations caractérise les propriétés dispersives de l'équation.

Caractère bien posé global

L'existence globale vient si on réussit à étendre les solutions pour tout $t \in \mathbb{R}_+$ en itérant le résultat d'existence locale. Cette extension requiert un contrôle a priori de la norme de u dans H^{s_1, s_2} . Si cette situation est réalisée, on dira que le problème de Cauchy est globalement bien posé. Par ailleurs, l'existence globale de la solution de notre équation vient de l'effet régularisant dû au terme dissipatif. En fait, l'effet régularisant lisse la solution dès que l'on décolle du temps initial. Ceci nous permet d'utiliser l'énergie de notre équation qui montre que $t \mapsto \|u(t)\|_{L^2}$ est non-croissante sur $]0, +\infty[$.

Le caractère mal-posé de KPBI

L'idée de base, dans la preuve du Théorème 0.7, repose sur la construction d'une suite de données initiales qui va nous assurer la non-régularité de l'application $\phi \mapsto u$ de $H^{s, 0}$ dans $C([0, T], H^{s, 0})$ pour $s < -1/2$. Nous raisonnons par l'absurde, nous supposons donc que l'application est C^2 , par conséquent nous pouvons utiliser un développement de Taylor autour de l'origine. Dans ces conditions, un choix convenable d'une suite de données initiales $(\phi_N)_N$, $N \gg 1$, nous permet d'aboutir à une contradiction pour $s < -1/2$ en montrant que

$$1 \sim \|\phi_N\|_{H^{s, 0}}^2 \geq \|u_{2, N}(t)\|_{H^{s, 0}} \geq CN^{-1-2s}$$

où $u_{2, N}$ désigne le second terme du développement de Taylor utilisé qui est la dérivée seconde de u en zéro dans la direction de ϕ_N . Soulignons que $u_{2, N}$ est aussi la seconde itération associée à ϕ_N dans une méthode itérative appliquée à notre équation.

Sur les mesures invariantes pour l'équation d'Ostrovsky

Dans ce dernier chapitre, nous construisons une mesure invariante pour le système dynamique défini par l'équation d'Ostrovsky périodique :

$$\begin{cases} \partial_t u - u_{xxx} + \partial_x^{-1} u + uu_x = 0, \\ u(0, x) = u_0(x), \end{cases} \quad (\text{Ost})$$

où $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ et u est à valeurs réelles.

L'opérateur ∂_x^{-1} dans l'équation désigne la primitive de moyenne nulle par rapport à la variable x . Cet opérateur est défini pour les fonctions 2π -périodiques de moyenne nulle par $(\widehat{\partial_x^{-1} f})(n) = \frac{\hat{f}(n)}{in}$, $n \in \mathbb{Z}^*$.

Les mesures invariantes jouent un rôle important dans la théorie des systèmes dynamiques (DS). Il est bien connu que toute la théorie ergodique est basée sur ce concept. D'autre part, elles sont nécessaires dans diverses considérations physiques.

Notons qu'une application bien connue des mesures invariantes est le principe de récurrence de Poincaré: pour presque toutes les conditions initiales, un système dynamique conservatif va repasser au cours du temps aussi près que l'on veut de sa condition initiale, et ce de façon répétée.

Récemment, plusieurs articles [Ars84],[Zhi95],[Zhi94] ont été publiés sur les mesures invariantes pour des systèmes dynamiques non linéaires.

Une série infinie de mesures invariantes associées aux lois de conservation a été construite par Zhidkov [Zhi01] pour l'équation:

$$\partial_t u + u_{xxx} + uu_x = 0. \quad (\text{KdV})$$

En particulier, une mesure invariante associée à la conservation de l'énergie a été construite pour cette équation. L'équation (Ost) est une perturbation de l'équation (KdV), avec un terme non local et a été introduite par Ostrovskii [Ost78]. Cette équation peut être considérée comme un modèle faiblement non linéaire d'ondes longues, dans un référentiel en rotation, pour décrire la propagation des ondes sur la surface de l'eau.

Nous allons construire une mesure invariante associée à la conservation de l'énergie:

$$H(u(t)) = \frac{1}{2} \int (u_x)^2 + \frac{1}{2} \int (\partial_x^{-1} u)^2 - \frac{1}{6} \int u^3.$$

Résultats principaux

Dans le but d'expliquer nos résultats principaux de ce chapitre, nous commençons par donner quelques notations essentielles.

Nous définissons la transformée de Fourier sur le cercle par

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \exp(-inx) dx$$

et $L_0^2(\mathbb{T})$ par

$$L_0^2(\mathbb{T}) = \{u \in L^2(\mathbb{T}); \int_{\mathbb{T}} u dx = 0\},$$

et nous introduisons l'espace de Sobolev espaces H_0^s défini par :

$$H_0^s(\mathbb{T}) =: \{u \in S'(\mathbb{T}); \hat{u}(0) = 0 \text{ et } \|u\|_{H^s} < +\infty\}, \quad (35)$$

où,

$$\|u\|_{H^s} = (2\pi)^{\frac{1}{2}} \|\langle \cdot \rangle^s \hat{u}\|_{l^2}. \quad (36)$$

Nous allons rappeler brièvement la construction de mesures de Gauss sur un espace de Hilbert. Soit X un espace de Hilbert, et $\{e_k\}$ une base orthonormée dans X constituée par les vecteurs propres pour un opérateur $S = S^* > 0$ qui admet $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \dots \leq \lambda_k \leq \dots$ comme valeurs propres. Nous disons qu'un ensemble $M \subset X$ est cylindrique ssi:

$$M = \{x \in X; [(x, e_1), (x, e_2), \dots, (x, e_r)] \in F\}$$

pour un certain Borélien $F \subset \mathbb{R}^r$, $r \in \mathbb{N}$. Nous définissons la mesure w par:

$$w(M) = (2\pi)^{-\frac{r}{2}} \prod_{j=1}^r \lambda_j^{\frac{1}{2}} \int_F e^{-\frac{1}{2} \sum_{j=1}^r \lambda_j y_j^2} dy. \quad (37)$$

Nous pouvons facilement vérifier que la classe \mathbb{A} de tous les ensembles cylindriques est une algèbre sur laquelle la mesure w est additive.

w s'appelle dans ce cas, la mesure gaussienne centrée sur X associée à l'opérateur S^{-1} .

Maintenant nous allons donner une définition pour une mesure invariante:

Definition 0.8. Soient M un espace métrique séparable complet et une fonction $h : \mathbb{R} \times M \mapsto M$ telle que pour chaque t , h est un homéomorphisme de l'espace M dans lui-même satisfaisant les propriétés suivantes:

1. $h(0, x) = x$ pour tout $x \in M$.
2. $h(t, h(\tau, x)) = h(t + \tau, x)$ pour $t, \tau \in \mathbb{R}$ et $x \in M$.

Alors la fonction h est appelée un système dynamique dans l'espace M . Si μ est une mesure définie sur M telle que $\mu(\Omega) = \mu(h(\Omega, t))$ pour tous $\Omega \subset M$ et $t \in \mathbb{R}$, nous dirons que μ est une mesure invariante par le système dynamique h .

Maintenant nous allons énoncé nos principaux résultats:

Theorem 0.9. Pour tout $s \geq -1/2$, le problème (Ost) est localement bien posé dans $H_0^s(\mathbb{T})$. De plus, pour tout $u_0 \in H_0^s$, il existe $T = T(u_0) > 0$ tel que l'application $\phi \mapsto u$ soit C^∞ de $H_0^s(\mathbb{T})$ dans $C([0, T], H_0^s(\mathbb{T}))$. \square

Theorem 0.10. Le problème (Ost) est globalement bien posé dans $L_0^2(\mathbb{T})$. La mesure μ sur $L_0^2(\mathbb{T})$ définie pour tout borélien $\Omega \subset L_0^2$ par

$$\mu(\Omega) = \int_{\Omega} e^{-g(u)} w(du)$$

où w est la mesure de Gauss associée à l'opérateur $S^{-1} = (-\Delta + \Delta^{-1})^{-1}$, et $g(u) = \frac{1}{3} \int u^3 dx$ est le terme non linéaire de l'Hamiltonien, est une mesure invariante pour l'équation (Ost).

Chapter 1

Blowup for the Damped L^2 -Critical Nonlinear Schrödinger Equation

1.1 Introduction

In this chapter, we study the blowup of solutions to the Cauchy problem for the L^2 -critical damped nonlinear Schrödinger equations:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + iau = 0, (t, x) \in [0, \infty) \times \mathbb{R}^d, d = 1, 2, 3, 4. \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (1.1)$$

with initial data $u(0) = u_0 \in H^1(\mathbb{R}^d)$ and where $a > 0$ is the coefficient of friction. Equation (1.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics. It is known that the Cauchy problem for 1.1 is locally well-posed in $H^1(\mathbb{R}^d)$ (see Kato [Kat87] and also Cazenave [Caz03]): For any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T \in (0, \infty]$ and a unique solution $u(t)$ of (1.1) with $u(0) = u_0$ such that $u \in C([0, T); H^1(\mathbb{R}^d))$. Moreover, T is the maximal existence time of the solution $u(t)$ in the sense that if $T < \infty$ then $\lim_{t \rightarrow T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$.

Ohta [OT09] and Tsutsumi [Tsu84] studied the supercritical case ($|u|^p u$ with $p > \frac{4}{d}$) and showed that blow-up in finite time can occur, using the virial method. However this method does not seem to apply in the critical case. Therefore, even if numerical simulations suggest the existence of finite time blowup solutions in this case (see Fibich [FM01]), there does not exist any mathematical proof of blow-up in the critical case.

Let us notice that for $a = 0$ (1.1) becomes the L^2 -critical nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (1.2)$$

This equation (1.2) admits a number of symmetries in the energy space H^1 : if $u(t, x)$ is a solution to (1.2) then $\forall \lambda_0 \in \mathbb{R}$, so is $\lambda_0^{\frac{d}{2}} u(\lambda_0 x, \lambda_0^2 t)$. Note that the L^2 -norm is left invariant by the the scaling symmetry and thus L^2 is the critical space associated with this symmetry.

The evolution of (1.2) admits the following conservation laws in the energy space H^1 :

L^2 norm : $\|u(t, x)\|_{L^2} = \|u(0, x)\|_{L^2} = \|u_0(x)\|_{L^2}$.

1.1. INTRODUCTION

Energy : $E(u(t, x)) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4+2d} \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = E(u_0)$.

Kinetic momentum : $P(u(t)) = \text{Im} \left(\int \nabla u \bar{u}(t, x) \right) = P(u_0)$.

Special solutions play a fundamental role for the description of the dynamics of (1.2). They are the solitary waves of the form $u(t, x) = \exp(it)Q(x)$, where Q solves:

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (1.3)$$

Equation (1.3) is a standard nonlinear elliptic equation, that possesses a unique positive solution (see [BL83], [Lio84],[Kwo89]).

For $u_0 \in H^1$, a sharp criterion for global existence has been exhibited by Weinstein [Wei83]:

a) For $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution of (1.2) is global in H^1 . This follows from the conservation of the energy and the L^2 norm and the sharp Gagliardo-Nirenberg inequality:

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right).$$

b) There exists blow-up solutions emanating from initial data $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. This follows from the pseudo-conformal symmetry applied to the solitary waves. In the series of papers [MR02a, MR06], Merle and Raphael have studied the blowup for the L^2 -critical nonlinear Schrödinger equation (1.2) and have proven the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \quad (1.4)$$

This regime has the advantage to be stable with respect to H^1 -perturbation and with respect some perturbations of the equation.

Remark 1.1. Based on the works [MR02a, MR06] we have the following result:

Let u_0 the initial data $\in H^1(\mathbb{R}^d)$ with small super-critical mass:

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0 \quad (1.5)$$

with nonpositive Hamiltonian $E(u_0) < 0$, then the corresponding solution to (1.2) blows up in finite time with the log-log speed.

In the case of (1.1), there does not exist conserved quantities anymore. However, it is easy to prove that if u is a solution of (1.1) then:

$$\|u(t)\|_{L^2} = \exp(-at)\|u_0\|_{L^2}, t \in [0, T], \quad (1.6)$$

$$\frac{d}{dt} E(u(t)) = -a \left(\|\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \right) \quad (1.7)$$

and

$$|P(u(t))| = \exp(-2at)|P(u_0)|, t \in [0, T]. \quad (1.8)$$

In this paper, we will show:

1.2. L^2 -CONCENTRATION

1. if $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, then the solution of (1.1) is global in H^1 .
2. The existence of finite time blowup solutions.

More precisely, we have the following theorem:

Theorem 1.2. *Let u_0 in $H^1(\mathbb{R}^d)$ with $d = 1, 2, 3, 4$:*

1. *if $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ then the solution of (1.1) is global in H^1 .*
2. *There exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in]0, \delta_0[$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (1.1) blows up in finite time in the log-log regime.*

To show the existence of the explosive solutions, we will put us in the log-log regime described by Merle and Raphael.

The global existence will be proved thanks to a L^2 -concentration phenomenon (see Proposition 1.4 in the next section).

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1.2 L^2 -concentration

In this section, we prove assertion (1) of Theorem 1.2 by extending the proof of the L^2 -concentration phenomenon, proved by Ohta and Todorova [OT09] in the radial case, to the non radial case.

Hmidi and Keraani showed in [HK05] the L^2 -concentration for the equation (1.2) without the hypothesis of radiality, using the following theorem:

Theorem 1.3. *Let $(v_n)_n$ be a bounded family of $H^1(\mathbb{R}^d)$, such that:*

$$\limsup_{n \rightarrow +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \|v_n\|_{L^{\frac{d}{d+2}}} \geq m. \quad (1.9)$$

Then, there exists $(x_n)_n \subset \mathbb{R}^d$ such that:

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly,}$$

$$\text{with } \|V\|_{L^2(\mathbb{R}^d)} \geq \left(\frac{d}{d+4}\right)^{\frac{d}{4}} \frac{m^{\frac{d}{2}+1}}{M^{\frac{d}{2}}} \|Q\|_{L^2(\mathbb{R}^d)}.$$

Now we have the following theorem:

Theorem 1.4. *Assume that $u_0 \in H^1(\mathbb{R}^d)$, and suppose that the solution of (1.1) with $u(0) = u_0$ blows up in finite time $T \in (0, +\infty)$. Then, for any function $w(t)$ satisfying $w(t) \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \rightarrow \infty$ as $t \rightarrow T$, there exists $x(t) \in \mathbb{R}^d$ such that,*

$$\limsup_{t \rightarrow T} \|u(t)\|_{L^2(|x-x(t)| < w(t))} \geq \|Q\|_{L^2(\mathbb{R}^d)}.$$

1.2. L^2 -CONCENTRATION

To show this theorem we shall need the following lemma:

Lemma 1.5. *Let $T \in (0, +\infty)$, and assume that a function $F : [0, T) \mapsto (0, +\infty)$ is continuous, and $\lim_{t \rightarrow T} F(t) = +\infty$. Then, there exists a sequence $(t_k)_k$ such that $t_k \rightarrow T$ and*

$$\lim_{t_k \rightarrow T} \frac{\int_0^{t_k} F(\tau) d\tau}{F(t_k)} = 0. \quad (1.10)$$

For the proof see [OT09].

Proof of Theorem 1.4:

By the energy identity (1.7), we have

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T[. \quad (1.11)$$

Where $K(u(t)) = \|\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{4}{d+2}}}^{\frac{4}{d+2}}$, and by the Gagliardo-Nirenberg inequality and (1.6), we have:

$$\begin{aligned} |K(u(t))| &\leq \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + \|u(t)\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \\ &\leq \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + C \|u(t)\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq (1 + C \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}) \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 \end{aligned}$$

for all $t \in [0, T[$. Moreover, we have $\lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$, thus by Lemma 1.5, there exists a sequence $(t_k)_k$ such that $t_k \rightarrow T$ and

$$\lim_{k \rightarrow \infty} \frac{\int_0^{t_k} K(u(\tau)) d\tau}{\|\nabla u(t_k)\|_{L^2(\mathbb{R}^d)}^2} = 0. \quad (1.12)$$

Let

$$\rho(t) = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}} \quad \text{and} \quad v(t, x) = \rho^{\frac{d}{2}} u(t, \rho x)$$

and $\rho_k = \rho(t_k)$, $v_k = v(t_k, \cdot)$. The family $(v_k)_k$ satisfies

$$\|v_k\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(\mathbb{R}^d)} = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

By (1.11) and (1.12), we have

$$E(v_k) = \rho_k^2 E(u_0) - a \rho_k^2 \int_0^{t_k} K(u(\tau)) d\tau \rightarrow 0, \quad (1.13)$$

which yields

$$\|v_k\|_{L^{\frac{4}{d+2}}}^{\frac{4}{d+2}} \rightarrow \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2. \quad (1.14)$$

1.3. STRATEGY OF THE PROOF OF THEOREM 1.2 PART 2.

The family $(v_k)_k$ satisfies the hypotheses of Theorem 1.3 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad \text{and} \quad M = \|\nabla Q\|_{L^2(\mathbb{R}^d)},$$

thus there exists a family $(x_k)_k \subset \mathbb{R}^d$ and a profile $V \in H^1(\mathbb{R}^d)$ with $\|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$, such that,

$$\rho_k^{\frac{d}{2}} u(t_k, \rho_k \cdot + x_k) \rightharpoonup V \in H^1 \quad \text{weakly.} \quad (1.15)$$

Using (1.15), $\forall A \geq 0$

$$\liminf_{n \rightarrow +\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{B(0,A)} |V|^2 dx,$$

but $\lim_{n \rightarrow +\infty} \frac{w(t_n)}{\rho_n} = +\infty$ thus $\frac{w(t_n)}{\rho_n} > A, \rho_n A < w(t_n)$. This gives immediately:

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq w(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx.$$

This is true for all $A > 0$ thus :

$$\liminf_{t \rightarrow T} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq w(t)} |u(t, x)|^2 dx \geq \int Q^2. \quad (1.16)$$

But for every $t \in [0, T[$, $y \mapsto \int_{|x-y| \leq w(t)} |u(t, x)|^2 dx$ is continuous and goes to 0 at infinity,

thus the sup is reached in a point $x(t) \in \mathbb{R}^d$, $\sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq w(t)} |u(t, x)|^2 dx = \int_{|x-x(t)| \leq w(t)} |u(t, x)|^2 dx$ and

Theorem 1.4 is proved.

Now the part one of Theorem 1.2 is a consequence of Theorem 1.4 and (1.6).

1.3 Strategy of the proof of Theorem 1.2 part 2.

We look for a solution of (1.1) such that for t close enough to blowup time, we shall have the following decomposition:

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} (Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad (1.17)$$

for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}$, here $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$, $b = -\lambda \lambda_t$ and the profiles Q_b are suitable deformations of Q related to some extra degeneracy of the problem.

Now we take u_0 in H^1 such that u_0 admits the following controls:

1.3. STRATEGY OF THE PROOF OF THEOREM 1.2 PART 2.

1. Control of the scaling parameter:

$$0 < b(0) \ll 1 \quad \text{and} \quad 0 < \lambda(0) < e^{-e^{\frac{2\pi}{3b(0)}}}. \quad (1.18)$$

2. L^2 control of the excess of mass:

$$\|\epsilon(0)\|_{L^2} \ll 1. \quad (1.19)$$

3. H^1 smallness of $\epsilon(0)$:

$$\int |\nabla \epsilon(0)|^2 + \int |\epsilon(0)|^2 e^{-|v|} < \Gamma_{\frac{3}{4}, b(0)}, \quad (1.20)$$

where $\Gamma_{b(0)} \sim e^{-\frac{c}{b(0)}}$.

4. Control of the energy and momentum:

$$|E(u_0)| < \frac{1}{\sqrt{\lambda(0)}} \quad (1.21)$$

$$|P(u_0)| < \frac{1}{\sqrt{\lambda(0)}}. \quad (1.22)$$

Remark 1.6. To prove that there exists u_0 in $H^1(\mathbb{R}^d)$ satisfying (1.18)-(1.22), we take \tilde{u}_0 in $H^1(\mathbb{R}^d)$ an initial data such that the corresponding solution to (1.2) blows up in the log-log regime as described by Merle and Raphael. Then from [MR06] there exists a time t_0 such that $\tilde{u}(t_0)$ admits a geometrical decomposition:

$$\tilde{u}(t_0, x) = \frac{1}{\lambda(t_0)^{\frac{d}{2}}} (Q_{b(t_0)} + \epsilon(t_0)) \left(\frac{x - x(t_0)}{\lambda(t_0)} \right) e^{i\gamma(t)}$$

such that (1.18)-(1.20) hold. Moreover by conservation of the Hamiltonian and the Kinetic momentum:

$$|E(\tilde{u}(t_0))| + |P(\tilde{u}(t_0))| = |E(\tilde{u}_0)| + |P(\tilde{u}_0)| \leq \frac{1}{\sqrt{\|\nabla \tilde{u}(t_0)\|_{L^2}}}$$

for t_0 close enough to blowup time, and hence (1.21) and (1.22) hold. We take $u_0 = \tilde{u}(t_0)$.

These conditions will be denoted by C.I. Now we have the following theorem:

Theorem 1.7. Let $u_0 \in H^1$ satisfying C.I, then for $0 < a < a_0$, a_0 small the corresponding solution $u(t)$ of (1.1) blows up in finite time in the log-log regime.

The set of initial data satisfying C.I is open in H^1 , using the continuity with regard to the initial data and the parameters, we can prove the following corollary(see the proof in section 5):

Corollary 1.8. Let $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (1.2) blows up in the loglog regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (1.1) with the initial data v_0 blowup in finite time.

Remark 1.9. 1. Combining Theorem 1.7 and Remark 1.1, we deduce that for $u_0 \in H^1(\mathbb{R}^d)$, having negative energy and satisfying (1.5), there exists $a_0 > 0$ such that the corresponding solution of (1.1) blows up in finite time providing $a \leq a_0$.

1.4. CHOICE OF THE BLOW UP PROFILE

2. It is easy to check that if u is a solution of NLS_a (Equation (1.1)), then $\lambda^{\frac{d}{2}}u(\lambda^2t, \lambda x)$ is a solution of NLS_{λ^2a} . Therefore Corollary 1.8 ensures the existence for any $a > 0$ of explosive solutions emanating from an initial data $u_{0,a} \in H^1$, where $u_{0,a}$ satisfies:

$$\|Q\|_{L^2} < \|u_{0,a}\|_{L^2} = \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0.$$

After the decomposition (1.17) of u , the log-log regime corresponds to the following asymptotic controls

$$b_s \sim Ce^{-\frac{c}{b}}, \quad -\frac{\lambda_s}{\lambda} \sim b \quad (1.23)$$

and

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}, \quad (1.24)$$

where we have introduced the rescaled time $\frac{ds}{dt} = \frac{1}{\lambda^2}$.

In fact, (1.24) is partly a consequence of the preliminary estimate:

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}} + \lambda^2 E(t). \quad (1.25)$$

One then observes that in the log-log regime, the integration of the laws (1.23) yields

$$\lambda \sim e^{-e^{\frac{c}{b}}} \ll e^{-\frac{c}{b}}, \quad b(t) \rightarrow 0, \quad t \rightarrow T. \quad (1.26)$$

Hence, the term involving the conserved Hamiltonian is asymptotically negligible with respect to the leading order term $e^{-\frac{c}{b}}$ which drives the decay (1.25) of b . This was a central observation made by Planchon and Raphael in [PR07]. In fact, any growth of the Hamiltonian algebraically below $\frac{1}{\lambda^2}$ would be enough. In this paper, we will prove that in the log-log regime, the growth of the energy is estimated by:

$$E(u(t)) \lesssim (\log(\lambda(t)))^2. \quad (1.27)$$

We deduce from (1.25) that:

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{c}{b}}. \quad (1.28)$$

An important feature of this estimate of H^1 flavor is that it relies on a flux computation in L^2 . This allows one to recover the asymptotic laws for the geometrical parameters (1.23) and to close the bootstrap estimates of the log-log regime.

This paper is organized as follows. In Sect. 1.4 and 1.5, we recall some nonlinear objects involved in the H^1 description of the log-log regime and set up the bootstrap argument, see Proposition 1.12. In Sect. 1.6, we will control in the bootstrap regime the growth of the energy and momentum, see Lemma 1.13. In Sect. 1.7, we close the bootstrap estimates and conclude the proof of Theorem 1.2 (part 2). Finally, the proof of Corollary 1.8 is postponed at the end of Section 1.8.

1.4 Choice of the blow up profile

Let us introduce the rescaled time :

$$s(t) = \int_0^t \frac{d\alpha}{\lambda^2(\alpha)}.$$

1.4. CHOICE OF THE BLOW UP PROFILE

If $u(t)$ blows up in finite time T , then $\frac{1}{\sqrt{T-t}} \lesssim \|\nabla u\|_{L^2}$ and one always has:
 $s([0, T[) = \mathbb{R}^+$.

Let us set:

$$v(s, y) = e^{-iy(t)} \lambda^{\frac{d}{2}} u(t, \lambda(t)x + x(t)),$$

where $y = \lambda(t)x + x(t)$, note that:

$$v_s = -iy_s v + \frac{d}{2} \frac{\lambda_s}{\lambda} v + e^{-iy(t)} \lambda^{2+\frac{d}{2}} u_t + \frac{\lambda_s}{\lambda} y \cdot \nabla v + \frac{x_s}{\lambda} \cdot \nabla v.$$

$$\Delta v = e^{-iy(t)} \lambda^{2+\frac{d}{2}} \Delta u(t, \lambda(t)x + x(t)) \quad \text{and} \quad v|v|^{\frac{4}{d}} = e^{-iy(t)} \lambda^{2+\frac{d}{2}} u|u|^{\frac{4}{d}}.$$

Now $u(t, x)$ solves (1.1) on $[0, T[$ iff $v(s, y)$ solves: $\forall s \geq 0$,

$$iv_s + \Delta v - v + v|v|^{\frac{4}{d}} = i \frac{\lambda_s}{\lambda} \left(\frac{d}{2} v + y \cdot \nabla v \right) + i \frac{x_s}{\lambda} \cdot \nabla v + \tilde{\gamma}_s v, \quad (1.29)$$

where $\tilde{\gamma}_s = -\gamma_s - 1 - ia\lambda^2$, and a is the coefficient of friction. Now $v(s, y) = Q(y) + \epsilon(s, y)$ and we linearize (1.29) close to Q . The obtained system has the form:

$$i\epsilon_s + L\epsilon = i \frac{\lambda_s}{\lambda} \left(\frac{d}{2} Q + x \cdot \nabla Q \right) + \tilde{\gamma}_s Q + i \frac{x_s}{\lambda} \cdot \nabla Q + R(\epsilon), \quad (1.30)$$

where $R(\epsilon)$ is formally quadratic in ϵ , and $L = (L_+, L_-)$ is the matrix linearized operator closed to Q which has components:

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{d}\right) Q^{\frac{4}{d}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{d}}.$$

A standard approach is to think of equation (1.30) in the following way: it is essentially a linear equation forced by terms depending on the law for the geometrical parameters.

Let us observe that the key geometrical parameter is λ which measures the size of the solution. Let us then set

$$b = -\frac{\lambda_s}{\lambda},$$

and study the simpler version of (1.29):

$$iv_s + \Delta v - v + v|v|^{\frac{4}{d}} + ib \left(\frac{d}{2} v + y \cdot \nabla v \right) = 0.$$

We look for solutions of the form $v(s, y) = \bar{Q}_{b(s)}(y)$ where the mapping $b \rightarrow Q_b$ and the laws for $b(s)$ are the unknown. We take b uniformly small and $Q_b|_{b=0} = Q$. Now injecting $v(s, y)$ into the equation, we get:

$$i \frac{db}{ds} \left(\frac{\partial \bar{Q}_b}{\partial b} \right) + \Delta \bar{Q}_{b(s)} - \bar{Q}_{b(s)} + ib(s) \left(\frac{d}{2} \bar{Q}_{b(s)} + y \cdot \nabla \bar{Q}_{b(s)} \right) + \bar{Q}_{b(s)} |\bar{Q}_{b(s)}|^{\frac{4}{d}} = 0.$$

We set $\bar{P}_{b(s)} = e^{i \frac{b(s)}{4}} |y|^2 \bar{Q}_{b(s)}$ and solve:

$$i \frac{db}{ds} \left(\frac{\partial \bar{P}_b}{\partial b} \right) + \Delta \bar{P}_{b(s)} - \bar{P}_{b(s)} + \left(\frac{db}{ds} + b^2(s) \right) \frac{|y|^2}{4} \bar{P}_{b(s)} + \bar{P}_{b(s)} |\bar{P}_{b(s)}|^{\frac{4}{d}} = 0. \quad (1.31)$$

Two remarkable solutions to (1.31) can be obtained as follows:

- Take $b(s) = 0$ and $\bar{P}_{b(s)} = Q$, that is the ground state itself.
- Take $b(s) = b$ and $\bar{P}_{b(s)} = \bar{P}$ for some non zero constant b and \bar{P}_b satisfying:

$$\Delta \bar{P}_b - \bar{P}_b + b^2(s) \frac{|y|^2}{4} \bar{P}_b + \bar{P}_b |\bar{P}_b|^{\frac{4}{d}} = 0. \quad (1.32)$$

The solutions to this non linear elliptic equation are those who produce the explicit self similar profiles solutions to this equation:

$$\Delta \bar{Q}_b - \bar{Q}_b + \bar{Q}_b |\bar{Q}_b|^{\frac{4}{d}} + ib \left(\frac{d}{2} \bar{Q}_b + y \cdot \nabla \bar{Q}_b \right) = 0. \quad (1.33)$$

A simple way to see this is to recall that we have set $b = -\frac{\lambda_s}{\lambda}$. Hence from $\frac{ds}{dt} = \frac{1}{\lambda^2}$,

$$b = -\frac{\lambda_s}{\lambda} = -\lambda \lambda_t \text{ ie } \lambda(t) = \sqrt{2b(T-t)},$$

which is the scaling law for the blow up speed.

1.5 Setting of the bootstrap

In this section, we recall some fundamental nonlinear objects central to the description of the log-log regime. We then set up the bootstrap argument, in the heart of the proof of Theorem 1.2. The conditions C.I will be to initialize the bootstrap.

Based on Propositions 8 and 9 of [MR04], we claim:

Proposition 1.10. *There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true: for all $0 < \eta < \eta^*$, there exist constants $\nu^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \rightarrow 0$ such that for all $|b| < b^*(\eta)$, setting*

$$R_b = \frac{2}{b} \sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta} R_b,$$

$B_{R_b} = \{y \in \mathbb{R}^d, |y| \leq R_b\}$, there exists a unique radial solution $\bar{Q}_b \in L^2(B(0, R))$ to

$$\begin{cases} \Delta \bar{Q}_b - \bar{Q}_b + \bar{Q}_b |\bar{Q}_b|^{\frac{4}{d}} + ib \left(\frac{d}{2} \bar{Q}_b + y \cdot \nabla \bar{Q}_b \right) = 0, \\ \bar{P}_b = \bar{Q}_b e^{i \frac{b|y|^2}{4}} > 0 \text{ in } B_{R_b}, \\ \bar{Q}_b(0) \in (Q(0) - \nu^*(\eta), Q(0) + \nu^*(\eta)), \bar{Q}_b(R_b) = 0. \end{cases} \quad (1.34)$$

Moreover, let ϕ_b be a smooth radially symmetric cut-off function such that $\phi_b(x) = 0$ for $|x| \geq R_b$ and $\phi_b(x) = 1$ for $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$ and set

$$Q_b(r) = \bar{Q}_b(r) \phi_b(r)$$

then

$$Q_b \rightarrow Q \text{ as } b \rightarrow 0$$

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in $L^2(\mathbb{R}^d)$, and Q_b satisfies

$$\Delta Q_b - Q_b + Q_b |Q_b|^{\frac{4}{d}} + ib \left(\frac{d}{2} Q_b + y \cdot \nabla Q_b \right) = -\Psi_b, \quad (1.35)$$

where $\Psi_b = 2\nabla\phi_b \nabla Q_b + Q_b(\Delta\phi_b) + iQ_b y \cdot \nabla\phi_b + (\phi_b^{1+\frac{4}{d}} - \phi_b)Q_b |Q_b|^{\frac{4}{d}}$,
with

$$\text{supp}(\Psi_b) \subset \{R_b^- \leq |y| \leq R_b\} \quad \text{and} \quad |\Psi_b|_{C^1} \leq e^{-\frac{c}{|b|}}.$$

Eventually, Q_b has supercritical mass:

$$\int |Q_b|^2 = \int Q^2 + c_0 b^2 + o(b^2) \quad \text{as} \quad b \rightarrow 0, \quad (1.36)$$

for some universal constant $c_0 > 0$.

The meaning of this proposition is that one can build localized Q_b on the ball B_{R_b} which is a smooth function of b and approximate Q in a very strong way as $b \rightarrow 0$. These profiles satisfy the self similar equation up to an exponentially small term Ψ_b supported around the turning point $\frac{2}{b}$. The proof of this proposition uses standard variational tools in the setting of non linear elliptic problems, and can be found in [MR04].

Now one can think of making a formal expansion of Q_b in terms of b , and the first term is non zero:

$$\frac{\partial Q_b}{\partial b} |_{b=0} = -\frac{i}{4} |y|^2 Q.$$

However, the energy of Q_b is degenerated in b at all orders:

$$|E(Q_b)| \leq e^{-\frac{C}{|b|}}, \quad (1.37)$$

for some universal constant $C > 0$.

Now given a well-localized function f , we set:

$$f_d = \frac{d}{2} f + y \cdot \nabla f \quad \text{and} \quad f_{dd} = (f_d)_d.$$

Note that integration by part yields:

$$(f_d, g)_{L^2} = -(f, g_d)_{L^2}.$$

We next introduce the outgoing radiation escaping the soliton core according to the following lemma(see Lemma 15 from [MR04]):

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Lemma 1.11. *(Linear outgoing radiation) There exist universal constants $C > 0$ and $\eta^* > 0$ such that $\forall 0 < \eta < \eta^*$, there exists $b^*(\eta) > 0$ such that $\forall |b| < b^*(\eta)$, the following holds true: there exists a unique radial solution ζ_b to*

$$\begin{cases} \Delta \zeta_b - \zeta_b + ib(\zeta_b)_d = -\Psi_b \\ \int |\nabla \zeta_b|^2 < +\infty. \end{cases} \quad (1.38)$$

Moreover, let

$$\Gamma_b = \lim_{|y| \rightarrow +\infty} |y|^d |\zeta_b(y)|^2, \quad (1.39)$$

then there holds

$$e^{-(1+c\eta)\frac{\pi}{|b|}} \leq \Gamma_b \leq e^{-(1-c\eta)\frac{\pi}{|b|}}. \quad (1.40)$$

We recall that the solution $u(t)$ admits the decomposition :

$$u(t, x) = \frac{1}{\lambda(t)} (Q_{b(t)} + \epsilon)(t, \frac{x-x(t)}{\lambda(t)}) e^{i\gamma(t)}$$

where the geometrical parameters are uniquely defined through some orthogonality conditions(see later): Let us assume the following uniform controls on $[0, T]$:

- Control of $b(t)$

$$b(t) > 0 \text{ and } b(t) < 10b(0). \quad (1.41)$$

- Control of λ :

$$\lambda(t) \leq e^{-e^{\frac{\pi}{100b(t)}}} \quad (1.42)$$

and the monotonicity of λ :

$$\lambda(t_2) \leq \frac{3}{2} \lambda(t_1), \forall 0 \leq t_1 \leq t_2 \leq T. \quad (1.43)$$

Let $k_0 \leq k_+$ be an integer and $T^+ \in [0, T]$ such that

$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0-1}}, \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+-1}} \quad (1.44)$$

and for $k_0 \leq k \leq k_+$, let t_k be a time such that

$$\lambda(t_k) = \frac{1}{2^k}, \quad (1.45)$$

then we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda^2(t_k). \quad (1.46)$$

- control of the excess of mass:

$$\int |\nabla \epsilon(t)|^2 + \int |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}. \quad (1.47)$$

The following proposition ensures that (1.42)-(1.47) determine a trapping region for the flow. \square

Proposition 1.12. *Assuming that (1.41)-(1.47) hold, then the following controls are also true:*

$$b > 0 \text{ and } b(t) < 5b(0). \quad (1.48)$$

$$\lambda(t) \leq e^{-\frac{\pi}{10b(t)}} \quad (1.49)$$

$$\lambda(t_2) \leq \frac{5}{4}\lambda(t_1), \forall \quad 0 \leq t_1 \leq t_2 \leq T \quad (1.50)$$

$$t_{k+1} - t_k \leq \sqrt{k}\lambda^2(t_k) \quad (1.51)$$

$$\int |\nabla \epsilon(t)|^2 + \int |\epsilon(t)|^2 e^{-|y|} \leq \Gamma \frac{2}{b(t)}. \quad (1.52)$$

1.6 Control of the energy and the kinetic momentum

We recall the Strichartz estimates. An ordered pair (q, r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 < q \leq \infty$. Let $r \geq 2$ and $2^+ < q \leq \infty$, we define the Strichartz norm of functions $u : [0, T] \times \mathbb{R}^d \mapsto \mathbb{C}$ by:

$$\|u\|_{S^0([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (1.53)$$

and

$$\|u\|_{S^1([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|\nabla u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (1.54)$$

We will sometimes abbreviate $S^i([0, T] \times \mathbb{R}^2)$ with S_T^i or $S^i[0, T]$, $i = 1, 2$. Let us denote the Hölder dual exponent of q by q' so that $\frac{1}{q} + \frac{1}{q'} = 1$. The Strichartz estimates may be expressed as:

$$\|u\|_{S_T^0} \lesssim \|u_0\|_{L^2} + \|(i\partial_t + \Delta)u\|_{L_t^{q'} L_x^{r'}} \quad (1.55)$$

where (q, r) is any admissible pair. Now we will derive an estimate on the energy, to check that it remains small with respect to λ^{-2} :

Lemma 1.13. *Assuming that (1.42)-(1.47) hold, then the energy and kinetic momentum are controlled on $[0, T^+]$ by:*

$$|E(u(t))| \lesssim (\log(\lambda(t)))^2, \quad (1.56)$$

$$|P(u(t))| \leq |P(u_0)|. \quad (1.57)$$

To prove this lemma, we shall need the following one:

Lemma 1.14. *Let u be a solution of (1.1) emanating for u_0 in H^1 . Then $u \in C([0, \Delta T], H^1)$ where $\Delta T = \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u_0\|_{H^1}^{-2}$, and we have the following control*

$$\|u\|_{S^0[0, \Delta T]} \leq 2 \|u_0\|_{L^2}, \quad \|u\|_{S^1[0, \Delta T]} \leq 2 \|u_0\|_{H^1(\mathbb{R}^d)}.$$

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proof: For all $v \in S(\mathbb{R}^d)$ we have:

$$\|v\|_{S^0[0,\Delta T]} \lesssim \|v(0)\|_{L^2} + \|(i\partial_t + \Delta)v\|_{L^q_{\Delta T} L^r_x},$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 < q < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

In particular,

$$\left\| \int_0^t e^{i(t-s)\Delta} |u|^{\frac{4}{d}} u \right\|_{S^1_t} \lesssim \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1_t L^2_x}.$$

Using the Hölder inequality we obtain:

$$\left(\int |u|^{\frac{8}{d}} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left(\int |u|^{\frac{2(4+d)}{d}} \right)^{\frac{2}{4+d}} \left(\int |\nabla u|^{\frac{2(4+d)}{d}} \right)^{\frac{d}{2(4+d)}}.$$

Integrating in time and applying again Hölder inequality we get:

$$\begin{aligned} \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([0,\Delta T])L^2(\mathbb{R}^d)} &\leq \left(\int \left(\int |u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{2}{4+d} \frac{4+d}{4}} dt \right)^{\frac{4}{4+d}} \\ &\quad \times \left(\int \left(\int |\nabla u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{d}{2(4+d)} \frac{4+d}{d}} dt \right)^{\frac{d}{4+d}}. \end{aligned}$$

Thus:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([0,\Delta T])L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([0,\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}^{\frac{4}{d}} \|\nabla u\|_{L^{\frac{4+d}{d}}([0,\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}.$$

But $(\frac{4+d}{d}, \frac{8+2d}{d})$ is admissible, thus we have:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([0,\Delta T])L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([0,\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}^{\frac{4}{d}} \|u\|_{S^1[0,\Delta T]}.$$

By Sobolev we have:

$$\begin{aligned} \|u\|_{L^{\frac{4+d}{d}}([0,\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} &\lesssim \|u\|_{L^{\frac{4+d}{d}}([0,\Delta T])H^{\frac{2d}{d+4}}(\mathbb{R}^d)} \\ &\leq (\Delta T)^{\frac{d}{d+4}} \|u\|_{L^\infty([0,\Delta T])H^{\frac{2d}{d+4}}(\mathbb{R}^d)}. \end{aligned}$$

Now by interpolation we obtain for $d = 1, 2, 3, 4$:

$$\|u\|_{L^{\frac{4+d}{d}}([0,\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} \leq (\Delta T)^{\frac{d}{d+4}} \|u\|_{L^\infty([0,\Delta T])L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{L^\infty([0,\Delta T])H^1(\mathbb{R}^d)}^{\frac{2d}{d+4}}.$$

But since according to (1.6), $\|u\|_{L^\infty([0,\Delta T])L^2(\mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}$, we finally get:

$$\|u\|_{S^1[0,\Delta T]} \leq \|u(t)\|_{H^1(\mathbb{R}^d)} + (\Delta T)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1_{\Delta T}}^{\frac{2d}{d+4}} \|u\|_{S^1[0,\Delta T]},$$

we deduce that for $\Delta T \leq C \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{d-4}{d}} \|u(t)\|_{H^1(\mathbb{R}^d)}^{-2}$:

$$\|u\|_{S^1[0,\Delta T]} \leq 2 \|u(t)\|_{H^1(\mathbb{R}^d)}.$$

In the same way, $\|u\|_{S^0[0,\Delta T]} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + (\Delta T)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[0,\Delta T]}^{\frac{2d}{d+4}} \|u\|_{S^0[0,\Delta T]}$,

but $(\Delta T)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[0,\Delta T]}^{\frac{2d}{d+4}} \leq \frac{1}{2}$ we obtain that:

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$$\|u\|_{S^0[0,\Delta T]} \leq 2\|u_0\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{S^1[0,\Delta T]} \leq 2\|u_0\|_{H^1(\mathbb{R}^d)}.$$

Let us return to the proof of the lemma 1.13:

According to (1.46) each interval $[t_k, t_{k+1}]$, can be divided into k intervals, $[\tau_k^j, \tau_k^{j+1}]$ such that the estimates of the previous lemma are true. From (1.7), we thus deduce that:

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \frac{d}{dt} E(u(t)) &\leq \int_{t_k}^{t_{k+1}} \int |u|^{\frac{4}{d}+2} \\ &\leq \sum_{j=1}^k \int_{\tau_k^j}^{\tau_k^{j+1}} \int |u|^{\frac{4}{d}+2} \\ &\leq \sum_{j=1}^k \|u\|_{S^0[\tau_k^j, \tau_k^{j+1}]}^{\frac{4}{d}+2}. \end{aligned}$$

Since $(\frac{4}{d} + 2, \frac{4}{d} + 2)$ is admissible. Using that $\|u\|_{S^0[0,\Delta T]} \leq 2\|u_0\|_{L^2(\mathbb{R}^d)}$ independantly of t , we obtain finally :

$$\int_{t_k}^{t_{k+1}} \frac{d}{dt} E(u(t)) \lesssim k.$$

Summing from k_0 to k_+ , we obtain

$$\int_0^{T^+} \frac{d}{dt} E(u(t)) \lesssim |\log(\lambda(T^+))|^2.$$

But $E(u(T^+)) = E(u(0)) + \int_0^T \frac{d}{ds} E(u(s)) ds$, since $|E(u(0))| \leq \frac{1}{\sqrt{\lambda(0)}}$ then we obtain $|E(u(T^+))| \lesssim |\log(\lambda(T^+))|^2$. This completes the proof of (1.56).

Now (1.57) follow directly from (1.8). For sake of completeness, let us prove (1.8):

Suppose first that $u(t)$ is very regular (for example $D(\mathbb{R}^d)$)

$$\begin{aligned} \frac{d}{dt} P(u(t)) &= \text{Im} \left(\int \bar{u}_t \nabla u dx + \int \bar{u} \nabla u_t \right) \\ &= \text{Im} \left(\int \bar{u}_t \nabla u dx - \int u_t \nabla \bar{u} \right) \\ &= \text{Im}(2i \text{Im} \int \bar{u}_t \nabla u) \\ &= 2 \text{Im} \int \bar{u}_t \nabla u dx = 2 \text{Im}(i \int \bar{i} \bar{u}_t \nabla u) = 2 \text{Re} \int \bar{i} \bar{u}_t \nabla u \\ &= 2 \text{Re} \left(\int (-\Delta \bar{u} - |u|^{\frac{4}{d}} \bar{u} + ia \bar{u}) \nabla u dx \right) \\ &= -2 \text{Re} \left(\int \Delta \bar{u} \nabla u \right) - 2 \text{Re} \int |u|^{\frac{4}{d}} \bar{u} \nabla u + 2a \text{Re} \int \bar{i} \bar{u} \nabla u. \end{aligned}$$

It is easy to prove that: $-2 \text{Re} \left(\int \Delta \bar{u} \nabla u \right) = \int \nabla |\nabla u|^2$, and $-2 \text{Re} \int |u|^{\frac{4}{d}} \bar{u} \nabla u = -\frac{d}{d+2} \int \nabla(|u|^{\frac{4}{d}+2})$.

But $2a \text{Re} \int \bar{i} \bar{u} \nabla u = -2a \text{Im} \int \bar{u} \nabla u = -2aP(u(t))$ we obtain:

$\frac{d}{dt}P(u(t)) = \int \nabla(|\nabla u|^2 - \frac{d}{d+2}|u|^{\frac{4}{d}+2}) - 2aP(u(t))$, But $\int \left(\nabla(|\nabla u|^2 - \frac{d}{d+2}|u|^{\frac{4}{d}+2}) \right) = 0$, we obtain finally:

$$\frac{d}{dt}P(u(t)) = -2ap(u(t)) \text{ and } P(u(t)) = e^{-2at}P(u_0).$$

Now we take u_0 in $H^1(\mathbb{R}^d)$, u_0 is the limit of a sequence (u_{0n}) in $D(\mathbb{R}^d)$, for each u_{0n} we denote by u_n the solution of (1.1) such that $u_n(0) = u_{0n}$, we have $P(u_n(t)) = e^{-2at}P(u_{0n})$, but $u_0 \mapsto u$ is continuous from $H^1(\mathbb{R}^d)$ to $C([0, T], H^1(\mathbb{R}^d))$ by passing to limit we obtain $P(u(t)) = e^{-2at}P(u_0)$.
□

1.7 Booting the log-log regime

Now we are going to prove the Proposition 1.12:

First of all we are going to prove the smallness of the L^2 norm of $\epsilon(t)$:

Lemma 1.15. *There exist $\alpha_0 \ll 1$, such that $\forall t \in [0, T]$, $\|\epsilon(t)\|_{L^2(\mathbb{R}^d)} < \alpha_0$.*

Proof: From (1.19) we have $\|u_0\|_{L^2} < \|Q\|_{L^2} + \gamma_0$, with γ_0 very small. By (1.6)

$$\begin{aligned} \|u_0\|_{L^2(\mathbb{R}^d)}^2 &\geq \|u(t)\|_{L^2(\mathbb{R}^d)}^2 \\ &= \|Q_b + \epsilon\|_{L^2(\mathbb{R}^d)}^2 \\ &= \|Q_b + \epsilon\|_{L^2(B(0,R))}^2 + \|Q_b + \epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))}^2. \\ \|Q_b + \epsilon\|_{L^2(B(0,R))} &\geq \|Q_b\|_{L^2(B(0,R))} - \|\epsilon\|_{L^2(B(0,R))} \\ &\geq \|Q\|_{L^2(\mathbb{R}^d)} - \|Q_b - Q\|_{L^2(\mathbb{R}^d)} \\ &\quad - \|Q_b\|_{L^2(\mathbb{R}^d \setminus B(0,R))} - \|\epsilon\|_{L^2(B(0,R))}. \end{aligned}$$

From (1.47), we have $\|\epsilon\|_{L^2(B(0,R))} < \beta$, where β is very small. Moreover $Q_b \rightarrow Q$ in $L^2(\mathbb{R}^d)$, and $\|Q_b\|_{L^2(\mathbb{R}^d \setminus B(0,R))} \leq \|Q_b - Q\|_{L^2(\mathbb{R}^d)} + \|Q\|_{L^2(\mathbb{R}^d \setminus B(0,R))}$, where $\|Q\|_{L^2(\mathbb{R}^d \setminus B(0,R))} \rightarrow 0$ as $R \rightarrow \infty$.
Therefore:

$$\|Q_b + \epsilon\|_{L^2(B(0,R))} \geq \|u_0\|_{L^2(\mathbb{R}^d)} - \gamma_0 - \beta - \delta,$$

and

$$\begin{aligned} \|Q_b + \epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))} &\geq \|\epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))} - \|Q_b\|_{L^2(\mathbb{R}^d \setminus B(0,R))} \\ &\geq \|\epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))} - \delta, \end{aligned}$$

where $\delta \rightarrow 0$ as $b \rightarrow 0$ and $R \rightarrow \infty$. We obtain finally:

$$\|u_0\|_{L^2(\mathbb{R}^d)}^2 \geq \|\epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))}^2 + \|u_0\|_{L^2(\mathbb{R}^d)}^2 - \alpha_0^2,$$

where $\alpha_0 \rightarrow 0$ as $\gamma_0 \rightarrow 0$.

This completes the proof. □

1.7.1 Control of the geometrical parameters

Let us now write down the equation satisfied by ϵ in rescaled variables. To simplify the notations, we note

$$Q_b = \Sigma + i\Theta, \quad \epsilon = \epsilon_1 + i\epsilon_2 \quad \text{and} \quad \Psi_b = \text{Re}(\Psi) + i\text{Im}(\Psi),$$

in terms of real and imaginary parts.

We have: $\forall s \in \mathbb{R}^+, \forall y \in \mathbb{R}^d$,

$$\begin{aligned} b_s \frac{\partial \Sigma}{\partial b} + \partial_s \epsilon_1 - M_-(\epsilon) + b(\epsilon_1)_d &= \left(\frac{\lambda_s}{\lambda} + b \right) \Sigma_d + \tilde{\gamma}_s \Theta + \frac{x_s}{\lambda} \cdot \nabla \Sigma \\ &+ \left(\frac{\lambda_s}{\lambda} + b \right) (\epsilon_1)_d + \tilde{\gamma}_s \epsilon_2 + \frac{x_s}{\lambda} \cdot \nabla \epsilon_1 \\ &+ \text{Im}(\Psi) - R_2(\epsilon). \end{aligned} \quad (1.58)$$

$$\begin{aligned} b_s \frac{\partial \Theta}{\partial b} + \partial_s \epsilon_2 + M_+(\epsilon) + b(\epsilon_2)_d &= \left(\frac{\lambda_s}{\lambda} + b \right) \Theta_d - \tilde{\gamma}_s \Sigma + \frac{x_s}{\lambda} \cdot \Theta \\ &+ \left(\frac{\lambda_s}{\lambda} + b \right) (\epsilon_2)_d - \tilde{\gamma}_s \epsilon_1 + \frac{x_s}{\lambda} \cdot \nabla \epsilon_2 \\ &- \text{Re}(\Psi) + R_1(\epsilon). \end{aligned} \quad (1.59)$$

With $\tilde{\gamma}_s(s) = -1 - \gamma_s(s) - ia\lambda^2$. The linear operator close to Q_b is now a deformation of the linear operator L close to Q and is $M = (M_+, M_-)$ with

$$M_+(\epsilon) = -\Delta \epsilon_1 + \epsilon_1 - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_2,$$

$$M_-(\epsilon) = -\Delta \epsilon_2 + \epsilon_2 - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_1.$$

The formally quadratic in ϵ interaction terms are:

$$R_1(\epsilon) = (\epsilon_1 + \Sigma) |\epsilon + Q_b|^{\frac{4}{d}} - \Sigma |Q_b|^{\frac{4}{d}} - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_2.$$

$$R_2(\epsilon) = (\epsilon_2 + \Sigma) |\epsilon + Q_b|^{\frac{4}{d}} - \Sigma |Q_b|^{\frac{4}{d}} - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_1.$$

We note $s(0)$ by s_0 and $s(T^+)$ by s^+ , now we have the following lemma:

Lemma 1.16. *(Control of the geometrical parameters) For all $s \in [s_0, s^+]$, there holds:*

- *Estimates induced by the control of energy and momentum:*

$$\begin{aligned} &|2(\epsilon_1, \Sigma + b\Theta_d - \text{Re}(\Psi_b)) + 2(\epsilon_2, \Theta - b\Sigma_d - \text{Im}(\Psi_b))| \\ &\leq \delta_0 \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-c\eta} + \lambda^2 |E(u(t))|. \end{aligned} \quad (1.60)$$

$$|(\epsilon_2, \nabla Q)| \leq \delta_0 \|\nabla \epsilon(s)\|_{L^2(\mathbb{R}^d)} + \lambda |P(u)|. \quad (1.61)$$

- *Estimates on the modulation parameters:*

$$\left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| \leq C \left(\int |\nabla \epsilon(s)|^2 + \int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-c\eta}. \quad (1.62)$$

$$\left| \tilde{\gamma}_s - \frac{(\epsilon_1, L_+ Q_{dd})}{\|Q_d\|_{L^2}^2} \right| + \left| \frac{x_s}{\lambda} \right| \leq \delta_0 \left(\int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-c\eta}. \quad (1.63)$$

Here δ_0 is a small constant $\delta_0 \ll 1$.

We will need the following lemma (for the proof see [MR03]).

Lemma 1.17. *(Control of nonlinear interactions). Let $P(y)$ a polynomial and integers $0 \leq k \leq 3$, $0 \leq l \leq 1$, $0 \leq m \leq 2$, then for some function $\delta(\alpha_0) \rightarrow 0$ as $\alpha_0 \rightarrow 0$,*

- $\left| (\epsilon, P(y) \frac{d^k}{dy^k} Q_b(y)) \right| \leq C_{P,k} \left(\int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}},$
- $\left| (\epsilon, P(y) \frac{d^k}{dy^k} (Q_b(y) - Q(y)) \right| \leq \delta(\alpha_0) \left(\int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}},$
- $\int |\epsilon| \left| P(y) \frac{d^m}{dy^m} \frac{\partial Q_b}{\partial b} \right| \leq \left(\int |\epsilon(s)|^2 e^{-|y|} + \int |\nabla \epsilon|^2 dy \right)^{\frac{1}{2}},$
- $\left| (R(\epsilon), P(y) \frac{d^k}{dy^k} Q_b(y)) \right| \leq C \left(\int |\epsilon(s)|^2 e^{-|y|} dy + \int |\nabla \epsilon|^2 dy \right)^{\frac{1}{2}},$
- $\int |F(\epsilon)| + |(\tilde{R}_1(\epsilon), \Sigma_d)| + |(\tilde{R}_2(\epsilon), \Theta_d)| \leq \delta(\alpha_0) \left(\int |\epsilon(s)|^2 e^{-|y|} dy + \int |\nabla \epsilon|^2 dy \right)^{\frac{1}{2}},$
- $\left| (P(y) \frac{d^l}{dy^l} \Psi, \frac{d^k}{dy^k} Q_b(y)) \right| + \left| (\epsilon, P(y) \frac{d^l}{dy^l} \Psi) \right| \leq e^{-\frac{c}{|b|}},$
- $\left| \left(\frac{\partial Q_b}{\partial b}, P(y) \frac{d^k}{dy^k} Q_b(y) \right) + \left(i \frac{|y|^2}{4} Q, P(y) \frac{d^k}{dy^k} Q(y) \right) \right| \leq \delta(\alpha_0).$

Proof of Lemma 1.16: To prove (1.60), we rewrite the expression of energy in the ϵ variable ($\epsilon = e^{i\gamma(t)} \lambda^{\frac{d}{2}}(t) u(t, \lambda(t)x + x(t)) - Q_b$):

$$\begin{aligned} & 2(\epsilon_1, \Sigma + b\Theta_d - \text{Re}(\Psi)) + 2(\epsilon_2, \Theta - b\Sigma_d - \text{Im}(\Psi)) \\ &= 2E(Q_b) - 2\lambda^2 E(u(t)) \\ &+ \int |\nabla \epsilon|^2 - \int \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1^2 \\ &- \int \left(\frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2^2 \\ &- 8 \int \frac{\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \epsilon_1 \epsilon_2 - \frac{2}{2 + \frac{4}{d}} \int F(\epsilon). \end{aligned} \quad (1.64)$$

With

$$\begin{aligned}
 F(\epsilon) &= |\epsilon + Q_b|^{\frac{4}{d}+2} - |Q_b|^{\frac{4}{d}+2} - \left(\frac{4}{d} + 2\right) \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^2} (\Sigma\epsilon_1 + \Theta\epsilon_2) \\
 &\quad - \epsilon_1^2 \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^4} \left(\left(\frac{2}{d} + 1\right) \left(\frac{4}{d} + 1\right) \Sigma^2 + \left(\frac{2}{d} + 1\right) \Theta^2 \right) \\
 &\quad - \epsilon_2^2 \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^4} \left(\left(\frac{2}{d} + 1\right) \left(\frac{4}{d} + 1\right) \Theta^2 + \left(\frac{2}{d} + 1\right) \Sigma^2 \right) \\
 &\quad - \epsilon_1 \epsilon_2 \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^4} \frac{8}{d} \left(\frac{2}{d} + 1\right) \Sigma \Theta.
 \end{aligned}$$

(1.60) then follows from Lemma 1.17 (we estimate the terms in (1.64) using Lemma 1.17, and we obtain (1.60)).

Now to prove (1.61), we rewrite the expression of the moment in the ϵ variable:

$$\begin{aligned}
 P(u(t)) &= \text{Im} \int (\nabla u \bar{u}) = \frac{1}{\lambda} \text{Im} \left(\int (\nabla \epsilon + \nabla Q_b) \overline{(\epsilon + Q_b)} \right) \\
 &= \frac{1}{\lambda} \left(\text{Im} \left(\int \nabla \epsilon \bar{\epsilon} \right) - 2(\epsilon_2, \nabla \Sigma) + 2(\epsilon_1, \nabla \Theta) \right),
 \end{aligned}$$

so that

$$2(\epsilon_2, \nabla \Sigma) = 2(\epsilon_1, \nabla \Theta) + \text{Im} \left(\int \nabla \epsilon \bar{\epsilon} \right) - \lambda P(u(t)).$$

From $\Theta_{b=0} = 0$ and the smallness of the L^2 norm of $\epsilon(t)$ and the control(1.8) of the momentum we obtain (1.61).

The prove (1.62)-(1.63), it suffices to follow the proof of Lemma 3 in [MR03]. \square

Now let

$$\begin{aligned}
 \tilde{R}_1(\epsilon) &= R_1(\epsilon) - \epsilon_1^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \left(\frac{4}{d} + 1\right) \Sigma^3 + \frac{6}{d} \Sigma \Theta^2 \right) \\
 &\quad - \epsilon_2^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \Sigma^3 + \frac{2}{d} \left(\frac{4}{d} - 1\right) \Sigma \Theta^2 \right) \\
 &\quad - \frac{4}{d} \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \epsilon_1 \epsilon_2 \left(\left(\frac{4}{d} - 1\right) \Sigma^2 \Theta + \Theta^3 \right), \tag{1.65}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{R}_2(\epsilon) &= R_2(\epsilon) - \epsilon_2^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \left(\frac{4}{d} + 1\right) \Theta^3 + \frac{6}{d} \Sigma^2 \Theta \right) \\
 &\quad - \epsilon_1^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \Theta^3 + \frac{2}{d} \left(\frac{4}{d} - 1\right) \Sigma^2 \Theta \right) \\
 &\quad - \frac{4}{d} \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \epsilon_1 \epsilon_2 \left(\left(\frac{4}{d} - 1\right) \Sigma \Theta^2 + \Sigma^3 \right). \tag{1.66}
 \end{aligned}$$

We define the two real Schrödinger operators:

$$L_1 = -\Delta + \frac{2}{d}\left(\frac{4}{d} + 1\right)Q^{\frac{4}{d}-1}y \cdot \nabla Q, \quad L_2 = -\Delta + \frac{2}{d}Q^{\frac{4}{d}-1}y \cdot \nabla Q.$$

To show the explosion, we will need to control b_s . Note that our continuous functions $(\lambda, \gamma, x(t), b)$ are chosen such that:

$$\epsilon = e^{i\gamma(t)}\lambda^{\frac{d}{2}}(t)u(t, \lambda(t)x + x(t)) - Q_b$$

satisfy the following conditions of orthogonality:

$$(\epsilon_1(t), \Sigma_d) + (\epsilon_2(t), \Theta_d) = 0, \quad (1.67)$$

$$(\epsilon_1(t), y\Sigma) + (\epsilon_2(t), y\Theta) = 0, \quad (1.68)$$

$$-(\epsilon_1(t), \Theta_{dd}) + (\epsilon_2(t), \Sigma_{dd}) = 0, \quad (1.69)$$

$$-(\epsilon_1(t), \Theta_d) + (\epsilon_2(t), \Sigma_d) = 0. \quad (1.70)$$

For the proof of these conditions see Lemma 2 in [MR03], the proof is based on the implicit function theorem using that $(Q_b)_{b=0} = Q$ and $(\frac{\partial Q_b}{\partial b})_{b=0} = -i\frac{|y|^2}{4}Q$.

Now we have the following one:

Proposition 1.18. *There exist $\delta_0 > 0$, $C > 0$ and $0 < \beta < 2$ such that:*

$$b_s \geq \delta_0 \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) - e^{-\frac{C}{b}} - \lambda^\beta(s). \quad (1.71)$$

Proof: To prove this proposition, it suffices to follow the proof of Proposition 3 in [MR03] and use the control of the energy. \square

We will need to refine Q_b because Q_b is not an exact self-similar solution. The basic idea is that the profile $Q_b + \zeta_b$ should be a better approximation of the solution. The problem is now that ζ_b is indeed in \dot{H}^1 , but not in L^2 , and we then are not able to estimate the main interaction terms. We therefore introduce a cut version of the radiation: leave a radial cutoff function: $\chi_A(r) = \chi(\frac{r}{A})$ with $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$.

The choice of the parameter $A(t)$ is a crucial issue in our analysis, and is roughly based on two constraints: we want A to be large in order first to enter the radiative zone, i.e., $\frac{2}{b} \ll A$, and to ensure the slowest possible variations of the L^2 -norm in the zone $|y| \geq A$. But we also want A not too large, in particular to keep a good control over local L^2 -terms of the form $\int_{|y| \leq A} |\epsilon|^2$.

A choice which balances these two constraints is:

$$A = A(t) = e^{2l\frac{1}{b(t)}} \text{ so that } \Gamma_b^{-\frac{1}{2}} \leq A \leq \Gamma_b^{-\frac{3l}{2}},$$

for some parameter $l > 0$ small enough to be chosen later and which depends on η . Now let

$$\tilde{\zeta} = \chi\left(\frac{r}{A}\right)\zeta_b,$$

Observe that $\tilde{\zeta}$ is now a small Schwartz function thanks to the A localization. we next consider the new variable

$$\tilde{\epsilon} = \epsilon - \tilde{\zeta}, \quad (1.72)$$

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$\tilde{\zeta}_b$ still satisfies the size estimates of Lemma 2.1 and is moreover in L^2 with an estimate

$$\int |\tilde{\zeta}|^2 \leq \Gamma_b^{1-C\eta}. \quad (1.73)$$

The equation satisfied by $\tilde{\zeta}$ is now

$$\Delta \tilde{\zeta} - \tilde{\zeta} + ib(\tilde{\zeta})_d = \Psi_b + F$$

with

$$F = (\Delta \chi_A) \zeta_b + 2\nabla \chi_A \cdot \nabla \zeta_b + iby \cdot \nabla \chi_A \zeta_b. \quad (1.74)$$

Now we have the following lemma (see Lemma 4.4 in [CR09] for further details):

Lemma 1.19. (*Virial dispersion in the radiative regime*) *There exist a constants $\delta_1 > 0$, $C > 0$ and $\alpha > 0$ such that:*

$$\begin{aligned} (f_1(s))_s &\geq \delta_1 \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\epsilon(s)|^2 e^{-|s|} + \Gamma_b \right) \\ &\quad - \frac{1}{\delta_1} \int_A^{2A} |\epsilon|^2 - C\lambda^2 E(u(t)). \end{aligned} \quad (1.75)$$

with

$$f_1(s) = \frac{b}{4} \|yQ_b\|_{L^2}^2 + \frac{1}{2} \text{Im} \left(\int (y \cdot \nabla \tilde{\epsilon}) \bar{\zeta} \right) + (\epsilon_2, \Lambda \tilde{\zeta}_{re}) - (\epsilon_1, \Lambda \tilde{\zeta}_{im}). \quad (1.76)$$

We now need to control the term $\int_A^{2A} |\epsilon|^2$ in (1.75). This is achieved by computing the flux of L^2 -norm escaping the radiative zone. We introduce a radial nonnegative cut off function $\phi(r)$ such that $\phi(r) = 0$ for $r \leq \frac{1}{2}$, $\phi(r) = 1$ for $r \geq 3$, $\frac{1}{4} \leq \phi'(r) \leq \frac{1}{2}$ for $1 \leq r \leq 2$, $\phi'(r) \geq 0$. We then set

$$\phi_A(s, r) = \phi\left(\frac{r}{A(s)}\right).$$

Moreover, we restrict the freedom on the choice of the parameters (η, l) by assuming $l > C\eta$. We have:

$$\begin{cases} \phi_A(r) = 0 & \text{for } 0 \leq r \leq \frac{A}{2}, \\ \frac{1}{4A} \leq \phi'_A(r) \leq \frac{1}{2A} & \text{for } A \leq r \leq 2A, \\ \phi_A(r) = 1 & \text{for } r \geq 3A \\ \phi'_A(r) \geq 0, 0 \leq \phi_A(r) \leq 1. \end{cases}$$

We now claim the following dispersive control at infinity in space (see Lemma 7 in [MR06] for the proof):

Lemma 1.20. (*L^2 dispersion at infinity in space*). *For some universal constant $C > 0$, if s large enough:*

$$\left(\int \phi_A |\epsilon|^2 \right)_s \geq \frac{b}{400} \int_A^{2A} |\epsilon|^2 - \Gamma_b^{1+C\eta} - \Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 - \frac{\lambda^2}{b^2} E(u(t)). \quad (1.77)$$

Note that $\lambda \leq e^{-e^{\frac{C}{b}}}$ thus $\frac{\lambda^2}{b^2} E(u(t)) \leq \lambda^\beta$ with $0 < \beta < 2$ close to 2, thus the last term in this estimation is small with respect to λ .

1.7.2 L^2 -dispersive constraint on the solution.

In this subsection, we derive the dispersive estimate needed for the proof of the blowup. The virial estimate (1.75) corresponds to nonlinear interactions on compact sets. The L^2 linear estimate (1.77) measures the interactions with the linear dynamic at infinity. We now couple these two facts through the smallness of the L^2 -norm, which is a global information in space.

Proposition 1.21. *For some universal constant $C > 0$ and for $s \geq 0$, the following holds:*

$$(\mathfrak{V})_s \leq -Cb \left(\Gamma_b + \int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|v|} + \int_A^{2A} |\epsilon|^2 \right) + C \frac{\lambda^2}{b^2} E(u(t)). \quad (1.78)$$

with:

$$\begin{aligned} \mathfrak{V}(s) = & \left(\int |Q_b|^2 - \int |Q|^2 \right) + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \\ & - \frac{\delta_1}{800} \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b((\epsilon_2, \Lambda \tilde{\zeta}_{re}) - (\epsilon_1, \Lambda \tilde{\zeta}_{im})) \right). \end{aligned} \quad (1.79)$$

Here $c > 0$ denotes some small enough universal constant and:

$$\tilde{f}_1(b) = \frac{b}{4} |y Q_b|_2^2 + \frac{1}{2} \text{Im} \left(\int (y \cdot \nabla \tilde{\zeta}) \bar{\tilde{\zeta}} \right). \quad (1.80)$$

Remark 1.22. *Here the range of parameters is more restricted and yields: there exist $\eta^*, l^*, C_0 > 0$ such that $\forall 0 < \eta < \eta^*, \forall 0 < l < l^*$ such that $l > C_0 \eta$, there exists $b^*(\eta, l)$ such that $\forall |b| \leq b^*(\eta, l)$, the estimates of this proposition hold with universal constants.*

Remark 1.23. *The gain is that we now have a Lyapunov function \mathfrak{V} in H^1 . Remark that in a regime when ϵ is small compared to b in a certain sense, $\mathfrak{V} \sim \int |Q_b|^2 - \int |Q|^2 \sim b^2$ from (1.36) and (1.78) forces b to decay.*

Proof: Multiply (1.75) by $\frac{\delta_1 b}{800}$ and sum with (1.77). We get

$$\begin{aligned} \left(\int \phi_A |\epsilon|^2 \right)_s + \frac{\delta_1 b}{800} (f_1)_s \geq & \frac{\delta_1^2 b}{800} \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|v|} \right) + \frac{b}{800} \int_A^{2A} |\epsilon|^2 \\ & + \frac{c \delta_1 b}{1000} \Gamma_b - C \frac{\lambda^2}{b^2} E(u(t)) - \Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2, \end{aligned} \quad (1.81)$$

We first integrate the left-hand side of (1.81) by parts in time:

$$\begin{aligned} b(f_1)_s = & \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)) \right)_s \\ & - b_s((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)), \end{aligned} \quad (1.82)$$

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where \tilde{f}_1 given by (1.80). (1.82) now yields

$$\begin{aligned} & \left(\int \phi_A |\epsilon|^2 + \frac{\delta_1}{800} (b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)dv + b((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d))) \right)_s \\ & \geq \frac{\delta_1^2 b}{800} \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} + \int_A^{2A} |\epsilon|^2 \right) + \frac{c\delta_1 b}{1000} \Gamma_b - C \frac{\lambda^2}{b^2} E(u(t)) \\ & - \Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 + \frac{\delta_1}{800} b_s ((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)). \end{aligned}$$

We now inject the expression of the L^2 -norm:

$$\int |\epsilon|^2 + \int |Q_b|^2 + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) = e^{-2at} \int |u_0|^2 dx,$$

but

$$\int \phi_A |\epsilon|^2 = \int |\epsilon|^2 - \int (1 - \phi_A) |\epsilon|^2,$$

we compute:

$$\begin{aligned} \left(\int \phi_A |\epsilon|^2 \right)_s &= - \left(\int |Q_b|^2 - \int |Q|^2 + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \right)_s \\ &+ \left(\int |u_0|^2 e^{-2at} \right)_s \\ &= - \left(\int |Q_b|^2 - \int |Q|^2 + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \right)_s \\ &- 2a\lambda^2 e^{-2at} \|u_0\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Thus, we get

$$\begin{aligned} (-\mathfrak{Y})_s &\geq \frac{\delta_1^2 b}{800} \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} + \int_A^{2A} |\epsilon|^2 \right) + \frac{c\delta_1 b}{1000} \Gamma_b - C \frac{\lambda^2}{b^2} E(u(t)) \\ &- \Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 + \frac{\delta_1}{800} b_s ((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)) - C_0 \lambda^2. \end{aligned} \quad (1.83)$$

We now have

$$\Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 \leq \Gamma_b^{\frac{1}{2}} (\Gamma_b^{1-C\eta} + \int |\nabla \tilde{\epsilon}|^2),$$

from the assumption $l > C\eta$. Next, we estimate from (1.62):

$$|b_s((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d))| \leq \Gamma_b^{\frac{1}{2}-C\eta} \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} + C\lambda^2 E(u(t)) \right).$$

Injecting these estimates into (1.83) yields (1.78). This concludes the proof of the proposition. \square

Note that now from (1.78) we obtain:

$$(\mathfrak{Y}(s))_s \leq -Cb\Gamma_b + C \frac{\lambda^2}{b^2} E(u(t)) \leq -\frac{1}{2}Cb\Gamma_b \leq 0.$$

1.7.3 Proof of the Bootstrap (Proposition 1.12)

Let f_2 be defined by:

$$f_2 = \left(\int |Q_b|^2 - \int Q^2 \right) - \frac{\delta_1}{800} \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv \right)$$

it satisfies (using the smallness (1.73) of ζ in L^2)

$$\frac{d_0}{C} < \frac{df_2}{db^2} \Big|_{b^2=0} < C d_0, \quad (1.84)$$

with d_0 defined by :

$$0 < \frac{d}{db^2} \left(\int |Q_b|^2 \right) \Big|_{b^2=0} = d_0 < +\infty. \quad (1.85)$$

Now as a consequence of the control (1.60) and of the coercivity of the linearized energy under the chosen set of orthogonality conditions, we have the bounds:

$$\left(\mathfrak{J}(s) - f_2(b(s)) \right) \begin{cases} \geq -\Gamma_b^{1-Cl} + \frac{1}{C} \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|v|} \right) \\ \leq CA^2 \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|v|} \right) + \Gamma_b^{1-Cl}. \end{cases} \quad (1.86)$$

Indeed,

$$\begin{aligned} \mathfrak{J}(s) - f_2(b(s)) &= 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \\ &\quad - \frac{\delta_1 b}{800} \left((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d) \right). \end{aligned} \quad (1.87)$$

From the estimates on $\tilde{\zeta}$ of Lemma 1.11, the choice of A , we have:

$$\begin{aligned} |(\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)| &\leq \Gamma_b^{\frac{1}{2}-C\eta} \left(\int_0^A |\epsilon|^2 \right)^{\frac{1}{2}} \\ &\leq A^2 \Gamma_b^{1-C\eta} + C \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|v|} \right) \\ &\leq \Gamma_b^{1-Cl} + C \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|v|} \right). \end{aligned}$$

The other term in (1.87) is estimated from the expression of energy:

$$\begin{aligned} &2(\epsilon_1, \Sigma + b\Theta_d - Re(\Psi)) + 2(\epsilon_2, \Theta - b\Sigma_d - Im(\Psi)) \\ &= 2E(Q_b) - 2\lambda^2 E(u(t)) \\ &+ \int |\nabla \epsilon|^2 - \int \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1^2 \\ &- \int \left(\frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2^2 \\ &- 8 \int \frac{\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \epsilon_1 \epsilon_2 - \frac{2}{2 + \frac{4}{d}} \int F(\epsilon), \end{aligned}$$

which can be rewritten as :

$$\begin{aligned}
 & 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 = (L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) - \int \phi_A |\epsilon|^2 \\
 & + 2(\epsilon_1, \operatorname{Re}(\Psi)) + 2(\epsilon_2, \operatorname{Im}(\Psi)) + 2E(Q_b) - 2\lambda^2 E(u(t)) \\
 & - \int \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} - \left(\frac{4}{d} + 1 \right) Q^{\frac{4}{d}} \epsilon_1^2 \\
 & - \int \left(\left(\frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} - Q^{\frac{4}{d}} \right) \epsilon_2^2 \\
 & - 8 \int \frac{\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \epsilon_1 \epsilon_2 - \frac{2}{2 + \frac{4}{d}} \int F(\epsilon).
 \end{aligned}$$

We first estimate:

$$\begin{aligned}
 & |(\epsilon_1, \operatorname{Re}(\Psi))| + |(\epsilon_2, \operatorname{Im}(\Psi))| + E(Q_b) + 2\lambda^2 |E(u(t))| \\
 & \leq \Gamma_b^{1-Cl} + \Gamma_b^l \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right).
 \end{aligned}$$

The cubic term $\int |F(\epsilon)|$ and the rest of the quadratic form are controlled by $\delta(\alpha_0) \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) + \Gamma_b^{1+l}$, we thus obtain:

$$\begin{aligned}
 & \left| \mathfrak{I}(s) - f_2(b(s)) - \left((L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) - \int (\phi_A) |\epsilon|^2 \right) \right| \\
 & \leq \delta(\alpha_0) \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) + \Gamma_b^{1-Cl}.
 \end{aligned}$$

The upper bound follows from:

$$\int (1 - \phi_A) |\epsilon|^2 \leq CA^2 \log A \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right). \quad (1.88)$$

For the lower bound, we use the elliptic estimate on $L = (L_+, L_-)$ (for the proof see Appendix D in [MR06]), this ends the proof of (1.86).

We are now in a position to prove the pointwise bound (1.52):

$$\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \lesssim \Gamma_{b(s)}^{\frac{2}{3}}.$$

Let $s_2 \in [s_0, s^+]$, if $b_s(s_2) \leq 0$, then (1.52) follows directly from (1.71). If $b_s(s_2) > 0$, let $s_1 \in [s_0, s^+]$ be from s_2 such that $b_s(s_1) = 0$, then either s_1 is attained or $s_1 = s_0$. In both cases, we have using (1.20):

$$\int |\nabla \epsilon(s_1)|^2 + \int |\epsilon(s_1)|^2 e^{-|y|} \leq \Gamma_{b(s_1)}^{\frac{3}{4}}$$

and thus

$$\mathfrak{I}(s_1) - f_2(b(s_1)) \leq \Gamma_{b(s_1)}^{\frac{17}{24}}, \quad (1.89)$$

from (1.86) and for $l > 0$ is small enough. Moreover, $b_s \geq 0$ on $[s_1, s_2]$ and thus:

$$b(s_2) \geq b(s_1). \quad (1.90)$$

1.7. BOOTING THE LOG-LOG REGIME

We now use the Lyapunov control (1.78) to derive:

$$\mathfrak{J}(s_2) \leq \mathfrak{J}(s_1),$$

we then inject (1.86), (1.90) and (1.89) to conclude:

$$\begin{aligned} f_2(b(s_2)) + \frac{1}{C} \left(\int |\nabla \epsilon(s_2)|^2 + \int |\epsilon(s_2)|^2 e^{-|y|} \right) &\leq \mathfrak{J}(s_2) + \Gamma_{b(s_2)}^{1-Cl} \\ &\leq f_2(b(s_1)) + \Gamma_{b(s_1)}^{1-Cl} + \Gamma_{b(s_2)}^{1-Cl} \leq f_2(b(s_1)) + 2\Gamma_{b(s_2)}^{\frac{2}{3}}. \end{aligned}$$

The monotonicity (1.84) of f_2 in b and (1.90) now imply:

$$\int |\nabla \epsilon(s_2)|^2 + \int |\epsilon(s_2)|^2 e^{-|y|} \lesssim \Gamma_{b(s_2)}^{\frac{2}{3}}$$

which implies that Equations (1.52) holds at s_2 . This concludes the proof of (1.52).

Now we are going to prove the upper bound on blowup rate:

From (1.71) we obtain:

$$b_s \geq -\Gamma_{b(s)}^{1-C\eta}. \quad (1.91)$$

In particular:

$$\left(e^{\frac{\pi}{2b(s)}} \right)_s \leq e^{\frac{\pi}{2b(s)}} \frac{\pi \Gamma_{b(s)}^{1-C\eta}}{2b^2} \leq 1$$

as $\Gamma_b \sim e^{-\frac{\pi}{b}}$, and therefore

$$e^{\frac{\pi}{2b(s)}} \leq e^{\frac{\pi}{b(0)}} + s - s_0 \leq s,$$

thus from (1.91) and the value of s_0 (we take $s_0 = e^{\frac{5\pi}{9b(0)}}$).

Finally, $\forall s \in [s_0, s^+]$,

$$b(s) \geq \frac{\pi}{2 \log(s)}, \quad (1.92)$$

we now rewrite the estimate (1.62) using (1.52) as follows

$$\left| \frac{\lambda_s}{\lambda} + b \right| \leq \Gamma_b^{\frac{1}{4}}. \quad (1.93)$$

Thus:

$$\frac{b}{2} \leq -\frac{\lambda_s}{\lambda} \leq 2b.$$

We integrate this in time and get: $\forall s \in [s_0, s^+]$,

$$-\log(\lambda(s)) \geq -\log(\lambda(s_0)) + \frac{1}{2} \int_{s_0}^s b \geq -\log(\lambda(s_0)) + \frac{\pi}{4} \left(\frac{s}{\log s} - \frac{s_0}{\log s_0} \right).$$

Now from (1.18):

$$-\log(\lambda(0)) \geq e^{\frac{2\pi}{3b(s_0)}} = s_0^{\frac{4}{3}},$$

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and thus

$$-\log(\lambda(s)) \geq -\frac{2}{3}\log(\lambda(0)) + \frac{\pi}{4}\frac{s}{\log(s)}, \quad \text{i.e. } \lambda(s) \leq \lambda^{\frac{2}{3}}(0)e^{-\frac{\pi}{4}\frac{s}{\log s}}. \quad (1.94)$$

This also implies: $\forall s \in [s_0, s_2[$,

$$-\log(\lambda(s)) \geq \frac{\pi}{4}\frac{s}{\log(s)} \geq \sqrt{s} \quad (1.95)$$

and taking the log of this inequality yields

$$\log|\log(\lambda(s))| \geq \frac{1}{2}\log(s), \quad \text{i.e. } b \geq \frac{\pi}{4\log|\log\lambda(s)|} \quad (1.96)$$

using (1.92). Therefore (1.49) is proved.

Now we are going to prove the monotonicity of λ : We turn to the proof of (1.50) and (1.46). From (1.49), (1.62) and (1.52), there holds:

$$-\frac{\lambda_s}{\lambda} \geq \frac{b}{2}$$

and thus: $\forall s_1, s_2 \in [s_0, s^+]$,

$$-\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \geq \frac{1}{2}\int_{s_1}^{s_2} b \geq \frac{1}{2}\int_{s_1}^{s_2} \frac{ds}{\log(s)}. \quad (1.97)$$

This prove (1.50). To prove (1.46), we let $[t_k, t_{k+1}]$ be a doubling time interval, then from (1.97):

$$\log(2) = -\log\left(\frac{\lambda(t_{k+1})}{\lambda(t_k)}\right) \geq \frac{1}{2}\int_{t_k}^{t_{k+1}} \frac{dt}{\lambda^2(t)\log(s(t))}$$

and thus

$$1 \geq \frac{C(t_{k+1} - t_k)}{\lambda^2(t_k)\log(s(t_{k+1}))} \geq \frac{C(t_{k+1} - t_k)}{\lambda^2(t_k)\log|\log(\lambda(t_{k+1}))|} \geq \frac{C(t_{k+1} - t_k)}{\lambda^2(t_k)\log k}$$

and (1.51) follows.

The upper bound on b is a direct consequence (1.36). The lower bound $b > 0$ follows from (1.96).

Now we prove the blowup in finite time. We observe from (1.95) that

$$T = \int_0^{+\infty} \lambda^2(s)ds \leq \lambda^{\frac{4}{3}}(0)(C + \int_2^{+\infty} e^{-\frac{\pi}{4}\frac{s}{\log s}} ds) < +\infty.$$

Moreover, from (1.52),

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \frac{1}{\lambda(t)},$$

and thus the local well-posedness theory in H^1 ensures $\lambda(t) \rightarrow 0$ as $t \rightarrow T$.

The convergence of the concentration point is a consequence of (1.63), (1.95) and (1.52) which imply:

$$\int_{s_0}^{+\infty} |x_s| ds \leq C \int_{s_0}^{+\infty} \lambda(s)ds < +\infty.$$

1.8 Determination of the blow-up speed

In this chapter, we prove that the blow-up holds with the log-log speed.

Observe from (1.86), (1.52) and (1.49) that:

$$\frac{b^2(s)}{C} \leq \mathfrak{Y}(s) \leq Cb^2(s).$$

Together with (1.78), this implies:

$$(\mathfrak{Y})_s \leq e^{-\frac{c}{\sqrt{s}}},$$

integrating this in time yields:

$$b(s) \leq C\sqrt{\mathfrak{Y}} \leq \frac{C}{\log(s)}$$

for s large enough. Integrating now (1.93) in time, we conclude that

$$-\log(\lambda(s)) \leq C \int_{s_0}^s b + C \leq C \frac{s}{\log(s)}$$

for s large enough, and thus together with (1.96):

$$\frac{1}{C} \leq b \log |\log(\lambda)| \leq C. \quad (1.98)$$

Now

$$\begin{aligned} -(\lambda^2 \log |\log \lambda|)_t &= -\lambda \lambda_t \log |\log(\lambda)| \left(2 + \frac{1}{|\log \lambda| \log |\log \lambda|} \right) \\ &= -\left(\frac{\lambda_s}{\lambda} + b \right) \log |\log \lambda| \left(2 + \frac{1}{|\log \lambda| \log |\log \lambda|} \right) \\ &\quad + b \log |\log \lambda| \left(2 + \frac{1}{|\log \lambda| \log |\log \lambda|} \right). \end{aligned} \quad (1.99)$$

From (1.93)

$$\int_t^T \left| \left(\frac{\lambda_s}{\lambda} + b \right) \log \log \lambda \right| \lesssim \int_t^T (b^2 \log |\log \lambda|) dt. \quad (1.100)$$

Injecting this into (1.99) integrated from t to T and using (1.98), we conclude that

$$\frac{T-t}{C} \leq \lambda^2(t) \log |\log \lambda(t)| \leq C(T-t)$$

from which

$$\frac{1}{C} \left(\frac{T-t}{\log |\log(T-t)|} \right)^{\frac{1}{2}} \leq \lambda(t) \leq C \left(\frac{T-t}{\log |\log(T-t)|} \right)^{\frac{1}{2}} \quad (1.101)$$

1.8. DETERMINATION OF THE BLOW-UP SPEED

for t close enough to T .

But

$$2b(T-t) = \lambda^2,$$

and thus:

$$\frac{1}{C} \frac{1}{\log |\log(T-t)|} \leq b(t) \leq C \frac{1}{\log |\log(T-t)|}.$$

From this we obtain:

$$\frac{1}{C} |\log(T-t)| \log |\log(T-t)| \leq s(t) \leq C |\log(T-t)| \log |\log(T-t)|.$$

This prove (1.4). \square

Remark 1.24. *Following the ideas in [MR05], we can also prove the existence of a L^2 -profile at blow-up point. More precisely there exists u^* in $L^2(\mathbb{R}^d)$ (L^2 -profile) such that:*

$$u(t, x) - \frac{1}{\lambda^{\frac{d}{2}}(t)} Q_{b(t)}(t, \frac{x-x(t)}{\lambda(t)}) e^{iy(t)} \rightarrow u^* \text{ in } L^2(\mathbb{R}^d), t \rightarrow T. \quad (1.102)$$

Proof of Corollary 1.8: Let $S(t)$ be the propagator for the linear equation:

$$i\partial_t u + \Delta u = 0, \quad (t, x) \in [0, \infty[\times \mathbb{R}^d.$$

The Cauchy problem for (1.1) with $u(0) = u_0 \in H^1(\mathbb{R}^d)$ is equivalent to the integral equation:

$$u(t) = S(t)u_0 + i \int_0^t S(t-s)(|u(s)|^{\frac{4}{d}}u(s) + iau)ds.$$

We know from Lemma 1.14: there exist $T(\|u_0\|_{H^1(\mathbb{R}^d)}) > 0$ such that:

$$\forall 0 \leq a \leq 1, \|u\|_{L^\infty([0,T];H^1)} \leq 2\|u_0\|_{H^1(\mathbb{R}^d)}.$$

Let u a solution for (1.1) and v solution for (1.2) we have:

$$\begin{aligned} u - v &= S(t)(u_0 - v_0) + i \int_0^t S(t-s)(|u(s)|^{\frac{4}{d}}u(s) - |v(s)|^{\frac{4}{d}}v(s))ds \\ &\quad + ia \int_0^t S(t-t')u(t')dt'. \end{aligned}$$

By Strichartz we obtain (see the proof of Lemma 1.14):

$$\begin{aligned} \|u - v\|_{L^\infty([0,T];H^1)} &\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} \\ &\quad + CT^\gamma (\|u\|_{L^\infty([0,T];H^1)}^{\frac{4}{d}} + \|v\|_{L^\infty([0,T];H^1)}^{\frac{4}{d}}) \|u - v\|_{L^\infty([0,T];H^1)} \\ &\quad + CaT \|u\|_{L^\infty([0,T];H^1)}. \end{aligned}$$

Thus for $T_1 = \text{Min}(T(\|u_0\|_{H^1(\mathbb{R}^d)}), T(\|v_0\|_{H^1(\mathbb{R}^d)}))$ we obtain $\forall 0 \leq t \leq T_1$:

$$\begin{aligned} \|u - v\|_{L^\infty([0,T];H^1)} &\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} \\ &\quad + CT^\gamma (\|u_0\|_{H^1}^{\frac{4}{d}} + \|v_0\|_{H^1}^{\frac{4}{d}}) \|u - v\|_{L^\infty([0,T];H^1)} \\ &\quad + CaT \|u\|_{L^\infty([0,T];H^1)}. \end{aligned}$$

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Now for $T_2 = \frac{1}{2} \text{Min}(\text{Max}^{-\frac{1}{\gamma}}(\|u_0\|_{H^1}^{\frac{4}{\gamma}}, \|v_0\|_{H^1}^{\frac{4}{\gamma}}), T_1)$:

$$\|u - v\|_{L^\infty([0,t];H^1)} \leq \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} + \frac{1}{2} \|u - v\|_{L^\infty([0,t];H^1)} + a,$$

thus

$$\|u - v\|_{L^\infty([0,t];H^1)} \lesssim \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} + a. \quad \forall 0 < t < T_2.$$

Thus the map $(a, \phi) \rightarrow u(\cdot, a, \phi)$ is continuous in $(0, u_0)$ from $\mathbb{R} \times H^1(\mathbb{R}^d)$ to $C([0, T_2], H^1(\mathbb{R}^d))$. Since T_2 only depends on $\|u_0\|_{H^1(\mathbb{R}^d)}$, this continuity extends to any interval $[0, T]$ in the maximal interval of existence of u .

We know after a time t_0 closed to blow-up time of u with the initial data u_0 , that $u(t_0)$ verifies C.I, and by continuity $v(t_0)$ verifies also C.I (the conditions C.I are stable by a small perturbations in H^1), then we obtain from Theorem 1.7 the blow up of v with the initial data $v(t_0)$ for (1.1). Therefore the solution of (1.1) emanating from v_0 blows up in finite time in the log-log regime. \square

1.8. DETERMINATION OF THE BLOW-UP SPEED

Chapter 2

On the L^2 -critical nonlinear Schrödinger equation with a nonlinear damping

2.1 Introduction

In this chapter, we study the blowup and the global existence of solutions for the focusing NLS equation with a nonlinear damping (NLS ap):

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia|u|^p u = 0, (t, x) \in [0, \infty[\times \mathbb{R}^d, d = 1, 2, 3, 4, \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (2.1)$$

with initial data $u(0) = u_0 \in H^1(\mathbb{R}^d)$ where $a > 0$ is the coefficient of friction and $p \geq 1$. Note that if we replace $+|u|^{\frac{4}{d}}u$ by $-|u|^{\frac{4}{d}}u$, (2.1) becomes the defocusing NLS equation.

Equation (2.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics. Fibich [FM01] noted that in the nonlinear optics context, the origin of the nonlinear damping is multiphoton absorption. For example, in the case of solids the number p corresponds to the number of photons it takes to make a transition from the valence band to the conduction band. Similar behavior can occur with free atoms, in this case p corresponds to the number of photons needed to make a transition from the ground state to some excited state or to the continuum.

The Cauchy problem for (2.1) was studied by Kato [Kat87] and Cazenave[Caz03] and it is known that if $p < \frac{4}{d-2}$, then the problem is locally well-posed in $H^1(\mathbb{R}^d)$: For any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T \in (0, \infty)$ and a unique solution $u(t)$ of (1.1) with $u(0) = u_0$ such that $u \in C([0, T]; H^1(\mathbb{R}^d))$. Moreover, T is the maximal existence time of the solution $u(t)$ in the sense that if $T < \infty$ then $\lim_{t \rightarrow T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$.

Let us notice that for $a = 0$ (2.1) becomes the L^2 -critical nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (2.2)$$

2.1. INTRODUCTION

For $u_0 \in H^1$, a sharp criterion for global existence for (2.2) has been exhibited by Weinstein [Wei83]: Let Q be the unique positive solution to

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (2.3)$$

For $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution of (2.2) is global in H^1 . This follows from the conservation of the energy and the L^2 norm and the sharp Gagliardo-Nirenberg inequality:

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right).$$

where

$$E(u) := \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4 + 2d} \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}. \quad (2.4)$$

On the other hand, there exist explicit solutions with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up in finite time $T \in \mathbb{R}$ with $\|u(t)\|_{H^1} \sim \frac{1}{T-t}$.

In the serie of papers [MR02a, Rap05], Merle and Raphael have studied the blowup for (2.2) with $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta$, δ small and have proven the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \quad (2.5)$$

In [Dar12a], Darwich has proved in case of the linear damping ($p = 0$), the global existence in H^1 for $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, and has showed that the log-log regime is stable by such perturbations (i.e. there exist solutions blows up in finite time with the log-log law).

Numerical observations suggest that this finite time blowup phenomena persists in the case of the nonlinear damping for $p < \frac{4}{d}$ (see Fibich [FM01] and [PSS05]). Passot and Sulem [PSS05] have proved that the solutions are global in $H^1(\mathbb{R}^2)$ in the case where the power of the damping term is strictly greater to the focusing nonlinearity. The case where the power of the damping term is equal to the focusing nonlinearity, "small damping prevents blow-up ? " was an open question for Sparber and Antonelli in their paper [AS10] and for Fibich and Klein in their paper [FK12]. Our results give an answer to their open problem, at least for the L^2 -critical case. In fact, our aim in this paper is study for each value of (d, p) , the existence of blow-up solutions as well as global existence criteria. In particular, we will be interested in the stability of the log-log blow-up regime by such perturbations.

Let us now state our results:

Theorem 2.1. *Let u_0 in $H^1(\mathbb{R}^d)$ with $d = 1, 2, 3, 4$*

1. *if $\frac{4}{d-2} > p \geq \frac{4}{d}$, then the solution of (2.1) is global in H^1 .*
2. *if $1 \leq p < \frac{4}{d}$ and $1 \leq p \leq 2$, then there exists $0 < \alpha < \|Q\|_{L^2}$ such that for any $u_0 \in H^1$ with $\|u_0\|_{L^2} < \alpha$, the emanating solution is global in H^1 .*

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3. if $1 \leq p < \frac{4}{d}$, then there does not exist an initial data u_0 with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ such that the solution u of (2.1) blow up in finite time $T > 0$ by satisfying

$$\frac{1}{(T-t)^{\beta-\epsilon}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^{\beta+\epsilon}},$$

for any $\beta \in]0, \frac{2}{pd}[$ and $0 < \epsilon < \frac{2-\beta pd}{8+pd}$.

4. if $1 \leq p < \frac{4}{d}$, then there exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in]0, \delta_0[$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (2.1) blows up in finite time in the log-log regime.

Remark 2.2. Note that, part (3) of Theorem 2.1 proves in particular that we don't have the blowup in the log-log regime for any $p \in [1, \frac{4}{d}[$ and in the regime $\frac{1}{l}$ for $d = 1$ and $1 \leq p < 2$, for initial data with critical or subcritical mass.

In the "critical" case $p = \frac{4}{d}$, we have more precisely :

Theorem 2.3. Let $p = \frac{4}{d}$, then the initial-value problem (2.1) is globally well posed in $H^s(\mathbb{R}^d)$, $s \geq 0$. Moreover, there exist unique u_+ in L^2 such that

$$\|u(\cdot, t) - e^{it\Delta} u_+\|_{L^2} \longrightarrow 0, \quad t \longrightarrow +\infty, \quad (2.6)$$

where $e^{it\Delta}$ is the free evolution.

Remark 2.4. Theorem 2.3 and parts (1) -(3) of Theorem 2.1 still hold in the defocusing case.

Remark 2.5. Note that if $u(t, \cdot)$ is a solution of $(NLS_{a,p})$ then $\overline{u(-t, \cdot)}$ is a solution of $(NLS_{-a,p})$, then we don't have the scattering in $-\infty$, because this changes the sign of the coefficient of friction.

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2.2 Global existence.

In this section, we prove assertions (1) and (2) of Theorem 2.1 and Theorem 2.3. To prove part (1), we will prove that the H^1 -norm of u is bounded for any time. To prove part (2), we use generalised Gagliardo-Nirenberg inequalities to show that the energy is non increasing. Finally to prove Theorem 2.3, we establish an a priori estimate on the critical Strichartz norm.

Theorem 2.6. Let $p \geq 1$ for $d = 1, 2$ or $1 \leq p \leq \frac{4}{d-2}$ for $d \geq 3$, then the initial-value problem (2.1) is locally well posed in $H^1(\mathbb{R}^d)$ (If $p < \frac{4}{d-2}$ the minimal time of the existence depends on $\|u_0\|_{H^1}$).

Proof: See [Caz03] page 93 Theorem 4.4.1.

To prove the following proposition, we will proceed in the same way as in the section 3.1 in [PSS05].

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Proposition 2.7. *Let u be a solution of (2.1) and $\frac{4}{d-2} \geq p > \frac{4}{d}$ then*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \leq \|\nabla u(0)\|_{L^2(\mathbb{R}^d)} e^{a^{\left(\frac{-4}{pd-4}\right)} t}.$$

Proof: Multiply Eq. (2.1) by $-\Delta \bar{u}$, integrate and take the imaginary part, this gives

$$\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + a \int |u|^p |\nabla u|^2 + a \Re \int u \nabla |u|^p \nabla \bar{u} dx = -\frac{4}{d} \Im \int u \nabla \bar{u} \Re(u \nabla u) |u|^{\frac{4}{d}-2}. \quad (2.7)$$

In the l.h.s, a simple calculation shows that the third term rewrites in the form $\frac{p}{4} \int |u|^{p-2} (\nabla |u|^2)^2$. Equation (2.7) becomes:

$$\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + a \int |u|^p |\nabla u|^2 + a \frac{p}{4} \int |u|^{p-2} (\nabla |u|^2)^2 \leq \frac{2}{d} \int |u|^{\frac{4}{d}} |\nabla u|^2. \quad (2.8)$$

To estimate the r.h.s of (2.8), we rewrite it as ($p > \frac{4}{d}$)

$$\int |u|^{\frac{4}{d}} |\nabla u|^2 = \int |u|^{\frac{4}{d}} |\nabla u|^{\frac{8}{pd}} |\nabla u|^{2-\frac{8}{pd}}.$$

Now by Hölder inequality we obtain that

$$\int |u|^{\frac{4}{d}} |\nabla u|^2 \leq \left(\int |u|^p |\nabla u|^2 \right)^{\frac{4}{pd}} \left(\int |\nabla u|^2 \right)^{1-\frac{4}{pd}}.$$

Then inequality (2.8) takes the form:

$$\frac{d}{dt} w(t) + 2av(t) \leq \frac{4}{d} v(t)^{\frac{4}{pd}} w(t)^{1-\frac{4}{pd}}.$$

where $w(t) = \int |\nabla u|^2$ and $v(t) = \int |u|^p |\nabla u|^2$.

Using Young's inequality $ab \leq \epsilon a^q + C \epsilon^{-\frac{1}{q-1}} b^q$, $\frac{1}{q} + \frac{1}{q'} = 1$, with $q = \frac{pd}{4}$ and $\epsilon = \frac{ad}{2}$ we obtain :

$$\frac{d}{dt} w(t) \leq C a^{-\frac{1}{\frac{pd}{4}-1}} w(t),$$

where $C = C_{p,d} = \frac{pd-4}{pd}$.

This ensures that:

$$w(t) \leq w(0) e^{Ca^{\left(-\frac{4}{pd-4}\right)} t}. \square$$

This show that the H^1 -norm of u is bounded for any time and gives directly the proof of part one of Theorem 2.1 in the case $p > 4/d$.

Now we will prove the global existence for small data, for this we will use the following generalized Gagliardo-Nirenberg inequalities (see for instance [Fri76]):

Lemma 2.8. *Let q, r be any real numbers satisfying $1 \leq q, r \leq \infty$, and let j, m be any integers satisfying $0 \leq j < m$. If u is any functions in $C_0^m(\mathbb{R}^d)$, then*

$$\|D^j u\|_{L^s} \leq C \|D^m u\|_r^a \|u\|_q^{1-a}$$

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where

$$\frac{1}{s} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1,$$

where C is a constant depending only on d, m, j, q, r and a .

As a direct consequence we get :

Lemma 2.9. *Let $1 \leq p \leq 2$ and $v \in C_0^\infty(\mathbb{R}^d)$ then:*

$$\int |v|^{\frac{4}{d}+2+p} \leq C \left(\int |\nabla(|v|^{\frac{p+2}{2}})|^2 \right) \times \left(\int |v|^2 \right)^{\frac{2}{d}}.$$

where $c > 0$ depending only on d and p .

Proof: Take $s = \frac{\frac{4}{d}+2+p}{1+\frac{p}{2}}$, $q = \frac{2}{1+\frac{p}{2}}$, $r = 2$, $j = 0$ and $m = 1$, then by Lemma 2.8 we obtain that:

$$|u|_{L^{\frac{4}{d}+2+p}} \leq C |u|_{L^2}^{\frac{4+2p}{\frac{8}{d}+4+2p}} |u|_{L^{\frac{2}{1+\frac{p}{2}}}}^{\frac{8}{d}}.$$

Taking $u = |v|^{1+\frac{p}{2}}$, we obtain our lemma. \square

Now we can prove the following proposition:

Proposition 2.10. *Let $1 \leq p \leq 2$. There exists $0 < \alpha = \alpha(p, d) < \|Q\|_{L^2}$, such that for any $u_0 \in H^1$ with $\|u_0\|_{L^2} < \alpha$, it holds*

$$\frac{d}{dt} E(u(t)) \leq 0, \quad \forall t > 0.$$

where the energy E is defined in (2.4).

Proof: We can check that

$$\frac{d}{dt} E(u(t)) = a \left(C_p \int |u|^{\frac{4}{d}+p+2} - \frac{4}{(p+2)^2} \int |\nabla(|u|^{\frac{p+2}{2}})|^2 \right),$$

and by Lemma 2.9 we obtain

$$\frac{d}{dt} E(u(t)) \leq a \left(\int |\nabla(|u|^{\frac{p+2}{2}})|^2 \right) \left(C_p C \left(\int |u|^2 \right)^{\frac{2}{d}} - \frac{4}{(p+2)^2} \right).$$

Choosing $\alpha^{\frac{2}{d}} < \frac{4}{(p+2)^2} \frac{1}{C_p C}$, and using that $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$ for all $t \geq 0$ (see 2.14 below) we get the result. \square

Now the proof of part (2) of Theorem 2.1 follows from the sharp Gagliardo-Nirenberg inequality :

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right).$$

Proposition 2.10 together with the above inequality ensure that the H^1 -norm of u is uniformly bounded in time. This leads to the global existence result for small initial data when $1 \leq p \leq 2$.

2.2.1 Critical case ($p = \frac{4}{d}$)

Now we will treat the critical case and prove Theorem 2.3. First let us prove that, if the solution blows up in finite time T , then $\|u\|_{L^{\frac{4}{d}+2}([0,T];L^{\frac{4}{d}+2}(\mathbb{R}^d))} = +\infty$.

Proposition 2.11. *Let u be the unique maximal solution of (2.1) in $[0, T^*)$; if $T^* < \infty$, then $\|u\|_{L^\sigma([0,T],L^\sigma)} = \infty$ where $\sigma = \frac{4}{d} + 2$.*

To prove this claim, denoting by $S(\cdot)$ the free evolution of the linear Schrödinger equation and defining the notion of admissible pair in the following way : An ordered pair (q, r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 < q \leq \infty$, we will use the following proposition:

Proposition 2.12. *There exists $\delta > 0$ with the following property. If $u_0 \in L^2(\mathbb{R}^d)$ and $T \in (0, \infty]$ are such that $\|S(\cdot)u_0\|_{L^\sigma([0,T],L^\sigma)} < \delta$, there exists a unique solution $u \in C([0, T], L^2(\mathbb{R}^d)) \cap L^\sigma([0, T], L^\sigma(\mathbb{R}^d))$ of (2.1). In addition, $u \in L^q([0, T], L^r(\mathbb{R}^d))$ for every admissible pair (q, r) ; for $t \in [0, T]$. Finally, u depends continuously in $C([0, T], L^2(\mathbb{R}^d)) \cap L^\sigma([0, T], L^\sigma(\mathbb{R}^d))$ on $u_0 \in L^2(\mathbb{R}^d)$. If $u_0 \in H^1(\mathbb{R}^d)$, then $u \in C([0, T], H^1(\mathbb{R}^d))$.*

See [CW89] for the proof.

We need also the following lemma (see [CW89]):

Lemma 2.13. *Let $T \in (0, \infty]$, let $\sigma = \frac{4}{d} + 2$, and let (q, r) be an admissible pair. Then, whenever $u \in L^\sigma([0, T], L^\sigma(\mathbb{R}^d))$, it follows that $F(u) = -i \int_0^t S(t-s)(|u|^{\frac{4}{d}}u + ia|u|^p u) ds \in C([0, T], H^{-1}(\mathbb{R}^d)) \cap L^q(0, T, L^r(\mathbb{R}^d))$. Furthermore, there exists K , independent of T , such that*

$$\|Fv - Fu\|_{L^q([0,T],L^r)} < K(\|u\|_{L^\sigma([0,T],L^\sigma(\mathbb{R}^d))}^{\frac{4}{d}} + \|v\|_{L^\sigma([0,T],L^\sigma(\mathbb{R}^d))}^{\frac{4}{d}})\|u - v\|_{L^\sigma([0,T],L^\sigma(\mathbb{R}^d))} \quad (2.9)$$

for every $u, v \in L^\sigma([0, T], L^\sigma(\mathbb{R}^d))$.

Proof of Proposition 2.11:

Let $u_0 \in L^2(\mathbb{R}^d)$. Observe that $\|S(\cdot)u_0\|_{L^\sigma(0,T,L^\sigma)} \rightarrow 0$ as $T \rightarrow 0$. Thus for sufficiently small T , the hypotheses of Proposition 2.12 are satisfied. Applying iteratively this proposition, we can construct the maximal solution $u \in C([0, T^*), L^2(\mathbb{R}^d)) \cap L^\sigma([0, T^*), L^\sigma(\mathbb{R}^d))$ of (2.1). We proceed by contradiction, assuming that $T^* < \infty$, and $\|u\|_{L^\sigma([0,T],L^\sigma)} < \infty$. Let $t \in [0, T^*)$. For every $s \in [0, T^* - t)$ we have

$$S(s)u(t) = u(t+s) - F(u(t+\cdot))(s).$$

From (2.9), we thus obtain

$$\|S(\cdot)u(t)\|_{L^\sigma([0,T^*-t],L^\sigma(\mathbb{R}^d))} \leq \|u\|_{L^\sigma([t,T^*],L^\sigma)} + K(\|u\|_{L^\sigma([t,T^*],L^\sigma)})^{\frac{4}{d}+1}$$

Therefore, for t fixed close enough to T^* , it follows that

$$\|S(\cdot)u(t)\|_{L^\sigma([0,T^*-t],L^\sigma(\mathbb{R}^d))} \leq \delta.$$

Applying Proposition 2.12, we find that u can be extended after T^* , which contradicts the maximality.

Corollary 2.14. *For $p = \frac{4}{d}$, the solution of (2.1) is global.*

Proof Multiply equation (2.1) by \bar{u} , and take the imaginary part to obtain:

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2a \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = 0.$$

Hence $\forall t \in \mathbb{R}_+$

$$\|u\|_{L^{\frac{4}{d}+2}_{[0,t][L^{\frac{4}{d}+2}(\mathbb{R}^d)]}}^{\frac{4}{d}+2} \leq \frac{1}{2a} \|u_0\|_{L^2}^2.$$

The global existence follows then directly from Proposition 2.11. Now to finish the proof of Theorem 2.3, we will prove the scattering:

Let $v(t) = e^{-it\Delta} u(t) := S(-t)u(t)$ then

$$v(t) = u_0 + i \int_0^t S(-s)(|u(s)|^{\frac{4}{d}} u(s) + ia|u(s)|^{\frac{4}{d}} u) ds.$$

Therefore for $0 < t < \tau$,

$$v(t) - v(\tau) = i \int_\tau^t S(-s)(|u(s)|^{\frac{4}{d}} u(s) + ia|u(s)|^{\frac{4}{d}} u) ds.$$

It follows from Strichartz's estimates that:

$$\|v(t) - v(\tau)\|_{L^2} = \|i \int_\tau^t S(-s)(|u(s)|^{\frac{4}{d}} u(s) + ia|u(s)|^{\frac{4}{d}} u) ds\|_{L^2} \leq C \|u\|_{L^{\frac{4}{d}+2}_{([t,\tau] \times \mathbb{R}^d)}}^{\frac{4}{d}+1}.$$

The right hand side goes to zero when $t, \tau \rightarrow +\infty$, then scattering follows from the Cauchy criterion.

This completes the proof of Theorem 2.3.

2.3 Proof of part (3) of Theorem 2.1

Special solutions play a fundamental role for the description of the dynamics of (NLS). They are the solitary waves of the form $u(t, x) = \exp(it)Q(x)$, where Q solves:

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (2.10)$$

The pseudo-conformal transformation applied to the stationary solution $e^{it}Q(x)$ yields an explicit solution for (NLS)

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}$$

which blows up at $T = 0$.

Note that

$$\|S(t)\|_{L^2} = \|Q\|_{L^2} \quad \text{and} \quad \|\nabla S(t)\|_{L^2} \sim \frac{1}{t} \quad (2.11)$$

It turns out that $S(t)$ is the unique minimal mass blow-up solution in H^1 up to the symmetries of the equation (see [Mer93]).

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On the other hand, a known lower bound (see [Caz03]) on the blow-up rate for (NLS) is

$$\|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}}. \quad (2.12)$$

Note that this lower bound is strictly smaller than the blow-up rate of $S(t)$ given by (2.11) and the log-log one given by (2.5). The assertion (3) of Theorem 2.1 enables to exclude some blow-up rates for initial data with critical or subcritical mass. To prove this assertion, we will need the following theorem (see [HK05]):

Theorem 2.15. *Let $(v_n)_n$ be a bounded family of $H^1(\mathbb{R}^d)$, such that:*

$$\limsup_{n \rightarrow +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m. \quad (2.13)$$

Then, there exists $(x_n)_n \subset \mathbb{R}^d$ such that:

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly,}$$

$$\text{with } \|V\|_{L^2(\mathbb{R}^d)} \geq \left(\frac{d}{d+4}\right)^{\frac{d}{4}} \frac{m^{\frac{d}{2}+1} + 1}{M^{\frac{d}{2}}} \|Q\|_{L^2(\mathbb{R}^d)}.$$

Let us recall the following quantities:

$$L^2\text{-norm : } \|u(t, x)\|_{L^2} = \int |u(t, x)|^2 dx.$$

$$\text{Energy : } E(u(t, x)) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4+2d} \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

$$\text{Kinetic momentum : } P(u(t)) = \text{Im} \left(\int \nabla u \bar{u}(t, x) \right).$$

Remark 2.16. *It is easy to prove that if u is a solution of (2.1) on $[0, T[$ given by Theorem 2.6, then for all $t \in [0, T[$ it holds*

$$\frac{d}{dt} \|u(t)\|_{L^2} = -2a \int |u|^{p+2}, \quad (2.14)$$

$$\frac{d}{dt} E(u(t)) = -a \left(\|u^{\frac{p}{2}} \nabla u\|_{L^2}^2 - C_p \|u\|_{L^{\frac{4}{d}+2+p}}^{\frac{4}{d}+2+p} \right) \quad (2.15)$$

and

$$\frac{d}{dt} P(u(t)) = -2a \text{Im} \int \bar{u} |u|^p \nabla u. \quad (2.16)$$

where $C_p = \frac{4+2d+pd}{4+2d}$.

Now we are ready to prove part (3) of Theorem 2.1:

Suppose that there exist an initial data u_0 with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, such that the corresponding solution $u(t)$ blows up with the following law

$$\frac{1}{(T-t)^{\beta-\epsilon}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^{\beta+\epsilon}},$$

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where β ; $0 < \beta < \beta(p, d) = \frac{2}{pd}$ and $0 < \epsilon < \frac{2-\beta pd}{8+pd}$.

Recall that

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T[, \quad (2.17)$$

where $K(u(t)) = (\|u^{\frac{p}{2}} \nabla u\|_{L^2}^2 - C_p \|u\|_{L^{\frac{4}{d}+2+p}}^{\frac{4}{d}+2+p})$.

By Gagliardo-Nirenberg inequality and (2.14), we have:

$$E(u(t)) \lesssim E(u_0) + \int_0^t \|u\|_{L^2}^{\frac{4}{d}+p-\frac{p}{2}} \|\nabla u\|_{L^2}^{2+\frac{pd}{2}} \lesssim E(u_0) + \int_0^t \|\nabla u\|_{L^2}^{2+\frac{pd}{2}}$$

By the choice of ϵ , we obtain that

$$0 \leq \lim_{t \rightarrow T} \frac{\int_0^t \|\nabla u(\tau)\|_{L^2}^{2+\frac{pd}{2}} d\tau}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2} \lesssim \lim_{t \rightarrow T} (T-t)^{\frac{1}{2}(2-\beta pd - \epsilon(8+pd))} = 0. \quad (2.18)$$

Let

$$\rho(t) = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}} \quad \text{and} \quad v(t, x) = \rho^{\frac{d}{2}} u(t, \rho x).$$

Moreover, let $(t_k)_k$ be a sequence such that $t_k \rightarrow T$ and set $\rho_k = \rho(t_k)$ and $v_k = v(t_k, \cdot)$. Clearly $\rho_k \rightarrow 0$ as $k \rightarrow \infty$ and the family $(v_k)_k$ satisfies

$$\|v_k\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \leq \|Q\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(\mathbb{R}^d)} = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

Remark that $\lim_{k \rightarrow +\infty} E(v_k) = 0$, because:

$$\begin{aligned} 0 \leq \frac{1}{2} \left(\int |\nabla v_k|^2 \right) \left(1 - \left(\frac{\int |v_k|^2}{\int |Q|^2} \right)^2 \right) &\leq E(v_k) = \rho_k^2 E(u_0) - a \rho_k^2 \int_0^{t_k} K(u(\tau)) d\tau \\ &\leq \rho_k^2 E(u_0) + \frac{1}{\|\nabla u(t_k)\|_{L^2(\mathbb{R}^d)}^2} \int_0^{t_k} \|\nabla u(\tau)\|_{L^2}^{2+\frac{pd}{2}} d\tau \end{aligned}$$

which tends to zero with k according to (2.18). This ensures that

$$\|v_k\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \rightarrow \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2. \quad (2.19)$$

The family $(v_k)_k$ satisfies the hypotheses of Theorem 2.15 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad \text{and} \quad M = \|\nabla Q\|_{L^2(\mathbb{R}^d)},$$

thus there exists a family $(x_k)_k \subset \mathbb{R}$ and a profile $V \in H^1(\mathbb{R})$ with $\|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$, such that,

$$\rho_k^{\frac{d}{2}} u(t_k, \rho_k \cdot + x_k) \rightharpoonup V \in H^1 \quad \text{weakly.} \quad (2.20)$$

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Using (2.20), $\forall A \geq 0$

$$\liminf_{n \rightarrow +\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{B(0,A)} |V|^2 dx.$$

But $\lim_{n \rightarrow +\infty} \frac{1}{\rho_n} = +\infty$. Hence, $\rho_n A < 1$ for n large enough. This gives immediately:

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \liminf_{n \rightarrow +\infty} \int_{|x-x_n| \leq \rho_n A} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx.$$

Since this is true for all $A > 0$ we finally obtain

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \int Q^2. \quad (2.21)$$

Noticing that (2.14) implies that $t \mapsto \|u(t)\|_{L^2}$ is decreasing as soon as u is not the null solution, we get

$$\|u_0\|_{L^2} > \liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq 1} |u(t_n, x)|^2 dx \geq \|Q\|_{L^2}^2.$$

This completes the proof of assertion (3) of Theorem 2.1 .

2.4 Blow up solutions.

In this section, we will prove the existence of the explosive solutions in the case $1 \leq p < 4/d$.

Theorem 2.17. *Let $1 \leq p < 4/d$. There exist a set of initial data Ω in H^1 , such that for any $0 < a < a_0$ with $a_0 = a_0(p)$ small enough, the emanating solution $u(t)$ to (2.1) blows up in finite time in the log-log regime.*

The set of initial data Ω is the set described in [MR02a] in order to initialize the log-log regime. It is open in H^1 . Using the continuity with regard to the initial data and the parameters, we easily obtain the following corollary:

Corollary 2.18. *Let $1 \leq p < 4/d$ and $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (2.2) blows up in the loglog regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (2.1) with the initial data v_0 blows up in finite time.*

Assertion (4) of Theorem 2.1 now follows directly from this corollary together with the results of [MR02a] on the L^2 -critical NLS equation and a scaling argument in order to drop the smallness condition on the damped coefficient $a > 0$.

To prove Theorem 2.17, we look for a solution of (2.1) such that for t close enough to blowup time, we shall have the following decomposition:

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} (Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad (2.22)$$

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for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}$, here $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$, and the profiles Q_b are suitable deformations of Q related to some extra degeneracy of the problem.

Note that we will abbreviate our proof because it is very very close to the case of linear damping ($p = 0$ see Darwich[Dar12a]). Actually, as noticed in [PR07], we only need to prove that in the log-log regime the L^2 norm does not grow, and the growth of the energy (resp the momentum) is below $\frac{1}{\lambda^2}$ (resp $\frac{1}{\lambda}$). In this chapter, we will prove that in the log-log regime, the growth of the energy and the momentum are bounded by:

$$E(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{-\frac{pd}{2}}, \quad P(u(t)) \lesssim \log(\lambda(t))\lambda(t)^{1-\frac{pd}{4}}.$$

Let us recall that a function $u : [0, T] \mapsto H^1$ follows the log-log regime if the following uniform controls on the decomposition (2.22) hold on $[0, T]$:

- Control of $b(t)$

$$b(t) > 0, \quad b(t) < 10b(0). \quad (2.23)$$

- Control of λ :

$$\lambda(t) \leq e^{-e^{-\frac{\pi}{100b(t)}}} \quad (2.24)$$

and the monotonicity of λ :

$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \quad \forall 0 \leq t_1 \leq t_2 \leq T. \quad (2.25)$$

Let $k_0 \leq k_+$ be integers and $T^+ \in [0, T]$ such that

$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0-1}}, \quad \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+-1}} \quad (2.26)$$

and for $k_0 \leq k \leq k_+$, let t_k be a time such that

$$\lambda(t_k) = \frac{1}{2^k}, \quad (2.27)$$

then we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda^2(t_k). \quad (2.28)$$

- control of the excess of mass:

$$\int |\nabla \epsilon(t)|^2 + \int |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}. \quad (2.29)$$

2.4.1 Control of the energy and the kinetic momentum in the log-log regime

We recall the Strichartz estimates. An ordered pair (q, r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 < q \leq \infty$. We define the Strichartz norm of functions $u : [0, T] \times \mathbb{R}^d \mapsto C$ by:

$$\|u\|_{S^0([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (2.30)$$

and

$$\|u\|_{S^1([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|\nabla u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (2.31)$$

We will sometimes abbreviate $S^i([0, T] \times \mathbb{R}^d)$ with S_T^i or $S^i[0, T]$, $i = 1, 2$. Let us denote the Hölder dual exponent of q by q' so that $\frac{1}{q} + \frac{1}{q'} = 1$. The Strichartz estimates may be expressed as:

$$\|u\|_{S_T^0} \leq \|u_0\|_{L^2} + \|(i\partial_t + \Delta)u\|_{L_t^{q'} L_x^{r'}} \quad (2.32)$$

where (q, r) is any admissible pair. Now we will derive an estimate on the energy, to check that it remains small with respect to λ^{-2} :

Lemma 2.19. *Assuming that (2.23)-(2.29) hold, then the energy and kinetic momentum of the solution u to (2.1) are controlled on $[0, T]$ by:*

$$|E(u(t))| \leq C(\log(\lambda(t))\lambda(t)^{-\frac{pd}{4}}), \quad (2.33)$$

$$|P(u(t))| \leq C(\log(\lambda(t))\lambda(t)^{1-\frac{pd}{4}}). \quad (2.34)$$

To prove this lemma, we shall need the following one that follows from the classical well-posedness theory (see for instance [Caz03]).

Lemma 2.20. *Let u be a solution of (2.1) emanating for u_0 in H^1 . Then $u \in C([0, \Delta T], H^1)$ where $\Delta T = \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u_0\|_{H^1}^{-2}$, and we have the following control*

$$\|u\|_{S^0([t, t+\Delta T])} \leq 2 \|u_0\|_{L^2}, \quad \|u\|_{S^1([t, t+\Delta T])} \leq 2 \|u_0\|_{H^1(\mathbb{R}^d)}.$$

Proof of Lemma 2.19: According to (1.51) each interval $[t_k, t_{k+1}]$, can be divided into k intervals, $[\tau_k^j, \tau_k^{j+1}]$ such that the estimates of the previous lemma are true. From (2.15) and the Gagliardo-Nirenberg inequality, we obtain that:

$$\frac{d}{dt} E(u(t)) \lesssim \|u\|_{L^2}^{\frac{4}{d} + p - \frac{pd}{2}} \|\nabla u\|_{L^2}^{2 + \frac{pd}{2}}$$

Using (2.14) this gives

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt} E(u(t)) dt \leq C \int_{\tau_k^j}^{\tau_k^{j+1}} \|\nabla u(t)\|_{L^2}^{2 + \frac{pd}{2}},$$

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then by Lemma 2.20, we obtain that:

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt} E(u(t)) dt \leq C(\tau_k^{j+1} - \tau_k^j) \lambda^{-2 - \frac{pd}{2}}(\tau_k^j)$$

Note that $\tau_k^{j+1} - \tau_k^j \sim \lambda^2(\tau_k^j) \sim \lambda^2(t_k)$, then

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt} E(u(t)) dt \leq C \lambda^{-\frac{pd}{2}}(t_k)$$

Summing from $j = 1$ to $J_k \leq CK$, we obtain that:

$$\sum_{j=1}^{J_k} \int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt} E(u(t)) dt \leq Ck \lambda^{-\frac{pd}{2}}(t_k)$$

Now taking $T^+ = T$ and summing from K_0 to K^+ , we obtain:

$$\int_0^{T^+} \frac{d}{dt} E(u(t)) dt \leq CK^+ \lambda^{-\frac{pd}{2}}(T^+) \lesssim C \log(\lambda(T)) \lambda^{-\frac{pd}{2}}(T).$$

Note that $\log(\lambda(T)) \lambda^{-\frac{pd}{2}}(T)$ is small with respect to $\frac{1}{\lambda^2}$ because $p < \frac{4}{d}$.

Now we prove **(2.34)**: From (2.16) we have :

$$\left| \frac{d}{dt} P(u(t)) \right| \leq \int |\bar{u}| |\nabla u| |u|^p$$

By Gagliardo-Nirenberg inequality we have:

$$\|u\|_{L^{2p+2}}^{2p+2} \leq \|u\|_{L^2}^{2p+2-dp} \|\nabla u\|_{L^2}^{dp}$$

then

$$\frac{d}{dt} P(u(t)) \leq \left(\int |u|^{2(p+1)} \right)^{\frac{1}{2}} \|\nabla u\|_{L^2} \leq C \|\nabla u\|_{L^2}^{1 + \frac{pd}{2}}.$$

Hence,

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt} P(u(t)) \leq C(\tau_k^{j+1} - \tau_k^j) \left\| \nabla u(\tau_k^j) \right\|_{L^2}^{1 + \frac{pd}{2}} \leq C \|\nabla u(t_k)\|_{L^2}^{-1 + \frac{pd}{2}}$$

Summing successively into j and k we obtain that:

$$\int_0^{T^+} \frac{d}{dt} P(u(t)) \lesssim \log(\lambda(T^+)) \lambda^{1 - \frac{pd}{2}}(T^+).$$

Remark that this quantity is small with respect to $\frac{1}{\lambda}$ because $p < \frac{4}{d}$. □

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Chapter 3

On the well-posedness for Kadomtsev-Petviashvili-Burgers I equation.

3.1 Introduction

In this chapter, we study the well-posedness of the initial value problem for the Kadomtsev-Petviashvili-Burgers (KPBI) equations in \mathbb{R}^2 :

$$\begin{cases} (\partial_t u + u_{xxx} - u_{xx} + uu_x)_x - u_{yy} = 0, \\ u(0, x, y) = \phi(x, y). \end{cases} \quad (3.1)$$

where u is a real-valued function of $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+$. Note that if we replace $-u_{yy}$ by $+u_{yy}$, (3.1) becomes the KPBI equation.

This equation, models in some regime the wave propagation in electromagnetic saturated zone (cf. [Leb02]). More generally, be considered as a toys model for two-dimensional wave propagation taking into account the effect of viscosity. Note that since we are interested in an almost unidirectional propagation, the dissipative term acts only in the main direction of the propagation in KPBI. This equation is a dissipative version of the Kadomtsev-Petviashvili-I equation (KPI) :

$$(\partial_t u + u_{xxx} + uu_x)_x - u_{yy} = 0. \quad (3.2)$$

which is a "universal" model for nearly one directional weakly nonlinear dispersive waves, with weak transverse effects and strong surface tension effects. Bourgain had developed a new method, clarified by Ginibre in [Gin95], for the study of Cauchy problem associated with non-linear dispersive equations. This method was successfully applied to the nonlinear Schrödinger, KdV as well as KPBI equations. It was shown by Molinet-Ribaud [MR02b] that the Bourgain spaces can be used to study the Cauchy problems associated to semi-linear equations with a linear part containing both dispersive and dissipative terms (and consequently this applies to KPBI equations).

By introducing a Bourgain space associated to the usual KPI equation (related only to the dispersive part of the linear symbol in the KPBI equation), Molinet-Ribaud [MR02b] had proved local existence for the Cauchy problem associated to the KPBI equation when the initial value in

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$H^{s_1, s_2}(\mathbb{R}^2)$, $s_1 > 0$ and $s_2 \geq 0$. These solutions can be globally extended in time whenever the initial data have finite energy. Recall that the energy for the KP-I equation reads :

$$E(\phi) = \frac{1}{2} \int (\phi_x)^2 + \frac{1}{2} \int (\partial_x^{-1} \phi_y)^2 - \frac{1}{6} \int \phi^3.$$

Kojok [Koj10] had proved the local and global existence for (3.1) for small initial data in $L^2(\mathbb{R}^2)$. In this chapter, we improve the results obtained by Molinet-Ribaud and Kojok, by proving the local and global existence for the KPBI equation , with initial value $\phi \in H^{s_1, 0}$ when $s_1 > -\frac{1}{2}$. Note that this proves, in particular, the global well-posedness for arbitrary data in $L^2(\mathbb{R}^2)$ and avoid to impose some constraints on the low frequencies of the initial data to have finite energy.

The main new ingredient is a trilinear estimate for the KPI equation proved in [IKT08]. Following [MR02c], we introduce a Bourgain space associated to the KPBI equation. This space is in fact the intersection of the space introduced in [Bou93c] and of a Sobolev space linked to the dissipative effect. The advantage of this space is that it contains both the dissipative and dispersive parts of the linear symbol of (3.1).

This paper is organized as follows. In Section 3.2, we introduce our notations and we give an extension of the semi-group of the KPBI equation by a linear operator defined on the whole real axis. In Section 3.3 we derive linear estimates and some smoothing properties for the operator L defined by (3.17) in the Bourgain spaces . In Section 3.4 we state Strichartz type estimates for the KP equation which yield bilinear estimates. In Section 3.5, using bilinear estimates, a standard fixed point argument and some smoothing properties, we prove uniqueness and global existence of the solution of KPBI equation in anisotropic sobolev space $H^{s, 0}$ with $s > -\frac{1}{2}$. Finally, in section 3.6, we ensures that our local existence result is optimal if one requires the smoothness of the flow-map.

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3.2 Notations and main results

We will use C to denote various time independent constants, usually depending only upon s . In case a constant depends upon other quantities, we will try to make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. similarly, we will write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We write $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2} \sim 1 + |\cdot|$. The notation a^+ denotes $a + \epsilon$ for an arbitrarily small ϵ . Similarly a^- denotes $a - \epsilon$. For a function $f = f(x)$ (resp $f = f(t)$) the space Fourier (time Fourier) transform is defined from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ as:

$$\begin{aligned} \hat{f}(\xi) &:= \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{i(x, \xi)} f(x) dx \\ (\text{resp } \hat{f}(\tau) &:= \mathcal{F}(f)(\tau) = \int_{\mathbb{R}} e^{i(t, \tau)} f(t) dt \end{aligned}$$

For $b \in \mathbb{R}$, we denote respectively by $H^b(\mathbb{R})$ and $\dot{H}^b(\mathbb{R})$ the nonhomogeneous and homogeneous Sobolev spaces which are endowed with the following norms :

$$\|u\|_{H^b}^2 = \int_{\mathbb{R}} \langle \tau \rangle^{2b} |\hat{u}(\tau)|^2 d\tau, \quad \|u\|_{\dot{H}^b}^2 = \int_{\mathbb{R}} |\tau|^{2b} |\hat{u}(\tau)|^2 d\tau \quad (3.3)$$

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For a function $f = f(t, x)$, we define the space-time Fourier transform from $\mathcal{S}'(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$ by :

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{i\langle \lambda, \xi \rangle} f(\lambda) d\lambda, \quad \forall f \in \mathcal{S}'(\mathbb{R}^2).$$

Moreover, we introduce the corresponding space (resp space-time) Sobolev spaces H^{s_1, s_2} (resp H^{b, s_1, s_2}) which are defined by :

$$H^{s_1, s_2}(\mathbb{R}^2) =: \{u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{H^{s_1, s_2}}(\mathbb{R}^2) < +\infty\}, \quad (3.4)$$

$$H^{b, s_1, s_2}(\mathbb{R}^2) =: \{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{H^{b, s_1, s_2}}(\mathbb{R}^3) < +\infty\} \quad (3.5)$$

where,

$$\|u\|_{H^{s_1, s_2}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\nu)|^2 d\nu, \quad (3.6)$$

$$\|u\|_{H^{b, s_1, s_2}}^2 = \int_{\mathbb{R}^2} \langle \tau \rangle^b \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\tau, \nu)|^2 d\nu d\tau, \quad (3.7)$$

and $\nu = (\xi, \eta)$. Let $U(\cdot)$ be the unitary group in H^{s_1, s_2} , $s_1, s_2 \in \mathbb{R}$, defining the free evolution of the (KP-II) equation, which is given by

$$U(t) = \exp(itP(D_x, D_y)), \quad (3.8)$$

where $P(D_x, D_y)$ is the Fourier multiplier with symbol $P(\xi, \eta) = \xi^3 - \eta^2/\xi$. By the Fourier transform, (23) can be written as :

$$\mathcal{F}_x(U(t)\phi) = \exp(itP(\xi, \eta))\hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \in \mathbb{R}. \quad (3.9)$$

Also, by the Fourier transform, the linear part of the equation (3.1) can be written as :

$$i(\tau - \xi^3 - \eta^2/\xi) + \xi^2 =: i(\tau - P(\eta, \xi)) + \xi^2. \quad (3.10)$$

We need to localize our solution, and the idea of Bourgain has been to consider this localisation, by defining the space $X^{b, s}$ equipped by the

$$\|u\|_{X^{b, s_1, s_2}} = \|\langle i(\tau - P(\eta, \xi)) + \xi^2 \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \tilde{u}(\tau, \xi, \eta)\|_{L^2(\mathbb{R}^3)}. \quad (3.11)$$

We will need to define the decomposition of Littlewood-Paley. Let $\eta \in C_0(\mathbb{R})$ be such that $\eta \geq 0$, $\text{supp } \eta \subset [-2, 2]$, $\eta = 1$ on $[-1, 1]$. We define next $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$.

Any summations over capitalized variables such as N, L are presumed to be dyadic, i.e. these variables range over numbers of the form $N = 2^j$, $j \in \mathbb{Z}$ and $L = 2^l$, $l \in \mathbb{N}$. We set $\varphi_N(\xi) = \varphi(\frac{\xi}{N})$ and define the operator P_N by $\mathcal{F}_x(P_N u) = \varphi_N \mathcal{F}_x(u)$. We introduce $\psi_L(\tau, \zeta) = \varphi_L(\tau - P(\zeta))$ and for any $u \in \mathcal{S}(\mathbb{R}^2)$,

$$\mathcal{F}_x(P_N u(t))(\xi) = \varphi_N(\xi) \mathcal{F}_x(u)(t, \xi), \quad \mathcal{F}(Q_L u)(\tau, \xi, \eta) = \psi_L(\tau, \zeta) \mathcal{F}(u)(\tau, \xi); L > 1$$

and $\mathcal{F}(Q_1 u)(\tau, \xi, \eta) = \eta(\tau - P(\zeta)) \mathcal{F}(u)(\tau, \xi)$. Roughly speaking, the operator P_N localizes in the annulus $\{|\xi| \sim N\}$ where as Q_L localizes in the region $\{|\tau - P(\zeta)| \sim L\}$. We denote $P_N u$ by u_N , $Q_L u$ by u_L and $P_N(Q_L u)$ by $u_{N, L}$.

For $T \geq 0$, we consider the localized Bourgain spaces X_T^{b, s_1, s_2} endowed with the norm

$$\|u\|_{X_T^{b, s_1, s_2}} = \inf_{w \in X^{b, s_1, s_2}} \{\|w\|_{X^{b, s_1, s_2}}, w(t) = u(t) \text{ on } [0, T]\}.$$

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We also use the space-time Lebesgue space $L_{t,x}^{p,q}$ endowed with the norm

$$\|u\|_{L_{t,x}^{q,r}} = \left\| \|u\|_{L_x^r} \right\|_{L_t^q},$$

and we will use the notation $L_{t,x}^2$ for $L_{t,x}^{2,2}$.

We denote by $W(\cdot)$ the semigroup associated with the free evolution of the KPB equations,

$$\mathcal{F}_x(W(t)\phi) = \exp(itP(\xi, \eta) - |\xi|^2 t)\hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \geq 0. \quad (3.12)$$

Also, we can extend W to a linear operator defined on the whole real axis by setting,

$$\mathcal{F}_x(W(t)\phi) = \exp(itP(\xi, \eta) - |\xi|^2 |t|)\hat{\phi}, \quad \forall \phi \in \mathcal{S}'(\mathbb{R}^2), \quad t \in \mathbb{R}. \quad (3.13)$$

By the Duhamel integral formulation, the equation (3.1) can be written as

$$u(t) = W(t)\phi - \frac{1}{2} \int_0^t W(t-t')\partial_x(u^2(t'))dt', \quad t \geq 0. \quad (3.14)$$

To prove the local existence result, we will apply a fixed point argument to the extension of (3.14), which is defined on whole the real axis by:

$$u(t) = \psi(t)[W(t)\phi - L(\partial_x(\psi_T^2 u^2))(x, t)], \quad (3.15)$$

where $t \in \mathbb{R}$, ψ indicates a time cutoff function :

$$\psi \in C_0^\infty(\mathbb{R}), \quad \sup \psi \subset [-2, 2], \quad \psi = 1 \text{ on } [-1, 1], \quad (3.16)$$

$\psi_T(\cdot) = \psi(\cdot/T)$, and

$$L(f)(x, t) = W(t) \int e^{ix\xi} \frac{e^{it\tau} - e^{-|t|\xi^2}}{it + \xi^2} \mathcal{F}(W(-t)f)(\xi, \tau) d\xi d\tau. \quad (3.17)$$

One easily sees that

$$\chi_{\mathbb{R}_+}(t)\psi(t)L(f)(x, t) = \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-\tau)f(\tau)d\tau. \quad (3.18)$$

Indeed, taking $w = W(\cdot)f$, the right hand side of (3.18) can be rewritten as

$$W(t)\left(\chi_{\mathbb{R}_+}(t)\psi(t) \int e^{ix\xi} \frac{e^{it\tau} - e^{-|t|\xi^2}}{it + \xi^2} \hat{w}(\xi, \tau') d\xi d\tau'\right).$$

In [MR02c], the authors performed the iteration process in the space $X^{s,b}$ equipped with the norm:

$$\|u\|_{X^{b,s_1,s_2}} = \|\langle i(\tau - P(\nu)) + \xi^2 \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{u}(\tau, \nu)\|_{L^2(\mathbb{R}^3)},$$

which take advantage of the mixed dispersive-dissipative part of the equation. We will rather work in its Besov version $X^{s,b,q}$ (with $q = 1$) defined as the weak closure of the test functions that are uniformly bounded by the norm

$$\|u\|_{X^{b,s,0,q}} = \left(\sum_N \left[\sum_L \langle L + N^2 \rangle^{bq} \langle N \rangle^{sq} \|P_N Q_L u\|_{L_{x,y,t}^2}^q \right]^{\frac{2}{q}} \right)^{\frac{1}{2}}.$$

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Remark 3.1. *It is clear that if u solves (3.15) then u is a solution of (3.14) on $[0, T]$, $T < 1$. Thus it is sufficient to solve (3.15) for a small time ($T < 1$ is enough).*

Definition 3.2. *The Cauchy problem (3.1) is locally well-posed in the space X if for any $\phi \in X$ there exists $T = T(\|\phi\|_X) > 0$ and a map F from X to $C([0, T]; X)$ such that $u = F(\phi)$ is the unique solution for the equation (3.1) in some space $Y \hookrightarrow C([0, T]; X)$ and F is continuous in the sense that*

$$\|F(\phi_1) - F(\phi_2)\|_{L^\infty([0, T]; X)} \leq M(\|\phi_1 - \phi_2\|_X, R)$$

for some locally bounded function M from $\mathbb{R}^+ \times \mathbb{R}^+$ to \mathbb{R}^+ such that $M(S, R) \rightarrow 0$ for fixed R when $S \rightarrow 0$ and for $\phi_1, \phi_2 \in X$ such that $\|\phi_1\|_X + \|\phi_2\|_X \leq R$.

Remark 3.3. *We obtain the global existence if we can extend the solutions to all $t \in \mathbb{R}^+$, by iterating the result of local existence, in this case we say that the Cauchy problem is globally well posed.*

The global existence of the solution to our equation will be obtained thanks to the regularizing effect of the dissipative term and the fact that the L^2 norm is not increasing for real-valued initial data.

Let us now state our results:

Theorem 3.4. *Let $s_1 > -1/2$ and $\phi \in H^{s_1, 0}$. Then there exists a time $T = T(\|\phi\|_{H^{-\frac{1}{2}, 0}}) > 0$ and a unique solution u of (3.1) in*

$$Y_T = X_T^{1/2, s_1, 0, 1}. \quad (3.19)$$

The map $\phi \mapsto u$ is C^∞ from the ball of $H^{s_1, 0}$ with radius $R > 0$ into $Y_{T(R)}$.

Moreover, in the case of real-valued initial data, the solution is global in time and belongs to $C(\mathbb{R}_+; H^{s_1, 0}) \cap C(\mathbb{R}_+^; H^{\infty, 0})$.* \square

Remark 3.5. *Proceeding as in [MR01], it can be easily seen that Theorem 3.4 also holds when replacing $H^{s_1, 0}$, $X_T^{1/2, s_1, 0, 1}$ and $H^{\infty, 0}$ by respectively H^{s_1, s_2} , $X_T^{1/2, s_1, s_2, 1}$ and H^{∞, s_2} , with $s_2 \geq 0$.*

Theorem 3.6. *Let $s < -1/2$. Then it does not exist a time $T > 0$ such that the equation (3.1) admits a unique solution in $C([0, T], H^{s, 0})$ for any initial data in some ball of $H^{s, 0}(\mathbb{R}^2)$ centered at the origin and such that the map*

$$\phi \mapsto u \quad (3.20)$$

is C^2 -differentiable at the origin from $H^{s, 0}$ to $C([0, T], H^{s, 0})$. \square

The principle of the proof of local existence result holds in two steps:

Step 1: In order to apply a standard argument of fixed point, we want to estimate the two terms: free term and the forcing term of equation (3.15). A first step is to show using Fourier analysis, that the map $\phi \mapsto \psi(t)W(t)\phi$ is bounded from $H^{s, 0}$ to $X^{\frac{1}{2}, s, 0, 1}$ and the map L is also bounded from $X^{-\frac{1}{2}, s, 0, 1}$ to $X^{\frac{1}{2}, s, 0, 1}$.

Step 2: We treat the bilinear term, by proving that the map $(u, v) \mapsto \partial_x(uv)$ is bounded from $X^{\frac{1}{2}, s, 0, 1} \times X^{\frac{1}{2}, s, 0, 1}$ to $X^{-\frac{1}{2}, s, 0, 1}$.

3.3 Linear Estimates

In this section, we mainly follow Molinet-Ribaud [MR02c] (see also [GW09] and [MV11] for the Besov version) to estimate the linear term in the space $X^{\frac{1}{2},s,0,1}$. We start by the free term:

3.3.1 Estimate for the free term

Proposition 3.7. *Let $s \in \mathbb{R}$, then $\forall \phi \in H^{s,0}$, we have:*

$$\|\psi(t)W(t)\phi\|_{X^{\frac{1}{2},s,0,1}} \lesssim \|\phi\|_{H^{s,0}}.$$

Proof. This is equivalent to prove that

$$\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|P_N Q_L(\psi(t)W(t)\phi)\|_{L^2_{x,y,t}} \lesssim \|P_N \phi\|_{L^2_{x,y}} \quad (3.21)$$

for each dyadic N. Using Plancherel, we obtain

$$\begin{aligned} & \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|P_N Q_L(\psi(t)W(t)\phi)\|_{L^2_{x,y,t}} \\ & \lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\hat{\phi}(\xi)\mathcal{F}_t(\psi(t)e^{-|t|\xi^2}e^{itP(v)})(\tau)\|_{L^2_{\xi,\eta,\tau}} \\ & \lesssim \|P_N \phi\|_{L^2} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)P_L(\psi(t)e^{-|t|\xi^2})\|_{L^\infty_{\xi}L^2_{\tau}}. \end{aligned} \quad (3.22)$$

Note that from Prop 4.1 in [MV11] we have:

$$\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)P_L(\psi(t)e^{-|t|\xi^2})\|_{L^\infty_{\xi}L^2_{\tau}} \lesssim 1. \quad (3.23)$$

Combining (3.23) and (3.22), we obtain the result. \square

3.3.2 Estimates for the forcing term

Now we shall study in $X^{\frac{1}{2},s,0,1}$ the linear operator L :

Proposition 3.8. *Let $f \in \mathcal{S}(\mathbb{R}^2)$, There exists $C > 0$ such that:*

$$\|\psi(t)L(f)\|_{X^{\frac{1}{2},s,0,1}} \leq C\|f\|_{X^{-\frac{1}{2},s,0,1}}.$$

Proof. Let

$$w(\tau) = W(-\tau)f(\tau), \quad K(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-|t|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau.$$

Therefore, by the definition, it suffices to prove that

$$\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\mathcal{F}_t(K)(\tau)\|_{L^2_{\xi,\eta,\tau}} \lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi)\varphi_L(\tau)\hat{w}(\xi, \eta, \tau)\|_{L^2_{\xi,\eta,\tau}}. \quad (3.24)$$

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We can break up K in $K = K_{1,0} + K_{1,\infty} + K_{2,0} + K_{2,\infty}$, where

$$K_{1,0} =: \psi(t) \int_{|\tau| \leq 1} \frac{e^{i\tau} - 1}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau, \quad K_{1,\infty} = \psi(t) \int_{|\tau| \geq 1} \frac{e^{i\tau}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau,$$

$$K_{2,0} = \psi(t) \int_{|\tau| \leq 1} \frac{1 - e^{-|\tau|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau, \quad K_{2,\infty} = \psi(t) \int_{|\tau| \geq 1} \frac{e^{-|\tau|\xi^2}}{i\tau + \xi^2} \hat{w}(\xi, \eta, \tau) d\tau.$$

Contribution of $K_{2,\infty}$.

Clearly we have

$$\begin{aligned} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{2,\infty})\|_{L_{\xi,\eta,\tau}^2} &\lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \sup_{\xi \in I_k} \|\varphi_N(\xi) Q_L(\psi(e^{-|\tau|\xi^2})) (t)\|_{L_{\xi,\tau}^2} \\ &\quad \times \int \frac{\|\varphi_N(\xi) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta}^2}}{\langle \tau \rangle} d\tau \\ &\lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta,\tau}^2}, \end{aligned}$$

where we use (3.23) in the last step.

Contribution of $K_{2,0}$.

We have for $|\xi| \geq 1$

$$\begin{aligned} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{2,0})\|_{L_{\xi,\eta,\tau}^2} &\lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \sup_{\xi \in I_k} \|\varphi_N(\xi) P_L(\psi(1 - e^{-|\tau|\xi^2})) (t)\|_{L_t^2} \\ &\quad \times \int \frac{\|\hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta}^2}}{\langle \tau \rangle} d\tau \\ &\lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta,\tau}^2}, \end{aligned}$$

where we used (3.23) in the last step.

For $|\xi| \leq 1$, using Taylors expansion, we have

$$\begin{aligned} &\sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{2,0})\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_n \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) \int_{|\tau| \leq 1} \frac{\hat{w}(\tau)}{i\tau + \xi^2} d\tau P_L(|t|^n \psi(t)) \frac{|\xi|^{2n}}{n!}\|_{L_{\xi,\eta,\tau}^2} \\ &\lesssim \sum_n \left\| \frac{t^n \psi(t)}{n!} \right\|_{B_{2,1}^{\frac{1}{2}}} \left\| \int_{|\tau| \leq 1} \frac{|\xi|^2 |\varphi_N(\xi) \hat{w}(\xi, \eta, \tau)|}{|i\tau + \xi^2|} d\tau \right\|_{L_{\xi,\eta}^2} \\ &\lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta,\tau}^2}, \end{aligned}$$

where in the last inequality we used the fact $\| |t|^n \psi(t) \|_{B_{2,1}^{\frac{1}{2}}} \leq \| |t|^n \psi(t) \|_{H^1} \leq C 2^n$.

Contribution of $K_{1,\infty}$.

By the identity $\mathcal{F}(u \star v) = \hat{u} \hat{v}$ and the triangle inequality $\langle i\tau + \xi^2 \rangle \leq \langle \tau_1 \rangle + |i(\tau - \tau_1) + \xi^2|$, Let

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$g(\xi, \eta, \tau) = \frac{|\hat{w}(\xi, \eta, \tau)|}{|i\tau + \xi^2|} \chi_{|\tau| \geq 1}$ we see that

$$\begin{aligned} & \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{1,\infty})\|_{L_{\xi,\eta,\tau}^2} \\ & \lesssim \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{\psi} *_{\tau_1} g(\xi, \eta, \tau_1)\|_{L_{\xi,\eta,\tau}^2} \\ & \lesssim \sum_L \langle L \rangle^{\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau_1) |\hat{\psi}(\tau_1)| \star g(\xi, \eta, \tau_1)\|_{L_{\xi,\eta,\tau}^2} \\ & + \sum_L \|\varphi_N(\xi) \varphi_L(\tau) \hat{\psi}(\tau) \star \left(\frac{\hat{w}(\xi, \eta, \tau)}{|i\tau + \xi^2|^{\frac{1}{2}}} \chi_{|\tau| \geq 1} \right)\|_{L_{\xi,\eta,\tau}^2}. \end{aligned}$$

Due to the convolution inequality $\|u \star v\|_{L_{\tau}^2} \lesssim \|u\|_{L_{\tau}^1} \|v\|_{L_{\tau}^2}$, we obtain

$$\begin{aligned} \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{1,\infty})\|_{L_{\xi,\eta,\tau}^2} & \lesssim \sum_L L \|\hat{\psi}(t)\|_{L_{\tau}^1} \|\varphi_N(\xi) \varphi_L(\tau) \frac{|\hat{w}(\tau)|}{|i\tau + \xi^2|} \chi_{\{|\tau| \geq 1\}}\|_{L_{\xi,\eta,\tau}^2} \\ & + \sum_L \|\psi(t)\|_{L_{\tau}^1} \|\varphi_N(\xi) \varphi_L(\tau) \frac{|\hat{w}(\tau)|}{|i\tau + \xi^2|^{1/2}} \chi_{\{|\tau| \geq 1\}}\|_{L_{\xi,\eta,\tau}^2} \\ & \leq C \sum_L \langle L + N^2 \rangle^{-1/2} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\tau)\|_{L_{\xi,\eta,\tau}^2}. \end{aligned}$$

Contribution of $K_{1,0}$.

Using Taylors expansion, we obtain that:

$$K_{1,0} = \psi(t) \int_{|\tau| \leq 1} \sum_{n \geq 1} \frac{(it\tau)^n}{n!(i\tau + \xi^2)^n} \hat{w}(\xi, \eta, \tau) d\tau.$$

Thus, we get

$$\begin{aligned} & \sum_L \langle L + N^2 \rangle^{\frac{1}{2}} \|\varphi_N(\xi) Q_L(K_{1,0})\|_{L_{\xi,\eta,\tau}^2} \\ & \lesssim \sum_{n \geq 1} \left\| \frac{t^n \psi(t)}{n!} \right\|_{B_{2,1}^{\frac{1}{2}}} \left\| \int_{|\tau| \leq 1} \frac{|\tau|}{|i\tau + \xi^2|} |\varphi_k(\xi) \hat{w}(\xi, \eta, \tau)| d\tau \right\|_{L_{\xi,\eta}^2} \\ & \lesssim \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} \|\varphi_N(\xi) \varphi_L(\tau) \hat{w}(\xi, \eta, \tau)\|_{L_{\xi,\eta,\tau}^2}, \end{aligned}$$

where we used $\| |t|^n \psi(t) \|_{B_{2,1}^{\frac{1}{2}}} \leq \| |t|^n \psi(t) \|_{H^1} \leq C 2^n$ in the last step.

Therefore, we complete the proof of the proposition. \square

3.4 Strichartz and bilinear estimates

The goal of this section is to establish the main bilinear estimate. This type of bilinear estimate is necessary to control the nonlinear term $\partial_x(u^2)$ in $X^{-\frac{1}{2}, s, 0, 1}$.

First following [GTV97] it is easy to check that for any $u \in X^{\frac{1}{2}, 0, 0, 1}$ supported in $[-T, T]$ and any $\theta \in [0, \frac{1}{2}]$ it holds:

$$\|u\|_{X^{\theta, s, 0, 1}} \leq T^{\frac{1}{2} - \theta} \|u\|_{X^{1/2, s, 0, 1}}. \quad (3.25)$$

The following lemma is prepared by Molinet-Ribaud in [MR02b].

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Lemma 3.9. *Let $2 \leq r$ and $0 \leq \beta \leq 1/2$. Then*

$$\left\| |D_x|^{-\frac{\beta\delta(r)}{2}} U(t)\varphi \right\|_{L_{t,x}^{q,r}} \leq C \|\varphi\|_{L^2} \quad (3.26)$$

where $\delta(r) = 1 - \frac{2}{r}$, and (q, r, β) fulfils the condition

$$0 \leq \frac{2}{q} \leq \left(1 - \frac{\beta}{3}\right) \delta(r) < 1. \quad (3.27)$$

Now we will prove the following one:

Lemma 3.10. *Let $v \in L^2(\mathbb{R}^3)$ with $\text{supp } v \subset \{(t, x, y) : |t| \leq T\}$, $\delta(r) = 1 - 2/r$ and $\hat{v}_N = \varphi_n \hat{v}$ for some dyadic integer N . Then for all (r, β, θ) with*

$$2 \leq r < \infty, \quad 0 \leq \beta \leq 1/2, \quad 0 \leq \delta(r) \leq \frac{\theta}{1 - \beta/3}, \quad (3.28)$$

$$\left\| \mathcal{F}_{t,x}^{-1} \left(|\xi|^{-\frac{\theta\beta\delta(r)}{2}} \langle \tau - P(v) \rangle^{-\frac{\theta}{2}} |\hat{v}_N(\tau, v)| \right) \right\|_{L_{t,x,y}^{q,r}} \leq C \|v_N\|_{L^2(\mathbb{R}^3)} \quad (3.29)$$

where q is defined by

$$2/q = (1 - \beta/3)\delta(r) + (1 - \theta). \quad (3.30)$$

□

Proof. Using Lemma 3.26 together with Lemma 3.3 of [Gin95], we see that

$$\left\| |D_x|^{-\frac{\beta\delta(r)}{2}} u_N \right\|_{L_{t,x}^{q,r}} \leq C \|u_N\|_{X^{1/2,0,0,1}}. \quad (3.31)$$

By the definition of $X^{b,s,0,1}$ we have

$$\|u_N\|_{L_{t,x}^2} = \|u_N\|_{X^{0,0,0,2}}. \quad (3.32)$$

Hence for $0 \leq \theta \leq 1$, by interpolation,

$$\left\| |D_x|^{-\frac{\theta\beta\delta(r)}{2}} u_N \right\|_{L_{t,x}^{q_1,r_1}} \leq C \|u_N\|_{X^{\frac{\theta}{2},0,0,1}} \quad (3.33)$$

where

$$\frac{1}{q_1} = \frac{\theta}{q} + \frac{1-\theta}{2}, \quad \frac{1}{r_1} = \frac{\theta}{r} + \frac{1-\theta}{2}.$$

Since $\delta(r_1) = \theta\delta(r)$, (3.28) follows from (3.27)

$$\frac{1}{q_1} = \left(1 - \frac{\beta}{3}\right) \delta(r_1) + (1 - \theta),$$

which can be rewritten as

$$\left\| \mathcal{F}_{t,x}^{-1} \left(|\xi|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_N \right) \right\|_{L_{t,x}^{q_1,r_1}} \leq C \left\| \langle \tau - P(v) \rangle^{\frac{\theta}{2}} \hat{u}_N \right\|_{L^2}.$$

This clearly completes the proof. □

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Now, we will estimate the bilinear terms using the following Lemma (see [IKT08]):

Lemma 3.11. *Let $k_1, k_2, k_3 \in \mathbb{Z}$, $j_1, j_2, j_3 \in \mathbb{Z}_+$, and $f_i : \mathbb{R}^3 \mapsto \mathbb{R}^+$ are L^2 functions supported in D_{k_i, j_i} , $i = 1, 2, 3$. Then*

$$\int (f_1 * f_2) f_3 \lesssim 2^{\frac{j_1+j_2+j_3}{2}} 2^{\frac{-(k_1+k_2+k_3)}{2}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \quad (3.34)$$

Where $D_{k,j} = \{(\xi, \mu, \tau) : |\xi| \in [2^{k-1}, 2^k], \mu \in \mathbb{R}, |\tau - P(\xi, \mu)| \leq 2^j\}$.

We are now in position to prove our main bilinear estimate:

Proposition 3.12. *For all $u, v \in X^{1/2, s, 0, 1}(\mathbb{R}^3)$, $s > -\frac{1}{2}$ with compact support in time included in the subset $\{(t, x, y) : t \in [-T, T]\}$, there exists $\mu > 0$ such that the following bilinear estimate holds*

$$\|\partial_x(uv)\|_{X^{-1/2, s, 0, 1}} \leq CT^\mu \|u\|_{X^{1/2, s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}}. \quad (3.35)$$

□

Remark 3.13. *We will mainly use the following version of (3.35), which is a direct consequence of Proposition 3.12, together with the triangle inequality*

$$\forall \beta \in]-\frac{1}{2}, 0], \forall s \geq \beta, \quad \langle \xi \rangle^s \leq \langle \xi \rangle^\beta \langle \xi_1 \rangle^{s-\beta} + \langle \xi \rangle^\beta \langle \xi - \xi_1 \rangle^{s-\beta},$$

$$\begin{aligned} \|\partial_x(uv)\|_{X^{-1/2, s, 0, 1}} &\leq CT^{\mu(\beta)} \left(\|u\|_{X^{1/2, \beta, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \right. \\ &\quad \left. + \|u\|_{X^{1/2, s, 0, 1}} \|v\|_{X^{1/2, \beta, 0, 1}} \right). \end{aligned} \quad (3.36)$$

with $\mu(\beta) > 0$.

Proof of Proposition 3.12. We proceed by duality. Let $w \in X^{1/2, -s, 0, \infty}$, we will estimate the following term

$$J = \sum_{N, N_1, N_2} \sum_{L, L_1, L_2} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \left| \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \right|$$

By symmetry we can assume that $N_1 \leq N_2$, note that $|\xi| \leq |\xi_1| + |\xi_2|$ then $N \lesssim N_2$.

From Lemma 3.11, we have:

$$\int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L^{\frac{1}{2}} N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N^{-\frac{1}{2}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \quad (3.37)$$

Case 1.: $1 \leq N, N_1 \geq 1$, and $N_2 \geq 1$.

We have clearly:

$$\int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim \|u_{N_1, L_1}\|_{L_{t, x, y}^4} \|v_{N_2, L_2}\|_{L_{t, x, y}^4} \|w_{N, L}\|_{L_{t, x, y}^2} \quad (3.38)$$

using Lemma 3.10 (with $\beta = \frac{1}{2}$, $r = 4$) we obtain that there exists $\alpha \in [\frac{6}{7}, \frac{12}{13}[$ such that:

$$\int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim L_1^{\frac{\alpha}{2}} N_1^{\frac{\alpha}{8}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} L_2^{\frac{\alpha}{2}} N_2^{\frac{\alpha}{8}} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \tau}^2} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \quad (3.39)$$

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By interpolating (3.37) with (3.39) we obtain that: there exist $\beta = \frac{\theta\alpha}{2} + \frac{1-\theta}{2} \in [\frac{\alpha}{2}, \frac{1}{2}]$ and $\theta = \frac{-8s+\alpha}{4+\alpha} \in]0, 1[$ such that:

$$\begin{aligned} \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim N_1^s L_1^\beta \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} \\ &\quad \times N_2^s L_2^\beta \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \\ &\quad \times L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

Then

$$\begin{aligned} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim N_1^s L_1^\beta \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} \\ &\quad \times N_2^s L_2^\beta \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \\ &\quad \times L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

Now we have:

$$\begin{aligned} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim N_1^{(\frac{1}{2}-\beta)} N_1^s \langle L_1 + N_1^2 \rangle^{\frac{1}{2} - (\frac{1}{2}-\beta)} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \tau}^2} \\ &\quad \times N_2^s \langle L_2 + N_2^2 \rangle^{\frac{1}{2}} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \\ &\quad \times L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N N_2^{\beta-\frac{1}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned} \tag{3.40}$$

Note that:

$$\begin{aligned} \sum_{L < N^2} L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N N_2^{\beta-\frac{1}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2} &\lesssim \sum_{L < N^2} \left(\frac{L}{N^2}\right)^{\frac{\theta}{2}} N^{\frac{\theta}{2} + s + \beta - \frac{1}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2} \\ &\lesssim \sum_{L < N^2} \left(\frac{L}{N^2}\right)^{\frac{\theta}{2}} N^\sigma \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2} \end{aligned}$$

where $\sigma = \frac{\alpha}{8} + \theta(\frac{3\alpha}{8} - \frac{1}{2}) < 0$.

By summing in L_1, N_1, L_2, N_2 and $L < N^2$, we get:

$$J \lesssim \|u\|_{X^{\frac{1}{2}-\mu, s, 0, 1}} \|v\|_{X^{\frac{1}{2}, s, 0, 1}} \|w\|_{L_{\xi, \eta, \tau}^2} \lesssim T^\mu \|u\|_{X^{\frac{1}{2}, s, 0, 1}} \|v\|_{X^{\frac{1}{2}, s, 0, 1}} \|w\|_{L_{\xi, \eta, \tau}^2},$$

where $\mu = \frac{1}{2} - \beta > 0$.

Now we have:

$$\begin{aligned} \sum_{L > N^2} L^{\frac{\theta}{2}} N^{-\frac{\theta}{2}} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N N_2^{\beta-\frac{1}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2} &\lesssim \sum_{L > N^2} \left(\frac{L}{N^2}\right)^{\frac{\theta-1}{2}} N^{\frac{\theta}{2} + s + \beta - \frac{1}{2}} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2} \\ &\lesssim \sum_{L > N^2} \left(\frac{N^2}{L}\right)^{\frac{1-\theta}{2}} N^\sigma \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2} \end{aligned}$$

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where $\sigma = \sigma(\alpha, \theta) < 0$. Thus by summing (3.40) in L_1, N_1, L_2, N_2 and $L \geq N^2$, we get the desired estimate.

Case 2.: $N_1 \leq 1$ and $N_2 \sim N \geq 1$.

By Cauchy-Schwarz we obtain:

$$\begin{aligned} & \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \\ & \leq \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \|u_{N_1, L_1}\|_{L_{t,x,y}^{4^+, 4^+}} \|v_{N_2, L_2}\|_{L_{t,x,y}^{4^-, 4^-}} \|w_{N, L}\|_{L_{t,x,y}^2}. \end{aligned}$$

But $|\xi_1| \sim N_1 \leq 1$ thus

$$\|u_{N_1, L_1}\|_{L_{t,x,y}^{4^+, 4^+}} \lesssim N_1^{\frac{\theta\beta\delta(r)}{2}} \|\mathcal{F}_{t,x}^{-1}(|\xi_1|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_{N_1, L_1})\|_{L_{t,x,y}^{4^+, 4^+}}.$$

By applying Lemma 3.10 with $r = 4^+, \beta = \frac{1}{2}$ and $\theta = 1$ we obtain that:

$$\begin{aligned} \|\mathcal{F}_{t,x}^{-1}(|\xi_1|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_{N_1, L_1})\|_{L_{t,x,y}^{4^+, 4^+}} & \lesssim N_1^\epsilon \|\mathcal{F}_{t,x}^{-1}(|\xi_1|^{-\frac{\theta\beta\delta(r)}{2}} \hat{u}_{N_1, L_1})\|_{L_{t,x,y}^{q, 4^+}} \\ & \lesssim N_1^\epsilon \|\langle \tau - P(v) + \xi^2 \rangle^{\frac{1}{2}} \hat{u}_{N_1, L_1}\|_{L_{t,x,y}^2} \\ & \lesssim N_1^\epsilon \langle N_1 \rangle^s \|\langle L_1 + N_1^2 \rangle^{\frac{1}{2}} \hat{u}_{N_1, L_1}\|_{L_{t,x,y}^2} \end{aligned}$$

where $\epsilon = \frac{\theta\beta\delta(r)}{2}$.

Now taking $r = 4^-, \beta = \frac{1}{2}$, and $\theta = \frac{1}{2}$ and using again Lemma 3.10 we obtain that:

$$\begin{aligned} \|v_{N_2, L_2}\|_{L_{t,x,y}^{4^-, 4^-}} & \lesssim N_2^{\frac{\theta\beta\delta(r)}{2}} \|\mathcal{F}_{t,x}^{-1}(|\xi_2|^{-\frac{\theta\beta\delta(r)}{2}} \hat{v}_{N_2, L_2})\|_{L_{t,x,y}^{4^-, 4^-}} \\ & \lesssim N_2^{\frac{1}{6^+}} \|\langle L_2 + N_2^2 \rangle^{\frac{1}{4}} \hat{v}_{N_2, L_2}\|_{L_{t,x,y}^2} \\ & \lesssim N^{-\gamma} \|\langle L_2 + N_2^2 \rangle^{\frac{1}{2} - \delta} \hat{v}_{N_2, L_2}\|_{L_{t,x,y}^2} \end{aligned}$$

where $0 < \delta < \frac{1}{2}$, and $\gamma > 0$ small. Thus:

$$\begin{aligned} & \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \\ & \lesssim N_1^\epsilon (\langle N_1 \rangle^s \langle L_1 + N_1^2 \rangle^{\frac{1}{2}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2}) \\ & \quad \times (\langle N_2 \rangle^s \langle L_2 + N_2^2 \rangle^{\frac{1}{2} - \delta} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2}) \\ & \quad \times \langle L + N^2 \rangle^{-\frac{1}{2}} N N^{-\gamma} \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

But $\langle L + N^2 \rangle^{-\frac{1}{2}} \leq L^{-\frac{\gamma}{4}} N^{-1 + \frac{\gamma}{2}}$, then :

$$\sum_N \sum_L \langle L + N^2 \rangle^{-\frac{1}{2}} N N^{-\gamma} \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2} \leq \sum_N \sum_L N N^{-\gamma} L^{-\frac{\gamma}{4}} N^{-1 + \frac{\gamma}{2}} \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2} \lesssim \|w\|_{L_{\xi, \eta, \tau}^2}.$$

This yields:

$$J \lesssim \|u\|_{X^{1/2, s, 0, 1}} \|v\|_{X^{1/2 - \delta, s, 0, 1}} \|w\|_{L^2} \lesssim T^\delta \|u\|_{X^{1/2, s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \|w\|_{L^2}.$$

Case 3.: N_1, N_2 and $N \lesssim 1$.

From (3.39) we have :

$$\int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau \lesssim L_1^{\frac{\alpha}{2}} N_1^{\frac{\alpha}{8}} L_2^{\frac{\alpha}{2}} N_2^{\frac{\alpha}{8}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2} \|\hat{w}_{N, L}\|_{L_{\xi, \eta, \tau}^2}.$$

Thus :

$$\begin{aligned} \langle L + N^2 \rangle^{-\frac{1}{2}} \langle N \rangle^s N \int (\hat{u}_{N_1, L_1} * \hat{v}_{N_2, L_2}) \hat{w}_{N, L} d\xi d\eta d\tau &\lesssim (\langle N_1 \rangle^s \langle L_1 + N_1^2 \rangle^{\frac{\alpha}{2}} N_1^{\frac{\alpha}{8}} \|\hat{u}_{N_1, L_1}\|_{L_{\xi, \eta, \tau}^2}) \\ &\times (\langle N_2 \rangle^s \langle L_2 + N_2^2 \rangle^{\frac{\alpha}{2}} N_2^{\frac{\alpha}{8}} \|\hat{v}_{N_2, L_2}\|_{L_{\xi, \eta, \tau}^2}) \\ &\times \langle L + N^2 \rangle^{-\frac{1}{2}} N N^s \|w_{N, L}\|_{L_{\xi, \eta, \tau}^2}. \end{aligned}$$

By summing we obtain that:

$$J \lesssim \|u\|_{X^{\frac{1}{2} - (\frac{1}{2} - \frac{\alpha}{2}), s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \|w\|_{L^2} \lesssim T^\mu \|u\|_{X^{\frac{1}{2}, s, 0, 1}} \|v\|_{X^{1/2, s, 0, 1}} \|w\|_{L^2},$$

where $\mu = \frac{1}{2} - \frac{\alpha}{2} > 0$. This completes the proof. \square

3.5 Proof of Theorem 3.4

3.5.1 Existence result

Let $\phi \in H^{s_1, 0}$ with $s_1 > -1/2$. For $T \leq 1$, if u is a solution of the integral equation (3.15), then u solve *KPB* - *I*- equation on $[0, T/2]$. We first prove the statement for $T = T(\|\phi\|_{H^{s_1, 0}})$.

Now we are going to solve (3.15) in a ball of the space $X_T^{1/2, s_1, 0, 1}$.

By Proposition 3.7 and Proposition 3.8, it results that,

$$\|L(u)\|_{X_T^{1/2, s_1, 0, 1}} \leq C\|\phi\|_{H^{s_1, 0}} + C\|\partial_x(u^2)\|_{X_T^{-1/2, s_1, 0, 1}}. \quad (3.41)$$

By the Proposition 3.12, we can deduce

$$\|L(u)\|_{X_T^{1/2, s_1, 0, 1}} \leq C\|\phi\|_{H^{s_1, 0}} + CT^\mu \|u\|_{X_T^{1/2, s_1, 0, 1}}^2. \quad (3.42)$$

Noticing that $\partial_x(u^2) - \partial_x(v^2) = \partial_x[(u - v)(u + v)]$, in the same way we get

$$\|L(u) - L(v)\|_{X_T^{1/2, s_1, 0, 1}} \leq CT^\mu \|u - v\|_{X_T^{1/2, s_1, 0, 1}} \|u + v\|_{X_T^{1/2, s_1, 0, 1}}. \quad (3.43)$$

Now take $T = (4C^2\|\phi\|_{H^{s_1, 0}})^{-1/\mu}$ we deduce from (3.42) and (3.43) that L is strictly contractive on the ball of radius $2C(\|\phi\|_{H^{s_1, 0}})$ in $X_T^{\frac{1}{2}, s_1, 0, 1}$. This proves the existence of a unique solution u_1 to (3.15) in $X_T^{1/2, s_1, 0, 1}$ with $T = T(\|\phi\|_{H^{s_1, 0}})$.

Note that our space $X_T^{\frac{1}{2}, s_1, 0, 1}$ is embedded in $C([0, T], H^{s_1, 0})$, thus u belongs $C([0, T_1], H^{s_1, 0})$.

3.5.2 Uniqueness.

The above contraction argument gives the uniqueness of the solution to the truncated integral equation (3.15). We give here the argument of [MR02c] to deduce easily the uniqueness of the solution to the integral equation (3.14).

Let $u_1, u_2 \in X_T^{1/2, s_1, 0, 1}$ be two solution of the integral equation (3.15) on the time interval $[0, T]$ and let $\tilde{u}_1 - \tilde{u}_2$ be an extension of $u_1 - u_2$ in $X^{1/2, s_1, 0, 1}$ such that $\tilde{u}_1 - \tilde{u}_2 = u_1 - u_2$ on $[0, \gamma]$ and

$$\|\tilde{u}_1 - \tilde{u}_2\|_{X^{1/2, s_1, 0, 1}} \leq 2\|u_1 - u_2\|_{X_\gamma^{1/2, s_1, 0, 1}}$$

with $0 < \gamma \leq T/2$. It results by Proposition 3.7 and 3.8 that,

$$\begin{aligned} \|u_1 - u_2\|_{X_\gamma^{1/2, s_1, 0, 1}} &\leq \|\psi(t)L[\partial_x(\psi_\gamma^2(t')(\tilde{u}_1(t') - \tilde{u}_2(t'))(u_1(t') + u_2(t')))]\|_{X^{1/2, s_1, 0, 1}} \\ &\leq C\|\partial_x(\psi_\gamma^2(t)(\tilde{u}_1(t) - \tilde{u}_2(t))(u_1(t) + u_2(t))\|_{X^{-1/2, s_1, 0, 1}} \\ &\leq C\gamma^{\mu/2}\|\tilde{u}_1 - \tilde{u}_2\|_{X^{1/2, s_1, 0, 1}}\|u_1 + u_2\|_{X_T^{1/2, s_1, 0, 1}} \end{aligned}$$

for some $\mu > 0$. Hence

$$\|u_1 - u_2\|_{X_\gamma^{1/2, s_1, 0, 1}} \leq 2C\gamma^{\mu/2}(\|u_1\|_{X_T^{1/2, s_1, 0, 1}} + \|u_2\|_{X_T^{1/2, s_1, 0, 1}})\|u_1 - u_2\|_{X_\gamma^{1/2, s_1, 0, 1}}.$$

Taking $\gamma \leq (4C(\|u_1\|_{X_T^{1/2, s_1, 0, 1}} + \|u_2\|_{X_T^{1/2, s_1, 0, 1}}))^{-\mu/2}$, this forces $u_1 \equiv u_2$ on $[0, \gamma]$. Iterating this argument, one extends the uniqueness result on the whole time interval $[0, T]$. \square

Now proceeding exactly (with (3.36) in hand) in the same way as above but in the space

$$Z = \{u \in X_T^{1/2, s_1, 0} / \|u\|_Z = \|u\|_{X_T^{1/2, \beta, 0, 1}} + \frac{\|\varphi\|_{H^{\beta, 0}}}{\|\varphi\|_{H^{s_1, 0}}}\|u\|_{X_T^{1/2, s_1, 0, 1}} < +\infty\},$$

where β is such that $\beta \in]-\frac{1}{2}, \min(0, s_1)[$, we obtain that for $T_1 = T_1(\|\varphi\|_{H^{\beta, 0}})$, L is also strictly contractive on a ball of Z . It follows that there exists a unique solution \tilde{u} to KPBI in $X_T^{1/2, s_1, 0, 1}$. If we indicate by $T_* = T_{max}$ the maximum time of the existence in $X^{1/2, s_1, 0, 1}$ then by uniqueness, we have $u = \tilde{u}$ on $[0, \min(T_1, T_*)[$ and this gives that $T_* \geq T(\|\phi\|_{H^{\beta, 0}})$.

The continuity of map $\phi \mapsto u$ from $H^{s_1, 0}$ to $X^{1/2, s_1, 0, 1}$ follows from classical argument, and in particular the map is continuous from $H^{s_1, 0}$ to $C([0, T_1], H^{s_1, 0})$. The analyticity of the flow-map is a direct consequence of the implicit function theorem. \square

3.5.3 Global existence.

First we take ϕ in $H_{-1}^{10, 10} \hookrightarrow H^{10, 10} \hookrightarrow L^2(\mathbb{R}^2)$ with

$$H_{-1}^{s_1, s_2} := \{\phi \in \mathcal{S}'(\mathbb{R}^2) / \int \langle |\xi|^{-1} \rangle^2 \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{\phi}(\xi, \eta)|^2 d\xi d\eta < \infty\}.$$

From the local well-posedness result (see also Remark 3.5) we know that there exists a unique solution $u \in X_T^{10, 10, 1/2, 1} \hookrightarrow C([0, T]; H^{10, 10})$, with $T = T(\|\phi\|_{H_{-1}^{10, 10}})$, to (3.1). By the Duhamel formula, u satisfies

$$u(t) = W(t)\phi - \frac{1}{2} \int_0^t W(t-t')\partial_x(u^2(t'))dt'$$

3.6. PROOF OF THEOREM 3.6

and thus

$$\partial_x^{-1}u = W(t)\partial_x^{-1}\phi - \frac{1}{2} \int_0^t W(t-t')(u^2(t'))dt', \quad (3.44)$$

where $\widehat{\partial_x^{-1}\phi} := \frac{1}{i\xi}\hat{\phi}$. The first term in the right hand side of (3.44) clearly belongs to $C([0, T], H^{10,10})$ and since

$$\begin{aligned} \left\| \int_0^t W(t-t')(u^2(t'))dt' \right\|_{C([0,T], H^{10}(\mathbb{R}^2))} &\lesssim \|u^2\|_{C([0,T], H^{10}(\mathbb{R}^2))} \\ &\lesssim \|u\|_{C([0,T], H^{10}(\mathbb{R}^2))}^2 \lesssim \|u\|_{C([0,T], H^{10,10})}^2, \end{aligned}$$

it follows that $\partial_x^{-1}u \in C([0, T], H^{10}(\mathbb{R}^2))$, where we denote by $H^s(\mathbb{R}^2)$ the usual isotropic Sobolev space. We can thus take the L^2 -scalar product of (3.1) with $\partial_x^{-1}u$ to obtain

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{L^2}^2 &= \int 2u(-u_{xxx} + u_{xx} - uu_x + \partial_x^{-1}u_{yy}) \\ &= \int \partial_x(u_x^2) - 2 \int (u_x)^2 - \frac{2}{3} \int \partial_x(u^3) - \int \partial_x(\partial_x^{-1}u_y)^2 \\ &= -2 \int (u_x)^2 \leq 0, \end{aligned}$$

This proves that $t \mapsto \|u(t)\|_{L^2}$ is non increasing and thus u can be extended for all positive times and belongs to $C(\mathbb{R}_+; H^{10,10})$. The result for $\phi \in L^2(\mathbb{R}^2)$, and more generally for $\phi \in H^{s,0}$ with $s \geq 0$, then follows by the density of $H_{-1}^{10,10}$ in $L^2(\mathbb{R}^2)$ and the continuity with respect to initial data in $L^2(\mathbb{R}^2)$.

Now, for $\phi \in H^{s,0}$ with $-\frac{1}{2} < s < 0$, we will use the regularizing effect of the dissipative term. Since $u \in X_T^{s,0,1/2,1} \hookrightarrow L_T^2 H^{s+1,0}$ with $T = T(\|\phi\|_{H^{-\frac{1}{2},0}})$ and $s+1 \geq 0$, there exists $t_0 \in]0, T[$ such that $u(t_0) \in L^2(\mathbb{R}^2)$. Taking t_0 as initial data and proceeding as above we obtain that u can be extended for all time $t \geq t_0$ and $u \in C(\mathbb{R}_+, H^{s,0}) \cap C([t_0, +\infty[, L^2)$. Since t_0 can be chosen arbitrary small, this ensures that $u \in C(\mathbb{R}_+, H^{s,0}) \cap C(\mathbb{R}_+^*, L^2)$.

Finally, for any $\phi \in H^{s,0}$ with $s > -\frac{1}{2}$, iterating the regularizing effect, we obtain that actually $u \in C(\mathbb{R}_+^*, H^{\infty,0})$. \square

3.6 Proof of Theorem 3.6

Let u be a solution of (3.1), we have

$$u(\phi, t, x, y) = W(t)\phi(x, y) - \frac{1}{2} \int_0^t W(t-t')\partial_x(u^2(\phi, t', x, y))dt'. \quad (3.45)$$

Suppose that the solution map is C^2 . Since $u(0, t, x, y) = 0$, it is easy to check that

$$\begin{aligned} u_1(t, x, y) &:= \frac{\partial u}{\partial \phi}(0, t, x, y)[h] = W(t)h \\ u_2(t, x, y) &:= \frac{\partial^2 u}{\partial \phi^2}(0, t, x, y)[h, h] \\ &= - \int_0^t W(t-t')\partial_x(W(t')h)^2 dt'. \end{aligned}$$

3.6. PROOF OF THEOREM 3.6

The assumption of C^2 -regularity of the solution map implies that

$$\|u_1(t, \cdot, \cdot)\|_{H^{s,0}} \lesssim \|h\|_{H^{s,0}}, \quad \|u_2(t, \cdot, \cdot)\|_{H^{s,0}} \lesssim \|h\|_{H^{s,0}}^2. \quad (3.46)$$

Now let $P(\xi, \eta) = \xi^3 + \eta^2/\xi$. A straightforward calculation reveals that

$$\begin{aligned} \mathcal{F}_{x \rightarrow \xi, y \rightarrow \eta}(u_2(t, \cdot, \cdot)) &= (i\xi)e^{itP(\xi, \eta)} \int_{\mathbb{R}^2} \hat{\phi}(\xi_1, \eta_1) \hat{\phi}(\xi - \xi_1, \eta - \eta_1) \\ &\quad \times \frac{e^{-t(\xi_1^2 + (\xi - \xi_1)^2)} e^{i\chi(\xi, \xi_1, \eta, \eta_1)} - e^{-\xi^2 t}}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 \end{aligned} \quad (3.47)$$

where $\chi(\xi, \xi_1, \eta, \eta_1) = P(\xi_1, \eta_1) + P(\xi - \xi_1, \eta - \eta_1) - P(\xi, \eta)$. Note that, from the definition of $P(\xi, \eta)$, we have that

$$\chi(\xi, \xi_1, \eta, \eta_1) = 3\xi\xi_1(\xi - \xi_1) - \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi\xi_1(\xi - \xi_1)}.$$

Let will use the counter-example constructed in [Koj07]. We define the sequence of initial data $(\phi_N)_N$, $N > 0$ by

$$\hat{\phi}_N(\xi, \eta) = N^{-3/2-s} (\chi_{A_N}(|\xi|, |\eta|) + \chi_{B_N}(|\xi|, |\eta|)) \quad (3.48)$$

where A_N, B_N are defined by

$$A_N = [N/2, 3N/4] \times [-6N^2, 6N^2], \quad B_N = [N, 2N] \times [\sqrt{3}N^2, (\sqrt{3} + 1)N^2].$$

It is simple to see that $\|\phi_N\|_{H^{s,0}} \sim 1$. We denote by $u_{2,N}$ the sequence of the second iteration u_2 associated with ϕ_N . Note that $\mathcal{F}_{x \rightarrow \xi, y \rightarrow \eta}(u_2(t))$ can be split into three parts:

$$\mathcal{F}_{x \rightarrow \xi, y \rightarrow \eta}(u_2(t)) = (g(t) + f(t) + h(t))$$

where g and f correspond to the contribution to (3.47) when respectively $(|\xi|, |\eta_i|) \in A_n$, $i = 1, 2$, and $(|\xi|, |\eta_i|) \in B_n$, $i = 1, 2$. Finally, h corresponds to the contribution to (3.47) of $D(\xi, \eta)$ Hence it is readily seen that where

$$\begin{aligned} D(\xi, \eta) &= \left\{ (\xi_1, \eta_1) : (\xi - \xi_1, \eta - \eta_1) \in A_N, (\xi_1, \eta_1) \in B_N \right\} \\ &\quad \cup \left\{ (\xi_1, \eta_1) : (\xi_1, \eta_1) \in A_N, (\xi - \xi_1, \eta - \eta_1) \in B_N \right\} \\ &:= D^1(\xi, \eta) \cup D^2(\xi, \eta) \end{aligned} \quad (3.49)$$

It is not too hard to check that g and f cancel when $\xi \in]3N/2, 2N[$. Therefore

$$\|u_2(t)\|_{H^{s,0}}^2 \geq \int_{3N/2}^{3N} \int_{(\sqrt{3}-5)N^2}^{(\sqrt{3}+5)N^2} |h|^2 (1 + |\xi|^2)^s d\xi d\eta.$$

This leads to

$$\begin{aligned} \|u_2(t)\|_{H^{s,0}}^2 &\geq CN^{-4s-6} \int_{3N/2}^{3N} \int_{(\sqrt{3}-5)N^2}^{(\sqrt{3}+6)N^2} |\xi|^2 (1 + |\xi|^2)^s \\ &\quad \times \left| \int_{D(\xi, \eta)} \frac{e^{-t(\xi_1^2 + (\xi - \xi_1)^2)} e^{i\chi(\xi, \xi_1, \eta, \eta_1)} - e^{-\xi^2 t}}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 \right|^2 d\xi d\eta. \end{aligned} \quad (3.50)$$

We need to find a lower bound for the right-hand side of (3.50). We will prove the following lemma:

3.6. PROOF OF THEOREM 3.6

Lemma 3.14. *Let $(\xi_1, \eta_1) \in D^1(\xi, \eta)$ or $(\xi_1, \eta_1) \in D^2(\xi, \eta)$. For $N \gg 1$ we have*

$$|\chi(\xi, \xi_1, \eta, \eta_1)| \lesssim N^3.$$

□

Proof of lemma 3.14. Let $\xi, \eta \in \mathbb{R}$ and $(\xi_1, \eta_1) \in D^1(\xi, \eta)$ such that $(\xi - \xi_1) \in [N/2, N]$. Let

$$\Lambda(\xi, \xi_1, \eta_1) = \eta_1 + \frac{(\xi - \xi_1)(\eta_1 - \sqrt{3}\xi\xi_1)}{\xi_1}.$$

Thus

$$|\Lambda(\xi, \xi_1, \eta_1) - \eta_1| \leq \frac{|\xi - \xi_1|}{|\xi_1|} |\eta_1 - \sqrt{3}\xi_1^2 - \sqrt{3}\xi_1(\xi - \xi_1)|.$$

We recall that $\eta_1 \in [\sqrt{3}N^2, (\sqrt{3} + 1)N^2]$ and $\xi_1 \in [N, 2N]$. Therefore, it follows that

$$\sqrt{3}\xi_1^2 \in [\sqrt{3}N^2, 4\sqrt{3}N^2]$$

and we have

$$|\eta_1 - \sqrt{3}N^2| \leq N^2.$$

Since $\xi_1 \in [N, 2N]$ and $\xi - \xi_1 \in [\frac{N}{2}, \frac{3N}{4}]$, it results that

$$|\Lambda(\xi, \xi_1, \eta_1) - \eta_1| \leq 1/4(3\sqrt{3}N^2 + 2\sqrt{3}N^2) \leq 6N^2.$$

Now by the mean value theorem we can write

$$\chi(\xi, \xi_1, \eta, \eta_1) = \chi(\xi, \xi_1, \Lambda(\xi, \xi_1, \eta_1), \eta_1) + (\eta - \Lambda(\xi, \xi_1, \eta_1)) \frac{\partial \chi}{\partial \eta}(\xi, \xi_1, \bar{\eta}, \eta_1)$$

where $\bar{\eta} \in [\eta, \Lambda(\xi, \xi_1, \eta_1)]$. Note that we choosed Λ such that $\chi(\xi, \xi_1, \Lambda(\xi, \xi_1, \eta_1), \eta_1) = 0$. Hence

$$|\chi(\xi, \xi_1, \eta, \eta_1)| = |\eta - \Lambda(\xi, \xi_1, \eta_1)| \left| \frac{2\xi_1(\bar{\eta}\xi_1 - \eta_1\xi)}{\xi\xi_1(\xi - \xi_1)} \right|.$$

Since $|\eta - \Lambda(\xi, \xi_1, \eta_1)| \leq |\eta - \eta_1| + |\eta_1 - \Lambda(\xi, \xi_1, \eta_1)| \leq CN^2$, it follows that

$$\begin{aligned} |\chi(\xi, \xi_1, \eta, \eta_1)| &\lesssim |\xi_1| |\eta - \Lambda(\xi, \xi_1, \eta_1)| \left| \frac{(\bar{\eta} - \eta_1)\xi_1 - \eta_1(\xi - \xi_1)}{\xi\xi_1(\xi - \xi_1)} \right| \\ &\lesssim N^3 \left(\frac{|(\bar{\eta} - \eta_1)\xi_1|}{|\xi\xi_1(\xi - \xi_1)|} + \frac{|\eta_1(\xi - \xi_1)|}{|\xi\xi_1(\xi - \xi_1)|} \right) \\ &\lesssim N^3 \left(\frac{(\sqrt{3} + 1)N^3}{N^3} + C \frac{N^3}{N^3} \right) \\ &\lesssim N^3. \end{aligned}$$

In the other case where $(\xi_1, \eta_1) \in D^2(\xi, \eta)$ i.e. $(\xi_1, \eta_1) \in A_N$ and $(\xi - \xi_1, \eta - \eta_1) \in B_N$, follows from first case since we can write $(\xi_1, \eta_1) = (\xi - (\xi - \xi_1), \eta - (\eta - \eta_1)) \in A_N$ and that

$$\chi(\xi, \xi_1, \eta, \eta_1) = \chi(\xi, \xi - \xi_1, \eta, \eta - \eta_1).$$

3.6. PROOF OF THEOREM 3.6

This completes the proof of the Lemma. \square

We return to the proof of the theorem, note that for any $\xi \in [3N/2, 2N]$ and $\eta \in [(\sqrt{3}-5)N^2, (\sqrt{3}+6)N^2]$, we have $\text{mes}(D(\xi, \eta)) \geq \frac{N^3}{2}$.

Now, for $0 < \epsilon \ll 1$ fixed, we choose a sequence of times $(t_N)_N$ defined by

$$t_N = N^{-3-\epsilon}.$$

For $N \gg 1$ it can be easily seen that

$$e^{-\xi^2 t_N} \sim e^{-N^2 t_N} > C. \quad (3.51)$$

By Lemma 3.14 we have $|-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)| \leq N^2 + N^3 \leq CN^3$. Hence

$$\left| \frac{e^{(-2\xi_1(\xi - \xi_1)t + i\chi(\xi, \xi_1, \eta, \eta_1))} - 1}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} \right| = \frac{1}{N^{3+\epsilon}} + O\left(\frac{1}{N^{6+2\epsilon}}\right). \quad (3.52)$$

By combining the relations (3.51) and (3.52), we obtain

$$\left| \int_{D(\xi, \eta)} \frac{e^{-\xi^2 t} \left[e^{(-2\xi_1(\xi - \xi_1)t + i\chi(\xi, \xi_1, \eta, \eta_1))} - 1 \right]}{-2\xi_1(\xi - \xi_1) + i\chi(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1 \right| \geq CN^{-\epsilon}. \quad (3.53)$$

It results that

$$\begin{aligned} \|u_{2,N}(t_N)\|_{H^{s,0}}^2 &\geq N^{-4s-6} \int_{3N/2}^{2N} \int_{(\sqrt{3}-5)N^2}^{(\sqrt{3}+6)N^2} |\xi|^2 (1 + |\xi|^2)^s d\xi d\eta \times N^{-2\epsilon} \\ &\geq N^{-1-2\epsilon-2s}. \end{aligned}$$

and, hence

$$1 \sim \|\phi_N\|_{H^{s,0}}^2 \geq \|u_{2,N}(t_N)\|_{H^{s,0}}^2 \geq N^{-1-2\epsilon-2s}.$$

This leads to a contradiction for $N \gg 1$, since we have $-1 - 2\epsilon - 2s > 0$ for $s \leq -1/2 - \epsilon$. This completes the proof of Theorem 3.6. \square

Chapter 4

On the invariant measures for the Ostrovsky equation.

4.1 Introduction

In this chapter, we construct an invariant measure for a dynamical system defined by the Ostrovsky equation (Ost)

$$\begin{cases} \partial_t u - u_{xxx} + \partial_x^{-1} u + uu_x = 0, \\ u(0, x) = u_0(x). \end{cases} \quad (4.1)$$

The operator ∂_x^{-1} in the equation denotes a certain antiderivative with respect to the variable x defined for 0-mean value periodic function the Fourier transform by $(\widehat{\partial_x^{-1} f}) = \frac{\hat{f}(\xi)}{i\xi}$.

Invariant measure play an important role in the theory of dynamical systems (DS). It is well known that the whole ergodic theory is based on this concept. On the other hand, they are necessary in various physical considerations.

Note that, one the well-known applications of invariant measures in the theory of dynamical is the Poincaré recurrence theorem : every flow which preserves a finite measure has the returning property modulo a set of measure zero.

Recently several papers([Zhi95],[Zhi94]) and [Bou96] have been published on invariant measures for dynamical system generated by nonlinear partial differential equations.

In [Zhi01] an infinite series of invariant measure associated with a higher conservation laws are constructed for the one-dimensional Korteweg de Vries (KdV) equation:

$$u_t + uu_x + u_{xxx} = 0,$$

by Zhidkov. In particular, invariant measure associated to the conservation of the energy are constructed for this equation.

Equation 4.1 is a perturbation of the Korteweg de Vries (KdV) equation with a nonlocal term and was deducted by Ostrovskii [Ost78] as a model for weakly nonlinear long waves, in a rotating frame of reference, to describe the propagation of surface waves in the ocean.

4.2. NOTATIONS AND MAIN RESULTS

We will construct invariant measures associated with the Hamiltonian:

$$H(u(t)) = \frac{1}{2} \int (u_x)^2 + \frac{1}{2} \int (\partial_x^{-1} u)^2 - \frac{1}{6} \int u^3.$$

The paper is organized as follows. In Section 4.2 the basic notation is introduced and the basic results are formulated. In Section 4.3 the invariant measure which corresponds to the conservation of the Hamiltonian is constructed.

In Section 4.4 we will prove the local well-posedness for our equation in H^s , $s > -\frac{1}{2}$.

4.2 Notations and main results

We will use C to denote various time independent constants, usually depending only upon s . In case a constant depends upon other quantities, we will try to make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we will write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We write $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2} \sim 1 + |\cdot|$. The notation a^+ denotes $a + \epsilon$ for an arbitrarily small ϵ . Similarly a^- denotes $a - \epsilon$. Let

$$L_0^2 = \{u \in L^2; \int_{\mathbb{T}} u dx = 0\}.$$

On the circle, for a function $f = f(x)$ the space Fourier transform is defined as

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \exp(-inx) dx.$$

For a function $u = u(x, t)$, we define the space-time Fourier transform by

$$\hat{u}(\tau, n) := \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} u(t, x) \exp(-inx) \exp(-it\tau) dt dx$$

We introduce the zero mean-value Sobolev spaces H_0^s defined by :

$$H_0^s = \{u \in \mathcal{S}'(\mathbb{T}); \|u\|_{H_0^s} < +\infty \text{ and } \int_{\mathbb{T}} u dx = 0\}, \quad (4.2)$$

where,

$$\|u\|_{H_0^s} = (2\pi)^{\frac{1}{2}} \|\langle \cdot \rangle^s \hat{u}(n)\|_{l_n^2}, \quad (4.3)$$

and $X^{s, \frac{1}{2}}$ by

$$\{u \in \mathcal{S}'(\mathbb{T} \times \mathbb{R}); \|u\|_{X^{s, \frac{1}{2}}} := \|\langle n \rangle^s \langle \tau + n^3 - \frac{1}{n} \rangle^{\frac{1}{2}} \hat{u}(\tau, n)\|_{l_n^2 L_\tau^1} < +\infty\}.$$

Let

$$Y^s =: \{u \in \mathcal{S}'(\mathbb{T} \times \mathbb{R}); \|u\|_{Y^s} < +\infty\},$$

where

$$\|u\|_{Y^s} = \|u\|_{X^{s, \frac{1}{2}}} + \|\langle n \rangle^s \hat{u}(\tau, n)\|_{l_n^1 L_\tau^1}.$$

We will briefly remind the general construction of a Gaussian measure on a Hilbert space. Let X be a Hilbert space, and $\{e_k\}$ be the orthonormal basis in X which consists of eigenvectors of

4.2. NOTATIONS AND MAIN RESULTS

some operator $S = S^* > 0$ with corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \dots \leq \lambda_k \leq \dots$. We call a set $M \subset X$ a cylindrical set iff:

$$M = \{x \in X; [(x, e_1), (x, e_2), \dots, (x, e_r)] \in F\}$$

for some Borel $F \subset \mathbb{R}^r$, and some integer r . We define the measure w as follows:

$$w(M) = (2\pi)^{-\frac{r}{2}} \prod_{j=1}^r \lambda_j^{\frac{1}{2}} \int_F e^{-\frac{1}{2} \sum_{j=1}^r \lambda_j y_j^2} dy. \quad (4.4)$$

One can easily verify that the class \mathbb{A} of all cylindrical sets is an algebra on which the function w is additive. The function w is called the centered Gaussian measure on X with the correlation operator S^{-1} .

Definition 4.1. *The measure w is called a countably additive measure on an algebra \mathbb{A} if $\lim_{n \rightarrow +\infty} w(A_n) = 0$ for any $A_n \in \mathbb{A}$ ($n = 1, 2, 3, \dots$) for which $A_1 \supset A_2 \supset A_3 \supset \dots \supset A_n \supset \dots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$*

Now we give the following Lemma:

Lemma 4.2. *The measure w is countably additive on the algebra \mathbb{A} iff S^{-1} is an operator of trace class, i.e. iff $\sum_{k=1}^{+\infty} \lambda_k^{-1} < +\infty$.*

Now we present some definitions related to invariant measure :

Definition 4.3. *Let M be a complete separable metric space and let a function $h : \mathbb{R} \times M \mapsto M$ for any fixed t be a homeomorphism of the space M into itself satisfying the properties:*

1. $h(0, x) = x$ for any $x \in M$.
2. $h(t, h(\tau, x)) = h(t + \tau, x)$ for any $t, \tau \in \mathbb{R}$ and $x \in M$.

Then, we call the function h a dynamical system with the space M . If μ is a Borel measure defined on the phase space M and $\mu(\Omega) = \mu(h(\Omega, t))$ for an arbitrary Borel set $\Omega \subset M$ and for all $t \in \mathbb{R}$, then it is called an invariant measure for the dynamical system h .

Let us now state our results:

Theorem 4.4. *Let $s \geq -1/2$, and $\phi \in H_0^s$. Then there exists a time $T = T(\|\phi\|_{H_0^s}) > 0$ and a unique solution u of (4.1) in $C([0, T], H_0^s) \cap Y^s$ and the map $\phi \mapsto u$ is C^∞ from H_0^s to Y^s . \square*

Theorem 4.5. *The Problem 4.1 is globally well-posedness in L^2 and the Borel measure μ on L^2 defined for any Borel set $\Omega \subset L^2$ by the rule*

$$\mu(\Omega) = \int_{\Omega} e^{-g(u)} dw(u)$$

where w is the centered Gaussian measure corresponding to the correlation operator $S^{-1} = (-\Delta + \Delta^{-1})^{-1}$, and $g(u) = \frac{1}{3} \int u^3 dx$ the nonlinear term of the Hamiltonian is an invariant measure for (4.1).

4.3 Invariance of Gibbs measure

In this section, we construct an invariant measure to Equation 4.1 with respect to the conservation of the Hamiltonian. Let us first present result on invariant measures for systems of autonomous ordinary differential equations. Consider the following system of ordinary differential equations:

$$\dot{x} = b(x), \quad (4.5)$$

where $x(t) : \mathbb{R} \mapsto \mathbb{R}^n$ is an unknown vector-function and $b(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a continuously differentiable map. Let $h(t, x)$ be the corresponding function (“dynamical system”) from $\mathbb{R} \times \mathbb{R}^n$ into \mathbb{R}^n transforming any $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ into the solution $x(t)$, taken at the moment of time t , of the above system supplied with the initial data $x(0) = x_0$.

Theorem 4.6. *Let $P(x)$ be a continuously differentiable function from \mathbb{R}^n into \mathbb{R} . For the Borel measure*

$$\nu(\Omega) = \int_{\Omega} P(x) dx$$

to be invariant for the function $h(t, x)$ in the sense that $\nu(h(t, \Omega)) = \nu(\Omega)$ for any bounded domain Ω and for any t , it is sufficient and necessary that

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} (P(x) b_i(x)) = 0,$$

for all $x \in \mathbb{R}^n$.

Proof See Zhidkov [Zhi95].

We shall construct an invariant measure for (4.1). Let $A > 0$, the space $L^2(0, A)$ be real equipped with the scalar product:

$$(u, v)_{L^2(0, A)} = \int_0^A u \bar{v} dx.$$

and $J = \frac{\partial}{\partial x} Q$ where the operator Q maps $v^* \in L^2$ into $v \in L^2$ such that $v^*(g) = (v, g)_{L^2(0, A)}$. Finally, let $S = -\Delta + \Delta^{-1}$. We set $H(u) = \frac{1}{2}(\int (u_x)^2 - \int (\partial_x^{-1} u)^2) + \frac{1}{3} \int u^3 = \frac{1}{2}(S u, u) + g(u)$. Note that System (4.1) takes the form:

$$\begin{cases} \frac{\partial u}{\partial t}(t) = J \frac{\delta}{\delta u} H(u(t)), t \in \mathbb{R} \\ u(t_0) = \phi, \end{cases} \quad (4.6)$$

Let $e_{2k-1}(x) = \frac{\sqrt{2}}{\sqrt{A}} \sin(\frac{2\pi n x}{A})$, $e_{2k} = \frac{\sqrt{2}}{\sqrt{A}} \cos(\frac{2\pi n x}{A})$ where $k = 1, 2, 3 \dots$. Then $(e_k)_{k=1, 2, \dots}$ is an orthonormal basis of the space $L_0^2(0, A)$ consisting of eigenfunctions of the operator Δ with the corresponding eigenvalues $0 < \lambda_1 = \lambda_2 < \dots < \lambda_{2k-1} = \lambda_{2k} < \dots$. Let P_m be the orthogonal projector in L_0^2 onto the subspace $L_m = \text{span}\{e_1, \dots, e_{2m}\}$ and P_m^\perp be the orthogonal projector in $L_0^2(0, A)$ onto the orthogonal complement L_m^\perp to the subspace L_m . Let also $\nu_i = -\lambda_i + \lambda_i^{-1}$, then ν_i are eigenvalues of S .

Consider the following problem:

$$\begin{cases} \partial_t u^m - u_{xxx}^m + \partial_x^{-1} u^m + P_m(u^m u_x^m) = 0, \\ u^m(0, x) = P_m u_0(x). \end{cases} \quad (4.7)$$

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The solution u of (4.1) is global in time (see the last section) and the solution of (4.7) converges to u in $C([0, T], L^2)$ for any fixed T . More precisely we have the following lemma:

Lemma 4.7. 1. For any fixed $T > 0$, the solution u_m of (4.7) converges in $C([0, T], L^2)$ to the solution u of (4.1).

2. For any $\epsilon > 0$, and $T > 0$ there exists $\delta > 0$ such that

$$\text{Max}_{t \in [t_0 - T, t_0 + T]} \|u_m(\cdot, t) - v_m(\cdot, t)\|_{L^2} < \epsilon,$$

for any two solutions u_m and v_m of the problem (4.7), satisfying the condition

$$\|u_m(\cdot, t_0) - v_m(\cdot, t_0)\|_{L^2} < \delta.$$

Proof: By the Duhamel formula, $u - u^m$ satisfies

$$u(t) - u^m(t) = e^{-itS}(u_0 - P_m u_0) - \frac{1}{2} \int_0^t e^{-i(t-t')S} (\partial_x(u^2(t')) - P_m(\partial_x((u^m)^2(t')))) dt'.$$

We can write that $R(t) := \partial_x(u^2(t')) - P_m(\partial_x((u^m)^2(t'))) = \partial_x(u^2 - (P_{\frac{m}{2}}u)^2) + P_m \partial_x((P_{\frac{m}{2}}u)^2 - u^2) + P_m \partial_x(u^2 - (u_m)^2)$. Now, using the linear and bilinear estimates proved in Section 4.4, we obtain that

$$\|u - u_m\|_{Y^s} \lesssim \|u_0 - P_m u_0\|_{H^s} + T^\gamma \|u - u_m\|_{Y^s} \|u + u_m\|_{Y^s} + \|u - P_{\frac{m}{2}}u\|_{Y^s} \|u + P_{m/2}u\|_{Y^s}, \quad (4.8)$$

then $u_m \rightarrow u$ in Y^s , but $Y^s \hookrightarrow L_t^\infty L_0^2$, this gives the uniform convergence in L^2 .

The proof of part (2) is similar to part (1).

By $h_m(u_0, t)$ we denote the function mapping any $u_0 \in L^2$ and $t \in \mathbb{R}$ into $u_m(\cdot, t + t_0)$ where $u_m(\cdot, t)$ is the solution of the problem (4.7). It is clear that the function h_m is a dynamical system with the phase space $X^m = \text{span}\{e_1, \dots, e_m\}$. In addition, a direct verification shows that $\frac{d}{dt} \|u_m(\cdot, t)\|_{L^2}^2 = 0$ and $\int u_m dx = 0$. For each $m = 1, 2, \dots$ let us consider in the space X^m the centered Gaussian measure w_m with the correlation operator S^{-1} . Since $S = S^*$ in X^m , the measure w_m is well-defined in X^m . Also, since $g(u) = \frac{1}{3} \int u^3$ is a continuous functional in X^m , the following Borel measures

$$\mu_m(\Omega) = \int_{\Omega} e^{-g(u)} dw_m(u).$$

(where Ω is an arbitrary Borel set in $L_0^2(\mathbb{T})$) are well defined.

Definition 4.8. A set Π of measures defined on the Borel sets of a topological space is called tight if, for each $\epsilon > 0$, there exist a compact set K such that

$$\mu(K) > 1 - \epsilon$$

For all $\mu \in \Pi$.

We will use the following theorem:

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Theorem 4.9. (Prokhorov) A tight set, Π , of measures on the Borel sets of a metric topological space, X , is relatively compact in the sense that for each sequence, P_1, P_2, \dots in Π there exists a subsequence that converges to a probability measure P , not necessarily in Π , in the sense that

$$\int g dP_{n_j} \rightarrow \int g dP$$

for all bounded continuous integrands. Conversely, if the metric space is separable and complete, then each relatively compact set is tight.

To prove Theorem 4.5, we will prove the following Lemma:

Lemma 4.10. μ_m is an invariant measure for the dynamical system h_m with the phase space X^m .

Proof: Let us rewrite the system (4.7) for the coefficients a_k , where $u^m(t) = \sum_{k=1}^{k=2m} a_k(t)e_k$. Let $h(a) = H(\sum_{k=1}^{k=2m} a_k e_k)$ and J is a skew-symmetric matrix, $(J_m)_{2k-1, 2k} = -\frac{2\pi k}{A} = -(J_m)_{2k, 2k-1}$ ($k=1, 2, \dots, m$) then the problem take the form

$$\begin{cases} a'(t) = J_m \nabla_a h(a(t)), \\ a_k(t_0) = (u_0, e_k), k = 1, 2, \dots, 2m \end{cases} \quad (4.9)$$

Using Theorem 4.6, we can easily verify that the Borel measure:

$$\mu'_m(A) = (2\pi)^{-\frac{2m+1}{2}} \prod_{j=1}^{2m} v_j^{\frac{1}{2}} \int_A e^{-\frac{1}{2} \sum_{j=1}^{2m} v_j a_j^2 - g(\sum_{j=1}^{2m} a_j e_j(x))} da,$$

(with $v_j = -\lambda_j + \lambda_j^{-1}$ the eigenvalues of S) is invariant for the problem (4.9). Also, we introduce the measures

$$w_m(A) = (2\pi)^{-\frac{2m+1}{2}} \prod_{j=1}^{2m} v_j^{\frac{1}{2}} \int_A e^{-\frac{1}{2} \sum_{j=1}^{2m} v_j a_j^2} da.$$

Let $\Omega_m \subset X^m$ and $\Omega_m = \{u \in L^2, u = \sum_{j=1}^{2m} a_j e_j, a \in A\}$ where $A \subset \mathbb{R}^{2m}$ is a Borel set. We set $\mu_m(\Omega_m) = \mu'_m(A)$. Since the measure μ'_m is invariant for (4.9), the measure μ_m is invariant for the problem (4.7).

Although the measure is defined on X^m , we can define it on the Borel sigma-algebra of L^2 by the rule: $\mu_m(\Omega) = \mu_m(\Omega \cap X^m)$. Since the set $\Omega \cap X^m$ is open as a set in X^m for any open set $\Omega \subset L^2$, this procedure is correct.

Lemma 4.11. $(w_m)_m$ weakly converges to w in L^2 .

Proof: S^{-1} is an operator of trace since the trace $Tr(S^{-1}) = \sum_k v_k^{-1} = \sum_k \frac{1}{\frac{1}{4\pi^2 k^2} + \frac{4\pi^2 k^2}{A^2}} < +\infty$. Thus we can find a continuous positive function $d(x)$ defined on $(0, \infty)$ with the property

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$\lim_{x \rightarrow +\infty} d(x) = +\infty$ such that $\sum_k v_k^{-1} d(\lambda_k) < +\infty$. We define the operator $T = d(S)$, the operator defined by $T(e_k) = d(v_k)e_k$ and let $B = S^{-1}T$. According to the definition of $d(x)$, $Tr(B) < +\infty$. Let $R > 0$ and $B_R = \{u \in L^2, T^{\frac{1}{2}}u \in L^2 \text{ and } \|T^{\frac{1}{2}}u\| \leq R\}$, it is clear that the closure of B_R is compact for any $R > 0$. Combined the following inequality (see [DF91] for the proof)

$$w_n(\overline{B_R}^C) = w_n(\{u; (Tu, u)_{L^2} > R\}) \leq \frac{Tr(B)}{R^2}.$$

with the Prokhorov theorem, this ensure that (w_n) is weakly compact on L^2 . In view of the definition $w_n(M) \rightarrow w(M)$ for any cylindrical set $M \subset L^2$ (because $w_n(M) = w(M)$ for all sufficiently large n). Hence, since the extension of a measure from an algebra to a minimal sigma-algebra is unique, we have proved that the sequence w_n converges to w weakly in L^2 and Lemma 4.11 is proved.

Lemma 4.12. $\liminf_m \mu_m(\Omega) \geq \mu(\Omega)$ for any open set $\Omega \subset L^2$.
 $\limsup_m \mu_m(K) \leq \mu(K)$ for any closed bounded set $K \subset L^2$.

Proof: Let $\Omega \subset L^2$ be open and let $B_R = \{u \in L^2, \|u\|_L^2 < R\}$ for some $R > 0$. Consider $\phi(u) : 0 < \phi(u) < 1$ with the support belonging to $\Omega_R = \Omega \cap B_R$ such that

$$\int_X \phi(u) e^{-g(u)} dw(u) \geq \mu(\Omega_R) - \epsilon.$$

Then,

$$\begin{aligned} \liminf_m \mu_m(\Omega_R) &= \liminf_m \int_{\Omega_R} e^{-g(u)} dw_m(u) \geq \liminf_m \int \phi(u) e^{-g(u)} dw_m(u) \\ &= \int \phi(u) e^{-g(u)} dw(u) \geq \mu(\Omega_R) - \epsilon. \end{aligned}$$

Therefore, due to the arbitrariness of $\epsilon > 0$ one has:

$$\liminf_m \mu_m(\Omega) \geq \limsup_m \mu_m(\Omega_R) \geq \mu(\Omega_R).$$

Taking $R \rightarrow +\infty$ in this inequality, we obtain the first statement the lemma.

Let K be a closed bounded set. Fix $\epsilon > 0$. We take a continuous function $\phi \in [0, 1]$ such that $\phi(u) = 1$ for any $u \in K$, $\phi(u) = 0$ if $\text{dist}(u, K) > \epsilon$ and $\int \phi(u) e^{-g(u)} w(du) < \mu(K) + \epsilon$. Then

$$\begin{aligned} \limsup_m \mu_m(K) &\leq \limsup_m \int \phi(u) e^{-g(u)} dw_m(u) \\ &= \int \phi(u) e^{-g(u)} dw(u) \leq \mu(K) + \epsilon. \end{aligned}$$

and due to the arbitrariness of $\epsilon > 0$, Lemma 4.12 is proved.

Lemma 4.13. Let $\Omega \subset L^2$ an open set and $t \in \mathbb{R}$. Then $\mu(\Omega) = \mu(h(\Omega, t))$.

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Proof: Let $\Omega_1 = h(\Omega, t)$. Fix an arbitrary $t \in \mathbb{R}$, then Ω_1 is open too. First, let us suppose that $\mu(\Omega) < \infty$.

Fix an arbitrary $\epsilon > 0$, by Prokhorov Theorem there exists a compact set $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \epsilon$, note that $K_1 = h(K, t)$ is a compact set, too, and $K_1 \subset \Omega_1$.

For any $A \subset L^2$, let ∂A be the boundary of the set A and let

$$\beta = \min\{\text{dist}(K, \partial\Omega); \text{dist}(K_1, \partial\Omega_1)\}$$

(where $\text{dist}(A, B) = \inf_{x \in A, y \in B} \|x - y\|_{L^2}$). Then, $\beta > 0$. According to Lemma 4.7, for any $z \in K$, there exists $\delta > 0$ such that for any $x, y \in B_\delta(z)$ one has $\|h_n(x, t) - h_n(y, t)\|_{L^2} < \frac{\beta}{3}$. Lets $\Omega^\alpha = \{q \in \Omega_1; \text{dist}(q, \partial\Omega_1) \geq \alpha\}$ and $B_{\delta_1}(z_1), \dots, B_{\delta_l}(z_l)$ be a finite covering of the compact set K by these balls and let $B = \bigcup_{i=1}^l B_{\delta_i}(z_i)$.

Since $h_n(z_i, t) \rightarrow h(z_i, t)$ ($n \rightarrow +\infty$) for any i we obtain that $\text{dist}(h_n(z, t), K_1) < \frac{\beta}{3}$, $\forall z \in B$ and large n . Thus, $h_n(B, t)$ belongs to a closed bounded subset of Ω_1 for all sufficiently large n .

Further, we get by the invariance of μ_n and Lemma 4.12

$$\mu(\Omega) \leq \mu(B) + \epsilon \leq \liminf \mu_n(B) + \epsilon \leq \liminf \mu_n(h_n(B, t)) + \epsilon \leq \mu(\Omega_1) + \epsilon$$

(because $\mu_n(B) = \mu_n(B \cap X_n) = \mu_n(h_n(B \cap X_n, t))$, and $h_n(B \cap X_n, t) \subset h_n(B, t)$). Hence, due to the arbitrariness of $\epsilon > 0$, we have $\mu(\Omega) \leq \mu(\Omega_1)$. By analogy $\mu(\Omega) \geq \mu(\Omega_1)$. Thus $\mu(\Omega) = \mu(\Omega_1)$.

Now if Ω is open and $\mu(\Omega) = +\infty$, then we take the sequence

$$\Omega^k = \Omega \cap \{u \in L^2; \|u\|_{L^2} + \|h(u, t)\| < k\}$$

and set $\Omega_1^k = h(\Omega^k, t)$. Then $\Omega = \bigcup \Omega^k$ and $\mu(\Omega^k) = \mu(\Omega_1^k) < \infty$. Taking $k \rightarrow +\infty$, we obtain the statement of the lemma.

4.4 Well-posedness in $X^{s, \frac{1}{2}}$

In this section, we prove a global wellposedness result for the Ostrovsky equation by following the idea of Kenig, Ponce, and Vega in [KPV96].

Our work space is Y^s , the completion of functions that are Schwartz in time and C^∞ in space with norm:

$$\|u\|_{Y^s} = \|u\|_{X^{s, \frac{1}{2}}} + \|\langle n \rangle^s \hat{u}(n, \tau)\|_{L_n^2 L_\tau^1}$$

Y^s is a slight modification of X^s such that $\|u\|_{L_t^\infty H_x^s} \lesssim \|u\|_{Y^s}$.

We see that the nonlinear part of the Ostrovsky equation is $u \partial_x u$, and by Fourier transform we write it in frequency as

$$n \sum_{n_1 \in \mathbb{Z}} \int_{\tau_1 \in \mathbb{R}} \hat{u}(n_1, \tau_1) \hat{u}(n - n_1, \tau - \tau_1) d\tau_1.$$

The resonance function is given by:

$$R(n, n_1) = \tau + m(n) - (\tau_1 + m(n_1)) - (\tau - \tau_1 + m(n - n_1)) = 3nn_1(n - n_1) - \frac{1}{n} \left(1 - \frac{n^3}{nn_1(n - n_1)}\right)$$

where $m(n) = n^3 - \frac{1}{n}$.

Now we have the following lower bound on the resonance function:

Lemma 4.14. *If $|n||n_1||n - n_1| \neq 0$, and $\frac{1}{|n|} < 1$, then:*

$$|\mathcal{R}(n, n_1)| \gtrsim |n||n_1||n - n_1|, \quad (4.10)$$

and

$$|n|^2 \leq 2|nn_1(n - n_1)|. \quad (4.11)$$

Proof: (4.11) is obvious.

Now

$$\begin{aligned} R^2(n, n_1) &= 9n^2n_1^2(n - n_1)^2 - 6n_1(n - n_1) + 6n^2 + \frac{1}{n^2} \left(1 - \frac{n^3}{n(n_1(n - n_1))}\right)^2 \\ &= n^2n_1^2(n - n_1)^2 + 8n^2n_1^2(n - n_1)^2 - 6n_1(n - n_1) + 6n^2 + \frac{1}{n^2} \left(1 - \frac{n^3}{n(n_1(n - n_1))}\right)^2 \\ &\geq n^2n_1^2(n - n_1)^2 + 8n^2n_1^2(n - n_1)^2 - 6n_1(n - n_1) \\ &= n^2n_1^2(n - n_1)^2 + |n_1(n - n_1)| (8n^2 |n_1(n - n_1)| - 6) \end{aligned}$$

Using (4.11) we obtain that:

$$R^2(n, n_1) \gtrsim n^2n_1^2(n - n_1)^2$$

Following the same argument employed in [KPV96], we state the following elementary estimates without proof.

Lemma 4.15. *For any $\epsilon > 0$, $\alpha \in \mathbb{R}$ and $0 < \rho < 1$, we have:*

$$\begin{aligned} \int_{\mathbb{R}} \frac{d\beta}{(1 + |\beta|)(1 + |\alpha - \beta|)} &\lesssim \frac{\log(2 + |\alpha|)}{(1 + |\alpha|)}. \\ \int_{\mathbb{R}} \frac{d\beta}{(1 + |\beta|)^\rho(1 + |\alpha - \beta|)} &\lesssim \frac{1 + \log(1 + |\alpha|)}{(1 + |\alpha|)^\rho}. \\ \int_{\mathbb{R}} \frac{d\beta}{(1 + |\beta|)^{1+\epsilon}(1 + |\alpha - \beta|)^{1+\epsilon}} &\lesssim \frac{1}{(1 + |\alpha|)^{1+\epsilon}}. \end{aligned}$$

Lemma 4.16. *There exists $C > 0$ such that for any $\rho > \frac{2}{3}$ and any $\tau, \tau_1 \in \mathbb{R}$, the following is true :*

$$\begin{aligned} \sum_{n_1 \neq 0} \frac{\log(2 + |\tau + m(n_1) + m(n - n_1)|)}{(1 + |\tau + m(n_1) + m(n - n_1)|)} &\leq C. \\ \sum_{n \neq 0} \frac{\log(2 + |\tau_1 + m(n_1) - m(n - n_1)|)}{(1 + |\tau_1 + m(n_1) - m(n - n_1)|)} &\leq C. \\ \sum_{n \neq 0} \frac{\log(1 + |\tau_1 + m(n_1) - m(n - n_1)|)}{(1 + |\tau_1 + m(n_1) - m(n - n_1)|)^\rho} &\leq C. \end{aligned}$$

Proposition 4.17. *Let $s \geq -\frac{1}{2}$, then for all f, g with compact support in time included in the subset $\{(t, x), t \in [-T, T]\}$, there exists $\theta > 0$ such that:*

$$\|\partial_x(fg)\|_{X^{s, -\frac{1}{2}}} \lesssim T^\theta \|f\|_{X^{s, \frac{1}{2}}} \|g\|_{X^{s, \frac{1}{2}}}.$$

Remark 4.18. *This proposition is false for $s < -\frac{1}{2}$. We can exhibit a counterexample to the bilinear estimate in the Proposition 4.17 inspired by the similar argument in [KPV96].*

We now use the lower bound of the resonance function to recover the derivative on the non-linear term $u\partial_x u$.

Lemma 4.19. *Let*

$$F_s = \frac{|n|^{2s+2} |n_1(n-n_1)|^{-2s}}{\sigma(\tau, \tau_1, n, n_1)}$$

and

$$F_{s,r} = \frac{|n|^{2s+2} |n_1(n-n_1)|^{-2s}}{\sigma^{2(1-r)}(\tau, \tau_1, n, n_1)}$$

where $\sigma(\tau, \tau_1, n, n_1) = \max\{|\tau + m(n)|, |\tau_1 + m(n_1)|, |\tau - \tau_1 + m(n-n_1)|\}$. Then, for $s \geq -\frac{1}{2}$, $0 < r < \frac{1}{4}$, we have

$$F_s \lesssim 1.$$

and

$$F_{s,r} \lesssim \frac{1}{|n|^{2-4r}}.$$

Proof: This follows from Lemma 4.14.

According to [GTV97] we have the following Lemma:

Lemma 4.20. *For any $u \in X^{s, \frac{1}{2}}$ supported in $[-T, T]$ and for any $0 < b < \frac{1}{2}$, it holds:*

$$\|u\|_{X^{s,b}} \lesssim T^{(\frac{1}{2}-b)-} \|u\|_{X^{s,1/2-}} \lesssim T^{(\frac{1}{2}-b)-} \|u\|_{X^{s,1/2}}. \quad (4.12)$$

Proof of Proposition 4.17 : Let

$$P_f^b(n, \tau) = |n|^s \langle \tau + m(n) \rangle^b |\hat{f}(n, \tau)|,$$

then we have

$$\|f\|_{X^{b,s}} = \left(\sum_n \int_{\mathbb{R}} (P_f^b(n, \tau))^2 d\tau \right)^{\frac{1}{2}} = \|P_f^b(n, \tau)\|_{l_n^2 L_{\tau}^2},$$

and

$$B(f, g)(n, \tau) = |n|^{s+1} \langle \tau + m(n) \rangle^{-\frac{1}{2}} \sum_{n_1 \neq 0, n_1 \neq n} \int_{\mathbb{R}} \frac{(n_1(n-n_1))^{-s} P_f^{\frac{1}{2}-\gamma}(n_1, \tau_1) P_g^{\frac{1}{2}}(n-n_1, \tau-\tau_1) d\tau_1}{\langle \tau_1 + m(n_1) \rangle^{\frac{1}{2}-\gamma} \langle \tau - \tau_1 + m(n-n_1) \rangle^{\frac{1}{2}}} \quad (4.13)$$

Denote

$$F(n, \tau, n_1, \tau_1) = \frac{|n|^{s+1} |n_1(n-n_1)|^{-s}}{\langle \tau + m(n) \rangle^{\frac{1}{2}} \langle \tau_1 + m(n_1) \rangle^{\frac{1}{2}-\gamma} \langle \tau - \tau_1 + m(n-n_1) \rangle^{\frac{1}{2}}}.$$

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Letting $E = \{(n, \tau, n_1, \tau_1) : |\tau - \tau_1 + m(n - n_1)| \leq |\tau_1 + m(n_1)|\}$, then by symmetry, (4.13) is reduced to estimate

$$\left(\sum_{n \neq 0} \int_{\mathbb{R}} \left(\sum_{n_1 \neq n, n_1 \neq 0} \int_{\mathbb{R}} (1_E F)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 \right)^2 d\tau \right)^{\frac{1}{2}}. \quad (4.14)$$

We separate the two cases.

Case I: $|\tau_1 + m(n_1)| \leq |\tau + m(n)|$

In this case, the set E is replaced by

$$E_I = \{(n, \tau, n_1, \tau_1) : |\tau - \tau_1 + m(n - n_1)| \leq |\tau_1 + m(n_1)| \leq |\tau + m(n)|\},$$

then by Cauchy-Schwarz inequality (4.14) is controlled by

$$\begin{aligned} & \left\| \left(\sum_{n_1 \neq n, n_1 \neq 0} \int_{\mathbb{R}} (1_{E_I} F)^2(n, \tau, n_1, \tau_1) d\tau_1 \right)^{\frac{1}{2}} \right. \\ & \quad \left. \times \left(\sum_{n_1 \neq n, n_1 \neq 0} \int_{\mathbb{R}} (P_f^{\frac{1}{2}-\gamma})^2(n - n_1, \tau - \tau_1) (P_g^{\frac{1}{2}})^2(n_1, \tau_1) d\tau_1 \right)^{\frac{1}{2}} \right\|_{l_n^2 L_\tau^2}. \end{aligned} \quad (4.15)$$

Remark that

$$F^2 \approx F_s \frac{1}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma} \langle \tau - \tau_1 + m(n - n_1) \rangle},$$

with $F_s = \frac{|m|^{2s+2} |m_1(n-n_1)|^{-2s}}{\sigma(\tau, \tau_1, n, n_1)}$, then by Lemma 4.19, for $s \geq -\frac{1}{2}$, $(n, \tau, n_1, \tau_1) \in E_I$, we have

$$\sup_{n, \tau} \sum_{n_1} \int_{\mathbb{R}} (1_{E_I} F)^2(n, \tau, n_1, \tau_1) d\tau_1 \lesssim \sup_{n, \tau} \sum_{n_1} \int_{\mathbb{R}} \frac{d\tau_1}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma} \langle \tau - \tau_1 + m(n - n_1) \rangle}$$

we can easily see that

$$(4.15) \leq \sup_{n, \tau} \sum_{n_1} \int_{\mathbb{R}} \frac{d\tau_1}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma} \langle \tau - \tau_1 + m(n - n_1) \rangle} \|P_f^{\frac{1}{2}-\gamma}(n, \tau)\|_{l_n^2 L_\tau^2} \|P_g^{\frac{1}{2}}(n, \tau)\|_{l_n^2 L_\tau^2}$$

then by Lemma 4.15, 4.16 (take $\alpha = \tau + m(n_1) + m(n - n_1)$ and $\beta = \tau_1 + m(n_1)$) and 4.20 we obtain that there exist $\theta > 0$ such that:

$$(4.14) \lesssim \|f\|_{X^{s, \frac{1}{2}-\gamma}} \|g\|_{X^{s, \frac{1}{2}}} \lesssim T^\theta \|f\|_{X^{s, \frac{1}{2}}} \|g\|_{X^{s, \frac{1}{2}}}.$$

Case II: $|\tau + m(n)| \leq |\tau_1 + m(n_1)|$ Here the set E becomes:

$$E_{II} = \{(n, \tau, n_1, \tau_1) : |\tau - \tau_1 + m(n - n_1)| \leq |\tau_1 + m(n_1)|, |\tau + m(n)| \leq |\tau_1 + m(n_1)|\}.$$

Then we will estimate

$$\left\| \sum_{n_1} \int_{\mathbb{R}} (1_{E_{II}} F)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 \right\|_{l_n^2 L_\tau^2} \quad (4.16)$$

By duality, (4.16) equals to

$$\sup_{\|w\|_{l_n^2 L_\tau^2} = 1} \sum_{n, n_1} \int_{\mathbb{R}^2} w(n, \tau) (1_{E_{II}} F)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 d\tau. \quad (4.17)$$

By Fubini's Theorem and Cauchy-Schwarz inequality, we could control (4.17) by

$$\begin{aligned} \sup_{\|w\|_{L_n^2 L_\tau^2} = 1} & \left(\sum_{n_1} \int_{\mathbb{R}} \left[\sum_n \int_{\mathbb{R}} (1_{E_{II}} F)^2(n, \tau, n_1, \tau_1) d\tau \right] \times \right. \\ & \left. \left[\sum_n \int_{\mathbb{R}} w^2 (P_f^{\frac{1}{2}-\gamma})^2(n - n_1, \tau - \tau_1) d\tau \right] d\tau_1 \right)^{\frac{1}{2}} \|g\|_{X^{s, \frac{1}{2}}}. \end{aligned} \quad (4.18)$$

Similarly to the previous case, we can show that:

$$\sup_{n_1, \tau_1} \sum_n \int_{\mathbb{R}} (1_{E_{II}} F)^2(n, \tau, n_1, \tau_1) d\tau \lesssim 1.$$

Finally we obtain that

$$(4.18) \lesssim \|f\|_{X^{s, \frac{1}{2}-\gamma}} \|g\|_{X^{s, \frac{1}{2}}} \lesssim T^\theta \|f\|_{X^{s, \frac{1}{2}}} \|g\|_{X^{s, \frac{1}{2}}}.$$

Now we have the following proposition:

Proposition 4.21. *Let $s \geq -\frac{1}{2}$ then for all f, g with compact support in time included in the subset $\{(t, x), t \in [-T, T]\}$, there exists $\theta > 0$ such that:*

$$\left(\sum_{n \in \mathbb{Z}} |n|^{2s} \left[\int_{\mathbb{R}} \frac{|nf * \hat{g}(n, \tau)|}{< \tau + m(n) >} d\tau \right]^2 \right)^{\frac{1}{2}} \lesssim T^\theta \|f\|_{X^{s, \frac{1}{2}}} \|g\|_{X^{s, \frac{1}{2}}}. \quad (4.19)$$

Proof: As in the proof of Proposition 4.17, we consider (4.19) in the same two cases. It could be written as:

$$\left\| \int_{\mathbb{R}} \sum_{n_1} \int_{\mathbb{R}} (1_E F)(\cdot, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(\cdot - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 d\tau \right\|_{l_n^2} \lesssim T^\theta \|f\|_{X^{s, \frac{1}{2}}} \|g\|_{X^{s, \frac{1}{2}}}, \quad (4.20)$$

where

$$F(n, \tau, n_1, \tau_1) = \frac{|n|^{s+1} |n_1(n - n_1)|^{-s}}{< \tau + m(n) >^{\frac{1}{2}} < \tau_1 + m(n_1) >^{\frac{1}{2}-\gamma} < \tau - \tau_1 + m(n - n_1) >^{\frac{1}{2}}}.$$

1) Case I: $|\tau_1 + m(n_1)| \leq |\tau + m(n)|$. As before, the set E is replaced by

$$E_I = \{|\tau - \tau_1 + m(n - n_1)| \leq |\tau_1 + m(n_1)| \leq |\tau + m(n)|\}.$$

By duality, we suffer to estimate

$$\sup_{\|w\|_{l_n^2} = 1} \sum_{n, n_1} \int_{\mathbb{R}^2} w(n) (1_{E_I} F)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 d\tau.$$

Now by Cauchy-Schwarz, we could control it by

$$\begin{aligned} \sup_{\|w\|_{l_n^2} = 1} & \left(\sum_{n_1} \int_{\mathbb{R}} \left[\sum_n \int_{\mathbb{R}} (1_{E_I} F)^2(n, \tau, n_1, \tau_1) d\tau \right] \times \right. \\ & \left. \left[\sum_n \int_{\mathbb{R}} w^2 (P_f^{\frac{1}{2}-\gamma})^2(n - n_1, \tau - \tau_1) d\tau \right] d\tau_1 \right)^{\frac{1}{2}} \|g\|_{X^{s, \frac{1}{2}}}, \end{aligned}$$

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then it is sufficient to show that, for $s \geq -\frac{1}{2}$

$$D = \sup_{n_1} \sum_n \int_{\mathbb{R}} \int_{\mathbb{R}} (1_{E_I} F)^2(n, \tau, n_1, \tau_1) d\tau d\tau_1 \lesssim 1.$$

For some $0 < r < \frac{1}{4}$, D can be rewritten as:

$$D = \sup_{n_1} \sum_n \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\langle \tau + m(n) \rangle^{2r}}, (1_{E_I} F_r)^2(n, \tau, n_1, \tau_1) d\tau d\tau_1$$

where

$$F_r^2 = \frac{|n|^{2s+2} |n_1(n-n_1)|^{-2s}}{\langle \tau + m(n) \rangle^{2(1-r)}} \frac{1}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma} \langle \tau - \tau_1 + m(n-n_1) \rangle}.$$

Remark that

$$F_r^2 = F_{s,r} \frac{1}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma} \langle \tau - \tau_1 + m(n-n_1) \rangle},$$

then by Lemma 4.19, D could be controlled by

$$D \lesssim \sup_{n_1} \sum_n \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|n|^{2-4r}} \frac{d\tau_1 d\tau}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma+r} \langle \tau - \tau_1 + m(n-n_1) \rangle^{1+r}}$$

by Lemma 4.15 we have:

$$\int_{\mathbb{R}} \frac{d\tau_1}{\langle \tau_1 + m(n_1) \rangle^{1-2\gamma+r} \langle \tau - \tau_1 + m(n-n_1) \rangle^{1+r}} \lesssim \frac{1}{(1 + |\tau + m(n_1) + m(n-n_1)|)^{1-2\gamma+r}}.$$

Hence

$$D \lesssim \sup_{n_1} \sum_n \frac{1}{|n|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{(1 + |\tau + m(n_1) + m(n-n_1)|)^{1-2\gamma+r}}.$$

Therefore, if $r < \frac{1}{4}$, we have $D \lesssim \sum_n \frac{1}{|n|^{2-4r}} < +\infty$.

2) Case II, $|\tau + m(n)| \leq |\tau_1 + m(n_1)|$. Now we replace E with

$$E_{II} = \{(n, \tau, n_1, \tau_1) : |\tau - \tau_1 + m(n-n_1)| \leq |\tau_1 + m(n_1)|, |\tau + m(n)| < |\tau_1 + m(n_1)|\}.$$

We write

$$1 + |\tau + m(n)| = (1 + |\tau + m(n)|)^r (1 + |\tau + m(n)|)^{1-r},$$

where $\frac{1}{2} < r < 1$. As in case I, F_r denotes

$$\frac{|n|^{s+1} |n_1(n-n_1)|^{-s}}{\langle \tau + m(n) \rangle^{(1-r)}} \frac{1}{\langle \tau_1 + m(n_1) \rangle^{\frac{1}{2}-\gamma} \langle \tau - \tau_1 + m(n-n_1) \rangle^{\frac{1}{2}}}.$$

It suffices to estimate

$$\left(\sum_n \left(\int_{\mathbb{R}} \sum_{n_1} \int_{\mathbb{R}} \frac{1}{\langle \tau + m(n) \rangle^r} (1_{E_{II}} F_r)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n-n_1, \tau-\tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 d\tau \right)^2 \right)^{\frac{1}{2}}. \quad (4.21)$$

Applying the Cauchy-Schwarz inequality in τ we see that (4.21) is bounded by

$$\left[\sum_n \left(\int_{\mathbb{R}} \frac{d\tau}{\langle \tau + m(n) \rangle^{2r}} \right) \times \left(\int_{\mathbb{R}} \left(\sum_{n_1} \int_{\mathbb{R}} (1_{E_{II}} F_r)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 \right)^2 d\tau \right)^{\frac{1}{2}} \right].$$

Since $2r > 1$, then (4.21) is dominated by

$$\left[\sum_n \int_{\mathbb{R}} \left(\sum_{n_1} \int_{\mathbb{R}} (1_{E_{II}} F_r)(n, \tau, n_1, \tau_1) P_f^{\frac{1}{2}-\gamma}(n - n_1, \tau - \tau_1) P_g^{\frac{1}{2}}(n_1, \tau_1) d\tau_1 \right)^2 d\tau \right]^{\frac{1}{2}},$$

then as the case II in the proof of Proposition 4.17, we obtain the estimate, and this end the proof.

Now we return to the proof of **Theorem 4.4**: Let L defined by

$$L(u) = \psi(t) \left[S(t) \phi - \int_0^t S(t-t') \partial_x (\psi_T^2 u^2(t')) dt' \right], \quad (4.22)$$

where $t \in \mathbb{R}$, ψ indicates a time cutoff function :

$$\psi \in C_0^\infty(\mathbb{R}), \quad \sup \psi \subset [-2, 2], \quad \psi = 1 \text{ on } [-1, 1], \quad (4.23)$$

$$\psi_T(\cdot) = \psi(\cdot/T),$$

we will apply a fixed point argument to (4.22), using the following estimates:

Proposition 4.22. *There exists a constant $C = C(\psi)$ such that:*

$$\|L(u)\|_{Y^s} \leq C \|\phi\|_{H^s} + CT^\gamma \|u\|_{Y^s}^2$$

and

$$\|L(u) - L(v)\|_{Y^s} \leq CT^\gamma \|u - v\|_{Y^s} \|u + v\|_{Y^s}.$$

Proof: It follows from Propositions 4.17, 4.21 and classical linear estimates (see [CKS⁺03]).

Note that if we take $T = (4C^2 \|\phi\|_{H^s})^{-1/\gamma}$ we deduce from Proposition 4.22 that L is strictly contractive in the ball $B(0, \frac{1}{8C^2})$ in Y^s . This proves the existence of a unique solution u to (4.22) in Y^s .

4.4.1 Global existence in $L_0^2(\mathbb{T})$

Its easy to see that the L^2 -norm is conserved ($\|u(t)\|_{L^2} = \|u_0\|_{L^2}$). Hence, if we take an initial data u_0 in $L_0^2(\mathbb{T})$, the solution u such that $u(0) = u_0$ can be extended for all positive times and the existence is global in $L_0^2(\mathbb{T})$.

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Etude qualitative de modèles dispersifs



Résumé. Dans cette thèse nous nous intéressons aux propriétés qualitatives des solutions de quelques équations d'ondes en milieux dispersifs ou dispersifs-dissipatifs. Dans le premier chapitre, nous étudions l'explosion de solutions dans le régime log-log et l'existence globale pour le problème de Cauchy de l'équation de Schrödinger L^2 -critique amortie. Dans un second chapitre, nous considérons l'équation de Schrödinger L^2 -critique avec un amortissement non linéaire. Selon la puissance du terme d'amortissement, nous montrons l'existence globale ou l'explosion en régime log-log. Dans le troisième chapitre, nous étudions le problème de Cauchy pour l'équation de Kadomtsev-Petviashvili-Burgers-I (KPBI) en deux dimensions, nous montrons que le problème est localement bien posé dans $H^s(\mathbb{R}^2)$ pour tout $s > -\frac{1}{2}$, et que l'existence est globale dans $L^2(\mathbb{R}^2)$ sans aucune condition sur la donnée initiale. Dans le dernier chapitre, nous considérons l'équation d'Ostrovsky sur le cercle, et nous construisons des mesures invariantes par le flot selon les quantités conservées par cette équation.

Mots clés. Equations dispersives et dissipatives, existence locale et globale, espaces de Bourgain, explosion, mesures invariantes.

Abstract. This thesis deals with the qualitative properties of solutions to some wave equations in dispersive or dispersive-dissipative media. In the first chapter, we study the blowup in the log-log regime and global existence of solutions to the Cauchy problem for the L^2 -critical damped nonlinear Schrödinger equation. In the second chapter, we consider the Cauchy problem for the L^2 -critical nonlinear Schrödinger equation with a nonlinear damping. According to the power of the damping term, we prove the global existence or the existence of finite time blowup dynamics with a log-log blow-up law. In the third chapter, we study the Cauchy problem for the Kadomtsev-Petviashvili-Burgers-I (KPBI) equations in two dimensions. We show that the problem is locally and globally well posed in $H^s(\mathbb{R}^2)$ for any $s > -\frac{1}{2}$, and that the existence is global in $L^2(\mathbb{R}^2)$ without any condition on the initial data. In the last chapter, we consider the Ostrovsky equation on the circle. We construct invariant measures under the flow for the conserved quantities of the equation.

Keywords. Dispersive and dissipative equation, local and global existence, Bourgain's spaces, invariants measures.