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Alexey Agaltsov

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THÈSE DE DOCTORAT  
DE L'UNIVERSITÉ PARIS-SACLAY  
PRÉPARÉE À L'ÉCOLE POLYTECHNIQUE

École doctorale n°574  
École doctorale de mathématiques Hadamard (EDMH)  
Spécialité de doctorat : mathématiques appliquées

par

**M. ALEXEY AGALTSOV**

Méthodes de reconstruction pour des problèmes inverses pour des  
équations de type Helmholtz

Thèse présentée et soutenue à Route de Saclay, 91128 Palaiseau, France - Amphithéâtre  
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# Introduction

Dans la présente thèse nous étudions différents aspects, variantes et cas particuliers du problème de Dirichlet-Neumann inverse et du problème de diffusion inverse pour l'équation de Helmholtz jauge-covariante motivés par des applications. La thèse est basée sur sept articles qui peuvent être divisés sur trois groupes selon la motivation pratique.

Les articles du premier groupe sont motivés par le problème de tomographie acoustique d'un fluide en écoulement. Dans les travaux I, II nous étudions les questions de l'unicité et dans les travaux III, IV, V nous développons un algorithme de reconstruction :

- I. Agaltsov A. D., *A global uniqueness result for acoustic tomography of moving fluid*, Bulletin des Sciences Mathématiques 139 (8), 2015, 937-942
- II. Agaltsov A. D., Novikov R. G., *Uniqueness and non-uniqueness in acoustic tomography of moving fluid*, Journal of Inverse and Ill-Posed Problems 24 (3), 2016, 333-340
- III. Agaltsov A. D., *Finding scattering data for a time-harmonic wave equation with first order perturbation from the Dirichlet-to-Neumann map*, Journal of Inverse and Ill-Posed Problems 23 (6), 2015, 627-645
- IV. Agaltsov A. D., Novikov R. G., *Riemann-Hilbert problem approach for two-dimensional flow inverse scattering*, Journal of Mathematical Physics 55 (10), 2014, id103502
- V. Agaltsov A. D., *On the reconstruction of parameters of a moving fluid from the Dirichlet-to-Neumann map*, Eurasian Journal of Mathematical and Computer Applications 4 (1), 2016, 4-11

Le deuxième groupe a pour motivation la tomographie utilisant des particules élémentaires. D'un point de vue mathématique il s'agit de la généralisation du problème de diffusion inverse classique au cas où seulement le module de l'amplitude de diffusion est connu (il n'y a pas d'information de phase) :

- VI. Agaltsov A. D., Novikov R. G., *Error estimates for phaseless inverse scattering in the Born approximation at high energies*, à paraître dans Journal of Geometric Analysis, [arxiv.org/abs/1604.06555](https://arxiv.org/abs/1604.06555)



Le troisième groupe est motivé par la tomographie d'impédance électrique. D'un point de vue mathématique, il s'agit du problème de Dirichlet-Neumann inverse dans le cas où la géométrie est non-triviale (le problème de Calderón) :

VII. Agaltsov A. D., Henkin G. M., *Explicit Reconstruction of Riemann Surface with Given Boundary in Complex Projective Space*, Journal of Geometric Analysis 25 (4), 2015, 2450-2473

# 1 Problèmes inverses pour l'équation de Helmholtz jauge-covariante

## 1.1 L'opérateur de Helmholtz jauge-covariant

Plusieurs modèles de la physique mathématique sont formulés sous forme d'équation aux dérivées partielles sur une certaine variété à bord. Les coefficients de cette équation décrivent certaines caractéristiques d'un phénomène naturel. Les problèmes inverses à valeurs au bord consistent en la détermination de ces coefficients (et, peut-être, de la variété qui est supposée inconnue) à partir des données au bord, qui correspondent aux quantités mesurables.

Dans la présente thèse, nous nous intéressons aux problèmes inverses pour l'opérateur de Helmholtz jauge-covariant. Dans le cas euclidien cet opérateur a deux formes équivalentes :

$$\begin{aligned} L_{A,V} &= -\Delta - 2i \sum_{j=1}^d A_j(x) \frac{\partial}{\partial x_j} + V(x), \\ \mathcal{L}_{A,Q} &= - \sum_{j=1}^d \left( \frac{\partial}{\partial x_j} + iA_j(x) \right)^2 + Q(x), \end{aligned} \tag{1}$$

où  $x = (x_1, \dots, x_d) \in D$ ,  $\Delta$  est l'opérateur de Laplace euclidien,  $A = (A_1, \dots, A_d)$ ,  $A_j$ ,  $V$ ,  $Q$  sont des fonctions assez régulières dans  $D$  à valeurs dans  $M_n(\mathbb{C})$ ,  $n \geq 1$ , et  $D$  est un domaine ouvert dans  $\mathbb{R}^d$ ,  $d \geq 2$ . Notons que

$$L_{A,V} = \mathcal{L}_{A,Q} \quad \text{pour } Q = V + i \sum_{j=1}^d \frac{\partial}{\partial x_j} A_j - \sum_{j=1}^d A_j^2. \tag{2}$$

L'opérateur  $L_{A,V}$  survient, en particulier, dans la mécanique des fluides :

1. L'équation  $L_{A,V}\psi = 0$  avec  $n = 1$  modélise la pression acoustique harmonique en temps  $p(x, t) = \text{Re}(\psi(x)e^{-i\omega t})$  dans un fluide en écoulement. Dans ce cadre,

$$A = \frac{\omega}{c^2}v + \frac{i}{2} \frac{\nabla \rho}{\rho}, \quad V = \frac{\omega^2}{c^2} + 2i\omega \frac{\alpha\omega}{c}, \tag{3}$$

où  $c = c(x)$  désigne la célérité du son,  $v = v(x)$  est la vitesse du fluide,  $\rho = \rho(x)$  est la densité et  $\alpha_\omega = \alpha_\omega(x)$  décrit l'absorption à la fréquence  $\omega$ . Cette équation dans sa forme générale a été récemment élaborée par A. S. Shurup et O. D. Rumyantseva de l'université de Moscou (article soumis à *Acoustical Physics*). Pour les cas particuliers, voir, e.g., [116, 118, 71, 24].

2. L'équation  $L_{A,V}\psi = 0$  avec  $n \geq 2$ ,  $d = 2$ , survient, en particulier, dans le cadre de tomographie acoustique océanique comme l'équation d'onde en représentation modale pour la pression acoustique dans un fluide en écoulement dans un domaine cylindrique trois-dimensionnel, voir [15] et [110, Section 2].

De plus, l'opérateur  $\mathcal{L}_{A,Q}$  apparaît souvent dans le cadre de la mécanique quantique :

1. Dans le cas où  $A_j$  et  $Q$  sont des fonctions scalaires réelles l'opérateur  $\mathcal{L}_{A,Q}$  survient comme le Hamiltonien d'une particule quantique non relativiste sans spin de charge et masse unitaire dans un champ électromagnétique, voir [50, chapitre 21-1]. Dans ce cadre,  $A$  est le potentiel magnétique et  $Q$  est le potentiel électrique.
2. Si  $n = 2$ ,  $d = 3$  et les matrices  $A_1, A_2, A_3$  sont des multiples scalaires réels des matrices de Pauli et  $Q$  est une matrice scalaire réelle, l'opérateur  $\mathcal{L}_{A,Q}$  apparaît dans l'équation de Pauli-Schrödinger pour un électron non relativiste dans un champ électromagnétique. Cette équation remonte à [114].
3. De manière plus générale, si  $A_1, \dots, A_d$  sont des matrices hermitiennes de trace nulle et  $Q$  est une matrice scalaire réelle, l'opérateur (1) survient comme le Hamiltonien d'une particule dans un champ de Yang-Mills externe, voir [122].

Notons également que l'équation  $L_{A,V}u = 0$  avec  $A = \frac{i}{2}\nabla \ln \sigma$ ,  $V = 0$ , décrit le potentiel électrique  $u$  dans un domaine conducteur de conductivité isotrope  $\sigma = \sigma(x) > 0$ . Des problèmes inverses à valeurs au bord pour cet opérateur surviennent dans le cadre de tomographie d'impédance électrique et ont une longue histoire, voir, e.g., [133, 25, 103, 137, 13, 80]. De plus notons que la substitution de jauge  $\psi = \sigma^{\frac{1}{2}}u$  mène à l'équation  $-\Delta\psi + v\psi = 0$ , où  $v = -\sigma^{-\frac{1}{2}}\Delta\sigma^{\frac{1}{2}}$ .

En outre, l'opérateur  $L_{A,V}$  apparaît dans un modèle qui décrit le plasma en équilibre thermique [34] et dans les autres modèles stationnaires de type advection-diffusion dans un milieu mobile, voir, e.g. [115].

## 1.2 Problème de Dirichlet-Neumann inverse

Soit  $D \subset \mathbb{R}^d$ ,  $d \geq 2$ , un domaine borné au bord  $\partial D$  assez lisse. Nous considérons le problème de Dirichlet

$$\begin{cases} L_{A,V}\psi = E\psi & \text{dans } D, \\ \psi|_{\partial D} = f, \end{cases} \quad (4)$$

où  $f$  est une fonction donnée sur  $\partial D$ ,  $E \in [0, +\infty) \setminus \mathcal{E}_{A,V}$ , et

$$\begin{aligned} \mathcal{E}_{A,V} \text{ désigne l'ensemble des } E \geq 0 \text{ tels que le problème (4)} \\ \text{admet une solution unique pour tout } f \text{ assez régulier,} \end{aligned} \quad (5)$$

dans le sens approprié qui sera précisé ci-dessous. Il est connu, en particulier, que dans le cas scalaire auto-adjoint l'ensemble  $\mathcal{E}_{A,V}$  est discret, voir [54, Théorème 8.37].

L'opérateur de Dirichlet-Neumann  $\Lambda_{A,V}$  est défini comme suit :

$$\Lambda_{A,V}f = \sum_{j=1}^d \nu_j \left( \frac{\partial}{\partial x_j} + iA_j \right) \psi \Big|_{\partial D}, \quad \text{où } \psi \text{ est la solution de (4),} \quad (6)$$

et  $\nu = (\nu_1, \dots, \nu_d)$  désigne le champ des vecteurs unitaires normaux (extérieurs) à  $\partial D$ . Notons que l'opérateur  $\Lambda_{A,V}$  peut être identifié avec son noyau intégral (au sens de Schwartz)  $\Lambda_{A,V}(x, y, E)$ ,  $x, y \in \partial D$  :

$$(\Lambda_{A,V}f)(x) = \int_{\partial D} \Lambda_{A,V}(x, y, E) f(y) dy, \quad x \in \partial D. \quad (7)$$

L'opérateur  $\Lambda_{A,V}$  est considéré comme une donnée mesurable à partir de laquelle il faut déterminer les coefficients  $A, V$  dans  $D$ .

Ce problème n'a jamais de solution unique. Plus précisément, en conjuguant l'opérateur  $L_{A,V}$  par une fonction  $g$  assez lisse dans  $\overline{D}$  à valeurs dans  $GL_n(\mathbb{C})$ , on obtient un opérateur  $L_{A',V'}$  de même forme :

$$\begin{cases} gL_{A,V}g^{-1} = L_{A',V'}, \\ A'_j = gA_jg^{-1} + i\frac{\partial g}{\partial x_j}g^{-1}, \quad j = 1, \dots, d, \\ Q' = gQg^{-1}, \quad \text{où } V, V' \text{ sont liés à } Q, Q' \text{ par (2).} \end{cases} \quad (8)$$

Il s'avère que les opérateurs  $\Lambda_{A,V}$  et  $\Lambda_{A',V'}$  sont conjugués par les valeurs au bord de la fonction  $g$  :

$$\Lambda_{A',V'} = g|_{\partial D} \Lambda_{A,V} g^{-1}|_{\partial D}.$$

En particulier, si  $g|_{\partial D} = \text{id}_n$ , on a  $\Lambda_{A,V} = \Lambda_{A',V'}$ . En tenant compte de cette non-unicité, nous considérons le problème suivant :

**Problème 1.** *Trouver les coefficients  $A, V$  dans  $D$  (à une transformation de jauge (8) avec  $g|_{\partial D} = \text{id}_n$  près) à partir de l'opérateur  $\Lambda_{A,V}$  correspondant à  $E$  fixé.*

Notons qu'en accord avec (7), l'opérateur  $\Lambda_{A,V}$  peut être considéré comme une fonction à  $2(d-1)$  variables réelles, tandis que les coefficients  $A, V$  sont des fonctions de  $d$  variables réelles. Le problème (1) est donc formellement non surdéterminé pour  $d = 2$  et surdéterminé pour  $d \geq 3$ .

Le problème 1 peut être considéré comme un problème de Gelfand pour l'opérateur de Helmholtz jauge-covariant à énergie fixée (voir [53]) ou comme une généralisation du problème de Calderón de tomographie d'impédance électrique (voir [25]).

Rappelons que l'opérateur  $L_{A,V}$  décrit, en particulier, la pression acoustique dans un fluide en écoulement. Dans ce cadre, l'opérateur  $\Lambda_{A,V}$  correspond aux mesures des ondes acoustiques diffusées associées aux ondes incidentes données. Donc, le problème 1 s'interprète comme le problème de détermination d'un diffuseur acoustique à partir des mesures des ondes diffusées.

Afin d'étudier le problème 1, il est utile d'introduire les notations suivantes :

$$\begin{aligned}\mathcal{A} &= iA_1 dx_1 + \cdots + iA_d dx_d, \\ \Omega_A &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\ &= -i \sum_{k < l} \left[ \left( \frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k} \right) + i(A_l A_k - A_k A_l) \right] dx_k \wedge dx_l.\end{aligned}\tag{9}$$

En géométrie différentielle  $\mathcal{A}$  et  $\Omega_A$  apparaissent comme une 1-forme de connexion et la 2-forme de courbure pour cette connexion. Notons également que dans le cas  $n = 1$ ,  $d = 3$ , la forme  $\Omega_A$  est étroitement lié au rotationnel  $\text{curl } A$  du champ  $A$ . Plus précisément, si  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  désignent la base canonique de  $\mathbb{R}^3$ , on a

$$\begin{aligned}\Omega_A &= -i \left[ \left( \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} \right) dx_2 \wedge dx_3 + \left( \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right) dx_3 \wedge dx_1 + \left( \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} \right) dx_1 \wedge dx_2 \right], \\ \text{curl } A &= \left( \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} \right) \mathbf{i} + \left( \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right) \mathbf{j} + \left( \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} \right) \mathbf{k}.\end{aligned}$$

On peut montrer que la transformation de jauge (8) mène à la loi de transformation suivante pour  $\Omega_A$  :

$$\Omega_{A'} = g \Omega_A g^{-1}.$$

Notons également que dans le cas scalaire  $n = 1$ , la fonction  $Q$  et la 2-forme  $\Omega_A$  sont invariantes par rapport aux transformations (8).

Dans le cas où l'opérateur  $\mathcal{L}_{A,Q}$  est le Hamiltonien d'une particule chargée dans un champ électromagnétique, la forme  $\Omega_A$  décrit le champ magnétique correspondant au potentiel magnétique  $A$ . Il découle de ce qui précède que les mesures au bord ne distinguent pas les coefficients d'une même classe de jauge (c.à.d. liés par une transformation (8)). D'un point de vue physique, cela reflète le principe de relativité de H. Weyl formulé en 1919, selon lequel toutes les configurations du potentiel électromagnétique de la même classe de jauge décrivent la même réalité physique, voir [127, section 1.1]. Notons que la proposition réciproque, à savoir que des configurations de différentes classes de jauge sont physiquement distinguables, est connu comme l'*effet Aharonov-Bohm* et remonte à [37, 6]. En particulier, même si le champ magnétique  $\Omega_A$  s'anule, dans le cas d'un domaine  $D$  non simplement connexe il est possible que le potentiel magnétique  $A$  n'est pas jauge-équivalent à zéro et peut donc influencer le comportement des particules chargées, voir par exemple [6, 142, 41].

### 1.3 Problèmes direct et inverse de diffusion

Nous allons formuler un analogue du problème 1 dans le cas où  $D$  est l'espace entier  $\mathbb{R}^d$ ,  $d \geq 2$ . Nous supposons que les coefficients  $A, V$  sont assez réguliers et décroissent assez rapidement (par exemple exponentiellement) à l'infini. Nous considérons le problème suivant pour  $\psi_{A,V}^+ = \psi^+(x, k)$ ,  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d$ ,  $k^2 = E$ , pour  $k$  fixé :

$$\begin{cases} L_{A,V}\psi_{A,V}^+ = E\psi_{A,V}^+ & \text{dans } \mathbb{R}^d, \\ \psi_{A,V}^+(x, k) = e^{ikx}\text{id}_n + \psi_{\text{sc}}^+(x, k), \\ \left(\frac{\partial}{\partial|x|} - ik\right)\psi_{\text{sc}}^+(x, k) = o(|x|^{-\frac{d-1}{2}}), & |x| \rightarrow \infty, \end{cases} \quad (10)$$

où  $E \in (0, +\infty) \setminus \mathcal{E}_{A,V}^+$ , et

$\mathcal{E}_{A,V}^+$  désigne l'ensemble des  $E > 0$  tels que le problème (10) admet une solution unique pour tout  $k \in \mathbb{R}^d$ ,  $k^2 = E$ ,

dans le sens approprié qui sera précisé ci-dessous. Nous appelons  $\mathcal{E}_{A,V}^+$  l'ensemble des *résonances réelles positives* de l'opérateur  $L_{A,V}$ . Il est connu que si  $A$  et  $Q$  vérifiant (1) et (2) sont des fonctions scalaires réelles décroissant exponentiellement à l'infini avec leur dérivées partielles, alors  $\mathcal{E}_{A,V}^+ = \emptyset$ , voir [42].

On peut montrer que le terme asymptotique principal à l'infini de la fonction  $\psi_{\text{sc}}^+$  s'écrit comme suit :

$$\begin{aligned} \psi_{\text{sc}}^+(x, k) &= c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f_{A,V}(k, |k| \frac{x}{|x|}) + o(|x|^{-\frac{d-1}{2}}), \quad |x| \rightarrow +\infty, \\ c(d, |k|) &= -\pi i (-2\pi i)^{\frac{d-1}{2}} |k|^{\frac{d-3}{2}}. \end{aligned} \quad (11)$$

La fonction  $f = f_{A,V}$  s'appelle *l'amplitude de diffusion*. Elle est définie sur le produit  $\mathcal{M}_E$  de deux sphères :

$$\mathcal{M}_E = \{(k, l) \in \mathbb{R}^d : k^2 = l^2 = E\} \simeq S^{d-1} \times S^{d-1}.$$

Notons que le problème de détermination de  $\psi_{A,V}^+$  et  $f_{A,V}$  à partir de  $L_{A,V}$  est connu comme *le problème de diffusion direct*. Dans des applications pratiques ce problème peut être résolu à l'aide d'une équation de type *Lippmann-Schwinger*<sup>1</sup>

$$\begin{aligned} \psi_{A,V}^+(x, k) &= e^{ikx}\text{id}_n + \int_{\mathbb{R}^d} G^+(x-y, k) (L_{A,V} - L_{0,0}) \psi_{A,V}^+(y, k) dy, \\ G^+(x, k) &= -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0}, \end{aligned} \quad (12)$$

1. les équations de ce type remontent à [88]

où  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d$ ,  $k^2 = E$ , et en utilisant la formule explicite

$$f_{A,V}(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} (L_{A,V} - L_{0,0}) \psi_{A,V}^+(x, k) dx, \quad (13)$$

où  $k, l \in \mathbb{R}^d$ ,  $k^2 = l^2 = E$ . Notons que, pour  $k$  fixé et  $A \neq 0$ , on peut considérer (12) et ses versions dérivées par rapport à  $x_1, \dots, x_d$  comme une équation intégrale vectorielle pour  $\psi_{A,V}^+(\cdot, k)$ ,  $\frac{\partial}{\partial x_1} \psi_{A,V}^+(\cdot, k)$ ,  $\dots$ ,  $\frac{\partial}{\partial x_d} \psi_{A,V}^+(\cdot, k)$ . En outre, notons qu'un algorithme efficace de résolution de l'équation (12) a été proposé par Vainikko [138].

Le problème réciproque de détermination des coefficients  $A$  et  $V$  à partir de  $f_{A,V}$  est connu comme *le problème de diffusion inverse*. Cependant, il s'avère que l'amplitude de diffusion est invariante par rapport aux transformations de jauge (8), où  $g$  tends vers  $\text{id}_n$  assez rapidement à l'infini. En tenant compte de cette non-unicité, on s'intéresse au problème suivant :

**Problème 2.** *Étant donné  $f_{A,V}$  sur  $\mathcal{M}_E$  pour  $E$  fixé, trouver  $A, V$  dans  $\mathbb{R}^d$  à une transformation de jauge (8) près, où  $g$  tends vers  $\text{id}_n$  assez rapidement à l'infini.*

Notons que l'amplitude de diffusion est une fonction de  $2(d-1)$  variables réelles tandis que les coefficients  $A$  et  $V$  dépendent de  $d$  variables réelles. Donc, le problème 2 est formellement non surdéterminé pour  $d = 2$  et surdéterminé pour  $d \geq 3$ .

Rappelons que dans la mécanique quantique l'opérateur  $\mathcal{L}_{A,Q}$  de (1), (2) survient en particulier comme le Hamiltonien d'une particule quantique dans un champ externe. Dans ce cadre,  $|f(k, l)|^2$ ,  $k \neq l$ , est la densité de probabilité de diffusion d'une particule avec impulsion initiale  $k$  dans la direction  $l$ , et la valeur  $\frac{\partial \arg f(k, l)}{\partial |k|}$ , dénommée *déphasage*, contribue à un retard de la particule diffusée, voir [30, p. 461-463]. Donc, dans ce cadre, le problème 2 s'interprète comme le problème de détermination du champ externe à partir des mesures des particules quantiques diffusées. Le problème similaire dans le cas des particules classiques a été aussi étudié, voir Jollivet [79] et les références qui y sont contenues.

## 1.4 L'état de l'art

### 1.4.1 Problème 1

De très nombreux ouvrages traitent le problème 1 dans le cas où  $A = 0$ , voir par exemple [91, 11, 20] (unicité,  $d = 2$ ,  $n = 1$ ); Novikov-Santacesaria [110, 111] (unicité et reconstruction,  $d = 2$ ,  $n \geq 1$ ), Novikov [100], Burov et al. [24] (reconstruction et aspects numériques,  $d = 2$ ,  $n = 1$ ), Hohage [72] (reconstruction numérique en utilisant les méthodes de type Newton), [131, 62] (unicité,  $d \geq 3$ ,  $n = 1$ ).

La majorité des résultats connus sur le problème 1 dans le cas où  $A \neq 0$  concernent l'unicité dans le cas des coefficients scalaires ( $n = 1$ ) en dimension  $d \geq 3$ . Dans ce cas, Sun [130] et Panchenko [113] ont démontré que si  $\Lambda_{A,V} = \Lambda_{A',V'}$  et  $\text{curl } A, \text{curl } A'$  sont

petits,  $\text{curl } A = \text{curl } A'$  et  $Q = Q'$ . Le même résultat sans la condition de petitesse a été obtenu par Nakamura-Sun-Uhlmann pour des coefficients de classe  $C^\infty$  [92], Tolmasky (coefficients de classe  $C^1$ ) [134], Salo (coefficients continus dans le sens de Dini) [119] et Krupchyk-Uhlmann (coefficients bornés) [85]. Comme corollaire, l'opérateur  $\Lambda_{A,V}$  détermine  $A, V$  uniquement sous condition que les coefficients  $A, V$  sont réels et que  $D$  est simplement connexe, voir l'article I de la présente thèse.

Le problème de reconstruction en dimension  $d \geq 3$  a été considérée pour la première fois par Salo [120] qui a proposé un algorithme basé sur les travaux de Novikov [97] et Nachman [90]. La stabilité de reconstruction de type  $\log \log$  a été démontrée par Tzou [136].

À notre connaissance, la question de l'unicité en dimension  $d = 2$  a été traitée dans le cas scalaire ( $n = 1$ ) avec  $A \neq 0$  pour la première fois par Guillarmou-Tzou [61]. Ils ont démontré que si  $A$  et  $A'$  sont réels et  $\Lambda_{A,V} = \Lambda_{A',V'}$ , les coefficients  $A, V$  et  $A', V'$  sont liés par une transformation (8) avec  $g|_{\partial D} = 1$ .

Notons également le travail [19] de Brown-Salo, dans lequel des formules pour la détermination des valeurs tangentes au bord du champ vectoriel  $A$  à partir de l'opérateur  $\Lambda_{A,V}$  ont été présentées en dimension  $d \geq 2$  dans le cas scalaire ( $n = 1$ ).

Le cas des coefficients matriciels ( $n \geq 2$ ) est beaucoup moins étudié. La question de l'unicité a été examiné par Eskin-Ralston [39, 45] en dimension  $d \geq 3$ , qui ont démontré que sous la condition de convexité de  $D$ , l'égalité des opérateurs  $\Lambda_{A,V}$  et  $\Lambda_{A',V'}$  implique que les coefficients  $A, V$  et  $A', V'$  sont liés par une transformation de jauge (8) avec  $g|_{\partial D} = \text{id}_n$ .

Notons également un cas particulier très étudié du problème 1, correspondant à l'opérateur  $L_{A,V}$  avec  $A = \frac{i}{2} \frac{\nabla \sigma}{\sigma}$ ,  $\sigma > 0$ ,  $V = 0$ . Comme noté ci-dessus, l'opérateur  $L_{A,V}$  survient en tomographie d'impédance électrique. Pour des résultats connus sur ce problème dans le cas euclidien, voir par exemple [103, 137, 13, 80]. En outre, une généralisation du problème 1 pour le cas d'une géométrie non euclidienne et qui est supposée inconnue a priori est étudié, en particulier, dans les articles [67, 69, 70] (voir aussi les références qui y sont contenues). Nous étudions un problème qui est étroitement lié à une modification du problème 1 pour le cas d'une géométrie inconnue et  $A = 0$  dans l'article VII.

Il est aussi important de noter qu'il existe un grand nombre de travaux qui étudient le problème 4 dans le cas d'information partielle. Ces problèmes sont très importants pour les applications, où il est impossible de mesurer des données sur une partie du bord (par exemple en tomographie médicale), voir par exemple [73, 84, 80].

### 1.4.2 Problème 2

En ce qui concerne le problème 2, notons que dans le cas des coefficients à support compact, il est équivalent au problème 1, voir [97] ( $A = 0$ ), [43] ( $A \neq 0$ ), [101] ( $A = 0$ , potentiel de fond non nul) et l'article III de la présente thèse ( $A \neq 0$ , coefficients

matriciels, potentiel de fond non nul).

Notons également que le problème 2 dans le cas scalaire avec  $A = 0$  est beaucoup étudié, voir par exemple [97, 63, 107] et les références qui y sont contenues. Le cas des coefficients matriciels  $n \geq 2$  avec  $A = 0$  a été étudié dans [111].

Dans le cas où  $A \neq 0$  en dimension  $d \geq 3$ , il est connu que si  $A_j$  et  $Q$  sont des fonctions réelles scalaires décroissant exponentiellement à l'infini, l'amplitude de diffusion  $f_{A,V}$  à énergie fixée détermine  $A$  et  $Q$  de (2) à une transformation de jauge (8) près, voir [109] (pour un petit curl  $A$ ), [42] ( $A \in C^{d+5}$ ,  $Q \in C^{d+3}$ ), [112] ( $A \in e^{-\gamma|x|}W^{1,\infty}$ ,  $V \in e^{-\gamma|x|}L^\infty$ ,  $\gamma > 0$ ). Ce résultat a été généralisé au cas matriciel dans les ouvrages [44] (pour  $A_j \in C^{d+5}$ ,  $Q \in C^{d+4}$  assez petits,  $A_j^* = A_j$ ,  $Q^* = Q$ ), [39] (pour  $A_j \in C^{d+3}$ ,  $Q \in L^\infty$  fonctions à support compact). Notons que la condition de décroissance exponentielle est naturelle car il existe un exemple de non unicité pour des coefficients de la classe Schwartz, voir [60].

L'unicité pour le problème 2 en dimension  $d = 2$  dans le cas matriciel  $n \geq 1$  a été démontrée par Xiaosheng [141], en supposant que  $A_j$ ,  $Q$  sont petits et décroissent exponentiellement à l'infini.

Le premier résultat concernant la question de reconstruction pour le problème 2 dans le cas où  $A = 0$ , est la formule, dénommée *approximation de Born*, qui remonte à [46] et qui permet de trouver une approximation au coefficient  $V$  à partir de  $f_{0,V}$  à énergie fixée :

$$\widehat{V}(p) = f_{0,V}(k, l) + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, \quad p = k - l, \quad (k, l) \in \mathcal{M}_E,$$

$$\widehat{V}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} V(x) dx.$$

Notons que les formules de ce type ne permettent pas de trouver les coefficients plus précisément qu'en  $O(E^{-\frac{1}{2}})$ .

Dans le cas où  $A \neq 0$ , des formules analogiques pour la détermination de curl  $A$  à partir de la limite de hautes énergies des données de diffusion ont été obtenues, en particulier, par Shiota [125] (coefficients à support compact), Henkin-Novikov [71] (coefficients lisses à décroissance rapide à l'infini), Arians [10] (le cas de curl  $A$  de courte portée), Nicoleau [93] (les cas de curl  $A$  de courte et longue portée). Dans l'article IV de la présente thèse nous établissons des formules analogiques pour la détermination de  $A$ ,  $Q$  à partir des données de diffusion à l'énergie fixée mais dans la limite des petits coefficients.

En dimension  $d \geq 3$  la question de reconstruction pour le problème 2 (à une énergie  $E$  fixée) a été étudié par Weder-Yafaev [139] (pour le comportement assez régulier de  $A$ ,  $Q$  à l'infini) et par Salo [120] (pour des coefficients à support compact).

En dimension  $d = 2$ , R. Novikov [100] a proposé un algorithme qui permet de trouver  $V$  à partir de  $f_{0,V}$  avec la précision  $O(E^{-\frac{N-2}{2}})$  au sens  $L^\infty$  pour  $V$  tel que  $(1 + |x|^2)^{\frac{\varepsilon}{2}} \frac{\partial^{|\alpha|} V}{\partial x^\alpha} \in L^1$ ,  $|\alpha| \leq N$ , pour quelque  $\varepsilon > 0$  fixé. Cet algorithme est basé sur



des idées des articles [89, 57, 96, 59]. Nous généralisons cet algorithme bidimensionnel au cas où  $A \neq 0$  dans l'article IV de la présente thèse.

De plus, notons que des modifications du problème 2 pour le cas des données partielles, quand il est seulement possible de mesurer le module de l'amplitude de diffusion, surviennent dans le cadre de la physique quantique, voir par exemple [26, 83, 105, 106] et l'article VI de la présente thèse (au moins, pour  $A = 0$ ).

## 2 Tomographie acoustique d'un fluide en écoulement. Résumé des articles I-V

### 2.1 Considérations préliminaires

Le premier groupe d'articles présentées dans le présent ouvrage sont consacrés à la solution du problème de tomographie acoustique de fluide en écoulement. Nous considérons un fluide dans un domaine borné  $D \subset \mathbb{R}^d$ ,  $d \geq 2$ , avec la célérité du son <sup>2</sup>  $c = c(x)$ , la densité  $\rho = \rho(x)$ , la vitesse du fluide  $v = v(x)$  et le coefficient d'absorption acoustique  $\alpha = \alpha(x, \omega)$ ,

$$\alpha(x, \omega) = \omega^{\zeta(x)} \alpha_0(x) \tag{14}$$

à la fréquence fixée  $\omega$ .

Sauf mention contraire, pour des raisons de simplicité, nous supposons que

$$v, c, \rho, \alpha_0, \zeta \text{ sont assez lisses dans } D; \tag{15}$$

$$\text{au voisinage de } \partial D, \text{ les paramètres } v, \alpha_0 \text{ s'annulent} \tag{16}$$

et les paramètres  $c, \rho$  sont égaux aux valeurs de fond connues  $c_0, \rho_0$ .

Décrivons en quelques mots l'expérience tomographique considéré ; pour plus d'informations, voir Burov et autres [24]. Des transducteurs acoustiques de type ponctuel sont placés sur le bord  $\partial D$ . Ils peuvent émettre et enregistrer des ondes acoustiques à une fréquence fixée  $\omega$  appartenant à l'ensemble des fréquences admissibles  $\Omega \subset (0, \infty)$ ,  $|\Omega| < \infty$ . Nous considérons l'expérience suivante : on fixe un transducteur qui émit une onde acoustique, cette onde se propage dans le fluide, et on enregistre l'onde diffusée en utilisant les autres transducteurs. Ensuite, on change le transducteur émettant et, peut être, la fréquence des ondes émises, et on répète l'expérience. Le but est de déterminer les paramètres du fluide à partir des données mesurées.

D'un point de vue mathématique, nous considérons le modèle de propagation du son dans un fluide, qui a été envisagé dans différents cas particuliers dans les articles [116, 118, 71, 24]. Dans le cas général, ce modèle a été élaboré par O. D. Rumyantseva et A. S. Shurup du département d'acoustique de l'université de Moscou (article soumis

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2. plus précisément, la vitesse de propagation des ondes acoustiques

à *Acoustical Physics*). Dans ce modèle, la pression acoustique harmonique en temps  $p(x, t) = \text{Re}(\psi(x)e^{-i\omega t})$ ,  $\omega > 0$ , satisfait à l'équation  $L_\omega \psi = F$  dans  $D$ , où  $L_\omega$  désigne l'opérateur différentiel suivant :

$$\begin{aligned} L_\omega &= -\Delta_x - 2iA_\omega(x)\nabla_x - U_\omega(x), \quad x = (x_1, \dots, x_d) \in D, \\ A_\omega(x) &= \frac{\omega v(x)}{c^2(x)} + \frac{i}{2} \frac{\nabla_x \rho(x)}{\rho(x)}, \\ U_\omega(x) &= \frac{\omega^2}{c^2(x)} + 2i\omega \frac{\alpha(x, \omega)}{c(x)}, \quad \alpha(x, \omega) = \omega^{\zeta(x)} \alpha_0(x), \end{aligned} \quad (17)$$

et  $F$  décrit des sources du son. Notons que pour des raisons physiques, on a

$$\begin{aligned} c \geq c_{\min} > 0, \quad \rho \geq \rho_{\min} > 0, \quad \alpha_0 \geq 0, \quad v = \bar{v}, \quad \zeta = \bar{\zeta} \text{ dans } D, \\ \text{pour des constantes } c_{\min}, \rho_{\min} \text{ appropriées.} \end{aligned} \quad (18)$$

Les définitions (1) et (17) impliquent que  $L_\omega = L_{A,V}$ , où  $A = A_\omega$ ,  $V = -U_\omega$ .

Notons par

$$\begin{aligned} \sigma &= \sigma(L_\bullet) \text{ l'ensemble des } \omega \in (0, +\infty) \text{ tels} \\ \text{que } 0 &\text{ est une valeur propre de Dirichlet pour } L_\omega \text{ dans } D. \end{aligned} \quad (19)$$

Pour l'opérateur  $L_\omega$ ,  $\omega \notin \sigma$ , on définit la fonction de Green  $G_\omega = G_\omega(x, y)$  comme la solution du problème

$$\begin{cases} L_\omega G_\omega(\cdot, y) = \delta_y & \text{dans } \mathbb{R}^d, \\ \left( \frac{\partial}{\partial |x|} - i \frac{\omega}{c_0} \right) G_\omega(x, y) = o(|x|^{-\frac{d-1}{2}}), & |x| \rightarrow \infty, \end{cases} \quad (20)$$

pour  $y \in \mathbb{R}^d$  fixé, où  $\delta_y(x) = \delta(x - y)$ , désignant la fonction  $\delta$  de Dirac. On considère  $G_\omega(\cdot, y)$  comme la pression acoustique à la fréquence  $\omega$  correspondant à une source du son ponctuelle située en  $y$ .

Très souvent le problème de tomographie acoustique est formulé comme suit (voir, e.g., [24]) :

**Problème 3.** *Étant donnés  $G_\omega(x, y)$  pour  $x, y \in \partial D$ ,  $\omega \in \Omega$ ,  $\Omega \subset (0, \infty)$ ,  $|\Omega| < \infty$ , et  $c_0, \rho_0$ , trouver  $v, c, \rho, \alpha_0, \zeta$  dans  $D$ .*

Le problème 3 survient en tomographie océanique, où on s'intéresse à l'estimation de la température de l'océan et des courants effectuant le transfert de chaleur. De plus, le problème 3 est motivé par des applications en imagerie médicale, où on s'intéresse à la détermination simultanée du flux sanguin et des caractéristiques scalaires d'un corps humain (par exemple la densité, la célérité du son ou l'absorption) sans danger pour le patient.

Nous considérons également l'opérateur de Dirichlet-Neumann  $\Lambda_\omega = \Lambda_{A,V}$ ,  $A = A_\omega$ ,  $V = -U_\omega$ , pour l'opérateur  $L_\omega = L_{A,V}$ ,  $\omega \notin \sigma$ , défini par les formules suivantes :

$$\Lambda_\omega(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D}, \quad L_\omega \psi = \psi \text{ dans } D, \quad (21)$$

où  $\partial/\partial\nu$  désigne la dérivée normale extérieure. Dans le cadre de la tomographie acoustique le problème de Dirichlet-Neumann inverse est formulé comme suit :

**Problème 4.** *Étant donnés  $\Lambda_\omega$  pour  $\omega \in \Omega$ ,  $\Omega \subset (0, \infty)$ ,  $|\Omega| < \infty$ , et  $c_0, \rho_0$ , trouver  $v, c, \rho, \alpha_0, \zeta$  dans  $D$ .*

Le problème 4 est une modification du problème 1.

On peut réduire le problème 3 au problème 4 comme suit (voir [16, 97, 90]). Soient  $A_\omega^0$  et  $U_\omega^0$  des coefficients définis comme dans (17) mais correspondant aux paramètres  $v = 0, \alpha = 0, c = c_0, \rho = \rho_0$  de fond connus. Posons  $L_\omega^0 = L_{A,V}$ , où  $A = A_\omega^0, V = -U_\omega^0$ . On suppose que  $\omega \notin \sigma(L_\bullet^0)$ . Alors, on peut définir la fonction de Green  $G_\omega^0$  et l'opérateur de Dirichlet-Neumann  $\Lambda_\omega^0 = \Lambda_{A,V}$  en accord avec (20) et (21) (avec  $L_\omega^0$  en place de  $L_\omega$ ). Il s'avère que les opérateurs  $\Lambda_\omega, \Lambda_\omega^0$  et les fonctions  $G_\omega, G_\omega^0$  sont liés par l'équation intégrale suivante sur  $\partial D$ , qui remonte à Nachman [90] (dans le cas où  $A = 0, n = 1$ ) :

$$G_\omega(x, y) - G_\omega^0(x, y) = \int_{\partial D} \int_{\partial D} G_\omega^0(x, z) (\Lambda_\omega - \Lambda_\omega^0)(z, w) G_\omega(w, y) dz dw, \quad x, y \in \partial D, \quad (22)$$

où la fonction  $(\Lambda_\omega - \Lambda_\omega^0)(z, w)$  est le noyau intégral de l'opérateur  $\Lambda_\omega - \Lambda_\omega^0$  défini de même façon que dans la formule (7). L'équation (22) découle des définitions des opérateurs  $\Lambda, \Lambda_\omega^0$  et des fonctions  $G_\omega, G_\omega^0$  et de la seconde formule de Green, voir [24, éq. 11]. L'équation (22) permet de réduire le problème 3 au problème 4, que l'on va étudier dans la présente thèse.

Dans la première partie du présent ouvrage nous étudions les questions d'unicité (articles I, II) et de reconstruction (articles III, IV, V) pour le problème 4, voir la Fig. 1.

Notons que le schéma de reconstruction présenté sur la Fig. 1 a été implémenté numériquement dans certains cas particuliers en dimension  $d = 2$  par le groupe de Burov du département acoustique de l'université de Moscou. En particulier, ce schéma dans le cas  $A = 0$  a été étudié numériquement dans [24].

Les résultats de la modélisation numérique pour le cas  $A = \bar{A}$  ont été annoncés dans l'exposé de Shurup-Rumyantseva [126] (article soumis à *Acoustical Physics*). En particulier, ils présentent un exemple de reconstruction approchée  $\hat{\mathbf{A}}$  du champ vectoriel  $\mathbf{A}$  (qui figure dans l'opérateur  $L_{A,V}$ ) de la Fig. 2 dans le cas des données non bruitées. Sur cette figure,  $\hat{\mathbf{A}}_1$  et  $\hat{\mathbf{A}}_2$  désignent, respectivement, des parties rotationnelle ( $\nabla \cdot \hat{\mathbf{A}}_1 = 0$ ) et potentielle ( $\text{curl } \hat{\mathbf{A}}_2 = 0$ ) de  $\hat{\mathbf{A}}$ . Notons que l'on utilise des données de sources ponctuelles pour deux fréquences différentes pour la reconstruction de  $\mathbf{A}_2$ , voir les articles II et V de la présente thèse. On peut voir que dans le cas des données non bruitées l'algorithme est très précis. Selon la recherche conduite par Shurup-Rumyantseva, l'algorithme est également suffisamment stable par rapport au bruit pour des applications pratiques.

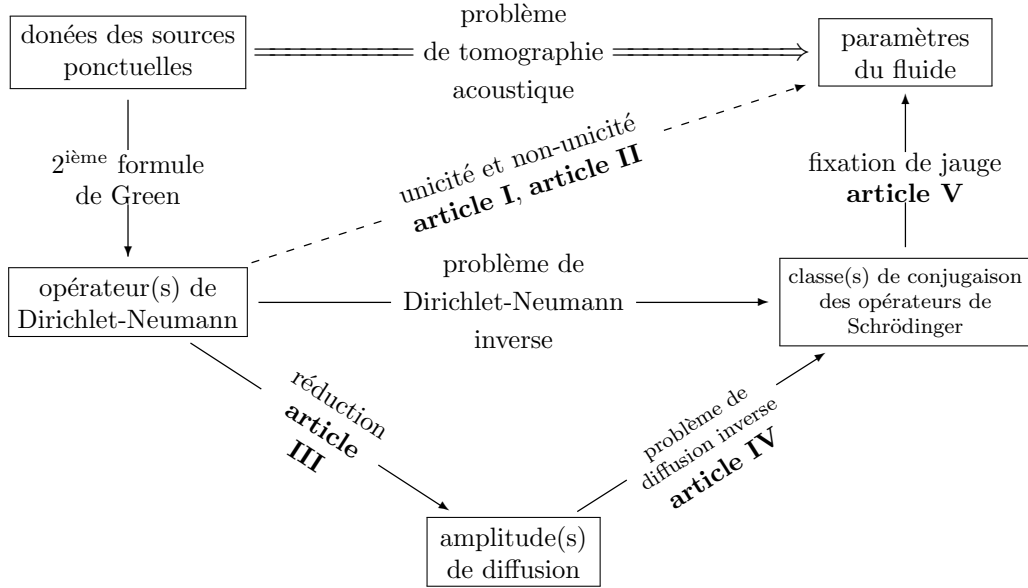


FIGURE 1 – Le schéma de résolution du problème de tomographie acoustique

## 2.2 Résumé de l'article I

Dans l'article I nous considérons le cas d'un fluide sans absorption ( $\alpha = 0$ ) et d'une densité constante ( $\rho = \rho_0$ ). Cela implique que les fonctions  $A_\omega$  et  $U_\omega$  de (17) sont réelles. Nous montrons que ces fonctions sont déterminées uniquement par l'opérateur  $\Lambda_\omega$  à une fréquence  $\omega \notin \sigma(L_\bullet)$  fixée. On peut voir que cela détermine également la vitesse du fluide  $v$  et la célérité du son  $c$ .

Plus précisément, nous considérons l'opérateur  $L_{A,V}$  de (1) avec  $A = \bar{A}$  et  $V = \bar{V}$ . Les résultats principaux de l'article I peuvent être formulés comme suit :

**Théorème 1.** *Soit  $D \subset \mathbb{R}^d$ ,  $d \geq 3$ , un domaine borné simplement connexe de bord  $\partial D \in C^1$  connexe par arcs. Soient  $A_1, A_2 \in W^{1,\infty}(D, \mathbb{R}^d)$  et  $V_1, V_2 \in L^\infty(D, \mathbb{R})$ . Supposons que 0 n'est pas une valeur propre de Dirichlet pour  $L_{A_1, V_1}$  et  $L_{A_2, V_2}$  dans  $D$ , et soient  $\Lambda_1$  et  $\Lambda_2$  les opérateurs de Dirichlet-Neumann pour  $L_{A_1, V_1}$  et  $L_{A_2, V_2}$ , respectivement. Alors,  $A_1 = A_2$  et  $V_1 = V_2$  lorsque  $\Lambda_1 = \Lambda_2$ .*

**Théorème 2.** *Soit  $D \subset \mathbb{R}^2$  un domaine borné simplement connexe de bord  $\partial D \in C^\infty$ . Soient  $A_1, A_2 \in W^{2,p}(D, \mathbb{R}^2)$  et  $V_1, V_2 \in W^{1,p}(D, \mathbb{R})$ ,  $p > 2$ . Supposons que 0 n'est pas une valeur propre de Dirichlet pour  $L_{A_1, V_1}$  et  $L_{A_2, V_2}$  dans  $D$ , et soient  $\Lambda_1$  et  $\Lambda_2$  les opérateurs de Dirichlet-Neumann pour  $L_{A_1, V_1}$  et  $L_{A_2, V_2}$ , respectivement. Alors,  $A_1 = A_2$  et  $V_1 = V_2$  si  $\Lambda_1 = \Lambda_2$ .*

Pour la démonstration des théorèmes 1 et 2 nous représentons l'opérateur  $L_{A,V}$  comme  $L_{A,V} = \mathcal{L}_{A,Q}$ , où  $Q = V + i\nabla \cdot A - A \cdot A$  (cf. (1), (2)). Il est connu que

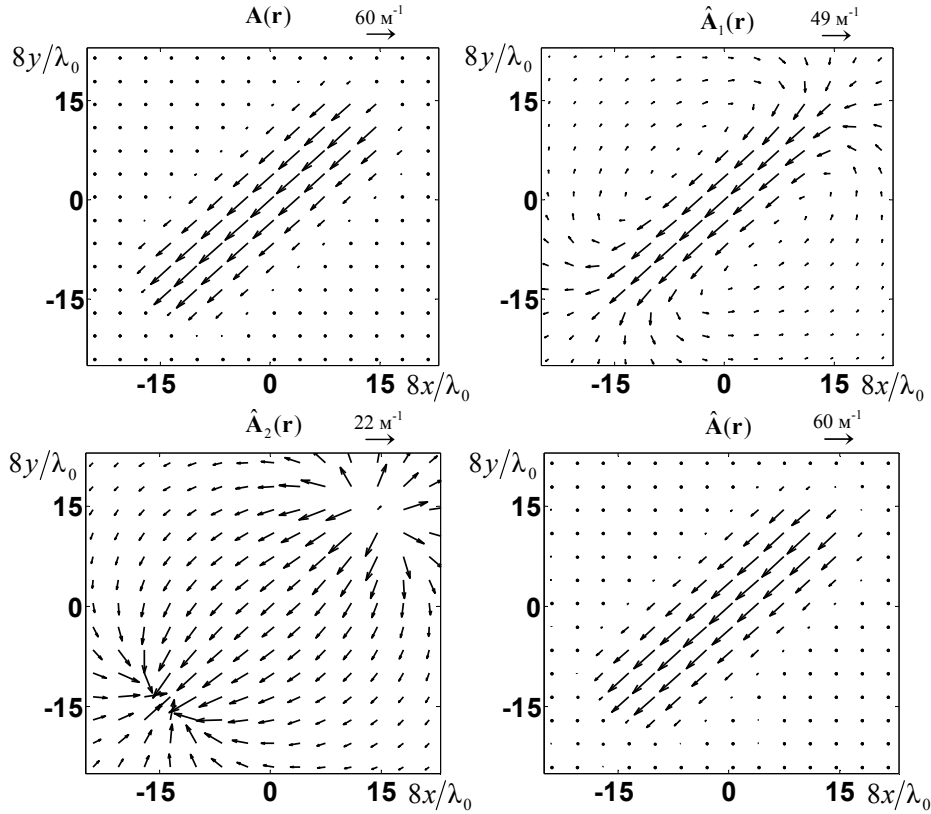


FIGURE 2 – Les champs vectoriels  $\mathbf{A}$ ,  $\widehat{\mathbf{A}}_1$ ,  $\widehat{\mathbf{A}}_2$ ,  $\widehat{\mathbf{A}} = \widehat{\mathbf{A}}_1 + \widehat{\mathbf{A}}_2$  de [126]

l'opérateur de Dirichlet-Neumann  $\Lambda_{A,V}$  détermine uniquement  $\text{curl } A$  et  $Q$  dans  $D$ , voir [61] pour  $d = 2$  et [85] pour  $d \geq 3$ . En general, les coefficients  $A$  et  $Q$  (ou, de façon équivalente,  $A$  et  $V$ ) sont déterminés à une transformation de jauge (8) près. Cependant, nous montrons que la condition  $A = \bar{A}$ ,  $V = \bar{V}$ , permet de se débarrasser de cette non-unicité et trouver  $A$  et  $V$  uniquement.

### 2.3 Résumé de l'article II

Dans l'article II, d'une part, nous continuons notre étude sur la question d'unicité pour le problème 4 de la sous-section 2.1. Nous montrons que les données mesurables à deux fréquences fixées déterminent la vitesse du fluide, la célérité du son et la densité du fluide si le fluide est supposé non absorbant. Nous montrons également que les données mesurables à trois fréquences fixées déterminent toutes les paramètres du fluide si on suppose que l'absorption n'est pas constante en fonction de fréquence. D'autre part, nous présentons des exemples des fluides qui ne sont pas distinguables à partir des mesures au bord dans le cadre du modèle considéré pour toutes les fréquences. En vue des résultats d'unicité susmentionnés, l'absorption de ces fluides

non distinguables ne depend pas de la fréquence (i.e.  $\alpha(x, \omega) = \alpha_0(x)$ ), ce qui montre que la dépendance à la fréquence est suffisante et nécessaire pour l'identifiabilité du fluide. Rappelons que  $\alpha$  est défini dans la formule (14).

D'un point de vue mathématique, nous supposons que

$$\begin{aligned} c \in W^{1,\infty}(D, \mathbb{R}), \quad \rho \in C(\overline{D}) \cup C^2(D), \quad v \in W^{1,\infty}(D, \mathbb{R}^d), \\ \alpha_0 \in C(\overline{D}), \quad \zeta \in C(\overline{D}), \quad \zeta \neq 0, \quad \text{où } d \geq 3, \end{aligned} \quad (23)$$

ou bien que

$$\begin{aligned} c \in W^{2,p}(D, \mathbb{R}), \quad \rho \in W^{3,p}(D, \mathbb{R}), \quad v \in W^{2,p}(D, \mathbb{R}^d), \\ \alpha_0 \in W^{1,p}(D, \mathbb{R}), \quad \zeta \in C(\overline{D}), \quad \zeta \neq 0, \quad \text{où } p > 2, \quad d = 2. \end{aligned} \quad (24)$$

Notons que l'on ne requiert pas (16). Les résultats principaux de l'article II concernant l'unicité peuvent être formulés comme suit :

**Théorème 3.** *Soit  $D \subset \mathbb{R}^d$  ( $d \geq 2$ ) un domaine simplement connexe borné de bord  $\partial D$  connexe par arcs, où  $\partial D \in C^\infty$  ( $d = 2$ ) ou  $\partial D \in C^1$  ( $d \geq 3$ ). Soient  $L_\omega^{(j)}$  et  $\Lambda_\omega^{(j)}$  les opérateurs de (17), (21) correspondant aux coefficients  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$ ,  $\zeta$ , satisfaisant à (18), (23) pour  $d \geq 3$  ou à (18), (24) pour  $d = 2$ . Supposons que  $\alpha_0^{(j)} = 0$ ,  $j = 1, 2$ . Soient  $\omega_1, \omega_2 \in [0, +\infty) \setminus (\sigma(L_{\bullet}^{(1)}) \cup \sigma(L_{\bullet}^{(2)}))$ ,  $\omega_1 \neq \omega_2$ . Alors, si  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ ,  $\omega \in \{\omega_1, \omega_2\}$ , on a  $c^{(1)} = c^{(2)}$ ,  $\rho^{(1)} = C\rho^{(2)}$ ,  $v^{(1)} = v^{(2)}$ , où  $C = \text{const} > 0$ .*

**Théorème 4.** *Soit  $D \subset \mathbb{R}^d$  ( $d \geq 2$ ) un domaine simplement connexe borné de bord  $\partial D$  connexe par arcs, où  $\partial D \in C^\infty$  ( $d = 2$ ) ou  $\partial D \in C^1$  ( $d \geq 3$ ). Soient  $L_\omega^{(j)}$  et  $\Lambda_\omega^{(j)}$  les opérateurs de (17), (21) correspondant aux coefficients  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$ ,  $\zeta$ , satisfaisant à (18), (23) pour  $d \geq 3$  ou à (18), (24) pour  $d = 2$ . Soient  $\omega_1, \omega_2, \omega_3 \in [0, +\infty) \setminus (\sigma(L_{\bullet}^{(1)}) \cup \sigma(L_{\bullet}^{(2)}))$  trois fréquences mutuellement différentes. Alors, si  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$ ,  $\omega \in \{\omega_1, \omega_2, \omega_3\}$ , on a  $c^{(1)} = c^{(2)}$ ,  $\rho^{(1)} = C\rho^{(2)}$ ,  $v^{(1)} = v^{(2)}$ ,  $\alpha^{(1)} = \alpha^{(2)}$ , où  $C = \text{const} > 0$  et  $\alpha^{(j)}(x, \omega) = \omega^{\zeta^{(j)}(x)} \alpha_0^{(j)}(x)$ .*

Pour formuler le résultat concernant la non-unicité, nous avons besoin d'introduire quelques notations. Soit

$$h \text{ une fonction réelle supportée dans } D, \quad h \in C^2(D), \quad |\nabla h|^2 < 1 \text{ dans } D. \quad (25)$$

Posons

$$\begin{aligned} c^{(1)} &\equiv \text{const} > 0, & \rho^{(1)} &\equiv \text{const} > 0, \\ v^{(1)} &\equiv 0, & \alpha_0^{(1)} &\equiv \text{const} > -\frac{1}{2} \min_{x \in D} \Delta h(x), \end{aligned} \quad (26)$$

et

$$\begin{aligned} c^{(2)} &= c^{(1)}(1 - |\nabla h|^2)^{-\frac{1}{2}}, & \rho^{(2)} &\equiv \text{const} > 0, \\ v^{(2)} &= c^{(1)}(1 - |\nabla h|^2)^{-1} \nabla h, & \alpha_0^{(2)} &= (1 - |\nabla h|^2)^{-\frac{1}{2}} (\alpha_0^{(1)} + \frac{1}{2} \Delta h). \end{aligned} \quad (27)$$

Notons que les paramètres  $v^{(j)}$ ,  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $\alpha_0^{(j)}$  avec  $\zeta^{(j)} = 0$  satisfont à (18), (23), (24) pour  $j = 1, 2$ .

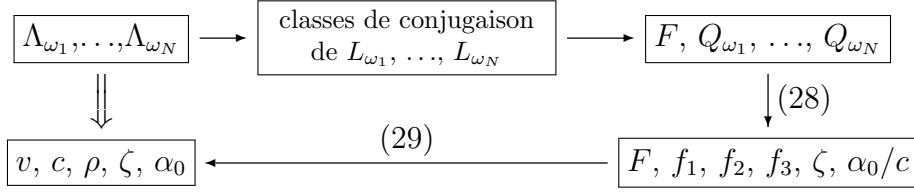


FIGURE 3 – Le schéma de démonstration de l'unicité

**Théorème 5.** Soit  $D \subset \mathbb{R}^d$  ( $d \geq 2$ ) un domaine borné de bord  $\partial D$  lisse. Supposons que  $h$  satisfait à (25),  $h \neq 0$ , et que les paramètres  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha^{(j)}$ ,  $j = 1, 2$ , sont définis par (26), (27). Soient  $L_\omega^{(j)}$ ,  $\Lambda_\omega^{(j)}$  les opérateurs correspondant aux paramètres  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$  et  $\zeta^{(j)} = 0$ . Alors,  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  pour tout  $\omega \in [0, \infty) \setminus \sigma$ , où  $\sigma = \sigma(L_\bullet^{(1)}) = \sigma(L_\bullet^{(2)})$ .

Le schéma de démonstration des théorèmes 3 et 4 est donné sur la Fig. 3. Pour ces démonstrations nous considérons les opérateurs  $L_{\omega_1}, \dots, L_{\omega_N}$  et  $\Lambda_{\omega_1}, \dots, \Lambda_{\omega_N}$ ,  $N \geq 2$ , de (17), (21) correspondant aux paramètres  $c, \rho, v, \alpha$  fixé et tels que  $\omega_j \notin \sigma(L_\bullet)$  pour tout  $j$ . Il est connu que les opérateurs  $\Lambda_{\omega_1}, \dots, \Lambda_{\omega_N}$  déterminent uniquement  $F, Q_{\omega_1}, \dots, Q_{\omega_N}$  dans  $D$ , où

$$F = \operatorname{curl} \frac{v}{c^2}, \quad (28)$$

$$Q_\omega = f_1 - \omega^2 f_2 + i\omega f_3 - 2i\omega^{1+\zeta} \frac{\alpha_0}{c},$$

$$f_1 = \rho^{\frac{1}{2}} \Delta \rho^{-\frac{1}{2}}, \quad f_2 = \frac{1}{c^2} + \frac{v}{c^2} \frac{v}{c^2}, \quad f_3 = \nabla \cdot \left( \frac{v}{c^2} \right) - \frac{v}{c^2} \nabla \ln \rho, \quad (29)$$

voir [61] ( $d = 2$ ) et [85] ( $d \geq 3$ ), en tenant compte de ce que  $L_\omega = L_{A_\omega, -U_\omega}$ ,  $\operatorname{curl} A_\omega = \omega F$ , et des formules (1), (2), (17). Autrement dit, l'opérateur  $\Lambda_{\omega_j}$  détermine la classe de conjugaison de l'opérateur  $L_{\omega_j}$ , et les fonctions  $F, Q_\omega$  sont des invariants de cette classe.

Si  $Q_\omega$  est connu pour plusieurs  $\omega$ , on peut considérer l'expression (28) pour  $Q_\omega$  comme un système déterminant  $f_1, f_2, f_3, \zeta, \alpha_0/c$ . Par exemple  $\operatorname{Re} Q_\omega = f_1 - \omega^2 f_2$ , et la connaissance de  $Q_\omega$  pour deux  $\omega$  différents est suffisante pour déterminer  $f_1$  et  $f_2$  à partir du système linéaire.

Ayant trouvé  $F, f_1, f_2, f_3$ , on peut déterminer  $\rho, c, v$  en utilisant les équations (29) et les valeurs tangentes au bord de  $\frac{v}{c^2}$  et  $\nabla \rho$ , qui sont également uniquement déterminées par l'opérateur  $\Lambda_\omega$  à une fréquence fixée (voir [19]).

D'autre part, on peut voir que si  $\zeta = 0$ , alors  $\operatorname{Im} Q_\omega = \omega(f_3 - 2\frac{\alpha_0}{c})$ . Il est donc impossible de trouver  $f_3$  et  $\frac{\alpha_0}{c}$  eux-mêmes de cette expression quand  $Q_\omega$  est connu pour plusieurs  $\omega$ . Cette observation nous amène au théorème 5.

## 2.4 Résumé de l'article III

Dans l'article III nous abordons la question de reconstruction pour le problème 4 de la sous-section 2.1. Nous établissons les formules qui permettent de réduire le problème 4 au problème 2 de la sous-section 1.3. Par souci de concision, nous allons formuler ces résultats dans le cas des données classiques de diffusion.

Revenons au problème 2 de la sous-section 1.3. Rappelons que la fonction  $\psi_{A,V}^+(\cdot, k)$  satisfait à l'équation de type Lippmann-Schwinger (12). Cette equation et ses versions dérivées par rapport à  $x_1, \dots, x_d$  sont considérées pour  $k$  fixé comme une équation vectorielle pour  $\psi_{A,V}^+(\cdot, k), \frac{\partial}{\partial x_1}\psi_{A,V}^+(\cdot, k), \dots, \frac{\partial}{\partial x_d}\psi_{A,V}^+(\cdot, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$ . Soit

$\mathcal{E}_{A,V}^+$  l'ensemble des  $k \in \mathbb{R}^d \setminus 0$  tels  
que l'équation (12) n'admet pas de solution unique.

On désigne par  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  l'espace des fonctions de classe  $C^1$  à valeurs dans  $M_n(\mathbb{C})$  sur  $\partial D$ , dont les premières dérivées partielles sont  $\beta$ -höldériennes. On munit cet espace avec la norme

$$\|\varphi\|_{C^{1,\beta}(\partial D)} = \|\varphi\|_{C^1(\partial D)} + \max_{\tau} \max_{i,j} \sup_{x_1 \neq x_2} \frac{|\partial_{\tau}\varphi_{ij}(x_1) - \partial_{\tau}\varphi_{ij}(x_2)|}{|x_1 - x_2|^{\beta}},$$

où  $\tau$  est un vecteur unitaire tangent à  $\partial D$ ,  $\varphi(x) = (\varphi_{ij}(x)) \in M_n(\mathbb{C})$ ,  $\partial_k = \partial/\partial x_k$ .

**Théorème 6.** *Soit  $D \subset \mathbb{R}^d$  un domaine borné au bord  $\partial D \in C^2$ , et soient  $A_1, \dots, A_d, V$  des fonctions höldériennes à valeurs dans  $M_n(\mathbb{C})$  et supportées dans  $D$ . Supposons que  $E$  n'est pas une valeur propre de Dirichlet pour les opérateurs  $L_{A,V}$  et  $-\Delta$  dans  $D$ , où  $A = (A_1, \dots, A_d)$ . Alors, la formule suivante est valable*

$$f_{A,V}(k, l) = (2\pi)^{-d} \int_{\partial D} \int_{\partial D} e^{-ix} (\Lambda_{A,V} - \Lambda_{0,0})(x, y, E) \psi_{A,V}^+(y, k) dy dx, \quad (30)$$

où  $k, l \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{A,V}^+)$ ,  $k^2 = l^2 = E$ , et la fonction  $\psi_{A,V}^+$  peut être trouvée à partir de l'équation

$$\begin{aligned} \psi_{A,V}^+(x, k) &= e^{ikx} \text{id}_n + \int_{\partial D} A^+(x, y, k) \psi_{A,V}^+(y, k) dy, \quad x \in \partial D, \\ A^+(x, y, k) &= \int_{\partial D} G^+(x - z, k) (\Lambda_{A,V} - \Lambda_{0,0})(z, y, E) dz, \quad x, y \in \partial D, \end{aligned} \quad (31)$$

où  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{A,V}^+)$ ,  $k^2 = E$ . De plus, l'équation (31) pour  $k$  fixé est une équation de Fredholm de seconde espèce qui admet la solution unique  $\psi_{A,V}^+(\cdot, k) \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  pour chaque  $\beta \in (0, 1)$  fixé.



Nous considérons (31) comme une équation pour trouver  $\psi_{A,V}^+$  à partir de l'opérateur  $\Lambda_{A,V} - \Lambda_{0,0}$ . De plus, nous considérons (30) comme une formule explicite pour trouver  $f_{A,V}$  à partir de  $\psi_{A,V}^+(x, \cdot)$ ,  $x \in \partial D$ , et de  $\Lambda_{A,V} - \Lambda_{0,0}$ .

Notons que du point de vue des applications, l'équation (31) peut être résolue d'une manière stable et efficace pour des  $A^+$  suffisamment petits. Cette situation correspond au cas des petits  $A$  et  $V$ . Cependant, des applications (par exemple en tomographie de l'océan) amènent au cas, où  $A$  est petit et  $V$  est proche d'un coefficient connu  $V^0$  tel que :

$$\begin{aligned} & \text{soit } V^0(x) \text{ une matrice diagonale pour chaque } x, \\ & \text{soit } V^0 = \mathcal{V}^0 v^0, \text{ où } \mathcal{V}^0 \in M_n(\mathbb{C}) \text{ et } v^0 \text{ est une fonction scalaire.} \end{aligned} \quad (32)$$

Il est possible de modifier le théorème 6 et d'établir des équations stables pour les cas (32). Pour formuler ces résultats, nous avons besoin d'introduire la fonction  $R_{0,V^0}^+(x, y, k)$  comme suit :

$$R_{0,V^0}^+(x, y, k) = G^+(x - y, k) \text{id}_n + \int_{\mathbb{R}^d} G^+(x - z, k) V^0(z) R_{0,V^0}^+(z, y, k) dz, \quad (33)$$

où  $x, y \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d \setminus 0$ . Dans la physique quantique l'équation (33) est connu comme *l'équation de Dyson* qui relie les fonctions de Green pour les opérateurs  $L_{0,0}$  et  $L_{0,V^0}$ . Notons que les équations de ce type remontent à [36] dans le cas particulier de l'électrodynamique quantique et à [124] dans un cadre plus général. Nous considérons (33) pour  $y, k$  fixés comme une équation intégrale pour

$$R_{0,V^0}^+(x, y, k) = G^+(x - y, k) \text{id}_n + e^{ik(x-y)} r_{0,V^0}^+(x, y, k),$$

où  $r_{0,V^0}^+(\cdot, y, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$ . Notons que l'équation (33) admet une solution unique pour chaque  $y \in \mathbb{R}^d$  si et seulement si  $k \notin \mathcal{E}_{0,V^0}^+$ . Notons également que  $R_{0,0}^+(x, y, k) = G^+(x - y, k) \text{id}_n$ .

**Théorème 7.** *Soit  $D \subset \mathbb{R}^d$  un domaine borné au bord  $\partial D \in C^2$  et soient  $A_1, \dots, A_d, V, V^0$  des fonctions höldériennes à valeurs dans  $M_n(\mathbb{C})$ , supportées dans  $D$  et telles que  $V^0$  satisfait à (32). Supposons que  $E$  n'est pas une valeur propre de Dirichlet pour les opérateurs  $L_{A,V}$  et  $L_{0,V^0}$  dans  $D$ , où  $A = (A_1, \dots, A_d)$ . Alors, la formule suivante est valable :*

$$\begin{aligned} & f_{A,V}(k, l) = f_{0,V^0}(k, l) \\ & + (2\pi)^{-d} \int_{\partial D} \psi_{0,V^0}^+(x, -l) (\Lambda_{A,V} - \Lambda_{0,V^0})(x, y, E) \psi_{A,V}^+(y, k) dy dx, \end{aligned} \quad (34)$$

où  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $k \notin \mathcal{E}_{A,V}^+ \cup \mathcal{E}_{0,V^0}^+$ , et la fonction  $\psi^+$  peut être trouvée de l'équation

$$\begin{aligned} \psi_{A,V}^+(x, k) &= \psi_{0,V^0}^+(x, k) + \int_{\partial D} A_{0,V^0}^+(x, y, k) \psi_{A,V}^+(y, k) dy, \quad x \in \partial D, \\ A_{0,V^0}^+(x, y, k) &= \int_{\partial D} R_{0,V^0}^+(x, z, k) (\Lambda_{A,V} - \Lambda_{0,V^0})(z, y, E) dz, \quad x, y \in \partial D, \end{aligned} \quad (35)$$

où  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{A,V}^+ \cup \mathcal{E}_{0,V^0}^+)$ ,  $k^2 = E$ . De plus, l'équation (35) pour  $k$  fixé est une équation de Fredholm de seconde espèce qui admet la solution unique  $\psi_{A,V}^+(\cdot, k) \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  pour chaque  $\beta \in (0, 1)$  fixé.

Les théorèmes 6, 7 dans le cas  $A = 0$ ,  $n = 1$  remontent à [97, 101].

La dérivation des formules (30), (34) et des équations (31), (35) est basée sur les identités de type Alessandrini, qui remontent à [8]. Plus précisément, nous établissons l'identité suivante :

$$\int_D u^0(x) (L_{A,V} - L_{0,V^0}) u(x) dx = \int_{\partial D} u^0(x) (\Lambda_{A,V} - \Lambda_{0,V^0})(u|_{\partial D})(x) dx, \quad (36)$$

pour des fonctions  $u^0$ ,  $u$  suffisamment régulières dans  $\bar{D}$  à valeurs dans  $M_n(\mathbb{C})$  telles que  $L_{A,V}u = Eu$ ,  $L_{0,V^0}u^0 = Eu^0$  et  $V^0(x)u^0(x) = u^0(x)V^0(x)$  dans  $D$ . L'identité (36) découle de la seconde formule de Green.

Les formules (30), (34) et les équations (31), (35) peuvent être dérivées de la formule (13), de l'équation (12) et de l'identité (36).

Pour démontrer que l'équation (31) est l'équation de Fredholm de seconde espèce (pour la démonstration de (35) on utilise la même idée), nous la réécrivons dans une forme opérationnelle :

$$\begin{aligned} \psi_{A,V}^+ &= e^{ikx} + G^+(\Lambda_{A,V} - \Lambda_{0,0})\psi^+, \\ \Lambda_{A,V} - \Lambda_{0,0} &= N_{A,V}S_{A,V}. \end{aligned} \quad (37)$$

Dans cette formule

$$\begin{aligned} G^+ &\text{ correspond au noyau intégral } G^+(x - y, k) \text{ (au sens de Schwartz),} \\ S_{A,V} &\text{ envoie une fonction } f \text{ sur } \partial D \text{ à la solution } \psi \text{ du problème (4),} \end{aligned} \quad (38)$$

et l'opérateur  $N_{A,V}$  est défini comme suit :

$$(N_{A,V}\psi)(x) = \int_D \frac{\partial \Gamma}{\partial \nu_x}(x, y, E) (L_{A,V} - L_{0,0})\psi(y) dy, \quad x \in \partial D,$$

où  $\Gamma$  est la fonction de Green pour le problème de Dirichlet pour l'opérateur  $\Delta + E$  dans  $D$ , et  $\partial/\partial \nu_x$  désigne la dérivée normale extérieure.

On peut montrer que les applications suivantes sont linéaires et continues :

$$C^{1,\beta}(\partial D) \xrightarrow{S_{A,V}} C^1(\overline{D}) \xrightarrow{N_{A,V}} C^2(\partial D) \xrightarrow{i} C^{1,\beta}(\partial D) \xrightarrow{G^+} C^{1,\beta}(\partial D), \quad (39)$$

où  $i$  désigne l'inclusion, et on suppose que des fonctions sont à valeurs dans  $M_n(\mathbb{C})$ . En tenant compte de la compacité de l'opérateur  $i$ , on voit que (37) est une équation de Fredholm de seconde espèce dans  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ .

## 2.5 Résumé de l'article IV

Dans l'article IV nous continuons à étudier la question de reconstruction pour le problème 4 de la sous-section 1.2 à une énergie  $E > 0$  fixée. Nous établissons un algorithme, qui permet de trouver une solution approchée du problème 2 en dimension  $d = 2$ . Cet algorithme est basé sur le problème de Riemann-Hilbert non local (également connu comme le problème Riemann-Hilbert-Manakov) et généralise l'algorithme de l'ouvrage [100] au cas où  $A \neq 0$ .

Notons qu'un algorithme en dimension  $d \geq 3$  avec  $n = 1$ ,  $A \neq 0$ , pour des coefficients à support compact a été proposé dans [120] comme une généralisation de l'algorithme de [97].

Nous considérons le cas scalaire  $n = 1$  en dimension  $d = 2$ . Pour formuler les résultats principaux, nous avons besoin d'introduire quelques notations. Soient  $(A, V)$  un couple de coefficients donnés décroissants à l'infini et soient  $(A^{\text{div}}, V^{\text{div}})$  le couple de coefficients décroissants à l'infini, liés aux coefficients  $A, V$  par une transformation de jauge (8) et satisfaisant à l'équation  $\nabla \cdot A^{\text{div}} = 0$ . Un tel couple de coefficients est déterminé uniquement. Dans le cadre de la physique quantique les couples  $(A, V)$  et  $(A^{\text{div}}, V^{\text{div}})$  décrivent la même réalité physique, mais la condition  $\nabla \cdot A^{\text{div}} = 0$ , dénommé *la jauge de Coulomb*, permet de se débarrasser des degrés de liberté redondants. Notre algorithme permet de trouver  $(A^{\text{div}}, V^{\text{div}})$  d'une manière approchée mais efficace à partir de l'amplitude  $f_{A,V}$  à  $E$  fixé.

Soit  $E > 0$  un nombre fixé qui n'est pas une résonance réelle pour  $L_{A,V}$ . L'amplitude de diffusion  $f_{A,V}$  peut être considérée comme une fonction sur le produit  $T^2 = T \times T$  de deux cercles unité  $T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$  comme suit :

$$\begin{aligned} f(e^{i\phi}, e^{i\theta}) &:= f_{A,V}(k, l), \quad \phi, \theta \in [0, 2\pi), \\ k &= E^{\frac{1}{2}}(\cos \phi, \sin \phi), \quad l = E^{\frac{1}{2}}(\cos \theta, \sin \theta). \end{aligned}$$

Nous introduisons les opérateurs suivants (voir [100]) :

$$\begin{aligned}
(P_{\pm}(\lambda)u)(\lambda') &= -\pi i \int_T u(\lambda'') \chi \left[ \pm i \left( \frac{\lambda}{\lambda''} - \frac{\lambda'}{\lambda} \right) \right] f(\lambda'', \lambda') |d\lambda''|, \\
(Q_{\pm}(x)u)(\lambda) &= \pi i \int_T h_{\pm}(\lambda, \lambda') e(\lambda, \lambda', x) \chi \left[ \pm i \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right] u(\lambda') |d\lambda'|, \\
e(\lambda, \lambda', x) &= \exp(-i\frac{\sqrt{E}}{2}((\lambda - \lambda')\bar{z} + (\lambda^{-1} - \lambda'^{-1})z)), \quad z = x + iy, \\
(C_{\pm}u)(\lambda) &= \frac{1}{2\pi i} \int_T \frac{u(\xi)}{\xi - \lambda(1 \mp 0)} d\xi, \\
B(x) &= C_+Q_-(x) - C_-Q_+(x),
\end{aligned}$$

où  $\lambda, \lambda' \in T$ ,  $x \in \mathbb{R}^2$ ,  $|d\lambda| = d\lambda/(i\lambda)$ ,  $\chi$  désigne la fonction de Heaviside, et les fonctions  $h_{\pm}(\lambda, \lambda')$ ,  $\lambda, \lambda' \in T$ , sont déterminées par l'équation suivante :

$$h_{\pm}(\lambda, \lambda') + (P_{\pm}(\lambda)h_{\pm}(\lambda, \cdot))(\lambda') = f(\lambda, \lambda'), \quad (\lambda, \lambda') \in T^2. \quad (40)$$

Le résultat principal de l'article IV consiste en un algorithme qui permet de trouver une approximation  $A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  aux coefficients  $A^{\text{div}}, V^{\text{div}}$ , et qui peut être formulé comme suit :

**Algorithme 1.** Soit  $f = f_{A,V}$  l'amplitude de diffusion pour l'opérateur  $L_{A,V}$  à l'énergie fixée  $E > 0$ , qui n'est pas une résonance réelle. Alors, on peut trouver une approximation  $A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  aux coefficients  $A^{\text{div}}, V^{\text{div}}$  en utilisant le schéma suivant :

$$f \longrightarrow h_{\pm} \longrightarrow \mu^+ \longrightarrow \mu_{\pm} \longrightarrow A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}. \quad (41)$$

Les fonctions  $h_{\pm}, \mu^+$  et  $\mu_{\pm}$  sont déterminées successivement en utilisant les équations (40), (42) et la formule explicite (43) :

$$\mu^+(x, \lambda) + (B(x)\mu^+(x, \cdot))(\lambda) = 1, \quad x \in \mathbb{R}^2, \lambda \in T, \quad (42)$$

$$\mu_{\pm}(x, \lambda) = \mu^+(x, \lambda) + (Q_{\pm}(x)\mu^+(x, \cdot))(\lambda), \quad x \in \mathbb{R}^2, \lambda \in T. \quad (43)$$

Finalement, les coefficients  $A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  sont définis comme suit :

$$A_{\text{appr}}^{\text{div}}(x) = \frac{1}{2} \text{curl} \left( \ln \int_T \mu_+(x, \zeta) |d\zeta| \right), \quad (44)$$

$$\begin{aligned}
V_{\text{appr}}^{\text{div}}(x) &= 2|A_{\text{appr}}^{\text{div}}(x)|^2 + \frac{\sqrt{E}}{2\pi} \int_T \frac{\partial \mu_-(x, \zeta)}{\partial z} d\zeta \\
&+ \sqrt{E} \frac{\partial}{\partial \bar{z}} \left( \int_T \mu_+(x, \zeta) \frac{d\zeta}{\zeta^2} \Big/ \int_T \mu_+(x, \zeta) |d\zeta| \right),
\end{aligned} \quad (45)$$

où  $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$ ,  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$ .

**Théorème 8.** Soient  $E > 0$  et  $x \in \mathbb{R}^2$  fixés. Soit  $f \in C^\infty(T^2)$  une fonction satisfaisant à  $\|f\|_{L^2(T^2)} < 1/(6\pi)$ . Alors l'équation (40) admet une solution unique  $h_\pm \in L^2(T^2)$  et l'équation (42) admet une solution unique  $\mu^+(x, \cdot) \in L^2(T)$ . De plus, le dénominateur dans la formule (45) ne s'annule pas, les fonctions  $A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  sont bornées, décroissent à l'infini et satisfont à  $\nabla \cdot A_{\text{appr}}^{\text{div}} = 0$ . Finalement, l'opérateur  $L_{A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}}$  admet  $f$  comme amplitude de diffusion à énergie fixée  $E$ .

Notons que les conditions requises sur la régularité et la petitesse de  $f$  sont surévaluées. Notons également que le calcul des coefficients  $A^{\text{div}}$  et  $V_{\text{appr}}^{\text{div}}$  aux points différents peut être effectué de façon parallèle. De plus, les résultats annoncés dans l'exposé [126] témoignent de la convergence rapide de  $A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  vers  $A^{\text{div}}, V^{\text{div}}$  quand  $E \rightarrow +\infty$ . Une étude théorique de la convergence sera menée dans un article ultérieur.

En outre, nous démontrons que la linéarisation de l'algorithme 1 pour le cas des petits coefficients  $A, V$  amène aux formules de l'approximation de Born.

Indiquons les idées principales sur lesquelles est basé l'algorithme 1. Nous considérons les solutions de diffusion généralisées  $\psi(\cdot, k)$ ,  $k \in \mathcal{K}_E$ ,

$$\mathcal{K}_E = \{k \in \mathbb{C}^2 \setminus \mathbb{R}^2 : k^2 = E\},$$

du problème 10 qui remontent à Faddeev [47]. Les fonctions  $\psi(x, k) = e^{ikx}\mu(x, k)$  admettent le développement asymptotique suivant :

$$\begin{aligned} \mu(x, k(\lambda)) &= \mu_0^\pm(x) + \mu_1^\pm(x)\lambda^{\pm 1} + o(|\lambda|^{\pm 1}), \quad |\lambda|^{\pm 1} \rightarrow 0, \\ k &= (k_1, k_2), \quad k_1 = \frac{1}{2}E^{\frac{1}{2}}(\lambda + \lambda^{-1}), \quad k_2 = \frac{i}{2}E^{\frac{1}{2}}(\lambda^{-1} - \lambda). \end{aligned} \quad (46)$$

Notons que une fixation de jauge permet d'obtenir des relations supplémentaires pour les coefficients  $\mu_0^\pm$ . En particulier,

$$\begin{aligned} \text{si } A_1 &= iA_2, \text{ alors } \mu_0^- = 1; \\ \text{si } A_1 &= -iA_2, \text{ alors } \mu_0^+ = 1. \end{aligned}$$

Ensuite, en utilisant le développement (46) dans la première équation de la formule (10) et en égalant les coefficients des mêmes puissances de  $\lambda$ , nous obtenons les équations pour trouver  $A$  et  $V$  à partir de  $\mu_0^\pm$  et  $\mu_1^\pm$ . Cette relation remonte à [58]. Il reste à trouver les coefficients  $\mu_0^\pm$  et  $\mu_1^\pm$  en fonction de l'amplitude  $f$ .

Il est connu que la fonction  $\mu(x, k(\lambda))$  est uniformément bornée pour  $x \in \mathbb{R}^2$ ,  $k \in \mathcal{K}_E$ , continue par rapport à  $\lambda \in \mathbb{C} \setminus T$ , et satisfait à l'équation de  $\bar{\partial}$  :

$$\frac{\partial}{\partial \bar{\lambda}} \mu(x, k(\lambda)) = r(x, \lambda) \mu(x, k(-1/\bar{\lambda})), \quad \lambda \in \mathbb{C} \setminus (T \cup 0), \quad (47)$$

où la fonction  $r$  est uniformément petite pour des grands  $E$ . De plus, les sauts  $\mu_\pm(x, \lambda) = \mu(x, k(\lambda \pm 0\lambda))$  sur le cercle  $T$  sont liés par l'équation (43), où  $\mu^+$  est supposé inconnu, voir [57, 100].

En imposant la condition de jauge  $A_1 = iA_2$  et en posant  $r = 0$  dans (47), on se ramène au problème de Riemann-Hilbert non local, pour trouver une approximation  $\mu_{\text{appr}}$  à  $\mu$  :

**Problème 5.** *Étant donné  $Q_{\pm}$ , trouver une fonction  $\mu_{\text{appr}}(x, k(\lambda))$ ,  $x \in \mathbb{R}^2$ ,  $\lambda \in \mathbb{C} \setminus T$ , uniformément bornée, continue,  $\mu_{\text{appr}}(x, k(\lambda)) = 1 + O(|\lambda|^{-1})$ ,  $\lambda \rightarrow \infty$ , analytique en  $\lambda \in \mathbb{C} \setminus (T \cup 0)$  pour tout  $x \in \mathbb{R}^2$ , et satisfaisant à la condition de saut*

$$\begin{aligned} \mu_{\pm, \text{appr}}(x, \lambda) &= \mu_{\text{appr}}^+(x, \lambda) + (Q_{\pm}(x)\mu_{\text{appr}}^+(x, \cdot))(\lambda), \quad x \in \mathbb{R}^2, \lambda \in T, \\ \mu_{\pm, \text{appr}}(x, \lambda) &= \mu_{\text{appr}}(x, k(\lambda \pm 0\lambda)), \quad \lambda \in T, \end{aligned} \quad (48)$$

où  $\mu_{\text{appr}}^+$  est une fonction uniformément bornée continue inconnue.

L'étude des tels problèmes remonte à [89, 52, 59], et la solution dans le cadre du problème de diffusion considéré remonte à [96] (dans le cas "auto-adjoint").

Ayant résolu le problème 5, on détermine des approximations  $\mu_{0, \text{appr}}^{\pm}$  et  $\mu_{1, \text{appr}}^{\pm}$  aux coefficients  $\mu_0^{\pm}$  et  $\mu_1^{\pm}$  correspondant au choix de jauge  $A_1 = iA_2$  à partir de  $\mu_{\pm, \text{appr}}$ , en tenant compte de (46) et en utilisant la formule de Cauchy pour des fonctions analytiques. Finalement, nous passons de ces coefficients aux coefficients correspondant à la jauge  $\nabla \cdot A^{\text{div}} = 0$  en utilisant une transformation de jauge (8).

Ce schéma de reconstruction a été utilisé dans l'article [100], où le cas  $A = 0$  a été considéré. De plus, dans l'article [100] il est démontré que la précision de reconstruction est en  $O(E^{-\frac{N-2}{2}})$  uniformément, si  $(1 + |x|^2)^{\frac{\varepsilon}{2}} \frac{\partial^{|\alpha|} Q}{\partial x^{\alpha}} \in L^1$ ,  $|\alpha| \leq N$ , pour quelque  $\varepsilon > 0$  fixé.

## 2.6 Résumé de l'article V

Dans l'article V nous établissons des formules pour trouver des paramètres du fluide à partir des fonctions  $F$  et  $Q_{\omega}$  de (28) connus pour plusieurs fréquences  $\omega$ . Notons que pour déterminer les fonctions  $F$ ,  $Q_{\omega}$  il suffit de reconstruire l'opérateur  $L_{\omega}$  à une transformation de jauge (8) près, parce que  $F$  et  $Q_{\omega}$  sont invariants par rapport à ces transformations. L'opérateur  $L_{\omega}$ , à son tour, peut être trouvé à une transformation de jauge près à partir de l'opérateur  $\Lambda_{\omega}$  en utilisant des formules des articles III et IV.

Pour des raisons de simplicité, dans la présente section nous supposons que  $D = \mathcal{B}_R^2$ ,  $\mathcal{B}_R^2 = \{x \in \mathbb{R}^2 : |x| < R\}$ ,  $R > 0$ . Nous supposons également que les coefficients  $c$ ,  $v$ ,  $\rho$ ,  $\alpha_0$ ,  $\zeta$  satisfont à (16) et à

$$\begin{aligned} c &\in C^2(D), & v &\in C^2(D, \mathbb{R}^2), \\ \rho &\in C^{2, \beta}(D), \beta \in (0, 1], & \zeta &\in C(\overline{D}), \alpha_0 \in C(\overline{D}). \end{aligned} \quad (49)$$

On utilise les notations suivantes :

$$G(x, y) = \frac{1}{2\pi} \ln \left( \frac{R|x-y|}{|y| \left| x - \frac{y}{|y|^2} R^2 \right|} \right) \quad (\text{fonction de Green-Dirichlet}),$$

$$P(x, \phi) = \frac{1}{2\pi} \frac{R^2 - |x|^2}{R^2 - 2R|x| \cos(\theta - \phi) + |x|^2} \quad (\text{noyau de Poisson}).$$

**Algorithme 2.** Soit  $D = \mathcal{B}_R^2$ ,  $R > 0$ . On considère l'opérateur  $L_\omega$  de (17) et on suppose que les conditions (16), (18), (49) sont satisfaites. Soient  $\omega_1, \omega_2, \omega_3 \in (0, +\infty) \setminus \sigma$  trois fréquences mutuellement différentes, où  $\sigma$  est défini dans (19). Alors, les paramètres  $c, v, \rho, \alpha_0, \zeta$  peuvent être trouvés à partir des fonctions  $F, Q_{\omega_1}, Q_{\omega_2}, Q_{\omega_3}$  de (28) comme suit :

1. Déterminer  $\rho^{-\frac{1}{2}}$  comme la solution unique de classe  $C(\overline{D})$  de l'équation

$$\rho^{-\frac{1}{2}}(x) = \rho_0^{-\frac{1}{2}} + \int_D G(x, y) f_1(y) \rho^{-\frac{1}{2}}(y) dy, \quad x \in \overline{D},$$

$$f_1 = \frac{\omega_2^2 \operatorname{Re} Q_{\omega_1} - \omega_1^2 \operatorname{Re} Q_{\omega_2}}{\omega_2^2 - \omega_1^2}.$$
(50)

2. Trouver  $D_0 = \{x \in D : \alpha_0(x) = 0\}$  de la formule explicite

$$D_0 = \{x \in D : \omega_1^{-1} \operatorname{Im} Q_{\omega_1} = \omega_2^{-1} \operatorname{Im} Q_{\omega_2}\},$$

et déterminer  $\zeta(x)$  pour  $x \in D \setminus D_0$  fixé comme la solution positive unique de l'équation

$$\frac{\omega_2^{-1} \operatorname{Im} Q_{\omega_2}(x) - \omega_1^{-1} \operatorname{Im} Q_{\omega_1}(x)}{\omega_3^{-1} \operatorname{Im} Q_{\omega_3}(x) - \omega_1^{-1} \operatorname{Im} Q_{\omega_1}(x)} = \frac{\left(\frac{\omega_2}{\omega_1}\right)^{\zeta(x)} - 1}{\left(\frac{\omega_3}{\omega_1}\right)^{\zeta(x)} - 1}.$$
(51)

3. Définir  $\eta \in C(\overline{D})$  comme la solution unique de l'équation

$$\eta(x) = \eta_0(x) + \eta_1(x) + \int_D G(x, y) f_1(y) \eta(y) dy, \quad x \in \overline{D},$$

$$\eta_0(x) = \int_D G(x, y) \rho^{-\frac{1}{2}}(y) (f_3(y) - W(y) \cdot \nabla \ln \rho(y)) dy,$$

$$\eta_1(x) = -R \rho_0^{-\frac{1}{2}} \int_0^{2\pi} \int_0^\phi P(x, \phi) W(\vartheta(\theta)) \cdot \vartheta^\perp(\theta) d\theta d\phi,$$
(52)

où  $\vartheta(\phi) = (\cos \phi, \sin \phi)$  et

$$f_3 = \omega_1 \operatorname{Im} Q_{\omega_1} \text{ dans } D_0, \quad f_3 = \frac{\frac{\omega_1^\zeta}{\omega_2} \operatorname{Im} Q_{\omega_2} - \frac{\omega_2^\zeta}{\omega_1} \operatorname{Im} Q_{\omega_1}}{\omega_1^\zeta - \omega_2^\zeta} \text{ dans } D \setminus D_0,$$
(53)

$$W(x) = \frac{1}{2\pi} \int_D \frac{(x-y)^\perp}{|x-y|^2} F(y) dy, \quad x \in \overline{D}.$$

4. Trouver  $c$  et  $v$  dans  $D$  et  $\alpha_0$  dans  $D \setminus D_0$  en utilisant les formules

$$\begin{aligned}
c^{-2} &= \frac{\operatorname{Re} Q_{\omega_1} - \operatorname{Re} Q_{\omega_2}}{\omega_2^2 - \omega_1^2} - (\nabla(\rho^{\frac{1}{2}}\eta) - W)^2 \quad \text{dans } D, \\
v &= c^2 (\nabla(\rho^{\frac{1}{2}}\eta) - W) \quad \text{dans } D, \\
\alpha_0 &= \frac{c}{2} \cdot \frac{\omega_2^{-1} \operatorname{Im} Q_{\omega_2} - \omega_1^{-1} \operatorname{Im} Q_{\omega_1}}{\omega_1^\zeta - \omega_2^\zeta} \quad \text{dans } D \setminus D_0.
\end{aligned} \tag{54}$$

Notons que l'algorithme 2 permet de trouver tous les paramètres du fluide à partir des données mesurables pour trois fréquences mutuellement différentes. Nous proposons également un algorithme simplifié qui requiert des données pour deux fréquences différentes pour le cas où il est connu a priori que  $\alpha_0 = 0$ . De plus, notons qu'il est possible d'améliorer l'algorithme 2 en utilisant des données pour des fréquences supplémentaires.

La dérivation de l'algorithme 2 suit le schéma présenté sur la Fig. 3. On considère la deuxième formule dans (28), dont le membre de gauche est connu pour  $\omega \in \{\omega_1, \dots, \omega_N\}$ , comme un système algébrique pour trouver  $f_1, f_2, f_3, \alpha_0/c, \zeta$ . Ensuite, on trouve  $c, \rho, v, \alpha_0$  des formules (29) que l'on considère comme des équations différentielles, dont le membre de gauche et les valeurs au bord sont connus. De plus, l'existence et l'unicité des solutions des équations (50) et (52) découlent de l'existence et l'unicité des solutions des équations de Schrödinger équivalentes avec un potentiel de type conductif.

### 3 Problème de diffusion inverse sans phase. Résumé de l'article VI

Nous étudions une version du problème 2 de la sous-section 1.3 motivée par des applications en physique quantique. Nous considérons l'équation

$$L_{0,v}\psi := -\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \tag{55}$$

où  $v \in L^\infty(\mathbb{R}^d)$  est une fonction à support compact. L'équation (55) est l'équation de Schrödinger pour une particule élémentaire à l'énergie fixée  $E$  interagissant avec un objet macroscopique décrit par le potentiel  $v$ .

Nous considérons l'amplitude de diffusion  $f$  pour l'équation (55) qui est définie par les formules (10), (11). En physique quantique  $|f(k, l)|^2$  décrit la densité de probabilité de diffusion d'une particule à l'impulsion initiale  $k$  dans la direction  $\frac{l}{|l|} \neq \frac{k}{|k|}$ , et  $\frac{\partial \arg f(k, l)}{\partial |k|}$  décrit le retard d'une particule due à la présence du diffuseur  $v$ , voir [30, p. 461-463] et [49]. Notons que dans une expérience réelle, il est beaucoup plus facile de



mesurer seulement  $|f|^2$  (il suffit de compter le nombre des particules diffusées dans des différentes directions).

Le problème de détermination du diffuseur  $v$  à partir de  $|f|^2$  (et ses modifications), dénommé le problème de diffusion inverse sans phase, est beaucoup moins étudié que le problème 2 (dans le cas où  $A = 0$ ,  $n = 1$ ), voir par exemple [26, 83, 105, 108] et les références qui y sont contenues. Notons que ce problème n'admet jamais de solution unique. En particulier, si on désigne par  $f_y$  l'amplitude de diffusion pour le potentiel  $v_y(x) = v(x - y)$  pour  $y \in \mathbb{R}^d$  fixé,  $|f_y|^2 = |f|^2$ .

Cependant, R. Novikov [106] a proposé une modification du problème pluridimensionnel de diffusion inverse sans phase qui permet de se débarrasser de la non-unicité et qui n'est pas restrictive d'un point de vue des applications. En suivant [106], nous définissons l'ensemble

$$S = \{|f|^2, |f_1|^2, \dots, |f_m|^2\},$$

où  $f_j$  désigne l'amplitude de diffusion pour le potentiel  $v_j = v + w_j$  et  $w_j$  est un potentiel de fond connu,  $j = 1, \dots, m$ . L'ensemble  $S$  est considéré comme une donnée mesurable pour trouver  $v$ . Le problème de diffusion inverse sans phase modifié se pose comme suit :

**Problème 6.** *Étant donné l'ensemble  $S$  des modules d'amplitudes de diffusion à un niveau d'énergie  $E$  fixé pour quelques potentiels de fond  $w_1, \dots, w_n$  connus, déterminer le potentiel  $v$ .*

L'étude du problème 6 a été commencé dans l'article [106] où des formules de type Born ont été présentées pour la solution approchée du problème 6 en dimension  $d \geq 2$ .

Notons que dans le cas unidimensionnel, un problème analogue avec  $m = 1$  a été posé et étudié par Aktosun-Sacks [7]. Notons également que dans l'article [7] il est supposé que  $S$  est connu pour tout  $E > 0$ .

Dans l'article VI nous continuons à étudier le problème 6. Nous établissons des estimations pour les reconstructions approchées de type Born pour le problème 6, qui sont requises pour le développement des méthodes de reconstruction précises. En particulier, ces estimations sont nécessaires pour généraliser au cas sans information de phase l'algorithme itératif de [107] pour le problème inverse de diffusion avec information de phase.

Notons que les estimations présentées dans l'article VI dépendent de la géométrie de l'ensemble des zéros communs des transformations de Fourier des potentiels de fond  $w_1, \dots, w_m$ . Pour des raisons de brièveté, nous allons présenter ces résultats dans les deux cas les plus simples.

Pour formuler des résultats principaux, nous avons besoin d'introduire un potentiel de fond de référence comme une fonction  $w$  telle que

$$w \in C_{\text{comp}}(\mathbb{R}^d, \mathbb{R}), \quad \widehat{w}(p) \geq c_1(1 + |p|)^{-\beta}, \quad \beta > d, \quad p \in \mathbb{R}^d, \quad (56)$$

pour quelque constante  $c_1 > 0$ . Dans l'article VI nous proposons une approche pour construire des fonctions  $w$  satisfaisant à (56).

Nous avons également besoin d'introduire des notations suivantes :

$$\begin{aligned}
U_{\widehat{w}_1, \widehat{w}_2} &= \operatorname{Re} U_{\widehat{w}_1, \widehat{w}_2} + i \operatorname{Im} U_{\widehat{w}_1, \widehat{w}_2}, \\
\begin{pmatrix} \operatorname{Re} U_{\widehat{w}_1, \widehat{w}_2}(p) \\ \operatorname{Im} U_{\widehat{w}_1, \widehat{w}_2}(p) \end{pmatrix} &= \frac{1}{2} M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) b_{\widehat{w}_1, \widehat{w}_2}(p, E), \quad |p| \leq 2\sqrt{E}, \\
M_{\widehat{w}_1, \widehat{w}_2}(p) &= \begin{pmatrix} \operatorname{Re} \widehat{w}_1(p) & \operatorname{Im} \widehat{w}_1(p) \\ \operatorname{Re} \widehat{w}_2(p) & \operatorname{Im} \widehat{w}_2(p) \end{pmatrix}, \\
b_{\widehat{w}_1, \widehat{w}_2}(p, E) &= \begin{pmatrix} |f_1(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_1(p)|^2 \\ |f_2(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_2(p)|^2 \end{pmatrix},
\end{aligned} \tag{57}$$

où  $f(p, E) = f(k_E(p), l_E(p))$ ,  $f_j(p, E) = f_j(k_E(p), l_E(p))$ ,  $j = 1, 2$ , et  $k_E(p)$ ,  $l_E(p)$  sont des fonctions à valeurs vectorielles continues par morceaux telles que  $k_E^2(p) = l_E^2(p) = E$ ,  $k_E(p) - l_E(p) = p$ ,  $p \in \mathcal{B}_{2\sqrt{E}}^d$ .

Le premier résultat principal de l'article VI est le théorème suivant :

**Théorème 9.** *Soit  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n > d$ , une fonction bornée à support compact. Considérons  $w_1 = w(\cdot - T_1)$ ,  $T_1 \in \mathbb{R}^d \setminus 0$ ,  $w_2 = iw_1$ , où  $w$  est une fonction satisfaisant à (56). Posons*

$$\begin{aligned}
u(x, E) &= \int_{\mathcal{B}_{r_1(E)}^d} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp, \quad x \in \mathbb{R}^d, \\
r_1(E) &= 2\tau E^{\frac{\alpha_1}{n-d}}, \quad \alpha_1 = \frac{n-d}{2(n+\beta)}, \quad \text{pour quelque } \tau \in (0, 1] \text{ fixé,}
\end{aligned}$$

où  $U_{\widehat{w}_1, \widehat{w}_2}$  est définie dans 57 et  $\beta$  est le nombre de (56). Alors,

$$u(\cdot, E) = v + O(E^{-\alpha_1}) \quad \text{dans } L^\infty(\mathbb{R}^d), \quad E \rightarrow +\infty.$$

Le théorème 9 permet de trouver une approximation  $u$  au potentiel inconnu  $v$  avec la précision uniforme en  $O(E^{-\alpha_1})$  à partir des données de diffusion sans phase à l'énergie  $E$  fixée pour le potentiel  $v$  et pour les potentiels perturbés  $v + w_1$  et  $v + w_2$ , où  $w_1$  et  $w_2$  sont des potentiels de fond connus,  $w_1$  est réel et  $w_2 = iw_1$ . Notons que le potentiel de fond  $w_2$  est purement imaginaire, ce qui permet de se débarrasser des quelques difficultés mathématiques.

Par contre, dans les applications en physique quantique il est souhaitable de n'utiliser que des potentiels de fond réels. Le résultat suivant permet de trouver une approximation au potentiel  $v$  à partir des données de diffusion sans phase pour le potentiel  $v$  et pour les potentiels perturbés  $v + w_1$  et  $v + w_2$ , où  $w_1$  et  $w_2 = w_1(\cdot - y)$  sont des potentiels de fond réels connus et  $y \in \mathbb{R}^d \setminus 0$  est fixé.

Pour formuler ce deuxième résultat principal, nous avons besoin d'introduire des notations supplémentaires. On suppose que  $w_1 = w(\cdot - T_1)$ ,  $w_2 = w(\cdot - T_2)$ ,  $y = T_2 - T_1 \neq 0$ , où  $w$  satisfait à (56). Posons

$$\begin{aligned}
U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) &= \frac{1}{2}(U_{\widehat{w}_1, \widehat{w}_2}(p_-^\varepsilon, E) + U_{\widehat{w}_1, \widehat{w}_2}(p_+^\varepsilon, E)), \\
p_\pm^\varepsilon &= p_\perp + \pi z(p) \frac{y}{|y|^2} \pm \varepsilon \frac{y}{|y|^2}, \quad p_\perp = p - (py) \frac{y}{|y|^2}, \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\
Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon &= \{p \in \mathbb{R}^d : py \in (-\varepsilon, \varepsilon) + \pi\mathbb{Z}\}, \quad \varepsilon \in (0, 1), \\
z(p) \in \mathbb{Z} &\text{ est le nombre unique tel que } |py - \pi z(p)| < \varepsilon.
\end{aligned} \tag{58}$$

**Théorème 10.** *Soit  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n > d$ , une fonction bornée à support compact. Considérons  $w_1 = w(\cdot - T_1)$ ,  $w_2 = w(\cdot - T_2)$ ,  $T_1 \neq T_2$ , où  $w$  est une fonction satisfaisant à (56). Posons*

$$\begin{aligned}
u(x, E) &= u_1(x, E) + u_2(x, E), \quad x \in \mathbb{R}^d, \\
u_1(x, E) &= \int_{\mathcal{B}_{r_2(E)} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp, \\
u_2(x, E) &= \int_{\mathcal{B}_{r_2(E)} \cap Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) dp, \\
r_2(E) &= 2\tau E^{\frac{\alpha_2}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_2}{2}}, \\
\alpha_2 &= \frac{n-d}{2(n+\beta+\frac{n-d}{2})}, \quad \text{pour quelque } \tau \in (0, 1] \text{ fixé,}
\end{aligned}$$

où  $U_{\widehat{w}_1, \widehat{w}_2}$  est définie dans (57),  $U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon$  est définie dans (58), et  $\beta$  est le nombre de (56). Alors,

$$u(\cdot, E) = v + O(E^{-\alpha_2}) \quad \text{dans } L^\infty(\mathbb{R}^d), \quad E \rightarrow +\infty.$$

Pour démontrer le théorème 9, nous montrons que  $U_{\widehat{w}_1, \widehat{w}_2}$  est une bonne approximation pour  $\widehat{v}$  dans  $\mathcal{B}_{r_1(E)}$ , puis nous utilisons l'analyse de Fourier pour transférer ce résultat dans l'espace de configuration et établir une estimation pour l'erreur  $u(\cdot, E) - v$ . L'idée est la même pour le théorème 10.

Dans l'article VI nous considérons également le cas, où  $m = d+1$  et  $w_1 = w(\cdot - T_1)$ ,  $w_{j+1} = w_1(\cdot - e_j)$ ,  $j = 1, \dots, d-1$ , où  $e_1, \dots, e_d$  désigne la base canonique pour  $\mathbb{R}^d$ . Dans ce cas, l'estimation est meilleure que dans le cas de théorème 10 mais du point de vue des applications, le cadre du théorème 10 est plus prometteur.

## 4 Problème de valeurs au bord pour une courbe complexe. Résumé de l'article VII

Soit  $(X, g)$  une variété riemannienne bidimensionnelle compacte au bord lisse  $\partial X$ . Sur cette variété on considère le problème de Dirichlet pour l'opérateur de Laplace-Beltrami :

$$\begin{cases} \Delta_g u = 0 & \text{sur } X, \\ u|_{\partial X} = u_0. \end{cases} \quad (59)$$

Rappelons que dans des coordonnées locales  $x_1, x_2$  sur  $X$  dans lesquelles la métrique  $g$  est représentée par une matrice  $(g_{ij})$  de déterminant  $|g|$  et d'inverse  $(g^{ij})$ , on a

$$\Delta_g u = |g|^{-\frac{1}{2}} \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} (|g|^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x_j}).$$

L'opérateur de Dirichlet-Neumann  $\Lambda_g$  pour l'opérateur de Laplace-Beltrami est défini comme suit :

$$\Lambda_g u_0 = \frac{\partial u}{\partial \nu} \Big|_{\partial X}, \quad \text{où } u \text{ est la solution de (59),} \quad (60)$$

et  $\partial/\partial \nu$  désigne la dérivée normale extérieure. On s'intéresse à la détermination de  $(X, g)$ , en supposant que  $\partial X$  et  $\Lambda_g$  sont connus.

Il y a deux sources de non-unicité pour ce problème :

1. Si  $F: (Y, h) \rightarrow (X, g)$  est une isométrie des variétés riemanniennes, on a  $\Delta_h \circ F^* = F^* \circ \Delta_g$ , où  $F^*$  désigne l'image réciproque. Alors, l'opérateur  $\Lambda_g$  est invariant par rapport aux isométries fixant le bord.
2. Si  $f$  est une fonction positive lisse sur  $X$ , on a  $\Delta_{fg} = f^{-1} \Delta_g$  (notons que cette propriété est spécifique pour le cas bidimensionnel). Cela implique que les métriques  $g$  et  $h = fg$ , où  $f > 0$ ,  $f|_{\partial X} = 1$ , ne sont pas distinguables par l'opérateur  $\Lambda_g$ .

En tenant compte de la non-unicité susmentionnée, on s'intéresse à la détermination de  $(X, g)$  à partir de  $\Gamma = \partial X$  et  $\Lambda_g$  à une application conforme fixant le bord  $\Gamma$  près. Rappelons qu'une application conforme est une application préservant les angles entre les vecteurs tangents tandis qu'une isométrie préserve le produit scalaire. Le problème inverse à valeurs au bord pour l'opérateur  $\Lambda_g$  est posé comme suit :

**Problème 7.** *Soit  $(X, g)$  une variété riemannienne bidimensionnelle au bord lisse  $\Gamma = \partial X$  et soit  $\Lambda_g$  l'opérateur de Dirichlet-Neumann défini dans (60). Étant donné  $\Gamma$  et  $\Lambda_g$ , déterminer  $(X, g)$  à une application conforme fixant le bord  $\Gamma$  près.*

Le problème 7 est lié au problème de Calderón sur les surfaces, qui consiste en détermination de la surface conductrice à partir des mesures au bord du courant électrique induit par un potentiel imposé, voir par exemple [67, 69, 70].

Lassas et Uhlmann [87] ont démontré que le problème (7) admet la solution unique (pour  $\Gamma$  de classe  $C^\infty$ ). Henkin et Michel [67] ont considéré la question de caractérisation des données non surdéterminées et ont débuté l'étude de la question de reconstruction.

Il faut souligner que Henkin-Michel [67] réduisent le problème 7 au problème de détermination d'une courbe complexe dans le plan projectif à partir de son bord comme suit. Soient  $x, y$  les coordonnées locales isothermes sur  $X$  (voir [129]). Posons  $z = x + iy$ ,  $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ ,  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ ,  $dz = dx + idy$ ,  $d\bar{z} = dx - idy$ . On peut montrer que si  $u$  est la solution du système (59), alors  $\partial u = \frac{\partial u}{\partial z} dz$  est une 1-forme analytique (i.e.  $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$ ) sur  $X$  qui satisfait à

$$\partial u|_{\partial X} = \frac{1}{2}(\Lambda_g u_0 - iTu_0)(\nu^* + i\tau^*),$$

où  $T$  désigne la dérivée le long du bord  $\partial X$ , et  $\nu^*, \tau^*$  désignent les covecteurs conormal et cotangent unitaires sur  $\partial X$ , respectivement. Alors, si  $u_1, u_2, u_3$  sont trois fonctions harmoniques sur  $X$  telles que  $\partial u_1, \partial u_2, \partial u_3$  ne s'annulent en même temps, on peut définir l'application

$$F: X \cup \Gamma \rightarrow \mathbb{C}P^2, \\ F(z) = (\partial u_1(z) : \partial u_2(z) : \partial u_3(z)),$$

où  $\mathbb{C}P^2$  désigne le plan projectif complexe muni des coordonnées homogènes  $(w_0 : w_1 : w_2)$ . Rappelons que des coordonnées homogènes sont déterminées à une multiplication par un nombre complexe non nul près. Cela justifie la définition de l'application  $F$ . Henkin et Michel montrent que si  $F|_{\partial X}$  est un plongement<sup>3</sup>, alors  $Y = F(X)$  est une courbe complexe de fermeture compacte de bord  $\gamma = F(\partial X)$  et sans composantes fermées,<sup>4</sup> avec un nombre fini de points singuliers, et  $F: X \rightarrow Y$  est sa normalisation. Pour des raisons de simplicité, on suppose que  $\gamma \subset \mathbb{C}^2$ , où  $\mathbb{C}^2 = \{(w_0 : w_1 : w_2) \in \mathbb{C}P^2 : w_0 \neq 0\}$  avec des coordonnées  $z_1 = w_1/w_0$ ,  $z_2 = w_2/w_0$ . Donc, on s'est ramené au problème suivant :

**Problème 8.** *Soit  $\mathcal{X} \subset \mathbb{C}P^2$  une courbe complexe compacte au bord  $\gamma = \partial X \subset \mathbb{C}^2$  et sans composantes fermées. Étant donnée  $\gamma$ , trouver  $\mathcal{X}$ .*

En utilisant le théorème de structure de Harvey-Shiffman [66], on peut démontrer qu'il existe au plus une courbe complexe<sup>5</sup>  $\mathcal{X} \subset \mathbb{C}P^2$  de bord donné  $\gamma \subset \mathbb{C}^2$  et sans

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3. Le plongement est une application injective dont la différentielle est aussi injective et qui de plus est un homéomorphisme sur son image

4. Par définition, une composante fermée est une composante connexe compacte et sans bord.

5. Plus généralement, il existe au plus une chaîne holomorphe bordée par un courant rectifiable réel donné.

composantes fermées, voir [32, Proposition 1.4.1]. Localement, cela correspond à l'unicité d'une fonction analytique dans un disque avec des valeurs données sur le bord.

Notons que  $\gamma$  est localement le graphe d'une fonction d'une variable réelle tandis que  $\mathcal{X}$  est localement le graphe d'une fonction analytique d'une variable complexe. En tenant compte de la formule de Cauchy pour des fonctions analytiques, on peut voir que le problème 8 est formellement non surdéterminé.

Dans l'article VII nous proposons un algorithme de reconstruction d'une courbe complexe compacte au bord donné dans le plan projectif. Cet algorithme peut être considéré comme une version constructive du célèbre résultat de Harwey-Shiffman [66] sur la détermination unique d'une chaîne holomorphe par son bord. Notre algorithme développe l'algorithme de Henkin-Michel [67].

Plus précisément, nous proposons des équations explicites pour trouver des polynômes  $P_m$  mentionnées dans le Théorème 2 de [67] qui jouent le rôle principal dans la reconstruction. Ces équations dépendent au nombre  $N_-$  de points à l'infini de la courbe complexe, et nous proposons des équations pour déterminer ce nombre dans les cas  $N_- \in \{0, 1, 2\}$ . Notons que malgré l'encombrante description proposée, il n'y a pas en principe d'obstacle pour la justification et l'implémentation numérique de l'algorithme pour  $N_- \geq 3$ .

Notre algorithme est basée sur des formules de type Cauchy, qui remontent à [31, 32], et sur le lemme de Darboux [29, p. 154]. Cet algorithme a été implémenté dans `Matlab` et a été illustré sur plusieurs exemples, voir l'article VII.

Soit  $X \subset \mathbb{C}P^2$  une courbe complexe compacte au bord  $\partial X \subset \mathbb{C}^2$  et sans composante fermée. Posons

$$\begin{aligned}\mathbb{C}_\xi^1 &= \{(z_1, z_2) \in \mathbb{C}^2 : \xi_0 + \xi_1 z_1 + z_2 = 0\}, \\ \mathbb{C}P_\infty^1 &= \{(0 : w_1 : w_2) \in \mathbb{C}P^2\},\end{aligned}$$

et désignons par  $N_+(\xi)$  et  $N_-$  le nombre de points d'intersection de  $X$  avec  $\mathbb{C}_\xi^1$  et  $\mathbb{C}P_\infty^1$ , respectivement :

$$N_+(\xi) = |X \cap \mathbb{C}_\xi^1|, \quad N_- = |X \cap \mathbb{C}P_\infty^1|.$$

Il est utile de décrire des intersections de  $X$  avec  $\mathbb{C}_\xi^1$  en utilisant les fonctions  $h_j$  définies comme suit :

$$X \cap \mathbb{C}_\xi^1 = \{(h_j(\xi), -\xi_0 - \xi_1 h_j(\xi)) \in \mathbb{C}^2, j = 1, \dots, N_+(\xi)\}.$$

On peut vérifier que dans un petit voisinage  $W_{\xi^*}$  de presque tout  $\xi^* \in \mathbb{C}^2$  les fonctions  $h_j$  sont analytiques et satisfont à l'équation de Riemann-Burgers (dénommée également l'équation de l'onde de choc), voir, e.g., [29] :

$$\frac{\partial h_j}{\partial \xi_1}(\xi) = h_j(\xi) \frac{\partial h_j}{\partial \xi_0}(\xi), \quad \xi \in W_{\xi^*}.$$

Posons également  $\pi_2(z_1, z_2) = -z_2$ . Notons que  $\mathbb{C} \setminus \pi_2(\partial X) = \cup_{l=0}^L \Omega_l$ , où  $\Omega_{l \geq 0}$  sont des domaines ouverts connexes,  $\Omega_i \cap \Omega_j = \emptyset$  pour  $i \neq j$ ,  $\Omega_{l > 0}$  sont bornés et  $N_+(\xi)$  est constante sur chaque  $\Omega_l$  et  $N_+(\xi) = N_-$  pour  $\xi \in \Omega_0$ .

Étant donné  $\partial X$ , on peut définir les fonctions  $G_k(\xi)$ ,  $k \geq 0$ ,  $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$ , dénommées *la transformation de Cauchy-Radon* de  $\partial X$  et qui remontent à Dolbeault-Henkin [31] :

$$G_k(\xi) = \frac{1}{2\pi i} \int_{\partial X} \frac{z_1^k d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2}, \quad k \geq 0. \quad (61)$$

La formule suivante de [32] joue un rôle principal dans la reconstruction de  $X$  à partir de  $\gamma$  :

$$G_k(\xi) = h_1^k(\xi) + \cdots + h_{N_+(\xi)}^k(\xi) + P_k(\xi), \quad k \geq 1, \quad \xi \in \mathbb{C}^2, \quad (62)$$

où  $P_k$  est analytique au voisinage de presque tout  $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$  et polynomial en  $\xi_0$  de degré au plus  $k$  pour  $\xi_1$  fixé.

Nous définissons des *moments de Wermer*  $\mathfrak{a}_{ij}$  comme suit (voir [140]) :

$$\mathfrak{a}_{ij} = \frac{1}{2\pi i} \int_{\partial X} z_1^i z_2^j dz_2, \quad i, j \geq 0.$$

Le résultat suivant de l'article VII permet de trouver les polynômes  $P_k$  nécessaires pour la reconstruction de  $X$  dans des cas, quand le nombre de points à l'infini  $N_-$  est égal à 0, 1 ou 2 :

**Théorème 11.** *Soit  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  une courbe complexe sans composante fermée de bord lisse  $\gamma \subset \mathbb{C}^2$ . Soient  $P_k$  les fonctions définies dans (62). Alors les assertions suivantes sont vraies :*

1. Si  $N_- = 0$ , on a  $P_k = 0$  pour tout  $k \geq 1$ .
2. Si  $N_- = 1$ , on a  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ , où les constantes  $c_{11}$  et  $c_{12}$  satisfont à l'équation suivante pour tout  $\xi = (\xi_0, 0)$ ,  $\xi_0 \in \Omega_0$  :

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi) + c_{12} (\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi) + G_1(\xi)) = G_1(\xi) \frac{\partial G_1}{\partial \xi_0}(\xi) - \frac{\partial G_1}{\partial \xi_1}(\xi). \quad (63)$$

3. Si  $N_- = 2$ , on a  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ ,  $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$ , où les constantes  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ ,  $c_{22}$ ,  $c_{23}$  satisfont à l'équation suivante pour tout  $(\xi_0, 0)$ ,  $\xi_0 \in \Omega_0$  :

$$\begin{aligned} \mathfrak{a}_{10}(c_{12}^2 + c_{23}) &= \frac{\partial G_2}{\partial \xi_1} - 2 \frac{\partial G_1}{\partial \xi_1} (G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0) \\ &+ \frac{\partial G_1}{\partial \xi_0} \cdot ((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2) \\ &+ (G_1^2 - 2c_{11}G_1 - 2c_{12}G_1\xi_0 - G_2) \cdot (-c_{12}). \end{aligned} \quad (64)$$

En supposant que le nombre de points à l'infini  $N_-$  est égal à 0, 1 ou 2, on peut proposer la méthode suivante pour trouver  $N_-$  :

**Théorème 12.** Soit  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  une courbe complexe au bord lisse  $\gamma \subset \mathbb{C}^2$ . Choisissons un point  $\xi^* = (\xi_0^*, 0)$ ,  $\xi_0^* \in \Omega_0$  et un voisinage  $W_{\xi^*}$  de  $\xi^*$  dans  $\mathbb{C}^2$ . Alors les assertions suivantes sont valables :

1. Si  $G_1 = 0$  dans  $W_{\xi^*}$ , alors soit  $N_- = 0$ , soit  $\partial X'$  le bord d'une courbe complexe compacte dans  $\mathbb{C}^2$ , où  $\partial X'$  désigne  $\partial X$  en orientation opposée.
2. S'il existe des constantes complexes  $c_{11}$ ,  $c_{12}$  telles que

$$\frac{\partial}{\partial \xi_1}(G_1 - P_1) = (G_1 - P_1) \frac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{dans } W_{\xi^*}, \quad (65)$$

où  $P_1(\xi_0, \xi_1) = \frac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}$ , alors  $N_- \leq 1$  et les constantes  $c_{11}$ ,  $c_{12}$  sont les mêmes que dans le théorème 11.

3. S'il existe des constantes complexes  $a_1, a_2, b_1, b_2, c_1, c_2$  telles que  $c_1 + c_2 = \mathfrak{a}_{10}$ ,

$$\begin{aligned} & \frac{\partial}{\partial \xi_1}((G_1 - P_1)^2 - G_2 - P_2) \\ &= ((G_1 - P_1)^2 - G_2 - P_2) \frac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{dans } W_{\xi^*}, \\ & P_1(\xi_0, \xi_1) = -\frac{a_1\xi_0 + b_1}{1 - a_1\xi_1} - \frac{a_2\xi_0 + b_2}{1 - a_2\xi_0}, \\ & P_2(\xi_0, \xi_1) = -\sum_{j=1}^2 \left[ \left( \frac{a_j\xi_0 + b_j}{1 - a_j\xi_1} \right)^2 + \frac{2a_j c_j}{1 - a_j\xi_1} \right], \end{aligned} \quad (66)$$

alors  $N_- \leq 2$  et ces constantes sont liées aux constantes du théorème 11 par les relations

$$\begin{aligned} a_1 + a_2 &= -c_{12}, & b_1 + b_2 &= -c_{11}, \\ a_1^2 + a_2^2 &= -c_{23}, & a_1 b_1 + a_2 b_2 &= -\frac{1}{2}c_{22}, \\ b_1^2 + b_2^2 + 2a_1 c_1 + 2a_2 c_2 &= -c_{21}, & c_1 + c_2 &= \mathfrak{a}_{10}. \end{aligned} \quad (67)$$

En se basant sur les théorèmes 11, 12, nous établissons l'algorithme suivant, qui permet de trouver  $X$  à partir de  $\partial X$  :

**Algorithme 3.** Soit  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  une courbe complexe sans composantes fermées au bord lisse  $\gamma \subset \mathbb{C}^2$  et telle que  $N_- \in \{0, 1, 2\}$ . Alors, étant donné  $\partial X$ , on peut déterminer  $X$  de la manière suivante :

1. Trouver  $N_-$  et  $N_+(\xi) = N_- + G_1(\xi)$ , en tenant compte que  $N_+(\xi)$  est constant sur chaque  $\Omega_{l \geq 0}$  et en utilisant le théorème 12.
2. (a) Si  $N_- = 1$ , trouver le polynôme  $P_1(\xi_0, 0)$  en utilisant le théorème 11.  
(b) si  $N_- = 2$ , trouver les polynômes  $P_1(\xi_0, 0)$ ,  $P_2(\xi_0, 0)$  à l'aide du théorème 11.
3. Soit  $R > 0$  un nombre assez grand tel que  $\pi_2(\gamma) \subset \{|z_2| < R\}$ . Posons  $\gamma'_R = X \cap \{|z_2| = R\}$ . Alors  $\gamma'_R = \{(h_j(\xi_0, 0), -\xi_0) : |\xi_0| = R, 1 \leq j \leq N_-\}$ .



(a) Si  $N_- = 1$ , déterminer  $h_1(\xi_0, 0)$ ,  $|\xi_0| = R$ , de la formule explicite :

$$h_1(\xi_0, 0) = G_1(\xi_0, 0) - P_1(\xi_0, 0), \quad |\xi_0| = R.$$

(b) Si  $N_- = 2$ , déterminer les polynômes symétriques élémentaires  $\sigma_1(\xi_0)$ ,  $\sigma_2(\xi_0)$  en  $h_1(\xi_0, 0)$ ,  $h_2(\xi_0, 0)$  pour chaque  $\xi_0 \in \mathbb{C}$ ,  $|\xi_0| = R$ , fixé en utilisant les formules explicites suivantes :

$$\begin{aligned} \sigma_1 &:= h_1 + h_2 = s_1, & \sigma_2 &:= h_1 h_2 = \frac{1}{2}(s_1^2 - s_2), \\ s_1(\xi_0) &:= h_1(\xi_0, 0) + h_2(\xi_0, 0) = G_1(\xi_0, 0) - P_1(\xi_0, 0), \\ s_2(\xi_0) &:= h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = G_2(\xi_0, 0) - P_2(\xi_0, 0). \end{aligned}$$

Trouver  $h_1(\xi_0, 0)$  et  $h_2(\xi_0, 0)$  (à l'ordre près) pour tout  $\xi_0 \in \mathbb{C}$ ,  $|\xi_0| = R$ , fixé comme les racines du polynôme suivant en  $t$  :

$$t^2 - \sigma_1(\xi_0)t + \sigma_2(\xi_0) = 0.$$

4. Soit  $X_R = X \cap \{|z_2| < R\}$ . Alors  $\partial X_R = \gamma_R$ ,  $\gamma_R = \gamma \sqcup \gamma'_R$  et  $X_R$  n'a pas de points à l'infini. Définir  $\mu_l = N_+(\xi_0, 0)$ ,  $\xi_0 \in \Omega_l$ , par la formule

$$\mu_l = G_1(\xi_0, 0) + N_-, \quad \xi_0 \in \Omega_l.$$

5. Trouver les sommes des puissances  $s_k(\xi_0)$  en  $h_{k \geq 1}(\xi_0, 0)$  pour chaque  $\xi_0 \in \Omega_l$ ,  $|\xi_0| < R$ , fixé et pour tout  $l$  :

$$s_k(\xi_0) := h_1^k(\xi_0, 0) + \dots + h_{\mu_l}^k(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad 1 \leq k \leq \mu_l.$$

6. Trouver des polynômes symétriques élémentaires  $\sigma_k(\xi_0)$  en  $h_{j \geq 1}(\xi_0, 0)$ ,  $|\xi_0| < R$ ,  $\xi_0 \in \Omega_l$ , pour tout  $l$  en utilisant des identités de Newton-Girard qui relient les polynômes élémentaires symétriques aux sommes des puissances :

$$\begin{aligned} \sigma_k(\xi_0) &= \sum_{i_1 < \dots < i_k} h_{i_1}(\xi_0, 0) \cdots h_{i_k}(\xi_0, 0), \quad 1 \leq k \leq \mu_l, \\ \sigma_1 &= s_1, \quad \sigma_2 = \frac{1}{2}(\sigma_1^2 - s_2), \quad \sigma_3 = \frac{1}{6}(s_1^3 - 3s_1 s_2 + 2s_3), \quad e.t.c. \end{aligned}$$

7. En tenant compte du théorème de Viète, trouver  $h_1(\xi_0, 0)$ ,  $\dots$ ,  $h_{\mu_l}(\xi_0, 0)$  (à l'ordre près),  $|\xi_0| < R$ ,  $\xi_0 \in \Omega_l$ , comme les racines du polynôme suivant en  $t$  :

$$t^{\mu_l} - \sigma_1(\xi_0)t^{\mu_l-1} + \dots + (-1)^{\mu_l} \sigma_{\mu_l}(\xi_0) = 0.$$

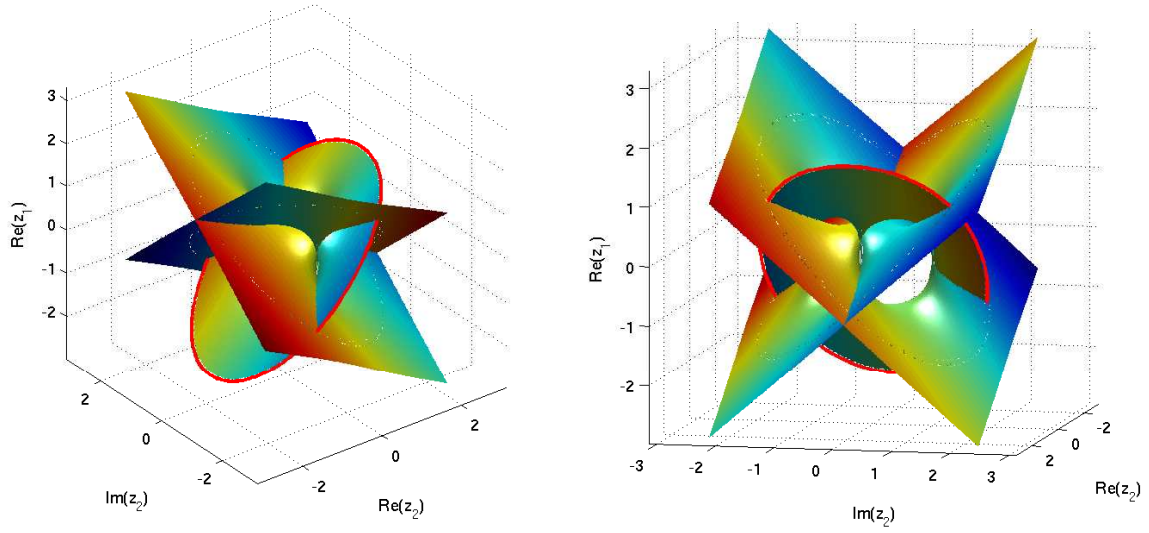


FIGURE 4 – La courbe complexe reconstruit à partir de son bord (la courbe réelle rouge). La couleur désigne  $\text{Im}(z_1)$ .

Un exemple de reconstruction d'une courbe complexe est présenté sur la Fig. 4. La courbe complexe sur la Fig. 4 est la partie de la courbe

$$X = \{(w_0 : w_1 : w_2) \in \mathbb{C}P^2 : w_1^3 = (w_2 - iw_0)(w_2 + iw_0)(w_2 - w_0)\}$$

bornée par la courbe réelle

$$\gamma = \{(z_1(t), z_2(t)) : t \in [0, 2\pi]\},$$

$$z_2(t) = 1.9e^{it}, \quad z_1(t) = \sqrt[3]{(z_2(t) - i)(z_2(t) + i)(z_2(t) - 1)}, \quad z_1(0) > 0.$$

Notons que dans cet exemple le nombre de points à l'infini est égal à  $N_- = 2$  et les points à l'infini sont  $(0 : 1 : e^{\pm \frac{2\pi i}{3}})$ .

# Article I

## A global uniqueness result for acoustic tomography of moving fluid

*A. D. Agaltsov*

We consider a model time-harmonic wave equation of acoustic tomography of moving fluid in an open bounded domain in dimension  $d \geq 2$ . We give global uniqueness theorems for related inverse boundary value problem at fixed frequency.

### 1 Introduction

Consider the operator

$$L_{A,V} = -\Delta - 2iA(x) \cdot \nabla + V(x), \quad (\text{I.1})$$

where  $\Delta$  is the standard Laplacian,  $x \in D$ ,  $A \in W^{1,\infty}(D, \mathbb{R}^d)$ ,  $V \in L^\infty(D, \mathbb{R})$ ,  $D$  is an open bounded domain in  $\mathbb{R}^d$  ( $d \geq 2$ ). In the present article we study an inverse boundary value problem for the equation  $L_{A,V}\psi = 0$  in  $D$ .

As in [4], [116], [118] we consider the equation  $L_{A,V}\psi = 0$  as a model equation for a time-harmonic ( $e^{-i\omega t}$ ) pressure  $\psi$  in moving fluid. In this setting

$$A(x) = \frac{\omega}{c^2(x)}v(x), \quad V(x) = -\frac{\omega^2}{c^2(x)},$$

where  $v$  is the fluid velocity vector,  $c$  is the sound speed,  $\omega$  is the frequency.

Suppose that 0 is not a Dirichlet eigenvalue for operator  $L_{A,V}$  in  $D$ . Then the Dirichlet problem

$$\begin{cases} L_{A,V}\psi = 0 & \text{in } D, \\ \psi|_{\partial D} = f, \end{cases} \quad (\text{I.2})$$

is uniquely solvable for  $\psi \in H^1(D)$  for any  $f \in H^{1/2}(\partial D)$ . The Dirichlet-to-Neumann map  $\Lambda_{A,V}$  sends  $f \in H^{1/2}(\partial D)$  to  $\Lambda_{A,V}f \in H^{-1/2}(\partial D)$  defined by the formula

$$\Lambda_{A,V}f = \frac{\partial\psi}{\partial\nu}\Big|_{\partial D} + i(A \cdot \nu)f, \quad (\text{I.3})$$

where  $\nu$  is the unit exterior normal to  $\partial D$  and  $\frac{\partial\psi}{\partial\nu}\Big|_{\partial D} \in H^{-1/2}(\partial D)$  can be defined, in particular, by the following formula :

$$\left\langle \frac{\partial\psi}{\partial\nu}\Big|_{\partial D}, u \right\rangle = \int_D \left( \nabla\psi(x)\nabla\tilde{u}(x) - 2i\tilde{u}(x)A(x)\nabla\psi(x) + \tilde{u}(x)V(x)\psi(x) \right) dx, \quad (\text{I.4})$$

for  $u \in H^{1/2}(\partial D)$  and arbitrary  $\tilde{u} \in H^1(D)$  with  $\tilde{u}|_{\partial D} = u$ . Note that since  $\psi$  satisfies  $L_{A,V}\psi = 0$ , the right hand side of the above formula doesn't depend on the choice of  $\tilde{u}$ .

The inverse boundary value problem for equation  $L_{A,V}\psi = 0$  in  $D$  consists in finding  $A, V$  from  $\Lambda_{A,V}$ . In the case when coefficients  $A, V$  can be complex-valued there is an obstruction to the unique solvability of this problem caused by the gauge invariance of the map  $\Lambda_{A,V}$  with respect to the gauge transformations

$$\begin{cases} A \rightarrow A + \nabla\varphi, \\ V \rightarrow V - i\Delta\varphi + (\nabla\varphi)^2 + 2A\nabla\varphi, \end{cases}$$

where  $\varphi \in W^{2,\infty}(D, \mathbb{C})$ ,  $\varphi|_{\partial D} = 0$ , see, e.g., [85] ( $d \geq 3$ ), [61] ( $d = 2$ ).

However, in the case of real-valued coefficients  $A, V$  there is no gauge non-uniqueness as it was observed, for example, in [4].

In addition, in general case, under some regularity assumptions on  $\partial D, A$  and  $V$ , the Dirichlet-to-Neumann map  $\Lambda_{A,V}$  uniquely determines the two-form  $dA$  and the function  $q$  in  $D$  and the tangential component of  $A$  on  $\partial D$ , where

$$\begin{aligned} dA &= \sum_{1 \leq k < l \leq d} \left( \frac{\partial A^l}{\partial x_k} - \frac{\partial A^k}{\partial x_l} \right) dx_k \wedge dx_l, \\ q &= V + i\nabla \cdot A - A \cdot A, \end{aligned} \quad (\text{I.5})$$

and  $A = (A^1, \dots, A^d)$ .

In particular, it was shown in [85] that in dimension  $d \geq 3$  the map  $\Lambda_{A,V}$  uniquely determines  $dA$  and  $q$  in  $D$  if  $A \in L^\infty(D, \mathbb{C}^d)$  and  $V \in L^\infty(D, \mathbb{C})$ . And in dimension  $d = 2$  it was shown in [61] that if  $D$  is a smooth Riemann surface with boundary (in particular, if  $D$  is a planar domain with  $\partial D \in C^\infty$ ) then the map  $\Lambda_{A,V}$  uniquely determines  $dA$  and  $q$  provided that  $A \in W^{2,p}(D, \mathbb{R}^d)$ ,  $V \in W^{1,p}(D, \mathbb{C})$ ,  $p > 2$ .

In addition, concerning the identifiability of tangential components of  $A$  on the boundary, it was proved in [19] that if  $\partial D \in C^1$  ( $d \geq 3$ ) or  $\partial D \in C^{1,\alpha}$ ,  $\alpha \in (0, 1)$

( $d = 2$ ) and if  $A \in C(\overline{D}, \mathbb{C}^d)$ ,  $V \in L^\infty(D, \mathbb{C})$  then  $\Lambda_{A,V}$  uniquely determines  $A - \nu(A \cdot \nu)$  on  $\partial D$ , where  $\nu$  is the unit exterior normal field to  $\partial D$ .

In the present article we combine the aforementioned results in order to obtain the following global uniqueness results in the case when coefficients  $A, V$  are real-valued.

**Theorem I.1.** *Let  $D$  be a bounded simply connected domain with path connected boundary in  $\mathbb{R}^d$  ( $d \geq 3$ ) with  $\partial D \in C^1$ . Let  $A_1, A_2 \in W^{1,\infty}(D, \mathbb{R}^d)$  and  $V_1, V_2 \in L^\infty(D, \mathbb{R})$ . If  $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$ , then  $A_1 = A_2, V_1 = V_2$ .*

**Theorem I.2.** *Let  $D$  be a bounded simply connected domain in  $\mathbb{R}^2$  with  $\partial D \in C^\infty$ . Let  $A_1, A_2 \in W^{2,p}(D, \mathbb{R}^d)$  and  $V_1, V_2 \in W^{1,p}(D, \mathbb{R})$  with  $p > 2$ . If  $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$  then  $A_1 = A_2$  and  $V_1 = V_2$ .*

Theorems I.1 and I.2 are proved in Section 3. In Section 2 we present formulas and equations for finding  $A, V$  from  $dA, q, A - \nu(A \cdot \nu)|_{\partial D}$ .

## 2 Formulas and equations for finding $A, V$

In this section we suppose that  $D$  is a bounded contractible domain with path connected  $C^2$  boundary in  $\mathbb{R}^d$  ( $d \geq 2$ ). By contractibility we mean that there exists  $F \in C^2(D \times [0, 1], D)$  such that  $F_0 \equiv \tilde{x}, F_1 = \text{id}_D$ , where  $F_t(x) = F(x, t)$ ,  $\tilde{x}$  is some fixed point in  $D$  and  $\text{id}_D$  is the identity mapping on  $D$ . We also suppose that  $A \in W^{2,\infty}(D, \mathbb{R}^d), V \in L^\infty(D, \mathbb{R})$ . Given  $dA, q$  as in (I.5) in  $D$  and  $A - \nu(A \cdot \nu)$  on  $\partial D$ , we can find  $A, V$  in the following way :

1. Define  $\tilde{A} = (\tilde{A}^1, \dots, \tilde{A}^d) \in W^{1,\infty}(D, \mathbb{R}^d)$  by the formula

$$\tilde{A}^k = \sum_{i < j} \int_0^1 \left( \frac{\partial F_t^i}{\partial t} \frac{\partial F_t^j}{\partial x_k} - \frac{\partial F_t^j}{\partial t} \frac{\partial F_t^i}{\partial x_k} \right) \left( \frac{\partial A^j}{\partial x_i} \circ F_t - \frac{\partial A^i}{\partial x_j} \circ F_t \right) dt, \quad (\text{I.6})$$

where  $k = 1, \dots, d$ ;  $A = (A^1, \dots, A^d), F_t = (F_t^1, \dots, F_t^d)$  and  $\circ$  denotes the composition of maps, i.e.  $\frac{\partial A^j}{\partial x_i} \circ F_t(y) = \frac{\partial A^j}{\partial x_i}(F_t(y)), y \in D$ .

2. Fix  $x^0 \in \partial D$ . Define  $\varphi_0 \in C^1(\partial D)$  by the formula

$$\varphi_0(x) = \sum_{k=1}^d \int_{x^0}^x (A_\tau^k(y) - \tilde{A}_\tau^k(y)) dy_k, \quad x \in \partial D, \quad (\text{I.7})$$

where  $A_\tau = A - \nu(A \cdot \nu), \tilde{A}_\tau = \tilde{A} - \nu(\tilde{A} \cdot \nu), A_\tau = (A_\tau^1, \dots, A_\tau^d), \tilde{A}_\tau = (\tilde{A}_\tau^1, \dots, \tilde{A}_\tau^d), \nu$  is the unit exterior normal field to  $\partial D$  and integration is over an arbitrary  $C^1$  curve on  $\partial D$  linking  $x^0$  to  $x$ .

3. Find the unique generalized solution  $\varphi \in W^{2,\infty}(D, \mathbb{R})$  to

$$\begin{cases} \Delta\varphi = \text{Im } q - \nabla \cdot \tilde{A} & \text{in } D, \\ \varphi|_{\partial D} = \varphi_0. \end{cases} \quad (\text{I.8})$$

4. Coefficients  $A, V$  are given by the following formulas :

$$\begin{cases} A = \tilde{A} + \nabla\varphi, \\ V = q - i\Delta\varphi - i\nabla \cdot \tilde{A} + \tilde{A} \cdot \tilde{A} + 2\tilde{A} \cdot \nabla\varphi + (\nabla\varphi)^2. \end{cases}$$

This algorithm will be justified in Section 4.

### 3 Proofs of Theorems I.1, I.2

We will prove Theorems I.1 and I.2 simultaneously. Let  $D, A_1, A_2, V_1, V_2$  satisfy the conditions of Theorem I.1 (resp. Theorem I.2) and suppose that  $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$ .

Using Theorem 1.1 of [19] we obtain that

$$(A_1 - \nu(A_1 \cdot \nu))|_{\partial D} = (A_2 - \nu(A_2 \cdot \nu))|_{\partial D}, \quad (\text{I.9})$$

where  $\nu$  is the unit exterior normal field to  $\partial D$ . Using Theorem 1.1 of [85] (resp. Theorem 1.1 of [61]) we get

$$dA_1 = dA_2 \text{ in } D, \quad (\text{I.10})$$

$$q_1 = q_2 \text{ in } D, \quad (\text{I.11})$$

where

$$dA_j = \sum_{1 \leq k < l \leq d} \left( \frac{\partial A_j^l}{\partial x_k} - \frac{\partial A_j^k}{\partial x_l} \right) dx_k \wedge dx_l,$$

$$q_j = V_j + i\nabla \cdot A_j - A_j \cdot A_j,$$

where  $A_j = (A_j^1, \dots, A_j^d)$ ,  $j = 1, 2$ .

Since the domain  $D$  is simply connected it follows from (I.10) that there exists  $\varphi \in W^{2,\infty}(D, \mathbb{R})$  such that

$$A_1 - A_2 = \nabla\varphi \text{ in } D. \quad (\text{I.12})$$

In dimension  $d = 2$  it follows from simple connectedness of  $D$  and from smoothness of  $\partial D$  that  $\partial D$  is path connected. Formulas (I.9), (II.30) and path connectedness of  $\partial D$  imply that  $\varphi$  is constant on  $\partial D$ .

Using (II.21), (II.30) we obtain that

$$V_1 - V_2 = -i\Delta\varphi - (\nabla\varphi)^2 + 2A_1 \nabla\varphi \text{ in } D.$$

Taking the imaginary part of this equation we obtain the equation  $\Delta\varphi = 0$  in  $D$ . Since  $\varphi$  is constant on  $\partial D$  and  $\varphi \in W^{2,\infty}(D)$  it follows that  $\varphi$  is constant in  $\overline{D}$ . Hence  $A_1 = A_2$  and  $V_1 = V_2$ . Theorems I.1 and I.2 are proved.

## 4 Justification of the algorithm of Section 2

It follows from formula (I.6) that  $d\tilde{A} = dA$ . More precisely, if we denote by  $F^*dA$  the pullback of the form  $dA$  by the map  $F$  and by  $\iota_{\partial_t}$  we denote the interior product with the vector field  $\frac{\partial}{\partial t}$  on  $\{(x, t) \in D \times [0, 1]\}$ , then

$$\sum_{k=1}^d \tilde{A}^k dx_k = \int_0^1 (\iota_{\partial_t} F^* dA) dt,$$

and the equality  $d\tilde{A} = dA$  follows from the Cartan magic formula  $\mathcal{L}_{\partial_t} = d \circ \iota_{\partial_t} + \iota_{\partial_t} \circ d$ , where  $\mathcal{L}_{\partial_t}$  is the Lie derivative along  $\frac{\partial}{\partial t}$ ,  $d$  is the exterior derivative on  $\{(x, t) \in D \times [0, 1]\}$  and  $\circ$  denotes the composition of maps.

Hence we can define  $\varphi \in W^{2,\infty}(D, \mathbb{R})$  by the formula

$$\varphi(x) = \int_{\tilde{x}}^x \sum_{k=1}^d (A_k - \tilde{A}_k) dx_k, \quad x \in \bar{D},$$

where  $\tilde{x} \in D$  is some fixed point and integration is over an arbitrary  $C^1$  curve in  $D$  linking  $\tilde{x}$  to  $x$ . Then  $\nabla\varphi = A - \tilde{A}$  in  $\bar{D}$  and this implies that  $\varphi|_{\partial D}$  differs by constant from  $\varphi_0$  defined in (I.7). We also obtain from (I.5) the equation

$$V = q - i\Delta\varphi - i\nabla \cdot \tilde{A} + \tilde{A} \cdot \tilde{A} + 2\tilde{A} \cdot \nabla\varphi + (\nabla\varphi)^2 \quad \text{in } D.$$

Taking into account that  $V$  is real-valued and separating the imaginary part in the latter equation we obtain (I.8). Since  $\text{Im } q$  and  $\nabla \cdot \tilde{A}$  belong to  $L^\infty(D, \mathbb{R})$ , the problem (I.8) is uniquely solvable for  $\varphi \in W^{2,\infty}(D, \mathbb{R})$ . Thus, the algorithm of Section 2 is justified.

# Article II

## Uniqueness and non-uniqueness in acoustic tomography of moving fluid

*A. D. Agaltsov, R. G. Novikov*<sup>1</sup>

We consider a model time-harmonic wave equation of acoustic tomography of moving fluid in an open bounded domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , with variable sound speed  $c$ , density  $\rho$ , fluid velocity  $v$  and absorption coefficient  $\alpha$ . We give global uniqueness results for related inverse boundary value problem for the cases of boundary measurements given for two and for three fixed frequencies. Besides, we also give a non-uniqueness result for this inverse problem for the case of boundary measurements given for all frequencies.

### 1 Introduction

We consider a moving fluid in an open bounded domain  $D \subset \mathbb{R}^d$  with sound speed  $c = c(x)$ , density  $\rho = \rho(x)$ , fluid velocity vector  $v = v(x)$  and the sound wave absorption coefficient  $\alpha = \alpha(x, \omega)$  at fixed frequency  $\omega$ , where  $x \in \bar{D} = D \cup \partial D$ . For this fluid we consider the following model equation for the time-harmonic ( $e^{-i\omega t}$ ) acoustic pressure  $\psi$  :

$$\begin{aligned} L_\omega \psi &= 0 \quad \text{in } D, \\ L_\omega &= -\Delta_x - 2iA_\omega(x)\nabla_x - U_\omega(x), \quad x = (x_1, \dots, x_d) \in D, \end{aligned} \tag{II.1}$$

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where

$$\Delta_x = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}, \quad \nabla_x = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right), \quad (\text{II.2})$$

$$\begin{aligned} A_\omega(x) &= \frac{\omega v(x)}{c^2(x)} + \frac{i}{2} \frac{\nabla_x \rho(x)}{\rho(x)}, \\ U_\omega(x) &= \frac{\omega^2}{c^2(x)} + 2i\omega \frac{\alpha(x, \omega)}{c(x)}, \\ \alpha(x, \omega) &= \omega^{\zeta(x)} \alpha_0(x). \end{aligned} \quad (\text{II.3})$$

In different particular cases this model was considered, for example, in [4, 15, 24, 71, 117, 116, 118].

In the present work we assume that the fluid parameters  $c, \rho, v, \alpha$  are such that

$$A_\omega \text{ and } V_\omega \text{ are sufficiently regular on } \bar{D} \text{ for any } \omega > 0, \quad (\text{II.4})$$

$$\begin{aligned} c \geq c_{\min} > 0, \quad \rho \geq \rho_{\min} > 0, \quad \alpha_0 \geq 0 \text{ in } \bar{D} \\ \text{for some constants } c_{\min}, \rho_{\min}. \end{aligned} \quad (\text{II.5})$$

For simplicity we consider equation (II.1) assuming that

$$\omega \notin \sigma(L_z), \quad (\text{II.6})$$

where

$$\begin{aligned} \sigma(L_z) \text{ consists of all } z \in \mathbb{C} \text{ such that} \\ 0 \text{ is a Dirichlet eigenvalue for operator } L_z \text{ in } D. \end{aligned} \quad (\text{II.7})$$

For equation (II.1), under assumptions (II.4), (II.6), we consider the Dirichlet-to-Neumann boundary map  $\Lambda_\omega$  defined by the relation

$$\Lambda_\omega(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D} + i(A_\omega \cdot \nu)\psi|_{\partial D}, \quad (\text{II.8})$$

fulfilled for all sufficiently regular solutions  $\psi$  of (II.1) in  $\bar{D}$ , where  $\nu$  is the unit exterior normal to  $\partial D$ .

We consider  $\Lambda_\omega$  as all possible boundary measurements for the model described by equation (II.1) at fixed  $\omega$ .

In the present work we consider the following inverse boundary value problem for equation (II.1) :

**Problem II.1.** *Given boundary data  $\Lambda_\omega$  for some frequencies  $\omega$ , find fluid parameters  $c, \rho, v, \alpha$  in  $\bar{D}$ .*

Under the assumption that

$$A_\omega \equiv 0 \quad \text{on } \partial D, \quad (\text{II.9})$$

Problem II.1 is closely related with the inverse scattering problem for the equation

$$L_\omega \psi = 0 \quad \text{on } \mathbb{R}^d, \quad (\text{II.10})$$

where

$$A_\omega \equiv 0, \quad U_\omega \equiv \frac{\omega^2}{c_0^2} \quad \text{on } \mathbb{R}^d \setminus D,$$

where  $c_0$  can be considered as the mean value of  $c$  on  $\partial D$ . By scattering data for equation (II.10) we mean, first of all, the scattering amplitude; see, e.g., [4, 2, 71, 97] for definitions of the scattering amplitude.

Due to results going back to [97], it is known that the inverse boundary value problems and the inverse scattering problems for operators like  $L_\omega$  are, actually, equivalent. For simplicity of exposition, in the present work we do not formulate the inverse scattering version of Problem II.1 in detail.

Note also that Problem II.1 at fixed  $\omega$  is closely related with inverse boundary value and inverse scattering problems for the Schrödinger equation in magnetic field at fixed energy. The reason is that the operator  $L_\omega$  at fixed  $\omega$  is closely related with the magnetic Schrödinger operator at fixed energy.

In the present work we are mainly focused on Problem II.1 and its inverse scattering version for the case when  $v \not\equiv 0$  in  $D$ , or, in other words, we are focused on the acoustic tomography of moving fluid in the framework of the wave propagation model (II.1), (II.3).

As regards results given in the literature on this acoustic tomography of moving fluid, see, e.g., [2, 3, 4, 15, 116, 118] and references therein including the case when  $v \equiv 0$ .

In particular, in [4] a Riemann–Hilbert problem approach to the inverse scattering version of Problem II.1 at fixed frequency  $\omega$  was developed for the case when  $\rho \equiv \text{const}$ ,  $\alpha \equiv 0$ ,  $d = 2$ .

In addition, in [3] global uniqueness theorems for Problem II.1 at fixed frequency  $\omega$  were proved for the case when  $\rho \equiv \text{const}$ ,  $\alpha \equiv 0$ ,  $d = 2$  or  $d \geq 3$ .

Note that in the present work we use, in particular, results developed in the literature for the case of the inverse boundary value problem for the Schrödinger equation in magnetic field at fixed energy; see [19, 61, 85] and references therein.

Note also that in the present work we use global uniqueness results on the Dirichlet problem for some linear and non-linear perturbations of the Laplace equation in  $D$ ; see systems (II.28), (II.34) of Section 3 and related results of [54].

The main results of the present work can be summarized as follows :

- (A1) We show that the boundary data  $\Lambda_\omega$  given for two different frequencies  $\omega = \omega_1, \omega_2$  uniquely determine the coefficients  $c, \rho, v$  under the assumptions that  $\omega_1, \omega_2 \notin \sigma(L_\omega)$  and  $\alpha \equiv 0$ , see Theorems II.1, II.2 of Section 2.
- (A2) We show that the boundary data  $\Lambda_\omega$  given for three different frequencies  $\omega = \omega_1, \omega_2, \omega_3$  uniquely determine  $c, \rho, v, \alpha$  under the assumptions that  $\omega_1,$

$\omega_2, \omega_3 \notin \sigma(L_\omega)$  and  $\zeta$  doesn't vanish anywhere, see Theorems II.3, II.4 of Section 2.

- (B) We give examples of coefficients  $c^{(1)}, \rho^{(1)}, v^{(1)}, \alpha_0^{(1)}$  and different coefficients  $c^{(2)}, \rho^{(2)}, v^{(2)}, \alpha_0^{(2)}$  such that  $\sigma(L_\omega^{(1)}) = \sigma(L_\omega^{(2)})$  and  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  for  $\omega \in \mathbb{C} \setminus \sigma(L_\omega^{(i)})$  for the case when  $\zeta \equiv 0$ , see Theorem II.5 of Section 2.

We recall that in the aforementioned results  $\sigma(L_\omega)$  is defined by (II.7).

The uniqueness results (A1), (A2) can be considered as results on global identifiability in acoustic tomography of moving fluid, whereas the non-uniqueness result (B) can be considered as a result on principal non-identifiability in this tomographical problem.

Note also that in these results the determination of  $\rho$  is considered modulo the transformations  $\rho \rightarrow C\rho$ , where  $C$  is a positive constant.

The main results of the present work are presented in detail in the next section.

## 2 Main results

Let  $W^{n,p}(D, \mathbb{C})$  denote the standard Sobolev space consisting of complex-valued functions which are  $n$  times differentiable in  $L^p(D)$ , where  $n \geq 0$ ,  $p \geq 1$  (including  $p = \infty$ ) and  $D$  is the domain of Section 1. We consider also  $W^{n,p}(D, \mathbb{R})$ ,  $W^{n,p}(D, \mathbb{R}^d)$  and  $W^{n,p}(D, \mathbb{C}^d)$  defined in the standard way.

We say that  $D$  is simply connected iff  $D$  is path connected and each continuous loop in  $D$  is contractible. In addition, we say that a set  $S$  in  $\mathbb{R}^d$  is path connected iff each pair of points in  $S$  can be joined by a continuous path in  $S$ . See, e.g., [35] in connection with these definitions.

In the present work we assume mainly that

$$\begin{aligned} c &\in W^{1,\infty}(D, \mathbb{R}), \quad c > 0, \quad \rho \in C(\bar{D}) \cup C^2(D), \quad \rho > 0, \\ v &\in W^{1,\infty}(D, \mathbb{R}^d), \quad \alpha_0 \in C(\bar{D}), \quad \zeta \in C(\bar{D}), \quad \zeta \neq 0, \\ &\alpha_0, \zeta \text{ are real-valued, for } d \geq 3, \end{aligned} \tag{II.11}$$

$$\begin{aligned} c &\in W^{2,p}(D, \mathbb{R}), \quad c > 0, \quad \rho \in W^{3,p}(D, \mathbb{R}), \quad \rho > 0, \\ v &\in W^{2,p}(D, \mathbb{R}^d), \quad \alpha_0 \in W^{1,p}(D, \mathbb{R}), \quad \zeta \in C(\bar{D}), \quad \zeta \neq 0, \\ &\alpha_0, \zeta \text{ are real-valued, where } p > 2, \quad d = 2. \end{aligned} \tag{II.12}$$

Let  $L_\omega$ ,  $\sigma(L_\omega)$  and  $\Lambda_\omega$  be defined as in Section 1.

In the present work we obtain, in particular, the following global uniqueness results for Problem II.1.

**Theorem II.1.** *Let  $D$  be an open bounded simply connected domain in  $\mathbb{R}^d$ ,  $d \geq 3$ , with path connected  $C^1$  boundary  $\partial D$ . Let  $L_\omega^{(j)}$  and  $\Lambda_\omega^{(j)}$  correspond to coefficients  $c^{(j)}, \rho^{(j)}, v^{(j)}, \alpha^{(j)}$ , where  $c^{(j)}, \rho^{(j)}, v^{(j)}$  satisfy (II.11),  $\alpha^{(j)} \equiv 0$ ,  $j = 1, 2$ . Let  $\omega_1$ ,*

$\omega_2 \in [0, +\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$ ,  $\omega_1 \neq \omega_2$ . Then the coincidence of the boundary data  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  for  $\omega \in \{\omega_1, \omega_2\}$  implies that  $c^{(1)} = c^{(2)}$ ,  $\rho^{(1)} = C\rho^{(2)}$ ,  $v^{(1)} = v^{(2)}$ , where  $C = \text{const} > 0$ .

**Theorem II.2.** Let  $D$  be an open bounded simply connected domain in  $\mathbb{R}^2$  with path connected  $C^\infty$  boundary  $\partial D$ . Let  $L_\omega^{(j)}$  and  $\Lambda_\omega^{(j)}$  correspond to coefficients  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha^{(j)}$ , where  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$  satisfy (II.12),  $\alpha^{(j)} \equiv 0$ ,  $j = 1, 2$ . Let  $\omega_1, \omega_2 \in [0, +\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$ ,  $\omega_1 \neq \omega_2$ . Then the coincidence of the boundary data  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  for  $\omega \in \{\omega_1, \omega_2\}$  implies that  $c^{(1)} = c^{(2)}$ ,  $\rho^{(1)} = C\rho^{(2)}$ ,  $v^{(1)} = v^{(2)}$ , where  $C = \text{const} > 0$ .

**Theorem II.3.** Let  $D$  be an open bounded simply connected domain in  $\mathbb{R}^d$ ,  $d \geq 3$ , with path connected  $C^1$  boundary  $\partial D$ . Let  $L_\omega^{(j)}$  and  $\Lambda_\omega^{(j)}$  correspond to coefficients  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$ ,  $\zeta^{(j)}$  satisfying (II.11),  $j = 1, 2$ . Let  $\omega_1, \omega_2, \omega_3 \in (0, +\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$  be three pairwise different frequencies. Then the coincidence of the boundary data  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  for  $\omega \in \{\omega_1, \omega_2, \omega_3\}$  implies that  $c^{(1)} = c^{(2)}$ ,  $\rho^{(1)} = C\rho^{(2)}$ ,  $v^{(1)} = v^{(2)}$ ,  $\alpha^{(1)} = \alpha^{(2)}$ , where  $C = \text{const} > 0$  and  $\alpha^{(j)}(x, \omega) = \omega^{\zeta^{(j)}(x)} \alpha_0^{(j)}(x)$ .

**Theorem II.4.** Let  $D$  be an open bounded simply connected domain in  $\mathbb{R}^2$  with path connected  $C^\infty$  boundary  $\partial D$ . Let  $L_\omega^{(j)}$  and  $\Lambda_\omega^{(j)}$  correspond to coefficients  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$ ,  $\zeta^{(j)}$  satisfying (II.12),  $j = 1, 2$ . Let  $\omega_1, \omega_2, \omega_3 \in (0, +\infty) \setminus (\sigma(L_\omega^{(1)}) \cup \sigma(L_\omega^{(2)}))$  be three pairwise different frequencies. Then the coincidence of the boundary data  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  for  $\omega \in \{\omega_1, \omega_2, \omega_3\}$  implies that  $c^{(1)} = c^{(2)}$ ,  $\rho^{(1)} = C\rho^{(2)}$ ,  $v^{(1)} = v^{(2)}$ ,  $\alpha^{(1)} = \alpha^{(2)}$ , where  $C = \text{const} > 0$  and  $\alpha^{(j)}(x, \omega) = \omega^{\zeta^{(j)}(x)} \alpha_0^{(j)}(x)$ .

Theorems II.1, II.2 and II.3, II.4 are proved in Sections 3 and 4, respectively.

Let

$$\begin{aligned} h & \text{ be a real-valued function supported in } D, \\ h & \in C^2(D) \text{ and } |\nabla h|^2 < 1 \text{ in } D, \end{aligned} \tag{II.13}$$

where  $D$  is an open bounded domain in  $\mathbb{R}^d$ .

We set

$$\begin{aligned} c^{(1)} & \equiv \text{const} > 0, \quad \rho^{(1)} \equiv \text{const} > 0, \\ v^{(1)} & \equiv 0, \quad \alpha_0^{(1)} \equiv \text{const} > -\frac{1}{2} \min_{x \in D} \Delta h(x), \end{aligned} \tag{II.14}$$

and

$$\begin{aligned} c^{(2)} & = c^{(1)}(1 - |\nabla h|^2)^{-1/2}, \quad \rho^{(2)} \equiv \text{const} > 0, \\ v^{(2)} & = c^{(1)}(1 - |\nabla h|^2)^{-1} \nabla h, \\ \alpha_0^{(2)} & = (1 - |\nabla h|^2)^{-1/2} (\alpha_0^{(1)} + \frac{1}{2} \Delta h). \end{aligned} \tag{II.15}$$

Note that for these fluid parameters  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$  with  $\zeta^{(j)} \equiv 0$  the conditions (II.4), (II.5) are fulfilled for both cases  $j = 1$  and  $j = 2$ .

In the present work, in addition to global uniqueness results of Theorems II.1, II.2, II.3, II.4 we give also the following non-uniqueness result.

**Theorem II.5.** *Let  $D$  be an open bounded domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , with smooth boundary. Let  $h$  satisfy (II.13),  $h \not\equiv 0$ , and  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha^{(j)}$ ,  $j = 1, 2$ , be defined by (II.14), (II.15). Let  $L_\omega^{(j)}$  and  $\Lambda_\omega^{(j)}$  correspond to coefficients  $c^{(j)}$ ,  $\rho^{(j)}$ ,  $v^{(j)}$ ,  $\alpha_0^{(j)}$  with  $\zeta^{(j)} \equiv 0$ . Then  $\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)}$  for all  $\omega \in \mathbb{C} \setminus \sigma$ , where  $\sigma = \sigma(L_z^{(1)}) = \sigma(L_z^{(2)})$ .*

Theorem II.5 is proved in Section 5.

## 3 Proofs of Theorems II.1 and II.2

### 3.1 Proof of Theorem II.1

Note that :

$$L_\omega^{(j)} = \sum_{k=1}^d \left( \frac{1}{i} \frac{\partial}{\partial x_k} + A_\omega^{(j),k} \right)^2 + q_\omega^{(j)}, \quad (\text{II.16})$$

where

$$\begin{aligned} A_\omega^{(j)} &= (A_\omega^{(j),1}, \dots, A_\omega^{(j),d}) = \frac{\omega v^{(j)}}{(c^{(j)})^2} + \frac{i \nabla \rho^{(j)}}{2 \rho^{(j)}}, \\ q_\omega^{(j)} &= -\frac{\omega^2}{(c^{(j)})^2} + i \nabla \cdot \left( \frac{\omega}{(c^{(j)})^2} v^{(j)} + \frac{i \nabla \rho^{(j)}}{2 \rho^{(j)}} \right) - \frac{\omega^2}{(c^{(j)})^4} (v^{(j)})^2 \\ &\quad + \frac{1}{4} (\rho^{(j)})^{-2} (\nabla \rho^{(j)})^2 - \frac{i \omega v^{(j)} \nabla \rho^{(j)}}{(c^{(j)})^2 \rho^{(j)}}, \end{aligned} \quad (\text{II.17})$$

$\nabla \cdot$  is the standard divergence,  $j = 1, 2$ .

By Theorem 1.1 of [19] we have that the tangential components of the fields  $A_\omega^{(1)}$  and  $A_\omega^{(2)}$  on  $\partial D$  are equal. And, as a corollary,

$$\text{tangential components of } \frac{\nabla \rho^{(1)}}{\rho^{(1)}} \text{ and } \frac{\nabla \rho^{(2)}}{\rho^{(2)}} \text{ on } \partial D \text{ are equal,} \quad (\text{II.18})$$

$$\text{tangential components of } \frac{v^{(1)}}{(c^{(1)})^2} \text{ and } \frac{v^{(2)}}{(c^{(2)})^2} \text{ on } \partial D \text{ are equal.} \quad (\text{II.19})$$

Using (II.18) and the path connectedness of  $\partial D$  we obtain

$$\begin{aligned} \ln \rho^{(2)} - \ln \rho^{(1)} &= \ln C \quad \text{on } \partial D, \\ \rho^{(2)}|_{\partial D} &= C \rho^{(1)}|_{\partial D} \quad \text{for some positive constant } C > 0. \end{aligned} \quad (\text{II.20})$$

Using Theorem 1.1 of [85] and the simple connectedness of  $D$  we get :

$$q_\omega^{(2)} - q_\omega^{(1)} = 0 \quad \text{in } D \quad (\text{II.21})$$

and

$$A_\omega^{(2)} - A_\omega^{(1)} = \nabla \varphi_\omega \quad \text{in } D, \quad (\text{II.22})$$

where  $\varphi_\omega \in W^{2,\infty}(D, \mathbb{C})$ ,  $\omega \in \{\omega_1, \omega_2\}$ .

Separating the real and the imaginary parts of (II.21) we get :

$$\begin{aligned} & \omega^2 \left[ \frac{1}{(c^{(1)})^2} - \frac{1}{(c^{(2)})^2} + \frac{(v^{(1)})^2}{(c^{(1)})^4} - \frac{(v^{(2)})^2}{(c^{(2)})^4} \right] \\ & + \left[ \left( \frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right)^2 - \left( \frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right)^2 - \nabla \cdot \left( \frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right) + \nabla \cdot \left( \frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right) \right] = 0, \end{aligned} \quad (\text{II.23})$$

where  $\omega \in \{\omega_1, \omega_2\}$  ;

$$\nabla \cdot \left( \frac{v^{(1)}}{(c^{(1)})^2} - \frac{v^{(2)}}{(c^{(2)})^2} \right) - \frac{\nabla \rho^{(1)}}{\rho^{(1)}} \frac{v^{(1)}}{(c^{(1)})^2} + \frac{\nabla \rho^{(2)}}{\rho^{(2)}} \frac{v^{(2)}}{(c^{(2)})^2} = 0. \quad (\text{II.24})$$

Using (II.23) and the assumptions that  $\omega_1, \omega_2 \geq 0$ ,  $\omega_1 \neq \omega_2$ , we obtain

$$\left( \frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right)^2 - \left( \frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right)^2 - \nabla \cdot \left( \frac{\nabla \rho^{(2)}}{2\rho^{(2)}} \right) + \nabla \cdot \left( \frac{\nabla \rho^{(1)}}{2\rho^{(1)}} \right) = 0, \quad (\text{II.25})$$

$$\frac{1}{(c^{(1)})^2} - \frac{1}{(c^{(2)})^2} + \frac{(v^{(1)})^2}{(c^{(1)})^4} - \frac{(v^{(2)})^2}{(c^{(2)})^4} = 0. \quad (\text{II.26})$$

Let

$$u^{(j)} = \frac{1}{2} \ln \rho^{(j)}, \quad j = 1, 2. \quad (\text{II.27})$$

Due to (II.20), (II.25) and (II.27), we have

$$\begin{cases} \Delta u^{(2)} - (\nabla u^{(2)})^2 = \Delta u^{(1)} - (\nabla u^{(1)})^2 & \text{in } D, \\ u^{(2)} = u^{(1)} + \frac{1}{2} \ln C & \text{on } \partial D. \end{cases} \quad (\text{II.28})$$

Since  $u^{(1)}, u^{(2)} \in C(\bar{D}) \cap C^2(D)$ , it follows from Theorem 10.1 of [54] that

$$u^{(2)} = u^{(1)} + \frac{1}{2} \ln C \quad \text{in } D$$

and, consequently,

$$\rho^{(2)} = C\rho^{(1)} \quad \text{in } D. \quad (\text{II.29})$$

Further, taking the real part of (II.22) and using (II.19) we obtain

$$\frac{v^{(2)}}{(c^{(2)})^2} - \frac{v^{(1)}}{(c^{(1)})^2} = \nabla \beta_\omega \quad \text{in } D \quad (\text{II.30})$$

and

$$\beta_\omega = \text{const} \quad \text{on } \partial D, \quad (\text{II.31})$$

where

$$\beta_\omega = \operatorname{Re} \varphi_\omega. \quad (\text{II.32})$$

Taking into account (II.29) we define

$$a := \frac{\nabla \rho^{(1)}}{\rho^{(1)}} = \frac{\nabla \rho^{(2)}}{\rho^{(2)}}. \quad (\text{II.33})$$

Formulas (II.24), (II.30), (II.31) and (II.33) imply that

$$\begin{cases} -\Delta \beta_\omega + a \nabla \beta_\omega = 0 & \text{in } D, \\ \beta_\omega = \text{const} & \text{on } \partial D. \end{cases} \quad (\text{II.34})$$

Now it follows from Theorem 8.1 of [54] that  $\beta_\omega = \text{const}$  in  $D$ . This result and formula (II.30) imply that

$$\frac{v^{(1)}}{(c^{(1)})^2} = \frac{v^{(2)}}{(c^{(2)})^2} \quad \text{in } D. \quad (\text{II.35})$$

Finally, using (II.26), (II.35) we obtain

$$c^{(2)} = c^{(1)} \quad \text{and} \quad v^{(2)} = v^{(1)} \quad \text{in } D. \quad (\text{II.36})$$

This completes the proof of Theorem II.1.

### 3.2 Proof of Theorem II.2

In a similar way with the proof of Theorem II.1, we have formulas (II.16)–(II.20) for  $d = 2$ .

Let

$$\mu^{(j)} = -\frac{i}{2} \ln \rho^{(j)}, \quad (\text{II.37})$$

and

$$\tilde{L}_\omega^{(j)} = e^{-i\mu^{(j)}} L_\omega^{(j)} e^{i\mu^{(j)}}, \quad (\text{II.38})$$

where  $e^{i\mu^{(j)}}$ ,  $e^{-i\mu^{(j)}}$  denote the multiplication operators by the functions  $e^{i\mu^{(j)}}$ ,  $e^{-i\mu^{(j)}}$ ,  $j = 1, 2$ .

Using (II.37), (II.38) one can see that

$$\sigma(\tilde{L}_z^{(j)}) = \sigma(L_z^{(j)}), \quad j = 1, 2, \quad (\text{II.39})$$

and

$$\tilde{\Lambda}_\omega^{(1)} = \tilde{\Lambda}_\omega^{(2)}, \quad (\text{II.40})$$

where  $\tilde{\Lambda}_\omega^{(1)}$ ,  $\tilde{\Lambda}_\omega^{(2)}$  are the Dirichlet-to-Neumann maps for operators  $\tilde{L}_\omega^{(1)}$ ,  $\tilde{L}_\omega^{(2)}$  in  $D$ , respectively,  $\omega \in \{\omega_1, \omega_2\}$ .

By direct computation we obtain that

$$\tilde{L}_\omega^{(j)} = \sum_{k=1}^2 \left( \frac{1}{i} \frac{\partial}{\partial x_k} + \tilde{A}_\omega^{(j),k} \right)^2 + \tilde{q}_\omega^{(j)}, \quad (\text{II.41})$$

where

$$\begin{aligned} \tilde{A}_\omega^{(j)} &= (\tilde{A}_\omega^{(j),1}, \tilde{A}_\omega^{(j),2}) = \frac{\omega}{(c^{(j)})^2} v, \\ \tilde{q}_\omega^{(j)} &= q_\omega^{(j)}. \end{aligned} \quad (\text{II.42})$$

Note that the fields  $\tilde{A}_\omega^{(1)}, \tilde{A}_\omega^{(2)}$  do not contain the imaginary part in contrast with  $A_\omega^{(1)}, A_\omega^{(2)}$ .

Now, using (II.39), (II.40), (II.42) and also Theorem 1.1 of [61] and simple connectedness of  $D$  we get the equalities (II.23), (II.24) for  $d = 2$ ,  $\omega \in \{\omega_1, \omega_2\}$ , and also the equality

$$\tilde{A}_\omega^{(2)} - \tilde{A}_\omega^{(1)} = \nabla \tilde{\varphi}_\omega \quad \text{in } D, \quad (\text{II.43})$$

where  $\tilde{\varphi}_\omega \in W^{2,\infty}(D, \mathbb{R})$ ,  $\omega \in \{\omega_1, \omega_2\}$ .

In a similar way with the proof of Theorem II.1, proceeding from (II.20), (II.23) for  $d = 2$  we obtain (II.29) for  $d = 2$ .

Now, using (II.19), (II.42) and (II.43) we obtain (II.30), (II.31) for  $d = 2$ , where

$$\beta_\omega = \tilde{\varphi}_\omega. \quad (\text{II.44})$$

Finally, using (II.24), (II.29), (II.30), (II.31) for  $d = 2$  and (II.44) we complete the proof of Theorem II.2 in a completely similar way to the corresponding part of the proof of Theorem II.1.

## 4 Proofs of Theorems II.3 and II.4

Note that under the assumptions of Theorems II.3, II.4, we have that  $L_\omega^{(j)}$  is given by (II.16), where

$$\begin{aligned} A_\omega^{(j)} &= (A_\omega^{(j),1}, \dots, A_\omega^{(j),d}) = \frac{\omega v^{(j)}}{(c^{(j)})^2} + \frac{i \nabla \rho^{(j)}}{2 \rho^{(j)}}, \\ q_\omega^{(j)} &= -\frac{\omega^2}{(c^{(j)})^2} + i \nabla \cdot \left( \frac{\omega}{(c^{(j)})^2} v^{(j)} + \frac{i \nabla \rho^{(j)}}{2 \rho^{(j)}} \right) - \frac{\omega^2}{(c^{(j)})^4} (v^{(j)})^2 \\ &\quad + \frac{1}{4} (\rho^{(j)})^{-2} (\nabla \rho^{(j)})^2 - \frac{i \omega v^{(j)} \nabla \rho^{(j)}}{(c^{(j)})^2 \rho^{(j)}} - 2i \omega^{1+\zeta^{(j)}} \frac{\alpha_0^{(j)}}{c^{(j)}}, \end{aligned} \quad (\text{II.45})$$

where  $j = 1, 2$ .



## 4.1 Proof of Theorem II.3

In a similar way with the proof of Theorem II.1, we have formulas (II.18)–(II.22), where in (II.21), (II.22) the functions  $q_\omega^{(1)}$ ,  $q_\omega^{(2)}$ ,  $A_\omega^{(1)}$ ,  $A_\omega^{(2)}$  are defined as in (II.45) and  $\omega \in \{\omega_1, \omega_2, \omega_3\}$ .

Now, separating the real and imaginary parts of (II.21) we obtain equality (II.23) and also the equality

$$\begin{aligned} & \left[ \nabla \cdot \left( \frac{v^{(1)}}{(c^{(1)})^2} - \frac{v^{(2)}}{(c^{(2)})^2} \right) - \frac{\nabla \rho^{(1)}}{\rho^{(1)}} \frac{v^{(1)}}{(c^{(1)})^2} + \frac{\nabla \rho^{(2)}}{\rho^{(2)}} \frac{v^{(2)}}{(c^{(2)})^2} \right] \\ & + 2\omega^{\zeta^{(2)}} \left[ \frac{\alpha_0^{(2)}}{(c^{(2)})^2} \right] - 2\omega^{\zeta^{(1)}} \left[ \frac{\alpha_0^{(1)}}{(c^{(1)})^2} \right] = 0, \end{aligned} \quad (\text{II.46})$$

where  $\omega \in \{\omega_1, \omega_2, \omega_3\}$ .

Using (II.46) and the assumption that  $\omega_1, \omega_2, \omega_3$  are positive and mutually different frequencies we obtain, in particular, that (II.24) holds.

In a similar way with the proof of Theorem II.1, proceeding from (II.19)–(II.24) we obtain (II.36).

Next, in order to show that  $\alpha_0^{(2)}(x) = \alpha_0^{(1)}(x)$  for fixed  $x \in D$  we consider two cases : (a)  $\zeta^{(1)}(x) \neq \zeta^{(2)}(x)$ ; (b)  $\zeta^{(1)}(x) = \zeta^{(2)}(x)$ .

For the case (a) using (II.46) and the assumption that  $\omega_1, \omega_2, \omega_3$  are positive and mutually different frequencies, in addition to (II.24), we obtain also that

$$\frac{\alpha_0^{(j)}}{(c^{(j)})^2} = 0 \quad \text{at point } x, \quad j = 1, 2, \quad (\text{II.47})$$

and, as a corollary,

$$\alpha_0^{(1)}(x) = \alpha_0^{(2)}(x) = 0. \quad (\text{II.48})$$

For the case (b) using (II.46) and the assumption that  $\omega_1, \omega_2, \omega_3$  are positive and mutually different frequencies, in addition to (II.24), we obtain also that

$$\frac{\alpha_0^{(2)}}{(c^{(2)})^2} - \frac{\alpha_0^{(1)}}{(c^{(1)})^2} = 0 \quad \text{at point } x. \quad (\text{II.49})$$

Using (II.36) and (II.49) we obtain

$$\alpha_0^{(2)}(x) = \alpha_0^{(1)}(x). \quad (\text{II.50})$$

Finally, the result that  $\alpha^{(2)} = \alpha^{(1)}$  in  $D$  follows from (II.48) for the case (a) and from (II.50) for the case (b).

This completes the proof of Theorem II.3.

## 4.2 Proof of Theorem II.4

In a similar way with the proofs of Theorems II.2, II.3 and we have formulas (II.16), (II.45), (II.18)–(II.21) for  $d = 2$  and formulas (II.37)–(II.43) where in (II.21), (II.40), (II.43)  $\omega \in \{\omega_1, \omega_2, \omega_3\}$ .

Now, separating the real and imaginary parts of (II.21) we obtain equality (II.23) and also equality (II.46) for  $d = 2$ , where  $\omega \in \{\omega_1, \omega_2, \omega_3\}$ .

Using equality (II.46) and the assumption that  $\omega_1, \omega_2, \omega_3$  are positive mutually different frequencies we obtain, in particular, (II.24) for  $d = 2$ .

As in the proof of Theorem II.1, we use (II.20), (II.23) to obtain (II.29).

Using (II.19), (II.24), (II.29) for  $d = 2$  and (II.43) as in the proof of Theorem II.2, we obtain (II.36) for  $d = 2$ .

Finally, using (II.46) we complete the proof of Theorem II.4 in a completely similar way to the proof of Theorem II.3.

## 5 Proof of Theorem II.5

Let

$$\mu = \frac{\omega}{c^{(1)}}h. \quad (\text{II.51})$$

One can check by direct computation that

$$e^{-i\mu}L_\omega^{(1)}e^{i\mu} = -\Delta - 2iA_\omega^{(2)}\nabla - U_\omega^{(2)}, \quad (\text{II.52})$$

where

$$\begin{aligned} A_\omega^{(2)} &= \frac{\omega}{c^{(1)}}\nabla h, \\ U_\omega^{(2)} &= \frac{\omega^2}{(c^{(1)})^2}(1 - |\nabla h|^2) + \frac{2i\omega}{c^{(1)}}(\alpha_0^{(1)} + \frac{1}{2}\Delta h), \end{aligned} \quad (\text{II.53})$$

and  $e^{i\mu}, e^{-i\mu}$  denote the multiplication operators by the functions  $e^{i\mu}, e^{-i\mu}$ , respectively. Using (II.52) one can see that

$$L_\omega^{(2)} = e^{-i\mu}L_\omega^{(1)}e^{i\mu}. \quad (\text{II.54})$$

Due to our assumptions, we have that

$$e^{\pm i\mu} - 1 \text{ is supported in } D. \quad (\text{II.55})$$

Using (II.54), (II.55) one can see that

$$\sigma(L_z^{(1)}) = \sigma(L_z^{(2)}), \quad (\text{II.56})$$

and

$$\Lambda_\omega^{(1)} = \Lambda_\omega^{(2)} \text{ for all } \omega \in \mathbb{C} \setminus \sigma, \quad (\text{II.57})$$

where  $\sigma = \sigma(L_z^{(1)}) = \sigma(L_z^{(2)})$ .

This completes the proof of Theorem II.5.

# Article III

## Finding scattering data for a time-harmonic wave equation with first order perturbation from the Dirichlet-to-Neumann map

*A. D. Agaltsov*

We present formulas and equations for finding scattering data from the Dirichlet-to-Neumann map for a time-harmonic wave equation with first order perturbation with compactly supported coefficients. We assume that the coefficients are matrix-valued in general. To our knowledge, these results are new even for the general scalar case.

### 1 Introduction

We consider the following equation :

$$L\psi \stackrel{\text{def}}{=} -\Delta\psi - 2i \sum_{j=1}^d A_j(x) \partial_{x_j} \psi + V(x)\psi = E\psi, \quad x \in D \subset \mathbb{R}^d, \quad (\text{III.1})$$

where  $x = (x_1, \dots, x_d)$ ,  $\partial_{x_j} = \partial/\partial x_j$ ,  $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_d}^2$ ,  $E \in \mathbb{C}$ ,  $d = 2, 3$ ,

$$D \text{ is a bounded open domain in } \mathbb{R}^d \text{ with } \partial D \in C^2, \quad (\text{III.2})$$

$A_1, \dots, A_d, V$  are sufficiently regular  $M_n(\mathbb{C})$ -valued functions on  $\overline{D}$  and  $M_n(\mathbb{C})$  is the set of  $n \times n$  complex matrices.

We also assume that

$$E \text{ is not a Dirichlet eigenvalue for operator } L \text{ in } D. \quad (\text{III.3})$$

Note that equation (III.1) can be written in the form

$$\sum_{j=1}^d (-i\partial_{x_j} + A_j(x))^2 \psi + v(x)\psi = E\psi, \quad (\text{III.4})$$

where

$$v(x) = V(x) - \sum_{j=1}^d A_j^2(x) + i \sum_{j=1}^d \partial_{x_j} A_j(x). \quad (\text{III.5})$$

For equation (III.1) (or (III.4)) we consider the maps  $\Phi(E)$ ,  $\Lambda(E)$  such that

$$\begin{aligned} \Phi(E)(\psi|_{\partial D}) &= \left. \frac{\partial \psi}{\partial \nu} \right|_{\partial D}, \\ \Lambda(E)(\psi|_{\partial D}) &= \left. \left( \frac{\partial \psi}{\partial \nu} + i \sum_{j=1}^d \nu_j A_j \psi \right) \right|_{\partial D}, \end{aligned}$$

for all sufficiently regular solutions  $\psi$  of (III.1) in  $\bar{D} = D \cup \partial D$ , for example for all  $\psi \in C^1(\bar{D}, M_n(\mathbb{C})) \cap C^2(D, M_n(\mathbb{C}))$  satisfying (III.1), where  $\nu = (\nu_1, \dots, \nu_d)$  is the unit exterior normal to  $\partial D$ . The map  $\Phi(E)$  is known as the Dirichlet-to-Neumann map for equation (III.1) in  $D$ .

In a similar way with [78], assumption (III.3) can be dropped by considering an appropriate Robin-to-Robin map instead of the Dirichlet-to-Neumann map.

Note that  $\Lambda(E)$  is invariant with respect to the gauge transformations

$$\begin{aligned} A_j &\rightarrow g A_j g^{-1} + i(\partial_{x_j} g) g^{-1}, \quad j = 1, \dots, d, \\ v &\rightarrow g v g^{-1}, \end{aligned} \quad (\text{III.6})$$

where  $g$  is a sufficiently regular  $M_n(\mathbb{C})$ -valued function on  $\bar{D}$  with  $\det g(x) \neq 0$ ,  $x \in \bar{D}$  and  $g(x) = \text{Id}_n$  on  $\partial D$ , where  $\text{Id}_n$  is the identity  $n \times n$  matrix. Note also that  $\Phi(E)$  is invariant with respect to the gauge transformations (III.6) under the additional assumption that  $\sum_{j=1}^d \nu_j \partial_{x_j} g = 0$  on  $\partial D$ . Furthermore, if  $\sum_{j=1}^d \nu_j A_j = 0$  on  $\partial D$  (in particular, if  $A_1, \dots, A_d$  have compact supports in  $D$ ), then  $\Lambda(E) = \Phi(E)$ . Besides, if  $A_1, \dots, A_d$  are known on  $\partial D$ , one can easily compute  $\Lambda(E)$  given  $\Phi(E)$  and vice versa.

For  $n = 1$  equation (III.4) can be considered as a Schrödinger equation at fixed energy  $E$  with magnetic potential  $A = (A_1, \dots, A_d)$  and electric potential  $v$ , see, e.g., [109, 42, 43].

Equation (III.4) for  $n \geq 2$  with Hermitian matrices  $A_1, \dots, A_d$  and with scalar matrix  $v$  can be considered as a Schrödinger equation for a particle in an external Yang–Mills field, see [122, 123, 132, 44].

Besides, equation (III.1) for  $n = 1$  is a model equation for the time-harmonic ( $e^{-i\omega t}$ ) acoustic pressure  $\psi$  in a moving fluid, see, e.g., [116, 118]. In this setting

$$E = \left(\frac{\omega}{c_0}\right)^2, \quad A_j(x) = \frac{\omega}{c_0} u_j(x), \quad V(x) = (1 - r^2(x)) \left(\frac{\omega}{c_0}\right)^2,$$

where  $j = 1, \dots, d$ ,  $c_0$  is a reference sound speed,  $r$  is a scalar index of refraction,  $u = (u_1, \dots, u_d)$  is a normalized fluid velocity vector.

In addition, for  $n \geq 2$ ,  $d = 2$ , equation (III.1) arises as a wave equation in the mode representation for a time-harmonic acoustic pressure  $\psi$  in a moving fluid in a three-dimensional cylindrical domain of finite height and with base  $D$ , see [15].

We also consider equation (III.1) in the entire space :

$$L\psi \equiv -\Delta\psi - 2i \sum_{j=1}^d A_j(x) \partial_{x_j} \psi + V(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad (\text{III.7})$$

where  $A_1, \dots, A_d, V$  are sufficiently regular  $M_n(\mathbb{C})$ -valued functions with sufficient decay at infinity.

There are scattering functions  $\psi^+$ ,  $f$  and Faddeev-type generalized scattering functions  $\psi$ ,  $h$  and  $\psi_\gamma$ ,  $h_\gamma$  associated with equation (III.7).

Functions  $\psi^+$ ,  $f$  can be defined as follows :

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} \text{Id}_n + \int_{\mathbb{R}^d} G^+(x - y, k) \times \\ &\times \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) \right) \psi^+(y, k) dy \end{aligned} \quad (\text{III.8})$$

$$G^+(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0}, \quad (\text{III.9})$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d \setminus 0$ ,  $k^2 = E$ ;

$$f(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) \right) \psi^+(x, k) dx, \quad (\text{III.10})$$

where  $k \in \mathbb{R}^d \setminus 0$ ,  $l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ . Actually, we consider (III.8) and its differentiated versions, where  $\partial_{x_j}$ ,  $j = 1, \dots, d$ , are applied to both sides of (III.8), as a system of coupled linear integral equations for  $\psi^+$ ,  $\partial_{x_j} \psi^+$ ,  $j = 1, \dots, d$ .

Functions  $\psi$  and  $h$  are defined as follows :

$$\begin{aligned} \psi(x, k) &= e^{ikx} \text{Id}_n + \int_{\mathbb{R}^d} G(x-y, k) \times \\ &\times \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) \right) \psi(y, k) dy, \end{aligned} \quad (\text{III.11})$$

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (\text{III.12})$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $k^2 = E$ ;

$$h(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) \right) \psi(x, k) dx, \quad (\text{III.13})$$

where  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $\text{Im } k = \text{Im } l$ ,  $k^2 = l^2 = E$ . In a similar way with (III.8), we consider (III.11) and its differentiated versions as a system of coupled linear integral equations for  $\psi$ ,  $\partial_{x_j} \psi$ , or, more precisely, for  $\mu$ ,  $\partial_{x_j} \mu$ ,  $j = 1, \dots, d$ , where  $\psi = e^{ikx} \mu$ .

Finally, functions  $\psi_\gamma$  and  $h_\gamma$  are defined as follows :

$$\psi_\gamma(x, k) = \psi(x, k + i0\gamma), \quad h_\gamma(k, l) = h(k + i0\gamma, l + i0\gamma), \quad (\text{III.14})$$

where  $x \in \mathbb{R}^d$ ,  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $\gamma \in S^{d-1}$ , and  $S^{d-1}$  is the unit sphere in  $\mathbb{R}^d$ .

Note that the history of functions  $\psi$ ,  $h$  and  $\psi_\gamma$ ,  $h_\gamma$  goes back to [Fa1], [Fa2].

Functions  $f(k, l)$  and  $h_\gamma(k, l)$ , where  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $\gamma \in S^{d-1}$ , and  $h(k, l)$ , where  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $\text{Im } k = \text{Im } l$ ,  $k^2 = l^2 = E$ , are considered as the scattering data  $S_E$  for equation (III.7) at fixed  $E \in (0, +\infty)$ . Function  $h(k, l)$ ,  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $\text{Im } k = \text{Im } l$ ,  $k^2 = l^2 = E$ , is considered as the scattering data  $S_E$  for equation (III.7) at fixed  $E \in \mathbb{C} \setminus (0, +\infty)$ .

In a similar way with the map  $\Lambda(E)$ , scattering data  $S_E$  is invariant with respect to gauge transformations (III.6), where  $g$  is a sufficiently regular  $M_n(\mathbb{C})$ -valued function on  $\mathbb{R}^d$  decaying fast enough at infinity with  $\det g(x) \neq 0$  for  $x \in \mathbb{R}^d$ , see, e.g., [4] for the case  $n = 1$ .

Let  $D$  be a fixed domain satisfying (III.2). Let

$$A_1, \dots, A_d, V \in C_c^{0,\alpha}(D, M_n(\mathbb{C})) \text{ for some } 0 < \alpha \leq 1, \quad (\text{III.15})$$

where  $C_c^{0,\alpha}(D, M_n(\mathbb{C}))$  denotes the space of  $M_n(\mathbb{C})$ -valued component-wise Hölder-continuous functions with compact support in  $D$ . As it was noted above, in the case of coefficients satisfying (III.15) the maps  $\Phi(E)$  and  $\Lambda(E)$  are the same. For coefficients  $A_1, \dots, A_d, V$  satisfying (III.15) we consider the Dirichlet-to-Neumann map  $\Phi(E)$  for equation (III.1) and the scattering data  $S_E$  for equation (III.7). In the latter case we define coefficients  $A_1, \dots, A_d, V$  outside of  $D$  by zero matrices.

**Problem III.1.** *Given  $\Phi(E)$  at fixed  $E$  (or for  $E$  in some fixed set) find  $A_1, \dots, A_d, V$  of (III.1) (modulo gauge transformations (III.6)).*

More precisely, we develop the approach of [97] (where this approach was suggested for  $n = 1, A_1, \dots, A_d \equiv 0$ ) and reduce Problem III.1 to the following inverse scattering problem for equation (III.7) :

**Problem III.2.** *Given  $S_E$  at fixed  $E$  (or for  $E$  in some fixed set) find  $A_1, \dots, A_d, V$  of (III.7) (modulo gauge transformations (III.6)).*

Concerning the results given in literature on Problem III.1 without the assumption that  $A_1 \equiv 0, \dots, A_d \equiv 0$ , see, e.g., [92, 113, 33, 84, 74, 85] for  $n = 1$  and [39] for  $n \geq 1$ . Besides, see [110, 111] for the case  $d = 2, A_1 \equiv 0, A_2 \equiv 0, n \geq 1$ . Concerning the results for the case  $n = 1, A_1 \equiv 0, \dots, A_d \equiv 0$ , see [101, 20, 103, 23, 77, 121] and references therein.

Concerning the results given in literature on Problem III.2 without the assumption  $A_1 \equiv 0, \dots, A_d \equiv 0$ , see, e.g., [125, 71, 98, 42, 43, 10, 93, 112, 4] for  $n = 1$  and [68, 44, 39, 141] for  $n \geq 1$ . The case  $A_1 \equiv 0, \dots, A_d \equiv 0, n \geq 1$ , was considered, e.g., in [111]. Concerning the results for the case  $n = 1, A_1 \equiv 0, \dots, A_d \equiv 0$ , see [107] and references therein.

The main results of the present work consist of Theorems III.1 and III.2 of Section 2. In Theorem III.1 we give, in particular, formulas and equations for finding  $S_E$  from  $\Phi(E) - \Phi^0(E)$ , where  $S_E$  and  $\Phi(E)$  correspond to coefficients  $A_1, \dots, A_d, V$  and  $\Phi^0(E)$  corresponds to zero coefficients  $A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0 \equiv 0$ . In Theorem III.2 we give a result on the solvability of equations of Theorem III.1.

In fact, the formulas and equations of Theorem III.1 are also valid if either  $V^0(x)$  is a diagonal matrix for all  $x \in D$  or  $V^0$  is a product of a constant matrix by a scalar function, see Theorems III.1' and III.2' of Section 2. In this case, the potential  $V^0$  is supposed to be known. This generalization to the case when  $V^0(x)$  is diagonal for all  $x \in D$  is useful, in particular, in the framework of Problem III.1 for the case of mode wave equation, see, e.g., [15] and Subsection 3.1 of [111].

Thus, due to the results of Theorems III.1, III.2, III.1', III.2' we reduced Problem III.1 to Problem III.2. As regards to methods of solving Problem III.2 we refer to [4, 10, 42, 43, 44, 39, 109, 71, 68, 93, 98, 107, 111, 112, 125, 141] and references therein.

For the case when  $n = 1, A_1 \equiv 0, \dots, A_d \equiv 0, A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0 \equiv 0$ , Theorems III.1' and III.2' were obtained for the first time in [97]. These theorems were generalized to the case when  $n = 1, A_1 \equiv 0, \dots, A_d \equiv 0, A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0 \neq 0$ , in [101]. In [111] the authors give formulas and equations for the case when  $d = 2, n \geq 1, A_1 \equiv 0, A_2 \equiv 0, A_1^0 \equiv 0, A_2^0 \equiv 0, V^0 \neq 0$ . In the present paper we generalize these results to the case when  $n \geq 1, A_1 \neq 0, \dots, A_d \neq 0, A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0 \neq 0$ . To our knowledge, these results are new even for the general scalar case when  $n = 1$  and  $V^0 \equiv 0$ .

The main results of the present work are presented in Section 2.

## 2 Main results

Consider equation (III.7) under assumption (III.15). We define the sets  $\mathcal{E}$ ,  $\mathcal{E}_\gamma$ ,  $\gamma \in S^{d-1}$ , and  $\mathcal{E}^+$ , as follows :

$$\mathcal{E} = \left\{ \zeta \in \mathbb{C}^d \setminus \mathbb{R}^d : \text{equation (III.11) at } k = \zeta \text{ is not uniquely solvable for } \psi = e^{ikx} \mu, \text{ where } \mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C})) \right\}, \quad (\text{III.16})$$

$$\mathcal{E}_\gamma = \left\{ \zeta \in \mathbb{R}^d \setminus 0 : \text{equation (III.11) at } k = \zeta + i0\gamma \text{ is not uniquely solvable for } \psi \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C})) \right\}, \quad (\text{III.17})$$

$$\mathcal{E}^+ = \left\{ \zeta \in \mathbb{R}^d \setminus 0 : \text{equation (III.8) at } k = \zeta + i0\zeta/|\zeta| \text{ is not uniquely solvable for } \psi \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C})) \right\}. \quad (\text{III.18})$$

The properties of sets  $\mathcal{E}$ ,  $\mathcal{E}_\gamma$ ,  $\mathcal{E}^+$  are similar to the properties of the analogs of sets  $\mathcal{E}$ ,  $\mathcal{E}_\gamma$ ,  $\mathcal{E}^+$  in the case when  $n = 1$ ,  $A_j \equiv 0$ ,  $j = 1, \dots, d$ . For the properties of the latter sets see, e.g., [101] and references therein. Restrictions in space and time prevent us from studying the properties of sets  $\mathcal{E}$ ,  $\mathcal{E}_\gamma$ ,  $\mathcal{E}^+$  in the present paper.

**Theorem III.1.** *Let  $D$  satisfy (III.2) and  $E$  be fixed. Suppose that  $E$  is not a Dirichlet eigenvalue for operators  $L$  and  $-\Delta$  in  $D$ . Let  $A_1, \dots, A_d, V$  satisfy (III.15). Let  $\Phi(E)$  correspond to coefficients  $A_1, \dots, A_d, V$  and  $\Phi^0(E)$  correspond to coefficients  $A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0 \equiv 0$ . Denote by  $(\Phi - \Phi^0)(x, y, E)$ ,  $x, y \in \partial D$ , the Schwartz kernel of operator  $\Phi(E) - \Phi^0(E)$ . Then the following formulas and equations hold :*

$$h(k, l) = (2\pi)^{-d} \int_{\partial D} \int_{\partial D} e^{-ilx} (\Phi - \Phi^0)(x, y, E) \psi(y, k) dy dx, \quad (\text{III.19})$$

where  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $\text{Im } k = \text{Im } l$ ,  $k^2 = l^2 = E$ ,  $k \notin \mathcal{E}$  ;

$$\psi(x, k) = e^{ikx} \text{Id}_n + \int_{\partial D} A(x, y, k) \psi(y, k) dy, \quad x \in \partial D, \quad (\text{III.20})$$

$$A(x, y, k) = \int_{\partial D} G(x - z, k) (\Phi - \Phi^0)(z, y, E) dz, \quad x, y \in \partial D, \quad (\text{III.21})$$

where  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E})$ ,  $k^2 = E$ , and  $G$  is defined in formula (III.12) ;

$$h_\gamma(k, l) = (2\pi)^{-d} \int_{\partial D} \int_{\partial D} e^{-ilx} (\Phi - \Phi^0)(x, y, E) \psi_\gamma(y, k) dy dx, \quad (\text{III.22})$$



where  $\gamma \in S^{d-1}$ ,  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $k \notin \mathcal{E}_\gamma$ ,

$$\psi_\gamma(x, k) = e^{ikx} \text{Id}_n + \int_{\partial D} A_\gamma(x, y, k) \psi_\gamma(y, k) dy, \quad x \in \partial D, \quad (\text{III.23})$$

$$A_\gamma(x, y, k) = \int_{\partial D} G_\gamma(x - z, k) (\Phi - \Phi^0)(z, y, E) dz, \quad x, y \in \partial D, \quad (\text{III.24})$$

$$G_\gamma(x, k) \stackrel{\text{def}}{=} G(x, k + i0\gamma), \quad x \in \mathbb{R}^d, \quad (\text{III.25})$$

where  $\gamma \in S^{d-1}$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma)$ ,  $k^2 = E$ ;

$$f(k, l) = (2\pi)^{-d} \int_{\partial D} \int_{\partial D} e^{-ilx} (\Phi - \Phi^0)(x, y, E) \psi^+(y, k) dy dx, \quad (\text{III.26})$$

where  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $k \notin \mathcal{E}^+$ ,

$$\psi^+(x, k) = e^{ikx} \text{Id}_n + \int_{\partial D} A^+(x, y, k) \psi^+(y, k) dy, \quad x \in \partial D, \quad (\text{III.27})$$

$$A^+(x, y, k) = \int_{\partial D} G^+(x - z, k) (\Phi - \Phi^0)(z, y, E) dz, \quad x, y \in \partial D, \quad (\text{III.28})$$

where  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^+)$ ,  $k^2 = E$ , and  $G^+$  is defined in formula (III.9).

Actually, we consider (III.20), (III.23), (III.27) as integral equations for finding  $\psi$ ,  $\psi_\gamma$ ,  $\psi^+$ , respectively, from  $\Phi(E) - \Phi^0(E)$ .

In addition, we consider (III.19), (III.22), (III.26) as explicit formulas for finding  $h$ ,  $h_\gamma$ ,  $f$  from  $\Phi(E) - \Phi^0(E)$  and  $\psi$ ,  $\psi_\gamma$ ,  $\psi^+$ , respectively.

For fixed  $0 < \beta \leq 1$  we denote by  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  the Banach space of functions from  $C^1(\partial D, M_n(\mathbb{C}))$  with component-wise Hölder-continuous derivatives.

**Theorem III.2.** *Let the assumptions of Theorem III.1 be fulfilled. Let  $0 < \beta < 1$  be fixed.*

1. *Fix  $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $k^2 = E$ . Then equation (III.20) is a Fredholm integral equation of second kind for  $\psi \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  which is uniquely solvable if and only if  $k \notin \mathcal{E}$ .*
2. *Fix  $\gamma \in S^{d-1}$ ,  $k \in \mathbb{R}^d \setminus 0$ ,  $k^2 = E$ . Then equation (III.23) is a Fredholm integral equation of second kind for  $\psi_\gamma \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  which is uniquely solvable if and only if  $k \notin \mathcal{E}_\gamma$ .*
3. *Fix  $k \in \mathbb{R}^d \setminus 0$ ,  $k^2 = E$ . Then equation (III.27) is a Fredholm integral equation of second kind for  $\psi^+ \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  which is uniquely solvable if and only if  $k \notin \mathcal{E}^+$ .*

In fact, Theorems III.1 and III.2 are particular cases of more general Theorems III.1' and III.2' given below. To formulate these results we need to introduce some notations.

Let coefficients  $A_1^0, \dots, A_d^0, V^0$  on  $\mathbb{R}^d$  satisfy

$$A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, \quad (\text{III.29})$$

and either

$$V^0(x) \text{ be a diagonal matrix for all } x, \quad (\text{III.30})$$

or

$$\begin{aligned} V^0 &= \mathcal{V}^0 v^0, \text{ where } \mathcal{V}^0 \in M_n(\mathbb{C}) \\ &\text{and } v^0 \text{ is a slalar function of } x. \end{aligned} \quad (\text{III.31})$$

We also suppose that  $V^0$  is zero outside of  $D$  and coefficients  $A_1^0, \dots, A_d^0, V^0$  restricted to  $D$  satisfy (III.15).

Define  $L_{V^0}, \mathcal{E}_{V^0}, \mathcal{E}_{V^0, \gamma}, \gamma \in S^{d-1}$ , and  $\mathcal{E}_{V^0}^+$  by formulas (III.1), (III.16), (III.17), (III.18), respectively, using coefficients  $A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0$  in (III.1), (III.11), (III.8) instead of  $A_1, \dots, A_d, V$ .

Note that, in fact, in the definition of set  $\mathcal{E}_{V^0}$  (or sets  $\mathcal{E}_{V^0, \gamma}, \mathcal{E}_{V^0}^+$ ) it is sufficient to consider the solvability of corresponding equations for  $\psi = e^{ikx}\mu$  with  $\mu \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$  (for  $\psi \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$ , respectively).

We consider the functions  $R^0, R_\gamma^0, \gamma \in S^{d-1}$ , and  $R^{+,0}$  defined as follows :

$$R^0(x, y, k) = G(x - y, k)\text{Id}_n + \int_{\mathbb{R}^d} G(x - z, k)V^0(z)R^0(z, y, k) dz, \quad (\text{III.32})$$

where  $x, y \in \mathbb{R}^d, k \in \mathbb{C}^d \setminus \mathbb{R}^d$  and  $G$  is defined in formula (III.12);

$$R_\gamma^0(x, y, k) \stackrel{\text{def}}{=} R^0(x, y, k + i0\gamma), \quad (\text{III.33})$$

$$R^{+,0}(x, y, k) \stackrel{\text{def}}{=} R_{k/|k|}^0(x, y, k), \quad (\text{III.34})$$

where  $x, y \in \mathbb{R}^d, k \in \mathbb{R}^d \setminus 0, \gamma \in S^{d-1}$ .

We consider (III.32) at fixed  $y, k$  as an integral equation for

$$R^0(x, y, k) = G(x - y, k)\text{Id}_n + e^{ik(x-y)}r^0(x, y, k), \quad (\text{III.35})$$

where  $r^0(\cdot, y, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$ .

It follows from (III.32), (III.35) that  $r^0$  satisfies the following equation :

$$\begin{aligned} r^0(x, y, k) &= \int_{\mathbb{R}^d} g(x - z, k)V^0(z)g(z - y, k) dz \\ &+ \int_{\mathbb{R}^d} g(x - z, k)V^0(z)r^0(z, y, k) dy, \end{aligned} \quad (\text{III.36})$$

where  $x, y \in \mathbb{R}^d$  and  $g$  is defined in formula (III.12).

Note that under assumption (III.15) for coefficients  $A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0$  the following statements are true :

1. Fix  $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ . Then equation (III.36) is uniquely solvable for  $r^0(\cdot, y, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$  for any  $y \in \mathbb{R}^d$  if and only if  $k \notin \mathcal{E}_{V^0}$ .
2. Fix  $\zeta \in \mathbb{R}^d \setminus 0, \gamma \in S^{d-1}$ . Then equation (III.36) with  $k = \zeta + i0\gamma$  is uniquely solvable for  $r^0(\cdot, y, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$  for any  $y \in \mathbb{R}^d$  if and only if  $\zeta \notin \mathcal{E}_{V^0, \gamma}$ .
3. Fix  $\zeta \in \mathbb{R}^d \setminus 0$ . Then equation (III.36) with  $k = \zeta + i0\zeta/|\zeta|$  is uniquely solvable for  $r^0(\cdot, y, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$  for any  $y \in \mathbb{R}^d$  if and only if  $\zeta \notin \mathcal{E}_{V^0}^+$ .

Besides, if equation (III.36) at fixed  $k$  is uniquely solvable for  $r^0(\cdot, y, k) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$  for any  $y \in \mathbb{R}^d$ , then function  $r^0$  has the following properties :

$$r^0(\cdot, \cdot, k) \in C(\mathbb{R}^d \times \mathbb{R}^d, M_n(\mathbb{C})) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d, M_n(\mathbb{C})) \text{ at fixed } k, \quad (\text{III.37})$$

$$\int_{\mathbb{R}^d} g(x-z, k)V^0(z)r^0(z, y, k) dz = \int_{\mathbb{R}^d} r^0(x, z, k)V^0(z)g(z-y, k) dz, \quad (\text{III.38})$$

where  $x, y \in \mathbb{R}^d$ .

We also consider the function  $\tilde{\psi}_\gamma^0$  defined as follows :

$$\tilde{\psi}_\gamma^0(x, k, l) = e^{ilx}\text{Id}_n + \int_{\mathbb{R}^d} G_\gamma(x-y, k)V^0(y)\tilde{\psi}_\gamma^0(y, k, l) dy, \quad (\text{III.39})$$

where  $x \in \mathbb{R}^d, k, l \in \mathbb{R}^d \setminus 0, k^2 = l^2, \gamma \in S^{d-1}$ . We consider (III.39) at fixed  $k, l, \gamma$  as an integral equation for  $\tilde{\psi}_\gamma^0(\cdot, k, l) \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$ .

**Theorem III.1'.** *Let  $D$  satisfy (III.2) and  $E$  be fixed. Suppose that  $E$  is not a Dirichlet eigenvalue for operators  $L, L_{V^0}$  and  $-\Delta$  in  $D$ . Consider two sets of coefficients  $A_1, \dots, A_d, V$  and  $A_1^0, \dots, A_d^0, V^0$ , satisfying (III.15). Let  $A_1^0, \dots, A_d^0, V^0$  satisfy (III.29) and either (III.30) or (III.31). Let  $\Phi, \psi, h, \psi_\gamma, h_\gamma, \psi^+, f, \mathcal{E}, \mathcal{E}_\gamma, \mathcal{E}^+$  correspond to  $A_1, \dots, A_d, V$  (as defined above) and  $\Phi_{V^0}, \psi^0, h^0, \psi_\gamma^0, \tilde{\psi}_\gamma^0, h_\gamma^0, \psi^{+,0}, f^0, R^0, R_\gamma^0, R^{+,0}, \mathcal{E}_{V^0}, \mathcal{E}_{V^0, \gamma}, \mathcal{E}_{V^0}^+$  correspond to  $A_1^0, \dots, A_d^0, V^0$  (as defined above with coefficients  $A_1^0, \dots, A_d^0, V^0$  instead of  $A_1, \dots, A_d, V$ ). Denote by  $(\Phi - \Phi_{V^0})(x, y, E), x, y \in \partial D$ , the Schwartz kernel of operator  $\Phi(E) - \Phi_{V^0}(E)$ . Then the following formulas and equations hold :*

$$h(k, l) = h^0(k, l) + (2\pi)^{-d} \int_{\partial D} \int_{\partial D} \psi^0(x, -l)(\Phi - \Phi_{V^0})(x, y, E)\psi(y, k) dy dx, \quad (\text{III.40})$$

where  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d, \text{Im } k = \text{Im } l, k^2 = l^2 = E, k \notin \mathcal{E} \cup \mathcal{E}_{V^0}$ ,

$$\psi(x, k) = \psi^0(x, k) + \int_{\partial D} A(x, y, k)\psi(y, k) dy, \quad x \in \partial D, \quad (\text{III.41})$$

$$A(x, y, k) = \int_{\partial D} R^0(x, z, k)(\Phi - \Phi_{V^0})(z, y, E) dz, \quad x, y \in \partial D, \quad (\text{III.42})$$

where  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E} \cup \mathcal{E}_{V^0})$ ,  $k^2 = E$ ;

$$\begin{aligned} h_\gamma(k, l) &= h_\gamma^0(k, l) \\ &+ (2\pi)^{-d} \int_{\partial D} \int_{\partial D} \tilde{\psi}_{-\gamma}^0(x, -k, -l) (\Phi - \Phi_{V^0})(x, y, E) \psi_\gamma(y, k) dy dx, \end{aligned} \quad (\text{III.43})$$

where  $\gamma \in S^{d-1}$ ,  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $k \notin \mathcal{E}_\gamma \cup \mathcal{E}_{V^0, \gamma}$ ,

$$\psi_\gamma(x, k) = \psi_\gamma^0(x, k) + \int_{\partial D} A_\gamma(x, y, k) \psi_\gamma(y, k) dy, \quad x \in \partial D, \quad (\text{III.44})$$

$$A_\gamma(x, y, k) = \int_{\partial D} R_\gamma^0(x, z, k) (\Phi - \Phi_{V^0})(z, y, E) dz, \quad x, y \in \partial D, \quad (\text{III.45})$$

where  $\gamma \in S^{d-1}$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma \cup \mathcal{E}_{V^0, \gamma})$ ,  $k^2 = E$ ;

$$f(k, l) = f^0(k, l) + (2\pi)^{-d} \int_{\partial D} \int_{\partial D} \psi^{+,0}(x, -l) (\Phi - \Phi_{V^0})(x, y, E) \psi^+(y, k) dy dx, \quad (\text{III.46})$$

where  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2 = E$ ,  $k \notin \mathcal{E}^+ \cup \mathcal{E}_{V^0}^+$ ,

$$\psi^+(x, k) = \psi^{+,0}(x, k) + \int_{\partial D} A^+(x, y, k) \psi^+(y, k) dy, \quad x \in \partial D, \quad (\text{III.47})$$

$$A^+(x, y, k) = \int_{\partial D} R^{+,0}(x, z, k) (\Phi - \Phi_{V^0})(z, y, E) dz, \quad x, y \in \partial D, \quad (\text{III.48})$$

where  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^+ \cup \mathcal{E}_{V^0}^+)$ ,  $k^2 = E$ .

In a similar way with Theorem III.1, we consider (III.41), (III.44), (III.47) as integral equations for finding  $\psi$ ,  $\psi_\gamma$ ,  $\psi^+$  from  $\Phi(E) - \Phi_{V^0}(E)$  and  $\psi^0$ ,  $R^0$ ;  $\psi_\gamma^0$ ,  $R_\gamma^0$ ;  $\psi^{+,0}$ ,  $R^{+,0}$ , respectively.

We also consider (III.40), (III.43), (III.46) as explicit formulas for finding  $h$ ,  $h_\gamma$ ,  $f$  from  $\Phi(E) - \Phi_{V^0}(E)$  and  $h^0$ ,  $\psi^0$ ,  $\psi$ ;  $h_\gamma^0$ ,  $\tilde{\psi}_{-\gamma}^0$ ,  $\psi_\gamma$ ;  $f^0$ ,  $\psi^{+,0}$ ,  $\psi^+$ , respectively.

Theorem III.1' is proved in Section 3.

**Theorem III.2'.** *Let the assumptions of Theorem III.1' be fulfilled. Let  $0 < \beta < 1$  be fixed.*

1. *Fix  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$ ,  $k^2 = E$ . Then equation (III.41) is a Fredholm integral equation of second kind for  $\psi \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  which is uniquely solvable if and only if  $k \notin \mathcal{E}$ .*
2. *Fix  $\gamma \in S^{d-1}$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0, \gamma})$ ,  $k^2 = E$ . Then equation (III.44) is a Fredholm integral equation of second kind for  $\psi_\gamma \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  which is uniquely solvable if and only if  $k \notin \mathcal{E}_\gamma$ .*

3. Fix  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0}^+)$ ,  $k^2 = E$ . Then equation (III.47) is a Fredholm integral equation of second kind for  $\psi^+ \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  which is uniquely solvable if and only if  $k \notin \mathcal{E}^+$ .

Theorem III.2' is proved in Section 4.

**Remark 1.** Note that the proofs of equations and formulas of Theorems III.1, III.1' remain valid without the assumption that coefficients  $A_1, \dots, A_d, V, V^0$  have compact supports in  $D$ . The assumption that the coefficients have compact supports in  $D$  was introduced in order to simplify the choice of functional spaces for solving equations (III.20), (III.23), (III.27), (III.41), (III.44), (III.47) and related proofs of Theorems III.2, III.2'.

It is important to note that equation (III.11) and formula (III.13) give much more stable way to find functions  $\psi^0, h^0$  from  $A_1^0, \dots, A_d^0, V^0$  than equation (III.20) and formula (III.19) if  $|\operatorname{Im} k|$  is sufficiently large.

On the other hand, it is known that the solution to equation (III.41) will be relatively stable if the norm of the integral operator involved in this equation is less than 1. If at fixed  $k$  coefficients  $A_1, \dots, A_d$  are sufficiently small whereas coefficient  $V$  is not small but is sufficiently close to coefficient  $V^0$ , then the integral operator in equation (III.41) will have much smaller norm than the integral operator in equation (III.20) (e.g., as a norm of operator on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ ,  $0 < \beta < 1$ ). In particular, the norm will be less than 1 and we will be able to use the method of successive approximations to solve (III.41). Hence equation (III.41) and formula (III.40) will give much more stable way to find  $\psi$  and  $h$  than equation (III.20) and formula (III.19), respectively. For more details, see pp. 262–263 of [101] and Section 3.2 of [111] for related discussion.

## 3 Proof of Theorem III.1'

### 3.1 Integral identity

Note that we have the identity

$$\begin{aligned} & \int_{\partial D} u^0(x) (\Phi(E) - \Phi_{V^0}(E))(u|_{\partial D})(x) dx \\ &= \int_D u^0(x) \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) u(x) dx. \end{aligned} \tag{III.49}$$

for any sufficiently regular  $M_n(\mathbb{C})$ -valued functions  $u, u^0$  on  $\bar{D}$  (for example, for  $u, u^0 \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$ ) satisfying

$$-\Delta u - 2i \sum_{j=1}^d A_j(x) \partial_{x_j} u + V(x)u = Eu, \quad x \in D, \quad (\text{III.50})$$

$$-\Delta u^0 + V^0(x)u^0 = Eu^0, \quad x \in D, \quad (\text{III.51})$$

where  $u^0$  also satisfies

$$V^0(x)u^0(x) = u^0(x)V^0(x), \quad x \in D. \quad (\text{III.52})$$

Identity (III.49) for the case when  $n = 1, A_1 \equiv 0, \dots, A_d \equiv 0$  first appeared in [8]. It was generalized to the case when  $n \geq 2, A_1 \equiv 0, \dots, A_d \equiv 0$  in [110].

Identity (III.49) can be deduced from the second Green formula. More precisely, formula (III.49) follows from the following chain of equalities :

$$\begin{aligned} & \int_D u^0(x) \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) u(x) dx \\ & \stackrel{(\text{III.50}), (\text{III.52})}{=} \int_D (u^0(x)(\Delta + E)u(x) - V^0(x)u^0(x)u(x)) dx \\ & \stackrel{(\text{III.51})}{=} \int_D (u^0(x)\Delta u(x) - \Delta u^0(x)u(x)) dx \\ & = \int_{\partial D} u^0(x)(\Phi(E) - \Phi_{V^0}(E))(u|_{\partial D})(x) dx \\ & + \int_{\partial D} \left( u^0(x)\Phi_{V^0}(E)(u|_{\partial D})(x) - \Phi_{V^0}(E)(u^0|_{\partial D})(x)u(x) \right) dx \\ & = \int_{\partial D} u^0(x)(\Phi(E) - \Phi_{V^0}(E))(u|_{\partial D})(x) dx \\ & + \int_D \left( u^0(x)V^0(x)\tilde{u}(x) - V^0(x)u^0(x)\tilde{u}(x) \right) dx \\ & \stackrel{(\text{III.52})}{=} \int_{\partial D} u^0(x)(\Phi(E) - \Phi_{V^0}(E))(u|_{\partial D})(x) dx, \end{aligned}$$

where  $\tilde{u}$  satisfies (III.51) and  $\tilde{u}|_{\partial D} = u|_{\partial D}$ .

### 3.2 Symmetries of the scattering functions

We denote by  $L_c^\infty(\mathbb{R}^d)$  the set of compactly supported functions from  $L^\infty(\mathbb{R}^d)$ .

**Lemma III.1.** *Let  $V^0 \in L_c^\infty(\mathbb{R}^d)$  satisfy either (III.30) or (III.31). Then the following identities hold :*

$$V^0(x)\psi^0(x, k) = \psi^0(x, k)V^0(x), \quad (\text{III.53})$$

$$V^0(x)R^0(x, y, k) = R^0(x, y, k)V^0(x), \quad (\text{III.54})$$

$$R^0(x, y, k) = R^0(y, x, -k), \quad (\text{III.55})$$

where  $x, y \in \mathbb{R}^d$ ,  $x \neq y$ ,  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$ .

*Démonstration.* Let  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$  be fixed. Then equation (III.11) with  $A_1 \equiv 0, \dots, A_d \equiv 0$ ,  $V \equiv V^0$  is uniquely solvable for  $\psi^0 = e^{ikx}\mu^0$  with  $\mu^0 \in L^\infty(\mathbb{R}^d, M_n(\mathbb{C}))$ .

Suppose, first, that (III.30) holds. In this case it follows from formula (III.11) that  $\psi^0(x, k)$  is a diagonal matrix for all  $x \in \mathbb{R}^d$ . Hence (III.53) holds.

Suppose now that (III.31) holds, so that  $V^0(x) = \mathcal{V}^0 v^0(x)$ ,  $x \in \mathbb{R}^d$ . Let  $U$  be a non-degenerated  $n \times n$  matrix such that

$$U\mathcal{V}^0 U^{-1} \stackrel{\text{def}}{=} \Lambda = \begin{bmatrix} \Lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_s \end{bmatrix}, \quad \Lambda_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{bmatrix},$$

where  $\Lambda_j \in M_{n_j}(\mathbb{C})$ ,  $j = 1, \dots, s$ . Define  $\psi' = U\psi^0 U^{-1}$ . Then  $\psi'$  satisfies the equation

$$\psi'(x, k) = e^{ikx} \text{Id}_n + \int_{\mathbb{R}^d} G(x-y, k) \Lambda v^0(y) \psi'(y, k) dy. \quad (\text{III.56})$$

Since the solution to (III.56) is unique it follows that  $\psi'$  has the block-diagonal form :

$$\psi' = \begin{bmatrix} \psi'_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \psi'_s \end{bmatrix},$$

where  $\psi'_j \in M_{n_j}(\mathbb{C})$ ,  $j = 1, \dots, s$ . Hence the following equations hold and have the unique solutions :

$$\psi'_j(x, k) = e^{ikx} \text{Id}_{n_j} + \int_{\mathbb{R}^d} G(x-y, k) \Lambda_j v^0(y) \psi'_j(y, k) dy, \quad (\text{III.57})$$

where  $j = 1, \dots, s$ .

We write  $\psi'_{j,il}$  for the element in position  $(i, l)$  in matrix  $\psi'_j$ . Fix  $j$  and consider the last row of matrix equation (III.57). We have the following equations :

$$\psi'_{j,il}(x, k) = \lambda_j \int_{\mathbb{R}^d} G(x - y, k) v^0(y) \psi'_{j,il}(y, k) dy, \quad i = n_j, \quad l < n_j. \quad (\text{III.58})$$

We claim that equation (III.58) has only the trivial solution. Suppose that, on the contrary, there is a nontrivial solution  $\phi$  to equation (III.58). Then we can construct a solution  $\tilde{\psi}'_j$  to (III.57) different from  $\psi'_j$ , putting  $\tilde{\psi}'_{j,11} = \psi'_{j,11} + \phi$  and  $\tilde{\psi}'_{j,il} = \psi'_{j,il}$  for all other  $i, l$ . This is a contradiction since we showed that equation (III.57) has the unique solution. This shows that equation (III.58) has only the trivial solution and  $\psi'_{j,il} \equiv 0$  for  $i = n_j$  and  $l < n_j$ .

Writing the equation (III.57) componentwise for rows with numbers  $i = n_j - 1, \dots, 2$  we show by induction that  $\psi'_{j,il} \equiv 0$  for  $i > l$ .

Fix  $i, l$  with  $i \neq l$ . Subtracting equation (III.57) for the element in position  $(l, l)$  from equation (III.57) for the element in position  $(i, i)$  we get the equation

$$\psi'_{j,ii}(x, k) - \psi'_{j,ll}(x, k) = \lambda_j \int_{\mathbb{R}^d} G(x - y, k) v^0(y) (\psi'_{j,ii}(y, k) - \psi'_{j,ll}(y, k)) dy.$$

Since equation (III.58) has only the trivial solution, it follows that  $\psi'_{j,ii} \equiv \psi'_{j,ll}$ .

Now fix  $i, l$  with  $i \neq l, i > 1, l > 1$ . Write equation (III.57) for the elements in positions  $(i - 1, i)$  and  $(l - 1, l)$  and subtract one from another. This leads to equation

$$\begin{aligned} \psi'_{j,i-1,i}(x, k) - \psi'_{j,l-1,l}(x, k) &= \lambda_j \int_{\mathbb{R}^d} G(x - y, k) \times \\ &\times v^0(y) (\psi'_{j,i-1,i}(y, k) - \psi'_{j,l-1,l}(y, k)) dy + \psi'_{j,ii}(x, k) - \psi'_{j,ll}(x, k). \end{aligned}$$

But we showed that  $\psi'_{j,ii} \equiv \psi'_{j,ll}$  and that equation (III.58) has only the trivial solution. This implies that  $\psi'_{j,i-1,i} \equiv \psi'_{j,l-1,l}$ .

Proceeding inductively, we obtain that  $\psi'_j$  has the following upper triangular form :

$$\psi'_j = \begin{bmatrix} \psi'_{j,11} & \psi'_{j,12} & \psi'_{j,13} & \cdots & \psi'_{j,1,n-1} & \psi'_{j,1n} \\ 0 & \psi'_{j,11} & \psi'_{j,12} & \cdots & \psi'_{j,1,n-2} & \psi'_{j,1,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \psi'_{j,11} & \psi'_{j,12} \\ 0 & 0 & 0 & \cdots & 0 & \psi'_{j,11} \end{bmatrix}.$$

It follows that  $\psi'_j(x, k)$  commutes with  $\Lambda_j$  for all  $x \in \mathbb{R}^d, j = 1, \dots, s$ . Hence  $\psi'(x, k)$  commutes with  $\Lambda$  and  $\psi(x, k)$  commutes with  $\mathcal{V}^0$  for all  $x \in \mathbb{R}^d$ . Property (III.53) is proved.



Formula (III.54) can be proved in a similar way.

The proof of (III.55) for the case when  $n = 1$  was given in [No3]. This proof also works in the case when  $n \geq 2$  if either (III.30) or (III.31) holds.  $\square$

**Remark 2.** In a similar way with Lemma III.1 it can be proved that if  $V^0 \in L_c^\infty(\mathbb{R}^d)$  satisfies either (III.30) or (III.31), then the following formulas hold :

$$V^0(x)\psi_\gamma^0(x, k) = \psi_\gamma^0(x, k)V^0(x), \quad (\text{III.59})$$

$$V^0(x)R_\gamma^0(x, y, k) = R_\gamma^0(x, y, k)V^0(x), \quad (\text{III.60})$$

$$R_\gamma^0(x, y, k) = R_{-\gamma}^0(y, x, -k), \quad (\text{III.61})$$

where  $\gamma \in S^{d-1}$ ,  $x, y \in \mathbb{R}^d$ ,  $x \neq y$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0, \gamma})$ ;

$$V^0(x)\psi^{+,0}(x, k) = \psi^{+,0}(x, k)V^0(x), \quad (\text{III.62})$$

$$V^0(x)R^{+,0}(x, y, k) = R^{+,0}(x, y, k)V^0(x), \quad (\text{III.63})$$

$$R^{+,0}(x, y, k) = R^{+,0}(y, x, -k), \quad (\text{III.64})$$

where  $x, y \in \mathbb{R}^d$ ,  $x \neq y$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0}^+)$ .

**Lemma III.2.** Let  $V^0 \in L_c^\infty(\mathbb{R}^d)$ . Then :

1. if  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $k^2 = l^2$ ,  $\text{Im } k = \text{Im } l$ ,  $k \notin \mathcal{E}_{V^0}$ , then  $l \notin \mathcal{E}_{V^0}$  and the following formula holds :

$$R^0(x, y, k) = R^0(x, y, l), \quad (\text{III.65})$$

where  $x, y \in \mathbb{R}^d$ ,  $x \neq y$ ;

2. if  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2$ ,  $k \notin \mathcal{E}_{V^0}^+$ , then  $l \notin \mathcal{E}_{V^0}^+$  and the following formula holds :

$$R^{+,0}(x, y, k) = R^{+,0}(x, y, l), \quad (\text{III.66})$$

where  $x, y \in \mathbb{R}^d$ ,  $x \neq y$ .

*Démonstration.* Part 1 follows from (III.32) and from the following formula of [109] :

$$G(x, k) = G(x, l), \quad x \in \mathbb{R}^d \setminus 0, \quad k, l \in \mathbb{C}^d \setminus \mathbb{R}^d, \quad k^2 = l^2, \quad \text{Im } k = \text{Im } l.$$

Part 2 follows from formulas (III.32), (III.34) and from identity

$$G^+(x, k) = G^+(x, l), \quad x \in \mathbb{R}^d \setminus 0, \quad k, l \in \mathbb{R}^d \setminus 0, \quad k^2 = l^2,$$

which is a consequence of the following well-known explicit formulas :

$$G^+(x, k) = -\frac{i}{4}H_0^{(1)}(|k||x|), \quad d = 2,$$

$$G^+(x, k) = -\frac{e^{i|k||x|}}{4\pi|x|}, \quad d = 3,$$

where  $H_0^{(1)}$  is the Hankel function of the first kind.  $\square$

### 3.3 Reformulation of equations (III.8), (III.11) and formulas (III.10), (III.13)

Subtracting equation (III.11) written for  $\psi$  from equation (III.11) written for  $\psi^0$  we obtain formula

$$\begin{aligned} & \psi(x, k) - \psi^0(x, k) - \int_{\mathbb{R}^d} G(x - y, k) V^0(y) (\psi(y, k) - \psi^0(y, k)) dy \\ &= \int_{\mathbb{R}^d} G(x - y, k) \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \psi(y, k) dy, \end{aligned}$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E} \cup \mathcal{E}_{V^0})$ . Comparing this equation with (III.32), we obtain

$$\begin{aligned} & \psi(x, k) = \psi^0(x, k) + \int_{\mathbb{R}^d} R^0(x, y, k) \times \\ & \times \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \psi(y, k) dy, \end{aligned} \tag{III.67}$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E} \cup \mathcal{E}_{V^0})$ .

Similarly, we can obtain the following equations :

$$\begin{aligned} & \psi_\gamma(x, k) = \psi_\gamma^0(x, k) + \int_{\mathbb{R}^d} R_\gamma^0(x, y, k) \times \\ & \times \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \psi_\gamma^+(y, k) dy, \end{aligned} \tag{III.68}$$

where  $\gamma \in S^{d-1}$ ,  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma \cup \mathcal{E}_{V^0, \gamma})$ ;

$$\begin{aligned} & \psi^+(x, k) = \psi^{+,0}(x, k) + \int_{\mathbb{R}^d} R^{+,0}(x, y, k) \times \\ & \times \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \psi^+(y, k) dy, \end{aligned} \tag{III.69}$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^+ \cup \mathcal{E}_{V^0}^+)$ .

From formula (III.67) with  $A_1 \equiv 0, \dots, A_d \equiv 0$ ,  $V \equiv 0$  and from formulas (III.54), (III.55), (III.65) we obtain

$$e^{-ilx} \text{Id}_n = \psi^0(x, -l) - \int_{\mathbb{R}^d} e^{-ily} V^0(y) R^0(y, x, k) dy, \tag{III.70}$$

where  $x \in \mathbb{R}^d$ ,  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $k^2 = l^2$ ,  $\text{Im } k = \text{Im } l$ ,  $k \notin \mathcal{E}_{V^0}$ .

Further, we have the following chain of equalities :

$$\begin{aligned}
& (2\pi)^d h(k, l) \stackrel{\text{(III.13)}}{=} \int_{\mathbb{R}^d} e^{-ilx} V^0(x) \psi(x, k) dx \\
& + \int_{\mathbb{R}^d} e^{-ilx} \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) \psi(x, k) dx \\
& \stackrel{\text{(III.70)}}{=} \int_{\mathbb{R}^d} e^{-ilx} V^0(x) \psi(x, k) dx + \int_{\mathbb{R}^d} \psi^0(x, -l) \times \\
& \times \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) \psi(x, k) dx \\
& \quad - \int_{\mathbb{R}^d} e^{-ily} V^0(y) \int_{\mathbb{R}^d} R^0(y, x, k) \times \\
& \times \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) \psi(x, k) dx dy,
\end{aligned}$$

where  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $k^2 = l^2$ ,  $\text{Im } k = \text{Im } l$ ,  $k \notin \mathcal{E} \cup \mathcal{E}_{V^0}$ .

From this formula and formula (III.67) it follows that

$$\begin{aligned}
h(k, l) &= h^0(k, l) + (2\pi)^{-d} \int_{\mathbb{R}^d} \psi^0(x, -l) \times \\
& \times \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) \psi(x, k) dx,
\end{aligned} \tag{III.71}$$

where  $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$ ,  $k^2 = l^2$ ,  $\text{Im } k = \text{Im } l$ ,  $k \notin \mathcal{E} \cup \mathcal{E}_{V^0}$ .

In a similar way with formula (III.71), using formulas (III.69), (III.63), (III.64), (III.66) instead of (III.67), (III.54), (III.55), (III.65), we can obtain formula

$$\begin{aligned}
f(k, l) &= f^0(k, l) + (2\pi)^{-d} \int_{\mathbb{R}^d} \psi^{+,0}(x, -l) \times \\
& \times \left( -2i \sum_{j=1}^d A_j(x) \partial_{x_j} + V(x) - V^0(x) \right) \psi^+(x, k) dx,
\end{aligned} \tag{III.72}$$

where  $k, l \in \mathbb{R}^d \setminus 0$ ,  $k^2 = l^2$ ,  $k \notin \mathcal{E}^+ \cup \mathcal{E}_{V^0}^+$ .

### 3.4 The final part of the proof of Theorem III.1'

It follows from formulas (III.53), (III.62) that we can apply identity (III.49) to (III.71) and (III.72). Applying (III.49) to (III.71) and (III.72), we obtain (III.40) and (III.46), respectively.

We recall that function  $G(x, k)$  satisfies the following equation at fixed  $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ , see, e.g., [47] :

$$\Delta_x G(x, k) + k^2 G(x, k) = \delta(x). \quad (\text{III.73})$$

It follows from formulas (III.32), (III.35), (III.38), (III.73) that  $R^0(x, y, k)$  at fixed  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$ , satisfies the equations :

$$\begin{aligned} (\Delta_x - V^0(x) + k^2)R(x, y, k) &= \delta_y(x)\text{Id}_n, & \text{at fixed } y \in \mathbb{R}^d, \\ (\Delta_y - V^0(y) + k^2)R(x, y, k) &= \delta_x(y)\text{Id}_n, & \text{at fixed } x \in \mathbb{R}^d. \end{aligned} \quad (\text{III.74})$$

Taking this into account, using formula (III.54), applying identity (III.49) to (III.67) with  $x \notin \bar{D}$  and passing to the limit in the resulting formula as  $x$  tends to a point at  $\partial D$ , we obtain formula (III.41). Formulas (III.44) and (III.47) can be obtained in a similar way, if we use equations (III.68) and (III.69) instead of (III.67).

We will prove formula (III.43) using the ideas of [101]. Note that formula (III.39) can be written in the form

$$\begin{aligned} e^{ilx}\text{Id}_n - \tilde{\psi}_\gamma^0(x, k, l) &- \int_{\mathbb{R}^d} G_\gamma(x - y, k)V^0(y)(e^{ily} - \tilde{\psi}_\gamma^0(y, k, l)) dy \\ &= - \int_{\mathbb{R}^d} G(x - y, k)V^0(y)e^{ily} dy. \end{aligned} \quad (\text{III.75})$$

From formulas (III.25), (III.32), (III.33) it follows that function  $R_\gamma^0(\cdot, y, k)$  is well-defined at fixed  $y \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0, \gamma})$  and satisfies

$$R_\gamma^0(x, y, k) = G_\gamma(x - y, k) + \int_{\mathbb{R}^d} G_\gamma(x - z, k)V^0(z)R_\gamma^0(z, y, k) dz, \quad (\text{III.76})$$

where  $x, y \in \mathbb{R}^d$ ,  $\gamma \in S^{d-1}$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0, \gamma})$ .

Comparing (III.75) with (III.76), we obtain the following formula :

$$e^{ilx}\text{Id}_n = \tilde{\psi}_\gamma^0(x, k, l) - \int_{\mathbb{R}^d} R_\gamma^0(x, y, k)V^0(y)e^{ily} dy, \quad (\text{III.77})$$

where  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_{V^0, \gamma})$ ,  $l \in \mathbb{R}^d$ ,  $k^2 = l^2$ ,  $\gamma \in S^{d-1}$ .

Replacing  $k, l, \gamma$  by  $-k, -l, -\gamma$  in formula (III.77) and using formulas (III.60), (III.61), we obtain the equality

$$e^{-ilx}\text{Id}_n = \tilde{\psi}_{-\gamma}^0(x, -k, -l) - \int_{\mathbb{R}^d} e^{-ily}V^0(y)R_\gamma^0(y, x, k) dy, \quad (\text{III.78})$$

where  $x \in \mathbb{R}^d, k, l \in \mathbb{R}^d \setminus 0, k^2 = l^2, k \notin \mathcal{E}_{V^0, \gamma}, \gamma \in S^{d-1}$ .

Formula (III.78) is an analog of formula (III.70). The remaining part of the proof of (III.43) is similar to the proof of formula (III.40).

## 4 Proof of Theorem III.2'

### 4.1 Auxiliary results

Consider the following Dirichlet problem for function  $\psi$  :

$$\begin{cases} L\psi \equiv -\Delta\psi - 2i \sum_{j=1}^d A_j(x)\partial_{x_j}\psi + V(x)\psi = E\psi, & x \in D, \\ \psi|_{\partial D} = \varphi, \end{cases} \quad (\text{III.79})$$

where  $\varphi$  is some given function on  $\partial D$ .

Our goal in this subsection is to prove the following lemma.

**Lemma III.3.** *Let  $A_1, \dots, A_d, V \in C_c^{0,\alpha}(D, M_n(\mathbb{C}))$  for some  $0 < \alpha \leq 1$ . Suppose that  $E$  is not a Dirichlet eigenvalue for operators  $L$  and  $-\Delta$  in  $D$ . Then :*

1. *for any  $\varphi \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ ,  $0 < \beta < 1$ , there exists the unique solution  $\psi \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$  to problem (III.79) ;*
2.  *$\psi$  is the unique solution of class  $C^1(\bar{D}, M_n(\mathbb{C}))$  to equation*

$$\psi(x) = \psi^0(x) + \int_D \Gamma(x, y, k) \left( -2i \sum_{j=1}^d A_j(y)\partial_{y_j} + V(y) \right) \psi(y) dy, \quad (\text{III.80})$$

where  $\Gamma(x, y, k)$  is the Green function for Dirichlet problem for operator  $\Delta + E$  in  $D$ ,  $E = k^2$ , and  $\psi^0 \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$ , is the unique solution to Dirichlet problem

$$\begin{cases} \Delta\psi^0 + E\psi^0 = 0 & \text{in } D, \\ \psi^0|_{\partial D} = \varphi; \end{cases} \quad (\text{III.81})$$

3. *the operator  $S: C^{1,\beta}(\partial D, M_n(\mathbb{C})) \rightarrow C^1(\bar{D}, M_n(\mathbb{C}))$ ,  $S(\varphi) = \psi$ , is a continuous linear operator.*

We need two other lemmas to prove Lemma III.3.

**Lemma III.4.** *Let  $A_1, \dots, A_d, V \in C_c^{0,\alpha}(D, M_n(\mathbb{C}))$  for some  $0 < \alpha \leq 1$ . Suppose that  $E$  is not a Dirichlet eigenvalue for operator  $L$  in  $D$ . Then :*

1. *for any  $\psi^0 \in C^1(\overline{D}, M_n(\mathbb{C})) \cap C^2(D, M_n(\mathbb{C}))$  satisfying  $(\Delta + E)\psi^0 = 0$  in  $D$  there exists the unique solution  $\psi \in C^1(\overline{D}, M_n(\mathbb{C}))$  to equation (III.80) ;*
2.  *$\psi$  belongs to  $C^2(D, M_n(\mathbb{C}))$  and satisfies (III.79) with  $\varphi = \psi^0|_{\partial D}$  ;*
3. *there exists a constant  $C > 0$ , not depending on  $\psi^0$ , such that*

$$\|\psi\|_{C^1(\overline{D})} \leq C\|\psi^0\|_{C^1(\overline{D})}. \quad (\text{III.82})$$

*Proof of Lemma III.4. 1. Reduction to a system of integral equations.* We introduce the following notations :

$$\begin{aligned} \psi_0(x) &= \psi(x), & \psi_j(x) &= \partial_{x_j}\psi(x), & j &= 1, \dots, d, \\ \psi_0^0(x) &= \psi^0(x), & \psi_j^0(x) &= \partial_{x_j}\psi^0(x), & j &= 1, \dots, d, \\ a_0(x) &= V(x), & a_j(x) &= -2iA_j(x), & j &= 1, \dots, d, \\ \Gamma_0(x, y, k) &= \Gamma(x, y, k), & \Gamma_j(x, y, k) &= \partial_{x_j}\Gamma(x, y, k), & j &= 1, \dots, d. \end{aligned}$$

Differentiating equation (III.80) we obtain the following system of coupled linear integral equations for  $\psi_j \in C(\overline{D})$ ,  $j = 0, \dots, d$  :

$$\psi_j(x) = \psi_j^0(x) + \sum_{m=0}^d \int_D \Gamma_j(x, y, k) a_m(y) \psi_m(y) dy, \quad j = 0, \dots, d. \quad (\text{III.83})$$

System (III.83) can be considered as a Fredholm equation of second kind in space  $(C(\overline{D}, M_n(\mathbb{C})))^{d+1}$ .

Suppose that functions  $\psi_j \in C(\overline{D}, M_n(\mathbb{C}))$ ,  $j = 0, \dots, d$ , solve (III.83). Denote  $\psi = \psi_0$ . The first equation of (III.83) implies that  $\psi \in C^1(\overline{D}, M_n(\mathbb{C}))$ . Differentiating the first equation with respect to  $x_1, \dots, x_d$ , we see that  $\partial_{x_j}\psi = \psi_j$ . Hence  $\psi$  satisfies (III.80).

One can see that solutions  $\psi$  of class  $C^1(\overline{D}, M_n(\mathbb{C}))$  to (III.80) are in bijective correspondence with solutions  $\psi_j \in C(\overline{D}, M_n(\mathbb{C}))$ ,  $j = 0, \dots, d$ , to system (III.83).

2. *Smoothness of solution to (III.80).* It follows from properties of fundamental solution  $\Gamma$  that any solution  $\psi \in C^1(\overline{D}, M_n(\mathbb{C}))$  to (III.80) belongs to  $C_{\text{loc}}^{1,\gamma}(D, M_n(\mathbb{C}))$  (the space of continuously differentiable  $M_n(\mathbb{C})$ -valued functions on  $D$  with locally Hölder continuous derivatives) for any  $0 < \gamma < 1$ .

In a similar way with Lemma 4.2, p. 55 of [54] it can be shown then that  $\psi \in C^2(D, M_n(\mathbb{C}))$  and that  $\psi$  satisfies (III.79) with  $\varphi = \psi^0|_{\partial D}$ .

3. *Existence and uniqueness.* As we noted in part 1 of the proof of this lemma, system of equations (III.83) can be considered as a Fredholm equation of second kind in

Banach space  $(C(\overline{D}, M_n(\mathbb{C})))^{d+1}$ . To show that this equation has the unique solution it is sufficient to show that the corresponding homogeneous equation has only the trivial solution. But it follows from parts 1, 2 of the proof that any solution to the homogeneous equation gives rise to a solution to (III.79) with zero boundary condition. Since  $E$  is not a Dirichlet eigenvalue for  $L$  in  $D$ , it follows that the homogeneous equation corresponding to (III.83) can have only the trivial solution.

Hence, using part 1 of the proof, we obtain that system (III.83) has the unique solution and this solution gives rise to the unique solution  $\psi \in C^1(\overline{D}, M_n(\mathbb{C}))$  to (III.80). By part 2 of the proof this function  $\psi$  belongs to  $C^2(D, M_n(\mathbb{C}))$  and satisfies (III.79) with  $\varphi = \psi^0|_{\partial D}$ .

Property 3 follows from Fredholm alternative for system (III.83) which asserts the existence of continuous inverse for Fredholm operator of (III.83).  $\square$

**Lemma III.5.** *Suppose that  $E$  is not a Dirichlet eigenvalue for operator  $-\Delta$  in  $D$ . Then for any  $\varphi \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ ,  $0 < \beta < 1$ , there exists the unique  $\psi^0 \in C^2(D, M_n(\mathbb{C})) \cap C^1(\overline{D}, M_n(\mathbb{C}))$  solving Dirichlet problem (III.81). Furthermore, there exists such  $C_\beta > 0$ , not depending on  $\varphi$ , that*

$$\|\psi^0\|_{C^1(\overline{D})} \leq C_\beta \|\varphi\|_{C^{1,\beta}(\partial D)}. \quad (\text{III.84})$$

*Proof of Lemma III.5.* The solution is unique since  $E$  is not a Dirichlet eigenvalue for  $-\Delta$ . To show existence, we can reduce problem (III.79) to a corresponding Fredholm integral equation of second kind as in Lemma III.4.

Existence of such  $C_\beta > 0$  that (III.84) holds follows from Lemmas 2.16, 2.23 of [28].  $\square$

*Proof of Lemma III.3.* One can see that solution to (III.79) is unique since  $E$  is not a Dirichlet eigenvalue for  $L$  in  $D$ .

Now let  $\psi^0$  be the unique solution to (III.81) constructed in Lemma III.5. Then Lemma III.4 gives us the solution to (III.79) with desired properties.

Property 3 is a consequence of formulas (III.82), (III.84).  $\square$

## 4.2 Compactness of operators in equations (III.41), (III.44), (III.47)

In this subsection we show that the integral operator in equation (III.41) is a compact linear operator on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  for any fixed  $0 < \beta < 1$ . The case of operators in equations (III.44), (III.47) can be considered in a similar way.

Throughout this subsection we assume that  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$  is fixed.

Rewrite equation (III.41) in the following form :

$$\psi = \psi^0 + R^0(k)(\Phi(E) - \Phi_{V^0}(E))\psi, \quad (\text{III.85})$$

where  $R^0(k)$  is the integral operator with Schwartz kernel  $R^0(x, y, k)$ ,  $x, y \in \partial D$ .

We are going to show that  $R^0(k)$  is a linear continuous operator on the space  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ . To show this we use the representation (III.35).

It follows from Theorems 2.12, 2.17 of [28] that the integral operator on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  with Schwartz kernel  $G(x - y, k)$ ,  $x, y \in \partial D$ , is continuous.

To see that the integral operator with Schwartz kernel  $e^{ik(x-y)}r^0(x, y, k)$ ,  $x, y \in \partial D$ , is continuous on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ , we will use formula (III.36).

It follows from (III.36), (III.37) (if we take into account that  $V^0$  has a compact support in  $D$ ) that derivatives  $\partial_{x_i}r^0(x, y, k)$ ,  $\partial_{x_i}\partial_{x_j}r^0(x, y, k)$ ,  $i, j = 1, \dots, d$ , exist and are continuous for  $x, y$  belonging to some neighborhood of set  $\partial D$ . Hence the operator with Schwartz kernel  $e^{ik(x-y)}r^0(x, y, k)$ ,  $x, y \in \partial D$ , is continuous on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ . Hence  $R^0(k)$  is also a linear continuous operator on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ .

If we show that  $\Phi(E) - \Phi_{V^0}(E)$  is a compact operator on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ , formula (III.85) will imply that integral operator in formula (III.41) is compact.

Let  $S$  be the operator defined in part 3 of Lemma III.3. Define  $S_{V^0}$  by the same formula using coefficients  $A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0$  in (III.80).

It follows from Lemma III.3, part 3, that  $S, S_{V^0}$  are linear continuous operators from  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  to  $C^1(\overline{D}, M_n(\mathbb{C}))$ .

Taking into account equation (III.80), we obtain the following formula :

$$\Phi(E) - \Phi_{V^0}(E) = NS - N_{V^0}S_{V^0}, \quad (\text{III.86})$$

where  $N, N_{V^0}$  are the linear continuous operators acting from  $C^1(\overline{D}, M_n(\mathbb{C}))$  to  $C^2(\partial D, M_n(\mathbb{C}))$  defined by the following formulas :

$$(N\psi)(x) = \int_D \frac{\partial \Gamma}{\partial \nu_x}(x, y, k) \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) \right) \psi(y) dy,$$

$$(N_{V^0}\psi)(x) = \int_D \frac{\partial \Gamma}{\partial \nu_x}(x, y, k) V^0(y) \psi(y) dy,$$

where  $\Gamma$  is the Green function for Dirichlet problem for operator  $\Delta + E$  in  $D$  and  $\nu_x$  denotes the unit exterior normal to  $\partial D$  at  $x \in \partial D$ .

Taking into account that inclusion  $C^2(\partial D, M_n(\mathbb{C})) \hookrightarrow C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  is compact, we obtain that  $N, N_{V^0}$  are compact operators from  $C^1(\overline{D}, M_n(\mathbb{C}))$  to  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ . It follows from continuity of  $S, S_{V^0}$ , from compactness of  $N, N_{V^0}$  and from formula (III.86) that  $\Phi(E) - \Phi_{V^0}(E)$  is a compact operator on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ .

Now formula (III.85) implies that the integral operator in equation (III.41) is compact on  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ .

In a similar way it can be shown that the integral operators in equations (III.44), (III.47) are compact.



### 4.3 Unique solvability of equations (III.41), (III.44), (III.47)

In this subsection we will finish the proof of the part 1 of Theorem III.2'. The proof of parts 2, 3 of Theorem III.2' can be finished in a similar way.

In fact, in this subsection we will prove the following lemma.

**Lemma III.6.** *Let  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$ ,  $k^2 = E$ , and  $0 < \beta < 1$  be fixed. Suppose that assumptions of Theorem III.1' are fulfilled. Then equation (III.41) is uniquely solvable for  $\psi \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  if and only if equation (III.67) is uniquely solvable for  $\psi(x, k) = e^{ikx}\mu(x, k)$  with  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$ .*

Lemma III.6 implies the statement of part 1 of Theorem III.2' since equation (III.67) at fixed  $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}_{V^0})$  is uniquely solvable for  $\psi(x, k) = e^{ikx}\mu(x, k)$  with  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$  if and only if  $k \notin \mathcal{E}$ .

Before passing to the proof of Lemma III.6, note that function  $\psi^0(x, k)$  of (III.41) is defined as the solution to (III.11) with coefficients  $A_1^0 \equiv 0, \dots, A_d^0 \equiv 0, V^0$ , such that  $\psi^0(x, k) = e^{ikx}\mu^0(x, k)$ ,  $\mu^0 \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$ . Note that function  $\psi^0(x, k)$  belongs to  $C^2(\mathbb{R}^d, M_n(\mathbb{C}))$  and satisfies the equation

$$-\Delta\psi^0 + V^0(x)\psi^0 = E\psi^0, \quad x \in \mathbb{R}^d. \quad (\text{III.87})$$

The proof of this fact is similar to the proof of parts 1, 2 of Lemma III.4.

We will prove Lemma III.6 in two steps.

1. *Extending a solution to equation (III.41) to a solution to equation (III.67).* Let  $\varphi \in C^{1,\beta}(D, M_n(\mathbb{C}))$ ,  $0 < \beta < 1$ , be a solution to equation (III.41). We will show that  $\varphi$  gives rise to a solution  $\psi(x, k) = e^{ikx}\mu(x, k)$ ,  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$  to equation (III.67).

Let  $\varphi^+ \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$  be the unique solution to (III.79) given by Lemma III.3. Define  $\varphi^-$  by formula

$$\varphi^-(x) = \psi^0(x, k) + \int_{\partial D} A^0(x, y, k)\varphi(y) dy, \quad x \in \mathbb{R}^d \setminus D. \quad (\text{III.88})$$

Using formulas (III.41), (III.42), (III.49), we obtain the following formula :

$$\begin{aligned} \int_D R^0(x, y, k) \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \varphi^+(y) dy \\ = \int_{\partial D} A(x, y, k) \varphi(y) dy, \end{aligned} \quad (\text{III.89})$$

where  $x \in \mathbb{R}^d \setminus \overline{D}$ . It follows from (III.35), (III.37) that formula (III.89) holds for  $x \in \mathbb{R}^d \setminus D$ . Using formulas (III.88), (III.89), we obtain formula

$$\begin{aligned} \varphi^-(x) &= \psi^0(x, k) + \int_D R^0(x, y, k) \times \\ &\times \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \varphi^+(y) dy, \end{aligned} \quad (\text{III.90})$$

where  $x \in \mathbb{R}^d \setminus D$ . It follows from (III.35), (III.36), (III.37), (III.90) that  $\varphi^- \in C^1(\mathbb{R}^d \setminus D, M_n(\mathbb{C}))$ .

Note that the following formula holds :

$$\begin{aligned} &\int_D R^0(x, y, k) \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \varphi^+(y) dy \\ &= \int_D \delta_x(y) \varphi^+(y) dy + \int_{\partial D} \left( R^0(x, y, k) \frac{\partial \varphi^+(y)}{\partial \nu_y} - \frac{\partial R^0}{\partial \nu_y}(x, y, k) \varphi(y) \right) dy, \end{aligned} \quad (\text{III.91})$$

where  $x \notin \partial D$ .

Formula (III.91) follows from the following chain of equalities :

$$\begin{aligned} &\int_D R^0(x, y, k) \left( -2i \sum_{j=1}^d A_j(y) \partial_{y_j} + V(y) - V^0(y) \right) \varphi^+(y) dy \\ &\stackrel{(\text{III.79})}{=} \int_D R^0(x, y, k) (\Delta + E - V^0(y)) \varphi^+(y) dy \\ &\stackrel{(\text{III.54})}{=} \int_D (\Delta_y + E - V^0(y)) R^0(x, y, k) \varphi^+(y) dy \\ &+ \int_{\partial D} \left( R^0(x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) - \frac{\partial R^0}{\partial \nu_y}(x, y, k) \varphi(y) \right) dy \\ &\stackrel{(\text{III.74})}{=} \int_D \delta_x(y) \varphi^+(y) dy + \int_{\partial D} \left( R^0(x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) - \frac{\partial R^0}{\partial \nu_y}(x, y, k) \varphi(y) \right) dy. \end{aligned}$$

Using formulas (III.90), (III.91), we obtain formula

$$\varphi^-(x) = \psi^0(x, k) + \int_{\partial D} \left( R^0(x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) - \frac{\partial R^0}{\partial \nu_y}(x, y, k) \varphi(y) \right) dy, \quad (\text{III.92})$$

where  $x \in \mathbb{R}^d \setminus \overline{D}$ .

It follows from formulas (III.32), (III.35), (III.37) and from the corresponding properties of single and double layer potentials that the following formulas are valid :

$$\begin{aligned} \int_{\partial D} R^0(x + 0\nu_x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) dy &= \int_{\partial D} R^0(x - 0\nu_x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) dy, \\ \int_{\partial D} \frac{\partial R^0}{\partial \nu_y}(x + 0\nu_x, y, k) \varphi(y) dy &= -\varphi(x) + \int_{\partial D} \frac{\partial R^0}{\partial \nu_y}(x - 0\nu_x, y, k) \varphi(y) dy, \end{aligned} \quad (\text{III.93})$$

where  $\nu_x$  is the unit exterior normal to  $\partial D$  at  $x \in \partial D$ , and the argument  $x + 0\nu_x$  (or  $x - 0\nu_x$ ) means that we evaluate function at  $x + \varepsilon\nu_x$  (or  $x - \varepsilon\nu_x$ ),  $\varepsilon > 0$ , and then pass  $\varepsilon \rightarrow +0$ .

Besides, the following equality follows from (III.74), (III.91) and from definition of function  $\varphi^+$  :

$$\begin{aligned} (\Delta_x - V^0(x) + E) \left( \int_{\partial D} R^0(x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) dy \right. \\ \left. - \int_{\partial D} \frac{\partial R^0}{\partial \nu_y}(x, y, k) \varphi(y) dy \right) = 0, \end{aligned} \quad (\text{III.94})$$

where  $x \in D$ .

Formulas (III.87), (III.94) imply that the function on the right hand side of equation (III.92) is annihilated by operator  $\Delta - V^0 + E$  in  $D$  and formula (III.93) together with property  $\varphi^-|_{\partial D} = \varphi$  imply that this function has limit zero as  $x$  approaches  $\partial D$  from  $D$ . Since  $E$  is not a Dirichlet eigenvalue for operator  $L_{V^0} \equiv -\Delta + V^0$  in  $D$  the following formula is valid :

$$\psi^0(x, k) + \int_{\partial D} \left( R^0(x, y, k) \frac{\partial \varphi^+}{\partial \nu_y}(y) - \frac{\partial R^0}{\partial \nu_y}(x, y, k) \varphi(y) \right) dy = 0, \quad (\text{III.95})$$

where  $x \in D$ .

Define

$$\psi(x) = \begin{cases} \varphi^-(x), & x \in \mathbb{R}^d \setminus \overline{D}, \\ \varphi(x), & x \in \partial D, \\ \varphi^+(x), & x \in D. \end{cases} \quad (\text{III.96})$$

It follows from formulas (III.91), (III.95) together with formula (III.90) that  $\psi$  satisfies (III.67) in  $\mathbb{R}^d$ .

Using (III.32), (III.35), (III.37), (III.67), we obtain that  $\psi \in C^1(\mathbb{R}^d, M_n(\mathbb{C}))$ .

From formulas (III.36), (III.37) it follows that functions  $r^0(x, y, k)$ ,  $\partial_{x_j} r^0(x, y, k)$ ,  $j = 1, \dots, d$ , are uniformly bounded for  $y \in D$ ,  $x \notin D$ ,  $\text{dist}(x, \partial D) > 1$ . This fact

together with the property that  $\psi^0(x, k) = e^{ikx}\mu^0(x, k)$ ,  $\mu^0 \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$  and with formula (III.67) imply that  $\psi(x, k) = e^{ikx}\mu(x, k)$  with  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$ .

Thus, we have shown that if equation (III.41) has a solution  $\varphi \in C^{1,\beta}(\partial D, M_n(\mathbb{C}))$ , then equation (III.67) has a solution  $\psi(x, k) = e^{ikx}\mu(x, k)$  with  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$  and  $\psi|_{\partial D} = \varphi$ . It follows from the latter property that different solutions to (III.41) give rise to different solutions to (III.67). More precisely, if  $\psi'$ ,  $\psi''$  are two solutions to (III.67), then  $\psi'|_{\partial D} = \psi''|_{\partial D} = \varphi$ . It follows that  $\psi'$ ,  $\psi''$  are two solutions to (III.79) and hence  $\psi'|_D = \psi''|_D$ . Finally, it follows from (III.67) that  $\psi' = \psi''$ .

2. *Restricting a solution to equation (III.67) to a solution to equation (III.41).* Let  $\psi(x, k) = e^{ikx}\mu(x, k)$ ,  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$ , be a solution to (III.67). It follows from inclusion  $W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C})) \subset C(\mathbb{R}^d, M_n(\mathbb{C}))$  that  $\varphi = \psi|_{\partial D}$  is a continuous function on  $\partial D$ . Repeating the proof of Theorem III.1', we can see that  $\varphi$  is a solution to (III.41). We are going to show that  $\varphi \in C^2(\partial D, M_n(\mathbb{C}))$ .

It follows from formula (III.67) that  $\psi \in C^1(\mathbb{R}^d, M_n(\mathbb{C}))$ . Since  $A_1, \dots, A_d, V$  have compact supports in  $D$ , formula (III.67) together with formulas (III.35), (III.37), (III.36) also imply that  $\psi \in C^2(\mathbb{R}^d \setminus D, M_n(\mathbb{C}))$ . Hence  $\varphi$  belongs to  $C^2(\partial D, M_n(\mathbb{C}))$ .

Now if  $\psi'$  and  $\psi''$  are two solutions to (III.67) such that  $\psi'|_{\partial D} \neq \psi''|_{\partial D}$ , then it is clear that they give rise to different solutions to (III.41). If, otherwise,  $\psi'|_{\partial D} = \psi''|_{\partial D}$  then  $\psi' = \psi''$  as was shown in the end of the preceding part of this subsection.

We have shown that solutions  $\varphi$  to (III.41) of class  $C^{1,\beta}(\partial D, M_n(\mathbb{C}))$  are in bijective correspondence with solutions  $\psi$  to (III.67) such that  $\psi(x, k) = e^{ikx}\mu(x, k)$ ,  $\mu \in W^{1,\infty}(\mathbb{R}^d, M_n(\mathbb{C}))$ . This finishes the proof of Lemma III.6.

# Article IV

## Riemann-Hilbert problem approach for two-dimensional flow inverse scattering

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We consider inverse scattering for the time-harmonic wave equation with first-order perturbation in two dimensions. This problem arises in particular in the acoustic tomography of moving fluid. We consider linearized and nonlinearized reconstruction algorithms for this problem of inverse scattering. Our nonlinearized reconstruction algorithm is based on the non-local Riemann–Hilbert problem approach. Comparisons with preceding results are given.

### 1 Introduction

We consider the equation

$$-\Delta\psi - 2iA(x)\nabla\psi + V(x)\psi = E\psi, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad E > 0, \quad (\text{IV.1})$$

where  $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ ,  $\nabla = (\partial_{x_1}, \partial_{x_2})$ ,  $\partial_{x_k} = \partial/\partial x_k$ ,  $k = 1, 2$ , and  $A = (A_1, A_2)$  and  $V$  are vector and scalar potentials on  $\mathbb{R}^2$ , respectively. In addition we assume that

$$\begin{aligned} A_1, A_2 \text{ and } V \text{ are sufficiently regular functions on } \mathbb{R}^2 \\ \text{with sufficient decay at infinity.} \end{aligned} \quad (\text{IV.2})$$

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Equation (IV.1) can be considered as a model equation for the time-harmonic  $\exp(-i\omega t)$  acoustic pressure  $\psi$  in a two-dimensional moving fluid, see, e.g., [116, 118]. In this setting

$$E = \left(\frac{\omega}{c_0}\right)^2, \quad A(x) = \frac{\omega}{c_0}u(x), \quad V(x) = (1 - n^2(x)) \left(\frac{\omega}{c_0}\right)^2, \quad (\text{IV.3})$$

where  $c_0$  is a reference sound speed,  $n(x)$  is a scalar index of refraction,  $u(x)$  is a normalized fluid velocity vector.

Equation (IV.1) can be also considered as the two-dimensional Schrödinger equation at fixed energy  $E$  with magnetic potential  $A$  and electric potential  $v$ , where

$$\begin{aligned} V(x) &= A^2(x) - i\nabla \cdot A(x) + v(x) \\ &= A_1^2(x) + A_2^2(x) - i \left( \frac{\partial A_1(x)}{\partial x_1} + \frac{\partial A_2(x)}{\partial x_2} \right) + v(x), \end{aligned}$$

see, e.g., [109, 43].

For equation (IV.1) we consider the classical scattering solutions  $\psi^+$  continuous and bounded on  $\mathbb{R}^2$  with their first derivatives and specified by the following asymptotics as  $|x| \rightarrow \infty$  :

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + C(|k|) \frac{e^{i|k||x|}}{|x|^{1/2}} f\left(k, |k| \frac{x}{|x|}\right) + o\left(\frac{1}{|x|^{1/2}}\right), \\ x \in \mathbb{R}^2, \quad k \in \mathbb{R}^2, \quad k^2 = E, \quad C(|k|) &= -\pi i \sqrt{2\pi} e^{-i\pi/4} |k|^{1/2}, \end{aligned} \quad (\text{IV.4})$$

where a priori unknown function  $f = f(k, l)$ ,  $k, l \in \mathbb{R}^2$ ,  $k^2 = l^2 = E$ , arising in (IV.4) is the classical scattering amplitude for (IV.1).

Given potentials  $A, V$ , to determine  $\psi^+$  and  $f$  one can use the following Lippmann–Schwinger integral equation (IV.5) and formula (IV.6); see, e.g., [109] :

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + \int_{\mathbb{R}^2} G^+(x - y, k) \times \\ &\quad \times (-2iA(y)\nabla_y \psi^+(y, k) + V(y)\psi^+(y, k)) dy, \\ G^+(x, k) &= -(2\pi)^{-2} \int_{\mathbb{R}^2} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0} = -\frac{i}{4} H_0^1(|x||k|), \end{aligned} \quad (\text{IV.5})$$

where  $x \in \mathbb{R}^2$ ,  $k \in \mathbb{R}^2$ ,  $k^2 = E$ ,  $H_0^1$  is the Hankel function of the first kind;

$$f(k, l) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{-ily} (-2iA(y)\nabla_y \psi^+(y, k) + V(y)\psi^+(y, k)) dy, \quad (\text{IV.6})$$

where  $k \in \mathbb{R}^2$ ,  $l \in \mathbb{R}^2$ ,  $k^2 = l^2 = E$ . Actually, we consider (IV.5) and its differentiated version, where  $\nabla$  is applied to both sides of (IV.5), as a system of linear integral equations for  $\psi^+$  and  $\nabla\psi^+$ .

One can see that the scattering amplitude  $f$  for equation (IV.1) at fixed  $E$  is defined on

$$M_E = \{k \in \mathbb{R}^2, l \in \mathbb{R}^2 : k^2 = l^2 = E\}, \quad E > 0. \quad (\text{IV.7})$$

Note that  $f$  on  $M_E$  is invariant with respect to the gauge transformations

$$\begin{aligned} A &\rightarrow A + \nabla\varphi, \\ V &\rightarrow V - i\Delta\varphi + (\nabla\varphi)^2 + 2A\nabla\varphi, \end{aligned} \quad (\text{IV.8})$$

where  $\varphi$  is a sufficiently regular function on  $\mathbb{R}^2$  with sufficient decay at infinity, see e.g. [71, 43]. In addition,  $\psi^+$  is transformed as

$$\psi^+ \rightarrow e^{-i\varphi}\psi^+ \quad (\text{IV.9})$$

with respect to (IV.8).

In this work we consider the following inverse scattering problem for equation (IV.1) under assumptions (IV.2) :

**Problem IV.1.** *Given scattering amplitude  $f$  on  $M_E$  at fixed  $E > 0$ , find potentials  $A$  and  $V$  on  $\mathbb{R}^2$  (at least approximately).*

Problem IV.1 for the case when  $A \equiv 0$  was studied, in particular, in [96, 57, 98, 60, 100, 18, 21] and in [97, 20].

Problem IV.1 for the general case was considered, in particular, on page 457 of [98] and in [141].

Problem IV.1 is also related with several other inverse problems for the Schrödinger equation in magnetic field (and for the time-harmonic wave equation with first-order perturbation) in dimension  $d \geq 2$ . Concerning these other inverse problems see [35, 125, 71, 95, 116, 118, 15, 42, 93, 139, 61, 74] and references therein.

Note that approximate finding  $A$  and  $V$  in Problem IV.1 means, in particular, finding modulo transformations (IV.8). However, for real-valued  $A$  and  $V$  there is no gauge nonuniqueness (IV.8) in Problem IV.1! In addition,  $A$  and  $V$  of formulas (IV.3) (of moving fluid acoustics) are real if  $n$  is real.

In this work we are mainly motivated by applications to the acoustic tomography of moving fluid described in [95, 116, 118, 15]. Note that in their reconstruction results the works [95, 116, 118, 15] proceed from the near-field scattering data (e.g., from some near field information on  $\psi^+$ ) instead of the scattering amplitude  $f$ . But it is also known that near-field scattering data can be transformed into values of  $f$ , see, e.g., [16, 15].

Results of the present work can be described as follows :

In Section 2 we give formulas for solving Problem IV.1 in the Born approximation. To our knowledge these formulas were not yet given explicitly in the literature for the case when  $A \neq 0$ . These formulas are proved in Section 5.

In Section 3 we give a nonlinearized reconstruction algorithm for Problem IV.1. For the case when  $A \equiv 0$  this algorithm is reduced to the algorithm of [100] and was implemented numerically in [21]. For the general case this algorithm can be also regarded as simplification and development of the algorithm mentioned on page 457 of [98] and based on the Riemann–Hilbert–Manakov problem approach of [58, 96]. A derivation of the reconstruction algorithm of Subsection 3.2 is given in Section 6.

In Section 4 we show that in the Born approximation the algorithm of Section 3 is reduced to formulas of Section 2. Related proofs are given in Section 7.

In a similar way with results of [111] the reconstruction algorithm of Subsection 3.2 can be generalized to the multi-channel case. This generalization will be given in a subsequent work.

In the present work we are focused on approximate reconstructions for Problem IV.1, admitting stable numerical implementation. Issues related with theorems of uniqueness and examples of nonuniqueness for Problem IV.1 will be considered in a subsequent work.

## 2 Inverse scattering in the Born approximation

If  $A = (A_1, A_2)$  and  $V$  satisfy (IV.2) and are sufficiently small, then proceeding from (IV.5), (IV.6) we have the following Born approximation formulas for direct scattering :

$$\begin{aligned} \psi^+(x, k) &\approx e^{ikx}, \quad \nabla\psi^+(x, k) \approx e^{ikx} ik, \\ f(k, l) &\approx f^{\text{lin}}(k, l), \end{aligned} \tag{IV.10}$$

$$f^{\text{lin}}(k, l) \stackrel{\text{def}}{=} (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(k-l)x} (2kA(x) + V(x)) dx, \tag{IV.11}$$

where  $x, k, l \in \mathbb{R}^2$ ,  $k^2 = l^2 = E$ . In particular, formulas (IV.10) can be specified as (IV.21).

Note that  $f^{\text{lin}}$  on  $M_E$  is invariant with respect to the gauge transformations

$$A \rightarrow A + \nabla\varphi, \quad V \rightarrow V - i\Delta\varphi, \tag{IV.12}$$

where  $\varphi$  is a sufficiently regular function on  $\mathbb{R}^2$  with sufficient decay at infinity. This invariance follows from (IV.11), (IV.12), integrating by parts and using that  $k^2 - l^2 = 0$ . We consider (IV.12) as a linearization of (IV.8) for small  $A, V$  and  $\varphi$ .

In this section we consider the following linearized inverse scattering problem for equation (IV.1) under assumptions (IV.2) :

**Problem IV.2.** *Given linearized scattering amplitude  $f^{\text{lin}}$  on  $M_E$  at fixed  $E > 0$ , find potentials  $A$  and  $V$  on  $\mathbb{R}^2$  (at least approximately).*



Note that approximate finding  $A$  and  $V$  in Problem IV.2 means, in particular, finding modulo transformations (IV.12). However, in a similar way with Problem IV.1, there is no gauge nonuniqueness (IV.12) in Problem IV.2 for the case of real-valued  $A$  and  $V$ .

Problem IV.2 is a linearization of Problem IV.1.

To study Problem IV.2 it is convenient to introduce  $\varphi^{\text{div}}$ ,  $A^{\text{div},0}$ ,  $V^{\text{div},0}$  and  $\varphi^\pm$ ,  $A^{\pm,0}$ ,  $V^{\pm,0}$  such that :

$$\begin{aligned} \Delta\varphi^{\text{div}}(x) &= -\nabla \cdot A(x), \quad \varphi^{\text{div}}(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ A^{\text{div},0}(x) &= A(x) + \nabla\varphi^{\text{div}}(x), \quad V^{\text{div},0}(x) = V(x) - i\Delta\varphi^{\text{div}}(x), \end{aligned} \quad (\text{IV.13})$$

where  $x \in \mathbb{R}^2$ ;

$$\begin{aligned} \partial_z\varphi^-(x) &= -\frac{1}{2}(A_1(x) - iA_2(x)), \quad \varphi^-(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ A^{-,0}(x) &= A(x) + \nabla\varphi^-(x), \quad V^{-,0}(x) = V(x) - i\Delta\varphi^-(x), \end{aligned} \quad (\text{IV.14})$$

$$\begin{aligned} \partial_{\bar{z}}\varphi^+(x) &= -\frac{1}{2}(A_1(x) + iA_2(x)), \quad \varphi^+(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ A^{+,0}(x) &= A(x) + \nabla\varphi^+(x), \quad V^{+,0}(x) = V(x) - i\Delta\varphi^+(x), \end{aligned} \quad (\text{IV.15})$$

where

$$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

One can see that

$$\nabla \cdot A^{\text{div},0}(x) = 0, \quad A_1^{-,0}(x) - iA_2^{-,0}(x) = 0, \quad A_1^{+,0}(x) + iA_2^{+,0}(x) = 0,$$

where  $x \in \mathbb{R}^2$ ,  $A^{\pm,0} = (A_1^{\pm,0}, A_2^{\pm,0})$ .

It is also convenient to transform formula (IV.11) to the form

$$\begin{aligned} f^{\text{lin}}(k, l) - f^{\text{lin}}(-l, -k) &= 2(k+l)\widehat{A}(k-l), \\ f^{\text{lin}}(k, l) + f^{\text{lin}}(-l, -k) &= 2(k-l)\widehat{A}(k-l) + 2\widehat{V}(k-l), \end{aligned} \quad (\text{IV.16})$$

$$\widehat{A}(p) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ipx} A(x) dx, \quad \widehat{V}(p) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ipx} V(x) dx, \quad (\text{IV.17})$$

where  $(k, l) \in M_E$ ,  $p \in \mathbb{R}^2$ .

Note that

$$\begin{aligned} (k, l) \in M_E &\implies k-l \in B_{2\sqrt{E}}, \\ p \in B_{2\sqrt{E}} &\implies p = k-l \text{ for some } (k, l) \in M_E, \end{aligned} \quad (\text{IV.18})$$

where

$$B_r = \{p \in \mathbb{R}^2: |p| \leq r\}, \quad r > 0. \quad (\text{IV.19})$$

We define

$$C^{N,\sigma}(\mathbb{R}^2) = \{u \in C^N(\mathbb{R}^2) : \|u\|_{N,\sigma} < +\infty\},$$

$$\|u\|_{N,\sigma} = \max_{|n| \leq N} \sup_{x \in \mathbb{R}^2} (1 + |x|^2)^{\sigma/2} |\partial^n u(x)|, \quad N \in \mathbb{N} \cup \{0\}, \quad \sigma \geq 0, \quad (\text{IV.20})$$

where  $C^N(\mathbb{R}^2)$  is the space of  $N$ -times continuously differentiable complex-valued functions on  $\mathbb{R}^2$ ,

$$\partial^n = \partial_{x_1}^{n_1} \partial_{x_2}^{n_2}, \quad n = (n_1, n_2) \in (\mathbb{N} \cup \{0\})^2, \quad |n| = n_1 + n_2.$$

Note that if  $A_1, A_2, V \in C^{0,\sigma}(\mathbb{R}^2)$  for some  $\sigma > 2$  and  $\|A_j\|_{0,\sigma} \leq q, j = 1, 2, \|V\|_{0,\sigma} \leq q$ , then

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + O(q), \quad \nabla \psi^+(x, k) = e^{ikx} ik + O(q), \\ f(k, l) &= f^{\text{lin}}(k, l) + O(q^2) \quad \text{as } q \rightarrow +0, \end{aligned} \quad (\text{IV.21})$$

uniformly with respect to  $x, k, l \in \mathbb{R}^2, k^2 = l^2 = E$ , at fixed  $E > 0$ .

**Theorem IV.1.** *Suppose that  $A_1, A_2, V$  are real-valued and  $A_1, A_2, V \in C^{N,\sigma}(\mathbb{R}^2)$  for some  $N \geq 3, \sigma > 2$ . Then the following formulas for solving Problem IV.2 are valid :*

$$\begin{aligned} \widehat{A}(k-l) &= \frac{f^{\text{lin}}(k, l) - \overline{f^{\text{lin}}(l, k)}}{2} \frac{k-l}{|k-l|^2} + \frac{f^{\text{lin}}(k, l) - f^{\text{lin}}(-l, -k)}{2} \frac{k+l}{|k+l|^2}, \\ \widehat{V}(k-l) &= \frac{\overline{f^{\text{lin}}(l, k)} + f^{\text{lin}}(-l, -k)}{2}, \end{aligned} \quad (\text{IV.22})$$

where  $\widehat{A}, \widehat{V}$  are defined by (IV.17) and  $(k, l) \in M_E$ ;

$$\begin{aligned} A(x) &= A_{\text{appr}}(x, E) + A_{\text{err}}(x, E), \quad x \in \mathbb{R}^2, \quad E > 0, \\ A_{\text{appr}}(x, E) &= \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \widehat{A}(p) dp, \quad A_{\text{err}}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{A}(p) dp, \end{aligned} \quad (\text{IV.23})$$

$$\begin{aligned} V(x) &= V_{\text{appr}}(x, E) + V_{\text{err}}(x, E), \quad x \in \mathbb{R}^2, \quad E > 0, \\ V_{\text{appr}}(x, E) &= \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \widehat{V}(p) dp, \quad V_{\text{err}}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{V}(p) dp, \end{aligned} \quad (\text{IV.24})$$

where  $\widehat{A}(p)$  and  $\widehat{V}(p)$  for  $|p| \leq 2\sqrt{E}$  are given in terms of  $f^{\text{lin}}$  on  $M_E$  according to (IV.18), (IV.22) and

$$|A_{\text{err},j}(x, E)| \leq c_1(N, \sigma) \|A_j\|_{N,\sigma} E^{-\frac{N-2}{2}}, \quad (\text{IV.25})$$

$$|V_{\text{err}}(x, E)| \leq c_1(N, \sigma) \|V\|_{N,\sigma} E^{-\frac{N-2}{2}}, \quad (\text{IV.26})$$

where  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $A_{err} = (A_{err,1}, A_{err,2})$ ,  $E \geq \frac{1}{4}$  and

$$c_1(N, \sigma) = \frac{4}{(N-2)(\sigma-2)}. \quad (\text{IV.27})$$

**Theorem IV.2.** *Suppose that  $A_1, A_2, V \in C^{N,\sigma}(\mathbb{R}^2)$  for some  $N \geq 4$  and  $\sigma > 2$ . Let  $A^{div,0}, V^{div,0}$  be defined according to (IV.13). Then the following formulas for solving Problem IV.2 are valid :*

$$\begin{aligned} \widehat{A}^{div,0}(k-l) &= \frac{f^{lin}(k,l) - f^{lin}(-l,-k)}{2} \frac{k+l}{|k+l|^2}, \\ \widehat{V}^{div,0}(k-l) &= \frac{f^{lin}(k,l) + f^{lin}(-l,-k)}{2}, \end{aligned} \quad (\text{IV.28})$$

where  $\widehat{A}^{div,0}, \widehat{V}^{div,0}$  are the Fourier transforms of  $A^{div,0}, V^{div,0}$  (see (IV.17)) and  $(k, l) \in M_E$ ;

$$A^{div,0}(x) = A_{appr}^{div,0}(x, E) + A_{err}^{div,0}(x, E), \quad x \in \mathbb{R}^2, \quad E > 0, \quad (\text{IV.29})$$

$$A_{appr}^{div,0}(x, E) = \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \widehat{A}^{div,0}(p) dp, \quad A_{err}^{div,0}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{A}^{div,0}(p) dp,$$

$$V^{div,0}(x) = V_{appr}^{div,0}(x, E) + V_{err}^{div,0}(x, E), \quad x \in \mathbb{R}^2, \quad E > 0, \quad (\text{IV.30})$$

$$V_{appr}^{div,0}(x, E) = \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \widehat{V}^{div,0}(p) dp, \quad V_{err}^{div,0}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{V}^{div,0}(p) dp,$$

where  $\widehat{A}^{div,0}(p)$  and  $\widehat{V}^{div,0}(p)$  for  $|p| \leq 2\sqrt{E}$  are given in terms of  $f^{lin}$  on  $M_E$  according to (IV.18), (IV.28) and

$$|A_{err,j}^{div,0}(x, E)| \leq (1 + \sqrt{2})c_1(N, \sigma) \|A\|_{N,\sigma} E^{-\frac{N-2}{2}}, \quad (\text{IV.31})$$

$$|V_{err}^{div,0}(x, E)| \leq c_1(N, \sigma) \left( \|V\|_{N,\sigma} E^{-\frac{N-2}{2}} + \sqrt{2} \|A\|_{N,\sigma} E^{-\frac{N-3}{2}} \right), \quad (\text{IV.32})$$

$$\|A\|_{N,\sigma} = \max(\|A_1\|_{N,\sigma}, \|A_2\|_{N,\sigma}), \quad (\text{IV.33})$$

where  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $E \geq \frac{1}{4}$ ,  $A_{err}^{div,0} = (A_{err,1}^{div,0}, A_{err,2}^{div,0})$  and  $c_1(N, \sigma)$  is defined by (IV.27). Furthermore, if  $\nabla \cdot A = 0$  then  $A^{div,0} = A$ ,  $V^{div,0} = V$ .

**Theorem IV.3.** *Suppose that  $A_1, A_2, V \in C^{N,\sigma}(\mathbb{R}^2)$  for some  $N \geq 4$  and  $\sigma > 2$ . Let  $A^{\pm,0}, V^{\pm,0}$  be defined according to (IV.14)–(IV.15). Then the following formulas for solving Problem IV.2 are valid :*

$$\begin{aligned} \widehat{A}_1^{\pm,0}(k-l) &= \frac{1}{2} \frac{f(k,l) - f(-l,-k)}{k_1 + l_1 \pm i(k_2 + l_2)}, \quad \widehat{A}_2^{\pm,0}(k-l) = \pm i \widehat{A}_1^{\pm,0}(k-l), \\ \widehat{V}^{\pm,0}(k-l) &= \frac{(l_1 \pm il_2)f(k,l) + (k_1 \pm ik_2)f(-l,-k)}{k_1 + l_1 \pm i(k_2 + l_2)}, \end{aligned} \quad (\text{IV.34})$$

where  $\widehat{A}^{\pm,0}$ ,  $\widehat{V}^{\pm,0}$  are the Fourier transforms of  $A^{\pm,0}$ ,  $V^{\pm,0}$  (see (IV.17)) and  $(k, l) \in M_E$ ;

$$A^{\pm,0}(x) = A_{appr}^{\pm,0}(x, E) + A_{err}^{\pm,0}(x, E), \quad x \in \mathbb{R}^2, \quad E > 0, \quad (\text{IV.35})$$

$$A_{appr}^{\pm,0}(x, E) = \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \widehat{A}^{\pm,0}(p) dp, \quad A_{err}^{\pm,0}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{A}^{\pm,0}(p) dp,$$

$$V^{\pm,0}(x) = V_{appr}^{\pm,0}(x, E) + V_{err}^{\pm,0}(x, E), \quad x \in \mathbb{R}^2, \quad E > 0, \quad (\text{IV.36})$$

$$V_{appr}^{\pm,0}(x, E) = \int_{|p| \leq 2\sqrt{E}} e^{-ipx} \widehat{V}^{\pm,0}(p) dp, \quad V_{err}^{\pm,0}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{V}^{\pm,0}(p) dp,$$

where  $\widehat{A}^{\pm,0}(p)$  and  $\widehat{V}^{\pm,0}(p)$  for  $|p| \leq 2\sqrt{E}$  are given in terms of  $f^{lin}$  on  $M_E$  according to (IV.18), (IV.34) and

$$|A_{err,j}^{\pm,0}(x, E)| \leq (1 + \sqrt{2})c_1(N, \sigma) \|A\|_{N,\sigma} E^{-\frac{N-2}{2}}, \quad (\text{IV.37})$$

$$|V_{err}^{\pm,0}| \leq c_1(N, \sigma) \left( \|V\|_{N,\sigma} E^{-\frac{N-2}{2}} + \sqrt{2} \|A\|_{N,\sigma} E^{-\frac{N-3}{2}} \right), \quad (\text{IV.38})$$

where  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $A_{err}^{\pm,0} = (A_{err,1}^{\pm,0}, A_{err,2}^{\pm,0})$ ,  $\|A\|_{N,\sigma}$  is defined by (2.23) and  $c_1(N, \sigma)$  is given by (IV.27). Furthermore, if  $A_1 \pm iA_2 = 0$  then  $A = A^{\pm,0}$ ,  $V = V^{\pm,0}$ .

Theorems IV.1–IV.3 are proved in Section 5.

## 3 Nonlinearized inverse scattering

### 3.1 Some notations

To study Problem IV.1 it is convenient to introduce  $\varphi^{\text{div}}$ ,  $A^{\text{div}}$ ,  $V^{\text{div}}$  and  $\varphi^{\pm}$ ,  $A^{\pm}$ ,  $V^{\pm}$ , where  $\varphi^{\text{div}}$  and  $\varphi^{\pm}$  are defined according to (IV.13)–(IV.15) and

$$A^{\text{div}} = A + \nabla \varphi^{\text{div}}, \quad V^{\text{div}} = V - i\Delta \varphi^{\text{div}} + (\nabla \varphi^{\text{div}})^2 + 2A \nabla \varphi^{\text{div}},$$

$$A^{\pm} = A + \nabla \varphi^{\pm}, \quad V^{\pm} = V - i\Delta \varphi^{\pm} + (\nabla \varphi^{\pm})^2 + 2A \nabla \varphi^{\pm}. \quad (\text{IV.39})$$

In this section we give a nonlinearized algorithm for approximate finding  $A^{\pm}$ ,  $V^{\pm}$  and  $A^{\text{div}}$ ,  $V^{\text{div}}$  on  $\mathbb{R}^2$  from  $f$  on  $M_E$ . This algorithm takes into account multiple scattering effects and can be regarded as a nonlinear version of formulas for  $A_{appr}^{\pm,0}$ ,  $V_{appr}^{\pm,0}$ ,  $A_{appr}^{\text{div},0}$ ,  $V_{appr}^{\text{div},0}$  of (IV.35), (IV.36), (IV.29), (IV.30).

It is convenient to use the following notations :

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad (\text{IV.40})$$

$$\lambda = E^{-1/2}(k_1 + ik_2), \quad \lambda' = E^{-1/2}(l_1 + il_2), \quad (\text{IV.41})$$

where  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $k = (k_1, k_2) \in \Sigma_E$ ,  $l = (l_1, l_2) \in \Sigma_E$ ,

$$\Sigma_E = \{m = (m_1, m_2) \in \mathbb{C}^2: m_1^2 + m_2^2 = E\}, \quad E > 0. \quad (\text{IV.42})$$

In these notations

$$k_1 = \frac{1}{2}E^{1/2}(\lambda + \lambda^{-1}), \quad k_2 = \frac{i}{2}E^{1/2}(\lambda^{-1} - \lambda), \quad (\text{IV.43})$$

$$l_1 = \frac{1}{2}E^{1/2}(\lambda' + \lambda'^{-1}), \quad l_2 = \frac{i}{2}E^{1/2}(\lambda'^{-1} - \lambda'), \quad (\text{IV.44})$$

$$\exp(ikx) = \exp\left(\frac{i}{2}E^{1/2}(\lambda\bar{z} + \lambda^{-1}z)\right),$$

where  $\lambda, \lambda' \in \mathbb{C} \setminus \{0\}$ ,  $z \in \mathbb{C}^2$ ,  $k, l \in \Sigma_E$ .

In addition, using formulas (IV.7), (IV.41)–(IV.44) one can see that

$$\begin{aligned} \Sigma_E &\cong \mathbb{C} \setminus \{0\}, \\ \Sigma_E \cap \mathbb{R}^2 &= \mathbb{S}_{\sqrt{E}}^1 \cong T, \\ M_E &\cong T \times T, \end{aligned}$$

where

$$\begin{aligned} \mathbb{S}_r^1 &= \{m \in \mathbb{R}^2: |m| = r\}, \quad r > 0, \\ T &= \{\lambda \in \mathbb{C}: |\lambda| = 1\}. \end{aligned}$$

In addition, the functions  $\psi^+$ ,  $f$  of (IV.4)–(IV.6) can be written as

$$\psi^+ = \psi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E),$$

where  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $E > 0$ .

### 3.2 Reconstruction algorithm

Our nonlinearized algorithm for approximate finding  $A^\pm$ ,  $V^\pm$  and  $A^{\text{div}}$ ,  $V^{\text{div}}$  on  $\mathbb{R}^2$  from  $f$  on  $M_E$  has the following scheme

$$f \longrightarrow h_\pm \longrightarrow \mu^+ \longrightarrow \mu_\pm \longrightarrow A_{\text{appr}}^\pm, V_{\text{appr}}^\pm \longrightarrow A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$$

and consists of the following steps :

Step 1. Find functions  $h_\pm(\lambda, \lambda', E)$ ,  $\lambda, \lambda' \in T$ , from the following linear integral equations :

$$\begin{aligned} h_\pm(\lambda, \lambda', E) - \pi i \int_T h_\pm(\lambda, \lambda'', E) \chi\left(\pm i \left[\frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda}\right]\right) \times \\ \times f(\lambda'', \lambda', E) |d\lambda''| = f(\lambda, \lambda', E), \end{aligned} \quad (\text{IV.45})$$

where

$$\chi(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \quad (\text{IV.46})$$

Step 2. Solve the following linear integral equation for  $\mu^+(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in T$ ,  $E > 0$  :

$$\mu^+(z, \lambda, E) + \int_T B(\lambda, \lambda', z, E) \mu^+(z, \lambda', E) |d\lambda'| = 1, \quad (\text{IV.47})$$

where

$$B(\lambda, \lambda', z, E) = \frac{1}{2} \int_T h_-(\zeta, \lambda', z, E) \chi \left( -i \left[ \frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1-0)} - \frac{1}{2} \int_T h_+(\zeta, \lambda', z, E) \chi \left( i \left[ \frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1+0)}, \quad (\text{IV.48})$$

$$h_{\pm}(\lambda, \lambda', z, E) \stackrel{\text{def}}{=} h_{\pm}(\lambda, \lambda', E) \times \exp \left( -i \frac{\sqrt{E}}{2} ((\lambda - \lambda') \bar{z} + (\lambda^{-1} - \lambda'^{-1}) z) \right), \quad (\text{IV.49})$$

and  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $E > 0$ .

Step 3. Define functions  $\mu_{\pm}(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in T$ ,  $E > 0$ , by formulas

$$\begin{aligned} \mu_{\pm}(z, \lambda, E) &= \mu^+(z, \lambda, E) + \pi i \int_T h_{\pm}(\lambda, \lambda', z, E) \times \\ &\times \chi \left( \pm i \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) \mu^+(z, \lambda', E) |d\lambda'|, \end{aligned} \quad (\text{IV.50})$$

where functions  $h_{\pm}(\lambda, \lambda', z, E)$  are given by (IV.49) and  $\chi$  is defined by (IV.46).

Step 4. Functions  $A_{\text{appr},j}^{\pm}(x, E)$ ,  $V_{\text{appr}}^{\pm}(x, E)$ ,  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $E > 0$ , are defined by formulas

$$\begin{aligned} A_{\text{appr},1}^{-}(x, E) &= \frac{i}{4} a_z^{-}(z, E), \quad A_{\text{appr},2}^{-}(x, E) = \frac{1}{4} a_z^{-}(z, E), \\ a_z^{-}(z, E) &= 4 \partial_{\bar{z}} \ln \int_T \mu_+(z, \zeta, E) |d\zeta|, \\ V_{\text{appr}}^{-}(x, E) &= \frac{\sqrt{E}}{\pi} \int_T \partial_z \mu_-(z, \zeta, E) d\zeta, \end{aligned} \quad (\text{IV.51})$$

and

$$\begin{aligned}
A_{\text{appr},1}^+(x, E) &= \frac{i}{4}a_{\bar{z}}^+(z, E), & A_{\text{appr},2}^+(x, E) &= -\frac{1}{4}a_{\bar{z}}^+(z, E), \\
a_{\bar{z}}^+(z, E) &= -4\partial_z \ln \int_T \mu_+(z, \zeta, E) |d\zeta|, \\
V_{\text{appr}}^+(z, E) &= 2i\sqrt{E}\partial_{\bar{z}} \left( \int_T \mu_+(z, \zeta, E) \frac{d\zeta}{\zeta^2} \Big/ \int_T \mu_+(z, \zeta, E) \frac{d\zeta}{\zeta} \right),
\end{aligned} \tag{IV.52}$$

where  $z$  is given by (IV.40).

Step 5. Find  $A_{\text{appr},j}^{\text{div}}(x, E)$ ,  $V_{\text{appr}}^{\text{div}}(x, E)$ ,  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $E > 0$ , from formulas

$$\begin{aligned}
A_{\text{appr},1}^{\text{div}}(x, E) &= \frac{i}{8}(a_z^-(z, E) + a_{\bar{z}}^+(z, E)), \\
A_{\text{appr},2}^{\text{div}}(x, E) &= \frac{1}{8}(a_z^-(z, E) - a_{\bar{z}}^+(z, E)), \\
V_{\text{appr}}^{\text{div}}(x, E) &= \frac{1}{2}(V_{\text{appr}}^-(x, E) + V_{\text{appr}}^+(z, E)) - \frac{1}{8}a_z^-(z, E)a_{\bar{z}}^+(z, E),
\end{aligned} \tag{IV.53}$$

where  $z$  is defined by (IV.40) and functions  $a_z^-$ ,  $a_{\bar{z}}^+$ ,  $V_{\text{appr}}^\pm$  are defined in (IV.51), (IV.52).

A derivation of this reconstruction algorithm is based on the method of the Riemann–Hilbert problem and on the  $\bar{\partial}$ -method. This derivation is given in Section 6.

For the case when  $A \equiv 0$  this algorithm is reduced to the algorithm of [100] for approximate finding  $V$  on  $\mathbb{R}^2$  from  $f$  on  $M_E$ . The algorithm of [100] consists of the same aforementioned steps 1, 2, 3 and the formula  $V_{\text{appr}} = V_{\text{appr}}^-$ , where  $V_{\text{appr}}^-$  is defined in (IV.51). This algorithm of [100] was implemented numerically in [21].

For the general case this algorithm can be also regarded as simplification and development of the algorithm mentioned (in few lines) on page 457 of [98]. Actually, in [98] the part of the algorithm consisting in finding  $\mu_\pm$  from  $h_\pm$  is realized in a more complicated way. In addition, in [98] the algorithm is mentioned for the case when

$$A_1 = \bar{A}_1, \quad A_2 = \bar{A}_2, \quad -2i\nabla \cdot \bar{A} + \bar{V} = V,$$

i.e. for the self-adjoint case, whereas this assumption is not necessary for the algorithm.

### 3.3 Properties of the algorithm

Let

$$\begin{aligned}
\|u_1\|_{L^2(T)} &= \left( \int_T |u_1(\lambda)|^2 |d\lambda| \right)^{1/2}, \\
\|u_2\|_{L^2(T^2)} &= \left( \int_{T^2} |u_2(\lambda, \lambda')|^2 |d\lambda| |d\lambda'| \right)^{1/2}, \quad T^2 = T \times T,
\end{aligned}$$

where  $u_1$  and  $u_2$  are test functions on  $T$  and  $T^2$ , respectively.

**Proposition IV.1.** *Let  $E > 0$  be fixed. Suppose that*

$$f \in L^2(T^2), \quad \|f\|_{L^2(T^2)} < \frac{1}{\pi}, \quad (\text{IV.54})$$

where  $f = f(\lambda, \lambda', E)$ . Then equation (IV.45) is uniquely solvable for  $h_{\pm} \in L^2(T^2)$  and

$$\begin{aligned} \|h_{\pm}\|_{L^2(T^2)} &< \frac{\|f\|_{L^2(T^2)}}{1 - \pi\|f\|_{L^2(T^2)}}, \\ \|B\|_{L^2(T^2)} &< \frac{2\pi\|f\|_{L^2(T^2)}}{1 - \pi\|f\|_{L^2(T^2)}}, \end{aligned}$$

where  $B$  is defined by (IV.48), (IV.49) (at fixed  $z, E$ ). In addition, if

$$\|f\|_{L^2(T^2)} < \frac{1}{3\pi}, \quad (\text{IV.55})$$

then  $\|B\|_{L^2(T^2)} < 1$ , equation (IV.47) at fixed  $z, E$  is uniquely solvable for  $\mu^+ \in L^2(T)$  and

$$\begin{aligned} \|\mu^+\|_{L^2(T)} &< \frac{(2\pi)^{1/2}}{1 - \|B\|_{L^2(T^2)}}, \quad \|\mu^+ - 1\|_{L^2(T)} < \frac{(2\pi)^{1/2}\|B\|_{L^2(T^2)}}{1 - \|B\|_{L^2(T^2)}}, \\ \|\mu_{\pm} - 1\|_{L^2(T)} &< \frac{3\pi(2\pi)^{1/2}\|f\|_{L^2(T^2)}}{1 - 3\pi\|f\|_{L^2(T^2)}}, \end{aligned}$$

where  $\mu_{\pm}$  are defined by (IV.50). In addition, at least, if

$$\|f\|_{L^2(T^2)} < \frac{1}{6\pi}, \quad (\text{IV.56})$$

then

$$\int_T \mu_+(z, \lambda, E) |d\lambda| \neq 0 \quad \text{for all } z \in \mathbb{C}, \quad (\text{IV.57})$$

and  $A_{\text{appr},j}^{\pm}$ ,  $A_{\text{appr},j}^{\text{div}}$ ,  $j = 1, 2$ , as well as  $V_{\text{appr}}^{\pm}$ ,  $V_{\text{appr}}^{\text{div}}$  are bounded on  $\mathbb{R}^2$ .

Proposition IV.1 is based on solving the linear integral equations (IV.45) and (IV.47) by the method of successive approximations in  $L^2(T^2)$  and  $L^2(T)$ , respectively, and on standard estimates of  $L^2$ -analysis for  $B$ ,  $h_{\pm}$ ,  $\mu_{\pm}$  of (IV.48)–(IV.50) and for the integral of (IV.57).

Note that assumptions (IV.54), (IV.55), (IV.56) of Proposition IV.1 are only some surplus sufficient conditions on  $f$  for unique solvability of integral equations (IV.45), (IV.47), fulfilment of (IV.57) and for boundedness of  $A_{\text{appr}}^{\text{div}}$ ,  $V_{\text{appr}}^{\text{div}}$ .



**Theorem IV.4.** *Let  $f \in L^2(T^2)$  at fixed  $E > 0$ . Suppose that  $f$  satisfies (IV.56) and is a smooth function on  $T^2$  and  $A_{\text{appr}}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  are constructed from  $f$  via the algorithm of Subsection 3.2. Then  $A_{\text{appr},1}^{\text{div}}, A_{\text{appr},2}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  are bounded functions on  $\mathbb{R}^2$ , decaying at infinity. In addition,  $f$  is the scattering amplitude for equation (IV.1) with  $A = A_{\text{appr}}^{\text{div}}(x, E)$ ,  $V = V_{\text{appr}}^{\text{div}}(x, E)$ .*

For simplicity one can assume that  $f \in C^\infty(T^2)$  in Theorem IV.4. However, very limited smoothness of  $f$  is already sufficient. As regards to smoothness of  $A_{\text{appr},1}^{\text{div}}, A_{\text{appr},2}^{\text{div}}, V_{\text{appr}}^{\text{div}}$  of Theorem IV.4 (which are complex-valued, in general), these functions are real-analytic functions of  $x \in \mathbb{R}^2$ . In addition, it is just for simplicity that we assume IV.56 in Theorem IV.4.

The proof of Theorem IV.4 is similar to the proof of Theorem 9.2 of [98] for the case when  $A \equiv 0$ . Results of this type go back to [96]. In the present work restrictions in time prevent us from proving Theorem IV.4 in details.

Finally, suppose that  $f = f(\lambda, \lambda', E)$  is the scattering amplitude for equation (IV.1) under assumptions (IV.2) and that  $A_{\text{appr}}^{\text{div}} = A_{\text{appr}}^{\text{div}}(x, E)$ ,  $V_{\text{appr}}^{\text{div}} = V_{\text{appr}}^{\text{div}}(x, E)$  are constructed from  $f$  via the algorithm of Subsection 3.2 at fixed  $E$ . In the present work restrictions on time prevent us from obtaining estimates for  $A^{\text{div}} - A_{\text{appr}}^{\text{div}}(\cdot, E)$  and  $V^{\text{div}} - V_{\text{appr}}^{\text{div}}(\cdot, E)$  for sufficiently large  $E$ . For the linearized case such error estimates are given by formulas (IV.31), (IV.32). For the nonlinearized case with  $A \equiv 0$  such error estimates were given in [100].

## 4 Reduction of the nonlinearized reconstruction algorithm to inversion formulas of the Born approximation

Suppose that we are given  $f$  on  $M_E \cong T \times T = T^2$  at fixed  $E$ , where

$$f \in L^2(T^2), \quad \|f\|_{L^2(T^2)} \leq \varepsilon. \quad (\text{IV.58})$$

**Proposition IV.2.** *Suppose that  $f$  satisfies (IV.58) at fixed  $E > 0$ . Then, for  $\varepsilon \rightarrow +0$ , the nonlinearized reconstruction algorithm of Subsection 3.2 is reduced to the following formulas at fixed  $E > 0$  :*

$$\begin{aligned} A_{\text{appr},j}^\pm(x, E) &= \mathcal{A}_{\text{appr},j}^\pm(x, E) + O(\varepsilon^2), \quad j = 1, 2, \\ V_{\text{appr}}^\pm(x, E) &= \mathcal{V}_{\text{appr}}^\pm(x, E) + O(\varepsilon^2), \end{aligned} \quad (\text{IV.59})$$

$$\begin{aligned} A_{\text{appr}}^{\text{div}}(x, E) &= \mathcal{A}_{\text{appr}}^{\text{div}}(x, E) + O(\varepsilon^2), \quad j = 1, 2, \\ V_{\text{appr}}^{\text{div}}(x, E) &= \mathcal{V}_{\text{appr}}^{\text{div}}(x, E) + O(\varepsilon^2), \end{aligned} \quad (\text{IV.60})$$

where  $O(\varepsilon^2)$  is considered in the uniform sense with respect to  $x \in \mathbb{R}^2$  and where functions  $\mathcal{A}_{appr,j}^\pm$ ,  $\mathcal{V}_{appr}^\pm$ ,  $j = 1, 2$ , and  $\mathcal{A}_{appr,j}^{div}$ ,  $\mathcal{V}_{appr}^{div}$ ,  $j = 1, 2$ , are defined by the following linear formulas with respect to  $f$  :

$$\begin{aligned} \mathcal{A}_{appr,1}^-(x, E) &= -\frac{i}{4}\sqrt{E} \int_{T^2} \operatorname{sgn} \left( \frac{1}{i} \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) (\lambda - \lambda') f(\lambda, \lambda', z, E) |d\lambda| |d\lambda'|, \\ \mathcal{A}_{appr,2}^-(x, E) &= -i\mathcal{A}_{appr,1}^-(x, E), \end{aligned} \quad (\text{IV.61})$$

$$\begin{aligned} \mathcal{V}_{appr}^-(x, E) &= i\frac{E}{2} \int_{T^2} (1 - \lambda\bar{\lambda}') \operatorname{sgn} \left( \frac{1}{i} \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) f(\lambda, \lambda', z, E) |d\lambda| |d\lambda'|, \\ \mathcal{A}_{appr,1}^+(x, E) &= \frac{i}{4}\sqrt{E} \int_{T^2} \operatorname{sgn} \left( \frac{1}{i} \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) \overline{(\lambda - \lambda')} f(\lambda, \lambda', z, E) |d\lambda| |d\lambda'|, \\ \mathcal{A}_{appr,2}^+(x, E) &= i\mathcal{A}_{appr,1}^+(x, E), \end{aligned} \quad (\text{IV.62})$$

$$\begin{aligned} \mathcal{V}_{appr}^+(x, E) &= -i\frac{E}{2} \int_{T^2} (1 - \bar{\lambda}\lambda') \operatorname{sgn} \left( \frac{1}{i} \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) f(\lambda, \lambda', z, E) |d\lambda| |d\lambda'|, \\ \mathcal{A}_{appr,j}^{div}(x, E) &= \frac{1}{2} (\mathcal{A}_{appr,j}^+(x, E) + \mathcal{A}_{appr,j}^-(x, E)), \quad j = 1, 2, \\ \mathcal{V}_{appr}^{div}(x, E) &= \frac{E}{2} \int_{T^2} \left| \frac{1}{2i} \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right| f(\lambda, \lambda', z, E) |d\lambda| |d\lambda'|, \end{aligned} \quad (\text{IV.63})$$

where

$$\begin{aligned} &f(\lambda, \lambda', z, E) \stackrel{\text{def}}{=} f(\lambda, \lambda', E) \times \\ &\times \exp \left( -i\frac{\sqrt{E}}{2} ((\lambda - \lambda')\bar{z} + (\lambda^{-1} - \lambda'^{-1})z) \right), \end{aligned} \quad (\text{IV.64})$$

$\lambda \in T$ ,  $\lambda' \in T$  and  $z, \bar{z}$  are given by (IV.40).

**Proposition IV.3.** *Suppose that  $A_1, A_2, V \in C^{2,\sigma}(\mathbb{R}^2)$  for some  $\sigma > 2$ , where  $C^{N,\sigma}(\mathbb{R}^2)$  is defined by (IV.20). Let  $f^{lin}$  be defined by (IV.11) and let  $A_{appr,j}^{div,0}$ ,  $V_{appr}^{div,0}$ ,  $A_{appr,j}^{\pm,0}$ ,  $V_{appr}^{\pm,0}$ ,  $j = 1, 2$ , be defined by (IV.28), (IV.29), (IV.30), (IV.34), (IV.35), (IV.36) in terms of  $f^{lin}$ . Suppose also that functions  $\mathcal{A}_{appr,j}^{div}$ ,  $\mathcal{V}_{appr}^{div}$ ,  $\mathcal{A}_{appr,j}^\pm$ ,  $\mathcal{V}_{appr}^\pm$ ,  $j = 1, 2$ , are given by (IV.63), (IV.61), (IV.62) with  $f = f^{lin}$ . Then the following equalities are valid :*

$$\begin{aligned} A_{appr,j}^{div,0}(x, E) &= \mathcal{A}_{appr,j}^{div}(x, E), \\ V_{appr}^{div,0}(x, E) &= \mathcal{V}_{appr}^{div}(x, E), \end{aligned} \quad (\text{IV.65})$$

$$\begin{aligned} A_{appr,j}^{\pm,0}(x, E) &= \mathcal{A}_{appr,j}^\pm(x, E), \\ V_{appr}^{\pm,0}(x, E) &= \mathcal{V}_{appr}^\pm(x, E), \end{aligned} \quad (\text{IV.66})$$

where  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $E > 0$ .

Propositions IV.2 and IV.3 are proved in Section 7.

## 5 Proofs of Theorems IV.1, IV.2, IV.3

Let us use the notations

$$\widehat{u}(p) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ipx} u(x) dx, \quad u_{\text{err}}(x, E) = \int_{|p| \geq 2\sqrt{E}} e^{-ipx} \widehat{u}(p) dp, \quad (\text{IV.67})$$

where  $p \in \mathbb{R}^2$ ,  $x \in \mathbb{R}^2$ ,  $E > 0$ .

**Lemma IV.1.** *Let  $u \in C^{N,\sigma}(\mathbb{R}^2)$ , where  $N \geq 3$ ,  $\sigma > 2$ . Then the following formula holds :*

$$|u_{\text{err}}(x, E)| \leq c_1(N, \sigma) \|u\|_{N,\sigma} E^{-\frac{N-2}{2}}, \quad (\text{IV.68})$$

where  $x \in \mathbb{R}^2$ ,  $E \geq 1/2$  and  $c_1(N, \sigma)$  is defined by (IV.27).

*Proof of Lemma IV.1.* We have that

$$\widehat{\partial^n u}(p) = (-ip_1)^{n_1} (-ip_2)^{n_2} \widehat{u}(p),$$

where  $\partial^n$  is defined in (2.13),  $p = (p_1, p_2) \in \mathbb{R}^2$ ,  $n = (n_1, n_2) \in (\mathbb{N} \cup \{0\})^2$ ,  $|n| \leq N$ . Using this equality we obtain that

$$|\widehat{u}(p)| \leq \frac{2^{N-1}}{\pi(\sigma-2)} \|u\|_{N,\sigma} (1 + |p|^2)^{-\frac{N}{2}}$$

for each  $p \in \mathbb{R}^2$ ,  $|p| \geq 1$ . Using the latter inequality we obtain (IV.68).  $\square$

*Proof of Theorem IV.1.* Since potentials  $A$  and  $V$  are real-valued it follows from (IV.16) that the following formula holds :

$$f^{\text{lin}}(k, l) - \overline{f^{\text{lin}}(l, k)} = 2(k-l) \widehat{A}(k-l), \quad (\text{IV.69})$$

where  $(k, l) \in M_E$ . We consider (IV.16), (IV.69) as a system of linear equations for finding  $\widehat{A}(k-l)$  and  $\widehat{V}(k-l)$ . In addition, we use that  $(k-l)(k+l) = 0$  for  $(k, l) \in M_E$ , i.e. that vectors  $(k-l)$  and  $(k+l)$  are orthogonal. As a result we obtain formulas (IV.22).

Formulas (IV.23), (IV.24) can be regarded as definitions of  $A_{\text{appr}}$ ,  $A_{\text{err}}$ ,  $V_{\text{appr}}$ ,  $V_{\text{err}}$ . Estimates (IV.25), (IV.26) follow from Lemma IV.1.  $\square$

**Lemma IV.2.** *Suppose that  $A_1, A_2 \in C^{N,\sigma}(\mathbb{R}^2)$ , where  $N \geq 4$ ,  $\sigma > 2$ . Let  $\varphi^{div}$  be the solution of (IV.13). Let  $\widehat{\nabla}\varphi^{div}$ ,  $(\nabla\varphi^{div})_{err}$ ,  $\widehat{\Delta}\varphi^{div}$ ,  $(\Delta\varphi^{div})_{err}$  be defined according to (IV.67). Then the following estimates hold :*

$$\begin{aligned} |(\partial_j\varphi^{div})_{err}(x, E)| &\leq \sqrt{2} c_1(N, \sigma) \|A\|_{N,\sigma} E^{-\frac{N-2}{2}}, \\ |(\Delta\varphi^{div})_{err}(x, E)| &\leq \sqrt{2} c_1(N, \sigma) \|A\|_{N,\sigma} E^{-\frac{N-3}{2}}, \end{aligned} \quad (\text{IV.70})$$

where  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $E \geq 1/4$ ,  $\|A\|_{N,\sigma}$  is defined by (IV.33) and  $c_1(N, \sigma)$  is defined by (IV.27).

*Proof of Lemma IV.2.* The solution  $\varphi^{div}$  of (IV.13) is given by :

$$\varphi^{div}(x) = -i \int_{\mathbb{R}^2} e^{-ipx} (p\widehat{A}(p)) |p|^{-2} dp, \quad x \in \mathbb{R}^2. \quad (\text{IV.71})$$

Using (IV.67), (IV.71) we obtain that

$$\widehat{\nabla}\varphi^{div}(p) = -p(p\widehat{A}(p)) |p|^{-2}, \quad \widehat{\Delta}\varphi^{div}(p) = ip\widehat{A}(p), \quad (\text{IV.72})$$

where  $p \in \mathbb{R}^2 \setminus \{0\}$ . Formulas (IV.72) imply the following inequalities :

$$|\widehat{\partial_j}\varphi^{div}(p)| \leq \sqrt{2} \max_{k=1,2} |\widehat{A}_k(p)|, \quad |\widehat{\Delta}\varphi^{div}(p)| \leq \sqrt{2} |p| \max_{k=1,2} |\widehat{A}_k(p)|, \quad (\text{IV.73})$$

where  $p \in \mathbb{R}^2 \setminus \{0\}$ ,  $j = 1, 2$ .

We have that

$$\widehat{\partial^n A_j}(p) = (-ip_1)^{n_1} (-ip_2)^{n_2} \widehat{A}_j(p),$$

where  $p = (p_1, p_2) \in \mathbb{R}^2$ ,  $n = (n_1, n_2) \in (\mathbb{N} \cup \{0\})^2$ ,  $|n| \leq N$ . Using this equality we obtain that

$$|\widehat{A}_j(p)| \leq \frac{2^{N-1}}{\pi(\sigma-2)} \|A_j\|_{N,\sigma} (1+|p|^2)^{-\frac{N}{2}} \quad (\text{IV.74})$$

for  $p \in \mathbb{R}^2$ ,  $|p| \geq 1$ ,  $j = 1, 2$ . Formulas (IV.73), (IV.74) imply the estimates

$$\begin{aligned} |\widehat{\partial_j}\varphi^{div}(p)| &\leq \sqrt{2} \frac{2^{N-1}}{\pi(\sigma-2)} \|A\|_{N,\sigma} (1+|p|^2)^{-\frac{N}{2}}, \\ |\widehat{\Delta}\varphi^{div}(p)| &\leq \sqrt{2} \frac{2^{N-1}}{\pi(\sigma-2)} \|A\|_{N,\sigma} (1+|p|^2)^{-\frac{N-1}{2}}, \end{aligned} \quad (\text{IV.75})$$

where  $p \in \mathbb{R}^2$ ,  $|p| \geq 1$ ,  $j = 1, 2$ . Using (IV.75) we obtain (IV.70).  $\square$

*Proof of Theorem IV.2.* Taking into account invariance of  $f^{\text{lin}}$  with respect to transformations (IV.12) and using (IV.13), (IV.16) we obtain the following equalities :

$$\begin{aligned} (k-l)\widehat{A}^{\text{div},0}(k-l) &= 0, \\ f^{\text{lin}}(k,l) - f^{\text{lin}}(-l,-k) &= 2(k+l)\widehat{A}^{\text{div},0}(k-l), \\ f^{\text{lin}}(k,l) + f^{\text{lin}}(-l,-k) &= 2\widehat{V}^{\text{div},0}(k-l), \end{aligned} \quad (\text{IV.76})$$

where  $(k,l) \in M_E$ . Using (IV.76) and orthogonality of vectors  $(k-l)$  and  $(k+l)$  we obtain (IV.28).

Formulas (IV.29), (IV.30) can be regarded as definitions of  $A_{\text{appr}}^{\text{div},0}$ ,  $A_{\text{err}}^{\text{div},0}$ ,  $V_{\text{appr}}^{\text{div},0}$ ,  $V_{\text{err}}^{\text{div},0}$ .

From (IV.13), (IV.29), (IV.30) we derive

$$\begin{aligned} A_{\text{err},j}^{\text{div},0}(x,E) &= (A_j)_{\text{err}}(x,E) + (\partial_j \varphi^{\text{div}})_{\text{err}}(x,E), \quad j = 1, 2, \\ V_{\text{err}}^{\text{div},0}(x,E) &= V_{\text{err}}(x,E) - i(\Delta \varphi^{\text{div}})_{\text{err}}(x,E), \end{aligned} \quad (\text{IV.77})$$

where  $x \in \mathbb{R}^2$ ,  $E > 0$ , and  $(A_1)_{\text{err}}$ ,  $(A_2)_{\text{err}}$ ,  $V_{\text{err}}$ ,  $(\nabla \varphi^{\text{div}})_{\text{err}}$ ,  $(\Delta \varphi^{\text{div}})_{\text{err}}$  are defined according to (IV.67).

From (IV.77) using inequalities (IV.68) for  $(A_1)_{\text{err}}$ ,  $(A_2)_{\text{err}}$ ,  $V_{\text{err}}$  and using inequalities (IV.70) we obtain formulas (IV.31), (IV.32).  $\square$

**Lemma IV.3.** *Suppose that  $A_1, A_2 \in C^{N,\sigma}(\mathbb{R}^2)$ , where  $N \geq 4$ ,  $\sigma > 2$ . Let  $\varphi^-$ ,  $\varphi^+$  be the solutions of (IV.14), (IV.15), respectively. Let  $\widehat{\nabla \varphi^\pm}$ ,  $(\nabla \varphi^\pm)_{\text{err}}$ ,  $\widehat{\Delta \varphi^\pm}$ ,  $(\Delta \varphi^\pm)_{\text{err}}$  be defined according to (IV.67). Then the following estimates hold :*

$$\begin{aligned} |(\partial_j \varphi^\pm)_{\text{err}}(x,E)| &\leq \sqrt{2} c_1(N,\sigma) \|A\|_{N,\sigma} E^{-\frac{N-2}{2}}, \\ |(\Delta \varphi^\pm)_{\text{err}}(x,E)| &\leq \sqrt{2} c_1(N,\sigma) \|A\|_{N,\sigma} E^{-\frac{N-3}{2}}, \end{aligned} \quad (\text{IV.78})$$

where  $x \in \mathbb{R}^2$ ,  $j = 1, 2$ ,  $E \geq 1/4$ ,  $\|A\|_{N,\sigma}$  is defined by (IV.33) and  $c_1(N,\sigma)$  is defined by (IV.27).

*Proof of Lemma IV.3.* The solutions  $\varphi^\pm$  of (IV.14), (IV.15) are given by :

$$\varphi^\pm(x) = -i \int_{\mathbb{R}^2} e^{-ipx} \frac{\widehat{A}_1(p) \pm i\widehat{A}_2(p)}{p_1 \pm ip_2} dp, \quad x \in \mathbb{R}^2. \quad (\text{IV.79})$$

Using (IV.67), (IV.79) we obtain that

$$\widehat{\nabla \varphi^\pm}(p) = -\frac{\widehat{A}_1(p) \pm i\widehat{A}_2(p)}{p_1 \pm ip_2} p, \quad \widehat{\Delta \varphi^\pm}(p) = i \frac{\widehat{A}_1(p) \pm i\widehat{A}_2(p)}{p_1 \pm ip_2} |p|^2, \quad (\text{IV.80})$$

where  $p \in \mathbb{R}^2 \setminus \{0\}$ . Formulas (IV.80) imply the following inequalities :

$$|\widehat{\partial_j \varphi^\pm}(p)| \leq \sqrt{2} \max_{k=1,2} |\widehat{A}_k(p)|, \quad |\widehat{\Delta \varphi^\pm}(p)| \leq \sqrt{2} |p| \max_{k=1,2} |\widehat{A}_k(p)|, \quad (\text{IV.81})$$

where  $p \in \mathbb{R}^2 \setminus \{0\}$ ,  $j = 1, 2$ .

As in the proof of Lemma IV.2 we have estimates (IV.74). Formulas (IV.74), (IV.81) imply the following estimates :

$$\begin{aligned} |\widehat{\partial_j \varphi^\pm}(p)| &\leq \sqrt{2} \frac{2^{N-1}}{\pi(\sigma-2)} \|A\|_{N,\sigma} (1+|p|^2)^{-\frac{N}{2}}, \\ |\widehat{\Delta \varphi^\pm}(p)| &\leq \sqrt{2} \frac{2^{N-1}}{\pi(\sigma-2)} \|A\|_{N,\sigma} (1+|p|^2)^{-\frac{N-1}{2}}, \end{aligned} \quad (\text{IV.82})$$

where  $p \in \mathbb{R}^2$ ,  $|p| \geq 1$ ,  $j = 1, 2$ . Using (IV.82) we obtain (IV.78).  $\square$

*Proof of Theorem IV.3.* Taking into account invariance of  $f^{\text{lin}}$  with respect to transformations (IV.12) and using (IV.14), (IV.15), (IV.16) we obtain the following equalities :

$$\begin{aligned} A_2^{\pm,0}(k-l) &= \pm i A_1^{\pm,0}(k-l), \\ f^{\text{lin}}(k,l) - f^{\text{lin}}(-l,-k) &= 2(k_1 + l_1 \pm i(k_2 + l_2)) \widehat{A}_1^{\pm,0}(k-l), \\ f^{\text{lin}}(k,l) + f^{\text{lin}}(-l,-k) &= 2(k_1 - l_1 \pm i(k_2 - l_2)) \widehat{A}_1^{\pm,0}(k-l) + 2\widehat{V}^{\pm,0}(k-l), \end{aligned} \quad (\text{IV.83})$$

where  $(k,l) \in M_E$ . Using (IV.83) and orthogonality of vectors  $(k-l)$  and  $(k+l)$  we obtain (IV.34).

Formulas (IV.35), (IV.36) can be regarded as definitions of  $A_{\text{appr}}^{\pm,0}$ ,  $A_{\text{err}}^{\pm,0}$ ,  $V_{\text{appr}}^{\pm,0}$ ,  $V_{\text{err}}^{\pm,0}$ . From (IV.14), (IV.15), (IV.35), (IV.36) we derive formulas

$$\begin{aligned} A_{\text{err},j}^{\pm,0}(x,E) &= (A_j)_{\text{err}}(x,E) + (\nabla \varphi^\pm)_{\text{err}}(x,E), \quad j = 1, 2, \\ V_{\text{err}}^{\pm,0}(x,E) &= V_{\text{err}}(x,E) - i(\Delta \varphi^\pm)_{\text{err}}(x,E), \end{aligned} \quad (\text{IV.84})$$

where  $x \in \mathbb{R}^2$ ,  $E > 0$ , and  $(A_1)_{\text{err}}$ ,  $(A_2)_{\text{err}}$ ,  $V_{\text{err}}$ ,  $(\nabla \varphi^\pm)_{\text{err}}$ ,  $(\Delta \varphi^\pm)_{\text{err}}$  are defined by (IV.67).

From (IV.84) using inequalities (IV.68) for  $(A_1)_{\text{err}}$ ,  $(A_2)_{\text{err}}$ ,  $V_{\text{err}}$  and using inequalities (IV.78) we obtain formulas (IV.37), (IV.38).  $\square$

## 6 Derivation of the reconstruction algorithm of Section 3

### 6.1 Faddeev functions

For equation (IV.1), under assumptions (IV.2), we consider the Faddeev functions  $\psi$ ,  $h$  (see, e.g., [47, 48] and Subsection 5.1 of [109]) :

$$\begin{aligned} \psi(x, k) &= e^{ikx} \mu(x, k), \\ \mu(x, k) &= 1 + \int_{\mathbb{R}^2} g(x-y, k) \times \\ &\times (-2iA(y)\nabla_y \mu(y, k) + (2A(y)k + V(y))\mu(y, k)) dy, \\ g(x, k) &= -(2\pi)^{-2} \int_{\mathbb{R}^2} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi, \end{aligned} \tag{IV.85}$$

where  $x \in \mathbb{R}^2$ ,  $k \in \Sigma_E \setminus \mathbb{R}^2$ ;

$$\begin{aligned} h(k, l) &= (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(k-l)y} \times \\ &\times (-2iA(y)\nabla_y \mu(y, k) + (2A(y)k + V(y))\mu(y, k)) dy, \end{aligned} \tag{IV.86}$$

where  $k, l \in \Sigma_E \setminus \mathbb{R}^2$ ,  $\text{Im } k = \text{Im } l$ . Here (IV.85) and its differentiated version, where  $\nabla$  is applied to both sides of (IV.85), are considered as a system of linear integral equations for bounded  $\mu$  and  $\nabla \mu$ ,  $\Sigma_E$  is defined by (IV.42).

We recall that  $\psi$  are “growing” solutions of (IV.1), in the sense of [47], parametrized by  $k \in \Sigma_E \setminus \mathbb{R}^2$ , and  $G = e^{ikx} g$  is the Faddeev’s Green function for the operator  $\Delta + k^2$ .

Equation (IV.85) for  $\mu$  and formula (IV.86) for  $h$  are analogs in the complex domain in  $k$  of equation (IV.5) for  $\psi^+$  and formula (IV.6) for  $f$ .

Note that

$$k, l \in \Sigma_E \setminus \mathbb{R}^2, \text{Im } k = \text{Im } l \implies l = k \text{ or } l = -\bar{k}.$$

Therefore, the function  $h$  of (IV.86) splits to the functions

$$a(k) = h(k, k), \quad b(k) = h(k, -\bar{k}), \quad k \in \Sigma_E \setminus \mathbb{R}^2. \tag{IV.87}$$

Note also that  $a$  and  $b$  are invariant with respect to transformations (IV.8) and  $\psi$ ,  $\mu$  are transformed as

$$\psi \rightarrow e^{-i\varphi} \psi, \quad \mu \rightarrow e^{-i\varphi} \mu \tag{IV.88}$$

with respect to (IV.8).

## 6.2 Analytic properties of $\psi$

Using notations of Section 3 coefficients  $A_1, A_2, V$  of equation (IV.1), functions  $\psi^+, f$  of (IV.5), (IV.6) and functions  $\psi, \mu, b$  of (IV.85), (IV.87) can be written as

$$\begin{aligned} A_1 &= A_1(z), \quad A_2 = A_2(z), \quad V = V(z), \\ \psi^+ &= \psi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E), \quad \lambda, \lambda' \in T, \\ \psi &= \psi(z, \lambda, E), \quad \mu = \mu(z, \lambda, E), \quad b = b(\lambda, E), \quad \lambda \in \mathbb{C} \setminus (T \cup 0), \end{aligned} \quad (\text{IV.89})$$

where  $z \in \mathbb{C}, E > 0$ .

It is known that the function  $\psi$  (or  $\mu$ ) has the following properties at fixed  $z \in \mathbb{C}$  and  $E > 0$  (see page 448 of [98]) :

$$\frac{\partial}{\partial \lambda} \mu(z, \lambda, E) = r(\lambda, z, E) \mu(z, -\frac{1}{\lambda}, E), \quad (\text{IV.90})$$

for  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ , where

$$\begin{aligned} r(\lambda, z, E) &= \exp\left(-i \frac{\sqrt{E}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} + \bar{\lambda} z + \frac{\bar{z}}{\lambda}\right)\right) \times \\ &\quad \times \frac{\pi}{\lambda} \operatorname{sgn}(\lambda \bar{\lambda} - 1) b(\lambda, E); \\ \mu(z, \lambda, E) &= \mu_0^-(z) + o(1) \quad \text{for } \lambda \rightarrow \infty, \\ \mu(z, \lambda, E) &= \mu_0^+(z) + o(1) \quad \text{for } \lambda \rightarrow 0, \end{aligned} \quad (\text{IV.91})$$

where

$$\begin{aligned} \partial_{\bar{z}} \mu_0^+(z) &= \frac{1}{2i} (A_1(z) + iA_2(z)) \mu_0^+(z), \\ \partial_z \mu_0^-(z) &= \frac{1}{2i} (A_1(z) - iA_2(z)) \mu_0^-(z), \\ \mu_0^\pm(z) &\rightarrow 1 \quad \text{as } z \rightarrow \infty; \end{aligned} \quad (\text{IV.92})$$

$$\begin{aligned} \psi_\pm(z, \lambda, E) &= \psi^+(z, \lambda, E) + \pi i \int_T h_\pm(\lambda, \lambda'', E) \chi\left(\pm i \left[\frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda}\right]\right) \times \\ &\quad \times \psi^+(z, \lambda'', E) |d\lambda''| \end{aligned} \quad (\text{IV.93})$$

for  $\lambda \in T$ , where

$$\begin{aligned} \psi_\pm(z, \lambda, E) &= \psi(z, \lambda(1 \mp 0), E) = \exp\left(i \frac{\sqrt{E}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda}\right)\right) \times \\ &\quad \times \mu_\pm(z, \lambda, E), \quad \mu_\pm(z, \lambda, E) = \mu(z, \lambda(1 \pm 0), E), \quad \lambda \in T, \end{aligned} \quad (\text{IV.94})$$

$\psi^+$  is the function of (IV.5), (IV.89),  $h_\pm$  are the functions related with the scattering amplitude  $f$  by equations (IV.45),  $\chi$  is defined by (IV.46).



More precisely, equation (IV.90) is fulfilled if the system of linear equations for  $\mu$  and  $\nabla\mu$  related with (IV.85) is uniquely solvable for  $k = (k_1(\lambda, E), k_2(\lambda, E))$  for fixed  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ , where  $k_1, k_2$  are given by (IV.43), and relation (IV.93) is fulfilled if the aforementioned system is uniquely solvable for  $k = (k_1(\lambda(1 \mp 0), E), k_2(\lambda(1 \mp 0), E))$  for fixed  $\lambda \in T$ . In particular, all these conditions are fulfilled if coefficients  $A_1, A_2, V$  of (IV.1) are sufficiently small for fixed  $E$ .

### 6.3 Inverse scattering from $f$ and $b$

Using the definitions of  $\varphi^\pm, A^\pm, V^\pm$  of (IV.14), (IV.15), (IV.39), the invariance of  $f$  and  $b$  with respect to (IV.8) and formulas (IV.9), (IV.88) one can see that

$$\begin{aligned} & \text{for } A, V \text{ transformed to } A^\pm, V^\pm \text{ (respectively)} \\ & \text{formulas (IV.90)–(IV.94) are fulfilled with } \mu_0^\pm \equiv 1 \text{ (respectively)}. \end{aligned} \quad (\text{IV.95})$$

Properties (IV.90)–(IV.95) of  $\psi, \mu$  yield the following approach to inverse scattering for equation (IV.1) from  $f$  and  $b$  :

1. find  $\psi, \mu$  satisfying (IV.90)–(IV.91), (IV.93), (IV.94) with a priori unknown  $\psi^+$  in (IV.93), where  $\mu_0^- \equiv 1, \mu(z, \cdot, E) \in C(\mathbb{C} \setminus T), h_\pm$  are related with  $f$  by (IV.45);
2. find  $A^-, V^-$  using that

$$A_1^-(z) - iA_2^-(z) = 0, \quad A_1^-(z) + iA_2^-(z) = 2i\partial_{\bar{z}} \ln \mu_0^+(z), \quad (\text{IV.96})$$

$$\begin{aligned} V^-(z)\psi(z, \lambda, E) &= (4\partial_z\partial_{\bar{z}} + \\ &+ 2i(A_1^-(z) + iA_2^-(z))\partial_z + E)\psi(z, \lambda, E), \end{aligned} \quad (\text{IV.97})$$

where  $z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus (T \cup 0)$ .

Or alternatively :

- 1'. find  $\psi, \mu$  satisfying (IV.90)–(IV.91), (IV.93), (IV.94) with a priori unknown  $\psi^+$  in (IV.93), where  $\mu_0^+ \equiv 1, \mu(z, \cdot, E) \in C((\mathbb{C} \cup \infty) \setminus T), h_\pm$  are related with  $f$  by (IV.45);
- 2'. find  $A^+, V^+$  using that

$$A_1^+(z) - iA_2^+(z) = 2i\partial_z \ln \mu_0^-(z), \quad A_1^+(z) + iA_2^+(z) = 0, \quad (\text{IV.98})$$

$$\begin{aligned} V^+(z)\psi(z, \lambda, E) &= (4\partial_z\partial_{\bar{z}} + \\ &+ 2i(A_1^+(z) - iA_2^+(z))\partial_{\bar{z}} + E)\psi(z, \lambda, E), \end{aligned} \quad (\text{IV.99})$$

where  $z \in \mathbb{C}, \lambda \in \mathbb{C} \setminus (T \cup 0)$ .

Note that (IV.96) arises from (IV.92) with  $\mu_0^- \equiv 1$ , (IV.98) arises from (IV.92) with  $\mu_0^+ \equiv 1$ , (IV.97) and (IV.99) arise from equation (IV.1) for the Faddeev functions  $\psi$  of Subsections 6.1, 6.2 in the gauge setting related with  $A^-, V^-$  and  $A^+, V^+$ , respectively. In addition,  $\psi, \mu, \mu_0^+$  of steps 1, 2 and  $\psi = \psi', \mu = \mu', \mu_0^-$  of steps 1', 2' are related by the formulas

$$\begin{aligned}\psi'(z, \lambda, E) &= (\mu_0^+(z))^{-1} \psi(z, \lambda, E), \\ \mu'(z, \lambda, E) &= (\mu_0^+(z))^{-1} \mu(z, \lambda, E), \quad \mu_0^-(z) = (\mu_0^+(z))^{-1},\end{aligned}\tag{IV.100}$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ .

As soon as  $A, V$  are recovered as  $A^-, V^-$  or  $A^+, V^+$ , then these coefficients can be transformed from  $A^-, V^-$  or from  $A^+, V^+$  to other possible gauge setting via (IV.8) and, in particular, to  $A^{\text{div}}, V^{\text{div}}$  of (IV.39).

Note that different ideas of the aforementioned approach to inverse scattering go back to [89, 1, 52, 58, 57, 98, 100]. In particular, finding  $\psi, \mu$  of the aforementioned steps 1 or 1' for the case when  $b \equiv 0$  at fixed  $E$  is reduced to solving a non-local Riemann–Hilbert problem for holomorphic functions, see Subsection 6.4. Such non-local Riemann–Hilbert problems go back to [89, 52]; see also [143, 51, 58, 57, 98, 100].

## 6.4 Inverse scattering with $b \equiv 0$ at fixed $E$

In the Born approximation at fixed  $E$  we have that

$$f(k, l) \approx f^{\text{lin}}(k, l) = 2k\widehat{A}(k-l) + \widehat{V}(k-l), \quad (k, l) \in M_E, \tag{IV.101}$$

$$b(k) \approx b^{\text{lin}}(k) = 2k\widehat{A}(2\text{Re } k) + \widehat{V}(2\text{Re } k), \quad k \in \Sigma_E \setminus \mathbb{R}^2, \tag{IV.102}$$

where  $\widehat{A}, \widehat{V}$  are defined by (IV.17). Here formula (IV.101) is equivalent to the formulas for  $f$  of (IV.10), (IV.11) and formula (IV.102) follows from (IV.85), (IV.86), (IV.87) in a similar way that (IV.101) follows from (IV.5), (IV.6). Note also that

$$k \in \Sigma_E \setminus \mathbb{R}^2 \implies 2\text{Re } k \in \mathbb{R}^2 \setminus B_{2\sqrt{E}}, \quad E > 0, \tag{IV.103}$$

where  $B_r$  is defined by (IV.19).

Using (IV.18), (IV.101) and (IV.102), (IV.103) one can see that the expression for  $f^{\text{lin}}$  involves  $\widehat{A}, \widehat{V}$  on  $B_{2\sqrt{E}}$ , only, and the expression for  $b^{\text{lin}}$  involves  $\widehat{A}, \widehat{V}$  on  $\mathbb{R}^2 \setminus B_{2\sqrt{E}}$ , only, at fixed  $E$ . Further, using also (IV.28)–(IV.30), (IV.34)–(IV.36) one can see that the expressions for  $A_{\text{appr}}^{\pm,0}, V_{\text{appr}}^{\pm,0}, A_{\text{appr}}^{\text{div},0}, V_{\text{appr}}^{\text{div},0}$  involve  $f^{\text{lin}}$ , only, and are independent of  $b^{\text{lin}}$  at fixed  $E$ .

In a similar way, in Section 3 in order to construct nonlinear analogs of  $A_{\text{appr}}^{\pm,0}, V_{\text{appr}}^{\pm,0}, A_{\text{appr}}^{\text{div},0}, V_{\text{appr}}^{\text{div},0}$  we use inverse scattering of Subsection 6.3 without  $b$  or, in other words, with  $b \equiv 0$  at fixed  $E$ .

In this case steps 1 and 1' of Subsection 6.3 consist in solving the following non-local Riemann–Hilbert problems for holomorphic functions :

1. find  $\psi = \exp((i/2)\sqrt{E}(\lambda\bar{z} + z/\lambda))\mu(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ , satisfying (IV.93), (IV.94) with a priori unknown  $\psi^+$  in (IV.93), where

$$\begin{aligned}\frac{\partial}{\partial \lambda} \mu(z, \lambda, E) &= 0, \quad \lambda \in \mathbb{C} \setminus (T \cup 0), \\ \mu(z, \lambda, E) &\rightarrow 1, \quad \text{as } \lambda \rightarrow \infty, \\ \mu(z, \cdot, E) &\in C(\mathbb{C} \setminus T),\end{aligned}$$

or, alternatively :

- 1'. find  $\psi = \exp((i/2)\sqrt{E}(\lambda\bar{z} + z/\lambda))\mu(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ , satisfying (IV.93), (IV.94) with a priori unknown  $\psi^+$  in (IV.93), where

$$\begin{aligned}\frac{\partial}{\partial \lambda} \mu(z, \lambda, E) &= 0, \quad \lambda \in \mathbb{C} \setminus (T \cup 0), \\ \mu(z, \lambda, E) &\rightarrow 1, \quad \text{as } \lambda \rightarrow 0, \\ \mu(z, \cdot, E) &\in C((\mathbb{C} \cup \infty) \setminus T).\end{aligned}$$

We recall that  $h_{\pm}$  of (IV.45) are related with  $f$  by (IV.45). Actually, it is also assumed that  $\mu(z, \cdot, E)$  admits continuous extension on  $T$  from its both sides.

Now due to considerations of Section 2 of [100] we have that finding  $\mu$  of step 1 is reduced to : (a) solving the linear integral equation (IV.47) for  $\mu^+(z, \cdot, E)$  on  $T$ , where  $\psi^+ = \exp((i/2)\sqrt{E}(\lambda\bar{z} + z/\lambda))\mu^+(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in T$ , (b) finding  $\mu_{\pm}(z, \cdot, E)$  on  $T$  by formulas (IV.50) (or, in other words, by formulas (IV.93) rewritten in terms of  $\mu_{\pm}$  and  $\mu^+$ ), (c) finding  $\mu(z, \cdot, E)$  on  $C \setminus T$  by the Cauchy formulas :

$$\begin{aligned}\mu(z, \lambda, E) &= \frac{1}{2\pi i} \int_T \frac{\mu_+(z, \zeta, E)}{\zeta - \lambda} d\zeta, \quad |\lambda| < 1, \\ \mu(z, \lambda, E) &= 1 - \frac{1}{2\pi i} \int_T \frac{\mu_-(z, \zeta, E)}{\zeta - \lambda} d\zeta, \quad |\lambda| > 1,\end{aligned}\tag{IV.104}$$

$z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus T$ .

In addition, due to (IV.100), (IV.104) finding  $\psi = \psi'$ ,  $\mu = \mu'$  of step 1' is reduced to finding  $\psi$ ,  $\mu$  of step 1 and to the formulas

$$\begin{aligned}\psi'(z, \lambda, E) &= (\mu_0^+(z))^{-1} \psi(z, \lambda, E), \quad \mu'(z, \lambda, E) = (\mu_0^+(z))^{-1} \mu(z, \lambda, E), \\ \mu_0^+(z) &= \frac{1}{2\pi i} \int_T \frac{\mu_+(z, \zeta, E)}{\zeta} d\zeta,\end{aligned}\tag{IV.105}$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ .

In addition, due to (IV.104) we have that

$$\begin{aligned}\mu(z, \lambda, E) &= 1 + \mu_1^-(z)\lambda^{-1} + O(\lambda^{-2}) \quad \text{as } \lambda \rightarrow \infty, \\ \mu(z, \lambda, E) &= \mu_0^+(z) + \mu_1^+(z)\lambda + O(\lambda^2) \quad \text{as } \lambda \rightarrow 0,\end{aligned}\tag{IV.106}$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ ,  $\mu_0^+(z)$  is given in (IV.105),

$$\begin{aligned}\mu_1^+(z) &= \frac{1}{2\pi i} \int_T \frac{\mu_+(z, \zeta, E)}{\zeta^2} d\zeta, \\ \mu_1^-(z) &= \frac{1}{2\pi i} \int_T \mu_-(z, \zeta, E) d\zeta.\end{aligned}\tag{IV.107}$$

Note that the non-local Riemann–Hilbert problems of steps 1 and 1' are better known in the literature (going back to [89]) for the case when relation (IV.93) between  $\psi_+$  and  $\psi_-$  on  $T$  with a priori unknown  $\psi^+$  is given in the form

$$\psi_+(\lambda) = \psi_-(\lambda) + \int_T \rho(\lambda, \lambda') \psi_-(\lambda') |d\lambda'|, \quad \lambda \in T,\tag{IV.108}$$

see, e.g., [89, 58, 57, 98]. Note also that in our case  $h_{\pm}$  of (IV.45) are related with  $\rho$  of (IV.108) by the following formulas and equations :

$$\begin{aligned}h_1(\lambda, \lambda', E) &= \chi\left(i \left[\frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'}\right]\right) h_+(\lambda, \lambda', E) - \\ &\quad - \chi\left(-i \left[\frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'}\right]\right) h_-(\lambda, \lambda', E), \\ h_2(\lambda, \lambda', E) &= \chi\left(i \left[\frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'}\right]\right) h_-(\lambda, \lambda', E) - \\ &\quad - \chi\left(-i \left[\frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'}\right]\right) h_+(\lambda, \lambda', E), \\ \rho(\lambda, \lambda', E) + \pi i \int_T \rho(\lambda, \lambda'', E) \chi\left(-i \left[\frac{\lambda'}{\lambda''} - \frac{\lambda''}{\lambda'}\right]\right) \times \\ &\quad \times h_1(\lambda'', \lambda', E) |d\lambda''| = -\pi i h_1(\lambda, \lambda', E), \\ \rho(\lambda, \lambda', E) + \pi i \int_T \chi\left(i \left[\frac{\lambda'}{\lambda''} - \frac{\lambda''}{\lambda'}\right]\right) \times \\ &\quad \times h_2(\lambda'', \lambda', E) |d\lambda''| = -\pi i h_2(\lambda, \lambda', E),\end{aligned}$$

where  $\lambda, \lambda' \in T$ , see [98].

Futher, due to results of [58, 98] and of Proposition IV.1 of the present work we have that, at least under assumption (IV.55), the non-local Riemann–Hilbert problems for  $\psi$  and for  $\psi = \psi'$  of steps 1 and 1' are uniquely solvable and

$$(-4\partial_z\partial_{\bar{z}} + a_{\bar{z}}^-(z)\partial_z + V^-(z))\psi(z, \lambda, E) = E\psi(z, \lambda, E), \quad (\text{IV.109})$$

$$a_{\bar{z}}^-(z) = 4\partial_{\bar{z}} \ln \mu_0^+(z), \quad V^-(z) = 2i\sqrt{E}\partial_z \mu_1^-(z), \quad (\text{IV.110})$$

$$(-4\partial_z\partial_{\bar{z}} + a_{\bar{z}}^+(z)\partial_z + V^+(z))\psi'(z, \lambda, E) = E\psi'(z, \lambda, E), \quad (\text{IV.111})$$

$$a_{\bar{z}}^+(z) = 4\partial_z \ln \frac{1}{\mu_0^+(z)}, \quad V^+(z) = 2i\sqrt{E}\partial_{\bar{z}} \frac{\mu_1^+(z)}{\mu_0^+(z)}, \quad (\text{IV.112})$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus (T \cup 0)$ ,  $\mu_0^+$ ,  $\mu_1^-$ ,  $\mu_1^+$  are the coefficients of (IV.106). Here (IV.109), (IV.110) correspond to (IV.96), (IV.97) and (IV.111), (IV.112) correspond to (IV.98), (IV.99).

Formulas (IV.51), (IV.52) follow from (IV.109)–(IV.111) and the integral expression for  $\mu_0^+$ ,  $\mu_1^-$ ,  $\mu_1^+$  of (IV.105), (IV.107).

Finally, formulas (IV.53) arise from considerations of the gauge transformations (IV.8) between  $A^-$ ,  $V^-$  and  $A^+$ ,  $V^+$  and  $A^{\text{div}}$ ,  $V^{\text{div}}$  of (IV.39). In particular, in these considerations we use that for equation (IV.1) written as

$$\begin{aligned} (-4\partial_z\partial_{\bar{z}} + a_z\partial_z + a_{\bar{z}}\partial_{\bar{z}} + V)\psi &= E\psi, \\ a_z &= -2i(A_1 + iA_2), \quad a_{\bar{z}} = -2i(A_1 - iA_2), \end{aligned}$$

the gauge transformations (IV.8), (IV.9) can be written as

$$\begin{aligned} a_z &\rightarrow a_z - 4i\partial_{\bar{z}}\varphi, \quad a_{\bar{z}} \rightarrow a_{\bar{z}} - 4i\partial_z\varphi, \\ V &\rightarrow V - 4i\partial_z\partial_{\bar{z}}\varphi + 4\partial_z\varphi\partial_{\bar{z}}\varphi + ia_z\partial_z\varphi + ia_{\bar{z}}\partial_{\bar{z}}\varphi, \\ \psi &\rightarrow e^{-i\varphi}\psi, \end{aligned}$$

and that the equations for  $\varphi^{\text{div}}$ ,  $\varphi^-$ ,  $\varphi^+$  of (IV.13)–(IV.15), (IV.39) can be written as

$$\begin{aligned} 8i\partial_z\partial_{\bar{z}}\varphi^{\text{div}} &= \partial_z a_z + \partial_{\bar{z}} a_{\bar{z}}, \quad \varphi^{\text{div}}(z) \rightarrow 0 \text{ as } |z| \rightarrow \infty, \\ 4i\partial_z\varphi^- &= a_{\bar{z}}, \quad \varphi^-(z) \rightarrow 0 \text{ as } |z| \rightarrow \infty, \\ 4i\partial_{\bar{z}}\varphi^+ &= a_z, \quad \varphi^+(z) \rightarrow 0 \text{ as } |z| \rightarrow \infty. \end{aligned}$$

## 7 Proofs of Propositions IV.2 and IV.3

*Proof of Proposition IV.2.* The method of successive approximations for solving (IV.45) with respect to  $h_{\pm} \in L^2(T^2)$  and assumptions (IV.58) imply that

$$h_{\pm} = f + O(\varepsilon^2), \quad \varepsilon \rightarrow +0, \quad (\text{IV.113})$$

where  $O(\varepsilon^2)$  is considered in the sense of  $\|\cdot\|_{L^2(T^2)}$ .

Consider the following operators acting in  $L^2(T)$

$$(C_{\pm}u)(\lambda) = \frac{1}{2\pi i} \int_T \frac{u(\zeta)}{\zeta - \lambda(1 \mp 0)} d\zeta, \quad \lambda \in T, \quad (\text{IV.114})$$

where  $u$  is a test function. We recall that

$$\|C_{\pm}u\|_{L^2(T)} \leq \|u\|_{L^2(T)}. \quad (\text{IV.115})$$

Using (IV.48), (IV.49), (IV.113), (IV.115) and the equality

$$\left| \exp\left(-i\frac{\sqrt{E}}{2}((\lambda - \lambda')\bar{z} + (\lambda^{-1} - \lambda'^{-1})z)\right) \right| = 1, \quad (\text{IV.116})$$

$$\lambda, \lambda' \in T, \quad z \in \mathbb{C}, \quad E > 0,$$

we obtain that

$$B(\lambda, \lambda', z, E) = \frac{1}{2} \int_T f(\zeta, \lambda', z, E) \chi\left(-i\left[\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta}\right]\right) \frac{d\zeta}{\zeta - \lambda(1-0)} -$$

$$-\frac{1}{2} \int_T f(\zeta, \lambda', z, E) \chi\left(i\left[\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta}\right]\right) \frac{d\zeta}{\zeta - \lambda(1+0)} + O(\varepsilon^2), \quad (\text{IV.117})$$

where  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $f(\zeta, \lambda', z, E)$  is given by (IV.64),  $O(\varepsilon^2)$  is considered in the sense of  $\|\cdot\|_{L^2(T^2)}$  and is uniform with respect to  $z \in \mathbb{C}$ .

From (IV.47) we derive the following equalities :

$$\begin{aligned} \partial_z \mu^+(z, \lambda, E) + \int_T B(\lambda, \lambda', z, E) \partial_z \mu^+(z, \lambda', E) |d\lambda'| &= \\ &= - \int_T \partial_z B(\lambda, \lambda', z, E) \mu^+(z, \lambda', E) |d\lambda'|, \\ \partial_{\bar{z}} \mu^+(z, \lambda, E) + \int_T B(\lambda, \lambda', z, E) \partial_{\bar{z}} \mu^+(z, \lambda', E) |d\lambda'| &= \\ &= - \int_T \partial_{\bar{z}} B(\lambda, \lambda', z, E) \mu^+(z, \lambda', E) |d\lambda'|, \end{aligned} \quad (\text{IV.118})$$

where  $\lambda \in T$ ,  $z \in \mathbb{C}$ ,  $E > 0$ .

The method of successive approximations for solving (IV.47) and (IV.118) with respect to  $\mu_+ \in L^2(T)$  and  $\partial_z \mu_+, \partial_{\bar{z}} \mu_+ \in L^2(T)$  and estimates (IV.58), (IV.117) imply that :

$$\begin{aligned}\mu^+(z, \lambda, E) &= 1 + O(\varepsilon), \\ \partial_z \mu^+(z, \lambda, E) &= - \int_T \partial_z B(\lambda, \lambda', z, E) |d\lambda'| + O(\varepsilon^2), \\ \partial_{\bar{z}} \mu^+(z, \lambda, E) &= - \int_T \partial_{\bar{z}} B(\lambda, \lambda', z, E) |d\lambda'| + O(\varepsilon^2),\end{aligned}\tag{IV.119}$$

where  $z \in \mathbb{C}$ ,  $\lambda \in T$ ,  $O(\varepsilon)$ ,  $O(\varepsilon^2)$  are considered in the sense of  $\|\cdot\|_{L^2(T)}$  and are uniform with respect to  $z \in \mathbb{C}$ .

Using (IV.50), (IV.113), (IV.119) we obtain that

$$\begin{aligned}\mu_{\pm}(z, \lambda, E) &= 1 + O(\varepsilon), \\ \partial_z \mu_{\pm}(z, \lambda, E) &= - \int_T \partial_z B(\lambda, \lambda', z, E) |d\lambda'| + \\ &\quad + \pi i \int_T \partial_z f(\lambda, \lambda', z, E) \chi\left(\pm i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right) |d\lambda'| + O(\varepsilon^2), \\ \partial_{\bar{z}} \mu_{\pm}(z, \lambda, E) &= - \int_T \partial_{\bar{z}} B(\lambda, \lambda', z, E) |d\lambda'| + \\ &\quad + \pi i \int_T \partial_{\bar{z}} f(\lambda, \lambda', z, E) \chi\left(\pm i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right) |d\lambda'| + O(\varepsilon^2),\end{aligned}\tag{IV.120}$$

where  $\lambda \in T$ ,  $f(\lambda, \lambda', z, E)$  is given by (IV.64) and  $O(\varepsilon)$ ,  $O(\varepsilon^2)$  are considered in the sense of  $\|\cdot\|_{L^2(T)}$  and are uniform with respect to  $z \in \mathbb{C}$ .

Using (IV.51), (IV.52), (IV.120) we obtain that :

$$\begin{aligned}
V_{\text{appr}}^-(x, E) &= -\frac{\sqrt{E}}{\pi} \int_{T^2} \partial_z B(\lambda, \lambda', z, E) d\lambda |d\lambda'| + \\
&\quad + i\sqrt{E} \int_{T^2} \partial_z f(\lambda, \lambda', z, E) \chi\left(-i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right) d\lambda |d\lambda'| + O(\varepsilon^2), \\
V_{\text{appr}}^+(x, E) &= \frac{\sqrt{E}}{\pi} \int_{T^2} \partial_{\bar{z}} B(\lambda, \lambda', z, E) \lambda^{-2} d\lambda |d\lambda'| - \\
&\quad - i\sqrt{E} \int_{T^2} \partial_{\bar{z}} f(\lambda, \lambda', z, E) \chi\left(i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right) \lambda^{-2} d\lambda |d\lambda'| + O(\varepsilon^2), \\
a_z^-(z, E) &= \frac{2i}{\pi} \int_{T^2} \partial_{\bar{z}} B(\lambda, \lambda', z, E) \lambda^{-1} d\lambda |d\lambda'| + \\
&\quad + 2 \int_{T^2} \partial_{\bar{z}} f(\lambda, \lambda', z, E) \chi\left(i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right) \lambda^{-1} d\lambda |d\lambda'| + O(\varepsilon^2), \\
a_z^+(z, E) &= -\frac{2i}{\pi} \int_{T^2} \partial_z B(\lambda, \lambda', z, E) \lambda^{-1} d\lambda |d\lambda'| - \\
&\quad - 2 \int_{T^2} \partial_z f(\lambda, \lambda', z, E) \chi\left(i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right) \lambda^{-1} d\lambda |d\lambda'| + O(\varepsilon^2),
\end{aligned} \tag{IV.121}$$

where  $x \in \mathbb{R}^2$ ,  $z$  is given by (IV.40), and  $O(\varepsilon^2)$  is considered in the uniform sense with respect to  $x \in \mathbb{R}^2$ .

Note that the following formulas hold for each  $u \in L^2(T)$  :

$$\begin{aligned}
\int_T (C_+ u)(\lambda) d\lambda &= 0, \quad \int_T (C_- u)(\lambda) d\lambda = - \int_T u(\lambda) d\lambda, \\
\int_T (C_+ u)(\lambda) \frac{d\lambda}{\lambda} &= \int_T u(\lambda) \frac{d\lambda}{\lambda}, \quad \int_T (C_- u)(\lambda) \frac{d\lambda}{\lambda} = 0, \\
\int_T (C_+ u)(\lambda) \frac{d\lambda}{\lambda^2} &= \int_T u(\lambda) \frac{d\lambda}{\lambda^2}, \quad \int_T (C_- u)(\lambda) \frac{d\lambda}{\lambda^2} = 0,
\end{aligned} \tag{IV.122}$$

where  $C_{\pm}$  are defined by (IV.114).



Formulas (IV.117), (IV.121), (IV.122) imply estimates

$$\begin{aligned}
V_{\text{appr}}^-(x, E) &= i\sqrt{E} \int_{T^2} s(\lambda, \lambda') \partial_z f(\lambda, \lambda', z, E) d\lambda |d\lambda'| + O(\varepsilon^2), \\
V_{\text{appr}}^+(x, E) &= i\sqrt{E} \int_{T^2} s(\lambda, \lambda') \partial_z f(\lambda, \lambda', z, E) \lambda^{-2} d\lambda |d\lambda'| + O(\varepsilon^2), \\
a_z^-(z, E) &= -2 \int_{T^2} s(\lambda, \lambda') \partial_{\bar{z}} f(\lambda, \lambda', z, E) \lambda^{-1} d\lambda |d\lambda'| + O(\varepsilon^2), \\
a_z^+(z, E) &= 2 \int_{T^2} s(\lambda, \lambda') \partial_z f(\lambda, \lambda', z, E) \lambda^{-1} d\lambda |d\lambda'| + O(\varepsilon^2), \\
s(\lambda, \lambda') &\stackrel{\text{def}}{=} \text{sgn} \left( \frac{1}{i} \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right),
\end{aligned} \tag{IV.123}$$

where  $x \in \mathbb{R}^2$ ,  $z$  is given by (IV.40), and  $O(\varepsilon^2)$  is considered in the uniform sense with respect to  $x \in \mathbb{R}^2$ .

In addition, due to (IV.64) we have that

$$\begin{aligned}
\partial_z f(\lambda, \lambda', z, E) &= -i \frac{\sqrt{E}}{2} (\lambda^{-1} - \lambda'^{-1}) f(\lambda, \lambda', z, E), \\
\partial_{\bar{z}} f(\lambda, \lambda', z, E) &= -i \frac{\sqrt{E}}{2} (\lambda - \lambda') f(\lambda, \lambda', z, E),
\end{aligned} \tag{IV.124}$$

for each  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $E > 0$ .

Formulas (IV.59), (IV.60), (IV.61), (IV.62), (IV.63) follow immediately from (IV.51), (IV.52), (IV.53), (IV.123), (IV.124).

Proposition IV.2 is proved.  $\square$

**Lemma IV.4.** *Let  $E > 0$  be fixed. Let  $u(\lambda, \lambda', E)$ ,  $(\lambda, \lambda') \in T^2$ , be a complex valued function such that  $u \in L^1(T^2)$  and*

$$u(\lambda, \lambda', E) = u(-\lambda', -\lambda, E), \quad (\lambda, \lambda') \in T^2. \tag{IV.125}$$

*Suppose that  $g(p, E)$ ,  $p = (p_1, p_2) \in \mathbb{R}^2$ ,  $|p| \leq 2\sqrt{E}$ , is the function defined by the formula*

$$g\left(\sqrt{E} \operatorname{Re}(\lambda - \lambda'), \sqrt{E} \operatorname{Im}(\lambda - \lambda'), E\right) = u(\lambda, \lambda', E), \tag{IV.126}$$

where  $(\lambda, \lambda') \in T^2$ . Then

$$\begin{aligned} \int_{|p| \leq 2\sqrt{E}} e^{-ipx} g(p, E) dp &= \frac{E}{2} \int_{T^2} u(\lambda, \lambda', z, E) \left| \frac{1}{2i} \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right| |d\lambda| |d\lambda'|, \\ &u(\lambda, \lambda', z, E) \stackrel{\text{def}}{=} u(\lambda, \lambda', E) \times \\ &\times \exp\left(-i \frac{\sqrt{E}}{2} ((\lambda - \lambda')\bar{z} + (\lambda^{-1} - \lambda'^{-1})z)\right), \end{aligned} \quad (\text{IV.127})$$

where  $x \in \mathbb{R}^2$ ,  $z$  is given by (IV.40).

Actually, (IV.127) arises from the following change of variables in the integration with respect to  $p$ :

$$\begin{aligned} p_1 &= \sqrt{E} \operatorname{Re}(\lambda - \lambda') = \sqrt{E}(\cos \phi - \cos \phi'), \\ p_2 &= \sqrt{E} \operatorname{Im}(\lambda - \lambda') = \sqrt{E}(\sin \phi - \sin \phi'), \end{aligned} \quad (\text{IV.128})$$

where  $\lambda = e^{i\phi}$ ,  $\lambda' = e^{i\phi'}$ .

*Proof of Proposition IV.3.* Let  $\lambda \in T$ ,  $\lambda' \in T$  be defined by (IV.41). It follows from (IV.43), (IV.44) that the following formulas are valid:

$$\begin{aligned} 2(k_1 + l_1) &= \sqrt{E}(\lambda + \lambda^{-1} + \lambda' + \lambda'^{-1}), \\ 2(k_2 + l_2) &= -i\sqrt{E}(\lambda - \lambda^{-1} + \lambda' - \lambda'^{-1}), \\ k_1 \pm ik_2 &= \sqrt{E}\lambda^{\pm 1}, \quad l_1 \pm il_2 = \sqrt{E}\lambda'^{\pm 1}, \\ k_1 + l_2 \pm i(k_2 + l_2) &= \sqrt{E}(\lambda^{\pm 1} + \lambda'^{\pm 1}), \\ |k + l|^2 &= E|\lambda + \lambda'|^2, \end{aligned} \quad (\text{IV.129})$$

where  $(k, l) \in M_E$ .

Using Lemma IV.4 and formulas (IV.129) we derive from (IV.28), (IV.29), (IV.30), (IV.34), (IV.35), (IV.36) the following formulas:

$$\begin{aligned} A_{\text{appr},j}^{\text{div},0}(x, E) &= \frac{E}{2} \int_{T^2} \tilde{A}_{\text{appr},j}^{\text{div},0}(\lambda, \lambda', z, E) \left| \frac{1}{2i} \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right| |d\lambda| |d\lambda'|, \\ V_{\text{appr}}^{\text{div},0}(x, E) &= \frac{E}{2} \int_{T^2} \tilde{V}_{\text{appr}}^{\text{div},0}(\lambda, \lambda', z, E) \left| \frac{1}{2i} \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right| |d\lambda| |d\lambda'|, \\ A_{\text{appr},1}^{\pm,0}(x, E) &= \frac{E}{2} \int_{T^2} \tilde{A}_{\text{appr},1}^{\pm,0}(\lambda, \lambda', z, E) \left| \frac{1}{2i} \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right| |d\lambda| |d\lambda'|, \\ V_{\text{appr}}^{\pm,0}(x, E) &= \frac{E}{2} \int_{T^2} \tilde{V}_{\text{appr}}^{\pm,0}(\lambda, \lambda', z, E) \left| \frac{1}{2i} \left( \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) \right| |d\lambda| |d\lambda'|, \end{aligned} \quad (\text{IV.130})$$

where  $j = 1, 2$ ,  $x \in \mathbb{R}^2$ ,  $z$  is given by (IV.40) and where

$$\begin{aligned}
\tilde{A}_{\text{appr},1}^{\text{div},0}(\lambda, \lambda', z, E) &= \frac{f^{\text{lin}}(\lambda, \lambda', z, E) - f^{\text{lin}}(-\lambda', -\lambda, z, E)}{4\sqrt{E}} \times \\
&\quad \times \left( \frac{1}{\lambda^{-1} + \lambda'^{-1}} + \frac{1}{\lambda + \lambda'} \right), \\
\tilde{A}_{\text{appr},2}^{\text{div},0}(\lambda, \lambda', z, E) &= \frac{f^{\text{lin}}(\lambda, \lambda', z, E) - f^{\text{lin}}(-\lambda', -\lambda, z, E)}{4i\sqrt{E}} \times \\
&\quad \times \left( \frac{1}{\lambda^{-1} + \lambda'^{-1}} - \frac{1}{\lambda + \lambda'} \right), \\
\tilde{V}_{\text{appr}}^{\text{div},0}(\lambda, \lambda', z, E) &= \frac{f^{\text{lin}}(\lambda, \lambda', z, E) + f^{\text{lin}}(-\lambda', -\lambda, z, E)}{2}, \\
\tilde{A}_{\text{appr},1}^{\pm,0}(\lambda, \lambda', z, E) &= \frac{1}{2\sqrt{E}} \frac{f(\lambda, \lambda', z, E) - f(-\lambda', -\lambda, z, E)}{\lambda^{\pm 1} + \lambda'^{\pm 1}}, \\
\tilde{V}_{\text{appr}}^{\pm,0}(\lambda, \lambda', z, E) &= \frac{\lambda'^{\pm 1} f^{\text{lin}}(\lambda, \lambda', z, E) + \lambda^{\pm 1} f^{\text{lin}}(-\lambda', -\lambda, z, E)}{\lambda^{\pm 1} + \lambda'^{\pm 1}},
\end{aligned} \tag{IV.131}$$

where  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $f^{\text{lin}}(\lambda, \lambda', z, E)$  is defined according to (IV.64).

Now using (IV.131) we represent each integral of (IV.130) as a sum of the integral containing  $f^{\text{lin}}(\lambda, \lambda', z, E)$  and the integral containing  $f^{\text{lin}}(-\lambda', -\lambda, z, E)$  within the integrands.

Making the change of variables  $(\lambda, \lambda') \rightarrow (-\lambda', -\lambda)$  in each integral containing  $f^{\text{lin}}(-\lambda', -\lambda, z, E)$  and taking into account (IV.34) for  $\widehat{A}_2^{\pm}$  we obtain formulas (IV.65), (IV.66).  $\square$

# Article V

## On the reconstruction of parameters of a moving fluid from the Dirichlet-to-Neumann map

*A. D. Agaltsov*

We consider an inverse boundary value problem for a model time-harmonic equation of acoustic tomography of moving fluid with variable current velocity, sound speed, density and absorption. In the present article it is assumed that at fixed frequency the coefficients of this equation are already recovered modulo an appropriate gauge transformation using some reconstruction method from boundary measurements presented in the literature. Our main result consists in formulas and equations that allow to get rid of this gauge non-uniqueness and recover the fluid parameters using boundary measurements at several frequencies.

### 1 Introduction

We consider a model equation for a time-harmonic acoustic pressure  $\psi = \psi(x)$  (time dependence  $e^{-i\omega t}$ ) in a moving fluid with sound speed  $c = c(x)$ , current velocity  $v = v(x)$ , density  $\rho = \rho(x)$  and absorption  $\alpha = \alpha(x, \omega) = \omega^{\zeta(x)}\alpha_0(x)$  at fixed frequency  $\omega$  :

$$L_\omega\psi = 0 \quad \text{in } D, \tag{V.1}$$

$$L_\omega = -\Delta - 2iA_\omega(x) \cdot \nabla - U_\omega(x), \quad x = (x_1, \dots, x_d), \tag{V.2}$$

$$\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right), \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2},$$

where

$$\begin{aligned} D & \text{ is an open bounded domain in } \mathbb{R}^d, \\ d & \in \{2, 3\}, \text{ with connected } C^\infty \text{ boundary } \partial D, \end{aligned} \quad (\text{V.3})$$

$$A_\omega(x) = \frac{\omega v(x)}{c^2(x)} + i\nabla \ln \rho^{\frac{1}{2}}(x), \quad U_\omega(x) = \frac{\omega^2}{c^2(x)} + 2i\omega^{1+\zeta(x)} \frac{\alpha_0(x)}{c(x)}. \quad (\text{V.4})$$

Note that the operator  $L_\omega$  is a special case of the so-called magnetic Schroedinger operator.

The model equation (V.1) was studied in different particular cases in [71, 116, 118, 117, 24, 3, 5].

In the present article we suppose that

$$c \in C^2(\overline{D}), \quad c > 0 \text{ in } \overline{D}, \quad (\text{V.5a})$$

$$v \in C^2(\overline{D}, \mathbb{R}^d), \quad (\text{V.5b})$$

$$\rho \in C^{2,\beta}(\overline{D}), \quad \beta \in (0, 1], \quad \rho > 0 \text{ in } \overline{D}, \quad (\text{V.5c})$$

$$\zeta \in C(\overline{D}), \quad \alpha_0 \in C(\overline{D}), \quad \zeta > 0, \quad \alpha_0 \text{ is real in } \overline{D}, \quad (\text{V.5d})$$

where  $\overline{D} = D \cup \partial D$  and  $C^{k,\beta}(\overline{D})$  denotes the space of  $k$  times continuously differentiable functions in  $\overline{D}$  whose  $k$ -th derivatives are  $\beta$ -Hölder continuous. Let

$$\tilde{c} = c|_{\partial D}, \quad \tilde{v} = v|_{\partial D}, \quad \tilde{\rho} = \rho|_{\partial D}. \quad (\text{V.6})$$

In what follows we always assume that

$$0 \text{ is not a Dirichlet eigenvalue for operator } L_\omega \text{ in } D. \quad (\text{V.7})$$

Note that the set of  $\omega$ 's for which (V.7) does not hold is locally finite. Besides, (V.7) always holds for  $\omega = 0$ .

For equation (V.1) under the assumption (V.7) we consider the Dirichlet-to-Neumann type operator  $\Lambda_\omega$  which maps a sufficiently regular function  $f$  on  $\partial D$  to the function

$$\Lambda_\omega f = \left( \frac{\partial \psi}{\partial \nu} + i(\nu \cdot A_\omega) \psi \right) \Big|_{\partial D},$$

where  $\psi$  is the solution of equation (V.1) in  $D$  with Dirichlet boundary condition  $\psi|_{\partial D} = f$  and  $\nu$  is the unit exterior normal field to  $\partial D$ .

Note that it is possible to get rid of the assumption (V.7) by considering a general Robin-to-Robin map instead of the Dirichlet-to-Neumann map, see [78].

We consider the following problem :

**Problem V.1.** Find  $c$ ,  $v$ ,  $\rho$  and  $\alpha$  in  $D$  from  $\Lambda_\omega$  given for  $\omega$  in some fixed set  $\Omega$  and from  $\tilde{c}$ ,  $\tilde{v}$  and  $\tilde{\rho}$ .

- This problem was studied, in particular, in [3, 5]. In these works it was shown that
- (a) if  $\rho \equiv \text{const}$ ,  $\alpha_0 \equiv 0$  and  $\Omega = \{\omega_1\}$  then the Problem V.1 is uniquely solvable;
  - (b) if  $\alpha_0 \equiv 0$ ,  $\Omega = \{\omega_1, \omega_2\}$ ,  $\omega_1 < \omega_2$ , then the Problem V.1 is uniquely solvable;
  - (c) if  $\Omega = \{\omega_1, \omega_2, \omega_3\}$ ,  $\omega_1 < \omega_2 < \omega_3$ , and  $\zeta \neq 0$  in  $D$  then the Problem V.1 is uniquely solvable.

Reconstruction results for Problem V.1 at fixed  $\omega$  can be summarized as follows :

1. Uniqueness modulo gauge transformations in the most general case follows from the results of [61] in dimension  $d = 2$  and of [85] in dimension  $d = 3$ .
2. An approximate reconstruction algorithm modulo gauge transformations in dimension  $d = 2$  was developed in [4, 2].
3. A possible reconstruction approach modulo gauge transformations in dimension  $d = 3$  was outlined in [71].

In the present article we assume that the coefficients  $A_\omega$  and  $U_\omega$  of equation (V.1) are already recovered from  $\Lambda_\omega$  up to a gauge transformation using some appropriate method at fixed  $\omega$ . The goal of the present work is to show how to get rid of the gauge non-uniqueness and recover  $c$ ,  $v$ ,  $\rho$  and  $\alpha$  using boundary measurements at several frequencies. In this respect, the present article can be considered as a development of the article [3] where the particular case  $\rho \equiv \text{const}$ ,  $\alpha_0 \equiv 0$ ,  $\Omega = \{\omega_1\}$  was considered, and of the article [5] where the corresponding uniqueness theorems for equation (V.1) with general operator of the form (V.2) were obtained.

For a vector field  $V = (V_1, \dots, V_d)$  and a function  $f$  in  $D$  we set by definition

$$\begin{aligned} \text{curl } V &= \begin{cases} \partial_1 V_2 - \partial_2 V_1, & d = 2, \\ (\partial_2 V_3 - \partial_3 V_2, \partial_3 V_1 - \partial_1 V_3, \partial_1 V_2 - \partial_2 V_1), & d = 3, \end{cases} & \text{(V.8)} \\ \text{curl } f &= (\partial_2 f, -\partial_1 f), \quad d = 2, \end{aligned}$$

where  $\partial_j = \partial/\partial x_j$ .

In the present article it is assumed that the following functions are already recovered from the operator  $\Lambda_\omega$  at fixed  $\omega$  :

$$F = \text{curl}\left(\frac{v}{c^2}\right) \quad \text{in } D, \quad \text{(V.9)}$$

$$q_\omega = f_1 - \omega^2 f_2 + i\omega f_3 - 2i\omega^{1+\zeta} \frac{\alpha_0}{c} \quad \text{in } D, \quad \text{(V.10)}$$

where

$$f_1 = \rho^{\frac{1}{2}} \Delta \rho^{-\frac{1}{2}}, \quad f_2 = \frac{1}{c^2} + \frac{v}{c^2} \cdot \frac{v}{c^2}, \quad f_3 = \nabla \cdot \left(\frac{v}{c^2}\right) - \frac{v \cdot \nabla \ln \rho}{c^2}. \quad \text{(V.11)}$$

For the corresponding identifiability results see [61] (for  $d = 2$ ) and [85] (for  $d = 3$ ); for an approximate reconstruction algorithm see [4, 2].

Thereby, in the present article we study the following problems.

**Problem V.2.** Find  $c$ ,  $v$  and  $\rho$  in  $D$  from  $q_\omega$  given for  $\omega \in \Omega = \{\omega_1, \omega_2\}$ ,  $\omega_1 < \omega_2$ , and from  $F$ ,  $\tilde{c}$ ,  $\tilde{v}$ ,  $\tilde{\rho}$ .

**Problem V.3.** Find  $c$ ,  $v$ ,  $\rho$ ,  $\zeta$  and  $\alpha_0$  in  $D$  from  $q_\omega$  given for  $\omega \in \Omega = \{\omega_1, \omega_2, \omega_3\}$ ,  $\omega_1 < \omega_2 < \omega_3$ , and from  $F$ ,  $\tilde{c}$ ,  $\tilde{v}$ ,  $\tilde{\rho}$ .

## 2 Solution of Problem V.2

We are going to derive the explicit formulas for solving the Problem V.2. We consider (V.10) with  $\omega \in \Omega$  as a system of linear equations for  $f_1$ ,  $f_2$  and  $f_3$ . Solving this system we obtain

$$f_1 = \frac{\omega_2^2 \operatorname{Re} q_{\omega_2} - \omega_1^2 \operatorname{Re} q_{\omega_1}}{\omega_2^2 - \omega_1^2}, \quad f_2 = \frac{\operatorname{Re} q_{\omega_1} - \operatorname{Re} q_{\omega_2}}{\omega_2^2 - \omega_1^2}, \quad f_3 = \omega_1^{-1} \operatorname{Im} q_{\omega_1}. \quad (\text{V.12})$$

Set  $g = \rho^{-\frac{1}{2}}$ . It follows from formula (V.11) that  $g$  satisfies the following equation

$$g(x) = g_0(x) + \int_D G(x, y) f_1(y) g(y) dy, \quad x \in \overline{D}, \quad (\text{V.13})$$

$$g_0(x) = \int_{\partial D} \frac{\partial G(x, y)}{\partial \nu_y} \tilde{\rho}^{-\frac{1}{2}}(y) dy, \quad (\text{V.14})$$

where  $G(x, y)$  is the (non-positive) Dirichlet Green's function for operator  $\Delta$  in  $D$  and  $\nu_y$  is the unit exterior normal to  $\partial D$  at point  $y$ . Note that  $g_0$  is just the harmonic extension of  $\tilde{\rho}^{-\frac{1}{2}}$  to  $D$ . The existence of function  $G$  follows from assumption (V.3) and from [12, Theorem 4.17, p. 112].

**Lemma V.1.** Equation (V.13) is uniquely solvable for  $g \in C(\overline{D})$ .

*Démonstration.* Suppose that  $g_1, g_2 \in C(\overline{D})$  are two solutions of equation (V.13). Then their difference  $h = g_1 - g_2$  satisfies

$$h(x) = \int_D G(x, y) f_1(y) h(y) dy, \quad x \in \overline{D}. \quad (\text{V.15})$$

Using formulas (V.5c) and (V.11) we obtain that  $f_1 \in C^{0,\beta}(D)$ . Taking this into account and using formula (V.15) and Lemma [54, Lemma 4.2], we obtain that  $h \in C^2(D) \cap C(\overline{D})$  and that

$$-\Delta h + f_1(x)h = 0 \quad \text{in } D, \quad (\text{V.16a})$$

$$h|_{\partial D} = 0. \quad (\text{V.16b})$$

Using formulas (V.12) and (V.16a) we can rewrite equation (V.16a) as

$$-\Delta(\rho^{\frac{1}{2}}h) + \nabla\rho \cdot \nabla(\rho^{\frac{1}{2}}h) = 0, \quad \text{in } D. \quad (\text{V.17})$$

It follows from [54, Lemma 4.2 and Corollary 8.2] and from formulas (V.16b) and (V.17) that  $h \equiv 0$ . Hence,  $\rho^{-\frac{1}{2}}$  is the unique solution of class  $C(\bar{D})$  to equation (V.13).  $\square$

For vectors  $a = (a_1, \dots, a_d)$  and  $b = (b_1, \dots, b_d)$  in  $D$  we put by definition

$$a \times b = \begin{cases} a_1b_2 - a_2b_1, & d = 2, \\ (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1), & d = 3. \end{cases} \quad (\text{V.18})$$

Recall that the Helmholtz decomposition of the vector field  $\frac{v}{c^2}$  is given by the following formula :

$$\frac{v}{c^2} = \nabla\Phi - \text{curl } V \quad \text{in } D, \quad (\text{V.19})$$

where

$$\Phi(x) = \int_D G_0(x-y) \nabla_y \cdot \left( \frac{v(y)}{c^2(y)} \right) dy - \int_{\partial D} G_0(x-y) \frac{\nu_y \cdot \tilde{v}(y)}{\tilde{c}^2(y)} dy, \quad (\text{V.20})$$

$$V(x) = \int_D G_0(x-y) F(y) dy - \int_{\partial D} G_0(x-y) \frac{\nu_y \times \tilde{v}(y)}{\tilde{c}^2(y)} dy, \quad (\text{V.21})$$

$$G_0(x) = -\frac{1}{2\pi} \ln|x|, \quad d = 2, \quad (\text{V.22})$$

$$G_0(x) = \frac{1}{4\pi} \frac{1}{|x|}, \quad d = 3, \quad (\text{V.23})$$

where  $\nu_y$  is the unit exterior normal to  $\partial D$  at point  $y$ . Note that the vector field  $V$  is known since  $F$ ,  $\tilde{v}$  and  $\tilde{c}$  are given.

Using formula (V.19) we can recover the function  $\Phi|_{\partial D}$  modulo an additive constant (which does not matter). Fix a point  $x^0 \in \partial D$ . Let  $x: [0, 1] \rightarrow \partial D$  be a smooth curve linking  $x^0$  to some given point  $x \in \partial D$ . Then

$$\Phi(x) - \Phi(x^0) = \int_0^1 \left( \frac{\tilde{v}}{\tilde{c}^2} + \text{curl } V \right) |_{x(t)} \cdot \dot{x}(t) dt, \quad \dot{x} = \frac{dx}{dt}. \quad (\text{V.24})$$

It follows from formulas (V.11) and (V.19) that  $\Phi$  satisfies the equation

$$\begin{aligned} -\Delta\Phi + \nabla \ln \rho \cdot \nabla\Phi &= -f_3 + \text{curl } V \cdot \nabla \ln \rho \quad \text{in } D, \\ \text{or } -\Delta\eta + f_1(y)\eta &= \rho^{-\frac{1}{2}}(-f_3 + \text{curl } V \cdot \nabla \ln \rho) \quad \text{in } D, \end{aligned}$$



where  $\eta = \rho^{-\frac{1}{2}}(\Phi - \Phi(x^0))$ . The function  $\eta$  can be found from the following integral equation :

$$\begin{aligned}\eta(x) &= \eta_0(x) + \eta_1(x) + \int_D G(x, y) f_1(y) \eta(y) dy, \quad x \in \overline{D}, \quad (\text{V.25}) \\ \eta_0(x) &= \int_D G(x, y) \rho^{-\frac{1}{2}}(y) (f_3(y) - \text{curl } V(y) \cdot \nabla \ln \rho(y)) dy, \\ \eta_1(x) &= \int_{\partial D} \frac{\partial G(x, y)}{\partial \nu_y} \tilde{\rho}^{-\frac{1}{2}}(\Phi(y) - \Phi(x^0)) dy,\end{aligned}$$

where  $G(x, y)$  is the (non-positive) Dirichlet Green's function for  $\Delta$  in  $D$ . Note that  $\eta_1$  is just the harmonic extension of  $\tilde{\rho}^{-\frac{1}{2}}(\Phi|_{\partial D} - \Phi(x^0))$  to  $D$ . Also note that equation (V.25) has the same kernel as equation (V.13). Therefore, it is uniquely solvable for  $\eta \in C(\overline{D})$ .

After recovering  $\Phi - \Phi(x^0)$  in  $D$  we can find  $\frac{v}{c^2}$  using formula (V.19). Finally, using formulas (V.11) and (V.19) we obtain

$$\frac{1}{c^2} = f_2 - (\nabla \Phi - \text{curl } V)^2, \quad v = c^2(\nabla \Phi - \text{curl } V). \quad (\text{V.26})$$

The described algorithm for solving Problem V.2 is summarized in the following theorem.

**Theorem V.1.** *Suppose that  $D$  satisfies (V.3),  $\Omega = \{\omega_1, \omega_2\}$ ,  $\omega_1 \neq \omega_2$  and (V.7) holds for all  $\omega \in \Omega$ , and suppose that  $c, v, \rho$  satisfy (V.5a)–(V.5c). Then Problem V.2 can be solved as follows :*

1. Define  $f_1, f_2, f_3$  and  $V$  using formulas (V.12) and (V.21).
2. Find  $g$  as the unique solution of class  $C(\overline{D})$  to equation (V.13). Set  $\rho = g^{-2}$ .
3. Fix  $x^0 \in \partial D$  and find  $\Phi|_{\partial D} - \Phi(x^0)$  using formula (V.24).
4. Find  $\eta$  as the unique solution of class  $C(\overline{D})$  to equation (V.25). Set  $\Phi - \Phi(x^0) = \rho^{\frac{1}{2}}\eta$ .
5. Find  $c$  and  $v$  using the explicit formulas (V.26).

**Remark 3.** Suppose that  $d = 2$  and  $D = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ . Then the function  $g_0$  from formula (V.13) and the function  $G(x, y)$  from formulas (V.13) and (V.25) can be found explicitly :

$$g_0(x) = \int_{S^1} \tilde{\rho}^{-\frac{1}{2}}(\vartheta) \frac{1 - |x|^2 d\vartheta}{|\vartheta - x|^2 2\pi}, \quad G(x, y) = \frac{1}{2\pi} \ln \frac{|x||y - x|}{|y|x|^2 - x|},$$

where  $S^1 = \partial D$ . Furthermore, if  $\|f_1\|_{C(\overline{D})} < 4$  then equations (V.13) and (V.25) can be solved using the method of successive approximations in  $C(\overline{D})$ .

### 3 Solution of Problem V.3

Define the sets  $D_0$  and  $D_1$  by the formulas

$$D_0 = \left\{ x \in D \mid \frac{\operatorname{Im} q_{\omega_1}(x)}{\omega_1} = \frac{\operatorname{Im} q_{\omega_2}(x)}{\omega_2} \right\}, \quad D_1 = D \setminus D_0. \quad (\text{V.27})$$

It follows from formulas (V.5d) and (V.10) that  $D_0 = \{x \in D \mid \alpha_0(x) = 0\}$  and that the functions  $f_1$ ,  $f_2$  and  $f_3$  defined in (V.10) can be found in  $D_0$  using formulas (V.12).

Using formula (V.10) for  $x \in D_1$  and  $\omega \in \Omega$  we obtain that

$$\frac{\omega_2^{-1} \operatorname{Im} q_{\omega_2}(x) - \omega_1^{-1} \operatorname{Im} q_{\omega_1}(x)}{\omega_3^{-1} \operatorname{Im} q_{\omega_3}(x) - \omega_1^{-1} \operatorname{Im} q_{\omega_1}(x)} = \frac{\left(\frac{\omega_2}{\omega_1}\right)^{\zeta(x)} - 1}{\left(\frac{\omega_3}{\omega_1}\right)^{\zeta(x)} - 1}. \quad (\text{V.28})$$

We consider (V.28) as an equation for finding  $\zeta(x)$  at fixed  $x \in D_1$ . This equation is uniquely solvable for  $\zeta(x)$  at fixed  $x \in D_1$  as the following lemma shows.

**Lemma V.2.** *The right side of equation (V.28) at fixed  $x \in D_1$  is a strictly decreasing function of  $\zeta(x) \in (0, +\infty)$ .*

*Démonstration.* It is sufficient to show that for any  $p, q$  such that  $1 < p < q$  and for any  $t > 1$  the equality

$$\frac{p^t - 1}{q^t - 1} = \frac{p - 1}{q - 1}. \quad (\text{V.29})$$

can not hold. Assuming the equality (V.29), we define

$$\lambda_1 = \frac{p - 1}{q - 1}, \quad \lambda_2 = \frac{q - p}{q - 1}. \quad (\text{V.30})$$

Using formulas (V.29) and (V.30) we obtain the formulas

$$\lambda_1 + \lambda_2 = 1, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad (\text{V.31a})$$

$$\lambda_1 q + \lambda_2 = p, \quad \lambda_1 q^t + \lambda_2 = p^t. \quad (\text{V.31b})$$

Since the function  $f(s) = s^t$  is strictly convex, relations (V.31a)–(V.31b) can not hold. Therefore the initial assumption that (V.29) holds must be false.  $\square$

Next we find functions  $f_1$ ,  $f_2$ ,  $f_3$  and  $\alpha_0/c$  in the domain  $D_1$ . It follows from formula (V.10) that  $f_1$  and  $f_2$  in  $D_1$  can be found using formulas (V.12). It also follows from (V.10) that  $f_3$  and  $\alpha_0/c$  in  $D_1$  can be found from the following formulas :

$$f_3 = \frac{\frac{\omega_1^\zeta}{\omega_2} \operatorname{Im} q_{\omega_2} - \frac{\omega_2^\zeta}{\omega_1} \operatorname{Im} q_{\omega_1}}{\omega_1^\zeta - \omega_2^\zeta}, \quad \frac{\alpha_0}{c} = \frac{1}{2} \frac{\omega_2^{-1} \operatorname{Im} q_{\omega_2} - \omega_1^{-1} \operatorname{Im} q_{\omega_1}}{\omega_1^\zeta - \omega_2^\zeta}. \quad (\text{V.32})$$

Using the values of  $f_1$ ,  $f_2$  and  $f_3$  in  $D$  we can find  $c$ ,  $\rho$  and  $v$  in  $D$  using the steps 2–5 mentioned in Theorem V.1. Finally, we find  $\alpha_0$  using  $\alpha_0/c$  and  $c$ . The algorithm for solving Problem V.3 is summarized in the following proposition.

**Theorem V.2.** *Suppose that  $D$  satisfies (V.3),  $\Omega = \{\omega_1, \omega_2, \omega_3\}$ ,  $\omega_1 < \omega_2 < \omega_3$ , and (V.7) holds for all  $\omega \in \Omega$ . Suppose also that  $c, v, \rho, \zeta$  and  $\alpha_0$  satisfy (V.5a)–(V.5d). Then Problem V.3 can be solved as follows :*

1. *Define  $D_0$  and  $D_1$  using formula (V.27). Find  $f_1$  and  $f_2$  in  $D$  using formulas (V.12). Find  $f_3$  using formula (V.12) in  $D_0$  and formula (V.32) in  $D_1$ .*
2. *Find  $\zeta(x)$  at fixed  $x \in D_1$  as the unique positive solution to equation (V.28).*
3. *Find  $\alpha_0/c$  in  $D_1$  using formula (V.32).*
4. *Find  $g$  as the unique solution of class  $C(\overline{D})$  to equation (V.13). Set  $\rho = g^{-2}$ .*
5. *Fix  $x^0 \in \partial D$  and find  $\Phi|_{\partial D} - \Phi(x^0)$  using formula (V.24).*
6. *Find  $\eta$  as the unique solution of class  $C(\overline{D})$  to equation (V.25). Set  $\Phi - \Phi(x^0) = \rho^{\frac{1}{2}}\eta$ .*
7. *Find  $c$  and  $v$  using the explicit formulas (V.26). Set  $\alpha_0$  to zero in  $D_0$  and find  $\alpha_0$  from  $\alpha_0/c$  and  $c$  in  $D_1$ .*

**Remark 4.** Note that the formulas and equations presented in Theorem V.1 (resp. V.2) require the knowledge of the function  $q_\omega$  at two (resp. three) frequencies  $\omega$ . These formulas are exact but they can be not very stable with respect to the noise in the initial data. However, if  $q_\omega$  is known for a bigger number of frequencies it is possible to increase the stability of reconstruction by replacing formulas (V.12) and (V.32) with their least squares analogues. A numerical study of reconstruction stability will be carried out in a subsequent paper.

# Article VI

## Error estimates for phaseless inverse scattering in the Born approximation at high energies

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We study explicit formulas for phaseless inverse scattering in the Born approximation at high energies for the Schrödinger equation with compactly supported potential in dimension  $d \geq 2$ . We obtain error estimates for these formulas in the configuration space.

### 1 Introduction

We consider the time-independent Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \quad (\text{VI.1})$$

where

$$v \in L^\infty(\mathbb{R}^d), \quad \text{supp } v \subset D, \quad (\text{VI.2})$$

where  $D$  is some fixed open bounded domain in  $\mathbb{R}^d$ .

In quantum mechanics equation (VI.1) describes an elementary particle interacting with a macroscopic object contained in  $D$  at fixed energy  $E$ .

For equation (VI.1) we consider the classical scattering solutions  $\psi^+ = \psi^+(x, k)$ ,  $x \in \mathbb{R}^d$ ,  $k \in \mathbb{R}^d$ ,  $k^2 = E$ . These solutions  $\psi^+$  can be specified by the following

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asymptotics as  $|x| \rightarrow \infty$  :

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + c(d, |k|) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k| \frac{x}{|x|}) + O(|x|^{-(d+1)/2}), \\ x \in \mathbb{R}^d, k \in \mathbb{R}^d, k^2 = E, c(d, |k|) &= -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \end{aligned} \quad (\text{VI.3})$$

for some a priori unknown  $f$ . The function  $f$  arising in (VI.3) is defined on

$$\mathcal{M}_E = \{(k, l) \in \mathbb{R}^d \times \mathbb{R}^d : k^2 = l^2 = E\}, \quad (\text{VI.4})$$

and is known as the classical scattering amplitude for equation (VI.1).

In quantum mechanics  $|f(k, l)|^2$  describes the probability density of scattering of particle with initial impulse  $k$  into direction  $l/|l| \neq k/|k|$ , and is known as differential scattering cross section for equation (VI.1); see, e.g., [49, Chapter 1, Section 6].

The problem of finding  $\psi^+$  and  $f$  from  $v$  is known as the direct scattering problem for equation (VI.1). For solving this problem, one can use, in particular, the Lippmann-Schwinger integral equation for  $\psi^+$  and an explicit integral formula for  $f$ , see, e.g., [17, 48, 108].

In turn, the problem of finding  $v$  from  $f$  is known as the inverse scattering problem (with phase information) and the problem of finding  $v$  from  $|f|^2$  is known as the phaseless inverse scattering problem for equation (VI.1).

There is a vast literature on the former inverse scattering problem with phase information; see [9, 14, 21, 26, 40, 46, 56, 64, 75, 76, 97, 99, 102, 104, 107] and references therein. In particular, it is well known that the scattering amplitude  $f$  uniquely determines  $v$  via the Born approximation formulas at high energies :

$$\widehat{v}(k-l) = f(k, l) + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, (k, l) \in \mathcal{M}_E, \quad (\text{VI.5})$$

$$\widehat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d, \quad (\text{VI.6})$$

and the inverse Fourier transform; see, e.g., [46, 107].

On the other hand, the literature for the phaseless case is much more limited; see [26, 108] and references therein for the case of the aforementioned phaseless inverse problem and see [82, 83, 105, 106, 108] and references therein for the case of some similar inverse problems without phase information. In addition, it is well known that the phaseless scattering data  $|f|^2$  does not determine  $v$  uniquely, even if  $|f|^2$  is given completely for all positive energies. In particular, it is known that

$$\begin{aligned} f_y(k, l) &= e^{i(k-l)y} f(k, l), \\ |f_y(k, l)|^2 &= |f(k, l)|^2, \quad k, l \in \mathbb{R}^d, k^2 = l^2 > 0, \end{aligned} \quad (\text{VI.7})$$

where  $f$  is the scattering amplitude for  $v$  and  $f_y$  is the scattering amplitude for  $v_y = v(\cdot - y)$ , where  $y \in \mathbb{R}^d$ ; see [108] and references therein.

In the present work, in view of the aforementioned non-uniqueness for the problem of finding  $v$  from  $|f|^2$ , we consider the modified phaseless inverse scattering problem formulated below as Problem VI.1. Let

$$S = \{|f|^2, |f_1|^2, \dots, |f_m|^2\}, \quad (\text{VI.8})$$

where  $f$  is the scattering amplitude for  $v$  and  $f_1, \dots, f_m$  are the scattering amplitudes for  $v_1, \dots, v_m$ , where

$$v_j = v + w_j, \quad j = 1, \dots, m, \quad (\text{VI.9})$$

where  $w_1, \dots, w_m$  are additional a priori known background scatterers such that

$$\begin{aligned} w_j &\in L^\infty(\mathbb{R}^d), \quad \text{supp } w_j \subset \Omega_j, \\ \Omega_j &\text{ is an open bounded domain in } \mathbb{R}^d, \quad \Omega_j \cap D = \emptyset, \\ w_j &\neq 0, \quad w_{j_1} \neq w_{j_2} \text{ if } j_1 \neq j_2 \text{ (in } L^\infty(\mathbb{R}^d)), \end{aligned} \quad (\text{VI.10})$$

where  $j, j_1, j_2 \in \{1, \dots, m\}$ . Thus,  $S$  consists of the phaseless scattering data  $|f|^2, |f_1|^2, \dots, |f_m|^2$  measured sequentially, first, for the unknown scatterer  $v$  and then for  $v$  in the presence of known scatterer  $w_j$  disjoint from  $v$  for  $j = 1, \dots, m$ .

Actually, in the present work we continue studies of [108] on the following inverse scattering problem for equation (VI.1) :

**Problem VI.1.** *Reconstruct potential  $v$  from the phaseless scattering data  $S$  for some appropriate background scatterers  $w_1, \dots, w_m$ .*

Studies of Problem VI.1 in dimension  $d \geq 2$  were started in [108]. In dimension  $d = 1$  for  $m = 1$  studies of Problem VI.1 were started earlier in [7], where phaseless scattering data was considered for all  $E > 0$ .

Actually, the key result of [108] consists in a proper extension of formula (VI.5) for the Fourier transform  $\hat{v}$  of  $v$  to the phaseless case of Problem VI.1; see Section 5.

In the present work we proceed from the aforementioned result of [108] and study related approximate reconstruction of  $v$  in the configuration space. In this connection our results consist in obtaining related error estimates in the configuration space at high energies  $E$ ; see Section 4.

In addition, results of the present work are necessary for extending the iterative algorithm of [107] to the phaseless case of Problem VI.1. The latter extension will be given in a subsequent work.

## 2 Extension of formula (VI.5) to the phaseless case

Actually, the key result of [108] consists in the following formulas for solving Problem VI.1 in dimension  $d \geq 2$  for  $m = 2$  at high energies  $E$  :

$$\begin{pmatrix} \operatorname{Re} \widehat{v} \\ \operatorname{Im} \widehat{v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \operatorname{Re} \widehat{w}_1 & \operatorname{Im} \widehat{w}_1 \\ \operatorname{Re} \widehat{w}_2 & \operatorname{Im} \widehat{w}_2 \end{pmatrix}^{-1} \begin{pmatrix} |\widehat{v}_1|^2 - |\widehat{v}|^2 - |\widehat{w}_1|^2 \\ |\widehat{v}_2|^2 - |\widehat{v}|^2 - |\widehat{w}_2|^2 \end{pmatrix}, \quad (\text{VI.11})$$

$$\begin{aligned} |\widehat{v}_j(p)|^2 &= |f_j(k, l)|^2 + O(E^{-\frac{1}{2}}), \quad E \rightarrow +\infty, \\ p \in \mathbb{R}^d, \quad (k, l) \in \mathcal{M}_E, \quad k - l = p, \quad j = 0, 1, 2, \end{aligned} \quad (\text{VI.12})$$

where :

- $v_0 = v$ ,  $v_j$  is defined by (VI.9),  $j = 1, 2$ , and  $f_0 = f$ ,  $f_1, f_2$  are the scattering amplitudes for  $v_0, v_1, v_2$ , respectively;
- $\widehat{v} = \widehat{v}(p)$ ,  $\widehat{v}_j = \widehat{v}_j(p)$ ,  $\widehat{w}_j = \widehat{w}_j(p)$ ,  $p \in \mathbb{R}^d$ , are the Fourier transforms of  $v, v_j, w_j$  (defined as in (VI.6));
- formula (VI.11) is considered for all  $p \in \mathbb{R}^d$  such that the determinant

$$\zeta_{\widehat{w}_1, \widehat{w}_2}(p) \stackrel{\text{def}}{=} \operatorname{Re} \widehat{w}_1(p) \operatorname{Im} \widehat{w}_2(p) - \operatorname{Im} \widehat{w}_1(p) \operatorname{Re} \widehat{w}_2(p) \neq 0. \quad (\text{VI.13})$$

The point is that using formulas (VI.12) for  $d \geq 2$  with

$$\begin{aligned} k &= k_E(p) = \frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \\ l &= l_E(p) = -\frac{p}{2} + \left(E - \frac{p^2}{4}\right)^{1/2} \gamma(p), \\ |\gamma(p)| &= 1, \quad \gamma(p)p = 0, \end{aligned} \quad (\text{VI.14})$$

where  $p \in \mathbb{R}^d$ ,  $|p| \leq 2\sqrt{E}$ , one can reconstruct  $|\widehat{v}|^2, |\widehat{v}_1|^2, |\widehat{v}_2|^2$  from  $S$  at high energies for any  $p \in \mathbb{R}^d$ . And then using formula (VI.11) one can reconstruct  $\widehat{v}$  completely, provided that condition (VI.13) is fulfilled for almost all  $p \in \mathbb{R}^d$ .

**Remark 5.** Formulas (VI.12) can be precised as formula (2.15) of [108] :

$$\begin{aligned} \left| |\widehat{v}_j(p)|^2 - |f_j(k, l)|^2 \right| &\leq c(D_j) N_j^3 E^{-\frac{1}{2}}, \\ p = k - l, \quad (k, l) \in \mathcal{M}_E, \quad E^{\frac{1}{2}} &\geq \rho(D_j, N_j), \quad j = 0, 1, 2, \end{aligned} \quad (\text{VI.15})$$

where  $\|v_j\|_{L^\infty(D_j)} \leq N_j$ ,  $j = 0, 1, 2$ , and  $D_0 = D$ ,  $D_j = D \cup \Omega_j$ ,  $j = 1, 2$ , and formulas for the constants  $c, \rho$  are given in [108].

In addition, from the experimental point of view it seems to be, in particular, convenient to consider Problem VI.1 with  $m = 2$  for the case when  $w_2$  is just a translation of  $w_1$  :

$$w_2(x) = w_1(x - y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d. \quad (\text{VI.16})$$

In this case

$$\widehat{w}_2(p) = e^{ipy} \widehat{w}_1(p), \quad \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = \sin(py) |\widehat{w}_1(p)|^2, \quad p \in \mathbb{R}^d. \quad (\text{VI.17})$$

On the level of analysis, the principal complication of (VI.11), (VI.12) in comparison with (VI.5) consists in possible zeros of the determinant  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  of (VI.13). For some simplest cases, we study these zeros in the next section.

### 3 Zeros of the determinant

Let

$$\begin{aligned} Z_{\widehat{w}_1, \widehat{w}_2} &= \{p \in \mathbb{R}^d : \zeta_{\widehat{w}_1, \widehat{w}_2}(p) = 0\}, \\ Z_{\widehat{w}_j} &= \{p \in \mathbb{R}^d : \widehat{w}_j(p) = 0\}, \quad j = 1, 2, \end{aligned} \quad (\text{VI.18})$$

where  $\zeta$  is defined by (VI.13). From (VI.13), (VI.18) it follows that

$$Z_{\widehat{w}_1} \cup Z_{\widehat{w}_2} \subseteq Z_{\widehat{w}_1, \widehat{w}_2}. \quad (\text{VI.19})$$

In view of (VI.19), in order to construct examples of  $w_1, w_2$  such that the set  $Z_{\widehat{w}_1, \widehat{w}_2}$  is as simple as possible, we use the following lemma :

**Lemma VI.1.** *Let*

$$w(x) = |x|^\nu K_\nu(|x|) \int_{\mathbb{R}^d} q(x-y)q(y) dy, \quad x \in \mathbb{R}^d, \nu > 0, \quad (\text{VI.20})$$

$$K_\nu(s) = \frac{\Gamma(\frac{1}{2} + \nu)}{\sqrt{\pi}} \left(\frac{2}{s}\right)^\nu \int_0^\infty \frac{\cos(st) dt}{(1+t^2)^{\frac{1}{2}+\nu}}, \quad s > 0, \quad (\text{VI.21})$$

$$\begin{aligned} q &\in L^\infty(\mathbb{R}^d), \quad q = \bar{q}, \quad q \neq 0 \text{ in } L^\infty(\mathbb{R}^d), \\ q(x) &= 0 \text{ if } |x| > r, \quad q(x) = q(-x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (\text{VI.22})$$

Then

$$\begin{aligned} w &\in C(\mathbb{R}^d), \quad w = \bar{w}, \quad w(x) = 0 \text{ if } |x| > 2r, \quad x \in \mathbb{R}^d, \\ \widehat{w}(p) &= \overline{\widehat{w}(p)} \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (\text{VI.23})$$

for  $\beta = d + 2\nu$  and some positive constant  $c_1 = c_1(q, \nu)$ , where  $\widehat{w}$  is the Fourier transform of  $w$ . In addition, if  $q \geq 0$ , then  $w \geq 0$ .

We recall that  $K_\nu$  defined by (VI.21) is the modified Bessel function of the second kind and order  $\nu$ . In addition,  $\Gamma$  denotes the gamma function.

Lemma VI.1 is proved in Subsection 5.4.

As a corollary of Lemma VI.1, functions

$$w_j(x) = w(x - T_j), \quad x \in \mathbb{R}^d, \quad T_j \in \mathbb{R}^d, \quad (\text{VI.24})$$

where  $w$  is constructed in Lemma VI.1, give us examples of  $w_j$  satisfying (VI.10) for fixed  $D, \Omega_j$  and for appropriate radius  $r$  of Lemma VI.1 and translations  $T_j$  of (VI.24), and such that

$$\begin{aligned} Z_{\widehat{w}_j} &= \emptyset, \\ |\widehat{w}_j(p)| &= \widehat{w}(p) \geq c_1(1+|p|)^{-\beta}, \quad p \in \mathbb{R}^d, \end{aligned} \quad (\text{VI.25})$$

where  $c_1, \beta$  are the same as in (VI.23). In addition,

$$\begin{aligned} \zeta_{\widehat{w}_1, \widehat{w}_2}(p) &= \sin(py)|\widehat{w}(p)|^2, \quad y = T_2 - T_1 \neq 0, \quad p \in \mathbb{R}^d, \\ Z_{\widehat{w}_1, \widehat{w}_2} &= \{p \in \mathbb{R}^d : \sin(py) = 0\} = \{p \in \mathbb{R}^d : py \in \pi\mathbb{Z}\}, \end{aligned} \quad (\text{VI.26})$$



for  $w_1, w_2$  of (VI.24).

As another corollary of Lemma VI.1, we have that

$$\begin{aligned} \zeta_{\widehat{w}_1, \widehat{w}_2}(p) &= |\widehat{w}(p)|^2 \geq c_1^2(1 + |p|)^{-2\beta}, \quad p \in \mathbb{R}^d, \\ Z_{\widehat{w}_1, \widehat{w}_2} &= \emptyset, \end{aligned} \tag{VI.27}$$

if  $w_1$  is defined as in (VI.24) and  $w_2 = iw_1$ .

Note that complex-valued  $v$  and  $w_j$  naturally arise if we interpret equation (VI.1) as a time-harmonic Helmholtz equation of acoustics or electrodynamics.

Finally, note that

$$\begin{aligned} Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}} &= \frac{\pi}{s} \mathbb{Z}^d, \quad \text{where} \\ Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}} &= Z_{\widehat{w}_1, \widehat{w}_2} \cap Z_{\widehat{w}_1, \widehat{w}_3} \cap \dots \cap Z_{\widehat{w}_1, \widehat{w}_{d+1}}, \end{aligned} \tag{VI.28}$$

if  $w_1$  is defined as in (VI.24), and

$$w_2(x) = w_1(x - se_1), \dots, w_{d+1}(x) = w_1(x - se_d), \tag{VI.29}$$

where  $(e_1, \dots, e_d)$  is the standard basis of  $\mathbb{R}^d$  and  $s > 0$ .

Thus, in principle, for Problem VI.1 with background scatterers  $w_1, \dots, w_{d+1}$  as in (VI.29), for each  $p \in \mathbb{R}^d \setminus \frac{\pi}{s} \mathbb{Z}^d$  formulas (VI.11), (VI.12) can be used with appropriate  $w_j$  in place of  $w_2$ , where  $j = 2, \dots, d+1$ .

## 4 Error estimates in the configuration space

We recall that for inverse scattering with phase information the scattering amplitude  $f$  on  $\mathcal{M}_E$  processed by (VI.5) and the inverse Fourier transform yield the approximate reconstruction

$$u(\cdot, E) = v + O(E^{-\alpha}) \quad \text{in } L^\infty(\mathbb{R}^d) \text{ as } E \rightarrow +\infty, \quad \alpha = \frac{n-d}{2n}, \tag{VI.30}$$

if  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n > d$  (in addition to the initial assumption (VI.2)), where  $W^{n,1}(\mathbb{R}^d)$  denotes the standard Sobolev space of  $n$ -times differentiable functions in  $L^1(\mathbb{R}^d)$ :

$$\begin{aligned} W^{n,1}(\mathbb{R}^d) &= \{u \in L^1(\mathbb{R}^d) : \|u\|_{n,1} < \infty\}, \\ \|u\|_{n,1} &= \max_{|J| \leq n} \left\| \frac{\partial^{|J|} u}{\partial x^J} \right\|_{L^1(\mathbb{R}^d)}, \quad n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{VI.31}$$

More precisely, the approximation  $u(\cdot, E)$  in (VI.30) is defined by

$$\begin{aligned} u(x, E) &= \int_{\mathcal{B}_{r(E)}} e^{-ipx} f(k_E(p), l_E(p)) dp, \quad x \in \mathbb{R}^d, \\ r(E) &= 2\tau E^{\frac{\alpha}{n-d}} \quad \text{for some fixed } \tau \in (0, 1], \end{aligned} \tag{VI.32}$$

where

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : |p| \leq r\}, \quad (\text{VI.33})$$

$\alpha$  is defined in (VI.30), and  $k_E(p)$ ,  $l_E(p)$  are defined as in (VI.14) with some piecewise continuous vector-function  $\gamma$  on  $\mathbb{R}^d$ ; see, e.g., [107].

Analogues of  $u(\cdot, E)$  for the phaseless case are given below in this section. In particular, related formulas depend on the zeros of determinant  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  of (VI.13).

We consider

$$\begin{aligned} U_{\widehat{w}_1, \widehat{w}_2} &= \text{Re } U_{\widehat{w}_1, \widehat{w}_2} + i \text{Im } U_{\widehat{w}_1, \widehat{w}_2}, \\ \begin{pmatrix} \text{Re } U_{\widehat{w}_1, \widehat{w}_2} \\ \text{Im } U_{\widehat{w}_1, \widehat{w}_2} \end{pmatrix} &= \frac{1}{2} M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) b_{\widehat{w}_1, \widehat{w}_2}(p, E), \end{aligned} \quad (\text{VI.34})$$

$$M_{\widehat{w}_1, \widehat{w}_2}(p) = \begin{pmatrix} \text{Re } \widehat{w}_1(p) & \text{Im } \widehat{w}_1(p) \\ \text{Re } \widehat{w}_2(p) & \text{Im } \widehat{w}_2(p) \end{pmatrix}, \quad (\text{VI.35})$$

$$M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) = \frac{1}{\zeta_{\widehat{w}_1, \widehat{w}_2}(p)} \begin{pmatrix} \text{Im } \widehat{w}_2(p) & -\text{Im } \widehat{w}_1(p) \\ -\text{Re } \widehat{w}_2(p) & \text{Re } \widehat{w}_1(p) \end{pmatrix}, \quad (\text{VI.36})$$

$$b_{\widehat{w}_1, \widehat{w}_2}(p, E) = \begin{pmatrix} |f_1(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_1(p)|^2 \\ |f_2(p, E)|^2 - |f(p, E)|^2 - |\widehat{w}_2(p)|^2 \end{pmatrix}, \quad (\text{VI.37})$$

$$f(p, E) = f(k_E(p), l_E(p)), \quad f_j(p, E) = f_j(k_E(p), l_E(p)), \quad j = 1, 2, \quad (\text{VI.38})$$

where  $\widehat{w}_1$ ,  $\widehat{w}_2$ ,  $f$ ,  $f_1$ ,  $f_2$  are the same as in (VI.11), (VI.12),  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  is defined by (VI.13),  $k_E(p)$ ,  $l_E(p)$  are the same as in (VI.14), (VI.32), and  $p \in \mathcal{B}_{2\sqrt{E}}$ ,  $d \geq 2$ .

For Problem VI.1 for  $d \geq 2$ ,  $m = 2$ , and for the case when  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  has no zeros (the case of (VI.27) in Section 3) we have the following result :

**Theorem VI.1.** *Let  $v$  satisfy (VI.2) and  $v \in W^{n,1}(\mathbb{R}^d)$  for some  $n > d$ . Let  $w_1$ ,  $w_2$  be the same as in (VI.27). Let*

$$u(x, E) = \int_{\mathcal{B}_{r_1(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp, \quad x \in \mathbb{R}^d, \quad (\text{VI.39})$$

$$r_1(E) = 2\tau E^{\frac{\alpha_1}{n-d}}, \quad \alpha_1 = \frac{n-d}{2(n+\beta)}, \quad \text{for some fixed } \tau \in (0, 1],$$

where  $U_{\widehat{w}_1, \widehat{w}_2}$  is defined by (VI.34),  $\mathcal{B}_r$  is defined by (VI.33),  $\beta$  is the number of (VI.23), (VI.27). Then

$$u(\cdot, E) = v + O(E^{-\alpha_1}) \quad \text{in } L^\infty(\mathbb{R}^d), \quad E \rightarrow +\infty. \quad (\text{VI.40})$$

Theorem VI.1 is proved in Subsection 5.1.

Next, we set

$$Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon = \{p \in \mathbb{R}^d : py \in (-\varepsilon, \varepsilon) + \pi\mathbb{Z}\}, \quad y \in \mathbb{R}^d \setminus 0, \quad 0 < \varepsilon < 1, \quad (\text{VI.41})$$

where  $\widehat{w}_1$ ,  $\widehat{w}_2$  and  $y$  are the same as in (VI.24)–(VI.26). One can see that  $Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon$  is the open  $\frac{\varepsilon}{|y|}$ -neighborhood of  $Z_{\widehat{w}_1, \widehat{w}_2}$  defined in (VI.26).

Note that

$$\begin{aligned} & \text{for any } p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon \text{ there exists} \\ & \text{the unique } z(p) \in \mathbb{Z} \text{ such that } |py - \pi z(p)| < \varepsilon. \end{aligned} \quad (\text{VI.42})$$

In addition to  $U_{\widehat{w}_1, \widehat{w}_2}$  of (VI.34), we define

$$\begin{aligned} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) &= \frac{1}{2} (U_{\widehat{w}_1, \widehat{w}_2}(p_-^\varepsilon, E) + U_{\widehat{w}_1, \widehat{w}_2}(p_+^\varepsilon, E)), \\ p_\pm^\varepsilon &= p_\perp + \pi z(p) \frac{y}{|y|^2} \pm \varepsilon \frac{y}{|y|^2}, \quad p_\perp = p - (py) \frac{y}{|y|^2}, \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \end{aligned} \quad (\text{VI.43})$$

where  $z(p)$  is the integer number of (VI.42).

For Problem VI.1 for  $d \geq 2$ ,  $m = 2$ , and for the case when  $\zeta_{\widehat{w}_1, \widehat{w}_2}$  has zeros on hyperplanes (the case of (VI.26) in Section 3) we have the following result :

**Theorem VI.2.** *Let  $v$  satisfy (VI.2) and  $v \in W^{n,1}(\mathbb{R}^d)$  for some  $n > d$ . Let  $w_1, w_2$  be the same as in (VI.24)–(VI.26). Let*

$$\begin{aligned} u(x, E) &= u_1(x, E) + u_2(x, E), \quad x \in \mathbb{R}^d, \\ u_1(x, E) &= \int_{\mathcal{B}_{r_2(E)} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp, \\ u_2(x, E) &= \int_{\mathcal{B}_{r_2(E)} \cap Z_{\widehat{w}_1, \widehat{w}_2}^{\varepsilon_2(E)}} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) dp, \\ r_2(E) &= 2\tau E^{\frac{\alpha_2}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_2}{2}}, \\ \alpha_2 &= \frac{n-d}{2(n+\beta+\frac{n-d}{2})}, \quad \text{for some fixed } \tau \in (0, 1], \end{aligned} \quad (\text{VI.44})$$

where  $U_{\widehat{w}_1, \widehat{w}_2}$  and  $U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon$  are defined by (VI.34), (VI.43),  $\mathcal{B}_r$  and  $Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon$  are defined by (VI.33), (VI.41), and  $\beta$  is the number of (VI.23). Then

$$u(\cdot, E) = v + O(E^{-\alpha_2}) \quad \text{in } L^\infty(\mathbb{R}^d), \quad E \rightarrow +\infty. \quad (\text{VI.45})$$

Theorem VI.2 is proved in Subsection 5.2.

Next, we set

$$\begin{aligned} Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon &= \mathcal{B}'_{\varepsilon/s} + \frac{\pi}{s} \mathbb{Z}^d, \quad 0 < \varepsilon < 1, \\ \mathcal{B}'_r &= \mathcal{B}_r \setminus \partial \mathcal{B}_r, \quad r > 0, \end{aligned} \quad (\text{VI.46})$$

where  $\widehat{w}_1, \dots, \widehat{w}_{d+1}$  are the same as in (VI.28), (VI.29), and  $\mathcal{B}_r$  is defined by (VI.33). One can see that  $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$  is the open  $\frac{\varepsilon}{s}$ -neighborhood of  $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}$  defined in (VI.28).

Note that

$$\begin{aligned} & \text{for any } p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon \text{ there exists} \\ & \text{the unique } z(p) \in \mathbb{Z}^d \text{ such that } |sp - \pi z(p)| < \varepsilon. \end{aligned} \quad (\text{VI.47})$$

In addition, we consider  $i'$  such that

$$\begin{aligned} i' &= i'(p, s), \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d \setminus \frac{\pi}{s}\mathbb{Z}^d, \quad s > 0, \\ & i' \text{ take values in } \{2, \dots, d+1\}, \\ |\sin(sp_{i'-1})| &\geq |\sin(sp_{i-1})| \quad \text{for all } i \in \{2, \dots, d+1\}. \end{aligned} \quad (\text{VI.48})$$

Let

$$U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) = U_{\widehat{w}_1, \widehat{w}_{i'}}(p, E), \quad p \in \mathbb{R}^d \setminus \frac{\pi}{s}\mathbb{Z}^d, \quad (\text{VI.49})$$

$$\begin{aligned} & U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) = \\ & \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}\left(\frac{\varepsilon}{s}\vartheta + \frac{\pi}{s}z(p), E\right) d\vartheta, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \end{aligned} \quad (\text{VI.50})$$

$$\mathbb{S}^{d-1} = \{p \in \mathbb{R}^d : |p| = 1\}, \quad (\text{VI.51})$$

where  $|\mathbb{S}^{d-1}|$  denotes the standard Euclidean volume of  $\mathbb{S}^{d-1}$ ,  $U_{\widehat{w}_1, \widehat{w}_{i'}}$  is defined in a similar way with  $U_{\widehat{w}_1, \widehat{w}_2}$  of (VI.34),  $\widehat{w}_1, \dots, \widehat{w}_{d+1}$  are the same as in (VI.28), (VI.29), and  $i'$  is the same as in (VI.48).

For Problem VI.1 for  $d \geq 2$ ,  $m = d + 1$ , we also have the following result :

**Theorem VI.3.** *Let  $v$  satisfy (VI.2) and  $v \in W^{n,1}(\mathbb{R}^d)$  for some  $n > d$ . Let  $w_1, \dots, w_{d+1}$  be the same as in (VI.29). Let*

$$\begin{aligned} u(x, E) &= u_1(x, E) + u_2(x, E), \quad x \in \mathbb{R}^d, \\ u_1(x, E) &= \int_{\mathcal{B}_{r_3(E)} \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^{\varepsilon_3(E)}} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) dp, \\ u_2(x, E) &= \int_{\mathcal{B}_{r_3(E)} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^{\varepsilon_3(E)}} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) dp, \\ r_3(E) &= 2\tau E^{\frac{\alpha_3}{n-d}}, \quad \varepsilon_2(E) = E^{-\frac{\alpha_3}{d+1}}, \\ \alpha_3 &= \frac{n-d}{2(n+\beta+\frac{n-d}{d+1})}, \quad \text{for some fixed } \tau \in (0, 1], \end{aligned} \quad (\text{VI.52})$$

where  $U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}$  and  $U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$  are defined by (VI.49), (VI.50),  $\mathcal{B}_r$  and  $Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon$  are defined by (VI.33), (VI.46), and  $\beta$  is the number of (VI.23). Then

$$u(\cdot, E) = v + O(E^{-\alpha_3}) \quad \text{in } L^\infty(\mathbb{R}^d), \quad E \rightarrow +\infty. \quad (\text{VI.53})$$

Theorem VI.3 is proved in Subsection 5.3.

## 5 Proofs of the main results

### 5.1 Proof of Theorem VI.1

**Proposition VI.1.** *Let  $v$  satisfy (VI.2) and  $w_1, w_2$  be the same as in (VI.27),  $d \geq 2$ . Then :*

$$\begin{aligned} |\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}(p, E)| &\leq c_2 |\widehat{w}(p)|^{-1} E^{-\frac{1}{2}} \quad \text{for } p \in \mathcal{B}_{2\sqrt{E}}, E \geq \rho_1, \\ c_2 &= 2c(D_0)N_0^3 + 2c(D_1)N_1^3, \\ \rho_1 &= \max_{j=0,1} \rho(D_j, N_j), \end{aligned} \tag{VI.54}$$

where  $w$  is the function of (VI.23), (VI.24), and  $c, \rho, N_j, D_j, j = 0, 1, 2$ , are the same as in estimates (VI.15).

Proposition VI.1 follows from formulas (VI.11), (VI.14), estimates (VI.15), definitions (VI.34)–(VI.38), and the properties that

$$\Omega_2 = \Omega_1, D_2 = D_1, N_2 = N_1. \tag{VI.55}$$

In turn, properties (VI.55) follow from (VI.9) for  $j = 1, 2$ , (VI.10) for  $j = 1$ , and from the equality  $w_2 = iw_1$  assumed in (VI.27).

Next, we represent  $v$  as follows :

$$\begin{aligned} v(x) &= v^+(x, r) + v^-(x, r), \quad x \in \mathbb{R}^d, r > 0, \\ v^+(x, r) &= \int_{\mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp, \\ v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp. \end{aligned} \tag{VI.56}$$

Since  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n > d$ , we have

$$\begin{aligned} |v^-(x, r)| &\leq c_3 \|v\|_{n,1} r^{d-n}, \quad x \in \mathbb{R}^d, r > 0, \\ c_3 &= |\mathbb{S}^{d-1}| \frac{(2\pi)^{-d} d^n}{n-d}, \end{aligned} \tag{VI.57}$$

where  $\|\cdot\|_{n,1}$  is defined in (VI.31), and  $|\mathbb{S}^{d-1}|$  is the standard Euclidean volume of  $\mathbb{S}^{d-1}$ . Indeed,

$$\begin{aligned} |p_1^{k_1} \cdots p_d^{k_d} \widehat{v}(p)| &\leq (2\pi)^{-d} \|v\|_{n,1}, \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d, \\ \text{for any } k_1, \dots, k_d &\in \mathbb{N} \cup \{0\}, k_1 + \cdots + k_d \leq n, \end{aligned} \tag{VI.58}$$

assuming also that  $p_j^0 = 1$ . Taking an appropriate sum in (VI.58) over all such  $k_1, \dots, k_d$  with  $k_1 + \cdots + k_d = m \leq n$ , we get

$$|p|^m |\widehat{v}(p)| \leq (|p_1| + \cdots + |p_d|)^m |\widehat{v}(p)| \leq (2\pi)^{-d} d^m \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \tag{VI.59}$$

The definition of  $v^-$  of (VI.56) and inequalities (VI.59) for  $m = n$  imply (VI.57).

In addition, using Proposition VI.1 and the estimate on  $\widehat{w}$  of (VI.23), we obtain :

$$\begin{aligned} & \left| v^+(x, r) - \int_{\mathcal{B}_r} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right| \leq c_2 E^{-\frac{1}{2}} \int_{\mathcal{B}_r} |\widehat{w}(p)|^{-1} dp \\ & \leq c_1^{-1} c_2 E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp \leq c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad c_4 = |\mathbb{S}^{d-1}|^{\frac{2d+\beta}{d+\beta}} c_1^{-1} c_2, \quad (\text{VI.60}) \\ & \quad x \in \mathbb{R}^d, \quad 1 \leq r \leq 2\sqrt{E}, \quad E \geq \rho_1. \end{aligned}$$

As a corollary of (VI.56), (VI.57), (VI.60), we have

$$\left| v(x) - \int_{\mathcal{B}_r} e^{-ipx} U_{\widehat{w}_1, \widehat{w}_2}(p, E) dp \right| \leq c_3 \|v\|_{n,1} r^{d-n} + c_4 E^{-\frac{1}{2}} r^{d+\beta}, \quad (\text{VI.61})$$

where  $x \in \mathbb{R}^d$ ,  $1 \leq r \leq 2\sqrt{E}$ ,  $E \geq \rho_1$ . In addition, if  $r = r_1(E)$ , where  $r_1(E)$  is defined in (VI.39), then

$$\begin{aligned} r^{d-n} &= (2\tau)^{d-n} E^{-\alpha_1}, \\ E^{-\frac{1}{2}} r^{d+\beta} &= (2\tau)^{d+\beta} E^{-\alpha_1}. \end{aligned} \quad (\text{VI.62})$$

Formula (VI.40) follows from formulas (VI.61) and (VI.62).

Theorem VI.1 is proved.

## 5.2 Proof of Theorem VI.2

**Proposition VI.2.** *Let  $v$  satisfy (VI.2) and  $w_1, w_2$  be the same as in (VI.24)–(VI.26),  $d \geq 2$ . Then :*

$$\begin{aligned} & |\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}(p, E)| \leq c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\ & \quad p \in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad E \geq \rho_2, \quad 0 < \varepsilon < 1, \\ & \quad c_5 = \frac{\pi}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + c(D_2)N_2^3) c_1, \\ & \quad \rho_2 = \max_{j=0,1,2} \rho(D_j, N_j), \end{aligned} \quad (\text{VI.63})$$

in addition, if  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n \geq 0$ , then :

$$\begin{aligned} & |\widehat{v}(p) - U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E)| \\ & \leq 2^\beta c_5 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}} + c_6 \varepsilon (1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|)^{-n}, \\ & \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad E \geq \rho_2, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}|y|\}; \\ & \quad c_6 = 2^n \frac{(d+1)^{n+1}}{(2\pi)^d |y|} \max_{j=1, \dots, d} \|x_j v\|_{n,1}, \end{aligned} \quad (\text{VI.64})$$

where  $c, \rho, N_j, D_j, j = 0, 1, 2$ , are the same as in estimates (VI.15),  $c_1, \beta$  are the same as in Lemma VI.1;  $z(p), p_\perp$  are defined in (VI.42), (VI.43),  $x_j v = x_j v(x)$ , and  $\|\cdot\|_{n,1}$  is defined in (VI.31).

*Proof of Proposition VI.2.* It follows from formulas (VI.24), (VI.26) and (VI.35), (VI.36) that

$$\begin{aligned} M_{\widehat{w}_1, \widehat{w}_2}(p) &= \widehat{w}(p) \begin{pmatrix} \cos(T_1 p) & \sin(T_1 p) \\ \cos(T_2 p) & \sin(T_2 p) \end{pmatrix}, \quad p \in \mathbb{R}^d, \\ M_{\widehat{w}_1, \widehat{w}_2}^{-1}(p) &= \frac{1}{\sin(py)\widehat{w}(p)} \begin{pmatrix} \sin(T_2 p) & -\sin(T_1 p) \\ -\cos(T_2 p) & \cos(T_1 p) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (\text{VI.65})$$

Also note that

$$|\sin(py)| \geq \frac{2\varepsilon}{\pi}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad 0 < \varepsilon < 1. \quad (\text{VI.66})$$

The estimate (VI.63) follows from (VI.11), (VI.15), (VI.23), (VI.34), (VI.37), (VI.38) and from (VI.65), (VI.66).

It remains to prove (VI.64). Using definition (VI.43), one can write

$$\begin{aligned} U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p, E) - \widehat{v}(p) &= \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\ \varphi_1^\varepsilon(p, E) &= \frac{1}{2}(U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p_-, E) - \widehat{v}(p_-)) \\ &+ \frac{1}{2}(U_{\widehat{w}_1, \widehat{w}_2}^\varepsilon(p_+, E) - \widehat{v}(p_+)), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \\ \varphi_2^\varepsilon(p) &= \frac{1}{2}(\widehat{v}(p_-) + \widehat{v}(p_+)) - \widehat{v}(p), \quad p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (\text{VI.67})$$

Using estimate (VI.63), formula (VI.67) and the definitions of  $p_\pm^\varepsilon$  in (VI.43), we get

$$\begin{aligned} |\varphi_1^\varepsilon(p, E)| &\leq \frac{1}{2}c_5\varepsilon^{-1}E^{-\frac{1}{2}}((1 + |p_-^\varepsilon|)^\beta + (1 + |p_+^\varepsilon|)^\beta) \\ &\leq c_5\varepsilon^{-1}(1 + |p| + 2\frac{\varepsilon}{|y|})^\beta E^{-\frac{1}{2}} \leq 2^\beta c_5\varepsilon^{-1}(1 + |p|)^\beta E^{-\frac{1}{2}}, \\ &\text{for } \varepsilon \text{ as in (VI.64), } p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (\text{VI.68})$$

Next, using the definition of  $\varphi_2^\varepsilon$  in (VI.67) and the mean value theorem, we obtain

$$|\varphi_2^\varepsilon(p)| \leq \frac{\varepsilon}{|y|} \max\{|\frac{y}{|y|} \nabla \widehat{v}(\xi)| : \xi \in [p_-^\varepsilon, p_+^\varepsilon]\}, \quad p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon, \quad (\text{VI.69})$$

where  $[p_-^\varepsilon, p_+^\varepsilon]$  denotes the segment joining  $p_-^\varepsilon$  to  $p_+^\varepsilon$ . Here, the mean value theorem was used for  $\widehat{v}(\xi)$  on  $[p_-^\varepsilon, p]$  and on  $[p, p_+^\varepsilon]$ .

Note also that

$$|\nabla \widehat{v}(\xi)| \leq d \max_{j=1, \dots, d} \left| \frac{\partial \widehat{v}}{\partial \xi_j}(\xi) \right|, \quad \xi = (\xi_1, \dots, \xi_d) \in [p_-^\varepsilon, p_+^\varepsilon]. \quad (\text{VI.70})$$

In addition, the following estimates hold :

$$\left| \frac{\partial \widehat{v}}{\partial \xi_j}(\xi) \right| \leq \frac{(1+d)^n}{(2\pi)^d (1+|\xi|)^n} \|x_j v\|_{n,1}, \quad \xi \in [p_-^\varepsilon, p_+^\varepsilon], \quad j = 1, \dots, d. \quad (\text{VI.71})$$

Indeed, taking the sum in (VI.59) over all  $m = 0, \dots, n$  with the binomial coefficients, we get

$$\begin{aligned} (1 + |p|)^n |\widehat{v}(p)| &\leq (1 + |p_1| + \dots + |p_d|)^n |\widehat{v}(p)| \\ &\leq (2\pi)^{-d} (1 + d)^n \|v\|_{n,1}, \quad p \in \mathbb{R}^d. \end{aligned} \quad (\text{VI.72})$$

Estimates (VI.71) follow from (VI.72), where we replace  $v$  by  $x_j v$  and use that  $v$  belongs to  $W^{n,1}(\mathbb{R}^d)$  and is compactly supported.

Estimates (VI.69)–(VI.71) imply

$$|\varphi_2^\varepsilon(p)| \leq 2^{-n} c_6 \varepsilon \max\{(1 + |\xi|)^{-n} : \xi \in [p_-^\varepsilon, p_+^\varepsilon]\}, \quad p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \quad (\text{VI.73})$$

Using also that

$$\xi = \tau \frac{y}{|y|} + p_\perp, \quad \text{where } |\tau - \frac{\pi}{|y|} z(p)| \leq \frac{\varepsilon}{|y|}, \quad \text{if } \xi \in [p_-^\varepsilon, p_+^\varepsilon], \quad (\text{VI.74})$$

and that  $\varepsilon < |y|$ , we obtain

$$\begin{aligned} |\varphi_2^\varepsilon(p)| &\leq 2^{-n} c_6 \varepsilon \left(1 + \frac{1}{2} \left(\frac{\pi}{|y|} |z(p)| - \frac{\varepsilon}{|y|} + |p_\perp|\right)\right)^{-n} \\ &\leq c_6 \left(1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|\right)^{-n}, \quad p \in Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon. \end{aligned} \quad (\text{VI.75})$$

Estimate (VI.64) follows from (VI.68) and (VI.75).

Proposition VI.2 is proved.  $\square$

The final part of the proof of Theorem VI.2 is as follows. In a similar way with (VI.56), we represent  $v$  as follows :

$$\begin{aligned} v(x) &= v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in \mathbb{R}^d, \quad r > 0, \\ v_1^+(x, r) &= \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\ v_2^+(x, r) &= \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \widehat{w}_2}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\ v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp. \end{aligned} \quad (\text{VI.76})$$

Since  $v \in W^{n,1}(\mathbb{R}^d)$ , estimate (VI.57) holds.



Using estimates (VI.63), (VI.64), we get :

$$\left| v_1^+(x, r) - \int_{\mathcal{B}_r \setminus Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} e^{-ipx} U_{\hat{w}_1, \hat{w}_2}(p, E) dp \right. \\ \left. + v_1^2(x, r) - \int_{\mathcal{B}_r \cap Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} e^{-ipx} U_{\hat{w}_1, \hat{w}_2}^\varepsilon(p, E) dp \right| \leq I_1 + I_2, \quad (\text{VI.77})$$

$$I_1 = 2^\beta c_5 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp, \quad (\text{VI.78})$$

$$I_2 = c_6 \varepsilon \int_{\mathcal{B}_r \cap Z_{\hat{w}_1, \hat{w}_2}^\varepsilon} \left(1 + \frac{\pi}{|y|} |z(p)| + |p_\perp|\right)^{-n} dp, \quad (\text{VI.79})$$

$$x \in \mathbb{R}^d, \quad 1 \leq r \leq 2\sqrt{E}, \quad E \geq \rho_2,$$

where  $\rho_2$  is the same as in Proposition VI.2. In addition :

$$I_1 \leq c_7 \varepsilon^{-1} E^{\frac{1}{2}} r^{d+\beta}, \quad c_7 = |\mathbb{S}^{d-1}|^{\frac{2d+2\beta}{d+\beta}} c_5; \quad (\text{VI.80})$$

$$I_2 = c_6 \varepsilon \sum_{z \in \mathbb{Z}} \int_{\left\{ \begin{array}{l} \tau^2 + p_\perp^2 \leq r^2 \\ \left| \tau - \frac{\pi}{|y|} z \right| \leq \frac{\varepsilon}{|y|} \end{array} \right\}} \left(1 + \frac{\pi}{|y|} |z| + |p_\perp|\right)^{-n} d\tau dp_\perp,$$

where  $\tau \in \mathbb{R}$ ,  $p_\perp \in \mathbb{R}^d$ ,  $p_\perp y = 0$ ,

$$I_2 \leq 2c_6 \frac{\varepsilon^2}{|y|} \sum_{z \in \mathbb{Z}} \int_{\xi \in \mathbb{R}^{d-1}, |\xi| \leq r} \left(1 + \frac{\pi}{|y|} |z| + |\xi|\right)^{-n} d\xi \leq c_6 c_8 \varepsilon^2, \quad (\text{VI.81}) \\ c_8 = \frac{2}{|y|} \frac{|\mathbb{S}^{d-2}|}{n-d+1} \sum_{z \in \mathbb{Z}} \left(1 + \frac{\pi}{|y|} |z|\right)^{d-n-1}.$$

In addition, if  $r = r_2(E)$ ,  $\varepsilon = \varepsilon_2(E)$ , where  $r_2(E)$ ,  $\varepsilon_2(E)$  are defined in (VI.44), then

$$r^{d-n} = (2\tau)^{d-n} E^{-\alpha_2}, \\ \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} = (2\tau)^{d+\beta} E^{-\alpha_2}, \quad (\text{VI.82}) \\ \varepsilon^2 = E^{-\alpha_2}.$$

Formula (VI.45) follows from representation (VI.76), estimates (VI.57), (VI.77), (VI.80), (VI.81) and formulas (VI.82).

Theorem VI.2 is proved.

### 5.3 Proof of Theorem VI.3

**Proposition VI.3.** *Let  $v$  satisfy (VI.2) and  $w_1, \dots, w_{d+1}$  be the same as in (VI.24), (VI.29),  $d \geq 2$ . Then :*

$$\begin{aligned} |\widehat{v}(p) - U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E)| &\leq c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \\ p &\in \mathcal{B}_{2\sqrt{E}} \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad E \geq \rho_3, \quad 0 < \varepsilon < 1, \\ c_9 &= \frac{\pi\sqrt{d}}{2} (2c(D_0)N_0^3 + c(D_1)N_1^3 + \max_{j=2, \dots, d+1} c(D_j)N_j^3) c_1, \\ \rho_3 &= \max_{j=0, \dots, d+1} \rho(D_j, N_j), \end{aligned} \quad (\text{VI.83})$$

in addition, if  $v \in W^{n,1}(\mathbb{R}^d)$ ,  $n \geq 0$ , then :

$$\begin{aligned} &|\widehat{v}(p) - U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E)| \\ &\leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}} + 2c_6 \varepsilon (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n}, \\ &p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad E \geq \rho_3, \quad 0 < \varepsilon < \min\{1, \frac{1}{2}s\}, \end{aligned} \quad (\text{VI.84})$$

where  $c, \rho, D_j, N_j, j = 0, \dots, d+1$ , are defined as in (VI.15);  $c_1, \beta$  are the same as in Lemma VI.1,  $c_6$  is the same as in Proposition VI.2,  $z(p)$  is defined in (VI.47) and  $\|z(p)\|_2$  is the standard Euclidean norm of  $z(p)$ .

*Proof of Proposition VI.3.* In a similar way with formulas (VI.65), one can write

$$\begin{aligned} M_{\widehat{w}_1, \widehat{w}_{i'}}(p) &= \widehat{w}_1(p) \begin{pmatrix} 1 & 0 \\ \cos(sp_{i'-1}) & \sin(sp_{i'-1}) \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus \frac{\pi}{s} \mathbb{Z}^d, \\ M_{\widehat{w}_1, \widehat{w}_{i'}}^{-1}(p) &= \frac{1}{\sin(sp_{i'-1}) \widehat{w}_1(p)} \begin{pmatrix} \sin(sp_{i'-1}) & 0 \\ -\cos(sp_{i'-1}) & 1 \end{pmatrix}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \end{aligned} \quad (\text{VI.85})$$

where  $i' = i'(p, s)$  is defined in (VI.48). Also note that

$$|\sin(sp_{i'-1})| \geq \frac{2\varepsilon}{\pi\sqrt{d}}, \quad p \in \mathbb{R}^d \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon, \quad 0 < \varepsilon < 1. \quad (\text{VI.86})$$

Estimate (VI.83) follows from (VI.11), (VI.15), (VI.23), (VI.34), (VI.37), (VI.38), (VI.48), (VI.49) and from (VI.85), (VI.86).

It remains to prove (VI.84). Using definition (VI.50), we represent

$$\begin{aligned} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) - \widehat{v}(p) &= \varphi_1^\varepsilon(p, E) + \varphi_2^\varepsilon(p), \quad p \in \mathcal{B}_{2\sqrt{E}} \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon \\ \varphi_1^\varepsilon(p, E) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(\eta, E) - \widehat{v}(\eta)) \Big|_{\eta = \frac{\varepsilon}{s} \vartheta + \frac{\pi}{s} z(p)} d\vartheta, \\ \varphi_2^\varepsilon(p) &= \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} (\widehat{v}(\frac{\varepsilon}{s} \vartheta + \frac{\pi}{s} z(p)) - \widehat{v}(p)) d\vartheta, \end{aligned} \quad (\text{VI.87})$$

where  $z(p)$  is defined in (VI.47).

Using formulas (VI.83), (VI.87), we obtain

$$\begin{aligned} |\varphi_1^\varepsilon(p, E)| &\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \left(1 + \left| \frac{\varepsilon}{s} \vartheta + \frac{\pi}{s} z(p) \right| \right)^\beta d\vartheta \\ &\leq c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \left(1 + |p| + 2\frac{\varepsilon}{s}\right)^\beta d\vartheta \leq 2^\beta c_9 \varepsilon^{-1} (1 + |p|)^\beta E^{-\frac{1}{2}}, \end{aligned} \quad (\text{VI.88})$$

for  $\varepsilon$  as in (VI.84).

Next, using the definition of  $\varphi_2^\varepsilon$  in formula (VI.87) and the mean value theorem, we get the following estimate :

$$|\varphi_2^\varepsilon(p)| \leq 2\frac{\varepsilon}{s} \max\{|\nabla \widehat{v}(\xi)| : \xi \in \mathbb{R}^d, |\xi - \frac{\pi}{s} z(p)| \leq \frac{\varepsilon}{s}\}, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon. \quad (\text{VI.89})$$

Here, the mean value theorem was used for  $\widehat{v}(\xi)$  on  $[p, \frac{\varepsilon}{s} \vartheta + \frac{\pi}{s} z(p)]$ ,  $\vartheta \in \mathbb{S}^{d-1}$ . One can see that

$$\begin{aligned} &\text{estimates (VI.70) and (VI.71) hold for all } \xi \in \mathbb{R}^d \\ &\text{such that } |\xi - \frac{\pi}{s} z(p)| \leq \frac{\varepsilon}{s}, \text{ where } p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon. \end{aligned} \quad (\text{VI.90})$$

It follows from (VI.89), (VI.90) and from the upper estimate on  $\varepsilon$  of (VI.84), that

$$\begin{aligned} |\varphi_2^\varepsilon(p)| &\leq 2^{1-n} c_6 \varepsilon \max\{(1 + |\xi|)^{-n} : |\xi - \frac{\pi}{s} z(p)| \leq \frac{\varepsilon}{s}\} \\ &\leq 2^{1-n} c_6 \varepsilon \left(1 + \frac{\pi}{s} \|z(p)\|_2 - \frac{\varepsilon}{s}\right)^{-n} \\ &\leq 2c_6 \varepsilon \left(1 + 2\frac{\pi}{s} \|z(p)\|_2\right)^{-n}, \quad p \in Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon. \end{aligned} \quad (\text{VI.91})$$

Estimate (VI.84) follows from estimates (VI.88) and (VI.91).

Proposition VI.3 is proved.  $\square$

The final part of the proof of Theorem VI.3 is as follows. In a similar way with (VI.76), we represent  $v$  as follows :

$$\begin{aligned} v(x) &= v_1^+(x, r) + v_2^+(x, r) + v^-(x, r), \quad x \in \mathbb{R}^d, \quad r > 0, \\ v_1^+(x, r) &= \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\ v_2^+(x, r) &= \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} \widehat{v}(p) dp, \\ v^-(x, r) &= \int_{\mathbb{R}^d \setminus \mathcal{B}_r} e^{-ipx} \widehat{v}(p) dp. \end{aligned} \quad (\text{VI.92})$$

Since  $v$  belongs to  $W^{n,1}(\mathbb{R}^d)$ , estimate (VI.57) is valid.

Using estimates (VI.83), (VI.84) we obtain

$$\begin{aligned}
& \left| v_1^+(x, r) - \int_{\mathcal{B}_r \setminus Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}(p, E) dp \right. \\
& \left. + v_2^+(x, r) - \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} e^{-ipx} U_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon(p, E) dp \right| \leq J_1 + J_2, \\
& J_1 = 2^\beta c_9 \varepsilon^{-1} E^{-\frac{1}{2}} \int_{\mathcal{B}_r} (1 + |p|)^\beta dp, \\
& J_2 = 2c_6 \varepsilon \int_{\mathcal{B}_r \cap Z_{\widehat{w}_1, \dots, \widehat{w}_{d+1}}^\varepsilon} (1 + 2\frac{\pi}{s} \|z(p)\|_2)^{-n} dp, \\
& x \in \mathbb{R}^d, \quad 1 \leq r \leq 2\sqrt{E}, \quad E \geq \rho_3,
\end{aligned} \tag{VI.93}$$

where  $\rho_3$  is the same as in Proposition VI.3. In addition,

$$\begin{aligned}
& J_1 \leq c_{10} \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta}, \quad c_{10} = |\mathbb{S}^{d-1}| \frac{2^{d+\beta}}{d+\beta} c_9, \\
& J_2 \leq c_{11} \varepsilon^{d+1}, \quad c_{11} = \left(\frac{1}{s}\right)^d |\mathcal{B}_1| \sum_{z \in \mathbb{Z}^d} (1 + 2\frac{\pi}{s} \|z\|_2)^{-n},
\end{aligned} \tag{VI.94}$$

where  $|\mathcal{B}_1|$  is the standard Euclidean volume of  $\mathcal{B}_1$ . Finally, if  $r = r_3(E)$ ,  $\varepsilon = \varepsilon_3(E)$ , where  $r_3(E)$ ,  $\varepsilon_3(E)$  are defined in (VI.52), then

$$\begin{aligned}
& r^{d-n} = (2\tau)^{d-n} E^{-\alpha_3}, \\
& \varepsilon^{-1} E^{-\frac{1}{2}} r^{d+\beta} = (2\tau)^{d+\beta} E^{-\alpha_3}, \\
& \varepsilon^{d+1} = E^{-\alpha_3}.
\end{aligned} \tag{VI.95}$$

Formula (VI.53) follows from representation (VI.92), estimates (VI.57), (VI.93), (VI.94) and formulas (VI.95).

Theorem VI.3 is proved.

## 5.4 Proof of Lemma VI.1

Note that

$$\widehat{w}(p) = \int_{\mathbb{R}^d} |\widehat{q}(\xi)|^2 \widehat{\omega}_\nu(p - \xi) d\xi, \quad p \in \mathbb{R}^d, \tag{VI.96}$$

$$\omega_\nu(x) = |x|^{\frac{d}{2}} K_\nu(|x|), \quad x \in \mathbb{R}^d, \tag{VI.97}$$

where  $\widehat{q}$ ,  $\widehat{\omega}_\nu$  are the Fourier transforms of  $q$ ,  $\omega_\nu$ . The Fourier transform  $\widehat{\omega}_\nu$  can be computed explicitly :

$$\widehat{\omega}_\nu(p) = \frac{c_{12}}{(1 + |p|^2)^{\frac{d}{2} + \nu}}, \quad c_{12} = \frac{\Gamma(\frac{d}{2} + \nu) 2^{\nu-1}}{\pi^{\frac{d}{2}}}. \tag{VI.98}$$

Indeed, formula (VI.98) follows from the Fourier inversion theorem and the following computations :

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{e^{-ipx} dp}{(1 + |p|^2)^{\frac{d}{2} + \nu}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \frac{e^{-i|x|t} dt d\xi}{(1 + t^2 + |\xi|^2)^{\frac{d}{2} + \nu}} \\
&= |\mathbb{S}^{d-2}| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i|x|t} r^{d-2} dt dr}{(1 + t^2 + r^2)^{\frac{d}{2} + \nu}} \\
&\stackrel{r=\sqrt{1+t^2}\tau}{=} |\mathbb{S}^{d-2}| \int_{\mathbb{R}} \frac{e^{-i|x|t} dt}{(1 + t^2)^{\frac{1}{2} + \nu}} \int_0^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} \\
&= c_{12}^{-1} |x|^\nu K_\nu(|x|), \quad x \in \mathbb{R}^d.
\end{aligned}$$

Here, it was used that

$$\begin{aligned}
\int_0^{+\infty} \frac{\tau^{d-2} d\tau}{(1 + \tau^2)^{\frac{d}{2} + \nu}} &= \frac{1}{2} B\left(\frac{d-1}{2}, \nu + \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{d-1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{d}{2} + \nu\right)}, \\
|\mathbb{S}^{d-2}| &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)},
\end{aligned}$$

where  $B$  and  $\Gamma$  denote the beta and gamma functions.

Using (VI.96), (VI.98), we obtain the estimates

$$\begin{aligned}
\widehat{w}(p) &\geq \int_{|\xi| \leq 1} \frac{c_{12} |\widehat{q}(\xi)|^2}{(1 + |p - \xi|)^{d+2\nu}} d\xi \geq \frac{c_1(q, \nu)}{(1 + |p|)^{d+2\nu}}, \quad p \in \mathbb{R}^d, \\
c_1(q, \nu) &= \frac{c_{12}}{2^{d+2\nu}} \int_{|\xi| \leq 1} |\widehat{q}(\xi)|^2 d\xi.
\end{aligned} \tag{VI.99}$$

Properties (VI.23) follow from (VI.20), (VI.22), (VI.96) and (VI.99).

Lemma VI.1 is proved.

# Article VII

## Explicit reconstruction of Riemann surface with given boundary in complex projective space

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In this paper we propose a numerically realizable method for reconstruction of a complex curve with known boundary and without compact components in complex projective space.

### 1 Introduction

Let  $\mathbb{C}P^2$  be the complex projective plane with homogeneous coordinates  $(w_0 : w_1 : w_2)$ . Let  $X \subset \mathbb{C}P^2$  be a complex curve with rectifiable boundary  $\gamma = \partial X$ . Without loss of generality, we suppose that the following conditions of general position hold :

$$(0 : 1 : 0) \notin X, \quad w_0|_\gamma \neq 0.$$

Put  $\mathbb{C}^2 = \{w \in \mathbb{C}P^2 : w_0 \neq 0\}$  with coordinates  $z_1 = \frac{w_1}{w_0}$ ,  $z_2 = \frac{w_2}{w_0}$ . For almost all  $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$  the points of intersection of  $X$  with complex line  $\mathbb{C}_\xi^1 = \{z \in \mathbb{C}^2 : \xi_0 + \xi_1 z_1 + z_2 = 0\}$  form a finite set of points

$$(z_1^{(j)}(\xi), z_2^{(j)}(\xi)) = (h_j(\xi), -\xi_0 - \xi_1 h_j(\xi)), \quad \xi = (\xi_0, \xi_1), \quad j = 1, \dots, N_+(\xi).$$

By Darboux lemma [29, 32] functions  $\{h_j\}$  satisfy the following equations

$$\frac{\partial h_j(\xi)}{\partial \xi_1} = h_j(\xi) \frac{\partial h_j(\xi)}{\partial \xi_0}, \quad \xi = (\xi_0, \xi_1), \quad j = 1, \dots, N_+(\xi), \quad (\text{VII.1})$$

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which are often called the shock-wave equations, the inviscid Burgers equations or the Riemann-Burgers equations. In this interpretation  $\xi_1$  is the time variable and  $\xi_0$  is the space variable.

The following Cauchy-type formula from [32] plays the essential role in the reconstruction of  $X$  from  $\gamma$  :

$$G_m(\xi) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^m d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi) + P_m(\xi), \quad m \geq 1, \quad (\text{VII.2})$$

where  $N_+(\xi) = |X \cap \mathbb{C}_{\xi}^1|$  and  $P_m(\xi_0, \xi_1)$  is a polynomial of degree at most  $m$  with respect to  $\xi_0$  at fixed  $\xi_1$ . In addition,  $P_0(\xi_0, \xi_1) = -N_-$ , where  $N_- = |X \cap \mathbb{C}P_{\infty}^1|$ ,  $\mathbb{C}P_{\infty}^1 = \{w \in \mathbb{C}P^2 : w_0 = 0\}$ , and

$$P_1(\xi_0, \xi_1) = - \sum_{k=1}^{N_-} \frac{a_k \xi_0 + b_k}{1 - a_k \xi_1}, \quad \text{if } h_k(\xi_0, 0) \sim a_k \xi_0 + b_k + O(\xi_0^{-1}) \text{ at } \infty. \quad (\text{VII.3})$$

In particular, it follows from (VII.2) that :

$$G_0(\xi) = \frac{1}{2\pi i} \int_{\gamma} \frac{d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = N_+(\xi) - N_-. \quad (\text{VII.4})$$

Let  $\pi_2: \mathbb{C}^2 \rightarrow \mathbb{C}$  be the projection onto the second coordinate :  $\pi_2(z_1, z_2) = -z_2$ . We have that  $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^L \Omega_l$ , where  $\Omega_{l \geq 0}$  are connected and  $\Omega_0$  is unbounded. From the definition of  $N_{\pm}$  it follows that

$$N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0. \quad (\text{VII.5})$$

Assume that complex curve  $X$  does not contain compact components without boundary, or equivalently, satisfies the following condition of minimality :

$$\begin{aligned} & \text{for any complex curve } \tilde{X} \subset \mathbb{C}P^2 \text{ such that } \partial\tilde{X} = \partial X = \gamma \\ & \text{and for almost all } \xi \in \mathbb{C}^2 \text{ we have } |\tilde{X} \cap \mathbb{C}_{\xi}^1| \geq |X \cap \mathbb{C}_{\xi}^1|. \end{aligned} \quad (*)$$

Condition of minimality (\*) is a condition of general position and is fulfilled for  $X$  if, for example, every irreducible component of  $X$  is a transcendental complex curve. Note that from theorems of Chow [27] and Harvey-Shiffman [66] it follows that an arbitrary complex curve  $\tilde{X} \subset \mathbb{C}P^2$  satisfying  $\partial\tilde{X} = \partial X$  admits the unique representation  $\tilde{X} = X \cup V$ , where  $X$  is a curve satisfying (\*), and  $V$  is a compact algebraic curve, possibly, with multiple components.

The main result of [31] gives a solution to the important problem of J. King [81], when a real curve  $\gamma \subset \mathbb{C}P^2$  is the boundary of a complex curve  $X \subset \mathbb{C}P^2$ . Let  $\gamma \subset \mathbb{C}^2$ .

Then  $\gamma = \partial X$  for some open connected complex curve  $X$  in  $\mathbb{C}P^2$  if and only if in a neighborhood  $W_{\xi^*}$  of some point  $\xi^* \in \mathbb{C}^2$  one can find mutually distinct holomorphic functions  $h_1, \dots, h_{N(\xi^*)}$  satisfying shock-wave equations (VII.1) and also the equation

$$\frac{\partial^2}{\partial \xi_0^2} (G_1(\xi_0, \xi_1) - \sum_{j=1}^p h_j(\xi_0, \xi_1)) = 0, \quad \xi = (\xi_0, \xi_1) \in W_{\xi^*}.$$

In this work in development of [31, 32] we propose a numerically realizable algorithm for reconstruction of a complex curve  $X \subset \mathbb{C}P^2$  from the known boundary and satisfying the condition of minimality. This algorithm permits, in particular, to make applicable the result of [67] about principal possibility to reconstruct the topology and the conformal structure of a two-dimensional bordered surface  $X$  in  $\mathbb{R}^3$  with constant scalar conductivity from measurements on  $\partial X$  of electric current densities, induced by three potentials in general position.

Our algorithm depends on the number of points at infinity  $N_-$  of the complex curve  $X$ . It was tested on many examples and admits a simple and complete justification for  $N_- = 0, 1, 2$ . Despite a cumbersome description for  $N_- \geq 3$ , we show that, in principle, there are no obstacles for the justification and numerical realization for any  $N_- \geq 0$ . Moreover, in Theorem VII.4 we propose a method for finding the parameter  $N_-$  in terms of  $\gamma$ . This makes the algorithm much more applicable.

## 2 Cauchy-type formulas and Riemann-Burgers equations

We begin by giving a new proof of the Cauchy type formulas (VII.2) from [32], which allows to obtain explicit expressions for functions  $P_m$ .

**Theorem VII.1.** *Let  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Suppose that for almost all  $\xi \in \mathbb{C}^2$  all the points of intersection of  $X$  with  $\mathbb{C}_\xi^1$  have multiplicity at most one. Then the following formulas hold for almost all  $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$  :*

$$G_m(\xi) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi) + P_m(\xi), \quad m \geq 1, \quad (\text{VII.6})$$

where  $P_m(\xi)$  is holomorphic in a neighborhood of almost all  $(\xi_0, 0)$ ,  $\xi_0 \in \mathbb{C} \setminus \pi_2(\gamma)$ , and is polynomial in  $\xi_0$  of degree at most  $m$  for any fixed  $\xi_1$ . Furthermore, the following



explicit formulas hold :

$$\begin{aligned}
P_m(\xi_0, \xi_1) = & \\
& \sum_{s=1}^{N_-} \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s} \\
& - \sum_{s=1}^{\mu_0} \sum_{i_1+\dots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s),
\end{aligned}$$

where  $X \cap \mathbb{C}P_\infty^1 = \{q_1, \dots, q_{N_-}\}$ . In particular, if  $N_- = 0$  then  $P_m = 0$ ,  $m \geq 1$ .

*Démonstration.* Put  $\tilde{g} = \xi_0 w_0 + \xi_1 w_1 + w_2$  and  $g = \frac{\tilde{g}}{w_0} = \xi_0 + \xi_1 z_1 + z_2$ . Consider differential forms

$$\omega_m \stackrel{\text{def}}{=} z_1^m \frac{dg}{g} = \frac{w_1^m w_0}{w_0^m \tilde{g}} d\left(\frac{\tilde{g}}{w_0}\right) = \frac{w_1^m d\tilde{g}}{w_0^m \tilde{g}} - \frac{w_1^m}{w_0^{m+1}} dw_0, \quad m = 0, 1, \dots$$

Then  $G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m$ . Let us compute this integral explicitly. Denote by  $p_j$ ,  $j = 1, \dots, N_+(\xi)$  the points of intersection of  $X$  with  $\mathbb{C}P_\xi^1$ , and by  $q_s$ ,  $s = 1, \dots, N_-$  the points of intersection of  $X$  with infinity  $\mathbb{C}P_\infty^1$ . Denote by  $B_j^\varepsilon$  the intersection of  $X$  with the ball of radius  $\varepsilon$  in  $\mathbb{C}P^2$  centered at  $p_j$  and by  $D_s^\varepsilon$  the intersection of  $X$  with the ball of radius  $\varepsilon$  centered at  $q_s$ . The restriction of form  $\omega_m$  on  $X$  is meromorphic with poles at points  $p_j$  and  $q_s$ . Thus the following equality is valid :

$$G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m = \sum_{j=1}^{N_+(\xi)} \frac{1}{2\pi i} \int_{bB_j^\varepsilon} \omega_m + \sum_{s=1}^{N_-} \frac{1}{2\pi i} \int_{bD_s^\varepsilon} \omega_m.$$

If  $N_- = 0$ , then the second group of terms is absent. The integral  $\int_{bB_j^\varepsilon} \omega_m$  can be calculated as a residue at the first order pole :

$$\int_{bB_j^\varepsilon} \omega_m = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} - \int_{bB_j^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} = 2\pi i h_j^m(\xi).$$

Let  $N_- > 0$ . Computation of integral  $\int_{bD_s^\varepsilon} \omega_1$  will be done in two steps. Let us calculate first  $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0$ . Consider the expansion of  $w_1(w_0)$  into power series in  $w_0$  in the neighborhood of point  $q_s$  :

$$w_1(w_0) = w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \dots$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^k.$$

The coefficient near  $w_0^m$  can be presented in the form

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = 2\pi i \sum_{i_1+\dots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral  $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}}$ . Using relation  $d\tilde{g} = \frac{d\tilde{g}}{dw_0} dw_0$  and expansion of  $w_1(w_0)$  into power series in  $w_0$  we obtain :

$$\begin{aligned} \int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}} &= \int_{bD_s^\varepsilon} \frac{1}{w_0^m} \left( w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \cdots \right)^m \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{\infty} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{m-1} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{m-1} \frac{2\pi i}{(m-k-1)!} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) \lim_{w_0 \rightarrow 0} \frac{d^{m-k-1}}{dw_0^{m-k-1}} \left( \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} \right). \end{aligned}$$

From here, taking into account the relation  $\frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} = \frac{d \ln \tilde{g}}{dw_0}$ , we obtain, finally

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}} = 2\pi i \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s}.$$

It is a polynomial of degree at most  $m$  with respect to  $\xi_0$ . □

**Remark 6.** We excluded the case  $(0 : 1 : 0) \in X$  because the point  $(0 : 1 : 0)$  can lead to a non-polynomial contribution in  $\xi_0$  in functions  $P_m$ .

Consider, for example, the projective curve  $\tilde{X} : w_0^2 = w_1 w_2$  and the meromorphic 1-form  $\Omega_{\xi_0}$  on it :

$$\Omega_{\xi_0} = \frac{z_1 dz_2}{z_2 + \xi_0}, \quad \xi_0 \in \mathbb{C} \setminus 0,$$

where  $z_1 = \frac{w_1}{w_0}$ ,  $z_2 = \frac{w_2}{w_0}$ . The form  $\Omega_{\xi_0}$  is holomorphic on  $X$  except the points  $q_1 = (0 : 0 : 1)$ ,  $q_2 = (0 : 1 : 0)$  and  $p_{\xi_0} = (-\xi_0 : 1 : \xi_0^2)$ .

One can see that the residues of  $\Omega_{\xi_0}$  at  $q_1, q_2$  are equal to 0 and  $1/\xi_0$ , respectively. It follows that the residue at  $p_{\xi_0}$  is equal to  $-1/\xi_0$ .

Now, consider the part  $X$  of  $\tilde{X}$  obtained by cutting off some small neighborhood of the point  $q_1$ . Let  $\gamma$  denote the boundary of  $X$ . Choose any  $\xi = (\xi_0, 0)$  such that  $X \cap \mathbb{C}_\xi^1 \neq \emptyset$ . Then  $X \cap \mathbb{C}_\xi^1 = \{(-1/\xi_0, -\xi_0)\}$  and, as far as the residue of  $\Omega_{\xi_0}$  at  $q_1$  is zero, we have that

$$0 = G_1(\xi_0, 0) = \frac{1}{2\pi i} \int_\gamma \Omega_{\xi_0} = -\frac{1}{\xi_0} + P_1(\xi_0, 0),$$

so that  $P_1(\xi_0, 0) = 1/\xi_0$ .

We will also use the following result of [67], which gives an effective characterization of functions  $h_{j \geq 1}$  of Theorem VII.1.

**Theorem VII.2** (Remark 4 to Theorem 3a of [67]). *Let  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Fix any  $\xi_0^* \in \Omega_0$  and let  $W_{\xi^*}$  be a neighborhood of  $\xi^* = (\xi_0^*, 0)$  in  $\mathbb{C}^2$ . Let  $G_{k \geq 1}, h_{j \geq 1}$  be defined as in (VII.2) and (VII.6). Suppose that there exist functions  $\tilde{h}_1, \dots, \tilde{h}_p$ , holomorphic in  $W_{\xi^*}$  and satisfying*

$$\frac{\partial^2}{\partial \xi_0^2} (G_1 - \tilde{h}_1 - \dots - \tilde{h}_p) = 0 \quad \text{in } W_{\xi^*}, \quad (\text{VII.7})$$

$$\frac{\partial \tilde{h}_k}{\partial \xi_1} = \tilde{h}_k \frac{\partial \tilde{h}_k}{\partial \xi_0} \quad \text{in } W_{\xi^*}, \quad k = 1, \dots, p. \quad (\text{VII.8})$$

*Then  $p \geq N_-$ . Furthermore, if  $p = N_- := |X \cap \mathbb{C}P_\infty^1|$ , then  $h_1, \dots, h_{N_-}$  coincide with  $\tilde{h}_1, \dots, \tilde{h}_{N_-}$  in  $W_{\xi^*}$  (up to order).*

### 3 Reconstruction algorithm

We now pass to the reconstruction algorithm for a complex curve  $X \subseteq \mathbb{C}P^2$  with given boundary  $\partial X$  and satisfying the condition of minimality (\*). Let us consider the cases  $N_- = 0, 1, 2$ .

The reconstruction algorithm is based on formulas (VII.6). The next theorem permits to find the functions  $P_m$  of (VII.6). We will use the notation

$$\varkappa_{kl} = \frac{1}{2\pi i} \int_\gamma z_1^k z_2^l dz_2, \quad k, l \geq 0. \quad (\text{VII.9})$$

**Theorem VII.3.** *Let  $X \subset \mathbb{C}P^2$ ,  $(0 : 1 : 0) \notin X$ , be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Let  $h_{j \geq 1}$  and  $P_{m \geq 1}$  be the functions defined in Theorem VII.1. Then the following statements are valid :*

1. If  $N_- = 0$ , then  $P_m = 0$  for all  $m \geq 1$ . Besides,  $G_1 = 0$  in a neighborhood of any  $\xi^* = (\xi_0^*, 0)$  with  $\xi_0^* \in \Omega_0$ .
2. If  $N_- = 1$ , then  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ , where constants  $c_{11}$  and  $c_{12}$  satisfy the following identity in  $\xi = (\xi_0, 0) \in \Omega_0 \times 0$  :

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi) + c_{12} \left( \xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi) + G_1(\xi) \right) = G_1(\xi) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi). \quad (\text{VII.10})$$

3. If  $N_- = 2$ , then  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ ,  $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$ , where constants  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ ,  $c_{22}$ ,  $c_{23}$  satisfy the following identity in  $\xi = (\xi_0, 0) \in \Omega_0 \times 0$  :

$$\begin{aligned} \varkappa_{10}(c_{12}^2 + c_{23}) &= \frac{\partial G_2}{\partial \xi_1} - 2 \frac{\partial G_1}{\partial \xi_1} (G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0) \\ &+ \frac{\partial G_1}{\partial \xi_0} \cdot \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \\ &+ (G_1^2 - 2c_{11}G_1 - 2c_{12}G_1\xi_0 - G_2) \cdot (-c_{12}), \end{aligned} \quad (\text{VII.11})$$

where all the functions are evaluated at point  $\xi = (\xi_0, 0)$ .

*Démonstration.* 1. By Theorem VII.1, if  $N_- = 0$  then  $P_m = 0$  for all  $m \geq 1$ . It also follows from formula (VII.6) that  $G_1 = 0$  in a neighborhood of any  $\xi^* = (\xi_0^*, 0)$  with  $\xi_0^* \in \Omega_0$ .

2. It follows from Theorem VII.1 that  $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$  in a neighborhood of almost all  $\xi^* = (\xi_0^*, 0) \in \Omega_l \times 0$ , where  $C_{11}$ ,  $C_{12}$  are holomorphic in a neighborhood of zero. We need to find constants  $c_{11} = C_{11}(0)$  and  $c_{12} = C_{12}(0)$ .

We differentiate the equation (VII.6) with respect to  $\xi_0$  and restrict this equation and its differentiated version to  $\Omega_l \times 0$  :

$$\begin{aligned} h_1(\xi_0, 0) &= G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0, \\ \frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0, \\ \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}, \end{aligned} \quad (\text{VII.12})$$

where  $\xi_0 \in \Omega_0$ . By (VII.1) there is the equality  $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0) \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$ . We substitute  $h_1(\xi_0, 0)$ ,  $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$ ,  $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$  in this equation by their expressions (VII.12), and we obtain the equation

$$\begin{aligned} &\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 = \\ &= \left( G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \left( \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12} \right). \end{aligned} \quad (\text{VII.13})$$

This equation is valid, in particular, for  $\xi_0 \in \Omega_0$ . We divide it by  $\xi_0$  and tend  $\xi_0 \rightarrow \infty$ . As a result, we obtain the equality  $\dot{C}_{12}(0) = -c_{12}^2$ . Taking into account this equality, we can rewrite equation (VII.13) in the form

$$\begin{aligned} & \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) = \\ & = \left( G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \left( G_1(\xi_0, 0) - c_{11} \right) c_{12}. \end{aligned} \quad (\text{VII.14})$$

Taking into account that  $\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \rightarrow 0$  as  $\xi_0 \rightarrow \infty$  and passing  $\xi_0 \rightarrow \infty$  in (VII.14), we obtain the equality  $\dot{C}_{11}(0) = -c_{11}c_{12}$ . Substituting this explicit expression for  $\dot{C}_{11}(0)$  into (VII.14), we get (VII.10).

3. By (VII.1) functions  $h_1(\xi)$  and  $h_2(\xi)$  satisfy the Riemann-Burgers equation in a neighborhood of any  $\xi^* = (\xi_0^*, 0) \in \Omega_0 \times 0$ , so that the following equalities are valid :

$$\frac{\partial(h_1 h_2)}{\partial \xi_1} = h_1 \frac{\partial h_2}{\partial \xi_1} + \frac{\partial h_1}{\partial \xi_1} h_2 = h_1 h_2 \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (\text{VII.15})$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2h_1 \frac{\partial h_1}{\partial \xi_0} + 2h_2 \frac{\partial h_2}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (\text{VII.16})$$

Note that  $h_1 h_2 = \frac{1}{2}(h_1 + h_2)^2 - \frac{1}{2}(h_1^2 + h_2^2)$ . Therefore the system (VII.15)–(VII.16) is equivalent to the system

$$\frac{\partial(h_1 + h_2)^2}{\partial \xi_1} - \frac{\partial(h_1^2 + h_2^2)}{\partial \xi_1} = \left( (h_1 + h_2)^2 - (h_1^2 + h_2^2) \right) \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (\text{VII.17})$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (\text{VII.18})$$

We substitute the expressions of (VII.6) for  $h_1^2 + h_2^2$  and  $h_1 + h_2$  into (VII.17), (VII.18), using the notations  $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$ ,  $P_2(\xi_0, \xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$ . Then, equation (VII.18) restricted to  $\Omega_0 \times 0$  takes the form

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0, 0) - c_{22} - 2c_{23}\xi_0 = 2 \left( \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right). \quad (\text{VII.19})$$

We divide this equation by  $\xi_0$  and tend  $\xi_0 \rightarrow \infty$ . It leads to the equality  $\dot{C}_{12}(0) = c_{23}$ . Taking this equality into account and passing  $\xi_0 \rightarrow \infty$  in (VII.19), we obtain the equality  $\dot{C}_{11}(0) = \frac{1}{2}c_{22}$ .

Next, we substitute the expressions of (VII.6) for  $h_1^2 + h_2^2$  and  $h_1 + h_2$  into (VII.17) and restrict the obtained formula to  $\Omega_0 \times 0$ . It leads to the equality

$$\begin{aligned} & 2(G_1 - c_{11} - c_{12}\xi_0) \left( \frac{\partial G_1}{\partial \xi_1} - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) + \dot{C}_{22}(0)\xi_0 + \dot{C}_{23}(0)\xi_0^2 = \\ & = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \left( \frac{\partial G_1}{\partial \xi_0} - c_{12} \right). \end{aligned} \quad (\text{VII.20})$$

We divide this equation by  $\xi_0^2$  and tend  $\xi_0 \rightarrow \infty$ . This leads to the equality

$$2c_{12}\dot{C}_{12}(0) + \dot{C}_{23}(0) = -(c_{12}^2 + c_{23})c_{12}.$$

Taking this equality into account, dividing (VII.20) by  $\xi_0$  and passing  $\xi_0 \rightarrow \infty$ , we obtain the equality

$$2c_{11}\dot{C}_{12}(0) + 2c_{12}\dot{C}_{11}(0) + \dot{C}_{22}(0) = -(2c_{11}c_{12} + c_{22})c_{12}.$$

Using the obtained equalities, one can rewrite (VII.20) in the form

$$\begin{aligned} & 2(G_1 - c_{11} - c_{12}\xi_0)\frac{\partial G_1}{\partial \xi_1} - 2G_1(\dot{C}_{11}(0) + \dot{C}_{12}(0)\xi_0) + 2c_{11}\dot{C}_{11}(0) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) = \\ & = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\ & \quad + \left( (G_1 - c_{11})^2 - 2G_1c_{12}\xi_0 - G_2 + c_{21} \right) (-c_{12}). \end{aligned} \quad (\text{VII.21})$$

We pass  $\xi_0 \rightarrow \infty$  and note that the following relations are valid

$$\begin{aligned} \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{\partial G_1}{\partial \xi_1} &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_1}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_1 = 0, \\ \lim_{\xi_0 \rightarrow \infty} \xi_0 G_1 &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2 = \mathfrak{a}_{10}, \\ \lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{\partial G_1}{\partial \xi_0} &= - \lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{(\xi_0 + z_2)^2} = - \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2 = -\mathfrak{a}_{10}. \end{aligned}$$

As a result, we obtain

$$-2\mathfrak{a}_{10}\dot{C}_{12}(0) + 2c_{11}\dot{C}_{11}(0) + \dot{C}_{21}(0) = -(c_{12}^2 + c_{23})\mathfrak{a}_{10} - (c_{11}^2 - 2c_{12}\mathfrak{a}_{10} + c_{21})c_{12}.$$

Due to the obtained relations, we can express constants  $\dot{C}_{ij}(0)$  as functions of  $c_{ij}$  :

$$\begin{aligned} \dot{C}_{11}(0) &= \frac{1}{2}c_{22}, \\ \dot{C}_{12}(0) &= c_{23}, \\ \dot{C}_{23}(0) &= -c_{12}^3 - 3c_{12}c_{23}, \\ \dot{C}_{22}(0) &= -2(c_{11}c_{12}^2 + c_{12}c_{22} + c_{11}c_{23}), \\ \dot{C}_{21}(0) &= \mathfrak{a}_{10}(c_{12}^2 + c_{23}) - c_{12}(c_{11}^2 + c_{21}) - c_{11}c_{22}. \end{aligned} \quad (\text{VII.22})$$

Substituting these constants to (VII.21), we obtain the third statement of Theorem VII.3.  $\square$

**Complement 1.** The statement of Theorem VII.3 admits a development for the case  $N_- \geq 3$ . In this case

$$P_k(\xi_0, \xi_1) = C_{k1}(\xi_1) + C_{k2}(\xi_1)\xi_0 + \cdots + C_{k,k+1}(\xi_1)\xi_0^k, \quad k = 1, \dots, N_-.$$

Denote  $\dot{C}_{ij}(0) = \frac{\partial C_{ij}}{\partial \xi_1}(0)$  and  $c_{ij} = C_{ij}(0)$  for  $i = 1, \dots, N_-$  and  $j = 1, \dots, i + 1$ .

Let us indicate the following general procedure for finding of constants  $c_{ij}$ . Due to the Riemann–Burgers equations (VII.1) the following identities in  $\xi_0 \in \Omega_0$  hold for  $k = 1, \dots, N_- - 1$  :

$$\begin{aligned} & -\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) + \dot{C}_{k1}(0) + \dot{C}_{k2}(0)\xi_0 + \cdots + \dot{C}_{k,k+1}(0)\xi_0^k \\ &= \frac{k}{k+1} \left( -\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) + c_{k+1,2} + 2c_{k+1,3}\xi_0 + \cdots + (k+1)c_{k+1,k+2}\xi_0^k \right). \end{aligned}$$

Taking into account that  $\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) \rightarrow 0$  and  $\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) \rightarrow 0$  as  $\xi_0 \rightarrow +\infty$  we obtain the equalities

$$\dot{C}_{k,m}(0) = \frac{km}{k+1}c_{k+1,m+1}, \quad k = 1, \dots, \mu_0 - 1, \quad m = 1, \dots, k + 1.$$

Due to the Riemann–Burgers equations (VII.1) the following identity in  $\xi_0 \in \Omega_0$  holds :

$$\frac{\partial e_{\mu_0}}{\partial \xi_1}(\xi_0, 0) = e_{\mu_0}(\xi_0, 0) \frac{\partial p_1}{\partial \xi_0}(\xi_0, 0), \quad (\text{VII.23})$$

where functions  $e_k$  are given by the following formulas :

$$\begin{aligned} ke_k(\xi_0, \xi_1) &= \sum_{i=1}^{k-1} (-1)^{i+1} e_{k-i}(\xi_0, \xi_1) p_i(\xi_0, \xi_1) + (-1)^{k+1} p_k(\xi_0, \xi_1), \\ p_k(\xi_0, \xi_1) &= G_k(\xi_0, \xi_1) - C_{k1}(\xi_1) - C_{k2}(\xi_1)\xi_0 - \cdots - C_{k,k+1}(\xi_1)\xi_0^k, \end{aligned} \quad (\text{VII.24})$$

where  $k = 1, \dots, \mu_0$ .

Equality (VII.23) allows to represent constants  $\{\dot{C}_{\mu_0,j}(0)\}$  as functions of constants  $\{c_{ij}\}$ . Finally, substituting the obtained expressions for constants  $\{\dot{C}_{ij}(0)\}$  via constants  $\{c_{ij}\}$  into equation (VII.23) we obtain the identity in  $\xi_0 \in \Omega_0$  for computation of constants  $\{c_{ij}\}$ .

For example, in the case  $N_- = 3$  the identity (VII.23) in  $\xi_0 \in \Omega_0$  for finding of constants  $c_{ij}$  takes the form

$$\begin{aligned} & \dot{C}_{31}(0) + \dot{C}_{32}(0)\xi_0 + \dot{C}_{33}(0)\xi_0^2 + \dot{C}_{34}(0)\xi_0^3 = \frac{\partial G_3}{\partial \xi_1} + \\ & + \frac{3}{4}(p_1^2 - p_2) \frac{\partial p_2}{\partial \xi_0} - p_1 \frac{\partial p_3}{\partial \xi_0} - \frac{1}{2}(p_1^3 - 3p_1 p_2 + 2p_3) \frac{\partial p_1}{\partial \xi_0}, \end{aligned}$$

where all functions are evaluated at point  $(\xi_0, 0)$ , the functions  $p_k$  are defined in formula (VII.24) and the constants  $\dot{C}_{31}(0)$ ,  $\dot{C}_{32}(0)$ ,  $\dot{C}_{33}(0)$ ,  $\dot{C}_{34}(0)$  are given by formulas

$$\begin{aligned}\dot{C}_{31}(0) &= \frac{1}{2}\mathfrak{a}_{10}(3c_{11}c_{12}^2 + 3c_{12}c_{22} + 3c_{11}c_{23} + 2c_{33}) \\ &\quad - \frac{1}{2}\mathfrak{a}_{11}(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}) - \frac{3}{2}\frac{1}{2\pi i} \int_{\gamma} z_1 z_2 dz_1 (c_{12}^2 + c_{23}) \\ &\quad - \frac{1}{2}c_{11}^3 c_{12} - \frac{3}{2}c_{11}c_{12}c_{21} - \frac{3}{4}c_{11}^2 c_{22} - \frac{3}{4}c_{21}c_{22} - c_{12}c_{31} - c_{11}c_{32}, \\ \dot{C}_{32}(0) &= \mathfrak{a}_{10}(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}) - \frac{3}{2}c_{11}^2 c_{12}^2 - \frac{3}{2}c_{12}^2 c_{21} - 3c_{11}c_{12}c_{22} \\ &\quad - \frac{3}{4}c_{22}^2 - \frac{3}{2}c_{11}^2 c_{23} - \frac{3}{2}c_{21}c_{23} - 2c_{12}c_{32} - 2c_{11}c_{33}, \\ \dot{C}_{33}(0) &= -\frac{3}{2}c_{11}c_{12}^3 - \frac{9}{4}c_{12}^2 c_{22} - \frac{9}{2}c_{11}c_{12}c_{23} - \frac{9}{4}c_{22}c_{23} - 3c_{12}c_{33} - 3c_{11}c_{34} \\ \dot{C}_{34}(0) &= -\frac{1}{2}c_{12}^4 - 3c_{12}^2 c_{23} - \frac{3}{2}c_{23}^2 - 4c_{12}c_{34},\end{aligned}$$

where  $\mathfrak{a}_{10}$  and  $\mathfrak{a}_{11}$  are defined in formula (VII.9).

The next theorem permits to find  $N_- = |X \cap \mathbb{C}P_{\infty}^1|$  from  $\gamma = \partial X$ .

**Theorem VII.4.** *Let  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Let  $G_{m \geq 1}$  be the functions defined in (VII.2) and let  $N_- = |X \cap \mathbb{C}P_{\infty}^1|$ . Fix any  $\xi_0^* \in \Omega_0$  and let  $W_{\xi^*}$  be a neighborhood of  $\xi^* = (\xi_0^*, 0)$  in  $\mathbb{C}^2$ . Then the following statements are valid :*

1. *If  $G_1 = 0$  in  $W_{\xi^*}$ , then either  $N_- = 0$ , or  $\gamma'$  bounds a complex curve in  $\mathbb{C}^2$ , where  $\gamma'$  denotes  $\gamma$  with the opposite orientation.*
2. *If there exist complex constants  $c_{11}$ ,  $c_{12}$  such that*

$$\frac{\partial}{\partial \xi_1}(G_1 - P_1) = (G_1 - P_1) \frac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{in } W_{\xi^*}, \quad (\text{VII.25})$$

where  $P_1(\xi_0, \xi_1) = \frac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}$ , then  $N_- \leq 1$ . Furthermore,  $c_{11}$ ,  $c_{12}$  are the same constants as in Theorem VII.3.

3. *If there exist complex constants  $a_1$ ,  $a_2$ ,  $b_1$ ,  $b_2$ ,  $c_1$ ,  $c_2$  such that  $c_1 + c_2 = \mathfrak{a}_{10}$ ,*

$$\begin{aligned}\frac{\partial}{\partial \xi_1}((G_1 - P_1)^2 - G_2 - P_2) &= ((G_1 - P_1)^2 - G_2 - P_2) \frac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{in } W_{\xi^*}, \\ P_1(\xi_0, \xi_1) &= -\frac{a_1\xi_0 + b_1}{1 - a_1\xi_1} - \frac{a_2\xi_0 + b_2}{1 - a_2\xi_0}, \\ P_2(\xi_0, \xi_1) &= -\sum_{j=1}^2 \left[ \left( \frac{a_j\xi_0 + b_j}{1 - a_j\xi_1} \right)^2 + \frac{2a_j c_j}{1 - a_j\xi_1} \right],\end{aligned} \quad (\text{VII.26})$$



then  $N_- \leq 2$ . Furthermore, these constants are related to the constants of Theorem VII.3 by the equations :

$$\begin{aligned} a_1 + a_2 &= -c_{12}, & b_1 + b_2 &= -c_{11}, \\ a_1^2 + a_2^2 &= -c_{23}, & a_1 b_1 + a_2 b_2 &= -\frac{1}{2}c_{22}, \\ b_1^2 + b_2^2 + 2a_1 c_1 + 2a_2 c_2 &= -c_{21}, & c_1 + c_2 &= \mathfrak{x}_{10}. \end{aligned} \quad (\text{VII.27})$$

*Démonstration.* 1. The equality  $G_1 = 0$  in  $W_{\xi^*}$  implies that  $G_1 = 0$  for all  $\xi \in \Omega_0 \times \mathbb{C}$ . In turn, it implies, according to [32], the moment condition

$$\int_{\gamma} z_1^{k_1} z_2^{k_2} dz_2 = 0, \quad k_1, k_2 \geq 0.$$

Then, according to [140] and [65], for an appropriate choice of orientation,  $\gamma$  is the boundary of a complex curve in  $\mathbb{C}^2$ .

2. Set  $h = G_1 - P_1$  in a neighborhood  $W_{\xi^*}$  of  $\xi^*$ . Then  $h$  satisfies

$$\begin{aligned} \frac{\partial^2}{\partial \xi_0^2} (G_1 - h) &= 0 \quad \text{in } W_{\xi^*}, \\ \frac{\partial h}{\partial \xi_1} &= h \frac{\partial h}{\partial \xi_0} \quad \text{in } W_{\xi^*}. \end{aligned}$$

It follows from Theorem VII.2 that  $N_- \leq 1$ . Note also that equation (VII.10) is the restriction of (VII.25) to  $W_{\xi^*} \cap (\Omega_0 \times 0)$ .

3. Consider the following quadratic equation in variable  $t$  :

$$t^2 - (G_1 - P_1)t + \frac{1}{2}((G_1 - P_1)^2 - G_2 - P_2) = 0. \quad (\text{VII.28})$$

Suppose that the discriminant is non-zero at  $\xi^*$ . Then, without loss of generality, it is non-zero in  $W_{\xi^*}$  (we can always choose a smaller neighborhood). We denote two different roots of this equation as  $\tilde{h}_1 = \tilde{h}_1(\xi)$ ,  $\tilde{h}_2 = \tilde{h}_2(\xi)$ . Clearly,  $\tilde{h}_1$  and  $\tilde{h}_2$  are holomorphic in  $W_{\xi^*}$ . Furthermore, by the Viète formulas we have

$$\begin{aligned} \tilde{h}_1 + \tilde{h}_2 &= \tilde{e}_1 := G_1 - P_1 \quad \text{in } W_{\xi^*}, \\ \tilde{h}_1 \tilde{h}_2 &= \tilde{e}_2 := \frac{1}{2}((G_1 - P_1)^2 - G_2 - P_2) \quad \text{in } W_{\xi^*}. \end{aligned}$$

Note that by definition  $\frac{\partial P_1}{\partial \xi_1} = \frac{1}{2} \frac{\partial P_2}{\partial \xi_0}$ . Note also that by Lemma 3.3.1 of [32] we have  $\frac{\partial G_1}{\partial \xi_1} = \frac{1}{2} \frac{\partial G_2}{\partial \xi_0}$ . It leads to the equation

$$\frac{\partial \tilde{e}_1}{\partial \xi_1} = \frac{1}{2} \frac{\partial}{\partial \xi_0} (\tilde{e}_1^2 - 2\tilde{e}_2) \quad \text{in } W_{\xi^*}. \quad (\text{VII.29})$$

Furthermore, equation (VII.26) can be rewritten in the form

$$\frac{\partial \tilde{e}_2}{\partial \xi_1} = \tilde{e}_2 \frac{\partial \tilde{e}_1}{\partial \xi_0} \quad \text{in } W_{\xi^*}. \quad (\text{VII.30})$$

Now denote by  $\widehat{h}_1, \widehat{h}_2$  the shock-wave extensions of  $\widetilde{h}_1(\cdot, 0)$  and  $\widetilde{h}_2(\cdot, 0)$  to  $W_{\xi^*}$  which exist and are unique by the Cauchy-Kowalevski theorem. Set  $\widehat{e}_1 = \widehat{h}_1 + \widehat{h}_2$ ,  $\widehat{e}_2 = \widehat{h}_1 \widehat{h}_2$ . Due to the shock-wave equations for  $\widehat{h}_1$  and  $\widehat{h}_2$ , the functions  $\widehat{e}_1$  and  $\widehat{e}_2$  satisfy

$$\begin{aligned}\frac{\partial \widehat{e}_1}{\partial \xi_1} &= \frac{1}{2} \frac{\partial}{\partial \xi_0} (\widehat{e}_1^2 - 2\widehat{e}_2) \quad \text{in } W_{\xi^*}, \\ \frac{\partial \widehat{e}_2}{\partial \xi_1} &= \widehat{e}_1 \frac{\partial \widehat{e}_2}{\partial \xi_0} \quad \text{in } W_{\xi^*}.\end{aligned}$$

Thus,  $\widetilde{e}_1, \widetilde{e}_2$  and  $\widehat{e}_1, \widehat{e}_2$  are holomorphic solutions to the same system with the same restrictions at  $\xi_1 = 0$ . By the Cauchy-Kowalevski theorem,  $\widetilde{e}_1 = \widehat{e}_1$  and  $\widetilde{e}_2 = \widehat{e}_2$ . It follows from the Viète formulas that  $\widetilde{h}_1, \widetilde{h}_2$  coincide with  $\widehat{h}_1, \widehat{h}_2$  (up to order). Hence,  $\widetilde{h}_1, \widetilde{h}_2$  satisfy the shock-wave equations.

Applying Theorem VII.2, we obtain that  $N_- \leq 2$ .

It remains to consider the case when the determinant of equation (VII.28) vanishes in  $W_{\xi^*}$ . Otherwise, it vanishes on (at most) a dimension one analytic set and in any neighborhood of  $\xi^*$  there are balls where it does not vanish.

The zero discriminant condition reads

$$(G_1 - P_1)^2 = 2(G_2 - P_1) \quad \text{in } W_{\xi^*}.$$

We set  $\widetilde{h} = \frac{1}{2}(G_1 - P_1)$ . Then by definition of  $P_1$  and from the discriminant condition we get

$$\begin{aligned}\frac{\partial}{\partial \xi_0^2} (G_1 - 2\widetilde{h}) &= 0 \quad \text{in } W_{\xi^*}, \\ \frac{\partial \widetilde{h}}{\partial \xi_1} &= \widetilde{h} \frac{\partial \widetilde{h}}{\partial \xi_0} \quad \text{in } W_{\xi^*}.\end{aligned}$$

By Theorem VII.2 it implies  $N_- \leq 2$ . Note also that if  $N_- = 2$ , then all intersections of  $X$  with  $\mathbb{C}_\xi^1$ ,  $\xi \in W_{\xi^*}$ , are double.

Finally, note that equation (VII.11) is the restriction of (VII.26) to the set  $W_{\xi^*} \cap (\Omega_0 \times 0)$ .  $\square$

**Complement 2.** The statement of Theorem VII.4 can be generalized to the case  $N_- \geq 3$  in the spirit of cases  $N_- \leq 2$ . Such a generalization will be developed in a separate paper together with a statement of Theorem VII.3 for  $N_- \geq 3$ , indicated in Complement 1.

We pass to the description of the algorithm of reconstruction of a complex curve  $X \subset \mathbb{C}P^2$  satisfying the minimality condition (\*) from the known boundary  $\gamma = \partial X \subset \mathbb{C}^2$ .

Let  $\{\xi_0^k\}_{k=1}^N$ ,  $\xi_0^k \in \mathbb{C}$  be an arbitrary grid in  $\mathbb{C}$ ,  $\xi_0^i \neq \xi_0^j$ ,  $i \neq j$ , and  $\xi_0^k \notin \pi_2 \gamma$ ,  $k = 1, \dots, N$ . The complex curve  $X$  intersects complex line  $\{z_2 = -\xi_0^k\}$  at points  $(h_s(\xi_0^k, 0), -\xi_0^k)$ ,  $1 \leq s \leq N_+(\xi_0^k, 0)$ . We are going to present the formulas for finding these points.

The algorithm takes as input the points  $\{\xi_0^k\}_{k=1}^N$  and the curve  $\gamma$  (for example, represented as a finite number of points belonging to  $\gamma$ ). On the output of the algorithm we obtain the set of points  $(h_s(\xi_0^k, 0), -\xi_0^k)$ ,  $1 \leq k \leq N$ ;  $1 \leq s \leq N_+(\xi_0^k, 0)$ .

## 1. No points at infinity

1. *Computation of  $N_+$ .* According to formula (VII.4), for every domain  $\Omega_{l \geq 0}$ , the number  $\mu_l = N_+(\xi_0, 0)$ ,  $\xi_0 \in \Omega_l$ , is equal to the winding number of the curve  $\pi_2\gamma$  with respect to a point  $\xi_0 \in \Omega_l$  :

$$\mu_l \equiv N_+(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz_2}{z_2 + \xi_0} \equiv \frac{1}{2\pi i} \int_{\pi_2\gamma} \frac{dz}{z - \xi_0}, \quad \xi_0 \in \Omega_l.$$

2. *Computation of power sums.* If  $N_- = 0$  then, according to Theorem VII.3, for every point  $\xi_0^k \in \Omega_{l \geq 0}$  we have that  $P_m(\xi_0^k, 0) = 0$ . Using formula (VII.6) we obtain the following formulas for the power sums of the functions to be determined :

$$s_m(\xi_0^k) \equiv h_1^m(\xi_0^k, 0) + \dots + h_{\mu_l}^m(\xi_0^k, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^m dz_2}{z_2 + \xi_0^k}, \quad 1 \leq m \leq \mu_l.$$

3. *Computation of symmetric functions.* For every point  $\xi_0^k \in \Omega_{\geq 1}$ , the Newton identities

$$k\sigma_k(\xi_0^k) = \sum_{i=1}^k (-1)^{i-1} \sigma_{k-i}(\xi_0^k) s_i(\xi_0^k), \quad 1 \leq k \leq N_+(\xi_0^k, 0).$$

allow to reconstruct the elementary symmetric functions :

$$\begin{aligned} \sigma_1(\xi_0^k) &= h_1(\xi_0^k, 0) + \dots + h_{\mu_l}(\xi_0^k, 0), \\ &\dots = \dots \\ \sigma_{\mu_l}(\xi_0^k) &= h_1(\xi_0^k, 0) \times \dots \times h_{\mu_l}(\xi_0^k, 0). \end{aligned}$$

4. *Desymmetrisation.* For every point  $\xi_0^k \in \Omega_l$ , using Viète formulas, one can find the complex numbers  $h_1(\xi_0^k, 0), \dots, h_{\mu_l}(\xi_0^k, 0)$  (up to order). The points  $(h_s(\xi_0^k, 0), -\xi_0^k)$ ,  $1 \leq s \leq N_+(\xi_0^k, 0)$ ,  $1 \leq k \leq N$ , are the required points of the complex curve  $X$ .

## 2. One or two points at infinity

These cases can be reduced to the case  $N_- = 0$  in the following way. Since  $\pi_2\gamma \subset \mathbb{C}$  is a compact real curve, there exists such  $R > 0$ , that the set  $B_R^c(0) = \{z \in \mathbb{C} \mid |z| \geq R\}$

$R\}$  in contained in  $\Omega_0$ . Without loss of generality, one can suppose that  $|\xi_0^k| < R$  for all  $k = 1, \dots, N$ .

Consider an auxiliary complex curve  $X_R = \{(z_1, z_2) \in X \mid |z_2| \leq R\}$ . Its boundary  $\gamma_R$  consists of two disjoint parts (possibly, multiconnected) : the first part is  $\gamma$  and the second part  $\gamma'_R$  is obtained by lifting the circle  $S_R = \{z \in \mathbb{C} \mid |z| = R\}$  to  $X$  via the projection  $\pi_2: X \rightarrow \mathbb{C}$ .

The complex curve  $X_R$  does not intersect infinity. Moreover, points of the form  $(a, -\xi_0^k)$ ,  $k = 1, \dots, N$  belong to  $X$  if and only if they belong to  $X_R$ .

Therefore, in order to reconstruct  $X$  it is sufficient to reconstruct  $\gamma'_R$  and then to reconstruct  $X_R$ , using the algorithm for the case when  $N_- = 0$ . The algorithm can be formulated as follows :

1. *New boundary.* Choose a sufficiently large  $R > 0$ , so that  $B_R^c \subset \Omega_0$  and all  $\xi_0^k$  belong to  $B_R$ . In the case of  $N_- = 1$ , by virtue of formulas (VII.6), we have that

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0), \quad |\xi_0| = R,$$

where  $P_1$  can be found using Theorem VII.3. This formula allows to recover  $\gamma'_R$  and, as a corollary,  $\gamma_R = \gamma \sqcup \gamma'_R = \partial X_R$ .

In the case of  $N_- = 2$  we have two equalities :

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0), \quad |\xi_0| = R,$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^2 dz_2}{z_2 + \xi_0} - P_2(\xi_0, 0), \quad |\xi_0| = R,$$

where  $P_1$  and  $P_2$  can be found using Theorem VII.3. Applying Newton identities and Viète formulas, we find  $h_1(\xi_0, 0)$  and  $h_2(\xi_0, 0)$ . Thus, we have recovered  $\gamma'_R$  and, as a corollary,  $\gamma_R = \gamma \sqcup \gamma'_R = \partial X_R$ .

2. *Reduction.* In order to find the complex curve  $X_R$  with boundary  $\partial X_R = \gamma_R$  we apply the algorithm of reconstruction for the case of  $N_- = 0$ .

## 4 Visualization

To our knowledge, there are two known algorithms for automatic visualisation of complex curves. The first one was proposed by Trott [135] and requires the knowledge of an analytic expression for the curve. The second algorithm was proposed by Nieser-Poelke-Polthier [94] and requires the knowledge of the branching points and their indices for some fixed projection to  $\mathbb{C}$ . Our algorithm requires the knowledge of the unordered set of points of intersections of the curve with complex lines  $\mathbb{C}_\xi^1$ .

Let us describe in few words the algorithm of visualisation of complex curves that we use in our examples. Denote by  $\pi_1: \mathbb{C}^2 \rightarrow \mathbb{C}$  the projection into the first factor :  $\pi_1(z_1, z_2) = z_1$ . Suppose that  $X$  is a complex curve in  $\mathbb{C}^2$  such that the covering  $\pi_1: X \setminus \{\text{ramification points}\} \rightarrow \mathbb{C}$  has multiplicity  $L$ . Consider, for simplicity, a rectangular grid  $\Lambda$  in  $\mathbb{C}$  :

$$\Lambda = \left\{ z_1^{ij} : \operatorname{Re} z_1^{ij} = \frac{i}{N}, \operatorname{Im} z_1^{ij} = \frac{j}{N}, i, j = 0, \dots, N \right\},$$

where  $N$  is a natural number. Suppose now that we are given the set  $X_\Lambda = \pi_1^{-1}(\Lambda) \cap X$  and we need to visualize the part of  $X$  lying above the rectangle  $0 \leq \operatorname{Re} z_1 \leq 1, 0 \leq \operatorname{Im} z_1 \leq 1$ .

Let us introduce some terminology. We define a path in  $\Lambda$  as a map  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  such that  $|\gamma(k+1) - \gamma(k)| = \frac{1}{N}$  for all admissible  $k$ , where  $M$  is some natural number.

Let  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  be a path in  $\Lambda$  and let  $i: \{1, \dots, M\} \rightarrow [1, M]$  be the inclusion map. Define the function  $i_*\gamma: [1, M] \rightarrow \mathbb{C}$  such that  $i_*\gamma(k) = \gamma(k)$  for integer  $k$  and  $i_*\gamma|_{[k, k+1]}$  is linear for all admissible  $k$ . It is clear that  $i_*\gamma$  is a continuous function and hence it can be lifted to  $X$  by the map  $\pi_1$ .

We define a path in  $X_\Lambda$  as a map  $\Gamma: \{1, \dots, M\} \rightarrow X_\Lambda$  such that  $\gamma = \pi_1 \circ \Gamma$  is a path in  $\Lambda$  and  $\Gamma = i^*L(i_*\gamma)$ , where  $i^*$  is the pullback map with respect to  $i$  and  $L(i_*\gamma)$  is some lift of  $i_*\gamma$  to  $X$  by  $\pi_1$ , i. e.  $L(i_*\gamma)$  is a continuous map from  $[1, M]$  to  $X$  such that  $\pi_1 \circ L(i_*\gamma) = i_*\gamma$ . We also say that  $\Gamma$  is obtained by lifting of  $\gamma$ .

We will call subsets of  $\Lambda$  and  $X_\Lambda$  path-connected if every two points of these sets can be connected by a path in  $\Lambda$  and  $X_\Lambda$ , respectively.

Let us describe the practical way to lift paths in  $\Lambda$  to paths in  $X_\Lambda$ . Suppose that  $N$  is sufficiently large. Let  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  be a path in  $\Lambda$  and let  $\Gamma(1) \in \pi^{-1}(\gamma(1)) \cap X$  be an arbitrary point. We select  $\Gamma(k) \in \pi^{-1}(\gamma(k)) \cap X$  in such a way that

$$|\Gamma(k) - \Gamma(k-1)| = \min\{|z - \Gamma(k-1)| : z \in \pi^{-1}(\gamma(k)) \cap X\}, \quad k = 2, \dots, M.$$

Then  $\Gamma$  is a path in  $X_\Lambda$  obtained by lifting of  $\gamma$ . All possible lifts of  $\gamma$  may be obtained by varying  $\Gamma(1)$ . Note that if  $\gamma$  is closed, i. e.  $\gamma(1) = \gamma(M)$ ,  $\Gamma$  need not to be closed.

**Finding of ramification points and making branch cuts.** The first step in visualization procedure consists in finding of ramification points of  $X$  with respect to projection  $\pi_1$ . Since we have only a finite number of points on  $X$  we can find ramification points only approximately. More precisely, we will localize them in small circles.

Without restriction of generality we suppose that all ramification points are projected by  $\pi_1$  into interior points of  $\Lambda$ . Take any interior point  $z_1 \in \Lambda$  and select a small closed path  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  around  $z_1$  so that there is at most one ramification

point inside the polygon  $\overline{\gamma(1) \dots \gamma(M)}$ . For example, one can take as  $\gamma$  the following path :

$$z_1 + \frac{1}{N} \rightarrow z_1 + \frac{1+i}{N} \rightarrow z_1 + \frac{i}{N} \rightarrow \dots \rightarrow z_1 + \frac{1-i}{N} \rightarrow z_1 + \frac{1}{N},$$

where  $i$  is the imaginary unit.

Now consider different lifts of  $\gamma$  to  $X_\Lambda$ . If at least one lift is not closed, mark  $z_1$  as a possible ramification point (meaning that it is situated near the projection of some ramification point of  $X$ ). Now vary  $z_1$  and mark all possible ramification points. The resulting set will consist of several path-connected components each of which localizes the position of one ramification point of  $X$  with respect to  $\pi_1$ .

Next, we need to cut the grid  $\Lambda$  with branching points removed into simply connected domains. The natural choice is the shortest cut graph, which can be computed by the algorithm proposed in [86], which, in turn, generalizes the algorithm for the closed surfaces of [38].

Denote the cut graph by  $\Lambda_c$ . An important observation is that every closed path in  $\Lambda \setminus \Lambda_c$  always lifts to a closed path in  $X_\Lambda$  since it doesn't contain  $\pi_1$ -projections of ramification points inside.

**Visualization.** Now denote  $\Lambda \setminus \Lambda_c = \cup_{s=1}^S \Lambda_s$ , where  $\Lambda_s$  are different path-connected components. Take any  $z_1^s \in \Lambda_s$  and  $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$ . Now take other  $z_1 \in \Lambda_s$  and connect  $z_1^s$  with  $z_1$  by some path  $\gamma$ . Then  $\gamma$  lifts to a path  $\Gamma$  with  $\Gamma(1) = (z_1^s, z_2^s)$  and  $\Gamma(2) = (z_1, z_2)$  for some  $z_2 \in \pi_1^{-1}(z_1) \cap X$  and  $z_2$  doesn't depend on  $\gamma$ . Varying  $z_1$  we thus obtain the map  $\Sigma(z_1^s, z_2^s): \Lambda_s \rightarrow X_\Lambda$  which allows us to visualize the part of  $X$ .

Varying  $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$  (the latter is the finite set, namely, it consists of  $L$  elements) we obtain the other maps  $\Sigma(z_1^s, z_2^s)$  which allow us to visualize other parts of  $X$ . Clearly, the set of obtained maps doesn't depend on the choice of  $z_1^s \in \Lambda_s$ . Hence we can denote the obtained maps by  $\Sigma_s^l$ ,  $l = 1, \dots, L$ . It is clear that  $\cup_{l=1}^L \Sigma_s^l(\Lambda_s) = \pi^{-1}(\Lambda_s) \cap X$ . Now vary  $s$  to visualize

$$\cup_{s=1}^S \cup_{l=1}^L \Sigma_s^l(\Lambda_s) = \pi^{-1}(\cup_{s=1}^S \Lambda_s) \cap X = X_\Lambda \setminus \pi^{-1}(\Lambda_c).$$

The part  $\pi^{-1}(\Lambda_c) \cap X_\Lambda$  consists of cuts and preimages of possible ramification points. The cuts can be visualized as the already visualized part of the surface. The only problem is the visualization of  $\pi_1$ -preimages of possible ramification points. But the latter take a little part of the surface when  $N$  is large and one can just forget about their visualization. On the other hand, in our examples they were visualized using low-level graphics approach.

Examples of application of this algorithm are given at Fig. VII.1 and in Section 5. The visualization algorithm can be easily generalized to the case of general grids. For instance, in our examples we have used a modification with periodic grid.

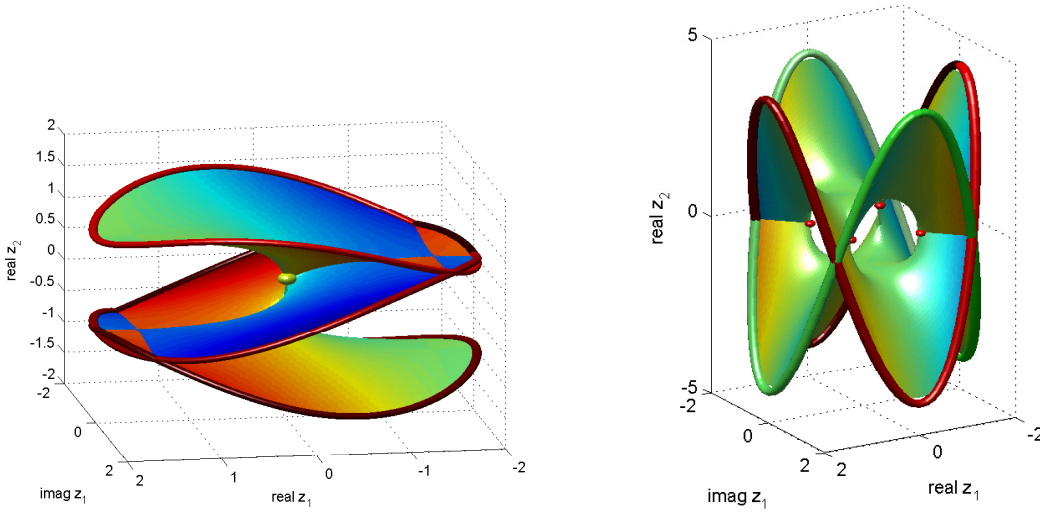


FIGURE VII.1 – Riemann surfaces of functions  $f(z) = \sqrt{\sin(z)}$ ,  $|z| \leq 2$  (left) and  $f(z) = \sqrt{z^4 + 1}$ ,  $|z| \leq 2$  (right) obtained by the visualization algorithm.

## 5 Examples

**1. No points at infinity.** We consider the complex curve given by the equation

$$X_1 = \{(z_1, z_2) \in \mathbb{C}^2: z_2 + 1 = (z_1^2 - \exp(\frac{z_2}{4}))^2, |z_2| \leq 1.9\}. \quad (\text{VII.31})$$

The boundary  $\gamma_1 = \partial X_1$  of this curve is the disjoint union of two real curves  $\gamma_1^r$  and  $\gamma_1^g$ , see Fig. VII.2. We are going to recover  $X_1$  from  $\gamma_1$ .

First, note that the real curve  $|z_2| = 1.9$  in  $\mathbb{C}$  divides  $\mathbb{C}$  into two parts  $\Omega_0 = \{|z_2| > 1.9\}$  and  $\Omega_1 = \{|z_2| < 1.9\}$ . We check numerically that  $G_1(\xi)$  is zero in a neighborhood  $W_{\xi^*}$  of some  $\xi^* = (\xi_0^*, 0)$ ,  $\xi_0 \in \Omega_0$ , and thus  $N_- = 0$  by Theorem VII.4.

Next, we compute the value

$$\mu_1 = \sigma_0(\xi_0) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{dz_2}{z_2 + \xi_0},$$

for some  $\xi_0 \in \Omega_1$  (it will be the same for any  $\xi_0 \in \Omega_1$ ). It turns out that  $\mu_1 = 4$  and thus  $\pi_2: X_1 \rightarrow \Omega_1$  is a (ramified) covering with 4 sheets. Denote  $X_1 \cap \mathbb{C}_\xi^1 = \{(h_j(\xi), -\xi_0)\}_{j=1}^4$ ,  $\xi = (\xi_0, 0)$ ,  $|\xi_0| < 1.9$ .

Next, we compute the values of the power sum functions  $s_k$  :

$$s_k(\xi_0) := \sum_{j=1}^4 h_j^k(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad |\xi_0| < 1.9, \quad 1 \leq k \leq 4.$$

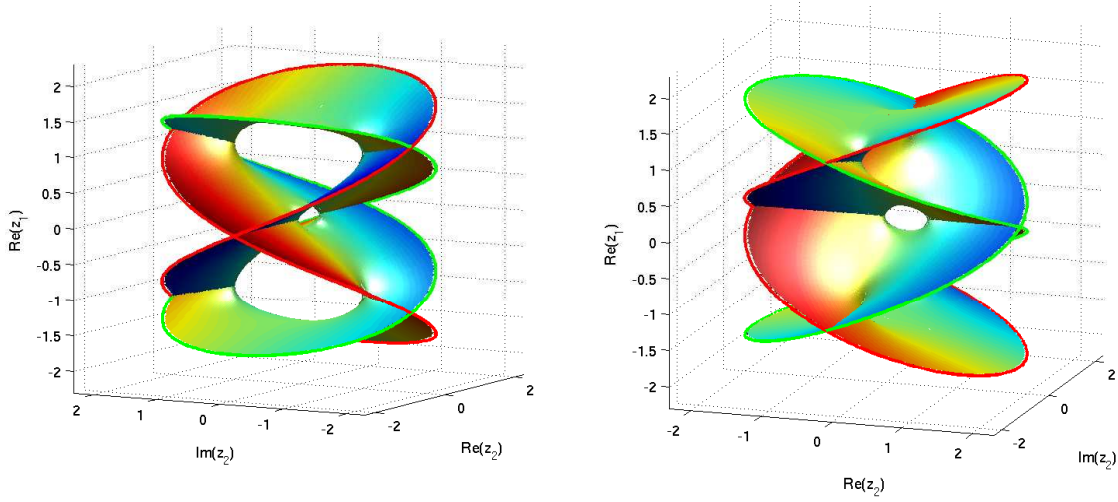


FIGURE VII.2 – The given boundary  $\gamma_1 = \gamma_1^r \sqcup \gamma_1^g$  (red and green) and the reconstructed complex curve  $X_1 \subset \mathbb{C}^2$ . The color indicates  $\text{Im}(z_1)$ .

Using Newton identities, we determine the values of symmetric functions  $\sigma_k$  :

$$\begin{aligned}\sigma_1 &= s_1, & \sigma_2 &= \frac{1}{2}(s_1^2 - s_2), \\ \sigma_3 &= \frac{1}{6}(s_1^3 - 3s_1s_2 + 2s_3), \\ \sigma_4 &= \frac{1}{24}(s_1^4 - 6s_1^2s_2 + 3s_2^2 + 8s_1s_3 - 6s_4).\end{aligned}$$

Finally, taking into account Viète formulas, we find  $h_1(\xi_0, 0), \dots, h_4(\xi_0, 0)$  as the roots of the following polynomial in  $t$  :

$$t^{N+(\xi_0)} - \sigma_1(\xi_0)t^{N+(\xi_0)-1} + \dots + (-1)^{N+(\xi_0)}\sigma_{N+(\xi_0,0)}(\xi_0) = 0.$$

Reconstructed surface is represented at Fig. VII.2.

**2. One point at infinity.** Consider the the complex curve  $X_2$  which is the part of the curve

$$\tilde{X}_2 = \{(w_0 : w_1 : w_2) \in \mathbb{C}P^2 : w_1^2 = w_2^2 + w_0^2\},$$

bounded by the real curve

$$\begin{aligned}\gamma_2 &= \{(z_1(t), z_2(t)) \in \mathbb{C}^2 : t \in [0, 2\pi]\}, \\ z_2(t) &= 1.9e^{it}, \quad z_1(t) = \sqrt{z_2^2(t) + 1}, \quad z_1(0) > 0,\end{aligned}$$

intersecting infinity at  $(0 : 1 : 1)$ . We suppose that the number of points at infinity is apriori unknown. We are going to reconstruct  $X_{2,R} = X_2 \cap \{z_2 < R\}$ , where  $R = 3$ , from  $\gamma_2$ .



Note that the curve  $\pi_2\gamma_2 = \{|z_2| = 1.9\}$  divides  $\mathbb{C}$  into two connected components  $\Omega_0 = \{z_2 > 1.9\}$  and  $\Omega_1 = \{z_2 < 1.9\}$ .

According to the algorithm, the first step is to determine the number of points at infinity. We choose some  $\xi^* = (\xi_0^*, 0)$  with  $\xi_0^* \in \Omega_0$  and check that in some small neighborhood  $W_{\xi^*}$  of  $\xi^*$  we have  $G_1 \not\equiv 0$  and thus  $N_- \geq 1$  by Theorem VII.4.

Next, we fix two generic  $\xi_0^1, \xi_0^2 \in \Omega_0$  and determine  $c_{11}, c_{12}$  from the linear system

$$\begin{aligned} c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + c_{12} \left( \xi_0^1 \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + G_1(\xi_0^1, 0) \right) &= G_1(\xi_0^1, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^2, 0), \\ c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + c_{12} \left( \xi_0^2 \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + G_1(\xi_0^2, 0) \right) &= G_1(\xi_0^2, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^2, 0). \end{aligned} \quad (\text{VII.32})$$

This system is uniquely solvable and gives  $c_{11} \approx 1, c_{12} \approx 0$ .

Next, we check numerically that in the neighborhood  $W_{\xi^*}$  of  $\xi^*$  the following identity holds :

$$\begin{aligned} \frac{\partial}{\partial \xi_1} (G_1 - P_1) &= (G_1 - P_1) \frac{\partial}{\partial \xi_0} (G_1 - P_1), \\ \text{where } P_1(\xi_0, \xi_1) &= \frac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}. \end{aligned} \quad (\text{VII.33})$$

Theorem VII.4 together with the estimate  $N_- \geq 1$  imply that  $N_- = 1$ .

Next, according to the algorithm, in order to reduce the problem to the case of no points at infinity, we need to recover the auxiliary curve  $\gamma'_{2,R} = X_2 \cap \{z_2 = R\}$ . Note that  $\gamma'_{2,R} = \{(h_1(\xi_0, 0), -\xi_0) : |\xi_0| = R\}$ , where  $h_1(\xi_0, 0)$  can be found from the explicit formula

$$h_1(\xi_0, 0) = G_1(\xi_0, 0) - c_{11}\xi_0 - c_{12}, \quad |\xi_0| = R.$$

Next, we set  $\gamma_{2,R} = \gamma_2 \sqcup \gamma'_{2,R}$  so that  $\partial X_{2,R} = \gamma_{2,R}$ ,  $X_{2,R} \cap \{|z_2| < R\} = X_2 \cap \{|z_2| < R\}$  and  $X_{2,R}$  does not intersect  $\mathbb{C}P_\infty^1$ . Thus, it remains to recover  $X_{2,R}$  from the known boundary  $\gamma_{2,R}$  using the algorithm for the case of no points at infinity, see the previous example. The reconstructed curve is depicted at Fig. VII.3.

**3. Two points at infinity.** Consider the complex curve  $X_3$ , which is the part of the algebraic curve

$$\tilde{X}_3 = \{(w_0 : w_1 : w_3) \in \mathbb{C}P^2 : w_1^3 = (w_2 - iw_0)(w_2 + iw_0)(w_2 - w_0)\},$$

bounded by the real curve

$$\begin{aligned} \gamma_3 &= \{(z_1(t), z_2(t)) : t \in [0, 2\pi]\}, \\ z_2(t) &= 1.9e^{it}, \quad z_1(t) = \sqrt[3]{(z_2(t) - i)(z_2(t) + i)(z_2(t) - 1)}, \quad z_1(0) > 0. \end{aligned}$$

The curve  $X_3$  has two points at infinity, namely,  $q_1 = (0 : 1 : e^{\frac{2\pi i}{3}})$ ,  $q_2 = (0 : 1 : e^{-\frac{2\pi i}{3}})$ . We suppose that the number of points at infinity is a priori unknown. We are going to reconstruct  $X_3$  from  $\gamma_3$ .

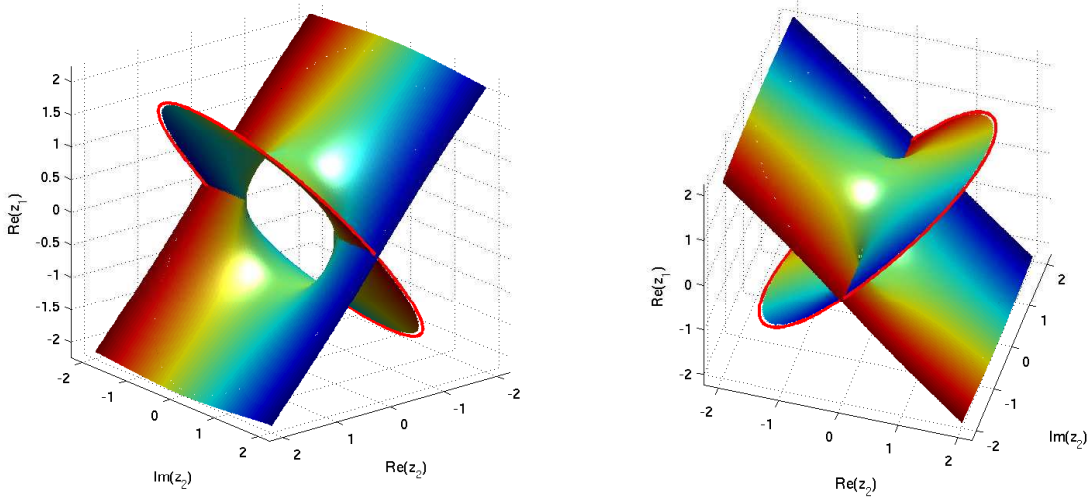


FIGURE VII.3 – The real curve  $\gamma_2$  (red) and the reconstructed complex curve  $X_2$  with one point at infinity. The color indicates  $\text{Im}(z_1)$ .

As in the previous examples, the real curve  $\pi_2\gamma_2$  divides  $\mathbb{C}$  into two parts  $\Omega_0 = \{|z_2| > 1.3\}$  and  $\Omega_1 = \{|z_2| < 1.9\}$ .

The first step is to determine the number of points at infinity. Choose a generic  $\xi^* = (\xi_0^*, 0)$ ,  $\xi_0^* \in \Omega_0$ , and some neighborhood  $W_{\xi^*}$  of  $\xi^*$ .

1. We check that  $G_1 \not\equiv 0$  in  $W_{\xi^*}$  so that  $N_- \geq 1$  by Theorem VII.4.
2. We choose two generic  $\xi_0^1, \xi_0^2 \in \Omega_0$  and solve (VII.32) for  $c_{11}, c_{12}$ . It turns out that system (VII.32) is uniquely solvable. However, identity (VII.33) does not hold in  $W_{\xi^*}$ . Using Theorem VII.4, we obtain that  $N_- \geq 2$ .
3. We consider (VII.11) as an identity in a neighborhood of  $\xi_0^*$ . We find numerically that

$$\begin{aligned} c_{11} &\approx 0.9974, & c_{12} &\approx -0.3335, \\ c_{21} &\approx 1.0133, & c_{22} &\approx -0.6699, & c_{23} &\approx 0.5567. \end{aligned}$$

We check that (VII.26) holds in a neighborhood of  $W_{\xi^*}$  with constants  $a_1, a_2, b_1, b_2, c_1, c_2$  given by equations (VII.27). It follows from Theorem VII.4 and from the inequality  $N_- \geq 2$  that  $N_- = 2$ .

Suppose that we want to recover  $X_{3,R} = X_3 \cap \{|z_2| < R\}$ ,  $R = 3$ . According to the algorithm, we need to find the auxiliary boundary  $\gamma'_{3,R} = X_3 \cap \{|z_2| = R\}$ . Denote  $\gamma'_{3,R} = \{(h_j(\xi_0, 0), -\xi_0) : |\xi_0| = R, j = 1, 2\}$ . We compute the values of symmetric functions  $\sigma_1(\xi_0) = h_1(\xi_0, 0) + h_2(\xi_0, 0)$ ,  $\sigma_2(\xi_0) = h_1(\xi_0, 0)h_2(\xi_0, 0)$  using the explicit

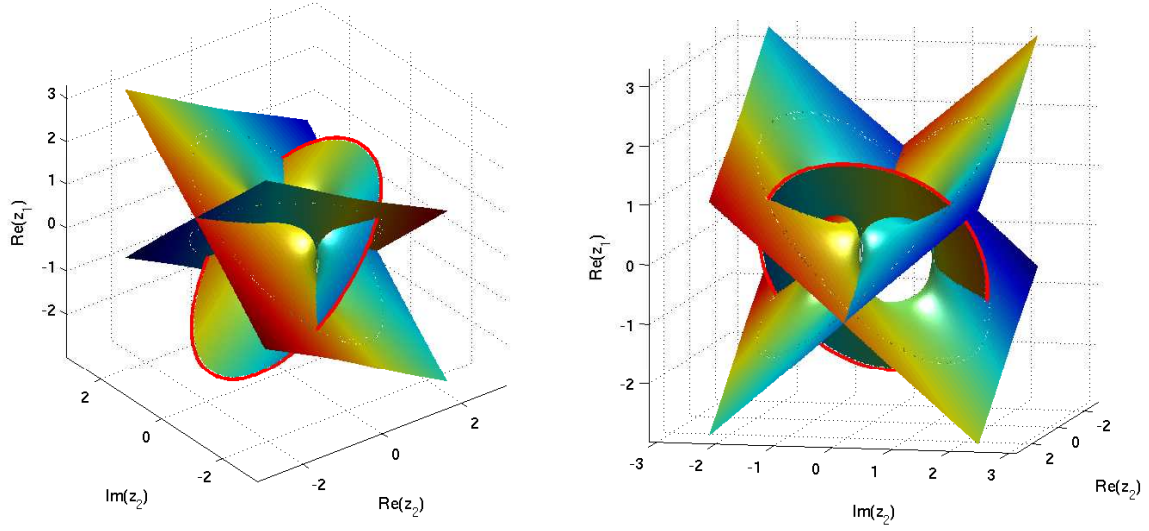


FIGURE VII.4 – The real curve  $\gamma_3$  (red) and the reconstructed complex curve  $X_3$  with two points at infinity. The color indicates  $\text{Im}(z_1)$ .

formulas

$$\begin{aligned} \sigma_1 &= s_1, & \sigma_2 &= \frac{1}{2}(s_1^2 - s_2), \\ s_1(\xi_0) &:= h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 + \xi_0} - c_{11} - c_{12}\xi_0, & |\xi_0| &= R, \\ s_2(\xi_0) &:= h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^2 dz_2}{z_2 + \xi_0} - c_{21} - c_{22}\xi_0 - c_{23}\xi_0^2, & |\xi_0| &= R. \end{aligned}$$

Next, taking into account the Viète formulas, we find the values  $h_1(\xi_0, 0)$ ,  $h_2(\xi_0, 0)$ ,  $|\xi_0| = R$ , as the roots of the following polynomial in  $t$  :

$$t^2 - \sigma_1(\xi_0)t + \sigma_2(\xi_0) = 0.$$

Then we put  $\gamma_{3,R} = \gamma_3 \sqcup \gamma'_{3,R}$  and recover  $X_{3,R}$  from  $\gamma_{3,R}$  using the algorithm for the case of no points at infinity. The result of reconstruction is depicted at Fig. VII.4.

# Conclusion

The main results of the present work can be summarized as follows :

1. We obtain global uniqueness theorems for acoustic tomography of moving fluid at one, two or three fixed frequencies depending on a chosen model.
2. We present examples of moving fluids which are not distinguishable in the setting of acoustic tomography even if the data is measured at all frequencies.
3. We give formulas and equations which reduce the inverse Dirichlet-to-Neumann problem for a gauge-covariant Helmholtz equation to an appropriate inverse scattering problem.
4. We propose a functional-analytic reconstruction algorithm for the above-mentioned inverse scattering problem, which allows to recover the gauge class of the operator from its scattering amplitude at fixed frequency.
5. We present equations which allow to get rid of the above gauge non-uniqueness and uniquely recover the operator in the setting of acoustic tomography of moving fluid, using measurements at several frequencies.
6. We obtain error estimates in the coordinate space for phaseless reconstructions in the Born approximation at high energies for the Schrödinger equation.
7. We give an explicit version of the Harvey-Shiffman result on the unique determination of a complex curve by its boundary, developing the algorithm of Henkin-Michel for the inverse boundary value problem for two-dimensional Riemannian manifolds.

One can underline the following principal directions of possible development of the subjects studied in the thesis :

1. Adapt the algorithm of Articles III-V for acoustic tomography of moving fluid to the case of partial data. In particular, the case where only the imaginary part of the Green function is known is of great importance for applications with cross-correlation-type measurements, e.g., in passive ocean tomography [22], in helioseismology [55] and in other settings, see, e.g., [128].
2. Study the stability and obtain the error estimates for the functional-analytic reconstruction algorithm of Article IV. Note that the stability was studied

numerically in many particular cases by O. D. Romyantseva, A. S. Shurup and D. I. Zotov (acoustic department of Moscow State University), see, e.g., [126].

3. Generalize the functional-analytic reconstruction algorithm of Article IV to the case of the two-dimensional gauge-covariant Helmholtz equation with matrix coefficients. This generalization is motivated by applications in mode tomography of moving ocean, see, e.g., [15].
4. Find an efficient and noise stable algorithm for the phaseless inverse scattering problem for the Schrödinger equation at fixed frequency using the error estimates of Article VI.
5. Find compact formulas and equations for recovering the complex curve in the complex projective plane from its boundary for the case of arbitrary number of points at infinity. Specify easy verifiable conditions for the unique solvability of these equations.

# Bibliographie

- [1] M. J. Ablowitz, D. B. Yaarov, and A. S. Fokas, “On the inverse scattering transform for the Kadomtsev-Petviashvili equation,” *Stud. Appl. Math.*, vol. 69, pp. 135–143, 1983.
- [2] A. D. Agaltsov, “Finding scattering data for a time-harmonic wave equation with first order perturbation from the Dirichlet-to-Neumann map,” *Journal of Inverse and Ill-Posed Problems*, vol. 23, no. 6, pp. 627–645, 2015.
- [3] —, “A global uniqueness result for acoustic tomography of moving fluid,” *Bulletin des Sciences Mathématiques*, vol. 139, no. 8, pp. 937–942, 2015.
- [4] A. D. Agaltsov and R. G. Novikov, “Riemann-Hilbert problem approach for two-dimensional flow inverse scattering,” *J. Math. Phys.*, vol. 55, no. 10, 2014, id 103502.
- [5] —, “Uniqueness and non-uniqueness in acoustic tomography of moving fluid,” *Journal of Inverse and Ill-Posed Problems*, 2015, doi :10.1515/jiip-2015-005.
- [6] Y. Aharonov and D. Bohm, “Significance of electromagnetic potentials in quantum theory,” *Phys. Rev.*, vol. 115, pp. 485–491, 1959.
- [7] T. Aktosun and P. E. Sacks, “Inverse problems on the line without phase information,” *Inverse Problems*, vol. 14, pp. 211–224, 1998.
- [8] G. Alessandrini, “Stable determination of conductivity by boundary measurements,” *Appl. Anal.*, vol. 27, pp. 153–172, 1988.
- [9] N. V. Alexeenko, V. A. Burov, and O. D. Rumyantseva, “Solution of the three-dimensional acoustical inverse scattering problem. The modified Novikov algorithm,” *Acoustical Physics*, vol. 54, no. 3, pp. 407–419, 2008.
- [10] S. Arians, “Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials,” *J. Math. Phys.*, vol. 38, no. 6, pp. 2761–2773, 1997.
- [11] K. Astala and L. Päivärinta, “Calderón’s inverse conductivity problem in the plane,” *Ann. of Math.*, vol. 163, pp. 265–299, 2006.
- [12] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*. New York : Springer-Verlag, 1982.

- [13] L. Baratchart, Y. Fischer, and J. Leblond, “Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation,” *Complex Variables and Elliptic Equations*, vol. 59, no. 4, pp. 504–538, 2014.
- [14] J. A. Barceló, C. Castro, and J. M. Reyes, “Numerical approximation of the potential in the two-dimensional inverse scattering problem,” *Inverse Problems*, vol. 32, no. 1, 2016, 015006 (19pp).
- [15] S. V. Baykov, V. A. Burov, and S. N. Sergeev, “Mode tomography of moving ocean,” in *European Conference on Underwater Acoustics*, vol. 2. Heraklion : Crete University Press, 1996, pp. 845–850.
- [16] Y. M. Berezansky, “On uniqueness theorem in inverse problem of spectral analysis for Schrödinger equation,” *Mosc. Math. Soc. Proc.*, vol. 7, pp. 3–62, 1958, (in Russian).
- [17] F. A. Berezin and M. A. Shubin, *The Schrödinger equation*, ser. Mathematics and its applications. Dordrecht : Kluwer Academic, 1991, vol. 66.
- [18] A. V. Bogatyrev, V. A. Burov, S. A. Morozov, O. D. Rumyantseva, and E. G. Sukhov, “Numerical realization of algorithm for exact solution of two-dimensional monochromatic inverse problem of acoustical scattering,” *Acoustic Imaging*, vol. 25, pp. 65–70, 2000.
- [19] R. M. Brown and M. Salo, “Identifiability at the boundary for first-order terms,” *Appl. Anal.*, vol. 85, no. 6-7, pp. 735–749, 2006.
- [20] A. L. Bukhgeim, “Recovering a potential from Cauchy data in the two-dimensional case,” *J. Inverse Ill-Posed Probl.*, vol. 16, no. 1, pp. 19–33, 2008.
- [21] V. A. Burov, N. V. Alekseenko, and O. D. Rumyantseva, “Multifrequency generalization of the Novikov algorithm for the two-dimensional inverse scattering problem,” *Acoustical Physics*, vol. 56, no. 6, pp. 843–856, 2009.
- [22] V. A. Burov, S. N. Sergeev, and A. S. Shurup, “The use of low-frequency noise in passive tomography of the ocean,” *Acoustical Physics*, vol. 54, no. 1, pp. 42–51, 2008.
- [23] V. A. Burov, S. N. Sergeev, A. S. Shurup, and O. D. Rumyantseva, “Application of the functional-analytical Novikov algorithm for the purposes of ocean tomography,” *Proceedings of Meetings on Acoustics*, vol. 17, 2012, 070065.
- [24] V. A. Burov, A. S. Shurup, D. I. Zotov, and O. D. Rumyantseva, “Simulation of a functional solution to the acoustic tomography problem for data from quasi-point transducers,” *Acoustical Physics*, vol. 59, no. 3, pp. 345–360, 2013.
- [25] A. P. Calderón, “On an inverse boundary value problem,” in *Seminar on numerical analysis and its applications to continuum physics*, W. H. Meyer and M. A. Raupp, Eds. Rio de Janeiro : Sociedade Brasileira de Matematica, 1980, pp. 65–73.

- [26] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory*, 2nd ed. Berlin : Springer, 1989.
- [27] W. L. Chow, “On compact complex analytic varieties,” *Amer. J. of Math.*, vol. 71, no. 4, pp. 893–914, 1949.
- [28] D. Colton and R. Kress, *Integral equation methods in scattering theory*. New York : John Wiley, 1983.
- [29] L. Darboux, *Théorie des surfaces*. Paris : Gauthier-Villars, 1914.
- [30] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten, *Quantum fields and strings : A course for mathematicians*, 1st ed. American Mathematical Society, 1999, vol. Vol.1.
- [31] P. Dolbeault and G. Henkin, “Surfaces de Riemann de bord donné dans  $\mathbb{C}P^n$ ,” in *Contributions to complex analysis and analytic geometry*, ser. Aspects Math. Braunschweig : Friedr. Vieweg, 1994, vol. E26, pp. 163–187.
- [32] —, “Chaînes holomorphes de bord donné dans  $\mathbb{C}P^n$ ,” *Bull. Sci. Math.*, vol. 125, pp. 383–445, 1997.
- [33] D. Dos Santos Ferreira, D. Kenig, J. Sjöstrand, and G. Uhlmann, “Determining a magnetic Schrödinger operator from partial Cauchy data,” *Comm. Math. Phys.*, vol. 271, no. 2, pp. 461–488, 2007.
- [34] K. Dressler, “Stationary solutions of the Vlasov-Fokker-Planck equation,” *Math. Meth. in the Appl. Sci.*, vol. 9, pp. 169–176, 1987.
- [35] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, “The Schrödinger equation in a periodic field and Riemann surfaces,” *Sov. Math. Dokl.*, vol. 17, pp. 947–951, 1976.
- [36] F. Dyson, “The S matrix in quantum electrodynamics,” *Phys. Rev.*, vol. 75, no. 11, pp. 1736–1755, 1949.
- [37] W. Ehrenberg and W. Siday, “The refractive index in electron optics and the principles of dynamics,” *Proceedings of the Physical Society. Series B*, vol. 62, pp. 8–21, 1949.
- [38] J. Erickson and K. Whittlesey, “Greedy optiman homotopy and homology generators,” in *SODA*, 2005, pp. 1038–1046.
- [39] G. Eskin, “Global uniqueness in the inverse scattering problem for the Schrödinger operator with external Yang-Mills potentials,” *Comm. Math. Phys.*, vol. 222, no. 3, pp. 503–531, 2001.
- [40] —, *Lectures on linear partial differential equations*, ser. Graduate studies in mathematics. American mathematical society, 2011, vol. 123.
- [41] —, “Aharonov-Bohm effect revisited,” *Reviews in Mathematical Physics*, vol. 27, no. 2, 2015, id 1530001.



- [42] G. Eskin and J. Ralston, “Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy,” *Commun. Math. Phys.*, vol. 173, pp. 173–199, 1995.
- [43] —, “Inverse scattering problems for Schrödinger operators with magnetic and electric potentials,” in *Inverse problems in wave propagation*, ser. IMA Vol. Math. Appl. New York : Springer, 1997, vol. 90, pp. 147–166.
- [44] —, “Inverse scattering problems for the Schrödinger operators with external Yang-Mills potentials,” in *Partial differential equations and their applications (Toronto 1995)*, ser. CRM. Proc. Lecture Notes. Providence : AMS, 1997, vol. 12, pp. 91–106.
- [45] —, “Inverse boundary value problems for systems of partial differential equations,” in *Recent Development in Theories and Numerics*. River Edge, NJ : World Sci. Publishing, 2003, pp. 105–113.
- [46] L. D. Faddeev, “Uniqueness of the solution of the inverse scattering problem,” *Vest. Leningrad Univ.*, vol. 7, pp. 126–130, 1956, (in Russian).
- [47] —, “Growing solutions of the Schrödinger equation,” *Sov. Phys. Dokl.*, vol. 10, pp. 1033–1035, 1966.
- [48] —, “Inverse problem of quantum scattering theory. II.” *Journal of Soviet Mathematics*, vol. 5, no. 3, pp. 334–396, 1976.
- [49] L. D. Faddeev and S. P. Merkuriev, *Quantum scattering theory for several particle systems*, ser. Mathematical physics and applied mathematics. Dordrecht : Kluwer Academic, 1993, vol. 11.
- [50] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics*. Reading : Addison-Wesley, 1965, vol. III.
- [51] A. S. Fokas, “Kadomtsev-Petviashvili equation revisited and integrability in  $4+2$  and  $3+1$ ,” *Stud. Appl. Math.*, vol. 122, no. 4, pp. 347–359, 2009.
- [52] A. S. Fokas and M. J. Ablowitz, “On the inverse scattering of the time-dependent Schrödinger equation and the associated Kadomtsev-Petviashvili equation,” *Stud. Appl. Math.*, vol. 69, no. 3, pp. 211–228, 1983.
- [53] I. M. Gelfand, “Some problems of functional analysis and algebra,” in *Proceedings of the International Congress of Mathematicians*, Amsterdam, 1954, pp. 253–276.
- [54] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, ser. Classics in Mathematics. Berlin Heidelberg : Springer-Verlag, 2001.
- [55] L. Gizon, H. Barucq, M. Duruflé, C. A. Birch, J. Chabassier, D. Fournier, and C. Hanson, “Solving the forward problem of helioseismology in the frequency domain,” preprint : <https://hal.archives-ouvertes.fr/hal-01264023>.

- [56] P. G. Grinevich, “The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy,” *Russian Math. Surveys*, vol. 55, no. 6, pp. 1015–1083, 2000.
- [57] P. G. Grinevich and S. V. Manakov, “Inverse scattering problem for the two-dimensional Schrödinger operator, the  $\bar{\partial}$ -method and nonlinear equations,” *Funct. Anal. Appl.*, vol. 20, no. 2, pp. 94–103, 1986.
- [58] P. G. Grinevich and R. G. Novikov, “Analogues of multisoliton potentials for the two-dimensional Schrödinger operator,” *Functional Analysis and Its Applications*, vol. 19, no. 4, pp. 276–285, 1985.
- [59] —, “Analogues of multisoliton potentials for the two-dimensional Schrödinger operator and the nonlocal Riemann problem,” *Dokl. Akad. Nauk SSSR*, vol. 286, pp. 19–22, 1986.
- [60] —, “Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials,” *Comm. Math. Phys.*, vol. 174, pp. 409–446, 1995.
- [61] C. Guillarmou and L. Tzou, “Identification of a connection from Cauchy data on a Riemann surface with boundary,” *Geom. Funct. Anal.*, vol. 21, no. 2, pp. 393–418, 2011.
- [62] H. Haberman and D. Tataru, “Uniqueness in Calderón’s problem with Lipschitz conductivities,” *Duke Math. J.*, vol. 162, pp. 497–516, 2013.
- [63] P. Hähner and T. Hohage, “New stability estimates for the inverse acoustic inhomogeneous medium problem and applications,” *SIAM J. Math. Anal.*, vol. 33, no. 3, pp. 670–685, 2001.
- [64] —, “New stability estimates for the inverse acoustic inhomogeneous medium problem and applications,” *SIAM J. Math. Anal.*, vol. 33, no. 3, pp. 670–685, 2001.
- [65] F. R. Harvey and H. Lawson, “Boundaries of complex analytic varieties,” *Ann. of Math.*, vol. 102, no. 2, pp. 223–290, 1975.
- [66] F. R. Harvey and B. Shiffman, “A characterization of holomorphic chains,” *Ann. of Math.*, vol. 99, no. 3, pp. 553–587, 1974.
- [67] G. Henkin and V. Michel, “On the explicit reconstruction of a Riemann surface from its Dirichlet-to-Neumann operator,” *Geom. Funct. Anal.*, vol. 17, no. 1, pp. 116–155, 2007.
- [68] G. Henkin and R. Novikov, “The Yang-Mills fields, the Radon-Penrose transform, and the Cauchy-Riemann equations,” in *Several Complex Variables V*, ser. Encyclopaedia of Mathematical Sciences, 1993, vol. 54, pp. 109–193.
- [69] —, “On the reconstruction of conductivity of bordered two-dimensional surface in  $\mathbb{R}^3$  from electrical currents measurements on its boundary,” *Journal of Geometric Analysis*, vol. 21, pp. 543–587, 2011.

- [70] G. Henkin and M. Santacesaria, “Gelfand-calderón inverse problem for anisotropic conductivities on bordered surfaces in  $\mathbb{R}^3$ ,” *Int. Math. Res. Not. IMRN*, no. 4, pp. 781–809, 2012.
- [71] G. M. Henkin and R. G. Novikov, “A multidimensional inverse problem in quantum and acoustic scattering,” *Inv. Problems*, vol. 4, pp. 103–121, 1988.
- [72] T. Hohage, “On the numerical solution of a three-dimensional inverse medium scattering problem,” *Inverse Problems*, vol. 17, pp. 1743–1763, 2001.
- [73] O. Y. Imanuvilov, G. Uhlmann, and M. Yamamoto, “Determination of second-order elliptic operators in two dimensions from partial Cauchy data,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 108, no. 2, pp. 467–472, 2011.
- [74] O. Y. Imanuvilov and M. Yamamoto, “Inverse problem by Cauchy data on an arbitrary sub-boundary for systems of elliptic equations,” *Inverse Problems*, vol. 28, 2012, id 095015.
- [75] M. I. Isaev, “Exponential instability in the inverse scattering problem on the energy interval,” *Funct. Anal. Appl.*, vol. 47, no. 3, pp. 187–194, 2013.
- [76] M. I. Isaev and R. G. Novikov, “New global stability estimates for monochromatic inverse acoustic scattering,” *SIAM J. Math. Analysis*, vol. 43, no. 5, pp. 1495–1504, 2013.
- [77] ———, “Effectivized Hölder-logarithmic stability estimates for the gelfand inverse problem,” *Inverse Problems*, vol. 30, no. 9, 2014, 095006.
- [78] M. I. Isayev and R. G. Novikov, “Reconstruction of a potential from the impedance boundary map,” *Eurasian Journal of Mathematical and Computer Applications*, vol. 1, no. 1, pp. 5–28, 2013.
- [79] A. Jollivet, “Inverse scattering at high energies for classical relativistic particles in a long-range electromagnetic field,” *AHP*, vol. 16, no. 11, pp. 2569–2602, 2015.
- [80] C. Kenig and M. Salo, “Recent progress in the Calderón problem with partial data,” in *Inverse problems and applications*, ser. Contemp. Math. Providence, RI : AMS, 2014, vol. 615, pp. 193–222.
- [81] J. King, “Open problems in geomtric function theory,” in *Conference on geometric function theory, Problem D-1*, Katata, September, 1-6, 1978, p. 4.
- [82] M. V. Klibanov, “Phaseless inverse scattering problems in three dimensions,” *SIAM J. Appl. Math.*, vol. 74, pp. 392–410, 2014.
- [83] M. V. Klibanov and V. G. Romanov, “Two reconstruction procedures for a 3D phaseless inverse scattering problem for the generalized Helmholtz equation,” *Inverse Problems*, vol. 32, no. 1, 2016, 015005 (16pp).

- [84] K. Krupchyk, M. Lassas, and G. Uhlmann, “Inverse problems with partial data for a magnetic Schrödinger operator in an infinite slab or on a bounded domain,” *Comm. Math. Phys.*, vol. 312, no. 1, pp. 87–126, 2012.
- [85] K. Krupchyk and G. Uhlmann, “Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential,” *Comm. Math. Phys.*, vol. 327, no. 3, pp. 993–1009, 2014.
- [86] F. Kälberer, M. Nieser, and K. Polthier, “Stripe parametrization of tubular surfaces,” in *Topology-Based Methods in Visualization III. Mathematics and Visualization*. Springer-Verlag, 2010.
- [87] M. Lassas and G. Uhlmann, “On determining a Riemannian manifold from the Dirichlet-to-Neumann map,” *Ann. Sci. Ecole Norm. Sup.*, vol. 34, no. 5, pp. 771–787, 2001.
- [88] B. A. Lippmann and J. Schwinger, “Variational principles for scattering processes. I,” *Physical Review*, vol. 79, no. 3, pp. 469–480, 1950.
- [89] S. V. Manakov, “The inverse scattering transform for the time dependent Schrödinger equation and Kadomtsev-Petviashvili equation,” *Physica D*, vol. 3, no. 1, 2, pp. 420–427, 1981.
- [90] A. Nachman, “Reconstructions from boundary measurements,” *Ann. of Math.*, vol. 128, pp. 531–587, 1988.
- [91] —, “Global uniqueness for a two-dimensional inverse boundary value problem,” *Ann. of Math.*, vol. 143, pp. 71–96, 1996.
- [92] G. Nakamura, Z. Q. Sun, and G. Uhlmann, “Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field,” *Math. Ann.*, vol. 303, no. 3, pp. 377–388, 1995.
- [93] F. Nicoleau, “A stationary approach to inverse scattering for Schrödinger operators with first order perturbation,” *Comm. Part. Diff. Eq.*, vol. 22, no. 3-4, pp. 527–553, 1997.
- [94] M. Nieser, K. Poelke, and K. Polthier, “Automatic generation of riemann surface meshes,” in *Advances in geometric modeling and processing*, ser. Lecture Notes in Computer Science. Berlin/Heidelberg : Springer, 2010, vol. 6130, pp. 161–178.
- [95] S. J. Norton, “Reconstruction of stratified fluid flow from reciprocal scattering measurements,” *J. Acoust. Soc. Am.*, vol. 89, no. 6, pp. 2567–2572, 1991.
- [96] R. G. Novikov, “Construction of two-dimensional Schrödinger operator with given scattering amplitude at fixed energy,” *Theoretical and Mathematical Physics*, vol. 66, no. 2, pp. 154–158, 1986.
- [97] —, “Multidimensional inverse spectral problem for the equation  $-\Delta\psi+(v(x)-Eu(x))\psi=0$ ,” *Funct. Anal. Appl.*, vol. 22, pp. 263–272, 1988.

- [98] —, “The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator,” *J. Funct. Anal.*, vol. 103, no. 2, pp. 409–469, 1992.
- [99] —, “Rapidly converging approximation in inverse quantum scattering in dimension 2,” *Physics Letters A*, vol. 238, pp. 73–78, 1998.
- [100] —, “Approximate solution of the inverse problem of quantum scattering theory with fixed energy in dimension 2,” *Proc. Steklov Inst. Math.*, vol. 225, pp. 285–302, 1999.
- [101] —, “Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential,” *Inv. Problems*, vol. 21, no. 1, pp. 257–270, 2005.
- [102] —, “The  $\bar{\delta}$ -approach to approximate inverse scattering at fixed energy in three dimensions,” *Int. Math. Res. Pap.*, no. 6, pp. 287–349, 2005.
- [103] —, “An effectivization of the global reconstruction in the Gelfand-Calderón inverse problem in three dimensions,” in *Imaging Microstructures*, ser. Contemp. Math., 2009, vol. 494, pp. 161–184.
- [104] —, “Approximate Lipschitz stability for non-overdetermined inverse scattering at fixed energy,” *J. Inverse Ill-Posed Probl.*, vol. 26, no. 6, pp. 813–823, 2013.
- [105] —, “Inverse scattering without phase information,” in *Séminaire Laurent Schwartz — EDP et applications*. Ecole Polytechnique, 2014-2015, exp. no. 16 (13pp).
- [106] —, “Formulas for phase recovering from phaseless scattering data at fixed frequency,” *Bull. Sci. Math.*, vol. 139, no. 8, pp. 923–936, 2015.
- [107] —, “An iterative approach to non-overdetermined inverse scattering at fixed energy,” *Sbornik : Mathematics*, vol. 206, no. 1, pp. 120–134, 2015.
- [108] —, “Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions,” *The Journal of Geometric Analysis*, vol. 26, no. 1, pp. 346–359, 2016.
- [109] R. G. Novikov and G. M. Khenkin, “The  $\bar{\delta}$ -equation in the multidimensional inverse scattering problem,” *Russian Mathematical Surveys*, vol. 42, pp. 109–180, 1987.
- [110] R. G. Novikov and M. Santacesaria, “Global uniqueness and reconstruction for the multi-channel Gelfand-Calderón inverse problem in two dimensions,” *Bull. Sci. Math.*, vol. 135, no. 5, pp. 421–434, 2011.
- [111] —, “Monochromatic reconstruction algorithms for two-dimensional multi-channel inverse problems,” *Int. Math. Res. Not. IMRN*, no. 6, pp. 1205–1229, 2013.

- [112] L. Päivärinta, M. Salo, and G. Uhlmann, “Inverse scattering for the magnetic Schrödinger operator,” *J. Funct. Anal.*, vol. 259, no. 7, pp. 1771–1798, 2010.
- [113] A. Panchenko, “An inverse problem for the magnetic Schrödinger equation and quasi-exponential solutions of nonsmooth partial differential equations,” *Inverse Problems*, vol. 18, no. 5, pp. 1421–1434, 2002.
- [114] W. Pauli, “Zur Quantenmechanik des magnetischen Elektrons,” *Zeitschrift für Physik*, vol. 43, pp. 601–623, 1927.
- [115] R. F. Probstein, *Physicochemical Hydrodynamics : An Introduction*, 2nd ed. New York : Wiley-Interscience, 1994.
- [116] D. Roussef and K. B. Winters, “Two-dimensional vector flow inversion by diffraction tomography,” *Inv. Problems*, vol. 10, pp. 687–697, 1994.
- [117] O. D. Rumyantseva, V. A. Burov, A. L. Konyushkin, and N. A. Sharapov, “Increased resolution of two-dimensional tomography imaging along the transverse coordinate and separate reconstruction of elastic and viscous scatterer characteristics,” *Acoustical Physics*, vol. 55, no. 4, pp. 613–629, 2009.
- [118] M. N. Rychagov and H. Ermert, “Reconstruction of fluid motion in acoustic diffraction tomography,” *J. Acoust. Soc. Am.*, vol. 99, no. 5, pp. 3029–3035, 1996.
- [119] M. Salo, “Inverse problems for nonsmooth first order perturbations of the Laplacian,” *Ann. Acad. Sci. Fenn. Math. Diss.*, vol. 139, 2004.
- [120] —, “Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field,” *Communications in Partial Differential Equations*, vol. 31, no. 11, pp. 1639–1666, 2006.
- [121] M. Santacesaria, “A Hölder-logarithmic stability estimate for an inverse problem in two dimensions,” *Journal of Inverse and Ill-Posed Problems*, vol. 23, no. 1, pp. 51–73, 2015.
- [122] R. Schrader and M. Taylor, “Small  $\hbar$ -asymptotics for quantum partition functions associated to particles in external Yang-Mills potentials,” *Comm. Math. Phys.*, vol. 92, no. 4, pp. 555–594, 1984.
- [123] —, “Semiclassical asymptotics, gauge fields and quantum chaos,” *J. Funct. Anal.*, vol. 83, no. 2, pp. 258–316, 1989.
- [124] J. Schwinger, “On Green functions of quantized fields I + II,” *PNAS*, vol. 37, pp. 452–459, 1951.
- [125] T. Shiota, “An inverse problem for the wave equation with first order perturbation,” *Amer. J. Math.*, vol. 107, no. 1, pp. 241–251, 1985.
- [126] A. S. Shurup and O. D. Rumyantseva, “Numerical simulation of the functional approach for recovering vector fields in acoustic tomography,” in *Quasilinear*

- equations, inverse problems and their applications*, ser. Conference handbook and proceedings. Dolgoprudny, Russia : Phystech-polygraph, 2015, p. 11.
- [127] A. A. Slavnov and L. D. Faddeev, *Gauge fields : an introduction to quantum theory*, 2nd ed., ser. Frontiers in Physics. Westview Press, 1993.
- [128] R. Snieder, K. Wapenaar, and U. Wegler, “Unified Green’s function retrieval by cross-correlation ; connection with energy principles,” *Physical Review E*, vol. 75, 2007, 036103.
- [129] G. Springer, *Introduction to Riemann surfaces*. Reading : Addison-Wesley Publishing, 1957.
- [130] Z. Sun, “An inverse boundary value problem for Schrödinger operators with vector potentials,” *Trans. Amer. Math. Soc.*, vol. 338, no. 2, pp. 953–969, 1993.
- [131] J. Sylvester and G. Uhlmann, “A global uniqueness theorem for an inverse boundary value problem,” *Ann. of Math.*, vol. 125, pp. 153–169, 1987.
- [132] M. Taylor and A. Uribe, “Semiclassical spectra of gauge fields,” *J. Funct. Anal.*, vol. 110, no. 1, pp. 1–46, 1992.
- [133] A. N. Tihonov, “On the uniqueness of the solution of the problem of electric prospecting,” *Doklady Akad. Nauk SSSR*, vol. 69, pp. 797–800, 1949, (Russian).
- [134] C. Tolmasky, “Exponentially growing solutions for nonsmooth first-order perturbations of the Laplacian,” *SIAM J. Math. Anal.*, vol. 29, pp. 116–133, 1998.
- [135] M. Trott, “Visualization of riemann surfaces of algebraic functions,” *Mathematica in education and research*, vol. 6, pp. 15–36, 1997.
- [136] L. Tzou, “Stability estimates for coefficients of magnetic Schrödinger equation from full and partial boundary measurements,” *Communications in Partial Differential Equations*, vol. 33, no. 11, pp. 1911–1952, 2008.
- [137] G. Uhlmann, “Electrical impedance tomography and Calderón’s problem,” *Inverse Problems*, vol. 25, no. 12, 2009, id 123011.
- [138] G. M. Vainikko, “Fast solvers of the Lippmann-Schwinger equation,” in *Direct and Inverse Problems of Mathematical Physics*, ser. International Society for Analysis, Applications and Computation, 1999, vol. 5, pp. 423–440.
- [139] R. Weder and D. Yafaev, “On inverse scattering at a fixed energy for potentials with a regular behaviour at infinity,” *Inverse Problems*, vol. 21, pp. 1937–1952, 2005.
- [140] J. Wermer, “The hull of a curve in  $\mathbb{C}^n$ ,” *Ann. of Math.*, vol. 58, no. 3, pp. 550–561, 1958.
- [141] L. Xiaosheng, “Inverse scattering problem for the Schrödinger operator with external Yang-Mills potentials in two dimensions at fixed energy,” *Comm. Part. Diff. Eq.*, vol. 30, pp. 451–482, 2005.

- [142] D. Yafaev, “Scattering by magnetic fields,” *St. Petersburg Math. J.*, vol. 17, no. 5, pp. 875–895, 2006.
- [143] X. Zhou, “Inverse scattering transform for the time dependent Schrödinger equation with applications to the KPI equation,” *Comm. Math. Phys.*, vol. 128, no. 3, pp. 551–564, 1990.







**Titre** : Méthodes de reconstruction pour des problèmes inverses pour des équations de type Helmholtz

**Mots clés** : équation de Helmholtz jauge-covariante, problèmes inverses de valeurs au bord, problème inverse de diffusion, tomographie acoustique d'un fluide en écoulement, problème inverse de diffusion sans phase

**Résumé** : La présente thèse est consacrée à l'étude de quelques problèmes inverses pour l'équation de Helmholtz jauge-covariante, dont des cas particuliers comprennent l'équation de Schrödinger pour une particule élémentaire chargée dans un champ magnétique et l'équation d'onde harmonique en temps qui décrit des ondes acoustiques dans un fluide en écoulement. Ces problèmes ont comme motivation des applications dans des tomographies différentes, qui comprennent la tomographie acoustique, la tomographie qui utilise des particules élémentaires et la tomographie d'impédance électrique. En particulier, nous étudions des problèmes inverses motivés par des applications en tomographie acoustique de fluide en écoulement. Nous proposons des formules et équations qui permettent de réduire le problème de tomographie acoustique à un problème de diffusion inverse approprié. En suivant, nous développons un algorithme fonctionnel-analytique pour la résolution de ce problème de diffusion inverse. Cependant, en général, la solution de ce problème n'est unique qu'à une transformation de jauge appropriée près. À cet égard, nous établissons des formules qui permettent de se débarrasser de cette non-unicité de jauge et retrouver des paramètres du fluide, en mesurant des ondes acoustiques à des plusieurs fréquences. Nous présentons également des exemples des fluides qui ne sont pas distinguables dans le cadre de tomographie acoustique considérée. En suivant, nous considérons le problème de diffusion inverse sans information de phase. Ce problème est motivé par des applications en tomographie qui utilise des particules élémentaires, où seulement le module de l'amplitude de diffusion peut être mesuré facilement. Nous établissons des estimations dans l'espace de configuration pour les reconstructions sans phase de type Borne, qui sont requises pour le développement des méthodes de diffusion inverse précises. Finalement, nous considérons le problème de détermination d'une surface de Riemann dans le plan projectif à partir de son bord. Ce problème survient comme une partie du problème de Dirichlet-Neumann inverse pour l'équation de Laplace sur une surface inconnue, qui est motivé par des applications en tomographie d'impédance électrique.

**Title** : Reconstruction methods for inverse problems for Helmholtz-type equations

**Keywords** : gauge-covariant Helmholtz equation, inverse boundary value problems, inverse scattering, acoustic tomography of moving fluid, phaseless inverse scattering

**Abstract** : This work is devoted to study of some inverse problems for the gauge-covariant Helmholtz equation, whose particular cases include the Schrödinger equation for a charged elementary particle in a magnetic field and the time-harmonic wave equation describing sound waves in a moving fluid. These problems are mainly motivated by applications in different tomographies, including acoustic tomography, tomography using elementary particles and electrical impedance tomography. In particular, we study inverse problems motivated by applications in acoustic tomography of moving fluid. We present formulas and equations which allow to reduce the acoustic tomography problem to an appropriate inverse scattering problem. Next, we develop a functional-analytic algorithm for solving this inverse scattering problem. However, in general, the solution to the latter problem is unique only up to an appropriate gauge transformation. In this connection, we give formulas and equations which allow to get rid of this gauge non-uniqueness and recover the fluid parameters, by measuring acoustic fields at several frequencies. We also present examples of fluids which are not distinguishable in this acoustic tomography setting. Next, we consider the inverse scattering problem without phase information. This problem is motivated by applications in tomography using elementary particles, where only the absolute value of the scattering amplitude can be measured relatively easily. We give estimates in the configuration space for the phaseless Born-type reconstructions, which are needed for the further development of precise inverse scattering algorithms. Finally, we consider the problem of determination of a Riemann surface in the complex projective plane from its boundary. This problem arises as a part of the inverse Dirichlet-to-Neumann problem for the Laplace equation on an unknown 2-dimensional surface, and is motivated by applications in electrical impedance tomography.