Champs quantiques massifs en espace-temps courbe et tenseur d'impulsion-énergie renormalisé

Andrei Belokogne

▶ **To cite this version:**

Andrei Belokogne. Champs quantiques massifs en espace-temps courbe et tenseur d'impulsion-énergie renormalisé. Relativité Générale et Cosmologie Quantique [gr-qc]. Université de Corse Pascal Paoli, 2016. Français. NNT : . tel-01445979

HAL Id: tel-01445979

https://hal.science/tel-01445979

Submitted on 25 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNIVERSITÉ DE CORSE-PASCAL PAOLI
ÉCOLE DOCTORALE ENVIRONNEMENT ET SOCIÉTÉ
UMR CNRS 6134 (SPE)

Thèse présentée pour l'obtention du grade de

DOCTEUR EN PHYSIQUE
Mention : **Constituants élémentaires**

Soutenue publiquement par

Andrei BELOKOGNE

le : 9 décembre 2016

CHAMPS QUANTIQUES MASSIFS EN ESPACE-TEMPS COURBE ET TENSEUR D'IMPULSION-ÉNERGIE RENORMALISÉ

Directeurs :

M Antoine FOLACCI, Professeur, Université de Corse
M Yves DECANINI, Professeur, Université de Corse

Rapporteurs :

M Eric HUGUET, MCF-HDR, Université de Paris VII
M Adrian OTTEWILL, Professeur, University College Dublin

Jury :

M Yves DECANINI, Professeur, Université de Corse
M Antoine FOLACCI, Professeur, Université de Corse
M Eric HUGUET, MCF-HDR, Université de Paris VII
M Adrian OTTEWILL, Professeur, University College Dublin
M Jacques RENAUD, Professeur, Université de Paris-Est
M Thomas SCHUCKER, Professeur, Université d'Aix-Marseille

« On ne peut plus expliquer le monde, faire ressentir sa beauté à ceux qui n'ont aucune connaissance profonde des mathématiques. »

Richard Feynman

Remerciements

J'aimerais tout d'abord remercier grandement mes directeurs de thèse, Antoine Folacci et Yves Décanini, de m'avoir proposé ce travail doctoral ainsi que les différents sujets qu'aborde ce manuscrit. Je leur exprime ma profonde gratitude pour tout ce qu'ils m'ont apporté sur le plan scientifique et humain. Je tiens tout particulièrement à remercier Antoine Folacci pour les discussions enrichissantes et fructueuses sur les sujets scientifiques pertinents ainsi que sa disponibilité durant toute cette période de thèse.

J'aimerais également remercier M. Eric Huguet et M. Adrian Ottewill d'avoir eu l'amabilité d'accepter de rapporter sur ce manuscrit. De même, je souhaite témoigner ma gratitude à M. Jacques Renaud et M. Thomas Schucker d'avoir acceptés de faire partie de mon jury de thèse. Je leur exprime toute ma reconnaissance.

J'aimerais remercier sincèrement mon collègue de bureau, Mohamed Ould El Hadj, de toutes les discussions scientifiques et amicales que l'on avait pendant ces années. Mes remerciements vont aussi aux docteurs et futurs docteurs, Damien, David, Eric, Gauthier, Jean-Baptiste, Lara, Romain, Sandrine, Tom et Wani ainsi que Aurélia et Damien, pour tous les moments agréables que l'on a partagés ensemble.

Au cours de la dernière année de thèse, j'ai aussi travaillé avec Julien Queva et je tiens à lui exprimer ma reconnaissance pour les discussions et les conseils. D'autre part, j'aimerais également remercier l'ensemble des chercheurs et personnels de la Faculté des Sciences de l'Université de Corse, du laboratoire SPE (UMR CNRS 6134) pour leur accueil et sympathie et, en particulier, Pierre Simonnet ainsi que Stéphane Ancey, Rachel Baïle, Frederic Bosseur, Benoit Cagnard, Bernard Di Martino, Catherine Ducourtioux, Vincent Ferreri, Jean Baptiste Filippi, Paul Gabrielli, Nicolas Heraud, David Moungar, Jean-François Muzy, Marie-Laure Nivet, Jean-Martin Paoli, Antoine Pieri, Sonia Ternengo et Yves Thibaudat.

Finalement, j'aimerais dédier ce manuscrit à mon frère, Ivan, ainsi qu'à ma soeur, Nicole, et à mes parents. Sans leur soutien et sans leurs encouragements jamais cette thèse n'aurait vu le jour.

Andrei Belokogne
Corte, décembre 2016.

Abstract

Quantum field theory in curved spacetime is a semiclassical approximation of quantum gravity which, by treating classically the spacetime metric $g_{\mu\nu}$ and considering from a quantum point of view all the other fields (including the graviton field to at least one-loop order for reasons of consistency), avoids the difficulties due to the nonrenormalizability of quantum gravity and provides a framework which permits us to study the low-energy consequences of a hypothetical "theory of everything". It should be recalled that this approach allowed theoretical physicists to obtain fascinating results concerning early universe cosmology and quantum black hole physics and, in particular, led to the discovery of particle creation in expanding universes by Parker and of black hole radiance by Hawking.

In quantum field theory in curved spacetime, it is conjectured that the backreaction of a quantum field in the normalized quantum state $|\psi\rangle$ on the spacetime geometry is governed by the semiclassical Einstein equations $G_{\mu\nu} = 8\pi \langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$. Here, $G_{\mu\nu}$ is the Einstein tensor $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}$ or some higher-order generalization of this geometrical tensor while $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ is the expectation value of the stress-energy tensor associated with the quantum field. The quantity $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ is ill-defined and formally infinite due to the "pathological" short-distance behavior of the Green functions of the quantum field. In order to extract from $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ a finite and physically acceptable contribution, it is necessary to regularize it and then to renormalize all the coupling constants of the theory. The corresponding renormalized expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ is of fundamental importance not only because it acts as the source in the semiclassical Einstein equations, but also because it permits us to analyze the quantum state $|\psi\rangle$ without any reference to its particle content.

In the first part of our manuscript, we have obtained a large-mass approximation for the renormalized expectation value of the stress-energy tensor associated with various massive fields (the massive scalar field, the massive Dirac field and the Proca field) propagating on Kerr-Newman spacetime. Our calculations are based on the DeWitt-Schwinger expansion of the Green functions and of the effective action. Our results could be useful because there exists no exact results on this gravitational background.

In the second part of our manuscript, we have discussed Stueckelberg massive electromagnetism on an arbitrary four-dimensional curved spacetime (gauge invariance of the classical theory and covariant quantization ; wave equations for the massive spin-1 field A_μ, for the auxiliary Stueckelberg scalar field Φ and for the ghost fields C and C^* ; Ward identities ; Hadamard representation of the various Feynman propagators and covariant Taylor series expansions of the corresponding coefficients). This has permitted us to construct, for a Hadamard quantum state $|\psi\rangle$, the renormalized expectation value of the stress-energy tensor associated with the Stueckelberg theory. As applications of our results, (i) we have considered, in the Minkowski spacetime, the Casimir effect outside a perfectly conducting medium with a plane boundary, and (ii) we have obtained, in de Sitter and anti-de Sitter spacetimes, an exact analytical expression for the vacuum expectation value of the renormalized stress-energy tensor of the massive vector field.

Keywords : Quantum field theory in curved spacetime ; Massive electromagnetism ; Stueckelberg mechanism ; Effective action ; DeWitt-Schwinger expansion ; Hadamard renormalization ; Renormalized stress-energy tensor ; Vacuum energy ; Kerr-Newman, de Sitter and anti-de Sitter spacetimes ; Casimir effect.

Résumé

La théorie quantique des champs en espace-temps courbe est une approximation semi-classique de la gravitation quantique qui, en traitant classiquement la métrique $g_{\mu\nu}$ de l'espace-temps et en considérant du point de vue quantique tous les autres champs (on inclut ici même le champ des gravitons à l'ordre au moins une boucle pour des raisons de cohérence), évite les difficultés dues à la non-renormalisabilité de la gravitation quantique et fournit un cadre qui permet d'étudier les conséquences, à basse énergie, d'une hypothétique "théorie du tout". Il faut rappeler que cette approche a permis aux physiciens d'obtenir des résultats fascinants concernant la cosmologie de l'univers primordial ainsi que la physique des trous noirs et, en particulier, a conduit Parker à la découverte de la création de particules dans des univers en expansion et Hawking à celle du rayonnement quantique des trous noirs.

En théorie quantique des champs en espace-temps courbe, il est admis que la réaction en retour d'un champ quantique dans un état quantique normalisé $|\psi\rangle$ sur la géométrie de l'espace-temps est régie par les équations d'Einstein semi-classiques $G_{\mu\nu} = 8\pi \langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$. Ici, $G_{\mu\nu}$ est le tenseur d'Einstein $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}$ ou une généralisation d'ordre supérieur de ce tenseur géométrique alors que $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ est la valeur moyenne du tenseur d'impulsion-énergie associé au champ quantique. La quantité $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ est formellement infinie du fait du comportement singulier à courte distance des fonctions de Green du champ quantique. Afin d'extraire une contribution finie et physiquement raisonnable de cette quantité, il est nécessaire de la régulariser puis de renormaliser toutes les constantes de couplage de la théorie. La valeur moyenne renormalisée $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ est d'une importance fondamentale non seulement parce qu'elle agit comme source dans les équations d'Einstein semi-classiques mais aussi parce qu'elle nous permet d'analyser l'état quantique $|\psi\rangle$ sans faire aucune référence à son contenu en particules.

Dans la première partie de notre manuscrit, nous avons obtenu, dans la limite des grandes masses, une approximation pour la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé à divers champs massifs (le champ scalaire massif, le champ de Dirac massif et le champ de Proca) se propageant sur l'espace-temps de Kerr-Newmann. Notre calcul est basé sur le développement de DeWitt-Schwinger des fonctions de Green et de l'action effective. Nos résultats sont intéressants parce qu'il n'existe aucun résultat exact sur ce background gravitationnel.

Dans la seconde partie de notre manuscrit, nous avons discuté l'électromagnétisme massif de Stueckelberg sur un espace-temps courbe arbitraire de dimension quatre (invariance de jauge de la théorie classique et quantification covariante ; équations d'ondes pour le champ massif de spin-1 A_μ, pour le champ scalaire auxiliaire de Stueckelberg Φ et pour les champs de fantômes C et C^* ; identité de Ward ; représentation de Hadamard des divers propagateurs de Feynman et développement en séries de Taylor covariantes des coefficients correspondants). Cela nous a permis de construire, pour un état quantique de type Hadamard $|\psi\rangle$, la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé à la théorie de Stueckelberg. Comme applications de nos résultats, (i) nous avons considéré, en espace-temps de Minkowski, l'effet Casimir en dehors d'un milieu parfaitement conducteur et de bord plan, et (ii) nous avons obtenu, dans les espaces-temps de de Sitter et anti-de Sitter, une expression analytique exacte pour la valeur moyenne dans le vide du tenseur d'impulsion-énergie renormalisé du champ vectoriel massif.

Mots clés : Théories quantiques des champs en espace-temps courbe ; Electromagnétisme massif ; Mécanisme de Stueckelberg ; Action effective ; Développement de DeWitt-Schwinger ; Renormalisation de Hadamard ; Tenseur d'impulsion-énergie renormalisé ; Energie du vide ; Espaces-temps de Kerr-Newman, de de Sitter et anti-de Sitter ; Effet Casimir.

TABLE DES MATIÈRES

Introduction

On peut considérer que c'est Michael Faraday qui, en 1849, a introduit en physique le concept de "*champ*". Dans la première moitié du XIX$^{\mathrm{e}}$ siècle, il a réalisé l'importance de cette notion après avoir mis expérimentalement en évidence des liens étroits entre les phénomènes électriques et magnétiques. C'est, par la suite, James Clerk Maxwell qui a construit la théorie mathématique de l'*électromagnétisme* et, plus précisément, la théorie dynamique du champ électromagnétique publiée dans son travail de 1865 [1] ainsi que dans son ouvrage de 1873 [2] où, pour la première fois, toutes les équations qui portent son nom sont apparues. L'une des prédictions majeures de cette théorie est que la lumière est une onde électromagnétique se propageant dans le vide avec une vitesse finie $c \approx 2.998 \times 10^8\,\mathrm{m.s}^{-1}$. En fait, c'est une théorie qui unifie les interactions associées aux champs électrique et magnétique et qui, de plus, rend compte des phénomènes optiques. C'est aussi une *théorie de jauge*, du fait de l'invariance de ses équations par les transformations associées à la symétrie de jauge locale correspondant au groupe abélien $U(1)$.

Malgré le succès remarquable de l'électromagnétisme de Maxwell aux niveaux théorique et expérimental, la fin de XIX$^{\mathrm{e}}$ siècle et le début de XX$^{\mathrm{e}}$ siècle ont été secoués par de nombreux problèmes de physique restant encore ouverts. Afin d'élucider ces questions, les physiciens ont dû abandonner l'ancienne conception du monde ce qui les a conduit à formuler les bases de deux théories révolutionnaires : (i) la *relativité restreinte* et (ii) la *mécanique quantique*.

(i) La formulation de la théorie de Maxwell a été suivie de nombreuses recherches théoriques et expérimentales concernant l'éther, ce support matériel introduit dans le but d'expliquer la propagation mécanique des ondes électromagnétiques. Elles n'ont pas donné de résultats convenables. C'est finalement la relativité restreinte formulée par Albert Einstein en 1905 [3] qui a mis fin à la nécessité de tous ces travaux. Il faut noter que, dans les équations de Maxwell, les transformations de Lorentz [3–7] qui sont à la base de la relativité restreinte sont présentes intrinsèquement. Cette théorie révolutionnaire fournit une nouvelle vision de l'espace et le temps qui sont réunis en un concept indissociable appelé l'*espace-temps* dont la formulation géométrique a été faite par Hermann Minkowski [8]. Dans l'espace-temps de Minkowski, l'invariant relativiste ds^2 correspondant à l'intervalle infinitésimal ds séparant deux évènements est donné par

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dT^2 + dX^2 + dY^2 + dZ^2$$

où $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ est le tenseur métrique et $x^\mu = (T, X, Y, Z)$ sont les coordonnées d'un évènement.

(ii) Les théories physiques du XIX$^{\mathrm{e}}$ siècle, malgré leurs innombrables succès, ne pouvaient pas expliquer deux problèmes d'une importance fondamentale : le rayonnement du corps noir et l'effet photoélectrique. Il fallut attendre l'année 1900 pour la résolution du premier problème par Max Planck dans un travail où il faisait l'hypothèse de la quantification de l'énergie dans les échanges entre rayonnement et matière [9, 10]. La valeur élémentaire du quantum d'énergie ε est reliée à la fréquence ν de l'onde électromagnétique par la relation

$$\varepsilon = h\nu$$

où $h \approx 6.626 \times 10^{-34}$ J.s est la constante de Planck. En 1905, Albert Einstein a publié un de ses trois fameux articles dans lequel il expliquait l'effet photoélectrique en postulant que le rayonnement électromagnétique est composé de quanta d'énergie localisés dans l'espace appelés aujourd'hui les photons [11]. L'extension du comportement dual "onde-corpuscule" à la matière a été faite par Louis de Broglie dans sa thèse publiée en 1924 [12]. C'est, principalement, à la base de ces considérations théoriques de de Broglie que, à partir des années 1925, la mécanique quantique a fait son apparition avec les travaux de Werner Heisenberg, Max Born, Pascual Jordan, Wolfgang Pauli, Paul Adrian Maurice Dirac et Erwin Schrödinger (voir les Réfs. [13–19] ainsi que les références qu'elles contiennent).

Toutefois, cette mécanique quantique développée entre 1925 et 1927 présentait une incohérence fondamentale. Lors de la conception des bases de la mécanique quantique, la considération des principes de la relativité restreinte était négligée. Ainsi, l'équation de Schrödinger décrit la dynamique d'une particule massive non relativiste par le moyen de la fonction d'onde associée. De la même façon, lorsque Dirac dans ses premiers travaux ayant pour but d'étudier les processus d'émission et absorption du rayonnement, a quantifié le système physique électrons-champ électromagnétique (voir la Réf. [20, 21]), il l'a fait d'une manière qui s'est avérée par la suite de ne pas être compatible avec la relativité restreinte. En fait, la relativité restreinte a été introduite dans des modèles quantiques avec les travaux, entre 1926 et 1928, de :

- Oskar Klein et Walter Gordon qui ont établi indépendamment l'équation [1] (elle porte maintenant leurs noms) décrivant des particules massives scalaires (spin 0) [22, 23],
- Heisenberg, Born, Jordan et Pauli qui ont appliqué les règles de la mécanique quantique au champ électromagnétique dans l'espace vide [17, 24],
- Dirac qui a établi l'équation (elle porte maintenant son nom) décrivant la dynamique de l'électron [25].

Il faut noter que la conséquence du dernier travail mentionné a été l'explication du spin-1/2 de l'électron comme résultat de l'unification de la mécanique quantique avec la relativité restreinte ainsi que la prédiction de son antiparticule appelée positron.

1. En réalité, Schrödinger est le premier qui a écrit l'équation de Klein-Gordon pour tenter de décrire, mais sans succès, l'électron dans l'atome d'hydrogène.

La mécanique quantique ainsi que sa version relativiste a permis d'expliquer les divers effets observés (tel que les raies spectrales, l'émission spontanée, la structure hyperfine de l'hydrogène, l'effet Zeeman et autres) en décrivant la matière quantifiée par le moyen de sa fonction d'onde (voir, par exemple, les Réfs. [14, 16–18]), c'est-à-dire en “première quantification”. Cependant, dans les travaux sur les processus d'émission et absorption du rayonnement (voir la Réf. [20]), Dirac a déjà commencé à considérer la description du champ électromagnétique du point de vue des processus de création et de destruction des photons, c'est-à-dire en “seconde quantification”. C'est une vision fondamentalement différente par rapport à la première vision de la mécanique quantique appliquée à la matière. Cette incohérence dans la quantification a disparu avec les travaux de Jordan et Eugene Wigner [26], de Heisenberg et Pauli [27, 28] ainsi que d'Enrico Fermi [29, 30] et de Dirac [31], ces auteurs ayant montré que, comme les photons qui sont les quanta du champ électromagnétique, les particules de matière doivent elles aussi être considérées comme les quanta associés à leurs champs. Wendell Furry et Robert Oppenheimer [32] ainsi que Pauli et Victor Weisskopf [33] ont montré que cette approche pouvait être étendue aux antiparticules. Ce concept fondamental de création et de destruction des particules dont la description mathématique est réalisée par des opérateurs est en fait à la base de ce que l'on appelle aujourd'hui la *théorie quantique relativiste des champs* dont les propriétés peuvent être résumées en citant Steven Weinberg : “*Thus, the inhabitants of the universe were conceived to be a set of fields – an electron field, a proton field, an electromagnetic field – and particles were reduced in status to mere epiphenomena. In its essentials, this point of view [...] forms the central dogma of quantum field theory : the essential reality is a set of fields, subject to the rules of special relativity and quantum mechanics ; all else is derived as a consequence of the quantum dynamics of these fields.*”. Le lecteur intéressé pourrait consulter les références [34–36] sur la description historique détaillée de la naissance de la théorie quantique des champs.

Dans tous les travaux sur l'électrodynamique quantique, les physiciens théoriciens ont rencontré, à un moment ou un autre de leurs calculs, différentes sortes d'infinis. Ceux-ci apparaissent, par exemple, (i) dans le calcul de la self-énergie de l'électron étudiée par Julius Robert Oppenheimer [37] et Ivar Waller [38, 39], (ii) dans l'effet appelé Lamb-shift correspondant à la différence des niveaux d'énergie $2s_{1/2} - 2p_{1/2}$ (voir les Réfs. [40–43]) ou bien encore (iii) dans la détermination de l'anomalie du moment magnétique de l'électron par rapport à celui prédit par Dirac [25] (voir les Réfs. [44–48]). La méthode astucieuse pour pouvoir éliminer ces infinis consiste à les absorber dans les paramètres du problème physique en redéfinissant ceux-ci pour qu'ils soient finis. Cette méthode est appelée *renormalisation* [2] et elle a été conceptualisée par Weisskopf [49] et Hans Kramers [50]. La formulation moderne de l'électrodynamique quantique, avec des méthodes plus adaptées pour les calculs, a été développée par Julian Schwinger [51–57], Sin-Itiro Tomonaga [58–63] et Richard Feynman [64–69]. Freeman Dyson [70, 71] a montré l'équivalence de trois approches et a donné le critère de *renormalisabilité* des théories quantiques des champs permettant d'absorber les infinis dans un nombre fini de paramètres physiques.

Il est important de rajouter que les succès de l'électrodynamique quantique sont aussi dus à son invariance de jauge $U(1)$ héritée de l'électromagnétisme de Maxwell. On a étendu ce modèle de théorie des champs pour formuler les théories de jauge non-abéliennes (voir, par exemple, la Réf. [72]) qui se sont révélées adaptées pour décrire les autres interactions fondamentales. Ainsi, aujourd'hui, les trois interactions fondamentales sont décrites par des champs quantiques vectoriels du type bosonique (spin

2. Dans le cas de l'électrodynamique quantique, la procédure de renormalisation consiste à absorber les infinis dans la masse et la charge de l'électron. En fait, ces grandeurs intervenant dans les équations de la théorie sont appelées nues et ce sont elles que l'on redéfinit afin d'avoir les grandeurs associées mesurables dans les expériences.

entier) et on a :

- l'interaction électromagnétique dont le groupe de jauge est $U(1)$ avec le photon comme médiateur,
- l'interaction faible [73–76] dont le groupe de jauge est $SU(2)$ avec trois bosons vectoriels Z^0 et $W^{\pm}$ comme médiateurs,
- l'interaction forte [77, 78] dont le groupe de jauge est $SU(3)$ avec huit gluons comme médiateurs.

Il faut noter que c'est dans le cadre de l'unification des deux premières interactions en une seule interaction électrofaible faite par Sheldon Glashow, Abdus Salam et Steven Weiberg [79–82] que l'interaction faible est mieux comprise. Effectivement, contrairement aux interactions électromagnétique et forte qui sont considérées non massives [3], les champs de jauge associés à trois bosons de l'interaction faible sont massifs [4]. Leurs masses sont acquises par l'intermédiaire du mécanisme de Brout-Englert-Higgs-Hagen-Guralnik-Kibble [88–90] où un nouveau boson massif appelé Higgs associé au champ quantique scalaire (spin 0) intervient (voir aussi la Réf. [91]). En résumé, les champs fermioniques (spin demi-entier) décrivant toute la matière sont en interaction entre eux par l'intermédiaire de ces champs bosoniques. C'est l'image actuelle du *modèle standard* des particules élémentaires [86]. Il est aussi naturel d'envisager une *théorie de grande unification* qui fusionne les interactions électrofaible et forte en une seule (voir, par exemple, les Réfs. [92, 93] pour les premiers travaux sur le sujet). Cependant, il ne reste que la gravitation, la quatrième interaction fondamentale, qui échappe à une telle unification.

En 1915, Einstein a publié sa version définitive de la *relativité générale* [94] qui associe la gravitation à la géométrie courbe et dynamique de l'espace-temps. Dans cette théorie, contrairement à l'espace-temps de Minkowski qui est plat et statique, l'invariant

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

est donné en fonction d'un tenseur métrique $g_{\mu\nu}$ de signature Lorentzienne vérifiant les équations d'Einstein

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}.$$

Ici, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}$ est le tenseur géométrique d'Einstein, Λ est la constante cosmologique tandis que le tenseur d'impulsion-énergie $T_{\mu\nu}$ décrit le contenu matériel [5] se trouvant dans cet espace-temps. Dans le facteur de proportionnalité interviennent la vitesse de la lumière c et la constante de la gravitation de Newton $G_N \approx 6.674 \times 10^{-11}\,\mathrm{m}^3.\mathrm{kg}^{-1}.\mathrm{s}^{-2}$. Ces équations nous montrent comment la matière déforme la géométrie tandis que la géométrie indique à la matière sa dynamique. La théorie de la relativité générale d'Einstein a permis de rendre compte de nombreux phénomènes gravitationnels (l'avancée du périhélie de Mercure [95], la déviation de la lumière par des corps massifs [95, 96], l'effet de la gravitation sur le temps [97, 98]). De plus, son formalisme s'est révélé bien adapté pour la construction des modèles d'espace-temps susceptibles de décrire l'univers [99]. Un de ces modèles est l'espace-temps de

3. La masse du photon est habituellement considérée comme nulle, mais, en fait, les expériences terrestres et spatiales ne permettent d'évaluer que sa limite supérieure qui est actuellement de l'ordre de $10^{-18}\,\mathrm{eV.c}^{-2} \approx 2 \times 10^{-54}\,\mathrm{kg}$ [83–86]. La situation est similaire pour la masse des gluons qui devrait être inférieure à $1\,\mathrm{MeV.c}^{-2} \approx 2 \times 10^{-42}\,\mathrm{kg}$ [86, 87].

4. La masse du boson Z^0 est de l'ordre de $90\,\mathrm{GeV.c}^{-2} \approx 160 \times 10^{-27}\,\mathrm{kg}$ alors que celle de deux bosons $W^{\pm}$ est de l'ordre de $80\,\mathrm{GeV.c}^{-2} \approx 140 \times 10^{-27}\,\mathrm{kg}$ [86].

5. Ici, le contenu matériel doit être compris dans le sens suivant : ce sont les champs (classiques) décrivant la matière ainsi que les interactions.

Friedmann-Lemaître-Robertson-Walker [100–103] qui rentre dans le cadre de la théorie du Big Bang développée avec la découverte accidentelle en 1964 du fond diffus cosmologique [104, 105]. Ce modèle permet de rendre compte de l'évolution de l'univers durant son expansion [106–108] et de discuter sa composition (la matière du modèle standard de la physique des particules, la matière noire et l'énergie sombre). En particulier, il faut noter que la géométrie de de Sitter [109] décrit l'univers très primordial et la période de l'inflation [110–113]. Un autre succès remarquable de la relativité générale est la prédiction de l'existence des ondes gravitationnelles [114] et des trous noirs [115–119] dont la preuve indirecte date des années 1970 [120–125] alors que la détection directe est récente [126]. Il reste que la relativité générale est une théorie classique du champ gravitationnel décrivant les phénomènes astrophysiques à basse énergie pour les corps macroscopiques.

Aujourd'hui, les physiciens théoriciens tentent de construire une *théorie quantique de la gravitation* et de réaliser son unification avec les autres interactions (voir, par exemple, la gravité quantique à boucles [127], la supergravité [128], la théorie des supercordes et la théorie M [129]), mais il semble que l'on en est encore loin. Cette difficulté prend son origine au niveau conceptuel. En effet, tous les champs quantiques vus auparavant, quelles que soient leurs natures bosoniques ou fermioniques, se propagent sur l'espace-temps tandis que le champ gravitationnel est en fait l'espace-temps lui-même. Une autre complication technique vient du fait que la relativité générale est aussi une théorie de jauge mais que son groupe de symétrie local est celui des difféomorphismes de la variété espace-temps (invariance par rapport aux transformations infinitésimales des coordonnées) dont la structure est plus complexe que celles des groupes intervenant dans les théories de Yang-Mills. Par ailleurs, la théorie quantique de la gravitation traitée de manière perturbative avec les méthodes de la théorie quantique des champs n'est pas renormalisable. D'autre part, en supposant que la théorie quantique de la gravitation ne fait intervenir que les trois constantes fondamentales que sont la vitesse de la lumière c, la constante de Planck réduite $\hbar = h/(2\pi)$ et la constante de la gravitation G_N, il est possible de construire d'une manière unique un temps $t_P = \left(G_N\hbar/c^5\right)^{1/2} \approx 10^{-43}\,\mathrm{s}$ (temps de Planck), une longueur $l_P = \left(G_N\hbar/c^3\right)^{1/2} \approx 10^{-35}\,\mathrm{m}$ (longueur de Planck) et une énergie $E_P = \left(c^5\hbar/G_N\right)^{1/2} \approx 10^{19}\,\mathrm{GeV}$ (énergie de Planck). Ces ordres de grandeur nous laissent croire que les effets de la gravitation quantique ne pourront pas être observés dans les années à venir (à comparer avec l'énergie actuelle atteinte dans l'accélérateur de particules LHC qui est de l'ordre de $10^4\,\mathrm{GeV}$). Cependant, il faut rappeler que, du point de vue théorique, la relativité générale prédit l'existence d'espaces-temps contenant des singularités géométriques (par exemple, la singularité du Big Bang ou celles au "centre" des trous noirs [130]) et que, de ce point de vue, elle semble être une théorie incomplète. En partant du principe que l'univers ne contient pas d'infinis, on pourrait espérer que la théorie quantique de la gravitation remédie à ce genre des problèmes.

En absence d'une théorie quantique de la gravitation, on peut étudier le comportement des champs quantiques (y compris le champ des gravitons [6]) sur un *background* gravitationnel considéré classique, i.e., l'espace-temps de la relativité générale. On peut ainsi espérer découvrir et comprendre les éventuels effets quantiques qui pourront exister en champ fort, par exemple, au voisinage des trous noirs ou dans l'univers primordial. Cette approche est dite *théorie quantique des champs en espace-temps courbe* (voir les Réfs. [131–137] pour des revues et monographies sur le sujet) et on peut la considérer comme une approximation semi-classique de la gravitation quantique. Il est intéressant de rappeler qu'une telle approche est

6. Ici, le champ des gravitons (spin 2) est considéré dans le sens de la quantification de l'interaction gravitationnelle dans le cadre de la théorie quantique des champs habituelle où la particule non massive appelée graviton est associée à ce champ bosonique.

similaire à celle utilisée pour résoudre certains problèmes de la mécanique quantique lorsqu'une théorie complète de l'électrodynamique quantique n'existait pas encore. En effet, dans certains cas, le champ électromagnétique pouvait être considéré classiquement et en interaction avec la matière quantifiée comme, par exemple, les électrons dans un atome. Une telle approche consiste en fait à faire une approximation semi-classique de l'électrodynamique quantique. Comme cela s'est avéré par la suite, la plupart des résultats obtenus se sont révélés être en relatif accord avec la théorie complète. Un exemple important qui met en évidence la polarisation du vide dans l'électrodynamique quantique est l'effet de décalage de Lamb mentionné plus haut. Il faut rappeler que la représentation de la polarisation du vide (mais aussi de l'énergie du vide) utilisant les diagrammes de Feynman ne fait intervenir que les boucles fermées associées aux propagateurs des champs de la théorie. Le lecteur pourra consulter la Réf. [138] qui est une revue sur le sujet de l'énergie du vide dans le cadre du problème de la constante cosmologique Λ. Par conséquent, on doit chercher à généraliser en espace-temps courbe les concepts et les méthodes de la théorie quantique des champs habituelles.

Les premières études sur les effets de la gravitation quantique remontent aux travaux de Schrödinger de 1932 [139]. Depuis, cette approche semi-classique de la gravitation quantique a permis aux physiciens théoriciens d'obtenir les résultats fascinants concernant la création de particules par des champs gravitationnels dans le domaine de la cosmologie de l'univers primordial ainsi dans la physique quantique des trous noirs (voir les Réfs. [131–137] ainsi que les références qu'elles contiennent). En particulier, il faut mentionner séparément la découverte de la création de particules dans des univers en expansion faite par Leonard Parker [140] et l'émission spontanée des particules par des trous noirs et son caractère thermique faite par Stephen Hawking [141]. Depuis le milieu des années soixante-dix, la théorie quantique des champs en espace-temps courbe a connu une avancée considérable sur des sujets comme la définition du vide quantique, le concept de particules et leur création en espace-temps courbe, le développement de méthodes efficaces permettant de construire et d'évaluer le tenseur d'énergie-impulsion renormalisé (voir ci-dessous), l'étude des phénomènes quantiques dans divers espaces-temps importants en cosmologie (les espaces-temps de de Sitter ou Friedmann-Lemaître-Robertson-Walker) ou en physique des trous noirs (les espaces-temps de Schwarzschild, Reissner-Nordstrom, Kerr ou Kerr-Newman).

En théorie quantique des champs en espace-temps courbe, il est admis que la réaction en retour d'un champ quantique dans un état quantique normalisé $|\psi\rangle$ sur la géométrie de l'espace-temps est régie par les équations d'Einstein semi-classiques

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4} \langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle.$$

Ici, $G_{\mu\nu}$ est le tenseur d'Einstein $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}$ ou une généralisation d'ordre supérieur de ce tenseur géométrique alors que $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ est la valeur moyenne de l'opérateur tenseur d'impulsion-énergie $\widehat{T}_{\mu\nu}$ (il est à valeur sur les distributions) construit à partir des champs quantiques [7] de la théorie (y compris celui des gravitons). La quantité $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ est formellement infinie du fait du comportement singulier à courte distance des fonctions de Green du champ quantique. Afin d'extraire une contribution finie et physiquement raisonnable de cette quantité, il est nécessaire de la régulariser puis de renormaliser toutes les constantes de couplage de la théorie. La valeur moyenne renormalisée $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ est d'une importance fondamentale en théorie quantique des champs en espace-temps courbe [131–136] non seulement parce

7. Les champs quantiques sont en fait des opérateurs définis au sens des distributions.

qu'elle agit comme source dans les équations d'Einstein semi-classiques [8] mais aussi parce qu'elle nous permet d'analyser l'état quantique $|\psi\rangle$ sans faire aucune référence à son contenu en particules [9]. A partir du milieu des années soixante-dix, de grands efforts ont permis de développer des techniques rigoureuses de construction formelle et de calcul de la valeur moyenne renormalisée du tenseur d'impulsion-énergie $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ (voir les Réfs. [131–135] et leurs références sur le sujet). Les méthodes principales valables sous certaines conditions sont les suivantes : la méthode de la régularisation adiabatique, la régularisation dimensionnelle, l'approche par les fonctions ζ, la méthode du *point-splitting*, l'approximation de Brown-Ottewill-Page, l'approximation de DeWitt-Schwinger et la renormalisation dite de Hadamard. Dans ce manuscrit, nous utilisons les deux dernières méthodes :

(i) L'approximation de DeWitt-Schwinger est basée sur le développement de DeWitt-Schwinger des fonctions de Green associées aux champs quantiques massifs et de l'action effective de la théorie considérée. Cette approximation est utilisée dans la limite des grandes masses pour les champs se propageant sur un espace-temps courbe arbitraire. Par exemple, le lecteur peut se reporter aux Réfs. [131–133, 149–154] pour les bases théoriques ainsi qu'aux Réfs. [155–173] en ce qui concerne ses applications aux trous noirs, aux trous de ver, aux cordes noires, à l'univers très primordial décrit par l'espace-temps de de Sitter et aux univers de Friedmann-Lemaître-Robertson-Walker. Il faut noter que, formellement, elle peut être utilisée lorsque la longueur d'onde de Compton associée à ce champ massif est plus petite qu'une longueur caractéristique construite à partir de la courbure de l'espace-temps. De plus, il est important de rappeler que le développement de DeWitt-Schwinger de l'action effective est purement géométrique et que, par conséquent, le tenseur d'impulsion-énergie renormalisé qui lui est associé ne prend pas en compte l'état quantique du champ.

(ii) La renormalisation de Hadamard (pour sa formulation axiomatique, nous nous référons aux monographies de Robert Wald [134] et de Stephen Fulling [133]) est une extension de la méthode du *point-splitting* [131, 150, 151] qui a été développée en liaison avec la représentation de Hadamard des fonctions de Green (voir, par exemple, les Réfs. [174–188] sur le sujet). Il faut noter que cette représentation des fonctions de Green consiste à considérer le comportement singulier du type Hadamard en espace-temps courbe ou encore un état quantique du type Hadamard, i.e., un état imitant dans le régime ultraviolet le comportement du vide de Poincaré dans l'espace-temps de Minkowski.

De plus, dans ce manuscrit, nous nous intéressons aux théories quantiques des champs massifs en espace-temps courbe à l'ordre une boucle. En particulier, nous ne considérons que la contribution associée aux champs quantiques massifs pour la valeur moyenne du tenseur d'impulsion-énergie en laissant de côté la partie associée au champ des gravitons. En plus du champ scalaire massif et du champ de Dirac massif, nous discutons ici l'électromagnétisme massif de de Broglie-Proca [189–194] et celui de Stueckelberg [195–198]. Contrairement à la théorie massive de de Broglie-Proca, la théorie proposée par Stueckelberg

8. Donnons quelques exemples d'applications pratiques de ces équations qui ont été utilisées

- par Alexei Starobinsky [142] pour montrer que, après l'ère de Planck, les effets quantiques conduisent à l'inflation de l'univers, i.e., à une phase d'expansion exponentielle décrite par l'espace-temps de de Sitter,
- pour analyser la dynamique de l'évaporation des trous noirs à cause du rayonnement de Hawking (voir les Réfs. [143, 144] pour des travaux pionniers sur ce sujet),
- pour expliquer l'expansion accélérée de l'univers (voir, par exemple, la Réf. [145]).

9. Il faut préciser que, bien que le contenu d'un état en termes de particules soit bien défini en espace-temps de Minkowski pour un observateur inertiel, la notion de particules n'a plus de sens en espace-temps courbe [141, 146, 147] ou même encore en espace-temps de Minkowski pour un observateur accéléré [148]. Pour comprendre le contenu d'un état, il est nécessaire d'utiliser la notion de détecteur de particules introduite initialement par William George Unruh [148].

préserve la symétrie de jauge locale $U(1)$ de Maxwell. Il faut noter que la théorie non-Maxwellienne (voir les Réfs. [83, 85] pour les revues récentes), où un photon massif très léger est considéré, est nécessaire pour restreindre expérimentalement la masse du photon. Par ailleurs, la théorie de Stuekelberg pourrait être incluse aisément dans le modèle standard des particules élémentaires. En outre, certaines extensions du modèle standard basées sur la théorie des cordes incluent l'existence d'un photon noir très massif pouvant expliquer la matière noire de l'univers (voir, par exemple, la Réf. [199]).

Ce manuscrit constitué des trois articles publiés [200–202] est organisé en deux parties. Dans la première partie I de ce manuscrit, nous nous intéressons à l'action effective à l'ordre une boucle associée aux champs quantiques massifs. Nous présentons le formalisme associé au développement de DeWitt-Schwinger des fonctions de Green et de l'action effective. Puis, nous utilisons ce développement dans la limite des grandes masses en liaison avec un package écrit sous *Mathematica* permettant d'effectuer les calculs tensoriels efficacement afin d'obtenir une approximation de la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé à divers champs massifs (le champ scalaire massif, le champ de Dirac massif et le champ de Proca) se propageant sur l'espace-temps de Kerr-Newman. Ce résultat nous permet de retrouver les résultats existant dans la littérature pour les trous noirs de Schwarzschild, de Reissner-Nordström et de Kerr (et d'en corriger certains). En absence de résultats exacts en espace-temps de Kerr-Newman, notre approximation de la valeur moyenne renormalisée du tenseur d'impulsion-énergie pourrait être utile, par exemple, pour étudier l'effet de *backreaction* des champs quantiques massifs sur cet espace-temps ou sur ses modes quasi-normaux. Nous étudions d'ailleurs un problème simplifié de *backreaction* consistant à déterminer les modifications de la masse et du moment angulaire du trou noir vues par un observateur à l'infini, et dues à la présence des champs quantiques. Dans la deuxième partie II de ce manuscrit, nous nous intéressons à l'électromagnétisme massif de Stueckelberg. Nous discutons cette théorie sur un espace-temps courbe arbitraire de dimension quatre (invariance de jauge de la théorie classique et quantification covariante ; équations d'ondes pour le champ massif de spin-1 A_μ, pour le champ scalaire auxiliaire de Stueckelberg Φ et pour les champs de fantômes C et C^* ; identité de Ward ; représentation de Hadamard des divers propagateurs de Feynman et développement en séries de Taylor covariantes des coefficients correspondants). Cela nous permet de construire, pour un état quantique de type Hadamard $|\psi\rangle$, la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé à la théorie de Stueckelberg. Nous avons donné deux expressions différentes mais équivalentes du résultat, l'une d'entre elles nous permettant de discuter la limite de masse nulle dans le but de retrouver certains résultats de la théorie de Maxwell. Comme applications de nos résultats, (i) nous considérons, en espace-temps de Minkowski, l'effet Casimir en dehors d'un milieu parfaitement conducteur et de bord plan, et (ii) nous obtenons, dans les espaces-temps de de Sitter et anti-de Sitter, une expression analytique exacte pour la valeur moyenne dans le vide du tenseur d'impulsion-énergie renormalisé du champ vectoriel massif.

Dans ce manuscrit, nous considérons des espaces-temps courbes de dimension quatre $(\mathscr{M}, g_{\mu\nu})$ sans bord ($\partial\mathscr{M} = \varnothing$) ayant pour signature de métrique $(-,+,+,+)$. Nous utilisons les conventions géométriques de Hawking et Ellis [130] en ce qui concerne les définitions de la courbure scalaire R, du tenseur de Ricci R_{pq} et du tenseur de Riemann R_{pqrs} ainsi que la commutation des dérivées covariantes sous la forme

$$T^{p\ \ \cdots}_{\ q\ldots;sr} - T^{p\ \ \cdots}_{\ q\ldots;rs} = +R^{p}_{\ trs} T^{t\ \ \cdots}_{\ q\ldots} + \cdots - R^{t}_{\ qrs} T^{p\ \ \cdots}_{\ t\ldots} - \cdots .$$

De plus, nous travaillons en unités telles que $c = \hbar = G_N = 1$. Tout au long de ce manuscrit, nous utilisons les concepts de bi-scalaires, de bi-vecteurs et, plus généralement, de bi-tenseurs développés par certains mathématiciens [203–206] ainsi que par Bryce DeWitt [131, 149, 154, 207] et ses collaborateurs [150, 151]. En espace-temps courbe arbitraire, ces outils sont fondamentaux pour les méthodes de régularisation et renormalisation basées sur les représentations des fonctions de Green en coordonnées spatiales [10]. En particulier, nous utilisons la distance géodétique $\sigma(x,x')$, le bi-vecteur de transport parallèle $g_{\mu\nu'}(x,x')$ et le déterminant de van Vleck-Morette $\Delta(x,x')$ (voir, par exemple, la Réf. [207]) ainsi que le concept de développement en séries de Taylor covariantes des bi-tenseurs [179, 183, 208, 209]. En ce qui concerne leur définition, le lecteur pourra consulter l'annexe 2.7 du Chap. 2 de ce manuscrit.

10. En général, l'absence des symétries et la courbure de l'espace-temps empêchent de travailler avec la transformation de Fourier.

Partie I:

Action effective et tenseur d'impulsion-énergie renormalisé

CHAPITRE 1:

TENSEUR D'IMPULSION-ÉNERGIE RENORMALISÉ ASSOCIÉ AUX CHAMPS MASSIFS EN ESPACE-TEMPS DE KERR-NEWMAN

En espace-temps de dimension quatre, le développement de DeWitt-Schwinger de l'action effective associée à un champ massif, après renormalisation et dans la limite des grandes masses, se réduit aux termes construits à partir du coefficient de Gilkey-DeWitt a_3 qui est de nature purement géométrique. Sa variation par rapport à la métrique nous donne une bonne approximation analytique pour la valeur moyenne renormalisée du tenseur d'impulsion-énergie d'un champ quantique massif. Ici, à partir de l'expression générale de ce tenseur, nous avons obtenu analytiquement la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé au champ scalaire massif, au champ de Dirac massif et au champ de Proca se propageant sur l'espace-temps de Kerr-Newman. Ce résultat nous a permis de retrouver les résultats existant dans la littérature pour les trous noirs de Schwarzschild, de Reissner-Nordström et de Kerr (et d'en corriger certains). En absence de résultats exacts en espace-temps de Kerr-Newman, notre approximation de la valeur moyenne renormalisée du tenseur d'impulsion-énergie pourrait être utile, par exemple, pour étudier l'effet de *backreaction* des champs quantiques massifs sur cet espace-temps ou sur ses modes quasi-normaux. Nous avons d'ailleurs étudié un problème simplifié de *backreaction* consistant à déterminer les modifications de la masse et du moment angulaire du trou noir vues par un observateur à l'infini, et dues à la présence des champs quantiques.

Renormalized stress tensor for massive fields in Kerr-Newman spacetime

Andrei Belokogne[*] and Antoine Folacci[†]
Equipe Physique Théorique, Projet COMPA,
SPE, UMR 6134 du CNRS et de l'Université de Corse,
Université de Corse, BP 52, F-20250 Corte, France
(Dated: September 3, 2014)

In a four-dimensional spacetime, the DeWitt-Schwinger expansion of the effective action associated with a massive quantum field reduces, after renormalization and in the large mass limit, to a single term constructed from the purely geometrical Gilkey-DeWitt coefficient a_3 and its metric variation provides a good analytical approximation for the renormalized stress-energy tensor of the quantum field. Here, from the general expression of this tensor, we obtain analytically the renormalized stress-energy tensors of the massive scalar field, the massive Dirac field and the Proca field in Kerr-Newman spacetime. It should be noted that, even if, at first sight, the expressions obtained are complicated, their structure is in fact rather simple, involving naturally spacetime coordinates as well as the mass M, the charge Q and the rotation parameter a of the Kerr-Newman black hole and permitting us to recover rapidly the results already existing in the literature for the Schwarzschild, Reissner-Nordström and Kerr black holes (and to correct them in the latter case). In the absence of exact results in Kerr-Newman spacetime, our approximate renormalized stress-energy tensors could be very helpful, in particular to study the backreaction of massive quantum fields on this spacetime or on its quasinormal modes.

PACS numbers: 04.62.+v, 04.70.Dy

I. INTRODUCTION

Quantum field theory in curved spacetime (for reviews and monographs on this subject, see Refs. [1–7]) is a semiclassical approximation of quantum gravity which, by treating classically the spacetime metric $g_{\mu\nu}$ and considering from a quantum point of view all the other fields (including the graviton field to at least one-loop order for reasons of consistency), avoids the difficulties due to the nonrenormalizability of quantum gravity and provides a framework which permits us to study the low-energy consequences of a hypothetical "theory of everything". It should be recalled that this approach allowed theoretical physicists to obtain fascinating results concerning early universe cosmology and quantum black hole physics (see Refs. [1–7] and references therein) and, in particular, led to the discovery of particle creation in expanding universes by Parker [8] and of black hole radiance by Hawking [9]. Furthermore, this approach also provides the natural theoretical framework to analyze the cosmic microwave background (CMB) observations made in recent years.

In quantum field theory in curved spacetime, it is conjectured that the backreaction of a quantum field on the spacetime geometry is governed by the semiclassical Einstein equations

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle_{\text{ren}}. \tag{1}$$

(In this article, we use the geometrical conventions of Misner, Thorne and Wheeler [10] and we adopt units such that $\hbar = c = G_N = 1$.) In Eq. (1), $G_{\mu\nu}$ is the Einstein tensor $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$ or some higher-order generalization of this geometrical tensor while $\langle T_{\mu\nu} \rangle_{\text{ren}}$ is the renormalized stress-energy tensor (RSET) of the quantum field or, more precisely, the renormalized expectation value of the stress-energy tensor operator associated with the quantum field.

The semiclassical Einstein equations (1) have permitted Starobinsky to show that, after the Planck era, quantum effects lead to an inflationary universe, i.e., a universe with an exponentially expanding de Sitter phase [11]. They have been also used by several authors to analyze the dynamics of evaporating black holes due to Hawking radiation (see, e.g., Refs. [12, 13] for pioneering work on this topic) and, more recently, they have been considered to explain the acceleration of the expansion of the universe (see, e.g., Ref. [14]). But unfortunately, and despite the impressive successes mentioned above, the backreaction problem is in general difficult to tackle, not only because the semiclassical Einstein equations (1) are a set of coupled nonlinear hyperbolic partial differential equations but also because it is in general difficult to define the right-hand side of these equations, i.e., to construct the RSET. Indeed, the expectation value of the stress-energy tensor operator is formally infinite and it is necessary to regularize it, i.e., to extract from this formally infinite quantity a meaningful finite part, and then to renormalize all the coupling constants appearing in the problem in order to remove the remaining infinite part. Much work has been done since the mid-1970s in connection with this subject (see Refs. [1–7] and references therein) and, currently, we have at our disposal

[*] belokogne.andrei@gmail.com
[†] folacci@univ-corse.fr

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

some powerful procedures (such as the adiabatic regularization method, the dimensional regularization method, the ζ-function approach, the point-splitting method...) permitting us to construct theoretically and without ambiguity the RSET.

It is however important to note that, in four-dimensional gravitational backgrounds, except if we work under very strong hypotheses (e.g., if we consider field theories in maximally symmetric spacetimes or if we study massless or conformally invariant field theories on very particular spacetimes), it is in general impossible to obtain an analytical expression for the RSET. In fact, in most cases, it is even impossible to construct, from a practical point of view, the RSET and, when this is possible, it is necessary in many cases to perform a numerical analysis in order to extract the physical content of the RSET and, of course, this does not simplify the backreaction problem. So, it is interesting to note that various approaches have been developed which permit us to deal with situations presenting a "lower degree of symmetry" and to construct, in this context, accurate analytical approximations of the RSET (see, e.g., Refs. [15–17] for the theoretical bases of the "Brown-Ottewill-Page approximation" which is limited to static Einstein spacetimes or Refs. [18–21] for the theoretical bases of the "DeWitt-Schwinger approximation" which can be used in an arbitrary spacetime but is limited to massive fields in the large mass limit [22]). From a theoretical point of view, such analytical approximations could be helpful to find self-consistent solutions to Eq. (1).

Here, we focus on the DeWitt-Schwinger approximation of the RSET. It is based on the DeWitt-Schwinger expansion of the effective action associated with a massive quantum field and, formally, it can be used only when the Compton length associated with the massive field is much less than a characteristic length constructed from the curvature of spacetime. In this context, an analytical expression for the RSET is directly obtained from the metric variation of the renormalized effective action associated with the massive field. Here, it is important to recall that the DeWitt-Schwinger expansion of the effective action is purely geometrical and to note that, as a consequence, the approximate RSET does not take into account the quantum state of the field. The literature concerning the DeWitt-Schwinger approximation and its applications is rich (see, e.g., Refs. [23–37] for applications to black holes, wormholes and black strings and Refs. [38, 39] for a recent work concerning the Friedmann-Lemaître-Robertson-Walker universes) and it is important to note that the approximate RSETs obtained from the DeWitt-Schwinger expansion of the effective action seem to be in good agreement with exact results (see Ref. [26] where this has been discussed for the Reissner-Nordström black hole).

In this article, we intend to use the DeWitt-Schwinger approximation in order to construct the approximate RSET for massive fields in Kerr-Newman spacetime. Despite its physical importance, this particular spacetime has never been considered previously and this is certainly due to the complexity of the calculations involved. Let us remark that the problems treated in the references previously mentioned mainly concern spherically or cylindrically symmetric spacetimes and/or Einstein spacetimes (i.e., spacetimes satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$) and that the Kerr-Newman spacetime does not belong to one of these categories. To facilitate our work, we shall write the general expression of the RSET in an irreducible form in order to reduce substantially the number of its terms and the size of calculations and we shall use *Mathematica* packages allowing us to perform tensor algebra very efficiently. We shall then obtain, in Sec. II (see also Appendix A for details), analytical expressions for the RSET of the massive scalar field, the massive Dirac field and the Proca field in Kerr-Newman spacetime. It should be noted that, even if, at first sight, the expressions we shall display are complicated, their structure is in fact rather simple, involving naturally spacetime coordinates as well as the mass M, the charge Q and the rotation parameter a of the Kerr-Newman black hole and permitting us to recover rapidly, in Sec. III, the results already existing in the literature for the Schwarzschild black hole [23, 40], for the Reissner-Nordström black hole [26, 30] and for the Kerr black hole [24, 25]. We shall moreover correct the result obtained for the Dirac field in the latter case. In Sec. IV, we shall conclude by mentioning some possible applications of our results and by considering the shift in mass and angular momentum of the Kerr-Newman black hole dressed with a massive quantum field.

Before entering into the technical part of our work, it seems to us necessary to recall that, due to its fundamental importance in connection with the Hawking effect, the construction of the RSET in black hole spacetimes has often been discussed in the past 40 years and, in Schwarzschild and Reissner-Norström spacetimes (i.e., in static spherically symmetric black holes), we have now at our disposal exact expressions (which, however, must be analyzed numerically) for the RSET associated with various quantum fields and various vacuum states (see, e.g., Refs. [41–46] for pioneering work concerning massless fields in Schwarzschild spacetime and Ref. [26] for results concerning both massless and massive fields in Schwarzschild and Reissner-Nordström spacetimes). For stationary axisymmetric black holes, the situation is less clear and much more complex. In Kerr spacetime (see, e.g., Refs. [47–51] and references therein), for massless fields, due both to problems linked with the quantization process in this spacetime [47, 52] and to the complexity of the mode solutions of the wave equation, the RSET has been calculated only in specific locations and, for massive fields, to our knowledge, there exists no result concerning the RSET. In Kerr-Newman spacetime, the situation is even worse: we have found no results for this tensor in the literature.

We now describe the formalism we shall use in the following. We work in a four-dimensional spacetime

$(\mathcal{M}, g_{ab})$ without boundary and we consider more particularly the massive scalar field ϕ solution of the Klein-Gordon equation

$$(\Box - m^2 - \xi R)\phi = 0, \tag{2}$$

the massive spinor field ψ solution of the Dirac equation

$$(\gamma^\mu \nabla_\mu + m)\psi = 0 \tag{3}$$

and the massive vector field A^μ solution of the Proca equation

$$(g_{\mu\nu}\Box - m^2 g_{\mu\nu} - \nabla_\mu\nabla_\nu - R_{\mu\nu})A^\nu = 0. \tag{4}$$

In the wave equations (2)-(4), m denotes the mass of the fields, in the Klein-Gordon equation (2), ξ is a dimensionless factor which accounts for the possible coupling between the scalar field and the gravitational background and in the Dirac equation (3), γ^μ denote the usual Dirac matrices. We recall that, after renormalization and in the large mass limit, the DeWitt-Schwinger expansion of the effective action associated with the massive scalar field, the massive Dirac field and the Proca field can be constructed from the Gilkey-DeWitt coefficient a_3 and reduces to [18–21]

TABLE I. The coefficients c_i defining the renormalized effective action (5).

	Scalar field ($s=0$)	Dirac field ($s=1/2$)	Proca field ($s=1$)
c_1	$(1/2)\xi^2 - (1/5)\xi + 1/56$	-3/280	-27/280
c_2	$1/140$	1/28	9/28
c_3	$-(\xi - 1/6)^3$	1/864	-5/72
c_4	$(1/30)(\xi - 1/6)$	-1/180	31/60
c_5	$-8/945$	-25/756	-52/63
c_6	$2/315$	47/1260	-19/105
c_7	$-(1/30)(\xi - 1/6)$	-7/1440	-1/10
c_8	$1/1260$	19/1260	61/140
c_9	$17/7560$	29/7560	-67/2520
c_{10}	$-1/270$	-1/108	1/18

$$\begin{aligned} W_{\text{ren}} = \frac{1}{192\pi^2 m^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\; \big(& c_1\, R\Box R + c_2\, R_{pq}\Box R^{pq} + c_3\, R^3 + c_4\, RR_{pq}R^{pq} + c_5\, R_{pq}R^{p}{}_{r}R^{qr} \\ & + c_6\, R_{pq}R_{rs}R^{prqs} + c_7\, RR_{pqrs}R^{pqrs} + c_8\, R_{pq}R^{p}{}_{rst}R^{qrst} + c_9\, R_{pqrs}R^{pquv}R^{rs}{}_{uv} + c_{10}\, R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv}\big). \end{aligned} \tag{5}$$

Here, the coefficients c_i depend on the field and are given in Table I. In Eq. (5) the integrand is constituted by ten Riemann polynomials of order six (in the derivatives of the metric tensor) and rank zero (number of free indices). It should be noted that terms in $1/m^4$, $1/m^6$ ... are also present in the full expression of the renormalized effective action W_{ren} (see, e.g., Refs. [19–21]) but here, because we assume a large enough mass m, we can neglect them. It is also important to recall that, in the large mass limit, the nonlocal contribution to W_{ren} associated with the quantum state of the massive field is not taken into account. So, the renormalized effective action W_{ren} is a purely geometrical object.

By functional derivation of the effective action (5) with respect to the metric tensor $g_{\mu\nu}$, we obtain a purely geometrical approximation for the RSET associated with the massive fields. We have

$$\langle\, T^{\mu\nu}\,\rangle_{\text{ren}} = \frac{2}{\sqrt{-g}}\frac{\delta W_{\text{ren}}}{\delta g_{\mu\nu}} \tag{6}$$

which can be written explicitly [53]

$$\begin{aligned} (96\pi^2 m^2)\langle\, T_{\mu\nu}\,\rangle_{\text{ren}} = {} & d_1\,(\Box R)_{;\mu\nu} + d_2\,\Box\Box R_{\mu\nu} + d_3\, RR_{;\mu\nu} + d_4\,(\Box R)R_{\mu\nu} + d_5\, R_{;p(\mu}R^{p}{}_{\nu)} + d_6\, R\Box R_{\mu\nu} \\ & + d_7\, R_{p(\mu}\Box R^{p}{}_{\nu)} + d_8\, R^{pq}R_{pq;(\mu\nu)} + d_9\, R^{pq}R_{p(\mu;\nu)q} + d_{10}\, R^{pq}R_{\mu\nu;pq} + d_{11}\, R^{;pq}R_{p\mu q\nu} + d_{12}\,(\Box R^{pq})R_{p\mu q\nu} \\ & + d_{13}\, R^{pq;r}{}_{(\mu}R_{|rqp|\nu)} + d_{14}\, R^{p}{}_{(\mu}{}^{;qr}R_{|pqr|\nu)} + d_{15}\, R^{pqrs}R_{pqrs;(\mu\nu)} + d_{16}\, R_{;\mu}R_{;\nu} + d_{17}\, R_{;p}R^{p}{}_{(\mu;\nu)} \\ & + d_{18}\, R_{;p}R_{\mu\nu}{}^{;p} + d_{19}\, R^{pq}{}_{;\mu}R_{pq;\nu} + d_{20}\, R^{pq}{}_{;(\mu}R_{\nu)p;q} + d_{21}\, R^{p}{}_{\mu;q}R_{p\nu}{}^{;q} + d_{22}\, R^{p}{}_{\mu;q}R^{q}{}_{\nu;p} + d_{23}\, R^{pq;r}R_{rqp(\mu;\nu)} \\ & + d_{24}\, R^{pq;r}R_{p\mu q\nu;r} + d_{25}\, R^{pqrs}{}_{;\mu}R_{pqrs;\nu} + d_{26}\, R^{pqr}{}_{\mu;s}R_{pqr\nu}{}^{;s} + d_{27}\, R^2R_{\mu\nu} + d_{28}\, RR_{p\mu}R^{p}{}_{\nu} + d_{29}\, R^{pq}R_{pq}R_{\mu\nu} \\ & + d_{30}\, R^{pq}R_{p\mu}R_{q\nu} + d_{31}\, RR^{pq}R_{p\mu q\nu} + d_{32}\, R^{pr}R^{q}{}_{r}R_{p\mu q\nu} + d_{33}\, R^{pq}R^{r}{}_{(\mu}R_{|rqp|\nu)} + d_{34}\, RR^{pqr}{}_{\mu}R_{pqr\nu} \\ & + d_{35}\, R_{\mu\nu}R^{pqrs}R_{pqrs} + d_{36}\, R^{p}{}_{(\mu}R^{qrs}{}_{|p|}R_{|qrs|\nu)} + d_{37}\, R^{pq}R^{rs}{}_{p\mu}R_{rsq\nu} + d_{38}\, R_{pq}R^{prqs}R_{r\mu s\nu} + d_{39}\, R_{pq}R^{prs}{}_{\mu}R^{q}{}_{rs\nu} \\ & + d_{40}\, R^{pqrs}R_{pqt\mu}R_{rs}{}^{t}{}_{\nu} + d_{41}\, R^{prqs}R^{t}{}_{pq\mu}R_{trs\nu} + d_{42}\, R^{pqr}{}_{s}R_{pqrt}R^{s}{}_{\mu}{}^{t}{}_{\nu} \\ & + g_{\mu\nu}[d_{43}\,\Box\Box R + d_{44}\, R\Box R + d_{45}\, R_{;pq}R^{pq} + d_{46}\, R_{pq}\Box R^{pq} + d_{47}\, R_{pq;rs}R^{prqs} + d_{48}\, R_{;p}R^{;p} + d_{49}\, R_{pq;r}R^{pq;r} \\ & + d_{50}\, R_{pq;r}R^{pr;q} + d_{51}\, R_{pqrs;t}R^{pqrs;t} + d_{52}\, R^3 + d_{53}\, RR_{pq}R^{pq} + d_{54}\, R_{pq}R^{p}{}_{r}R^{qr} + d_{55}\, R_{pq}R_{rs}R^{prqs} \\ & + d_{56}\, RR_{pqrs}R^{pqrs} + d_{57}\, R_{pq}R^{p}{}_{rst}R^{qrst} + d_{58}\, R_{pqrs}R^{pquv}R^{rs}{}_{uv} + d_{59}\, R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv}]. \end{aligned} \tag{7}$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

TABLE II. The coefficients d_i defining the expansion of the approximate RSET (7) on the FKWC basis.

FKWC basis		Coefficients d_i	Scalar field ($s=0$)	Dirac field ($s=1/2$)	Proca field ($s=1$)
$\mathcal{R}^2_{6,1}$	$(\Box R)_{;\mu\nu}$	d_1	$\xi^2-(2/5)\xi+3/70$	1/70	9/70
	$\Box\Box R_{\mu\nu}$	d_2	$-1/140$	-1/28	-9/28
$\mathcal{R}^2_{\{2,0\}}$	$RR_{;\mu\nu}$	d_3	$-6\,(\xi-1/6)[\xi^2-(1/3)\xi+1/30]$	-1/120	-1/10
	$(\Box R)R_{\mu\nu}$	d_4	$-(\xi-1/6)(\xi-1/5)$	1/120	-7/30
	$R_{;p(\mu}R^{p}{}_{\nu)}$	d_5	$(1/15)(\xi-1/7)$	23/840	13/35
	$R\Box R_{\mu\nu}$	d_6	$(1/10)(\xi-1/6)$	1/40	-7/60
	$R_{p(\mu}\Box R^{p}{}_{\nu)}$	d_7	$1/42$	29/420	337/210
	$R^{pq}R_{pq;(\mu\nu)}$	d_8	$(1/15)(\xi-2/7)$	-19/420	22/105
	$R^{pq}R_{p(\mu;\nu)q}$	d_9	$2/105$	61/420	34/105
	$R^{pq}R_{\mu\nu;pq}$	d_{10}	$-1/70$	-11/105	-107/210
	$R^{;pq}R_{p\mu q\nu}$	d_{11}	$(2/15)(\xi-3/14)$	-1/105	1/21
	$(\Box R^{pq})R_{p\mu q\nu}$	d_{12}	$-1/105$	-17/210	-22/35
	$R^{pq;r}{}_{(\mu}R_{\lvert rqp\rvert\nu)}$	d_{13}	$4/105$	13/105	46/35
	$R^{p}{}_{(\mu}{}^{;qr}R_{\lvert pqr\rvert\nu)}$	d_{14}	$2/35$	16/105	116/105
	$R^{pqrs}R_{pqrs;(\mu\nu)}$	d_{15}	$-(1/15)(\xi-3/14)$	1/210	-1/42
$\mathcal{R}^2_{\{1,1\}}$	$R_{;\mu}R_{;\nu}$	d_{16}	$-6(\xi-1/4)(\xi-1/6)^2$	0	-1/24
	$R_{;p}R^{p}{}_{(\mu;\nu)}$	d_{17}	$-(1/5)(\xi-3/14)$	19/840	83/210
	$R_{;p}R_{\mu\nu}{}^{;p}$	d_{18}	$(1/5)(\xi-17/84)$	-1/420	-41/84
	$R^{pq}{}_{;\mu}R_{pq;\nu}$	d_{19}	$(1/15)(\xi-1/4)$	0	31/60
	$R^{pq}{}_{;(\mu}R_{\nu)p;q}$	d_{20}	0	-1/60	-14/15
	$R^{p}{}_{\mu;q}R_{p\nu}{}^{;q}$	d_{21}	$-1/210$	-1/140	221/210
	$R^{p}{}_{\mu;q}R^{q}{}_{\nu;p}$	d_{22}	$1/42$	3/35	113/210
	$R^{pq;r}R_{rqp(\mu;\nu)}$	d_{23}	$-1/105$	-1/21	5/21
	$R^{pq;r}R_{p\mu q\nu;r}$	d_{24}	$-1/70$	-11/105	-107/210
	$R^{pqrs}{}_{;\mu}R_{pqrs;\nu}$	d_{25}	$-(1/15)(\xi-13/56)$	1/420	-17/840
	$R^{pqr}{}_{\mu;s}R_{pqr\nu}{}^{;s}$	d_{26}	$-1/70$	-4/105	-29/105
$\mathcal{R}^2_{6,3}$	$R^2R_{\mu\nu}$	d_{27}	$3(\xi-1/6)^3$	-1/288	5/24
	$RR_{p\mu}R^{p}{}_{\nu}$	d_{28}	$-(2/15)(\xi-1/6)$	-7/360	-2/5
	$R^{pq}R_{pq}R_{\mu\nu}$	d_{29}	$-(1/30)(\xi-1/6)$	1/180	-31/60
	$R^{pq}R_{p\mu}R_{q\nu}$	d_{30}	$-2/315$	-1/252	1/21
	$RR^{pq}R_{p\mu q\nu}$	d_{31}	$(1/15)(\xi-1/6)$	11/360	-19/30
	$R^{pr}R^{q}{}_{r}R_{p\mu q\nu}$	d_{32}	$1/315$	13/1260	33/35
	$R^{pq}R^{r}{}_{(\mu}R_{\lvert rqp\rvert\nu)}$	d_{33}	$1/315$	97/1260	-139/105
	$RR^{pqr}{}_{\mu}R_{pqr\nu}$	d_{34}	$(1/15)(\xi-1/6)$	7/720	1/5
	$R_{\mu\nu}R^{pqrs}R_{pqrs}$	d_{35}	$(1/30)(\xi-1/6)$	7/1440	1/10
	$R^{p}{}_{(\mu}R^{qrs}{}_{\lvert p\rvert}R_{\lvert qrs\rvert\nu)}$	d_{36}	$-4/315$	-73/1260	-74/105
	$R^{pq}R^{rs}{}_{p\mu}R_{rsq\nu}$	d_{37}	$-2/315$	19/504	-5/42
	$R_{pq}R^{prqs}R_{r\mu s\nu}$	d_{38}	$4/315$	73/1260	74/105
	$R_{pq}R^{prs}{}_{\mu}R^{q}{}_{rs\nu}$	d_{39}	$-1/315$	-97/1260	-71/105
	$R^{pqrs}R_{pqt\mu}R_{rs}{}^{t}{}_{\nu}$	d_{40}	$2/315$	73/2520	37/105
	$R^{prqs}R^{t}{}_{pq\mu}R_{trs\nu}$	d_{41}	$4/63$	239/1260	97/105
	$R^{pqr}{}_{s}R_{pqrt}R^{s}{}_{\mu}{}^{t}{}_{\nu}$	d_{42}	$-2/315$	-73/2520	-37/105
$\mathcal{R}^0_{6,1}$	$\Box\Box R$	d_{43}	$-\xi^2+(2/5)\,\xi-11/280$	1/280	9/280
$\mathcal{R}^0_{\{2,0\}}$	$R\Box R$	d_{44}	$6(\xi-1/6)[\xi^2-(1/3)\,\xi+1/40]$	-1/240	19/120
	$R_{;pq}R^{pq}$	d_{45}	$-(1/30)(\xi-3/14)$	1/420	-1/84
	$R_{pq}\Box R^{pq}$	d_{46}	$-(1/15)(\xi-5/28)$	1/105	-223/420
	$R_{pq;rs}R^{prqs}$	d_{47}	$(4/15)(\xi-1/7)$	3/70	79/105
$\mathcal{R}^0_{\{1,1\}}$	$R_{;p}R^{;p}$	d_{48}	$6[\xi^3-(13/24)\xi^2+(17/180)\xi-53/10080]$	-1/672	163/1680
	$R_{pq;r}R^{pq;r}$	d_{49}	$-(1/15)(\xi-13/56)$	3/280	-17/56
	$R_{pq;r}R^{pr;q}$	d_{50}	$-1/420$	-1/280	11/420
	$R_{pqrs;t}R^{pqrs;t}$	d_{51}	$(1/15)(\xi-19/112)$	1/168	51/560
$\mathcal{R}^0_{6,3}$	R^3	d_{52}	$-(1/2)(\xi-1/6)^3$	1/1728	-5/144
	$RR_{pq}R^{pq}$	d_{53}	$(1/60)(\xi-1/6)$	-1/360	31/120
	$R_{pq}R^{p}{}_{r}R^{qr}$	d_{54}	$1/1890$	-1/945	1/630
	$R_{pq}R_{rs}R^{prqs}$	d_{55}	$-1/630$	1/315	-53/105
	$RR_{pqrs}R^{pqrs}$	d_{56}	$-(1/60)(\xi-1/6)$	-7/2880	-1/20
	$R_{pq}R^{p}{}_{rst}R^{qrst}$	d_{57}	$(2/15)(\xi-1/6)$	7/360	2/5
	$R_{pqrs}R^{pquv}R^{rs}{}_{uv}$	d_{58}	$-(1/15)(\xi-47/252)$	-61/15120	-263/2520
	$R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv}$	d_{59}	$-(4/15)(\xi-41/252)$	-43/1512	-106/315

This formula is the basic cornerstone of our calculations and it is necessary to comment on it briefly. It should be noted that the expression of the RSET has been written in an "irreducible form". Indeed, if we do not carefully take into account the symmetries of the Riemann tensor as well as the Bianchi identities, the metric variation of the renormalized effective action (5) could lead to an expression with many of the terms which are linearly dependent in a nontrivial way. As a consequence, the resulting expression contains too many terms (this is the case in most of the articles dealing with the DeWitt-Schwinger approximation) and, in practice, this increases significantly the size of calculations. For that reason, in Ref. [53], we have expanded the RSET on a standard basis constituted by Riemann polynomials of order six in the derivatives of the metric tensor and constructed from group theoretical considerations by Fulling, King, Wybourne and Cummings (FKWC) [54]. This basis is described in Secs. 2.1 and 2.2 of Ref. [53] and displayed in Table II where we use, furthermore, the FKWC notations $\mathcal{R}^r_{s,q}$ and $\mathcal{R}^r_{\{\lambda_1\dots\}}$ to denote the various subspaces of the space of Riemann polynomials of rank r (see Ref. [54] for more precision). It should be finally noted that there also exist two geometrical identities [see Eqs. (3.21) and (3.22) in Ref. [53]] coming from a topological and a geometrical constraint due to the four-dimensional nature of spacetime. We could have used them in order to eliminate two other terms in (7) but we have chosen to work with the FKWC basis of Riemann polynomials of order six which can be used in any dimension.

II. APPROXIMATE RENORMALIZED STRESS TENSORS IN KERR-NEWMAN SPACETIME

In this section, we consider the massive scalar field, the massive Dirac field and the Proca field in the Kerr-Newman spacetime and we provide, in the large mass limit, the associated explicit expressions of the approximate RSET $\langle T^\mu{}_\nu\rangle_{\rm ren}$. We work with Boyer-Lindquist coordinates and the spacetime metric then takes the form [10]

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\,dtd\varphi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta\,d\varphi^2 + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 \quad (8)$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Here M, Q and $J = aM$ are the mass, the charge and the angular momentum of the black hole while a is the so-called rotation parameter and we assume $M^2 \geq a^2 + Q^2$. The outer horizon is located at $r_+ = M + \sqrt{M^2 - (a^2+Q^2)}$, the largest root of Δ.

By using the general formula (7) and Table II in connection with (8), we can obtain explicitly $\langle T^\mu{}_\nu\rangle_{\rm ren}$. Of course, due to the complexity of the Kerr-Newman metric, the calculations involved cannot be done by hand. For this reason, we have written the package *SETArbitraryST* (available upon request from the first author). It is based on the suite of *Mathematica* packages *xAct* [55] which permits us to perform tensor algebra very efficiently.

The explicit expressions of the nonzero components of $\langle T^\mu{}_\nu\rangle_{\rm ren}$ can be written in the same form for the different massive fields. We have

$$\langle T^t{}_t\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2\,(r^2+a^2\cos^2\theta)^9}\sum_{p=0}^{5}\left\{\sum_{q=0}^{3} A^{tt}{}_{p,q}\,[\theta, M/r]\left(\frac{Q^2}{M^2}\right)^q\right\}\left(\frac{a}{r}\right)^{2p}, \quad (9)$$

$$\langle T^r{}_r\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2\,(r^2+a^2\cos^2\theta)^9}\sum_{p=0}^{5}\left\{\sum_{q=0}^{3} A^{rr}{}_{p,q}\,[\theta, M/r]\left(\frac{Q^2}{M^2}\right)^q\right\}\left(\frac{a}{r}\right)^{2p}, \quad (10)$$

$$\langle T^\theta{}_\theta\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2\,(r^2+a^2\cos^2\theta)^9}\sum_{p=0}^{5}\left\{\sum_{q=0}^{3} A^{\theta\theta}{}_{p,q}\,[\theta, M/r]\left(\frac{Q^2}{M^2}\right)^q\right\}\left(\frac{a}{r}\right)^{2p}, \quad (11)$$

$$\langle T^\varphi{}_\varphi\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2\,(r^2+a^2\cos^2\theta)^9}\sum_{p=0}^{5}\left\{\sum_{q=0}^{3} A^{\varphi\varphi}{}_{p,q}\,[\theta, M/r]\left(\frac{Q^2}{M^2}\right)^q\right\}\left(\frac{a}{r}\right)^{2p}, \quad (12)$$

$$\langle T^t{}_\varphi\rangle_{\rm ren} = \frac{M^2 r^{11}\sin^2\theta}{20160\pi^2 m^2\,(r^2+a^2\cos^2\theta)^9}\sum_{p=0}^{5}\left\{\sum_{q=0}^{3} A^{t\varphi}{}_{p,q}\,[\theta, M/r]\left(\frac{Q^2}{M^2}\right)^q\right\}\left(\frac{a}{r}\right)^{2p+1}, \quad (13)$$

$$\langle T^{\varphi}{}_{t}\rangle_{\rm ren} = \frac{M^2 r^9}{20160\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{4} \left\{ \sum_{q=0}^{3} A^{\varphi t}{}_{p,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^{2p+1}, \tag{14}$$

$$\langle T^{r}{}_{\theta}\rangle_{\rm ren} = \frac{M^2 r^{11}\sin\theta\cos\theta}{360\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{4} \left\{ \sum_{q=0}^{3} A^{r\theta}{}_{p,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^{2p+2}, \tag{15}$$

$$\langle T^{\theta}{}_{r}\rangle_{\rm ren} = \frac{M^2 r^{9}\sin\theta\cos\theta}{360\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{3} \left\{ \sum_{q=0}^{2} A^{\theta r}{}_{p,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q \right\} \left(\frac{a}{r}\right)^{2p+2}. \tag{16}$$

Here, the coefficients $A^{\mu\nu}{}_{p,q}$ are polynomials of the variables $\cos^2\theta$ and M/r with coefficients depending on the field. The interested reader can find these coefficients for the massive scalar field, the massive Dirac field and the Proca field in the subsections A 1, A 2 and A 3 of the Appendix.

III. SPECIAL CASES : APPROXIMATE RENORMALIZED STRESS TENSORS IN SCHWARZSCHILD, REISSNER-NORDSTRÖM AND KERR SPACETIMES

The structure of the expressions (9)–(16) permits us to recover very quickly the results corresponding to the special cases of the Schwarzschild, Reissner-Nordström and Kerr spacetimes.

By taking $Q=0$ and $a=0$ we recover the results obtained in Schwarzschild spacetime by Frolov and Zelnikov (see Ref. [23] and Sec. 11.3.7 of Ref. [40]). The nonzero components of $\langle T^{\mu}{}_{\nu}\rangle_{\rm ren}$ are

$$\langle T^{t}{}_{t}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} A^{tt}{}_{0,0}\left[\theta, M/r\right], \tag{17}$$

$$\langle T^{r}{}_{r}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} A^{rr}{}_{0,0}\left[\theta, M/r\right], \tag{18}$$

$$\langle T^{\theta}{}_{\theta}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} A^{\theta\theta}{}_{0,0}\left[\theta, M/r\right], \tag{19}$$

$$\langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} A^{\varphi\varphi}{}_{0,0}\left[\theta, M/r\right], \tag{20}$$

where the coefficients $A^{\mu\nu}{}_{0,0}$ do not depend explicitly of θ. Due to the spherical symmetry of the Schwarzschild black hole, $A^{\theta\theta}{}_{0,0} = A^{\varphi\varphi}{}_{0,0}$ and, as a consequence, $\langle T^{\theta}{}_{\theta}\rangle_{\rm ren} = \langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren}$. It should be noted that, for the Dirac field, Frolov and Zelnikov have forgotten a multiplicative factor 1/2. The absence of this factor seems to be due to an error of these authors in their construction of the effective action for the massive Dirac field from the (bosonic) Lichnerowicz operator. This error does not exist in Refs. [20, 21] and in Table I.

Similarly, by putting $a=0$ into the expressions (9)–(16), we recover the results obtained in Reissner-Nordström spacetime by Anderson, Hiscock and Samuel [26] (for the scalar field) and by Matyjasek [30] (for the Dirac and Proca fields). The nonzero components of $\langle T^{\mu}{}_{\nu}\rangle_{\rm ren}$ are

$$\langle T^{t}{}_{t}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} \sum_{q=0}^{3} A^{tt}{}_{0,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q, \tag{21}$$

$$\langle T^{r}{}_{r}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} \sum_{q=0}^{3} A^{rr}{}_{0,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q, \tag{22}$$

$$\langle T^{\theta}{}_{\theta}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} \sum_{q=0}^{3} A^{\theta\theta}{}_{0,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q, \tag{23}$$

$$\langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren} = \frac{M^2}{40320\pi^2 m^2 r^8} \sum_{q=0}^{3} A^{\varphi\varphi}{}_{0,q}\left[\theta, M/r\right] \left(\frac{Q^2}{M^2}\right)^q, \tag{24}$$

where the coefficients $A^{\mu\nu}{}_{0,q}$ do not depend explicitly of θ. Due to the spherical symmetry of the Reissner-Nordström black hole, $A^{\theta\theta}{}_{0,q} = A^{\varphi\varphi}{}_{0,q}$ and, as a consequence, $\langle T^{\theta}{}_{\theta}\rangle_{\rm ren} = \langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren}$.

Finally, by putting $Q=0$ into the expressions (9)–(16), we can find the results in Kerr spacetime. The nonzero components of $\langle T^{\mu}{}_{\nu}\rangle_{\rm ren}$ are

$$\langle T^{t}{}_{t}\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{5} A^{tt}{}_{p,0}\left[\theta, M/r\right] \left(\frac{a}{r}\right)^{2p}, \tag{25}$$

$$\langle T^r{}_r\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{5} A^{rr}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p}, \tag{26}$$

$$\langle T^\theta{}_\theta\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{5} A^{\theta\theta}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p}, \tag{27}$$

$$\langle T^\varphi{}_\varphi\rangle_{\rm ren} = \frac{M^2 r^{10}}{40320\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{5} A^{\varphi\varphi}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p}, \tag{28}$$

$$\langle T^t{}_\varphi\rangle_{\rm ren} = \frac{M^2 r^{11}\sin^2\theta}{20160\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{5} A^{t\varphi}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p+1}, \tag{29}$$

$$\langle T^\varphi{}_t\rangle_{\rm ren} = \frac{M^2 r^{9}}{20160\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{4} A^{\varphi t}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p+1}, \tag{30}$$

$$\langle T^r{}_\theta\rangle_{\rm ren} = \frac{M^2 r^{11}\sin\theta\cos\theta}{360\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{4} A^{r\theta}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p+2}, \tag{31}$$

$$\langle T^\theta{}_r\rangle_{\rm ren} = \frac{M^2 r^{9}\sin\theta\cos\theta}{360\pi^2 m^2 \left(r^2+a^2\cos^2\theta\right)^9} \sum_{p=0}^{3} A^{\theta r}{}_{p,0}\left[\theta, M/r\right]\left(\frac{a}{r}\right)^{2p+2}. \tag{32}$$

It should be noted that the approximate RSETs for the massive scalar field, the massive Dirac field and the Proca field in Kerr spacetime have been obtained a long time ago by Frolov and Zelnikov [24, 25]. At first sight, their results and ours are different because theirs are given in terms of the complex spin coefficient $\rho = -(r - ia\cos\theta)^{-1}$ and its powers. In our opinion, our formulas are clearer. Moreover, we have checked that both results are in agreement for the scalar and vector fields while, for the Dirac field, they differ by the multiplicative factor 1/2 previously discussed.

IV. CONCLUSION

In this article, we have obtained an analytical approximation for the RSET of the massive scalar, spinor and vector fields in Kerr-Newman spacetime. To our knowledge, no other result concerning this tensor in Kerr-Newman spacetime can be found within the literature. The *Mathematica* package *SETArbitraryST* which permits us to derive the expressions of the RSET $\langle T_{\mu\nu}\rangle_{\rm ren}$ in Kerr-Newman spacetime as well as another package which contains these explicit expressions are available upon request from the first author.

The approximate expressions obtained are based on the DeWitt-Schwinger expansion of the effective action associated with a massive quantum field. As a consequence, they do not take into account the quantum state of the field and are only valid in the large mass limit. In particular, it is important to note that they neglect the existence of supperradiance instabilities for massive fields in rotating black holes (see Refs. [56–58] for pioneering work on this subject and, e.g., Ref. [59] for a more recent article concerning more particularly the Kerr-Newman black hole) which seems quite reasonable because the instability time scale is very long in the large mass limit. We have also shown that our results permit us to recover those already existing in the literature for the RSET in Schwarzschild, Reissner-Nordström and Kerr spacetimes and we hope they will be useful to people who will want, in the future, to make the exact calculations by taking into account, in particular, the quantum state of the massive field. But, in our opinion, the complexity of our results leads us to think that an exact expression for the RSET of a massive field in Kerr-Newman spacetime is completely out of reach.

However, this should not prevent us from discussing the following fundamental question : what exact result(s) may be associated with our approximation? This is far from obvious and, here, we shall only provide a partial answer to this important question. Let us first consider the case of Schwarzschild spacetime. It is well known that, in this gravitational background, three Hadamard

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

TABLE III. The coefficients α_i and β_i appearing in the shift formulas (35) and (36).

	Scalar field ($s=0$)	Dirac field ($s=1/2$)	Proca field ($s=1$)
α_0	$-392(9\xi-2)$	196	-1176
α_1	$14(63\xi-16)$	7	210
α_2	-66	-162	978
β_0	$-60(84\xi-17)$	480	-1980
β_1	$-9(252\xi-53)$	180	-837
β_2	$20(714\xi-145)$	-640	19580

vacua are physically relevant (see Ref. [41] and references therein): the so-called Hartle-Hawking ($|H\rangle$), Unruh ($|U\rangle$) and Boulware ($|B\rangle$) vacua. In Ref [26], it has been shown that the DeWitt-Schwinger approximation (17)–(20) is a good approximation of $\langle H|T^{\mu}{}_{\nu}|H\rangle_{\text{ren}}$ (i.e., of the exact RSET in the Hartle-Hawking vacuum) for small and intermediate values of $s=r/(2M)-1$. As noted by Frolov and Zelnikov [25], the differences between this mean value and the mean values in the Unruh vacuum $|U\rangle$ and the Boulware vacuum $|B\rangle$ are proportional to the factor $\exp(-m/T_{\text{BH}})$ and this difference can be neglected everywhere except close to the horizon. As a consequence, the DeWitt-Schwinger approximation (17)–(20) is certainly a good approximation of $\langle U|T^{\mu}{}_{\nu}|U\rangle_{\text{ren}}$ and $\langle B|T^{\mu}{}_{\nu}|B\rangle_{\text{ren}}$ for intermediate values of s. Similar considerations apply in the Reissner-Nordström spacetime. By contrast, in Kerr and Kerr-Newman spacetimes the problem is much more complicated due, in particular, to the superradiant modes (see, e.g., Ref. [48]) and to the nonexistence of Hadamard states which respect the symmetries of the spacetime and are regular everywhere [52]. However, it seems that these difficulties are naturally eliminated for the fermionic fields [51] and can be, in some sense, circumvented for bosonic fields [47–50] and, in fact, one can consider that nonconventional Hartle-Hawking, Unruh and Boulware vacua exist in Kerr spacetime (and probably in Kerr-Newman spacetime). In our opinion, the DeWitt-Schwinger approximation (9)–(16) is certainly a good approximation of the RSETs associated with these nonconventional vacua, but, of course, the region of space where this approximation can be used must strongly depend on the considered vacuum.

In the absence of exact results in Kerr-Newman spacetime, our approximate RSETs could be very helpful, in particular to study the backreaction of massive quantum fields on this gravitational background. In Refs. [28, 29, 31, 33, 35, 36] which deal with massive field theories on the Schwarzschild and Reissner-Nordström black holes, some aspects of the backreaction problem have been considered. Unfortunately, here, for the Kerr-Newman black hole, due to the complexity of the RSET, it seems impossible to find self-consistent solutions to the semiclassical Einstein equations (1). However, it is possible to simplify considerably the backreaction problem by limiting us to the determination of the shift in mass and angular momentum of the black hole (measured by a distant observer) due to the RSET. Such an approach has already been considered by Frolov and Zelnikov for the Kerr black hole [24, 25] and its extension to the Kerr-Newman black hole is tractable and permits us to emphasize the role of the black hole charge. For a stationary axisymmetric black hole, we recall that the mass M_D and the angular momentum J_D of the black hole dressed with a quantum field can be expressed in terms of its mass M and its angular momentum J by (see, e.g., Ref. [60])

$$M_D - M = 2\int_{\mathcal{S}} \left(\langle T^{\mu}{}_{\nu}\rangle_{\text{ren}} - \frac{1}{2} g^{\mu}{}_{\nu} \langle T^{\rho}{}_{\rho}\rangle_{\text{ren}}\right) \xi^{\nu} d\mathcal{S}_{\mu} \tag{33}$$

and

$$J_D - J = -\int_{\mathcal{S}} \langle T^{\mu}{}_{\nu}\rangle_{\text{ren}} \psi^{\nu} d\mathcal{S}_{\mu} \tag{34}$$

where $\langle T_{\mu\nu}\rangle_{\text{ren}}$ is the RSET of the quantum field, $\xi^{\mu} = (\partial/\partial t)^{\mu}$ and $\psi^{\mu} = (\partial/\partial\varphi)^{\mu}$ denote the two Killing vectors of the Kerr-Newman black hole, $\mathcal{S}$ is any spacelike hypersurface that extends from the outer horizon at r_+ to spatial infinity and $d\mathcal{S}_{\mu}$ is the associated surface element. In the following, we take for $\mathcal{S}$ the hypersurface defined by $t=$ const and we have therefore $d\mathcal{S}_{\mu} = -(r^2+a^2\cos^2\theta)\sin\theta dr d\theta d\varphi\,(dt)_{\mu}$. By using the expressions (9) - (16) and assuming furthermore $a \ll M$ and $Q \ll M$ (for arbitrary values of the parameters a and Q, the expressions obtained are too complicated to be interesting), we obtain

$$\begin{aligned} M_D - M &= \frac{M}{336\times 7!\pi m^2 M^4} \\ &\times \left\{\alpha_0 + \alpha_1\left(\frac{a}{M}\right)^2 + \alpha_2\left(\frac{Q}{M}\right)^2 + \dots\right\} \end{aligned} \tag{35}$$

and

$$\begin{aligned} J_D - J &= \frac{1}{480\times 7!\pi m^2 M^2}\left(\frac{a}{M}\right) \\ &\times \left\{\beta_0 + \beta_1\left(\frac{a}{M}\right)^2 + \beta_2\left(\frac{Q}{M}\right)^2 + \dots\right\} \end{aligned} \tag{36}$$

where the dots denote terms of fourth-order [i.e., in $(a/M)^4$ or in $(Q/M)^4$ or in a^2Q^2/M^4]. The coefficients α_i and β_i appearing in Eqs. (35) and (36) are given in Table III. For $Q=0$ (i.e., for the Kerr black hole), our results correct some errors made in Refs. [24, 25] for the Dirac field (all the coefficients α_i and β_i are incorrect) but also for the scalar and vector fields (the coefficient α_1 is incorrect).

Our results could be also used to study the quasinormal modes of the Kerr-Newman black hole dressed by a massive quantum field. Similar problems have been considered in Refs. [35, 36] for spherically symmetric black holes. The extension to the Kerr-Newman black hole is far from obvious, not only because of the RSET complexity but also because the subject of massive quasinormal modes in this spacetime is rather difficult and has been little studied (see, however, for recent articles dealing with this subject and which could be helpful, Ref. [61] where the uncharged massless scalar field is considered and Ref. [62] where the charged massive scalar field is studied).

ACKNOWLEDGMENTS

We wish to thank Yves Décanini, Mohamed Ould El Hadj and Julien Queva for various discussions and the "Collectivité Territoriale de Corse" for its support through the COMPA project.

Appendix A: Coefficients $A^{\mu\nu}{}_{p,q}\,[\theta, M/r]$

1. Massive scalar field

For the massive scalar field, the coefficients $A^{tt}{}_{p,q}$ appearing in the expression (9) of $\langle T^t{}_t\rangle_{\text{ren}}$ are

$$A^{tt}{}_{0,0}\,[\theta, M/r] = 180(112\xi - 25) - 8(5544\xi - 1237)\,(M/r) \tag{A1a}$$
$$A^{tt}{}_{0,1}\,[\theta, M/r] = 1080 - 192(294\xi - 41)\,(M/r) + 4(36456\xi - 6845)\,(M/r)^2 \tag{A1b}$$
$$A^{tt}{}_{0,2}\,[\theta, M/r] = 8(5096\xi - 883)\,(M/r)^2 - 48(3164\xi - 613)\,(M/r)^3 \tag{A1c}$$
$$A^{tt}{}_{0,3}\,[\theta, M/r] = 52(882\xi - 179)\,(M/r)^4 \tag{A1d}$$
$$A^{tt}{}_{1,0}\,[\theta, M/r] = -4860(112\xi - 25)\cos^2\theta + 288\left[2(2303\xi - 513)\cos^2\theta + 14\xi - 3\right](M/r) \tag{A1e}$$
$$\begin{aligned}A^{tt}{}_{1,1}\,[\theta, M/r] = &-1080\left(7\cos^2\theta - 12\right) + 144\left[5(1344\xi - 275)\cos^2\theta - 224\xi - 219\right](M/r)\\ &-16\left[7(26556\xi - 5903)\cos^2\theta + 2(462\xi - 769)\right](M/r)^2\end{aligned} \tag{A1f}$$
$$\begin{aligned}A^{tt}{}_{1,2}\,[\theta, M/r] = &-16\left[49(472\xi - 95)\cos^2\theta - 16(140\xi + 43)\right](M/r)^2\\ &+16\left[28(4553\xi - 1016)\cos^2\theta + 1120\xi - 1559\right](M/r)^3\end{aligned} \tag{A1g}$$
$$A^{tt}{}_{1,3}\,[\theta, M/r] = -4\left[(109942\xi - 24683)\cos^2\theta + 4(455\xi - 421)\right](M/r)^4 \tag{A1h}$$
$$A^{tt}{}_{2,0}\,[\theta, M/r] = 7560(112\xi - 25)\cos^4\theta - 144\cos^2\theta\left[(24836\xi - 5525)\cos^2\theta + 18(14\xi - 3)\right](M/r) \tag{A1i}$$
$$\begin{aligned}A^{tt}{}_{2,1}\,[\theta, M/r] = &-2160\left(4\cos^4\theta + 6\cos^2\theta - 15\right)\\ &-144\left[8(546\xi - 149)\cos^4\theta - 3(672\xi - 173)\cos^2\theta + 335\right](M/r)\\ &+24\cos^2\theta\left[5(43680\xi - 9941)\cos^2\theta + 4(1470\xi + 61)\right](M/r)^2\end{aligned} \tag{A1j}$$
$$\begin{aligned}A^{tt}{}_{2,2}\,[\theta, M/r] = &-96\left[180\cos^4\theta + 2(784\xi - 227)\cos^2\theta - 253\right](M/r)^2\\ &-16\cos^2\theta\left[7(20612\xi - 4833)\cos^2\theta + 2(4424\xi + 377)\right](M/r)^3\end{aligned} \tag{A1k}$$
$$A^{tt}{}_{2,3}\,[\theta, M/r] = 4\cos^2\theta\left[(72310\xi - 17861)\cos^2\theta + 56(175\xi + 17)\right](M/r)^4 \tag{A1l}$$
$$A^{tt}{}_{3,0}\,[\theta, M/r] = 7560(112\xi - 25)\cos^6\theta + 96\cos^4\theta\left[2(8631\xi - 1916)\cos^2\theta - 15(14\xi - 3)\right](M/r) \tag{A1m}$$
$$\begin{aligned}A^{tt}{}_{3,1}\,[\theta, M/r] = &\,2160\left(4\cos^6\theta - 24\cos^4\theta + 15\cos^2\theta + 10\right)\\ &-240\cos^2\theta\left[7(768\xi - 169)\cos^4\theta - 3(224\xi + 47)\cos^2\theta + 318\right](M/r)\\ &-16\cos^4\theta\left[(90636\xi - 20021)\cos^2\theta + 6(630\xi - 101)\right](M/r)^2\end{aligned} \tag{A1n}$$
$$\begin{aligned}A^{tt}{}_{3,2}\,[\theta, M/r] = &\,16\cos^2\theta\left[(23128\xi - 5195)\cos^4\theta - 24(392\xi - 1)\cos^2\theta + 2700\right](M/r)^2\\ &+16\cos^4\theta\left[14(1414\xi - 291)\cos^2\theta + 3472\xi - 939\right](M/r)^3\end{aligned} \tag{A1o}$$
$$A^{tt}{}_{3,3}\,[\theta, M/r] = -4\cos^4\theta\left[(2506\xi - 9)\cos^2\theta + 4(455\xi - 253)\right](M/r)^4 \tag{A1p}$$
$$A^{tt}{}_{4,0}\,[\theta, M/r] = -4860(112\xi - 25)\cos^8\theta - 72\cos^6\theta\left[(1456\xi - 321)\cos^2\theta - 20(14\xi - 3)\right](M/r) \tag{A1q}$$
$$A^{tt}{}_{4,1}\,[\theta, M/r] = 1080\cos^2\theta\left(7\cos^6\theta - 12\cos^4\theta - 30\cos^2\theta + 40\right)$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$+720\cos^4\theta\left[2(252\xi-65)\cos^4\theta-(224\xi-107)\cos^2\theta-39\right](M/r)$$
$$+4\cos^6\theta\left[(9240\xi-2069)\cos^2\theta-8(42\xi-17)\right](M/r)^2 \tag{A1r}$$
$$A^{tt}{}_{4,2}\left[\theta,M/r\right]=-8\cos^4\theta\left[13(392\xi-151)\cos^4\theta-8(560\xi-503)\cos^2\theta-2364\right](M/r)^2 \tag{A1s}$$
$$A^{tt}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A1t}$$
$$A^{tt}{}_{5,0}\left[\theta,M/r\right]=180(112\xi-25)\cos^{10}\theta \tag{A1u}$$
$$A^{tt}{}_{5,1}\left[\theta,M/r\right]=-1080\cos^4\theta\left(\cos^2\theta-2\right)\left(\cos^4\theta-10\cos^2\theta+10\right) \tag{A1v}$$
$$A^{tt}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A1w}$$
$$A^{tt}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A1x}$$

For the massive scalar field, the coefficients $A^{rr}{}_{p,q}$ appearing in the expression (10) of $\langle T^r{}_r\rangle_{\rm ren}$ are

$$A^{rr}{}_{0,0}\left[\theta,M/r\right]=-252(32\xi-7)+56(216\xi-47)(M/r) \tag{A2a}$$
$$A^{rr}{}_{0,1}\left[\theta,M/r\right]=216+128(147\xi-40)(M/r)-4(8232\xi-2081)(M/r)^2 \tag{A2b}$$
$$A^{rr}{}_{0,2}\left[\theta,M/r\right]=-8(1456\xi-383)(M/r)^2+112(252\xi-65)(M/r)^3 \tag{A2c}$$
$$A^{rr}{}_{0,3}\left[\theta,M/r\right]=-4(1638\xi-421)(M/r)^4 \tag{A2d}$$
$$A^{rr}{}_{1,0}\left[\theta,M/r\right]=252(32\xi-7)\left(31\cos^2\theta-4\right)-2016(124\xi-27)\cos^2\theta\,(M/r) \tag{A2e}$$
$$A^{rr}{}_{1,1}\left[\theta,M/r\right]=-216\left(3\cos^2\theta-8\right)-288\left[7(186\xi-41)\cos^2\theta-2(91\xi-27)\right](M/r)$$
$$+48(9744\xi-2053)\cos^2\theta\,(M/r)^2 \tag{A2f}$$
$$A^{rr}{}_{1,2}\left[\theta,M/r\right]=8\left[7(2288\xi-503)\cos^2\theta-2800\xi+867\right](M/r)^2-16(16268\xi-3303)\cos^2\theta\,(M/r)^3 \tag{A2g}$$
$$A^{rr}{}_{1,3}\left[\theta,M/r\right]=4(11326\xi-2197)\cos^2\theta\,(M/r)^4 \tag{A2h}$$
$$A^{rr}{}_{2,0}\left[\theta,M/r\right]=-3528(32\xi-7)\cos^2\theta\left(11\cos^2\theta-8\right)+1008(336\xi-73)\cos^4\theta\,(M/r) \tag{A2i}$$
$$A^{rr}{}_{2,1}\left[\theta,M/r\right]=-432\left(6\cos^4\theta-6\cos^2\theta-5\right)$$
$$+96\cos^2\theta\left[(14826\xi-3187)\cos^2\theta-6(2107\xi-444)\right](M/r)-8(44520\xi-9851)\cos^4\theta\,(M/r)^2 \tag{A2j}$$
$$A^{rr}{}_{2,2}\left[\theta,M/r\right]=-8\cos^2\theta\left[(50064\xi-10603)\cos^2\theta-50064\xi+10279\right](M/r)^2$$
$$+16(7196\xi-1709)\cos^4\theta\,(M/r)^3 \tag{A2k}$$
$$A^{rr}{}_{2,3}\left[\theta,M/r\right]=-4(3346\xi-951)\cos^4\theta\,(M/r)^4 \tag{A2l}$$
$$A^{rr}{}_{3,0}\left[\theta,M/r\right]=3528(32\xi-7)\cos^4\theta\left(17\cos^2\theta-20\right)+672(84\xi-19)\cos^6\theta\,(M/r) \tag{A2m}$$
$$A^{rr}{}_{3,1}\left[\theta,M/r\right]=-432\cos^2\theta\left(4\cos^4\theta+6\cos^2\theta-15\right)$$
$$-32\cos^4\theta\left[(59010\xi-13061)\cos^2\theta-18(4025\xi-893)\right](M/r)-16(4704\xi-1049)\cos^6\theta\,(M/r)^2 \tag{A2n}$$
$$A^{rr}{}_{3,2}\left[\theta,M/r\right]=8\cos^4\theta\left[(55664\xi-12563)\cos^2\theta-68880\xi+15649\right](M/r)^2+16(1036\xi-235)\cos^6\theta\,(M/r)^3 \tag{A2o}$$
$$A^{rr}{}_{3,3}\left[\theta,M/r\right]=-4(182\xi-81)\cos^6\theta\,(M/r)^4 \tag{A2p}$$
$$A^{rr}{}_{4,0}\left[\theta,M/r\right]=-252(32\xi-7)\cos^6\theta\left(85\cos^2\theta-112\right)-72(392\xi-87)\cos^8\theta\,(M/r) \tag{A2q}$$
$$A^{rr}{}_{4,1}\left[\theta,M/r\right]=216\cos^4\theta\left(3\cos^4\theta-28\cos^2\theta+30\right)$$
$$+96\cos^6\theta\left[(4410\xi-953)\cos^2\theta-18(315\xi-68)\right](M/r)+4(1848\xi-367)\cos^8\theta\,(M/r)^2 \tag{A2r}$$
$$A^{rr}{}_{4,2}\left[\theta,M/r\right]=-8\cos^6\theta\left[2(2912\xi-577)\cos^2\theta-7280\xi+1429\right](M/r)^2 \tag{A2s}$$
$$A^{rr}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A2t}$$
$$A^{rr}{}_{5,0}\left[\theta,M/r\right]=252(32\xi-7)\cos^8\theta\left(3\cos^2\theta-4\right) \tag{A2u}$$
$$A^{rr}{}_{5,1}\left[\theta,M/r\right]=216\cos^6\theta\left(3\cos^4\theta-12\cos^2\theta+10\right) \tag{A2v}$$
$$A^{rr}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A2w}$$
$$A^{rr}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A2x}$$

For the massive scalar field, the coefficients $A^{\theta\theta}{}_{p,q}$ appearing in the expression (11) of $\langle T^\theta{}_\theta\rangle_{\rm ren}$ are

$$A^{\theta\theta}{}_{0,0}\left[\theta, M/r\right] = 756(32\xi - 7) - 8(7056\xi - 1543)\,(M/r) \tag{A3a}$$

$$A^{\theta\theta}{}_{0,1}\left[\theta, M/r\right] = -648 - 16(4116\xi - 1093)\,(M/r) + 12(15288\xi - 3649)\,(M/r)^2 \tag{A3b}$$

$$A^{\theta\theta}{}_{0,2}\left[\theta, M/r\right] = 16(2912\xi - 739)\,(M/r)^2 - 16(11928\xi - 2851)\,(M/r)^3 \tag{A3c}$$

$$A^{\theta\theta}{}_{0,3}\left[\theta, M/r\right] = 28(2106\xi - 497)\,(M/r)^4 \tag{A3d}$$

$$A^{\theta\theta}{}_{1,0}\left[\theta, M/r\right] = -252(32\xi - 7)\left(85\cos^2\theta - 4\right) + 288(5516\xi - 1207)\cos^2\theta\,(M/r) \tag{A3e}$$

$$A^{\theta\theta}{}_{1,1}\left[\theta, M/r\right] = -648\left(\cos^2\theta + 4\right) + 144\left[(8204\xi - 1763)\cos^2\theta - 588\xi + 169\right](M/r)$$
$$-16(223104\xi - 48331)\cos^2\theta\,(M/r)^2 \tag{A3f}$$

$$A^{\theta\theta}{}_{1,2}\left[\theta, M/r\right] = -8\left[(55664\xi - 11807)\cos^2\theta - 7280\xi + 1969\right](M/r)^2 + 112(22048\xi - 4741)\cos^2\theta\,(M/r)^3 \tag{A3g}$$

$$A^{\theta\theta}{}_{1,3}\left[\theta, M/r\right] = -4(134414\xi - 28727)\cos^2\theta\,(M/r)^4 \tag{A3h}$$

$$A^{\theta\theta}{}_{2,0}\left[\theta, M/r\right] = 3528(32\xi - 7)\cos^2\theta\left(17\cos^2\theta - 8\right) - 1008(4088\xi - 895)\cos^4\theta\,(M/r) \tag{A3i}$$

$$A^{\theta\theta}{}_{2,1}\left[\theta, M/r\right] = 432\left(4\cos^4\theta - 14\cos^2\theta - 5\right)$$
$$-48\cos^2\theta\left[7(5796\xi - 1261)\cos^2\theta - 3(10444\xi - 2181)\right](M/r) + 8(766920\xi - 168349)\cos^4\theta\,(M/r)^2 \tag{A3j}$$

$$A^{\theta\theta}{}_{2,2}\left[\theta, M/r\right] = 8\cos^2\theta\left[7(7152\xi - 1561)\cos^2\theta - 68880\xi + 14029\right](M/r)^2$$
$$-272(10136\xi - 2241)\cos^4\theta\,(M/r)^3 \tag{A3k}$$

$$A^{\theta\theta}{}_{2,3}\left[\theta, M/r\right] = 4(88802\xi - 19981)\cos^4\theta\,(M/r)^4 \tag{A3l}$$

$$A^{\theta\theta}{}_{3,0}\left[\theta, M/r\right] = -3528(32\xi - 7)\cos^4\theta\left(11\cos^2\theta - 20\right) + 672(2772\xi - 607)\cos^6\theta\,(M/r) \tag{A3m}$$

$$A^{\theta\theta}{}_{3,1}\left[\theta, M/r\right] = 1296\cos^2\theta\left(2\cos^4\theta - 2\cos^2\theta - 5\right)$$
$$+16\cos^4\theta\left[(50820\xi - 11489)\cos^2\theta - 9(14980\xi - 3363)\right](M/r) - 16(105168\xi - 23071)\cos^6\theta\,(M/r)^2 \tag{A3n}$$

$$A^{\theta\theta}{}_{3,2}\left[\theta, M/r\right] = -8\cos^4\theta\left[(16016\xi - 3845)\cos^2\theta - 50064\xi + 11899\right](M/r)^2$$
$$+16(24304\xi - 5305)\cos^6\theta\,(M/r)^3 \tag{A3o}$$

$$A^{\theta\theta}{}_{3,3}\left[\theta, M/r\right] = -4(3962\xi - 773)\cos^6\theta\,(M/r)^4 \tag{A3p}$$

$$A^{\theta\theta}{}_{4,0}\left[\theta, M/r\right] = 252(32\xi - 7)\cos^6\theta\left(31\cos^2\theta - 112\right) - 3528(32\xi - 7)\cos^8\theta\,(M/r) \tag{A3q}$$

$$A^{\theta\theta}{}_{4,1}\left[\theta, M/r\right] = 648\cos^4\theta\left(\cos^4\theta + 4\cos^2\theta - 10\right)$$
$$-48\cos^6\theta\left[2(1260\xi - 253)\cos^2\theta - 3(2660\xi - 559)\right](M/r) + 4(10248\xi - 2267)\cos^8\theta\,(M/r)^2 \tag{A3r}$$

$$A^{\theta\theta}{}_{4,2}\left[\theta, M/r\right] = 8\cos^6\theta\left[(1456\xi - 167)\cos^2\theta - 2800\xi + 327\right](M/r)^2 \tag{A3s}$$

$$A^{\theta\theta}{}_{4,3}\left[\theta, M/r\right] = 0 \tag{A3t}$$

$$A^{\theta\theta}{}_{5,0}\left[\theta, M/r\right] = -252(32\xi - 7)\cos^8\theta\left(\cos^2\theta - 4\right) \tag{A3u}$$

$$A^{\theta\theta}{}_{5,1}\left[\theta, M/r\right] = -216\cos^6\theta\left(\cos^4\theta - 8\cos^2\theta + 10\right) \tag{A3v}$$

$$A^{\theta\theta}{}_{5,2}\left[\theta, M/r\right] = 0 \tag{A3w}$$

$$A^{\theta\theta}{}_{5,3}\left[\theta, M/r\right] = 0. \tag{A3x}$$

For the massive scalar field, the coefficients $A^{\varphi\varphi}{}_{p,q}$ appearing in the expression (12) of $\langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren}$ are

$$A^{\varphi\varphi}{}_{0,0}\left[\theta, M/r\right] = 756(32\xi - 7) - 8(7056\xi - 1543)\,(M/r) \tag{A4a}$$

$$A^{\varphi\varphi}{}_{0,1}\left[\theta, M/r\right] = -648 - 16(4116\xi - 1093)\,(M/r) + 12(15288\xi - 3649)\,(M/r)^2 \tag{A4b}$$

$$A^{\varphi\varphi}{}_{0,2}\left[\theta, M/r\right] = 16(2912\xi - 739)\,(M/r)^2 - 16(11928\xi - 2851)\,(M/r)^3 \tag{A4c}$$

$$A^{\varphi\varphi}{}_{0,3}\left[\theta, M/r\right] = 28(2106\xi - 497)\,(M/r)^4 \tag{A4d}$$

$$A^{\varphi\varphi}{}_{1,0}\left[\theta, M/r\right] = -20412(32\xi - 7)\cos^2\theta + 288\left[10(553\xi - 121)\cos^2\theta - 14\xi + 3\right](M/r) \tag{A4e}$$

$$A^{\varphi\varphi}{}_{1,1}\left[\theta, M/r\right] = 216\left(41\cos^2\theta - 56\right) + 288\left[(3920\xi - 953)\cos^2\theta - 4(28\xi - 39)\right](M/r)$$
$$-16\left[9(24892\xi - 5541)\cos^2\theta - 2(462\xi - 769)\right](M/r)^2 \tag{A4f}$$

$$A^{\varphi\varphi}{}_{1,2}\left[\theta, M/r\right] = -16\left[2(13216\xi - 3305)\cos^2\theta - 2240\xi + 1691\right](M/r)^2$$
$$+16\left[2(77728\xi - 17373)\cos^2\theta - 1120\xi + 1559\right](M/r)^3 \tag{A4g}$$

$$A^{\varphi\varphi}{}_{1,3}\left[\theta, M/r\right] = -4\left[(136234\xi - 30411)\cos^2\theta - 4(455\xi - 421)\right](M/r)^4 \tag{A4h}$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$A^{\varphi\varphi}{}_{2,0}\left[\theta,M/r\right]=31752(32\xi-7)\cos^4\theta-144\cos^2\theta\left[(28868\xi-6319)\cos^2\theta-18(14\xi-3)\right](M/r) \tag{A4i}$$

$$\begin{aligned}A^{\varphi\varphi}{}_{2,1}\left[\theta,M/r\right]&=432\left(22\cos^4\theta+38\cos^2\theta-75\right)\\&-48\left[13(1176\xi-205)\cos^4\theta-126(48\xi-11)\cos^2\theta-1005\right](M/r)\\&+8\cos^2\theta\left[(784560\xi-167617)\cos^2\theta-12(1470\xi+61)\right](M/r)^2\end{aligned} \tag{A4j}$$

$$\begin{aligned}A^{\varphi\varphi}{}_{2,2}\left[\theta,M/r\right]&=48\left[414\cos^4\theta-7(448\xi-87)\cos^2\theta-506\right](M/r)^2\\&-16\cos^2\theta\left[(181160\xi-37343)\cos^2\theta-2(4424\xi+377)\right](M/r)^3\end{aligned} \tag{A4k}$$

$$A^{\varphi\varphi}{}_{2,3}\left[\theta,M/r\right]=4\cos^2\theta\left[(98602\xi-19029)\cos^2\theta-56(175\xi+17)\right](M/r)^4 \tag{A4l}$$

$$A^{\varphi\varphi}{}_{3,0}\left[\theta,M/r\right]=31752(32\xi-7)\cos^6\theta+96\cos^4\theta\left[2(9597\xi-2102)\cos^2\theta+15(14\xi-3)\right](M/r) \tag{A4m}$$

$$\begin{aligned}A^{\varphi\varphi}{}_{3,1}\left[\theta,M/r\right]&=-432\left(22\cos^6\theta-132\cos^4\theta+75\cos^2\theta+50\right)\\&-32\cos^2\theta\left[2(23520\xi-5293)\cos^4\theta-18(280\xi-199)\cos^2\theta-2385\right](M/r)\\&-16\cos^4\theta\left[(108948\xi-23677)\cos^2\theta-6(630\xi-101)\right](M/r)^2\end{aligned} \tag{A4n}$$

$$\begin{aligned}A^{\varphi\varphi}{}_{3,2}\left[\theta,M/r\right]&=16\cos^2\theta\left[2(13216\xi-2927)\cos^4\theta-3(3136\xi-1509)\cos^2\theta-2700\right](M/r)^2\\&+16\cos^4\theta\left[28(992\xi-223)\cos^2\theta-3472\xi+939\right](M/r)^3\end{aligned} \tag{A4o}$$

$$A^{\varphi\varphi}{}_{3,3}\left[\theta,M/r\right]=-4\cos^4\theta\left[7(826\xi-255)\cos^2\theta-4(455\xi-253)\right](M/r)^4 \tag{A4p}$$

$$A^{\varphi\varphi}{}_{4,0}\left[\theta,M/r\right]=-20412(32\xi-7)\cos^8\theta-72\cos^6\theta\left[(1288\xi-283)\cos^2\theta+20(14\xi-3)\right](M/r) \tag{A4q}$$

$$\begin{aligned}A^{\varphi\varphi}{}_{4,1}\left[\theta,M/r\right]&=-216\cos^2\theta\left(41\cos^6\theta-76\cos^4\theta-150\cos^2\theta+200\right)\\&+48\cos^4\theta\left[10(882\xi-169)\cos^4\theta-6(560\xi+11)\cos^2\theta+585\right](M/r)\\&+4\cos^6\theta\left[(9912\xi-2131)\cos^2\theta+8(42\xi-17)\right](M/r)^2\end{aligned} \tag{A4r}$$

$$A^{\varphi\varphi}{}_{4,2}\left[\theta,M/r\right]=-16\cos^4\theta\left[(2912\xi-253)\cos^4\theta-(2240\xi+1009)\cos^2\theta+1182\right](M/r)^2 \tag{A4s}$$

$$A^{\varphi\varphi}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A4t}$$

$$A^{\varphi\varphi}{}_{5,0}\left[\theta,M/r\right]=756(32\xi-7)\cos^{10}\theta \tag{A4u}$$

$$A^{\varphi\varphi}{}_{5,1}\left[\theta,M/r\right]=216\cos^4\theta\left(\cos^2\theta-2\right)\left(3\cos^4\theta-50\cos^2\theta+50\right) \tag{A4v}$$

$$A^{\varphi\varphi}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A4w}$$

$$A^{\varphi\varphi}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A4x}$$

For the massive scalar field, the coefficients $A^{t\varphi}{}_{p,q}$ appearing in the expression (13) of $\langle T^t{}_\varphi\rangle_{\rm ren}$ are

$$A^{t\varphi}{}_{0,0}\left[\theta,M/r\right]=-72(84\xi-17)(M/r) \tag{A5a}$$

$$A^{t\varphi}{}_{0,1}\left[\theta,M/r\right]=-2160+7056\,(M/r)+28(672\xi-293)(M/r)^2 \tag{A5b}$$

$$A^{t\varphi}{}_{0,2}\left[\theta,M/r\right]=-3444\,(M/r)^2-32(609\xi-253)(M/r)^3 \tag{A5c}$$

$$A^{t\varphi}{}_{0,3}\left[\theta,M/r\right]=72(91\xi-32)(M/r)^4 \tag{A5d}$$

$$A^{t\varphi}{}_{1,0}\left[\theta,M/r\right]=144\left[(910\xi-181)\cos^2\theta-14\xi+3\right](M/r) \tag{A5e}$$

$$\begin{aligned}A^{t\varphi}{}_{1,1}\left[\theta,M/r\right]&=2160\left(\cos^2\theta-6\right)-72\left(33\cos^2\theta-397\right)(M/r)\\&-16\left[(18606\xi-3505)\cos^2\theta-462\xi+769\right](M/r)^2\end{aligned} \tag{A5f}$$

$$A^{t\varphi}{}_{1,2}\left[\theta,M/r\right]=-12\left(13\cos^2\theta+1191\right)(M/r)^2+8\left[7(3836\xi-677)\cos^2\theta-1120\xi+1559\right](M/r)^3 \tag{A5g}$$

$$A^{t\varphi}{}_{1,3}\left[\theta,M/r\right]=-8\left[(6118\xi-1011)\cos^2\theta-455\xi+421\right](M/r)^4 \tag{A5h}$$

$$A^{t\varphi}{}_{2,0}\left[\theta,M/r\right]=-144\cos^2\theta\left[10(189\xi-37)\cos^2\theta-9(14\xi-3)\right](M/r) \tag{A5i}$$

$$\begin{aligned}A^{t\varphi}{}_{2,1}\left[\theta,M/r\right]&=4320\left(2\cos^4\theta-2\cos^2\theta-5\right)-72\left(240\cos^4\theta-353\cos^2\theta-335\right)(M/r)\\&+8\cos^2\theta\left[(55860\xi-9617)\cos^2\theta-6(1470\xi+61)\right](M/r)^2\end{aligned} \tag{A5j}$$

$$\begin{aligned}A^{t\varphi}{}_{2,2}\left[\theta,M/r\right]&=12\left(671\cos^4\theta-1381\cos^2\theta-1012\right)(M/r)^2\\&-16\cos^2\theta\left[(14014\xi-2133)\cos^2\theta-4424\xi-377\right](M/r)^3\end{aligned} \tag{A5k}$$

$$A^{t\varphi}{}_{2,3}\left[\theta,M/r\right]=8\cos^2\theta\left[(4123\xi-530)\cos^2\theta-14(175\xi+17)\right](M/r)^4 \tag{A5l}$$

$$A^{t\varphi}{}_{3,0}\left[\theta,M/r\right]=144\cos^4\theta\left[(714\xi-139)\cos^2\theta+5(14\xi-3)\right](M/r) \tag{A5m}$$

$$A^{t\varphi}{}_{3,1}\left[\theta, M/r\right] = 2160\left(\cos^6\theta + 9\cos^4\theta - 15\cos^2\theta - 5\right) - 72\cos^2\theta\left(73\cos^4\theta + 261\cos^2\theta - 530\right)(M/r)$$
$$-16\cos^4\theta\left[(7266\xi - 1525)\cos^2\theta - 3(630\xi - 101)\right](M/r)^2 \quad \text{(A5n)}$$
$$A^{t\varphi}{}_{3,2}\left[\theta, M/r\right] = 12\cos^2\theta\left(289\cos^4\theta + 531\cos^2\theta - 1800\right)(M/r)^2$$
$$+8\cos^4\theta\left[(4508\xi - 1231)\cos^2\theta - 3472\xi + 939\right](M/r)^3 \quad \text{(A5o)}$$
$$A^{t\varphi}{}_{3,3}\left[\theta, M/r\right] = -8\cos^4\theta\left[(364\xi - 191)\cos^2\theta - 455\xi + 253\right](M/r)^4 \quad \text{(A5p)}$$
$$A^{t\varphi}{}_{4,0}\left[\theta, M/r\right] = -72\cos^6\theta\left[(56\xi - 11)\cos^2\theta + 10(14\xi - 3)\right](M/r) \quad \text{(A5q)}$$
$$A^{t\varphi}{}_{4,1}\left[\theta, M/r\right] = -2160\cos^2\theta\left(\cos^6\theta - 6\cos^4\theta + 10\right) + 72\cos^4\theta\left(36\cos^4\theta - 217\cos^2\theta + 195\right)(M/r)$$
$$+4\cos^6\theta\left[9(56\xi - 11)\cos^2\theta + 4(42\xi - 17)\right](M/r)^2 \quad \text{(A5r)}$$
$$A^{t\varphi}{}_{4,2}\left[\theta, M/r\right] = -12\cos^4\theta\left(108\cos^4\theta - 721\cos^2\theta + 788\right)(M/r)^2 \quad \text{(A5s)}$$
$$A^{t\varphi}{}_{4,3}\left[\theta, M/r\right] = 0 \quad \text{(A5t)}$$
$$A^{t\varphi}{}_{5,0}\left[\theta, M/r\right] = 0 \quad \text{(A5u)}$$
$$A^{t\varphi}{}_{5,1}\left[\theta, M/r\right] = -2160\cos^4\theta\left(\cos^4\theta - 5\cos^2\theta + 5\right) \quad \text{(A5v)}$$
$$A^{t\varphi}{}_{5,2}\left[\theta, M/r\right] = 0 \quad \text{(A5w)}$$
$$A^{t\varphi}{}_{5,3}\left[\theta, M/r\right] = 0. \quad \text{(A5x)}$$

For the massive scalar field, the coefficients $A^{\varphi t}{}_{p,q}$ appearing in the expression (14) of $\langle T^{\varphi}{}_{t}\rangle_{\text{ren}}$ are

$$A^{\varphi t}{}_{0,0}\left[\theta, M/r\right] = 144(14\xi - 3)(M/r) \quad \text{(A6a)}$$
$$A^{\varphi t}{}_{0,1}\left[\theta, M/r\right] = 2160 - 9648(M/r) - 16(462\xi - 769)(M/r)^2 \quad \text{(A6b)}$$
$$A^{\varphi t}{}_{0,2}\left[\theta, M/r\right] = 4740(M/r)^2 + 8(1120\xi - 1559)(M/r)^3 \quad \text{(A6c)}$$
$$A^{\varphi t}{}_{0,3}\left[\theta, M/r\right] = -8(455\xi - 421)(M/r)^4 \quad \text{(A6d)}$$
$$A^{\varphi t}{}_{1,0}\left[\theta, M/r\right] = -1296(14\xi - 3)\cos^2\theta\,(M/r) \quad \text{(A6e)}$$
$$A^{\varphi t}{}_{1,1}\left[\theta, M/r\right] = -2160\left(\cos^2\theta - 5\right) - 72\left(39\cos^2\theta + 335\right)(M/r) + 48(1470\xi + 61)\cos^2\theta\,(M/r)^2 \quad \text{(A6f)}$$
$$A^{\varphi t}{}_{1,2}\left[\theta, M/r\right] = 12\left(229\cos^2\theta + 1012\right)(M/r)^2 - 16(4424\xi + 377)\cos^2\theta\,(M/r)^3 \quad \text{(A6g)}$$
$$A^{\varphi t}{}_{1,3}\left[\theta, M/r\right] = 112(175\xi + 17)\cos^2\theta\,(M/r)^4 \quad \text{(A6h)}$$
$$A^{\varphi t}{}_{2,0}\left[\theta, M/r\right] = -720(14\xi - 3)\cos^4\theta\,(M/r) \quad \text{(A6i)}$$
$$A^{\varphi t}{}_{2,1}\left[\theta, M/r\right] = -2160\left(4\cos^4\theta - 5\cos^2\theta - 5\right) + 720\cos^2\theta\left(24\cos^2\theta - 53\right)(M/r)$$
$$-48(630\xi - 101)\cos^4\theta\,(M/r)^2 \quad \text{(A6j)}$$
$$A^{\varphi t}{}_{2,2}\left[\theta, M/r\right] = -12\cos^2\theta\left(671\cos^2\theta - 1800\right)(M/r)^2 + 8(3472\xi - 939)\cos^4\theta\,(M/r)^3 \quad \text{(A6k)}$$
$$A^{\varphi t}{}_{2,3}\left[\theta, M/r\right] = -8(455\xi - 253)\cos^4\theta\,(M/r)^4 \quad \text{(A6l)}$$
$$A^{\varphi t}{}_{3,0}\left[\theta, M/r\right] = 720(14\xi - 3)\cos^6\theta\,(M/r) \quad \text{(A6m)}$$
$$A^{\varphi t}{}_{3,1}\left[\theta, M/r\right] = -2160\cos^2\theta\left(\cos^4\theta + 5\cos^2\theta - 10\right) + 360\cos^4\theta\left(29\cos^2\theta - 39\right)(M/r)$$
$$-16(42\xi - 17)\cos^6\theta\,(M/r)^2 \quad \text{(A6n)}$$
$$A^{\varphi t}{}_{3,2}\left[\theta, M/r\right] = -12\cos^4\theta\left(505\cos^2\theta - 788\right)(M/r)^2 \quad \text{(A6o)}$$
$$A^{\varphi t}{}_{3,3}\left[\theta, M/r\right] = 0 \quad \text{(A6p)}$$
$$A^{\varphi t}{}_{4,0}\left[\theta, M/r\right] = 0 \quad \text{(A6q)}$$
$$A^{\varphi t}{}_{4,1}\left[\theta, M/r\right] = 2160\cos^4\theta\left(\cos^4\theta - 5\cos^2\theta + 5\right) \quad \text{(A6r)}$$
$$A^{\varphi t}{}_{4,2}\left[\theta, M/r\right] = 0 \quad \text{(A6s)}$$
$$A^{\varphi t}{}_{4,3}\left[\theta, M/r\right] = 0. \quad \text{(A6t)}$$

For the massive scalar field, the coefficients $A^{r\theta}{}_{p,q}$ appearing in the expression (15) of $\langle T^{r}{}_{\theta}\rangle_{\text{ren}}$ are

$$A^{r\theta}{}_{0,0}\left[\theta, M/r\right] = 72(32\xi - 7) - 144(32\xi - 7)(M/r) \quad \text{(A7a)}$$
$$A^{r\theta}{}_{0,1}\left[\theta, M/r\right] = -36(109\xi - 24)(M/r) + 72(141\xi - 31)(M/r)^2 \quad \text{(A7b)}$$
$$A^{r\theta}{}_{0,2}\left[\theta, M/r\right] = 10(168\xi - 37)(M/r)^2 - 4(1821\xi - 401)(M/r)^3 \quad \text{(A7c)}$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$A^{r\theta}{}_{0,3}\,[\theta, M/r] = 10(168\xi - 37)\,(M/r)^4 \tag{A7d}$$
$$A^{r\theta}{}_{1,0}\,[\theta, M/r] = -72(32\xi - 7)\left(7\cos^2\theta - 1\right) + 1008(32\xi - 7)\cos^2\theta\,(M/r) \tag{A7e}$$
$$A^{r\theta}{}_{1,1}\,[\theta, M/r] = 12\left[5(339\xi - 74)\cos^2\theta - 3(109\xi - 24)\right](M/r) - 24(2367\xi - 517)\cos^2\theta\,(M/r)^2 \tag{A7f}$$
$$A^{r\theta}{}_{1,2}\,[\theta, M/r] = -2\left[2(1464\xi - 319)\cos^2\theta - 5(168\xi - 37)\right](M/r)^2 + 4(8013\xi - 1748)\cos^2\theta\,(M/r)^3 \tag{A7g}$$
$$A^{r\theta}{}_{1,3}\,[\theta, M/r] = -4(1464\xi - 319)\cos^2\theta\,(M/r)^4 \tag{A7h}$$
$$A^{r\theta}{}_{2,0}\,[\theta, M/r] = 504(32\xi - 7)\cos^2\theta(\cos^2\theta - 1) - 1008(32\xi - 7)\cos^4\theta\,(M/r) \tag{A7i}$$
$$A^{r\theta}{}_{2,1}\,[\theta, M/r] = -60\cos^2\theta\left[(201\xi - 44)\cos^2\theta - 339\xi + 74\right](M/r) + 24(1677\xi - 367)\cos^4\theta\,(M/r)^2 \tag{A7j}$$
$$A^{r\theta}{}_{2,2}\,[\theta, M/r] = 2\cos^2\theta\left[5(168\xi - 37)\cos^2\theta - 2(1464\xi - 319)\right](M/r)^2 - 20(771\xi - 169)\cos^4\theta\,(M/r)^3 \tag{A7k}$$
$$A^{r\theta}{}_{2,3}\,[\theta, M/r] = 10(168\xi - 37)\cos^4\theta\,(M/r)^4 \tag{A7l}$$
$$A^{r\theta}{}_{3,0}\,[\theta, M/r] = -72(32\xi - 7)\cos^4\theta\left(\cos^2\theta - 7\right) + 144(32\xi - 7)\cos^6\theta\,(M/r) \tag{A7m}$$
$$A^{r\theta}{}_{3,1}\,[\theta, M/r] = 60\cos^4\theta\left[(9\xi - 2)\cos^2\theta - 201\xi + 44\right](M/r) - 24(141\xi - 31)\cos^6\theta\,(M/r)^2 \tag{A7n}$$
$$A^{r\theta}{}_{3,2}\,[\theta, M/r] = 10(168\xi - 37)\cos^4\theta\,(M/r)^2 + 60(9\xi - 2)\cos^6\theta\,(M/r)^3 \tag{A7o}$$
$$A^{r\theta}{}_{3,3}\,[\theta, M/r] = 0 \tag{A7p}$$
$$A^{r\theta}{}_{4,0}\,[\theta, M/r] = -72(32\xi - 7)\cos^6\theta \tag{A7q}$$
$$A^{r\theta}{}_{4,1}\,[\theta, M/r] = 60(9\xi - 2)\cos^6\theta\,(M/r) \tag{A7r}$$
$$A^{r\theta}{}_{4,2}\,[\theta, M/r] = 0 \tag{A7s}$$
$$A^{r\theta}{}_{4,3}\,[\theta, M/r] = 0. \tag{A7t}$$

For the massive scalar field, the coefficients $A^{\theta r}{}_{p,q}$ appearing in the expression (16) of $\langle T^{\theta}{}_{r}\rangle_{\rm ren}$ are

$$A^{\theta r}{}_{0,0}\,[\theta, M/r] = 72(32\xi - 7) \tag{A8a}$$
$$A^{\theta r}{}_{0,1}\,[\theta, M/r] = -36(109\xi - 24)\,(M/r) \tag{A8b}$$
$$A^{\theta r}{}_{0,2}\,[\theta, M/r] = 10(168\xi - 37)\,(M/r)^2 \tag{A8c}$$
$$A^{\theta r}{}_{1,0}\,[\theta, M/r] = -504(32\xi - 7)\cos^2\theta \tag{A8d}$$
$$A^{\theta r}{}_{1,1}\,[\theta, M/r] = 60(339\xi - 74)\cos^2\theta\,(M/r) \tag{A8e}$$
$$A^{\theta r}{}_{1,2}\,[\theta, M/r] = -4(1464\xi - 319)\cos^2\theta\,(M/r)^2 \tag{A8f}$$
$$A^{\theta r}{}_{2,0}\,[\theta, M/r] = 504(32\xi - 7)\cos^4\theta \tag{A8g}$$
$$A^{\theta r}{}_{2,1}\,[\theta, M/r] = -60(201\xi - 44)\cos^4\theta\,(M/r) \tag{A8h}$$
$$A^{\theta r}{}_{2,2}\,[\theta, M/r] = 10(168\xi - 37)\cos^4\theta\,(M/r)^2 \tag{A8i}$$
$$A^{\theta r}{}_{3,0}\,[\theta, M/r] = -72(32\xi - 7)\cos^6\theta \tag{A8j}$$
$$A^{\theta r}{}_{3,1}\,[\theta, M/r] = 60(9\xi - 2)\cos^6\theta\,(M/r) \tag{A8k}$$
$$A^{\theta r}{}_{3,2}\,[\theta, M/r] = 0. \tag{A8l}$$

2. Massive Dirac field

For the massive Dirac field, the coefficients $A^{tt}{}_{p,q}$ appearing in the expression (9) of $\langle T^{t}{}_{t}\rangle_{\rm ren}$ are

$$A^{tt}{}_{0,0}\,[\theta, M/r] = -1080 + 2384\,(M/r) \tag{A9a}$$
$$A^{tt}{}_{0,1}\,[\theta, M/r] = 5400 - 22464\,(M/r) + 21832\,(M/r)^2 \tag{A9b}$$
$$A^{tt}{}_{0,2}\,[\theta, M/r] = 10544\,(M/r)^2 - 21496\,(M/r)^3 \tag{A9c}$$
$$A^{tt}{}_{0,3}\,[\theta, M/r] = 4917\,(M/r)^4 \tag{A9d}$$
$$A^{tt}{}_{1,0}\,[\theta, M/r] = 29160\cos^2\theta - 288\left(251\cos^2\theta + 1\right)(M/r) \tag{A9e}$$
$$A^{tt}{}_{1,1}\,[\theta, M/r] = -5400\left(7\cos^2\theta - 12\right) + 288\left(185\cos^2\theta - 691\right)(M/r) + 16\left(7805\cos^2\theta + 7277\right)(M/r)^2 \tag{A9f}$$
$$A^{tt}{}_{1,2}\,[\theta, M/r] = -64\left(497\cos^2\theta - 1616\right)(M/r)^2 - 8\left(10010\cos^2\theta + 15373\right)(M/r)^3 \tag{A9g}$$

$$A^{tt}{}_{1,3}\left[\theta,M/r\right]=\left(17765\cos^2\theta+32546\right)(M/r)^4 \tag{A9h}$$

$$A^{tt}{}_{2,0}\left[\theta,M/r\right]=-45360\cos^4\theta+2592\cos^2\theta\left(76\cos^2\theta+1\right)(M/r) \tag{A9i}$$

$$A^{tt}{}_{2,1}\left[\theta,M/r\right]=-10800\left(4\cos^4\theta+6\cos^2\theta-15\right)+288\left(628\cos^4\theta-111\cos^2\theta-855\right)(M/r) \\ -240\cos^2\theta\left(1658\cos^2\theta-535\right)(M/r)^2 \tag{A9j}$$

$$A^{tt}{}_{2,2}\left[\theta,M/r\right]=-96\left(900\cos^4\theta-422\cos^2\theta-1349\right)(M/r)^2+8\cos^2\theta\left(30793\cos^2\theta-20378\right)(M/r)^3 \tag{A9k}$$

$$A^{tt}{}_{2,3}\left[\theta,M/r\right]=-\cos^2\theta\left(48149\cos^2\theta-49476\right)(M/r)^4 \tag{A9l}$$

$$A^{tt}{}_{3,0}\left[\theta,M/r\right]=-45360\cos^6\theta-96\cos^4\theta\left(967\cos^2\theta-15\right)(M/r) \tag{A9m}$$

$$A^{tt}{}_{3,1}\left[\theta,M/r\right]=10800\left(4\cos^6\theta-24\cos^4\theta+15\cos^2\theta+10\right) \\ +1440\cos^2\theta\left(21\cos^4\theta+247\cos^2\theta-258\right)(M/r)+16\cos^4\theta\left(4751\cos^2\theta-645\right)(M/r)^2 \tag{A9n}$$

$$A^{tt}{}_{3,2}\left[\theta,M/r\right]=-64\cos^2\theta\left(178\cos^4\theta+2742\cos^2\theta-3375\right)(M/r)^2+24\cos^4\theta\left(140\cos^2\theta-687\right)(M/r)^3 \tag{A9o}$$

$$A^{tt}{}_{3,3}\left[\theta,M/r\right]=-\cos^4\theta\left(8773\cos^2\theta-11042\right)(M/r)^4 \tag{A9p}$$

$$A^{tt}{}_{4,0}\left[\theta,M/r\right]=29160\cos^8\theta+144\cos^6\theta\left(43\cos^2\theta-10\right)(M/r) \tag{A9q}$$

$$A^{tt}{}_{4,1}\left[\theta,M/r\right]=5400\cos^2\theta\left(7\cos^6\theta-12\cos^4\theta-30\cos^2\theta+40\right) \\ -1440\cos^4\theta\left(52\cos^4\theta-131\cos^2\theta+87\right)(M/r)-8\cos^6\theta\left(325\cos^2\theta-158\right)(M/r)^2 \tag{A9r}$$

$$A^{tt}{}_{4,2}\left[\theta,M/r\right]=16\cos^4\theta\left(2041\cos^4\theta-7036\cos^2\theta+5406\right)(M/r)^2 \tag{A9s}$$

$$A^{tt}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A9t}$$

$$A^{tt}{}_{5,0}\left[\theta,M/r\right]=-1080\cos^{10}\theta \tag{A9u}$$

$$A^{tt}{}_{5,1}\left[\theta,M/r\right]=-5400\cos^4\theta\left(\cos^2\theta-2\right)\left(\cos^4\theta-10\cos^2\theta+10\right) \tag{A9v}$$

$$A^{tt}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A9w}$$

$$A^{tt}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A9x}$$

For the massive Dirac field, the coefficients $A^{rr}{}_{p,q}$ appearing in the expression (10) of $\langle T^r{}_r\rangle_{\rm ren}$ are

$$A^{rr}{}_{0,0}\left[\theta,M/r\right]=504-784\,(M/r) \tag{A10a}$$

$$A^{rr}{}_{0,1}\left[\theta,M/r\right]=1080-6336\,(M/r)+8440\,(M/r)^2 \tag{A10b}$$

$$A^{rr}{}_{0,2}\left[\theta,M/r\right]=3560\,(M/r)^2-8680\,(M/r)^3 \tag{A10c}$$

$$A^{rr}{}_{0,3}\left[\theta,M/r\right]=2253\,(M/r)^4 \tag{A10d}$$

$$A^{rr}{}_{1,0}\left[\theta,M/r\right]=-504\left(31\cos^2\theta-4\right)+16128\cos^2\theta\,(M/r) \tag{A10e}$$

$$A^{rr}{}_{1,1}\left[\theta,M/r\right]=-1080\left(3\cos^2\theta-8\right)+144\left(189\cos^2\theta-169\right)(M/r)-16160\cos^2\theta\,(M/r)^2 \tag{A10f}$$

$$A^{rr}{}_{1,2}\left[\theta,M/r\right]=-8\left(1141\cos^2\theta-1563\right)(M/r)^2+88\cos^2\theta\,(M/r)^3 \tag{A10g}$$

$$A^{rr}{}_{1,3}\left[\theta,M/r\right]=2015\cos^2\theta\,(M/r)^4 \tag{A10h}$$

$$A^{rr}{}_{2,0}\left[\theta,M/r\right]=7056\cos^2\theta\left(11\cos^2\theta-8\right)-22176\cos^4\theta\,(M/r) \tag{A10i}$$

$$A^{rr}{}_{2,1}\left[\theta,M/r\right]=-2160\left(6\cos^4\theta-6\cos^2\theta-5\right)-144\cos^2\theta\left(429\cos^2\theta-193\right)(M/r) \\ +31440\cos^4\theta\,(M/r)^2 \tag{A10j}$$

$$A^{rr}{}_{2,2}\left[\theta,M/r\right]=8\cos^2\theta\left(1313\cos^2\theta+307\right)(M/r)^2-18840\cos^4\theta\,(M/r)^3 \tag{A10k}$$

$$A^{rr}{}_{2,3}\left[\theta,M/r\right]=4887\cos^4\theta\,(M/r)^4 \tag{A10l}$$

$$A^{rr}{}_{3,0}\left[\theta,M/r\right]=-7056\cos^4\theta\left(17\cos^2\theta-20\right)-2688\cos^6\theta\,(M/r) \tag{A10m}$$

$$A^{rr}{}_{3,1}\left[\theta,M/r\right]=-2160\cos^2\theta\left(4\cos^4\theta+6\cos^2\theta-15\right)+9360\cos^4\theta\left(15\cos^2\theta-19\right)(M/r) \\ +4064\cos^6\theta\,(M/r)^2 \tag{A10n}$$

$$A^{rr}{}_{3,2}\left[\theta,M/r\right]=-8\cos^4\theta\left(5275\cos^2\theta-7013\right)(M/r)^2-1496\cos^6\theta\,(M/r)^3 \tag{A10o}$$

$$A^{rr}{}_{3,3}\left[\theta,M/r\right]=773\cos^6\theta\,(M/r)^4 \tag{A10p}$$

$$A^{rr}{}_{4,0}\left[\theta,M/r\right]=504\cos^6\theta\left(85\cos^2\theta-112\right)+1584\cos^8\theta\,(M/r) \tag{A10q}$$

$$A^{rr}{}_{4,1}\left[\theta,M/r\right]=1080\cos^4\theta\left(3\cos^4\theta-28\cos^2\theta+30\right)-720\cos^6\theta\left(31\cos^2\theta-39\right)(M/r)$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$+248\cos^8\theta\,(M/r)^2 \tag{A10r}$$
$$A^{rr}{}_{4,2}\,[\theta, M/r] = -40\cos^6\theta\left(22\cos^2\theta - 41\right)(M/r)^2 \tag{A10s}$$
$$A^{rr}{}_{4,3}\,[\theta, M/r] = 0 \tag{A10t}$$
$$A^{rr}{}_{5,0}\,[\theta, M/r] = -504\cos^8\theta\left(3\cos^2\theta - 4\right) \tag{A10u}$$
$$A^{rr}{}_{5,1}\,[\theta, M/r] = 1080\cos^6\theta\left(3\cos^4\theta - 12\cos^2\theta + 10\right) \tag{A10v}$$
$$A^{rr}{}_{5,2}\,[\theta, M/r] = 0 \tag{A10w}$$
$$A^{rr}{}_{5,3}\,[\theta, M/r] = 0. \tag{A10x}$$

For the massive Dirac field, the coefficients $A^{\theta\theta}{}_{p,q}$ appearing in the expression (11) of $\langle T^{\theta}{}_{\theta}\rangle_{\rm ren}$ are

$$A^{\theta\theta}{}_{0,0}\,[\theta, M/r] = -1512 + 3536\,(M/r) \tag{A11a}$$
$$A^{\theta\theta}{}_{0,1}\,[\theta, M/r] = -3240 + 20016\,(M/r) - 30808\,(M/r)^2 \tag{A11b}$$
$$A^{\theta\theta}{}_{0,2}\,[\theta, M/r] = -12080\,(M/r)^2 + 33984\,(M/r)^3 \tag{A11c}$$
$$A^{\theta\theta}{}_{0,3}\,[\theta, M/r] = -9933\,(M/r)^4 \tag{A11d}$$
$$A^{\theta\theta}{}_{1,0}\,[\theta, M/r] = 504\left(85\cos^2\theta - 4\right) - 99072\cos^2\theta\,(M/r) \tag{A11e}$$
$$A^{\theta\theta}{}_{1,1}\,[\theta, M/r] = -3240\left(\cos^2\theta + 4\right) - 144\left(359\cos^2\theta - 243\right)(M/r) + 186272\cos^2\theta\,(M/r)^2 \tag{A11f}$$
$$A^{\theta\theta}{}_{1,2}\,[\theta, M/r] = 40\left(299\cos^2\theta - 499\right)(M/r)^2 - 106960\cos^2\theta\,(M/r)^3 \tag{A11g}$$
$$A^{\theta\theta}{}_{1,3}\,[\theta, M/r] = 18817\cos^2\theta\,(M/r)^4 \tag{A11h}$$
$$A^{\theta\theta}{}_{2,0}\,[\theta, M/r] = -7056\cos^2\theta\left(17\cos^2\theta - 8\right) + 256032\cos^4\theta\,(M/r) \tag{A11i}$$
$$A^{\theta\theta}{}_{2,1}\,[\theta, M/r] = 2160\left(4\cos^4\theta - 14\cos^2\theta - 5\right) + 144\cos^2\theta\left(749\cos^2\theta - 139\right)(M/r)$$
$$-393360\cos^4\theta\,(M/r)^2 \tag{A11j}$$
$$A^{\theta\theta}{}_{2,2}\,[\theta, M/r] = -8\cos^2\theta\left(2933\cos^2\theta + 1087\right)(M/r)^2 + 196128\cos^4\theta\,(M/r)^3 \tag{A11k}$$
$$A^{\theta\theta}{}_{2,3}\,[\theta, M/r] = -31959\cos^4\theta\,(M/r)^4 \tag{A11l}$$
$$A^{\theta\theta}{}_{3,0}\,[\theta, M/r] = 7056\cos^4\theta\left(11\cos^2\theta - 20\right) - 115584\cos^6\theta\,(M/r) \tag{A11m}$$
$$A^{\theta\theta}{}_{3,1}\,[\theta, M/r] = 6480\cos^2\theta\left(2\cos^4\theta - 2\cos^2\theta - 5\right) - 720\cos^4\theta\left(113\cos^2\theta - 269\right)(M/r)$$
$$+108448\cos^6\theta\,(M/r)^2 \tag{A11n}$$
$$A^{\theta\theta}{}_{3,2}\,[\theta, M/r] = 8\cos^4\theta\left(2761\cos^2\theta - 7793\right)(M/r)^2 - 23888\cos^6\theta\,(M/r)^3 \tag{A11o}$$
$$A^{\theta\theta}{}_{3,3}\,[\theta, M/r] = -549\cos^6\theta\,(M/r)^4 \tag{A11p}$$
$$A^{\theta\theta}{}_{4,0}\,[\theta, M/r] = -504\cos^6\theta\left(31\cos^2\theta - 112\right) + 7056\cos^8\theta\,(M/r) \tag{A11q}$$
$$A^{\theta\theta}{}_{4,1}\,[\theta, M/r] = 3240\cos^4\theta\left(\cos^4\theta + 4\cos^2\theta - 10\right) - 720\cos^6\theta\left(4\cos^2\theta + 13\right)(M/r)$$
$$-3032\cos^8\theta\,(M/r)^2 \tag{A11r}$$
$$A^{\theta\theta}{}_{4,2}\,[\theta, M/r] = 8\cos^6\theta\left(635\cos^2\theta - 1137\right)(M/r)^2 \tag{A11s}$$
$$A^{\theta\theta}{}_{4,3}\,[\theta, M/r] = 0 \tag{A11t}$$
$$A^{\theta\theta}{}_{5,0}\,[\theta, M/r] = 504\cos^8\theta\left(\cos^2\theta - 4\right) \tag{A11u}$$
$$A^{\theta\theta}{}_{5,1}\,[\theta, M/r] = -1080\cos^6\theta\left(\cos^4\theta - 8\cos^2\theta + 10\right) \tag{A11v}$$
$$A^{\theta\theta}{}_{5,2}\,[\theta, M/r] = 0 \tag{A11w}$$
$$A^{\theta\theta}{}_{5,3}\,[\theta, M/r] = 0. \tag{A11x}$$

For the massive Dirac field, the coefficients $A^{\varphi\varphi}{}_{p,q}$ appearing in the expression (12) of $\langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren}$ are

$$A^{\varphi\varphi}{}_{0,0}\,[\theta, M/r] = -1512 + 3536\,(M/r) \tag{A12a}$$
$$A^{\varphi\varphi}{}_{0,1}\,[\theta, M/r] = -3240 + 20016\,(M/r) - 30808\,(M/r)^2 \tag{A12b}$$
$$A^{\varphi\varphi}{}_{0,2}\,[\theta, M/r] = -12080\,(M/r)^2 + 33984\,(M/r)^3 \tag{A12c}$$
$$A^{\varphi\varphi}{}_{0,3}\,[\theta, M/r] = -9933\,(M/r)^4 \tag{A12d}$$
$$A^{\varphi\varphi}{}_{1,0}\,[\theta, M/r] = 40824\cos^2\theta - 288\left(345\cos^2\theta - 1\right)(M/r) \tag{A12e}$$

$$A^{\varphi\varphi}{}_{1,1}\left[\theta, M/r\right] = 1080\left(41\cos^2\theta - 56\right) - 576\left(370\cos^2\theta - 341\right)(M/r) + 16\left(18919\cos^2\theta - 7277\right)(M/r)^2 \quad \text{(A12f)}$$

$$A^{\varphi\varphi}{}_{1,2}\left[\theta, M/r\right] = 160\left(617\cos^2\theta - 667\right)(M/r)^2 - 8\left(28743\cos^2\theta - 15373\right)(M/r)^3 \quad \text{(A12g)}$$

$$A^{\varphi\varphi}{}_{1,3}\left[\theta, M/r\right] = \left(51363\cos^2\theta - 32546\right)(M/r)^4 \quad \text{(A12h)}$$

$$A^{\varphi\varphi}{}_{2,0}\left[\theta, M/r\right] = -63504\cos^4\theta + 288\cos^2\theta\left(898\cos^2\theta - 9\right)(M/r) \quad \text{(A12i)}$$

$$A^{\varphi\varphi}{}_{2,1}\left[\theta, M/r\right] = 2160\left(22\cos^4\theta + 38\cos^2\theta - 75\right) - 288\left(382\cos^4\theta + 168\cos^2\theta - 855\right)(M/r) \\ -240\cos^2\theta\left(1104\cos^2\theta + 535\right)(M/r)^2 \quad \text{(A12j)}$$

$$A^{\varphi\varphi}{}_{2,2}\left[\theta, M/r\right] = 96\left(1035\cos^4\theta - 21\cos^2\theta - 1349\right)(M/r)^2 + 16\cos^2\theta\left(2069\cos^2\theta + 10189\right)(M/r)^3 \quad \text{(A12k)}$$

$$A^{\varphi\varphi}{}_{2,3}\left[\theta, M/r\right] = 3\cos^2\theta\left(5839\cos^2\theta - 16492\right)(M/r)^4 \quad \text{(A12l)}$$

$$A^{\varphi\varphi}{}_{3,0}\left[\theta, M/r\right] = -63504\cos^6\theta - 96\cos^4\theta\left(1189\cos^2\theta + 15\right)(M/r) \quad \text{(A12m)}$$

$$A^{\varphi\varphi}{}_{3,1}\left[\theta, M/r\right] = -2160\left(22\cos^6\theta - 132\cos^4\theta + 75\cos^2\theta + 50\right) \\ +2880\cos^2\theta\left(53\cos^4\theta - 143\cos^2\theta + 129\right)(M/r) + 16\cos^4\theta\left(6133\cos^2\theta + 645\right)(M/r)^2 \quad \text{(A12n)}$$

$$A^{\varphi\varphi}{}_{3,2}\left[\theta, M/r\right] = -32\cos^2\theta\left(1195\cos^4\theta - 6687\cos^2\theta + 6750\right)(M/r)^2 \\ -8\cos^4\theta\left(5047\cos^2\theta - 2061\right)(M/r)^3 \quad \text{(A12o)}$$

$$A^{\varphi\varphi}{}_{3,3}\left[\theta, M/r\right] = \cos^4\theta\left(10493\cos^2\theta - 11042\right)(M/r)^4 \quad \text{(A12p)}$$

$$A^{\varphi\varphi}{}_{4,0}\left[\theta, M/r\right] = 40824\cos^8\theta + 144\cos^6\theta\left(39\cos^2\theta + 10\right)(M/r) \quad \text{(A12q)}$$

$$A^{\varphi\varphi}{}_{4,1}\left[\theta, M/r\right] = -1080\cos^2\theta\left(41\cos^6\theta - 76\cos^4\theta - 150\cos^2\theta + 200\right) \\ +720\cos^4\theta\left(41\cos^4\theta - 232\cos^2\theta + 174\right)(M/r) - 8\cos^6\theta\left(221\cos^2\theta + 158\right)(M/r)^2 \quad \text{(A12r)}$$

$$A^{\varphi\varphi}{}_{4,2}\left[\theta, M/r\right] = -16\cos^4\theta\left(1675\cos^4\theta - 6830\cos^2\theta + 5406\right)(M/r)^2 \quad \text{(A12s)}$$

$$A^{\varphi\varphi}{}_{4,3}\left[\theta, M/r\right] = 0 \quad \text{(A12t)}$$

$$A^{\varphi\varphi}{}_{5,0}\left[\theta, M/r\right] = -1512\cos^{10}\theta \quad \text{(A12u)}$$

$$A^{\varphi\varphi}{}_{5,1}\left[\theta, M/r\right] = 1080\cos^4\theta\left(\cos^2\theta - 2\right)\left(3\cos^4\theta - 50\cos^2\theta + 50\right) \quad \text{(A12v)}$$

$$A^{\varphi\varphi}{}_{5,2}\left[\theta, M/r\right] = 0 \quad \text{(A12w)}$$

$$A^{\varphi\varphi}{}_{5,3}\left[\theta, M/r\right] = 0. \quad \text{(A12x)}$$

For the massive Dirac field, the coefficients $A^{t\varphi}{}_{p,q}$ appearing in the expression (13) of $\langle T^t{}_\varphi\rangle_{\rm ren}$ are

$$A^{t\varphi}{}_{0,0}\left[\theta, M/r\right] = 576\,(M/r) \quad \text{(A13a)}$$

$$A^{t\varphi}{}_{0,1}\left[\theta, M/r\right] = -10800 + 37296\,(M/r) - 26320\,(M/r)^2 \quad \text{(A13b)}$$

$$A^{t\varphi}{}_{0,2}\left[\theta, M/r\right] = -20160\,(M/r)^2 + 27740\,(M/r)^3 \quad \text{(A13c)}$$

$$A^{t\varphi}{}_{0,3}\left[\theta, M/r\right] = -7425\,(M/r)^4 \quad \text{(A13d)}$$

$$A^{t\varphi}{}_{1,0}\left[\theta, M/r\right] = -144\left(93\cos^2\theta - 1\right)(M/r) \quad \text{(A13e)}$$

$$A^{t\varphi}{}_{1,1}\left[\theta, M/r\right] = 10800\left(\cos^2\theta - 6\right) - 576\left(39\cos^2\theta - 256\right)(M/r) + 8\left(3837\cos^2\theta - 7277\right)(M/r)^2 \quad \text{(A13f)}$$

$$A^{t\varphi}{}_{1,2}\left[\theta, M/r\right] = 96\left(47\cos^2\theta - 817\right)(M/r)^2 - 4\left(3360\cos^2\theta - 15373\right)(M/r)^3 \quad \text{(A13g)}$$

$$A^{t\varphi}{}_{1,3}\left[\theta, M/r\right] = \left(526\cos^2\theta - 16273\right)(M/r)^4 \quad \text{(A13h)}$$

$$A^{t\varphi}{}_{2,0}\left[\theta, M/r\right] = 144\cos^2\theta\left(205\cos^2\theta - 9\right)(M/r) \quad \text{(A13i)}$$

$$A^{t\varphi}{}_{2,1}\left[\theta, M/r\right] = 21600\left(2\cos^4\theta - 2\cos^2\theta - 5\right) - 144\left(635\cos^4\theta - 774\cos^2\theta - 855\right)(M/r) \\ +120\cos^2\theta\left(19\cos^2\theta - 535\right)(M/r)^2 \quad \text{(A13j)}$$

$$A^{t\varphi}{}_{2,2}\left[\theta, M/r\right] = 48\left(949\cos^4\theta - 1616\cos^2\theta - 1349\right)(M/r)^2 - 4\cos^2\theta\left(6277\cos^2\theta - 20378\right)(M/r)^3 \quad \text{(A13k)}$$

$$A^{t\varphi}{}_{2,3}\left[\theta, M/r\right] = \cos^2\theta\left(8095\cos^2\theta - 24738\right)(M/r)^4 \quad \text{(A13l)}$$

$$A^{t\varphi}{}_{3,0}\left[\theta, M/r\right] = -144\cos^4\theta\left(79\cos^2\theta + 5\right)(M/r) \quad \text{(A13m)}$$

$$A^{t\varphi}{}_{3,1}\left[\theta, M/r\right] = 10800\left(\cos^6\theta + 9\cos^4\theta - 15\cos^2\theta - 5\right) \\ -1440\cos^2\theta\left(13\cos^4\theta + 74\cos^2\theta - 129\right)(M/r) + 8\cos^4\theta\left(2027\cos^2\theta + 645\right)(M/r)^2 \quad \text{(A13n)}$$

$$A^{t\varphi}{}_{3,2}\left[\theta, M/r\right] = 288\cos^2\theta\left(50\cos^4\theta + 143\cos^2\theta - 375\right)(M/r)^2 - 4\cos^4\theta\left(3406\cos^2\theta - 2061\right)(M/r)^3 \quad \text{(A13o)}$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$A^{t\varphi}{}_{3,3}\,[\theta, M/r] = \cos^4\theta\left(4112\cos^2\theta - 5521\right)(M/r)^4 \tag{A13p}$$
$$A^{t\varphi}{}_{4,0}\,[\theta, M/r] = 144\cos^6\theta\left(3\cos^2\theta + 5\right)(M/r) \tag{A13q}$$
$$A^{t\varphi}{}_{4,1}\,[\theta, M/r] = -10800\cos^2\theta\left(\cos^6\theta - 6\cos^4\theta + 10\right) + 720\cos^4\theta\left(18\cos^4\theta - 98\cos^2\theta + 87\right)(M/r) \\ -8\cos^6\theta\left(27\cos^2\theta + 79\right)(M/r)^2 \tag{A13r}$$
$$A^{t\varphi}{}_{4,2}\,[\theta, M/r] = -48\cos^4\theta\left(135\cos^4\theta - 840\cos^2\theta + 901\right)(M/r)^2 \tag{A13s}$$
$$A^{t\varphi}{}_{4,3}\,[\theta, M/r] = 0 \tag{A13t}$$
$$A^{t\varphi}{}_{5,0}\,[\theta, M/r] = 0 \tag{A13u}$$
$$A^{t\varphi}{}_{5,1}\,[\theta, M/r] = -10800\cos^4\theta\left(\cos^4\theta - 5\cos^2\theta + 5\right) \tag{A13v}$$
$$A^{t\varphi}{}_{5,2}\,[\theta, M/r] = 0 \tag{A13w}$$
$$A^{t\varphi}{}_{5,3}\,[\theta, M/r] = 0. \tag{A13x}$$

For the massive Dirac field, the coefficients $A^{\varphi t}{}_{p,q}$ appearing in the expression (14) of $\langle T^{\varphi}{}_{t}\rangle_{\rm ren}$ are

$$A^{\varphi t}{}_{0,0}\,[\theta, M/r] = -144\,(M/r) \tag{A14a}$$
$$A^{\varphi t}{}_{0,1}\,[\theta, M/r] = 10800 - 50256\,(M/r) + 58216\,(M/r)^2 \tag{A14b}$$
$$A^{\varphi t}{}_{0,2}\,[\theta, M/r] = 26640\,(M/r)^2 - 61492\,(M/r)^3 \tag{A14c}$$
$$A^{\varphi t}{}_{0,3}\,[\theta, M/r] = 16273\,(M/r)^4 \tag{A14d}$$
$$A^{\varphi t}{}_{1,0}\,[\theta, M/r] = 1296\cos^2\theta\,(M/r) \tag{A14e}$$
$$A^{\varphi t}{}_{1,1}\,[\theta, M/r] = -10800\left(\cos^2\theta - 5\right) - 432\left(8\cos^2\theta + 285\right)(M/r) + 64200\cos^2\theta\,(M/r)^2 \tag{A14f}$$
$$A^{\varphi t}{}_{1,2}\,[\theta, M/r] = 48\left(176\cos^2\theta + 1349\right)(M/r)^2 - 81512\cos^2\theta\,(M/r)^3 \tag{A14g}$$
$$A^{\varphi t}{}_{1,3}\,[\theta, M/r] = 24738\cos^2\theta\,(M/r)^4 \tag{A14h}$$
$$A^{\varphi t}{}_{2,0}\,[\theta, M/r] = 720\cos^4\theta\,(M/r) \tag{A14i}$$
$$A^{\varphi t}{}_{2,1}\,[\theta, M/r] = -10800\left(4\cos^4\theta - 5\cos^2\theta - 5\right) + 720\cos^2\theta\left(127\cos^2\theta - 258\right)(M/r) \\ -5160\cos^4\theta\,(M/r)^2 \tag{A14j}$$
$$A^{\varphi t}{}_{2,2}\,[\theta, M/r] = -48\cos^2\theta\left(949\cos^2\theta - 2250\right)(M/r)^2 - 8244\cos^4\theta\,(M/r)^3 \tag{A14k}$$
$$A^{\varphi t}{}_{2,3}\,[\theta, M/r] = 5521\cos^4\theta\,(M/r)^4 \tag{A14l}$$
$$A^{\varphi t}{}_{3,0}\,[\theta, M/r] = -720\cos^6\theta\,(M/r) \tag{A14m}$$
$$A^{\varphi t}{}_{3,1}\,[\theta, M/r] = -10800\cos^2\theta\left(\cos^4\theta + 5\cos^2\theta - 10\right) + 720\cos^4\theta\left(62\cos^2\theta - 87\right)(M/r) \\ +632\cos^6\theta\,(M/r)^2 \tag{A14n}$$
$$A^{\varphi t}{}_{3,2}\,[\theta, M/r] = -48\cos^4\theta\left(570\cos^2\theta - 901\right)(M/r)^2 \tag{A14o}$$
$$A^{\varphi t}{}_{3,3}\,[\theta, M/r] = 0 \tag{A14p}$$
$$A^{\varphi t}{}_{4,0}\,[\theta, M/r] = 0 \tag{A14q}$$
$$A^{\varphi t}{}_{4,1}\,[\theta, M/r] = 10800\cos^4\theta\left(\cos^4\theta - 5\cos^2\theta + 5\right) \tag{A14r}$$
$$A^{\varphi t}{}_{4,2}\,[\theta, M/r] = 0 \tag{A14s}$$
$$A^{\varphi t}{}_{4,3}\,[\theta, M/r] = 0. \tag{A14t}$$

For the massive Dirac field, the coefficients $A^{r\theta}{}_{p,q}$ appearing in the expression (15) of $\langle T^{r}{}_{\theta}\rangle_{\rm ren}$ are

$$A^{r\theta}{}_{0,0}\,[\theta, M/r] = -144 + 288\,(M/r) \tag{A15a}$$
$$A^{r\theta}{}_{0,1}\,[\theta, M/r] = 279\,(M/r) - 702\,(M/r)^2 \tag{A15b}$$
$$A^{r\theta}{}_{0,2}\,[\theta, M/r] = -125\,(M/r)^2 + 529\,(M/r)^3 \tag{A15c}$$
$$A^{r\theta}{}_{0,3}\,[\theta, M/r] = -125\,(M/r)^4 \tag{A15d}$$
$$A^{r\theta}{}_{1,0}\,[\theta, M/r] = 144\left(7\cos^2\theta - 1\right) - 2016\cos^2\theta\,(M/r) \tag{A15e}$$
$$A^{r\theta}{}_{1,1}\,[\theta, M/r] = -9\left(135\cos^2\theta - 31\right)(M/r) + 3438\cos^2\theta\,(M/r)^2 \tag{A15f}$$
$$A^{r\theta}{}_{1,2}\,[\theta, M/r] = (326\cos^2\theta - 125)\,(M/r)^2 - 1867\cos^2\theta\,(M/r)^3 \tag{A15g}$$

$$A^{r\theta}{}_{1,3}\left[\theta,M/r\right]=326\cos^2\theta\,(M/r)^4 \tag{A15h}$$
$$A^{r\theta}{}_{2,0}\left[\theta,M/r\right]=-1008\cos^2\theta\left(\cos^2\theta-1\right)+2016\cos^4\theta\,(M/r) \tag{A15i}$$
$$A^{r\theta}{}_{2,1}\left[\theta,M/r\right]=45\cos^2\theta\left(17\cos^2\theta-27\right)(M/r)-2538\cos^4\theta\,(M/r)^2 \tag{A15j}$$
$$A^{r\theta}{}_{2,2}\left[\theta,M/r\right]=-\cos^2\theta\left(125\cos^2\theta-326\right)(M/r)^2+1015\cos^4\theta\,(M/r)^3 \tag{A15k}$$
$$A^{r\theta}{}_{2,3}\left[\theta,M/r\right]=-125\cos^4\theta\,(M/r)^4 \tag{A15l}$$
$$A^{r\theta}{}_{3,0}\left[\theta,M/r\right]=144\cos^4\theta\left(\cos^2\theta-7\right)-288\cos^6\theta\,(M/r) \tag{A15m}$$
$$A^{r\theta}{}_{3,1}\left[\theta,M/r\right]=-45\cos^4\theta\left(\cos^2\theta-17\right)(M/r)+234\cos^6\theta\,(M/r)^2 \tag{A15n}$$
$$A^{r\theta}{}_{3,2}\left[\theta,M/r\right]=-125\cos^4\theta\,(M/r)^2-45\cos^6\theta\,(M/r)^3 \tag{A15o}$$
$$A^{r\theta}{}_{3,3}\left[\theta,M/r\right]=0 \tag{A15p}$$
$$A^{r\theta}{}_{4,0}\left[\theta,M/r\right]=144\cos^6\theta \tag{A15q}$$
$$A^{r\theta}{}_{4,1}\left[\theta,M/r\right]=-45\cos^6\theta\,(M/r) \tag{A15r}$$
$$A^{r\theta}{}_{4,2}\left[\theta,M/r\right]=0 \tag{A15s}$$
$$A^{r\theta}{}_{4,3}\left[\theta,M/r\right]=0. \tag{A15t}$$

For the massive Dirac field, the coefficients $A^{\theta r}{}_{p,q}$ appearing in the expression (16) of $\langle T^{\theta}{}_{r}\rangle_{\rm ren}$ are

$$A^{\theta r}{}_{0,0}\left[\theta,M/r\right]=-144 \tag{A16a}$$
$$A^{\theta r}{}_{0,1}\left[\theta,M/r\right]=279\,(M/r) \tag{A16b}$$
$$A^{\theta r}{}_{0,2}\left[\theta,M/r\right]=-125\,(M/r)^2 \tag{A16c}$$
$$A^{\theta r}{}_{1,0}\left[\theta,M/r\right]=1008\cos^2\theta \tag{A16d}$$
$$A^{\theta r}{}_{1,1}\left[\theta,M/r\right]=-1215\cos^2\theta\,(M/r) \tag{A16e}$$
$$A^{\theta r}{}_{1,2}\left[\theta,M/r\right]=326\cos^2\theta\,(M/r)^2 \tag{A16f}$$
$$A^{\theta r}{}_{2,0}\left[\theta,M/r\right]=-1008\cos^4\theta \tag{A16g}$$
$$A^{\theta r}{}_{2,1}\left[\theta,M/r\right]=765\cos^4\theta\,(M/r) \tag{A16h}$$
$$A^{\theta r}{}_{2,2}\left[\theta,M/r\right]=-125\cos^4\theta\,(M/r)^2 \tag{A16i}$$
$$A^{\theta r}{}_{3,0}\left[\theta,M/r\right]=144\cos^6\theta \tag{A16j}$$
$$A^{\theta r}{}_{3,1}\left[\theta,M/r\right]=-45\cos^6\theta\,(M/r) \tag{A16k}$$
$$A^{\theta r}{}_{3,2}\left[\theta,M/r\right]=0. \tag{A16l}$$

3. Proca field

For the Proca field, the coefficients $A^{tt}{}_{p,q}$ appearing in the expression (9) of $\langle T^{t}{}_{t}\rangle_{\rm ren}$ are

$$A^{tt}{}_{0,0}\left[\theta,M/r\right]=6660-14664\,(M/r) \tag{A17a}$$
$$A^{tt}{}_{0,1}\left[\theta,M/r\right]=48600-276096\,(M/r)+374148\,(M/r)^2 \tag{A17b}$$
$$A^{tt}{}_{0,2}\left[\theta,M/r\right]=167416\,(M/r)^2-430064\,(M/r)^3 \tag{A17c}$$
$$A^{tt}{}_{0,3}\left[\theta,M/r\right]=124228\,(M/r)^4 \tag{A17d}$$
$$A^{tt}{}_{1,0}\left[\theta,M/r\right]=-179820\cos^2\theta+288\left(1528\cos^2\theta+5\right)(M/r) \tag{A17e}$$
$$\begin{aligned}A^{tt}{}_{1,1}\left[\theta,M/r\right]&=-48600\left(7\cos^2\theta-12\right)+144\left(10435\cos^2\theta-13313\right)(M/r)\\&\quad-48\left(36771\cos^2\theta-23602\right)(M/r)^2\end{aligned} \tag{A17f}$$
$$A^{tt}{}_{1,2}\left[\theta,M/r\right]=-16\left(47215\cos^2\theta-69264\right)(M/r)^2+16\left(91476\cos^2\theta-81229\right)(M/r)^3 \tag{A17g}$$
$$A^{tt}{}_{1,3}\left[\theta,M/r\right]=-4\left(86153\cos^2\theta-92104\right)(M/r)^4 \tag{A17h}$$
$$A^{tt}{}_{2,0}\left[\theta,M/r\right]=279720\cos^4\theta-144\cos^2\theta\left(8261\cos^2\theta+90\right)(M/r) \tag{A17i}$$
$$A^{tt}{}_{2,1}\left[\theta,M/r\right]=-97200\left(4\cos^4\theta+6\cos^2\theta-15\right)+144\left(6824\cos^4\theta+3147\cos^2\theta-15845\right)(M/r)$$

$$+120\cos^2\theta\left(8033\cos^2\theta+6732\right)(M/r)^2 \tag{A17j}$$
$$A^{tt}{}_{2,2}\left[\theta,M/r\right]=-96\left(8100\cos^4\theta+290\cos^2\theta-13289\right)(M/r)^2+16\cos^2\theta\left(13713\cos^2\theta-73046\right)(M/r)^3 \tag{A17k}$$
$$A^{tt}{}_{2,3}\left[\theta,M/r\right]=-4\cos^2\theta\left(51301\cos^2\theta-97664\right)(M/r)^4 \tag{A17l}$$
$$A^{tt}{}_{3,0}\left[\theta,M/r\right]=279720\cos^6\theta+288\cos^4\theta\left(1922\cos^2\theta-25\right)(M/r) \tag{A17m}$$
$$A^{tt}{}_{3,1}\left[\theta,M/r\right]=97200\left(4\cos^6\theta-24\cos^4\theta+15\cos^2\theta+10\right)$$
$$-240\cos^2\theta\left(4403\cos^4\theta-14871\cos^2\theta+13386\right)(M/r)-144\cos^4\theta\left(2837\cos^2\theta+1030\right)(M/r)^2 \tag{A17n}$$
$$A^{tt}{}_{3,2}\left[\theta,M/r\right]=80\cos^2\theta\left(4583\cos^4\theta-24648\cos^2\theta+24300\right)(M/r)^2$$
$$+16\cos^4\theta\left(10654\cos^2\theta-2425\right)(M/r)^3 \tag{A17o}$$
$$A^{tt}{}_{3,3}\left[\theta,M/r\right]=-172\cos^4\theta\left(337\cos^2\theta-376\right)(M/r)^4 \tag{A17p}$$
$$A^{tt}{}_{4,0}\left[\theta,M/r\right]=-179820\cos^8\theta-72\cos^6\theta\left(493\cos^2\theta-100\right)(M/r) \tag{A17q}$$
$$A^{tt}{}_{4,1}\left[\theta,M/r\right]=48600\cos^2\theta\left(7\cos^6\theta-12\cos^4\theta-30\cos^2\theta+40\right)$$
$$-720\cos^4\theta\left(362\cos^4\theta-1665\cos^2\theta+1293\right)(M/r)+12\cos^6\theta\left(675\cos^2\theta+584\right)(M/r)^2 \tag{A17r}$$
$$A^{tt}{}_{4,2}\left[\theta,M/r\right]=8\cos^4\theta\left(27673\cos^4\theta-104472\cos^2\theta+83532\right)(M/r)^2 \tag{A17s}$$
$$A^{tt}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A17t}$$
$$A^{tt}{}_{5,0}\left[\theta,M/r\right]=6660\cos^{10}\theta \tag{A17u}$$
$$A^{tt}{}_{5,1}\left[\theta,M/r\right]=-48600\cos^4\theta\left(\cos^2\theta-2\right)\left(\cos^4\theta-10\cos^2\theta+10\right) \tag{A17v}$$
$$A^{tt}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A17w}$$
$$A^{tt}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A17x}$$

For the Proca field, the coefficients $A^{rr}{}_{p,q}$ appearing in the expression (10) of $\langle T^r{}_r\rangle_{\rm ren}$ are

$$A^{rr}{}_{0,0}\left[\theta,M/r\right]=-2772+4200\,(M/r) \tag{A18a}$$
$$A^{rr}{}_{0,1}\left[\theta,M/r\right]=9720-41792\,(M/r)+51628\,(M/r)^2 \tag{A18b}$$
$$A^{rr}{}_{0,2}\left[\theta,M/r\right]=25768\,(M/r)^2-67984\,(M/r)^3 \tag{A18c}$$
$$A^{rr}{}_{0,3}\left[\theta,M/r\right]=21460\,(M/r)^4 \tag{A18d}$$
$$A^{rr}{}_{1,0}\left[\theta,M/r\right]=2772\left(31\cos^2\theta-4\right)-86688\cos^2\theta\,(M/r) \tag{A18e}$$
$$A^{rr}{}_{1,1}\left[\theta,M/r\right]=-9720\left(3\cos^2\theta-8\right)-288\left(301\cos^2\theta+624\right)(M/r)+271088\cos^2\theta\,(M/r)^2 \tag{A18f}$$
$$A^{rr}{}_{1,2}\left[\theta,M/r\right]=8\left(3633\cos^2\theta+13213\right)(M/r)^2-236144\cos^2\theta\,(M/r)^3 \tag{A18g}$$
$$A^{rr}{}_{1,3}\left[\theta,M/r\right]=64076\cos^2\theta\,(M/r)^4 \tag{A18h}$$
$$A^{rr}{}_{2,0}\left[\theta,M/r\right]=-38808\cos^2\theta\left(11\cos^2\theta-8\right)+117936\cos^4\theta\,(M/r) \tag{A18i}$$
$$A^{rr}{}_{2,1}\left[\theta,M/r\right]=-19440\left(6\cos^4\theta-6\cos^2\theta-5\right)+96\cos^2\theta\left(7673\cos^2\theta-8766\right)(M/r)$$
$$-76280\cos^4\theta\,(M/r)^2 \tag{A18j}$$
$$A^{rr}{}_{2,2}\left[\theta,M/r\right]=-8\cos^2\theta\left(34243\cos^2\theta-48823\right)(M/r)^2-34096\cos^4\theta\,(M/r)^3 \tag{A18k}$$
$$A^{rr}{}_{2,3}\left[\theta,M/r\right]=24796\cos^4\theta\,(M/r)^4 \tag{A18l}$$
$$A^{rr}{}_{3,0}\left[\theta,M/r\right]=38808\cos^4\theta\left(17\cos^2\theta-20\right)+18144\cos^6\theta\,(M/r) \tag{A18m}$$
$$A^{rr}{}_{3,1}\left[\theta,M/r\right]=-19440\cos^2\theta\left(4\cos^4\theta+6\cos^2\theta-15\right)-160\cos^4\theta\left(2963\cos^2\theta-3384\right)(M/r)$$
$$-18192\cos^6\theta\,(M/r)^2 \tag{A18n}$$
$$A^{rr}{}_{3,2}\left[\theta,M/r\right]=8\cos^4\theta\left(5571\cos^2\theta-2977\right)(M/r)^2-1872\cos^6\theta\,(M/r)^3 \tag{A18o}$$
$$A^{rr}{}_{3,3}\left[\theta,M/r\right]=4836\cos^6\theta\,(M/r)^4 \tag{A18p}$$
$$A^{rr}{}_{4,0}\left[\theta,M/r\right]=-2772\cos^6\theta\left(85\cos^2\theta-112\right)-9432\cos^8\theta\,(M/r) \tag{A18q}$$
$$A^{rr}{}_{4,1}\left[\theta,M/r\right]=9720\cos^4\theta\left(3\cos^4\theta-28\cos^2\theta+30\right)+480\cos^6\theta\left(341\cos^2\theta-450\right)(M/r)$$
$$+5676\cos^8\theta\,(M/r)^2 \tag{A18r}$$
$$A^{rr}{}_{4,2}\left[\theta,M/r\right]=-8\cos^6\theta\left(4126\cos^2\theta-5765\right)(M/r)^2 \tag{A18s}$$

$$A^{rr}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A18t}$$
$$A^{rr}{}_{5,0}\left[\theta,M/r\right]=2772\cos^8\theta\left(3\cos^2\theta-4\right) \tag{A18u}$$
$$A^{rr}{}_{5,1}\left[\theta,M/r\right]=9720\cos^6\theta\left(3\cos^4\theta-12\cos^2\theta+10\right) \tag{A18v}$$
$$A^{rr}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A18w}$$
$$A^{rr}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A18x}$$

For the Proca field, the coefficients $A^{\theta\theta}{}_{p,q}$ appearing in the expression (11) of $\langle T^{\theta}{}_{\theta}\rangle_{\rm ren}$ are

$$A^{\theta\theta}{}_{0,0}\left[\theta,M/r\right]=8316-19416\left(M/r\right) \tag{A19a}$$
$$A^{\theta\theta}{}_{0,1}\left[\theta,M/r\right]=-29160+126832\left(M/r\right)-123524\left(M/r\right)^2 \tag{A19b}$$
$$A^{\theta\theta}{}_{0,2}\left[\theta,M/r\right]=-83632\left(M/r\right)^2+176272\left(M/r\right)^3 \tag{A19c}$$
$$A^{\theta\theta}{}_{0,3}\left[\theta,M/r\right]=-55916\left(M/r\right)^4 \tag{A19d}$$
$$A^{\theta\theta}{}_{1,0}\left[\theta,M/r\right]=-2772\left(85\cos^2\theta-4\right)+545760\cos^2\theta\left(M/r\right) \tag{A19e}$$
$$A^{\theta\theta}{}_{1,1}\left[\theta,M/r\right]=-29160\left(\cos^2\theta+4\right)+4176\left(143\cos^2\theta+59\right)\left(M/r\right)-1519024\cos^2\theta\left(M/r\right)^2 \tag{A19f}$$
$$A^{\theta\theta}{}_{1,2}\left[\theta,M/r\right]=-8\left(39591\cos^2\theta+18535\right)\left(M/r\right)^2+1292144\cos^2\theta\left(M/r\right)^3 \tag{A19g}$$
$$A^{\theta\theta}{}_{1,3}\left[\theta,M/r\right]=-340212\cos^2\theta\left(M/r\right)^4 \tag{A19h}$$
$$A^{\theta\theta}{}_{2,0}\left[\theta,M/r\right]=38808\cos^2\theta\left(17\cos^2\theta-8\right)-1414224\cos^4\theta\left(M/r\right) \tag{A19i}$$
$$\begin{aligned}A^{\theta\theta}{}_{2,1}\left[\theta,M/r\right]=&\,19440\left(4\cos^4\theta-14\cos^2\theta-5\right)-48\cos^2\theta\left(17171\cos^2\theta-24471\right)\left(M/r\right)\\&+2057800\cos^4\theta\left(M/r\right)^2\end{aligned} \tag{A19j}$$
$$A^{\theta\theta}{}_{2,2}\left[\theta,M/r\right]=8\cos^2\theta\left(19663\cos^2\theta-75877\right)\left(M/r\right)^2-766160\cos^4\theta\left(M/r\right)^3 \tag{A19k}$$
$$A^{\theta\theta}{}_{2,3}\left[\theta,M/r\right]=34908\cos^4\theta\left(M/r\right)^4 \tag{A19l}$$
$$A^{\theta\theta}{}_{3,0}\left[\theta,M/r\right]=-38808\cos^4\theta\left(11\cos^2\theta-20\right)+639072\cos^6\theta\left(M/r\right) \tag{A19m}$$
$$\begin{aligned}A^{\theta\theta}{}_{3,1}\left[\theta,M/r\right]=&\,58320\cos^2\theta\left(2\cos^4\theta-2\cos^2\theta-5\right)-80\cos^4\theta\left(67\cos^2\theta+3159\right)\left(M/r\right)\\&-522864\cos^6\theta\left(M/r\right)^2\end{aligned} \tag{A19n}$$
$$A^{\theta\theta}{}_{3,2}\left[\theta,M/r\right]=8\cos^4\theta\left(10947\cos^2\theta-24077\right)\left(M/r\right)^2+110928\cos^6\theta\left(M/r\right)^3 \tag{A19o}$$
$$A^{\theta\theta}{}_{3,3}\left[\theta,M/r\right]=-12956\cos^6\theta\left(M/r\right)^4 \tag{A19p}$$
$$A^{\theta\theta}{}_{4,0}\left[\theta,M/r\right]=2772\cos^6\theta\left(31\cos^2\theta-112\right)-38808\cos^8\theta\left(M/r\right) \tag{A19q}$$
$$\begin{aligned}A^{\theta\theta}{}_{4,1}\left[\theta,M/r\right]=&\,29160\cos^4\theta\left(\cos^4\theta+4\cos^2\theta-10\right)-240\cos^6\theta\left(542\cos^2\theta-993\right)\left(M/r\right)\\&+9756\cos^8\theta\left(M/r\right)^2\end{aligned} \tag{A19r}$$
$$A^{\theta\theta}{}_{4,2}\left[\theta,M/r\right]=8\cos^6\theta\left(6499\cos^2\theta-11087\right)\left(M/r\right)^2 \tag{A19s}$$
$$A^{\theta\theta}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A19t}$$
$$A^{\theta\theta}{}_{5,0}\left[\theta,M/r\right]=-2772\cos^8\theta\left(\cos^2\theta-4\right) \tag{A19u}$$
$$A^{\theta\theta}{}_{5,1}\left[\theta,M/r\right]=-9720\cos^6\theta\left(\cos^4\theta-8\cos^2\theta+10\right) \tag{A19v}$$
$$A^{\theta\theta}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A19w}$$
$$A^{\theta\theta}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A19x}$$

For the Proca field, the coefficients $A^{\varphi\varphi}{}_{p,q}$ appearing in the expression (12) of $\langle T^{\varphi}{}_{\varphi}\rangle_{\rm ren}$ are

$$A^{\varphi\varphi}{}_{0,0}\left[\theta,M/r\right]=8316-19416\left(M/r\right) \tag{A20a}$$
$$A^{\varphi\varphi}{}_{0,1}\left[\theta,M/r\right]=-29160+126832\left(M/r\right)-123524\left(M/r\right)^2 \tag{A20b}$$
$$A^{\varphi\varphi}{}_{0,2}\left[\theta,M/r\right]=-83632\left(M/r\right)^2+176272\left(M/r\right)^3 \tag{A20c}$$
$$A^{\varphi\varphi}{}_{0,3}\left[\theta,M/r\right]=-55916\left(M/r\right)^4 \tag{A20d}$$
$$A^{\varphi\varphi}{}_{1,0}\left[\theta,M/r\right]=-224532\cos^2\theta+1440\left(380\cos^2\theta-1\right)\left(M/r\right) \tag{A20e}$$
$$A^{\varphi\varphi}{}_{1,1}\left[\theta,M/r\right]=9720\left(41\cos^2\theta-56\right)-8352\left(115\cos^2\theta-216\right)\left(M/r\right)$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$-16\left(24133\cos^2\theta+70806\right)(M/r)^2 \tag{A20f}$$
$$A^{\varphi\varphi}{}_{1,2}\left[\theta,M/r\right]=16\left(36322\cos^2\theta-65385\right)(M/r)^2-16\left(470\cos^2\theta-81229\right)(M/r)^3 \tag{A20g}$$
$$A^{\varphi\varphi}{}_{1,3}\left[\theta,M/r\right]=4\left(7051\cos^2\theta-92104\right)(M/r)^4 \tag{A20h}$$
$$A^{\varphi\varphi}{}_{2,0}\left[\theta,M/r\right]=349272\cos^4\theta-144\cos^2\theta\left(9911\cos^2\theta-90\right)(M/r) \tag{A20i}$$
$$A^{\varphi\varphi}{}_{2,1}\left[\theta,M/r\right]=19440\left(22\cos^4\theta+38\cos^2\theta-75\right)-48\left(32549\cos^4\theta+7686\cos^2\theta-47535\right)(M/r)$$
$$+40\cos^2\theta\left(71641\cos^2\theta-20196\right)(M/r)^2 \tag{A20j}$$
$$A^{\varphi\varphi}{}_{2,2}\left[\theta,M/r\right]=48\left(18630\cos^4\theta-1421\cos^2\theta-26578\right)(M/r)^2$$
$$-16\cos^2\theta\left(120931\cos^2\theta-73046\right)(M/r)^3 \tag{A20k}$$
$$A^{\varphi\varphi}{}_{2,3}\left[\theta,M/r\right]=4\cos^2\theta\left(106391\cos^2\theta-97664\right)(M/r)^4 \tag{A20l}$$
$$A^{\varphi\varphi}{}_{3,0}\left[\theta,M/r\right]=349272\cos^6\theta+288\cos^4\theta\left(2194\cos^2\theta+25\right)(M/r) \tag{A20m}$$
$$A^{\varphi\varphi}{}_{3,1}\left[\theta,M/r\right]=-19440\left(22\cos^6\theta-132\cos^4\theta+75\cos^2\theta+50\right)$$
$$+160\cos^2\theta\left(970\cos^4\theta-22662\cos^2\theta+20079\right)(M/r)-144\cos^4\theta\left(4661\cos^2\theta-1030\right)(M/r)^2 \tag{A20n}$$
$$A^{\varphi\varphi}{}_{3,2}\left[\theta,M/r\right]=-16\cos^2\theta\left(2302\cos^4\theta-117237\cos^2\theta+121500\right)(M/r)^2$$
$$+16\cos^4\theta\left(4508\cos^2\theta+2425\right)(M/r)^3 \tag{A20o}$$
$$A^{\varphi\varphi}{}_{3,3}\left[\theta,M/r\right]=4\cos^4\theta\left(12929\cos^2\theta-16168\right)(M/r)^4 \tag{A20p}$$
$$A^{\varphi\varphi}{}_{4,0}\left[\theta,M/r\right]=-224532\cos^8\theta-72\cos^6\theta\left(439\cos^2\theta+100\right)(M/r) \tag{A20q}$$
$$A^{\varphi\varphi}{}_{4,1}\left[\theta,M/r\right]=-9720\cos^2\theta\left(41\cos^6\theta-76\cos^4\theta-150\cos^2\theta+200\right)$$
$$+240\cos^4\theta\left(2626\cos^4\theta-6054\cos^2\theta+3879\right)(M/r)+12\cos^6\theta\left(1397\cos^2\theta-584\right)(M/r)^2 \tag{A20r}$$
$$A^{\varphi\varphi}{}_{4,2}\left[\theta,M/r\right]=-16\cos^4\theta\left(16643\cos^4\theta-56115\cos^2\theta+41766\right)(M/r)^2 \tag{A20s}$$
$$A^{\varphi\varphi}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A20t}$$
$$A^{\varphi\varphi}{}_{5,0}\left[\theta,M/r\right]=8316\cos^{10}\theta \tag{A20u}$$
$$A^{\varphi\varphi}{}_{5,1}\left[\theta,M/r\right]=9720\cos^4\theta\left(\cos^2\theta-2\right)\left(3\cos^4\theta-50\cos^2\theta+50\right) \tag{A20v}$$
$$A^{\varphi\varphi}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A20w}$$
$$A^{\varphi\varphi}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A20x}$$

For the Proca field, the coefficients $A^{t\varphi}{}_{p,q}$ appearing in the expression (13) of $\langle T^t{}_\varphi\rangle_{\rm ren}$ are

$$A^{t\varphi}{}_{0,0}\left[\theta,M/r\right]=-2376\,(M/r) \tag{A21a}$$
$$A^{t\varphi}{}_{0,1}\left[\theta,M/r\right]=-97200+359856\,(M/r)-248836\,(M/r)^2 \tag{A21b}$$
$$A^{t\varphi}{}_{0,2}\left[\theta,M/r\right]=-219660\,(M/r)^2+303168\,(M/r)^3 \tag{A21c}$$
$$A^{t\varphi}{}_{0,3}\left[\theta,M/r\right]=-90072\,(M/r)^4 \tag{A21d}$$
$$A^{t\varphi}{}_{1,0}\left[\theta,M/r\right]=144\left(367\cos^2\theta-5\right)(M/r) \tag{A21e}$$
$$A^{t\varphi}{}_{1,1}\left[\theta,M/r\right]=97200\left(\cos^2\theta-6\right)-72\left(4467\cos^2\theta-19223\right)(M/r)+16\left(7687\cos^2\theta-35403\right)(M/r)^2 \tag{A21f}$$
$$A^{t\varphi}{}_{1,2}\left[\theta,M/r\right]=12\left(8221\cos^2\theta-66601\right)(M/r)^2-8\left(10717\cos^2\theta-81229\right)(M/r)^3 \tag{A21g}$$
$$A^{t\varphi}{}_{1,3}\left[\theta,M/r\right]=8\left(275\cos^2\theta-23026\right)(M/r)^4 \tag{A21h}$$
$$A^{t\varphi}{}_{2,0}\left[\theta,M/r\right]=-2160\cos^2\theta\left(52\cos^2\theta-3\right)(M/r) \tag{A21i}$$
$$A^{t\varphi}{}_{2,1}\left[\theta,M/r\right]=194400\left(2\cos^4\theta-2\cos^2\theta-5\right)-72\left(12200\cos^4\theta-11363\cos^2\theta-15845\right)(M/r)$$
$$+440\cos^2\theta\left(1243\cos^2\theta-918\right)(M/r)^2 \tag{A21j}$$
$$A^{t\varphi}{}_{2,2}\left[\theta,M/r\right]=12\left(39001\cos^4\theta-53339\cos^2\theta-53156\right)(M/r)^2$$
$$-16\cos^2\theta\left(30799\cos^2\theta-36523\right)(M/r)^3 \tag{A21k}$$
$$A^{t\varphi}{}_{2,3}\left[\theta,M/r\right]=8\cos^2\theta\left(15007\cos^2\theta-24416\right)(M/r)^4 \tag{A21l}$$
$$A^{t\varphi}{}_{3,0}\left[\theta,M/r\right]=144\cos^4\theta\left(297\cos^2\theta+25\right)(M/r) \tag{A21m}$$
$$A^{t\varphi}{}_{3,1}\left[\theta,M/r\right]=97200\left(\cos^6\theta+9\cos^4\theta-15\cos^2\theta-5\right)$$

$$-360\cos^2\theta\left(223\cos^4\theta+3091\cos^2\theta-4462\right)(M/r)-144\cos^4\theta\left(397\cos^2\theta-515\right)(M/r)^2 \tag{A21n}$$

$$A^{t\varphi}{}_{3,2}\left[\theta,M/r\right]=12\cos^2\theta\left(7615\cos^4\theta+40317\cos^2\theta-81000\right)(M/r)^2$$
$$-8\cos^4\theta\left(3721\cos^2\theta-2425\right)(M/r)^3 \tag{A21o}$$

$$A^{t\varphi}{}_{3,3}\left[\theta,M/r\right]=8\cos^4\theta\left(2813\cos^2\theta-4042\right)(M/r)^4 \tag{A21p}$$

$$A^{t\varphi}{}_{4,0}\left[\theta,M/r\right]=-72\cos^6\theta\left(23\cos^2\theta+50\right)(M/r) \tag{A21q}$$

$$A^{t\varphi}{}_{4,1}\left[\theta,M/r\right]=-97200\cos^2\theta\left(\cos^6\theta-6\cos^4\theta+10\right)$$
$$+360\cos^4\theta\left(324\cos^4\theta-1519\cos^2\theta+1293\right)(M/r)+12\cos^6\theta\left(69\cos^2\theta-292\right)(M/r)^2 \tag{A21r}$$

$$A^{t\varphi}{}_{4,2}\left[\theta,M/r\right]=-12\cos^4\theta\left(4860\cos^4\theta-27055\cos^2\theta+27844\right)(M/r)^2 \tag{A21s}$$

$$A^{t\varphi}{}_{4,3}\left[\theta,M/r\right]=0 \tag{A21t}$$

$$A^{t\varphi}{}_{5,0}\left[\theta,M/r\right]=0 \tag{A21u}$$

$$A^{t\varphi}{}_{5,1}\left[\theta,M/r\right]=-97200\cos^4\theta\left(\cos^4\theta-5\cos^2\theta+5\right) \tag{A21v}$$

$$A^{t\varphi}{}_{5,2}\left[\theta,M/r\right]=0 \tag{A21w}$$

$$A^{t\varphi}{}_{5,3}\left[\theta,M/r\right]=0. \tag{A21x}$$

For the Proca field, the coefficients $A^{\varphi t}{}_{p,q}$ appearing in the expression (14) of $\langle T^{\varphi}{}_{t}\rangle_{\text{ren}}$ are

$$A^{\varphi t}{}_{0,0}\left[\theta,M/r\right]=720\,(M/r) \tag{A22a}$$

$$A^{\varphi t}{}_{0,1}\left[\theta,M/r\right]=97200-476496\,(M/r)+566448\,(M/r)^2 \tag{A22b}$$

$$A^{\varphi t}{}_{0,2}\left[\theta,M/r\right]=277980\,(M/r)^2-649832\,(M/r)^3 \tag{A22c}$$

$$A^{\varphi t}{}_{0,3}\left[\theta,M/r\right]=184208\,(M/r)^4 \tag{A22d}$$

$$A^{\varphi t}{}_{1,0}\left[\theta,M/r\right]=-6480\cos^2\theta\,(M/r) \tag{A22e}$$

$$A^{\varphi t}{}_{1,1}\left[\theta,M/r\right]=-97200\left(\cos^2\theta-5\right)+72\left(1227\cos^2\theta-15845\right)(M/r)+403920\cos^2\theta\,(M/r)^2 \tag{A22f}$$

$$A^{\varphi t}{}_{1,2}\left[\theta,M/r\right]=12\left(1499\cos^2\theta+53156\right)(M/r)^2-584368\cos^2\theta\,(M/r)^3 \tag{A22g}$$

$$A^{\varphi t}{}_{1,3}\left[\theta,M/r\right]=195328\cos^2\theta\,(M/r)^4 \tag{A22h}$$

$$A^{\varphi t}{}_{2,0}\left[\theta,M/r\right]=-3600\cos^4\theta\,(M/r) \tag{A22i}$$

$$A^{\varphi t}{}_{2,1}\left[\theta,M/r\right]=-97200\left(4\cos^4\theta-5\cos^2\theta-5\right)+720\cos^2\theta\left(1220\cos^2\theta-2231\right)(M/r)$$
$$-74160\cos^4\theta\,(M/r)^2 \tag{A22j}$$

$$A^{\varphi t}{}_{2,2}\left[\theta,M/r\right]=-12\cos^2\theta\left(39001\cos^2\theta-81000\right)(M/r)^2-19400\cos^4\theta\,(M/r)^3 \tag{A22k}$$

$$A^{\varphi t}{}_{2,3}\left[\theta,M/r\right]=32336\cos^4\theta\,(M/r)^4 \tag{A22l}$$

$$A^{\varphi t}{}_{3,0}\left[\theta,M/r\right]=3600\cos^6\theta\,(M/r) \tag{A22m}$$

$$A^{\varphi t}{}_{3,1}\left[\theta,M/r\right]=-97200\cos^2\theta\left(\cos^4\theta+5\cos^2\theta-10\right)+360\cos^4\theta\left(871\cos^2\theta-1293\right)(M/r)$$
$$+3504\cos^6\theta\,(M/r)^2 \tag{A22n}$$

$$A^{\varphi t}{}_{3,2}\left[\theta,M/r\right]=-12\cos^4\theta\left(17335\cos^2\theta-27844\right)(M/r)^2 \tag{A22o}$$

$$A^{\varphi t}{}_{3,3}\left[\theta,M/r\right]=0 \tag{A22p}$$

$$A^{\varphi t}{}_{4,0}\left[\theta,M/r\right]=0 \tag{A22q}$$

$$A^{\varphi t}{}_{4,1}\left[\theta,M/r\right]=97200\cos^4\theta\left(\cos^4\theta-5\cos^2\theta+5\right) \tag{A22r}$$

$$A^{\varphi t}{}_{4,2}\left[\theta,M/r\right]=0 \tag{A22s}$$

$$A^{\varphi t}{}_{4,3}\left[\theta,M/r\right]=0. \tag{A22t}$$

For the Proca field, the coefficients $A^{r\theta}{}_{p,q}$ appearing in the expression (15) of $\langle T^{r}{}_{\theta}\rangle_{\text{ren}}$ are

$$A^{r\theta}{}_{0,0}\left[\theta,M/r\right]=792-1584\,(M/r) \tag{A23a}$$

$$A^{r\theta}{}_{0,1}\left[\theta,M/r\right]=-972\,(M/r)+2736\,(M/r)^2 \tag{A23b}$$

$$A^{r\theta}{}_{0,2}\left[\theta,M/r\right]=250\,(M/r)^2-1472\,(M/r)^3 \tag{A23c}$$

$$A^{r\theta}{}_{0,3}\left[\theta,M/r\right]=250\,(M/r)^4 \tag{A23d}$$

Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

$$A^{r\theta}{}_{1,0}\,[\theta, M/r] = -792\left(7\cos^2\theta - 1\right) + 11088\cos^2\theta\,(M/r) \tag{A23e}$$
$$A^{r\theta}{}_{1,1}\,[\theta, M/r] = 12\left(635\cos^2\theta - 81\right)(M/r) - 20784\cos^2\theta\,(M/r)^2 \tag{A23f}$$
$$A^{r\theta}{}_{1,2}\,[\theta, M/r] = -2\left(1334\cos^2\theta - 125\right)(M/r)^2 + 12956\cos^2\theta\,(M/r)^3 \tag{A23g}$$
$$A^{r\theta}{}_{1,3}\,[\theta, M/r] = -2668\cos^2\theta\,(M/r)^4 \tag{A23h}$$
$$A^{r\theta}{}_{2,0}\,[\theta, M/r] = 5544(\cos\theta - 1)\cos^2\theta(\cos\theta + 1) - 11088\cos^4\theta\,(M/r) \tag{A23i}$$
$$A^{r\theta}{}_{2,1}\,[\theta, M/r] = -60\cos^2\theta\left(67\cos^2\theta - 127\right)(M/r) + 13584\cos^4\theta\,(M/r)^2 \tag{A23j}$$
$$A^{r\theta}{}_{2,2}\,[\theta, M/r] = 2\cos^2\theta\left(125\cos^2\theta - 1334\right)(M/r)^2 - 4520\cos^4\theta\,(M/r)^3 \tag{A23k}$$
$$A^{r\theta}{}_{2,3}\,[\theta, M/r] = 250\cos^4\theta\,(M/r)^4 \tag{A23l}$$
$$A^{r\theta}{}_{3,0}\,[\theta, M/r] = -792\cos^4\theta\left(\cos^2\theta - 7\right) + 1584\cos^6\theta\,(M/r) \tag{A23m}$$
$$A^{r\theta}{}_{3,1}\,[\theta, M/r] = 60\cos^4\theta\left(\cos^2\theta - 67\right)(M/r) - 912\cos^6\theta\,(M/r)^2 \tag{A23n}$$
$$A^{r\theta}{}_{3,2}\,[\theta, M/r] = 250\cos^4\theta\,(M/r)^2 + 60\cos^6\theta\,(M/r)^3 \tag{A23o}$$
$$A^{r\theta}{}_{3,3}\,[\theta, M/r] = 0 \tag{A23p}$$
$$A^{r\theta}{}_{4,0}\,[\theta, M/r] = -792\cos^6\theta \tag{A23q}$$
$$A^{r\theta}{}_{4,1}\,[\theta, M/r] = 60\cos^6\theta\,(M/r) \tag{A23r}$$
$$A^{r\theta}{}_{4,2}\,[\theta, M/r] = 0 \tag{A23s}$$
$$A^{r\theta}{}_{4,3}\,[\theta, M/r] = 0. \tag{A23t}$$

For the Proca field, the coefficients $A^{\theta r}{}_{p,q}$ appearing in the expression (16) of $\langle T^\theta{}_r\rangle_{\rm ren}$ are

$$A^{\theta r}{}_{0,0}\,[\theta, M/r] = 792 \tag{A24a}$$
$$A^{\theta r}{}_{0,1}\,[\theta, M/r] = -972\,(M/r) \tag{A24b}$$
$$A^{\theta r}{}_{0,2}\,[\theta, M/r] = 250\,(M/r)^2 \tag{A24c}$$
$$A^{\theta r}{}_{1,0}\,[\theta, M/r] = -5544\cos^2\theta \tag{A24d}$$
$$A^{\theta r}{}_{1,1}\,[\theta, M/r] = 7620\cos^2\theta\,(M/r) \tag{A24e}$$
$$A^{\theta r}{}_{1,2}\,[\theta, M/r] = -2668\cos^2\theta\,(M/r)^2 \tag{A24f}$$
$$A^{\theta r}{}_{2,0}\,[\theta, M/r] = 5544\cos^4\theta \tag{A24g}$$
$$A^{\theta r}{}_{2,1}\,[\theta, M/r] = -4020\cos^4\theta\,(M/r) \tag{A24h}$$
$$A^{\theta r}{}_{2,2}\,[\theta, M/r] = 250\cos^4\theta\,(M/r)^2 \tag{A24i}$$
$$A^{\theta r}{}_{3,0}\,[\theta, M/r] = -792\cos^6\theta \tag{A24j}$$
$$A^{\theta r}{}_{3,1}\,[\theta, M/r] = 60\cos^6\theta\,(M/r) \tag{A24k}$$
$$A^{\theta r}{}_{3,2}\,[\theta, M/r] = 0. \tag{A24l}$$

[1] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[3] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-time* (Cambridge University Press, Cambridge, 1989).

[4] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1995).

[5] L. H. Ford, (2005), arXiv:gr-qc/0504096.

[6] L. E. Parker and D. J. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, Cambridge, 2009).

[7] S. Hollands and R. M. Wald, (2014), arXiv:1401.2026 [gr-qc].

[8] L. E. Parker, Phys. Rev. **183**, 1057 (1969).

[9] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[10] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).

[11] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).

[12] J. M. Bardeen, Phys. Rev. Lett. **46**, 382 (1981).
[13] W. A. Hiscock, Phys. Rev. D **23**, 2813 (1981).
[14] L. Parker and D. A. T. Vanzella, Phys. Rev. D **69**, 104009 (2004), arXiv:gr-qc/0312108.
[15] D. N. Page, Phys. Rev. D **25**, 1499 (1982).
[16] M. R. Brown and A. C. Ottewill, Phys. Rev. D **31**, 2514 (1985).
[17] M. R. Brown, A. C. Ottewill, and D. N. Page, Phys. Rev. D **33**, 2840 (1986).
[18] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[19] B. S. DeWitt, *The Global Approach to Quantum Field Theory* (Oxford University Press, Oxford, 2003).
[20] I. G. Avramidi, (1995), arXiv:hep-th/9510140.
[21] I. G. Avramidi, *Heat Kernel and Quantum Gravity* (Springer, Berlin, 2000).
[22] The expression DeWitt-Schwinger approximation could confuse the reader. Here, we intend the DeWitt-Schwinger approximation of the RSET. This approximation is constructed from the renormalized effective action (5), is given explicitly by (7), is only valid in the large mass limit and is purely geometrical. It is important to note that the DeWitt-Schwinger representation of the Feynman propagator permitted Christensen [63, 64] and other authors (see, e.g., Refs. [1, 2]) to develop the so-called DeWitt-Schwinger approach of the renormalization of the stress tensor. Such an approach provides, for all mass values, exact results (in general, they cannot be put in an analytical form) which take into account the geometry of the gravitational background as well as the quantum state of the field.
[23] V. P. Frolov and A. I. Zelnikov, Phys. Lett. B **115**, 372 (1982).
[24] V. P. Frolov and A. I. Zelnikov, Phys. Lett. B **123**, 197 (1983).
[25] V. P. Frolov and A. I. Zelnikov, Phys. Rev. D **29**, 1057 (1984).
[26] P. R. Anderson, W. A. Hiscock, and D. A. Samuel, Phys. Rev. D **51**, 4337 (1995).
[27] B. E. Taylor, W. A. Hiscock, and P. R. Anderson, Phys. Rev. D **55**, 6116 (1997), arXiv:gr-qc/9608036.
[28] W. A. Hiscock, S. L. Larson, and P. R. Anderson, Phys. Rev. D **56**, 3571 (1997), arXiv:gr-qc/9701004.
[29] B. E. Taylor, W. A. Hiscock, and P. R. Anderson, Phys. Rev. D **61**, 084021 (2000), arXiv:gr-qc/9911119.
[30] J. Matyjasek, Phys. Rev. D **61**, 124019 (2000), arXiv:gr-qc/9912020.
[31] P. R. Anderson, W. A. Hiscock, and B. E. Taylor, Phys. Rev. Lett. **85**, 2438 (2000), arXiv:gr-qc/0002007.
[32] J. Matyjasek, Phys. Rev. D **63**, 084004 (2001), arXiv:gr-qc/0010097.
[33] J. Matyjasek, Phys. Rev. D **74**, 104030 (2006), arXiv:gr-qc/0610020.
[34] O. P. Fernández Piedra and A. Cabo Montes de Oca, Phys. Rev. D **75**, 107501 (2007), arXiv:0707.0706 [gr-qc].
[35] O. P. Fernández Piedra and J. de Oliveira, Int. J. Mod. Phys. D **19**, 63 (2010), arXiv:0902.1487 [gr-qc].
[36] O. P. Fernández Piedra and J. de Oliveira, Classical Quantum Gravity **28**, 085003 (2011), arXiv:1006.3802 [gr-qc].
[37] J. Matyjasek and K. Zwierzchowska, Phys. Rev. D **85**, 024009 (2012), arXiv:1110.0041 [gr-qc].
[38] J. Matyjasek and P. Sadurski, Phys. Rev. D **88**, 104015 (2013), arXiv:1309.0552 [gr-qc].
[39] J. Matyjasek, P. Sadurski, and M. Telecka, Phys. Rev. D **89**, 084055 (2014).
[40] V. P. Frolov and I. D. Novikov, *Black Hole Physics* (Kluwer Academic Publishers, Dordrecht, 1998).
[41] P. Candelas, Phys. Rev. D **21**, 2185 (1980).
[42] M. S. Fawcett, Commun. Math. Phys. **89**, 103 (1983).
[43] K. W. Howard and P. Candelas, Phys. Rev. Lett. **53**, 403 (1984).
[44] K. W. Howard, Phys. Rev. D **30**, 2532 (1984).
[45] B. P. Jensen and A. C. Ottewill, Phys. Rev. D **39**, 1130 (1989).
[46] B. P. Jensen, J. G. McLaughlin, and A. C. Ottewill, Phys. Rev. D **43**, 4142 (1991).
[47] V. P. Frolov and K. S. Thorne, Phys. Rev. D **39**, 2125 (1989).
[48] A. C. Ottewill and E. Winstanley, Phys. Rev. D **62**, 084018 (2000), arXiv:gr-qc/0004022.
[49] A. C. Ottewill and E. Winstanley, Phys. Lett. A **273**, 149 (2000), arXiv:gr-qc/0005108.
[50] G. Duffy and A. C. Ottewill, Phys. Rev. D **77**, 024007 (2008), arXiv:gr-qc/0507116.
[51] M. Casals, S. R. Dolan, B. C. Nolan, A. C. Ottewill, and E. Winstanley, Phys. Rev. D **87**, 064027 (2013), arXiv:1207.7089 [gr-qc].
[52] B. S. Kay and R. M. Wald, Phys. Rept. **207**, 49 (1991).
[53] Y. Decanini and A. Folacci, Classical Quantum Gravity **24**, 4777 (2007), arXiv:0706.0691 [gr-qc].
[54] S. A. Fulling, R. C. King, B. G. Wybourne, and C. J. Cummins, Classical Quantum Gravity **9**, 1151 (1992).
[55] J. M. Martín-García, *xAct: Efficient Tensor Computer Algebra for Mathematica* (http://www.xact.es/).
[56] T. Damour, N. Deruelle, and R. Ruffini, Lett. Nuovo Cim. **15**, 257 (1976).
[57] T. J. M. Zouros and D. M. Eardley, Ann. Phys. (N.Y.) **118**, 139 (1979).
[58] S. L. Detweiler, Phys. Rev. D **22**, 2323 (1980).
[59] H. Furuhashi and Y. Nambu, Prog. Theor. Phys. **112**, 983 (2004), arXiv:gr-qc/0402037.
[60] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).
[61] E. Berti and K. D. Kokkotas, Phys. Rev. D **71**, 124008 (2005), arXiv:gr-qc/0502065.
[62] R. A. Konoplya and A. Zhidenko, Phys. Rev. D **88**, 024054 (2013), arXiv:1307.1812 [gr-qc].
[63] S. M. Christensen, Phys. Rev. D **14**, 2490 (1976).
[64] S. M. Christensen, Phys. Rev. D **17**, 946 (1978).

 Published in Phys. Rev. D 90, 044045 (2014), eprint : arXiv:1404.7422

Partie II:

Electromagnétisme massif de Stueckelberg et renormalisation du tenseur d'impulsion-énergie associé

Chapitre 2:

Electromagnétisme massif de Stueckelberg en espace-temps courbe : renormalisation de Hadamard du tenseur d'impulsion-énergie et effet Casimir

Dans ce chapitre, nous nous sommes intéressés à l'électromagnétisme massif de Stueckelberg. Contrairement à la théorie massive de de Broglie-Proca qui est une généralisation directe de l'électromagnétisme de Maxwell obtenue en ajoutant un terme de masse qui brise la symétrie de jauge locale $U(1)$ originelle, la théorie massive proposée par Stueckelberg la préserve par l'intermédiaire d'un couplage approprié d'un champ scalaire auxiliaire avec le champ vectoriel massif. Il faut noter aussi que cette théorie est unitaire et renormalisable et peut être incluse dans le modèle standard des particules. De plus, certaines extensions du modèle standard basées sur la théorie des cordes incluent l'existence d'un photon noir très massif pouvant expliquer la matière noire de l'univers. Ici, nous avons développé le formalisme général de la théorie de Stueckelberg sur un espace-temps arbitraire de dimension quatre (invariance de jauge de la théorie classique et quantification covariante ; équations d'ondes pour le champ massif de spin-1 A_μ, pour le champ scalaire auxiliaire de Stueckelberg Φ et pour les champs de fantômes C et C^* ; identité de Ward ; représentation de Hadamard des divers propagateurs de Feynman et développement en séries de Taylor covariantes des coefficients correspondants). Cela nous a permis de construire, pour un état quantique de type Hadamard $|\psi\rangle$, la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé à la théorie de Stueckelberg. Nous avons donné deux expressions différentes mais équivalentes du résultat. La première expression ne fait intervenir que les caractéristiques du champ vectoriel A_μ. Dans la seconde expression, nous avons séparé de manière artificielle en deux parties les contributions indépendamment conservées du champ vectoriel A_μ et du champ scalaire auxiliaire Φ dans le but de discuter la limite de masse nulle des résultats obtenus dans le cadre de la théorie de Stueckelberg et les comparer avec ceux issus de celle de Maxwell. Notons que l'on peut passer de la deuxième expression à la première en éliminant la contribution du champ scalaire de Stueckelberg Φ par l'intermédiaire de deux identités de Ward, ce champ pouvant être considéré comme un fantôme.

Comme application de ce résultat, nous avons considéré, en espace-temps de Minkowski, l'effet de Casimir en dehors d'un milieu parfaitement conducteur et de bord plan. Nous avons retrouvé le résultat existant déjà dans la littérature pour la théorie de de Broglie-Proca. Notons toutefois que nous ne sommes pas vraiment satisfaits des conditions aux limites prises sur le bord du milieu conducteur et que des conditions plus réalistes pourraient être envisagées.

Pour finir, nous avons comparé les formalismes de de Broglie-Proca et de Stueckelberg et discuté les avantages de l'approche de Stueckelberg bien qu'à notre avis ces deux descriptions de l'électromagnétisme massif sont les deux faces d'une même théorie.

Stueckelberg massive electromagnetism in curved spacetime: Hadamard renormalization of the stress-energy tensor and the Casimir effect

Andrei Belokogne* and Antoine Folacci†
Equipe Physique Théorique - Projet COMPA,
SPE, UMR 6134 du CNRS et de l'Université de Corse,
Université de Corse, BP 52, F-20250 Corte, France
(Dated: October 13, 2016)

We discuss Stueckelberg massive electromagnetism on an arbitrary four-dimensional curved spacetime (gauge invariance of the classical theory and covariant quantization; wave equations for the massive spin-1 field A_μ, for the auxiliary Stueckelberg scalar field Φ and for the ghost fields C and C^*; Ward identities; Hadamard representation of the various Feynman propagators and covariant Taylor series expansions of the corresponding coefficients). This permits us to construct, for a Hadamard quantum state, the expectation value of the renormalized stress-energy tensor associated with the Stueckelberg theory. We provide two alternative but equivalent expressions for this result. The first one is obtained by removing the contribution of the "Stueckelberg ghost" Φ and only involves state-dependent and geometrical quantities associated with the massive vector field A_μ. The other one involves contributions coming from both the massive vector field and the auxiliary Stueckelberg scalar field, and it has been constructed in such a way that, in the zero-mass limit, the massive vector field contribution reduces smoothly to the result obtained from Maxwell's theory. As an application of our results, we consider the Casimir effect outside a perfectly conducting medium with a plane boundary. We discuss the results obtained using Stueckelberg but also de Broglie-Proca electromagnetism and we consider the zero-mass limit of the vacuum energy in both theories. We finally compare the de Broglie-Proca and Stueckelberg formalisms and highlight the advantages of the Stueckelberg point of view, even if, in our opinion, the de Broglie-Proca and Stueckelberg approaches of massive electromagnetism are two faces of the same field theory.

PACS numbers: 04.62.+v, 11.10.Gh, 03.70.+k

I. INTRODUCTION

It is generally assumed that the electromagnetic interaction is mediated by a massless photon. This seems largely justified (i) by the countless theoretical and practical successes of Maxwell's theory of electromagnetism and of its extension in the framework of quantum field theory as well as (ii) by the stringent upper limits on the photon mass (see p. 559 of Ref. [1] and references therein) which have been obtained by various terrestrial and extraterrestrial experiments (currently, one of the most reliable results provides for the photon mass m the limit $m \leq 10^{-18}$ eV $\approx 2 \times 10^{-54}$ kg [2]).

Despite this, physicists are seriously considering the possibility of a massive but, of course, ultralight photon and are very interested by the associated non-Maxwellian theories of electromagnetism (for recent reviews on the subject, see Refs. [3, 4]). Indeed, the incredibly small value mentioned above does not necessarily imply that the photon mass is exactly zero, and from a theoretical point of view, massive electromagnetism can be rather easily included in the Standard Model of particle physics. Moreover, in order to test the masslessness of the photon or, more precisely, to impose experimental constraints on its mass, it is necessary to have a good understanding of the various massive non-Maxwellian theories. Among these, two theories are particularly important, and we intend to discuss them at more length in our article:

(i) The most popular one, which is the simplest generalization of Maxwell's electromagnetism, is mainly due to de Broglie (note that the idea of an ultralight massive photon is already present in de Broglie's doctoral thesis [5, 6] and has been developed by him in modern terms in a series of works [7–9] where he has considered the theory from a Lagrangian point of view and has explicitly shown the modifications induced by the photon mass for Maxwell's equations) but is attributed in the literature to its "PhD student" Proca (for the series of his original articles dating from 1930 to 1938 which led him to introduce in Ref. [10] the so-called Proca equation for a massive vector field, see Ref. [11], but note, however, that the main aim of Proca was the description of spin-1/2 particles inspired by the neutrino theory of light due to de Broglie). Here, it is worth pointing out that, due to the mass term, the de Broglie-Proca theory is not a gauge theory, and this has some important consequences when we compare, in the limit $m^2 \to 0$, the results obtained via the de Broglie-Proca theory with those derived from Maxwell's electromagnetism. It is also important to recall that, in general, it is the de Broglie-Proca theory that is used to impose experimental constraints on the photon mass [2–4].

* belokogne.andrei@gmail.com
† folacci@univ-corse.fr

(ii) The most aesthetically appealing one which, contrarily to the de Broglie-Proca theory preserves the local $U(1)$ gauge invariance of Maxwell's electromagnetism, has been proposed by Stueckelberg (see Refs. [12, 13] for the original articles on the subject and also Ref. [14] for a nice recent review). The construction of such a massive gauge theory can be achieved by coupling appropriately an auxiliary scalar field to the massive spin-1 field. This theory is unitary and renormalizable and can be included in the Standard Model of particle physics [14]. Moreover, it is interesting to note that extensions of the Standard Model based on string theory predict the existence of a hidden sector of particles which could explain the nature of dark matter. Among these exotic particles, there exists in particular a dark photon, the mass of which arises also via the Stueckelberg mechanism (see, e.g., Ref. [15]). This "heavy" photon may be detectable in low energy experiments (see, e.g., Refs [16–19]). It is also worth pointing out that the Stueckelberg procedure is not limited to vector fields. It has been recently extended to "restore" the gauge invariance of various massive field theories (see, e.g., Refs. [20, 21] which discuss the case of massive antisymmetric tensor fields and, e.g., Ref. [22] where massive gravity is considered).

In the two following paragraphs, we shall briefly review these two theories at the classical level.

De Broglie-Proca massive electromagnetism is described by a vector field A_μ, and its action $S = S[A_\mu, g_{\mu\nu}]$, which is directly obtained from the original Maxwell Lagrangian by adding a mass contribution, is given by

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \right]. \tag{1}$$

Here, m is the mass of the vector field A_μ, and the associated field strength $F_{\mu\nu}$ is defined as usual by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

Let us note that, while Maxwell's theory is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \nabla_\mu \Lambda \tag{3}$$

for an arbitrary scalar field Λ, this gauge invariance is broken for the de Broglie-Proca theory due to the mass term. The extremization of (1) with respect to A_μ leads to the Proca equation $\nabla^\nu F_{\mu\nu} + m^2 A_\mu = 0$. Applying ∇^μ to this equation, we obtain the Lorenz condition

$$\nabla^\mu A_\mu = 0 \tag{4a}$$

which is here a dynamical constraint (and not a gauge condition) as well as the wave equation

$$\Box A_\mu - m^2 A_\mu - R_\mu{}^\nu A_\nu = 0. \tag{4b}$$

It should be noted that the action (1) is also directly relevant at the quantum level because the de Broglie-Proca theory is not a gauge theory.

Stueckelberg massive electromagnetism is described by a vector field A_μ and an auxiliary scalar field Φ, and its action $S_{\text{Cl}} = S_{\text{Cl}}[A_\mu, \Phi, g_{\mu\nu}]$, which can be constructed from the de Broglie-Proca action (1) by using the substitution

$$A_\mu \to A_\mu + \frac{1}{m} \nabla_\mu \Phi, \tag{5}$$

is given by

$$S_{\text{Cl}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 \left(A^\mu + \frac{1}{m} \nabla^\mu \Phi \right) \left(A_\mu + \frac{1}{m} \nabla_\mu \Phi \right) \right] \tag{6a}$$

$$= \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla^\mu A^\nu \nabla_\mu A_\nu + \frac{1}{2} (\nabla^\mu A_\mu)^2 - \frac{1}{2} m^2 A^\mu A_\mu - \frac{1}{2} R_{\mu\nu} A^\mu A^\nu - \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi - m A^\mu \nabla_\mu \Phi \right]. \tag{6b}$$

It should be noted that, at the classical level, the vector field A_μ and the scalar field Φ are coupled [see, in Eq. (6b), the last term $-mA^\mu \nabla_\mu \Phi$]. Here, it is important to note that Stueckelberg massive electromagnetism is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \nabla_\mu \Lambda, \tag{7a}$$

$$\Phi \to \Phi' = \Phi - m\Lambda, \tag{7b}$$

for an arbitrary scalar field Λ, so the local $U(1)$ gauge symmetry of Maxwell's electromagnetism remains unbroken for the spin-1 field of the Stueckelberg theory. As a consequence, in order to treat this theory at the quantum level (see below), it is necessary to add to the action (6) a gauge-breaking term and the compensating ghost contribution.

Here, it seems important to highlight some considerations which will play a crucial role in this article. Let us note that the de Broglie-Proca theory can be obtained from Stueckelberg electromagnetism by taking

$$\Phi = 0. \tag{8}$$

We can therefore consider that the de Broglie-Proca theory is nothing other than the Stueckelberg gauge theory in the particular gauge (8). However, it is worth noting that this is a "bad" choice of gauge leading to some complications. In particular:

(i) Due to the constraint (4a), the Feynman propagator associated with the vector field A_μ does not admit a Hadamard representation (see below), and, as a consequence, the quantum states of the de Broglie-Proca theory are not of Hadamard type. This complicates the regularization and renormalization procedures.

(ii) In the limit $m^2 \to 0$, singularities occur, and a lot of physical results obtained in the context of the de Broglie-Proca theory do not coincide with the corresponding results obtained with Maxwell's theory.

In this article, we intend to focus on the Stueckelberg theory at the quantum level, and we shall analyze its energetic content with possible applications to the Casimir effect (in this paper) and to cosmology of the very early universe (in a next paper) in mind. More precisely, we shall develop the formalism permitting us to construct, for a normalized Hadamard quantum state $|\psi\rangle$ of the Stueckelberg theory, the quantity $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ which denotes the renormalized expectation value of the stress-energy-tensor operator. It is well known that such an expectation value is of fundamental importance in quantum field theory in curved spacetime (see, e.g., Refs. [23–27]). Indeed, it permits us to analyze the quantum state $|\psi\rangle$ without any reference to its particle content, and, moreover, it acts as a source in the semiclassical Einstein equations $G_{\mu\nu} = 8\pi\,\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ which govern the backreaction of the quantum field theory on the spacetime geometry.

Let us recall that the stress-energy tensor $\widehat{T}_{\mu\nu}$ is an operator quadratic in the quantum fields which is, from the mathematical point of view, an operator-valued distribution. As a consequence, this operator is ill defined and the associated expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ is formally infinite. In order to extract from this expectation value a finite and physically acceptable contribution which could act as the source in the semiclassical Einstein equations, it is necessary to regularize it and then to renormalize all the coupling constants. For a description of the various techniques of regularization and renormalization in the context of quantum field theory in curved spacetime (adiabatic regularization method, dimensional regularization method, ζ-function approach, point-splitting methods, ...), see Refs. [23–27] and references therein.

In this paper, we shall deal with Stueckelberg electromagnetism by using the so-called Hadamard renormalization procedure (for a rigorous axiomatic presentation of this approach, we refer to the monographs of Wald [26] and Fulling [25]). Here, we just recall that it is an extension of the point-splitting method [23, 28, 29] which has been developed in connection with the Hadamard representation of the Green functions (see, e.g., Refs. [30–42] and, more particularly, Refs. [32, 33, 35, 38, 39] where gauge theories are considered).

Our article is organized as follows. In Sec. II, we review the covariant quantization of Stueckelberg massive electromagnetism on an arbitrary four-dimensional curved spacetime (gauge-breaking action and associated ghost contribution; wave equations for the massive spin-1 field A_μ, for the auxiliary Stueckelberg scalar field Φ and for the ghost fields C and C^*; Feynman propagators and Ward identities). In Sec. III, we focus on the particular gauge for which the various Feynman propagators and the associated Hadamard Green functions admit Hadamard representation, or, in other words, we consider quantum states of Hadamard type. We also construct the covariant Taylor series expansions of the geometrical and state-dependent coefficients involved in the Hadamard representation of the Green functions. In Sec. IV, we obtain, for a Hadamard quantum state, the renormalized expectation value of the stress-energy-tensor operator, and we discuss carefully its geometrical ambiguities. In fact, we provide two alternative but equivalent expressions for this renormalized expectation value. The first one is obtained by removing the contribution of the auxiliary scalar field Φ (here, it plays the role of a kind of ghost field) and only involves state-dependent and geometrical quantities associated with the massive vector field A_μ. The other one involves contributions coming from both the massive vector field and the auxiliary Stueckelberg scalar field, and it has been constructed in such a way that, in the zero-mass limit, the massive vector field contribution reduces smoothly to the result obtained from Maxwell's theory. In Sec. V, as an application of our results, we consider in the Minkowski spacetime the Casimir effect outside of a perfectly conducting medium with a plane boundary wall separating it from free space. We discuss the results obtained using Stueckelberg but also de Broglie-Proca electromagnetism, and we consider the zero-mass limit of the vacuum energy in both theories. Finally, in a conclusion (Sec. VI), we provide a step-by-step guide for the reader wishing to use our formalism, we briefly discuss and compare the de Broglie-Proca and Stueckelberg approaches in the light of the results obtained in our paper, and we highlight the advantages of the latter. In a short Appendix, we have gathered some important results which are helpful to do the calculations of Secs. III and IV, and, in particular, (i) we define the geodetic interval $\sigma(x,x')$, the Van Vleck-Morette determinant $\Delta(x,x')$ and the bivector of parallel transport $g_{\mu\nu'}(x,x')$ which play a crucial role along our article, and (ii) we discuss the concept of covariant Taylor series expansions.

It should be noted that, in this paper, we consider a four-dimensional curved spacetime $(\mathcal{M}, g_{\mu\nu})$ with no boundary ($\partial\mathcal{M} = \varnothing$), and we use units with $\hbar = c = G = 1$ and the geometrical conventions of Hawking and Ellis [43] concerning the definitions of the scalar curvature R, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$ as well as the commutation of covariant derivatives. It is moreover important to note that we provide the covariant Taylor series expansions of the Hadamard coefficients in irreducible form by using the algebraic proprieties of the Riemann tensor (and more particularly the cyclicity relation and its consequences) as well as the Bianchi identity.

II. QUANTIZATION OF STUECKELBERG ELECTROMAGNETISM

In this section, we review the covariant quantization of Stueckelberg electromagnetism on an arbitrary four-dimensional curved spacetime. The gauge-breaking term considered includes an arbitrary gauge parameter ξ, and all the results concerning the wave equations for the massive vector field A_μ, for the auxiliary scalar field Φ, for the ghost fields C and C^* and for all the associated Feynman propagators as well as the Ward identities are expressed in terms of ξ.

A. Quantum action

At the quantum level, the action defining Stueckelberg massive electromagnetism is given by (see, e.g., Ref. [14])

$$\begin{aligned} S[A_\mu, \Phi, C, C^*, g_{\mu\nu}] &= S_{\text{Cl}}[A_\mu, \Phi, g_{\mu\nu}] \\ &+ S_{\text{GB}}[A_\mu, \Phi, g_{\mu\nu}] + S_{\text{Gh}}[C, C^*, g_{\mu\nu}], \end{aligned} \tag{9}$$

where we have added to the classical action (6) the gauge-breaking term

$$S_{\text{GB}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{2\xi} (\nabla^\mu A_\mu + \xi m \Phi)^2 \right] \tag{10}$$

and the compensating ghost action

$$S_{\text{Gh}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\nabla^\mu C^* \nabla_\mu C + \xi m^2 C^* C \right]. \tag{11}$$

By collecting the fields in the explicit expression (9), the quantum action can be written in the form

$$\begin{aligned} S\left[A_\mu, \Phi, C, C^*, g_{\mu\nu}\right] &= S_A\left[A_\mu, g_{\mu\nu}\right] \\ &+ S_\Phi\left[\Phi, g_{\mu\nu}\right] + S_{\text{Gh}}\left[C, C^*, g_{\mu\nu}\right], \end{aligned} \tag{12}$$

where

$$\begin{aligned} S_A = \int_{\mathcal{M}} d^4x \sqrt{-g} \Bigg[&-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &- \frac{1}{2} m^2 A^\mu A_\mu - \frac{1}{2\xi} \left(\nabla^\mu A_\mu\right)^2 \Bigg] \end{aligned} \tag{13a}$$

$$\begin{aligned} = \int_{\mathcal{M}} d^4x \sqrt{-g} \Bigg[&-\frac{1}{2} \nabla^\mu A^\nu \nabla_\mu A_\nu - \frac{1}{2} R_{\mu\nu} A^\mu A^\nu \\ &- \frac{1}{2} m^2 A^\mu A_\mu + \frac{1}{2} \left(1 - \frac{1}{\xi}\right) \left(\nabla^\mu A_\mu\right)^2 \Bigg] \end{aligned} \tag{13b}$$

and

$$S_\Phi = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi - \frac{1}{2} \xi m^2 \Phi^2 \right], \tag{14}$$

S_{Gh} remaining unchanged and still given by Eq. (11). It is worth noting that the term $-mA^\mu \nabla_\mu \Phi$ coupling the fields A_μ and Φ in the classical action (6b) has disappeared; because spacetime is assumed with no boundary, it is neutralized by the term $-m\Phi \nabla^\mu A_\mu$ in the gauge-breaking action (10).

The functional derivatives with respect to the fields A_μ, Φ, C and C^* of the quantum action (9) or (12) will allow us to obtain, in Sec. II B, the wave equations for all the fields and to discuss, in Sec. IV A, the conservation of the stress-energy tensor associated with Stueckelberg electromagnetism. They are given by

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A_\mu} = \big[&g^{\mu\nu} \Box - (1 - 1/\xi) \nabla^\mu \nabla^\nu \\ &- R^{\mu\nu} - m^2 g^{\mu\nu} \big] A_\nu \end{aligned} \tag{15}$$

for the vector field A_μ,

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \Phi} = \left[\Box - \xi m^2 \right] \Phi \tag{16}$$

for the auxiliary scalar field Φ, as well as

$$\frac{1}{\sqrt{-g}} \frac{\delta_{\text{R}} S}{\delta C} = -\left[\Box - \xi m^2 \right] C^* \tag{17}$$

and

$$\frac{1}{\sqrt{-g}} \frac{\delta_{\text{L}} S}{\delta C^*} = -\left[\Box - \xi m^2 \right] C \tag{18}$$

for the ghost fields C and C^*. It should be noted that, due to the fermionic behavior of the ghost fields, we have introduced in Eq. (17) the right functional derivative and in Eq. (18) the left functional derivative.

B. Wave equations

The extremization of the quantum action (9) or (12) permits us to obtain the wave equations for the fields A_μ, Φ, C and C^*. The vanishing of the functional derivatives (15)–(18) provides

$$\left[g^{\mu\nu} \Box - (1 - 1/\xi) \nabla^\mu \nabla^\nu - R^{\mu\nu} - m^2 g^{\mu\nu} \right] A_\nu = 0 \tag{19}$$

for the vector field A_μ,

$$\left[\Box - \xi m^2 \right] \Phi = 0 \tag{20}$$

for the auxiliary scalar field Φ, as well as

$$\left[\Box - \xi m^2 \right] C = 0 \quad \text{and} \quad \left[\Box - \xi m^2 \right] C^* = 0 \tag{21}$$

for the ghost fields C and C^*.

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

C. Feynman propagators and Ward identities

From now on, we shall assume that the Stueckelberg field theory previously described has been quantized and is in a normalized quantum state $|\psi\rangle$. The Feynman propagator

$$G^A_{\mu\nu'}(x,x') = i\langle\psi|TA_\mu(x)A_{\nu'}(x')|\psi\rangle \tag{22}$$

associated with the field A_μ (here, T denotes time ordering) is, by definition, a solution of

$$\left[g_\mu{}^\nu\Box_x - (1-1/\xi)\,\nabla^\nu\nabla_\mu - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] G^A_{\nu\rho'}(x,x') = -g_{\mu\rho'}\,\delta^4(x,x') \tag{23}$$

with $\delta^4(x,x') = [-g(x)]^{-1/2}\,\delta^4(x-x')$. Similarly, the Feynman propagator

$$G^\Phi(x,x') = i\langle\psi|T\Phi(x)\Phi(x')|\psi\rangle \tag{24}$$

associated with the scalar field Φ satisfies

$$\left[\Box_x - \xi m^2\right] G^\Phi(x,x') = -\delta^4(x,x'), \tag{25}$$

and the Feynman propagator

$$G^{\mathrm{Gh}}(x,x') = i\langle\psi|TC^*(x)C(x')|\psi\rangle \tag{26}$$

associated with the ghost fields C and C^* satisfies

$$\left[\Box_x - \xi m^2\right] G^{\mathrm{Gh}}(x,x') = -\delta^4(x,x'). \tag{27}$$

The three propagators are related by two Ward identities. The first one is a nonlocal relation linking the propagators $G^A_{\mu\nu'}(x,x')$ and $G^{\mathrm{Gh}}(x,x')$. It can be obtained by extending the approach of DeWitt and Brehme in Ref. [44] as follows: we take the covariant derivative ∇^μ of Eq. (23) and the covariant derivative $\nabla_{\rho'}$ of Eq. (27); then, by commuting suitably the various covariant derivatives involved and by using the relation $\nabla^\mu[g_{\mu\rho'}\,\delta^4(x,x')] = -\nabla_{\rho'}\delta^4(x,x')$, we obtain the formal relation

$$(1/\xi)\,\nabla^\mu G^A_{\mu\nu'}(x,x') + \nabla_{\nu'}G^{\mathrm{Gh}}(x,x') = (1-1/\xi)\left[\Box_x - \xi m^2\right]^{-1}\left[\nabla^\mu\left\{R_\mu{}^\rho G^A_{\rho\nu'}(x,x')\right\}\right]. \tag{28}$$

It should be noted that the nonlocal term in the right-hand side of this equation is associated with the nonminimal term $(1-1/\xi)\,\nabla^\nu\nabla_\mu$ appearing in the wave equation (23) and includes appropriate boundary conditions. The second Ward identity can be obtained directly from the wave equations (25) and (27) by using arguments of uniqueness. We have

$$G^\Phi(x,x') - G^{\mathrm{Gh}}(x,x') = 0. \tag{29}$$

III. HADAMARD EXPANSIONS OF THE GREEN FUNCTIONS OF STUECKELBERG ELECTROMAGNETISM

From now on, we assume that $\xi = 1$. (For $\xi \neq 1$, the various Feynman propagators cannot be represented in the Hadamard form.) For this choice of gauge parameter, the wave equations (23), (25) and (27) for the Feynman propagators $G^A_{\mu\nu'}(x,x')$, $G^\Phi(x,x')$ and $G^{\mathrm{Gh}}(x,x')$ reduce to

$$\left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] G^A_{\nu\rho'}(x,x') = -g_{\mu\rho'}\,\delta^4(x,x'), \tag{30}$$

$$\left[\Box_x - m^2\right] G^\Phi(x,x') = -\delta^4(x,x') \tag{31}$$

and

$$\left[\Box_x - m^2\right] G^{\mathrm{Gh}}(x,x') = -\delta^4(x,x'). \tag{32}$$

As far as the Ward identity (28) is concerned, it takes now the local form

$$\nabla^\mu G^A_{\mu\nu'}(x,x') + \nabla_{\nu'}G^{\mathrm{Gh}}(x,x') = 0, \tag{33}$$

while the Ward identity (29) remains unchanged. Because this last relation expresses the equality of the Feynman propagators associated with the auxiliary scalar field and the ghost fields, we shall often use a generic form for these propagators (and for their Hadamard representation discussed below) where the labels Φ and Gh are omitted, and we shall write

$$G(x,x') = G^\Phi(x,x') = G^{\mathrm{Gh}}(x,x'). \tag{34}$$

For $\xi = 1$ the nonminimal term in the wave equation for $G^A_{\mu\nu'}(x,x')$ has disappeared [compare Eq. (30) with Eq. (23)]. As consequence, we can consider a Hadamard representation for this propagator as well as for the propagators $G^\Phi(x,x')$ and $G^{\mathrm{Gh}}(x,x')$. In other words, we can assume that all fields of Stueckelberg theory are in a normalized quantum state $|\psi\rangle$ of Hadamard type.

A. Hadamard representation of the Feynman propagators

The Feynman propagator $G^A_{\mu\nu'}(x,x')$ associated with the vector field A_μ can be now represented in the Hadamard form

$$G^A_{\mu\nu'}(x,x') = \frac{i}{8\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon}\,g_{\mu\nu'}(x,x') + V^A_{\mu\nu'}(x,x')\ln[\sigma(x,x')+i\epsilon] + W^A_{\mu\nu'}(x,x')\right\}, \tag{35}$$

where the bivectors $V^A_{\mu\nu'}(x,x')$ and $W^A_{\mu\nu'}(x,x')$ are symmetric in the sense that $V^A_{\mu\nu'}(x,x') = V^A_{\nu'\mu}(x',x)$ and $W^A_{\mu\nu'}(x,x') = W^A_{\nu'\mu}(x',x)$ and are regular for $x' \to x$. Furthermore, these bivectors have the following expansions

$$V^A_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} V^A_n{}_{\mu\nu'}(x,x')\,\sigma^n(x,x'), \tag{36a}$$

$$W^A_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} W^A_n{}_{\mu\nu'}(x,x')\,\sigma^n(x,x'). \tag{36b}$$

Similarly, the Hadamard expansion of the Feynman propagator $G(x,x')$ associated with the auxiliary scalar field Φ or the ghost fields is given by

$$G(x,x') = \frac{i}{8\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon} + V(x,x')\ln[\sigma(x,x')+i\epsilon] + W(x,x')\right\}, \quad (37)$$

where the biscalars $V(x,x')$ and $W(x,x')$ are symmetric, i.e., $V(x,x') = V(x',x)$ and $W(x,x') = W(x',x)$, regular for $x' \to x$ and possess expansions of the form

$$V(x,x') = \sum_{n=0}^{+\infty} V_n(x,x')\,\sigma^n(x,x'), \quad (38a)$$

$$W(x,x') = \sum_{n=0}^{+\infty} W_n(x,x')\,\sigma^n(x,x'). \quad (38b)$$

In Eqs. (35) and (37), the factor $i\epsilon$ with $\epsilon \to 0_+$ ensures the singular behavior prescribed by the time-ordered product introduced in the definition of the Feynman propagators [see Eqs. (22), (24) and (26)].

The Hadamard coefficients $V^A_n{}_{\mu\nu'}(x,x')$ and $W^A_n{}_{\mu\nu'}(x,x')$ introduced in Eq. (36) are also symmetric and regular bivector functions. The coefficients $V^A_n{}_{\mu\nu'}(x,x')$ satisfy the recursion relations

$$\begin{aligned} &2(n+1)(n+2)\,V^A_{n+1\mu\nu'} + 2(n+1)\,V^A_{n+1\mu\nu';a}\sigma^{;a} \\ &-2(n+1)\,V^A_{n+1\mu\nu'}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+\left[g_\mu{}^\rho\Box_x - R_\mu{}^\rho - m^2 g_\mu{}^\rho\right] V^A_n{}_{\rho\nu'} = 0 \end{aligned} \quad (39a)$$

for $n \in \mathbb{N}$ with the boundary condition

$$\begin{aligned} &2\,V^A_0{}_{\mu\nu'} + 2\,V^A_0{}_{\mu\nu';a}\sigma^{;a} - 2\,V^A_0{}_{\mu\nu'}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+\left[g_\mu{}^\rho\Box_x - R_\mu{}^\rho - m^2 g_\mu{}^\rho\right](g_{\rho\nu'}\Delta^{1/2}) = 0, \end{aligned} \quad (39b)$$

while the coefficients $W^A_n{}_{\mu\nu'}(x,x')$ satisfy the recursion relations

$$\begin{aligned} &2(n+1)(n+2)\,W^A_{n+1\mu\nu'} + 2(n+1)\,W^A_{n+1\mu\nu';a}\sigma^{;a} \\ &-2(n+1)\,W^A_{n+1\mu\nu'}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+2(2n+3)\,V^A_{n+1\mu\nu'} + 2\,V^A_{n+1\mu\nu';a}\sigma^{;a} \\ &-2\,V^A_{n+1\mu\nu'}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+\left[g_\mu{}^\rho\Box_x - R_\mu{}^\rho - m^2 g_\mu{}^\rho\right] W^A_n{}_{\rho\nu'} = 0 \end{aligned} \quad (40)$$

for $n \in \mathbb{N}$. It should be noted that from the recursion relations (39) and (40) we can show that

$$\left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] V^A_{\nu\rho'} = 0 \quad (41)$$

and

$$\begin{aligned} &\sigma\left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] W^A_{\nu\rho'} = \\ &-\left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right](g_{\nu\rho'}\Delta^{1/2}) \\ &-2\,V^A_{\mu\rho'} - 2\,V^A_{\mu\rho';a}\sigma^{;a} + 2\,V^A_{\mu\rho'}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a}. \end{aligned} \quad (42)$$

These two "wave equations" permit us to prove that the Feynman propagator (35) solves the wave equation (30).

Similarly, the Hadamard coefficients $V_n(x,x')$ and $W_n(x,x')$ are also symmetric and regular biscalar functions. The coefficients $V_n(x,x')$ satisfy the recursion relations

$$\begin{aligned} &2(n+1)(n+2)\,V_{n+1} + 2(n+1)\,V_{n+1;a}\sigma^{;a} \\ &-2(n+1)\,V_{n+1}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+\left[\Box_x - m^2\right] V_n = 0 \end{aligned} \quad (43a)$$

for $n \in \mathbb{N}$ with the boundary condition

$$\begin{aligned} &2\,V_0 + 2\,V_{0;a}\sigma^{;a} - 2\,V_0\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+\left[\Box_x - m^2\right]\Delta^{1/2} = 0, \end{aligned} \quad (43b)$$

while the coefficients $W_n(x,x')$ satisfy the recursion relations

$$\begin{aligned} &2(n+1)(n+2)\,W_{n+1} + 2(n+1)\,W_{n+1;a}\sigma^{;a} \\ &-2(n+1)\,W_{n+1}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+2(2n+3)\,V_{n+1} + 2\,V_{n+1;a}\sigma^{;a} \\ &-2\,V_{n+1}\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a} \\ &+\left[\Box_x - m^2\right] W_n = 0 \end{aligned} \quad (44)$$

for $n \in \mathbb{N}$. It should be also noted that from the recursion relations (43) and (44) we can show that

$$\left[\Box_x - m^2\right] V = 0 \quad (45)$$

and

$$\begin{aligned} &\sigma\left[\Box_x - m^2\right] W = -\left[\Box_x - m^2\right]\Delta^{1/2} \\ &-2\,V - 2\,V_{;a}\sigma^{;a} + 2\,V\Delta^{-1/2}(\Delta^{1/2})_{;a}\sigma^{;a}. \end{aligned} \quad (46)$$

These two "wave equations" permit us to prove that the Feynman propagator (37) solves the wave equation (31) or (32).

The Hadamard representation of the Feynman propagators permits us to straightforwardly identify their singular and regular parts (when the coincidence limit $x' \to x$ is considered). We can write

$$G^A_{\mu\nu'}(x,x') = G^A_{\text{sing}\,\mu\nu'}(x,x') + G^A_{\text{reg}\,\mu\nu'}(x,x') \quad (47)$$

with

$$G^A_{\text{sing}\,\mu\nu'}(x,x') = \frac{i}{8\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon}\,g_{\mu\nu'}(x,x') + V^A_{\mu\nu'}(x,x')\ln[\sigma(x,x')+i\epsilon]\right\} \quad (48a)$$

and

$$G^A_{\text{reg}\,\mu\nu'}(x,x') = \frac{i}{8\pi^2}\,W^A_{\mu\nu'}(x,x') \quad (48b)$$

as well as

$$G(x,x') = G_{\text{sing}}(x,x') + G_{\text{reg}}(x,x') \quad (49)$$

with

$$G_{\text{sing}}(x,x') = \frac{i}{8\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon} + V(x,x')\ln[\sigma(x,x')+i\epsilon]\right\} \quad (50a)$$

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

and

$$G_{\rm reg}(x,x') = \frac{i}{8\pi^2}\, W(x,x'). \tag{50b}$$

Here, it is important to note that, due to the geometrical nature of $\sigma(x,x')$, $g_{\mu\nu'}(x,x')$, $\Delta^{1/2}(x,x')$ (see the Appendix) and of $V^A_{\mu\nu'}(x,x')$ and $V(x,x')$ (see Sec. III C), the singular parts (48a) and (50a) are purely geometrical objects. By contrast, the regular parts (48b) and (50b) are state dependent (see Sec. III D).

B. Hadamard Green functions

In the context of the regularization of the stress-energy-tensor operator, instead of working with the Feynman propagators, it is more convenient to use the associated so-called Hadamard Green functions. Their representations can be derived from those of the Feynman propagators by using the formal identities

$$\frac{1}{\sigma + i\epsilon} = \mathcal{P}\frac{1}{\sigma} - i\pi\delta(\sigma) \tag{51}$$

and

$$\ln(\sigma + i\epsilon) = \ln|\sigma| + i\pi\Theta(-\sigma). \tag{52}$$

Here, $\mathcal{P}$ is the symbol of the Cauchy principal value, and Θ denotes the Heaviside step function. Indeed, these identities permit us to rewrite the expression (35) of the Feynman propagator associated with the massive vector field A_μ as

$$G^A_{\mu\nu'}(x,x') = \overline{G}^A_{\mu\nu'}(x,x') + \frac{i}{2}\, G^{(1)A}_{\mu\nu'}(x,x'), \tag{53}$$

where the average of the retarded and advanced Green functions is represented by

$$\overline{G}^A_{\mu\nu'}(x,x') = \frac{1}{8\pi}\Big\{\Delta^{1/2}(x,x')\, g_{\mu\nu'}(x,x')\,\delta[\sigma(x,x')] - V^A_{\mu\nu'}(x,x')\,\Theta[-\sigma(x,x')]\Big\} \tag{54}$$

and the Hadamard Green function has the representation

$$G^{(1)A}_{\mu\nu'}(x,x') = \frac{1}{4\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')}\, g_{\mu\nu'}(x,x') + V^A_{\mu\nu'}(x,x')\ln|\sigma(x,x')| + W^A_{\mu\nu'}(x,x')\right\}. \tag{55}$$

Similarly, we have for the Feynman propagator (37) associated with the auxiliary scalar field Φ or the ghost fields

$$G(x,x') = \overline{G}(x,x') + \frac{i}{2}\, G^{(1)}(x,x'), \tag{56}$$

where

$$\overline{G}(x,x') = \frac{1}{8\pi}\Big\{\Delta^{1/2}(x,x')\delta[\sigma(x,x')] - V(x,x')\Theta[-\sigma(x,x')]\Big\} \tag{57}$$

and

$$G^{(1)}(x,x') = \frac{1}{4\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln|\sigma(x,x')| + W(x,x')\right\}. \tag{58}$$

It is important to recall that the Hadamard Green function associated with the massive vector field A_μ is defined as the anticommutator

$$G^{(1)A}_{\mu\nu'}(x,x') = \langle\psi|\,\{A_\mu(x), A_{\nu'}(x')\}\,|\psi\rangle \tag{59}$$

and satisfies the wave equation

$$\left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] G^{(1)A}_{\nu\rho'}(x,x') = 0. \tag{60}$$

Similarly, the Hadamard Green function associated with the auxiliary scalar field Φ is defined as the anticommutator

$$G^{(1)\Phi}(x,x') = \langle\psi|\,\{\Phi(x), \Phi(x')\}\,|\psi\rangle \tag{61}$$

which is a solution of

$$\left[\Box_x - m^2\right] G^{(1)\Phi}(x,x') = 0, \tag{62}$$

while the Hadamard Green function associated with the ghost fields is defined as the commutator

$$G^{(1)\rm Gh}(x,x') = \langle\psi|\,[C^*(x), C(x')]\,|\psi\rangle \tag{63}$$

and satisfies the wave equation

$$\left[\Box_x - m^2\right] G^{(1)\rm Gh}(x,x') = 0. \tag{64}$$

It should be noted that the expressions (59), (61) and (63) can be obtained from the definitions (22), (24) and (26) by noting that a Hadamard Green function is twice the imaginary part of the corresponding Feynman propagator [see also Eqs. (53) and (56)]. The transition from (22) to (59) or from (24) to (61) is rather immediate but it is not so obvious to obtain (63) from (26): it is necessary to use the fact that the time-ordered product appearing in Eq. (26) is defined with a minus sign due to the anticommuting nature of the ghost fields [i.e., we have $TC^*(x)C(x') = \theta(x^0 - x'^0)\, C^*(x)C(x') - \theta(x'^0 - x^0)\, C(x')C^*(x)$] and to consider that the ghost field C is hermitian while the anti-ghost field C^* is anti-hermitian (see, e.g., Ref. [45]).

The Ward identities (33) and (29) satisfied by the Feynman propagators are also valid for the Hadamard Green functions. We have

$$\nabla^\mu G^{(1)A}_{\mu\nu'}(x,x') + \nabla_{\nu'} G^{(1)\rm Gh}(x,x') = 0 \tag{65}$$

and

$$G^{(1)\Phi}(x,x') - G^{(1)\rm Gh}(x,x') = 0. \tag{66}$$

Similarly, as it has been previously noted in the case of the Feynman propagators, the Hadamard representaion of the Hadamard Green functions permits us to straightforwardly identify their singular and purely geometrical parts as well as their regular and state-dependent parts (when the coincidence limit $x' \to x$ is considered). We can write

$$G^{(1)A}_{\mu\nu'}(x,x') = G^{(1)A}_{\text{sing}\ \mu\nu'}(x,x') + G^{(1)A}_{\text{reg}\ \mu\nu'}(x,x') \quad (67)$$

with

$$G^{(1)A}_{\text{sing}\ \mu\nu'}(x,x') = \frac{1}{4\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')}\, g_{\mu\nu'}(x,x') + V^A_{\mu\nu'}(x,x')\ln|\sigma(x,x')|\right\} \quad (68a)$$

and

$$G^{(1)A}_{\text{reg}\ \mu\nu'}(x,x') = \frac{1}{4\pi^2}\, W^A_{\mu\nu'}(x,x') \quad (68b)$$

as well as

$$G^{(1)}(x,x') = G^{(1)}_{\text{sing}}(x,x') + G^{(1)}_{\text{reg}}(x,x') \quad (69)$$

with

$$G^{(1)}_{\text{sing}}(x,x') = \frac{1}{4\pi^2}\left\{\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln|\sigma(x,x')|\right\} \quad (70a)$$

and

$$G^{(1)}_{\text{reg}}(x,x') = \frac{1}{4\pi^2}\, W(x,x'). \quad (70b)$$

It should be pointed out that the regular part of the Hadamard Green function given by Eq. (68b) [respectively, by Eq. (70b)] is proportional to that of the Feynman propagator given by Eq. (48b) [respectively, by Eq. (50b)].

C. Geometrical Hadamard coefficients and associated covariant Taylor series expansions

Formally, the Hadamard coefficients $V^A_n{}_{\mu\nu'}(x,x')$ or $V_n(x,x')$ can be determined uniquely by solving the recursion relations (39) or (43), i.e., by integrating these recursion relations along the unique geodesic joining x to x' (it is unique for x' near x or more generally for x' in a convex normal neighborhood of x). As a consequence, all these coefficients as well as the sums given by Eqs. (36a) and (38a) are of purely geometric nature, i.e., they only depend on the geometry along the geodesic joining x to x'.

From the point of view of the practical applications considered in this work, it is sufficient to know the expressions of the two first geometrical Hadamard coefficients. Furthermore, their covariant Taylor series expansions are needed up to order σ^1 for $n = 0$ and σ^0 for $n = 1$. The covariant Taylor series expansions of the bivector coefficients $V^A_0{}_{\mu\nu}(x,x')$ and $V^A_1{}_{\mu\nu}(x,x')$ are given by [see Eqs. (A.7) and (A.8)]

$$\begin{aligned} V^A_0{}_{\mu\nu} &= g_\nu{}^{\nu'}\, V^A_0{}_{\mu\nu'} = v^A_0{}_{(\mu\nu)} \\ &- \left\{(1/2)\, v^A_0{}_{(\mu\nu);a} + v^A_0{}_{[\mu\nu]a}\right\}\sigma^{;a} \\ &+ \frac{1}{2!}\left\{v^A_0{}_{(\mu\nu)ab} + v^A_0{}_{[\mu\nu]a;b}\right\}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \end{aligned} \quad (71a)$$

and

$$V^A_1{}_{\mu\nu} = g_\nu{}^{\nu'}\, V^A_1{}_{\mu\nu'} = v^A_1{}_{(\mu\nu)} + O(\sigma^{1/2}). \quad (71b)$$

Here, the explicit expressions of the Taylor coefficients can be determined from the recursion relations (39). We have

$$v^A_0{}_{(\mu\nu)} = (1/2)\, R_{\mu\nu} + g_{\mu\nu}\left\{(1/2)\, m^2 - (1/12)\, R\right\}, \quad (72a)$$

$$v^A_0{}_{[\mu\nu]a} = (1/6)\, R_{a[\mu;\nu]}, \quad (72b)$$

$$\begin{aligned} v^A_0{}_{(\mu\nu)ab} &= (1/6)\, R_{\mu\nu;(ab)} + (1/12)\, R_{\mu\nu}\, R_{ab} \\ &+ (1/12) R_{\mu pq(a|}\, R_\nu{}^{pq}{}_{|b)} \\ &+ g_{\mu\nu}\Big\{(1/12)\, m^2\, R_{ab} - (1/40)\, R_{;ab} - (1/120)\,\Box R_{ab} \\ &- (1/72)\, R\, R_{ab} + (1/90)\, R_{ap}\, R_b{}^p \\ &- (1/180)\, R_{pq}\, R_a{}^p{}_b{}^q - (1/180)\, R_{apqr}\, R_b{}^{pqr}\Big\}, \end{aligned} \quad (72c)$$

$$\begin{aligned} v^A_1{}_{(\mu\nu)} &= (1/4)\, m^2\, R_{\mu\nu} - (1/24)\,\Box R_{\mu\nu} - (1/24)\, R\, R_{\mu\nu} \\ &+ (1/8)\, R_{\mu p}\, R_\nu{}^p - (1/48)\, R_{\mu pqr}\, R_\nu{}^{pqr} \\ &+ g_{\mu\nu}\left\{(1/8)\, m^4 - (1/24)\, m^2\, R + (1/120)\,\Box R \right. \\ &+ (1/288)\, R^2 - (1/720)\, R_{pq}\, R^{pq} \\ &\left. + (1/720)\, R_{pqrs}\, R^{pqrs}\right\}. \end{aligned} \quad (72d)$$

The covariant Taylor series expansions of the biscalar coefficients $V_0(x,x')$ and $V_1(x,x')$ are given by [see Eqs. (A.5) and (A.6)]

$$\begin{aligned} V_0 &= v_0 - \{(1/2)\, v_{0;a}\}\,\sigma^{;a} \\ &+ \frac{1}{2!}\, v_{0ab}\,\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \end{aligned} \quad (73a)$$

and

$$V_1 = v_1 + O(\sigma^{1/2}). \quad (73b)$$

Here, the explicit expressions of the Taylor coefficients can be determined from the recursion relations (43). We have

$$v_0 = (1/2)\, m^2 - (1/12)\, R, \quad (74a)$$

$$\begin{aligned} v_{0ab} &= (1/12)\, m^2\, R_{ab} - (1/40)\, R_{;ab} - (1/120)\,\Box R_{ab} \\ &- (1/72)\, R\, R_{ab} + (1/90)\, R_{ap}\, R_b{}^p \\ &- (1/180)\, R_{pq}\, R_a{}^p{}_b{}^q - (1/180)\, R_{apqr}\, R_b{}^{pqr}, \end{aligned} \quad (74b)$$

$$\begin{aligned} v_1 &= (1/8)\, m^4 - (1/24)\, m^2\, R + (1/120)\,\Box R \\ &+ (1/288)\, R^2 - (1/720)\, R_{pq}\, R^{pq} \\ &+ (1/720)\, R_{pqrs}\, R^{pqrs}. \end{aligned} \quad (74c)$$

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

In order to obtain the expressions of the Taylor coefficients given by Eqs. (72) and (74), we have used some of the properties of $\sigma(x,x')$, $g_{\mu\nu'}(x,x')$, $\Delta^{1/2}(x,x')$ mentioned in the Appendix as well as the algebraic properties of the Riemann tensor.

D. State-dependent Hadamard coefficients and associated covariant Taylor series expansions

1. General considerations

Unlike the geometrical Hadamard coefficients, the coefficients $W^A_{n\;\mu\nu'}(x,x')$ and $W_n(x,x')$ are neither uniquely defined nor purely geometrical. Indeed, the coefficient $W^A_{0\;\mu\nu'}(x,x')$ [respectively, $W_0(x,x')$] is unrestrained by the recursion relations (42) [respectively, by the recursion relations (46)]. As a consequence, this is also true for all the coefficients $W^A_{n\;\mu\nu'}(x,x')$ and $W_n(x,x')$ with $n \geq 1$ and for the sums (36b) and (38b). This arbitrariness is in fact very interesting, and it can be used to encode the quantum state dependence of the theory in the coefficients $W^A_{0\;\mu\nu'}(x,x')$ and $W_0(x,x')$. Once they have been specified, the coefficients $W^A_{n\;\mu\nu'}(x,x')$ and $W_n(x,x')$ with $n \geq 1$ as well as the bivector $W^A_{\mu\nu'}(x,x')$ and the biscalar $W(x,x')$ are uniquely determined.

In the following, instead of working with the state-dependent Hadamard coefficients, we shall consider the sums $W^A_{\mu\nu'}(x,x')$ and $W(x,x')$, and, more precisely, we shall use their covariant Taylor series expansions up to order $\sigma^{3/2}$. We have [see Eqs. (A.7) and (A.8)]

$$\begin{aligned} W^A_{\mu\nu} = g_\nu{}^{\nu'} W^A_{\mu\nu'} &= s_{\mu\nu} - \{(1/2)\, s_{\mu\nu;a} + a_{\mu\nu a}\}\, \sigma^{;a} \\ &+ \frac{1}{2!}\, \{s_{\mu\nu ab} + a_{\mu\nu a;b}\}\, \sigma^{;a}\sigma^{;b} - \frac{1}{3!}\, \{(3/2)\, s_{\mu\nu ab;c} \\ &- (1/4)\, s_{\mu\nu;abc} + a_{\mu\nu abc}\}\, \sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2) \end{aligned} \tag{75}$$

and [see Eqs. (A.5) and (A.6)]

$$\begin{aligned} W &= w - \{(1/2)\, w_{;a}\}\, \sigma^{;a} + \frac{1}{2!}\, w_{ab}\, \sigma^{;a}\sigma^{;b} \\ &- \frac{1}{3!}\, \{(3/2)\, w_{ab;c} - (1/4)\, w_{;abc}\}\, \sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2). \end{aligned} \tag{76}$$

In the expansion (75) we have introduced the notations

$$s_{\mu\nu a_1\cdots a_p} \equiv w^A_{(\mu\nu)a_1\cdots a_p} \tag{77a}$$

for the symmetric part of the Taylor coefficients and

$$a_{\mu\nu a_1\cdots a_p} \equiv w^A_{[\mu\nu]a_1\cdots a_p} \tag{77b}$$

for their antisymmetric part.

It is important to note that, with practical applications in mind, it is interesting to express some of the Taylor coefficients appearing in Eqs. (75) and (76) in terms of the bitensors $W^A_{\mu\nu'}(x,x')$ and $W(x,x')$. This can be done by inverting these equations. From Eq. (75), we obtain

$$s_{\mu\nu}(x) = \lim_{x'\to x} W^A_{\mu\nu'}(x,x'), \tag{78a}$$

$$a_{\mu\nu a}(x) = \frac{1}{2} \lim_{x'\to x} \left[W^A_{\mu\nu';a'}(x,x') - W^A_{\mu\nu';a}(x,x')\right], \tag{78b}$$

$$s_{\mu\nu ab}(x) = \frac{1}{2} \lim_{x'\to x} \left[W^A_{\mu\nu';(a'b')}(x,x') + W^A_{\mu\nu';(ab)}(x,x')\right]. \tag{78c}$$

(Here, the coefficient $a_{\mu\nu abc}$ is not relevant because it does not appear in the final expressions of the renormalized stress-energy-tensor operator given in Sec. IV C). Similarly, from Eq. (76), we straightforwardly establish that

$$w(x) = \lim_{x'\to x} W(x,x'), \tag{79a}$$

$$w_{ab}(x) = \lim_{x'\to x} W_{;(a'b')}(x,x'). \tag{79b}$$

We shall now rewrite the wave equations (60), (62) and (64) as well as the Ward identity (65) in terms of the Taylor coefficients of $W^A_{\mu\nu}(x,x')$ and $W(x,x')$. To achieve the calculations, we shall use extensively the properties of $\sigma(x,x')$, $g_{\mu\nu'}(x,x')$, $\Delta^{1/2}(x,x')$ mentioned in the Appendix.

2. Wave equations

By inserting the Hadamard representation (55) of the Green function $G^{(1)A}_{\mu\nu'}(x,x')$ into the wave equation (60), we obtain a wave equation with source for the state-dependent Hadamard coefficient $W^A_{\mu\nu'}(x,x')$. We have

$$\begin{aligned} & g_\rho{}^{\rho'}\left[g_\mu{}^\nu \Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] W^A_{\nu\rho'} \\ &= -6\, V^A_{1\;\mu\rho} - 2\, g_\rho{}^{\rho'} V^A_{1\;\mu\rho';a}\sigma^{;a} + O(\sigma) \\ &= -6\, v^A_{1\;(\mu\rho)} + \left(2\, v^A_{1\;(\mu\rho);a} + 8\, v^A_{1\;[\mu\rho]a}\right)\sigma^{;a} + O(\sigma). \end{aligned} \tag{80}$$

Here, we have used the expansions of the geometrical Hadamard coefficients given by Eqs. (36a) and (71). By inserting the expansion (75) of $W^A_{\mu\nu}(x,x')$ into the left-hand side of Eq. (80), we find the following relations:

$$s_{\mu\rho\nu}{}^{\nu} = R^p{}_{(\mu}s_{\rho)p} + m^2 s_{\mu\rho} - 6\, v^A_{1\;(\mu\rho)}, \tag{81a}$$

$$s_\mu{}^\mu{}_\nu{}^\nu = R^{pq}s_{pq} + m^2 s_p{}^p - 6\, v^A_1{}_p{}^p, \tag{81b}$$

$$a_{\mu\rho\nu}{}^{;\nu} = -R^p{}_{[\mu}s_{\rho]p}, \tag{81c}$$

$$\begin{aligned} s_{\mu\rho\nu a}{}^{;\nu} &= (1/4)\,(\Box s_{\mu\rho})_{;a} + (1/2)\, R^p{}_{a;(\mu}s_{\rho)p} \\ &- (1/2)\, R_{(\mu|a}{}^{;p}s_{|\rho)p} - (1/2)\, R^p{}_{(\mu}s_{\rho)p;a} \\ &+ (1/2)\, R^p{}_a s_{\mu\rho;p} - R^p{}_{(\mu|}{}^q{}_a s_{|\rho)p;q} \\ &- R^p{}_{(\mu|}a_{p|\rho)a} - R^p{}_{(\mu|}{}^q{}_a a_{|\rho)pq} + (1/2)\, s_{\mu\rho p}{}^p{}_{;a} \\ &- (1/2)\, m^2 s_{\mu\rho;a} + 2\, v^A_{1\;(\mu\rho);a}, \end{aligned} \tag{81d}$$

$$s_\mu{}^\mu{}_{\nu a}{}^{;\nu} = (1/4)\,(\Box s_p{}^p)_{;a} + (1/2)\, R^q{}_a s_p{}^p{}_{;q}$$

$$-(1/2)\, R^{pq} s_{pq;a} + R^{pqr}{}_a a_{pqr} + (1/2)\, s_p{}^p{}_q{}^q{}_{;a}$$
$$-(1/2)\, m^2 s_p{}^p{}_{;a} + 2\, v^A_1{}_p{}^p{}_{;a}. \tag{81e}$$

Furthermore, by combining Eq. (81d) with Eq. (81a) and Eq. (81e) with Eq. (81b), we also establish that

$$\begin{aligned} s_{\mu\rho\nu a}{}^{;\nu} &= (1/4)\,(\Box s_{\mu\rho})_{;a} + (1/2)\, R^p{}_{a;(\mu} s_{\rho)p} \\ &-(1/2)\, R_{(\mu|a}{}^{;p} s_{|\rho)p} + (1/2)\, R^p{}_{(\mu|;a} s_{|\rho)p} \\ &+(1/2)\, R^p{}_a s_{\mu\rho;p} - R^p{}_{(\mu|}{}^q{}_a s_{|\rho)p;q} \\ &-R^p{}_{(\mu|} a_{p|\rho)a} - R^p{}_{(\mu|}{}^q{}_a a_{|\rho)pq} - v^A_1{}_{(\mu\rho);a} \end{aligned} \tag{81f}$$

and

$$\begin{aligned} s_\mu{}^\mu{}_{\nu a}{}^{;\nu} &= (1/4)\,(\Box s_p{}^p)_{;a} + (1/2)\, R^q{}_a s_p{}^p{}_{;q} \\ &+(1/2)\, R^{pq}{}_{;a} s_{pq} + R^{pqr}{}_a a_{pqr} - v^A_1{}_p{}^p{}_{;a}. \end{aligned} \tag{81g}$$

Mutatis mutandis, by inserting the Hadamard representation (58) of the Green function $G^{(1)}(x,x')$ into the wave equation (62) or (64), we obtain a wave equation with source for the state-dependent Hadamard coefficient $W(x,x')$. We have

$$\begin{aligned} \left(\Box_x - m^2\right) W &= -6\, V_1 - 2\, V_{1;a}\sigma^{;a} + O(\sigma) \\ &= -6\, v_1 + 2\, v_{1;a}\sigma^{;a} + O(\sigma). \end{aligned} \tag{82}$$

Here, we have used the expansions of the geometrical Hadamard coefficients given by Eqs. (38a) and (73). By inserting the expansion (76) of $W(x,x')$ into the left-hand side of Eq. (82), we find the following relations:

$$w_\rho{}^\rho = m^2 w - 6\, v_1, \tag{83a}$$
$$\begin{aligned} w_{\rho a}{}^{;\rho} &= (1/4)\,(\Box w)_{;a} + (1/2)\, w_p{}^p{}_{;a} \\ &+(1/2)\, R^p{}_a w_{;p} - (1/2)\, m^2 w_{;a} + 2\, v_{1;a}. \end{aligned} \tag{83b}$$

Furthermore, by combining Eq. (83b) with Eq. (83a), we also establish that

$$w_{\rho a}{}^{;\rho} = (1/4)\,(\Box w)_{;a} + (1/2)\, R^p{}_a w_{;p} - v_{1;a}. \tag{83c}$$

3. Ward identities

The first Ward identity given by Eq. (65) expressed in terms of the Hadamard representation of the Green functions $G^{(1)A}_{\mu\nu'}(x,x')$ [see Eq. (55)] and $G^{(1)}(x,x')$ [see Eq. (58)] permits us to write a relation between the geometrical Hadamard coefficients $V^A_{\mu\nu'}(x,x')$ and $V(x,x')$ as well as another one between the state-dependent Hadamard coefficients $W^A_{\mu\nu'}(x,x')$ and $W(x,x')$. We obtain

$$g_\nu{}^{\nu'}\left(V^A_{\mu\nu'}{}^{;\mu} + V_{;\nu'}\right) = 0 \tag{84}$$

which is an identity between the geometrical Taylor coefficients (72a)–(72d) and (74a)–(74c) and

$$\begin{aligned} &g_\nu{}^{\nu'}\left(W^A_{\mu\nu'}{}^{;\mu} + W_{;\nu'}\right) \\ &= -V^A_1{}_{\mu\nu}\sigma^{;\mu} + V_1\sigma_{;\nu} + O\left(\sigma\right) \\ &= -\left(v^A_1{}_{(\nu a)} - v_1 g_{\nu a}\right)\sigma^{;a} + O\left(\sigma\right). \end{aligned} \tag{85}$$

To establish Eq. (85), we have used the expansions of the geometrical Hadamard coefficients given by Eqs. (36a), (38a), (71) and (73). By inserting the expansions (75) of $W^A_{\mu\nu}(x,x')$ and (76) of $W(x,x')$ into the left-hand side of Eq. (85), we find the following relations:

$$a_{\mu\nu}{}^\mu = (1/2)\, s_{p\nu}{}^{;p} + (1/2)\, w_{;\nu}, \tag{86a}$$
$$\begin{aligned} s_{\mu\nu}{}^\mu{}_a &= (1/2)\, s_{p\nu}{}^{;p}{}_a + (1/2)\, R^p{}_a s_{\nu p} + a^p{}_{\nu[a;p]} \\ &+w_{\nu a} - v^A_1{}_{(\nu a)} + v_1 g_{\nu a}. \end{aligned} \tag{86b}$$

Furthermore, by combining Eq. (86b) with Eq. (86a), we also establish that

$$\begin{aligned} s_{\mu\nu}{}^\mu{}_a &= (1/4)\, s_{p\nu}{}^{;p}{}_a + (1/2)\, R^p{}_a s_{\nu p} + (1/2)\, a_{p\nu a}{}^{;p} \\ &-(1/4)\, w_{;\nu a} + w_{\nu a} - v^A_1{}_{(\nu a)} + v_1 g_{\nu a}. \end{aligned} \tag{86c}$$

Of course, the second Ward identity given by Eq. (66) provides trivially the equality of the Taylor coefficients of the Hadamard coefficients associated with the auxiliary scalar field and the ghost fields. We have

$$V^\Phi = V^{\rm Gh} \tag{87}$$

and

$$W^\Phi = W^{\rm Gh}. \tag{88}$$

IV. RENORMALIZED STRESS-ENERGY TENSOR OF STUECKELBERG ELECTROMAGNETISM

A. Stress-energy tensor

The functional derivation of the quantum action of the Stueckelberg theory with respect to the metric tensor $g_{\mu\nu}$ permits us to obtain the associated stress-energy tensor $T_{\mu\nu}$. By definition, we have

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\, S[A_\mu, \Phi, C, C^*, g_{\mu\nu}], \tag{89}$$

and its explicit expression can be obtain by using that, in the elementary variation

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{90}$$

of the metric tensor, we have (see, for example, Ref. [46])

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}, \tag{91a}$$
$$\sqrt{-g} \to \sqrt{-g} + \delta\sqrt{-g}, \tag{91b}$$
$$\Gamma^\rho{}_{\mu\nu} \to \Gamma^\rho{}_{\mu\nu} + \delta\Gamma^\rho{}_{\mu\nu} \tag{91c}$$

with

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}, \tag{91d}$$
$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}, \tag{91e}$$
$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\left(-\delta g_{\mu\nu}{}^{;\rho} + \delta g^\rho{}_{\mu;\nu} + \delta g^\rho{}_{\nu;\mu}\right). \tag{91f}$$

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

The stress-energy tensor derived from the action (9) is given by

$$T^{\mu\nu} = T^{\mu\nu}_{\text{cl}} + T^{\mu\nu}_{\text{GB}} + T^{\mu\nu}_{\text{Gh}}, \tag{92}$$

where the contributions of the classical and gauge-breaking parts take the forms

$$\begin{aligned} T^{\mu\nu}_{\text{cl}} &= F^{\mu}{}_{\rho}F^{\nu\rho} + m^2 A^\mu A^\nu \\ &+\nabla^\mu\Phi\nabla^\nu\Phi + 2\,mA^{(\mu}\nabla^{\nu)}\Phi \\ &-(1/4)\,g^{\mu\nu}\left\{F_{\rho\tau}F^{\rho\tau} + 2\,m^2 A_\rho A^\rho \right.\\ &\quad\left. +2\,\nabla_\rho\Phi\nabla^\rho\Phi + 4\,mA_\rho\nabla^\rho\Phi\right\} \\ &= \nabla_\rho A^\mu\nabla^\rho A^\nu - 2\,\nabla_\rho A^{(\mu}\nabla^{\nu)}A^\rho + \nabla^\mu A_\rho\nabla^\nu A^\rho \\ &+m^2 A^\mu A^\nu + \nabla^\mu\Phi\nabla^\nu\Phi + 2\,mA^{(\mu}\nabla^{\nu)}\Phi \\ &-(1/2)\,g^{\mu\nu}\left\{\nabla_\rho A_\tau\nabla^\rho A^\tau - \nabla_\rho A_\tau\nabla^\tau A^\rho\right. \\ &\quad\left.+m^2 A_\rho A^\rho + \nabla_\rho\Phi\nabla^\rho\Phi + 2\,mA_\rho\nabla^\rho\Phi\right\} \end{aligned} \tag{93a}$$

and

$$\begin{aligned} T^{\mu\nu}_{\text{GB}} &= -2\,A^{(\mu}\nabla^{\nu)}\nabla_\rho A^\rho - 2\,mA^{(\mu}\nabla^{\nu)}\Phi \\ &\quad -(1/2)\,g^{\mu\nu}\Big\{-2\,A_\rho\nabla^\rho\nabla_\tau A^\tau - (\nabla_\rho A^\rho)^2 \\ &\qquad +m^2\Phi^2 - 2\,mA_\rho\nabla^\rho\Phi\Big\}, \end{aligned} \tag{93b}$$

while the contribution associated with the ghost fields is given by

$$\begin{aligned} T^{\mu\nu}_{\text{Gh}} &= -2\,\nabla^{(\mu|}C^*\nabla^{|\nu)}C \\ &\quad + g^{\mu\nu}\left\{\nabla_\rho C^*\nabla^\rho C + m^2 C^* C\right\}. \end{aligned} \tag{93c}$$

We can note the existence of terms coupling the fields A_μ and Φ in the expression of $T^{\mu\nu}_{\text{cl}}$ [see Eq. (93a)] as well as in the expression of $T^{\mu\nu}_{\text{GB}}$ [see Eq. (93b)].

We also give an alternative expression for the stress-energy tensor which can be derived from the action (12) or by summing $T^{\mu\nu}_{\text{cl}}$ and $T^{\mu\nu}_{\text{GB}}$. This eliminates any coupling between the fields A_μ and Φ and permits us to straightforwardly infer that the stress-energy tensor has three independent contributions corresponding to the massive vector field A_μ, the auxiliary scalar field Φ and the ghost fields C and C^*. We can write

$$T^{\mu\nu} = T^{\mu\nu}_{A} + T^{\mu\nu}_{\Phi} + T^{\mu\nu}_{\text{Gh}} \tag{94}$$

with

$$\begin{aligned} T^{\mu\nu}_{A} &= F^{\mu}{}_{\rho}F^{\nu\rho} + m^2 A^\mu A^\nu - 2\,A^{(\mu}\nabla^{\nu)}\nabla_\rho A^\rho \\ &-(1/4)\,g^{\mu\nu}\Big\{F_{\rho\tau}F^{\rho\tau} + 2\,m^2 A_\rho A^\rho \\ &\quad -4\,A_\rho\nabla^\rho\nabla_\tau A^\tau - 2\,(\nabla_\rho A^\rho)^2\Big\} \\ &= \nabla_\rho A^\mu\nabla^\rho A^\nu - 2\,\nabla_\rho A^{(\mu}\nabla^{\nu)}A^\rho + \nabla^\mu A_\rho\nabla^\nu A^\rho \\ &+m^2 A^\mu A^\nu - 2\,A^{(\mu}\nabla^{\nu)}\nabla_\rho A^\rho \\ &-(1/2)\,g^{\mu\nu}\Big\{\nabla_\rho A_\tau\nabla^\rho A^\tau - \nabla_\rho A_\tau\nabla^\tau A^\rho \\ &\quad +m^2 A_\rho A^\rho - 2\,A_\rho\nabla^\rho\nabla_\tau A^\tau - (\nabla_\rho A^\rho)^2\Big\} \end{aligned} \tag{95a}$$

and

$$T^{\mu\nu}_{\Phi} = \nabla^\mu\Phi\nabla^\nu\Phi - (1/2)\,g^{\mu\nu}\left\{\nabla_\rho\Phi\nabla^\rho\Phi + m^2\Phi^2\right\}, \tag{95b}$$

while the contribution associated with the ghost fields remains unchanged [see Eq. (93c)].

By construction, the stress-energy tensor (89) [see also its explicit expressions (92) and (94)] is conserved, i.e.,

$$\nabla_\nu T^{\mu\nu} = 0. \tag{96}$$

Indeed, this property is due to the invariance of the action (9) or (12) under spacetime diffeomorphisms and therefore under the infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \epsilon^\mu \quad \text{with} \quad |\epsilon^\mu| \ll 1. \tag{97}$$

Under this transformation, the vector, scalar and ghost fields as well as the background metric transform as

$$A_\mu \to A_\mu + \delta A_\mu, \tag{98a}$$
$$\Phi \to \Phi + \delta\Phi, \tag{98b}$$
$$C \to C + \delta C, \tag{98c}$$
$$C^* \to C^* + \delta C^*, \tag{98d}$$
$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}. \tag{98e}$$

The variations associated with the field transformations (98) are obtained by Lie derivation with respect to the vector $-\epsilon^\mu$:

$$\delta A_\mu = \mathcal{L}_{-\epsilon}A_\mu = -\epsilon^\rho\nabla_\rho A_\mu - (\nabla_\mu\epsilon^\rho)A_\rho, \tag{99a}$$
$$\delta\Phi = \mathcal{L}_{-\epsilon}\Phi = -\epsilon^\rho\nabla_\rho\Phi, \tag{99b}$$
$$\delta C = \mathcal{L}_{-\epsilon}C = -\epsilon^\rho\nabla_\rho C, \tag{99c}$$
$$\delta C^* = \mathcal{L}_{-\epsilon}C^* = -\epsilon^\rho\nabla_\rho C^*, \tag{99d}$$
$$\delta g_{\mu\nu} = \mathcal{L}_{-\epsilon}g_{\mu\nu} = -\nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu. \tag{99e}$$

The invariance of the action (9) or (12) leads to

$$\begin{aligned} \int_{\mathcal{M}} d^4x\sqrt{-g}\,\Bigg[&\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu}\right)\delta A_\mu + \left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Phi}\right)\delta\Phi \\ &+\left(\frac{1}{\sqrt{-g}}\frac{\delta_{\text{R}}S}{\delta C}\right)\delta C + \delta C^*\left(\frac{1}{\sqrt{-g}}\frac{\delta_{\text{L}}S}{\delta C^*}\right) \\ &+\frac{1}{2}\left(\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right)\delta g_{\mu\nu}\Bigg] = 0 \end{aligned} \tag{100}$$

which implies

$$\begin{aligned} \nabla_\nu T^{\mu\nu} = &\\ &[\nabla^\mu A_\alpha - \nabla_\alpha A^\mu - A^\mu\nabla_\nu]\left(\Box A^\nu - R^\nu{}_\rho A^\rho - m^2 A^\nu\right) \\ &+[\nabla^\mu\Phi]\left(\Box\Phi - m^2\Phi\right) \\ &-\left(\Box C^* - m^2 C^*\right)[\nabla^\mu C] - [\nabla^\mu C^*]\left(\Box C - m^2 C\right) \end{aligned} \tag{101}$$

by using Eq. (99) as well as Eqs. (15)–(18). From the wave equations associated with the massive vector field A_μ [see Eq. (19)], the auxiliary scalar field Φ [see Eq. (20)] and the ghost fields C and C^* [see Eq. (21)], we then obtain Eq. (96).

B. Expectation value of the stress-energy tensor

At the quantum level, all the fields involved in the Stueckelberg theory as well as the associated stress-energy tensor [see Eqs. (92) and (94)] are operators. From now on, we shall denote the stress-energy-tensor operator by $\widehat{T}_{\mu\nu}$ and we shall focus on the quantity $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ which denotes its expectation value with respect to the Hadamard quantum state $|\psi\rangle$ discussed in Sec. III.

The expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ corresponding to the expression (92) of the stress-energy tensor is decomposed as follows:

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle = \langle\psi|\widehat{T}^{\text{cl}}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\text{GB}}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\text{Gh}}_{\mu\nu}|\psi\rangle. \quad (102)$$

The three terms in the right-hand side of this equation are explicitly given by

$$\langle\psi|\widehat{T}^{\text{cl}}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{\text{cl}_A\,\rho\sigma'}_{\mu\nu}(x,x')\left[G^{(1)A}_{\rho\sigma'}(x,x')\right] + \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{\text{cl}_\Phi}_{\mu\nu}(x,x')\left[G^{(1)\Phi}(x,x')\right], \quad (103a)$$

$$\langle\psi|\widehat{T}^{\text{GB}}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{\text{GB}_A\,\rho\sigma'}_{\mu\nu}(x,x')\left[G^{(1)A}_{\rho\sigma'}(x,x')\right] + \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{\text{GB}_\Phi}_{\mu\nu}(x,x')\left[G^{(1)\Phi}(x,x')\right] \quad (103b)$$

and

$$\langle\psi|\widehat{T}^{\text{Gh}}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{\text{Gh}}_{\mu\nu}(x,x')\left[G^{(1)\text{Gh}}(x,x')\right], \quad (103c)$$

where $\mathcal{T}^{\text{cl}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\text{cl}_\Phi}_{\mu\nu}$, $\mathcal{T}^{\text{GB}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\text{GB}_\Phi}_{\mu\nu}$ and $\mathcal{T}^{\text{Gh}}_{\mu\nu}$ are the differential operators constructed by point splitting from the expressions (93a), (93b) and (93c). We have

$$\begin{aligned}\mathcal{T}^{\text{cl}_A\,\rho\sigma'}_{\mu\nu} &= g_\nu{}^{\alpha'} g^{\rho\sigma'} \nabla_\mu \nabla_{\alpha'} + g_\mu{}^\rho g_\nu{}^{\sigma'} g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} \\ &\quad -2\, g_\mu{}^\rho g_\nu{}^{\alpha'} g^{\beta\sigma'} \nabla_\beta \nabla_{\alpha'} + m^2 g_\mu{}^\rho g_\nu{}^{\sigma'} \\ &\quad -\frac{1}{2} g_{\mu\nu} \Big\{ g^{\rho\sigma'} g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} - g^{\rho\alpha'} g^{\beta\sigma'} \nabla_\beta \nabla_{\alpha'} \\ &\qquad + m^2 g^{\rho\sigma'} \Big\}, \end{aligned} \quad (104a)$$

$$\mathcal{T}^{\text{cl}_\Phi}_{\mu\nu} = g_\nu{}^{\nu'} \nabla_\mu \nabla_{\nu'} - \frac{1}{2} g_{\mu\nu}\left\{ g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} \right\}, \quad (104b)$$

$$\begin{aligned}\mathcal{T}^{\text{GB}_A\,\rho\sigma'}_{\mu\nu} &= -2\, g_\mu{}^\rho g_\nu{}^{\alpha'} \nabla_{\alpha'} \nabla^{\sigma'} \\ &\quad -\frac{1}{2} g_{\mu\nu} \left\{ -\nabla^\rho \nabla^{\sigma'} - 2\, g^{\rho\alpha'} \nabla_{\alpha'} \nabla^{\sigma'} \right\}, \end{aligned} \quad (104c)$$

$$\mathcal{T}^{\text{GB}_\Phi}_{\mu\nu} = -\frac{1}{2} m^2 g_{\mu\nu} \quad (104d)$$

and

$$\mathcal{T}^{\text{Gh}}_{\mu\nu} = -2\, g_\nu{}^{\nu'} \nabla_\mu \nabla_{\nu'} + g_{\mu\nu} \left\{ g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} + m^2 \right\}. \quad (104e)$$

It should be noted that we have not included in Eqs. (103a) and (103b) the contributions which can be obtained by point splitting from the terms coupling A_μ and Φ in Eqs. (93a) and (93b). Such contributions are not present because, due to the absence of coupling between A_μ and Φ in the quantum action (12), two-point correlation functions involving both A_μ and Φ vanish identically. It should be noted that the absence of these contributions can be also justified in another way: in the quantum stress-energy-tensor operator (94), any coupling between A_μ and Φ has disappeared.

Here, some remarks are in order:

(i) When we use the point-splitting method, it is more convenient to define the expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ from Hadamard Green functions rather than from Feynman propagators. Indeed, this avoids us having to deal with additional singular terms due to the time-ordered product.

(ii) Of course, because of the short-distance behavior of the Hadamard Green functions, the expressions (103) as well as the expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ given in Eq. (102) are divergent and therefore meaningless. In Sec. IV C we will regularize these quantities.

(iii) Even if the formal expression (102) of the expectation value of the stress-energy-tensor operator is divergent, it is interesting to note that

$$\langle\psi|\widehat{T}^{\text{GB}}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\text{Gh}}_{\mu\nu}|\psi\rangle = 0. \quad (105)$$

Indeed, from the definitions (103b) and (103c), we can obtain Eq. (105) by using Eqs. (65) and (66) as well as the wave equation (64). It should be noted that, as a consequence of Eq. (105), Eq. (102) reduces to

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle = \langle\psi|\widehat{T}^{\text{cl}}_{\mu\nu}|\psi\rangle. \quad (106)$$

We can also give the alternative expression of the expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ obtained from Eq. (94). It takes the following form

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle = \langle\psi|\widehat{T}^{A}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\Phi}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\text{Gh}}_{\mu\nu}|\psi\rangle, \quad (107)$$

where the contributions associated with the massive vector field A_μ and the auxiliary scalar field Φ are separated and given by

$$\langle\psi|\widehat{T}^{A}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{A\,\rho\sigma'}_{\mu\nu}(x,x')\left[G^{(1)A}_{\rho\sigma'}(x,x')\right] \quad (108a)$$

and

$$\langle\psi|\widehat{T}^{\Phi}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathcal{T}^{\Phi}_{\mu\nu}(x,x')\left[G^{(1)\Phi}(x,x')\right]. \quad (108b)$$

Here, the differential operators $\mathcal{T}^{A\,\rho\sigma'}_{\mu\nu}$ and $\mathcal{T}^{\Phi}_{\mu\nu}$ are constructed by point splitting from the expressions (95a) and

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

(95b). We have

$$\begin{aligned} \mathcal{T}^{A\,\rho\sigma'}_{\mu\nu} &= g_\nu{}^{\alpha'} g^{\rho\sigma'} \nabla_\mu \nabla_{\alpha'} + g_\mu{}^\rho g_\nu{}^{\sigma'} g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} \\ &-2\, g_\mu{}^\rho g_\nu{}^{\alpha'} g^{\beta\sigma'} \nabla_\beta \nabla_{\alpha'} + m^2 g_\mu{}^\rho g_\nu{}^{\sigma'} \\ &-2\, g_\mu{}^\rho g_\nu{}^{\alpha'} \nabla_{\alpha'} \nabla^{\sigma'} \\ &-\frac{1}{2}\, g_{\mu\nu} \left\{ g^{\rho\sigma'} g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} - g^{\rho\alpha'} g^{\beta\sigma'} \nabla_\beta \nabla_{\alpha'} \right. \\ &\left. \quad + m^2 g^{\rho\sigma'} - \nabla^\rho \nabla^{\sigma'} - 2\, g^{\rho\alpha'} \nabla_{\alpha'} \nabla^{\sigma'} \right\} \end{aligned} \quad (109a)$$

and

$$\mathcal{T}^{\Phi}_{\mu\nu} = g_\nu{}^{\nu'} \nabla_\mu \nabla_{\nu'} - \frac{1}{2}\, g_{\mu\nu} \left\{ g^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} + m^2 \right\}. \quad (109b)$$

It should be noted that the expressions (102) and (107) of the expectation value $\langle \psi | \widehat{T}_{\mu\nu} | \psi \rangle$ are identical because the various differential operators $\mathcal{T}^{\mathrm{cl}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu}$, $\mathcal{T}^{\mathrm{GB}_A\,\rho\sigma'}_{\mu\nu}$ and $\mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}$ appearing in (103) and $\mathcal{T}^{A\,\rho\sigma'}_{\mu\nu}$ and $\mathcal{T}^{\Phi}_{\mu\nu}$ appearing in (108) are related by

$$\mathcal{T}^{A\,\rho\sigma'}_{\mu\nu} = \mathcal{T}^{\mathrm{cl}_A\,\rho\sigma'}_{\mu\nu} + \mathcal{T}^{\mathrm{GB}_A\,\rho\sigma'}_{\mu\nu}, \quad (110a)$$

$$\mathcal{T}^{\Phi}_{\mu\nu} = \mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu} + \mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}. \quad (110b)$$

C. Renormalized stress-energy tensor

1. *Definition and conservation*

As we have already noted, the expectation value $\langle \psi | \widehat{T}_{\mu\nu} | \psi \rangle$ given by Eq. (102) is divergent due to the short-distance behavior of the Green functions or, more precisely, to the singular purely geometrical part of the Hadamard functions given in Eqs. (68a) and (70a) (see the terms in $1/\sigma$ and $\ln|\sigma|$). It is possible to construct the renormalized expectation value of the stress-energy-tensor operator with respect to the Hadamard quantum state $|\psi\rangle$ by using the prescription proposed by Wald in Refs. [26, 30, 31]. In Eqs. (103a)–(103c) we first discard the singular contributions, i.e., we make the replacements

$$G^{(1)A}_{\mu\nu'}(x,x') \to G^{(1)A}_{\mathrm{reg}\ \mu\nu'}(x,x') = \frac{1}{4\pi^2}\, W^A_{\mu\nu'}(x,x'), \quad (111a)$$

$$G^{(1)\Phi}(x,x') \to G^{(1)\Phi}_{\mathrm{reg}}(x,x') = \frac{1}{4\pi^2}\, W^{\Phi}(x,x'), \quad (111b)$$

$$G^{(1)\mathrm{Gh}}(x,x') \to G^{(1)\mathrm{Gh}}_{\mathrm{reg}}(x,x') = \frac{1}{4\pi^2}\, W^{\mathrm{Gh}}(x,x'), \quad (111c)$$

and we add to the result a state-independent tensor $\widetilde{\Theta}_{\mu\nu}$ which only depends on the mass parameter and on the local geometry and which ensures the conservation of the final expression. In other words, we consider that the renormalized expectation value of the stress-energy-tensor operator is given by

$$\begin{aligned} \langle \psi | \widehat{T}_{\mu\nu} | \psi \rangle_{\mathrm{ren}} &= \frac{1}{8\pi^2} \left\{ \mathcal{T}^{\mathrm{cl}_A}_{\mu\nu} \left[W^A \right] + \mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu} \left[W^\Phi \right] \right\} \\ &+ \frac{1}{8\pi^2} \left\{ \mathcal{T}^{\mathrm{GB}_A}_{\mu\nu} \left[W^A \right] + \mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu} \left[W^\Phi \right] \right\} \\ &+ \frac{1}{8\pi^2}\, \mathcal{T}^{\mathrm{Gh}}_{\mu\nu} \left[W^{\mathrm{Gh}} \right] \\ &+ \widetilde{\Theta}_{\mu\nu} \end{aligned} \quad (112)$$

with

$$\mathcal{T}^{\mathrm{cl}_A}_{\mu\nu} \left[W^A \right](x) = \lim_{x' \to x} \mathcal{T}^{\mathrm{cl}_A\,\rho\sigma'}_{\mu\nu}(x,x') \left[W^A_{\rho\sigma'}(x,x') \right], \quad (113a)$$

$$\mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu} \left[W^\Phi \right](x) = \lim_{x' \to x} \mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu}(x,x') \left[W^\Phi(x,x') \right], \quad (113b)$$

$$\mathcal{T}^{\mathrm{GB}_A}_{\mu\nu} \left[W^A \right](x) = \lim_{x' \to x} \mathcal{T}^{\mathrm{GB}_A\,\rho\sigma'}_{\mu\nu}(x,x') \left[W^A_{\rho\sigma'}(x,x') \right], \quad (113c)$$

$$\mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu} \left[W^\Phi \right](x) = \lim_{x' \to x} \mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}(x,x') \left[W^\Phi(x,x') \right] \quad (113d)$$

and

$$\mathcal{T}^{\mathrm{Gh}}_{\mu\nu} \left[W^{\mathrm{Gh}} \right](x) = \lim_{x' \to x} \mathcal{T}^{\mathrm{Gh}}_{\mu\nu}(x,x') \left[W^{\mathrm{Gh}}(x,x') \right]. \quad (113e)$$

Here, the differential operators $\mathcal{T}^{\mathrm{cl}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu}$, $\mathcal{T}^{\mathrm{GB}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}$ and $\mathcal{T}^{\mathrm{Gh}}_{\mu\nu}$ are given by Eqs. (104a)–(104e). In Eqs. (113a)–(113e), the coincidence limits $x' \to x$ are obtained from the covariant Taylor series expansions (75) and (76) by using extensively some of the results displayed in the Appendix. The final expressions can be simplified by using the relations (81a), (81b) and (83a) we have previously obtained from the wave equations. We have

$$\begin{aligned} \mathcal{T}^{\mathrm{cl}_A}_{\mu\nu} \left[W^A \right] &= (1/2)\, s_\rho{}^\rho{}_{;\mu\nu} + (1/2)\, \Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^\rho \\ &- a_\mu{}^\rho{}_{[\nu;\rho]} - a_\nu{}^\rho{}_{[\mu;\rho]} - s_\rho{}^\rho{}_{\mu\nu} + 2\, s_{\rho(\mu\nu)}{}^\rho \\ &- (1/2)\, g_{\mu\nu} \left\{ (1/2)\, \Box s_\rho{}^\rho - (1/2)\, s_{\rho\tau}{}^{;\rho\tau} \right. \\ &\quad \left. - (1/2)\, R^{\rho\tau} s_{\rho\tau} - a_{\rho\tau}{}^{\rho;\tau} + s_{\rho\tau}{}^{\rho\tau} \right\} \\ &+ 6\, v^A_1{}_{\mu\nu} - 3\, g_{\mu\nu}\, v^A_1{}_\rho{}^\rho, \end{aligned} \quad (114a)$$

$$\begin{aligned} \mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu} \left[W^\Phi \right] &= (1/2)\, w^\Phi_{;\mu\nu} - w^\Phi_{\mu\nu} \\ &- (1/2)\, g_{\mu\nu} \left\{ (1/2)\, \Box w^\Phi - m^2 w^\Phi \right\} - 3\, g_{\mu\nu}\, v_1, \end{aligned} \quad (114b)$$

$$\begin{aligned} \mathcal{T}^{\mathrm{GB}_A}_{\mu\nu} \left[W^A \right] &= R^\rho{}_{(\mu} s_{\nu)\rho} - a_\mu{}^\rho{}_{(\nu;\rho)} - a_\nu{}^\rho{}_{(\mu;\rho)} - 2\, s_{\rho(\mu\nu)}{}^\rho \\ &- (1/2)\, g_{\mu\nu} \left\{ -(1/2)\, s_{\rho\tau}{}^{;\rho\tau} + (1/2)\, R^{\rho\tau} s_{\rho\tau} \right. \\ &\quad \left. + a_{\rho\tau}{}^{\rho;\tau} - s_{\rho\tau}{}^{\rho\tau} \right\}, \end{aligned} \quad (114c)$$

$$\mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}\left[W^\Phi\right] = -(1/2)\, g_{\mu\nu}\, m^2 w^\Phi \tag{114d}$$

and

$$\mathcal{T}^{\mathrm{Gh}}_{\mu\nu}\left[W^{\mathrm{Gh}}\right] = -w^{\mathrm{Gh}}_{;\mu\nu} + 2\, w^{\mathrm{Gh}}_{\mu\nu} \\ +(1/2)\, g_{\mu\nu}\, \Box w^{\mathrm{Gh}} + 6\, g_{\mu\nu}\, v_1. \tag{114e}$$

Let us now consider the divergence of the terms given by Eqs. (114a)–(114e). By taking into account Eqs. (81) and (83), we obtain

$$\left(\mathcal{T}^{\mathrm{cl}_A}_{\mu\nu}\left[W^A\right] + \mathcal{T}^{\mathrm{GB}_A}_{\mu\nu}\left[W^A\right]\right)^{;\nu} = 6\, {v_1^A}_{\mu\nu}{}^{;\nu} - 2\, {v_1^A}_{\rho}{}^{\rho}{}_{;\mu}, \tag{115a}$$

$$\left(\mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu}\left[W^\Phi\right] + \mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}\left[W^\Phi\right]\right)^{;\nu} = -2\, v_{1;\mu} \tag{115b}$$

and

$$\left(\mathcal{T}^{\mathrm{Gh}}_{\mu\nu}\left[W^{\mathrm{Gh}}\right]\right)^{;\nu} = 4\, v_{1;\mu}, \tag{115c}$$

and we then have

$$\left(\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\mathrm{ren}}\right)^{;\nu} = \frac{1}{8\pi^2}\left\{6\, {v_1^A}_{\mu\nu} - 2\, g_{\mu\nu}\, {v_1^A}_{\rho}{}^{\rho} \right. \\ \left. +2\, g_{\mu\nu}\, v_1\right\}^{;\nu} + \widetilde{\Theta}_{\mu\nu}{}^{;\nu} = 0. \tag{116}$$

It is therefore suitable to redefine the purely geometrical tensor $\widetilde{\Theta}_{\mu\nu}$ by

$$\widetilde{\Theta}_{\mu\nu} \to \Theta_{\mu\nu} - \frac{1}{8\pi^2}\left\{6\, {v_1^A}_{\mu\nu} - 2\, g_{\mu\nu}\, {v_1^A}_{\rho}{}^{\rho} + 2\, g_{\mu\nu}\, v_1\right\}, \tag{117}$$

where the new local tensor $\Theta_{\mu\nu}$ is assumed to be conserved, i.e.,

$$\Theta_{\mu\nu}{}^{;\nu} = 0. \tag{118}$$

As a consequence, the renormalized expectation value of the stress-energy-tensor operator takes the following form:

$$\begin{aligned}\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\mathrm{ren}} = &\frac{1}{8\pi^2}\left\{\mathcal{T}^{\mathrm{cl}_A}_{\mu\nu}\left[W^A\right] + \mathcal{T}^{\mathrm{GB}_A}_{\mu\nu}\left[W^A\right] \right.\\ &\left. - 6\, {v_1^A}_{\mu\nu} + 2\, g_{\mu\nu}\, {v_1^A}_{\rho}{}^{\rho}\right\} \\ &+ \frac{1}{8\pi^2}\left\{\mathcal{T}^{\mathrm{cl}_\Phi}_{\mu\nu}\left[W^\Phi\right] + \mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}\left[W^\Phi\right] \right.\\ &\left. + 2\, g_{\mu\nu}\, v_1\right\} \\ &+ \frac{1}{8\pi^2}\left\{\mathcal{T}^{\mathrm{Gh}}_{\mu\nu}\left[W^{\mathrm{Gh}}\right] - 4\, g_{\mu\nu}\, v_1\right\} \\ &+ \Theta_{\mu\nu},\end{aligned} \tag{119}$$

where the various state-dependent contributions are given by Eqs. (114a)–(114e).

2. Cancellation of the gauge-breaking and ghost contributions

In Sec. IV B we have mentioned that the formal contributions of the gauge-breaking term $\langle\psi|\widehat{T}^{\mathrm{GB}}_{\mu\nu}|\psi\rangle$ and the ghost term $\langle\psi|\widehat{T}^{\mathrm{Gh}}_{\mu\nu}|\psi\rangle$ cancel each other out [see Eq. (105)]. This still remains valid for the corresponding regularized expectation values up to purely geometrical terms. Indeed, by using the first Ward identity in the form (86) as well as the second Ward identity in the form (88), we obtain

$$\mathcal{T}^{\mathrm{GB}_A}_{\mu\nu}\left[W^A\right] + \mathcal{T}^{\mathrm{GB}_\Phi}_{\mu\nu}\left[W^\Phi\right] + \mathcal{T}^{\mathrm{Gh}}_{\mu\nu}\left[W^{\mathrm{Gh}}\right] = \\ 2\, {v_1^A}_{\mu\nu} + g_{\mu\nu}\left\{-(1/2)\, {v_1^A}_{\rho}{}^{\rho} + 3\, v_1\right\}. \tag{120}$$

Now, by using this relation in connection with Eqs. (114a) and (114b), we can rewrite the renormalized expectation value of the stress-energy-tensor operator given by Eq. (119) in the form

$$\begin{aligned}&\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\mathrm{ren}} = \\ &\quad\frac{1}{8\pi^2}\left\{(1/2)\, s_\rho{}^\rho{}_{;\mu\nu} + (1/2)\, \Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^\rho \right.\\ &\quad -a_\mu{}^\rho{}_{[\nu;\rho]} - a_\nu{}^\rho{}_{[\mu;\rho]} - s_\rho{}^\rho{}_{\mu\nu} + 2\, s_{\rho(\mu\nu)}{}^\rho \\ &\quad -(1/2)\, g_{\mu\nu}\left[(1/2)\, \Box s_\rho{}^\rho - (1/2)\, s_{\rho\tau}{}^{;\rho\tau}\right.\\ &\quad\quad \left.-(1/2)\, R^{\rho\tau} s_{\rho\tau} - a_{\rho\tau}{}^{\rho;\tau} + s_{\rho\tau}{}^{\rho\tau}\right] \\ &\quad +(1/2)\, w^\Phi_{;\mu\nu} - w^\Phi_{\mu\nu} \\ &\quad -(1/2)\, g_{\mu\nu}\left[(1/2)\, \Box w^\Phi - m^2 w^\Phi\right] \\ &\quad \left.+2\, {v_1^A}_{\mu\nu} - (3/2)\, g_{\mu\nu} {v_1^A}_\rho{}^\rho - 2\, g_{\mu\nu} v_1\right\} \\ &\quad + \Theta_{\mu\nu}.\end{aligned} \tag{121}$$

This expression only involves state-dependent and geometrical quantities associated with the quantum fields A_μ and Φ. We could consider it as our final result, but, in fact, it is very important here to note that, due to the first Ward identity, the decomposition into a part involving the massive vector field and another part involving the auxiliary scalar field is not unique. In the next sections, we shall provide two alternative expressions which, in our opinion, are much more interesting from the physical point of view.

From now, in order to simplify the notations and because this does not lead to any ambiguity, we shall omit the label Φ for the Taylor coefficients w^Φ and $w^\Phi_{\mu\nu}$.

3. Substitution of the auxiliary scalar field contribution and final result

It is possible to remove in Eq. (121) any reference to the auxiliary scalar field Φ. In some sense, it plays the role of a kind of ghost field (the so-called Stueckelberg ghost [47]), but its contribution must be carefully taken into account. By using Eqs. (83a), (86a) and (86b) in the form

$$\begin{aligned}m^2 w &= w_\rho{}^\rho + 6\, v_1 \\ &= -(1/2)\, s_{\rho\tau}{}^{;\rho\tau} - (1/2)\, R^{\rho\tau} s_{\rho\tau} \\ &\quad + a_{\rho\tau}{}^{\rho;\tau} + s_{\rho\tau}{}^{\rho\tau} + {v_1^A}_\rho{}^\rho + 2\, v_1,\end{aligned} \tag{122a}$$

$$w_{;\mu\nu} = -s_{\rho(\mu|}{}^{;\rho}{}_{|\nu)} + 2\, a_{\rho(\mu|}{}^{\rho}{}_{;|\nu)} \tag{122b}$$

and

$$\begin{aligned} w_{\mu\nu} = &-(1/2)\, s_{\rho(\mu|}{}^{;\rho}{}_{|\nu)} - (1/2)\, R^{\rho}{}_{(\mu}s_{\nu)\rho} \\ &+(1/2)\, a_{\mu}{}^{\rho}{}_{[\nu;\rho]} + (1/2)\, a_{\nu}{}^{\rho}{}_{[\mu;\rho]} \\ &+s_{\rho(\mu\nu)}{}^{\rho} + v^{A}_{1\ \mu\nu} - g_{\mu\nu}v_1, \end{aligned} \tag{122c}$$

we obtain

$$\begin{aligned} \langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\rm ren} = &\\ &\frac{1}{8\pi^2}\Big\{(1/2)\, s_{\rho}{}^{\rho}{}_{;\mu\nu} + (1/2)\,\Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^{\rho} \\ &+(1/2)\, R^{\rho}{}_{(\mu}s_{\nu)\rho} - (1/2)\, a_{\mu}{}^{\rho}{}_{(\nu;\rho)} - (1/2)\, a_{\nu}{}^{\rho}{}_{(\mu;\rho)} \\ &-a_{\mu}{}^{\rho}{}_{[\nu;\rho]} - a_{\nu}{}^{\rho}{}_{[\mu;\rho]} - s_{\rho}{}^{\rho}{}_{\mu\nu} + s_{\rho(\mu\nu)}{}^{\rho} \\ &-(1/2)\, g_{\mu\nu}\left[(1/2)\,\Box s_{\rho}{}^{\rho} - (1/2)\, s_{\rho\tau}{}^{;\rho\tau} - a_{\rho\tau}{}^{\rho;\tau}\right] \\ &+v^{A}_{1\ \mu\nu} - g_{\mu\nu}v^{A}_{1\ \rho}{}^{\rho}\Big\} \\ &+\Theta_{\mu\nu}. \end{aligned} \tag{123}$$

We have now at our disposal an expression for the renormalized expectation value of the stress-energy-tensor operator associated with the full Stueckelberg theory which only involves state-dependent and geometrical quantities associated with the massive vector field A_μ. It is the main result of our article.

It is interesting to note that Eq. (123) combined with Eq. (81b) leads to

$$\begin{aligned} \langle\psi|\widehat{T}_{\rho}{}^{\rho}|\psi\rangle_{\rm ren} = &\frac{1}{8\pi^2}\left\{-m^2 s_{\rho}{}^{\rho} - (1/2)\, R^{\rho\tau}s_{\rho\tau}\right. \\ &\left.+s_{\rho\tau}{}^{\rho\tau} + 3\, v^{A}_{1\ \rho}{}^{\rho}\right\} + \Theta_{\rho}{}^{\rho}. \end{aligned} \tag{124}$$

4. *Another final expression involving both the vector field A_μ and the auxiliary scalar field Φ*

Even if we are satisfied with our final expression (123), it is worth nothing that it does not reduce, in the limit $m^2 \to 0$, to the result obtained from Maxwell's theory. This is not really surprising because it involves implicitly the contribution of the auxiliary scalar field. In fact, by replacing in Eq. (121) the term $m^2 w$ given by Eq. (122a), it is possible to split the renormalized expectation value of the stress-energy-tensor operator in the form

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\rm ren} = \mathcal{T}^{A}_{\mu\nu} + \mathcal{T}^{\Phi}_{\mu\nu} + \Theta_{\mu\nu}, \tag{125}$$

where the terms associated with the vector and scalar fields are given by

$$\begin{aligned} \mathcal{T}^{A}_{\mu\nu} = &\frac{1}{8\pi^2}\Big\{(1/2)\, s_{\rho}{}^{\rho}{}_{;\mu\nu} + (1/2)\,\Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^{\rho} \\ &-a_{\mu}{}^{\rho}{}_{[\nu;\rho]} - a_{\nu}{}^{\rho}{}_{[\mu;\rho]} - s_{\rho}{}^{\rho}{}_{\mu\nu} + 2\, s_{\rho(\mu\nu)}{}^{\rho} \\ &-(1/2)\, g_{\mu\nu}\left[(1/2)\,\Box s_{\rho}{}^{\rho} - 2\, a_{\rho\tau}{}^{\rho;\tau}\right] \\ &+2\, v^{A}_{1\ \mu\nu} - g_{\mu\nu}v^{A}_{1\ \rho}{}^{\rho}\Big\} \end{aligned} \tag{126a}$$

and

$$\begin{aligned} \mathcal{T}^{\Phi}_{\mu\nu} = &\frac{1}{8\pi^2}\left\{(1/2)\, w_{;\mu\nu} - w_{\mu\nu}\right. \\ &\left.-(1/4)\, g_{\mu\nu}\Box w - g_{\mu\nu}v_1\right\}. \end{aligned} \tag{126b}$$

The stress-energy tensors $\mathcal{T}^{A}_{\mu\nu}$ and $\mathcal{T}^{\Phi}_{\mu\nu}$ are separately conserved (this can be checked from relations obtained in Sec. III D), and, moreover, the expression of $\mathcal{T}^{\Phi}_{\mu\nu}$ corresponds exactly to the renormalized expectation value of the stress-energy-tensor operator associated with the quantum action (14) for ξ=1 (see, e.g., Refs. [35, 36]). As a consequence, it could be rather natural to consider $\mathcal{T}^{A}_{\mu\nu}$ given by Eq. (126a) as the renormalized expectation value of the stress-energy-tensor operator associated with the massive vector field A_μ. This physical interpretation is strengthened by noting that, in the limit $m^2 \to 0$, $\mathcal{T}^{A}_{\mu\nu}$ reduces to the result obtained from Maxwell's theory (see Sec. IV D). However, despite this, we are not really satisfied by this artificial way to split the contributions of the vector and scalar fields because, as we have already noted, the first Ward identity allows us to move terms from one contribution to the other. So, we consider that the only nonambiguous result is the one given by Eq. (123).

It is interesting to note that Eq. (125) combined with Eqs. (81b) and (83a) leads to

$$\langle\widehat{T}_{\rho}{}^{\rho}\rangle_{\rm ren} = \mathcal{T}^{A}{}_{\rho}{}^{\rho} + \mathcal{T}^{\Phi}{}_{\rho}{}^{\rho} + \Theta_{\rho}{}^{\rho} \tag{127}$$

with

$$\begin{aligned} \mathcal{T}^{A}{}_{\rho}{}^{\rho} = &\frac{1}{8\pi^2}\left\{-s_{\rho\tau}{}^{;\rho\tau} - m^2 s_{\rho}{}^{\rho} - R^{\rho\tau}s_{\rho\tau}\right. \\ &\left.+2\, a_{\rho\tau}{}^{\rho;\tau} + 2\, s_{\rho\tau}{}^{\rho\tau} + 4\, v^{A}_{1\ \rho}{}^{\rho}\right\} \end{aligned} \tag{128a}$$

and

$$\mathcal{T}^{\Phi}{}_{\rho}{}^{\rho} = \frac{1}{8\pi^2}\left\{-(1/2)\,\Box w - m^2 w + 2\, v_1\right\}. \tag{128b}$$

D. Maxwell's theory

Let us now consider the limit $m^2 \to 0$ of $\mathcal{T}^{A}_{\mu\nu}$ given by Eq. (126a). By using Eq. (122a), it reduces to

$$\begin{aligned} \mathcal{T}^{\rm Maxwell}_{\mu\nu} = &\frac{1}{8\pi^2}\Big\{(1/2)\, s_{\rho}{}^{\rho}{}_{;\mu\nu} + (1/2)\,\Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^{\rho} \\ &-a_{\mu}{}^{\rho}{}_{[\nu;\rho]} - a_{\nu}{}^{\rho}{}_{[\mu;\rho]} - s_{\rho}{}^{\rho}{}_{\mu\nu} + 2\, s_{\rho(\mu\nu)}{}^{\rho} \\ &-(1/2)\, g_{\mu\nu}\left[(1/2)\,\Box s_{\rho}{}^{\rho} - (1/2)\, s_{\rho\tau}{}^{;\rho\tau}\right. \\ &\left.-(1/2)\, R^{\rho\tau}s_{\rho\tau} - a_{\rho\tau}{}^{\rho;\tau} + s_{\rho\tau}{}^{\rho\tau}\right] \\ &+2\, v^{A}_{1\ \mu\nu} - g_{\mu\nu}\left[(3/2)\, v^{A}_{1\ \rho}{}^{\rho} + v_1\right]\Big\}. \end{aligned} \tag{129}$$

This last expression is nothing other than the renormalized expectation value of the stress-energy-tensor operator associated with Maxwell's electromagnetism (see Eq. (3.41b) of Ref. [39]).

It is interesting to note that Eq. (129) combined with Eq. (81b) leads to

$$g^{\mu\nu}T^{\text{Maxwell}}_{\mu\nu} = \frac{1}{8\pi^2}\left\{2\, v_1{}^{A}{}_{\rho}{}^{\rho} - 4\, v_1\right\}$$
$$= \frac{1}{8\pi^2}\left\{-(1/20)\,\Box R - (5/72)\, R^2 + (11/45)\, R_{pq}\, R^{pq} - (13/360)\, R_{pqrs}\, R^{pqrs}\right\}. \tag{130}$$

We recover the trace anomaly for Maxwell's theory.

E. Ambiguities in the renormalized stress-energy tensor

1. General expression of the ambiguities

The renormalized expectation value $\langle\psi|\widehat{T}_{\mu\nu}(x)|\psi\rangle_{\text{ren}}$ is unique up to the addition of a geometrical conserved tensor $\Theta_{\mu\nu}$. In other words, even if it takes perfectly into account the quantum state dependence of the theory, it is ambiguously defined (see, Sec. III of Ref. [31] as well as, e.g., comments in Refs. [25, 26, 41, 48, 49]).

As noted by Wald [31], $\Theta_{\mu\nu}$ is a local conserved tensor of dimension $(\text{mass})^4 = (\text{length})^{-4}$ which remains finite in the massless limit. As a consequence, it can be constructed by functional derivation with respect to the metric tensor from a geometrical Lagrangian of dimension $(\text{mass})^4 = (\text{length})^{-4}$. Such a Lagrangian is necessarily a linear combination of the following four terms: m^4, m^2R, R^2, $R_{pq}R^{pq}$. It should be noted that we could also take into account the term $R_{pqrs}R^{pqrs}$. But, in fact, we can eliminate this term because, in a four-dimensional background, the Euler number

$$\chi = \int_{\mathcal{M}} d^4x\sqrt{-g}\,\left[R^2 - 4\, R_{pq}R^{pq} + R_{pqrs}R^{pqrs}\right] \tag{131}$$

associated with the quadratic Gauss-Bonnet action is a topological invariant.

The functional derivation of the action terms previously discussed provides the conserved tensors

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int_{\mathcal{M}} d^4x\sqrt{-g}\, m^4 = (1/2)\, m^4 g^{\mu\nu}, \tag{132a}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int_{\mathcal{M}} d^4x\sqrt{-g}\, m^2 R = -m^2\left[R^{\mu\nu} - (1/2)\, R\, g^{\mu\nu}\right], \tag{132b}$$

$${}^{(1)}H^{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int_{\mathcal{M}} d^4x\sqrt{-g}\, R^2 = 2\, R^{;\mu\nu} - 2\, RR^{\mu\nu} + g^{\mu\nu}\left[-2\,\Box R + (1/2)\, R^2\right], \tag{132c}$$

$${}^{(2)}H^{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int_{\mathcal{M}} d^4x\sqrt{-g}\, R_{pq}R^{pq} = R^{;\mu\nu} - \Box R^{\mu\nu} - 2\, R_{pq}R^{\mu p\nu q} + g^{\mu\nu}\left[-(1/2)\,\Box R + (1/2)\, R_{pq}R^{pq}\right]. \tag{132d}$$

The general expression of the local conserved tensor $\Theta_{\mu\nu}$ can be therefore written in the form

$$\Theta_{\mu\nu} = \frac{1}{8\pi^2}\Big\{\alpha\, m^4 g_{\mu\nu} + \beta\, m^2\left[R_{\mu\nu} - (1/2)\, R\, g_{\mu\nu}\right] + \gamma_1\, {}^{(1)}H_{\mu\nu} + \gamma_2\, {}^{(2)}H_{\mu\nu}\Big\}, \tag{133}$$

where α, β, γ_1 and γ_2 are constants which can be fixed by imposing additional physical conditions on the renormalized expectation value of the stress-energy-operator tensor, these conditions being appropriate to the problem treated.

2. Ambiguities associated with the renormalization mass

So far, in order to simplify the calculations, we have dropped the scale length λ (or, equivalently, the mass scale $M = 1/\lambda$, i.e., the so-called renormalization mass) that should be introduced in order to make dimensionless the argument of the logarithm in the Hadamard representation of the Green functions. In fact, in Eqs. (35) and (37) it is necessary to make the substitution $\ln[\sigma(x,x') + i\epsilon] \to \ln[\sigma(x,x')/\lambda^2 + i\epsilon]$ which leads in Eqs. (55) and (58) to the substitution

$$\ln|\sigma(x,x')| \to \ln|\sigma(x,x')/\lambda^2|. \tag{134}$$

This scale length induces an indeterminacy in the bitensors $W^A_{\mu\nu'}(x,x')$ and $W(x,x')$ which corresponds to the replacements

$$W^A_{\mu\nu'}(x,x') \to W^A_{\mu\nu'}(x,x') - V^A_{\mu\nu'}(x,x')\ln(M^2), \tag{135a}$$
$$W^{\Phi}(x,x') \to W^{\Phi}(x,x') - V^{\Phi}(x,x')\ln(M^2), \tag{135b}$$
$$W^{\text{Gh}}(x,x') \to W^{\text{Gh}}(x,x') - V^{\text{Gh}}(x,x')\ln(M^2), \tag{135c}$$

i.e., in terms of the associated Taylor coefficients, to replacements

$$s_{\mu\nu} \to s_{\mu\nu} - v_0^A{}_{(\mu\nu)}\ln(M^2), \tag{136a}$$
$$a_{\mu\nu a} \to a_{\mu\nu a} - v_0^A{}_{[\mu\nu]a}\ln(M^2), \tag{136b}$$
$$s_{\mu\nu ab} \to s_{\mu\nu ab} - \left(v_0^A{}_{(\mu\nu)ab} + v_1^A{}_{(\mu\nu)}g_{ab}\right)\ln(M^2) \tag{136c}$$

for the vector field A_μ and

$$w \to w - v_0\ln(M^2), \tag{137a}$$
$$w_{ab} \to w_{ab} - (v_{0ab} + v_1 g_{ab})\ln(M^2) \tag{137b}$$

for the scalar field Φ or the ghost fields. By substituting Eqs. (135a)–(135c) into the general expression (112) of the renormalized expectation value of the stress-energy-tensor operator, we obtain the general form of the ambiguity associated with the scale length (or with the renormaization mass). It is given by

$$\Theta_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\left\{\Theta^{\text{cl}_A}_{\mu\nu}\left[V^A\right] + \Theta^{\text{cl}_\Phi}_{\mu\nu}\left[V^\Phi\right]\right\} - \frac{\ln(M^2)}{8\pi^2}\left\{\Theta^{\text{GB}_A}_{\mu\nu}\left[V^A\right] + \Theta^{\text{GB}_\Phi}_{\mu\nu}\left[V^\Phi\right]\right\} - \frac{\ln(M^2)}{8\pi^2}\,\Theta^{\text{Gh}}_{\mu\nu}\left[V^{\text{Gh}}\right] \tag{138}$$

with

$$\Theta^{\text{cl}_A}_{\mu\nu}\left[V^A\right](x) = \lim_{x'\to x} \mathcal{T}^{\text{cl}_A\,\rho\sigma'}_{\mu\nu}(x,x')\left[V^A_{\rho\sigma'}(x,x')\right], \tag{139a}$$

$$\Theta^{\text{cl}_\Phi}_{\mu\nu}\left[V^\Phi\right](x) = \lim_{x'\to x} \mathcal{T}^{\text{cl}_\Phi}_{\mu\nu}(x,x')\left[V^\Phi(x,x')\right], \tag{139b}$$

$$\Theta^{\text{GB}_A}_{\mu\nu}\left[V^A\right](x) = \lim_{x'\to x} \mathcal{T}^{\text{GB}_A\,\rho\sigma'}_{\mu\nu}(x,x')\left[V^A_{\rho\sigma'}(x,x')\right], \tag{139c}$$

$$\Theta^{\text{GB}_\Phi}_{\mu\nu}\left[V^\Phi\right](x) = \lim_{x'\to x} \mathcal{T}^{\text{GB}_\Phi}_{\mu\nu}(x,x')\left[V^\Phi(x,x')\right] \tag{139d}$$

and

$$\Theta^{\text{Gh}}_{\mu\nu}\left[V^{\text{Gh}}\right](x) = \lim_{x'\to x} \mathcal{T}^{\text{Gh}}_{\mu\nu}(x,x')\left[V^{\text{Gh}}(x,x')\right], \tag{139e}$$

where the differential operators $\mathcal{T}^{\text{cl}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\text{cl}_\Phi}_{\mu\nu}$, $\mathcal{T}^{\text{GB}_A\,\rho\sigma'}_{\mu\nu}$, $\mathcal{T}^{\text{GB}_\Phi}_{\mu\nu}$ and $\mathcal{T}^{\text{Gh}}_{\mu\nu}$ are given in Eqs. (104a)–(104e). It should be noted that $\Theta_{\mu\nu}(M)$ is a purely geometrical object. This is due to the geometrical nature of the Hadamard coefficients $V^A_{\mu\nu'}(x,x')$ and $V(x,x')$.

In order to obtain the explicit expression of the stress-energy tensor $\Theta_{\mu\nu}(M)$, we can repeat the calculations of Sec. IV C by replacing $W^A_{\mu\nu'}$ by $V^A_{\mu\nu'}$, W^Φ by V^Φ and W^{Gh} by V^{Gh}. From Eqs. (114a)–(114e) it is straightforward to obtain explicitly $\Theta^{\text{cl}_A}_{\mu\nu}\left[V^A\right]$, $\Theta^{\text{cl}_\Phi}_{\mu\nu}\left[V^\Phi\right]$, $\Theta^{\text{GB}_A}_{\mu\nu}\left[V^A\right]$, $\Theta^{\text{GB}_\Phi}_{\mu\nu}\left[V^\Phi\right]$ and $\Theta^{\text{Gh}}_{\mu\nu}\left[V^{\text{Gh}}\right]$ by using the replacements (136) and (137). We can then show that

$$\left(\Theta^{\text{cl}_A}_{\mu\nu}\left[V^A\right] + \Theta^{\text{GB}_A}_{\mu\nu}\left[V^A\right]\right)^{;\nu} = 0, \tag{140a}$$

$$\left(\Theta^{\text{cl}_\Phi}_{\mu\nu}\left[V^\Phi\right] + \Theta^{\text{GB}_\Phi}_{\mu\nu}\left[V^\Phi\right]\right)^{;\nu} = 0 \tag{140b}$$

and

$$\left(\Theta^{\text{Gh}}_{\mu\nu}\left[V^{\text{Gh}}\right]\right)^{;\nu} = 0. \tag{140c}$$

Equations (140a)–(140c) are similar to Eqs. (115a)–(115c) but now with the right-hand sides vanishing. This is due to the fact that, unlike the wave equations (42) and (46) for $W^A_{\mu\nu'}$, W^Φ and W^{Gh}, the wave equations for $V^A_{\mu\nu'}$, V^Φ and V^{Gh} [cf. Eqs. (41) and (45)] have no source terms. As a consequence $\Theta_{\mu\nu}(M)$ is a conserved geometrical tensor. We can also check that

$$\Theta^{\text{GB}_A}_{\mu\nu}\left[V^A\right] + \Theta^{\text{GB}_\Phi}_{\mu\nu}\left[V^\Phi\right] + \Theta^{\text{Gh}}_{\mu\nu}\left[V^{\text{Gh}}\right] = 0. \tag{141}$$

Equation (141) is similar to Eq. (120) but now with the right-hand side vanishing. This is due to the fact that, unlike the Ward identity (85) linking $W^A_{\mu\nu'}$ and W^{Gh}, the Ward identity (84) linking $V^A_{\mu\nu'}$ and V^{Gh} has no right-hand side. As a consequence, from Eqs. (138) and (141), we obtain

$$\Theta_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\left\{\Theta^{\text{cl}_A}_{\mu\nu}\left[V^A\right] + \Theta^{\text{cl}_\Phi}_{\mu\nu}\left[V^\Phi\right]\right\}. \tag{142}$$

The ambiguities associated with the scale length can now be obtained explicitly from the replacements (136) and (137). If we use the form (123) without taking into account the geometrical terms, we obtain

$$\begin{aligned}\Theta_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\Big\{&(1/2)\, v^A_0{}_\rho{}^\rho{}_{;\mu\nu} + (1/2)\,\Box v^A_0{}_{\mu\nu} \\ &- v^A_0{}_{\rho(\mu;\nu)}{}^\rho + (1/2)\, R^\rho{}_{(\mu} v^A_0{}_{\nu)\rho} - (1/2)\, g^{\rho\tau} v^A_0{}_{[\mu\rho](\nu;\tau)} \\ &-(1/2)\, g^{\rho\tau} v^A_0{}_{[\nu\rho](\mu;\tau)} - g^{\rho\tau} v^A_0{}_{[\mu\rho][\nu;\tau]} - g^{\rho\tau} v^A_0{}_{[\nu\rho][\mu;\tau]} \\ &- v^A_0{}_\rho{}^\rho{}_{\mu\nu} + (1/2)\, v^A_0{}_{(\rho\mu)\nu}{}^\rho + (1/2)\, v^A_0{}_{(\rho\nu)\mu}{}^\rho + v^A_1{}_{\mu\nu} \\ &- g_{\mu\nu}\Big[(1/4)\,\Box v^A_0{}_\rho{}^\rho - (1/4)\, v^A_0{}_{\rho\tau}{}^{;\rho\tau} - (1/2)\, v^A_0{}_{[\rho\tau]}{}^{\rho;\tau} \\ &+ v^A_1{}_\rho{}^\rho\Big]\Big\}.\end{aligned} \tag{143}$$

Similarly, if we use the alternative form (125) where the contributions corresponding to the vector field A_μ and the auxiliary scalar field Φ are highlighted [see Eqs. (126a) and (126b)], we obtain

$$\Theta_{\mu\nu}(M) = \Theta^A_{\mu\nu}(M) + \Theta^\Phi_{\mu\nu}(M) \tag{144}$$

with

$$\begin{aligned}\Theta^A_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\Big\{&(1/2)\, v^A_0{}_\rho{}^\rho{}_{;\mu\nu} + (1/2)\,\Box v^A_0{}_{\mu\nu} \\ &- v^A_0{}_{\rho(\mu;\nu)}{}^\rho - g^{\rho\tau} v^A_0{}_{[\mu\rho][\nu;\tau]} - g^{\rho\tau} v^A_0{}_{[\nu\rho][\mu;\tau]} \\ &- v^A_0{}_\rho{}^\rho{}_{\mu\nu} + v^A_0{}_{(\rho\mu)\nu}{}^\rho + v^A_0{}_{(\rho\nu)\mu}{}^\rho + 2\, v^A_1{}_{\mu\nu} \\ &- g_{\mu\nu}\Big[(1/4)\,\Box v^A_0{}_\rho{}^\rho - v^A_0{}_{[\rho\tau]}{}^{\rho;\tau} + v^A_1{}_\rho{}^\rho\Big]\Big\}\end{aligned} \tag{145a}$$

and

$$\begin{aligned}\Theta^\Phi_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\{&(1/2)\, v_{0;\mu\nu} - v_{0\mu\nu} \\ &- g_{\mu\nu}\left[(1/4)\,\Box v_0 + v_1\right]\}.\end{aligned} \tag{145b}$$

Now, by using the explicit expressions (72) and (74) of the Taylor coefficients of the purely geometrical Hadamard coefficients, we can show that Eq. (143) reduces to

$$\begin{aligned}\Theta_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\{&(1/4)\, m^2 R_{\mu\nu} - (1/20)\, R_{;\mu\nu} \\ &+(13/120)\,\Box R_{\mu\nu} - (1/8)\, R R_{\mu\nu} + (2/15)\, R_{\mu p} R_\nu{}^p \\ &+(7/20)\, R_{pq} R_\mu{}^p{}_\nu{}^q - (1/15)\, R_{\mu pqr} R_\nu{}^{pqr} \\ &+ g_{\mu\nu}\left[-(3/8)\, m^4 - (1/8)\, m^2 R - (1/240)\,\Box R\right. \\ &\quad +(1/32)\, R^2 - (29/240)\, R_{pq} R^{pq} \\ &\quad \left. +(1/60)\, R_{pqrs} R^{pqrs}\right]\},\end{aligned} \tag{146}$$

while Eqs. (145a) and (145b) provide

$$\begin{aligned}\Theta^A_{\mu\nu}(M) = -\frac{\ln(M^2)}{8\pi^2}\{&(1/3)\, m^2 R_{\mu\nu} - (1/30)\, R_{;\mu\nu} \\ &+(1/10)\,\Box R_{\mu\nu} - (5/36)\, R R_{\mu\nu} + (13/90)\, R_{\mu p} R_\nu{}^p \\ &+(31/90)\, R_{pq} R_\mu{}^p{}_\nu{}^q - (13/180)\, R_{\mu pqr} R_\nu{}^{pqr} \\ &+ g_{\mu\nu}\left[-(1/4)\, m^4 - (1/6)\, m^2 R - (1/60)\,\Box R\right.\end{aligned}$$

$$+(5/144)\,R^2-(11/90)\,R_{pq}R^{pq}$$
$$+(13/720)\,R_{pqrs}R^{pqrs}\big]\big\} \tag{147a}$$

and

$$\Theta^{\Phi}_{\mu\nu}(M)=-\frac{\ln(M^2)}{8\pi^2}\,\{-(1/12)\,m^2R_{\mu\nu}-(1/60)\,R_{;\mu\nu}$$
$$+(1/120)\,\Box R_{\mu\nu}+(1/72)\,RR_{\mu\nu}-(1/90)\,R_{\mu p}R_{\nu}{}^{p}$$
$$+(1/180)\,R_{pq}R_{\mu}{}^{p}{}_{\nu}{}^{q}+(1/180)\,R_{\mu pqr}R_{\nu}{}^{pqr}$$
$$+g_{\mu\nu}\,\big[-(1/8)\,m^4+(1/24)\,m^2R+(1/80)\,\Box R$$
$$-(1/288)\,R^2+(1/720)\,R_{pq}R^{pq}$$
$$-(1/720)\,R_{pqrs}R^{pqrs}\big]\big\}\,. \tag{147b}$$

Of course, it is easy to check that the sum of $\Theta^{A}_{\mu\nu}(M)$ and $\Theta^{\Phi}_{\mu\nu}(M)$ is equal to $\Theta_{\mu\nu}(M)$.

It is possible to obtain a more compact form for the stress-energy tensors (146), (147a) and (147b) by using the conserved tensors (132a)–(132d). It should be noted that the terms in $R_{\mu p}R_{\nu}{}^{p}$, $R_{\mu pqr}R_{\nu}{}^{pqr}$ and $R_{pqrs}R^{pqrs}$ which are not present in ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ can be eliminated by introducing

$${}^{(3)}H^{\mu\nu}\equiv\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int_{\mathcal{M}}d^4x\sqrt{-g}\,R_{pqrs}R^{pqrs}$$
$$=2\,R^{;\mu\nu}-4\,\Box R^{\mu\nu}+4\,R^{\mu}{}_{p}R^{\nu p}-4\,R_{pq}R^{\mu p\nu q}$$
$$-2\,R^{\mu}{}_{pqr}R^{\nu pqr}+g^{\mu\nu}\,[(1/2)\,R_{pqrs}R^{pqrs}] \tag{148}$$

and by noting that, due to Eq. (131),

$${}^{(1)}H_{\mu\nu}-4\,{}^{(2)}H_{\mu\nu}+{}^{(3)}H_{\mu\nu}=0. \tag{149}$$

We then have

$$\Theta_{\mu\nu}(M)=\frac{\ln(M^2)}{8\pi^2}\,\Big\{(3/8)\,m^4g_{\mu\nu}$$
$$-(1/4)\,m^2\,[R_{\mu\nu}-(1/2)\,R\,g_{\mu\nu}]$$
$$-(7/240)\,{}^{(1)}H_{\mu\nu}+(13/120)\,{}^{(2)}H_{\mu\nu}\Big\}, \tag{150}$$

$$\Theta^{A}_{\mu\nu}(M)=\frac{\ln(M^2)}{8\pi^2}\,\Big\{(1/4)\,m^4g_{\mu\nu}$$
$$-(1/3)\,m^2\,[R_{\mu\nu}-(1/2)\,R\,g_{\mu\nu}]$$
$$-(1/30)\,{}^{(1)}H_{\mu\nu}+(1/10)\,{}^{(2)}H_{\mu\nu}\Big\} \tag{151a}$$

and

$$\Theta^{\Phi}_{\mu\nu}(M)=\frac{\ln(M^2)}{8\pi^2}\,\Big\{(1/8)\,m^4g_{\mu\nu}$$
$$+(1/12)\,m^2\,[R_{\mu\nu}-(1/2)\,R\,g_{\mu\nu}]$$
$$+(1/240)\,{}^{(1)}H_{\mu\nu}+(1/120)\,{}^{(2)}H_{\mu\nu}\Big\}. \tag{151b}$$

As expected, we can note that the ambiguities associated with the scale length (or with the renormalization mass) are of the form (133).

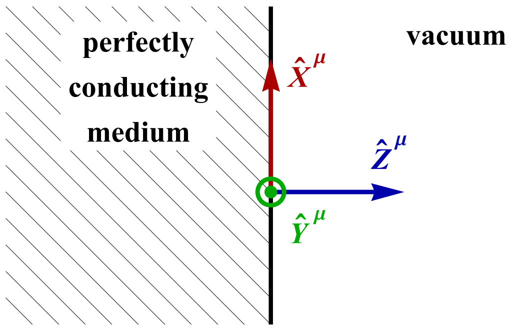

FIG. 1. **Geometry of the Casimir effect**

V. CASIMIR EFFECT

A. General considerations

In this section, we shall consider the Casimir effect for Stueckelberg massive electromagnetism in the Minkowski spacetime $(\mathbb{R}^4,\eta_{\mu\nu})$ with $\eta_{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1)$. We denote by (T,X,Y,Z) the coordinates of an event in this spacetime. We shall provide the renormalized vacuum expectation value of the stress-energy-tensor operator outside of a perfectly conducting medium with a plane boundary wall at $Z=0$ separating it from free space (see Fig. 1). It is worth pointing out that this problem has been studied a long time ago by Davies and Toms in the framework of de Broglie-Proca electromagnetism [50]. We shall revisit this problem in order to compare, at the quantum level and in the case of a simple example, de Broglie-Proca and Stueckelberg theories and to discuss their limit for $m^2\to 0$. It should be noted that the Casimir effect in connection with a massive photon has been considered for various geometries (see, e.g., Refs. [51–55]).

From symmetries and physical considerations, we can observe that, outside of the perfectly conducting medium, the renormalized stress-energy tensor takes the form (see Chap. 4 of Ref. [24])

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\mathrm{ren}}=\frac{1}{3}\,\langle 0|\widehat{T}_{\rho}{}^{\rho}|0\rangle_{\mathrm{ren}}\,\Big(\eta_{\mu\nu}-\hat{Z}_{\mu}\hat{Z}_{\nu}\Big)\,, \tag{152}$$

where $\hat{Z}^{\mu}$ is the spacelike unit vector orthogonal to the wall. As a consequence, it is sufficient to determine the trace of the renormalized stress-energy tensor. From Eq. (124), we have

$$\langle 0|\widehat{T}_{\rho}{}^{\rho}|0\rangle_{\mathrm{ren}}=\frac{1}{8\pi^2}\,\{-m^2s_{\rho}{}^{\rho}+s_{\rho\tau}{}^{\rho\tau}+(3/2)\,m^4\}$$
$$+\,\Theta_{\rho}{}^{\rho}. \tag{153}$$

The term $\Theta_{\rho}{}^{\rho}$ encodes the usual ambiguities discussed in Sec. IV E. In the Minkowski spacetime it reduces to

$$\Theta_{\rho}{}^{\rho}=\frac{1}{8\pi^2}\,\{\alpha\,m^4\}\,, \tag{154}$$

where α is a constant. From Eq. (153) it is clear that in order to evaluate $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$, it is sufficient to take the coincidence limit $x' \to x$ of $W^A_{\mu\nu}(x,x')$ and $W^A_{\mu\nu;(ab)}(x,x')$ [see Eqs. (78a) and (78c) and note that, in the Minkowski spacetime, the bivector of parallel transport $g_\mu{}^{\nu'}(x,x')$ is equal to the unit matrix $\delta_\mu{}^{\nu'}(x,x')$], where $W^A_{\mu\nu}(x,x')$ is the regular part of the Feynman propagator $G^A_{\mu\nu}(x,x')$ corresponding to the geometry of the Casimir effect.

B. Stress-energy tensor in the Minkowski spacetime

Let us first consider the vacuum expectation value of the stress-energy-tensor operator in the ordinary Minkowski spacetime (i.e., without the boundary wall). This will permit us to establish some notations and, moreover, to fix the constant α appearing in Eq. (154). Due to symmetry considerations, we have

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\rm ren} = \frac{1}{4}\,\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}\,\eta_{\mu\nu}, \tag{155}$$

where $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$ is still given by Eqs. (153) and (154). Of course, we must have $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\rm ren} = 0$, and we have therefore

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren} = 0 \tag{156}$$

which plays the role of a constraint for α.

In the Minkowski spacetime, the Feynman propagator $G^A_{\mu\nu}(x,x')$ associated with the vector field A_μ satisfies the wave equation (23), i.e.,

$$\left[\Box_x - m^2\right] G^A_{\mu\nu}(x,x') = -\eta_{\mu\nu}\delta^4(x,x'), \tag{157}$$

and its explicit expression is given in terms of a Hankel function of the second kind by (see, e.g., Chap. 27 of Ref. [56])

$$G^A_{\mu\nu}(x,x') = -\frac{m^2}{8\pi}\,\frac{1}{\mathcal{Z}(x,x')}H_1^{(2)}\left[\mathcal{Z}(x,x')\right]\eta_{\mu\nu}. \tag{158}$$

Here, $\mathcal{Z}(x,x') = \sqrt{-2\,m^2[\sigma(x,x')+i\epsilon]}$ with $2\,\sigma(x,x') = -(T-T')^2 + (X-X')^2 + (Y-Y')^2 + (Z-Z')^2$.

We have (see Chap. 9 of Ref. [57])

$$H_1^{(2)}(z) = J_1(z) - iY_1(z), \tag{159}$$

where $J_1(z)$ and $Y_1(z)$ are the Bessel functions of the first and second kinds. By using the series expansions for $z \to 0$ (see Eqs. (9.1.10) and (9.1.11) of Ref. [57])

$$J_1(z) = \frac{z}{2}\sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{k!(k+1)!} \tag{160a}$$

and

$$Y_1(z) = -\frac{2}{\pi z} + \frac{2}{\pi}\ln\left(\frac{z}{2}\right)J_1(z) - \frac{z}{2\pi}\sum_{k=0}^{\infty}[\psi(k+1)+\psi(k+2)]\,\frac{(-z^2/4)^k}{k!(k+1)!} \tag{160b}$$

[we note that Eq. (160b) is valid for $|\arg(z)| < \pi$, and we recall that the digamma function ψ is defined by the recursion relation $\psi(z+1) = \psi(z) + 1/z$ with $\psi(1) = -\gamma$], we can provide the Hadamard representation of $G^A_{\mu\nu}(x,x')$ given by Eq. (158). We can write

$$-\frac{m^2}{8\pi}\,\frac{1}{\mathcal{Z}(x,x')}H_1^{(2)}\left[\mathcal{Z}(x,x')\right] = \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon} + V(x,x')\ln[\sigma(x,x')+i\epsilon] + W(x,x')\right], \tag{161}$$

where

$$\Delta^{1/2}(x,x') = 1, \tag{162a}$$

$$V(x,x') = \sum_{k=0}^{\infty} V_k\,\sigma^k(x,x') \tag{162b}$$

with

$$V_k = \frac{(m^2/2)^{k+1}}{k!(k+1)!} \tag{162c}$$

and

$$W(x,x') = \sum_{k=0}^{\infty} W_k\sigma^k(x,x') \tag{162d}$$

with

$$W_k = -\frac{(m^2/2)^{k+1}}{k!(k+1)!}\left[\psi(k+1)+\psi(k+2)-\ln\left(\frac{m^2}{2}\right)\right]. \tag{162e}$$

By noting that

$$W^A_{\mu\nu}(x,x') = W(x,x')\,\eta_{\mu\nu}, \tag{163}$$

where $W(x,x')$ is given by Eqs. (162d) and (162e), we are now able to express $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$. From Eqs. (78a) and (78c) we have, respectively,

$$s_{\mu\nu} = m^2\left[-1/2+\gamma+(1/2)\,\ln(m^2/2)\right]\eta_{\mu\nu} \tag{164}$$

and

$$s_{\mu\nu ab} = m^4\left[-5/16+(1/4)\,\gamma+(1/8)\ln(m^2/2)\right]\eta_{\mu\nu}\eta_{ab}. \tag{165}$$

Then, from Eq. (153), we obtain

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren} = \frac{m^4}{8\pi^2}\left\{\alpha+9/4-3\,\gamma-(3/2)\,\ln(m^2/2)\right\}, \tag{166}$$

and, necessarily, by using Eq. (156), we have the constraint

$$\alpha = -9/4+3\,\gamma+(3/2)\,\ln(m^2/2). \tag{167}$$

C. Stress-energy tensor for the Casimir effect

Let us now come back to our initial problem. The Feynman propagator previously considered is modified by the presence of the plane boundary wall. The new Feynman propagator $\widetilde{G}^{A}_{\mu\nu}(x,x')$ can be constructed by the method of images if we assume, in order to simplify our problem, a perfectly reflecting wall. It should be noted that this particular boundary condition is questionable from the physical point of view. It is logical for the transverse components of the electromagnetic field but much less natural for its longitudinal component. Indeed, for this component, we could also consider perfect transmission instead of complete reflection (see Refs. [50, 51, 58]). We shall now consider that the Feynman propagator is given by

$$\widetilde{G}^{A}_{\mu\nu}(x,x') = G^{A}_{\mu\nu}(x,x') - q_\nu\, G^{A}_{\mu\nu}(x,\tilde{x}'). \tag{168}$$

Here, $x'^\mu = (T', X', Y', Z')$ and $\tilde{x}'^\mu = (T', X', Y', -Z')$, while $q_\nu = (1 - 2\,\delta_{3\nu})$. It is important to note that, in Eq. (168), the index ν is not summed. Furthermore, we remark that the term $G^{A}_{\mu\nu}(x,\tilde{x}')$ which is obtained by replacing x' by $\tilde{x}'$ in Eq. (158) as well as its derivatives are regular in the limit $x' \to x$.

By following the steps of Sec. V B and using the relation

$$K_\nu(z) = -(1/2)\, i\pi\, e^{-i\pi\nu/2}\, H^{(2)}_\nu(z\, e^{-i\pi/2}) \tag{169}$$

which is valid for $-\pi/2 \le \arg(z) \le \pi$ as well as the properties of the modified Bessel functions of the second kind K_1, K_2 and K_3 (see Chap. 9 of Ref. [57]), it is easy to show that the Taylor coefficients $s_{\mu\nu}$ and $s_{\mu\nu ab}$ involved in $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}}$ are now given by

$$\begin{aligned} s_{\mu\nu} &= m^2 \left[-1/2 + \gamma + (1/2)\,\ln(m^2/2)\right] \eta_{\mu\nu} \\ &\quad -(m/Z)\, K_1(2mZ)\, q_\nu\, \eta_{\mu\nu} \end{aligned} \tag{170}$$

and

$$\begin{aligned} s_{\mu\nu ab} &= m^4 \left[-5/16 + (1/4)\,\gamma + (1/8)\ln(m^2/2)\right] \eta_{\mu\nu}\eta_{ab} \\ &\quad - \big[(m^2/Z^2)\, K_2(2mZ)\, q_\nu\, \eta_{\mu\nu}\, (2\,\eta_{3a}\,\eta_{3b} - (1/2)\,\eta_{ab}) \\ &\quad +(m^3/Z)\, K_1(2mZ)\, q_\nu\, \eta_{\mu\nu}\eta_{3a}\eta_{3b}\big]\,. \end{aligned} \tag{171}$$

By inserting Eqs. (170) and (171) in the expression (153) and using the value of α fixed by Eq. (167), we obtain

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}} = \frac{3}{8\pi^2}\left\{\frac{m^2}{Z^2}\, K_2(2mZ) + \frac{m^3}{Z}\, K_1(2mZ)\right\}, \tag{172}$$

and from Eq. (152) we have

$$\begin{aligned} \langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}} &= \frac{1}{8\pi^2}\left\{\frac{m^2}{Z^2}\, K_2(2mZ) + \frac{m^3}{Z}\, K_1(2mZ)\right\} \\ &\quad \times \left(\eta_{\mu\nu} - \hat{Z}_\mu \hat{Z}_\nu\right). \end{aligned} \tag{173}$$

It is very important to note that this result coincides exactly with the result obtained by Davies and Toms in the framework of de Broglie-Proca electromagnetism [50].

In the limit $m^2 \to 0$ and for $Z \neq 0$, we obtain

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}} = \frac{1}{16\pi^2}\,\frac{1}{Z^4}\left(\eta_{\mu\nu} - \hat{Z}_\mu \hat{Z}_\nu\right). \tag{174}$$

In the massless limit, the vacuum expectation value of the renormalized stress-energy tensor associated with the Stueckelberg theory diverges like Z^{-4} as the boundary surface is approached. This result contrasts with that obtained from Maxwell's theory (see also Ref. [50]). Indeed, for this theory, the renormalized stress-energy-tensor operator vanishes identically. In order to extract that result from the Stuckelberg theory, we will now repeat the previous calculations from the expressions (125) and (126) [as well as (127) and (128)] given in Sec. IV C 4, where we have proposed an artificial separation of the contributions associated with the vector field A_μ and the auxiliary scalar field Φ.

D. Separation of the contributions associated with the vector field A_μ and the auxiliary scalar field Φ

In the Minkowski spacetime, Eqs. (127) and (128) reduce to

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}} = \mathcal{T}^A{}_\rho{}^\rho + \mathcal{T}^\Phi{}_\rho{}^\rho + \Theta_\rho{}^\rho \tag{175}$$

with

$$\begin{aligned} \mathcal{T}^A{}_\rho{}^\rho = \frac{1}{8\pi^2}\,\big\{&-s_{\rho\tau}{}^{;\rho\tau} - m^2 s_\rho{}^\rho \\ &+2\,a_{\rho\tau}{}^{\rho;\tau} + 2\,s_{\rho\tau}{}^{\rho\tau} + 2\,m^4\big\} \end{aligned} \tag{176}$$

and

$$\mathcal{T}^\Phi{}_\rho{}^\rho = \frac{1}{8\pi^2}\left\{-(1/2)\,\Box w - m^2 w + (1/4)\, m^4\right\}. \tag{177}$$

The term $\Theta_\rho{}^\rho$ encodes the usual ambiguities discussed in Sec. IV E. We can split it in the form

$$\Theta_\rho{}^\rho = \Theta^A{}_\rho{}^\rho + \Theta^\Phi{}_\rho{}^\rho \tag{178a}$$

with

$$\Theta^A{}_\rho{}^\rho = \frac{1}{8\pi^2}\left\{\alpha^A\, m^4\right\} \tag{178b}$$

and

$$\Theta^\Phi{}_\rho{}^\rho = \frac{1}{8\pi^2}\left\{\alpha^\Phi\, m^4\right\}, \tag{178c}$$

where α^A and α^Φ are two constants associated, respectively, with the contributions of the vector field A_μ and the auxiliary scalar field Φ. We can then replace Eq. (175) by

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}} = \langle 0|\widehat{T}^A{}_\rho{}^\rho|0\rangle_{\text{ren}} + \langle 0|\widehat{T}^\Phi{}_\rho{}^\rho|0\rangle_{\text{ren}} \tag{179}$$

with

$$\langle 0|\widehat{T}^A{}_\rho{}^\rho|0\rangle_{\text{ren}} = \mathcal{T}^A{}_\rho{}^\rho + \Theta^A{}_\rho{}^\rho \tag{180}$$

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

and

$$\langle 0|\widehat{T}^{\Phi}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren} = \mathcal{T}^{\Phi}{}_{\rho}{}^{\rho} + \Theta^{\Phi}{}_{\rho}{}^{\rho}, \qquad (181)$$

where the contributions associated with the vector field A_μ and the auxiliary scalar field Φ are separated. At first sight, $\mathcal{T}^{A}{}_{\rho}{}^{\rho}$ seems complicated because it involves Taylor coefficients of orders $\sigma^{1/2}$ and σ^1 of $W^A_{\mu\nu}(x,x')$. In fact, its expression can be simplified by replacing the sum $a_{\rho\tau}{}^{\rho;\tau} + s_{\rho\tau}{}^{\rho\tau}$ from the relation (86b), and we obtain

$$\mathcal{T}^{A}{}_{\rho}{}^{\rho} = \frac{1}{8\pi^2}\left\{-m^2 s_\rho{}^\rho + 2\,m^2 w + (1/2)\,m^4\right\} \qquad (182)$$

which only involves the first Taylor coefficients $s_{\mu\nu}$ and w of order σ^0. So, in order to evaluate $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$ given by Eq. (175), it is sufficient to take the coincidence limit $x' \to x$ of the state-dependent Hadamard coefficients $W^A_{\mu\nu}(x,x')$ and $W(x,x')$ associated with the Feynman propagators $G^A_{\mu\nu}(x,x')$ and $G^\Phi(x,x')$ corresponding to the geometry of the Casimir effect.

At first, we must fix the constants α^A and α^Φ appearing in Eq. (178). This can be achieved by imposing, in the Minkowski spacetime without boundary, the vanishing of $\langle 0|\widehat{T}^{A}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren}$ given by Eq. (180) and $\langle 0|\widehat{T}^{\Phi}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren}$ given by Eq. (181). In this spacetime, everything related to the Feynman propagator $G^A_{\mu\nu}(x,x')$ has been already given in Sec. V B, while the Feynman propagator $G^\Phi(x,x')$ associated with the scalar field Φ satisfies the wave equation (25) and is explicitly given by

$$G^\Phi(x,x') = -\frac{m^2}{8\pi}\,\frac{1}{\mathcal{Z}(x,x')}H_1^{(2)}\left[\mathcal{Z}(x,x')\right]. \qquad (183)$$

By using Eqs. (161) and (162), it is easy to see that this propagator can be represented in the Hadamard form and to obtain

$$w = m^2\left[-1/2 + \gamma + (1/2)\,\ln(m^2/2)\right]. \qquad (184)$$

We are now able to express $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$. From Eqs. (180), (181), (182), (177) and (178), we obtain

$$\langle 0|\widehat{T}^{A}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren} = \frac{m^4}{8\pi^2}\left\{\alpha^A + 3/2 - 2\,\gamma - \ln(m^2/2)\right\} \qquad (185)$$

and

$$\langle 0|\widehat{T}^{\Phi}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren} = \frac{m^4}{8\pi^2}\left\{\alpha^\Phi + 3/4 - \gamma - (1/2)\,\ln(m^2/2)\right\}, \qquad (186)$$

and, necessarily, the vanishing of these traces provides the two constraints

$$\alpha^A = -3/2 + 2\,\gamma + \ln(m^2/2) \qquad (187a)$$

and

$$\alpha^\Phi = -3/4 + \gamma + (1/2)\,\ln(m^2/2). \qquad (187b)$$

We now come back to the Casimir effect. The two Feynman propagators previously considered are modified by the presence of the plane boundary wall. The new Feynman propagators can be constructed by the method of images. Of course, the propagator of the vector field A_μ is still given by Eq. (168), while we have

$$\widetilde{G}^\Phi(x,x') = G^\Phi(x,x') - G^\Phi(x,\tilde{x}') \qquad (188)$$

for the propagator of the scalar field Φ. In the context of the Casimir effect, Eq. (184) must be replaced by

$$\begin{aligned} w = {} & m^2\left[-1/2 + \gamma + (1/2)\,\ln(m^2/2)\right] \\ & -(m/Z)\,K_1(2mZ), \end{aligned} \qquad (189)$$

and $s_{\mu\nu}$ is given by Eq. (170). By inserting Eqs. (170) and (189) in Eqs. (182) and (177) and taking into account the constraints (187a) and (187b), we obtain from Eq. (180)

$$\langle 0|\widehat{T}^{A}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren} = 0 \qquad (190)$$

and from Eq. (181)

$$\langle 0|\widehat{T}^{\Phi}{}_{\rho}{}^{\rho}|0\rangle_{\rm ren} = \frac{3}{8\pi^2}\left\{\frac{m^2}{Z^2}\,K_2(2mZ) + \frac{m^3}{Z}\,K_1(2mZ)\right\}. \qquad (191)$$

From Eq. (152) we can then see that the vacuum expectation value of the stress-energy-tensor operator associated with the vector field A_μ is such that

$$\langle 0|\widehat{T}^A_{\mu\nu}|0\rangle_{\rm ren} = 0, \qquad (192)$$

while the vacuum expectation value of the stress-energy-tensor operator associated with the auxiliary scalar field Φ is given by

$$\begin{aligned} \langle 0|\widehat{T}^\Phi_{\mu\nu}|0\rangle_{\rm ren} = {} & \frac{1}{8\pi^2}\left\{\frac{m^2}{Z^2}\,K_2(2mZ) + \frac{m^3}{Z}\,K_1(2mZ)\right\} \\ & \times\left(\eta_{\mu\nu} - \hat{Z}_\mu\hat{Z}_\nu\right). \end{aligned} \qquad (193)$$

Of course, the sum of these two contributions permits us to recover the result (173) of Sec. V C which is also the result obtained by Davies and Toms in the framework of de Broglie-Proca electromagnetism [50]. It is moreover interesting to note that the contribution (192) associated with the vector field A_μ and which has been artificially separated from the scalar field contribution (see Sec. IV C 4) vanishes identically for any value of the mass parameter m. This result coincides exactly with that obtained from Maxwell's theory (see also Ref. [50]).

VI. CONCLUSION

In the context of quantum field theory in curved spacetime and with possible applications to cosmology and to black hole physics in mind, the massive vector field is frequently studied. It should be, however, noted that, in this particular domain, it is its description via the de Broglie-Proca theory which is mostly considered and that there are very few works dealing with the Stueckelberg point of view (see, e.g., Refs. [59–64], but remark that these papers are restricted to de Sitter and anti-de Sitter spacetimes or to Roberstson-Walker backgrounds with spatially flat sections). In this article, in order to fill a void, we have developed the general formalism of the Stueckelberg theory on an arbitrary four-dimensional spacetime (quantum action, Feynman propagators, Ward identities, Hadamard representation of the Green functions), and we have particularly focussed on the aspects linked with the construction, for a Hadamard quantum state, of the expectation value of the renormalized stress-energy-tensor operator. It is important to note that we have given two alternative but equivalent expressions for this result. The first one has been obtained by eliminating from a Ward identity the contribution of the auxiliary scalar field Φ (the so-called Stueckelberg ghost [47]) and only involves state-dependent and geometrical quantities associated with the massive vector field A_μ [see Eq. (123)]. The other one involves contributions coming from both the massive vector field and the auxiliary Stueckelberg scalar field [see Eqs. (125)–(126)], and it has been constructed artificially in such a way that these two contributions are independently conserved and that, in the zero-mass limit, the massive vector field contribution reduces smoothly to the result obtained from Maxwell's electromagnetism. It is also important to note that, in Sec. IV E, we have discussed the geometrical ambiguities of the expectation value of the renormalized stress-energy-tensor operator. They are of fundamental importance (see, e.g., in Sec. V, their role in the context of the Casimir effet).

We intend to use our results in the near future in cosmology of the very early universe, but we hope they will be useful for other authors. This is why we shall now provide a step-by-step guide for the reader who is not especially interested by the technical details of our work but who wishes to calculate the expectation value of the renormalized stress-energy tensor from the expression (123), i.e., from the expression where any reference to the Stueckelberg auxiliary scalar field Φ has disappeared. We shall describe the calculation from the Feynman propagator as well as from the anticommutator function :

(i) We assume that the Feynman propagator $G^A_{\mu\nu'}(x,x')$ which is given by Eq. (22) and satisfies the wave equation (30) [or that the anticommutator $G^{(1)A}_{\mu\nu'}(x,x')$ which is given by Eq. (59) and satisfies the wave equation (60)] has been determined in a particular gravitational background and for a Hadamard quantum state. In other words, we consider that the Feynman propagator $G^A_{\mu\nu'}(x,x')$ can be represented in the Hadamard form (35) [or that the anticommutator $G^{(1)A}_{\mu\nu'}(x,x')$ can be represented in the Hadamard form (55)].

(ii) We need the regular part of the Feynman propagator $G^A_{\mu\nu'}(x,x')$ [or that of the anticommutator $G^{(1)A}_{\mu\nu'}(x,x')$] at order σ. To extract it, we subtract from the Feynman propagator $G^A_{\mu\nu'}(x,x')$ its singular part (48a) in order to obtain its regular part (48b) [or we subtract from the anticommutator $G^{(1)A}_{\mu\nu'}(x,x')$ its singular part (68a) in order to obtain its regular part (68b)]. We have then at our disposal the state-dependent Hadamard bivector $W^A_{\mu\nu'}(x,x')$. Here, it is important to note that we do not need the full expression of the singular part of the Green function considered, but we can truncate it by neglecting the terms vanishing faster than $\sigma(x,x')$ for $x' \to x$. As a consequence, we can construct the singular part (48a) [or the singular part (68a)] by using the covariant Taylor series expansion (A.9) of $\Delta^{1/2}$ up to order σ^2, the covariant Taylor series expansion (71a) of ${V_0^A}_{\mu\nu}$ up to order σ^1 [see Eqs. (72a)–(72c)] and the covariant Taylor series expansion (71b) of ${V_1^A}_{\mu\nu}$ up to order σ^0 [see Eq. (72d)].

(iii) Finally, by using Eqs. (78a)–(78c), we can construct the expectation value of the renormalized stress-energy tensor given by Eq. (123).

It is interesting to note that, in the literature concerning Stueckelberg electromagnetism, some authors only focus on the part of the action associated with the massive vector field A_μ and which is given by Eq. (13a) (see, e.g., Refs. [59, 62, 65]). Of course, this is sufficient because they are mainly interested, in the context of canonical quantization, by the determination of the Feynman propagator associated with this field. However, in order to calculate physical quantities, it is necessary to take into account the contribution of the auxiliary scalar field Φ. It cannot be discarded. This is very clear in the context of the construction of the renormalized stress-energy-tensor operator as we have shown in our article and remains true for any other physical quantity.

To conclude this article, we shall briefly compare the de Broglie-Proca and Stueckelberg formulations of massive electomagnetism and discuss the advantages of the Stueckelberg formulation over the de Broglie-Proca one. It is interesting to note the existence of a nice paper by Pitts [66], where de Broglie-Proca and Stueckelberg approaches of massive electromagnetism are discussed from a philosophical point of view based on the machinery of the Hamiltonian formalism (primary and secondary constraints, Poisson brackets, ...). Here, we adopt a more pragmatic point of view. We discuss the two formulations in light of the results obtained in our article. In our opinion:

(i) De Broglie-Proca and Stueckelberg approaches of massive electromagnetism are two faces of the same theory. Indeed, the transition from de Broglie-Proca to Stueckelberg theory is achieved via the

Stueckelberg trick (5) which permits us, by introducing an auxiliary scalar field Φ, to artificially restore Maxwell's gauge symmetry in massive electromagnetism, but, reciprocally, the transition from Stueckelberg to de Broglie-Proca theory is achieved by imposing the gauge choice $\Phi = 0$ [see Eq. (8)]. As a consequence, it is not really surprising to obtain the same result for the renormalized stress-energy-tensor operator associated with the Casimir effect (see Sec. V) when we consider this problem in the framework of the de Broglie-Proca and Stueckelberg formulations of massive electromagnetism. Indeed, we can expect that this remains true for any other quantum quantity.

(ii) However, we can note that with regularization and renormalization in mind, it is much more interesting to work in the framework of the Stueckelberg formulation of massive electromagnetism. Indeed, this permits us to have at our disposal the machinery of the Hadamard formalism which is not the case in the framework of the de Broglie-Proca formulation. Indeed, due to the constraint (4a), the Feynman propagator $G^{A}_{\mu\nu'}(x,x')$ associated with the vector field A_μ cannot be represented in the Hadamard form (35).

ACKNOWLEDGMENTS

We wish to thank Yves Décanini, Mohamed Ould El Hadj and Julien Queva for various discussions and the "Collectivité Territoriale de Corse" for its support through the COMPA project.

Appendix: Biscalars, bivectors and their covariant Taylor series expansions

Regularization and renormalization of quantum field theories in the Minkowski spacetime are most times based on the representation of Green functions in momentum space, and, in general, this greatly simplifies reasoning and calculations. The use of such a representation is not possible in an arbitrary gravitational background where the lack of symmetries as well as spacetime curvature prevent us from working within the framework of the Fourier transform. As a consequence, regularization and renormalization in curved spacetime are necessarily based on representations of Green functions in coordinate space, and, moreover, they require extensively the concepts of biscalars, bivectors and, more generally, bitensors. Thanks to the work of some mathematicians [67–70] and of DeWitt [23, 44, 56, 71] and coworkers [28, 29], we have at our disposal all the tools necessary to deal with this subject.

In this short Appendix, in order to make a self-consistent paper (i.e., to avoid the reader needing to consult the references mentioned above), we have gathered some important results which are directly related with the representations of Green functions in coordinate space and, more particularly, with the Hadamard representations of the Green functions appearing in Stueckelberg electromagnetism [see Eqs. (35), (37), (55) and (58)] which is the main subject of Sec. III and which plays a crucial role in Sec. IV. In particular, we define the geodetic interval $\sigma(x,x')$, the Van Vleck-Morette determinant $\Delta(x,x')$ and the bivector of parallel transport from x to x' denoted by $g_{\mu\nu'}(x,x')$ (see, e.g., Ref. [44]), and we moreover discuss the concept of covariant Taylor series expansions for biscalars and bivectors.

We first recall that $2\sigma(x,x')$ is the square of the geodesic distance between x and x' which satisfies

$$2\sigma = \sigma^{;\mu}\sigma_{;\mu}. \tag{A.1}$$

We have $\sigma(x,x') < 0$ if x and x' are timelike related, $\sigma(x,x') = 0$ if x and x' are null related and $\sigma(x,x') > 0$ if x and x' are spacelike related. We furthermore recall that $\Delta(x,x')$ is given by

$$\Delta(x,x') = -[-g(x)]^{-1/2}\det(-\sigma_{;\mu\nu'}(x,x'))[-g(x')]^{-1/2} \tag{A.2}$$

and satisfies the partial differential equation

$$\Box_x\sigma = 4 - 2\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \tag{A.3a}$$

as well as the boundary condition

$$\lim_{x'\to x}\Delta(x,x') = 1. \tag{A.3b}$$

The bivector of parallel transport from x to x' is defined by the partial differential equation

$$g_{\mu\nu';\rho}\sigma^{;\rho} = 0 \tag{A.4a}$$

and the boundary condition

$$\lim_{x'\to x} g_{\mu\nu'}(x,x') = g_{\mu\nu}(x). \tag{A.4b}$$

The Hadamard coefficients $V^A_n{}_{\mu\nu'}(x,x')$ and $W^A_n{}_{\mu\nu'}(x,x')$ introduced in Eq. (36) and which are bivectors involved in the Hadamard representation of the Green functions (35) and (55) or the Hadamard coefficients $V_n(x,x')$ and $W_n(x,x')$ introduced in Eq. (38) and which are biscalars involved in the Hadamard representation of the Green functions (37) and (58) cannot in general be determined exactly. They are solutions of the recursion relations (39a), (39b) and (40) or (43a), (43b) and (44), and, following DeWitt [44, 71], we can look for the solutions of these equations in the form of covariant Taylor series expansions for x' in the neighborhood of x. This is the method we use in Sec. III. The series defining the biscalars $V_n(x,x')$ and $W_n(x,x')$ can be written in the form

$$\begin{aligned} T(x,x') &= t(x) - t_{a_1}(x)\,\sigma^{;a_1}(x,x') + \frac{1}{2!}\,t_{a_1a_2}(x)\,\sigma^{;a_1}(x,x')\,\sigma^{;a_2}(x,x') \\ &\quad -\frac{1}{3!}\,t_{a_1a_2a_3}(x)\,\sigma^{;a_1}(x,x')\,\sigma^{;a_2}(x,x')\,\sigma^{;a_3}(x,x') \\ &\quad +\frac{1}{4!}\,t_{a_1a_2a_3a_4}(x)\,\sigma^{;a_1}(x,x')\,\sigma^{;a_2}(x,x')\,\sigma^{;a_3}(x,x')\,\sigma^{;a_4}(x,x') + \cdots . \end{aligned} \tag{A.5}$$

By construction, the coefficients $t_{a_1\dots a_p}(x)$ are symmetric in the exchange of the indices $a_1\dots a_p$, i.e., $t_{a_1\dots a_p}(x) = t_{(a_1\dots a_p)}(x)$, and, moreover, by requiring the symmetry of $T(x,x')$ in the exchange of x and x', i.e., $T(x,x') = T(x',x)$, the coefficients $t(x)$ and $t_{a_1\dots a_p}(x)$ with $p = 1,2,\dots$ are constrained. The symmetry of $T(x,x')$ permits us to express the odd coefficients of the covariant Taylor series expansion of $T(x,x')$ in terms of the even ones. We have for the odd coefficients of lowest orders (see, e.g., Ref. [72])

$$t_{a_1} = (1/2)\,t_{;a_1}, \tag{A.6a}$$
$$t_{a_1a_2a_3} = (3/2)\,t_{(a_1a_2;a_3)} - (1/4)\,t_{;(a_1a_2a_3)}. \tag{A.6b}$$

Similarly, the series defining the bivectors $V_n^A{}_{\mu\nu'}(x,x')$ and $W_n^A{}_{\mu\nu'}(x,x')$ can be written in the form

$$\begin{aligned} T_{\mu\nu}(x,x') &= g_\nu{}^{\nu'}(x,x')\,T_{\mu\nu'}(x,x') \\ &= t_{\mu\nu}(x) - t_{\mu\nu a_1}(x)\,\sigma^{;a_1}(x,x') \\ &\quad +\frac{1}{2!}\,t_{\mu\nu a_1a_2}(x)\,\sigma^{;a_1}(x,x')\,\sigma^{;a_2}(x,x') \\ &\quad -\frac{1}{3!}\,t_{\mu\nu a_1a_2a_3}(x)\,\sigma^{;a_1}(x,x')\,\sigma^{;a_2}(x,x')\,\sigma^{;a_3}(x,x') \\ &\quad +\cdots . \end{aligned} \tag{A.7}$$

By construction, the coefficients $t_{\mu\nu a_1\dots a_p}(x)$ are symmetric in the exchange of indices $a_1\dots a_p$, i.e., $t_{\mu\nu a_1\dots a_p}(x) = t_{\mu\nu(a_1\dots a_p)}(x)$, and by requiring the symmetry of $T_{\mu\nu'}(x,x')$ in the exchange of x and x', i.e., $T_{\mu\nu'}(x,x') = T_{\nu'\mu}(x',x)$, the coefficients $t_{\mu\nu}(x)$ and $t_{\mu\nu a_1\dots a_p}(x)$ with $p = 1,2,\dots$ are constrained. The symmetry of $T_{\mu\nu'}(x,x')$ permits us to express the coefficients of the covariant Taylor series expansion of $T_{\mu\nu'}(x,x')$ in terms of their symmetric and antisymmetric parts in μ and ν. We have for the coefficients of lowest orders (see, e.g., Refs. [35, 39])

$$t_{\mu\nu} = t_{(\mu\nu)}, \tag{A.8a}$$
$$t_{\mu\nu a_1} = (1/2)\,t_{(\mu\nu);a_1} + t_{[\mu\nu]a_1}, \tag{A.8b}$$
$$t_{\mu\nu a_1a_2} = t_{(\mu\nu)a_1a_2} + t_{[\mu\nu](a_1;a_2)}, \tag{A.8c}$$
$$\begin{aligned} t_{\mu\nu a_1a_2a_3} &= (3/2)\,t_{(\mu\nu)(a_1a_2;a_3)} \\ &\quad -(1/4)\,t_{(\mu\nu);(a_1a_2a_3)} + t_{[\mu\nu]a_1a_2a_3}. \end{aligned} \tag{A.8d}$$

In order to solve the recursion relations (39a), (39b), (40), (43a), (43b) and (44) but also to do most of the calculations in Secs. III and IV and, in particular, to obtain the explicit expression of the renormalized stress-energy-tensor operator, it is necessary to have at our disposal the covariant Taylor series expansions of the biscalars $\Delta^{1/2}$, $\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$ and $\Box\Delta^{1/2}$ and of the bivectors $\sigma_{;\mu\nu'}$ and $\Box g_{\mu\nu'}$ but also of some bitensors such as $\sigma_{;\mu\nu}$, $g_{\mu\nu';\rho}$ and $g_{\mu\nu';\rho'}$. Here, we provide these expansions up to the orders necessary in this article (for higher orders, see Refs. [72, 73]). We have

$$\begin{aligned} \Delta^{1/2} &= 1 + \frac{1}{12}\,R_{a_1a_2}\sigma^{;a_1}\sigma^{;a_2} - \frac{1}{24}\,R_{a_1a_2;a_3}\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} \\ &\quad + \left[\frac{1}{80}\,R_{a_1a_2;a_3a_4} + \frac{1}{360}\,R^p{}_{a_1qa_2}R^q{}_{a_3pa_4} + \frac{1}{288}\,R_{a_1a_2}R_{a_3a_4}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4} \\ &\quad - \left[\frac{1}{360}\,R_{a_1a_2;a_3a_4a_5} + \frac{1}{360}\,R^p{}_{a_1qa_2}R^q{}_{a_3pa_4;a_5} + \frac{1}{288}\,R_{a_1a_2}R_{a_3a_4;a_5}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3}\sigma^{;a_4}\sigma^{;a_5} + O(\sigma^3), \end{aligned} \tag{A.9}$$

$$\begin{aligned} \Box\Delta^{1/2} &= \frac{1}{6}\,R \\ &\quad + \left[\frac{1}{40}\,\Box R_{a_1a_2} - \frac{1}{120}\,R_{;a_1a_2} + \frac{1}{72}\,RR_{a_1a_2} - \frac{1}{30}\,R^p{}_{a_1}R_{pa_2} + \frac{1}{60}\,R^{pq}R_{pa_1qa_2} + \frac{1}{60}\,R^{pqr}{}_{a_1}R_{pqra_2}\right]\sigma^{;a_1}\sigma^{;a_2} \\ &\quad - \left[-\frac{1}{360}\,R_{;a_1a_2a_3} + \frac{1}{120}\,(\Box R_{a_1a_2})_{;a_3} + \frac{1}{144}\,RR_{a_1a_2;a_3} - \frac{1}{45}\,R^p{}_{a_1}R_{pa_2;a_3} + \frac{1}{180}\,R^p{}_{q;a_1}R^q{}_{a_2pa_3} \right. \\ &\qquad \left. +\frac{1}{180}\,R^p{}_qR^q{}_{a_1pa_2;a_3} + \frac{1}{90}\,R^{pqr}{}_{a_1}R_{pqra_2;a_3}\right]\sigma^{;a_1}\sigma^{;a_2}\sigma^{;a_3} + O(\sigma^2), \end{aligned} \tag{A.10}$$

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

$$\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} = \frac{1}{6} R_{a_1 a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}), \tag{A.11}$$

$$\sigma_{;\mu\nu} = g_{\mu\nu} - \frac{1}{3} R_{\mu a_1 \nu a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}), \tag{A.12}$$

$$g_\nu{}^{\nu'}\sigma_{;\mu\nu'} = -g_{\mu\nu} - \frac{1}{6} R_{\mu a_1 \nu a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}), \tag{A.13}$$

$$g_\nu{}^{\nu'} g_{\mu\nu';\rho} = -\frac{1}{2} R_{\mu\nu\rho a_1}\sigma^{;a_1} + \frac{1}{6} R_{\mu\nu\rho a_1;a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}), \tag{A.14}$$

$$g_\nu{}^{\nu'} g_\rho{}^{\rho'} g_{\mu\nu';\rho'} = -\frac{1}{2} R_{\mu\nu\rho a_1}\sigma^{;a_1} + \frac{1}{3} R_{\mu\nu\rho a_1;a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}) \tag{A.15}$$

and

$$g_\nu{}^{\nu'}\Box g_{\mu\nu'} = \frac{2}{3} R_{a_1[\mu;\nu]}\,\sigma^{a_1} + \left[-\frac{1}{6} R_{a_1[\mu;\nu]a_2} + \frac{1}{6} R_{\mu\nu p a_1} R^p{}_{a_2} - \frac{1}{4} R_{\mu p q a_1} R_\nu{}^{pq}{}_{a_2}\right]\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}). \tag{A.16}$$

[1] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).
[2] D. D. Ryutov, Plasma Phys. Control. Fusion **49**, B429 (2007).
[3] L.-C. Tu, J. Luo, and G. Gillies, Rept. Prog. Phys. **68**, 77 (2005).
[4] A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. **82**, 939 (2010).
[5] L. de Broglie, *Recherches sur la théorie des quantas - seconde thèse* (Université de Paris-Sorbonne, Paris, 1924).
[6] L. de Broglie, J. Phys. Radium **3**, 422 (1922).
[7] L. de Broglie, *Une nouvelle conception de la lumière.* (Hermann, Paris, 1934).
[8] L. de Broglie, *Nouvelles recherches sur la lumière.* (Hermann, Paris, 1936).
[9] L. de Broglie, *Une nouvelle théorie de la lumière, la mécanique ondulatoire du photon.* (Hermann, Paris, 1940).
[10] A. Proca, J. Phys. Radium **7**, 347 (1936).
[11] A. Proca and G. A. Proca, *Alexandre Proca 1897-1955: Oeuvre scientifique publiée.* (Editions Georges A. Proca, Paris, 1988).
[12] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 225 (1938).
[13] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 299 (1938).
[14] H. Ruegg and M. Ruiz-Altaba, Int. J. Mod. Phys. A **19**, 3265 (2004), arXiv:hep-th/0304245.
[15] J. Jaeckel and A. Ringwald, Ann. Rev. Nucl. Part. Sci. **60**, 405 (2010), arXiv:1002.0329 [hep-ph].
[16] R. Essig *et al.* (2013) arXiv:1311.0029 [hep-ph].
[17] H. An, M. Pospelov, and J. Pradler, Phys. Rev. Lett. **111**, 041302 (2013), arXiv:1304.3461 [hep-ph].
[18] J. Balewski *et al.* (2014) arXiv:1412.4717 [physics.ins-det].
[19] G. Eigen (BaBar Collaboration), J. Phys. Conf. Ser. **631**, 012034 (2015), arXiv:1503.02860 [hep-ex].
[20] S. V. Kuzmin and D. G. C. McKeon, Can. J. Phys. **80**, 767 (2002).
[21] I. L. Buchbinder, E. N. Kirillova, and N. G. Pletnev, Phys. Rev. D **78**, 084024 (2008), arXiv:0806.3505 [hep-th].
[22] C. de Rham, Living Rev. Rel. **17**, 7 (2014), arXiv:1401.4173 [hep-th].
[23] B. S. DeWitt, Phys. Rep. **19**, 295 (1975).
[24] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[25] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-time* (Cambridge University Press, Cambridge, England, 1989).
[26] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1995).
[27] L. E. Parker and D. J. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, Cambridge, England, 2009).
[28] S. M. Christensen, Phys. Rev. D **14**, 2490 (1976).
[29] S. M. Christensen, Phys. Rev. D **17**, 946 (1978).
[30] R. M. Wald, Commun. Math. Phys. **54**, 1 (1977).
[31] R. M. Wald, Phys. Rev. D **17**, 1477 (1978).
[32] S. L. Adler, J. Lieberman, and Y. J. Ng, Annals Phys. **106**, 279 (1977).
[33] S. L. Adler and J. Lieberman, Annals Phys. **113**, 294 (1978).
[34] M. A. Castagnino and D. D. Harari, Annals Phys. **152**, 85 (1984).
[35] M. R. Brown and A. C. Ottewill, Phys. Rev. D **34**, 1776 (1986).
[36] D. Bernard and A. Folacci, Phys. Rev. D **34**, 2286 (1986).
[37] S. Tadaki, Prog. Theor. Phys. **77**, 671 (1987).
[38] B. Allen, A. Folacci, and A. C. Ottewill, Phys. Rev. D **38**, 1069 (1988).
[39] A. Folacci, J. Math. Phys. **32**, 2813 (1991).
[40] Y. Decanini and A. Folacci, Phys. Rev. D **78**, 044025 (2008), arXiv:gr-qc/0512118.
[41] V. Moretti, Commun. Math. Phys. **232**, 189 (2003), arXiv:gr-qc/0109048.
[42] T.-P. Hack and V. Moretti, J. Phys. A **45**, 374019 (2012), arXiv:1202.5107 [gr-qc].
[43] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[44] B. S. DeWitt and R. W. Brehme, Annals Phys. **9**, 220 (1960).
[45] T. Kugo and I. Ojima, Prog. Theor. Phys. **60**, 1869

(1978).
[46] N. H. Barth and S. M. Christensen, Phys. Rev. D **28**, 1876 (1983).
[47] H. van Hees, (2003), arXiv:hep-th/0305076 [hep-th].
[48] W. Tichy and E. E. Flanagan, Phys. Rev. D **58**, 124007 (1998), arXiv:gr-qc/9807015 [gr-qc].
[49] S. Hollands and R. M. Wald, Rev. Math. Phys. **17**, 227 (2005), arXiv:gr-qc/0404074 [gr-qc].
[50] P. C. W. Davies and D. J. Toms, Phys. Rev. D **31**, 1363 (1985).
[51] P. C. W. Davies and S. D. Unwin, Phys. Lett. B **98**, 274 (1981).
[52] G. Barton and N. Dombey, Nature **311**, 336 (1984).
[53] G. Barton and N. Dombey, Annals Phys. **162**, 231 (1985).
[54] L. P. Teo, Phys. Rev. D **82**, 105002 (2010), arXiv:1007.4397 [quant-ph].
[55] L. P. Teo, Phys. Lett. B **696**, 529 (2011), arXiv:1012.2196 [quant-ph].
[56] B. S. DeWitt, *The Global Approach to Quantum Field Theory* (Oxford University Press, Oxford, 2003).
[57] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New-York, 1965).
[58] L. Bass and E. Schrödinger, Proceedings of the Royal Society of London Series A **232**, 1 (1955).
[59] H. Janssen and C. Dullemond, J. Math. Phys. **28**, 1023 (1987).
[60] L. P. Chimento and A. E. Cossarini, Phys. Rev. D **41**, 3101 (1990).
[61] N. C. Tsamis and R. P. Woodard, J. Math. Phys. **48**, 052306 (2007), arXiv:gr-qc/0608069.
[62] M. B. Fröb and A. Higuchi, J. Math. Phys. **55**, 062301 (2014), arXiv:1305.3421 [gr-qc].
[63] Ö. Akarsu, M. Arik, N. Katirci, and M. Kavuk, JCAP **1407**, 009 (2014), arXiv:1404.0892 [gr-qc].
[64] S. Kouwn, P. Oh, and C.-G. Park, (2015), arXiv:1512.00541 [gr-qc].
[65] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[66] J. B. Pitts, (2009), arXiv:0911.5400 [physics.hist-ph].
[67] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).
[68] A. Lichnerowicz, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **10**, 5 (1961).
[69] P. R. Garabedian, *Partial Differential Equations* (Wiley, New York, 1964).
[70] F. G. Friedlander, *The Wave Equation on a Curved Space-Time* (Cambridge University Press, Cambridge, England, 1975).
[71] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[72] Y. Decanini and A. Folacci, Phys. Rev. D **73**, 044027 (2006), arXiv:gr-qc/0511115.
[73] A. C. Ottewill and B. Wardell, Phys. Rev. D **84**, 104039 (2011), arXiv:0906.0005 [gr-qc].

Published in Phys. Rev. D 93, 044063 (2016), eprint : arXiv:1512.06326

Chapitre 3:

Electromagnétisme massif de Stueckelberg dans les espaces-temps de de Sitter et anti-de Sitter : fonctions à deux points et tenseur d'impulsion-énergie renormalisé

Dans ce chapitre, nous avons considéré de nouvelles applications de l'électromagnétisme massif de Stueckelberg, dont le formalisme a été développé dans le chapitre précédent. Nous avons travaillé dans les espaces-temps maximalement symétriques de de Sitter et anti-de Sitter. Le fait d'avoir la symétrie maximale nous permet de construire analytiquement toutes les fonctions de corrélations à deux points associées à l'électromagnétisme de Stueckelberg. A partir des fonctions de Hadamard, nous avons obtenu les expressions analytiques de la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé au champ vectoriel massif en espaces-temps de de Sitter et anti-de Sitter. Des résultats analogues étaient déjà connus pour le champ scalaire et pour le champ de Dirac mais aucun n'avait été obtenu pour le champ vectoriel massif.

Stueckelberg massive electromagnetism in de Sitter and anti-de Sitter spacetimes: Two-point functions and renormalized stress-energy tensors

Andrei Belokogne,* Antoine Folacci,† and Julien Queva‡
Equipe Physique Théorique - Projet COMPA, SPE, UMR 6134 du CNRS et de l'Université de Corse, Université de Corse, BP 52, F-20250 Corte, France
(Dated: December 8, 2016)

By considering Hadamard vacuum states, we first construct the two-point functions associated with Stueckelberg massive electromagnetism in de Sitter and anti-de Sitter spacetimes. Then, from the general formalism developed in [A. Belokogne and A. Folacci, Phys. Rev. D **93**, 044063 (2016)], we obtain an exact analytical expression for the vacuum expectation value of the renormalized stress-energy tensor of the massive vector field propagating in these maximally symmetric spacetimes.

I. INTRODUCTION

In a recent article, we discussed the covariant quantization of Stueckelberg massive electromagnetism on an arbitrary four-dimensional curved spacetime $(\mathcal{M}, g_{\mu\nu})$ with no boundary and we constructed, for Hadamard quantum states, the expectation value of the renormalized stress-energy tensor (RSET) [1]. Here, we do not return to the motivations leading us to consider Stueckelberg massive electromagnetism in curved spacetime. The interested reader is invited to consult our previous article, in particular its introduction as well as references therein. The formalism developed in Ref. [1] permitted us to discuss, as an application, the Casimir effect outside a perfectly conducting medium with a plane boundary.

In the present paper, we shall address a much more difficult problem which could have interesting implications in cosmology of the very early Universe or in the context of the AdS/CFT correspondence: we shall obtain an exact analytical expression for the vacuum expectation value of the RSET of the massive vector field propagating in de Sitter and anti-de Sitter spacetimes. It is interesting to note that such results do not exist in the literature while the RSETs associated with the massive scalar field and the massive spinor field have been obtained quite a long time ago (see, e.g., Refs. [2–8] for the case of the massive scalar field and Refs. [9–11] for the case of the massive spinor field). In fact, this void in the literature can easily be explained. Indeed, even if there exist numerous works concerning the massive vector field in de Sitter and anti-de Sitter spacetimes [12–22], the two-point functions are in general constructed in the framework of the de Broglie-Proca theory and, as a consequence, do not display the usual Hadamard singularity (see the last remark in the conclusion of Ref. [1]) which is a fundamental ingredient of regularization and renormalization techniques in curved spacetime.

In our article, we shall focus on Stueckelberg electromagnetism defined, at the quantum level, by the action [1, 23]

$$S\left[A_\mu, \Phi, C, C^*, g_{\mu\nu}\right] = S_A\left[A_\mu, g_{\mu\nu}\right] + S_\Phi\left[\Phi, g_{\mu\nu}\right] + S_{\text{Gh}}\left[C, C^*, g_{\mu\nu}\right], \tag{1}$$

where

$$S_A = \int_{\mathcal{M}} d^4x \sqrt{-g}\left[-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\, m^2 A^\mu A_\mu - \frac{1}{2}\left(\nabla^\mu A_\mu\right)^2\right] \tag{2}$$

denotes the action associated with the massive vector field A_μ with mass m (here, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the associated field strength) and

$$S_\Phi = \int_{\mathcal{M}} d^4x \sqrt{-g}\left(-\frac{1}{2}\,\nabla^\mu\Phi\nabla_\mu\Phi - \frac{1}{2}\, m^2\Phi^2\right) \tag{3}$$

is the action governing the auxiliary Stueckelberg scalar field Φ. The last action term in Eq. (1) is the compensating ghost contribution given by

$$S_{\text{Gh}} = \int_{\mathcal{M}} d^4x \sqrt{-g}\left(\nabla^\mu C^* \nabla_\mu C + m^2 C^* C\right), \tag{4}$$

where C and C^* are two fermionic ghost fields. Here, it is important to note that some authors dealing with Stueckelberg electromagnetism (see, e.g., Refs. [16, 21, 24]) have considered Stueckelberg electromagnetism defined from the sole action

$$S_A = \int_{\mathcal{M}} d^4x \sqrt{-g}\left[-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\, m^2 A^\mu A_\mu - \frac{1}{2\xi}\left(\nabla^\mu A_\mu\right)^2\right], \tag{5}$$

where ξ is a gauge parameter. In fact, these authors were mainly interested by the determination of the Feynman propagator associated with the massive vector field A_μ. Of course, in order to calculate physical quantities such as the RSET associated with Stueckelberg electromagnetism, the full action must be considered, i.e., it is necessary to take also into account, in addition to the contribution of the massive vector field, those of the auxiliary Stueckelberg field and of the ghost fields. In our

* belokogne.andrei@gmail.com
† folacci@univ-corse.fr
‡ queva@univ-corse.fr

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

article, we shall not consider the case of an arbitrary gauge parameter ξ. Indeed, as we have already noted in Ref. [1], if we want to work with Hadamard quantum states, it is necessary to take $\xi = 1$. However, it is interesting to recall that Fröb and Higuchi in a recent article [21] have provided, for an arbitrary value of ξ, a mode-sum construction of the two-point functions for the massive vector field by working in the Poincaré patch of de Sitter space. Their results have permitted them to recover, as particular cases, the two-point functions obtained by Allen and Jacobson in Ref. [15] (they correspond to $\xi \to \infty$, i.e., to the de Broglie-Proca theory) and those obtained by Tsamis and Woodard in Ref. [19] (they correspond to $\xi \to 0$).

Our article is organized as follows. In Sec. II, we construct the Wightman functions associated with the massive vector field A_μ, the Stueckelberg auxiliary scalar field Φ and the ghost fields C and C^* and, from them, we deduce by analytic continuation all the other two-point functions. We do not use a mode-sum construction as in Ref. [21], but we extend the approach of Allen and Jacobson in Ref. [15] (see also Refs. [16, 25, 26]). More precisely, by assuming that the vacuum is a maximally symmetric quantum state, we solve the wave equations for the various Wightman functions involved by taking into account, as constraints, two Ward identities; we then fix the remaining integration constants by imposing (i) Hadamard-type singularities at short distance and (ii) in de Sitter spacetime, the regularity of the solutions at the antipodal point or (iii) in anti-de Sitter spacetime, that the solutions fall off as fast as possible at spatial infinity. In Sec. III, from the general formalism developed in Ref. [1], we obtain an exact analytical expression for the vacuum expectation value of the RSET of the massive vector field propagating in de Sitter and anti-de Sitter spacetimes. The geometrical ambiguities are fixed by considering the flat-space limit and, moreover, we consider the two alternative but equivalent expressions for the renormalized expectation value given in Ref. [1] in order to discuss the zero-mass limit of our results. Finally, in a conclusion (Sec. IV), we briefly consider the possible extension of our work to Stueckelberg electromagnetism in an arbitrary ξ gauge.

It should be noted that, in this article, we use units such that $\hbar = c = G = 1$ and the geometrical conventions of Hawking and Ellis [27] concerning the definitions of the scalar curvature R, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$ as well as the commutation of covariant derivatives. Moreover, we will frequently refer to our previous article [1] and we assume that the reader has "in hand" a copy of it.

II. TWO-POINT FUNCTIONS OF STUECKELBERG ELECTROMAGNETISM

In this section, we shall construct the various two-point functions involved in Stueckelberg massive electromagnetism from the Wightman functions associated with the vector field A_μ, the Stueckelberg scalar field Φ and the ghost fields C and C^*.

A. de Sitter and anti-de Sitter spacetimes

Here, we gather some results concerning (i) the geometry of the four-dimensional de Sitter spacetime (dS^4) and the four-dimensional anti-de Sitter spacetime (AdS^4) as well as (ii) the properties of some geometrical objects defined on these maximally symmetric gravitational backgrounds. We have minimized the information on these topics (for more details and proofs, see Refs. [15, 26–29]). Those results are necessary to construct the Wightman functions of Stueckelberg electromagnetism and, in Sec. III, will permit us to simplify in dS^4 and AdS^4 the formalism developed in Ref. [1].

dS^4 and AdS^4 are maximally symmetric spacetimes of constant scalar curvature (positive for the former and negative for the latter) which are locally characterized by the relations

$$R_{\mu\nu\rho\tau} = (R/12)\,(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho})\,, \tag{6a}$$
$$R_{\mu\nu} = (R/4)\,g_{\mu\nu}, \tag{6b}$$

and

$$R = \begin{cases} +12H^2 & \text{for dS}^4, \\ -12K^2 & \text{for AdS}^4. \end{cases} \tag{6c}$$

Here H and K are two positive constants of dimension (length)$^{-1}$. The relations (6a)–(6c) are useful in order to simplify the various covariant Taylor series expansions involved in the Hadamard renormalization process (see Ref. [1]) and will be extensively used in Sec. III.

dS^4 and AdS^4 can be realized as the four-dimensional hyperboloids

$$\eta_{ab}X^aX^b = 12/R \tag{7}$$

embedded in the flat five-dimensional space $\mathbb{R}^5$ equipped with the metric

$$\eta_{ab} = \begin{cases} \text{diag}(-1,+1,+1,+1,+1) & \text{for dS}^4, \\ \text{diag}(-1,-1,+1,+1,+1) & \text{for AdS}^4. \end{cases} \tag{8}$$

Equations (7) and (8) make it obvious that $O(1,4)$ is the symmetry group of dS^4 and that its topology is that of $\mathbb{R} \times S^3$, while $O(2,3)$ is the symmetry group of AdS^4 whose topology is that of $S^1 \times \mathbb{R}^3$. It is important to recall that, in order to avoid closed timelike curves in AdS^4, it is necessary to "unwrap" the circle S^1 to go onto its universal covering space $\mathbb{R}^1$ and then AdS^4 has the topology of $\mathbb{R}^4$.

In the context of field theories in curved spacetime, the geodetic interval $\sigma(x,x')$, defined as one-half the square of the geodesic distance between the points x and x', is of fundamental interest (see, e.g., Refs. [30, 31]) and the

Hadamard renormalization process developed in Ref. [1], which we will exploit in Sec. III, is based on an extensive use of this geometrical object. However, in this section, even though $\sigma(x,x')$ is invariant under the symmetry group of dS^4 or AdS^4, it is advantageous to consider instead the real quadratic form

$$z(x,x') = \frac{1}{2}\left[1 + (R/12)\,\eta_{ab} X^a(x) X^b(x')\right] \qquad (9)$$

in order to construct the two-point functions of Stueckelberg electromagnetism. In Eq. (9), $X^a(x)$ and $X^b(x')$ are the coordinates of the points x and x' on the hyperboloid (7) defining dS^4 or AdS^4 and η_{ab} is the corresponding metric given by (8). This quadratic form is obviously invariant under the symmetry group of dS^4 or AdS^4 and is moreover defined on the whole spacetime, while $\sigma(x,x')$ is not defined everywhere because there is not always a geodesic between two arbitrary points in these maximally symmetric spacetimes. However, when $\sigma(x,x')$ is defined, we have

$$z(x,x') = \frac{1}{2}\left[1 + \cos\sqrt{(R/6)\,\sigma(x,x')}\right] \qquad (10a)$$
$$= \cos^2\sqrt{(R/24)\,\sigma(x,x')} \qquad (10b)$$

or, equivalently,

$$z(x,x') = \cos^2\sqrt{(H^2/2)\,\sigma(x,x')} \qquad (11)$$

in dS^4 and

$$z(x,x') = \cosh^2\sqrt{(K^2/2)\,\sigma(x,x')} \qquad (12)$$

in AdS^4. The previous relations can be inverted and used to define $\sigma(x,x')$ globally. In fact, it will chiefly help us, in Sec. III, to reexpress the two-point functions obtained here in terms of $\sigma(x,x')$.

We shall now point out some useful properties of $z(x,x')$. With respect to the antipodal transformation which sends the point x with coordinates $X^a(x)$ on the hyperboloid (7) to its antipodal point Px with coordinates

$$X^a(Px) = -X^a(x), \qquad (13)$$

we have

$$z(x,Px') = 1 - z(x,x'). \qquad (14)$$

We have also

$$z(x,x') > 1 \text{ if } x \text{ and } x' \text{ are timelike related,} \qquad (15a)$$
$$z(x,x') = 1 \text{ if } x \text{ and } x' \text{ are null related,} \qquad (15b)$$
$$0 < z(x,x') < 1 \text{ if } x \text{ and } x' \text{ are spacelike related,} \qquad (15c)$$
$$z(x,x') \leq 0 \text{ if } x \text{ and } x' \text{ cannot be joined by a geodesic,} \qquad (15d)$$

in dS^4 and

$$z(x,x') > 1 \text{ if } x \text{ and } x' \text{ are spacelike related,} \qquad (16a)$$

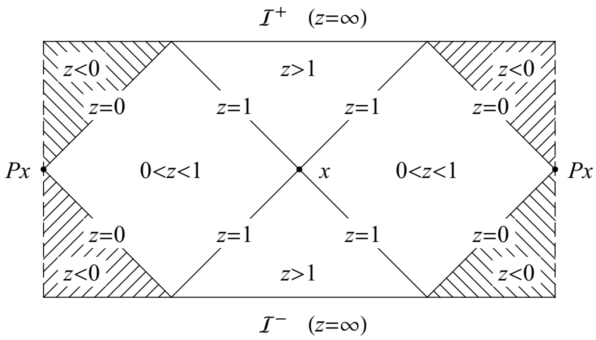

FIG. 1. Carter-Penrose diagram of dS^4. Without loss of generality, the point x can be taken to be any point in spacetime. Px denotes its associated antipodal point. The left and right edges of this diagram must be identified along the dashed lines. $\mathcal{I}^+$ and $\mathcal{I}^-$ denote, respectively, the future and past spacelike infinities for timelike and null geodesics. The hatched area is the set of points x' which cannot be reached by geodesics from x and for which $\sigma(x,x')$ is not defined.

$$z(x,x') = 1 \text{ if } x \text{ and } x' \text{ are null related,} \qquad (16b)$$
$$0 < z(x,x') < 1 \text{ if } x \text{ and } x' \text{ are timelike related,} \qquad (16c)$$
$$z(x,x') \leq 0 \text{ if } x \text{ and } x' \text{ cannot be joined by a geodesic,} \qquad (16d)$$

in AdS^4. All these results can be visualized in the Carter-Penrose diagrams of dS^4 (see Fig. 1) and AdS^4 (see Fig. 2).

To conclude this subsection, we recall that in dS^4 and AdS^4, any bitensor which is invariant under the spacetime symmetry group (a maximally symmetric tensor) can be expressed only in terms of the bitensors z, $z_{;\mu}$, $z_{;\mu'}$, $g_{\mu\nu}$, $g_{\mu'\nu'}$ and $g_{\mu\nu'}$ [15]. Here, $g_{\mu\nu'}$ is the usual bivector of parallel transport from x to x' which is defined by the differential equation $g_{\mu\nu';\rho}\sigma^{;\rho} = 0$ and the boundary condition $\lim_{x'\to x} g_{\mu\nu'}(x,x') = g_{\mu\nu}(x)$. For example, a two-point function $G(x,x')$ associated with a scalar field is necessarily of the form $G(z)$ and a two-point function $G_{\mu\nu'}(x,x')$ associated with a vector field can be written in the form $A(z)g_{\mu\nu'} + B(z)z_{;\mu}z_{;\nu'}$. This very important remark will simplify the construction of the two-point functions of Stueckelberg electromagnetism. Besides, in order to handle these functions in connection with the wave equations and the Ward identities, it will be necessary to use the following geometrical relations

$$z_{;\mu}z^{;\mu} = (R/12)\,z(1-z), \qquad (17a)$$
$$g_{\mu\nu';\rho}z^{;\rho} = 0, \qquad (17b)$$
$$g_{\mu\nu'}z^{;\mu} = -z_{;\nu'}, \qquad (17c)$$
$$g_{\mu\nu'}z^{;\nu'} = -z_{;\mu}, \qquad (17d)$$
$$z_{;\mu\nu} = (R/24)\,(1-2z)g_{\mu\nu}, \qquad (17e)$$
$$z_{;\mu\nu'} = (R/24)\,g_{\mu\nu'} + (1/z)\,z_{;\mu}z_{;\nu'}, \qquad (17f)$$
$$g_{\mu\nu';\rho} = -(1/z)\,(g_{\mu\rho}z_{;\nu'} + g_{\rho\nu'}z_{;\mu}), \qquad (17g)$$
$$g_{\mu\nu';\rho'} = -(1/z)\,(g_{\mu\rho'}z_{;\nu'} + g_{\nu'\rho'}z_{;\mu}), \qquad (17h)$$

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

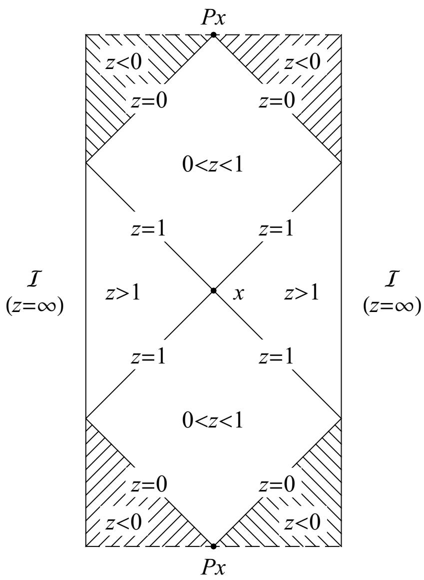

FIG. 2. Carter-Penrose diagram of AdS^4. Without loss of generality, the point x can be taken to be any point in spacetime. Px denotes its associated antipodal point. Here, we only display an elementary cell of the universal covering of anti-de Sitter spacetime. The whole manifold is obtained by gluing along the dashed lines the top edge of one cell with the bottom edge of another one and by replicating, ad nauseam, this process. $\mathcal{I}$ denotes timelike infinity. The hatched area is the set of points x' which cannot be reached by geodesics from x and for which $\sigma(x,x')$ is not defined.

$$g_{\mu\nu';\rho}{}^{;\rho} = (R/12)\left[(z-1)/z\right] g_{\mu\nu'} + (2/z^2)\, z_{;\mu} z_{;\nu'}, \quad \text{(17i)}$$

and to note that the d'Alembertian operator acting on biscalar functions is given by

$$\Box = \frac{R}{12}\left[z(1-z)\,\frac{d^2}{dz^2} + (2-4z)\,\frac{d}{dz}\right]. \quad (18)$$

B. Stueckelberg theory, wave equations and Ward identities for the Wightman functions

From the quantum action (1), we can easily obtain the wave equations satisfied by the massive vector field A_μ, the auxiliary scalar field Φ and the ghost fields C and C^*. They have been derived in our previous article [see Eqs. (19)–(21) in Ref. [1]] and, in dS^4 or in AdS^4, due to Eq. (6b), they reduce to

$$\left(\Box - R/4 - m^2\right) A_\mu = 0, \quad \text{(19a)}$$
$$\left(\Box - m^2\right) \Phi = 0, \quad \text{(19b)}$$
$$\left(\Box - m^2\right) C = 0, \quad \text{(19c)}$$
$$\left(\Box - m^2\right) C^* = 0. \quad \text{(19d)}$$

From now on, we shall assume that the Stueckelberg theory is quantized in a normalized vacuum state $|0\rangle$ and, in addition, that this quantum state is (i) maximally symmetric and (ii) of Hadamard type. We recall that, in the context of the calculation of the renormalized expectation value of the stress-energy-tensor operator with respect to a vacuum $|0\rangle$, it is convenient to work with the Feynman propagators or, equivalently, with the so-called Hadamard Green functions associated with the fields of the theory [1]. So, we need in dS^4 and AdS^4 the explicit expressions of these two-point functions. In fact, it is possible to construct the zoo of the two-point functions of the theory from the Wightman functions and, in a first step, we shall focus on these particular two-point functions.

We recall that the Wightman function associated with the massive vector field A_μ is given by

$$G^{(+)A}_{\mu\nu'}(x,x') = \langle 0|A_\mu(x)A_{\nu'}(x')|0\rangle \quad (20)$$

and satisfies the wave equation [see also Eq. (19a)]

$$\left(\Box_x - R/4 - m^2\right) G^{(+)A}_{\mu\nu'}(x,x') = 0. \quad (21)$$

Similarly, the Wightman function associated with the auxiliary scalar field Φ is given by

$$G^{(+)\Phi}(x,x') = \langle 0|\Phi(x)\Phi(x')|0\rangle \quad (22)$$

and is a solution of [see also Eq. (19b)]

$$\left(\Box_x - m^2\right) G^{(+)\Phi}(x,x') = 0, \quad (23)$$

while the Wightman function associated with the ghost fields is defined by

$$G^{(+)\mathrm{Gh}}(x,x') = \langle 0|C^*(x)C(x')|0\rangle \quad (24)$$

and satisfies the wave equation [see also Eq. (19d)]

$$\left(\Box_x - m^2\right) G^{(+)\mathrm{Gh}}(x,x') = 0. \quad (25)$$

Moreover, we have two Ward identities that relate these three Wightman functions. We can write

$$\nabla^\mu G^{(+)A}_{\mu\nu'}(x,x') + \nabla_{\nu'} G^{(+)\mathrm{Gh}}(x,x') = 0 \quad (26)$$

and

$$G^{(+)\Phi}(x,x') - G^{(+)\mathrm{Gh}}(x,x') = 0. \quad (27)$$

It should be noted that, in our previous article, we have derived the Ward identities for the Feynman propagators [see Eqs. (33) and (29) in Ref. [1]] and for the Hadamard Green functions [see Eqs. (65) and (66) in Ref. [1]]. Here we have written them for the Wightman functions. They can be derived in the same way, i.e., from the wave equations by using arguments of uniqueness [30]. The Ward identity (27) expresses the equality of the Wightman functions associated with the auxiliary scalar field and the ghost fields. So, thereafter we shall omit the labels Φ and Gh and use a generic form for these two Green functions by writing

$$G^{(+)}(x,x') = G^{(+)\Phi}(x,x') = G^{(+)\mathrm{Gh}}(x,x'). \quad (28)$$

C. Explicit expression for the Wightman functions in dS^4 and AdS^4

1. General form of the Wightman functions in maximally symmetric backgrounds

We have previously assumed that the vacuum $|0\rangle$ is a maximally symmetric state. As a consequence, we can express the Wightman functions $G^{(+)A}_{\mu\nu'}(x,x')$ and $G^{(+)}(x,x')$ as a function of the quadratic form $z(x,x')$ [see also the last paragraph of the subsection II A] and write

$$G^{(+)}(x,x') = G^{(+)}(z) \tag{29}$$

for the scalar Wightman functions (22) and (24) and

$$G^{(+)A}_{\mu\nu'}(x,x') = G^{(+)A}_{\mu\nu'}(z) = \alpha(z)\, g_{\mu\nu'} + 4\,\beta(z)\, z_{;\mu} z_{;\nu'} \tag{30}$$

for the vector Wightman function (20).

By inserting (29) into the wave equation (23) or (25) and by taking into account the relations (17) and (18), we obtain the differential equation

$$\left[z(1-z)\,\frac{d^2}{dz^2} + (2-4z)\,\frac{d}{dz} - \frac{12m^2}{R}\right] G^{(+)}(z) = 0. \tag{31}$$

Similarly, inserting (30) into the wave equation (21) leads to a system of two coupled differential equations for the functions $\alpha(z)$ and $\beta(z)$ given by

$$\left[z(1-z)\,\frac{d^2}{dz^2} + (2-4z)\,\frac{d}{dz} - \frac{2z+1}{z} - \frac{12m^2}{R}\right]\alpha(z) + \frac{R}{6}\,(1-2z)\,\beta(z) = 0 \tag{32a}$$

and

$$\left[z(1-z)\,\frac{d^2}{dz^2} + (4-8z)\,\frac{d}{dz} - \frac{10z-1}{z} - \frac{12m^2}{R}\right]\beta(z) + \frac{6}{R}\,\frac{1}{z^2}\,\alpha(z) = 0, \tag{32b}$$

while, from the Ward identity (26), we obtain

$$\left[\frac{d}{dz} + \frac{3}{z}\right]\alpha(z) - \left[\frac{R}{3}\,z(1-z)\,\frac{d}{dz} + \frac{5R}{6}\,(1-2z)\right]\beta(z) - \frac{d}{dz}G^{(+)}(z) = 0. \tag{33}$$

Equation (31) is a hypergeometric differential equation of the form [32–34]

$$\left[z(1-z)\,\frac{d^2}{dz^2} + [c-(a+b+1)z]\,\frac{d}{dz} - ab\right] F(a,b;c;z) = 0 \tag{34}$$

with $a = 3/2+\kappa$, $b = 3/2-\kappa$ and $c = 2$ where

$$\kappa = \sqrt{9/4 - 12\,m^2/R}. \tag{35}$$

This equation is invariant under the transformation $z \to 1-z$ because the parameters a, b and c satisfy $a+b+1 = 2c$. As a consequence, we can write

$$G^{(+)}(x,x') = C^1_G\, F(3/2+\kappa, 3/2-\kappa; 2; z) + C^2_G\, F(3/2+\kappa, 3/2-\kappa; 2; 1-z) \tag{36}$$

where C^1_G and C^2_G are two integration constants.

The differential equations (32a) and (32b) which provide the Wightman function (30) are much more complicated to solve. In order to do so, we introduce the new function $\gamma(z)$ defined by

$$\gamma(z) = \alpha(z) - (R/3)\,z(1-z)\,\beta(z), \tag{37}$$

and rewrite the Ward identity (33) in the form

$$\left[\frac{d}{dz} + \frac{3}{z}\right]\gamma(z) + \frac{R}{2}\,\beta(z) - \frac{d}{dz}G^{(+)}(z) = 0. \tag{38}$$

Then, by using Eqs. (37) and (38), we can combine the differential equations (32a) and (32b) and we have

$$\left[z(1-z)\,\frac{d^2}{dz^2} + (3-6z)\,\frac{d}{dz} - 6 - \frac{12m^2}{R}\right]\gamma(z) = (1-2z)\,\frac{d}{dz}G^{(+)}(z). \tag{39}$$

The general solution $\gamma(z)$ of this nonhomogeneous hypergeometric differential equation is the sum

$$\gamma(z) = \gamma_c(z) + \gamma_p(z) \tag{40}$$

of the complementary solution $\gamma_c(z)$ and a particular solution $\gamma_p(z)$. $\gamma_c(z)$ is the general solution of the hypergeometric differential equation (34) with $a = 5/2+\lambda$, $b = 5/2-\lambda$ and $c = 3$ where

$$\lambda = \sqrt{1/4 - 12\,m^2/R}. \tag{41}$$

Since the coefficients a, b and c fulfill again $a+b+1 = 2c$, we can therefore write

$$\gamma_c(z) = C^1_\gamma\, F(5/2+\lambda, 5/2-\lambda; 3; z) + C^2_\gamma\, F(5/2+\lambda, 5/2-\lambda; 3; 1-z) \tag{42}$$

where C^1_γ and C^2_γ are two new integration constants. Furthermore, it is rather easy to check that it is possible to take as a particular solution of (39)

$$\gamma_p(z) = \frac{1}{9/4-\kappa^2} \times \left[z(1-z)\,\frac{d^2}{dz^2} + (1/2)\,(1-2z)\,\frac{d}{dz}\right] G^{(+)}(z). \tag{43}$$

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

We have now at our disposal the general solution of the nonhomogeneous differential equation (39). This permits us to establish the expression of the Wightman function (30) by determining $\beta(z)$ from Eq. (38) and then $\alpha(z)$ from Eq. (37). After a long but straightforward calculation using systematically, in order to remove higher-order derivatives, the differential relation [32–34]

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z) \tag{44}$$

as well as the hypergeometric differential equation (34), we obtain

$$\begin{aligned}
G^{(+)A}_{\mu\nu'}(x,x') = \Big[&-(2/3)C^1_\gamma\, z(1-z)\, F'(5/2+\lambda,5/2-\lambda;3;z) + (2/3)C^2_\gamma\, z(1-z)\, F'(5/2+\lambda,5/2-\lambda;3;1-z) \\
&- C^1_\gamma\,(1-2z)\, F(5/2+\lambda,5/2-\lambda;3;z) - C^2_\gamma\,(1-2z)\, F(5/2+\lambda,5/2-\lambda;3;1-z) \\
&- (1/4)C^1_G\, F(5/2+\kappa,5/2-\kappa;3;z) + (1/4)C^2_G\, F(5/2+\kappa,5/2-\kappa;3;1-z)\Big]\, g_{\mu\nu'} \\
+\frac{8}{R}\Big[&- C^1_\gamma\, F'(5/2+\lambda,5/2-\lambda;3;z) + C^2_\gamma\, F'(5/2+\lambda,5/2-\lambda;3;1-z) \\
&- 3C^1_\gamma\,(1/z)\, F(5/2+\lambda,5/2-\lambda;3;z) - 3C^2_\gamma\,(1/z)\, F(5/2+\lambda,5/2-\lambda;3;1-z) \\
&- (3/4)C^1_G\, F'(5/2+\kappa,5/2-\kappa;3;z) - (3/4)C^2_G\, F'(5/2+\kappa,5/2-\kappa;3;1-z) \\
&- (3/4)C^1_G\,(1/z)\, F(5/2+\kappa,5/2-\kappa;3;z) + (3/4)C^2_G\,(1/z)\, F(5/2+\kappa,5/2-\kappa;3;1-z)\Big]\, z_{;\mu} z_{;\nu'}.
\end{aligned} \tag{45}$$

It is important to note that, in Eq. (45) as well in the following of the article, the derivative of the hypergeometric function $F(a,b;c;z)$ with respect to its argument z is denoted by $F'(a,b;c;z)$.

In summary, the general form of the Wightman function associated with the massive vector field A_μ is explicitly given by Eq. (45) while the Wightman function associated with the scalar field Φ and the ghost fields C and C^* is explicitly given by Eq. (36). In the following subsections, we shall fix the integration constants C^1_G, C^2_G, C^1_γ and C^2_γ appearing in the expression of these two-point functions.

2. *Wightman functions for Hadamard vacua*

Previously, we have assumed that the vacuum $|0\rangle$ of Stueckelberg electromagnetism is of Hadamard type. Here, we shall consider that this assumption can be realized by imposing that, at short distance, i.e. for $x' \to x$, the Wightman function (20) associated with the massive vector field A_μ and the Wightman function (28) associated with both the scalar field Φ and the ghost fields C and C^* satisfy

$$g^{\mu\nu'} G^{(+)A}_{\mu\nu'}(x,x') \underset{x'\to x}{\sim} \frac{1}{2\pi^2}\frac{1}{\sigma(x,x')} \tag{46}$$

and

$$G^{(+)}(x,x') \underset{x'\to x}{\sim} \frac{1}{8\pi^2}\frac{1}{\sigma(x,x')}. \tag{47}$$

It should be noted that, at first sight, the conditions (46) and (47) are less constraining than assuming that the Feynman propagators associated with all the fields of the theory can be represented in the Hadamard form [1]. (Without loss of generality, in this discussion, we focus on Feynman propagators but it would be possible to consider, equivalently, Hadamard Green functions.) Indeed, this last assumption provides stronger constraints on the geometrical coefficients of the singular terms in $1/[\sigma(x,x')+i\epsilon]$ and $\ln[\sigma(x,x')+i\epsilon]$ of the Hadamard representations of the Feynman propagators. In fact, due to the choice $\xi = 1$ of the gauge parameter, we know that the two-point functions of the Stueckelberg theory can be represented in the Hadamard form and, as a consequence, if we fix the dominant term of the coefficient of $1/[\sigma(x,x')+i\epsilon]$, all the other terms are unambiguously determined [see the differential equation (A3a) and the boundary condition (A3b) as well as the recursion relations (39a), (39b), (43a) and (43b) in Ref. [1]].

By inserting (10a) or (10b) into (45) and (36), we obtain the short distance expansions

$$\begin{aligned}
g^{\mu\nu'} G^{(+)A}_{\mu\nu'}(x,x') = &\left[-\frac{3072\, C^1_\gamma}{\Gamma(5/2+\lambda)\Gamma(5/2-\lambda)} - \frac{2304\, C^1_G}{\Gamma(5/2+\kappa)\Gamma(5/2-\kappa)}\right]\frac{1}{R\sigma^2(x,x')} \\
&+\left[-\frac{32\,(85/4-\lambda^2)\, C^1_\gamma}{\Gamma(5/2+\lambda)\Gamma(5/2-\lambda)} - \frac{72\,(19/4+\kappa^2)\, C^1_G}{\Gamma(5/2+\kappa)\Gamma(5/2-\kappa)}\right]\frac{1}{R\sigma(x,x')} + \underset{x'\to x}{O}[\ln\sigma(x,x')]
\end{aligned} \tag{48}$$

and

$$G^{(+)}(x,x') = \frac{24\, C^1_G}{\Gamma(3/2+\kappa)\Gamma(3/2-\kappa)} \, \frac{1}{R\sigma(x,x')} + \underset{x'\to x}{O}\, [\ln \sigma(x,x')] \tag{49}$$

and, by comparing with (46) and (47), we can fix the two integration constants C^1_γ and C^1_G. We have

$$C^1_\gamma = -\frac{R}{256\pi^2} \, \frac{\Gamma(5/2+\lambda)\Gamma(5/2-\lambda)}{1/4-\lambda^2} \tag{50a}$$

$$= -\frac{R}{256\pi} \, \frac{9/4-\lambda^2}{\cos(\pi\lambda)} \tag{50b}$$

and

$$C^1_G = \frac{R}{192\pi^2} \, \Gamma(3/2+\kappa)\Gamma(3/2-\kappa) \tag{51a}$$

$$= \frac{R}{192\pi} \, \frac{1/4-\kappa^2}{\cos(\pi\kappa)}. \tag{51b}$$

Here, in order simplify the expressions (50a) and (51a) which involve the Gamma function $\Gamma(z)$, we have used the reflection formula [32–34]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{52}$$

3. Wightman functions in dS^4

In dS^4, in order to fix the remaining integration constants C^2_γ and C^2_G, we require the regularity of the Wightman functions (45) and (36) at the antipodal point of x, $x' = Px$ and therefore on its light cone), or, in other words, for $z \to 0$ (see also Fig. 1) [29, 35]. We obtain immediately

$$C^2_\gamma = 0 \tag{53a}$$

and

$$C^2_G = 0. \tag{53b}$$

By inserting now the integration constants (50), (51), (53a) and (53b) into the general expressions (45) and (36), we obtain in dS^4 the explicit expressions of the Wightman function associated with the massive vector field A_μ and the Wightman function associated with both the Stueckelberg scalar field Φ and the ghost fields C and C^*. We have

$$\begin{aligned} G^{(+)A}_{\mu\nu'}(x,x') = \frac{H^2}{32\pi} \Bigg[& \frac{9/4-\lambda^2}{\cos(\pi\lambda)}\, z(1-z)\, F'(5/2+\lambda, 5/2-\lambda; 3; z) + \frac{3}{2}\, \frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\, (1-2z)\, F(5/2+\lambda, 5/2-\lambda; 3; z) \\ & - \frac{1}{2}\, \frac{(1/4-\kappa^2)}{\cos(\pi\kappa)}\, F(5/2+\kappa, 5/2-\kappa; 3; z)\Bigg]\, g_{\mu\nu'} \\ + \frac{1}{32\pi} \Bigg[& \frac{9/4-\lambda^2}{\cos(\pi\lambda)}\, F'(5/2+\lambda, 5/2-\lambda; 3; z) + 3\, \frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\, (1/z)\, F(5/2+\lambda, 5/2-\lambda; 3; z) \\ & - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\, F'(5/2+\kappa, 5/2-\kappa; 3; z) - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\, (1/z)\, F(5/2+\kappa, 5/2-\kappa; 3; z)\Bigg]\, z_{;\mu} z_{;\nu'} \end{aligned} \tag{54}$$

and

$$G^{(+)}(x,x') = \frac{H^2}{16\pi} \, \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\, F(3/2+\kappa, 3/2-\kappa; 2; z). \tag{55}$$

It should be noted that (54) is in accordance with the result obtained recently by Fröb and Higuchi from a mode-sum construction [see Eq. (25) in Ref. [21]] while (55) is a closed form which can be found in various works concerning the massive scalar field in dS^4 [2, 3, 15, 29, 35].

4. Wightman functions in AdS^4

In AdS^4, in order to fix the remaining integration constants C^2_γ and C^2_G, we require that the Wightman functions (45) and (36) fall off as fast as possible at spatial infinity, i.e., for $z \to \infty$ (see also Fig. 2). We recall that this condition is imposed because the Cauchy problem is not well posed in AdS^4, this gravitational background being not globally hyperbolic [27]. Such a condition permits us to control the flow of information through spatial infinity [28, 36].

In the expressions (45) and (36) of the Wightman functions, the hypergeometric functions are expressed in term of the variables z and $1-z$. In order to impose the boundary condition previously mentioned, it is helpful

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

to reexpress them in term of the variable $1/z$. This can be achieved thanks to the connection formulas [32–34]

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,(-z)^{-a}F(a,a-c+1;a-b+1;1/z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,(-z)^{-b}F(b,b-c+1;b-a+1;1/z) \tag{56a}$$

which is valid for $|\arg(-z)| < \pi$ and

$$F(a,b;c;1-z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,z^{-a}F(a,c-b;a-b+1;1/z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,z^{-b}F(b,c-a;b-a+1;1/z) \tag{56b}$$

which is valid for $|\arg(z)| < \pi$. We can then observe that the Wightman functions (45) and (36) approach zero as fast as possible at spatial infinity if we eliminate the terms in $z^{-(5/2-\kappa)}$ and $z^{-(5/2-\lambda)}$ in the expression of the former and the term in $z^{-(3/2-\kappa)}$ in the expression of the latter. (Here, since $m > 0$, we have that $\lambda = \sqrt{1/4 + m^2/K^2} > 0$ and $\kappa = \sqrt{9/4 + m^2/K^2} > 0$.) We then obtain immediately

$$C^2_\gamma = \pm\, i\, e^{\pm i\pi\lambda} C^1_\gamma \tag{57a}$$

and

$$C^2_G = \mp\, i\, e^{\pm i\pi\kappa} C^1_G. \tag{57b}$$

In Eqs. (57a) and (57b), the upper sign (the lower sign) must be chosen if, in the expressions (45) and (36) of the Wightman functions, z lies in the upper half plane (in the lower half plane). This follows from the relation $(-z)^{-a} = e^{\mp i\pi a} z^{-a}$ which is a consequence of $(-z)^{-a} = \exp[-a\ln(-z)]$.

By inserting now the integration constants (50), (51), (57a) and (57b) into the general expressions (45) and (36), we can obtain in AdS^4 the explicit expressions of the Wightman function associated with the massive vector field A_μ and the Wightman function associated with both the Stueckelberg scalar field Φ and the ghost fields C and C^*. Making use of the reflection formula (52) and of the duplication formula [32–34]

$$\Gamma(2z) = \frac{2^{2z}}{2\sqrt{\pi}}\,\Gamma(z)\Gamma(z+1/2) \tag{58}$$

to deal with the Γ-function, we have

$$\begin{aligned}
G^{(+)A}_{\mu\nu'}(x,x') = &-\frac{K^2}{32\pi}\left\{\frac{9/4-\lambda^2}{\cos(\pi\lambda)}\,z(1-z)\,\left[F'(5/2+\lambda,5/2-\lambda;3;z) \mp i\,e^{\pm i\pi\lambda}F'(5/2+\lambda,5/2-\lambda;3;1-z)\right]\right.\\
&\quad+\frac{3}{2}\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\,(1-2z)\,\left[F(5/2+\lambda,5/2-\lambda;3;z) \pm i\,e^{\pm i\pi\lambda}F(5/2+\lambda,5/2-\lambda;3;1-z)\right]\\
&\quad\left.-\frac{1}{2}\,\frac{(1/4-\kappa^2)}{\cos(\pi\kappa)}\,\left[F(5/2+\kappa,5/2-\kappa;3;z) \pm i\,e^{\pm i\pi\kappa}F(5/2+\kappa,5/2-\kappa;3;1-z)\right]\right\} g_{\mu\nu'}\\
&+\frac{1}{32\pi}\left\{\frac{9/4-\lambda^2}{\cos(\pi\lambda)}\,\left[F'(5/2+\lambda,5/2-\lambda;3;z) \mp i\,e^{\pm i\pi\lambda}F'(5/2+\lambda,5/2-\lambda;3;1-z)\right]\right.\\
&\quad+3\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\,(1/z)\,\left[F(5/2+\lambda,5/2-\lambda;3;z) \pm i\,e^{\pm i\pi\lambda}F(5/2+\lambda,5/2-\lambda;3;1-z)\right]\\
&\quad-\frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,\left[F'(5/2+\kappa,5/2-\kappa;3;z) \mp i\,e^{\pm i\pi\kappa}F'(5/2+\kappa,5/2-\kappa;3;1-z)\right]\\
&\quad\left.-\frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,(1/z)\,\left[F(5/2+\kappa,5/2-\kappa;3;z) \pm i\,e^{\pm i\pi\kappa}F(5/2+\kappa,5/2-\kappa;3;1-z)\right]\right\} z_{;\mu}z_{;\nu'}
\end{aligned} \tag{59a}$$

$$\begin{aligned}
= -\frac{K^4}{32\pi^2 m^2}&\left\{\frac{\Gamma(7/2+\lambda)\Gamma(1/2+\lambda)}{\Gamma(1+2\lambda)}\,z^{-(5/2+\lambda)}(1-z)\,F(7/2+\lambda,1/2+\lambda;1+2\lambda;1/z)\right.\\
&-3\,\frac{\Gamma(5/2+\lambda)\Gamma(1/2+\lambda)}{\Gamma(1+2\lambda)}\,z^{-(5/2+\lambda)}(1-2z)\,F(5/2+\lambda,1/2+\lambda;1+2\lambda;1/z)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\Gamma(5/2+\kappa)\Gamma(1/2+\kappa)}{\Gamma(1+2\kappa)}\, z^{-(5/2+\kappa)}\, F(5/2+\kappa, 1/2+\kappa; 1+2\kappa; 1/z) \bigg\} g_{\mu\nu'} \\
&+ \frac{K^2}{32\pi^2 m^2} \bigg\{ \frac{\Gamma(7/2+\lambda)\Gamma(1/2+\lambda)}{\Gamma(1+2\lambda)}\, z^{-(7/2+\lambda)}\, F(7/2+\lambda, 1/2+\lambda; 1+2\lambda; 1/z) \\
&\quad - 6\, \frac{\Gamma(5/2+\lambda)\Gamma(1/2+\lambda)}{\Gamma(1+2\lambda)}\, z^{-(7/2+\lambda)}\, F(5/2+\lambda, 1/2+\lambda; 1+2\lambda; 1/z) \\
&\quad - \frac{\Gamma(7/2+\kappa)\Gamma(1/2+\kappa)}{\Gamma(1+2\kappa)}\, z^{-(7/2+\kappa)}\, F(7/2+\kappa, 1/2+\kappa; 1+2\kappa; 1/z) \\
&\quad + 2\, \frac{\Gamma(5/2+\kappa)\Gamma(1/2+\kappa)}{\Gamma(1+2\kappa)}\, z^{-(7/2+\kappa)}\, F(5/2+\kappa, 1/2+\kappa; 1+2\kappa; 1/z) \bigg\} z_{;\mu} z_{;\nu'}
\end{aligned} \tag{59b}
$$

and

$$
G^{(+)}(x,x') = -\frac{K^2}{16\pi}\, \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\, \left[F(3/2+\kappa, 3/2-\kappa; 2; z) \mp i\, e^{\pm i\pi\kappa} F(3/2+\kappa, 3/2-\kappa; 2; 1-z) \right] \tag{60a}
$$

$$
= \frac{K^2}{16\pi^2}\, \frac{\Gamma(3/2+\kappa)\Gamma(1/2+\kappa)}{\Gamma(1+2\kappa)}\, z^{-(3/2+\kappa)}\, F(3/2+\kappa, 1/2+\kappa; 1+2\kappa; 1/z). \tag{60b}
$$

Here, it is important to recall that, in Eqs. (59a) and (60a), the upper sign (the lower sign) must be chosen if z lies in the upper half plane (in the lower half plane). It should be noted that we have provided two equivalent expressions for these two-point functions: the hypergeometric functions appearing in formulas (59a) and (60a) are given in term of the variables z and $1-z$ while those appearing in formulas (59b) and (60b) are expressed in term of $1/z$. The expression (60) is a result which can be found in some other works concerning the massive scalar field in AdS^4 (see, e.g., Refs. [8, 9, 15, 37]). To our knowledge, the Wightman function (59) associated with the massive vector field A_μ of the Stueckelberg theory is not in the literature. In Ref. [16], Janssen and Dullemond have considered that problem but we are unable to link their results with ours.

D. Feynman propagators and Hadamard Green functions in dS^4 and AdS^4

It is well known that, in quantum field theory in flat spacetime, we can construct all the interesting two-point functions, i.e., the retarded and advanced Green functions, the Feynman propagator and the Hadamard Green function, from the Wightman function by taking its real or its imaginary part and, when it is necessary, by using multiplication by a step function in time. In some sense, this remains true in curved spacetime [30, 31, 38] and, in this subsection, we shall provide the expressions of the Feynman propagators and the Hadamard Green functions of Stueckelberg massive electromagnetism from the Wightman functions obtained previously. Here, we shall adopt a pragmatic point of view by following the approach and the arguments of Allen and Jacobson [15, 29]. It is however interesting to note the existence of a more rigorous point of view exposed in impressive articles by Bros, Epstein and Moschella which concern the scalar field in de Sitter spacetime [39] and general quantum field theories in anti-de Sitter spacetime [40].

1. In dS^4

The expressions (54) and (55) of the Wightman functions involve the hypergeometric function $F(a,b;c;z)$ which, in general, has a branch point at $z=1$ (i.e., for x' on the light cone of x) and a branch cut which runs along the real axis from $z=1$ to $+\infty$ (i.e., for x' in the light cone of x) [see also Eq. (15)]. As a consequence, these Wightman functions are perfectly defined when x and x' are spacelike related or cannot be joined by a geodesic but, when they are timelike related, it is important to specify how to approach the branch cut. In fact, it is necessary to replace in Eqs. (54) and (55) the biscalar z by the biscalar $z \mp i\epsilon$ (here $\epsilon \to 0_+$) where the minus sign (respectively the plus sign) is chosen when x' lies in the past (respectively the future) of x. Indeed, in dS^4, due to the relation (10) [note that $z = 1-(R/24)\sigma + \dots$], the prescription $z \to z \mp i\epsilon$ induces the change $\sigma \to \sigma \pm i\epsilon$ which permits us to encode the usual behavior of the Wightman functions in curved spacetime (see also Chap. 4 of Ref. [41]) and to recover, in the flat-space limit, the Wightman functions of Minkowski quantum field theory.

The Feynman propagator $G^A_{\mu\nu'}(x,x') = i\langle 0|TA_\mu(x)A_{\nu'}(x')|0\rangle$ associated with the massive vector field A_μ (here, T denotes the time-ordering operator) is obtained from the Wightman function (54) by writing $G^A_{\mu\nu'}(x,x') = i\, G^{(+)A}_{\mu\nu'}(z-i\epsilon)$ with $\epsilon \to 0_+$. Indeed, in dS^4, the prescription $z \to z - i\epsilon$ induces the change $\sigma \to \sigma + i\epsilon$ which permits us to encode the time ordering (see also Secs. II C and III A of Ref. [1]). Similarly, the Feynman propagators $G^\Phi(x,x') = i\langle\psi|T\Phi(x)\Phi(x')|\psi\rangle$

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

and $G^{\mathrm{Gh}}(x,x') = i\langle\psi|TC^*(x)C(x')|\psi\rangle$ associated respectively with the scalar field Φ and the ghost fields C and C^* are equals to $i\,G^{(+)}(z-i\epsilon)$.

The expression of the Hadamard Green function

$$G^{(1)A}_{\mu\nu'}(x,x') = \langle 0|A_\mu(x)A_{\nu'}(x') + A_{\nu'}(x')A_\mu(x)|0\rangle \quad (61)$$

associated with the massive vector field A_μ is obviously obtained from (54) by noting that

$$G^{(1)A}_{\mu\nu'}(x,x') = G^{(+)A}_{\mu\nu'}(x,x') + G^{(+)A}_{\nu'\mu}(x',x). \quad (62)$$

In the same way, the Hadamard Green function

$$G^{(1)\Phi}(x,x') = \langle 0|\Phi(x)\Phi(x') + \Phi(x')\Phi(x)|0\rangle \quad (63)$$

associated with the auxiliary scalar field Φ and the Hadamard Green function

$$G^{(1)\mathrm{Gh}}(x,x') = \langle 0|C^*(x)C(x') - C(x')C^*(x)|0\rangle \quad (64)$$

associated with the ghost fields C and C^* can be obtained from (55). We have

$$G^{(1)\Phi}(x,x') = G^{(1)\mathrm{Gh}}(x,x') = G^{(1)}(x,x') \quad (65)$$

with

$$G^{(1)}(x,x') = G^{(+)}(x,x') + G^{(+)}(x',x). \quad (66)$$

Formulas (62) and (66) must be taken with a grain of salt. Indeed, it is important to recall that Hadamard Green functions are real-valued while Wightman functions are complex-valued and the prescription permitting us to define the former and the latter on the branch cut are different. In fact, Hadamard Green functions are average across the cut, i.e., we have

$$G^{(1)A}_{\mu\nu'}(x,x') = G^{(+)A}_{\mu\nu'}(z+i\epsilon) + G^{(+)A}_{\mu\nu'}(z-i\epsilon) \quad (67)$$

and

$$G^{(1)}(x,x') = G^{(+)}(z+i\epsilon) + G^{(+)}(z-i\epsilon) \quad (68)$$

with $\epsilon \to 0_+$. This prescription, which is in total agreement with that defining the Wightman functions, permits us to have at our disposal Hadamard Green functions which are real-valued. More explicitly, by inserting (54) into (67) and (55) into (68), we obtain

$$\begin{aligned}
G^{(1)A}_{\mu\nu'}(x,x') = \frac{H^2}{16\pi}\Bigg[&\frac{9/4-\lambda^2}{\cos(\pi\lambda)}\, z(1-z)\,(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;z) + \frac{3}{2}\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\,(1-2z)\,(\mathrm{Re}F)(5/2+\lambda,5/2-\lambda;3;z)\\
&-\frac{1}{2}\,\frac{(1/4-\kappa^2)}{\cos(\pi\kappa)}\,(\mathrm{Re}F)(5/2+\kappa,5/2-\kappa;3;z)\Bigg]\, g_{\mu\nu'}\\
+\frac{1}{16\pi}\Bigg[&\frac{9/4-\lambda^2}{\cos(\pi\lambda)}\,(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;z) + 3\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\,(1/z)\,(\mathrm{Re}F)(5/2+\lambda,5/2-\lambda;3;z)\\
&-\frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,(\mathrm{Re}F)'(5/2+\kappa,5/2-\kappa;3;z) - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,(1/z)\,(\mathrm{Re}F)(5/2+\kappa,5/2-\kappa;3;z)\Bigg]\, z_{;\mu}z_{;\nu'}.
\end{aligned} \quad (69)$$

and

$$G^{(1)}(x,x') = \frac{H^2}{8\pi}\,\frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,(\mathrm{Re}F)(3/2+\kappa,3/2-\kappa;2;z). \quad (70)$$

Here, we have introduced the average across the cut of the hypergeometric function $F(a,b;c;z)$ defined by

$$(\mathrm{Re}F)(a,b;c;z) = \frac{F(a,b;c;z+i\epsilon)+F(a,b;c;z-i\epsilon)}{2}. \quad (71a)$$

This function is nothing else than the real part of $F(a,b;c;z)$ on the branch cut. We have also used its derivative $(\mathrm{Re}F)'(a,b;c;z)$ with respect to its argument z. It should be noted that below we shall need also its imaginary part

$$(\mathrm{Im}F)(a,b;c;z) = \frac{F(a,b;c;z+i\epsilon)-F(a,b;c;z-i\epsilon)}{2i} \quad (71b)$$

and we shall use its derivative $(\mathrm{Im}F)'(a,b;c;z)$ with respect to its argument z.

2. *In* AdS^4

Mutatis mutandis, the previous discussion can be adapted to obtain the Feynman propagators and the Hadamard Green functions in AdS^4. We first note that the branch cut of the Wightman functions (59) and (60) runs along the real axis from $z=-\infty$ to $z=1$. This appears clearly if we consider the expressions (59b) and (60b). Indeed, the functions of the form

$z^{-a} = \exp(-a \ln z)$ and the hypergeometric functions of the form $F(a, b; c; 1/z)$ involved in these expressions are respectively cut along the negative axis and the segment $[0, 1]$. As a consequence, the Wightman functions (59) and (60) are perfectly defined if $z > 1$, i.e., when x and x' are spacelike related [see also Eq. (16)] whereas, if $z \leq 1$, and in particular when x and x' are timelike related (i.e., if $z \in]0, 1[$), it is important to specify how to approach the branch cut. In fact, it is necessary to replace in Eqs. (59) and (60) the biscalar z by the biscalar $z \pm i\epsilon$ (here $\epsilon \to 0_+$) where the plus sign (respectively the minus sign) is chosen when x' lies in the past (respectively the future) of x. Indeed, in AdS^4, due to the relation (10), it is now the prescription $z \to z \pm i\epsilon$ which induces the change $\sigma \to \sigma \pm i\epsilon$ permitting us to encode the usual behavior of the Wightman functions in curved spacetime.

In AdS^4, the Feynman propagator $G^A_{\mu\nu'}(x, x') = i\langle 0|TA_\mu(x)A_{\nu'}(x')|0\rangle$ associated with the massive vector field A_μ is obtained from the Wightman function (59) by writing $G^A_{\mu\nu'}(x, x') = i\, G^{(+)A}_{\mu\nu'}(z + i\epsilon)$ with $\epsilon \to 0_+$. Indeed, in this gravitational background, the prescription $z \to z + i\epsilon$ induces the change $\sigma \to \sigma + i\epsilon$ which permits us to encode the time ordering. Similarly, the Feynman propagators $G^\Phi(x, x') = i\langle\psi|T\Phi(x)\Phi(x')|\psi\rangle$ and $G^{\mathrm{Gh}}(x, x') = i\langle\psi|TC^*(x)C(x')|\psi\rangle$ associated respectively with the scalar field Φ and the ghost fields C and C^* are equals to $i\, G^{(+)}(z + i\epsilon)$.

In AdS^4, the expression of the Hadamard Green function (61) associated with the massive vector field A_μ is obtained by inserting (59) into (67) while the Hadamard Green function (65) associated with both the massive scalar field Φ and the ghost fields C and C^* is obtained by inserting (60) into (68). In fact, here we shall construct the Hadamard Green functions from (59a) and (60a) only. Indeed, the resulting expressions are easily tractable in the context of the renormalization of the stress-energy tensor, or more precisely, their regular parts can be naturally extracted. Of course, in order to obtain the Hadamard Green functions, it is then important to take carefully into account the upper sign (in that case, we use the variable $z + i\epsilon$ and we are working in the upper half plane) or the lower sign (in that case, we use the variable $z - i\epsilon$ and we are working in the lower half plane) in (59a) and (60a). A straightforward calculation leads to expressions which are explicitly real-valued and given by

$$
\begin{aligned}
G^{(1)A}_{\mu\nu'}(x, x') = -\frac{K^2}{16\pi}\Bigg\{ & \frac{9/4 - \lambda^2}{\cos(\pi\lambda)}\, z(1-z)\, \big[(\mathrm{Re}F)'(5/2+\lambda, 5/2-\lambda; 3; z) + \sin(\pi\lambda)(\mathrm{Re}F)'(5/2+\lambda, 5/2-\lambda; 3; 1-z) \\
& \qquad - \cos(\pi\lambda)(\mathrm{Im}F)'(5/2+\lambda, 5/2-\lambda; 3; 1-z)\big] \\
& + \frac{3}{2}\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\,(1-2z)\, \big[(\mathrm{Re}F)(5/2+\lambda, 5/2-\lambda; 3; z) - \sin(\pi\lambda)(\mathrm{Re}F)(5/2+\lambda, 5/2-\lambda; 3; 1-z) \\
& \qquad + \cos(\pi\lambda)(\mathrm{Im}F)(5/2+\lambda, 5/2-\lambda; 3; 1-z)\big] \\
& - \frac{1}{2}\,\frac{(1/4-\kappa^2)}{\cos(\pi\kappa)}\, \big[(\mathrm{Re}F)(5/2+\kappa, 5/2-\kappa; 3; z) - \sin(\pi\kappa)(\mathrm{Re}F)(5/2+\kappa, 5/2-\kappa; 3; 1-z) \\
& \qquad + \cos(\pi\kappa)(\mathrm{Im}F)(5/2+\kappa, 5/2-\kappa; 3; 1-z)\big]\Bigg\}\, g_{\mu\nu'} \\
+\frac{1}{16\pi}\Bigg\{ & \frac{9/4-\lambda^2}{\cos(\pi\lambda)}\, \big[(\mathrm{Re}F)'(5/2+\lambda, 5/2-\lambda; 3; z) + \sin(\pi\lambda)(\mathrm{Re}F)'(5/2+\lambda, 5/2-\lambda; 3; 1-z) \\
& \qquad - \cos(\pi\lambda)(\mathrm{Im}F)'(5/2+\lambda, 5/2-\lambda; 3; 1-z)\big] \\
& + 3\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}\,(1/z)\, \big[(\mathrm{Re}F)(5/2+\lambda, 5/2-\lambda; 3; z) - \sin(\pi\lambda)(\mathrm{Re}F)(5/2+\lambda, 5/2-\lambda; 3; 1-z) \\
& \qquad + \cos(\pi\lambda)(\mathrm{Im}F)(5/2+\lambda, 5/2-\lambda; 3; 1-z)\big] \\
& - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\, \big[(\mathrm{Re}F)'(5/2+\kappa, 5/2-\kappa; 3; z) + \sin(\pi\kappa)(\mathrm{Re}F)'(5/2+\kappa, 5/2-\kappa; 3; 1-z) \\
& \qquad - \cos(\pi\kappa)(\mathrm{Im}F)'(5/2+\kappa, 5/2-\kappa; 3; 1-z)\big] \\
& - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,(1/z)\, \big[(\mathrm{Re}F)(5/2+\kappa, 5/2-\kappa; 3; z) - \sin(\pi\kappa)(\mathrm{Re}F)(5/2+\kappa, 5/2-\kappa; 3; 1-z) \\
& \qquad + \cos(\pi\kappa)(\mathrm{Im}F)(5/2+\kappa, 5/2-\kappa; 3; 1-z)\big]\Bigg\}\, z_{;\mu} z_{;\nu'}
\end{aligned}
\tag{72}
$$

and

$$G^{(1)}(x,x') = -\frac{K^2}{8\pi}\,\frac{1/4-\kappa^2}{\cos(\pi\kappa)}\,\left[(\mathrm{Re}F)(3/2+\kappa,3/2-\kappa;2;z) + \sin(\pi\kappa)(\mathrm{Re}F)(3/2+\kappa,3/2-\kappa;2;1-z) - \cos(\pi\kappa)(\mathrm{Im}F)(3/2+\kappa,3/2-\kappa;2;1-z)\right]. \tag{73}$$

III. RENORMALIZED STRESS-ENERGY TENSOR OF STUECKELBERG ELECTROMAGNETISM

In this section, from the general formalism developed in Ref. [1], we shall obtain exact analytical expressions for the vacuum expectation value of the RSET of the massive vector field propagating in dS^4 and AdS^4. We shall, in particular, fix the geometrical ambiguities in the results (see Refs. [42, 43] for interesting remarks on the ambiguity problem as well as Sec. IV E of Ref. [1] and references therein) by considering the flat-space limit and, moreover, we shall discuss the zero-mass limit of the expressions found.

A. General considerations

In this subsection, we have gathered some results established in Ref. [1] which will be necessary to construct, in the next three subsections, the RSETs associated with Stueckelberg electromagnetism in dS^4 and AdS^4. By doing so, we hope to alleviate the task of the reader and to prevent him from drowning in the heavy formalism developed in our previous article.

We have seen in Ref. [1] that, in the context of the renormalization of the stress-energy tensor of Stueckelberg electromagnetism, it is necessary to extract the regular and state-dependent parts of the Hadamard Green functions $G^{(1)A}_{\mu\nu'}(x,x')$ and $G^{(1)}(x,x')$. They can be obtained by removing from $G^{(1)A}_{\mu\nu'}(x,x')$ and $G^{(1)}(x,x')$ their singular and purely geometrical parts $G^{(1)A}_{\text{sing}\ \mu\nu'}(x,x')$ and $G^{(1)}_{\text{sing}}(x,x')$. We have

$$G^{(1)A}_{\text{reg}\ \mu\nu'}(x,x') = G^{(1)A}_{\mu\nu'}(x,x') - G^{(1)A}_{\text{sing}\ \mu\nu'}(x,x') \tag{74}$$

and

$$G^{(1)}_{\text{reg}}(x,x') = G^{(1)}(x,x') - G^{(1)}_{\text{sing}}(x,x') \tag{75}$$

where

$$G^{(1)A}_{\text{sing}\ \mu\nu'}(x,x') = \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')}\,g_{\mu\nu'}(x,x') + V^A_{\mu\nu'}(x,x')\ln\left|M^2\,\sigma(x,x')\right|\right] \tag{76}$$

and

$$G^{(1)}_{\text{sing}}(x,x') = \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln\left|M^2\sigma(x,x')\right|\right]. \tag{77}$$

The expressions of $G^{(1)A}_{\text{sing}\ \mu\nu'}(x,x')$ and $G^{(1)}_{\text{sing}}(x,x')$ involve the geodetic interval $\sigma(x,x')$, the bivector of parallel transport $g_{\mu\nu'}(x,x')$, the Van Vleck-Morette determinant $\Delta(x,x')$ (see Ref. [30] or the Appendix of Ref. [1] for its definition and properties) as well as the geometrical bivector $V^A_{\mu\nu'}(x,x')$ and the geometrical biscalar $V(x,x')$ which are defined by the expansions $V^A_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} V^A_n{}_{\mu\nu'}(x,x')\,\sigma^n(x,x')$ and $V(x,x') = \sum_{n=0}^{+\infty} V_n(x,x')\,\sigma^n(x,x')$ and by the recursion relations satisfied by the coefficients $V^A_n{}_{\mu\nu'}(x,x')$ and $V_n(x,x')$ [see Eqs. (39a), (39b), (43a) and (43b) in Ref. [1]]. Moreover, we have introduced the renormalization mass M permitting us to make dimensionless the argument of the logarithm.

In fact, in order to construct the RSET, we only need the lower coefficients of the covariant Taylor series expansions for $x' \to x$ of the bitensor

$$W^A_{\mu\nu}(x,x') = 4\pi^2\, g_\nu{}^{\nu'}(x,x')G^{(1)A}_{\text{reg}\ \mu\nu'}(x,x') \tag{78}$$

and of the biscalar

$$W(x,x') = 4\pi^2\, G^{(1)}_{\text{reg}}(x,x') \tag{79}$$

or, more precisely, we only need the covariant Taylor series expansions of these two quantities up to order $\sigma^1(x,x')$ [see Sec. IV of Ref. [1]]. As a consequence, we are not really interested by the full expressions of (76) and (77) but by their covariant Taylor series expansions truncated by neglecting the terms vanishing faster than $\sigma(x,x')$ for $x' \to x$. They can be obtained from the covariant Taylor series expansion of $\Delta^{1/2}(x,x')$ up to order $\sigma^2(x,x')$ [see Eq. (A9) in Ref. [1]], the covariant Taylor series expansion of $V^A_{\mu\nu'}(x,x')$ up to order $\sigma^1(x,x')$ [see Eqs. (71a)–(72d) in Ref. [1]] and the covariant Taylor series expansion of $V(x,x')$ up to order $\sigma^1(x,x')$ [see Eqs. (73a)–(74c) in Ref. [1]]. By inserting these covariant Taylor series expansions into (76) and (77), we can derive the covariant Taylor series expansions of $W^A_{\mu\nu}(x,x')$ defined by (78) [see Eq. (75) in Ref. [1] for its general expression] and of $W(x,x')$ defined by (79) [see Eq. (76) in Ref. [1] for its general expression].

Fortunately, in maximally symmetric spacetimes, due to the relations (6a)–(6c), the regularization of the Hadamard Green functions $G^{(1)A}_{\mu\nu'}(x,x')$ and $G^{(1)}(x,x')$ greatly simplifies. Indeed :

(i) The covariant Taylor series expansions of the various bitensors involved in the singular parts of the

Hadamard Green functions reduce to

$$\Delta^{1/2} = 1 + (1/24)\, R\sigma + (19/17280)\, R^2\sigma^2 + O(\sigma^3), \tag{80a}$$

$$V^A_{\mu\nu} = \big\{ \big[(1/2)\, m^2 + (1/24)\, R\big] + \big[(1/8)\, m^4 + (1/24)\, m^2 R + (1/432)\, R^2\big]\, \sigma \big\}\, g_{\mu\nu} + (1/1728) R^2\, \sigma_{;\mu}\sigma_{;\nu} + O(\sigma^{3/2}), \tag{80b}$$

and

$$V = \big[(1/2)\, m^2 - (1/12)\, R\big] + \big[(1/8)\, m^4 - (1/48)\, m^2 R\big]\, \sigma + O(\sigma^{3/2}). \tag{80c}$$

(ii) The covariant Taylor series expansions of the bitensors $W^A_{\mu\nu}(x,x')$ and $W(x,x')$ reduce to

$$W^A_{\mu\nu} = s_{\mu\nu} + \tfrac{1}{2}\, s_{\mu\nu ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \tag{81}$$

and

$$W = w + \frac{1}{2}\, w_{ab}\, \sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}). \tag{82}$$

In the following, these considerations will facilitate our task.

In our previous article, we have provided two different expressions for the RSET of Stueckelberg electromagnetism:

(i) A first expression which only involves state-dependent as well as geometrical quantities associated with the massive vector field A_μ [see Eq. (123) in Ref. [1]]. Here, the contribution of the quantum massive scalar field Φ has been removed thanks to a Ward identity and the result obtained is given in terms of the first coefficients of the covariant Taylor series expansions for $x' \to x$ of the bitensor $W^A_{\mu\nu}(x,x')$ [see Eq. (75) in Ref. [1]].

(ii) A second expression where the contributions of the massive vector field A_μ and of the massive scalar field Φ have been artificially separated [see Eqs. (125) and (126) in Ref. [1]] and which has been constructed in such a way that the zero-mass limit of the first contribution reduces to the RSET of Maxwell's electromagnetism. The result obtained is given in terms of the first coefficients of the covariant Taylor series expansions for $x' \to x$ of the bitensors $W^A_{\mu\nu}(x,x')$ and $W(x,x')$ [see Eqs. (75) and (76) in Ref. [1]].

Of course, these expressions are equivalent but it will be necessary to consider both in order to clearly discuss the zero-mass limit of the Stueckelberg theory (see Sec. III D). In maximally symmetric spacetimes, these expressions simplify considerably. Indeed, the RSET of Stueckelberg electromagnetism can be expressed in term of its trace as

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}} = \frac{1}{4}\, \langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}}\, g_{\mu\nu} \tag{83}$$

and this one reduces to

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}} = \frac{1}{8\pi^2}\, \big\{ \big[-m^2 - (1/8)\, R\big]\, s_\rho{}^\rho + s_{\rho\tau}{}^{\rho\tau} + 3\, v^A_1{}_\rho{}^\rho \big\} + \Theta_\rho{}^\rho \tag{84}$$

if we focus on the expression which only involves the characteristics of the quantum massive vector field A_μ [see, in Ref. [1], Eq. (123) as well as Eq. (124)]. If we alternatively focus on the expression which involves both the characteristics of the massive vector field A_μ and of the massive scalar field Φ [see, in Ref. [1], Eqs. (125) and (126) as well as Eqs. (127) and (128)], we have

$$\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}} = \mathcal{T}^A{}_\rho{}^\rho + \mathcal{T}^\Phi{}_\rho{}^\rho + \Theta_\rho{}^\rho \tag{85}$$

with

$$\mathcal{T}^A{}_\rho{}^\rho = \frac{1}{8\pi^2}\, \big\{ \big[-m^2 - (1/4)\, R\big]\, s_\rho{}^\rho + 2\, s_{\rho\tau}{}^{\rho\tau} + 4\, v^A_1{}_\rho{}^\rho \big\} \tag{86a}$$

and

$$\mathcal{T}^\Phi{}_\rho{}^\rho = \frac{1}{8\pi^2}\, \big(-m^2 w + 2\, v_1\big)\, . \tag{86b}$$

It should be noted that the term $\Theta_\rho{}^\rho$ appearing in Eqs. (84) and (85) is a purely geometrical term which encodes the ambiguities in the definition of the RSET [see Sec. IV E of Ref. [1]] and which involves, in particular, a contribution associated with the renormalization mass M. We also recall that the term $v^A_1{}_\rho{}^\rho$ in Eqs. (84) and (86a) and the term v_1 in Eq. (86b) are purely geometrical quantities which appear in the covariant Taylor series expansions of $V^A_{\mu\nu}(x,x')$ and $V(x,x')$ [see Eqs. (71b) and (73b) in Ref. [1]]. From Eqs. (72d) and (74c) of Ref. [1], we can show that, in maximally symmetric backgrounds, they reduce to

$$v^A_1{}_\rho{}^\rho = (1/2)\, m^4 + (1/12)\, m^2 R - (1/2160)\, R^2 \tag{87}$$

and

$$v_1 = (1/8)\, m^4 - (1/24)\, m^2 R + (29/8640)\, R^2. \tag{88}$$

It is crucial to discuss the form of the trace term $\Theta_\rho{}^\rho$ appearing in Eqs. (84) and (85). In an arbitrary gravitational background, such a term is given by Eq. (133) of Ref. [1] which reduces, in maximally symmetric spacetimes, to

$$\Theta_\rho{}^\rho = \frac{1}{8\pi^2}\, \big(\alpha\, m^4 + \beta\, m^2 R\big)\, . \tag{89}$$

Here, α and β are constants which can be fixed by imposing additional physical conditions on the RSET (see below) or which can be redefined by renormalization of Newton's gravitational constant and of the cosmological constant. Indeed, let us recall that, in Eq. (89), the terms $\alpha\, m^4$ and $\beta\, m^2 R$ come from the Einstein-Hilbert action defining the dynamics of the gravitational field (see also the discussion in Sec. IV E 1 of Ref. [1]). Of course, the ambiguities associated with the renormalization mass M are necessarily of this form but their expressions are totally determined. In maximally symmetric spacetimes, the renormalization mass induces in (84) a contribution given by

$$\Theta_\rho{}^\rho(M) = \frac{\ln(M^2)}{8\pi^2}\left[(3/2)\, m^4 + (1/4)\, m^2 R\right]. \tag{90}$$

It can be derived from Eq. (146) of Ref. [1] which is valid in an arbitrary gravitational background. Similarly, the renormalization mass induces in (85) a contribution given by

$$\Theta_\rho{}^\rho(M) = \Theta^A{}_\rho{}^\rho(M) + \Theta^\Phi{}_\rho{}^\rho(M) \tag{91}$$

with

$$\Theta^A{}_\rho{}^\rho(M) = \frac{\ln(M^2)}{8\pi^2}\left[m^4 + (1/3)\, m^2 R\right] \tag{92a}$$

and

$$\Theta^\Phi{}_\rho{}^\rho(M) = \frac{\ln(M^2)}{8\pi^2}\left[(1/2)\, m^4 - (1/12)\, m^2 R\right]. \tag{92b}$$

It can be derived from Eqs. (144), (147a) and (147b) of Ref. [1] which are valid in an arbitrary gravitational background.

To conclude this subsection, we would like to remark that the Taylor coefficient $s_{\rho\tau}{}^{\rho\tau}$ appearing in the expressions (84) and (86a) of the RSET can be related with lower-order Taylor coefficients. Indeed, due to the "Ward identity" linking the bitensors $W^A_{\mu\nu}(x,x')$ and $W(x,x')$ [see Eq. (85) in Ref. [1]], we can write in a maximally symmetric spacetime [see Eq. (86b) in Ref. [1]]

$$s_{\rho\tau}{}^{\rho\tau} = (1/8)\, R s_\rho{}^\rho + w_\rho{}^\rho - v^A_1{}_\rho{}^\rho + 4\, v_1. \tag{93}$$

Moreover, due to the "wave equation" satisfied by the biscalar $W(x,x')$ [see Eq. (82) in Ref. [1]], we have the constraint [see Eq. (83a) in Ref. [1]]

$$w_\rho{}^\rho = m^2\, w - 6\, v_1. \tag{94}$$

By inserting (94) into (93), we then obtain

$$s_{\rho\tau}{}^{\rho\tau} = (1/8)\, R s_\rho{}^\rho + m^2\, w - v^A_1{}_\rho{}^\rho - 2\, v_1. \tag{95}$$

This last equation is very interesting. Indeed, it permits us to realize that, in maximally symmetric spacetimes, the construction of the RSET of Stueckelberg electromagnetism can be accomplished using only the coefficients $s_\rho{}^\rho$ and w, i.e., the lowest-order coefficients of the covariant Taylor series expansions (81) and (82). As a consequence, in the regularization process, it would be sufficient to consider the covariant Taylor series expansions of (76) and (77) truncated by neglecting the terms vanishing faster than $\sigma^0(x,x') = 1$ for $x' \to x$. Moreover, we can remark that the Taylor coefficient $s_\rho{}^\rho$ is a trace term. So, in order to obtain its expression, it would be sufficient to regularize the trace $g^{\mu\nu'}(x,x')G^{(1)A}_{\mu\nu'}(x,x')$ of the Hadamard Green functions (69) or (72) and this could be done rather easily by taking into account the relation (17a). In fact, in the following, we shall not use these considerations even if it is obvious they could greatly simplify our job. We intend to determine the full covariant Taylor series expansions (81) and (82) because this will permit us to control the internal consistency of our calculations and, in particular, that the singular terms in $\ln|\sigma(x,x')|$ hidden in the expressions (69), (70), (72) and (73) of the Hadamard Green functions are of Hadamard type. We shall use the constraints (94) and (95) only to check our results.

B. The renormalized stress-energy tensor in dS^4

In dS^4, the covariant Taylor series expansions (81) and (82) can be obtained by inserting the expressions (69) and (70) of the Hadamard Green functions into (74) and (75) taking into account (i) the relation (11) which links the quadratic form $z(x,x')$ with the geodetic interval $\sigma(x,x')$ as well as (ii) the expression of the singular parts (76) and (77) constructed by using the covariant Taylor series expansions (80a), (80b) and (80c). We then obtain

$$\begin{aligned} s_{\mu\nu} = \Big\{ &-(1/2)\, m^2 - (19/144)\, R + \left[(3/8)\, m^2 + (1/16)\, R\right]\left[\ln(R/24) + \Psi(5/2+\lambda) + \Psi(5/2-\lambda) + 2\gamma\right] \\ &+ \left[(1/8)\, m^2 - (1/48)\, R\right]\left[\ln(R/24) + \Psi(5/2+\kappa) + \Psi(5/2-\kappa) + 2\gamma\right]\Big\}\, g_{\mu\nu}, \end{aligned} \tag{96a}$$

$$\begin{aligned} s_{\mu\nu ab} = \Big\{ &-(5/16)\, m^4 - (43/288)\, m^2 R - (691/51840)\, R^2 + \\ &+ \left[(5/48)\, m^4 + (5/72)\, m^2 R + (5/576)\, R^2\right]\left[\ln(R/24) + \Psi(7/2+\lambda) + \Psi(7/2-\lambda) + 2\gamma\right] \\ &- \left[(1/32)\, m^2 R + (1/192)\, R^2\right]\left[\ln(R/24) + \Psi(5/2+\lambda) + \Psi(5/2-\lambda) + 2\gamma\right] \\ &+ \left[(1/48)\, m^4 + (1/288)\, m^2 R - (1/864)\, R^2\right]\left[\ln(R/24) + \Psi(7/2+\kappa) + \Psi(7/2-\kappa) + 2\gamma\right]\Big\}\, g_{\mu\nu} g_{ab} \end{aligned}$$

$$
\begin{aligned}
&+\Big\{(1/144)\,m^2R+(19/5184)\,R^2\\
&\quad-\big[(1/24)\,m^4+(1/36)\,m^2R+(1/288)\,R^2\big]\big[\ln(R/24)+\Psi(7/2+\lambda)+\Psi(7/2-\lambda)+2\gamma\big]\\
&\quad+\big[(1/32)\,m^2R+(1/192)\,R^2\big]\big[\ln(R/24)+\Psi(5/2+\lambda)+\Psi(5/2-\lambda)+2\gamma\big]\\
&\quad+\big[(1/24)\,m^4+(1/144)\,m^2R-(1/432)\,R^2\big]\big[\ln(R/24)+\Psi(7/2+\kappa)+\Psi(7/2-\kappa)+2\gamma\big]\\
&\quad-\big[(1/96)\,m^2R-(1/576)\,R^2\big]\big[\ln(R/24)+\Psi(5/2+\kappa)+\Psi(5/2-\kappa)+2\gamma\big]\Big\}\,g_{\mu(a|}g_{\nu|b)}
\end{aligned}
\tag{96b}
$$

and

$$
w=-(1/2)\,m^2+(1/18)\,R+\big[(1/2)\,m^2-(1/12)\,R\big]\big[\ln(R/24)+\Psi(3/2+\kappa)+\Psi(3/2-\kappa)+2\gamma\big],
\tag{96c}
$$

$$
\begin{aligned}
w_{ab}=&\Big\{-(5/16)\,m^4+(13/288)\,m^2R+(1/5760)\,R^2\\
&+\big[(1/8)\,m^4-(1/48)\,m^2R\big]\big[\ln(R/24)+\Psi(5/2+\kappa)+\Psi(5/2-\kappa)+2\gamma\big]\Big\}\,g_{ab}.
\end{aligned}
\tag{96d}
$$

In the previous expressions, we have introduced the Digamma function $\Psi(z)=(d/dz)\ln\Gamma(z)$ and we have used systematically its properties [32–34] and, more particularly, the recurrence formula

$$
\Psi(z+1)=\Psi(z)+1/z
\tag{97}
$$

in order to simplify them. We have also introduced the Euler-Mascheroni constant $\gamma=-\Psi(1)$. From these results, we can now write

$$
\begin{aligned}
s_\rho{}^\rho=&-2\,m^2-(19/36)\,R+\big[(3/2)\,m^2+(1/4)\,R\big]\big[\ln(R/24)+\Psi(5/2+\lambda)+\Psi(5/2-\lambda)+2\gamma\big]\\
&+\big[(1/2)\,m^2-(1/12)\,R\big]\big[\ln(R/24)+\Psi(5/2+\kappa)+\Psi(5/2-\kappa)+2\gamma\big],
\end{aligned}
\tag{98a}
$$

$$
\begin{aligned}
s_{\rho\tau}{}^{\rho\tau}=&-(5/4)\,m^4-(19/36)\,m^2R-(1/60)\,R^2\\
&+\big[(3/16)\,m^2R+(1/32)\,R^2\big]\big[\ln(R/24)+\Psi(5/2+\lambda)+\Psi(5/2-\lambda)+2\gamma\big]\\
&+\big[(1/2)\,m^4+(1/12)\,m^2R-(1/36)\,R^2\big]\big[\ln(R/24)+\Psi(7/2+\kappa)+\Psi(7/2-\kappa)+2\gamma\big]\\
&-\big[(5/48)\,m^2R-(5/288)\,R^2\big]\big[\ln(R/24)+\Psi(5/2+\kappa)+\Psi(5/2-\kappa)+2\gamma\big]
\end{aligned}
\tag{98b}
$$

and

$$
\begin{aligned}
w_\rho{}^\rho=&-(5/4)\,m^4+(13/72)\,m^2R+(1/1440)\,R^2\\
&+\big[(1/2)\,m^4-(1/12)\,m^2R\big]\big[\ln(R/24)+\Psi(5/2+\kappa)+\Psi(5/2-\kappa)+2\gamma\big].
\end{aligned}
\tag{98c}
$$

It should be noted that the constraints (94) and (95) are satisfied by these coefficients. This can be easily checked using the recurrence formula (97).

By inserting now the expressions (98a) and (98b) into (84) taking into account the geometrical term (87) as well as the geometrical ambiguities (89) and (90), we have for the trace of the RSET

$$
\begin{aligned}
8\pi^2\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren dS}^4}=&(\alpha+9/4)\,m^4+(\beta+17/24)m^2R+(19/1440)\,R^2\\
&-\big[(3/2)\,m^4+(1/4)\,m^2R\big]\big[\ln(R/(24M^2))+\Psi(5/2+\lambda)+\Psi(5/2-\lambda)+2\gamma\big]
\end{aligned}
\tag{99}
$$

and, of course, the RSET $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}}$ can be obtained immediately from (83). This expression could be considered as the final result of our work in dS^4. Indeed, by construction, it fully includes the state-dependence of the Stueckelberg theory and, moreover, it takes into account all the geometrical ambiguities. However, it is possible to go further and to fix the renormalization mass M and the coefficient α by requiring the vanishing of this expression in the flat-space limit, i.e. for $R\to 0$. We first absorb the term 2γ into the renormalization mass M and we then obtain $M=m/\sqrt{2}$ and $\alpha=-9/4$ which leads to

$$
\begin{aligned}
8\pi^2\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren dS}^4}=&(\beta+17/24)\,m^2R+(19/1440)\,R^2\\
&-\big[(3/2)\,m^4+(1/4)\,m^2R\big]\big[\ln(R/(12m^2))+\Psi(5/2+\lambda)+\Psi(5/2-\lambda)\big].
\end{aligned}
\tag{100}
$$

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

This last result is not free of ambiguities due to the arbitrary coefficient β remaining in the expression of the trace of the RSET. However, it is worth noting that it would be possible to cancel it or, more precisely, to cancel the term $(\beta+17/24)m^2R$ by a finite renormalization of the Einstein-Hilbert action of the gravitational field.

C. The renormalized stress-energy tensor in AdS^4

Mutatis mutandis, the calculations of Sec. III B can be adapted to obtain the RSET of Stueckelberg electromagnetism in AdS^4. We first determine the covariant Taylor series expansions (81) and (82) from the expressions (72) and (73) of the Hadamard Green functions. We have

$$\begin{aligned}
s_{\mu\nu} = \Big\{ &-(1/2)\,m^2 + (5/144)\,R \\
&+ \big[(3/8)\,m^2 + (1/16)\,R\big]\big[\ln(-R/24) + 2\Psi(1/2+\lambda) + 2\gamma\big] \\
&+ \big[(1/8)\,m^2 - (1/48)\,R\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big]\Big\}\, g_{\mu\nu},
\end{aligned} \tag{101a}$$

$$\begin{aligned}
s_{\mu\nu ab} = &\Big\{ -(5/16)\,m^4 - (1/18)\,m^2R + (119/51840)\,R^2 \\
&\quad + \big[(5/48)\,m^4 + (11/288)\,m^2R + (1/288)\,R^2\big]\big[\ln(-R/24) + 2\Psi(1/2+\lambda) + 2\gamma\big] \\
&\quad + \big[(1/48)\,m^4 + (1/288)\,m^2R - (1/864)\,R^2\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big]\Big\}\, g_{\mu\nu}g_{ab} \\
&+ \Big\{(1/144)\,m^2R + (1/5184)\,R^2 \\
&\quad - \big[(1/24)\,m^4 - (1/288)\,m^2R - (1/576)\,R^2\big]\big[\ln(-R/24) + 2\Psi(1/2+\lambda) + 2\gamma\big] \\
&\quad + \big[(1/24)\,m^4 - (1/288)\,m^2R - (1/1728)\,R^2\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big]\Big\}\, g_{\mu(a|}g_{\nu|b)}
\end{aligned} \tag{101b}$$

and

$$w = -(1/2)\,m^2 + (7/72)\,R + \big[(1/2)\,m^2 - (1/12)\,R\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big], \tag{101c}$$

$$\begin{aligned}
w_{ab} = \Big\{ &-(5/16)\,m^4 + (25/288)\,m^2R - (29/5760)\,R^2 \\
&+ \big[(1/8)\,m^4 - (1/48)\,m^2R\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big]\Big\} g_{ab}.
\end{aligned} \tag{101d}$$

In order to simplify the previous expressions and, in particular, to eliminate the terms in $\tan(\pi\kappa)$ and in $\tan(\pi\lambda)$ occurring in the expressions (72) and (73) of the Hadamard Green functions, we have used systematically the relation

$$\Psi(n+1/2+z) + \Psi(n+1/2-z) + \pi\tan(\pi z) = 2\Psi(1/2+z) + 2\,(1-\delta_{n0}) \sum_{p=0}^{n-1} \frac{p+1/2}{(p+1/2)^2 - z^2} \tag{102}$$

(here δ_{nm} is the Kronecker delta) which is valid for $n \in \mathbb{N}$. This relation can be derived from the reflection formula [32–34]

$$\Psi(1-z) = \Psi(z) + \pi\cot(\pi z) \tag{103}$$

making use of $\tan(\pi z) = -\cot[\pi(z+1/2)]$ and of the recurrence formula (97). From the results (101a), (101b) and (101d), we can write

$$\begin{aligned}
s_\rho{}^\rho = &-2\,m^2 + (5/36)\,R \\
&+ \big[(3/2)\,m^2 + (1/4)\,R\big]\big[\ln(-R/24) + 2\Psi(1/2+\lambda) + 2\gamma\big] \\
&+ \big[(1/2)\,m^2 - (1/12)\,R\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big],
\end{aligned} \tag{104a}$$

$$\begin{aligned}
s_{\rho\tau}{}^{\rho\tau} = &-(5/4)\,m^4 - (11/72)\,m^2R + (1/90)\,R^2 \\
&+ \big[(3/16)\,m^2R + (1/32)\,R^2\big]\big[\ln(-R/24) + 2\Psi(1/2+\lambda) + 2\gamma\big] \\
&+ \big[(1/2)\,m^4 - (1/48)\,m^2R - (1/96)\,R^2\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big]
\end{aligned} \tag{104b}$$

and

$$\begin{aligned}
w_\rho{}^\rho = &-(5/4)\,m^4 + (25/72)\,m^2R - (29/1440)\,R^2 \\
&+ \big[(1/2)\,m^4 - (1/12)\,m^2R\big]\big[\ln(-R/24) + 2\Psi(1/2+\kappa) + 2\gamma\big].
\end{aligned} \tag{104c}$$

It should be noted that the constraints (94) and (95) are satisfied by these coefficients.

By inserting now the expressions (104a) and (104b) into (84) taking into account the geometrical term (87) as well as the geometrical ambiguities (89) and (90), we have for the trace of the RSET

$$8\pi^2 \langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren AdS}^4} = (\alpha + 9/4)\, m^4 + (\beta + 5/24) m^2 R - (11/1440)\, R^2 \\ - \left[(3/2)\, m^4 + (1/4)\, m^2 R\right]\left[\ln(-R/(24M^2)) + 2\Psi(1/2+\lambda) + 2\gamma\right]. \tag{105}$$

Finally, by requiring the vanishing of this expression in the flat-space limit, i.e. for $R \to 0$, we can fix the renormalization mass M (after absorption of the term 2γ) and the coefficient α. We then obtain $M = m/\sqrt{2}$ and $\alpha = -9/4$ which leads to

$$8\pi^2 \langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren AdS}^4} = (\beta + 5/24)\, m^2 R - (11/1440)\, R^2 \\ - \left[(3/2)\, m^4 + (1/4)\, m^2 R\right]\left[\ln(-R/(12m^2)) + 2\Psi(1/2+\lambda)\right]. \tag{106}$$

D. Remarks concerning the zero-mass limit of the renormalized stress-energy tensor

Because Stueckelberg massive electromagnetism is a $U(1)$ gauge theory which generalizes Maxwell's theory, we could naively expect to recover, by considering the zero-mass limit of the previous results, the usual trace anomaly for Maxwell's theory given by [see, e.g., Eq. (130) in Ref. [1] or Eq. (3.25) in Ref. [44]]

$$8\pi^2 \langle 0|\widehat{T}^A{}_\rho{}^\rho|0\rangle_{\text{ren}} = 2v_1^A{}_\rho{}^\rho - 4v_1 \tag{107}$$

which reduces, in a maximally symmetric gravitational background, to

$$8\pi^2 \langle 0|\widehat{T}^A{}_\rho{}^\rho|0\rangle_{\text{ren}} = -(31/2160)\, R^2. \tag{108}$$

In fact, this is not the case. In dS^4, for $m^2 \to 0$, Eq. (100) provides

$$8\pi^2 \langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren dS}^4} = (19/1440)\, R^2, \tag{109}$$

while, in AdS^4, for $m^2 \to 0$, we obtain from Eq. (106)

$$8\pi^2 \langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren AdS}^4} = -(11/1440)\, R^2. \tag{110}$$

This "discontinuity" is not really surprising. Indeed, as we have already noted in Ref. [1], in an arbitrary spacetime, due to the contribution of the auxiliary scalar field Φ, the full RSET of Stueckelberg electromagnetism never permits us to recover the RSET of Maxwell's theory. This can be alternatively interpreted by noting that the presence of a mass term in the Stueckelberg theory breaks the conformal invariance of Maxwell's theory. To circumvent this difficulty, we have proposed in Ref. [1] to split the RSET of Stueckelberg electromagnetism into two separately conserved RSETs, a contribution directly associated with the vector field A_μ and another one corresponding to the scalar field Φ (see also our discussion in Sec. III A), the zero-mass limit of the first contribution reducing to the RSET of Maxwell's electromagnetism. Even if we consider that this separation is rather artificial because, in our opinion, only the full RSET is physically relevant, the auxiliary scalar field Φ playing the role of a kind of ghost field, it is interesting to test our proposal which has nevertheless provided correct results in the context of the Casimir effect (see Sec. V of Ref. [1]).

In dS^4, we can obtain the separated RSETs associated with the massive vector field A_μ and the massive scalar field Φ which vanish in the flat-space limit by inserting into Eqs. (86a) and (86b) the Taylor coefficients (98a), (98b) and (96c) and by moreover taking into account the geometrical ambiguities (92a), (92b) and (89). We have

$$8\pi^2 \langle 0|\widehat{T}^A{}_\rho{}^\rho|0\rangle_{\text{ren dS}^4} = (\beta_A + 13/18)\, m^2 R + (59/2160)\, R^2 \\ - \left[(3/2)\, m^4 + (1/4)\, m^2 R\right]\left[\ln(R/(12m^2)) + \Psi(5/2+\lambda) + \Psi(5/2-\lambda)\right] \\ + \left[(1/2)\, m^4 - (1/12)\, m^2 R\right]\left[\ln(R/(12m^2)) + \Psi(5/2+\kappa) + \Psi(5/2-\kappa)\right] \tag{111}$$

for the RSET associated with the massive vector field A_μ and

$$8\pi^2 \langle 0|\widehat{T}^\Phi{}_\rho{}^\rho|0\rangle_{\text{ren dS}^4} = (\beta_\Phi - 5/36)\, m^2 R + (29/4320)\, R^2 \\ - \left[(1/2)\, m^4 - (1/12)\, m^2 R\right]\left[\ln(R/(12m^2)) + \Psi(3/2+\kappa) + \Psi(3/2-\kappa)\right] \tag{112}$$

for the RSET associated with the massive scalar field Φ. It is worth pointing out that the sum of these two RSETs

Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

coincides with the RSET given by Eq. (100). This is a trivial consequence of Eq. (97). We can moreover note that the RSET (112) associated with the scalar field Φ is nothing else than the result derived by Bunch and Davies in Ref. [3] (see also Refs. [2, 4, 5]. If we now consider the limit $m^2 \to 0$ of the RSET (111) associated with the vector field A_μ, we obtain

$$8\pi^2 \langle 0|\widehat{T}^{A}{}_{\rho}{}^{\rho}|0\rangle_{\text{ren dS}^4} = (59/2160)\, R^2. \tag{113}$$

Here, we do not recover the usual trace anomaly (108) of Maxwell's electromagnetism. In fact, this can be explained easily. Indeed, in Secs. IV C 4 and IV D. of Ref. [1], we have constructed the RSET associated with the massive vector field A_μ which reduces, in the zero-mass limit, to the RSET of Maxwell's electromagnetism by assuming tacitely the regularity for $m^2 \to 0$ of the Taylor coefficients involved in the calculations and, in particular, of the coefficient w. This assumption is certainly valid for a large class of spacetimes but, in the context of field theory in dS^4, it is not founded. For example, from Eq. (96c), we can show that $w \sim R^2/(48\, m^2)$ for $m^2 \to 0$. We here encounter the well-known infrared divergence problem which plagues quantum field theories in de Sitter spacetime (see, e.g., Ref [29] and references therein).

In AdS^4, we can obtain the separated RSETs associated with the massive vector field A_μ and the massive scalar field Φ which vanish in the flat-space limit by inserting into Eqs. (86a) and (86b) the Taylor coefficients (104a), (104b) and (101c) and by moreover taking into account the geometrical ambiguities (92a), (92b) and (89). We have

$$\begin{aligned} 8\pi^2 \langle 0|\widehat{T}^{A}{}_{\rho}{}^{\rho}|0\rangle_{\text{ren AdS}^4} = {} & (\beta_A + 7/18)\, m^2 R - (31/2160)\, R^2 \\ & - \left[(3/2)\, m^4 + (1/4)\, m^2 R\right] \left[\ln(-R/(12m^2)) + 2\Psi(1/2+\lambda)\right] \\ & + \left[(1/2)\, m^4 - (1/12)\, m^2 R\right] \left[\ln(-R/(12m^2)) + 2\Psi(1/2+\kappa)\right] \end{aligned} \tag{114}$$

for the RSET associated with the massive vector field A_μ and

$$\begin{aligned} 8\pi^2 \langle 0|\widehat{T}^{\Phi}{}_{\rho}{}^{\rho}|0\rangle_{\text{ren AdS}^4} = {} & (\beta_\Phi - 13/72)\, m^2 R + (29/4320)\, R^2 \\ & - \left[(1/2)\, m^4 - (1/12)\, m^2 R\right] \left[\ln(-R/(12m^2)) + 2\Psi(1/2+\kappa)\right] \end{aligned} \tag{115}$$

for the RSET associated with the massive scalar field Φ. We can note that the sum of these two RSETs coincides with the RSET given by Eq. (106) and that the RSET (115) associated with the scalar field Φ is in agreement with the result derived by Camporesi and Higuchi in Ref. [9] (see also Ref. [8]). If we now consider the limit $m^2 \to 0$ of the RSET (114) associated with the vector field A_μ, we obtain

$$8\pi^2 \langle 0|\widehat{T}^{A}{}_{\rho}{}^{\rho}|0\rangle_{\text{ren AdS}^4} = -(31/2160)\, R^2. \tag{116}$$

In AdS^4, we recover the usual trace anomaly (108) of Maxwell's electromagnetism.

IV. CONCLUSION

In the present article, by focusing on Hadamard vacuum states, we have first constructed the various two-point functions associated with Stueckelberg massive electromagnetism in de Sitter and anti-de Sitter spacetimes. Then, from the general formalism developed in Ref. [1], we have obtained an exact analytical expression for the vacuum expectation value of the RSET of the massive vector field propagating in de Sitter and anti-de Sitter spacetimes. It is worth pointing out that these results have been obtained by working in the unique gauge for which the machinery of Hadamard renormalization can be used (i.e., $\xi = 1$). However, in the literature, Stueckelberg massive electromagnetism is often considered for an arbitrary gauge parameter ξ, i.e., by describing the massive vector field from the action (5) instead of the action (2). Here, we refer to the articles by Fröb and Higuchi [21] who consider the massive vector field in de Sitter spacetime and by Janssen and Dullemond [16] who work in anti-de Sitter spacetime. The propagators constructed in this context are ξ dependent but have good physical properties and, even if they do not have a Hadamard-type singularity at short distance (for $\xi \neq 1$), they reproduce in the flat-space limit the standard Minkowski two-point functions given by Itzykson and Zuber in Ref. [24]. It would be interesting to analyze the physical content of these propagators by constructing their associated RSETs. Due to the expected gauge independence of the RSET, it is quite likely that the results obtained will be identical to those given in the previous section but a proof of this claim certainly requires a careful study. We plan to provide it in the near future.

ACKNOWLEDGMENTS

We wish to thank Yves Décanini and Mohamed Ould El Hadj for various discussions, Philippe Spindel for providing us with a copy of Ref. [13] and the "Collectivité Territoriale de Corse" for its support through the COMPA project.

[1] A. Belokogne and A. Folacci, "Stueckelberg massive electromagnetism in curved spacetime: Hadamard renormalization of the stress-energy tensor and the Casimir effect," Phys. Rev. D **93**, 044063 (2016), arXiv:1512.06326 [gr-qc].
[2] J. S. Dowker and R. Critchley, "Effective Lagrangian and energy-momentum tensor in de Sitter space," Phys. Rev. D **13**, 3224 (1976).
[3] T. S. Bunch and P. C. W. Davies, "Quantum field theory in de Sitter space: Renormalization by point-splitting," Proc. Roy. Soc. Lond. **A360**, 117–134 (1978).
[4] D. Bernard and A. Folacci, "Hadamard function, stress tensor and de Sitter space," Phys. Rev. D **34**, 2286 (1986).
[5] S. Tadaki, "Stress tensor in de Sitter space," Prog. Theor. Phys. **80**, 654–662 (1988).
[6] R. Camporesi, "zeta function regularization of one loop effective potentials in anti-de Sitter space-time," Phys. Rev. D **43**, 3958–3965 (1991).
[7] M. M. Caldarelli, "Quantum scalar fields on anti-de Sitter space-time," Nucl. Phys. B **549**, 499–515 (1999), arXiv:hep-th/9809144.
[8] C. Kent and E. Winstanley, "Hadamard renormalized scalar field theory on anti-de Sitter spacetime," Phys. Rev. D **91**, 044044 (2015), arXiv:1408.6738 [gr-qc].
[9] R. Camporesi and A. Higuchi, "Stress-energy tensors in anti-de Sitter space-time," Phys. Rev. D **45**, 3591–3603 (1992).
[10] A. Landete, J. Navarro-Salas, and F. Torrenti, "Adiabatic regularization for spin-1/2 fields," Phys. Rev. D **88**, 061501 (2013), arXiv:1305.7374 [gr-qc].
[11] V. E. Ambrus and E. Winstanley, "Renormalised fermion vacuum expectation values on anti-de Sitter spacetime," Phys. Lett. B **749**, 597–602 (2015), arXiv:1505.04962 [hep-th].
[12] G. Boerner and H. P. Duerr, "Classical and quantum fields in de Sitter space," Nuovo Cim. A **64**, 669–714 (1969).
[13] C. Schomblond and P. Spindel, "Propagateurs des champs spinoriels et vectoriels dans l'univers de de Sitter," Bulletin de l'Académie Royale de Belgique (Classe des Sciences) **62**, 124 (1976).
[14] W. Drechsler and R. Sasaki, "Solutions of invariant field equations in (4,1) de Sitter space," Nuovo Cim. A **46**, 527 (1978).
[15] B. Allen and T. Jacobson, "Vector two-point functions in maximally symmetric spaces," Commun. Math. Phys. **103**, 669 (1986).
[16] H. Janssen and C. Dullemond, "Propagators for massive vector fields in anti-de Sitter space-time using Stueckelberg's Lagrangian," J. Math. Phys. **28**, 1023 (1987).
[17] J.-P. Gazeau and M. Hans, "Integral-spin fields on (3+2)-de Sitter space," J. Math. Phys. **29**, 2533 (1988).
[18] J.-P. Gazeau and M. V. Takook, " 'Massive' vector field in de Sitter space," J. Math. Phys. **41**, 5920–5933 (2000), [J. Math. Phys. 43, 6379 (2002)], arXiv:gr-qc/9912080.
[19] N. C. Tsamis and R. P. Woodard, "Maximally symmetric vector propagator," J. Math. Phys. **48**, 052306 (2007), arXiv:gr-qc/0608069.
[20] S. P. Miao, N. C. Tsamis, and R. P. Woodard, "De Sitter breaking through infrared divergences," J. Math. Phys. **51**, 072503 (2010), arXiv:1002.4037 [gr-qc].
[21] M. B. Fröb and A. Higuchi, "Mode-sum construction of the two-point functions for the Stueckelberg vector fields in the Poincaré patch of de Sitter space," J. Math. Phys. **55**, 062301 (2014), arXiv:1305.3421 [gr-qc].
[22] G. Narain, "Green's function of the vector fields on de Sitter background," (2014), arXiv:1408.6193 [gr-qc].
[23] H. Ruegg and M. Ruiz-Altaba, "The Stueckelberg field," Int. J. Mod. Phys. A **19**, 3265–3348 (2004), arXiv:hep-th/0304245.
[24] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[25] C. Dullemond and E. van Beveren, "Scalar field propagators in anti-de Sitter space-time," J. Math. Phys. **26**, 2050–2058 (1985).
[26] A. Folacci, "Two-point functions and stress-energy tensors of p-forms in de Sitter and anti-de Sitter spaces," J. Math. Phys. **32**, 2828–2838 (1991), [Erratum: J. Math. Phys.33,1932(1992)].
[27] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).
[28] S. J. Avis, C. J. Isham, and D. Storey, "Quantum field theory in anti-de Sitter space-time," Phys. Rev. D **18**, 3565 (1978).
[29] B. Allen, "Vacuum states in de Sitter space," Phys. Rev. D **32**, 3136 (1985).
[30] B. S. DeWitt and R. W. Brehme, "Radiation damping in a gravitational field," Annals Phys. **9**, 220–259 (1960).
[31] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[32] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New-York, 1965).
[33] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions* (Cambridge University Press, New York, NY, 2010) print companion to [34].
[34] DLMF, "NIST Digital Library of Mathematical Functions," http://dlmf.nist.gov/, Release 1.0.10 of 2015-08-07, online companion to [33].
[35] C. Schomblond and P. Spindel, "Conditions d'unicité pour le propagateur $\Delta^1(x,y)$ du champ scalaire dans l'univers de de Sitter," Ann. Inst. H. Poincaré (Phys. Theor.) **25**, 67–78 (1976).
[36] P. Breitenlohner and D. Z. Freedman, "Stability in gauged extended supergravity," Annals Phys. **144**, 249 (1982).
[37] C. P. Burgess and C. A. Lutken, "Propagators and effective potentials in anti-de Sitter space," Phys. Lett. B **153**, 137–141 (1985).

 Published in Phys. Rev. D 94, 105028 (2016), eprint : arXiv:1610.00244

[38] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, 1989).

[39] J. Bros and U. Moschella, "Two-point functions and quantum fields in de Sitter universe," Rev. Math. Phys. **08**, 327–392 (1996), arXiv:gr-qc/9511019.

[40] J. Bros, H. Epstein, and U. Moschella, "Towards a general theory of quantized fields on the anti-de Sitter space-time," Commun. Math. Phys. **231**, 481–528 (2002), arXiv:hep-th/0111255.

[41] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1995).

[42] R. M. Wald, "Trace anomaly of a conformally invariant quantum field in curved spacetime," Phys. Rev. D **17**, 1477–1484 (1978).

[43] W. Tichy and E. E. Flanagan, "How unique is the expected stress-energy tensor of a massive scalar field?" Phys. Rev. D **58**, 124007 (1998), arXiv:gr-qc/9807015.

[44] M. R. Brown and A. C. Ottewill, "Photon propagators and the definition and approximation of renormalized stress tensors in curved space-time," Phys. Rev. D **34**, 1776–1786 (1986).

Conclusion et perspectives

Ce manuscrit est relatif au domaine de la théorie quantique des champs en espace-temps courbe. Cette théorie est une approximation semi-classique de la gravitation quantique qui consiste à traiter classiquement la métrique $g_{\mu\nu}$ de l'espace-temps et à considérer du point de vue quantique tous les autres champs. Il faut préciser que, pour avoir une théorie cohérente, nous devons aussi inclure le champ des gravitons à l'ordre au moins une boucle. Cette approche évite les difficultés dues à la non-renormalisabilité de la théorie quantique de la gravitation et fournit un cadre qui permet d'étudier les conséquences, à basse énergie, d'une hypothétique "théorie du tout". Il faut rappeler que cette approche a permis aux physiciens d'obtenir des résultats fascinants concernant la cosmologie de l'univers primordial ainsi que la physique des trous noirs et, en particulier, a conduit Parker à la découverte de la création de particules dans des univers en expansion et Hawking à celle du rayonnement quantique des trous noirs.

En théorie quantique des champs en espace-temps courbe, il est admis que la réaction en retour d'un champ quantique dans un état quantique normalisé $|\psi\rangle$ sur la géométrie de l'espace-temps est régie par les équations d'Einstein semi-classiques $G_{\mu\nu} = 8\pi \langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$. Ici, $G_{\mu\nu}$ est le tenseur d'Einstein $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}$ ou une généralisation d'ordre supérieur de ce tenseur géométrique alors que $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ est la valeur moyenne du tenseur d'impulsion-énergie associé au champ quantique. La quantité $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ est formellement infinie du fait du comportement singulier à courte distance des fonctions de Green du champ quantique. Afin d'extraire une contribution finie et physiquement raisonnable de cette quantité, il est nécessaire de la régulariser puis de renormaliser toutes les constantes de couplage de la théorie. La valeur moyenne renormalisée $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ est d'une importance fondamentale non seulement parce qu'elle agit comme source dans les équations d'Einstein semi-classiques mais aussi parce qu'elle nous permet d'analyser l'état quantique $|\psi\rangle$ sans faire aucune référence à son contenu en particules.

Dans ce manuscrit, nous avons considéré les théories quantiques des champs massifs en espace-temps courbe à l'ordre une boucle. Par conséquent, la contribution des gravitons négligée doit être rajoutée pour avoir une théorie cohérente. En outre, nous nous sommes intéressés à la contribution "matériel-le" associée aux champs quantiques massifs des théories considérées. Elle est représentée par la valeur moyenne du tenseur d'impulsion-énergie. Comme cette quantité est mal définie, afin d'en extraire une valeur moyenne physiquement raisonnable, nous avons utilisé deux méthodes de renormalisation, à savoir, celle fondée sur le développement de DeWitt-Schwinger de l'action effective et la renormalisation dite de Hadamard.

Dans la première partie de ce manuscrit, nous nous sommes intéressés à l'action effective à l'ordre une boucle associée aux champs quantiques massifs. Nous avons plus particulièrement considéré les théories quantiques libres du champ scalaire massif, du champ de Dirac massif et du champ de Proca. La valeur moyenne renormalisée du tenseur d'impulsion-énergie associé aux champs massifs considérés peut être formellement exprimée en utilisant le développement de DeWitt-Schwinger de l'action effective. Il faut rappeler que l'état quantique associé au champ est négligé pour ce développement ce qui est dû à la représentation de DeWitt-Schwinger des fonctions de Green. En espace-temps de dimension quatre, le développement de DeWitt-Schwinger de l'action effective associée à un champ massif, après renormalisation et dans la limite des grandes masses, se réduit aux termes construits à partir du coefficient de Gilkey-DeWitt a_3 qui est de nature purement géométrique. Sa variation par rapport à la métrique nous donne une bonne approximation analytique pour la valeur moyenne renormalisée du tenseur d'impulsion-énergie d'un champ quantique massif. A partir de l'expression générale de ce tenseur et en utilisant des *packages* écrits sous *Mathematica* qui nous permettent de faire les calculs d'algèbre tensorielle de manière efficace, nous avons obtenu analytiquement la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé au champ scalaire massif, au champ de Dirac massif et au champ de Proca se propageant sur l'espace-temps de Kerr-Newman. Ce résultat nous a permis de retrouver les résultats existant dans la littérature pour les trous noirs de Schwarzschild, de Reissner-Nordström et de Kerr (et d'en corriger certains). Il est important aussi de noter que les expressions associées aux trous noirs en rotation pour les champs massifs négligent l'existence d'instabilités liées à la *superradiance*. En absence de résultats exacts en espace-temps de Kerr-Newman, notre approximation de la valeur moyenne renormalisée du tenseur d'impulsion-énergie pourrait être utile, par exemple, pour étudier l'effet de *backreaction* des champs quantiques massifs sur cet espace-temps ou sur ses modes quasi-normaux. Nous avons d'ailleurs étudié un problème simplifié de *backreaction* consistant à déterminer les modifications de la masse et du moment angulaire du trou noir vues par un observateur à l'infini, et dues à la présence des champs quantiques.

Dans la deuxième partie de ce manuscrit, nous nous sommes intéressés à l'électromagnétisme massif de Stueckelberg. Contrairement à la théorie massive de de Broglie-Proca qui est une généralisation directe de l'électromagnétisme de Maxwell obtenue en ajoutant un terme de masse qui brise la symétrie de jauge locale $U(1)$ originelle, la théorie massive proposée par Stueckelberg la préserve par l'intermédiaire d'un couplage approprié d'un champ scalaire auxiliaire avec le champ vectoriel massif. Nous avons développé le formalisme général de la théorie de Stueckelberg sur un espace-temps arbitraire de dimension quatre (invariance de jauge de la théorie classique et quantification covariante ; équations d'ondes pour le champ massif de spin-1 A_μ, pour le champ scalaire auxiliaire de Stueckelberg Φ et pour les champs de fantômes C et C^* ; identité de Ward ; représentation de Hadamard des divers propagateurs de Feynman et développement en séries de Taylor covariantes des coefficients correspondants). Cela nous a permis de construire, pour un état quantique de type Hadamard $|\psi\rangle$ et en utilisant la méthode de la renormalisation de Hadamard, la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé à la théorie de Stueckelberg. Nous avons donné deux expressions différentes mais équivalentes du résultat. La première expression ne fait intervenir que les caractéristiques du champ vectoriel A_μ. La seconde expression a été construite dans le but de discuter la limite de masse nulle des résultats obtenus dans le cadre de la théorie de Stueckelberg et les comparer avec ceux issus de celle de Maxwell ; ainsi, nous avons séparé de manière artificielle en deux parties les contributions indépendamment conservées du champ vectoriel A_μ et du champ scalaire auxiliaire Φ. Notons que l'on peut passer de la deuxième expression à la première en éliminant la contribution du champ scalaire de Stueckelberg Φ par l'intermédiaire de deux identités de Ward, ce champ pouvant être considéré comme un fantôme. De plus, nous avons discuté des ambiguïtés géométriques de la valeur moyenne renormalisée du tenseur d'impulsion-énergie qui sont d'une importance fondamentale. Comme première application de ce résultat, nous avons considéré, en espace-temps de Minkowski, l'effet de Casimir en dehors d'un milieu parfaitement conducteur et de bord plan. Nous avons retrouvé le résultat existant déjà dans la littérature pour la théorie de de Broglie-Proca. Notons toutefois que nous ne sommes pas vraiment satisfaits des conditions aux limites prises sur le bord du milieu conducteur et que des conditions plus réalistes pourraient être envisagées. Nous avons aussi comparé les formalismes de de Broglie-Proca et de Stueckelberg et discuté les avantages de l'approche de Stueckelberg bien qu'à notre avis ces deux descriptions de l'électromagnétisme massif ne sont que les deux faces d'une même théorie. Cependant, du point de vue pratique, c'est la théorie de Stueckelberg qui est bien adaptée pour pouvoir utiliser la renormalisation de Hadamard. Par la suite, nous avons considéré de nouvelles applications de l'électromagnétisme massif de Stueckelberg. Nous avons travaillé dans les espaces-temps maximalement symétriques de de Sitter et anti-de Sitter. Le fait d'avoir la symétrie maximale nous a permis de construire analytiquement toutes les fonctions de corrélations à deux points associées à l'électromagnétisme de Stueckelberg. A partir des fonctions de Hadamard, nous avons obtenu les expressions analytiques de la valeur moyenne renormalisée du tenseur d'impulsion-énergie associé au champ vectoriel massif en espaces-temps de de Sitter et anti-de Sitter. Des résultats analogues étaient déjà connus pour le champ scalaire et pour le champ de Dirac mais aucun n'avait été obtenu pour le champ vectoriel massif.

Pour finir, nous concluons par quelques prolongements perspectifs du travail présenté.

– Nous pourrions étudier la théorie quantique de l'électromagnétisme massif de Stueckelberg sur les espaces-temps de Friedmann-Lemaître-Robertson-Walker qui sont d'une importance considérable dans le domaine de la cosmologie.
– Il serait très intéressant de considérer la théorie de l'électrodynamique quantique en espace-temps courbe où le champ de jauge deviendrait massif par l'intermédiaire du mécanisme de Stueckelberg. Dans le contexte de la renormalisation de Hadamard, on construirait le tenseur d'impulsion-énergie renormalisé total qui pourrait être considéré par la suite sur l'espace-temps de de Sitter et celui de Friedmann-Lemaître-Robertson-Walker.

BIBLIOGRAPHIE

[1] J. C. MAXWELL, “A Dynamical Theory of the Electromagnetic Field”, Philosophical Transactions of the Royal Society of London **155**, 459 (1865).

[2] J. C. MAXWELL, *A treatise on electricity and magnetism* (Clarendon Press, 1873).

[3] A. EINSTEIN, “Zur Elektrodynamik bewegter Körper”, Annalen der Physik **322**, 891 (1905).

[4] W. VOIGT, “Ueber das Doppler'sche Princip”, Nachr. K. Gesel. Wiss. George-August-Universität **2**, 41 (1887), eprint : https://en.wikisource.org/wiki/Translation:On_the_Principle_of_Doppler.

[5] H. A. LORENTZ, “Electromagnetic phenomena in a system moving with any velocity less than that of light”, Proceedings of the Academy of Sciences of Amsterdam **6**, 809 (1904).

[6] H. POINCARÉ, “Sur la dynamique de l'électron”, Note de H. Poincaré aux C.R. Acad. Sci. Paris **140**, 1504 (1905).

[7] H. POINCARÉ, “Sur la dynamique de l'électron”, Rend. Circ. Matem. Palermo **21**, 17 (1906).

[8] H. MINKOWSKI, “Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern”, Mathematische Annalen **68**, 472 (1910).

[9] M. K. E. L. PLANCK, “Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum”, Verhandl. Dtsc. Phys. Ges. **2**, 237 (1900).

[10] M. K. E. L. PLANCK, “Ueber das Gesetz der Energieverteilung im Normalspectrum”, Annalen der Physik **309**, 553 (1901).

[11] A. EINSTEIN, “Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt”, Annalen der Physik **322**, 132 (1905).

[12] L. DE BROGLIE, “Recherches sur la théorie des Quanta”, Theses (1924), eprint : https://tel.archives-ouvertes.fr/tel-00006807.

[13] W. HEISENBERG, “Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen”, Zeitschrift für Physik **33**, 879 (1925).

[14] M. BORN et P. JORDAN, “Zur quantenmechanik”, Zeitschrift für Physik **34**, 858 (1925).

[15] P. A. M. DIRAC, “The Fundamental Equations of Quantum Mechanics”, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **109**, 642 (1925).

[16] P. A. M. DIRAC, “On the Theory of Quantum Mechanics”, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **112**, 661 (1926).

[17] M. BORN, W. HEISENBERG et P. JORDAN, "Zur Quantenmechanik. II.", Zeitschrift für Physik **35**, 557 (1926).

[18] E. SCHRÖDINGER, "An Undulatory Theory of the Mechanics of Atoms and Molecules", Phys. Rev. **28**, 1049 (1926).

[19] W. PAULI, "Zur Quantenmechanik des magnetischen Elektrons", Zeitschrift für Physik **43**, 601 (1927).

[20] P. A. M. DIRAC, "The Quantum Theory of the Emission and Absorption of Radiation", Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **114**, 243 (1927).

[21] P. A. M. DIRAC, "The Quantum Theory of Dispersion", Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **114**, 710 (1927).

[22] O. KLEIN, "Quantentheorie und fünfdimensionale Relativitätstheorie", Zeitschrift für Physik **37**, 895 (1926).

[23] W. GORDON, "Der Comptoneffekt nach der Schrödingerschen Theorie", Zeitschrift für Physik **40**, 117 (1926).

[24] P. JORDAN et W. PAULI, "Zur Quantenelektrodynamik ladungsfreier Felder", Zeitschrift für Physik **47**, 151 (1928).

[25] P. A. M. DIRAC, "The Quantum Theory of the Electron", Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **117**, 610 (1928).

[26] P. JORDAN et E. WIGNER, "Über das Paulische Äquivalenzverbot", Zeitschrift für Physik **47**, 631 (1928).

[27] W. HEISENBERG et W. PAULI, "Zur Quantendynamik der Wellenfelder", Zeitschrift für Physik **56**, 1 (1929).

[28] W. HEISENBERG et W. PAULI, "Zur Quantentheorie der Wellenfelder. II.", Zeitschrift für Physik **59**, 168 (1930).

[29] E. FERMI, "Sopra l'elettrodinamica quantistica", Atti della Reale Academia Nazionale dei Lincei **9**, 881 (1929).

[30] E. FERMI, "Quantum Theory of Radiation", Rev. Mod. Phys. **4**, 87 (1932).

[31] P. DIRAC, *The Principles of Quantum Mechanics* (Clarendon Press, 1930).

[32] W. H. FURRY et J. R. OPPENHEIMER, "On the Theory of the Electron and Positive", Phys. Rev. **45**, 245 (1934).

[33] W. PAULI et V. WEISSKOPF, ""Uber die Quantisierung der skalaren relativistischen Wellengleichung", Helv. Phys. Acta **7**, 709 (1934).

[34] S. WEINBERG, "The Search for Unity : Notes for a History of Quantum Field Theory", Daedalus **106**, 17 (1977).

[35] S. WEINBERG, "What is quantum field theory, and what did we think it is ?", in Conceptual foundations of quantum field theory. Proceedings, Symposium and Workshop, Boston, USA, March 1-3, 1996 (1996), p. 241–251, arXiv :hep-th/9702027.

[36] S. WEINBERG, *The quantum theory of fields*, t. 1 (Cambridge university press, 1996).

[37] J. R. OPPENHEIMER, "Note on the Theory of the Interaction of Field and Matter", Phys. Rev. **35**, 461 (1930).

[38] I. WALLER, "Die Streuung von Strahlung durch gebundene und freie Elektronen nach der Diracschen relativistischen Mechanik", Zeitschrift für Physik **61**, 837 (1930).

[39] I. WALLER, "Bemerkungen über die Rolle der Eigenenergie des Elektrons in der Quantentheorie der Strahlung", Zeitschrift für Physik **62**, 673 (1930).

[40] E. A. UEHLING, "Polarization Effects in the Positron Theory", Phys. Rev. **48**, 55 (1935).

[41] S. PASTERNACK, "Note on the Fine Structure of Hα and Dα", Phys. Rev. **54**, 1113 (1938).

[42] H. A. BETHE, "The Electromagnetic Shift of Energy Levels", Phys. Rev. **72**, 339 (1947).

[43] W. E. LAMB et R. C. RETHERFORD, "Fine Structure of the Hydrogen Atom by a Microwave Method", Phys. Rev. **72**, 241 (1947).

[44] J. E. NAFE, E. B. NELSON et I. I. RABI, "The Hyperfine Structure of Atomic Hydrogen and Deuterium", Phys. Rev. **71**, 914 (1947).

[45] D. E. NAGLE, R. S. JULIAN et J. R. ZACHARIAS, "The Hyperfine Structure of Atomic Hydrogen and Deuterium", Phys. Rev. **72**, 971 (1947).

[46] P. KUSCH et H. M. FOLEY, "Precision Measurement of the Ratio of the Atomic 'g Values' in the $^2P_{\frac{3}{2}}$ and $^2P_{\frac{1}{2}}$ States of Gallium", Phys. Rev. **72**, 1256 (1947).

[47] G. BREIT, "Does the Electron Have an Intrinsic Magnetic Moment ?", Phys. Rev. **72**, 984 (1947).

[48] J. SCHWINGER, "On Quantum-Electrodynamics and the Magnetic Moment of the Electron", Phys. Rev. **73**, 416 (1948).

[49] V. F. WEISSKOPF, "Über die elektrodynamik des vakuums auf grund des quanten-theorie des elektrons", Dan. Mat. Fys. Medd. **14**, 1 (1936).

[50] H. KRAMERS, "The interaction between charged particles and the radiation field", Nuovo Cim. **15**, 108 (1938).

[51] J. SCHWINGER, "Quantum Electrodynamics. I. A Covariant Formulation", Phys. Rev. **74**, 1439 (1948).

[52] J. SCHWINGER, "Quantum Electrodynamics. II. Vacuum Polarization and Self-Energy", Phys. Rev. **75**, 651 (1949).

[53] J. SCHWINGER, "Quantum Electrodynamics. III. The Electromagnetic Properties of the Electron–Radiative Corrections to Scattering", Phys. Rev. **76**, 790 (1949).

[54] J. SCHWINGER, "On Gauge Invariance and Vacuum Polarization", Phys. Rev. **82**, 664 (1951).

[55] J. SCHWINGER, "On the Green's functions of quantized fields. I", Proceedings of the National Academy of Sciences **37**, 452 (1951).

[56] J. SCHWINGER, "The Theory of Quantized Fields. I", Phys. Rev. **82**, 914 (1951).

[57] J. SCHWINGER, "The Theory of Quantized Fields. II", Phys. Rev. **91**, 713 (1953).

[58] S. TOMONAGA, "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields", Progress of Theoretical Physics **1**, 27 (1946).

[59] Z. KOBA, T. TATI et S. TOMONAGA, "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. II : Case of Interacting Electromagnetic and Electron Fields", Progress of Theoretical Physics **2**, 101 (1947).

[60] S. KANESAWA et S. TOMONAGA, "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields V : Case of Interacting Electromagnetic and Meson Fields", Progress of Theoretical Physics **3**, 101 (1948).

[61] S. TOMONAGA et J. R. OPPENHEIMER, "On Infinite Field Reactions in Quantum Field Theory", Phys. Rev. **74**, 224 (1948).

[62] D. ITÔ, Z. KOBA et S. TOMONAGA, "Corrections due to the Reaction of "Cohesive Force Field" to the Elastic Scattering of an Electron. I", Progress of Theoretical Physics **3**, 276 (1948).

[63] Z. KOBA et S. TOMONAGA, "On Radiation Reactions in Collision Processes. I : Application of the "Self-Consistent" Subtraction Method to the Elastic Scattering of an Electron", Progress of Theoretical Physics **3**, 290 (1948).

[64] R. P. FEYNMAN, "Space-Time Approach to Non-Relativistic Quantum Mechanics", Rev. Mod. Phys. **20**, 367 (1948).

[65] R. P. FEYNMAN, "A Relativistic Cut-Off for Classical Electrodynamics", Phys. Rev. **74**, 939 (1948).

[66] R. P. FEYNMAN, "Relativistic Cut-Off for Quantum Electrodynamics", Phys. Rev. **74**, 1430 (1948).

[67] R. P. FEYNMAN, "The Theory of Positrons", Phys. Rev. **76**, 749 (1949).

[68] R. P. FEYNMAN, "Space-Time Approach to Quantum Electrodynamics", Phys. Rev. **76**, 769 (1949).

[69] R. P. FEYNMAN, "Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction", Phys. Rev. **80**, 440 (1950).

[70] F. J. DYSON, "The Radiation Theories of Tomonaga, Schwinger, and Feynman", Phys. Rev. **75**, 486 (1949).

[71] F. J. DYSON, "The S Matrix in Quantum Electrodynamics", Phys. Rev. **75**, 1736 (1949).

[72] C. N. YANG et R. L. MILLS, "Conservation of Isotopic Spin and Isotopic Gauge Invariance", Phys. Rev. **96**, 191 (1954).

[73] E. FERMI, "Versuch einer Theorie der β-Strahlen. I", Zeitschrift für Physik **88**, 161 (1934).

[74] E. C. G. SUDARSHAN et R. E. MARSHAK, "The nature of the four-fermion interaction", (1994).

[75] R. P. FEYNMAN et M. GELL-MANN, "Theory of the Fermi Interaction", Phys. Rev. **109**, 193 (1958).

[76] M. KOBAYASHI et T. MASKAWA, "CP-Violation in the Renormalizable Theory of Weak Interaction", Progress of Theoretical Physics **49**, 652 (1973).

[77] D. J. GROSS et F. WILCZEK, "Ultraviolet Behavior of Non-Abelian Gauge Theories", Phys. Rev. Lett. **30**, 1343 (1973).

[78] H. D. POLITZER, "Reliable Perturbative Results for Strong Interactions ?", Phys. Rev. Lett. **30**, 1346 (1973).

[79] S. L. GLASHOW, "Partial-symmetries of weak interactions", Nuclear Physics **22**, 579 (1961).

[80] A. SALAM et J. C. WARD, "Weak and electromagnetic interactions", Il Nuovo Cimento (1955-1965) **11**, 568 (1959).

[81] A. SALAM et J. WARD, "Electromagnetic and weak interactions", Physics Letters **13**, 168 (1964).

[82] S. WEINBERG, "A Model of Leptons", Phys. Rev. Lett. **19**, 1264 (1967).

[83] L.-C. TU, J. LUO et G. GILLIES, "The mass of the photon", Rept. Prog. Phys. **68**, 77 (2005).

[84] D. D. RYUTOV, "Using Plasma Physics to Weigh the Photon", Plasma Phys. Control. Fusion **49**, B429 (2007).

[85] A. S. GOLDHABER et M. M. NIETO, "Photon and Graviton Mass Limits", Rev. Mod. Phys. **82**, 939 (2010).

[86] C. PATRIGNANI et AL. (PARTICLE DATA GROUP), "Review of Particle Physics", Chinese Physics C **40**, 100001 (2016).

[87] S. NUSSINOV et R. SHROCK, "Upper limits on a possible gluon mass", Phys. Rev. D **82**, 034031 (2010), arXiv :1005.0850 [hep-ph].

[88] F. ENGLERT et R. BROUT, "Broken Symmetry and the Mass of Gauge Vector Mesons", Phys. Rev. Lett. **13**, 321 (1964).

[89] P. W. HIGGS, "Broken Symmetries and the Masses of Gauge Bosons", Phys. Rev. Lett. **13**, 508 (1964).

[90] G. S. GURALNIK, C. R. HAGEN et T. W. B. KIBBLE, "Global Conservation Laws and Massless Particles", Phys. Rev. Lett. **13**, 585 (1964).

[91] T. W. B. KIBBLE, "History of electroweak symmetry breaking", Journal of Physics: Conference Series **626**, 012001 (2015), arXiv :1502.06276 [physics.hist-ph].

[92] H. GEORGI et S. L. GLASHOW, "Unity of All Elementary-Particle Forces", Phys. Rev. Lett. **32**, 438 (1974).

[93] A. BURAS, J. ELLIS, M. GAILLARD et al., "Aspects of the grand unification of strong, weak and electromagnetic interactions", Nuclear Physics B **135**, 66 (1978).

[94] A. EINSTEIN, "Die Feldgleichungen der Gravitation", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 844 (1915).

[95] A. EINSTEIN, "Erklarung der Perihelionbewegung der Merkur aus der allgemeinen Relativitatstheorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin) **47**, 831 (1915).

[96] A. S. EDDINGTON, "The total eclipse of 1919 May 29 and the influence of gravitation on light", The Observatory **42**, 119 (1919).

[97] A. EINSTEIN, "ÜBER DEN EINFLUSS DER SCHWERKRAFT AUF DIE AUSBREITUNG DES LICHTES", Annalen der Physik **340**, 898 (1911).

[98] R. V. POUND et G. A. REBKA, "Gravitational Red-Shift in Nuclear Resonance", Phys. Rev. Lett. **3**, 439 (1959).

[99] A. EINSTEIN, "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 142 (1917).

[100] A. FRIEDMAN, "Über die Krümmung des Raumes", Zeitschrift für Physik **10**, 377 (1922).

[101] A. G. LEMAÎTRE, "Un Univers homogéne de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques", Annales de la Société Scientifique de Bruxelles **47**, 49 (1927).

[102] H. P. ROBERTSON, "Kinematics and World-Structure", The Astrophysical Journal **82**, 284 (1935).

[103] A. G. WALKER, "On Milne's Theory of World-Structure", Proceedings of the London Mathematical Society **s2-42**, 90 (1937).

[104] R. H. RALPH A. ALPHER, George Gamow, "Thermal Cosmic Radiation and the Formation of Protogalaxies", Proceedings of the National Academy of Sciences of the United States of America **58**, 2179 (1967).

[105] A. A. PENZIAS et R. W. WILSON, "A Measurement of Excess Antenna Temperature at 4080 Mc/s.", The Astrophysical Journal **142**, 419 (1965).

[106] V. M. SLIPHER, "The radial velocity of the Andromeda Nebula", Lowell Observatory Bulletin **2**, 56 (1913).

[107] V. M. SLIPHER, "Spectrographic Observations of Nebulae", Popular Astronomy **23**, 21 (1915).

[108] E. HUBBLE, "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae", Proceedings of the National Academy of Science **15**, 168 (1929).

[109] W. de SITTER, "Einstein's theory of gravitation and its astronomical consequences. Third paper", Monthly Notices of the Royal Astronomical Society **78**, 3 (1917).

[110] A. A. STAROBINSKII, "Spectrum of relict gravitational radiation and the early state of the universe", ZhETF Pisma Redaktsiiu **30**, 719 (1979).

[111] A. H. GUTH, "Inflationary universe : A possible solution to the horizon and flatness problems", Phys. Rev. D **23**, 347 (1981).

[112] A. LINDE, "A new inflationary universe scenario : A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems", Physics Letters B **108**, 389 (1982).

[113] A. ALBRECHT et P. J. STEINHARDT, "Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking", Phys. Rev. Lett. **48**, 1220 (1982).

[114] A. EINSTEIN, "Näherungsweise Integration der Feldgleichungen der Gravitation", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 688 (1916).

[115] K. SCHWARZSCHILD, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), 189 (1916), arXiv :physics/9905030.

[116] H. REISSNER, "Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie", Annalen der Physik **355**, 106 (1916).

[117] G. NORDSTRÖM, "On the Energy of the Gravitation field in Einstein's Theory", Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences **20**, 1238 (1918).

[118] R. P. KERR, "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics", Phys. Rev. Lett. **11**, 237 (1963).

[119] E. T. NEWMAN, E. COUCH, K. CHINNAPARED et al., "Metric of a Rotating, Charged Mass", Journal of Mathematical Physics **6**, 918 (1965).

[120] R. A. HULSE et J. TAYLOR, "Discovery of a pulsar in a binary system", The Astrophysical Journal, Letters **195**, 51 (1975).

[121] J. H. TAYLOR et J. M. WEISBERG, "A new test of general relativity - Gravitational radiation and the binary pulsar PSR 1913+16", The Astrophysical Journal **253**, 908 (1982).

[122] M. ODA, P. GORENSTEIN, H. GURSKY et al., “X-Ray Pulsations from Cygnus X-1 Observed from UHURU”, The Astrophysical Journal, Letters **166**, 1 (1971).

[123] D. M. EARDLEY et W. H. PRESS, “Astrophysical processes near black holes”, Annual Review of Astronomy & Astrophysics **13**, 381 (1975).

[124] B. BALICK et R. L. BROWN, “Intense sub-arcsecond structure in the galactic center”, The Astrophysical Journal **194**, 265 (1974).

[125] W. M. GOSS, R. L. BROWN et K. Y. LO, “The Discovery of Sgr A*”, Astronomische Nachrichten **324**, 497 (2003), arXiv :astro-ph/0305074 [astro-ph].

[126] B. P. ABBOTT et al., “Observation of Gravitational Waves from a Binary Black Hole Merger”, Phys. Rev. Lett. **116**, 061102 (2016), arXiv :1602.03837 [gr-qc].

[127] C. ROVELLI, “Loop Quantum Gravity”, Living Reviews in Relativity **1** (1998) 10.1007/lrr-1998-1.

[128] R. GREGORY, “Supergravity”, Classical and Quantum Gravity **30**, 039001 (2013).

[129] J. H. SCHWARZ, “Introduction to superstring theory”, NATO Sci. Ser. C **566**, 143 (2001), arXiv :hep-ex/0008017 [hep-ex].

[130] S. W. HAWKING et G. F. R. ELLIS, *The large scale structure of space-time* (Cambridge University Press, Cambridge, England, 1973).

[131] B. S. DEWITT, “Quantum field theory in curved spacetime”, Physics Reports **19**, 295 (1975).

[132] N. D. BIRRELL et P. C. W. DAVIES, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[133] S. A. FULLING, *Aspects of Quantum Field Theory in Curved Space-time* (Cambridge University Press, Cambridge, 1989).

[134] R. M. WALD, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1995).

[135] L. E. PARKER et D. J. TOMS, *Quantum Field Theory in Curved Spacetime : Quantized Fields and Gravity* (Cambridge University Press, Cambridge, 2009).

[136] L. H. FORD, “Spacetime in semiclassical gravity”, (2005), arXiv :gr-qc/0504096.

[137] S. HOLLANDS et R. M. WALD, “Quantum fields in curved spacetime”, (2014), arXiv :1401.2026 [gr-qc].

[138] J. MARTIN, “Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask)”, Comptes Rendus Physique **13**, 566 (2012), arXiv :1205.3365 [astro-ph.CO].

[139] E. SCHRÖDINGER, “Dirac’sches Elektron im Schwerefeld. I. (German) [Dirac’s electron in the gravitational field. I]”, Sitzungsberichte der Preussischen Akademie der Wissenschaften : Physikalisch-Mathematische Klasse, 105 (1932).

[140] L. E. PARKER, “Quantized fields and particle creation in expanding universes. 1.”, Phys. Rev. **183**, 1057 (1969).

[141] S. W. HAWKING, “Particle Creation by Black Holes”, Commun. Math. Phys. **43**, 199 (1975).

[142] A. A. STAROBINSKY, “A New Type of Isotropic Cosmological Models Without Singularity”, Phys. Lett. B **91**, 99 (1980).

[143] J. M. BARDEEN, “Black Holes Do Evaporate Thermally”, Phys. Rev. Lett. **46**, 382 (1981).

[144] W. A. HISCOCK, “Models of Evaporating Black Holes”, Phys. Rev. D **23**, 2813 (1981).

[145] L. PARKER et D. A. T. VANZELLA, “Acceleration of the universe, vacuum metamorphosis, and the large time asymptotic form of the heat kernel”, Phys. Rev. D **69**, 104009 (2004), arXiv :gr-qc/0312108.

[146] G. W. GIBBONS et S. W. HAWKING, “Cosmological event horizons, thermodynamics, and particle creation”, Phys. Rev. D **15**, 2738 (1977).

[147] P. DAVIES, "Particles do not Exist", in *Quantum theory of gravity*, sous la dir. de S. M. CHRISTENSEN (Adam Hilger Ltd, 1984), p. 66.

[148] W. G. UNRUH, "Notes on black-hole evaporation", Phys. Rev. D **14**, 870 (1976).

[149] B. S. DEWITT, *Dynamical Theory of Groups and Fields* (Gordon et Breach, New York, 1965).

[150] S. M. CHRISTENSEN, "Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background : The Covariant Point Separation Method", Phys. Rev. D **14**, 2490 (1976).

[151] S. M. CHRISTENSEN, "Regularization, Renormalization, and Covariant Geodesic Point Separation", Phys. Rev. D **17**, 946 (1978).

[152] I. G. AVRAMIDI, "Covariant methods for the calculation of the effective action in quantum field theory and investigation of higher derivative quantum gravity", (1995), arXiv :hep-th/9510140.

[153] I. G. AVRAMIDI, *Heat Kernel and Quantum Gravity* (Springer, Berlin, 2000).

[154] B. S. DEWITT, *The Global Approach to Quantum Field Theory* (Oxford University Press, Oxford, 2003).

[155] J. S. DOWKER et R. CRITCHLEY, "Effective Lagrangian and energy-momentum tensor in de Sitter space", Phys. Rev. D **13**, 3224 (1976).

[156] V. P. FROLOV et A. I. ZELNIKOV, "Vacuum polarization by a massive scalar field in Schwarzschild space-time", Phys. Lett. B **115**, 372 (1982).

[157] V. P. FROLOV et A. I. ZELNIKOV, "Vacuum polarization of massive fields in Kerr space-time", Phys. Lett. B **123**, 197 (1983).

[158] V. P. FROLOV et A. I. ZELNIKOV, "Vacuum polarization of massive fields near rotating black holes", Phys. Rev. D **29**, 1057 (1984).

[159] V. P. FROLOV et I. D. NOVIKOV, *Black Hole Physics* (Kluwer Academic Publishers, Dordrecht, 1998).

[160] P. R. ANDERSON, W. A. HISCOCK et D. A. SAMUEL, "Stress-energy tensor of quantized scalar fields in static spherically symmetric space-times", Phys. Rev. D **51**, 4337 (1995).

[161] B. E. TAYLOR, W. A. HISCOCK et P. R. ANDERSON, "Stress energy of a quantized scalar field in static wormhole space-times", Phys. Rev. D **55**, 6116 (1997), arXiv :gr-qc/9608036.

[162] W. A. HISCOCK, S. L. LARSON et P. R. ANDERSON, "Semiclassical effects in black hole interiors", Phys. Rev. D **56**, 3571 (1997), arXiv :gr-qc/9701004.

[163] B. E. TAYLOR, W. A. HISCOCK et P. R. ANDERSON, "Semiclassical charged black holes with a quantized massive scalar field", Phys. Rev. D **61**, 084021 (2000), arXiv :gr-qc/9911119.

[164] J. MATYJASEK, "Stress energy tensor of neutral massive fields in the Reissner-Nordstrom space-time", Phys. Rev. D **61**, 124019 (2000), arXiv :gr-qc/9912020.

[165] P. R. ANDERSON, W. A. HISCOCK et B. E. TAYLOR, "Do semiclassical zero temperature black holes exist ?", Phys. Rev. Lett. **85**, 2438 (2000), arXiv :gr-qc/0002007.

[166] J. MATYJASEK, "Vacuum polarization of massive scalar fields in the space-time of the electrically charged nonlinear black hole", Phys. Rev. D **63**, 084004 (2001), arXiv :gr-qc/0010097.

[167] J. MATYJASEK, "Entropy of quantum-corrected black holes", Phys. Rev. D **74**, 104030 (2006), arXiv :gr-qc/0610020.

[168] J. MATYJASEK et K. ZWIERZCHOWSKA, "Semiclassical lukewarm black holes", Phys. Rev. D **85**, 024009 (2012), arXiv :1110.0041 [gr-qc].

[169] O. P. FERNÁNDEZ PIEDRA et A. Cabo Montes de OCA, "Quantization of massive scalar fields over static black string backgrounds", Phys. Rev. D **75**, 107501 (2007), arXiv :0707.0706 [gr-qc].

[170] O. P. FERNÁNDEZ PIEDRA et J. de OLIVEIRA, "Vacuum polarization effects on quasinormal modes in electrically charged black hole spacetimes", Int. J. Mod. Phys. D **19**, 63 (2010), arXiv :0902.1487 [gr-qc].

[171] O. P. FERNÁNDEZ PIEDRA et J. de OLIVEIRA, "Quasinormal modes of semiclassical electrically charged black holes", Classical Quantum Gravity **28**, 085003 (2011), arXiv :1006.3802 [gr-qc].

[172] J. MATYJASEK et P. SADURSKI, "Stress-Energy Tensor of the Quantized Massive Fields in Friedman-Robertson-Walker Spacetimes", Phys. Rev. D **88**, 104015 (2013), arXiv :1309.0552 [gr-qc].

[173] J. MATYJASEK, P. SADURSKI et M. TELECKA, "Vacuum polarization of the quantized massive fields in Friedman-Robertson-Walker spacetime", Phys. Rev. D **89**, 084055 (2014).

[174] R. M. WALD, "The Back Reaction Effect in Particle Creation in Curved Space-Time", Commun. Math. Phys. **54**, 1 (1977).

[175] R. M. WALD, "Trace Anomaly of a Conformally Invariant Quantum Field in Curved Space-Time", Phys. Rev. D **17**, 1477 (1978).

[176] S. L. ADLER, J. LIEBERMAN et Y. J. NG, "Regularization of the Stress Energy Tensor for Vector and Scalar Particles Propagating in a General Background Metric", Annals Phys. **106**, 279 (1977).

[177] S. L. ADLER et J. LIEBERMAN, "Trace Anomaly of the Stress - Energy Tensor for Massless Vector Particles Propagating in a General Background Metric", Annals Phys. **113**, 294 (1978).

[178] M. A. CASTAGNINO et D. D. HARARI, "Hadamard Renormalization in Curved Space-time", Annals Phys. **152**, 85 (1984).

[179] M. R. BROWN et A. C. OTTEWILL, "Photon Propagators and the Definition and Approximation of Renormalized Stress Tensors in Curved Space-time", Phys. Rev. D **34**, 1776 (1986).

[180] D. BERNARD et A. FOLACCI, "Hadamard Function, Stress Tensor and De Sitter Space", Phys. Rev. D **34**, 2286 (1986).

[181] S. TADAKI, "Hadamard Regularization in Curved Space-time", Prog. Theor. Phys. **77**, 671 (1987).

[182] B. ALLEN, A. FOLACCI et A. C. OTTEWILL, "The Renormalized Graviton Stress - Energy Tensor in Curved Vacuum Space-times", Phys. Rev. D **38**, 1069 (1988).

[183] A. FOLACCI, "Quantum field theory of p forms in curved space-time", J. Math. Phys. **32**, 2813 (1991).

[184] A. FOLACCI, "Two-point functions and stress-energy tensors of p-forms in de Sitter and anti-de Sitter spaces", J. Math. Phys. **32**, [Erratum: J. Math. Phys.33,1932(1992)], 2828 (1991).

[185] Y. DECANINI et A. FOLACCI, "Hadamard renormalization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension", Phys. Rev. D **78**, 044025 (2008), arXiv :gr-qc/0512118.

[186] V. MORETTI, "Comments on the stress energy tensor operator in curved space-time", Commun. Math. Phys. **232**, 189 (2003), arXiv :gr-qc/0109048.

[187] T.-P. HACK et V. MORETTI, "On the Stress-Energy Tensor of Quantum Fields in Curved Spacetimes - Comparison of Different Regularization Schemes and Symmetry of the Hadamard/Seeley-DeWitt Coefficients", J. Phys. A **45**, 374019 (2012), arXiv :1202.5107 [gr-qc].

[188] C. KENT et E. WINSTANLEY, "Hadamard renormalized scalar field theory on anti-de Sitter spacetime", Phys. Rev. D **91**, 044044 (2015), arXiv :1408.6738 [gr-qc].

[189] L. de BROGLIE, *Recherches sur la théorie des quantas - seconde thèse* (Université de Paris-Sorbonne, Paris, 1924).

[190] L. de BROGLIE, "Rayonnement noir et quanta de lumière", J. Phys. Radium **3**, 422 (1922).

[191] L. de BROGLIE, *Une nouvelle conception de la lumière.* (Hermann, Paris, 1934).

[192] L. de BROGLIE, *Nouvelles recherches sur la lumière.* (Hermann, Paris, 1936).

[193] L. de BROGLIE, *Une nouvelle théorie de la lumière, la mécanique ondulatoire du photon.* (Hermann, Paris, 1940).

[194] A. PROCA, "Sur la theorie ondulatoire des electrons positifs et negatifs", J. Phys. Radium **7**, 347 (1936).

[195] E. C. G. STUECKELBERG, "Interaction energy in electrodynamics and in the field theory of nuclear forces", Helv. Phys. Acta **11**, 225 (1938).

[196] E. C. G. STUECKELBERG, "Interaction forces in electrodynamics and in the field theory of nuclear forces", Helv. Phys. Acta **11**, 299 (1938).

[197] E. C. G. STUECKELBERG, "Theory of the radiation of photons of small arbitrary mass", Helv. Phys. Acta **30**, 209 (1957).

[198] H. RUEGG et M. RUIZ-ALTABA, "The Stueckelberg field", Int. J. Mod. Phys. A **19**, 3265 (2004), arXiv :hep-th/0304245.

[199] J. JAECKEL et A. RINGWALD, "The Low-Energy Frontier of Particle Physics", Ann. Rev. Nucl. Part. Sci. **60**, 405 (2010), arXiv :1002.0329 [hep-ph].

[200] A. BELOKOGNE et A. FOLACCI, "Renormalized stress tensor for massive fields in Kerr-Newman spacetime", Phys. Rev. D **90**, 044045 (2014), arXiv :1404.7422 [gr-qc].

[201] A. BELOKOGNE et A. FOLACCI, "Stueckelberg massive electromagnetism in curved spacetime : Hadamard renormalization of the stress-energy tensor and the Casimir effect", Phys. Rev. D **93**, 044063 (2016), arXiv :1512.06326 [gr-qc].

[202] A. BELOKOGNE, A. FOLACCI et J. QUEVA, "Stueckelberg massive electromagnetism in de Sitter and anti-de Sitter spacetimes : Two-point functions and renormalized stress-energy tensors", Phys. Rev. D **94**, 105028 (2016), arXiv :1610.00244 [gr-qc].

[203] J. HADAMARD, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).

[204] A. LICHNEROWICZ, "Propagateurs et commutateurs en relativité générale", French, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **10**, 5 (1961).

[205] P. R. GARABEDIAN, *Partial Differential Equations* (Wiley, New York, 1964).

[206] F. G. FRIEDLANDER, *The wave equation on a curved space-time* (Cambridge University Press, Cambridge, England, 1975).

[207] B. S. DEWITT et R. W. BREHME, "Radiation damping in a gravitational field", Annals Phys. **9**, 220 (1960).

[208] Y. DECANINI et A. FOLACCI, "Off-diagonal coefficients of the Dewitt-Schwinger and Hadamard representations of the Feynman propagator", Phys. Rev. D **73**, 044027 (2006), arXiv :gr-qc/0511115.

[209] A. C. OTTEWILL et B. WARDELL, "A Transport Equation Approach to Calculations of Hadamard Green functions and non-coincident DeWitt coefficients", Phys. Rev. D **84**, 104039 (2011), arXiv :0906.0005 [gr-qc].