On positive loops of loose Legendrian embeddings

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UNIVERSITE BRETAGNE LOIRE

UNIVERSITÉ DE NANTES

Thèse de Doctorat

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Mémoire présenté en vue de l'obtention du
grade de Docteur de l'Université de Nantes
sous le sceau de l'Université Bretagne Loire

École doctorale : Sciences et technologies de l'information, et mathématiques

Discipline : Mathématiques et leurs interactions, section CNU 25
Unité de recherche : Laboratoire de mathématiques Jean Leray(LMJL)

Soutenue le 12 septembre 2016

Sur les lacets positifs des plongements legendriens lâches

JURY

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Remerciements

Je remercie d'abord mon directeur de thèse, Vincent Colin, pour tout ce qu'il m'a enseigné, mais aussi pour sa grande disponibilité et son éternel optimisme. Sans lui je ne peut pas avance sur la voie de recherche. Merci à François Laudenbach, Paolo Ghiggini et Baptiste Chantraine pour m'aide dans le monde de recherche. Merci à Paolo Ghiggini pour des cours et seminaires sur Floer homology. Merci à Gaël Meigniez pour le cour M2 sur la foliation.

Merci à Claude Viterbo et à Francosic Presas d'avoir rapporté ce texte. Je suis très honoré qu'il aient accepté de s'interesser à mon travail. Merci à François Laudenbach et klaus Niederküger er Baptiste Chantraine d'avoir fait partie de mon jury.

Merci à tout ceux, au Laboratoire de Mathématiques Jean Leray ou ailleurs, qui m'ont en diverses occasions consacré un partie de leur temps. Merci à tout les collegues doctorals pour les seminaires des doctorants, tout particulièrement Gilbelto, Thomas, Gulliaume et Noémie.

Merci à Urs Frauenfelder pour m'avoir invité à la conference CAST et Vincent Huimillière pour m'avoir invité à IHP, ou je peut present mon travaux. Merci à tout les auditeurs pour leur attentions et les valuable questions.

Merci aux mes voisins et voisines qui sont très gentills à ma famille, particulièment Madamme et Monsieur Fruchard. Merci aux tout les Nantais, Nantes est une ville très agreeable et les Nantais sont très gentills.

Merci à ma famille, merci à ma femme pour toujour m'accompagner et m'encourager, surtout au moment difficile. Merci à mon fils qui est très sage et m'offre infinis bonheurs en ma vie.

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Introduction

Dans cette thèse, on étudie les isotopies de contact et legendriennes positives sur une variété de contact (M, ξ) coorientée.

Une variété de contact (M^{2n+1}, ξ) est une variété lisse M de dimension $2n+1$ munie d'un champ d'hyperplans ξ non-intégrable qui s'appelle une structure de contact. Lorsque le champ ξ est coorienté, il est donné par le noyau d'une forme différentielle α de degré 1 appelée forme de contact. Par exemple, dans $\mathbb{R}^4$ muni des coordonnées usuelles (x_1, x_2, y_1, y_2), la sphère $\mathbb{S}^3$ admet une forme de contact $\alpha_{std} = (y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2)|_{\mathbb{S}^3}$. On note ξ_{std} la structure de contact décrite par α_{std}. Celle-ci passe au quotient sur $\mathbb{R}P^3$ pour donner une structure de contact également notée ξ_{std}.

Les sous-variétés de (M^{2n+1}, ξ) exhibant un comportement intéressant sont les sous-variétés legendriennes $L \subset M^{2n+1}$ qui ont dimension n et satisfont $\alpha|_L = 0$. Un contactomorphisme de (M, ξ) est un difféomorphisme de M qui préserve ξ et une isotopie de contact ϕ_t est un chemin de contactomorphismes issu de $\phi_0 = id$. On dit qu'une isotopie ϕ_t est positive si $\alpha(\partial_t \phi_t) > 0$, c'est-à-dire si le générateur infinitésimal de l'isotopie est partout positivement transversal au champ d'hyperplans ξ. Une isotopie legendrienne φ_t basée sur une sous-variété legendrienne L est contrainte par le fait que $\varphi_t(L)$ est une sous-variété legendrienne pour tout t. Similairement, on dit φ_t est positive si $\alpha(\partial_t \varphi_t) > 0$.

Avec le concept d'isotopie de contact positive, Eliashberg et Polterovich définissent un ordre partiel sur le revêtement universel $\widetilde{Cont}_0(M, \xi)$ de la composante neutre du groupe des contactomorphismes de la variété (M, ξ) (i.e. des contactomorphismes isotopes à l'identité). Une classe d'isotopie de contact $[(\psi_t)_{t\in[0,1]}]$ est supérieure à une autre $[(\phi_t)_{t\in[0,1]}]$ si il existe une isotopie positive de ϕ_1 vers ψ_1 homotope à la concaténation de l'opposé de $[(\phi_t)_{t\in[0,1]}]$ avec $[(\psi_t)_{t\in[0,1]}]$.

Proposition 0.1. *[EP99] Soit (M, ξ) une variété de contact. Les conditions suivantes sont équivalentes :*

(i). (M, ξ) est non-ordonnable ;

(ii). Il existe un lacet positif contractile de contactomorphismes pour (M, ξ).

Cet ordre aide à étudier la géométrie de (M, ξ). Il est également associé aux propriétés de *squeezing* en géométrie de contact [EP99] ainsi qu'à l'existence de métriques bi-invariantes sur $\widetilde{Cont}_0(M, \xi)$ ou sur l'espace des sous-variétés legendriennes [CS12].

Depuis le début des années 80, on sait que le monde des structures de contact en dimension trois se scinde en deux classes aux comportements opposés. Suivant Eliashberg, on dit qu'une structure de contact ξ sur M^3 est vrillée si elle contient un disque vrillé $D_{OT} \subset M$, c'est-à-dire un disque plongé dans M qui est tangent à la structure de contact le long de son bord. Celles-ci sont flexibles et se laissent classifier par un h-principe adéquat [Eli89]. On note α_{OT} une forme de contact pour la structure vrillée modèle ξ définie au voisinage d'un disque vrillé. Plus récemment, les travaux de Niederkrüger [Nie06] et Murphy[Mur12] notamment ont permis d'étendre cette dichotomie à la dimension supérieure. Ainsi, de façon similaire, en dimension plus grande que trois et en suivant une suggestion de Niederkrüger, on dit ξ est vrillée si (M^{2n+1}, α) contient $D^3 \times T^* D^{n-1}(r)$ avec $\alpha|_{D^3\times T^*D^{n-1}(r)} = \alpha_{OT} - (ydx - xdy)$ pour une certaine constante $r > 0$ assez grande [CMP15] dépendant de la dimension.

À nouveau, les structures vrillées sont des objets purement topologiques et flexibles d'après [BEM].

Au contraire, on dit que ξ est une structure de contact tendue si elle n'est pas vrillée. Par exemple, les variétés $(\mathbb{S}^3, \xi_{std})$ et $(\mathbb{R}P^3, \xi_{std})$ sont tendues d'après le résultat fondateur de Bennequin [Ben83]. Les structures tendues possèdent de nombreuses propriétés rigides qui les apparentent à la géométrie complexe.

La propriété d'ordonnabilité n'est pas possédée par toutes les variétés de contact :

Théorème 0.2. *(i). $(\mathbb{R}P^3, \xi_{std})$ est ordonnable, mais pas $(\mathbb{S}^3, \xi_{std})$. [EKP06]*

(ii). Il existe des variétés de contact vrillées qui sont non-ordonnables.[CPS14]

On peut trouver beaucoup d'autres exemples des variétés de contact ordonnables dans les travaux de Albers, Frauenfelder, Fuchs et Merry [AF12, AM13, AFM15].

Il intéressant de voir que les variétés de contact tendues peuvent être ordonnables ou pas, malgré leur caractère rigide. En même temps, on imagine que toutes les variétés de contact vrillées sont non-ordonnables.

Question 0.3. *Est-que toutes les variétés de contact vrillées sont non-ordonnables ?*

Pour étudier la question précédente, on transfère l'étude des isotopies de contact positives vers l'étude des isotopies legendriennes positives en prenant leur graphe dans le produit de contact. En effet, une isotopie de contact positive de (M, α) se transforme en une isotopie legendrienne négative de la diagonale $\Delta_{M\times M} \times \{0\}$ dans le produit de contact $(M \times M \times \mathbb{R}, \alpha_1 - e^s\alpha_2)$. L'avantage est que l'étude des isotopies legendriennes positives ou négatives devrait être plus facile que celle des contactomorphismes.

Dans ce contexte, on pose une question naturelle sur les isotopies legendriennes positives :

Question 0.4. *Soit (M, ξ) une variété de contact, L_0 et L_1 deux sous-variétés legendriennes dans (M, ξ) qui sont isotopes par une isotopie legendrienne. Est-ce qu'il existe une isotopie legendrienne positive entre les deux ?*

Exemple 0.5. *Soit $(\mathbb{S}^2, g)$ la 2-sphère avec la métrique ronde g et soit $ST^*\mathbb{S}^2$ l'espace des éléments de contact sur $\mathbb{S}^2$. Notons S et N les pôles de $\mathbb{S}^2$. Alors le flot géodésique de g induit une isotopie legendrienne positive L_t connectant les deux fibres legendriennes $ST^*_N\mathbb{S}^2$ et $ST^*_S\mathbb{S}^2$.*

Pourtant, en général, la réponse à la question 0.4 est négative.

Théorème 0.6. *Soit M^n, $n > 1$ une variété dont le revêtement universel est ouvert. Alors*

(i). les fibres de ST^*M ne sont pas dans des lacets positifs de plongements legendriens [CFP10, CN10, GKS12] ;

(ii). la section nulle de $(T^*M \times \mathbb{R}, dz - ydx)$ n'est pas dans un lacet positif de plongements legendriens [CFP10].

On note F l'application $(T^*M \times \mathbb{R}, dz - ydx) \to M \times \mathbb{R} : (x, y, z) \mapsto (x, z)$, qui est appelée la projection frontale. Pour une sous-variété legendrienne $L \subset (T^*M \times \mathbb{R}, dz - ydx)$, le sous-ensemble $L_F := F(L) \subset M \times \mathbb{R}$ est appelé le front de L. On identifie souvent le front L_F avec L, car la coordonnée manquante est donnée par la pente de son espace tangent. Quand la dimension de M est 1, on peut remplacer un segment lisse de L_F par un zig-zag. La sous-variété obtenue par cette opération est notée $S(L)$ et est appelée une stabilisation de L. Alors on a :

Théorème 0.7. *[CFP10] Soient L la section nulle de $T^*\mathbb{S}^1 \times \mathbb{R}$ et $S(L)$ une stabilisation de L. Alors il existe un lacet positif de plongements legendriens basé en $S(L)$.*

Pour une variété de contact (M, ξ) de dimension strictement plus grande que trois, Murphy [Mur12] introduit une classe de sous-variétés legendriennes appelées sous-variétés legendriennes lâches. Les sous-variétés lâches sont une généralisation en dimension supérieure des sous-variétés stabilisées $S(L)$ en dimension trois. Elles satisfont un h-principe découvert par Murphy qui les rend flexibles. Le résultat principal de ce travail de thèse renforce encore cette flexibilité.

Théorème 0.8. *Soient (M^{2n+1}, ξ), $n > 1$ une variété de contact et $L \subset (M, \xi)$ une sous-variété legendrienne. Si L est lâche, alors il existe un lacet positif de plongements legendriens basé en L.*

On prouve le Théorème 0.8 de la manière suivante :

Dans les cas des sphères en dimension 1 et 2, on donne des démonstration à la main pour illustrer les idées géométriques. Pour une 1-sphère legendrienne avec des zigzags sur le front un lacet positif est obtenu explicitement comme une rotation du front dans une direction transversale (c'est une observation

de [CFP10]). Pour une 2-sphère legendrienne L lâche, on construit deux lacets ϕ_t^1 et ϕ_t^2. L'isotopie ϕ_t^1 est un lacet de plongements legendriens basé en L qui est positif sur un voisinage des pôles. Une observation de base est en effet que chaque pôle peut être placé sur un nœud positivement transversal à la structure de contact par non-intégrabilité du plan de contact ; ceci donne une façon d'incorporer les pôles dans un lacet positif. L'isotopie ϕ_t^2 est un lacet de contactomorphismes qui est non négatif sur L et assez grand en dehors d'un autre voisinage plus petit des pôles. Elle est obtenue en faisant tourner très rapidement, sur le front, une *ride* (généralisation du zig-zag) autour de l'équateur. Une image concrète est celle d'une vague s'étendant entre les deux pôles le long d'une longitude, dont la coupe est un zig-zag. Elle est d'amplitude nulle aux pôles et positive en dehors des pôles, et tourne très rapidement autour de la terre. On montre que la composition $\phi_t^2 \circ \phi_t^1$ est un lacet de plongements (après désingularisation de la vague près des pôles) legendriens positif basé en L.

Le cas général est obtenu par des techniques de h-principe, qui résument le fait de recouvrir le front par des rides (ou vagues) se propageant rapidement pour incorporer tous les points dans un lacet positif. On utilise l'approximation holonome ridée et l'intégration convexe pour relations différentielles non-amples. En bref, lorsque la sous-variété L est fermée, l'approximation holonome classique ne marche pas. On doit considérer des singularités et puis essayer de les résoudre. Par l'approximation holonome ridée, on obtient un lacet positif pour L sauf sur un ensemble fini de disques. Il reste à prouver que si une isotopie legendrienne de disques est positive au bord, alors on peut la rendre positive. Notons que l'approximation holonome ne marche toujours pas, donc on besoin d'une sorte d'intégration convexe pour obtenir la positivité. Mais l'intégration convexe classique est seulement valable pour les relations différentielles amples. Dans notre cas la condition de positivité n'est pas ample, donc on a besoin d'une nouvelle technique. Heureusement, inspiré par les cas de basse dimension, on peut encore utiliser la technique de *wrinkling*. On utilise de façon intensive le résultat de Murphy [Mur12] qui permet d'ajouter des rides à une Legendrienne lâche sans changer sa classe d'isotopie legendrienne.

Comme application du Théorème 0.8, on obtient une preuve de l'existence des variétés de contact tendues (i.e. non vrillée au sens de Borman-Eliashberg-Murphy [BEM]) en toute dimension sans utilisation des courbes holomorphes. La partie "dure" utilise le Théorème 0.7 dont la preuve repose sur l'existence d'une fonction génératrice pour une classe de sous-variétés legendriennes spécifiques (dans ce cas les fibres legendriennes de la fibration $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$).

Corollaire 0.9. *[MNPS13] La variété de contact $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$ est tendue.*

Le corollaire précédent est prouvé dans la sous-section 5.1.

Dans la dernière section, on définit un nouvel ordre partiel sur certains groupes $\widetilde{Cont}_0(M, \xi)$, appelé ordonnabilité forte, basé sur le transfert d'une isotopie de contactomorphismes en une isotopie legendrienne de leurs graphes dans le produit de contact. On relâche la condition de graphe pour s'en tenir à des isotopies legendriennes et on obtient une notion d'ordre fort (peut-être) différent que celui d'Eliashberg-Polterovich [EP99].

Proposition 0.10. *Soit (M, ξ) une variété de contact. Alors (M, ξ) est fortement ordonnable si et seulement si il n'existe pas de lacet positif contractile de plongements legendriens basé sur la diagonale du produit de contact de (M, ξ).*

En exemple, on montre que la variété de contact $(\mathbb{S}^1, d\theta)$ est fortement ordonnable.

Dans ce contexte, on prouve dans la Proposition 5.6 que si (M^{2n+1}, ξ) est une variété de contact vrillée, alors le produit de contact $(M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ est aussi vrillé. De plus, la diagonale est lâche. En particulier, par le Théorème 0.8, la diagonale est le point base d'un lacet positif. Que ce lacet (ou un autre) soit contractile déterminerait la non ordonnabilité faible de la variété vrillée originale (M, ξ).

Organisation du manuscrit. Cette thèse comprend trois parties. La première partie est principalement consacrée à des rappels de géométrie de contact et au h-principe. Le chapitre 1 rappelle les notion de base de

géométrie de contact : isotopies positives de contact/legendriennes et sous-variétés legendriennes lâches et ridées. Le chapitre 3 présente le h-principe. On rappelle l'approximation holonome classique et la technique d'intégration convexe. On présente également l'approximation holonome ridée et la méthode de résolution des rides.

La deuxième partie est consacrée à la preuve du Théorème 0.8. Le chapitre 2 traite les cas de la sphère lâche en dimension 1 et 2 à la main. Le chapitre 4 traite le cas général par h-principe. D'abord, on rappelle les idées de Murphy sur le h-principe pour les plongements de legendriennes lâches, puis on donne une démonstration détaillée du Théorème 0.8.

Enfin, la troisième partie présente des applications et des développements. D'abord, on prouve le Corollaire 0.9. Après on traite le cas du produit de contact et prouve la Proposition 5.6. Puis on définit un nouvel ordre partiel sur les groupes $\widetilde{Cont}_0(M,\xi)$ que l'on illustre par l'exemple de $(S^1, d\theta)$.

Introduction

In this thesis, we focus on the study of positive contact and Legendrian isotopies in a co-oriented contact manifold (M, ξ).

A contact manifold (M^{2n+1}, ξ) is a $2n+1$ dimensional smooth manifold M with a non-integrable hyperplane field ξ which is called a contact structure. When ξ is co-oriented, it is given by the kernel of a differential 1-form α which is called a contact form. For example, in $\mathbb{R}^4$ with the usual coordinates (x_1, x_1, y_1, y_2), the sphere $\mathbb{S}^3$ carries a contact form $\alpha_{std} = (y_1 dx_1 - x_1 dy_1 + y_2 dx_2 - x_2 dy_2)|_{\mathbb{S}^3}$. We denote ξ_{std} the contact structure defined by α_{std}. It passes to a contact structure on the quotient $\mathbb{R}P^3$ which is also denoted by ξ_{std}.

One class of submanifolds of (M^{2n+1}, ξ) with an interesting behavior is that of Legendrian submanifolds. A n-dimensional submanifold $L \subset M$ is called a Legendrian submanifold if $\alpha|_L = 0$. A contactomorphsim of (M, ξ) is a diffeomorphism which preserves ξ and a contact isotopy ϕ_t is a path of contactomorphisms with $\phi_0 = id$. We say a contact isotopy ϕ_t is positive if $\alpha(\partial_t \phi_t) > 0$. That is to say, the infinitesimal generator of the isotopy is positively transverse to ξ everywhere. An isotopy φ_t based in a Legendrian submanifold L is said to be a Legendrian isotopy if $\varphi_t(L)$ is a Legendrian submanifold for all t. Similarly, We say φ_t is positive if $\alpha(\partial_t \varphi_t) > 0$.

With the concept of positive contact isotopy, Eliashberg and Polterovich defined a partial order on the universal cover $\widetilde{Cont}_0(M, \xi)$ of the identity component of the contactomorphisms group of (M, ξ) which helps to investigate its geometry. A class of contact isotopy $[(\psi_t)_{t\in[0,1]}]$ is greater than another class $[(\phi_t)_{t\in[0,1]}]$ if there exists a positive contact isotopy from ϕ_1 to ψ_1 which is homotopic to the concatenation of the opposite of ϕ_t and ψ_t.

Proposition 0.1. *[EP99] If (M, ξ) is a contact manifold, the following conditions are equivalent :*

(i). (M, ξ) is non-orderable ;

(ii). There exists a contractible positive loop of contactomorphisms for (M, ξ).

This order helps us to study the geometry of (M, ξ). It is also closely related to *squeezing* properties in contact geometry [EP99] as well as to the existence of bi-invariant metrics on $\widetilde{Cont}_0(M, \xi)$ or on the space of Legendrian submanifolds [CS12].

From the beginning of the eighties, it is known that the world of contact structures splits into two classes with opposite behaviors. Following Bennequin and Eliashberg, we say that a contact structure ξ on M^3 is overtwisted if there exists an overtwisted disk $D_{OT} \subset M$, that is to say a embedded disk which is tangent to ξ along its boundary. The overtwisted contact structures are flexible and classified by an adequate h-principle [Eli89]. We denote α_{OT} a contact form for an overtwisted contact structure ξ defined on a neighborhood of an overtwisted disk. More recently, the work of Niederküger [Nie06] and Murphy [Mur12] permit us to reach that dichotomy in the higher dimensional case. That is, similarly, in dimension greater than three, following a suggestion of Niederküger, we say ξ is overtwisted if (M^{2n+1}, α) contains $D^3 \times T^*D^{n-1}(r)$ with $\alpha|_{D^3\times T^*D^{n-1}(r)} = \alpha_{OT} - (ydx - xdy)$ for some constant $r > 0$ large enough depending on the dimension of M [CMP15].

Again, the overtwisted structures are purely topological objects and are flexible following from [BEM].

On the contrary, we say ξ is a tight contact structure if it is not overtwisted. For examples, the contact manifolds $(\mathbb{S}^3, \xi_{std})$ and $(\mathbb{R}P^3, \xi_{std})$ are tight according to the fundamental result of Bennequin.[Ben83] Tight contact structures possess many rigidity properties similar to complex geometry.

The orderability property is not shared by all contact manifolds :

Theorem 0.2. *(i). $(\mathbb{S}^3, \xi_{std})$ is non-orderable while $(\mathbb{R}P^3, \xi_{std})$ is orderable [EKP06] ;*

(ii). There are some overtwisted contact manifolds which are non-orderable [CPS14].

There are many more examples of orderable contact manifolds from the work of Albers, Frauenfelder, Fuchs and Merry [AF12, AM13, AFM15].

It is interesting to see that tight contact manifolds can be orderable or not despite their rigid nature. At the same time we guess overtwisted contact manifolds are non-orderable.

Question 0.3. *Are all overtwisted contact manifolds non-orderable ?*

In order to answer the above question, we transfer the study of positive contact isotopies to that of positive Legendrian isotopies by the trick of contact product. Indeed, a positive contact isotopy of (M, ξ) can be lifted to a negative Legendrian isotopy of the diagonal $\Delta_{M\times M} \times \{0\}$ in the contact product $(M \times M \times \mathbb{R}, \alpha_1 - e^s\alpha_2)$. The advantage is that the study of positive Legendrian isotopies should be easier than that of contact isotopies.

In that context, we have a natural question concerning positive Legendrian isotopies :

Question 0.4. *Let (M, ξ) be a contact manifold and let L_0 and L_1 be Legendrian submanifolds in (M, ξ) which are Legendrian isotopic. Does there exist a positive Legendrian isotopy connecting them ?*

Example 0.5. *Let $(\mathbb{S}^2, g)$ be the 2-sphere with the round metric g, and let $ST^*\mathbb{S}^2$ be the space of contact elements on $\mathbb{S}^2$. Denoting S, N the poles, then the geodesic flow of g induces a positive Legendrian isotopy L_t connecting the Legendrian fibers $ST^*_N\mathbb{S}^2$ and $ST^*_S\mathbb{S}^2$.*

Generally, the answer to Question 0.4 is negative.

Theorem 0.6. *Let M^n, $n > 1$ be a manifold with open universal cover. Then*

(i). the fibers of ST^*M are not in a positive loop of Legendrian embeddings [CFP10, CN10, GKS12] ;

(ii). the zero-section of $(T^*M \times \mathbb{R}, dz - ydx)$ is not in a positive loop of Legendrian embeddings [CFP10].

We denote F the map $(T^*M \times \mathbb{R}, dz - ydx) \to M \times \mathbb{R} : (x, y, z) \mapsto (x, z)$, which is called the front projection. For a Legendrian submanifold $L \subset (T^*M \times \mathbb{R}, dz - ydx)$, the subset $L_F := F(L) \subset M \times \mathbb{R}$ is the front of L. We usually identify L_F to L, since the y coordinates are given by the slopes of the front. We can replace a smooth segment of L_F by a zigzag with two cusps. The submanifold obtained by this operation is denoted by $S(L)$ and is called a stabilization of L. We have :

Theorem 0.7. *[CFP10] Let L be the zero-section of $T^*\mathbb{S}^1 \times \mathbb{R}$ and $S(L)$ a stabilization of L. Then there exists a loop of positive Legendrian embeddings based in $S(L)$.*

For a contact manifold (M, ξ) of dimension strictly higher than three, Murphy [Mur12] introduced the class of loose Legendrian submanifolds. This is a higher dimensional generalization of the stabilized $S(L)$ in dimension three. Loose Legendrian submanifolds satisfy a h-principle discovered by Murphy which make them flexible. The main result of the work of this thesis extends this flexibility.

Theorem 0.8. *Let (M, ξ) be a contact manifold and $L \subset (M, \xi)$ be a Legendrian submanifold. If L is loose then there exists a positive loop of Legendrian embeddings based in L.*

We prove Theorem 0.8 in the following way :

In the cases of the one and two dimensional spheres, we give a proof by hand to illustrate the geometric ideas. For a one dimension Legendrian sphere with zigzags shape on the front, a positive loop is obtained explicitly as a rotation of the front in a transverse direction (it's an observation of [CFP10]). For a two dimensional loose sphere L, we construct two loops ϕ^1_t and ϕ^2_t. The isotopy ϕ^1_t is a loop of Legendrian embeddings which is positive on a neighborhood of the poles. In fact, it is a basic observation that each pole can be placed in a knot which is positively transverse to ξ since the field ξ is non-integrable. This gives a way to incorporate the poles into a positive loop. We extend ϕ^1_t smoothly to a contact loop which is also denoted by ϕ^1_t. The isotopy ϕ^2_t is a loop of contactomorphisms which is non negative on L and large enough

outside another smaller neighborhood of the poles. It is obtained by turning very fast, on the front, a *wrinkle* (a generalization of zigzag) around the equator. A concrete image is that of a wave expanding between the two poles along a longitude whose sections are zigzags with amplitudes zero at the poles and positive away from the poles. The wave is turning very fast around the earth. We prove that the composition $(\phi_t^1 \circ \phi_t^2))|_L$ is a positive loop of (after resolving the singularities) Legendrian embeddings based in L.

The general case is obtained via h-principle techniques. Roughly, we cover the front by wrinkles (waves) which propagate fast to incorporate all the points into a positive loop. We use the wrinkled holonomic approximation and the convex integration for non-ample differential relations. Briefly, since L is closed, classical holonomic approximation doesn't work. We have to consider singularities and then try to resolve them. By the wrinkled holonomic approximation, we obtain a loop which is positive outside a finite number of disks. This way it remains to prove that if a Legendrian isotopy of a disk is positive near the boundary, then we can make it positive everywhere. Notice that holonomic approximation does no longer work and we expect some kind of convex integration lemma to reach positivity. However the classical convex integration is only true for ample relations. In our case the positivity condition is not ample, thus we need another technique for this problem. Luckily, inspired by the lower dimensional cases, we can apply the wrinkling technique again. We apply the result of Murphy [Mur12] intensively which permits us to add wrinkles to a loose Legendrian without changing the Legendrian isotopy class.

As an application of Theorem 0.8, we obtain a holomorphic curve free proof of the existence of tight (i.e. non overtwisted in the Borman-Eliashberg-Murphy sense[BEM]) contact structures in every dimensions. The "hard part" of the argument uses Theorem 0.6 whose proof relies on the existence of a generating function for a specific class of Legendrians (in that case the Legendrian fibers of the Legendrian fibration in $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$). This contact structure is the natrual one on $ST^*(\mathbb{R}^n)$.

Corollary 0.9. *[MNPS13] The contact manifold $(\mathbb{R}^n \times \mathbb{S}^{n-1}, \xi_{std})$ is tight.*

This corollary is proved in Subsection 5.1.

Moreover, in the last section, we define a new partial order on certain groups $\widetilde{Cont}_0(M, \xi)$, called strong orderability, based on the transfer of an isotopy of contactomorphisms to a Legendrian isotopy of their graphs in the contact product. We then drop the graph condition to stick to Legendrian isotopies and get a (possibly) different notion than Eliashberg-Polterovich's [EP99].

Proposition 0.10. *Let (M, ξ) be a contact manifold. Then (M, ξ) is strongly orderable if and only if there does not exist a contractible positive loop of Legendrian embeddings based in the diagonal of the contact product of (M, ξ).*

For example, we prove the contact manifold $(\mathbb{S}^1, d\theta)$ is strongly orderable.

In that context, we prove in Proposition 5.6 that if (M^{2n+1}, ξ) is an overtwisted contact manifold, then the contact product $(M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ is also overtwisted and its diagonal is loose. In particular, by Theorem 0.8, it is contained in a positive loop. Whether this loop (or another !) is contractible or not would determine the strong orderability of the original (M, ξ).

Organisation of the manuscript. This thesis includes three parts. The first part is mainly devoted to recall contact geometry and the h-principle. In chapter 1, we recall the basic notions of contact geometry : positive contact/Legendrian isotopies, loose and wrinkled Legendrian submanifolds. In chapter 3, we present the h-principle. We recall the classical holonomic approximation and convex integration techniques. We also present the wrinkled holonomic approximation and the technique for resolving wrinkles.

The second part is devoted to prove Theorem 0.8. In chapter 2, we handle the cases of loose spheres of dimension one and two by hand. In chapter 4, we deal with the general case via h-principle. First of all, we recall Murphy's ideas on h-principle for loose Legendrian embeddings. Then we give a detailed proof of Theorem 0.8.

Finally, in the third part, we present some applications and developments. First of all, we prove Corollary 0.9. After that, we work on the contact product and prove Proposition 5.6. In the end, we define a new partial order on the group $\widetilde{Cont}_0(M,\xi)$ which is illustrated by the example of $(\mathbb{S}^1, d\theta)$.

1

Contact Geometry

In this chapter we recall the basic notions and results we need in the thesis. For more in-depth treatment we refer to [Gei08].

1.1 Contact structures

Let M be a differential manifold, TM its tangent bundle, and $\xi \subset TM$ a co-oriented field of hyperplanes on M. It is useful to represent ξ as the kernel of a differential 1-form, that is to say $\xi = ker\alpha$ for some differential 1-form α.

We say a hyperplane ξ is integrable when it has the property that at each point $p \in M$ there is a codimensional 1 submanifold N passing through p with $T_pN = \xi_p$. It turns out that $\xi = ker\alpha$ is integrable precisely if α satisfies the Frobenius integrability condition

$$\alpha \wedge d\alpha \equiv 0.$$

In terms of Lie brackets of vector fields, the integrability condition is equivalent to

$$[X, Y] \in \xi \text{ for all } X, Y \in \xi.$$

Contact structures are in some sense the exact opposite of integrable hyperplane fields.

Definition 1.1. Let M be a differential manifold of odd dimension $2n + 1$. A **contact structure** is a maximally non-integrable hyperplane field $\xi = ker\alpha \subset TM$, that is, the differential 1-form is required to satisfy

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Such a 1-form α is called a **contact form**. The pair (M, ξ) is called a **contact manifold**.

Example 1.2. *On $\mathbb{R}^{2n+1}$ with Cartesian coordinates*

$$(x_1, y_1, \ldots, x_n, y_n, z),$$

the 1-form

$$\alpha = dz - \sum_{j=1}^{n} y_j dx_j$$

*is a contact form. The contact structure defined by this α is called the **standard contact structure** on $\mathbb{R}^{2n+1}$.*

Locally, every contact structure looks like the standard one, more precisely,

Theorem 1.3 (Darboux's theorem). *Let α be a contact form on the $(2n+1)$-dimensional manifold M and p a point on M. Then there are coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ on a neighborhood $U \subset M$ of p such that $p = (0, \ldots, 0)$ and*

$$\alpha|_U = dz - \sum_{j=1}^{n} y_j dx_j.$$

Example 1.4. *Let B be a n-dimensional smooth manifold, T^*B its cotangent bundle with standard coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, and let $M = ST^*B$ be the radius one sphere bundle for some Riemannian metric on B. Then*

$$\alpha = \sum_{i=1}^{n} p_j dq_j|_M$$

*is a contact form on M. The pair (M, α) is called the **space of contact elements** of B.*

Example 1.5. *Let L be a n-dimensional smooth manifold, T^*L its cotangent bundle, and let $M = T^*L \times \mathbb{R}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$. Then*

$$\alpha = dz - \sum_{j=1}^{n} y_j dx_j$$

*is a contact form on M. This contact manifold is denoted by $J^1(L, \mathbb{R})$ and called the 1**-jet space** of smooth functions on L.*

1.2 Legendrian submanifolds

Given a contact manifold (M, ξ), we want to understand the global features of the contact structure. A natural idea is to consider some class of submanifolds and to see how the contact structure changes along

such a submanifold. A fundamental class is the one of submanifolds tangent to ξ.

1.2.1 Basic facts about Legendrian submanifolds

In this section, we summarize basic definitions, neighborhood theorems and give several examples.

Definition 1.6. Let (M, ξ) be a contact manifold. A submanifold L of M is called an **isotropic submanifold** if $T_pL \subset \xi_p$ for all points $p \in L$.

In fact, if $L \subset (M, \xi)$ is isotropic, and if the dimension of M is $2n+1$, then $dimL \leq n$.

Definition 1.7. An isotropic submanifold $L \subset (M^{2n+1}, \xi)$ of maximal possible dimension n is called a **Legendrian submanifold**.

Example 1.8. *Consider the* 1*-jet space* $J^1(L, \mathbb{R})$ *with the standard contact structure. Then for every smooth function* $f : L \to \mathbb{R}$*, the subset*

$$L_f = \{(x, df(x), f(x)),\ x \in L\}$$

is a Legendrian submanifold. More generally, for some $N \geq 0$*, consider the product* $L \times \mathbb{R}^N$ *with coordinate* $z = (x, y)$*. Let* $f : L \times \mathbb{R}^N \to \mathbb{R}$ *be a function such that the image of* df *in* $T^*(L \times \mathbb{R}^N)$ *intersects* $T^*L \times \mathbb{R}^N \times \{0\}$ *transversely. Then the subset*

$$L_f = \{(x, df(z), f(z)) \mid \frac{\partial f}{\partial y}(z) = 0\}$$

is an immersed Legendrian of $J^1(L, \mathbb{R})$ *and this function* f *is called a* ***generating function*** *for* L_f*.*

Example 1.9. *Consider the space of contact elements* ST^*B *with its standard contact structure.*

i) For all submanifold $X \subset B$ *the spherical conormal bundle* SN^*X *consisting of all* 1*-forms on* T_LB *vanishing on* T_LL *is a Legendrian submanifold. Especially, for a point* $p \in B$ *we have* $SN^*\{p\} = ST^*_pB$*, which is the fiber of* ST^*B *through* p*.*

ii) Consider the product $B \times \mathbb{R}^N$ *and take a function* $f : B \times \mathbb{R}^N \to \mathbb{R}$ *such that* 0 *is not a critical value and the two subsets* $\{f = 0\}$ *and* $\{df|_{\mathbb{R}^N} = 0\}$ *intersect transversely. Let*

$$L = \{f = 0\} \cap \{df|_{\mathbb{R}^N} = 0\},$$

then the subset

$$L_f = \left\{ \frac{df(x)}{|df(x)|} \mid x \in L \right\} \subset ST^*B$$

is a Legendrian submanifold and $\{f = 0\}$ *is called a* ***generating hypersurface*** *for* L_f.

Given a Legendrian submanifold $L \subset (M, \xi)$, it is reasonable to ask how should the contact structure look like along L. As the following neighborhood theorem indicates, ξ is standard.

Theorem 1.10 (Weinstein neighborhood theorem). *Let* $(M, \xi = ker\alpha)$ *be a contact manifold and let* $L \subset M$ *be a Legendrian submanifold. Then there is a neighborhood* U *of L, such that* $(U, \alpha|_U)$ *is contactomorphic to a neighborhood of* $O_{T*L} \times \{0\}$ *in* $(J^1(L, \mathbb{R}), \alpha_{std})$.

Remark 1.11. Beware that a priori we do not have a lower bound of the size of U.

It is good to understand a Legendrian submanifold by drawing pictures, especially when the Legendrian submanifold is a curve.

Definition 1.12. Let M be a n-dimensional manifold with coordinates $(x_1, \ldots, x_n)$, and $T^*M \times \mathbb{R}$ the 1-jet space with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ and the standard contact form $\alpha_{std} = dz - \sum_{j=1}^{n} y_j dx_j$. Then the map

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \longmapsto (x_1, \ldots, x_n, z)$$

is called the **front projection**, and the image of a Legendrian submanifold in $M \times \mathbb{R}$ is called its **front**. The map

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \longmapsto (x_1, \ldots, x_n, y_1, \ldots, y_n,)$$

is called the **Lagrangian projection**.

Remark 1.13. Since a Legendrian submanifold $L \subset (T^*M \times \mathbb{R}, \alpha_{std})$ satisfies the equation

$$dz - \sum_{j=1}^{n} y_j dx_j = 0,$$

L can be easily recovered from its front by setting $y_j = \frac{\partial z}{\partial x_j}$. Therefore, when talking about a Legendrian submanifold $L \subset (T^*M \times \mathbb{R}, \alpha_{std})$, we usually regard the front as L itself, which is much simpler.

When M is $\mathbb{R}$, then $T^*M \times \mathbb{R}$ equals to $\mathbb{R}^3$. Therefore, the front of a Legendrian curve is just a curve in $\mathbb{R}^2$ with well-defined tangents (including cusp points) which are never vertical.

1.2.2 Positive contact/Legendrian isotopy

Let (M, ξ) be a contact manifold and $L \subset M$ a Legendrian submanifold. It is natural to investigate the group of contactomorphisms of (M, ξ) and the set of of Legendrian embeddings of L. A class of special paths in these sets is the one of positive isotopies.

Definition 1.14. [EP99](**Positive contact isotopy**) Let $(M, \xi = ker\alpha)$ be a contact manifold, $\varphi : M \times [0,1] \to M$ be a contact isotopy and let $X_t = \frac{d\varphi}{dt}$ where $t \in [0,1]$. We say φ is **positive** if X_t is transverse to ξ positively, that is to say,

$$\alpha(X_t) > 0.$$

Moreover, φ is said to be a **positive loop** if in addition $\varphi_0 = \varphi_1$.

Similarly, we can talk about positive Legendrian isotopies.

Definition 1.15. [CFP10, CN10](**Positive Legendrian isotopy**) Let $(M, \xi = ker\alpha)$ be a contact manifold, $L \subset M$ a Legendrian submanifold, $\varphi : L \times [0,1] \to M$ a Legendrian isotopy and let $X_t = \frac{d\varphi}{dt}$ where $t \in [0,1]$. We say φ is **positive** if X_t is transverse to ξ positively, i.e.

$$\alpha(X_t) > 0.$$

Moreover, φ is said to be a **positive loop** if in addition $\varphi_0 = \varphi_1$.

Positive contact/Legendrian isotopies sometimes have rigid behaviours, since it is not quite possible to make a general isotopy positive. When do they exist ? What are the consequences of the existence of a positive loop ?

If there is no contractible positive loop in $Cont_0(M, \xi)$, there is a well defined partial order on $\widetilde{Cont}_0(M, \xi)$ the universal cover of $Cont_0(M, \xi)$, according to Proposition 0.1.

Definition 1.16. [EP99] Let (M, ξ) be a closed contact manifold. For $f, g \in \widetilde{Cont}_0(M, \xi)$, write $f \succeq g$ if fg^{-1} can be represented by a positive path $\phi_t \in Cont_0(M, \xi)$. (M, ξ) is said to be **orderable** if $\succeq$ defines a partial order on $\widetilde{Cont}_0(M, \xi)$.

From [EKP06], the existence of positive contractible loops is a manifestation of symplectic flexibility. In some cases any homotopy of a positive contractible loop to the constant loop can serve as a squeezing tool of some subset.

There does not always exist a positive Legendrian isotopy connecting two Legendrians which are legendrian isotopic.

Theorem 1.17. *[CFP10, CN10, GKS12] Let M be a manifold, ST^*M with the standard contact structure, then there is no positive Legendrian isotopy connecting any two fibers.*

If we consider the weaker notion of **Positive Legendrian regular homotopy**, then the rigidity properties disappear and we have the following existence theorem.

Theorem 1.18. *[Lau07] Let (M, ξ) be contact manifold and $L \subset M$ be a Legendrian submanifold. Then there always exist positive loops of Legendrian immersions based in L.*

1.2.3 Loose Legendrian submanifolds

When we want to construct or to prove the existence of positive loops of Legendrian embeddings, it is not possible to do it for all classes of Legendrian submanifolds due to Theorem 1.17. Therefore it is more reasonable to consider some flexible class. Luckily, there is a good candidate, the loose one, discovered by Emmy Murphy [Mur12], which is somehow a generalization of the notion of a stabilized Legendrian curve.

In the 1-jet space $T^*\mathbb{S}^1 \times \mathbb{R}$, from Example 1.8 there are graphic-like Legendrian submaifolds, that is, those with generating functions. There exists an other class that turns around the zero-section, more precisely,

Definition 1.19. Consider $T^*\mathbb{S}^1 \times \mathbb{R}$ with the standard contact structure. Let $L \subset T^*\mathbb{S}^1 \times \mathbb{R}$ be a Legendrian curve, and denote its front in $\mathbb{S}^1 \times \mathbb{R}$ by L_F. The **stabilization** is an operation that replaces a portion of L_F by a **zigzag**. Denote the resulting Legendrian by $S(L)$. It is called a **stabilization** of L (there are two such stabilizations going upward or downward).

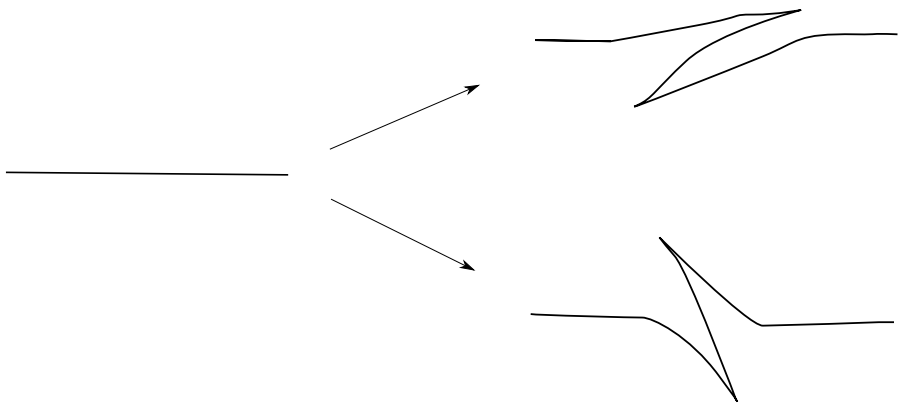

FIGURE 1.1 – Stabilization in the front.

In the Lagrangian projection, a stabilization corresponds to adding a small loop, and thus introducing a double point that corresponds to a Reeb chord. The action of this Reeb chord (the difference of altitude between the two branches) is also equal to the area of the loop and is called the *action* of the stabilization.

Proposition 1.20. *[FR11, Eli87] If $L \subset (T^*\mathbb{S}^1 \times \mathbb{R}, \xi_{std})$ is a closed Legendrian curve, then a stabilization $S(L)$ of L does not admit a generating function.*

Non-existence of generating function suggests flexibility of $S(L)$. This is confirmed by the following :

Theorem 1.21. *[Che03, p. 10] Let $L \subset (T^*\mathbb{S}^1 \times \mathbb{R}, \xi_{std})$ be a closed Legendrian curve, then the Legendrian contact homology of $S(L)$ is zero.*

It is interesting to generalize the notion of **Zigzag** and **Stabilization** to higher dimensional situations. There is one such generalization discovered by Emmy Murphy [Mur12] when she proved some Legendrian embeddings satisfy h-principle. She gives it the name **Loose** to emphasize its flexible nature.

Definition 1.22. [Mur12] Suppose $n > 1$. Let $B \subseteq (\mathbb{R}^3, \xi_{std})$ be an open ball containing $S(L)$, a stabilization of action a, and let $V_\rho = \{|x| < \rho, |y| < \rho\} \subseteq T^*\mathbb{R}^{n-1}$. Note that $B \times V_\rho$ is an open convex set in $(\mathbb{R}^{2n+1}, \xi_{std})$. Let Δ be the cartesian product of the stabilization and the zero section of $T^*\mathbb{R}^{n-1}$, which is a Legendrian in $B \times V_\rho$. We call the pair $(B \times V_\rho, \Delta)$ a Legendrian twist. A Legendrian twist satisfying $a < \rho^2$ is called a loose chart. Finally, let L be a Legendrian submanifold in (M, ξ). If there is a Darboux chart $U \subseteq M$ such that $(U, U \cap L)$ is a loose chart then L is called **loose**.

For our purpose, we give the following equivalent definition of loose.

Definition 1.23. Let $L : Y^n \hookrightarrow (J^1(Y^n), \xi_{std})$ be a Legendrian embedding. Let Λ be a one dimensional zigzag and N a $n-1$ dimensional manifold. We say L is loose if its front contains contact image of $\Lambda \times N$, which is called a **Zigzag**.

Proposition 1.24. *The above two definitions are equivalent.*

Proof: First of all, given a Legendrian with a Zigzag $\Lambda \times N \subset B \times N$, we fix a Riemannian metric on N and a disk $D(r) \subset N$ for some r. Next, we can shrink $\Lambda \times N$ to $\Lambda' \times N$ such that the action $a'^2 < r$. Then, we get a loose chart $B \times D(r)$. Conversely, given a loose Legendrian L, its front $F(L)$ contains $\Lambda \times \mathbb{D}^{n-1}$, and we can take a submanifold $N \subset F(L)$ with trivial normal bundle such that $\partial N = \partial \mathbb{D}^{n-1}$. Then we replace a normal neighborhood of N by $\Lambda \times N$. After this operation, we get another Legendrian L' which is formally isotopic to L. Thus, by Murphy's h-principle[Mur12] L' and L are in the same isotopy class of Legendrian embeddings. □

In the higher dimensional situation, loose Legendrian submanifolds have the following flexible property :

Theorem 1.25. *[Mur12] If $(M^{2n+1}, \xi), n \geq 2$ is a contact manifold, then any loose Legendrian embedding satisfies a parametric h-principle. This means that*

(i) for every formal Legendrian embedding $f : L \hookrightarrow M$, there exists a loose Legendrian embedding f' close to it ,

(ii) for any two loose Legendrian embeddings f_0 and f_1, if there is a formal Legendrian isotopy connecting them, eventually f_0 and f_1 are Legendrian isotopic.

Remark 1.26. We can refer the definition of formal isotopies in Chapter 3. Notice that we assume $n \geq 2$ here, otherwise, the theorem is no longer true. However, the same technique still works and produces the following folklore result : if two Legendrian knots are formally Legendrian isotopic, they become Legendrian isotopic after sufficiently many stabilizations.

One of the main ideas for proving the above theorem is to consider wrinkled Legendrian embeddings. Roughly, we could say that loose Legendrians come from wrinkled Legendrians by resolving wrinkles.

Definition 1.27. [EM11](**Wrinkled embeddings**) Let $W : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a smooth, proper map, which is a topological embedding. Suppose W is a smooth embedding away from a finite collection of spheres $\{S_j^{n-1}\}$. Suppose, in some coordinates near these spheres, that W can be parametrized by

$$W(u, v) = (v, u^3 - 3u(1 - |v|^2), \frac{1}{5}u^5 - \frac{2}{3}u^3(1 - |v|^2) + u(1 - |v|^2)^2,$$

where our domain coordinates lies in a small neighborhood of the sphere $\{|v|^2 + u^2 = 1\} \subset \mathbb{R}^n$. Then W is called a *wrinkled embedding*, and the spheres S_j^{n-1} are called the *wrinkles*.

Definition 1.28. [Mur12](**Wrinkled Legendrians**) Let Y^n be a closed and connected manifold and (M^{2n+1}, ξ) be a contact manifold. A wrinkled Legendrian is a smooth map $L : Y \to M$, which is a topological embedding, satisfying the following properties : The image of dL is contained in ξ everywhere and dL is full rank outside a subset of codimension 2. This singular set is required to be diffeomorphic to a disjoint union of $(n-2)$-spheres $\{S_j^{n-2}\}$, whose images are called *Legendrian wrinkles*. We assume the image of each S_j^{n-2} is contained in a Darboux chart U_j, so that the front projection of $L(Y) \cap U_j$ is a wrinkled embedding, smooth outside of a compact set. (In particular, the front projection of each Legendrian wrinkle is the unfurled swallowtail singularities of a single wrinkle in the front.)

Definition 1.29. [Mur12](**twist marking**) Let $L : Y \to (M, \xi)$ be a wrinkled Legendrian embedding, and $\{S_j^{n-2}\}$ be the set of singular spheres. Let $N \subset Y$ be a submanifold with $\partial N = \cup_j S_j^{n-2}$. Denote $\Phi := L|_N$. Then (Φ, N) is called a *twist marking*.

Remark 1.30. We will put the C^∞-topology on the space of wrinkled Legendrian embeddings. Thus we can talk about a smooth family of wrinkled embeddings (L_t, Φ_t, N_t).

We can resolve wrinkles along twist markings as in [Mur12].

Theorem 1.31. *Let L_t^w be a smooth family of wrinkled Legendrian embeddings, let (Φ_t, N_t) be the twist markings. Then there is a smooth family of Legendrian embeddings L_t, such that L_t is identical to L_t^w*

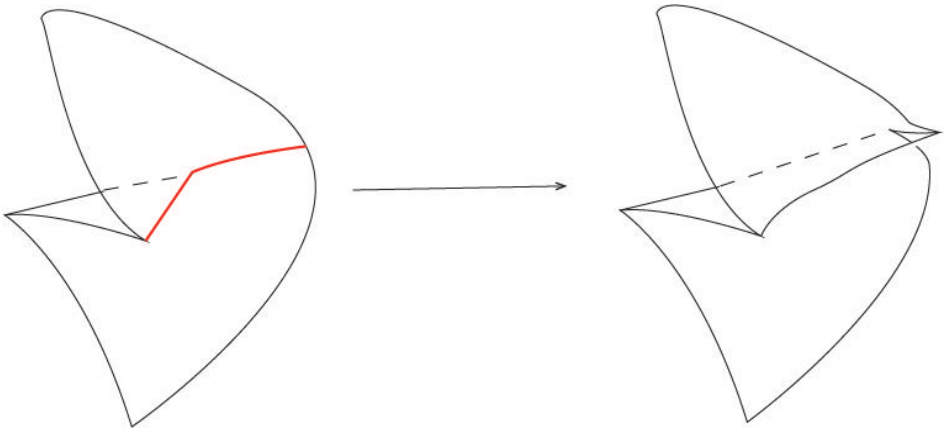

FIGURE 1.2 – Local resolution of a wrinkle.

outside of any small neighborhood of N_t for all t. Also, the resolution L_t can be taken to be as C^0 close as we want from L_t^w.

2

Elementary constructions in lower dimensions

In this chapter we construct positive loops of Legendrian embeddings for loose Legendrian surfaces in an elementary way. There are at least two advantages : firstly, we can get a simple explanation why we need the looseness assumption ; secondly, it will inspire us to find the idea for solving the general problem.

2.1 Positive Legendrian isotopy in dimension one

We start with the elementary case of dimension 1 Legendrian knots in 3-dimensional contact manifolds that are stabilizations. A positive loop is found in a semi-local neighborhood of the knot L, i.e. a Weinstein neighborhood of the destabilization of L.

First recall that there is no local positive loop :

Proposition 2.1. *[CFP10] Let L be the zero section of $J^1(\mathbb{S}^1), \xi_{std})$. Then there does not exist a positive loop of Legendrian embeddings based in L.*

It is a critical observation in [CFP10] that the slopes of a Legendrian knot can all be made to be positive (or negative) when it contains a Zigzag. With some preparations we can construct positive loop for stabilized Legendrian knots.

Definition 2.2. A Legendrian embedding $L : \mathbb{S}^1 \hookrightarrow (J^1(\mathbb{S}^1), \xi_{std})$ is said to be loose if it is one stabilization of the zero-section.

Proposition 2.3. *Let $L : \mathbb{S}^1 \hookrightarrow (J^1(\mathbb{S}^1), \xi_{std})$ be a loose Legendrian embedding whose front have positive slopes everywhere. Then there exists a positive Legendrian loop based in L.*

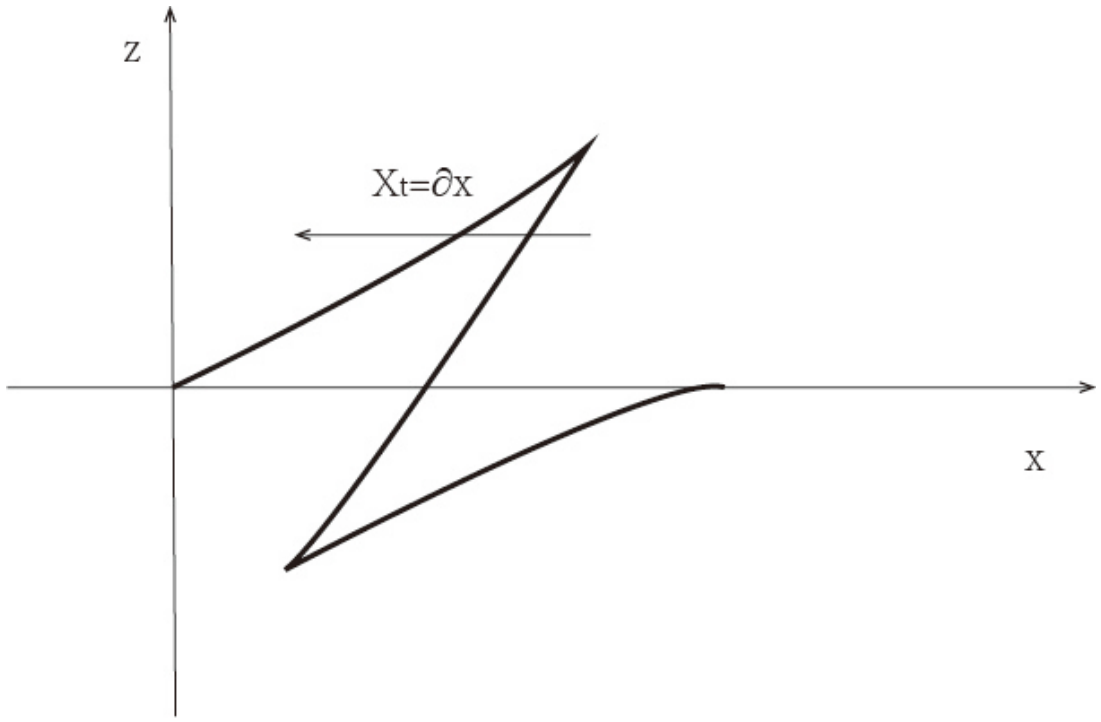

FIGURE 2.1 – Rotation

Proof: Regard $\mathbb{S}^1$ as $\mathbb{R}/\mathbb{Z}$ with coordinate x. Let $L_F : \mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{R}$ be the front of L. Denote $Z = L_F(\mathbb{S}^1)$. On Z, the slopes $\partial z/\partial x > 0$ are positive. Consider the vector field $v := -\partial_x$ on $J^1(\mathbb{S}^1)$ and its flow φ_t.

Because $\alpha(v) > 0$ on $\varphi_t(Z)$ for every $t \in [0, 1]$, then φ_t is a positive Legendrian isotopy. Since $\varphi_1 = Id$, then we have a positive loop.

□

Remark 2.4. If the front of L has negative slopes everywhere, we can choose $v = \partial_x$ so that its flow is a positive loop.

Notation conventions : Let $L : Y \hookrightarrow (J^1(Y), \alpha)$ be a smooth Legendrian embedding. We denote its front map by $L_F : Y \to Y \times \mathbb{R}$. If $\phi_t : Y \to Y \times \mathbb{R}$ is an isotopy with $\phi_t(Y)$ transverse to the $\mathbb{R}$ factor, we denote $\widetilde{\phi}_t$ its Legendrian lift and write v_{ϕ_t} and $v_{\widetilde{\phi}_t}$ the corresponding generating time dependent vector fields.

Given a loose Legendrian curve, we can change the front projection such that there is only one zigzag on the front. However, we present Lemma 2.5 to warm up.

Lemma 2.5. *Let $L : \mathbb{S}^1 \hookrightarrow (J^1(\mathbb{S}^1), \xi_{std})$ be a smooth loose Legendrian embedding. Then there is an isotopy φ_t of $L_F(\mathbb{S}^1)$ such that the slopes of $\varphi_1 (L_F(\mathbb{S}^1))$ are positive or negative everywhere.*

Proof: Let $Z = L_F(\mathbb{S}^1)$ in the (x, z)-plane. Without loss of generality, we may assume there is at least one positive zigzag and, to symplify the presentation, we also assume there is only one negative[1] zigzag that we denote Λ.

Then we construct φ_t in the following steps.

Step 1. **Stretch the zigzags** by φ_t^1 to make the positive zigzags large and the negative one small. That is $\varphi_1^1(x, z) = (cx, cz)$ restricted to the positive stabilization and $\varphi_1^1(x, z)|_\Lambda = (\frac{1}{c}x, \frac{1}{c}z)$ for a given a constant $c > 1$.

[1] it means the slopes are negative.

Step 2. **Rotate** the zigzag Λ by φ_t^2 to make the slopes positive. For example, take $\phi_t^1(w) = e^{i\theta t}w$, with $w = x + iz$. Clearly Λ will not be vertical if θ is not too large.

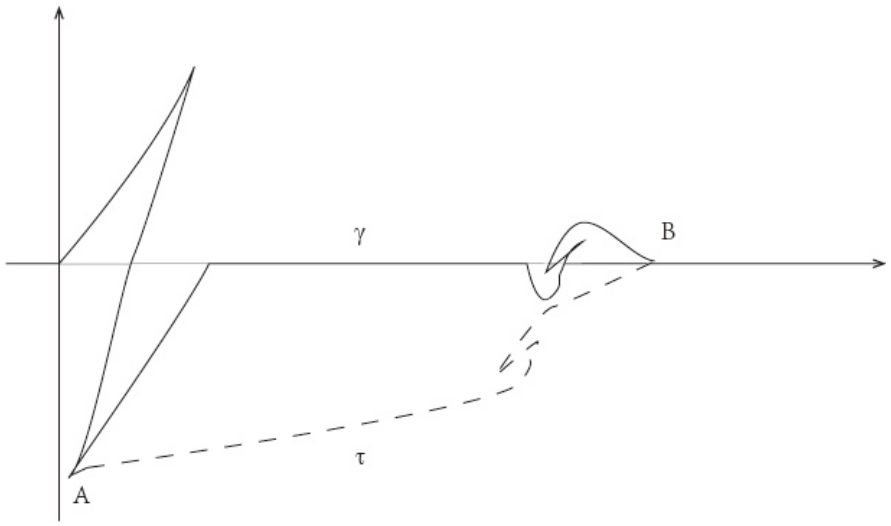

FIGURE 2.2 – Adjust other parts of Z.

Step 3. Adjust other parts of Z by φ_t^3, which is just a translation. See figure 2.2.
Then $\varphi_t = \varphi_t^3 \circ \varphi_t^2 \circ \varphi_t^1$ makes the slopes of Z positive.

□

Immediately we have the main result of this section.

Proposition 2.6. *Let $L : \mathbb{S}^1 \hookrightarrow (J^1(\mathbb{S}^1), \xi_{std})$ be a smooth loose Legendrian embedding. Then there exists a positive loop of Legendrian embeddings based in L.*

Remark 2.7. Proposition 2.6 follows directly from Proposition 2.3. We give a more involved proof which will be helpful to understand the dimension two case.

Proof: Let $Z = L_F(\mathbb{S}^1)$ in the (x, z)-plane. Then we construct an isotopy for Z in the following steps.

Step 1. **Make the slopes of Z positive** by ψ_t^1 according to Lemma 2.5. Denote $Z_1 = \psi_1^1(Z)$.

Step 2. **Move Z_1 up** by ψ_t^2 such that $\psi_t^2 \circ \psi_t^1$ lifts to a positive Legendrian isotopy. Take a number $k_1 > \max\left(|\alpha_F(v_{\psi_t^1})|\right)$, then set $v_{\psi_t^1} = k_1\partial_z$, so $\psi_t^2 \circ \psi_t^1$ is a desired isotopy. Denote $Z_2 = \psi_1^2(Z_1)$.

Step 3. **Move Z_2 down** positively by ψ_t^3 to Z_3. Precisely take $v_{\psi_t^3} = -k_2\partial_z - n\partial_x$, with $0 \ll n \in \mathbb{N}$ and $0 < k_2 < n\min(slope(Z_2))$. Here $slope(Z_2)$ denote the slopes of Z_2. We can see that Z_3 is a copy of Z_2 lying below Z_2 in z coordinate for a large k_2.

Step 4. **Change the slopes of Z_3** by ψ_t^4 such that the shape of $Z_4 = \psi_1^4(Z_3)$ is the same as Z, that is $slope(Z_4) = slope(Z)$. By Lemma 2.5 such a ψ_t^4 exists.

Step 5. **Move Z_4 up** back to Z by ψ_t^5 such that $\psi_t^5 \circ \psi_t^4$ corresponds to a positive Legendrian isotopy as *Step 2*.

Thus, $(\psi_t)_{t\in[0,1]} = (\psi_t^2 \circ \psi_t^1)_{t\in[0,1]} * (\psi_t^3)_{t\in[0,1]} * (\psi_t^5 \circ \psi_t^4)_{t\in[0,1]}$ corresponds to a positive loop of Legendrian embeddings based in L. Here $\circ$ means composition and $*$ means concatenation of paths in the space of embeddings. □

We can see that if $L(\mathbb{S}^1)$ stays in a small neighborhood of the zero section of $J^1(\mathbb{S}^1)$. Then the isotopy may be supported in that neighborhood.

2.2 Positive Legendrian isotopy in dimension two

We generalize the strategy for knots to the case of surfaces. We start with the simplest but non trivial case.

2.2.1 Construction for $\mathbb{S}^2$ case

We endow the 2-sphere $\mathbb{S}^2$ with the usual spherical coordinates (θ, β). According to Definition 1.23, given a smooth Legendrian sphere in $(J^1(\mathbb{S}^2), \xi_{std})$, we say it is loose if the front contains $Z \times \mathbb{S}^1$. See figure 2.3.

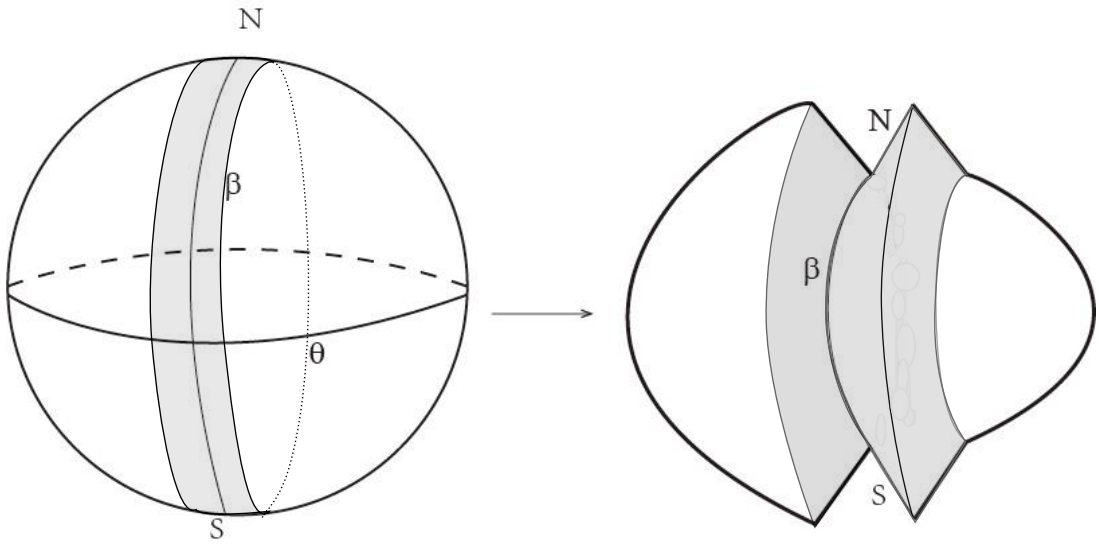

FIGURE 2.3 – The front of a loose Legendrian sphere.

It is convenient for us to represent a Zigzag in coordinates, that is on the cylinder $U = \{(\theta, \beta)| - \delta < \theta < \delta\} \subset \mathbb{S}^2$ the image $L_F(U) = Z \times \mathbb{S}^1$ where Z is contained in (θ, z)-plane. Then on each circle $\mathbb{S}^1_\beta = \{(\theta, \beta)|\beta \neq 0, \pi\}$, there are two zigzags with opposite slopes, where $\beta = 0$ and $\beta = \pi$ correspond to the north and south pole respectively. Here slopes means $\partial z/\partial\theta$. We will put $\mathbb{S}^1_\beta$ in (θ, z)-plane. where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ by rescaling. See figure 2.4.

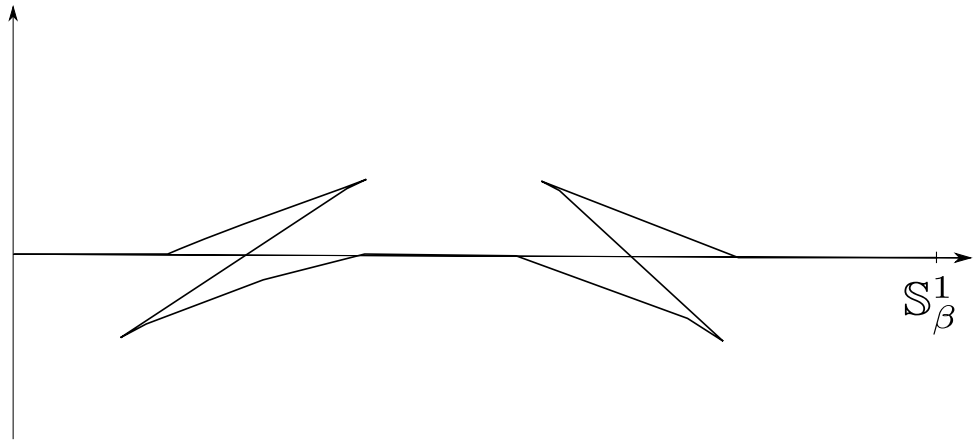

FIGURE 2.4 – See $\mathbb{S}^1_\beta$ in (θ, z) plane.

Notation conventions : We denote $D_\delta\left(J^1(\mathbb{S}^2)\right) \subset (J^1(\mathbb{S}^2), \xi_{std})$ a disk bundle with $\mathbb{D}^3_\delta$ fiber where δ is the radius of the disk. For some $\delta' < \delta$, we denote $L^w_{\delta'} : \mathbb{S}^2 \hookrightarrow D_\delta(J^1(\mathbb{S}^2)$ a wrinkled Legendrian embedding

with one wrinkle $S = \{(\theta, \beta) \mid \theta \in [-1, 1]\}$ such that $0 \leq Slope(L^W_{\delta'}(\mathbb{S}^1_\beta)) \leq \delta'$ and be zero exactly when $\beta = 0$ and $\beta = \pi$. We will write $L^w_{\delta'}$ as L^w when there is no confusion.

Take the *twist marking* $N = \{(\pi, \beta)\}$. According to Theorem 1.31, for any $\eta > 0$, let W_η^{-1} be the operation of resolving the wrinkle along N such that $\| W_\eta^{-1}(L^w) - L^w \|_{C^0} \leq \eta$.

Now we state the main result in this section.

Theorem 2.8. *Given $\delta > 0$, let $L : \mathbb{S}^2 \hookrightarrow D_\delta(J^1(\mathbb{S}^2)$ be a loose Legendrian embedding. Then there exists a positive loop of Legendrian embeddings based in L.*

Remark 2.9. Roughly speaking, let us regard L as $W_\eta^{-1}(L^w)$. Then the above Theorem can be proved in two steps :

(i). Construct a loop of contact isotopy φ^1_t which is non-negative on L^w and as positive as we want outside a small neighborhood D of the poles.

(ii). Construct a contact isotopy φ^2_t which is supported in a neighborhood of the poles and positive in a smaller neighborhood $D' \supseteq D$. Thus, if φ^1_t is positive enough, the composition $\varphi^2_t \circ \varphi^1_t$ is a positive isotopy of wrinkled Legendrians based in L^w.

Then, for η small enough, the composition $W_\eta^{-1} \circ \varphi^2_t \circ \varphi^1_t$ is a positive loop based in L.

We explain how to obtain a loop which is positive near the poles in Lemma 2.11. First of all, we study the dimension one case in Lemma 2.10.

Lemma 2.10. *Given constants $\delta > 0$ and $a \in (0, 1)$, let $L : D^1 \hookrightarrow D_\delta(J^1(D^1)$ be the zero-section. Then, for any $0 < \epsilon < \delta$, there exist a loop of Legendrian embeddings $\widetilde{\phi}_t : D^1 \hookrightarrow D_\delta(J^1(D^1)$ such that :*

(i). $\widetilde{\phi}_t|_{\partial D^1} = L|_{\partial D^1}$ *and* $\| \widetilde{\phi}_t - L \|_{C^0} < \epsilon$,

(ii). $\alpha(v_{\widetilde{\phi}_t})|_{D(a)} > 0$.

We define a box $B(a, b) := [-a, a] \times [-b, b]$.

Proof: Let τ be the coordinate for D^1 and (x, y, z) the coordinates for $D_\delta(J^1(D^1))$, thus, $L_F(\tau) = (\tau, 0)$. Let's construct ϕ_t for L_F such that its lift satisfies the above conditions.

Given a family of parametric curves $\tau \mapsto (x_t(\tau), z_t(\tau)$, define the slopes by $A_t(\tau) = \frac{z_t(\tau)\prime}{x_t(\tau)\prime}$. Denote $Z = L_F([-1, 1])$ and $\Lambda = L_F([-a, a])$. Now we will move Z in the following steps and fix our attention on $B_0 = B(a, \delta a)$.

Step 1. **Rotate Λ counterclockwise to Λ_1** by ϕ^1_t such that $Slope(\Lambda_1) < \epsilon$. More precisely, denote (x_0, z_0) the center of B_0, define $r = sign(x - x_0)\sqrt{(x - x_0)^2 + (z - z_0)^2}$, where $sign(x)$ is the the sign function. Let $\theta \in (0, arc(\epsilon))$, in the box B_0, thus

$$R = \frac{-\theta t r}{\sqrt{\tan^2(\theta t) + 1}}(\tan(\theta t)\partial_x - \partial_z)$$

is the rotation of angle θ, we have to extend R to the rest of Z. Denote Z_t be this isotopy. We can choose it to be graphical with small slopes, since

$$z_t(a) = \int_1^a A_t(\tau) < \epsilon a,$$

we can just take such $Z_t[a,1]$ that $-\frac{\epsilon a}{1-a} < A_t(\tau) \leq 0$. Note that R is not positive. However, we have $R' = R + \epsilon\partial_z$ positive on Λ_1, because

$$\alpha(\widetilde{R}') = \epsilon - \frac{r\theta\tan\theta}{\sqrt{\tan^2\theta+1}} > \epsilon(1-a) > 0,$$

where $\widetilde{R}'$ is the lift of R'. Then we take $v_{\phi_t^1} = R'$. We can see that the norm $\| \widetilde{\phi}_t^1 - L \|_{C^0} < \epsilon$, since $|A_t(\tau)| < \epsilon$ and $\| \phi_t^1 - L \|_{C^0} < \epsilon$.

Let $Z^1 = \phi_1^1(Z), B_1 = \phi_1^1(B_0)$, see figure 2.5 for Z^1.

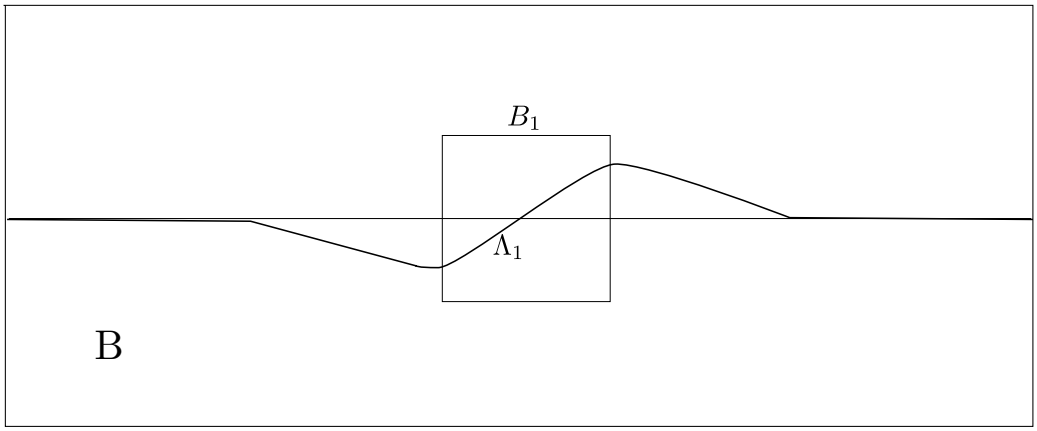

FIGURE 2.5 – Λ_1 lives in a small box B_1.

Step 2. **Move Λ_1 down to the left** positively by ϕ_t^2 near Z^1. Given a constant $0 < k < Slope(\Lambda_1)$, let $Z_t^2 = \phi_t^2(Z^1)$ such that

$$\begin{aligned} Z_t^2(\tau) &= (x_t(\tau), z_t(\tau)) \\ &= (x(\tau) - \epsilon\sigma(\tau)t, z(\tau) - \epsilon k\sigma(\tau)t) \end{aligned}$$

with $\sigma(\tau)$ being a C^0-close smoothing of the following function

$$\widetilde{\sigma}(\tau) = \begin{cases} \frac{1}{1-a}(\tau+1) & , \quad \tau \in [-1,-a] \\ 1 & , \quad \tau \in (-a,a) \\ \frac{-1}{1-a}(\tau-1) & , \quad \tau \in [a,1] \end{cases}$$

such that $\sigma(\pm 1) = 0$ and $|\sigma'(\tau)| \leq \frac{1}{1-a}$.

Let $\Lambda_2 = \phi_1^2(\Lambda_1), B_2 = \phi_1^2(B_1), Z^2 = \phi_1^2(Z^1)$. See figure 2.6.

Let us check that the lift $\widetilde{\phi}_t^2$ satisfies all the conditions.

Firstly, note that

$$A_t(\tau) = \frac{z'(\tau) - \epsilon k t\sigma'(\tau)}{x'(\tau) - \epsilon t\sigma'(\tau)}.$$

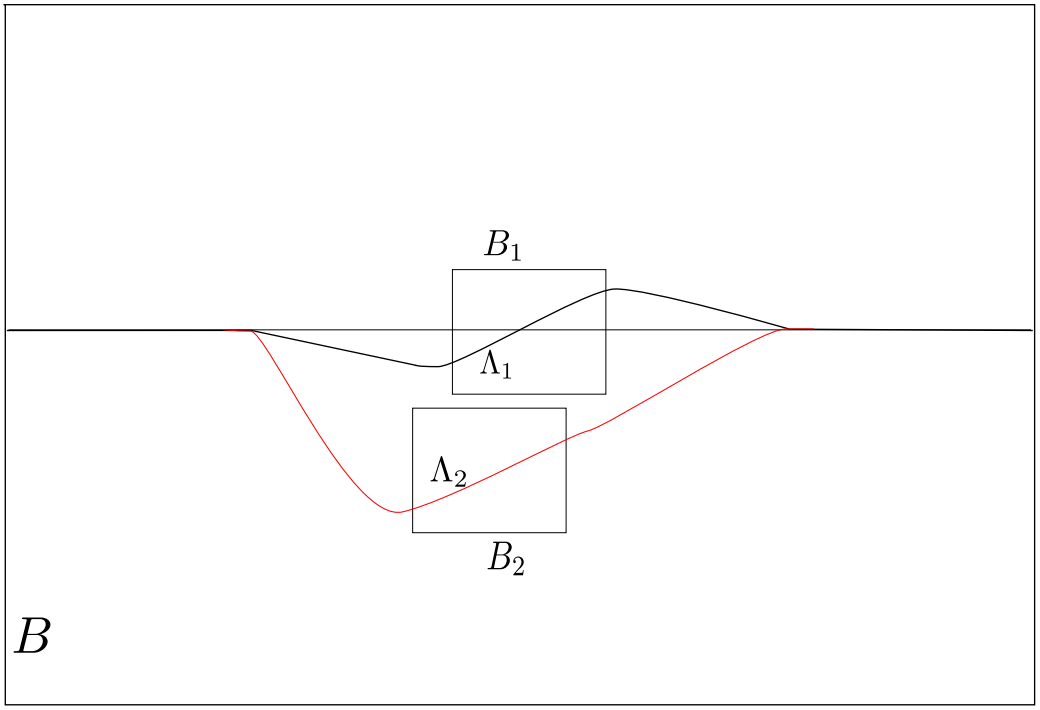

FIGURE 2.6 – Move B_0 down to the left.

Thus, for $|\tau| \leq a$, we have $|A_t(\tau)| = \tan\theta < \epsilon$; for $a < |\tau| \leq 1$, we have

$$|A_t(\tau)| < |A_0(\tau)| + \frac{2(|A_0(\tau)| + \epsilon)\epsilon}{1 - 2\epsilon} < \epsilon(4a\epsilon + 4\epsilon + a) < \epsilon.$$

Secondly,

$$\alpha(v_{\widetilde{\phi_t^2}}) = \epsilon\sigma(\tau)(A_t - k),$$

thus, we have $\alpha(v_{\widetilde{\phi_t^2}}) > 0$ for $\tau \in [-a, a]$.

Step 3. **Move Λ_2 down to the right** positively by ϕ_t^3. First of all, we repeat *Step 1*, that is to say, rotate Λ^2 to Λ' with opposite slopes and compose it with a upward move, denote it by ϕ_t'. Besides, similar to *step 2*, let us take $\phi_t'^3$ such that $v_{\phi_t'^3} = \epsilon(-\lambda(\tau)\partial_z + \mu(\tau)\partial_x)$ with $v_{\phi_t'^3} = \epsilon(-k\partial_z + 2\partial_x)$ on Λ'. Then take $\phi_t^3 = \phi_t'^3 * \phi_t'$. Let $\Lambda_3 = \phi_1^3(\Lambda_3), B_3 = \phi_1^3$.

Step 4. **Move B_3 back to B_0** positively by ϕ_t^4. Firstly, rotate Λ_3 clockwise to Λ_4 by X' such that the shape of Λ_4 is the same as Λ, then move the whole curve back to Z.

Summary. Define $\phi_t := \phi_t^1 * \phi_t^2 * \phi_t^3 * \phi_t^4$. It is a loop supported in a small Weinstein neighborhood. For a suitable choice of v_{ϕ_t}, we have $\alpha_F(v_{\phi_t}) > 0$ on Λ while $\alpha_F(v_{\phi_t}) > -c_0$ on the other part of Z for some constant $c_0 > 0$, since our isotopies are compactly supported. Figure 2.7 indicate the isotopy ϕ_t.

□

Now, we are ready to present the main two-dimensional lemma.

Lemma 2.11. *Given constants $\delta > 0$ and $a \in (0, 1)$, let $L : D^2 \hookrightarrow D_\delta(J^1(D^2)$ be the zero-section. Then, for any $0 < \epsilon < \delta$, there exist and a compactly supported loop of contactomorphisms $\widetilde{\phi}_t : D_\delta(J^1(D^2)) \to D_\delta(J^1(D^2))$ such that :*

(i). $\alpha(v_{\widetilde{\phi}_t})|_{L(D(a))} > 0$,

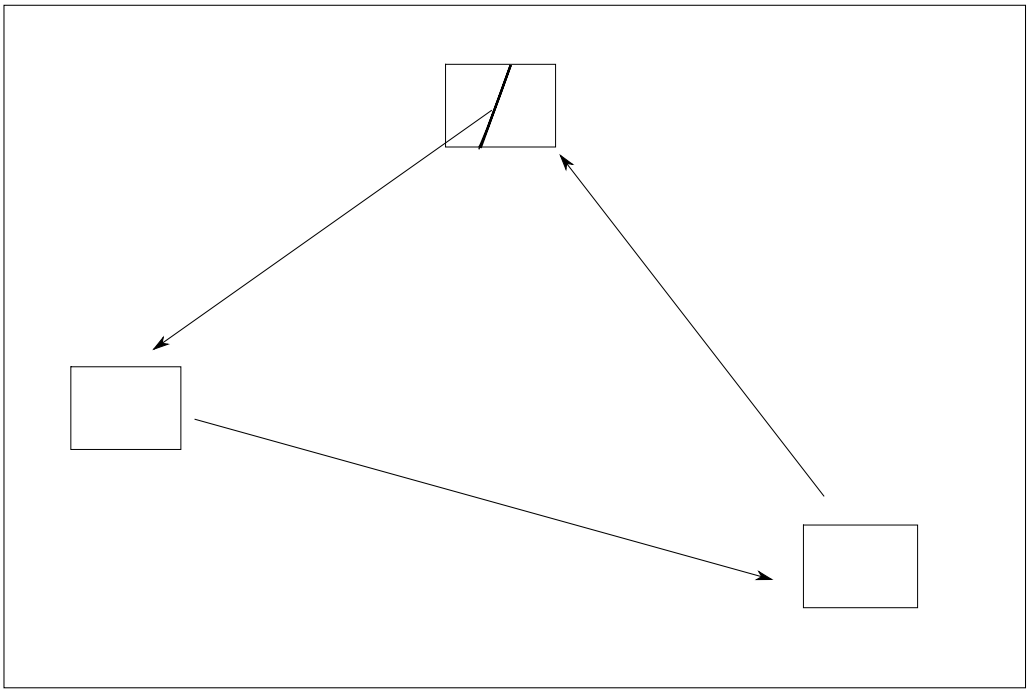

FIGURE 2.7 – Schematic diagram for ϕ_t

(ii). $(\widetilde{\phi}_t \circ L)|_{\partial D^2} = L|_{\partial D^2}$ *and* $\| \widetilde{\phi}_t \circ L - L \|_{C^0} < \epsilon$.

Note that for every singular Legendrian L' close to L, the properties (i) and (ii) still hold if L is replaced by L'.

Proof: Regard D^2 as a disk with radius $\sqrt{2}$ in $\mathbb{R} \times \mathbb{R}$ with coordinates (τ, u), let (x, u, y, w, z) be the coordinates for $D_\delta(J^1(\mathbb{D}^2)$, write $\alpha = dz - ydx - wdu$. We will construct a loop $\widetilde{\phi}_t$ of Legendrian embeddings. This can be extended to a loop of contactomorphisms.

Step 1. Let $Z = L_F(\tau, 0)$, we have a loop $Z_t(\tau) = (x_t(\tau), z_t(\tau))$ based in Z satisfying the conditions of lemma 2.10.

Step 2. Take a cut-off function $f(u)$ such that $f(u) = 1$ on $[-a, a]$ and $max|f'(u)| \leq \frac{1}{1-a}$. Then define $Z_t(\tau, u) = (f(u)x_t(\tau), u, f(u)z_t(\tau))$.

Step 3. Extend $Z_t(\tau, u)$ to a loop of diffeomorphisms in the front, then lift it to a loop of contactomorphisms $\widetilde{\phi}_t$. We can see $\widetilde{\phi}_t$ satisfies all the conditions.

□

Let $D_N(r)$ and $D_S(r)$ be disks of radius r around the pole N and S respectively. Now, we are ready to prove Theorem 2.8.

Proof: We assume there is only one Zigzag for L. Let us regard L as $W_\eta^{-1} \circ L^w_{\delta'}$ for some constant $\delta' < \delta$. Then, we will proof the theorem in the following steps.

Step 1. Take a constant $\delta'' \in (\delta', \delta)$. For any $K > 0$, let $\widetilde{\psi}_t$ be the compactly supported contact isotopy such that $\widetilde{\psi}_t(\theta, \beta, z) = (\theta - 2\pi Kt, \beta, z)$ on $D_{\delta''}(J^1(\mathbb{S}^2))$. Note that $\alpha(\partial_t \widetilde{\psi}_t) = KSlope(L^w(\mathbb{S}_\beta))$. Thus, $\widetilde{\psi}_t$ can be arbitrary positive (depending on K) away from the poles.

Step 2. Given a constant $0 < a < 1$, according to Lemma 2.11, there exists a loop of contactomorphism $\widetilde{\phi}_t$ compactly supported in $D_\delta(J^1(D_N(\sqrt{2}) \cup D_S(\sqrt{2})))$ such that $\alpha(v_{\widetilde{\phi}_t})|_{L^w(D_N(a) \cup D_S(a))} > 0$, if L^w is close

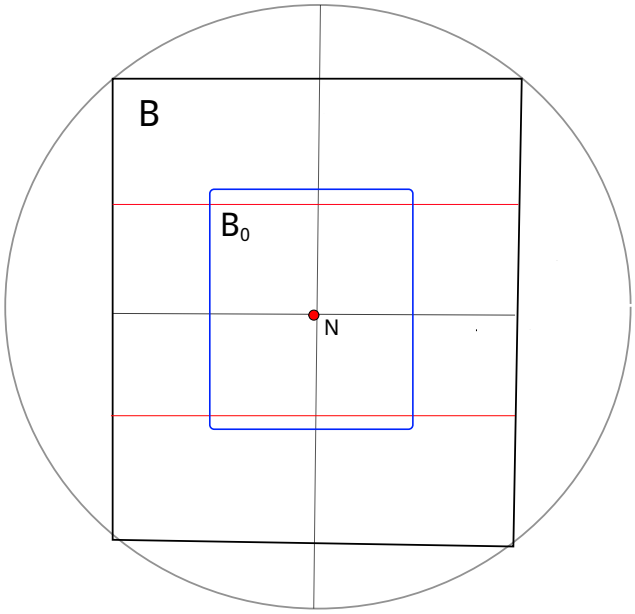

FIGURE 2.8 – Change of coordinates near the north pole.

to the zero-section. Note that there exist some k_1, $k_2 > 0$ independent of K such that $\alpha(v_{\widetilde{\phi}_t}) > -k_1$ and $\widetilde{\phi}_t^*\alpha > k_2\alpha$ since $\widetilde{\phi}_t$ is compactly supported.

Step 3. We make sure that $\widetilde{\phi}_t \circ \widetilde{\psi}_t$ is a positive loop based in L^w. Since

$$v_{\widetilde{\phi}_t\circ\widetilde{\psi}_t}(x) = v_{\widetilde{\phi}_t}(\widetilde{\psi}_t(x)) + d\widetilde{\phi}_t(v_{\widetilde{\psi}_t}(x)),$$

we have

$$\alpha(v_{\widetilde{\phi}_t\circ\widetilde{\psi}_t}(x)) = \alpha(v_{\widetilde{\phi}_t}(\widetilde{\psi}_t(x))) + \widetilde{\phi}_t^*\alpha(v_{\widetilde{\psi}_t}(x))$$

for $x \in D_\delta(J^1(\mathbb{S}^2))$. We can see that $\alpha(v_{\widetilde{\phi}_t\circ\widetilde{\psi}_t})|_{L^w(D_N(a)\cup D_S(a))} > 0$, since $\widetilde{\psi}_t \circ L^w$ stays in a small neighborhood of the zero-section. Besides, away from $D_N(a) \cup D_S(a)$ we have $\alpha(v_{\widetilde{\phi}_t\circ\widetilde{\psi}_t})|_{L^w} > -k_1 + k_2K$. Thus, for K large enough $\widetilde{\phi}_t \circ \widetilde{\psi}_t$ is a positive loop based in L^w (see figure 2.9 for the composition near N).

Step 4. Note that $W_\eta^{-1}\left(\widetilde{\phi}_t \circ \widetilde{\psi}_t(L^w)\right)$ is a loop based in L. It is C^0 close to $\widetilde{\phi}_t \circ \widetilde{\psi}_t(L^w)$ with respect to η. Thus it is also positive for η small enough, because positivity is an open condition with respect to the C^0-topology. □

Remark 2.12. For any $n \in \mathbb{N}^+$, let L be a loose sphere such that L_F contains n Zigzags for some front projection F. Then, there exists another front projection F' such that $L_{F'}$ contains only one Zigzag. Thus, it is enough to prove the case of one Zigzag.

We have another way to make a neighborhood of the poles in a positive loop : Take another $D^1 \times \mathbb{S}^1 \subset \mathbb{S}^2$ which contains a small neighborhood of the poles. Then we add a wrinkle on the front $\widetilde{\psi}_F(\mathbb{S}^2)$ along the image of $D^1 \times \{0\}$ for all t. We rotate the wrinkles fast in a small neighborhood of $\widetilde{\psi}_F(\mathbb{S}^2)$ for every t such that the poles are in a positive loop. After resolving wrinkles, we get a positive loop of Legendrian

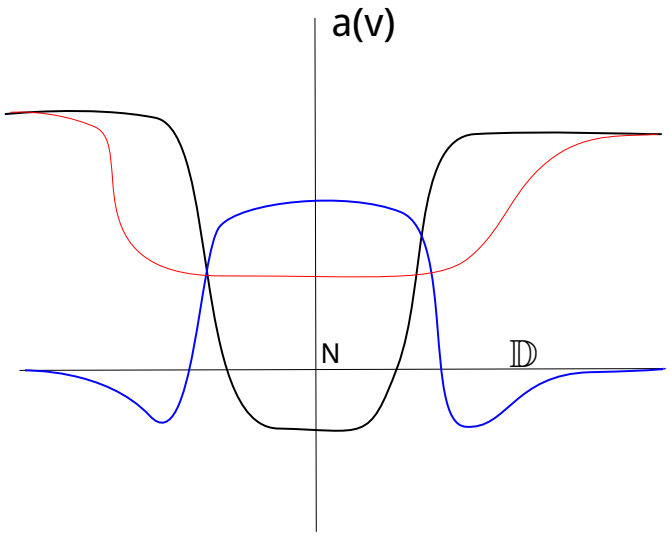

FIGURE 2.9 – Composition of isotopies

embeddings. Note that the new Legendrians are in the same Legendrian isotopy class of L.

3

Basic h-principle

In the case of surfaces, we can construct positive loops by hand. It is not so easy to repeat the same strategy in the higher dimensional situation. Luckily, positive Legendrian isotopy can be regarded as a differential relation, therefore we could hope to solve the problem by h-principle techniques.

A least, there are two facts that advocate in favor of those techniques :

(i) Looseness implies a flexible nature of the Legendrian.

(ii) Finding a positive loop of Legendrian embedding is a **global** problem, it can be localized near a formal solution.

In this chapter, we will recall the basic definitions and techniques in h-principle. The reference are [Gro86, EM02, Spr98, Bor].

3.1 Differential relations

Given a differentiable map $f : M \to N$, informally a differential relation on f are conditions on f and its differentials $d^r f$.

Example 3.1. *(i) differential equations,*

(ii) immersion(smooth, Legendrian),

(iii) curvature restrictions on Riemannian metrics.

For precise definition of differential relations, we need jet-space formalization.

Definition 3.2. Let M and N be manifolds, the 1**-jet space** $J^1(M,N)$ is defined to be the space

$$\{(x, f(x), F(x))\},$$

where $f : M \to N$ is a C^1-function and $F : T_xM \to T_{f(x)}N$ is a linear map.

Example 3.3. $J^1(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times \mathbb{R}^n \times M_{m\times n}(\mathbb{R})$.

With the language of jet-spaces, we can formulate the definition of a differential relation.

Definition 3.4. Let M and N be manifolds. A first-order **differential relation** $\mathcal{R}$ on the first-order differentiable functions $C^1(M,N)$ is defined to be a subset of $J^1(M,N)$. The relation $\mathcal{R}$ is called **open** if it is an open subset of $J^1(M,N)$.

A **formal solution** of the differential relation $\mathcal{R}$ is a point $(f,F) \in \mathcal{R}$. Such a point (f,F) is called a **solution** if $F = df$.

We will only consider first-order differential relations.

It is interesting to interpret a formal solution in a geometric way, which will help us to find a real solution. For example,

Example 3.5. *Let $\mathcal{R}$ be the relation of immersion of $\mathbb{R}$ into $\mathbb{R}^2$, figure 3.1 represents a formal solution.*

FIGURE 3.1 – Formal solution : geometric interpretation.

3.1.1 Idea of h-principle

When a problem is formulated in term of a differential relation $\mathcal{R}$, solving the problem means to find a real solution. A way to do it is to start from a formal solution and try to deform it into a real one (see figure 3.2). Luckily, in some good situations that is possible.

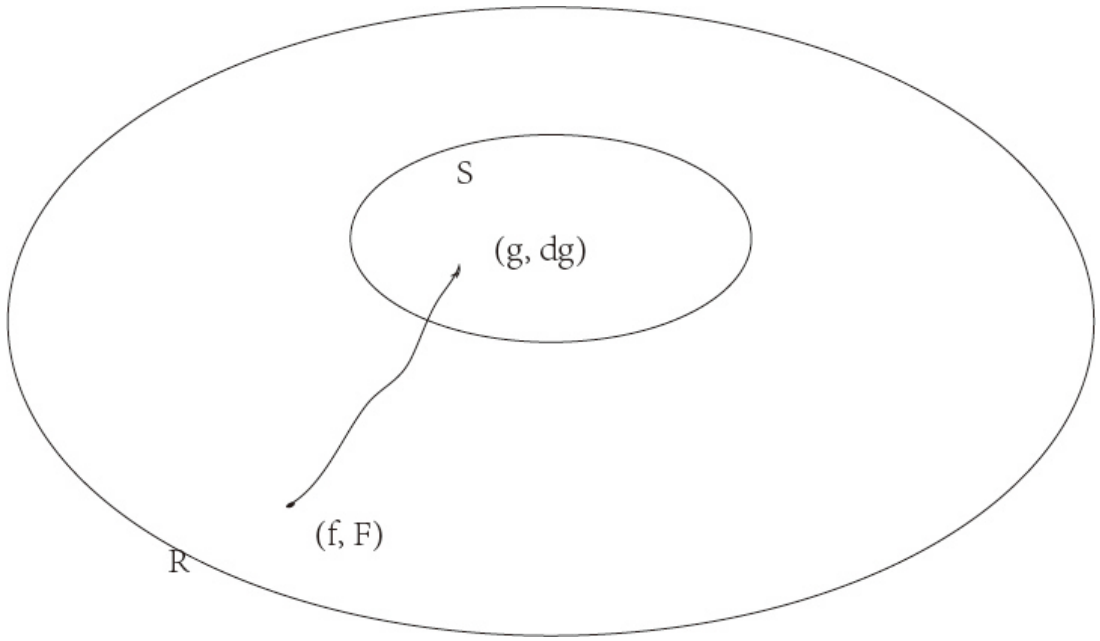

FIGURE 3.2 – homotopy to a solution.

We have the following formal definition of h-principle.

Definition 3.6. Let $\mathcal{R}$ be a differential relation, $S \subset \mathcal{R}$ the subset of real solutions, then we say $\mathcal{R}$ satisfies **h-principle** if S and $\mathcal{R}$ are homotopy equivalent. For example

(i) every formal solution $(f, F) \in \mathcal{R}$ is homotopic to a solution $(g, dg) \in S$,

(ii) any two solutions homotopic as formal solutions can be connected by a homotopy of solutions.

Up to now, there are at least two general ways to do h-principle : holonomic approximation and convex integration. We will present the basic ideas and results in the following sections.

3.2 Holonomic approximation

Let $\mathcal{R}$ be a differential relation, we want to know whether there is a real solution. A natural idea is a priori to take a formal solution and then to find a solution which is close to it. A simple example (see picture 3.1) shows this is not quite possible. However, this idea is still interesting and useful.

3.2.1 Holonomic approximation without singularities

Although, given a general formal solution (f, F) for a differential relation $\mathcal{R}$ there is no hope to have a global solution almost tangent to F. It is sometime possible to find one near a codimension one skeleton, that is Y.Eliashberg and M.Mishachev's Holonomic Approximation Theorem.

Theorem 3.7 (**Holonomic Approximation Theorem**). *[EM02, Gro86] Let $\mathcal{R}$ be an open differential relation in $j^1(M^m, N^n)$, let $(f, F) \in \mathcal{R}$ be a formal solution. Then there exists a formal solution $(f', F') \in \mathcal{R}$ and a triangulation Δ of M, such that*

(i) F' *is* C^0 *close to* F,

(ii) (f', F') *is a solution near* $(m-1)$*-skeleton of* Δ.

Remark 3.8. It is easy to image the above theorem like the following way : Think about a road winding up to a mountain, the horizontal planes to the boundary surface of the mountain can be regarded as a formal solution. The road going up to the mountain can be chosen to be arbitrarily close to the horizontal planes.

The above theorem says that given a differential relation $\mathcal{R}$, we always have a solution outside some balls of M, if we consider natural relations, i.e. invariant by diffeomorphisms. . It also means that for an open manifold M, basically, a solution always exists .

The relation $\mathcal{R}$ being open is a technical condition, otherwise we have to take a limiting process.

When (f', F') approximates (f, F), it does not mean f and f' are C^0-close, usually they are not.

3.2.2 Holonomic approximation with wrinkles

Usually, the source manifold M is closed and there is no trick to make an open extension for M. If we still want to do holonomic approximation then we have to allow singularities (see figure 3.3).

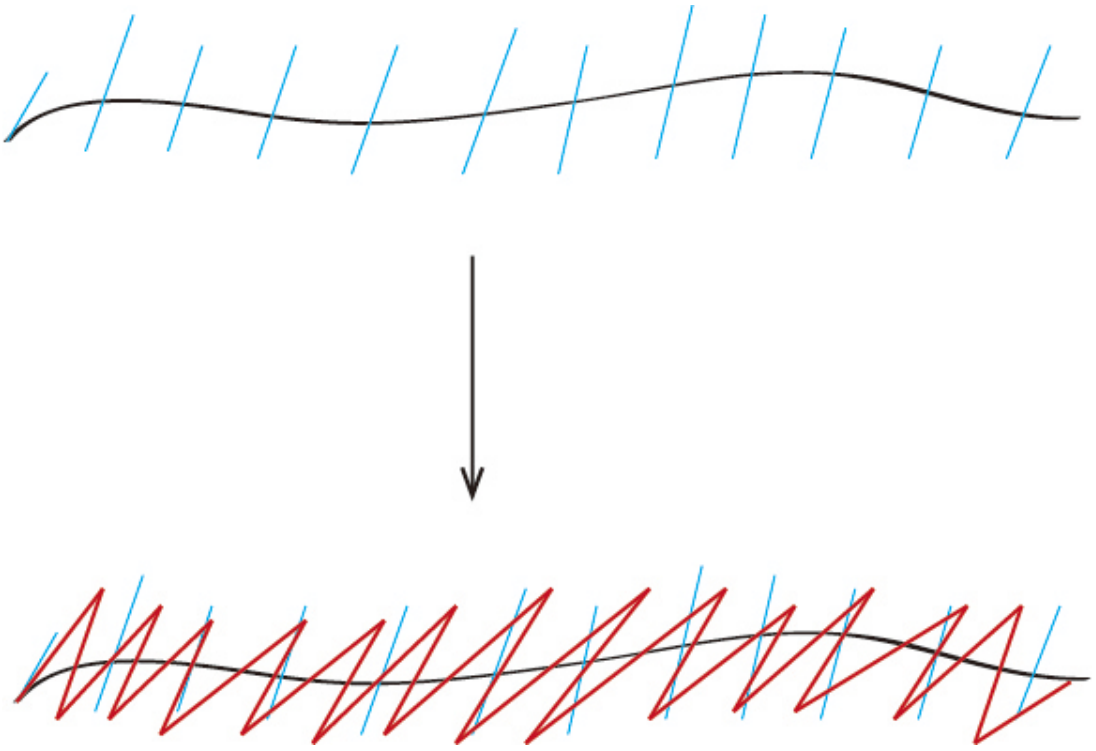

FIGURE 3.3 – Approximation with singularities.

In that case, it is possible to introduce only simple singularities, that is to say folds and cusps. We have the following generalized Holonomic Approximation Theorem of Y.Eliashberg and M.Mishachev.

Theorem 3.9. *[EM11](**Wrinkled Holonomic Approximation Theorem**) Let $\mathcal{R}$ be an open differential relation in $j^1(M^m, N^n)$ with $n > m$, and $(f, F) \in \mathcal{R}$ a formal solution. Then there exists a wrinkled solution $(f', F') \in \mathcal{R}^1$ such that F and F' are C^0-close.*

Remark 3.10. We can see the above theorem more closely in the figure 3.4. All singularities are on the boundary sphere of a highest dimensional disk and the cusps are on a codimension two sphere.

The above theorem is crucial for us to construct positive Legendrian isotopy.

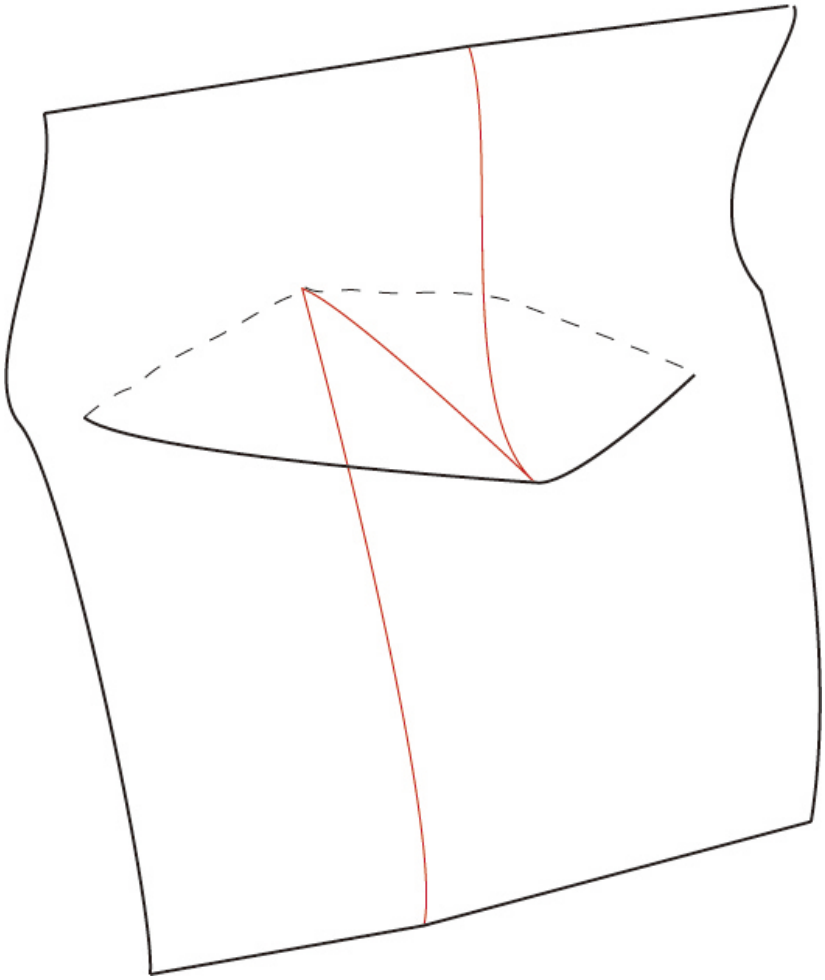

FIGURE 3.4 – A wrinkle.

3.3 Convex integration

Holonomic approximation is not sufficient to treat our problem. We need another technique, called convex integration.

It is another natural way to do h-principle : given a differential relation $\mathcal{R} \subset J^1(M, N)$ and a formal solution (f, F), instead of finding a real solution (g, dg) with dg close to F, we can try to find a map g such

[1] f' is a wrinkled embedding

that f and g are C^0-close and $(g, dg) \in \mathcal{R}$. Notice that g and f are usually not C^1-close, otherwise, (f, F) itself a solution when $\mathcal{R}$ is open. The author benefited a lot from Vincent Borrelli [Bor] when learning convex integration and the presentation here is mainly from his online lecture notes.

Intuitively, g should oscillate around f . Let's see the following example.

Example 3.11. *Let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^3)$ be a differential relation whose fiber is*

$$\mathcal{R}_x = \{v \in \mathbb{R}^3 \mid |\angle(v, \partial_z)| > \frac{\pi}{4}\}.$$

Let $f(x) = (0, 0, x)$, then the map $g(x) = (k \sin(2N\pi x), k \cos(2N\pi x), x)$ satisfies $\mathcal{R}$ for large N and k.

The relation $\mathcal{R}$ should be some special subset of $J^1(M, N)$ so that solutions exist. A special case is when $\mathcal{R}$ is ample.

3.3.1 H-principle for ample relations

We recall the following definitions before stating the h-principle for ample differential relations.

Definition 3.12. Let A be a subset of $\mathbb{R}^n$, and $a \in A$. The **convex hull** $Conv(A, a) \subset A$ containing a is defined to be the union of all simplexes in $\mathbb{R}^n$ with vertex in the component of A containing a. We denote $IntConv(A, a)$ the interior of $Conv(A, a)$. A is said to be **ample** if for every point $a \in A$ we have $IntConv(A, a) = \mathbb{R}^n$.

We talk about ample relations in $J^1(M, N)$. Locally, we can identify $J^1(M, N)$ with

$$J^1(U, V) = U \times V \times \prod_{i=1}^{m} \mathbb{R}^n,$$

where U and V are charts of M and N. Denote $J^1(U, V) = \{(x, y, v_1, v_2, \cdots, v_m)\}$. Let $J^1(U, V)^\perp = \{(x, y, v_1, v_2, \cdots, v_{m-1})\}$, thus $J^1(U, V) = J^1(U, V)^\perp \times \mathbb{R}^n$. Therefore we have the following diagram

$$\begin{array}{ccc} \mathcal{R}_{U,V} & \xrightarrow{i} & J^1(U, V) \\ & & \downarrow P^\perp \\ & & J^1(U, V)^\perp \end{array}$$

where $P^\perp$ is the obvious projection.

Definition 3.13. Let $\mathcal{R}$ be a differential relation in $J^1(M, N)$. We say $\mathcal{R}$ is **ample** if for every local identification $J^1(U, V)$ and every $z \in J^1(U, V)^\perp$, the subset $P^{\perp -1}(z) \cap \mathcal{R}_{U,V}$ is ample in $\mathbb{R}^n$.

Example 3.14. *The differential relation of immersion from M^m to N^n is ample if $m < n$.*

Let's state the h-principle for ample differential relations.

Theorem 3.15. *[Gro86] Let $\mathcal{R} \subset J^1(M, N)$ be an open and ample differential relation, then it satisfies a parametric h-principle.*

In practice, our differential relations are usually not ample, because the ample condition is too strong, for example, the conditions for PDEs are hardly ample. Besides, given a formal solution $(f, F) \in \mathcal{R}$, our aim is to construct a solution g such that g and f are C^0-close. Essentially, the existence of such g depends on the shape of $\mathcal{R}$ and the position of (f, f') with respect to $\mathcal{R}$. A concrete situation is when (f, f') is *surrounded* by $\mathcal{R}$. We will explain this in detail in the following subsection.

3.3.2 Fundamental lemma

First of all, we give an example to see the importance of being *surrounded*.

Example 3.16. *Let f be the map as in example 3.11. We have a constant family of loops*

$$\begin{array}{rcl} h : [0,1] \times \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathcal{R} \\ (x, s) & \longmapsto & (2\pi N k \cos(2\pi s), -2\pi N k \sin(2\pi s), 1)\,. \end{array}$$

We can see that the loop h_x surrounds the point $(f(x), f'(x)) \in \mathcal{R}$. We also have the important integral representation

$$\int_0^1 h_x(s)ds = f'(x).$$

Then we can define

$$F(x) = f(0) + \int_0^x h_u(Nu)du,$$

and check that $F(x) = g(x)$ where $g(x)$ is the same one in example 3.11.

Generally, if the family of surrounding loops $h_u(s)$ exists for (f, f'), and

$$\int_0^1 h_u(s)ds = f'(x),$$

then we can define

$$g(x) := f(0) + \int_0^x h_u(s(u))du,$$

where $s(u)$ is a suitable parametrization of s, such that g is a solution C^0-close to f.

Therefore, it is time to talk about the existence of surrounding loops $h_u(s)$ with an integral representation property.

Definition 3.17. Let $h : [0, 1] \to \mathbb{R}^n$ be a loop, and $z \in \mathbb{R}^n$ be a point. If

$$z \in IntConv(h[0, 1]),$$

we say z is strictly surrounded by h.

We now present the fundamental lemma of convex integration.

Lemma 3.18 (integral representation lemma)**.** *Let $\mathcal{R}^n \subset \mathbb{R}^n$ be an open subset, $z_0 \in \mathcal{R}$ and $z \in IntConv(\mathcal{R}, z_0)$. Then there exists a continuous loop $h : [0,1] \to \mathcal{R}$ with base point z_0 that strictly surrounds z and*

$$z = \int_0^1 h(s)ds.$$

The figure 3.5 shows a surrounding loop.

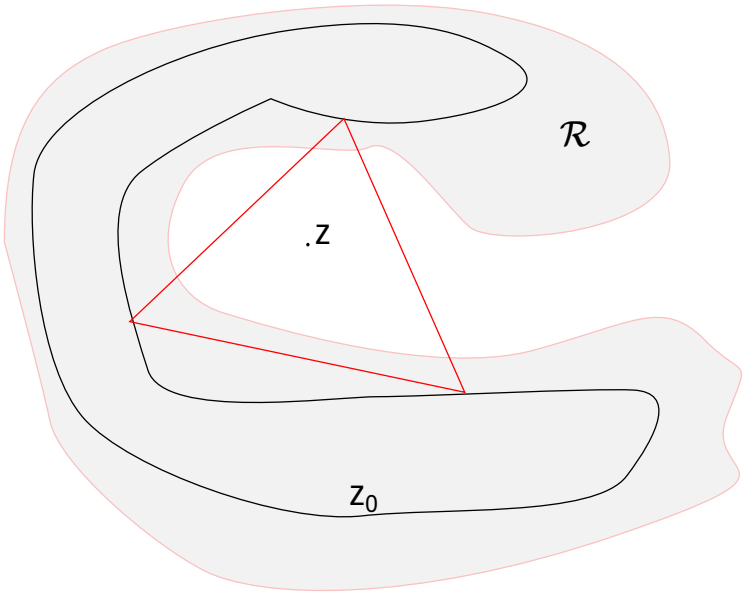

FIGURE 3.5 – A surrounding loop for z.

When the dimension of our source manifold is greater than 1, we need a parametric version of the fundamental lemma to do convex integration.

Lemma 3.19 (Parametric integral representation lemma)**.** *Let B be a compact manifold, $E = B \times \mathbb{R} \xrightarrow{\pi} B$ a trivial bundle, and $\mathcal{R} \subset E$ be a set such that*

$$\forall b \in B, \text{ the set } \mathcal{R}_b := \pi^{-1}b \cap \mathcal{R} \text{ is open in } \mathbb{R}^n.$$

Let $z_0 \in \Gamma(\mathcal{R})$[2] and $z \in \Gamma(E)$ such that

$$\forall b \in B,\ z(p) \in IntConv\left(\mathcal{R}_b, z_0(b)\right).$$

[2] $\Gamma(-)$ means a section of a fibration

Then there exist loops $h : B \times [0,1] \to \mathcal{R}$ *such that*

$$\forall b \in B,\ z(p) = \int_0^1 h_b(s)ds.$$

Remark 3.20. In the fundamental lemma the the reversed path h *i.e.* $h(s) = h(1-s)$ can also be regarded as a loop.

A C^∞ parametric version of fundamental lemma also holds.

With the help of the parametric version of the fundamental lemma, we can work in the higher dimensional case $\mathcal{R} \subset J^1([0,1]^m, \mathbb{R}^n)$ with $m > 1$. Given $f : [0,1]^m \to \mathbb{R}^n$, where $(x_1, x_2, \cdots, x_m) \in [0,1]^m$, if there are parametric surrounding loops h such that

$$\partial_{x_m} f = \int_0^1 h(x_1, x_2, \cdots, x_m, s)ds,$$

Then we can easily define

$$g(x_1, x_2, \cdots, x_m) := f(x_1, x_2, \cdots, x_{m-1}, 0) + \int_0^{x_m} h(x_1, x_2, \cdots, x_{m-1}, s, Ns)ds,$$

which satisfies the properties stated in the following proposition.

Given $f : [0,1]^m \to \mathbb{R}^n$, where $(x_1, x_2, \cdots, x_m) \in [0,1]^m$, let

$$\| f \|_{C^{1,\hat{m}}} = max\{f, \partial_{x_1} f, \cdots, \partial_{x_{m-1}} f\}$$

.

Proposition 3.21. *Let* $E = [0,1]^m \times \mathbb{R}^n \xrightarrow{\pi} [0,1]^m$ *be the trivial bundle over* $[0,1]^m$*, let* $\mathcal{R} \subset E$ *be an open set and* $z_0 \in \Gamma(\mathcal{R})$*. Let* $f : [0,1]^m \to \mathbb{R}^n$ *be a* C^1 *map such that*

$$\forall x = (x_1, x_2, \cdots, x_m) \in [0,1]^m,\ \partial_P x_m f(x) \in IntConv\left(\mathcal{R}_x, z_0(x)\right)$$

where $\mathcal{R}_x = \pi^{-1}(x) \cap \mathcal{R}$*. Then there exists* $g : [0,1]^m \to \mathbb{R}^n$ *such that*

(i). $\partial_{x_m} f \in \Gamma(\mathcal{R})$*,*

(ii). $\| g - f \|_{C^{1,\hat{m}}} = O(\frac{1}{N})$*.*

Remark 3.22. We can apply this result to prove the h-principle for ample relations and to do convex integration for more general situations.

4

Positive Legendrian isotopies via h-principle

With the preparations of h-principle from chapter 3, we can demonstrate the existence of positive loops of Legendrian embeddings in the general case.

The proof essentially consists of two parts : a wrinkled holonomic approximation and a convex integration for non-ample differential relations.

When we try to apply the wrinkled holonomic approximation we are confronted with the same difficulty as Emmy Murphy had been faced of when she worked on Legendrian embeddings. Fortunately, she has already paved the road for us [Mur12].

4.1 E.Murphy's loose Legendrian embeddings

Since the first part of the proof almost repeats Murphy's ideas, it is better to summarize them before going ahead.

Now let us enjoy the journey of constructing Legendrian embeddings via h-principle.

Problem 4.1. *Let (M^{2n+1}, ξ) be a $(2n+1)$-dimensional contact manifold, and L be a n-dimensional closed manifold. Then does h-principle hold for Legendrian embeddings of L ?*

Solution : Let's summarize the solution of the above problem by h-principle to see how Murphy's ideas work.

Obviously, Legendrian embeddings satisfy a differential relation $\mathcal{R}$ which consists of pairs (f, F) such that $f : L \to M$ is a smooth embedding and $F : TL \to TM$ is injective with image in ξ. A formal Legendrian embedding is a pair (f_s, F_s) such that for every $s \in [0, 1]$, f_s is an embedding, F_s is injective and $(f_1, F_1) \in \mathcal{R}$.

To do holonomic approximation, we want to start from a formal solution that is not far from being a real one, so that the technique is simpler. At the same time, the relation $\mathcal{R}$ is not open. Therefore, we take

an open extension $\mathcal{R}^\varepsilon$ which consists of points $\{(f, F)\}$ such that the image of F is ε-close to a Legendrian subspace of ξ. Eventually, the extension $\mathcal{R}^\varepsilon$ is ample and thus it satisfies a h-principle. That means every formal solution of $\mathcal{R}$ can be homotoped to a solution in $\mathcal{R}^\varepsilon$.

Basically, we can solve the relation $\mathcal{R}$ near a $n-1$-skeleton of L by the classical holonomic approximation. Therefore, we can get a formal Legendrian L' which is Legendrian near the $n-1$-skeleton, the problem left is to make L' to be a real one at the level of n-skeleton. We have better to allow singularities, that is to say after perturbation we get L'' which is Legendrian with singularities.

If the singularities are good enough, we can resolve them to turn L'' into a smooth Legendrian L'''. The good singularities Murphy has considered are the so called wrinkles we have mentioned before. The Legendrian L''' is loose.

It is more difficult to reach parametric h-principle, that is to ask whether two formal isotopic loose Legendrians L_0 and L_1 could be isotopic through Legendrians. Of course, we can add wrinkles and resolve them parametrically. The problem is that, for example, when we do this for L_0 to get L_0', we do not know whether the two Legendrians L_0 and L_0' are in the same formal class or not.

Murphy has solved this problem successfully. First of all, she defines loose Legendrians as those obtained by resolving inside-out wrinkles. Besides, If L is a wrinkled Legendrian with one inside-out wrinkle, she proved that the Legendrian L' obtained by resolving all wrinkles is formally Legendrian isotopic to the one obtained by only resolving the inside-out wrinkle.

If we consider a path of wrinkled Legendrians L_t connecting two loose Legendrians L_0 and L_1 which are obtained from L_0' and L_1' with inside-out wrinkles, there is a path L_t' between L_0' and L_1'. Then resolving the wrinkles on L_t' parametrically, we get a path of smooth Legendrians $\widetilde{L_t'}$ between L_0 and L_1 which is in the same formal isotopy class of L_t . Finally, we have the parametric h-principle for loose Legendrian embeddings.

4.2 The existence of positive loops

4.2.1 Basic definitions and results

Notation convention : Given a map $f(x,t)$, we will write it f_t or f_x just to regard t or x as parameters.

Definition 4.2 (formal positive Legendrian isotopy)**.** Let $(M^{2n+1}, \xi = ker\alpha), n > 1$ be a contact manifold, consider a map

$$(f, F^s) : T(L \times [0,1]) \to TM,$$

where $s \in [0,1]$ is regarded as a parameter. We say (f, F^s) is a **formal positive Legendrian isotopy** if

i). $f(x,t) : L \times [0,1] \to M$ is an isotopy ;

ii). $F^s = (P_t^s, v_t^s)$ where P_t^s is a homotopy from Tf_t to a Legendrian plane through n-planes and v_t^s is a homotopy from $\partial_t f_t$ to a vector field positively transverse to the contact plane ξ.

In addition, (f, F^s) is called a formal positive **loop** if $t \in \mathbb{R}/\mathbb{Z}$.

Now, we will define the differential relation $\mathcal{R}$ for positive Legendrian isotopy
Firstly, we want to fix some kind of decomposition of $L^n \times \mathbb{R}/\mathbb{Z}$ to project an isotopy to the Front.

Definition 4.3. (Good decomposition) Let $f : L^n \times \mathbb{R}/\mathbb{Z} \to (M^{2n+1}, \alpha)$ be an isotopy. We decompose $L^n \times \mathbb{R}/\mathbb{Z}$ into $\Delta = \{\Delta^k_{i_k} \times I_j\}$ such that

a). $\mathbb{R}/\mathbb{Z} = \cup_j I_j$ where $I_j = [t_{j-1}, t_j]$ are small intervals ;

b). $\{\Delta^k_{i_k}\}$ is a triangulation of L.

Then Δ is said to be **Good** if $f(\Delta^n_{i_n} \times I_j)$ is contained in a Darboux ball.

Definition 4.4 (Differential relation for positive Legendrian isotopy)**.** Let $f(x,t) : L^n \times \mathbb{R}/\mathbb{Z} \to M^{2n+1}$ be an isotopy, and $\{\Delta^k_j \times I_i\}$ be a good decomposition of $L^n \times \mathbb{R}/\mathbb{Z}$. Denote f_F the composition of f and the Front projection. At every point (x,t), let $\mathcal{R}_{(x,t)}$ be the set

$$\{(P, v) \in Gr_n(\mathbb{R}^{n+1}) \times Gr_1(\mathbb{R}^{n+1}) \mid P \pitchfork \partial z, v \text{ and } \partial z \text{ define the same co-orientation}\}$$

where $Gr_n(\mathbb{R}^{n+1})$ is the bundle of n-planes in $\mathbb{R}^{n+1}$. We say

$$\mathcal{R} = \{(x, t, f_F(x,t)) \times \mathcal{R}_{(x,t)}\} \subset J^1(\Delta^n_j \times I_i, \mathbb{R}^{n+1})$$

is the **differential relation for positive Legendrian isotopy** near (x,t).

Let $f : L \to J^1(L)$ be a Legendrian isotopy and f_F its front. Let ν be orthonormal to $T_x f_F$. Then $\alpha(\partial_t f) = \langle \nu, \partial_t f_F \rangle = |\partial_t f_F| \cos\theta$, where $\theta = \angle(\nu, \partial_t f_F)$. Thus positivity means $|\theta| < \frac{\pi}{2}$.

Lemma 4.5. *The relation $\mathcal{R}$ is open and not empty.*

Proof: We can check it point-wise, say that $\mathcal{R}_{(x,t)} \subset Gr_n(\mathbb{R}^{n+1}) \times Gr_1(\mathbb{R}^{n+1})$ is open and not empty. For every point $(P, v) \in \mathcal{R}_{(x,t)}$, let (P', v') be a point near (P, v) then $P' \pitchfork \partial z$ and v' is in the same co-orientation class as ∂z, that means $(P', v') \in \mathcal{R}_{(x,t)}$, so $\mathcal{R}_{(x,t)}$ is open. Non-emptyness is obvious.
□

When we try to construct a positive Legendrian isotopy from a formal Legendrian isotopy by h-principle, we have to make sure that the formal one really exists.

Lemma 4.6. *There exist formal positive Legendrian isotopies.*

Proof: The existence of a formal positive Legendrian isotopy is totally a homotopy problem. We want to prove that the formal solutions (f, F^s) exist. Note that it is enough to consider $M = (\mathbb{R}^{2n+1}, \xi_{std})$. Let $f(x,t) : L \times [0,1] \to (\mathbb{R}^{2n+1}, \xi_{std})$ be an isotopy connecting two Legendrian embeddings $f(x,0)$ and $f(x,1)$.

First of all, we explain the existence of a formal Legendrian isotopy for some f, which has been proven by Murphy[Mur12]. Let's repeat her idea. Denote $U_n \subseteq V_{2n+1,n}$ the subset of Legendrian frames. Then $d_x f \in Map(L, V_{2n+1,n})$ defines an element in $\pi_1\left(Map(L, V_{2n+1,n}), Map(L, U_n)\right)$, say β. The existence of such (f, P^s) is equivalent to $\beta = 0$. A priori, given a path γ_t in $Map(L, V_{2n+1,n})$ with endpoints in $Map(L, U_n)$ we don't know whether $[\gamma_t] \in \pi_1\left(Map(L, V_{2n+1,n}), Map(L, U_n)\right)$ is trivial or not. We have to assume that $f(x, 0)$ and $f(x, 1)$ are in the same rotation class that we get a lift $\tilde{\beta}$ of β, by connecting the endpoints of $d_x f$. Let $(A, B) = (Map(L, V_{2n+1,n}), Map(L, U_n))$, we have the exact sequence of homotopy groups $\pi_1(B) \xrightarrow{i_*} \pi_1(A) \xrightarrow{j_*} \pi_1(A, B)$. Murphy has proved that $i_* = 0$, therefore for every loop $\sigma_t \in B$ there exists a loop $\tilde{\sigma}_t \in A (\tilde{\sigma}_t \notin B)$ such that $\tilde{\sigma}_t \sim \sigma_t$ in A and $[\tilde{\sigma}_t] = 0$ in $\pi_1(A)$. Besides she has also proved that every element of $\pi_1(A)$ can be represented by some $d_x f$. Thus we can choose such isotopy f with $d_x f \sim \sigma_t$. Then the corresponding $\tilde{\beta}$ is zero, and thus $\beta = j_* \tilde{\beta} = 0$. We obtain a formal Legendrian isotopy for a carefully chosen f.

Moreover, we prove there is no obstruction for the existence of formal positive isotopy. Note that V^s is independent of the choice of P^s. Let $V \subseteq V_{2n+1,1}$ be the subset of positive normal vectors at a point. So $d_t f$ defines an element $\alpha \in \pi_1\left(Map(L, V_{2n+1,1}), Map(L, V)\right)$. We always have $\alpha = 0$ since $\pi_1\left(Map(L, V_{2n+1,1}), Map(L, V)\right) = \pi_{n+1}(V_{2n+1,1}, V)$ and $(V_{2n+1,1}, V) \simeq (\mathbb{S}^{2n}, \mathbb{D}^{2n}_+)$.

□

We now define an extension $\mathcal{R}^\varepsilon$ of the relation $\mathcal{R}$, consisting of ε-positive ε-Legendrian isotopies. In particular, a solution for $\mathcal{R}^\varepsilon$ is not far from being positive Legendrian isotopy.

Definition 4.7 (The ε-extension $\mathcal{R}^\varepsilon$ of $\mathcal{R}$)**.** Let $\mathcal{R}^\varepsilon$ be the extension of $\mathcal{R}$ such that the fibers are

$$\mathcal{R}^\varepsilon_{(x,t)} = \{(P, v) \in Gr_n(\mathbb{R}^{n+1}) \times Gr_1(\mathbb{R}^{n+1}) \mid \exists (P', v') \in \mathcal{R}, |(P, v) - (P', v')| < \varepsilon\}.$$

Thus a Legendrian isotopy f is ε-positive if $|\theta| < \frac{\pi}{2} + \varepsilon$.

Lemma 4.8. *The relation R^ε as above satisfies the h-principle.*

Proof: Let (f, F^s) be a formal positive Legendrian isotopy, and write the relation $\mathcal{R}^\varepsilon$ as $\mathcal{R}^\varepsilon_1 \times \mathcal{R}^\varepsilon_2$ where $\mathcal{R}^\varepsilon_1$ is the ε-Legendrian relation and $\mathcal{R}^\varepsilon_2$ is the ε-positive relation. First of all, the relation $\mathcal{R}^\varepsilon_1$ is ample, so (f, F^s) is homotopic to (f_1, F^s_1) which satisfies $\mathcal{R}^\varepsilon_1$ by convex integration for ample relations. Besides, the relation $\mathcal{R}^\varepsilon_2$ is also ample(see figure 4.1), then we can get (f_2, F^s_2) satisfying $\mathcal{R}^\varepsilon_2$ with $||f_2 - f_1||_{C^\infty(x)} < \varepsilon$ by convex integration, that is to say (f_2, F^s_2) also satisfy $\mathcal{R}^\varepsilon_1$. Therefore our (f_2, F^s_2) is a solution for $\mathcal{R}^\varepsilon$. The parametric version of h-principle is similar.

□

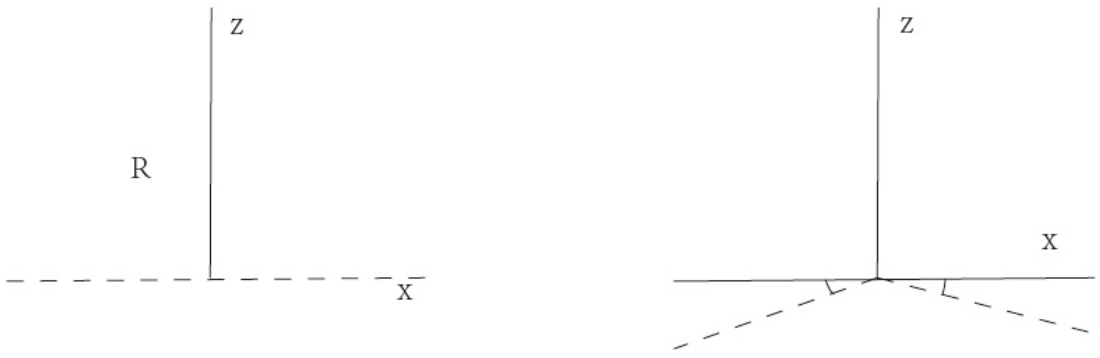

FIGURE 4.1 – the relation $\mathcal{R}^\varepsilon$.

4.2.2 The proof the main theorem

In this subsection we prove the main theorem :

Theorem 4.9. *Let (M,ξ) be a contact manifold and $f_0 : L \to (M,\xi)$ be a Legendrian embedding. If f_0 is loose then there exists a positive loop of Legendrian embeddings based in f_0.*

Theorem 4.9 follows from Proposition 4.12.

Before proving Proposition 4.12, we present Lemma 4.10. Roughly speaking, after applying wrinkled holonomic approximation, we obtain a positive loop of winkled Legendrian away from a collection of cubes $\{\Delta_i^n \times I_j\} \subseteq L \times \mathbb{R}/\mathbb{Z}$.

Given non-negative constants k, K and $\delta \in (0, \frac{\pi}{2})$. For any non-negative ϵ, we denote $(f_k, F^s_{k,K,\delta,\epsilon})$ a formal solution of $\mathcal{R}$ such that $|\partial_t f| < k$ and $|\angle(v^1,\nu)| < \delta$. For a subset $A \subset L \times \mathbb{R}/\mathbb{Z}$, denote $\mathcal{O}_p(A)$ a non specific open neighborhood of A. Then we have the following main lemma :

Lemma 4.10. *Let $(f_k, F^s_{k,K,\delta,\epsilon}) : L \times \mathbb{R}/\mathbb{Z} \to (M,\alpha)$ be a formal positive loop of Legendrian embeddings for some $k,\ K,\ \delta$ and ε. Then there exists an ε-positive loop of wrinkled Legendrian embeddings f^w, a finite family of cubes $\{\Delta_l^n \times I_m\}_{l,m}$ and $D(a_l) \subset \Delta_l^n$ such that*

(i). f^w is positive outside $\{\Delta_l^n \times I_m\}_{l,m}$,

(ii). $\alpha(\partial_t f^w) > K$ on $\{(\Delta_l^n \setminus D(a_l)) \times I_m\}_{l,m}$,

(iii). $\alpha(\partial_t f^w) > -k$ on $\{\Delta_l^n \times I_m\}_{l,m}$.

Proof: We construct f^w in the following steps.

Step 1. Construct a solution $(\widetilde{f}_{\frac{k}{\sin\varepsilon}}, \widetilde{F}^s_{k,K,\delta,\varepsilon})$ for the relation $\mathcal{R}^\varepsilon$ satisfying $\alpha(\partial_t \widetilde{f}_{\frac{k}{\sin\varepsilon}}) > -k$. We apply the convex integration as in Lemma 4.8. Choosing surrounding loops $h_{x,t} \in \mathcal{R}_2^\varepsilon$ for $\partial_t f_k$ with $|h_{x,t}| < \frac{k}{\sin\varepsilon}$, we get $\widetilde{f}_{\frac{k}{\sin\varepsilon}}$ satisfying $|\widetilde{f}_{\frac{k}{\sin\varepsilon}}| < \frac{k}{\sin\varepsilon}$. Thus $\alpha(\partial_t \widetilde{f}_{\frac{k}{\sin\varepsilon}}) > \frac{k}{\sin\varepsilon}\cos(\frac{\pi}{2}+\varepsilon) = -k$. We can assign a $\widetilde{F}^s_{k,K,\delta,\varepsilon}$ for $\widetilde{f}_{\frac{k}{\sin\varepsilon}}$.

Step 2. Given a decomposition Δ of $L \times \mathbb{R}/\mathbb{Z}$, deform $(\widetilde{f}_{\frac{k}{\sin\varepsilon}}, \widetilde{F}^s_{k,K,\delta,\varepsilon})$ near the n-skeleton Δ^n of Δ by holonomic approximation. Let φ_t be a C^0-small diffeomorphism of $L \times \mathbb{R}/\mathbb{Z}$ supported in $\mathcal{O}_p(\Delta^n)$. According to holonomic approximation, for any ε_0, we can deform $\widetilde{f}_{\frac{k}{\sin\varepsilon}}$ on $\mathcal{O}_p(\varphi_1(\Delta^n))$ to $\bar{f}^0$ such that $|d\bar{f}^0 - \widetilde{F}^s_{k,K,\delta,\varepsilon}| < \varepsilon$. We can extend $\bar{f}^0$ to $\mathcal{O}_p(\Delta^n) \setminus \mathcal{O}_p(\varphi_1(\Delta^n))$ matching to $\widetilde{f}_{\frac{k}{\sin\varepsilon}}$ near $\mathcal{O}_p(\partial\Delta^n)$ such that the extended $(\bar{f}^0, d\bar{f}^0)$ satisfies the relation $\mathcal{R}^\varepsilon$. Thus, pasting $\bar{f}^0$ and $\widetilde{f}_{\frac{k}{\sin\varepsilon}}$ along the boundary, and denoting the new map by $\bar{f}$, we have $\alpha(\partial_t \bar{f}) > -k$ and $\alpha(\partial_t \bar{f}) > K$ on $\mathcal{O}_p(\phi_1(\Delta^n))$. We can also assign a $\bar{F}_{k,K,\delta,\varepsilon}$ for $\bar{f}$ such that $(\bar{f}, \bar{F}_{k,K,\delta,\varepsilon})$ satisfies the relation $\mathcal{R}^\varepsilon$. Note that there are finitely many cubes $\{\Delta_l^n \times I_m\}_{l,m}$ where $(\bar{f}, d\bar{f})$ is not a real solution for $\mathcal{R}$ and for every l there exists $D(a_l) \subset \Delta_l^n$ such that (ii) is satisfied.

Step 3. Apply holonomic approximation with wrinkles for these cubes. We regard I_j as parameters for every j. Then, we add wrinkles parametrically such that the C^1-norm $||f^w - f||_{C^1(t)} < \varepsilon' < \varepsilon$ for some small ε'. Therefore, f^w satisfies all the above conditions. □

As Murhpy [Mur12], we are going to resolve the wrinkles along a family of *twist markings*. Let $S_t \subset f_t^w(L)$ be a family of twist markings. Then we replace a small neighborhood of S_t by $Z \times S_t$, where Z is a zigzag, such that the new manifold is also a Legendrian for all t. Recall that we regard this operation as a smooth map $W_\eta^{-1} : M \to M$. When f_0 is loose, we require $W_\eta^{-1} \circ f_0^w = f_0$ (see [Mur12][Proposition 7.1]).

Lemma 4.11. *Let $f^w : L \times [0,1] \to M$ be a positive wrinkled Legendrian isotopy. Then $W_\eta^{-1} \circ f^w$ is a positive Legendrian isotopy for η small enough.*

Proof: Since the C^1 norm $\| W_\eta^{-1} \|_{C^1} \leq \eta$ and the positivity condition is open, we have that $W_\eta^{-1} \circ f^w$ is positive for η small enough.

□

Therefore, combining Lemma 4.10 and Lemma 4.11, we have the following result :

Proposition 4.12. *Let $(f_k, F^s_{k,K,\delta,\epsilon}) : L \times \mathbb{R}/\mathbb{Z} \to M$ be a formal positive loop of Legendrian embeddings with f_0 being loose for some k, K, δ and ε. Then there exists a loop of ε-positive Legendrian embeddings $\widetilde{f}$, a finite family of cubes $\{\Delta^n_l \times I_m\}_{l,m}$ and $D(a_l) \subset \Delta^n_l$ such that :*

(i). $\widetilde{f}$ *is positive outside* $\{\Delta^n_l \times I_m\}_{l,m}$,

(ii). $\alpha(\partial_t \widetilde{f}) > K$ *on* $\{(\Delta^n_l \setminus D(a_l)) \times I_m\}_{l,m}$,

(iii). $\alpha(\partial_t \widetilde{f}) > -k$ *on* $\{\Delta^n_l \times I_m\}_{l,m}$,

(iv). $\widetilde{f}_0 = f_0$.

Now we are ready to prove the main result (Theorem 4.9). The idea is to add wrinkles again on the the highest skeletons parametrically.

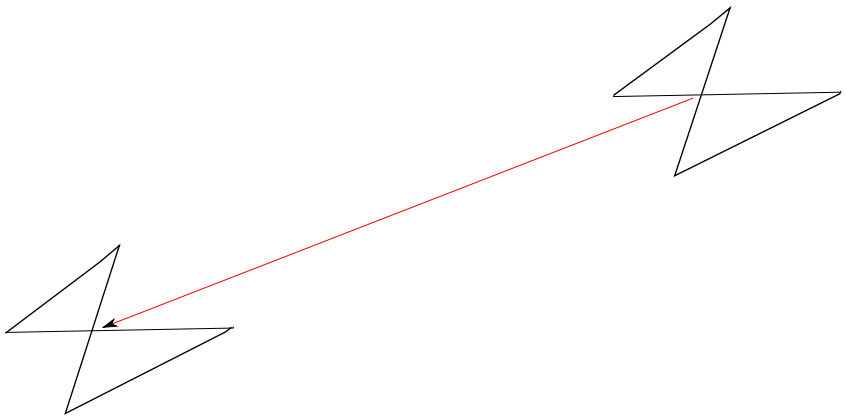

FIGURE 4.2 – f^w_t is smooth.

Proof: (The first version) According to Proposition 4.12, assume we have a loop of ε-positive loose Legendrian embeddings f_t which is not positive on a finite family of cubes $\{D(a_l) \times I_m\}$. Take $D_l \supset D(a_l)$ for every l. We construct a wrinkled Legendrian loop f^w_t with wrinkles on the image of D_l in the following way. If $\angle(\partial_t f_F, \nu_{f_F}) > -(\frac{\pi}{2} + \varepsilon)$ on $D(a_l) \times I_m$ for some $I_m = (t_{m-1}, t_m)$, we add wrinkles to f_F such that $\angle(\nu_{f_F}, \nu_{f^w_F}) > 2\varepsilon$ and f^w_t stays in the Weinstein neighborhood of f_t. The wrinkles is added parametrically such that $|\partial_t f^w_F - \partial_t f_F| < \varepsilon$. Thus we have $\angle(\partial_t f^w_F, \nu_{f^w_F}) > -\frac{\pi}{2}$. That means f^w_t is positive (see figure 4.2). There exists some $\delta > 0$ such that $\angle(\partial f_F, \nu_{f_F}) < 0$ for $t \in (t_m, t_m + \delta]$. We add wrinkles parametrically such that $\angle(\nu_{f_F}, \nu_{f^w_F})$ decreases when t grows and the wrinkle dies away at $t = t_m + \delta$. Note that $f^w_{t_m+\delta}$ is a Legendrian embryo. Similarly, if $\angle(\partial_t f_F, \nu_{f_F}) < (\frac{\pi}{2} + \varepsilon)$ on $D(a_l) \times I_m$, we add wrinkles to f_F such that $\angle(\nu_{f_F}, \nu_{f^w_F}) > -2\varepsilon$. After these operations for all l, we get a positive loop f^w. Then we resolve the

wrinkles (include Legendrian embryos) such that $W_\eta^{-1} f^w$ is a positive loop for η small enough. Note that we do not change the formal isotopy class with a suitable choice of *twist marking*. □

We give another proof which was suggested by V.Colin.

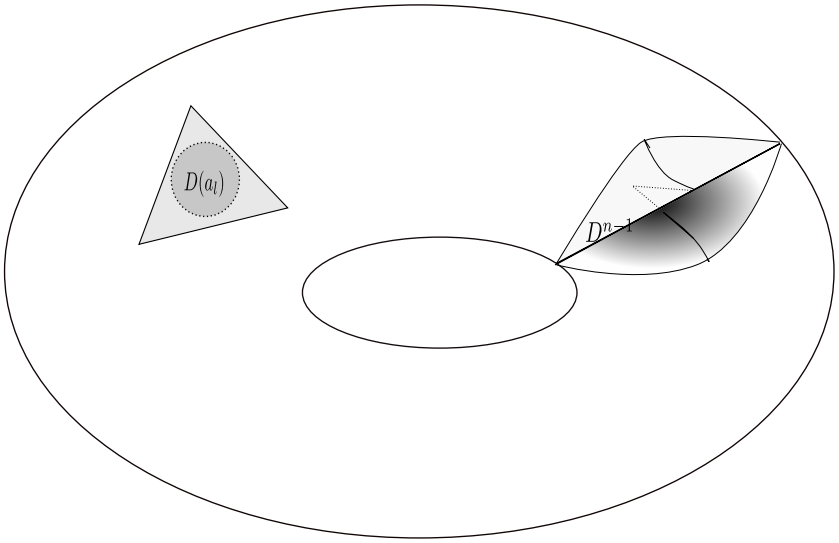

FIGURE 4.3 – The wrinkled $D^{n-1} \times \mathbb{S}^1$.

Proof: (The second version) Assume we have a loop of ε-positive loose Legendrian embeddings f_t which is not positive on a finite family of cubes $\{D(a_l) \times I_m\}$. For a disk $D(a_l)$, take a $D^{n-1} \times \mathbb{S}^1 \subset L$ such that $D(a_l) \subset D^{n-1} \times \mathbb{S}^1$. Then we add wrinkles to f_t along the image of $D^{n-1} \times \{0\}$ for all $t \in [0,1]$ (see figure 4.3). Denote f^w the wrinkled loop. We assume that f^w is C^0-close to f. Then we can turn the wrinkles arbitrary fast in a small neighborhood of $f_F(D^{n-1} \times \mathbb{S}^1)$ to incorporate $D(a_l)$ into a positive loop. More explicitly, take some $t_0 \in [0,1]$, there exists some $\delta > 0$ such that $(f_t^w(L))_{t \in [t_0-\delta, t_0+\delta]}$ are contained in a Weinstein neighborhood U_{t_0} of $f_{t_0}(L)$. Take the front projection and regard $f_{Ft_0}(L)$ as the zero-section $L \times \{0\}$. Note that there exists some δ' such that $D^{n-1} \times \mathbb{S}^1 \times (-\delta', \delta')$ is contained in the front of U_{t_0}. Now we construct a compactly supported loop of diffeomorphisms ϕ_t in the front such that $\phi_t(x, \theta, z) = (x, \theta - 2K_l \pi t, z)$ on $D^{n-1} \times \mathbb{S}^1 \times (-\delta', \delta')$ for some $K_l > 0$. Like the case of the 2-sphere, we take K_l large enough, then the composition $\phi_t \circ f_{Ft}^w$ defines a loop which is positive on $D(a_l)$ for all $t \in [t_0 - \delta, t_0 + \delta]$. As t_0 varies, we can incorporate $D(a_l)$ into a positive loop. Then we resolve the wrinkles without changing the Legendrian isotopy class. We do the same operation for all the disks $\{D(a_l)\}$ one by one and turn the wrinkles faster and faster. Finally, we get a positive loop of Legendrian embeddings which is in the same Legendrian isotopy class of f_0.

□

5

Applications

In this chapter, we give some applications of our main theorem. Firstly, we reprove the tightness of $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$. Secondly, we define a partial order on the universal cover $\widetilde{Cont}_0(M, \xi)$ of the identity component of the group of contactomorphisms of a contact manifold (M, ξ).

5.1 Tightness of $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$

In this section we prove Corollary 0.9. A similar proof for $\mathbb{S}^1 \times \mathbb{R}^2$ was given in [CFP10].

Proof: Assume $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is overtwisted, and $D_{OT} \subset (\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is an overtwisted disk. Denote $\pi : \mathbb{S}^{n-1} \times \mathbb{R}^n \to \mathbb{R}^n$ the projection. There exists some point $x \in \mathbb{R}^n$ such that the fiber $\pi^{-1}(x) \cap D_{OT} = \emptyset$. According to [CMP15], the fiber $\pi^{-1}(x)$ is loose. Thus, there exists a positive loop based in it by Theorem 0.8. That contradicts to Theorem 0.6. Therefore, the manifold $(\mathbb{S}^{n-1} \times \mathbb{R}^n, \xi_{std})$ is tight. □

5.2 Positive loops and orderings

Definition 5.1. Given (M, α) a contact manifold, The manifold $(\Gamma_M, \tilde{\alpha}) = (M \times M \times \mathbb{R}, \alpha_1 - e^s \alpha_2)$ is called a contact product. Here $\alpha_i = \pi_i^* \alpha$ where π_i projet Γ_M to the i-th factor. The Legendrian submanifold of $(\Gamma_M, \tilde{\alpha})$ defined by $\Delta = \{(x, x, 0)\}$ is called the diagonal.

The contact product Γ_M is a special case of contact fibration where we can talk about contact connection, let's recall the definition from [Pre07].

Definition 5.2. Let $(E, \xi = ker\alpha)$ be a contact manifold, and $E \longrightarrow B$ is a fibration with fiber F. Then $(E, \xi = ker\alpha) \longrightarrow B$ is called a contact fibration if $(F, \alpha|_F)$ is a contact manifold. Let $(E, \xi = ker\alpha) \longrightarrow B$ be contact fibration we say that the horizontal distribution $H = (TF \cap \xi)^{\perp d\alpha}$ is the contact connection associated to the fibration.

Remark 5.3. The horizontal distribution is dependent on the contact form α.

The connection defined above has the following properties :

Proposition 5.4. *[Pre07] For a path $\gamma : [0,1] \to B$, the monodromy $m_\gamma : F(\gamma(0)) \to F(\gamma(1))$ induced by γ is a contactomorphism.*

Corollary 5.5. *Let $\phi \in Diff_0(B)$. Then it lifts to a contactomorphism $\widetilde{\phi}$.*

Γ_M is a contact fibration with $F = (M, \alpha)$ and $B = M \times \mathbb{R}$. In this case $R_{\tilde{\alpha}} = (R_\alpha, 0, 0)$, and

$$\xi = ker\alpha_1 \oplus ker\alpha_2 \oplus < \partial_s > \oplus < e^s R_{\alpha_1} + R_{\alpha_2} >,$$

and

$$d\tilde{\alpha} = d\alpha_1 - e^s(d\alpha_2 + ds \wedge \alpha_2),$$

such that

$$H = ker\alpha_2 \oplus < \partial_s > \oplus < e^s R_{\alpha_1} + R_{\alpha_2} > .$$

We now explain the following result which was first suggested by Klaus Niederkrüger and also observed by Casals and Presas.

Proposition 5.6. *Let (M^{2n+1}, α) be a compact overtwisted contact manifold and let $(\Gamma_M, \tilde{\alpha})$ be the associated contact product. Then $(\Gamma_M, \tilde{\alpha})$ is also overtwisted and the diagonal $\Delta \subset \Gamma_M$ is loose.*

Proof: We apply the overtwisted criterion from [CMP15]. If $\lambda = ydx - xdy$, it is enough to construct a higher dimensional overtwisted ball $D = (B^{2n+1}_{OT} \times D^{2n+2}(r), \alpha_{OT} - \lambda) \subset (\Gamma_M, \tilde{\alpha})$ for some r large enough, such that D does not intersect Δ.

Let $\mathbb{S}^{2n+1} = \{(x, y) \mid x^2 + y^2 = 1\}$ with its standard contact form α_{std}, and let $\varphi_0 : \mathbb{S}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+2}, (x, y, s) \mapsto (e^s x, e^s y)$. Note that $\varphi_0^* \lambda = \alpha_{std}$. We take a Darboux ball $B \subset (M, \alpha)$ and we regard it as a subset of $(\mathbb{S}^{2n+1}, \alpha_{std})$. Then we can construct a contact embedding $\varphi : (M \times B \times \mathbb{R}, \tilde{\alpha}) \hookrightarrow (M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda)$ by the following series of contact embeddings

$$(M \times B \times \mathbb{R}, \tilde{\alpha}) \overset{i}{\hookrightarrow} (M \times \mathbb{S}^{2n+1} \times \mathbb{R}, \alpha_1 - e^s \alpha_{std}) \overset{id \times \varphi_0}{\longrightarrow} (M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda).$$

Let $B^{2n+1}_{OT} \subset M$ be a overtwisted ball, then $D_0 = (B^{2n+1}_{OT} \times \mathbb{D}^{2n+2}(r), \alpha_1 - \lambda)$ is the overwisted ball in $(M \times \mathbb{R}^{2n+2}, \alpha_1 - \lambda)$. We can move D_0 away from $\varphi(\Delta)$ by Corollary 5.5. More precisely, we take the vector field $V = 2r\partial_x + 2r\partial_y$ on $\mathbb{R}^{2n+2}$, then lift it to a contact vector field $V' = V + 2r(y - x)R_\alpha$ on $M \times \mathbb{R}^{2n+2}$ where R_α is the Reeb vector field of (M, α). Let ϕ_t be the contact isotopy of V'. Denote

$C = \{rx \mid x \in B, r > 0\}$ the cone defined by B. Then $D_1 = \phi_1(D_0) \subset M \times (C \setminus \{0\}) = \varphi(M \times B)$ does not intersect $\varphi(\Delta)$. Therefore $D = \varphi^{-1}(D_1)$ is an overtwisted ball we want.

□

Corollary 5.7. *Let (M, α) be a compact overtwisted contact manifold and $(\Gamma_M, \tilde{\alpha})$ the contact product. Then there exists a positive loop of Legendrian embeddings based in Δ.*

Let $Leg(M, \Gamma_M)$ be the set of Legendrian embeddings $M \hookrightarrow (\Gamma_M, \widetilde{\alpha})$. Given $\phi \in Cont_0(M, \xi = ker\alpha)$ with $\phi^*\alpha = e^{g(x)}\alpha$, it induces a contactomorphism

$$\bar{\phi}(x, y, s) := (x, \phi(x), s - g(y))$$

on $(\Gamma_M, \widetilde{\alpha})$. We denote $gr(\phi) = \bar{\phi}|_\Delta$ which is in $Leg(M, \Gamma_M)$. In fact, given a positive contact isotopy ϕ_t, we can see that $gr(\phi_t)$ is a negative Legendrian isotopy. Therefore, we would like to transfer the study of positive contact isotopies to that of negative Legendrian isotopies.

Definition 5.8. Let $f = [f_t]$ and $g = [g_t]$ be two elements in $\widetilde{Cont}_0(M, \xi)$. We say $f \succeq g$ if there exists a non-positive path $L_t \in Leg(M, \Gamma_M)$ from $gr(g_1)$ to $gr(f_1)$ and $gr(g_t) * L_t$ is homotopic to $gr(f_t)$. The space $\widetilde{Cont}_0(M, \xi)$ and (M, ξ) are said to be **strongly orderable** if $\succeq$ defines a partial order [1] on it. Otherwise, they are said to be non strongly orderable.

Remark 5.9. Let C be the set generated by all the homotopy classes of non-positive paths in $Leg(M, \Gamma_M)$. Then $f \succeq g$ equals to $gr(g^{-1}f) \in C$. Given $[L_t] \in C$ and $\phi \in Cont_0(M, \xi)$, then we have $[\bar{\phi}L_t] \in C$. Therefore, the order $\succeq$ is left invariant, that is to say, given f and g in $\widetilde{Cont}_0(M, \xi)$, if $f \succeq g$, then $hf \succeq hg$ for all $h \in \widetilde{Cont}_0(M, \xi)$. Because if L_t is a non-positive path from g_1 to f_1, then $\bar{h}_1 L_t$ is a non-positive path from $h_1 g_1$ to $h_1 f_1$.

Proposition 5.10. *Let (M, ξ) be a contact manifold. Then (M, ξ) is strongly orderable if and only if there does not exist a contractible negative loop of Legendrian embeddings based in Δ.*

Proof: Let $f = [f_t]$, $g = [g_t]$ and $h = [h_t]$ be elements in $\widetilde{Cont}_0(M, \xi)$. Firstly, the order $\succeq$ is reflective, since we have $f \succeq f$ by the definition of $\succeq$. If there are two non-positive paths L_t^1 from $gr(g_1)$ to $gr(f_1)$ and L_t^2 from $gr(h_1)$ to $gr(g_1)$, then $L_t^2 * L_t^1$ is a non-positive path from $gr(h_1)$ to $gr(f_1)$. Thus, the order is transitive. Secondly, we check the antisymmetry of $\succeq$. According to [CN13][Propostion 4.5], the existence of contractible non-positive non-trivial loop of Legendrian embeddings is equivalent to the existence of contractible negative loop of Legendrian embeddings. Thus, for any $f \neq 1$, on one hand, if there does not exist any negative loop based in Δ, we can not find a non-negative path L_t^1 and a non-positive path L_t^1 in the homotopy class of $gr(f_t)$ at the same times. Otherwise, $L_t^1 * L_t^2$ is a contractible non-negative loop. On the other hand, if there exists a non-positive loop f_t based in Δ, then $f_{1/2} \succeq 1$ and $1 \succeq f_{1/2}$. That means (M, ξ) is not strongly orderable. □

Our definition is stronger than that of [EP99], since we do not require the path of Legendrian embeddings $\widetilde{\phi}_t$ to be graphical for all t.

[1] in the sense of a partial order on sets

Corollary 5.11. *Let (M, ξ) be a contact manifold. If (M, ξ) is strongly orderable, then it is orderable.*

We have the following example of strong orderability.

Proposition 5.12. *$(\mathbb{S}^1, \xi_{std})$ is strongly orderable.*

Proof: Denote $d\theta$ the standard contact form for $\mathbb{S}^1$. We have a contactomorphism $\varphi : (\Gamma_{\mathbb{S}^1}, d\theta_1 - e^s d\theta_2) \to (\mathbb{S}^1 \times T^*\mathbb{S}^1, dz - ydx), (\theta_1, \theta_2, s) \mapsto (z = \theta_1 - \theta_2, x = \theta_2, y = e^s - 1)$ such that $\varphi(\Delta)$ is the zero-section. Assume there exists a contractible positive loop based in the zero-section of $(\mathbb{S}^1 \times T^*\mathbb{S}^1, dz - ydx)$, then it lifts to a positive loop based in the zero-section of $(\mathbb{R}^1 \times T^*\mathbb{S}^1, dz - ydx)$. However, such loop does not exist according to [CFP10]. Thus $(\mathbb{S}^1, \xi_{std})$ is strongly orderable. □

Question 5.13. *Is $(\mathbb{R}P^3, \xi_{std})$ strongly orderable ?*

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UNIVERSITE BRETAGNE LOIRE

UNIVERSITÉ DE NANTES

Thèse de Doctorat

Guogang LIU

Sur les lacets positifs des plongements legendriens lâches

On positive loops of loose Legendrian embeddings

Résumé
Dan la thèse, on a étudié le problème des isotopies legendriennes positif. C'est-à-dire que les isotopies presérvent le structure de contact et les fonctions Hamiltoniennes associés sont positif. On a montré que si une sou-variété legendrienne est lâche, il existe un lacet positif des plongements legendriennes basé sur lui. On a le trait en deux cas, le cas en dimension un et deux, l'autre en grandes dimensions. Dans les cas en bases dimensions, on a construit des lacets positive par la main. Dans les autres cas, on a utilisé les techniques de h-principe advancé, c'est-à-dire, la approximation holonome ridé et la intégration convexe pour les relations «non-ample». Avec la approximation holonome ridé, on a obtenue un lacet de plongements Legendriennes qui est positive sauf que en un ensemble fini des discs. Puis, on a le deformé à un lacet positif par l'idée de la intégration convexe. Ce resulat a deux applications immédiates. On donne une simple démonstraion san les techniques de courbes holomorphes pour le Théorème : les espaces des elements de contact, muni de la structure standard sont tendues. On a aussi montré le produit contact de une variété de contact vrillées est vrillées et la diagonale est lâche, de puis la diagonal est dans un lacet positif. Isotopies positif legendriennes relient aux ordres de le revêtement universel de la groupe de contactomorphisme. On a définit un ordre par isotopies positif legendriennes dan le produit contact. Il nous aide de étudié les propriétés de contactomorphisme en manière de isotopies positif legendriennes.

Mots clés
géométrie de contct, variété de contact, sou-variété legendrienne, isotopie positif legendrienne, lacet positif de legendrienne, h-principe, sou-variété legendrienne lâche, ordre partiel.

Abstract
In the thesis, we have studied the problem of positive Lengendrian isotopies. That is to say, the isotopies preservepo the contact structure and the hamiltonnian functions of the isotopies are positive. We have proved that for a loose Legendrian there exists a positive loop of Legendrian embeddings based in it. We treated this result in two cases. In lower dimensions cases, we constructed positive loops by hand. In higher dimensions cases, we applied the advanced h-principle techniques. Given a loose Legendrian embedding, firstly, by the holonomic approximation, we constructed a loop of Legendrian embeddings based in it which is positive away from a finite number of disks. Secondly, we deformed it to a positive loop by the idea of convex integration. The result has two immediate applications. Firstly, we reprove the theorem that the spaces of contact elements are tight without holomorphic curves techniques. Secondly, we proved the contact product of an overtwisted contact manifold is overtwisted and the diagonal is loose, furthermore, the diagonal is in positive loop. In the end, we have defined a partial order on the universal cover of the contactomorphism group by positive Legendrian isotopies in the contact product. It will help us to study the properties of contactomorphism via positive Legendrian isotopies.

Key Words
contact geometry, contact manifold, Legendrian submanifold, positive Legendrian isotopy, loop of positive Legendrian isotopy, h-principle, loose Legendrian embedding, partial order.