



Sur des problèmes aux conditions aux limites et à dérivées fractionnaires

Ali Benabbes

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UNIVERSITÉ DJILLALI LIABÈS
FACULTÉ DES SCIENCES EXACTES
SIDI BEL ABBÈS

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de

ALI BENLABBES

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Intitulée

SUR DES PROBLÈMES AUX CONDITIONS AUX LIMITES
ET À DÉRIVÉES FRACTIONNAIRES

Soutenue

à titre posthume

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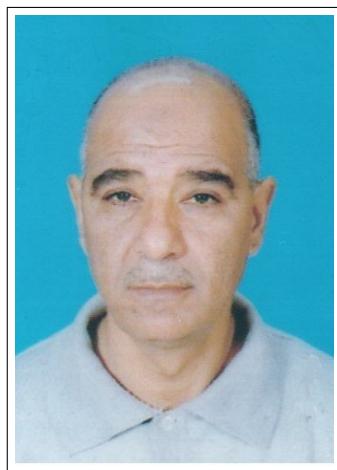
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A la mémoire du regretté Ali BENLABBES
A sa femme et ses enfants
A sa famille et ses amis.

Mustapha Lakrib et Maamar Benbachir
Sidi Bel Abbès, décembre 2016.

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INTRODUCTION

CETTE thèse est une (modeste) contribution au développement de la théorie émergente du calcul fractionnaire. On y étudie l'existence de solutions pour certaines classes d'équations différentielles d'ordre non entier (ordre fractionnaire). Les résultats obtenus sont basés sur la théorie du point fixe.

Le manuscrit de thèse est constitué de quatre chapitres répartis comme suit :

Chapitre 1 : C'est un rappel de quelques définitions, notions de base et résultats sur la dérivation fractionnaire.

Chapitre 2 : Dans ce chapitre nous présentons les résultats que nous obtenons pour le problème aux limites suivant :

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t)), & t \in J = [0, 1], 2 < \alpha \leq 3 \\ u(0) = 0, u'(0) = u'(1) = \beta u(\eta). \end{cases} \quad (1)$$

Ces résultats concernent l'existence et l'unicité de solutions pour le problème (1).

Le premier de ces résultats s'énonce comme ci-après. Il est obtenu moyennant le principe de contraction de Banach.

Théorème 1 *On suppose que les conditions suivantes sont vérifiées :*

(H₁) *La fonction $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ est continue.*

(H₂) *Il existe une constante $k > 0$ telle que :*

$$|f(t, u(t)) - f(t, v(t))| \leq k |u(t) - v(t)|,$$

pour tout $t \in J$ et tous $u, v \in C(J, \mathbb{R})$.

Si de plus la condition suivante

$$\frac{3}{2\Gamma(\alpha)} k \left(\frac{\beta}{|\beta\eta - 1|} + 1 \right) < 1$$

est satisfaite, alors le problème (1) admet une unique solution dans $C(J, \mathbb{R})$.

Le second résultat de ce chapitre est obtenu par utilisation du théorème de point fixe de Schaefer. Il s'écrit :

Théorème 2 *Supposons que la condition (H_1) du Théorème 1 est satisfaite. On suppose de plus que la condition suivante le soit aussi :*

(H_3) *Il existe une constante $N > 0$ telle que $|f(t, 0)| \leq N$, pour tout $t \in J$.*

Alors, le problème (1) admet au moins une solution définie sur J .

Un exemple illustre le résultat du Théorème 1.

Chapitre 3 : Ce chapitre traite le problème aux limites suivant :

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t)), t \in J = [0, 1], 2 < \alpha \leq 3 \\ D_{0+}^{\alpha-1}u(0) = 0, D_{0+}^{\alpha-2}u(1) = 0, u(1) = 0. \end{cases} \quad (2)$$

Nous montrons deux résultats d'existence de solutions pour le problème (2).

Le premier résultat, moyennant le principe de contraction de Banach, affirme que le problème (2) admet une solution unique.

Théorème 3 *Soit $f : J \times \mathbb{R}_+ \rightarrow \mathbb{R}$ une fonction continue satisfaisant la condition suivante :*

(H_1) *Il existe une constante $k > 0$ telle que :*

$$|f(t, u) - f(t, v)| < k |u - v|, \forall u, v \in \mathbb{R}_+, \forall t \in J.$$

Si en outre la condition suivante est réalisée :

$$k \left(\frac{2}{\Gamma(\alpha + 1)} + \frac{1}{2\Gamma(\alpha - 1)} \right) < 1$$

alors le problème (2) admet une solution unique définie sur J .

Le second résultat ici est montré par utilisation du théorème de point fixe de Krasnoselskii :

Théorème 4 *Soit $f : J \times \mathbb{R}_+ \rightarrow \mathbb{R}$ une fonction continue satisfaisant la condition (H_1) du Théorème 3. On suppose que la conditions suivante est aussi vérifiée :*

(H_2) *Il existe une constante $\beta > 0$ telle que*

$$|f(t, u)| \leq \beta, \quad \forall t \in J, \forall u \geq 0.$$

Si en outre il existe $\delta > 0$ tel que

$$\beta \frac{\alpha^2 - \alpha + 2}{\Gamma(\alpha + 1)} \leq \delta, \quad (3)$$

alors la problème (2) admet au moins une solution définie sur J .

Le Théorème 3 est illustré par un exemple.

Chapitre 4 : Il concerne les problèmes à trois points au bord du type :

$$\begin{cases} {}^cD_0^\alpha u(t) = f(t, u(t)), & n-1 < \alpha \leq n, \quad t \in J = [0, 1] \\ u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}(0) = 0, & 1 \leq i \leq n-2. \end{cases} \quad (4)$$

Moyennant un théorème de point fixe sur les cones, le résultat suivant sur l'existence de solution du problème (4) est montré. Il est illustré par un exemple.

Théorème 5 *Supposons que la fonction f est continue et super-linéaire, c'est-à-dire qu'elle vérifie :*

$$f_0 = \liminf_{\|u\| \rightarrow 0} \min_{t \in J} \frac{f(t, u(t))}{u(t)} = 0, \quad f_\infty = \limsup_{\|u\| \rightarrow +\infty} \max_{t \in J} \frac{f(t, u(t))}{u(t)} = \infty.$$

Alors, le problème (4) admet au moins une solution définie sur J .

DÉRIVATION FRACTIONNAIRE : NOTIONS PRÉLIMINAIRES

1

DANS ce chapitre nous rappelons quelques définitions, notions, propriétés et résultats sur les différentes approches de la dérivation fractionnaire (voir [1, 4, 15, 16, 17, 20, 29]).

1.1 Fonctions de base

Définition 1.1 (Fonction Gamma) *Pour tout nombre complexe α , de partie réelle positive, on définit la fonction Gamma par :*

$$\Gamma(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-t} dt. \quad (1.1)$$

La fonction Γ prolonge la fonction factorielle aux valeurs réelles et complexes.

La fonction Γ est analytique pour $\operatorname{Re}(\alpha) > 0$. Par prolongement analytique on peut définir $\Gamma(\alpha)$, pour $\alpha \in \mathbb{C} \setminus \mathbb{Z}_-^*$.

Propriété 1.1 (Relations fonctionnelles remarquables) *Nous avons la relation suivante :*

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha). \quad (1.2)$$

En particulier, nous avons

$$\Gamma(n + 1) = n!.$$

Nous avons aussi

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Définition 1.2 (Fonction Beta) *On appelle fonction Beta la fonction définie par :*

$$B(\alpha, \beta) = \int_0^1 \tau^{\alpha-1} (1 - \tau)^{\beta-1} d\tau, \quad \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0.$$

Les fonctions Γ et B sont liées par la relation suivante :

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}. \quad (1.3)$$

1.2 Quelques approches de dérivations fractionnaires

1.2.1 Approche de Riemann-Liouville

Définition 1.3 (Intégrale de Riemann-Liouville) Soit $f : [a, b] \rightarrow \mathbb{R}$ une fonction continue. On appelle intégrale de Riemann-Liouville de f l'intégrale suivante :

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x \in [a, b] \quad (1.4)$$

où α est un nombre réel ou complexe donné.

Exemple 1.1 On considère la fonction f définie par :

$$f(x) = (x-a)^\beta, \quad x \in [a, b], \quad \beta \in \mathbb{R}.$$

On a alors

$$\begin{aligned} I_a^\alpha (x-a)^\beta &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (t-a)^\beta dt \\ &= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_a^x (1-\tau)^{\alpha-1} \tau^\beta d\tau. \end{aligned}$$

D'après (1.2) et (1.3) on obtient

$$I_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}. \quad (1.5)$$

Cas particulier où $\alpha = 1$. D'après (1.2) on déduit que

$$I_a^1 (x-a)^\beta = \frac{1}{\beta+1} (x-a)^{1+\beta}.$$

Cas particulier où $\beta = 0$. On a dans ce cas

$$(I_a^\alpha 1)(x) = \frac{1}{\Gamma(\alpha+1)} (x-a)^\alpha (I_a^\alpha 1)(x) \longrightarrow 1 \quad \text{lorsque} \quad \alpha \longrightarrow 0.$$

Proposition 1.1 Soit $f \in \mathcal{C}([a, b])$. Pour α et β des nombres complexes avec : $\operatorname{Re}(\alpha) > 0$ et $\operatorname{Re}(\beta) > 0$, on a

$$I_a^\alpha (I_a^\beta f) = I_a^{\alpha+\beta} f = I_a^{\beta+\alpha} f \quad (1.6)$$

et pour $\operatorname{Re}(\alpha) > 1$ on a

$$\frac{d}{dx} I_a^\alpha f = I_a^{\alpha-1} f \quad (1.7)$$

et, pour tout $x \in [a, b]$

$$\lim_{\alpha \rightarrow 0} (I_a^\alpha f)(x) = f(x). \quad (1.8)$$

Démonstration. Soit $x \in [a, b]$.

- On a par définition de I_a^α

$$\begin{aligned}
 I_a^\alpha (I_a^\beta f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} (I_a^\beta f)(s) ds \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} f(t) dt \right) ds \\
 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_a^s (x-s)^{\alpha-1} (s-t)^{\beta-1} f(t) dt ds \\
 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(t) \left[\int_t^x (x-s)^{\alpha-1} (s-t)^{\beta-1} ds \right] dt.
 \end{aligned} \tag{1.9}$$

On pose alors $s = t + (x-t)\tau$ dans l'expression entre crochets du second membre de (1.9) pour obtenir

$$\begin{aligned}
 &\int_t^x (x-s)^{\alpha-1} (s-t)^{\beta-1} ds \\
 &= \int_0^1 (x - (t + (x-t)\tau))^{\alpha-1} (t + (x-t)\tau - t)^{\beta-1} (x-t) d\tau \\
 &= \int_0^1 [(x-t)(1-\tau)]^{\alpha-1} [(x-1)\tau]^{\beta-1} (x-t) d\tau \\
 &= (x-t)^{\alpha+\beta-1} \int_0^1 (1-\tau)^{\alpha-1} \tau^{\beta-1} d\tau.
 \end{aligned}$$

Moyennant la définition de la fonction Béta, on déduit que

$$B(\beta, \alpha) = \int_0^1 (1-\tau)^{\alpha-1} \tau^{\beta-1} d\tau = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

En retournant à la formule (1.9), on obtient alors

$$\begin{aligned}
 I_a^\alpha (I_a^\beta f)(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(t) \left[(x-t)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \right] dt \\
 &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-t)^{(\alpha+\beta)-1} f(t) dt \\
 &= (I_a^{\alpha+\beta} f)(x).
 \end{aligned}$$

• Pour l'identité (1.7) on utilise la dérivation classique d'une intégrale dépendant d'un paramètre.

- Pour l'identité (1.8), on suppose que $f \in \mathcal{C}^0([a, b])$. Alors on a

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

D'où

$$\begin{aligned}
 & \left| (I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x) \right| \\
 &= \left| \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt - \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^\alpha f(x) dt \right| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t) - f(x)| dt \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^{x-\delta} (x-t)^{\alpha-1} |f(t) - f(x)| dt + \frac{1}{\Gamma(\alpha)} \int_{x-\delta}^x (x-t)^{\alpha-1} |f(t) - f(x)| dt.
 \end{aligned}$$

Puisque la fonction $f \in \mathcal{C}^0([a, b])$ alors on a

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, t \in [a, b], \quad |x-t| \leq \delta \implies |f(x) - f(t)| \leq \varepsilon.$$

D'où

$$\begin{aligned}
 & \left| (I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x) \right| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_a^{x-\delta} (x-t)^{\alpha-1} |f(t) - f(x)| dt + \frac{1}{\Gamma(\alpha)} \int_{x-\delta}^x (x-t)^{\alpha-1} |f(t) - f(x)| dt \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_a^{x-\delta} (x-t)^{\alpha-1} (|f(t)| + |f(x)|) dt + \frac{\varepsilon}{\Gamma(\alpha)} \int_{x-\delta}^x (x-t)^{\alpha-1} dt \\
 &\leq 2 \frac{M}{\Gamma(\alpha)} \int_a^{x-\delta} (x-t)^{\alpha-1} dt + \frac{\varepsilon}{\Gamma(\alpha+1)} \delta^\alpha \\
 &\leq 2 \frac{M}{\Gamma(\alpha+1)} [(x-a)^\alpha - \delta^\alpha] + \frac{\varepsilon}{\Gamma(\alpha+1)} \delta^\alpha \\
 &\leq \frac{1}{\Gamma(\alpha+1)} (2M((x-a)^\alpha - \delta^\alpha) + \varepsilon \delta^\alpha)
 \end{aligned}$$

où $M = \sup_{x \in [a, a-\delta]} |f(x)|$. Donc, lorsque $\alpha \rightarrow 0$ on a

$$\left| (I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x) \right| \leq \varepsilon;$$

ce qui entraîne que

$$\lim_{\alpha \rightarrow 0} \left| (I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x) \right| = \left| \lim_{\alpha \rightarrow 0} (I_a^\alpha f)(x) - f(x) \right| \leq \varepsilon.$$

Comme ε est arbitraire, il s'en suit que : $\lim_{\alpha \rightarrow 0} (I_a^\alpha f)(x) = f(x)$, et achève ainsi la preuve de la proposition \square

Remarque 1.1 Nous avons, pour tout $x \in [a, b]$:

$$\begin{aligned}
 (I_a^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{d}{dt} \right) \left(-\frac{(x-t)}{\alpha} \right)^\alpha f(t) dt \\
 &= \frac{1}{\Gamma(\alpha)} \left[\left(-\frac{(x-t)}{\alpha} \right)^\alpha f(t) \right]_a^x - \int_a^x \left(-\frac{(x-t)}{\alpha} \right)^\alpha f'(t) dt \quad (1.10) \\
 &= \frac{1}{\Gamma(\alpha+1)} (x-a)^\alpha f(a) + (I_a^{\alpha+1} f')(x).
 \end{aligned}$$

De (1.10) on déduit,

- pour $\alpha = 0$:

$$(I_a^0 f)(x) = f(a) + (I_a^1 f)(x) = f(a) + \int_a^x f'(t) dt = f(x).$$

- Pour $\alpha = -1$:

$$(I_a^{-1} f)(x) = f'(x) = \frac{d}{dx} f.$$

D'où

$$I_a^{-1} f = \frac{d}{dx}.$$

Remarque 1.2

- L'identité (1.6) n'est valable que si $\operatorname{Re}(\alpha) > 0$ et $\operatorname{Re}(\beta) > 0$ car nous avons, par exemple (pour $\alpha = 1$ et $\beta = -1$) :

$$I_a^1 (I_a^{-1} f)(x) = I_a^1 \left(\frac{d}{dx} f \right)(x) = f(x) - f(a) \neq I_a^0 f(x) = f(x), \quad x \in [a, b].$$

- Une simple itération de la formule (1.10) donne :

$$(I_a^\alpha f)(x) = \sum_{j=0}^{n-1} \frac{(x-a)^{\alpha+j}}{\Gamma(\alpha+1+j)} f^{(j)}(a) + I_a^{\alpha+1} f^{(n)}(x), \quad x \in [a, b].$$

Définition 1.4 (Dérivée de Riemann-Liouville) Soit $\alpha \in]m-1, m[$, $m \in \mathbb{N}^*$. On appelle dérivée d'ordre α au sens de Riemann-Liouville de la fonction f , la fonction définie par :

$$({}^{RL}D_a^\alpha f)(x) = \left(\frac{d}{dx} \right)^m [(I_a^{m-\alpha} f)(x)], \quad x \in [a, b].$$

Exemple 1.2 Soit $f(x) = (x-a)^\beta$, $x \in [a, b]$. Nous avons

$$\begin{aligned} {}^{RL}D_a^\alpha (x-a)^\beta &= \left(\frac{d}{dx} \right)^m \left[I_a^{m-\alpha} (x-a)^\beta \right] \\ &= \left(\frac{d}{dx} \right)^m \left[\frac{\Gamma(\beta+1)}{\Gamma(m-\alpha+\beta+1)} (x-a)^{m-\alpha+\beta} \right] \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}. \end{aligned} \tag{1.11}$$

Si on pose $\beta = 0$ dans l'identité (1.11) on obtient :

$${}^{RL}D_a^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} (x-a)^{-\alpha}.$$

Ce qui montre que la dérivée de Riemann-Liouville d'une constante n'est pas nulle.

Exemple 1.3 Nous avons les relations suivantes :

$${}^{RL}D_a^0 1 = 1, \quad {}^{RL}D_a^{-1} 1 = x, \quad {}^{RL}D_a^{\frac{1}{2}} 1 = \frac{1}{\sqrt[2]{\pi x}}, \quad x \in [a, b].$$

Proposition 1.2 Soit $f(x) = x^\beta$ pour $x > 0$, avec : $\beta - \alpha + 1 > 0$. Alors

$${}^{RL}D_a^\alpha x^\beta = c \in \mathbb{R}_+ \text{ si et seulement si } \alpha = \beta.$$

Démonstration. Soit $x > 0$. Nous avons :

$$\begin{aligned} {}^{RL}D_a^\alpha x^\beta = c &\Leftrightarrow \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta - \alpha} = c \\ &\Leftrightarrow x^{\beta - \alpha} = \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta + 1)} c > 0 \\ &\Leftrightarrow (\beta - \alpha) \ln x = \ln \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta + 1)} c \\ &\Leftrightarrow (\beta - \alpha) = 0 \text{ et } \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta + 1)} c = 1 \\ &\Leftrightarrow (\beta = \alpha) \text{ et } c = \Gamma(\beta + 1). \end{aligned}$$

□

Remarque 1.3 (Dérivée de fonctions exponentielles) La dérivée d'ordre $n \in \mathbb{N}$, d'une fonction exponentielle e^{bx} où b est une constante, est donnée par l'expression suivante :

$$\left(\frac{d}{dx} \right)^n e^{bx} = b^n e^{bx}.$$

Liouville a étendu cette définition pour inclure les dérivées d'ordre arbitraire α :

$${}^{RL}D_a^\alpha e^{bx} = b^\alpha e^{bx}. \quad (1.12)$$

Il a aussi utilisé le développement en série pour collecter toutes les fonctions exponentielles :

$$f(x) = \sum_{n=0}^{\infty} c_n e^{b_n x},$$

où $\operatorname{Re}(b_n) > 0$ et $c_n \in \mathbb{R}$.

D'où

$${}^{RL}D_a^\alpha (f(x)) = {}^{RL}D_a^\alpha \left(\sum_{n=0}^{\infty} c_n e^{b_n x} \right) = \sum_{n=0}^{\infty} c_n b^\alpha e^{b_n x}, \quad \forall \alpha \in \mathbb{C}.$$

Remarque 1.4 (Dérivée de fonctions trigonométriques) On a d'après (1.11) :

$$\begin{aligned} {}^{RL}D_a^\alpha \left(e^{ibx} \right) &= (ib)^\alpha e^{ibx} \\ &= \left(\cos \left(\alpha \frac{\pi}{2} \right) = i \sin \left(\alpha \frac{\pi}{2} \right) \right) (\cos(bx) + i \sin(bx)) b^\alpha \\ &= b^\alpha \left[\cos \left(bx + \alpha \frac{\pi}{2} \right) + i \sin \left(bx + \alpha \frac{\pi}{2} \right) \right]. \end{aligned}$$

D'où l'on déduit les identités suivantes :

$${}^{RL}D_a^\alpha \cos(bx) = b^\alpha \cos\left(bx + \alpha \frac{\pi}{2}\right) \quad (1.13)$$

$${}^{RL}D_a^\alpha \sin(bx) = b^\alpha \sin\left(bx + \alpha \frac{\pi}{2}\right). \quad (1.14)$$

Lemme 1.1 Soient $\alpha \in]m-1, m[$ et f une fonction appartenant au noyau de l'opérateur ${}^{RL}D_a^\alpha$. Alors

$$f(x) = \sum_{j=0}^{m-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-m)} (x-a)^{\alpha+j-m}, \quad x \in [a, b] \quad (1.15)$$

où les c_j , $j = 0, \dots, m-1$, sont des constantes réelles.

Démonstration. Soit $x \in [a, b]$. On a

$$\begin{aligned} {}^{RL}D_a^\alpha f(x) = 0 &\Leftrightarrow \left(\frac{d}{dx}\right)^m [(I_a^{m-\alpha} f)(x)] = 0 \\ &\Leftrightarrow (I_a^{m-\alpha} f)(x) = \sum_{j=0}^{m-1} c_j (x-a)^j \\ &\Leftrightarrow (I_a^m f)(x) = \sum_{j=0}^{m-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)} (x-a)^{\alpha+j} \\ &\Leftrightarrow f(x) = \sum_{j=0}^{m-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-m)} (x-a)^{\alpha+j-m}. \end{aligned}$$

□

Exemple 1.4 Les fonctions que nous voulons maintenant discuter et illustrer sont les fonctions puissances, les fonctions exponentielles et les fonctions cosinus et sinus.

- **Fonctions de la forme** : $f(x) = x^{\frac{m}{n}}$, avec $m \in \mathbb{Z}$ et $n \in \mathbb{Z}^*$.

D'après (1.8), on a :

$${}^{RL}D_a^{\frac{1}{4}} x^{\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} x^{\frac{1}{4}} = 0,97774 x x^{\frac{1}{2}},$$

$${}^{RL}D_a^{\frac{1}{2}} x^{\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)} x^0 = \frac{1}{2} \sqrt[2]{\pi},$$

$${}^{RL}D_a^{\frac{3}{4}} x^{\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} x^{-\frac{1}{4}} = 0,7232 x^{-\frac{1}{4}},$$

$${}^{RL}D_a^1 x^{\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}} = 0.5 x^{-\frac{1}{2}}.$$

- **Fonctions de la forme :** $f(x) = e^{bx}$ où b est une constante donnée.
D'après (1.12), on a :

$${}^{RL}D_a^{\frac{1}{4}}e^{2x} = \sqrt[4]{2}e^{2x}, \quad {}^{RL}D_a^{\frac{1}{2}}e^{2x} = \sqrt[2]{2}e^{2x}$$

$${}^{RL}D_a^1e^{2x} = 2e^{2x}, \quad {}^{RL}D_a^{\frac{5}{4}}e^{2x} = \sqrt[4]{32}e^{2x}.$$

- **Fonctions de la forme :** $f(x) = \sin(bx)$ et $g(x) = \cos(bx)$, où b est une constante donnée.

Pour $f(x) = \sin(bx)$, d'après (1.13) et (1.14), on a :

$${}^{RL}D_a^{\frac{1}{4}}\sin(x) = \sin(x + \frac{\pi}{8}), \quad {}^{RL}D_a^{\frac{1}{2}}\sin(x) = \sin(x + \frac{\pi}{4}),$$

$${}^{RL}D_a^{\frac{3}{4}}\sin(x) = \sin(x + \frac{3\pi}{8}), \quad {}^{RL}D_a^1\sin(x) = \cos(x).$$

Pour $f(x) = \cos(bx)$, d'après (1.13) et (1.14), on a :

$${}^{RL}D_a^{\frac{1}{4}}\cos(x) = \cos(x + \frac{\pi}{8}), \quad {}^{RL}D_a^{\frac{1}{2}}\cos(x) = \cos(x + \frac{\pi}{4})$$

$${}^{RL}D_a^{\frac{3}{4}}\cos(x) = \cos(x + \frac{3\pi}{8}), \quad {}^{RL}D_a^1\cos(x) = -\sin(x).$$

1.2.2 Approche de Caputo

Définition 1.5 Soit $\alpha \in]m-1, m[$ et $f \in C^m([a, b])$. On appelle dérivée de f au sens de Caputo la fonction définie par :

$$({}^C D_0^\alpha f)(x) = (I_0^{m-\alpha} f^{(m)})(x), \quad x \in [a, b]. \quad (1.16)$$

Remarque 1.5 La dérivée fractionnaire de Riemann-Liouville d'ordre $\alpha \in]m-1, m[$ s'obtient par une application de $I_0^{m-\alpha}$ suivie par une dérivation classique d'ordre m , c'est-à-dire, $\left((\frac{d}{dx})^m I_0^{m-\alpha} \right)$. Par contre la dérivée fractionnaire de Caputo est le résultat de la permutation de ces deux opérations, c'est-à-dire, $\left(I_0^{m-\alpha} (\frac{d}{dx})^m \right)$.

Exemple 1.5

- On a ${}^C D_0^\alpha 1 = 0$ mais ${}^{RL}D_0^\alpha 1 \neq 0$.
- On a ${}^C D_a^\alpha (x-a)^\beta = I_a^{m-\alpha} [(x-a)^\beta]^{(m)}$, $x \in [a, b]$, avec

$$[(x-a)^\beta]^{(m)} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} (x-a)^{\beta-m}$$

et donc

$$\begin{aligned}
 I_a^{m-\alpha} [(x-a)^\beta]^{(m)} &= I_a^{m-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} (x-a)^{\beta-m} \right) \\
 &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} (t-a)^{\beta-m} dt \\
 &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} \frac{(x-a)^{\beta-\alpha}}{\Gamma(m-\alpha)} \int_0^1 (1-\tau)^{m-\alpha-1} \tau^{\beta-m} d\tau \\
 &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} \frac{(x-a)^{\beta-\alpha}}{\Gamma(m-\alpha)} B(m-\alpha, \beta-m+1) \\
 &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} \frac{(x-a)^{\beta-\alpha}}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha) \Gamma(\beta-m+1)}{\Gamma(\beta-\alpha+1)} \\
 &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}.
 \end{aligned}$$

Remarque 1.6 Nous avons : ${}^C D_a^\alpha (x-a)^\beta = {}^{RL} D_a^\alpha (x-a)^\beta$ si $\beta \notin \mathbb{N}$ mais si $\beta \in \mathbb{N}$, $\beta < m$, on a ${}^C D_a^\alpha (x-a)^\beta = 0$ et ${}^{RL} D_a^\alpha (x-a)^\beta \neq 0$.

1.2.3 Lien entre les dérivées fractionnaires de Caputo et de Riemann-Liouville

On a, pour $x \in [0, b]$ ($a = 0$) :

$$\begin{aligned}
 (I_0^{m-\alpha} f)(x) &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f(t) dt \\
 &= \frac{1}{\Gamma(m-\alpha)} \int_0^x d \left(\frac{-(x-t)^{m-\alpha}}{m-\alpha} \right) f(t) dt \\
 &= \frac{1}{\Gamma(m-\alpha)} \left[f(t) \frac{-(x-t)^{m-\alpha}}{m-\alpha} \right]_0^x \\
 &\quad - \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{-(x-t)^{m-\alpha}}{m-\alpha} f'(t) dt \\
 &= \frac{1}{\Gamma(m-\alpha+1)} f(0) x^{m-\alpha} \\
 &\quad - \frac{1}{\Gamma(m-\alpha+2)} f'(0) x^{m-\alpha+1} + (I_0^{m-\alpha+2} f'')(x).
 \end{aligned}$$

Par une simple itération on trouve

$$(I_0^{m-\alpha} f)(x) = \sum_{j=0}^{m-1} \frac{x^{m-\alpha+j}}{\Gamma(m-\alpha+j+1)} f^{(j)}(0) + (I_0^{2m-\alpha} f^{(m)})(x). \quad (1.17)$$

En appliquant $\left(\frac{d}{dx}\right)^m$ on obtient :

$$\begin{aligned} & \left(\frac{d}{dx}\right)^m (I_0^{m-\alpha} f)(x) \\ &= \left(\frac{d}{dx}\right)^m \left(\sum_{j=0}^{m-1} \frac{x^{m-\alpha+j}}{\Gamma(m-\alpha+j+1)} f^{(j)}(0) + (I_0^{2m-\alpha} f^{(m)}) (x) \right) \\ &= \sum_{j=0}^{m-1} \frac{x^{-\alpha+j}}{\Gamma(-\alpha+j+1)} f^{(j)}(0) + (I_0^{m-\alpha} f^{(m)}) (x). \end{aligned}$$

Ainsi, la relation entre les deux approches est donnée par :

$$({}^{RL}D_0^\alpha f)(x) = ({}^C D_0^\alpha f)(x) + \sum_{j=0}^{m-1} \frac{x^{-\alpha+j}}{\Gamma(-\alpha+j+1)} f^{(j)}(0). \quad (1.18)$$

Proposition 1.3 *Nous avons les propriétés suivantes :*

- 1) ${}^C D_0^\alpha (I_0^\alpha f) = f$.
- 2) Si ${}^C D_0^\alpha f(x) = 0$ alors $f(x) = \sum_{j=0}^{m-1} c_j (x-a)^j$, $x \in [0, b]$.
- 3) $I_0^\alpha [{}^C D_0^\alpha f](x) = f(x) - \sum_{j=0}^{m-1} \frac{(x)^j}{\Gamma(j+1)} f^{(j)}(0)$, $x \in [0, b]$.

Démonstration. Soit $x \in [0, b]$.

- 1) On a : ${}^C D_0^\alpha (I_0^\alpha f)(x) = I_0^{m-\alpha} \left(\frac{d}{dx} \right)^m (I_0^\alpha f)(x) = I_0^0 f(x) = f(x)$.

2) Il suffit de remarquer que :

$$\left(\frac{d}{dx} \right)^m \left(\sum_{j=0}^{m-1} c_j (x-a)^j \right) = 0.$$

3) D'après la définition de ${}^C D_0^\alpha$ on a :

$$\begin{aligned} I_0^\alpha [{}^C D_0^\alpha f](x) &= I_0^\alpha [I_0^{m-\alpha} (f)^{(m)}(x)] \\ &= (I_0^m f^{(m)})(x) \\ &= \frac{1}{\Gamma(m)} \int_0^x (x-t)^{m-1} f^{(m)}(t) dt \\ &= \frac{1}{\Gamma(m)} \int_0^x (x-t)^{m-1} (f^{(m-1)})'(t) dt \\ &= -\frac{1}{\Gamma(m)} (x)^{m-1} f(0) + \frac{1}{\Gamma(m-1)} \int_0^x (x-t)^{m-2} (f^{(m-2)})'(t) dt \\ &= -\frac{1}{\Gamma(m)} (x)^{m-1} f(0) - \frac{1}{\Gamma(m-1)} (x)^{m-1} f'(0) \\ &\quad + \frac{1}{\Gamma(m-2)} \int_0^x (x-t)^{m-3} (f^{(m-3)})'(t) dt. \end{aligned}$$

Par une intégration par partie successive, on obtient finalement :

$$\begin{aligned} I_0^\alpha \left[{}^C D_0^\alpha f \right] (x) &= -\frac{1}{\Gamma(m)} (x)^{m-1} f(0) - \frac{1}{\Gamma(m-1)} (x)^{m-2} f'(0) \\ &\quad - \cdots + f(x) - f(0) \\ &= f(x) - \sum_{j=0}^{m-1} \frac{(x)^j}{\Gamma(j+1)} f^{(j)}(0). \end{aligned}$$

□

Lemme 1.2 Si $f \in \mathcal{C}([0, b])$ et $\alpha > 0$, alors on a :

$$\lim_{x \rightarrow 0} (I_0^\alpha f)(x) = 0.$$

Démonstration. Soit $x \in [0, b]$. On a :

$$\begin{aligned} 0 \leq |(I_0^\alpha f)(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t)| dt \\ &\leq \frac{\|f\|_\infty}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} dt \leq \frac{\|f\|_\infty}{\Gamma(\alpha+1)} x^\alpha. \end{aligned}$$

Au passage à la limite lorsque x tend vers zéro, on trouve le résultat recherché. □

1.2.4 Approche de Grunwald-Letnikov

L'idée de cette approche est de généraliser la définition classique de la dérivation entière d'une fonction à des ordres de dérivée arbitraire. Ce qui permet d'exprimer la dérivée d'ordre entier p (si p est positif) et l'intégrale répétée $(-p)$ fois (si p est négatif), d'une fonction f comme ceci : Pour une fonction f donnée, d'après la définition classique de la dérivation en un point t on a :

$$f'(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t-h)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} (f(t) - f(t-h)). \quad (1.19)$$

On utilise le même concept pour la dérivée seconde pour trouver

$$\begin{aligned} f''(t) &= \lim_{h \rightarrow 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^2} (f(t) - 2f(t-h) + f(t-2h)). \end{aligned} \quad (1.20)$$

Moyennant (1.19) et (1.20) on obtient

$$f^{(3)}(t) = \lim_{h \rightarrow 0} \frac{1}{h^3} (f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)).$$

Par itération on trouve la formule générale

$$f^{(n)}(t) = \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} f(t - kh)$$

où

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}.$$

Pour un entier p arbitraire on a

$$f^{(p)}(t) = \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{p}{k} f(t - kh).$$

Remarquons que si $p \leq n$ on a

$$\begin{aligned} f^{(p)}(t) &= \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{p}{k} f(t - kh) \\ &= \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^p (-1)^k \binom{p}{k} f(t - kh) \\ &\quad + \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=p+1}^n (-1)^k \binom{p}{k} f(t - kh) \\ &= \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^p (-1)^k \binom{p}{k} f(t - kh), \end{aligned}$$

en tenant compte du fait que $\binom{p}{p+j} = 0$ pour tout $j = 1, \dots, n$.

Définition 1.6 La dérivée fractionnaire d'ordre $\alpha > 0$ de Grunwald-Letnikov est donnée par

$${}^{GL}D^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} f(t - kh)$$

où $0 \leq n-1 < \alpha < n$.

Remarque 1.7 Remarquons que

$$\begin{aligned} (-1)^k \binom{\alpha}{k} &= (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \\ &= (-1)^k \frac{\alpha(\alpha-1)\dots(\alpha-k+1)(\alpha-k)!}{k!(\alpha-k)!} \\ &= \frac{-\alpha(-\alpha+1)\dots(-\alpha-k+1)}{k!} \\ &= \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}. \end{aligned}$$

Donc

$${}^{GL}D^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)} f(t - kh).$$

L'intégrale fractionnaire se traduit par l'expression suivante :

$${}^{GL}I^{\alpha}f(t) = {}^{GL}D^{-\alpha}f(t) = \lim_{h \rightarrow 0} h^{\alpha} \sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)} f(t-kh).$$

Cas particuliers :

Cas $\alpha = 1$:

$$\begin{aligned} {}^{GL}I^1f(t) &= \lim_{h \rightarrow 0} h \sum_{k=0}^{+\infty} \frac{\Gamma(k+1)}{\Gamma(k+1)\Gamma(1)} f(t-kh) \\ &= \lim_{h \rightarrow 0} h \sum_{k=0}^{+\infty} f(t-kh) \\ &= \int_0^{t-a} f(t-y) dy = \int_0^t f(\tau) d\tau. \end{aligned}$$

Cas $\alpha = 2$:

$$\begin{aligned} {}^{GL}I^2f(t) &= \lim_{h \rightarrow 0} h \sum_{k=0}^{+\infty} \frac{\Gamma(k+2)}{\Gamma(k+1)\Gamma(2)} f(t-kh) \\ &= \lim_{h \rightarrow 0} h \sum_{k=0}^{+\infty} (k+1) f(t-kh). \end{aligned}$$

En faisant le changement de variable $t+h = z$, on déduit que

$${}^{GL}I^2f(t) = \lim_{h \rightarrow 0} h \sum_{k=1}^{+\infty} kf(z-kh).$$

EXISTENCE OF SOLUTIONS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

2

THIS chapter essentially contains the paper [7] intitled : "*Existence solutions of a nonlinear fractional differential equation*" and published in *Journal of Advanced Research in Dynamical and Control Systems, Volume 6, 2014, pp. 1-12.*

Abstract. In this work, we give an improvement of the results in [23, 27, 28]. By use of some fixed point theorems, we investigate the existence and uniqueness of solutions for the following problem :

$$\begin{cases} D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 2 < \alpha \leq 3, \quad 0 < t < 1 \\ u(0) = 0, \quad u'(0) = u'(1) = \beta u(\eta) \end{cases}$$

where D_{0+}^α denotes the fractional derivative of order α in the sense of Caputo. $\eta \in (0, 1)$ and $\beta \in [1, 1/\eta]$ are some arbitrary constants.

2.1 Introduction

There are many works which deal with the existence and multiplicity of solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis, especially the theory of fixed point.

In [30], by use of some fixed point arguments, Zhang proved the existence of solutions for the following nonlinear fractional boundary value problem involving Caputo's derivative :

$$\begin{cases} D_t^\alpha u + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha < 2 \\ u(0) = \nu, \quad u(1) = \rho, \quad \nu\rho \neq 0. \end{cases}$$

In another paper, by use of a fixed point theorem in cones, Zhang in [29] studied the existence and multiplicity of positive solutions of the nonlinear fractional boundary value problem :

$$\begin{cases} D_t^\alpha u + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha < 2, \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0, \end{cases}$$

where D_t^α is the Caputo's fractional derivative.

In [5], Benchohra, Hamani, Ntouyas and Ouahab, by means of the Banach's fixed point theorem and the nonlinear alternative of Leray–Schauder type, proved the existence of solutions for the first order boundary value problem for a fractional order differential equation :

$$\begin{cases} D_t^\alpha u = f(t, u(t)) = 0, \quad 0 < t < 1, \quad 0 < \alpha < 1, \\ au(0) + bu(1) = c, \end{cases}$$

where D_t^α is the Caputo's fractional derivative, f is a continuous function and a, b, c are real constants with $a + b \neq 0$.

Houas and Dahmani in [14] considered the following nonlinear fractional boundary value problem :

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t), D_{0+}^\beta u(t)), \quad 2 < \alpha \leq 3, \quad \beta \leq \alpha - 1, \quad 0 < t < 1 \\ u(0) = u(1) = 0, \quad \lambda_1 u(\eta) + \lambda_2 u'(\eta) = \lambda_3 u(\xi) + \lambda_4 u'(\xi) = 0 \end{cases}$$

where D_{0+}^α and D_{0+}^β are the Caputo's fractional derivative, $\lambda_i, i = 1, 4$ are real constants with $\lambda_2 \lambda_3 + \lambda_1 \lambda_4 + \lambda_1 \lambda_3 (\xi - \eta) \neq 0$ and f is a continuous function on $[0, 1]$.

Qui and Bai in [23] discussed the existence of positive solutions for boundary value problems associated to nonlinear fractional differential equations of the form :

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t)), \quad 2 < \alpha \leq 3, \quad 0 < t < 1 \\ u(0) = u'(1) = u''(0) = 0 \end{cases}$$

where D_{0+}^α is the Caputo's differentiation, $f : (0, 1] \times [0, +\infty[\rightarrow [0, +\infty[$ is continuous, with $\lim_{t \rightarrow 0} f(t, \cdot) = +\infty$ (that is f is singular at $t = 0$).

In this work, we investigate existence and uniqueness of solutions for the following boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 2 < \alpha \leq 3, \quad 0 < t < 1 \\ u(0) = 0, \quad u'(0) = u'(1) = \beta u(\eta) \end{cases} \tag{2.1}$$

where D_{0+}^α denotes the fractional derivative of order α in the sense of Caputo. $\eta \in (0, 1)$ and $\beta \in [1, 1/\eta]$ are some arbitrary constants. Our results improve those in [23, 27, 28].

2.2 Preliminaries

From the definition of Caputo's derivative we can obtain the following lemmas which will play major roles in our analysis.

Lemma 2.1 *Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^\alpha u(t) = 0$$

has as solution

$$u(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 1, \dots, n-1,$$

where n is the smallest integer greater than or equal to α .

Lemma 2.2 *Let $\alpha > 0$. Then*

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 1, \dots, n-1,$$

where n is the smallest integer greater than or equal to α .

Lemma 2.3 *Let $\beta\eta \neq 1$ and $h \in L^1([0, 1])$. Then the three-point boundary-value problem*

$$\begin{cases} D_{0+}^\alpha u(t) + h(t) = 0, & 2 < \alpha \leq 3, 0 < t < 1 \\ u(0) = 0, u'(0) = u'(1) = \beta u(\eta) \end{cases}$$

has the unique solution

$$u(t) = \int_0^1 G_1(t, s) h(s) ds + \frac{\beta t}{1 - \beta\eta} \int_0^1 G_2(t, s) h(s) ds$$

where

$$G_1(\eta, s) = \frac{1}{2\Gamma(\alpha-1)} \begin{cases} \frac{2}{\alpha-1} (t-s)^{\alpha-1} - t^2 (1-s)^{\alpha-2}, & s \leq t \\ -t^2 (1-s)^{\alpha-2}, & s \geq t. \end{cases}$$

and

$$G_2(\eta, s) = \frac{\beta t}{2\Gamma(\alpha-1)(\beta\eta-1)} \begin{cases} \frac{2}{\alpha-1} (\eta-s)^{\alpha-1} - \eta^2 (1-s)^{\alpha-2}, & s \leq \eta \\ -\eta^2 (1-s)^{\alpha-2}, & s \geq \eta. \end{cases}$$

Proof. By applying Lemma 2.1 and Lemma 2.2, the equation $D_{0+}^\alpha u(t) + h(t) = 0$ is equivalent to the following integral equation

$$u(t) = -c_0 - c_1 t - c_2 t^2 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \quad (2.2)$$

for some constants $c_0, c_1, c_2 \in \mathbb{R}$. The boundary-value condition $u(0) = 0$ implies that $c_0 = 0$.

From

$$u'(t) = -c_1 - 2c_2 t - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s) ds,$$

condition $u'(0) = u'(1)$ implies that

$$-c_1 = -c_1 - 2c_2 - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds.$$

Therefore

$$c_2 = -\frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds.$$

For c_1 we have

$$\begin{aligned} u'(0) = \beta u(\eta) &\implies \\ -c_1 &= \beta \left(-c_1 \eta - c_2 \eta^2 - \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s) ds \right) \implies \\ -c_1 &= \beta \left(-c_1 \eta + \frac{\eta^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s) ds \right) \end{aligned}$$

and then

$$c_1 = \frac{\beta \eta^2}{(\beta \eta - 1)} \int_0^1 \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} h(s) ds - \frac{\beta}{(\beta \eta - 1)} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds.$$

Substituting the values of c_1 and c_2 in (2.2), we obtain the desired relation in Lemma 2.3, that is :

$$\begin{aligned} u(t) &= -\frac{\beta \eta^2 t}{2\Gamma(\alpha-1)(\beta \eta - 1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds \\ &\quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta \eta - 1)} \int_0^\eta (\eta-s)^{\alpha-1} h(s) ds \\ &\quad + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &= \frac{\beta t}{2\Gamma(\alpha-1)(\beta \eta - 1)} \left[\int_0^\eta \frac{2}{\alpha-1} (\eta-s)^{\alpha-1} h(s) ds \right. \\ &\quad \left. - \int_0^1 \eta^2 (1-s)^{\alpha-2} h(s) ds \right] \\ &\quad + \frac{1}{2\Gamma(\alpha-1)} \int_0^1 t^2 (1-s)^{\alpha-2} h(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &= \frac{\beta t}{2\Gamma(\alpha-1)(\beta \eta - 1)} \int_0^1 G_2(\eta, s) h(s) ds + \int_0^1 G_1(t, s) h(s) ds. \end{aligned}$$

□

Let us now present the fundamental tools on which the proofs of our main results are based.

Definition 2.1 Let $(\mathbb{E}, \|\cdot\|)$ be a normed space. A contraction of \mathbb{E} is a mapping $T : \mathbb{E} \rightarrow \mathbb{E}$ that satisfies :

$$\|Tx_1 - Tx_2\| \leq \beta \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{E}$$

for some real number $\beta < 1$.

Theorem 2.1 (Banach's contraction principle, [10]) Every contraction mapping on a complete metric space has a unique fixed point.

Theorem 2.2 (Schaefer's fixed point theorem, [26]) Let \mathbb{E} be a normed space and let $T : \mathbb{E} \rightarrow \mathbb{E}$ be a completely continuous map, that is, it is a continuous mapping which is relatively compact on each bounded subset of \mathbb{E} . If the set

$$\mathcal{E} = \{x \in \mathbb{E} : \lambda x = Tx \text{ for some } \lambda > 1\}$$

is bounded, then T has a fixed point.

2.3 Main Results

We introduce the following conditions :

- (H₁) The function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.
- (H₂) There exists a constant $k > 0$ such that,

$$|f(t, u(t)) - f(t, v(t))| \leq k |u(t) - v(t)|,$$

for each $t \in [0, 1]$ and all $u, v \in C([0, 1], \mathbb{R})$.

- (H₃) There exists a constant $N > 0$ such that $|f(t, 0)| \leq N$, for each $t \in [0, 1]$.

Our first result in this work is based on the Banach's contraction principle and reads as follows.

Theorem 2.3 Assume that conditions (H₁) and (H₂) hold. If

$$\frac{3}{2\Gamma(\alpha)}k \left(\frac{\beta}{|\beta\eta - 1|} + 1 \right) < 1 \tag{2.3}$$

then problem (2.1) has a unique solution in $C([0, 1])$.

Proof. Consider the operator $T : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ given by

$$\begin{aligned} Tu(t) &= -\frac{\beta\eta^2 t}{2\Gamma(\alpha-1)(\beta\eta-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\ &\quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u(s)) ds \\ &\quad + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds. \end{aligned} \tag{2.4}$$

For all $u, v \in C([0, 1], \mathbb{R})$, $t \in [0, 1]$, we have

$$\begin{aligned} |Tu(t) - Tv(t)| &= \left| -\frac{\beta\eta^2 t}{2\Gamma(\alpha-1)(\beta\eta-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \right. \\ &\quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u(s)) ds \\ &\quad + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \\ &\quad + \frac{\beta\eta^2 t}{2\Gamma(\alpha-1)(\beta\eta-1)} \int_0^1 (1-s)^{\alpha-2} f(s, v(s)) ds \\ &\quad - \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, v(s)) ds \\ &\quad - \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, v(s)) ds \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s)) ds \right|. \end{aligned}$$

By (H_2) , we obtain

$$\begin{aligned} &|Tu(t) - Tv(t)| \\ &\leq \frac{\beta\eta^2 t}{2\Gamma(\alpha-1)|\beta\eta-1|} k \|u-v\| \int_0^1 (1-s)^{\alpha-2} ds \\ &\quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)|\beta\eta-1|} k \|u-v\| \int_0^\eta (\eta-s)^{\alpha-1} ds \\ &\quad + \frac{t^2}{2\Gamma(\alpha-1)} k \|u-v\| \int_0^1 (1-s)^{\alpha-2} ds \\ &\quad + \frac{1}{\Gamma(\alpha)} k \|u-v\| \int_0^t (t-s)^{\alpha-1} ds. \end{aligned}$$

Consequently, we have

$$\begin{aligned} &|Tu(t) - Tv(t)| \\ &\leq \|u-v\| k \left[\frac{\beta}{2\Gamma(\alpha)|\beta\eta-1|} + \frac{\beta}{\Gamma(\alpha+1)|\beta\eta-1|} \right. \\ &\quad \left. + \frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right] \\ &\leq \|u-v\| k \left[\frac{\beta}{2\Gamma(\alpha)|\beta\eta-1|} \right. \\ &\quad \left. + \frac{\beta}{\Gamma(\alpha)|\beta\eta-1|} + \frac{1}{2\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \right] \\ &\leq \frac{3}{2\Gamma(\alpha)} k \left[\frac{\beta}{|\beta\eta-1|} + 1 \right] \|u-v\| \end{aligned}$$

from which we deduce that

$$\|Tu - Tv\| \leq \frac{3}{2\Gamma(\alpha)} k \left[\frac{\beta}{|\beta\eta - 1|} + 1 \right] \|u - v\|.$$

Finally, thanks to (2.3) we deduce that T is a contraction mapping. As a consequence of Banach contraction principle, problem (2.1) has a unique solution defined on $[0, 1]$. \square

The second result of this work is the following.

Theorem 2.4 Suppose that conditions (H_1) and (H_3) are satisfied. Then problem (2.1) has at least one solution on $[0, 1]$.

We use Scheafer's fixed point theorem to prove that the operator $T : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ given by (2.4) has at least a fixed point on $C([0, 1], \mathbb{R})$.

Proof. We divide the proof in four steps :

Step 1 : Let $(u_n)_{n \in \mathbb{N}}$ be a sequence such that $u_n \rightarrow u$ in $C([0, 1], \mathbb{R})$. Then, for each $t \in [0, 1]$, we have

$$\begin{aligned} & |Tu_n(t) - Tu(t)| \\ &= \left| -\frac{\beta\eta^2 t}{2\Gamma(\alpha-1)(\beta\eta-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u_n(s)) ds \right. \\ &\quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u_n(s)) ds \\ &\quad + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u_n(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_n(s)) ds \\ &\quad + \frac{\beta\eta^2 t}{2\Gamma(\alpha-1)(\beta\eta-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\ &\quad - \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u(s)) ds \\ &\quad - \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \right|. \end{aligned}$$

After a simple calculation we find

$$\begin{aligned}
 & |Tu_n(t) - Tu(t)| \\
 & \leq \frac{\beta\eta^2 t}{2\Gamma(\alpha-1)|\beta\eta-1|} \int_0^1 (1-s)^{\alpha-2} (f(s, u_n(s)) - f(s, u(s))) ds \\
 & \quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} (f(s, u_n(s)) - f(s, u(s))) ds \\
 & \quad + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} (f(s, u_n(s)) - f(s, u(s))) ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, u_n(s)) - f(s, u(s))) ds.
 \end{aligned}$$

Using (H_1) , we can find $\|Tu_n - Tu\| \rightarrow 0$ when $n \rightarrow +\infty$. Then the operator T is continuous on $C([0, 1], \mathbb{R})$.

Step 2 : Let $\rho > 0$ and $B_\rho = \{u \in C([0, 1], \mathbb{R}) : \|u\| \leq \rho\}$. For all $u \in B_\rho$ we have

$$\begin{aligned}
 |Tu(t)| &= \left| -\frac{\beta\eta^2 t}{2\Gamma(\alpha-1)(\beta\eta-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \right. \\
 &\quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)(\beta\eta-1)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, u(s)) ds \\
 &\quad \left. + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \right|.
 \end{aligned} \tag{2.5}$$

We can see that

$$\begin{aligned}
 |f(s, u(s))| &= |f(s, u(s)) - f(s, 0) + f(s, 0)| \\
 &\leq |f(s, u(s)) - f(s, 0)| + |f(s, 0)| \\
 &\leq k|u(s)| + N \\
 &\leq k\rho + N.
 \end{aligned} \tag{2.6}$$

Using (2.6) in (2.5), we obtain

$$\begin{aligned}
 & |Tu(t)| \\
 & \leq \frac{\beta\eta^2 t}{2\Gamma(\alpha-1)|\beta\eta-1|} \int_0^1 (1-s)^{\alpha-2} (k|u(s)| + N) ds \\
 & \quad + \frac{\beta t}{(\alpha-1)\Gamma(\alpha-1)|\beta\eta-1|} \int_0^\eta (\eta-s)^{\alpha-1} (k|u(s)| + N) ds \\
 & \quad + \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} (k|u(s)| + N) ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (k|u(s)| + N) ds.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 |Tu(t)| &\leq \frac{(k\rho + N) \beta \eta^2 t}{2\Gamma(\alpha - 1) |\beta\eta - 1|} \int_0^1 (1-s)^{\alpha-2} ds \\
 &\quad + \frac{(k\rho + N) \beta t}{(\alpha - 1) \Gamma(\alpha - 1) |\beta\eta - 1|} \int_0^\eta (\eta - s)^{\alpha-1} ds \\
 &\quad + \frac{(k\rho + N) t^2}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} ds + \frac{(k\rho + N)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
 &\leq \frac{(k\rho + N) \beta}{2\Gamma(\alpha) |\beta\eta - 1|} + \frac{(k\rho + N) \beta}{\Gamma(\alpha + 1) |\beta\eta - 1|} + \frac{(k\rho + N)}{2\Gamma(\alpha)} + \frac{(k\rho + N)}{\Gamma(\alpha + 1)}
 \end{aligned}$$

so that

$$|Tu(t)| \leq \frac{3(k\rho + N)}{2\Gamma(\alpha)} \left(\frac{\beta}{|\beta\eta - 1|} + 1 \right). \quad (2.7)$$

Thus,

$$\|Tu(t)\| < \infty, \quad \forall t \in [0, 1]$$

Hence, the operator T maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$.

Step3 : We show that T is equicontinuous on $C([0, 1], \mathbb{R})$.

Let us take $u \in B_\rho$, $t_1, t_2 \in [0, 1]$, with $t_1 < t_2$. We have :

$$\begin{aligned}
 |Tu(t_1) - Tu(t_2)| &\leq \left| -\frac{\beta \eta^2 t_1}{2\Gamma(\alpha - 1) (\beta\eta - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \right. \\
 &\quad + \frac{\beta t_1}{(\alpha - 1) \Gamma(\alpha - 1) (\beta\eta - 1)} \int_0^\eta (\eta - s)^{\alpha-1} f(s, u(s)) ds \\
 &\quad + \frac{t_1^2}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s)) ds \\
 &\quad - \frac{\beta \eta^2 t_2}{2\Gamma(\alpha - 1) (\beta\eta - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\
 &\quad - \frac{\beta t_2}{(\alpha - 1) \Gamma(\alpha - 1) (\beta\eta - 1)} \int_0^\eta (\eta - s)^{\alpha-1} f(s, u(s)) ds \\
 &\quad - \frac{t_2^2}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\
 &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \right|.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & |Tu(t_1) - Tu(t_2)| \\
 & \leq \frac{\beta\eta^2|t_1 - t_2|}{2\Gamma(\alpha - 1)(|\beta\eta - 1|)} \int_0^1 (1-s)^{\alpha-2} |f(s, u(s))| ds \\
 & \quad + \frac{\beta|t_1 - t_2|}{(\alpha - 1)\Gamma(\alpha - 1)(|\beta\eta - 1|)} \int_0^\eta (\eta - s)^{\alpha-1} |f(s, u(s))| ds \\
 & \quad + \frac{|t_1 - t_2|(t_1 + t_2)}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, u(s))| ds \\
 & \quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} |f(s, u(s))| ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} |f(s, u(s))| ds.
 \end{aligned}$$

Thanks to (H_3) and (2.6) , we obtain

$$\begin{aligned}
 & |Tu(t_1) - Tu(t_2)| \\
 & \leq \frac{\beta\eta^2|t_1 - t_2|}{2\Gamma(\alpha - 1)(|\beta\eta - 1|)} \int_0^1 (1-s)^{\alpha-2} (k|u(s)| + N) ds \\
 & \quad + \frac{\beta|t_1 - t_2|}{(\alpha - 1)\Gamma(\alpha - 1)(|\beta\eta - 1|)} \int_0^\eta (\eta - s)^{\alpha-1} (k|u(s)| + N) ds \\
 & \quad + \frac{|t_1 - t_2|(t_1 + t_2)}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} (k|u(s)| + N) ds \\
 & \quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} (k|u(s)| + N) ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} (k|u(s)| + N) ds.
 \end{aligned}$$

After a simple calculation we find that

$$\begin{aligned}
 |Tu(t_1) - Tu(t_2)| & \leq \frac{\beta\eta^2|t_1 - t_2|(k\rho + N)}{2\Gamma(\alpha)(|\beta\eta - 1|)} \\
 & \quad + \frac{\beta|t_1 - t_2|(k\rho + N)\eta^\alpha}{\Gamma(\alpha + 1)(|\beta\eta - 1|)} \\
 & \quad + \frac{|t_1 - t_2|(t_1 + t_2)(k\rho + N)}{2\Gamma(\alpha)} \\
 & \quad + \frac{(k\rho + N)}{\Gamma(\alpha + 1)} (t_1^\alpha - t_2^\alpha)
 \end{aligned}$$

which implies that $\|Tu(t_1) - Tu(t_2)\| \rightarrow 0$ as $t_1 \rightarrow t_2$. By Arzela-Ascoli theorem, we deduce that T is a completely continuous operator

Step4 : Let $\Omega = \{u \in C([0, 1], \mathbb{R}) : u = \rho T(u), 0 < \rho < 1\}$.

We show that the set Ω is bounded. Let $u \in \Omega$. Then $u = \rho T(u)$ for some $0 < \rho < 1$ and, for each $t \in [0, 1]$, we have

$$\begin{aligned} |u(t)| &= \rho |Tu(t)| \\ &= \left| \frac{\rho \beta \eta^2 t}{2\Gamma(\alpha - 1)(\beta\eta - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \right. \\ &\quad + \frac{\rho \beta t}{(\alpha - 1) \Gamma(\alpha - 1) (\beta\eta - 1)} \int_0^\eta (\eta - s)^{\alpha-1} f(s, u(s)) ds \\ &\quad + \frac{\rho t^2}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\ &\quad \left. - \frac{\rho}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \right|. \end{aligned}$$

Since $0 < \rho < 1, 0 \leq t \leq 1$ and $\eta \in (0, 1)$ then

$$\begin{aligned} |u(t)| &\leq \frac{\beta}{2\Gamma(\alpha - 1)(|\beta\eta - 1|)} \int_0^1 (1-s)^{\alpha-2} |f(s, u(s))| ds \\ &\quad + \frac{\beta}{(\alpha - 1) \Gamma(\alpha - 1) (|\beta\eta - 1|)} \int_0^\eta (\eta - s)^{\alpha-1} |f(s, u(s))| ds \\ &\quad + \frac{1}{2\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, u(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, u(s))| ds. \end{aligned}$$

Thanks to (2.7) we obtain

$$|u(t)| \leq \frac{3(k+N)}{2\Gamma(\alpha)} \left(\frac{\beta}{|\beta\eta - 1|} + 1 \right).$$

Hence $\|u\| < \infty$. This shows that Ω is bounded.

As a consequence of Schaefer's fixed point theorem, problem (2.1) has at least one solution defined on $[0, 1]$. \square

2.4 Example

Consider the following boudary value problem :

$$\begin{cases} D_{0^+}^\alpha u(t) + \frac{u(t)}{e^{-\pi t} + \pi} + \frac{\cos u(t)}{(t+4)^2} + 1 = 0, & 2 < \alpha \leq 3, \quad 0 \leq t \leq 1 \\ u(0) = 0, \quad u'(0) = u'(1) = 2u(1/2). \end{cases} \quad (2.8)$$

We have

$$f(t, u(t)) = \frac{u(t)}{e^{-\pi t} + \pi} + \frac{\cos u(t)}{(t+4)^2} + 1$$

with

$$\begin{aligned}
 |f(t, u(t)) - f(t, v(t))| &= \left| \frac{u(t)}{e^{-\pi t} + \pi} + \frac{\cos u(t)}{(t+4)^2} - \frac{v(t)}{e^{-\pi t} + \pi} + \frac{\cos v(t)}{(t+4)^2} \right| \\
 &\leq \left| \frac{u(t)}{e^{-\pi t} + \pi} - \frac{v(t)}{e^{-\pi t} + \pi} \right| + \left| \frac{\cos u(t)}{(t+4)^2} - \frac{\cos v(t)}{(t+4)^2} \right| \\
 &\leq \frac{1}{e^{-\pi t} + \pi} |u(t) - v(t)| + \frac{1}{(t+4)^2} |\cos u(t) - \cos v(t)| \\
 &\leq \frac{1}{e^{-\pi t} + \pi} |u(t) - v(t)| \\
 &\quad + \frac{1}{(t+4)^2} \left| -2 \sin \frac{u(t) - v(t)}{2} \sin \frac{u(t) + v(t)}{2} \right|.
 \end{aligned}$$

Tacking into account that $|\sin x| \leq 1$ and $|\sin x| \leq |x|$, we deduce that

$$\begin{aligned}
 |f(t, u(t)) - f(t, v(t))| &\leq \frac{1}{e^{-\pi t} + \pi} |u(t) - v(t)| \\
 &\quad + \frac{2}{(t+4)^2} \left| \sin \frac{u(t) - v(t)}{2} \right| \\
 &\leq \frac{1}{e^{-\pi t} + \pi} |u(t) - v(t)| \\
 &\quad + \frac{2}{(t+4)^2} \left| \frac{u(t) - v(t)}{2} \right| \\
 &\leq \left(\frac{1}{e^{-\pi t} + \pi} + \frac{1}{(t+4)^2} \right) |u(t) - v(t)|.
 \end{aligned}$$

The fact that $0 \leq t \leq 1$ implies that

$$\frac{1}{(t+4)^2} \leq \frac{1}{16}$$

and

$$\frac{1}{e^{-\pi t} + \pi} \leq 0.3139$$

so that

$$\frac{1}{e^{-\pi t} + \pi} + \frac{1}{(t+4)^2} \leq 0.3764.$$

Finanlly, by Theorem 2.3 we can deduce that problem (2.8) has a unique solution defined on $[0, 1]$ \square

BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

3

THIS chapter essentially contains the paper [8] intitled : "Boundary value problems for nonlinear fractional differential equations" and published in *Facta Universitatis (NIS), Ser. Math. Inform.* Vol. 30, No 2 (2015), pp. 157-168.

Abstract. Sufficient conditions are given for the existence of solutions for the following boundary value problem for a nonlinear fractional differential equation

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t)), & 0 < t < 1, \ 2 < \alpha \leq 3 \\ D_{0+}^{\alpha-1} u(0) = 0, \ D_{0+}^{\alpha-2} u(1) = 0, \ u(1) = 0 \end{cases}$$

where f is a given function and D_{0+}^α is the Riemann-Liouville fractional derivative operator of order α . The results are proved using Banach contraction principle and Krasnoselskii's fixed point theorem on a cone.

3.1 Introduction

Usually, the fundamental tool used in the literature to prove the existence of positive solutions for boundary value problems for ordinary and fractional differential equations, difference equations, and dynamic equations on time scales, is the theory of fixed point (see, for example, [2, 4, 5, 16, 18, 20] and the works cited below).

In [3], Bai and Lü considered the nonlinear fractional boundary value problem :

$$\begin{cases} D_{0+}^\alpha u + f(t, u(t)) = 0, & 0 < t < 1, \ 1 < \alpha < 2, \\ u(0) = u(1) = 0, \end{cases}$$

where D_{0+}^α is the Riemann-Liouville fractional differential operator of order α and $f : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function. They obtained the existence of positive solutions by means of some fixed point theorems on cones.

In [6], Benchohra, Henderson, Ntouyas and Ouahab used the Banach fixed point Theorem and the nonlinear alternative of Leray-Schauder principle to investigate the existence of solutions for fractional order functional and neutral functional differential equations with infinite delay, of the form :

$$\begin{cases} D_0^\alpha u = f(t, u(t)), & 0 \leq t < 1, 0 < \alpha < 1 \\ u(t) = \phi(t), & t \leq 0, \end{cases}$$

where D_0^α is the Riemann-Liouville differential operator of fractional calculus of order α , $f : [0, 1] \times B \rightarrow \mathbb{R}$ is a given function, $\phi \in B$ with $\phi(0) = 0$, and B is the so-called phase space.

In [12], Delbosco and Rodino proved existence and uniqueness of solutions of some classes of nonlinear fractional differential equations of the form :

$$D^\alpha u = f(t, u), \quad 0 \leq t \leq a, \quad 0 < \alpha < 1,$$

where $f : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$, $0 < a \leq +\infty$, is some given continuous function. In this work, the authors used the Banach's contraction principle.

By using a Krasnoselskii's fixed point theorem in cones, El-Shahed in [13] proved the existence and non-existence of positive solutions for the following nonlinear fractional boundary value problem :

$$\begin{cases} D_{0+}^\alpha u + \lambda a(t)f(u(t)) = 0, & 0 < t < 1, 2 < \alpha \leq 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where D_{0+}^α is the Riemann-Liouville fractional differential operator of order α .

In [24], Saadi and Benbachir obtained sufficient conditions for the existence and non-existence of positive solutions for the following nonlinear fractional boundary value problem :

$$\begin{cases} D_{0+}^\alpha u + a(t)f(u(t)) = 0, & 0 < t < 1, 2 < \alpha \leq 3, \\ u(0) = u'(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda, \end{cases}$$

where D_{0+}^α is the Riemann-Liouville fractional differential operator of order α , $f : [0, \infty) \rightarrow [0, \infty)$ and $a : (0, 1) \rightarrow [0, \infty)$ are continuous functions, $\eta \in (0, 1)$, $\mu \in [0, 1/\eta^{\alpha-2})$ and $\lambda \in [0, \infty)$ are some fixed constants. By use of the Guo-Krasnoselskii's fixed point theorem and Schauder's fixed point theorem the existence of positive solutions of this problem is obtained in the case when either f is superlinear or sublinear.

Recently, in [17], Saadi, Benmezai and Benbachir gave some sufficient conditions for the existence of positive solutions of the nonlinear fractional semi-positone boundary value problem :

$$\begin{cases} D_{0+}^\alpha u + f(t, u(t)) = 0, & 0 < t < 1, 2 < \alpha \leq 3, \\ u(0) = u'(0) = 0, \quad u'(1) = \mu u'(\eta), \end{cases}$$

where D_{0+}^α is the Riemann-Liouville differential operator of order α and the non-linear term $f : [0, 1] \times [0, +\infty) \rightarrow \mathbb{R}$ satisfies a L^1 -Carathéodory condition. By use of a fixed point index theorem, the existence of at least two positive solutions is obtained.

Motivated by [25], this work deals with the existence of solutions for the following nonlinear fractional boundary value problem :

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t)), t \in J = [0, 1], 2 < \alpha \leq 3 \\ D_{0+}^{\alpha-1} u(0) = 0, D_{0+}^{\alpha-2} u(1) = 0, u(1) = 0, \end{cases} \quad (3.1)$$

where D_{0+}^α is the Riemann-Liouville fractional differential operator of order α and $f : J \times [0, \infty) \rightarrow \mathbb{R}$ is a given function.

3.2 Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this work.

Remark 3.1

- For $\alpha < 0$, we use the convention that $D_{0+}^\alpha u(t) = I_0^{-\alpha} u(t)$, for $t \geq 0$.
- For $\beta \in [0, \alpha)$, we have $D_{0+}^\beta I_{0+}^\alpha u = I_{0+}^{\alpha-\beta} u$, with $D_{0+}^\alpha I_{0+}^\alpha u = u$.
- For $\lambda > -1, \lambda \neq \alpha - 1, \alpha - 2, \dots, \alpha - n$, we have, for $t \geq 0$,

$$D_{0+}^\alpha t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda - \alpha} \quad \text{and} \quad D_{0+}^\alpha t^{\alpha-i} = 0, \quad \forall i = 1, 2, \dots, n. \quad (3.2)$$

Lemma 3.1 Let $\alpha > 0$. The general solution of the homogeneous equation

$$D_{0+}^\alpha u(t) = 0,$$

in $\mathcal{C}(J, \mathbb{R}) \cap \mathbf{L}^1(J, \mathbb{R})$ is

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad t \in J.$$

Lemma 3.2 Let $n - 1 < \alpha < n$ and $u \in \mathcal{C}(J, \mathbb{R})$. We have

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - \dots - c_n t^{\alpha-n}, \quad t \in J.$$

Now, we present the fundamental tools on which the proofs of our main results are based.

Theorem 3.1 (Banach's contraction principle, [10]) Every contraction mapping on a complete metric space has a unique fixed point.

Theorem 3.2 (Krasnoselskii's fixed point theorem, [26]) Let \mathcal{B} be a closed convex non-empty subset of a Banach space \mathbb{E} . Suppose that T_1, T_2 map \mathcal{B} into \mathbb{E} such that

1. $T_1 u + T_2 v \in \mathcal{B}, \forall u, v \in \mathcal{B}$,
2. T_1 is a contraction mapping,
3. T_2 is continuous and $T_2(\mathcal{B})$ is contained in a compact set.

Then there exists $u \in \mathcal{B}$ such that $T_1 u + T_2 u = u$.

3.3 Existence of solutions

Let us consider the following problem

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t)), t \in J = [0, 1], 2 < \alpha \leq 3, \\ D_{0+}^{\alpha-1} u(0) = 0, D_{0+}^{\alpha-2} u(1) = 0, u(1) = 0. \end{cases} \quad (3.3)$$

A function $u \in \mathcal{C}(J, \mathbb{R})$ is said to be a solution of problem (3.3) if u satisfies the fractional differential equation

$$D_{0+}^\alpha u(t) = f(t, u(t)), t \in J,$$

and the conditions

$$D_{0+}^{\alpha-1} u(0) = 0, D_{0+}^{\alpha-2} u(1) = 0, u(1) = 0.$$

Lemma 3.3 *Let $2 < \alpha < 3$ and let $h : J \rightarrow \mathbb{R}$ be a continuous function. A function u is a solution of the fractional integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

if and only if it is a solution of the initial value problem

$$\begin{cases} D_{0+}^\alpha u(t) = h(t), t \in J \\ u(0) = u_0. \end{cases}$$

Lemma 3.4 *Let $2 < \alpha < 3$ and let $h : J \rightarrow \mathbb{R}$ be a continuous function. A function u is a solution of the initial value problem*

$$\begin{cases} D_{0+}^\alpha u(t) = h(t), t \in J, \\ D_{0+}^{\alpha-1} u(0) = 0, D_{0+}^{\alpha-2} u(1) = 0, u(1) = 0, \end{cases} \quad (3.4)$$

if and only if it is a solution of the fractional integral equation

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{t^{\alpha-3} (1-t)}{\Gamma(\alpha-1)} \int_0^1 (1-s) h(s) ds \\ & - \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds. \end{aligned} \quad (3.5)$$

Proof. Let u be a solution of problem (3.4). By Lemma 3.2 we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \quad (3.6)$$

If we consider the boundary conditions, we have

$$D_{0+}^{\alpha-1} u(0) = 0 \Rightarrow c_1 = 0,$$

$$D_{0+}^{\alpha-2}u(1) = 0 \Rightarrow c_2 = -\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s) h(s) ds$$

and

$$u(1) = 0 \Rightarrow c_3 = \frac{1}{\Gamma(\alpha)} \int_0^1 \left[(1-\alpha)(1-s) - (1-s)^{\alpha-1} \right] ds.$$

Now, if we replace the values of c_2 and c_3 in (3.6) we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)} \int_0^1 (1-s) h(s) ds \\ &\quad - \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds, \end{aligned}$$

that is, u is a solution of problem (3.5). Conversely, if u is a solution of problem (3.5), it can be written as follows

$$u(t) = I_0^\alpha h(t) + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)} I_0^2 h(1) - t^{\alpha-3} I_0^\alpha h(1),$$

so that, by (3.2) we deduce that : $D_{0+}^\alpha u(t) = h(t)$, $D_{0+}^{\alpha-1} u(0) = I_0^1 h(0) = 0$, $D_{0+}^{\alpha-2} u(1) = I_0^2 h(1) - I_0^2 h(1) = 0$ and $u(1) = I_0^\alpha h(1) - I_0^\alpha h(1) = 0$, which finishes to prove that u is a solution of problem (3.4). \square

The first main result of this work states that problem (3.3) admits a unique solution. It reads as follows.

Theorem 3.3 *Let $f : J \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function such that :*

(H₁) There exists a constant $k > 0$ such that

$$|f(t, u) - f(t, v)| < k |u - v|, \quad \forall u, v \in [0, \infty), \quad \forall t \in J.$$

If the following condition is satisfied

$$k \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} \right) < 1 \tag{3.7}$$

then problem (3.3) has a unique solution on J .

Proof. We transform problem (3.3) into a fixed point problem. For this, consider the operator $T : \mathcal{C}(J, \mathbb{R}) \rightarrow \mathcal{C}(J, \mathbb{R})$ defined by

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \\ &\quad + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)} \int_0^1 (1-s) f(s, u(s)) ds \\ &\quad - \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds. \end{aligned} \tag{3.8}$$

Fixed points of T are solutions of problem (3.3).

Now, according to Theorem 3.1 it is enough to prove that T is a contraction. Let $u \in \mathcal{C}(J, \mathbb{R})$ and $t \in J$. We have

$$\begin{aligned}
 & |Tu_1(t) - Tu_2(t)| \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u_1(s)) - f(s, u_2(s))| ds \\
 & \quad + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)} \int_0^1 (1-s) |f(s, u_1(s)) - f(s, u_2(s))| ds \\
 & \quad + \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, u_1(s)) - f(s, u_2(s))| ds \\
 & \leq \frac{k \|u_1 - u_2\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + k \|u_1 - u_2\|_\infty \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)} \int_0^1 (1-s) ds \\
 & \quad + k \|u_1 - u_2\|_\infty \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
 & \leq \frac{k \|u_1 - u_2\|_\infty t^\alpha}{\alpha \Gamma(\alpha)} + \frac{k \|u_1 - u_2\|_\infty t^{\alpha-3}(1-t)}{2 \Gamma(\alpha-1)} + \frac{k \|u_1 - u_2\|_\infty t^{\alpha-3}}{\alpha \Gamma(\alpha)} \\
 & \leq \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{2 \Gamma(\alpha-1)} \right) k \|u_1 - u_2\|_\infty
 \end{aligned}$$

from which we deduce that

$$\|Tu_1 - Tu_2\|_\infty \leq \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{2 \Gamma(\alpha-1)} \right) k \|u_1 - u_2\|_\infty.$$

By (3.7), this proves that T is a contraction. As a consequence of Theorem 3.1, T has a unique fixed point which is the unique solution of problem (3.3). \square

Now, we will consider the following condition that we use in the second main result of this work, that is Theorem 3.4 below.

(H₂) There exists a constant $\beta > 0$ such that

$$|f(t, u)| \leq \beta, \quad \forall t \in J, \forall u \geq 0.$$

Theorem 3.4 *Let $f : J \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. Suppose that (H₁) and (H₂) are satisfied. If there exists $\delta > 0$ such that*

$$\beta \frac{\alpha^2 - \alpha + 2}{\Gamma(\alpha+1)} \leq \delta, \tag{3.9}$$

then problem (3.3) has at least one solution on J .

Proof. We will use Theorem 3.2 to prove the conclusion of Theorem 3.4. For this, consider again the operator T defined in (3.8) and define operators T_1 and T_2 as follows : For $u \in \mathcal{C}(J, \mathbb{R})$ and $t \in J$,

$$T_1 u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds + \frac{t^{\alpha-3}}{\Gamma(\alpha-1)} \int_0^1 (1-s) f(s, u(s)) ds$$

and

$$T_2 u(t) = -\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) f(s, u(s)) ds - \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds.$$

- Let $\delta > 0$ and $\mathcal{B} = \{u \in \mathcal{C}(J, \mathbb{R}) : \|u\|_\infty \leq \delta\}$. We will prove that $T_1 u_1 + T_2 u_2 \in \mathcal{B}$, for any $u_1, u_2 \in \mathcal{B}$. Let $u_1, u_2 \in \mathcal{B}$ and $t \in J$. We have

$$\begin{aligned} & |T_1 u_1(t) + T_2 u_2(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u_1(s))| ds + \frac{t^{\alpha-3}}{\Gamma(\alpha-1)} \int_0^1 (1-s) |f(s, u_1(s))| ds \\ & \quad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) |f(s, u_2(s))| ds + \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, u_2(s))| ds. \\ & \leq \frac{\beta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{\beta}{\Gamma(\alpha-1)} \int_0^1 (1-s) ds \\ & \quad + \frac{\beta}{\Gamma(\alpha-1)} \int_0^1 (1-s) ds + \frac{\beta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds, \\ & \leq \frac{\beta}{\Gamma(\alpha)} \left(\frac{t^\alpha}{\alpha} + \alpha - 1 + \frac{1}{\alpha} \right) \leq \frac{\beta}{\Gamma(\alpha)} \left(\frac{\alpha^2 - \alpha + 2}{\alpha} \right) \end{aligned}$$

from which and by condition (3.9) we deduce that

$$\|T_1 u_1 - T_2 u_2\|_\infty \leq \frac{\beta}{\Gamma(\alpha)} \left(\frac{\alpha^2 - \alpha + 2}{\alpha} \right) \leq \delta$$

which finish to prove that $Tu_1 + Tu_2 \in \mathcal{B}$.

- Now, we will prove that T_1 is a contraction on $\mathcal{C}(J, \mathbb{R})$. Let $u_1, u_2 \in \mathcal{C}(J, \mathbb{R})$ and $t \in J$. We have

$$\begin{aligned} & |T_1 u_1(t) - T_1 u_2(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u_1(s)) - f(s, u_2(s))| ds \\ & \quad + \frac{t^{\alpha-3}}{\Gamma(\alpha-1)} \int_0^1 (1-s) |f(s, u_1(s)) - f(s, u_2(s))| ds. \\ & \leq \frac{k \|u_1 - u_2\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + t^{\alpha-3} \frac{k \|u_1 - u_2\|_\infty}{\Gamma(\alpha-1)} \int_0^1 (1-s) ds \\ & \leq \frac{k \|u_1 - u_2\|_\infty}{\Gamma(\alpha+1)} t^\alpha + t^{\alpha-3} \frac{k \|u_1 - u_2\|_\infty}{2\Gamma(\alpha-1)} \\ & \leq k \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} \right) \|u_1 - u_2\|_\infty \end{aligned}$$

from which we deduce that

$$\|T_1 u_1 - T_1 u_2\|_\infty \leq k \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} \right) \|u_1 - u_2\|_\infty.$$

Using the condition (3.7), we conclude that T_1 is a contraction.

- We will prove that T_2 is continuous. Let $(u_n)_n$ be a sequence such that $u_n \rightarrow u$ in $\mathcal{C}(J, \mathbb{R})$. For $n \in \mathbb{N}$ and $t \in J$, we have

$$\begin{aligned} & |T_2 u_n(t) - T_2 u(t)| \\ & \leq \frac{1}{\Gamma(\alpha-1)} \left[\int_0^1 ((1-s) - (1-s)^{\alpha-1}) |f(s, u_n(s)) - f(s, u(s))| ds \right] \\ & \leq \frac{1}{\Gamma(\alpha-1)} k \|u_n - u\| \int_0^1 ((1-s) - (1-s)^{\alpha-1}) ds \\ & \leq \frac{\alpha-2}{2\alpha\Gamma(\alpha-1)} k \|u_n - u\|. \end{aligned}$$

Therefore

$$\|T_2 u_n - T_2 u\| \rightarrow 0 \text{ as } \|u_n - u\| \rightarrow 0.$$

- Finally, we will prove the compactness of T_2 . From the Ascoli-Arzela Theorem, it is sufficient to prove that for each bounded subset \mathcal{B} of $\mathcal{C}(J, \mathbb{R})$, the set $T\mathcal{B}$ is bounded and is equicontinuous. Let \mathcal{B} be a bounded subset of $\mathcal{C}(J, \mathbb{R})$.

◊ We prove that $T_2\mathcal{B}$ is a bounded subset of $\mathcal{C}(J, \mathbb{R})$. Let $u \in \mathcal{B}$ and $t \in J$. We have

$$\begin{aligned} |T_2 u(t)| & \leq \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s) |f(s, u(s))| ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, u(s))| ds \\ & \leq \frac{\beta}{\Gamma(\alpha-1)} \int_0^1 (1-s) ds + \frac{\beta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\ & \leq \beta \left(\frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)} \right), \end{aligned}$$

from which we deduce that

$$\|T_2 u\|_\infty \leq \beta \left(\frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)} \right) = cte.$$

◊ Now, we prove that $T_2\mathcal{B}$ is equicontinuous. Let $u \in \mathcal{B}$ and $t_1, t_2 \in J$ with $t_1 < t_2$. We have

$$\begin{aligned} & |T_2 u(t_1) - T_2 u(t_2)| \\ & \leq \frac{t_1^{\alpha-2} - t_2^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) |f(s, u(s))| ds \\ & \quad + \frac{t_2^{\alpha-3} - t_1^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, u(s))| ds \\ & \leq \beta \frac{t_1^{\alpha-2} - t_2^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) ds + \beta \frac{t_2^{\alpha-3} - t_1^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\ & \leq \beta \frac{t_1^{\alpha-2} - t_2^{\alpha-2}}{2\Gamma(\alpha-1)} + \beta \frac{t_2^{\alpha-3} - t_1^{\alpha-3}}{\Gamma(\alpha+1)}, \end{aligned}$$

from which we deduce the desired property. Finally as a consequence of Theorem 3.2 we deduce that P has at least one fixed point, which is a solution of problem (3.3). □

3.4 Example

Let $f : [0, 1] \times [0, +\infty) \rightarrow \mathbb{R}$ be defined by

$$f(t, x) = \frac{x}{3e^t + 2}, \quad (t, x) \in [0, 1] \times [0, +\infty). \quad (3.10)$$

For $t \in [0, 1]$ and $x \geq 0$ we have

$$|f(t, x) - f(t, y)| = \frac{1}{3e^t + 2} |x - y| \leq \frac{1}{5} |x - y|$$

which proves that f satisfies condition (H1) in Theorem 3.3.

Now, to apply Theorem 3.3 it reminds to verify that

$$\frac{1}{5} \left(\frac{2}{\Gamma(\alpha + 1)} + \frac{1}{2\Gamma(\alpha - 1)} \right) < 1,$$

which is the case since we have :

For $\alpha = \frac{14}{5}$ we get

$$\frac{1}{5} \left(\frac{2}{\Gamma\left(\frac{19}{5}\right)} + \frac{1}{2\Gamma\left(\frac{9}{5}\right)} \right) = 0.19258 < 1,$$

and, for $\alpha = \frac{7}{3}$ we get

$$\frac{1}{5} \left(\frac{2}{\Gamma\left(\frac{10}{5}\right)} + \frac{1}{2\Gamma\left(\frac{4}{3}\right)} \right) = 0.51198 < 1.$$

In conclusion, by Theorem 3.3 we conclude that for f given by (3.10) problem (3.3) admits at least one solution. \square

AN EXISTENCE RESULT FOR n^{TH} -ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

4

THIS chapter essentially contains the paper [9] intitled : "*An existence result for n^{th} -order nonlinear fractional differential equations*", submitted for publication in 2015.

Abstract. By use of a fixed point Theorem on cones as a fundamental tool, the aim of this work is to investigate the existence of solutions for the following three-point boundary value problem for n^{th} -order nonlinear fractional differential equations with higher boundary values

$$\begin{cases} {}^cD_0^\alpha u(t) = f(t, u(t)), & n - 1 < \alpha \leq n, \quad 0 \leq t \leq 1 \\ u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}(0) = 0, & 1 \leq i \leq n - 2, \end{cases}$$

where $n > 4$ is an integer, $p \in (\frac{1}{2}, 1)$ is a constant, ${}^cD_0^\alpha u$ is the Caputo's fractional derivative of order α of u and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function.

4.1 Introduction

Different kind of fixed point theorems are widely used as fundamental tools in order to prove the existence of positive solutions for boundary value problems associated to some differential equations, difference equations, and dynamic equations on scales (see, for example, [2, 3, 6, 18, 19, 24, 25, ?, 31] and the references cited therein).

Motivated by the problem treated in [29], this work deals with the existence of solutions for the following three-point boundary value problem for n^{th} -order nonlinear fractional differential equations with higher boundary values

$$\begin{cases} {}^cD_0^\alpha u(t) = f(t, u(t)), & t \in J = [0, 1], \quad n - 1 < \alpha \leq n \\ u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}(0) = 0, & 1 \leq i \leq n - 2, \end{cases} \quad (4.1)$$

where $n > 4$ is an integer, $p \in \left(\frac{1}{2}, 1\right)$ is a constant and ${}^cD_0^\alpha$ is the Caputo differential operator of fractional calculus of order α and $f : J \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function.

4.2 Preliminaries

The hereafter fixed point theorem on cones is the fundamental tool on which the proof of our main theorem is based.

Theorem 4.1 ([16]) *Let \mathbb{E} be a Banach space and let $\mathcal{K} \subset \mathbb{E}$ be a cone. Assume that Ω_1 and Ω_2 are open subsets of \mathbb{E} with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{K}$ be a completely continuous operator. In addition, suppose either*

- (B1) $\|Tu\| \leq \|u\|, \forall u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in \mathcal{K} \cap \partial\Omega_2$, or
- (B2) $\|Tu\| \leq \|u\|, \forall u \in \mathcal{K} \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|, \forall u \in \mathcal{K} \cap \partial\Omega_1$, holds.

Then T has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Before to prove our existence result for the three-point boundary value problem (4.1), we first will determine the Green's function associated to this problem.

Lemma 4.1 ([1]) *Let $n - 1 < \alpha \leq n$ and $h \in \mathcal{C}(J, \mathbb{R})$. A function u is a solution of the initial value problem*

$$\begin{cases} {}^cD_0^\alpha u(t) = h(t), & t \in J \\ u(0) = u_0 \end{cases}$$

if and only if it is a solution of the fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \quad t \in J.$$

Lemma 4.2 *Let $n - 1 < \alpha \leq n$ and $h \in \mathcal{C}(J, \mathbb{R})$ with $h(t) \geq 0$, for all $t \in J$. The following three-point boundary value problem*

$$\begin{cases} {}^cD_0^\alpha u(t) = h(t), & t \in J \\ u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}(0) = 0, & 1 \leq i \leq n-2 \end{cases} \quad (4.2)$$

has the unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds + \int_0^1 \bar{G}_p(t,s)h(s)ds, \quad t \in J$$

where

$$-G(t,s) = \begin{cases} \frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)} (1-s)^{\alpha-n}, & t \leq s \\ \frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1}, & t \geq s \end{cases}$$

and

$$\overline{G}_p(t, s) = \begin{cases} \frac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} \\ \quad -\frac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+2)}(p-s)^{\alpha-n+1}, & p \geq s \\ \frac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}, & s \geq p. \end{cases}$$

Proof. By applying lemma 4.1, the equation ${}^cD_0^\alpha u(t) = h(t)$ is equivalent to the following integral equation

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} + I_{0^+}^\alpha h(t)$$

for some arbitrary constants $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n-1$. The three-point boundary conditions permit us to deduce the exact values $c_0 = c_1 = \cdots = c_{n-3} = 0$ so that

$$u(t) = c_{n-2} t^{n-2} + c_{n-1} t^{n-1} + I_{0^+}^\alpha h(t), \quad t \in J \quad (4.3)$$

with

$$c_{n-1} = -\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s) ds.$$

Indeed, for $t \in J$

$$u^{(n-1)}(t) = (n-1)! c_{n-1} + I_{0^+}^{\alpha-n+1} h(t)$$

implies that

$$u^{(n-1)}(1) = (n-1)! c_{n-1} + I_{0^+}^{\alpha-n+1} h(1) = 0,$$

so that

$$c_{n-1} = -\frac{1}{\Gamma(n)} I_{0^+}^{\alpha-n+1} h(1), \quad \text{with } \Gamma(n) = (n-1)!.$$

Also we have

$$u^{(n-2)}(p) = 0 \Leftrightarrow (n-2)! c_{n-2} + (n-1)! c_{n-1} p + I_{0^+}^{\alpha-n+2} h(p) = 0$$

from which we deduce that

$$\begin{aligned} c_{n-2} &= -\frac{(n-1)}{(n-2)} p c_{n-1} - \frac{I_{0^+}^{\alpha-n+2} h(p)}{(n-2)!} \\ &= \frac{(n-1)p}{\Gamma(n)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s) ds \\ &\quad - \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s) ds \\ &= \frac{p}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s) ds \\ &\quad - \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s) ds. \end{aligned}$$

After replacing c_{n-2} and c_{n-1} in (4.3), we obtain

$$\begin{aligned}
 u(t) &= \frac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s) ds \\
 &\quad - \frac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s) ds \\
 &\quad - \frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\
 &= \frac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^p (1-s)^{\alpha-n} h(s) ds \\
 &\quad - \frac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s) ds \\
 &\quad + \frac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_p^1 (1-s)^{\alpha-n} h(s) ds \\
 &\quad - \frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)} \int_0^t (1-s)^{\alpha-n} h(s) ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - \frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)} \int_t^1 (1-s)^{\alpha-n} h(s) ds.
 \end{aligned}$$

This completes the proof. \square

Hereafter, we give some properties of the functions $G(t,s)$ and $\bar{G}_p(t,s)$ in Lemma 4.2, and of the (unique) solution of the three-point boundary value problem (4.2), which we use later.

Lemma 4.3 *For all $(t,s) \in [\tau, 1] \times J$ with $\tau \geq 0$ and $p \in (\frac{1}{2}, 1)$, we have :*

$$(E_1) \quad \bar{G}_{\frac{1}{2}}(t,s) \leq \bar{G}_p(t,s) \leq \bar{G}_1(t,s).$$

$$(E_2) \quad 0 \leq \bar{G}_p(t,s) \leq \frac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)} (1-s)^{\alpha-n}.$$

$$(E_3) \quad \tau^{\alpha-1} \bar{G}_p(1,s) \leq \min_{\tau \leq t \leq 1} \bar{G}_p(t,s) = \tau^{n-2} \bar{G}_p(1,s).$$

$$(E_4) \quad \gamma G(1,s) \leq G(t,s) \leq G(1,s), \text{ where}$$

$$\gamma = \left(1 - \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha)} \right) \tau^{\alpha-1}.$$

Proof.

• *Property (E₁)*. We have

$$\frac{\partial \bar{G}_p(t,s)}{\partial p} = \begin{cases} \frac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)} (1-s)^{\alpha-n} \\ \quad - \frac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+2)} (p-s)^{\alpha-n}, \quad p \geq s, \\ \frac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)} (1-s)^{\alpha-n}, \quad p \leq s. \end{cases}$$

Taking into account that $p \in \left(\frac{1}{2}, 1\right)$, it is easy to see that $\frac{\partial \bar{G}_p(t,s)}{\partial p} \geq 0$. Hence $\bar{G}_p(t,s)$ is nondecreasing with respect to p and then

$$\bar{G}_{\frac{1}{2}}(t,s) \leq \bar{G}_p(t,s) \leq \bar{G}_1(t,s).$$

- *Properties (E₂) and (E₃)*. The proofs are obvious and then omitted.
- *Property (E₄)*. For $s \geq t$, taking into account that $t \in [\tau, 1]$ with $\tau \geq 0$ and $\alpha \leq n$, we have

$$\gamma \leq \tau^{n-1} \leq \frac{G(t,s)}{G(1,s)} = t^{n-1} \leq 1.$$

For $s \leq t$, we have

$$\begin{aligned} \frac{G(t,s)}{G(1,s)} &= \frac{\frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{G(1,s)} \\ &\geq \min_{\tau \leq t \leq 1} \frac{\frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}}{G(1,s)} - \max_{\tau \leq t \leq 1} \frac{\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{G(1,s)} \\ &\geq \frac{\frac{\tau^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{G(1,s)} \\ &= \frac{\frac{\tau^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}} \\ &\geq \frac{\frac{\tau^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}} \\ &\geq \tau^{\alpha-1} - \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha)}(1-s)^{\alpha-1} \end{aligned}$$

so that when $1-s \leq \tau$, we get

$$\frac{G(t,s)}{G(1,s)} \geq \left(1 - \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha)}\right) \tau^{\alpha-1} =: \gamma.$$

Now, taking into account that

$$0 \leq 1 - \frac{s}{t} \leq 1 - s \Rightarrow -(1 - \frac{s}{t})^{\alpha-1} \geq -(1 - s)^{\alpha-1},$$

we have

$$\begin{aligned} \frac{G(t,s)}{G(1,s)} &= t^{\alpha-1} \left[\frac{\frac{t^{n-\alpha}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(1 - \frac{s}{t}\right)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1}} \right] \\ &\leq t^{\alpha-1} \left[\frac{\frac{t^{n-\alpha}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1}} \right] \\ &\leq t^{\alpha-1} \leq 1. \end{aligned}$$

This completes the proof. \square

Lemma 4.4 Let $\tau \leq 1$, $p \in \left(\frac{1}{2}, 1\right)$ and $h \in \mathcal{C}(J, R)$ with $h(t) \geq 0$. The unique solution of the three-point boundary value problem (1.3) is such that

$$\min_{t \in [0,\tau]} u(t) \geq \gamma \|u\|, \quad (4.4)$$

where $\gamma = \tau^{\alpha-1}$.

Proof. Let $t \in [\tau, 1]$. Taking into account that $\gamma \leq 1$, by Lemma 3.3 we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)h(s)ds + \int_0^1 \bar{G}_p(t,s)h(s)ds \\ &\geq \gamma \int_0^1 G(1,s)h(s)ds + \int_0^1 \bar{G}_p(t,s)h(s)ds \\ &\geq \gamma \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \bar{G}_p(1,s)h(s)ds \right] \end{aligned}$$

so that

$$\min_{\tau \leq t \leq 1} u(t) \geq \gamma \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \bar{G}_p(1,s)h(s)ds \right]. \quad (4.5)$$

On the other hand, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)h(s)ds + \int_0^1 \bar{G}_p(t,s)h(s)ds \\ &\leq \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \bar{G}_p(1,s)h(s)ds \right] \end{aligned}$$

from which we deduce that

$$\|u\| \leq \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \bar{G}_p(1,s)h(s)ds \right]. \quad (4.6)$$

Finally, by (4.5) and (4.6) we get the desired inequality (4.4). \square

4.3 Existence of solutions

Now, we will apply Theorem 4.1 to prove the existence of solutions for the three-point boundary value problem (4.1).

First, we recall that a function $u \in \mathcal{C}(J, \mathbb{R})$ is a solution of (4.1) if u satisfies the equation

$${}^cD_0^\alpha u(t) = f(t, u(t)), \quad t \in J$$

and the conditions

$$u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}(0) = 0, \quad 1 \leq i \leq n-2.$$

By Lemma 4.2, we see that u is a solution of (4.1) if and only if

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds, \quad t \in J.$$

Let us consider the Banach space $\mathbb{E} = \{u \in \mathcal{C}(J, \mathbb{R}) : u(t) \geq 0, t \in J\}$ equipped with the norm $\|u\| = \max\{u(t) : t \in J\}$. We define on \mathbb{E} the integral operator $T : \mathbb{E} \rightarrow \mathbb{E}$ given by

$$Tu(t) = \int_0^1 G(t, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds, \quad u \in \mathbb{E}, t \in J. \quad (4.7)$$

T is well defined and fixed points of T are solutions of problem (4.1).

Let \mathcal{K} be the cone in \mathcal{E} given by $\mathcal{K} = \{u \in \mathbb{E} : \min\{u(t) : t \in [0, \tau]\} \geq \gamma \|u\|\}$. The following result gives some properties of the restriction of T to \mathcal{K} .

Lemma 4.5 *Let $f : J \times (0, +\infty) \rightarrow (0, +\infty)$ be continuous. The operator $T : \mathbb{E} \rightarrow \mathbb{E}$ defined by the relation (4.7) is such that :*

- (1) $T(\mathcal{K}) \subset \mathcal{K}$
- (2) $T_K := T|_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{K}$ is completely continuous.

Proof.

- *Property (1).* Let $u \in \mathcal{K}$ and $t \in J$. We have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds \\ &\leq \gamma \int_0^1 G(1, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds \\ &\leq \int_0^1 G(1, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds \end{aligned}$$

which implies that

$$\|Tu\| \leq \int_0^1 G(1, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds. \quad (4.8)$$

On the other hand, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \bar{G}_p(1,s)f(s,u(s))ds \\ &\geq \int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \bar{G}_p(1,s)f(s,u(s))ds \\ &\geq \gamma \left[\int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \bar{G}_p(1,s)f(s,u(s))ds \right]. \end{aligned} \quad (4.9)$$

From inequalities (4.8) and (4.9) we deduce that $\min_{t \in [0,\tau]} Tu(t) \geq \gamma \|Tu\|$ and consequently we have $T(\mathcal{K}) \subset \mathcal{K}$.

• *Property (2).* We divide the proof into several steps.

Step 1. T_K is continuous. Let $(u_n)_n$ be a sequence such that $u_n \rightarrow u_0$ in \mathcal{K} , and let

$$M = 2 \left(\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+2)} - \frac{1}{\Gamma(\alpha+1)} \right).$$

Fix $\varepsilon > 0$. For n sufficiently large, we have

$$\begin{aligned} |Tu_n(t) - Tu_0(t)| &= \left| \int_0^1 G(t,s)f(s,u_n(s))ds + \int_0^1 \bar{G}_p(1,s)f(s,u_n(s))ds \right. \\ &\quad \left. - \int_0^1 G(t,s)f(s,u_0(s))ds - \int_0^1 \bar{G}_p(1,s)f(s,u_0(s))ds \right| \\ &= \left| \int_0^1 G(t,s)(f(s,u_n(s)) - f(s,u_0(s)))ds \right. \\ &\quad \left. + \int_0^1 \bar{G}_p(1,s)(f(s,u_n(s)) - f(s,u_0(s)))ds \right| \\ &\leq \left| \gamma \int_0^1 G(1,s)(f(s,u_n(s)) - f(s,u_0(s)))ds \right. \\ &\quad \left. + \int_0^1 \bar{G}_1(1,s)(f(s,u_n(s)) - f(s,u_0(s)))ds \right| \\ &\leq \left| \int_0^1 G(1,s)(f(s,u_n(s)) - f(s,u_0(s)))ds \right. \\ &\quad \left. + \int_0^1 \bar{G}_1(1,s)(f(s,u_n(s)) - f(s,u_0(s)))ds \right| \\ &\leq \frac{\varepsilon}{M} \left| \int_0^1 [G(1,s) + \bar{G}_1(1,s)] ds \right|. \end{aligned}$$

If we replace $G(1,s) + \bar{G}_1(1,s)$ by its value, we obtain

$$\begin{aligned}
 & |Tu_n(t) - Tu_0(t)| \\
 & \leq \frac{\varepsilon}{M} \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
 & \quad + t^{n-2} \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 \left[(1-s)^{\alpha-n} - (1-s)^{\alpha-n+1} \right] ds \\
 & \leq \frac{\varepsilon}{M} \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
 & \quad + \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 \left[(1-s)^{\alpha-n} - (1-s)^{\alpha-n+1} \right] ds \\
 & \leq \frac{\varepsilon}{M} \frac{2}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
 & \quad - \frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-1} ds.
 \end{aligned}$$

After some calculations we get

$$\|Tu_n - Tu_0\| \leq \frac{\varepsilon}{M} \cdot M = \varepsilon$$

which finish to prove that T is continuous.

Now, we will prove that T_K takes bounded sets into relatively compact sets in \mathcal{K} . From the Ascoli-Arzela Theorem, it is sufficient to prove that for each bounded subset \mathcal{B} of \mathcal{K} , the set $T_K(\mathcal{B})$ is bounded and is equicontinuous. Let \mathcal{B} be a bounded set in \mathcal{K} . Then there exists a real number $m > 0$ such that $\|u\| \leq m$, for all $u \in \mathcal{B}$.

Step 2. The set $T_K(\mathcal{B})$ is bounded. For $u \in \mathcal{B}$ and $t \in J$, we have

$$\begin{aligned}
 |Tu(t)| &= \left| \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \bar{G}_p(p,s)f(s,u(s))ds \right| \\
 &\leq \max_{t \in J, 0 \leq u(t) \leq m} |f(t,u(t))| \left| \int_0^1 G(t,s)ds + \int_0^1 \bar{G}_p(p,s)ds \right| \\
 &\leq c_m M
 \end{aligned}$$

where $c_m = \max \{|f(t,u)| : t \in J, \|u\| \leq m\}$. This implies that $\|Tu\| \leq c_m M$, for all $u \in \mathcal{B}$, and finish to prove that $T(\mathcal{B})$ is bounded.

Step 3. The set $T_K(\mathcal{B})$ is equicontinuous on J . Let $u \in \mathcal{B}$ and $t_1, t_2 \in J$, $t_1 < t_2$. We have

$$\begin{aligned}
 & |Tu(t_1) - Tu(t_2)| \\
 &= \left| \int_0^1 G(t_1, s) f(s, u(s)) ds - \int_0^1 G(t_2, s) f(s, u(s)) ds \right| \\
 &\leq c_m \left| \int_0^1 (G(t_1, s) - G(t_2, s)) ds \right| \\
 &\leq c_m \left| \int_0^{t_1} (G(t_1, s) - G(t_2, s)) ds + \int_{t_1}^{t_2} (G(t_1, s) - G(t_2, s)) ds \right. \\
 &\quad \left. + \int_{t_2}^1 (G(t_1, s) - G(t_2, s)) ds \right| \\
 &\leq c_m \left| \int_0^{t_1} \left[\left(\frac{t_1^{n-1}}{\Gamma(n) \Gamma(\alpha - n + 1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t_1 - s)^{\alpha-1} \right) \right. \right. \\
 &\quad \left. \left. - \left(\frac{t_2^{n-1}}{\Gamma(n) \Gamma(\alpha - n + 1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t_2 - s)^{\alpha-1} \right) \right] ds \right. \\
 &\quad \left. + \int_{t_1}^{t_2} \left[\left(\frac{t_1^{n-1}}{\Gamma(n) \Gamma(\alpha - n + 1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t_1 - s)^{\alpha-1} \right) \right. \right. \\
 &\quad \left. \left. - \left(\frac{t_2^{n-1}}{\Gamma(n) \Gamma(\alpha - n + 1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t_2 - s)^{\alpha-1} \right) \right] ds \right. \\
 &\quad \left. + \int_{t_2}^1 \left[\left(\frac{t_1^{n-1}}{\Gamma(n) \Gamma(\alpha - n + 1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t_1 - s)^{\alpha-1} \right) \right. \right. \\
 &\quad \left. \left. - \left(\frac{t_2^{n-1}}{\Gamma(n) \Gamma(\alpha - n + 1)} (1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)} (t_2 - s)^{\alpha-1} \right) \right] ds \right|.
 \end{aligned}$$

After some standard calculations we get

$$\begin{aligned}
 |Tu(t_1) - Tu(t_2)| &\leq c_m \left| \frac{1}{\Gamma(n) \Gamma(\alpha - n + 1)} (t_1^{n-1} - t_2^{n-1}) \right. \\
 &\quad \left. + \frac{1}{\Gamma(\alpha + 1)} ((t_1 - 1)^\alpha - (t_2 - 1)^\alpha) - \frac{1}{\Gamma(\alpha + 1)} (t_1^\alpha - t_2^\alpha) \right|.
 \end{aligned}$$

When $t_1 \rightarrow t_2$ the right-hand side of the above inequality tends to zero. This property holds also for $t_2 < t_1$. This finish to prove the equicontinuity of T . \square

In what follows, we will use the following notations :

$$f_0 = \liminf_{\|u\| \rightarrow 0} \min_{t \in J} \frac{f(t, u(t))}{u(t)}, \quad f_\infty = \limsup_{\|u\| \rightarrow +\infty} \max_{t \in J} \frac{f(t, u(t))}{u(t)}.$$

f is called superlinear if $f_0 = 0$ and $f_\infty = \infty$.

Finally, our main result reads as follows.

Theorem 4.2 Suppose that f is continuous and superlinear. Then the three-point boundary value problem (4.1) has at least one solution.

Proof. Since $f_0 = 0$, for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that, for $u \in \mathcal{C}(J)$ and $t \in J$,

$$\|u\| \leq \eta(\varepsilon) \Rightarrow \frac{f(t, u(t))}{u(t)} \leq \varepsilon \Rightarrow f(t, u(t)) \leq \varepsilon u(t). \quad (4.10)$$

Let $\Omega_1 = \{u \in \mathcal{C}(J) : \|u\| \leq \eta(\varepsilon)\}$. For any $u \in \mathcal{K} \cap \partial\Omega_1$ and $t \in J$, it follows from Lemma 4.3 and relation (4.7) that

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(p, s)f(s, u(s))ds \\ &\leq \gamma \int_0^1 G(1, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds \\ &\leq \int_0^1 G(1, s)f(s, u(s))ds + \int_0^1 \bar{G}_p(1, s)f(s, u(s))ds. \end{aligned}$$

By use of (4.10) we get

$$\begin{aligned} Tu(t) &\leq \varepsilon \left(\int_0^1 G(1, s)u(s)ds + \int_0^1 \bar{G}_1(1, s)u(s)ds \right) \\ &\leq \varepsilon \left(\int_0^1 G(1, s)ds + \int_0^1 \bar{G}_1(1, s)ds \right) \|u\|. \end{aligned}$$

Now, if we let

$$\varepsilon = \left(\int_0^1 G(1, s)ds + \int_0^1 \bar{G}_1(1, s)ds \right)^{-1}$$

we get

$$\|Tu\| \leq \|u\|. \quad (4.11)$$

In the same way, since $f_\infty = \infty$, for any $A > 0$ there exists $B > \eta(\varepsilon) > 0$ such that, for $u \in \mathcal{C}(J)$ and $t \in J$,

$$\|u\| \geq \gamma B \Rightarrow \frac{f(t, u(t))}{u(t)} > A \Rightarrow f(t, u(t)) > Au(t). \quad (4.12)$$

Let $\Omega_2 = \{u \in \mathcal{C}(J) : \|u\| \leq B\}$. For any $u \in \mathcal{K} \cap \partial\Omega_2$ and $t \in J$, we have $\|u\| = B$ and by Lemma 4.4, we get

$$\min_{t \in [0, \tau]} u(t) \geq \gamma \|u\| = \gamma B.$$

Thus, from Lemma 4.3 and relation (4.7) we can conclude that

$$Tu(t) \geq \int_0^1 G(1, s)f(s, u(s))ds + \int_0^1 \bar{G}_{\frac{1}{2}}(1, s)f(s, u(s))ds.$$

By use of (4.12) we get

$$\begin{aligned} Tu(t) &\geq A \left(\int_0^1 G(1,s)u(s)ds + \int_0^1 \bar{G}_{\frac{1}{2}}(1,s)u(s)ds \right) \\ &\geq A\gamma \left(\int_0^1 G(1,s)ds + \int_0^1 \bar{G}_{\frac{1}{2}}(1,s)ds \right) \|u\|. \end{aligned}$$

At this stage, if we let

$$A = \left[\gamma \left(\int_0^1 G(1,s)ds + \int_0^1 \bar{G}_{\frac{1}{2}}(1,s)ds \right) \right]^{-1}$$

we get

$$\|Tu\| \geq \|u\|. \quad (4.13)$$

From inequalities (4.11), (4.13) and Theorem 4.1, case (B1), we deduce the existence of a solution for the three-point boundary value problem (4.1). \square

4.4 Example

Let $f : J \times (0, +\infty) \rightarrow (0, +\infty)$ be the continuous function defined by

$$f(t, x) = t \ln(2e^{x^2} - 1), \quad t \in J, x > 0. \quad (4.14)$$

It is easy to verify that

$$f_0 = \liminf_{\|u\| \rightarrow 0} \min_{t \in J} \frac{f(t, u(t))}{u(t)} = 0$$

and

$$f_\infty = \limsup_{\|u\| \rightarrow +\infty} \max_{t \in J} \frac{f(t, u(t))}{u(t)} = +\infty$$

so that f is a superlinear function. Thus, by Theorem 48, the three-point boundary value problem (4.1), associated to f , has at least one solution. \square

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