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François Maucourant

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École doctorale Matisse

présentée par

François Maucourant

préparée à l'Institut de recherche
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UMR 6625 du CNRS

CONCERNING THE LINEAR ACTION OF DISCRETE SUBGROUPS OF $SL(2, \mathbb{R})$ ON THE PLANE

soutenue le 2 décembre 2014

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*Les intellectuels, ne l'oublions pas, ont bien intégré l'idée qu'il faut que les choses paraissent compliquées. Sinon, à quoi servent-ils ?*¹

NOAM CHOMSKY (1928-)

¹ Let's not forget that intellectuals have very well grasped the idea that things have to look complicated. Otherwise, what are they good for ?

Publications

- (1) *Approximation diophantienne, dynamique des chambres de Weyl, et répartition d'orbites de réseaux*, PhD Thesis, 2002, Université Lille 1.(TEL)
- (2) *Sur les spectres de Lagrange et de Markoff des corps imaginaires quadratiques*, Ergodic Theory and Dyn. Sys. 23 (2003), 193-205. (article)
- (3) with A. Gorodnik, *Proximity and Equidistribution on the Furstenberg boundary*, Geometriae Dedicata, 113 (2005), p 197-213. (Lien arXiv)
- (4) *Dynamical Borel-Cantelli lemma for hyperbolic spaces*, Israel Journal of mathematics, 152 (2006), p 143-155. (article).
- (5) *Homogeneous asymptotic limits of Haar measures of semisimple Lie groups and their lattices*, Duke Mathematical Journal, vol 136 (2), 2007, p 357-399. (arXiv).
- (6) With A. Gorodnik et H. Oh, *Manin's and Peyre's conjectures on rational points and adelic mixing.*, Annales scientifiques de l'ENS, vol 41, 2008, p 383-435. (arXiv).
- (7) *Arithmetical and geometrical aspects of homogeneous diophantine approximation by algebraic numbers in a given number field*, Proceedings "Dynamical systems and Diophantine Approximations", Séminaires et congrès vol 22, S.M.F.
- (8) *A non-homogeneous orbit closure of a diagonal subgroup*, Annals of Maths, 171 (1), 2010, p557-570. (arXiv).
- (9) With Y. Coudène, *Horocycles récurrents sur des surfaces de volume infini*, Geometriae Dedicata, Volume 149.1 (2010), p 231-242. (article)
- (10) Appendix of an article of S. Lim et H. Oh, *On the distribution of orbits of geometrically finite hyperbolic groups on the boundary*, Ergod. Th. and Dynam. Sys., Vol 32 (2012), 173-189. (arXiv).
- (11) With B. Weiss, *Lattice action on the plane revisited*, Geometriae Dedicata, Vol 157.1 (2012), p1-21. (Lien arXiv).
- (12) With B. Schapira, *Distribution of orbits in \mathbf{R}^2 of a finitely generated group of $SL(2, \mathbf{R})$* , American Journal of Mathematics(ArXiv).
- (13) With S. Gouëzel et F. Mathéus, *Sharp lower bounds for the asymptotic entropy of symmetric random walks*, to appear in Groups, Geometry and Dynamics. (ArXiv).

In preparation

- (14) With Yves Guivarc'h, *Linear random walks on the plane*.
- (15) With Barbara Schapira, *Nondivergent unipotent orbits for convex-concompact hyperbolic manifolds*
- (16) With S. Gouëzel et F. Mathéus, *Entropy and drift in word-hyperbolic groups*

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Introduction

This HDR ² is constructed around three papers, and a work-in-progress. These four papers represent a large part of my work after the PhD, but not everything; this choice is motivated by the fact that all four can be considered partial answers to the same question.

My goal here is not to reproduce those articles, but rather to put them into the same context. I tried to explain the main ideas without diving (too much) into squalid details. Overall, I tried to write something easy to read but that gives a precise idea of the actual technical problems and the tools involved to deal with them. For more precise statements, one can always refer to the original article, but hopefully that will not be necessary.

Also, this is an HDR and not a review article; it focuses on my own work together with my coauthors Barak, Barbara and Yves. This means that very relevant results of others around this topic will often only be alluded to. In some cases, my results have been later improved by others (in chapter 1) but it's the 'old' result that is discussed, also because it is often simpler to state and understand.

The initial (vague) question is:

Take a (discrete) subgroup $\Gamma \subset \mathrm{SL}(2, \mathbf{R})$, a nonzero vector $\mathbf{u} \in \mathbf{R}^2$ and choose an element $\gamma \in \Gamma$ at random. What is $\gamma\mathbf{u}$, where is it on the plane ?

My belief is that this question is not *that* interesting, but that the methods to answer it are.

The typical objective in this memoir (at least in the first three chapters) will be to find asymptotic formulas for expressions of the form

$$\sum_{\gamma \in \Gamma, \|\gamma\| \leq T} f\left(\frac{\gamma\mathbf{u}}{T^\alpha}\right),$$

²short for 'Habilitation à Diriger des Recherches', that is, the present text.

where α is a parameter, and f a continuous, compactly supported function on \mathbf{R}^2 .

Chapter 1. The first chapter, based on [31] - (5) in the publication list -, describes the 'large scale' answer for lattices, through the analysis of homogeneous limits of Haar measure in representations. The link with the initial problem is the following: assume $\Gamma = \mathrm{SL}(2, \mathbf{R})$ (it's not discrete, but the question can be asked anyway) and that 'randomly' means taken with respect to Haar measure, restricted to a ball of radius $T > 0$, and normalized to be a probability. To understand the limit behavior of this probability measure, first apply an homothety of ratio $1/T$, so the probability is supported in a ball of radius 1 and we can hope (and prove) some convergence.

Once we have found a limit in the space of matrices, we can use the evaluation map $M \mapsto M\mathbf{u}$ (which commutes indeed with homotheties) to compute the limit distribution of vectors.

One can do exactly same thing for any semisimple, connected, non-compact linear Lie group. A very practical problem arise in this context: what is the normalization required on the Haar measure restricted to a ball a radius T , that is, what is the asymptotic growth of the Haar measure ? I explain the procedure to find it and illustrate it on the automorphism representation of the split form of the exceptional Lie group of type G_2 . I chose G_2 because it was (at my knowledge) not done before, the 2-dimensional Cartan subalgebra allows nice pictures, and, once one admits some facts about the representation theory of G_2 , it is not more complicated than, say, the standard representation of $\mathrm{SL}(3, \mathbf{R})$.

Having understood the large-scale behaviour of Haar measures of Lie groups, we then turn our attention on lattices; the result is (under some hypothesis), at large scale, the lattice and the ambient Lie group look exactly the same (Theorem I.2). This allows an answer to the initial question at large-scale for lattices - Theorem I.3. Note that this chapter gives a theoretic description of the limit measure, it won't be before Chapter 3 that we will understand how to compute it in a more explicit way.

Chapter 2. This chapter is based on the article [33] with Barak Weiss, where we obtain effective remainder terms to Ledrappier's Theorem in [26]. The hypothesis here is that Γ is a lattice in $\mathrm{SL}(2, \mathbf{R})$, and we are interested in the local distribution. We explain the introduction of a large-deviation parameter α , and give simplified (and less precise) versions of the results of [33].

The study of orbit distribution of lattices on \mathbf{R}^2 was initiated by Ledrappier [26]. Nogueira [34] had an alternate approach, which led him also to remainder terms in [35].

This Theorem of Ledrappier received a lot of attention, and was generalized or strengthened in a number of cases: on \mathbf{C}^2 and Clifford algebras (Ledrappier-Pollicott [25]), on the p -adic plane (Ledrappier-Pollicott) [28], on \mathbf{R}^n for $n \geq 3$ (Gorodnik [16]), on other homogeneous manifolds (Gorodnik-Weiss [15], Guiloux [21], in my thesis [30]), with remainder terms (Nogueira [35], Pollicott [37] on \mathbf{C}^2). These are instances of the duality principle : to understand the left-action of Γ on G/N , one translates properties of the right-action of N on $\Gamma \backslash G$. The most general result of this type was developed recently by Gorodnik and Nevo in [19], where they treat the general case of a lattice acting on a homogeneous case, with a remainder term in L^p mean.

What was new in our work with Barak was the introduction of a scaling parameter $\alpha \in (-1, 1)$ (Ledrappier's Theorem correspond to $\alpha = 0$) and the consideration of remainder terms, which follow after careful analysis from the remainder terms in the ergodic Theorem for horocycle flow. Remainder terms were also obtained in [35], but our hypotheses were more general, although in some (important) particular cases, Nogueira's remainder term is more precise.

The proof sketched follows Ledrappier's, and relates the sum considered to an ergodic sum for the horocycle flow. This part does not assume the group Γ to be a lattice, only a discrete subgroup, so will be reused in Chapter 3.

We then discuss briefly the problem of handling these ergodic sums in the case of cocompact lattices, and non-cocompact lattices.

Chapter 3. This chapter is based on the article [32] with Barbara Schapira, where we leave the realm of lattices and address the case of finitely generated subgroup $\Gamma \subset \mathrm{SL}(2, \mathbf{R})$.

As the group is not a lattice anymore, the ergodic theory used is a bit more involved; there are not one, but two relevant measures for the horocycle flow (although they are not unrelated, their weak stable conditionals are the same). One finite but not invariant, the other one invariant but infinite. We briefly recall their construction and properties.

We state the main ergodic result of [32], namely Theorem III.1, and explain ideas of the proof.

In Section III.2, we use this result to obtain distribution result for the orbits of Γ on the plane, for a parameter $\alpha \in (-1, 1)$; the results are stronger if the group does not contain parabolics. If it does, we find that the behaviour changes whether the critical exponent is bigger or less than $2/3$ (a rather curious and unexpected condition).

In Section III.3, we treat the large scale case $\alpha = 1$ and find a (somewhat surprising) formula for the limit.

To the best of my knowledge, only Ledrappier [27] and Ledrappier-Pollicott [25] obtained similar results for groups which are not lattices. They address the case of a subgroup $\bar{\Gamma}$ of a cocompact lattice Γ such that $\Gamma/\bar{\Gamma}$ is abelian. This case is disjoint from ours, as the group $\bar{\Gamma}$ is not finitely generated, and the analysis of ergodic properties is different.

Chapter 4. This chapter is based on a work-in-progress made with Yves Guivarc'h, and maybe asks more questions than gives answers. The motivation behind is to find results analogous to those of the previous chapters, but for a non-degenerate symmetric random walk of generator μ at time n on a convex-cocompact subgroup of $SL(2, \mathbf{R})$. However, what will be proved might seem very far off this goal.

The topic will be to consider a family of operator $(P_s)_{s \in \mathbf{R}}$ on $C^0(\mathbb{P}^1)$, or rather their duals P_s^* on the space of signed measures. In section IV.1 will be explained why finding eigenvalues and eigenvectors for these operators seems relevant to the distribution problem we discussed in the previous chapters.

Denote by $\sigma(\gamma, \xi) = |\gamma'(\xi)|_o^{-1/2}$ where $|\gamma'(\xi)|_o$ stands for the derivative of γ at ξ for the $SO(2)$ -invariant probability on \mathbb{P}^1 . Consider the following twisted convolution operator on $C^0(\mathbb{P}^1)$:

$$P_{s,\mu}(f)(\xi) = \int_{\Gamma} \sigma(\gamma, \xi)^s f(\gamma\xi) d\mu(\gamma).$$

Note that several well-known measures are eigenmeasures for P_s^* for particular s : the Lebesgue measure ($SO(2)$ -invariant) \mathcal{L} for $s = -2$, more generally any δ' -conformal density for $s = -2\delta'$, and the stationary measure ν_o of the random walk for $s = 0$.

This operator is not new at all ³. It was studied for sufficiently small s by Le Page [24] to prove limit Theorems for products of random matrices, and in

³ One may also notice that, for the particular value $s = -1$, this operator is nothing else than the random walk operator associated to the unitary representation of G on $L^2(\mathbb{P}^1)$.

particular large deviations not too far away from the average. In [22], Guivarc'h and Le Page studied these operators for nonnegative s ; they showed (among other things) that P_s has a spectral gap on Hölder functions, a unique positive continuous eigenfunction f_s , and a unique eigenprobability ν_s , with eigenvalue $e^{k(s)}$ for some analytic function k .⁴

However, for negative s , the spectral analysis was not done; one (technical) reason behind the difference between positive or negative s , is the following. For positive s , $\sigma(\gamma, \xi)^s$ is approximately $\|\gamma\|^s$ for most ξ , and in any case $\sigma(\gamma, \xi)^s \leq \|\gamma\|^s$. For negative s , this inequality is reversed, and although $\sigma(\gamma, \xi)^s$ is usually small, it can be of the order of $\|\gamma\|^{-s}$, i.e. very big on a small set, thus hard to estimate properly.

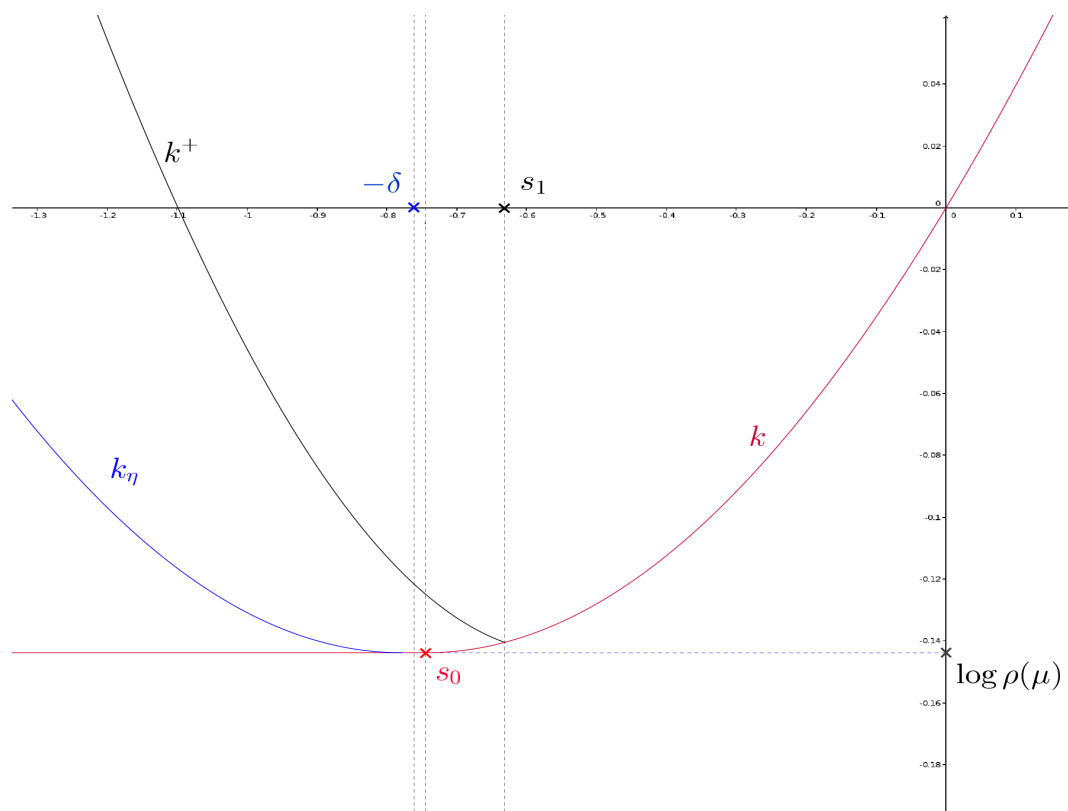


FIGURE 1. One glimpse of the functions k (red), k^+ (black then red), k_η for the Patterson Measure (blue then red)

After introducing necessary notations and basic facts about random walks on groups in Section IV.2, we define in Section IV.3, two real functions $s \mapsto k(s)$,

⁴the function $k(s)$ in this memoir differs from the one in [22] by a logarithm.

$s \mapsto k^+(s)$ as in [22], and a (new) family of functions $s \mapsto k_\eta(s)$ indexed by a probability $\eta \in \mathcal{M}(\mathbb{P}^1)$. These functions are natural candidates to be the logarithm of some eigenvalues of P_s^* for the parameter s . These three functions are equal for $s \geq 0$ (and in fact for $s > -\epsilon$, ϵ sufficiently small), but here are expected (and in some cases, shown) to be distinct.

The idea behind the analysis is to show existence of a *phase transition*, that is, a modification as s decreases of the spectral properties of P_s^* . Some special values of s are interesting: s_0 is the largest s such that k is constant up to s_0 . Another remarkable value is s_1 , the point where k and k^+ separate. The parameter $s = -\delta$, where δ is the critical exponent of Γ , also plays a special role.

The picture 1, based on explicit computations on a toy example, gives an idea of what the functions are expected to look like in general.

We show in particular by an abstract argument that both $e^{k^+(s)}$ and $e^{k_\eta(s)}$ are eigenvalues of P_s^* . (Theorem IV.6). When $k(s) > \log \rho(\mu)$, that is when $s > s_0$, we give a less abstract proof of the fact that $e^{k(s)}$ is an eigenvalue of P_s^* (Theorem IV.10). This is done by a construction à la Patterson, using a series ('the Green-Poincaré series') that mimics the usual Poincaré series. The eigenmeasures obtained have specific Radon-Nikodym derivatives (' (Γ, μ) -conformal measure').

In Section IV.6, we explicit some examples. We briefly explain the case where the measure μ is absolutely continuous, and why the phenomena expected in the discrete case do not occur. We then turn to a toy model, the simple random walk on the free group on two generators, but with a distance that differs from the word distance, more precisely where length of edges depends on the label in the Cayley graph. This model is a good approximation of free groups inside $SL(2, \mathbf{R})$ where the two generators have quite different norms. In this model, we compute the interval on which k and k^+ do not agree, and show that k stays constant on a strictly larger interval than the predicted $(-\infty, -\delta)$.

This imply the phase transition mentioned above: for $s < s_1$, $k^+(s) \neq k_\eta(s)$ when η is the Patterson measure on the boundary. Thus P_s^* has several eigenvalues associated to positive eigenmeasures, the operator P_s does not have a positive eigenfunction, contrasting with the situation $s \geq 0$. The above picture of k, k^+, k_η was done using the explicit formulas in this case.

CHAPTER I

Growth in semi-simple Lie groups

I.1. Asymptotic growth of Haar measure

LET G be a connected, semisimple, noncompact, matrix Lie group. In other words, this is an abstract Lie group G endowed with a particular (faithful) representation $\rho : G \rightarrow \mathrm{GL}(V)$, where V is a finite dimensional real vector space. Let μ be a Haar measure, and $\|\cdot\|$ be a norm. For $T > 0$, let

$$G_T = \{g \in G : \|\rho(g)\| \leq T\}$$

A very natural question in this setting is to ask for an asymptotic for $\mu(G_T)$, which turns out to be of the form

$$\mu(G_T) \sim_{T \rightarrow +\infty} c.T^a(\ln T)^b,$$

for some $a > 0, b \geq 0, c > 0$. This asymptotic is a consequence of the following:

THEOREM I.1. [**31**, Theorem 1] *Under the previous hypotheses, there is a rational positive number a , an integer b between 0 and $\mathrm{rank}_{\mathbf{R}} G - 1$, and a nonzero Radon measure μ_∞ on the space $\mathrm{End}(V)$ of endomorphisms of V such that we have: for any continuous function f of compact support from $\mathrm{End}(V)$ to \mathbf{C} ,*

$$\lim_{T \rightarrow +\infty} \frac{1}{T^a \ln(T)^b} \int_G f\left(\frac{\rho(g)}{T}\right) d\mu(g) = \int_{\mathrm{End}(V)} f d\mu_\infty.$$

The limit measure μ_∞ is homogeneous of degree a , that is $\mu_\infty(t.E) = t^a \mu_\infty(E)$ for any Borel subset E and $t > 0$.

SOME IDEAS OF THE PROOF. The strategy of proof of Theorem I.1 is quite straightforward: one express explicitly the Haar measure μ using the Cartan decomposition KA^+K , where K is a maximal compact subgroup of G , and $A^+ = e^{\mathfrak{a}^+}$ a Weyl chamber in a Cartan subgroup. Since the action of K commutes with the scaling, one has only to deal with (linear combinations of) integrals of the form

$$\int_{\mathfrak{a}^+} f(\rho(e^a)/T) e^{\chi(a)} da,$$

where $\chi \in \mathfrak{a}^*$. An appropriate (but complicated) change of variable gives then the correct asymptotic. Nowadays, I would rather use the result given in [**20**, Theorem 6.1], which is a bit less clumsy way to deal with this kind of integral.

One can also give remainder terms in Theorem I.1, provided some regularity on the test function f ; this was done in [31], but only in the case where $b = 0$ - no logarithmic terms. In fact, it should be possible to find an expansion in all cases.

Note that following the proof step by step gives an explicit expression for the limit measure μ_∞ . \square

The equivalent $T^a(\ln T)^b$ was proved independently by Gorodnik-Weiss [15, Theorem 2.7]; however they did not obtain the procedure to find a, b explained below. This form of asymptotic is in fact very general (see [17, Theorem 7.17]). This result was generalized to affine symmetric varieties by Gorodnik, Oh and Shah in [20].

I.2. A procedure to compute the volume growth

The change of variables alluded to above is based on a geometric understanding of the interplay between the weights of the representations (which have to do with the norm of a matrix), and the data of the root system (which have to do with the Haar measure). That is why maybe one of the most important things is left out this statement: how to find the numbers a, b . This is given by the following procedure:

- (1) Find out the root system Σ , that is, the weights of the adjoint representation. That might involve finding first a Cartan subalgebra \mathfrak{a} , as $\Sigma \subset \mathfrak{a}^*$.
- (2) Choose a system of positive roots Σ^+ , and compute the value of

$$\beta = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha.$$

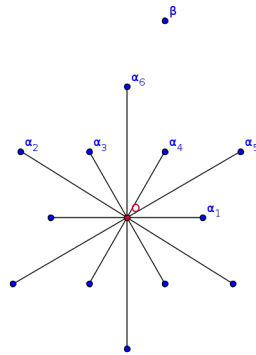
- (3) Find out the weights Ψ of the representation ρ , and consider the convex hull \mathcal{C} of Ψ .
- (4) There is a unique $a > 0$ such that $2\beta/a \in \partial\mathcal{C}$.
- (5) The vector $2\beta/a$ is a (minimal) face of the polyhedron \mathcal{C} . Write its codimension as $b + 1$.

I.3. An example : G_2 as an automorphism group

To illustrate the previous procedure, we apply it to an exceptional Lie group : the real split form of G_2 ; more precisely, its matrix realization as automorphism group of split octonions. The asymptotic that we will find is T^6 .

I.3.1. First step : The root system. The root system is drawn on Figure 1.

Classical Lie Theory textbooks (for example [12]) tell us that the root system Σ consists of twelve roots, which we decompose as $\Sigma^+ = \{\alpha_1, \dots, \alpha_6\}$ and their opposites $-\Sigma^+$. With this choice of positive roots, the two simple roots are α_1, α_2 , which correspond to the two dots of the Dynkin diagram.

FIGURE 1. The Root System of G_2

I.3.2. Second step : Half-sum of positive roots. Since

$$\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1\},$$

we have

$$\beta = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha = 5\alpha_1 + 3\alpha_2,$$

as drawn on Picture 1.

I.3.3. Third step : The representation of G_2 . As existence of G_2 is itself a non-trivial fact, we will rely on the following description of the split form G_2 .

Let \mathbf{O} be the algebra of *split octonions*; this is a 8-dimensional vector space, together with an (non-associative, non-commutative) product and a quadratic form Q of signature $(4, 4)$, which satisfy

$$Q(xy) = Q(x)Q(y).$$

The group G_2 is by definition the automorphism group of \mathbf{O} ; this defines a representation ρ of G_2 in $GL(\mathbf{O})$. We have to understand what are the weights of this representation, so we will have to understand a little bit more the split octonions.

Note that this algebra is not to be confused with the (non-split) octonions, for which the quadratic form is positive definite; the automorphism group of the latter algebra, a subgroup of $SO(8, \mathbf{R})$, is the real compact form of G_2 .

Let us describe a little bit more the algebra \mathbf{O} , following the description in [6].

Let \mathbf{H} be the usual quaternionic algebra, the 4-dimensional non-commutative division algebra over the reals with basis $1, i, j, k$, and the usual relationships :

$i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik$, together with the conjugation

$$\overline{a + bi + cj + dk} = a - bi - cj - dk, (a, b, c, d) \in \mathbf{R}^4.$$

As a vector space, $\mathbf{O} = \mathbf{H}^2$, endowed with the product

$$(a, b)(c, d) = (ac + \bar{d}b, da + b\bar{c}), a, b, c, d \in \mathbf{H}.$$

The conjugate of a split octonion is defined as

$$\overline{(a, b)} = (\bar{a}, -b),$$

which defines in turn the real and imaginary part of an octonion x by $\mathcal{R}(x) = (x + \bar{x})/2 \in \mathbf{R}(1, 0)$, $\mathcal{I}(x) = (x - \bar{x})/2$. The inner product

$$\langle x, y \rangle = \mathcal{R}(x\bar{y}),$$

is of signature $(4, 4)$, and is invariant under G_2 .

This implies that the real octonions $\mathbf{R}(1, 0)$ and the purely imaginary octonions $(\mathbf{R}(1, 0))^\perp$ are two G_2 -invariant subspaces, of respective dimension 1 and 7. Since there is only one non-trivial (irreducible) representation of dimension 7 of G_2 , the *standard* representation of \mathfrak{g}_2 , (see [12]), and G_2 acts trivially on the real octonions, we know exactly the decomposition of \mathbf{O} as sum of irreducible representations.

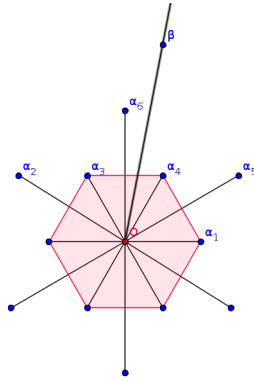


FIGURE 2. The convex hull \mathcal{C} of the weights of ρ

The eight weights Ψ of ρ are thus (still following [12]) the six short roots, and zero (with multiplicity 2). The convex hull of Ψ is represented on Figure 2.

I.3.4. Last steps. One glance at the above picture tells us that the half-line from 0 to β intersects $\partial\mathcal{C}$ on the edge $[\alpha_3, \alpha_4] = \alpha_2 + [1, 2]\alpha_1$, which is a face of codimension 1 of the 2-dimensional polygon \mathcal{C} . Indeed,

$$\frac{2\beta}{6} = \frac{5}{3}\alpha_1 + \alpha_2,$$

Therefore, the volume growth in the representation ρ is asymptotic to T^6 .

I.4. And for lattices ?

I.4.1. A general result. The original motivation for the large-scale study of Haar measure of connected Lie groups came with the hope that, at large scale, the Haar (counting) measure on a lattice would very much look like the Haar (continuous) measure of the ambient group.

A thing that might be surprising at first is that it may fail to be true in full generality, namely if the lattice is reducible and the representation has a particular property relative to the decomposition of the lattice; an example of that is given in [31]. However, assume that at least one of the two following hypotheses is true:

- The lattice $\Gamma \subset G$ is irreducible.
- For every proper normal subgroup $H \subset G$, the growth rate of H is strictly smaller than the growth rate of G .

Then

THEOREM I.2. [31, Theorem 2] *Under the previous hypotheses and notations, for any continuous function f of compact support from $\text{End}(V)$ to \mathbf{C} ,*

$$\lim_{T \rightarrow +\infty} \frac{1}{T^a \ln(T)^b} \sum_{\gamma \in \Gamma} f\left(\frac{\rho(\gamma)}{T}\right) = \frac{1}{\mu(\Gamma \backslash G)} \int_{\text{End}(V)} f d\mu_\infty.$$

ABOUT THE PROOF. Theorem I.2 was proved by a modification of the argument of Eskin-McMullen [9], following ideas of Eskin-Mozes-Shah [10], but without use of Ratner's theory.

More precisely, let $\Delta \subset (\Gamma \backslash G)^2$ be the diagonal. As in [9], [10], counting integer points reduces to understand how the translate measured submanifold $(g, e)\Delta$ equidistributes in $(\Gamma \backslash G)^2$ as $g \rightarrow +\infty$.

This is expressed mathematically as : let f be a test function on $(\Gamma \backslash G)^2$, how close is $\int_\Delta f(g(x, x))dx$ to $\int_{(\Gamma \backslash G)^2} f(x, y)dx dy$? Now, notice that $L^1(\Gamma \backslash G) \otimes L^1(\Gamma \backslash G)$ is dense in $L^1((\Gamma \backslash G)^2)$; therefore, one can assume that our test function f is a product of functions with separate variables: $f(x, y) = f_1(x)f_2(y)$. Now $\int_\Delta f((gx, x))dx = \int_{\Gamma \backslash G} f_1(gx)f_2(x)dx$, that is, equidistribution of the translate of Δ is expressed through *mixing*.

This sketch also applies in the adelic setting, once one has proved a result about adelic mixing. This was done in [14]. \square

I.4.2. Application to orbits of lattices. An application (and in fact, the initial motivation to study asymptotic limits of Haar measures in the first place) is to understand the large scale behaviour of an orbit of a lattice $\Gamma \subset \mathrm{SL}(2, \mathbf{R})$. We thus apply Theorem 1 and 2 to the group $G = \mathrm{SL}(2, \mathbf{R})$, and ρ the standard representation on \mathbf{R}^2 . As is well known, the growth rate here is T^2 - one can recover it from the procedure explained in the beginning of the chapter, using Figure 3, where $(\xi, -\xi)$ are the weights of the standard representation, and $(-2\xi, 0, 2\xi)$ the weights of the adjoint representation.



FIGURE 3. The Root System and standard representation of type A_1

A first thing to do is to understand the measure μ_∞ on the space $M(2, \mathbf{R})$ of 2×2 matrices. One easily checks that this measure must satisfy the following:

- It is supported on the set of matrices of rank 1,
- It is $\mathrm{SO}(2, \mathbf{R})$ -invariant by multiplication on left and on the right,
- It is homogeneous of degree 2.

These properties imply directly that μ_∞ is equal to (up to renormalisation by a multiplicative constant):

$$\int_{M(2, \mathbf{R})} f d\mu_\infty = \int_{\mathbf{R}^+} \int_0^{2\pi} \int_0^{2\pi} f \left(\begin{bmatrix} r \sin \theta \sin \phi & r \sin \theta \cos \phi \\ r \cos \theta \sin \phi & r \cos \theta \cos \phi \end{bmatrix} \right) r dr d\theta d\phi.$$

Denote by Γ_T the ball of radius T :

$$\Gamma_T = G_T \cap \Gamma = \{\gamma \in \Gamma : \|\gamma\| \leq T\},$$

where $\|\cdot\|$ is an arbitrary norm on $M(2, \mathbf{R})$.

Let \mathbf{u} be any initial vector of norm 1. One wish to understand how the set $\Gamma_T \mathbf{u}$ looks like - see Pictures 4 and 5. These pictures seem to indicate that, when renormalized in a unit disk, the points of the cloud behave as if chosen randomly with respect to an absolutely continuous measure on the disk, of the form $\Theta(\mathbf{v}) d\mathbf{v}$. In the pictures, where the initial vector \mathbf{u} is of norm 1 and the

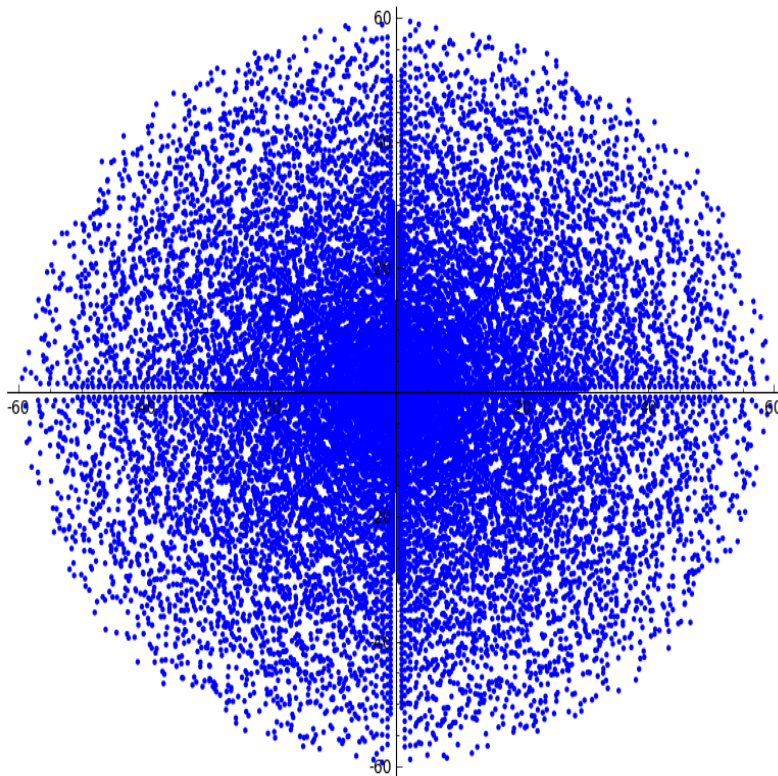


FIGURE 4. The cloud for $SL(2, \mathbf{Z})$ for $T = 60$, ℓ^2 -norm and $\mathbf{u} = (\sqrt{3}/2, 1/2)$.

ℓ^2 -norm on $M(2, \mathbf{R})$ is chosen, it will turn out that the density in question is

$$\Theta(\mathbf{v}) = \frac{\sqrt{1 - |\mathbf{v}|^2}}{|\mathbf{v}|} d\mathbf{v}.$$

However, in general, the norm $\|\cdot\|$ on $M(2, \mathbf{R})$ is not $SO(2)$ -invariant (left or right), therefore the shape of the "disk" and the density will both depend on the initial vector \mathbf{u} , thus the density will be of the form $\Theta(\mathbf{u}, \mathbf{v}) d\mathbf{v}$ on a set $\mathcal{D}(\mathbf{u})$. See Picture 6 for an example with ℓ^4 -norm.

More precisely, let $f : \mathbf{R}^2 \rightarrow \mathbf{C}$ be a (continuous, nonnegative) test function. After an homothety of ratio $1/T$ to take into account the full picture, the relevant quantity is:

$$S_T(f) = \frac{1}{T^2} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T}\right).$$

This measure (as a function of f) tends to an absolutely continuous measure. More precisely, the result is the following and strengthens [31, Corollary 1.2] :

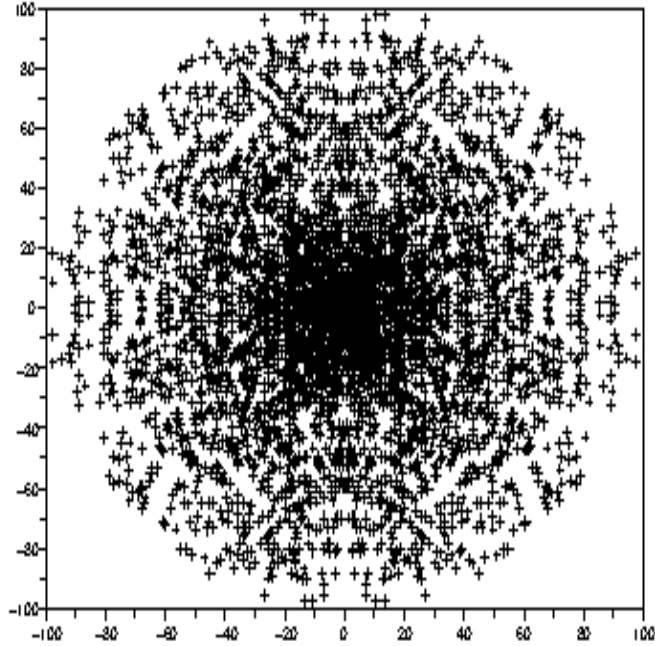


FIGURE 5. The cloud for a cocompact lattice for $T = 100$ and $\mathbf{u} = (1, 0)$.

THEOREM I.3. *For any norm $\|\cdot\|$ on $M(2, \mathbf{R})$, there is a map $\Theta : (\mathbf{R}^2)^2 \rightarrow \mathbf{R}^+$ such that the following holds. For any $\mathbf{u} \in \mathbf{R}^2 - \{0\}$, the set $\mathcal{D}(\mathbf{u}) = \{\mathbf{v} : \Theta(\mathbf{u}, \mathbf{v}) > 0\}$ is relatively compact and star-shaped around 0, and there is a $c(\mathbf{u}) > 0$ such that for any continuous f ,*

$$\lim_{T \rightarrow +\infty} \frac{1}{|\Gamma_T|} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T}\right) = c(\mathbf{u}) \int_{\mathcal{D}(\mathbf{u})} f(\mathbf{v}) \Theta(\mathbf{u}, \mathbf{v}) d\mathbf{v}.$$

SOME IDEAS. The constant $c(\mathbf{u})$ plays no role other than normalizing the right-hand side to be a probability measure. Since $|\Gamma_T| \sim c' T^2$ for some $c' > 0$, one can work either with $1/T^2$ or $1/|\Gamma_T|$ normalizations for $S_T(f)$.

One can write

$$S_T(f) = \frac{1}{T^2} \sum_{\gamma \in \Gamma} f\left(\frac{\gamma}{T} \mathbf{u}\right) \mathbf{1}_{B(0,1)}\left(\frac{\gamma}{T}\right),$$

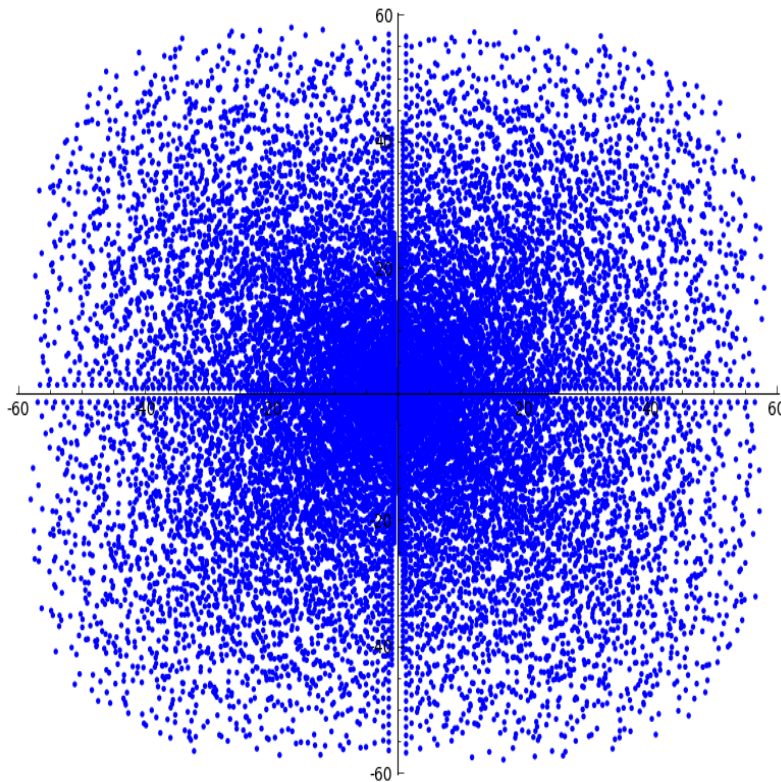


FIGURE 6. The cloud for $\text{SL}(2, \mathbf{Z})$ for $T = 50$, ℓ^4 -norm and $\mathbf{u} = (\sqrt{3}/2, 1/2)$.

where $B(0, 1) \subset M(2, \mathbf{R})$ is the ball of center 0 and radius 1 for the chosen norm. By Theorem I.2,

$$S_T(f) \rightarrow_{T \rightarrow +\infty} \int_{B(0,1)} f(g\mathbf{u}) d\mu_\infty.$$

Thus, one would like to know the expression of the image measure of $\mathbf{1}_{B(0,1)}\mu_\infty$ via the evaluation map by \mathbf{u} . This turns out to be an absolutely continuous measure with compact support, i.e. of the form $\Theta(\mathbf{u}, \mathbf{v})d\mathbf{v}$. It can be computed explicitly in some particular cases; however, the general computation (i.e. with an arbitrary norm on $M(2, \mathbf{R})$) seems difficult at best.

Therefore, we will rely on the description obtained in [32], which was proved by a different method. We will come back later on how to obtain this description.

Consider the map

$$\kappa(\mathbf{u}, \mathbf{v}, s) = \left\| \frac{1}{|\mathbf{u}|^2} \begin{pmatrix} \mathbf{u}_x \mathbf{v}_x & \mathbf{u}_y \mathbf{v}_x \\ \mathbf{u}_x \mathbf{v}_y & \mathbf{u}_y \mathbf{v}_y \end{pmatrix} + s \begin{pmatrix} -\mathbf{u}_y \mathbf{v}_x & \mathbf{u}_x \mathbf{v}_x \\ -\mathbf{u}_y \mathbf{v}_y & \mathbf{u}_x \mathbf{v}_y \end{pmatrix} \right\|$$

Here, $|\cdot|$ stands for the ℓ^2 -norm on \mathbf{R}^2 , and $\|\cdot\|$ the arbitrary norm chosen on $M(2, \mathbf{R})$. By convexity and properness of the norm, either $\kappa(\mathbf{u}, \mathbf{v}, s) > 1$ for all s , or there is a compact interval $[s_1, s_2]$ which is precisely the set of s such that $\kappa(\mathbf{u}, \mathbf{v}, s) \leq 1$. In the first case, put $\Theta(\mathbf{u}, \mathbf{v}) = 0$, and in the second case, put $\Theta(\mathbf{u}, \mathbf{v}) = (s_2 - s_1)/2$.

Although this is not straightforward, this function Θ does the job. This will be proved in Section III.3.

In the case of ℓ^2 -norm on $M(2, \mathbf{R})$, one has the following formula:

$$\Theta(\mathbf{u}, \mathbf{v}) = \frac{1}{|\mathbf{u}| \cdot |\mathbf{v}|} \sqrt{1 - \frac{|\mathbf{v}|^2}{|\mathbf{u}|^2}},$$

with $\mathcal{D}(\mathbf{u})$ the euclidean disk of center 0 and radius $|\mathbf{u}|$. This formula can be obtained either by the previous description using κ , or by computing the image measure of $\mathbf{1}_{B(0,1)}\mu_\infty$ using the $SO(2, \mathbf{R})$ -invariance of μ_∞ and $B(0, 1)$, which implies that the formula may depend only on the norms of the vectors \mathbf{u}, \mathbf{v} . \square

I.5. Other related results

In fact, the article [31] also gave in some cases a remainder term. These were better handled later by Gorodnik and Nevo [17], [18]. These results were also generalized to affine symmetric varieties by Gorodnik, Oh and Shah in [20].

The adelic analogue was dealt in [14]; the volume estimate used there is due to Shalika, Takloo-Bighash and Tschinkel [41].

CHAPTER II

Distribution of orbits of lattices on the plane

LET $G = \mathrm{SL}(2, \mathbf{R})$ and $\Gamma \subset G$ a lattice. In the previous chapter, we have seen the large scale distribution of the cloud $\Gamma_T \mathbf{u}$, where Γ_T is the ball of radius T in Γ for an arbitrary norm $\|\cdot\|$ on $M(2, \mathbf{R})$. Now, we are interested in understanding its *local* distribution, that is, at different scales. This chapter is based on the article [33], which is joint work with Barak Weiss.

The point of view chosen here needs a little bit of explanation. Rephrasing the result of the previous chapter, choosing a point at random with respect to the uniform probability in the set $\frac{1}{T}\Gamma_T \mathbf{u}$ is almost the same as choosing a point at random in something that looks like a disk with respect to some absolutely continuous measure. Therefore, the typical point of $\Gamma_T \mathbf{u}$ has norm of order T . More precisely, there exists $C > 0$, such that for any $\epsilon > 0$, there exists a constant $c > 0$ such that a proportion $1 - \epsilon$ of points in $\Gamma_T \mathbf{u}$ satisfies

$$cT \leq |\gamma \mathbf{u}| \leq CT.$$

It happens that some (rare) points have small norm, say approximately T^α , here $\alpha < 1$. Those points are far from the average behaviour, how far being quantified by a parameter: this is a *large deviation* study.

Therefore, in this chapter and the next, we will always use a parameter α , and will look at those points of norm approximately T^α . Because a 2×2 unimodular matrix of norm T cannot shrink a vector more than T , the relevant range for α is $\alpha \in (-1, 1)$, the limit case $\alpha = 1$ being already treated by Theorem I.3.

This large deviation point of view will motivate Chapter IV.

II.1. Local distributions results

II.1.1. The \star -product. It turns out that a quantity naturally appears in the problem, which will be denoted by a 'product' \star on \mathbf{R}^2 , and depends on the norm chosen $\|\cdot\|$ on $M(2, \mathbf{R})$. For \mathbf{u}, \mathbf{v} two vectors of \mathbf{R}^2 , write:

$$\mathbf{v} \star \mathbf{u} = \left\| \left[\begin{array}{cc} -\mathbf{u}_y \mathbf{v}_x & \mathbf{u}_x \mathbf{v}_x \\ -\mathbf{u}_y \mathbf{v}_y & \mathbf{u}_x \mathbf{v}_y \end{array} \right] \right\|,$$

where $\mathbf{u}_x, \mathbf{u}_y, \mathbf{v}_x, \mathbf{v}_y$ are the (x, y) coordinates of \mathbf{u}, \mathbf{v} . We already encountered this matrix in the expression of κ in section I.4.2.

This is denoted by a product because it behaves like a product of norms: there is a constant $c > 1$ such that

$$\frac{1}{c} \leq \frac{\mathbf{v} \star \mathbf{u}}{|\mathbf{v}| \cdot |\mathbf{u}|} \leq c,$$

where $|\cdot|$ is the euclidean norm on \mathbf{R}^2 .

In fact, one can check that if one considers ℓ^p -norms on both $M(2, \mathbf{R})$ and \mathbf{R}^2 , $1 \leq p \leq +\infty$, $\mathbf{v} \star \mathbf{u}$ is precisely the product of the norms of \mathbf{v} and \mathbf{u} ([26]). For arbitrary norms, this quantity was introduced by Gorodnik and Weiss [15].

II.1.2. Local distribution, cocompact case. In the next result, we will assume some regularity on our test function f , in order to obtain a remainder term. The test functions will be Hölder of exponent θ , and will be required to vanish outside a compact annulus. The result is the following.

THEOREM II.1. [33, Corollary 1.3] *Given a cocompact lattice Γ in G and $-1 < \alpha < 1$, $0 < \theta \leq 1$, there is $\delta > 0$ such that for any $\mathbf{u} \in \mathbf{R}^2 - \{0\}$ and any compactly supported θ -Hölder function f on $\mathbf{R}^2 - \{0\}$,*

$$\frac{1}{T^{1+\alpha}} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) = \frac{2}{\mu(\Gamma \backslash G)} \int_{\mathbf{R}^2 - \{0\}} \frac{f(\mathbf{v})}{\mathbf{v} \star \mathbf{u}} d\mathbf{v} + O(T^{-\delta}).$$

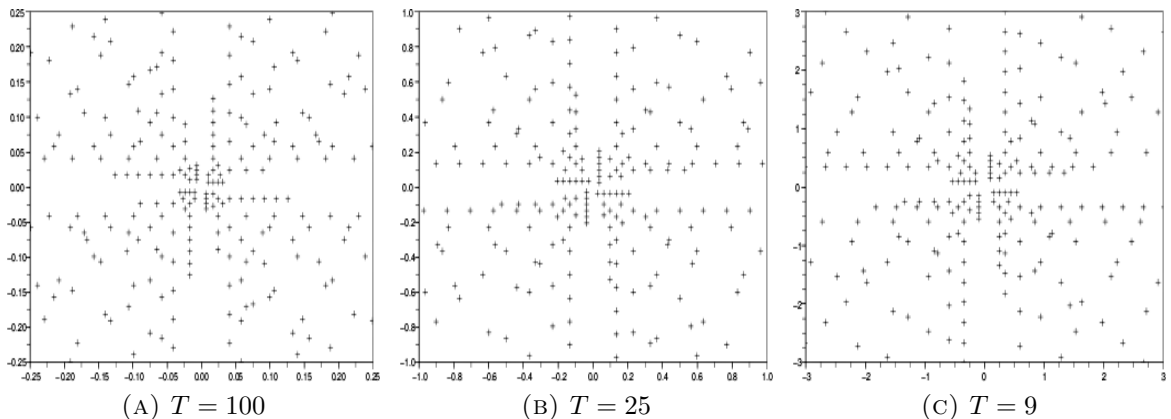


FIGURE 1. The ‘scaling property’. For $\Gamma = \text{SL}(2, \mathbf{Z})$, parts of the orbit of three different points are shown, at different scales and with different values of T . Each contains approximately 200 points.

The proportion of points of $\Gamma_T \mathbf{u}$ at scale T^α depends on α , and is $T^{1+\alpha}/T^2 = T^{\alpha-1}$. The remarkable thing is that the limit measure, $\frac{d\mathbf{v}}{\mathbf{v} \star \mathbf{u}}$, is independent of the scale α , something we will refer to as the ‘scaling property’ of this cloud of

points: it looks the same at completely different scales.

In Figure 1, this property is illustrated for $\Gamma = \text{SL}(2, \mathbf{Z})$. The latter is not cocompact, but is less computer-time consuming; we will see that it satisfies analogous properties, unconditionally if $\alpha > 0$, and as soon as we choose the slope of \mathbf{u} to be quadratic irrational if $\alpha \leq 0$.

Let us say a few more words about these pictures. In Figure 1, for each picture, the relevant α is determined by the scale of the picture itself: it is the α such that $T^\alpha = 0.25$ (resp. 1, 3). The scale being T^α , the number of points is approximately some constant (depending on the lattice) times $T^{1+\alpha} = T * \text{scale}$. For the first picture, $100 * 0.25 = 25$; in the second one, $25 * 1 = 25$. In the third one, $9 * 3 = 27 \simeq 25$.

II.1.3. Local distribution for $\text{SL}(2, \mathbf{Z})$. For $\Gamma = \text{SL}(2, \mathbf{Z})$ - or more generally a non-uniform lattice -, one cannot expect such a nice distribution without any condition on \mathbf{u} . If \mathbf{u} is a vector with integer coordinates, the whole orbit $\Gamma\mathbf{u}$ is a *discrete* set, so $\frac{1}{T^\alpha}\Gamma_T\mathbf{u}$ cannot equidistribute to an absolutely continuous measure if $\alpha \leq 0$. If \mathbf{u} is very close to a vector \mathbf{u}' of rational slope and T is 'small' with respect to this approximation, then the set $\Gamma_T\mathbf{u}$ will be close to the set $\Gamma_T\mathbf{u}'$, and the same thing will happen. It is therefore natural to introduce diophantine conditions for 'small' scales (ie $\alpha \leq 0$).

For $\beta > 0$, say $z \in \mathbf{R}$ is β -diophantine if there is $c > 0$ such that $|z - p/q| \geq cq^{-\beta}$ for all $p, q \in \mathbf{Z}$. Note that quadratic irrationals are 2-diophantine, and by Roth's theorem, all algebraic numbers are $2 + \eta$ -diophantine for any $\eta > 0$, like Lebesgue almost any real number.

THEOREM II.2. [**33**, Corollary 1.7] *Let $\Gamma = \text{SL}(2, \mathbf{Z})$. Given $\beta \in [2, +\infty]$, $\theta \in (0, 1]$ and $-\frac{1}{\beta-1} < \alpha < 1$, there is $\delta > 0$ such that for any vector $\mathbf{u} \in \mathbf{R}^2 - \{0\}$ with a β -diophantine slope and any compactly supported θ -Hölder function f on $\mathbf{R}^2 - \{0\}$,*

$$\frac{1}{T^{1+\alpha}} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma\mathbf{u}}{T^\alpha}\right) = \frac{2}{\mu(\Gamma \backslash G)} \int_{\mathbf{R}^2 - \{0\}} \frac{f(\mathbf{v})}{\mathbf{v} \star \mathbf{u}} d\mathbf{v} + O(T^{-\delta}).$$

As claimed before, vectors \mathbf{u} of quadratic irrational slope (or, more generally, Lebesgue almost every number) behave very much as if the lattice were cocompact, and the picture is the same at any scale T^α , $\alpha \in (-1, 1)$. On the other hand, even vectors of rational slope behave nicely at large scale, that is when $\alpha > 0$.

II.2. Some remarks

In the previous Theorems II.1 and II.2, there are several issues, that we will now discuss more in details.

II.2.1. What happened to very well approximable vectors ? Theorem II.2 does not fully recover Ledrappier's Theorem - it does not imply, for example, that for every initial vector \mathbf{u} of irrational slope (for example a Liouville slope, ie ∞ -diophantine),

$$\frac{1}{T} \sum_{\gamma \in \Gamma_T} f(\gamma \mathbf{u}) = \frac{2}{\mu(\Gamma \backslash G)} \int_{\mathbf{R}^2} \frac{f(\mathbf{v})}{\mathbf{v} \star \mathbf{u}} d\mathbf{v} + o(1).$$

The reason is that in [33], the goal is to have remainder terms, and the statements we obtain are effective: the difference between the integral and the sum is bounded by *effective* constants and valid for *finite time* T , depending on 'easy-to-compute' data about the continued fraction expansion. The result is a bit suboptimal, but gives satisfying bounds for 'most' numbers¹; one may dive into the proof to obtain a estimate for numbers whose continued fraction expansion coefficients are growing fast; the result would be that 'most of the times' T , the remainder term will be quite nice, but for some subsequence $T_k \rightarrow +\infty$, the remainder term will be awfully large (but still tends to zero). In short, it is possible, but complicated to state.

II.2.2. Other non-uniform lattices. One could wonder if Theorem II.2 is valid for other non-uniform lattices in general. The answer is yes, but one would have to define what is to be diophantine with respect to such a abstract lattice, and this is probably less natural, except in the case of congruence subgroups. For theses lattices, Theorem II.2 applies.

II.2.3. Near Zero. Ledrappier's result is valid for tests functions f which may not vanish at zero, that is, continuous compactly supported on \mathbf{R}^2 functions. In [33], we did not manage to authorize this; it may be because it is difficult to express the dependency of the remainder terms near zero with help of Hölder norms.

II.2.4. A related question. A related question, on which I won't say much, is the diophantine approximation of a fixed vector \mathbf{v} by the orbit of a vector \mathbf{u} , i.e., is there infinitely many solution $\gamma \in \Gamma$ to the inequality

$$|\gamma \mathbf{u} - \mathbf{v}| \leq \|\gamma\|^{-\epsilon},$$

and which is the best ϵ possible ? A first answer was given in [33, Corollary 1.9], but it's far from optimal; a better bound for $\mathrm{SL}(2, \mathbf{Z})$ was given by Laurent and Nogueira [23], namely $\epsilon = 1/3$. See also the preprint [13] for more on the subject.

¹One could argue that Liouville numbers, being a G^δ -set, are not *that* scarce.

II.3. Some ideas of the proofs

In what follows, we will make use of the symbol \simeq , which means 'under the right set of hypotheses, is approximately equal to...'. Working with actual equalities or inequalities is necessary, consubstantial with a mathematician's job, and was done in [33]. But here it would only hide the ideas and repel the potential reader, along with being utterly tedious. So I ask for the reader's trust that behind a \simeq -like 'equality', there is an actual equality whose main terms are the one given here.

II.3.1. The cocycle.

II.3.1.1. *Identifying G as a product.* Since G acts transitively on $\mathbf{R}^2 - \{0\}$, and the stabilizer of the first vector e_1 of the canonical basis of \mathbf{R}^2 is the subgroup

$$N = \left\{ n_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}; s \in \mathbf{R} \right\},$$

of upper triangular unipotent matrices, one can of course identify $\mathbf{R}^2 - \{0\}$ to G/N by the quotient of the map $g \mapsto g(e_1)$. One can also identify G to $(\mathbf{R}^2 - \{0\}) \times N$, in a non-canonical way. First, fix a section $\Psi : \mathbf{R}^2 - \{0\} \rightarrow G$, that is a map that satisfies $\Psi(\mathbf{u})e_1 = \mathbf{u}$. The usual choice here is

$$\Psi \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x & -y/(x^2 + y^2) \\ y & x/(x^2 + y^2) \end{bmatrix},$$

this choice being linked with the usual Iwasawa decomposition $G = KAN$; indeed, with natural notations, $\Psi(kan(e_1)) = ka$. It satisfies the useful equation

$$\Psi(e^t \mathbf{v}) = \Psi(\mathbf{v})a_{2t},$$

where

$$a_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}.$$

Now, it is clear that $(\mathbf{R}^2 - \{0\}) \times N$ is homeomorphic to G , by the map $(\mathbf{u}, n) \mapsto \Psi(\mathbf{u})n$. This homeomorphism allows to normalize the Haar measure μ on G : $d\mu = d\mathcal{L}_{\mathbf{R}^2} \otimes dn$, which is important if one wishes to keep track of the constants.

If one wishes to describe the actions of G on itself by left multiplications using these coordinates, one is led to consider the *cocycle*

$$c(\gamma, \mathbf{u}) = \Psi(\gamma\mathbf{u})^{-1}\gamma\Psi(\mathbf{u}) \in N,$$

so that in the $(\mathbf{R}^2 - \{0\}) \times N$ -coordinates, the left multiplication by $\gamma \in G$ is

$$\gamma(\mathbf{u}, n) = (\gamma\mathbf{u}, c(\gamma, \mathbf{u})n).$$

As its name suggests, the cocycle satisfies the cocycle identity $c(\gamma', \gamma \mathbf{u})c(\gamma, \mathbf{u}) = c(\gamma'\gamma, \mathbf{u})$.²

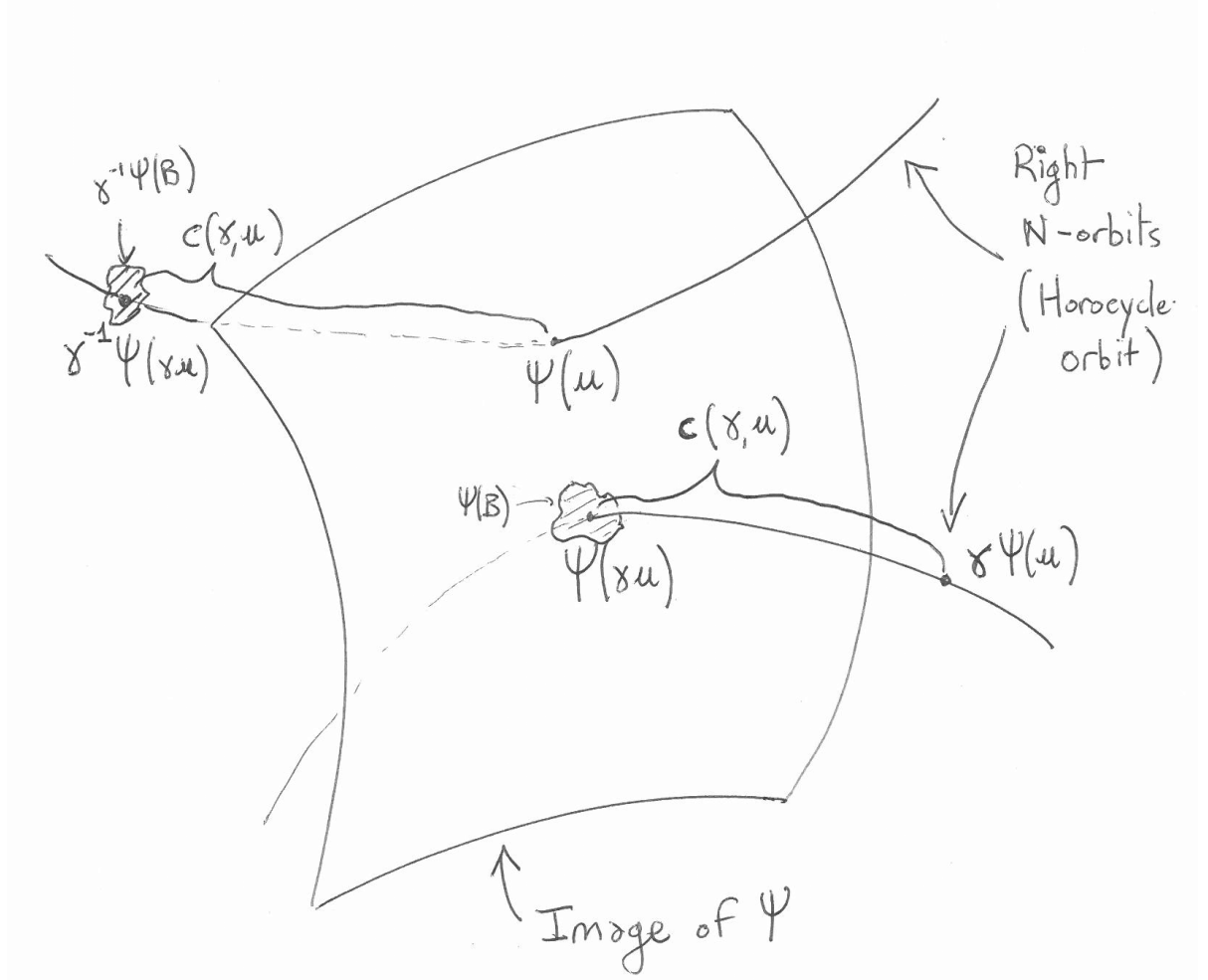


FIGURE 2. The cocycle, in the 3-dimensional space $SL(2, \mathbf{R})$

Interpretation of the cocycle can be found on Picture 2.

Since N is isomorphic to \mathbf{R} , it is convenient to write this cocycle as a real number, differentiating this by a slight change of notation: we define the real number $c_{\mathbf{u}}(\gamma)$ by the equation

$$c(\gamma, \mathbf{u}) = \begin{bmatrix} 1 & c_{\mathbf{u}}(\gamma) \\ 0 & 1 \end{bmatrix}.$$

²Changing the choice of the section changes of course the cocycle by a coboundary, but we will stick to the Ψ given, as some computations are easier with this choice.

II.3.1.2. *Fact.* The following fact is the basis of all the subsequent analysis; if the matrix γ sends \mathbf{u} very close to \mathbf{v} , and both have 'comparable norms compared to the size of γ itself', then we have the estimate

$$(1) \quad \boxed{\|\gamma\| \simeq |c_{\mathbf{u}}(\gamma)| \mathbf{v} \star \mathbf{u},}$$

See [33, Lemma 2.1] for a precise statement. Let us have a glance at the proof. We have

$$\gamma = \Psi(\gamma\mathbf{u}) \begin{bmatrix} 1 & c_{\mathbf{u}}(\gamma) \\ 0 & 1 \end{bmatrix} \Psi(\mathbf{u})^{-1},$$

which is, under the above hypothesis, 'almost the same' as

$$\gamma \simeq \Psi(\mathbf{v}) \begin{bmatrix} 0 & c_{\mathbf{u}}(\gamma) \\ 0 & 0 \end{bmatrix} \Psi(\mathbf{u})^{-1}.$$

Therefore,

$$\|\gamma\| \simeq |c_{\mathbf{u}}(\gamma)| \left\| \Psi(\mathbf{v}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Psi(\mathbf{u})^{-1} \right\|$$

A simple matrix computation shows that the second term of the right-hand side does not depend on the choice of the section, and is in fact equal to $\mathbf{v} \star \mathbf{u}$.

II.3.2. Algebraic and geometric notations. Let us explain briefly the relationship between the notations used here, and the geometric interpretation. A point $\Gamma x \in \Gamma \backslash G$ is also a unit vector in the tangent bundle $T^1 S$ of the hyperbolic surface $S = \Gamma \backslash G / \text{SO}(2)$. This point pushed by the geodesic flow of time t is then written $\Gamma x a_t$, and when pushed by the horocycle flow of time s , $\Gamma x n_s$.

II.3.3. Counting elements of the group and horocyclic flow. Let us sketch the approach. We consider a test function f and we wish to estimate $\sum_{\gamma \in \Gamma_T} f(T^{-\alpha} \gamma \mathbf{u})$. Decomposing f as a sum of functions of small support, we reduce to the case where the test function f has support in a small ball B around a non-zero vector \mathbf{v} , and is (essentially) constant, equal to one.³

Thus, $\sum_{\gamma \in \Gamma_T} f(T^{-\alpha} \gamma \mathbf{u})$ is the number of matrices $\gamma \in \Gamma_T$ which send \mathbf{u} in $T^{-\alpha} B$.

Now, each time a matrix γ of norm $\simeq T$ sends \mathbf{u} near $T^{-\alpha} \mathbf{v}$, it also sends $T^{-\alpha} \mathbf{u}$ near \mathbf{v} and it can be checked that the estimate on the cocycle applies:

$$|c_{T^{-\alpha} \mathbf{u}}(\gamma)| \simeq \frac{\|\gamma\|}{\mathbf{v} \star (T^{-\alpha} \mathbf{u})} \simeq \frac{T}{T^{-\alpha} \mathbf{v} \star \mathbf{u}} \simeq \frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}.$$

³In [33], the decomposition is made along annuli for the norm $\cdot \star \mathbf{u}$, in order to obtain slightly better error estimates. The decomposition alluded to here must be done with more care than the above sketch, because one has to retain some control on Sobolev norms of the functions in the decomposition

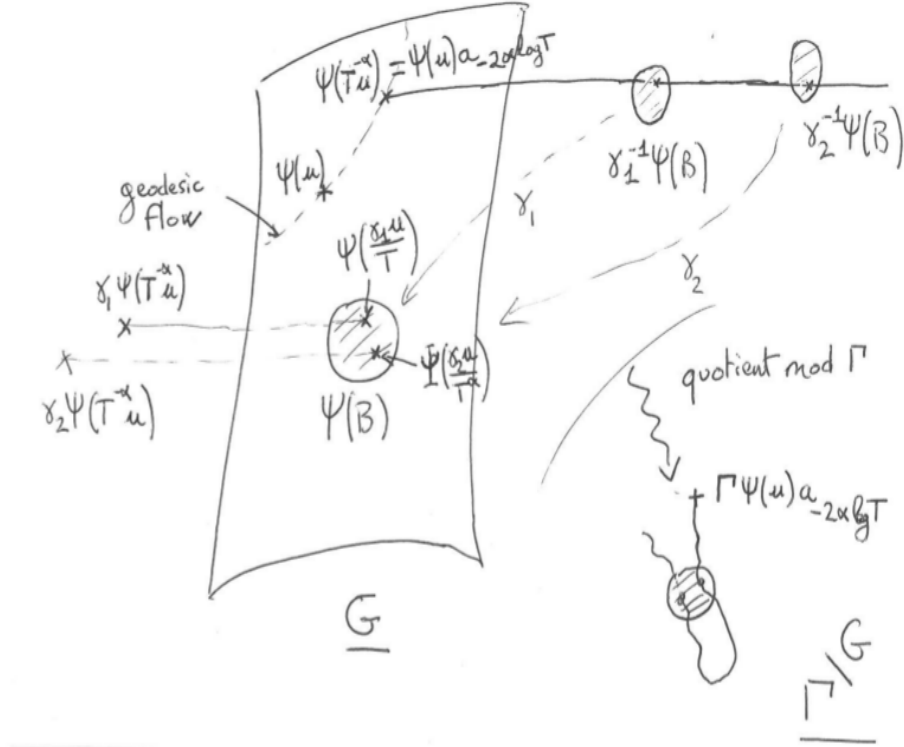


FIGURE 3. Some intersections of a horocycle with $\Gamma\Psi(B)$

This can be done the other way: a γ such that $\gamma(T^{-\alpha}\mathbf{u}) \in B$ has a norm approximately $T^{-\alpha}|c_{T^{-\alpha}\mathbf{u}}(\gamma)|\mathbf{v} \star \mathbf{u}$.

Now consider the horocycle segment centered on $\Gamma\Psi(T^{-\alpha}\mathbf{u}) = \Gamma\Psi(\mathbf{u})a_{-2\alpha \log T}$, of length $T^{1+\alpha}/(\mathbf{v} \star \mathbf{u})$, in the manifold $\Gamma \backslash G$. For each intersection with $\Gamma\Psi(B)$, there is a corresponding γ such that

$$\Psi(\mathbf{u})a_{-2\alpha \log T} n_s \in \gamma^{-1}\Psi(B),$$

where $|s|$ is smaller than $T^{1+\alpha}/(\mathbf{v} \star \mathbf{u})$, and is precisely equal to the cocycle $|c_{T^{-\alpha}\mathbf{u}}(\gamma)|$.

See Picture 3.

This means that we reduced the problem to counting such intersections with $\Gamma\Psi(B)$ on a horocycle segment; to do so, let us introduce the indicatrix function \bar{f} of the set $\Gamma\Psi(B)n_{[-1/2,1/2]}$. Its integral is equal to the surface of B , and integration on N -right orbits (that is, the horocyclic flow) counts the number of intersections.⁴ Therefore,

$$(2) \quad \boxed{\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) \simeq \int_{-\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}}^{\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}} \bar{f}(\Gamma\Psi(\mathbf{u})a_{-2\alpha \log T}n_s) ds.}$$

This means we have reduced the problem to finding a estimate of the ergodic sum along a piece of horocycle centered on $\Gamma\Psi(\mathbf{u})$, pushed (or pulled) by the geodesic flow of time $-2\alpha \log T$.

II.3.4. Cocompact case.

II.3.4.1. *Consequences.* In the cocompact case, the horocycle flow is uniquely ergodic (Furstenberg) so ergodic averages of continuous functions converge uniformly to their integral with respect to the Liouville measure. Because of this uniformity, the fact that the horocycle is centered around a moving point $\Gamma\Psi(\mathbf{u})a_{-2\alpha \log T}$ is irrelevant, as long as the horocycle segment is long enough.

The effect on the main terms we try to compute are:

$$\begin{aligned} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) &\simeq \int_{-\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}}^{\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}} \bar{f}(\Gamma\Psi(\mathbf{u})a_{-2\alpha \log T}n_s) ds, \\ &\simeq \frac{2T^{1+\alpha}}{\mathbf{v} \star \mathbf{u} \mu(\Gamma \backslash G)} \int_{\Gamma \backslash G} \bar{f} d\mu, && \text{(Unique ergodicity)} \\ &\simeq \frac{2T^{1+\alpha}}{\mathbf{v} \star \mathbf{u} \mu(\Gamma \backslash G)} \int f(\mathbf{w}) d\mathbf{w}, && \text{(Properties of } \bar{f}) \\ &\simeq \frac{2T^{1+\alpha}}{\mu(\Gamma \backslash G)} \int \frac{f(\mathbf{w})}{\mathbf{w} \star \mathbf{u}} d\mathbf{w}. && \text{(Support of } f) \end{aligned}$$

II.3.4.2. *Ergodic Theorem with remainder terms.* Now that we have determined the main term, we only need effective versions of every \simeq -equation to obtain Theorem II.1. The most interesting one is of course the ergodic Theorem. So we need to understand what looks like the ergodic Theorem with remainder term for the horocycle flow on compact surfaces.

⁴This construction of \bar{f} is not quite precise, as it may happen that $\Psi(B)n_{[-1/2,1/2]}$ does not embed injectively in $\Gamma \backslash G$. In practice, in [33] we take a smooth thickening to be able to control the Sobolev norm of \bar{f} .

Let us recall briefly the thickening argument that allows to pass from mixing of the geodesic flow to equidistribution of horocycles - one usual way uses some equicontinuity, but knowledge of the speed of mixing allows to bypass this argument. Assume we already know (exponential) mixing: for every smooth functions f, g on $\Gamma \backslash G$,

$$\int_{\Gamma \backslash G} f(\Gamma x) g(\Gamma x a_t) d\mu(\Gamma x) = \left(\int_{\Gamma \backslash G} f \right) \left(\int_{\Gamma \backslash G} g \right) + O(e^{-\beta|t|} \|f\|_{Sob} \|g\|_{Sob}).$$

for some appropriate Sobolev norm, and $\beta > 0$. Let f be the function we want to integrate along an horocycle $\Gamma x_0 n_{[-S, S]}$ of length $2S > 0$, define g_ϵ to be a smooth approximation of the integral against the small piece of horocycle $\Gamma x_0 a_{\log S} n_{[-1, 1]}$, say on a neighborhood of size ϵ . If g_ϵ is well-chosen, its Sobolev norm will be of order ϵ^{-N} , for some fixed N .

The usual trick is to see that $g_\epsilon \circ a_{-\log S}$ is also a smooth approximation of the integral against $\Gamma x_0 n_{[-S, S]}$. Therefore,

$$\frac{1}{2S} \int_{-S}^S f(\Gamma x_0 n_s) ds = \int_{\Gamma \backslash G} f(\Gamma x) g_\epsilon(\Gamma x a_{-\log S}) d\mu(\Gamma x) + O(\epsilon \|f\|_{Sob}),$$

and now exponential mixing tells us that

$$\frac{1}{2S} \int_{-S}^S f(\Gamma x_0 n_s) ds = \int_{\Gamma \backslash G} f + O(S^{-\beta} \|f\|_{Sob} \epsilon^{-N}) + O(\epsilon \|f\|_{Sob}).$$

We choose ϵ to minimize the remainder term - and that will hurt the exponent β , but still gives a remainder term of the form $S^{-\beta'} \|f\|_{Sob}$. So an exponential speed for mixing of geodesic flow implies a polynomial speed for ergodic averages of horocycle flow. Something we must be careful about is that g_ϵ in fact depends on S ; but since the lattice is cocompact, one can give an uniform bound for the Sobolev norm of a family of smooth approximations of horocycle $\Gamma y n_{[-1, 1]}$ for all points Γy .

The bad news is that this strategy would give suboptimal exponents (as smoothing process often do). In [33], we appeal to the results of Ratner, Burger [7], Flaminio-Forni [11] and Strömbergson [42], in which they analyse the decomposition of $L^2(\Gamma \backslash G)$ into irreducible representations of G and work in each component separately. Recall that the Casimir operator is basically the Laplacian when estimated on right-SO(2)-invariant functions, so knowledge of the spectrum of the Laplacian gives essentially the decomposition.

This last approach is not very different from Ratner's results about exponential mixing [38] (or Matheus [29]), and this is not surprising, as we have seen that

exponential mixing of geodesic flow and polynomial equidistribution of horocycles are more or less synonymous.

II.3.5. Non-uniform lattices. The previous rough analysis gives us a hint about what a good estimate for ergodic average should be in this case. Consider a large piece of horocycle $\Gamma x n_{[-S,S]}$, then if $\Gamma x a_{\log S}$ is in a compact part of the manifold $\Gamma \backslash G$, one still can have a uniform bound for the Sobolev norm of the function g , and the same kind of estimate follows. However, if $\Gamma x a_{\log S}$ is far away in the cusp(s), the Sobolev norm of g will be bigger; estimating properly by how much, depending on the distance of $\Gamma x a_{\log S}$ to the compact part, seems difficult. But it is a way to understand why, in Strömbergsson's result [42], this distance appears as an important term. More precisely, if K is any compact subset of $\Gamma \backslash G$, then

THEOREM II.3 (Strömbergsson [42]). *Let Γ be a lattice in G there are positive $\delta = \delta_\Gamma$ and c such that for any $S \geq 1$, any C^4 -map F on $\Gamma \backslash G$ supported in K and any $x \in \Gamma \backslash G$,*

$$\left| \frac{1}{2S} \int_{-S}^S F(xn_s) ds - \frac{1}{\mu(\Gamma \backslash G)} \int_{\Gamma \backslash G} F d\mu \right| \leq c \|F\|_{2,4} S^{-\delta} e^{\delta \operatorname{dist}(x a_{\log S}, K)}.$$

Returning to our original problem, we had to estimate an ergodic average along an horocycle of length $T^{1+\alpha}$ around a point $\Gamma \Psi(\mathbf{u}) a_{-2\alpha \log T}$. The point we need to understand how far in the cusp(s) is $\Gamma \Psi(\mathbf{u}) a_{(1-\alpha) \log T}$, and this distance has to be compared to the length $T^{1+\alpha}$ in the quantitative way given by the above result, to obtain a remainder term.

If $\Gamma = \mathrm{SL}(2, \mathbf{Z})$, and \mathbf{u} has β -diophantine slope, one can give a priori bounds on this distance, namely $\operatorname{dist}(\Gamma \Psi(\mathbf{u}) a_{(1-\alpha) \log T}, K) \ll \frac{(1-\alpha)\beta}{\beta+2} \log T$, which lead to Theorem II.2.

CHAPTER III

Action of finitely generated groups on the plane

IN this chapter, $G = \mathrm{SL}(2, \mathbf{R})$, $G_0 = \mathrm{PSL}(2, \mathbf{R})$, and \mathbb{H} is the hyperbolic plane. We want to analyse the linear action of a finitely generated, non-elementary subgroup $\Gamma \subset G$, containing $-I$ as unique torsion element. We will denote by Γ_0 its image in G_0 , and $S = \Gamma_0 \backslash \mathbb{H}$ the corresponding surface, where \mathbb{H} is the hyperbolic plane. The assumption on Γ means that the surface S is geometrically finite.

The crucial estimate (2)

$$\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) \simeq \int_{-\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}}^{\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}} \bar{f}(\Gamma \Psi(\mathbf{u}) a_{-2\alpha \log T n_s}) ds.$$

for a test function f whose support is a small set near \mathbf{v} , and a related function \bar{f} , still applies; however, the Liouville/Haar measure cannot be used anymore to estimate the right-hand side ergodic sum (it is infinite, and not even ergodic anymore).

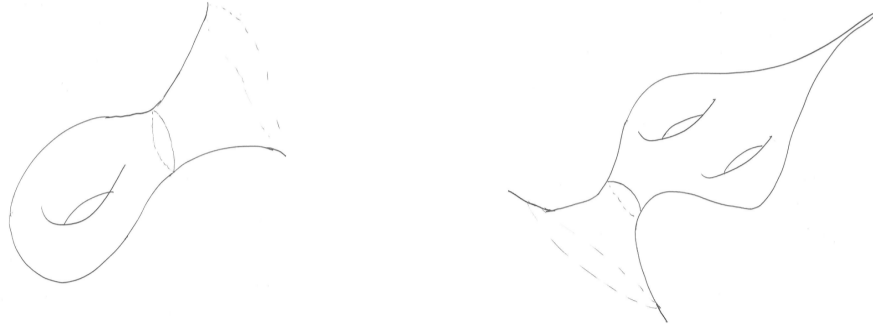
We will proceed in the opposite order relatively to the previous chapter, first focusing on ergodic properties, and then apply them to get results about the orbit distribution on the plane. We will retain the (algebraic) notations, therefore for $v \in T^1 S$, vn_s and va_t will be respectively the vector v pushed by the horocycle flow of time s (resp. geodesic flow of time t).

We will distinguish between the convex-cocompact case (where the results will be the strongest) and the geometrically finite with cusps case. The main reference for the setup of this chapter is the book of Roblin [39].

III.1. Ergodic theory on geometrically finite surfaces

III.1.1. Patterson measures. Let $\Lambda \subset \mathbb{P}^1$ be the limit set of Γ_0 , and δ be the critical exponent. For geometrically finite surfaces in constant curvature, it was proved by Sullivan that there is, up to normalisation, a unique family of conformal densities $(\nu_x)_{x \in \mathbb{H}}$ of exponent δ , called the Patterson measures, satisfying $\gamma_* \nu_x = \nu_{\gamma x}$, and for any $x, y \in \mathbb{H}, \xi \in \mathbb{P}^1 = \partial \mathbb{H}$,

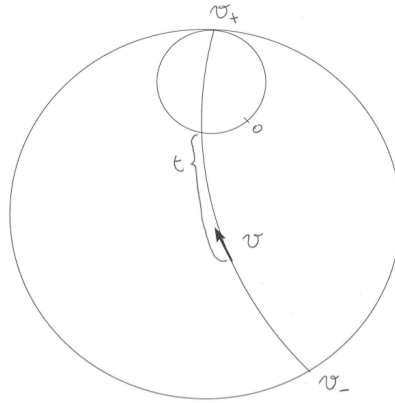
$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\delta \beta_\xi(x,y)},$$



(A) Convex-cocompact surface

(B) Geometrically finite with a cusp

where β_ξ is the Busemann function. Let o be the origin in \mathbb{H} , that is, the point fixed by the action of $\text{SO}(2)$.

FIGURE 1. The Hopf coordinates on $T^1\mathbb{H}$

III.1.2. Bowen-Margulis-Patterson-Sullivan measure. We use the classical Hopf coordinates, explicitly the map $T^1\mathbb{H} \rightarrow (\partial\mathbb{H} \times \partial\mathbb{H} - \text{diag}) \times \mathbf{R}$, $v \mapsto (v_+, v_-, \beta_{v_+}(o, \pi(v)))$, which to any vector v associate the endpoints of the geodesic it generates, and a time parameter. Here, π is the base-point projection from $T^1\mathbb{H}$ to \mathbb{H} .

Let $\tilde{\Omega} = (\Lambda \times \Lambda - \text{diag}) \times \mathbf{R}$, this is a Γ -invariant set, invariant under the geodesic flow. Put $\Omega = \Gamma \backslash \tilde{\Omega}$. It turns out to be the non-wandering set for the geodesic

flow on T^1S . When Γ_0 does not contain parabolic elements, the group is convex-cocompact and Ω is a compact set; it is however non-compact if S has cusps.

Still in the Hopf coordinates, define

$$d\tilde{m}_{PS}(v) = e^{\delta(\beta_{v_-}(o,\pi(v)) + \beta_{v_+}(o,\pi(v)))} d\nu_o(v_+) d\nu_o(v_-) dt.$$

It is clearly invariant under the geodesic flow. Because of the properties of conformal densities, this is a Γ -invariant measure, so descend to a measure m_{PS} on T^1S , the Bowen-Margulis-Sullivan measure. This measure turns out to be finite. It is the unique measure of maximal entropy (namely, δ) for the geodesic flow. It is ergodic and mixing.

III.1.3. Burger-Roblin measure. Define $\tilde{\mathcal{E}} = (\Lambda \times \partial\mathbb{H} - \text{diag}) \times \mathbf{R}$, this is a Γ -invariant set, and $\mathcal{E} = \Gamma \backslash \tilde{\mathcal{E}}$. This set turns out to be the non-wandering set for horocycle flow on T^1S .

An trivial but important observation is that any $v \in \mathcal{E}$ is getting closer to Ω when pushed by the geodesic flow of positive time.

The so-called *Burger-Roblin measure* is constructed the following way. Put

$$d\tilde{m}_{BR} = e^{\delta\beta_{v_+}(o,\pi(v)) + \beta_{v_-}(o,\pi(v))} d\nu_o(v_+) d\theta(v_-) dt,$$

where $d\theta$ is the $SO(2)$ -invariant probability measure on $\partial\mathbb{H}$. Similarly, this measure is Γ -invariant and defines a measure m_{BR} on T^1S , the Burger-Roblin measure. Not completely obvious in the above formula is the fact that the conditional measure on the stable manifold $e^{\beta_{v_-}(o,\pi(v))} d\theta(v_-)$ is none other than the Lebesgue measure, so m_{BR} is invariant under the horocycle flow.

It turns out that this is the only ergodic, conservative measure of full support in \mathcal{E} . In the convex-cocompact case, it is the only ergodic, conservative measure. In the presence of cusps, ergodic conservative invariant measures also include the (finite) measures on periodic horocycles.

Notice that m_{BR} is not invariant under the geodesic flow, but merely quasi-invariant; it decreases by a factor $e^{(\delta-1)t}$ when flowed.

III.1.4. Measures on Horocycles. For $v \in T^1\mathbb{H}$, denote by $H^-(v) = \{vn_s\}_{s \in \mathbf{R}}$ the horocycle passing through v . On each horocycle $H^-(v)$, we will consider two measures $\lambda_{H^-(v)}$ and $\mu_{H^-(v)}$. The measure $\lambda_{H^-(v)}$ will simply be the Lebesgue measure when $H^-(v)$ is identified with \mathbf{R} through the map $s \mapsto vn_s$ (it does not depends on the chosen v in the horocycle). The measure $\mu_{H^-(v)}$ is given by

$$d\mu_{H^-(v)}(w) = e^{\delta\beta_{w_-}(o,\pi(w))} d\nu_o(w_-),$$

is infinite if $v_+ \in \Lambda$, and correspond to the conditional measure of \tilde{m}_{PS} along the horocycles. It is Γ -invariant in the sense that $\gamma_*\mu_{H^-(v)} = \mu_{H^-(\gamma v)}$, so can be thought as a measure on an horocycle on S (we omit the case where the horocycle is periodic).

The set $\mathcal{H} \simeq \partial\mathbb{H} \times \mathbf{R}$ of all horocycles of \mathbb{H} can be endowed with a measure $\hat{\mu}$ written as

$$d\hat{\mu}(\xi, t) = e^{\delta t} d\nu_o(\xi) dt,$$

which is Γ -invariant and ergodic.

The measure \tilde{m}_{PS} (respectively \tilde{m}_{BR}) is the measure obtained by integration of the family of measure $\mu_{H^-(\cdot)}$ (resp. $\lambda_{H^-(\cdot)}$) on the horocycle foliation with transverse measure $\hat{\mu}$.

We define, for $v \in T^1S$,

$$\tau(v) = \mu_{H^-(v)}((vn_s)_{s \in [-1,1]}),$$

the $\mu_{H^-(v)}$ -measure of an horocycle segment of radius 1. Understanding the properties of this function will be crucial in the sequel. It is positive on Ω , and turns out to grow exponentially in cusps [32, Proposition 5.1]. However, if Γ is convex-cocompact, then it is bounded from above and below by positive constants on any compact neighbourhood of Ω ; since for any $v \in \mathcal{E}$, there is a T such that for all $t \geq T$, va_t is in the chosen neighbourhood of Ω , hence $\tau(va_t)$ oscillates between two positive constants.

III.1.5. Ergodic Theorems. The main ergodic result is the following.

THEOREM III.1. [32, Theorem 1.1] *Let S be a nonelementary geometrically finite hyperbolic surface. There is a nonnegative continuous function τ on T^1S , such that the following holds. Let $u \in \mathcal{E}$ be a nonwandering and non-periodic vector for the horocycle flow. If $f : T^1S \rightarrow \mathbf{R}$ is continuous with compact support, then*

$$\lim_{t \rightarrow +\infty} \frac{1}{t^\delta \tau(ua_{\log t})} \int_{-t}^t f(un_s) ds = \frac{1}{m_{PS}(T^1S)} \int_{T^1S} f dm_{BR}.$$

Moreover, $t^\delta \tau(ua_{\log t}) \rightarrow +\infty$ as $t \rightarrow +\infty$.

If the surface S is convex-cocompact, the nonwandering set $\Omega \subset \mathcal{E}$ of the geodesic flow is compact, the map τ is bounded from below and from above on Ω , and the above convergence is uniform in $u \in \Omega$.

In the convex-cocompact setting, this is a strong statement, once one knows that the mysterious function $\tau(ua_{\log t})$ is not so mysterious. For large enough t , it oscillates at a decaying rate between two positive constants - one could think of it as something like $2 + \cos(\log t)$.

One happy consequence is that it did show that the famous Theorem of Aaronson forbidding the existence of normalizing sequence in infinite ergodic theory [1, Theorem 2.4.2] is a result specific to *one-sided* ergodic sums. This motivated the paper [2].

III.1.6. Sketch of the proof of the Theorem. Although this Theorem was new, its proof followed known strategies. An old idea is to first consider similar integrals but with respect to the measures μ_{H^-} on the leaves. These measures behave nicely under the geodesic flow : the measure of the image of a set under the geodesic flow of time t is $e^{-\delta t}$ smaller than the measure on the original horocycle.

III.1.6.1. *The integral with respect to μ_{H^-} .* For f continuous with compact support on T^1S , consider

$$\begin{aligned} M_{t,v}(f) &= \frac{1}{\mu_{H^-(v)}(vn_s)_{s \in (-t,t)}} \int_{-t}^t f(vn_s) d\mu_{H^-(v)}(s) \\ &= \frac{1}{t^\delta \tau(va_{\log t})} \int_{-t}^t f(vn_s) d\mu_{H^-(v)}(s). \end{aligned}$$

For all $v \in \Omega$ a non-periodic vector for n_s , the statement is that

$$(3) \quad \lim_{t \rightarrow +\infty} M_{t,v}(f) = \frac{1}{m_{PS}(T^1S)} \int f dm_{PS}.$$

In the convex-cocompact case, this is due to Roblin. The pointwise convergence of $M_{t,v}(f \circ a_{-\log t})$ is essentially the mixing property of the finite measure m_{PS} (see for example Babillot [4]), in a quite similar fashion than in the lattice case sketched in Chapter 2. Then one can use a equicontinuity argument to show the pointwise convergence of $M_{t,v}(f)$.

In the geometrically finite case (with cusps), this is due to Schapira [40], and the proof is a bit different, as equicontinuity does not occur in this case. Another difficulty is to control the tightness of horocycles, i.e. check there is no loss of mass at infinity.

III.1.6.2. *From μ_{H^-} to λ_{H^-} .* One would like to interpret the convergence (3). Consider f the indicatrix function of a flow box B , that is a small set which is locally a product of a transversal T to the horocycle foliation, and a small piece of horocycle. The integral

$$\int_{-t}^t f(vn_s) d\mu_{H^-(v)}(vn_s)$$

counts how many times $N_{t,v}$ the horocycle $(vn_s)_{s \in (-t,t)}$ hits the box B , each time being counted with a weight which is the measure μ_{H^-} of the piece of corresponding leaf. See picture 2. The crucial fact here is that μ_{H^-} is a continuous

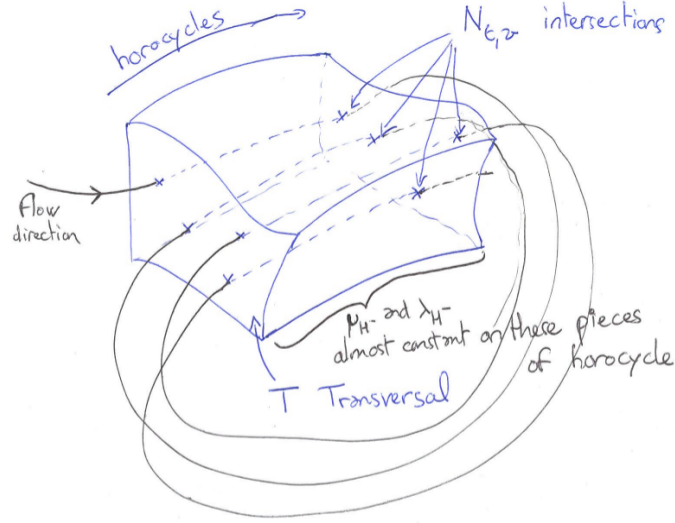


FIGURE 2. The Box B: from μ_{H^-} to λ_{H^-}

family of measures on the horocycle foliation on T^1S . In particular, the weight of each piece of leaf is almost constant c . Therefore,

$$\int_{-t}^t f(vn_s) d\mu_{H^-(v)}(s) \simeq N_{t,v} c \simeq N_{t,v} \frac{\int f dm_{PS}}{\hat{\mu}(T)}.$$

The same things occur for the Lebesgue measure on horocycles, with a positive number c' :

$$\int_{-t}^t f(vn_s) d\lambda_{H^-(v)}(s) \simeq N_{t,v} c'.$$

Hence,

$$\int_{-t}^t f(vn_s) ds \simeq \frac{c' \hat{\mu}(T)}{\int f dm_{PS}} \int_{-t}^t f(vn_s) d\mu_{H^-(v)}(vn_s).$$

Notice that $c' \hat{\mu}(T)$ is an approximation of $\int f dm_{BR}$. Together with equation (3), this implies Theorem III.1.

III.1.7. Complementary results. The two following observations will be useful.

LEMMA III.2. [32, Lemma 4.5] *If Γ is not a lattice, τ is not constant on any orbit of the geodesic flow in Ω .*

This follows from the fact that the support of $\mu_{H^{-}(v)}$ locally looks like a cantor set, so the mass of an increasing interval is very often locally constant.

The second observation is less elementary.

THEOREM III.3. [**32**, Theorem 1.7] *Assume the group Γ is finitely generated and contains at least one parabolic element, or equivalently that the surface S is geometrically finite with at least one cusp. Then $\tau \in L^1(m_{PS})$ if and only if $\delta > 2/3$.*

This is proven by estimating τ and m_{PS} in the cusps. These computations are somewhat similar to those of Dal'bo-Otal-Peigné in [8].

III.2. The action on the plane

Let $\mathcal{C}(\Gamma_0) \subset \mathbf{R}^2 - \{0\}$ be the cone of vectors whose projective component lies in Λ .

This set carries a unique (up to scalar multiple) Γ_0 -invariant ergodic measure $\bar{\mu}$ of full support, which in polar coordinates is written $d\bar{\mu} = 2r^{2\delta-1}drd\bar{v}_o$, where \bar{v}_o is the symmetric lift of the Patterson measure. It is also the symmetric lift of the measure $\hat{\mu}$, if one identifies horocycles with $(\mathbf{R}^2 - \{0\})/\pm$.

III.2.1. Convex-cocompact case. In the case of a convex-cocompact group Γ , we show that an analogue of Ledrappier's result holds 'up to multiplicative constants'. A first modification to consider is to assume that the initial vector \mathbf{u} lies in the cone $\mathcal{C}(\Gamma_0)$. Indeed, for any vector $\mathbf{u} \notin \mathcal{C}(\Gamma_0)$, there is a constant $c > 0$ (depending on \mathbf{u}) such that $|\gamma\mathbf{u}| \geq c\|\gamma\|$, hence forbidding any local distribution result for any parameter $\alpha < 1$.

THEOREM III.4. *Let $\Gamma_0 < \text{SL}(2, \mathbf{R})$ a convex-cocompact group of critical exponent δ , which contains $-I$ as unique element of torsion. Let $\alpha \in (-1, 1)$ be a scaling factor. For all $\mathbf{u} \in \mathcal{C}(\Gamma_0)$, for all nonzero, nonnegative, continuous and compactly supported functions f on $\mathbf{R}^2 - \{0\}$, we have as $T \rightarrow +\infty$:*

$$(4) \quad \frac{1}{T^{(1+\alpha)\delta}} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma\mathbf{u}}{T^\alpha}\right) \asymp \int_{\mathbf{R}^2} \frac{f(\mathbf{v})}{(\mathbf{v} \star \mathbf{u})^\delta} d\bar{\mu}(\mathbf{v}),$$

where the implied constants do not depend on \mathbf{u} nor on f .

The symbol $a(T) \asymp b(T)$ means that the ratio $a(T)/b(T)$ lies between two positive constants for T sufficiently large.

IDEA OF THE PROOF. The computation begins in a similar fashion to the one in Chapter II, except we have to use Theorem III.1 instead of the (uniquely) ergodic Theorem. For a test function f supported in a small ball around \mathbf{v} , the estimates are the following.

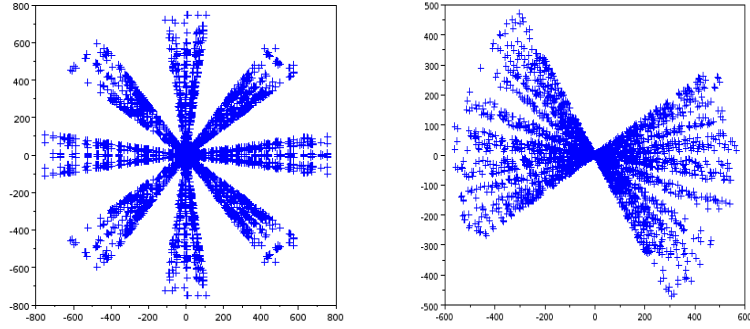


FIGURE 3. The cloud for a convex-cocompact free group (left) and for a free group with a parabolic element (right)

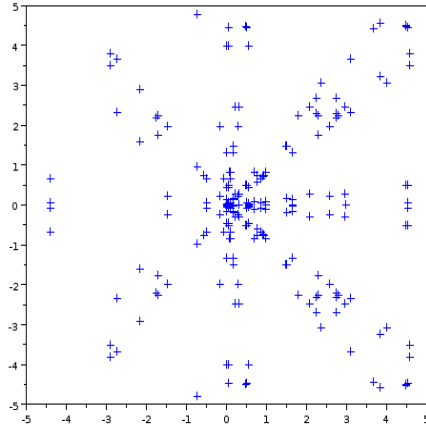


FIGURE 4. Local distribution for a convex-cocompact free group

$$\begin{aligned}
\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) &\simeq \int_{-\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}}^{\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}} \bar{f}(\Gamma \Psi(\mathbf{u}) a_{-2\alpha \log T} n_s) ds, \\
&\simeq \left(\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}\right)^\delta \frac{\tau(\Gamma \Psi(\mathbf{u}) a_{(1+\alpha) \log T - \log \mathbf{v} \star \mathbf{u}})}{m_{PS}(T^1 S)} \int_{T^1 S} \bar{f} dm_{BR}, && \text{(Theorem III.1)} \\
&\asymp \frac{T^{(1+\alpha)\delta}}{(\mathbf{v} \star \mathbf{u})^\delta m_{PS}(T^1 S)} \int_{\mathbf{R}^2} f(\mathbf{w}) d\bar{\mu}(\mathbf{w}), && \text{(Properties of } \bar{f} \text{ and } \tau) \\
&\asymp \frac{T^{(1+\alpha)\delta}}{m_{PS}(T^1 S)} \int \frac{f(\mathbf{w})}{(\mathbf{w} \star \mathbf{u})^\delta} d\bar{\mu}(\mathbf{w}). && \text{(Support of } f)
\end{aligned}$$

We used the uniformity in Theorem III.1 to ignore the fact that the middle of our horocycle segment is moving with T .

Here, we simply dismissed the oscillating function τ with the use of the symbol \asymp . This might seem a very crude way to deal with it. However, notice that here τ is computed at a point which depends on \mathbf{v} (the center of the support of f) and on T , explicitly at $\Gamma\Psi(\mathbf{u})a_{(1+\alpha)\log T - \log \mathbf{v}\star\mathbf{u}}$. Therefore, if we tried to decompose a continuous function F as a sum of functions f_i of small support, we would be quite embarrassed in trying to sum the different estimates for the functions f_i . \square

Being embarrassed about how to deal with a sum is not a very convincing argument that one cannot do much better. One could imagine that there exists a better measure than $d\bar{\mu}(\mathbf{v})/(\mathbf{v}\star\mathbf{u})^\delta$, or a better normalization than $T^{(1+\alpha)\delta}$, for which a true limit could be proved. In fact, this cannot happen simply because there is not even a ratio limit Theorem; here is the reason.

Consider two test functions f_1 and f_2 of small supports around different points \mathbf{v}_1 and \mathbf{v}_2 , such that $\mathbf{v}_1\star\mathbf{u} \neq \mathbf{v}_2\star\mathbf{u}$. The sum over f_i will behave accordingly to $\tau(\Gamma\Psi(\mathbf{u})a_{(1+\alpha)\log T - \log \mathbf{v}_i\star\mathbf{u}})$, therefore there would be a fixed difference in time for the points where τ is computed for $i = 1, 2$, namely $\log \frac{\mathbf{v}_1\star\mathbf{u}}{\mathbf{v}_2\star\mathbf{u}}$. It is therefore sufficient to notice that for any $t_0 \neq 0$ and for m_{BR} -almost every vector $v \in T^1S$, the ratio

$$\frac{\tau(va_{t+t_0})}{\tau(va_t)},$$

has no limit as $t \rightarrow +\infty$, because of Lemma III.2. We have just sketched the proof of:

PROPOSITION III.5. *Assume that Γ_0 is non-elementary, geometrically finite, with infinite volume, and contains $-I$ as unique element of torsion. There exists f_1, f_2 as in Theorem III.4 with $\int f_i d\bar{\mu} > 0$, such that for $\bar{\mu}$ -almost every \mathbf{u} , the ratio*

$$\frac{\sum_{\gamma \in \Gamma_T} f_1(\gamma\mathbf{u})}{\sum_{\gamma \in \Gamma_T} f_2(\gamma\mathbf{u})},$$

has no limit as $T \rightarrow +\infty$.

III.2.2. Geometrically finite case. The measure $d\bar{\mu}(\mathbf{v})/(\mathbf{v}\star\mathbf{u})^\delta$ is indeed the correct limit, but in a weaker sense, namely under a log-Cesaro average. The statement is the following.

THEOREM III.6. *Assume that Γ_0 is a nonelementary group containing $-I$ as unique element of torsion. Write $S = \Gamma_0 \backslash \mathbb{H}$.*

- (1) If Γ_0 is convex-cocompact, then, with the same notations as in Theorem III.4, we have for $\bar{\mu}$ -almost every $\mathbf{u} \in \mathcal{C}(\Gamma_0)$,

$$\lim_{S \rightarrow +\infty} \frac{1}{\log S} \int_1^S \frac{1}{T^{(1+\alpha)\delta}} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) \frac{dT}{T} = \frac{2 \int_{T^1 S} \tau dm_{PS}}{(m_{PS}(T^1 S))^2} \int_{\mathbf{R}^2} \frac{f(\mathbf{v})}{(\mathbf{v} \star \mathbf{u})^\delta} d\bar{\mu}(\mathbf{v}).$$

- (2) If Γ_0 is geometrically finite with cusps, with critical exponent $\delta > 2/3$, and $\alpha = 0$, then we have for $\bar{\mu}$ -almost every $\mathbf{u} \in \mathcal{C}(\Gamma_0)$,

$$\lim_{S \rightarrow +\infty} \frac{1}{\log S} \int_1^S \frac{1}{T^\delta} \sum_{\gamma \in \Gamma_T} f(\gamma \mathbf{u}) \frac{dT}{T} = \frac{2 \int_{T^1 S} \tau dm_{PS}}{(m_{PS}(T^1 S))^2} \int_{\mathbf{R}^2} \frac{f(\mathbf{v})}{(\mathbf{v} \star \mathbf{u})^\delta} d\bar{\mu}(\mathbf{v}).$$

SKETCH OF PROOF. First consider the convex-cocompact case. From the proof of Theorem III.4, recall that for a function f of small support around a point \mathbf{v} , we have the estimate:

$$\frac{1}{T^{(1+\alpha)\delta}} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) \simeq \tau(\Gamma\Psi(\mathbf{u})a_{(1+\alpha)\log T - \log \mathbf{v} \star \mathbf{u}}) \frac{1}{m_{PS}(T^1 S)} \int \frac{f(\mathbf{w})}{(\mathbf{w} \star \mathbf{u})^\delta} d\bar{\mu}(\mathbf{w}).$$

If we integrate under a log-Cesaro average, the (non-constant part of) the expression on the right-hand side becomes

$$\frac{1}{\log S} \int_1^S \tau(\Gamma\Psi(\mathbf{u})a_{(1+\alpha)\log T - \log \mathbf{v} \star \mathbf{u}}) \frac{dT}{T}$$

is nothing else than an ergodic average of τ along the geodesic flow, as is seen using the change of variable $t = \log T$. By the Birkhoff ergodic Theorem, it converges to $\frac{1}{m_{PS}(T^1 S)} \int \tau dm_{PS}$, m_{PS} -almost surely. Notice that it is a almost sure limit, and certainly not an everywhere limit (one may indeed chose \mathbf{u} such that $\Gamma\Psi(\mathbf{u})$ is in the stable manifold of a periodic vector for the geodesic flow, for instance). This proves the first point.

For a general geometrically finite surface, the problem in estimating the right-hand side of

$$\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T^\alpha}\right) \simeq \int_{-\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}}^{\frac{T^{1+\alpha}}{\mathbf{v} \star \mathbf{u}}} \bar{f}(\Gamma\Psi(\mathbf{u})a_{-2\alpha \log T} n_s) ds,$$

is that the middle point $\Gamma\Psi(\mathbf{u})a_{-2\alpha \log T}$ of the horocycle segment considered moves with T , as soon as $\alpha \neq 0$. This was not a problem in the convex-cocompact case because of the uniformity in the limit of Theorem III.1, but here we cannot apply Theorem III.1. Thus, we will assume from now on that $\alpha = 0$.¹

¹It should be possible to provide analogues of Theorem III.1 with a moving point, but diophantine conditions should appear if the point is moved forward along the geodesic flow - as in the lattice case, this correspond to $\alpha < 0$ here. An instance of this is Theorem III.7, where the point is moved backward.

Under that hypothesis, the same computations are valid, and we are left to estimate

$$\frac{1}{\log S} \int_1^S \tau(\Gamma\Psi(\mathbf{u})a_{\log T - \log \mathbf{v} \star \mathbf{u}}) \frac{dT}{T}.$$

Here, there is a difference, however; the map τ might not be in $L^1(m_{PS})$. Theorem III.3 tells us that it happens exactly when $\delta > 2/3$, which is precisely the hypothesis we assumed. This concludes the sketch of proof of Theorem III.6. \square

III.3. Large scale $\alpha = 1$

We now turn our attention to the large scale $\alpha = 1$. As in the lattice case, there is no condition on initial vector \mathbf{u} , which can be outside the cone $\mathcal{C}(\Gamma_0)$. The presence or not of cusp(s) will also be irrelevant.

III.3.1. The size of the cocycle. The estimate $\|\gamma\| \simeq |c_{\mathbf{u}}(\gamma)|\mathbf{v} \star \mathbf{u}$ is not valid any more in the case $\alpha = 1$, but we still need to understand the implications on the size of the cocycle of the condition $\|\gamma\| \leq T$. Recall that by definition

$$\gamma = \Psi(\gamma\mathbf{u}/T) \begin{bmatrix} 1 & c_{T^{-1}\mathbf{u}}(\gamma) \\ 0 & 1 \end{bmatrix} \Psi(\mathbf{u}/T)^{-1},$$

and define $\mathbf{v} = \gamma\mathbf{u}/T$, which is expected to be in a fixed compact part of the plane. By the property $\Psi(e^t\mathbf{w}) = \Psi(\mathbf{w})a_{2t}$, the previous equation becomes

$$\gamma = \Psi(\mathbf{v}) \begin{bmatrix} 1 & c_{T^{-1}\mathbf{u}}(\gamma) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-1} \end{bmatrix} \Psi(\mathbf{u})^{-1},$$

which is

$$\gamma = \Psi(\mathbf{v}) \begin{bmatrix} T & T^{-1}c_{T^{-1}\mathbf{u}}(\gamma) \\ 0 & T^{-1} \end{bmatrix} \Psi(\mathbf{u})^{-1},$$

Here, \mathbf{v} and \mathbf{u} are in some compact part of $\mathbf{R}^2 - \{0\}$, so $\Psi(\mathbf{u}), \Psi(\mathbf{v})$ are bounded; the cocycle is expected to be large, of the order of T^2 , so

$$\gamma \simeq T\Psi(\mathbf{v}) \begin{bmatrix} 1 & T^{-2}c_{T^{-1}\mathbf{u}}(\gamma) \\ 0 & 0 \end{bmatrix} \Psi(\mathbf{u})^{-1},$$

therefore

$$(5) \quad \frac{\gamma}{T} \simeq \left(\Psi(\mathbf{v}) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Psi(\mathbf{u})^{-1} + \frac{c_{T^{-1}\mathbf{u}}(\gamma)}{T^2} \Psi(\mathbf{v}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Psi(\mathbf{u})^{-1} \right).$$

Recall the definition of κ given in Chapter 1, section I.4.2:

$$\kappa(\mathbf{u}, \mathbf{v}, s) = \left\| \frac{1}{|\mathbf{u}|^2} \begin{pmatrix} \mathbf{u}_x \mathbf{v}_x & \mathbf{u}_y \mathbf{v}_x \\ \mathbf{u}_x \mathbf{v}_y & \mathbf{u}_y \mathbf{v}_y \end{pmatrix} + s \begin{pmatrix} -\mathbf{u}_y \mathbf{v}_x & \mathbf{u}_x \mathbf{v}_x \\ -\mathbf{u}_y \mathbf{v}_y & \mathbf{u}_x \mathbf{v}_y \end{pmatrix} \right\|.$$

For $s = \frac{c_{T^{-1}\mathbf{u}}(\gamma)}{T^2}$, one can check that the matrices involved in Equation (5) are identical to the ones in the expression of κ . Therefore, the condition $\|\gamma\| \leq T$ is

satisfied (essentially) when the cocycle is in the interval $T^2[s_1, s_2]$, where $[s_1, s_2]$ is the interval on which $\kappa(\mathbf{u}, \mathbf{v}, s) \leq 1$ (it may be empty, in which case we will take $s_2 = s_1 = 0$). Define $\Theta(\mathbf{u}, \mathbf{v}) = (s_2 - s_1)/2$, the half-length of this interval, and $\Theta_m(\mathbf{u}, \mathbf{v}) = (s_2 + s_1)/2$ the middle.

III.3.2. The geometry. Like in the previous cases, it is now possible to relate the sum we are investigating, for a test function f of small support around \mathbf{v} , with an ergodic sum of a related function \bar{f} on $\Gamma \backslash G$. The integration interval is determined by the condition on the cocycle given by the requirement $\|\gamma\| \leq T$. The approximation is the following:

$$\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T}\right) \simeq \int_{-T^2\Theta(\mathbf{u}, \mathbf{v})}^{T^2\Theta(\mathbf{u}, \mathbf{v})} \bar{f}(\Gamma\Psi(\mathbf{u})n_{\Theta_m(\mathbf{u}, \mathbf{v})}a_{-2\log T}n_s)ds.$$

To estimate this integral, we will use the following result.

THEOREM III.7 (Roblin [39]). *Let S and τ be as in Theorem III.1. For any $v \in T^1S$, and all $f : T^1S \rightarrow \mathbf{R}$ continuous with compact support, we have*

$$\lim_{t \rightarrow +\infty} \frac{1}{t^\delta} \int_{-t}^t f(va_{-\log t}n_s)ds = \frac{\tau(v)}{m_{PS}(T^1S)} \int_{T^1S} f dm_{BR}.$$

Therefore, by Theorem III.7 applied to the time $t = T^2\Theta(\mathbf{u}, \mathbf{v})$,

$$\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T}\right) \simeq T^{2\delta}(\Theta(\mathbf{u}, \mathbf{v}))^\delta \frac{\tau(\Gamma\Psi(\mathbf{u})n_{\Theta_m(\mathbf{u}, \mathbf{v})}a_{\log \Theta(\mathbf{u}, \mathbf{v})})}{m_{PS}(T^1S)} \int_{T^1S} \bar{f} dm_{BR},$$

so

$$\sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T}\right) \simeq T^{2\delta}(\Theta(\mathbf{u}, \mathbf{v}))^\delta \frac{\tau(\Gamma\Psi(\mathbf{u})n_{\Theta_m(\mathbf{u}, \mathbf{v})}a_{\log \Theta(\mathbf{u}, \mathbf{v})})}{m_{PS}(T^1S)} \int_{\mathbf{R}^2} f d\bar{\mu}.$$

The previous estimate being valid for a function f of small support around \mathbf{v} . For a (general) continuous function f , we have to somehow 'integrate' over \mathbf{v} . Recall that $\mathcal{D}(\mathbf{u}) = \{\mathbf{v}; \Theta(\mathbf{u}, \mathbf{v}) > 0\}$ (we include 0 in this definition). The final result, which generalize Theorem I.3, is the following statement:

THEOREM III.8. [32, Theorem 1.10] *Let Γ_0 be a finitely generated, non-elementary subgroup of $\mathrm{SL}(2, \mathbf{R})$, containing $-I$ as the unique element of torsion. For all $\mathbf{u} \in \mathbf{R}^2 \setminus \{0\}$ and all continuous functions $f : \mathbf{R}^2 \rightarrow \mathbf{R}$, we have*

$$\begin{aligned} \lim_{T \rightarrow +\infty} \frac{1}{T^{2\delta}} \sum_{\gamma \in \Gamma_T} f\left(\frac{\gamma \mathbf{u}}{T}\right) = \\ \frac{2}{m_{PS}(T^1S)} \int_{\mathcal{D}(\mathbf{u})} \Theta(\mathbf{u}, \mathbf{v})^\delta \tau(\Gamma\Psi(\mathbf{u})n_{\Theta_m(\mathbf{u}, \mathbf{v})}a_{\log \Theta(\mathbf{u}, \mathbf{v})}) f(\mathbf{v}) d\bar{\mu}(\mathbf{v}), \end{aligned}$$

and the right-hand side is a finite integral, whose total mass does not depend of \mathbf{u} .

In [32, Theorem 1.10], there is an additional hypothesis that the norm is strictly convex. This is useful in the proof because it implies the continuity of Θ , but it can be removed by approximating the norm with a strictly convex one.

CHAPTER IV

Random orbits on the plane

IN probabilistic terms, the two previous chapters dealt with picking an element Γ at random in $\mathrm{SL}(2, \mathbf{R})$ following a uniform law in a ball of radius T , and looking at the law of the evaluation at $\mathbf{u} \in \mathbf{R}^2$, at a certain scale. This chapter is about what might happen if the element in question was chosen using a random walk.

IV.1. Heuristic and motivations

Let μ be a probability on $G = \mathrm{SL}(2, \mathbf{R})$. We will assume through the chapter (except in the examples) that μ is symmetric, and has a finite support that generates a non-elementary, convex-cocompact subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$. These hypotheses are suboptimal at several places and unnecessary in several results, but will allow a more transparent exposition.

IV.1.1. Interpretation of the previous results. Consider the random walk on G (beginning at e) generated by μ . The random walk at step n has law μ^n , the n -th convolution power of μ . Let $\alpha \in \mathbf{R}$ be a parameter, f a continuous test function on \mathbf{R}^2 , and $\mathbf{u} \in \mathbf{R}^2 - \{0\}$ an initial vector. What can we expect of the behavior of

$$\int_G f(e^{-\alpha n} \gamma \mathbf{u}) d\mu^n(\gamma) ?$$

Interpreting the results of the previous chapters for $T = e^n$ and the uniform measure u_n on Γ_T , where in this case $\alpha \in (-1, 1)$,

- (1) The above quantity was exponentially small, of the order of $T^{\alpha-1} = e^{(\alpha-1)n}$. Write $\beta = \alpha - 1$.

- (2) When renormalized by this exponent, there was some convergence

$$e^{\beta n} \int_G \mathcal{D}_{e^{-\alpha n} \gamma \mathbf{u}} du_n(\gamma) \rightarrow \frac{d\bar{\mu}}{(\mathbf{v} \star \mathbf{u})^\delta},$$

where $\mathcal{D}_{\mathbf{v}}$ stands for a Dirac mass at a point \mathbf{v} (often under some additional conditions). The convergence could be a genuine convergence for the weak-* topology on measures, or in a weaker sense (log-Cesaro average).

- (3) The limit measure obtained $\frac{d\bar{\mu}}{(\mathbf{v} \star \mathbf{u})^\delta}$ was homogeneous of degree δ .
- (4) The above convergence occurred only if some hypothesis on the initial vector \mathbf{u} satisfied some conditions. The smaller the parameter α , the more restrictive the condition were - diophantine condition in the case $\text{SL}(2, \mathbf{Z})$, being in $\mathcal{C}(\Gamma)$ for convex-cocompact Γ whereas the large-scale result did not care for that hypothesis, and so on.

IV.1.2. What if.. ? For the purpose of a heuristic calculation, assume the situation for a random walk is similar. More precisely, assume that there exist an exponent β (depending on α), and an infinite, Radon measure $\bar{\nu}$ on $\mathbf{R}^2 - \{0\}$, homogeneous of degree d for some real number d , such that

$$e^{\beta n} \int_G f(e^{-\alpha n} \gamma \mathbf{u}) d\mu^n(\gamma) \rightarrow \int_{\mathbf{R}^2} f d\bar{\nu}.$$

While we will not prove any result of this kind, our goal in this chapter will be to guess what $\beta, \bar{\nu}$ and d could be, and produce reasonable candidates.

So, let us find what necessary conditions this assumption implies. We would have ¹

$$\int_{\mathbf{R}^2} f d(\mu \star \bar{\nu}) = \int_G \int_{\mathbf{R}^2} f(\gamma \mathbf{v}) d\mu(\gamma) d\bar{\nu}(\mathbf{v}) \simeq \int_G \left(e^{\beta n} \int_G f(e^{-\alpha n} \gamma \gamma' \mathbf{u}) d\mu^n(\gamma') \right) d\mu(\gamma),$$

$$\int_{\mathbf{R}^2} f d(\mu \star \bar{\nu}) \simeq e^{-\beta} e^{\beta(n+1)} \int_G f(e^\alpha e^{-\alpha(n+1)} \gamma'' \mathbf{u}) d\mu^{n+1}(\gamma'') \simeq e^{-d\alpha-\beta} \int_{\mathbf{R}^2} f d\bar{\nu}.$$

Put $\lambda = e^{-d\alpha-\beta}$. Such a measure $\bar{\nu}$ would thus have to satisfy the convolution equation

$$(6) \quad \mu \star \bar{\nu} = \lambda \bar{\nu}.$$

Assume moreover that $\bar{\nu}$ is symmetric around zero, or equivalently, that the equation of homogeneity $\bar{\nu}(t.E) = t^d \bar{\nu}(E)$ is also valid for $t < 0$. Then $\bar{\nu}$ can be recovered from the data of a (finite) measure ν on \mathbb{P}^1 and the exponent of homogeneity d , in polar coordinates

$$d\bar{\nu}(r, \theta) = |r|^{d-1} dr d\nu(\theta).$$

We would like to translate Equation (6) into a condition on ν , which should be simpler to solve/understand, because ν is a finite measure on a compact space.

¹This heuristic computation can also be done with an additional (and probably more realistic) multiplicative polynomial term like $n^{-3/2}$ to the assumed asymptotic.

IV.1.3. The twisted convolution operator P_s . For $\gamma \in G$ and $\xi \in \mathbb{P}^1$, we will denote by

$$\sigma(\gamma, \xi) = \frac{|\gamma \mathbf{v}|}{|\mathbf{v}|} = e^{\frac{1}{2}\beta_\xi(\gamma^{-1}o, o)} = |\gamma'(\xi)|_o^{-1/2},$$

where $\mathbf{v} \in \mathbf{R}^2 - \{0\}$ is any representative of ξ , the exponential of half the Busemann cocycle, or equivalently the inverse of the square root of the derivative of γ at ξ with respect to the $\text{SO}(2)$ -invariant measure. It satisfies the cocycle identity

$$\sigma(\gamma\gamma', \xi) = \sigma(\gamma, \gamma'\xi) \cdot \sigma(\gamma', \xi).$$

For a parameter $s \in \mathbf{R}$, introduce the operator on $C^0(\mathbb{P}^1)$,

$$P_{s,\mu}(f)(\xi) = \int_G \sigma(\gamma, \xi)^s f(\gamma\xi) d\mu(\gamma).$$

Its dual acts on finite (signed) measures, viewed as the dual of $C^0(\mathbb{P}^1)$,

$$P_{s,\mu}^*(\nu)(f) = \int_G \int_{\mathbb{P}^1} \sigma(\gamma, \xi)^s f(\gamma\xi) d\mu(\gamma) d\nu(\xi).$$

From the cocycle identity, one has $P_{s,\mu}^n = P_{s,\mu^n}$. We will now omit the μ in the notation P_s . The link between Equation (6) and this operator is the following.

LEMMA IV.1. *Let ν be a finite measure on \mathbb{P}^1 , and $\bar{\nu}$ the measure on $\mathbf{R}^2 - \{0\}$ defined by $d\bar{\nu} = |r|^{d-1} dr d\nu(\theta)$, for some $d \in \mathbf{R}$. Then, for any $\lambda > 0$, the following are equivalent.*

(1)

$$P_{-d}^*(\nu) = \lambda\nu,$$

(2)

$$\mu * \bar{\nu} = \lambda\bar{\nu}.$$

PROOF. The radial integration $C_c^0(\mathbf{R}^2 - \{0\}) \rightarrow C^0(\mathbb{P}^1)$ defined by

$$K(f)(\theta) = \int_{\mathbf{R}^*} f(r, \theta) |r|^{d-1} dr,$$

satisfies $\nu(K(f)) = \bar{\nu}(f)$, but also

$$K(f \circ \gamma)(\theta) = \int_{\mathbf{R}^*} f(r\sigma(\gamma, \theta), \gamma\theta) |r|^{d-1} dr = \sigma(\gamma, \theta)^{-d} \int_{\mathbf{R}^*} f(r, \gamma\theta) |r|^{d-1} dr,$$

so $K(f \circ \gamma)(\theta) = \sigma(\gamma, \theta)^{-d} K(f)(\gamma\theta)$. A short computation shows this operator satisfies

$$P_{-d}^*(\nu)(K(f)) = \mu * \bar{\nu}(f).$$

Then one notices that K is surjective to conclude. \square

IV.1.4. Recipe for the computation of d, β . From the above analysis, our candidate $\bar{\nu}$ should be constructed using an eigenmeasure ν of an operator P_s^* , but with s unknown (equal to $-d$, but we don't know d a priori, only the parameter α).

Let us conclude this heuristic with a recipe on how to guess our candidates for $d, \beta, \bar{\nu}$ from a good knowledge of the spectral properties of P_s . We won't explain why, but it should give further motivation for the study of the operators P_s and P_s^* as a family (and not individually).

Assume we know a 'good' family of eigenvalues $e^{k_0(s)}$ of P_s^* , with eigenmeasures ν_s , and that $s \mapsto k_0(s)$ is convex. In good cases, there is a unique s such that $k_0'(s) = \alpha$. The candidates are $d = -s$, $\nu = \nu_s$, and $\beta = s\alpha - k(s)$.

IV.2. Random walk setup

In this section we recall some basic facts about random walks on groups.

IV.2.1. Random walks on Γ . For f a real-valued function on Γ , define the random walk operator on $T_\mu(f) : \Gamma \rightarrow \mathbf{R}$ by

$$T_\mu(f)(g) = \int_{\Gamma} f(\gamma^{-1}g) d\mu(\gamma).$$

The *spectral radius* of the random walk is by definition the spectral radius of T_μ acting on $\ell^2(\Gamma)$, and is denoted by $\rho(\mu)$. Since μ is symmetric, it is related to the probability of return to e by a Theorem of Kesten:

$$\rho(\mu) = \limsup_n (\mu^n(e))^{1/n}.$$

Since Γ is non-amenable, $\rho(\mu) < 1$, by another Theorem of Kesten.

A function f on Γ is said to be λ -harmonic if $T_\mu f = \lambda f$ for all $\gamma \in \Gamma$. Let $c(\gamma, \xi)$ be a multiplicative cocycle on $\Gamma \times X$, where X is a Γ -space. Notice that, thanks to the cocycle property, for the family $(c(\cdot, \xi))_{\xi \in X}$ of functions of $\ell^2(\Gamma)$ to be λ -harmonic is equivalent to be λ -harmonic only at the point e , in the sense that

$$\forall \xi, \int_G c(\gamma, \xi) d\mu(\gamma) = \lambda.$$

IV.2.2. The Green function. The *Green function* is defined for $x, y \in \Gamma$, and $z \in \mathbf{C}$, $|z| \leq 1/\rho(\mu)$, by

$$G_z(x, y) = \sum_{n \geq 0} \mu^n(x^{-1}y) z^n.$$

The radius of convergence of G_z as a power series in z is precisely $1/\rho(\mu)$, by the characterisation of $\rho(\mu)$. Note that $G_z(\gamma x, \gamma y) = G_z(x, y)$.

We will also need the fact that for all $0 < z < 1/\rho(\mu)$, the map $\gamma \mapsto G_z(e, \gamma)$ is in $\ell^2(\Gamma)$; indeed, let $R_z = \sum_{n \geq 0} T_\mu^n z^n$ be the resolvent, we have

$$G_z(e, \gamma) = \langle R_z \mathcal{D}_e, \mathcal{D}_\gamma \rangle,$$

where \mathcal{D}_γ is the indicatrix function of $\{\gamma\}$. Therefore,

$$\sum_{\gamma \in \Gamma} G_z(e, \gamma)^2 = \|R_z \mathcal{D}_e\|^2 < +\infty.$$

IV.2.3. The Martin Kernel. For $\xi \in \Lambda_\Gamma$ the limit set of Γ , $z \in (0, 1/\rho(\mu))$, and $\gamma \in \Gamma$, the Martin Kernel is by definition

$$K_z(\gamma, \xi) = \lim_{\gamma' \rightarrow \xi} \frac{G_z(\gamma, \gamma')}{G_z(e, \gamma')}.$$

The fact that the above limit exists (and is finite) is a nontrivial fact due to Ancona [3], which reflects that the Martin boundary of an hyperbolic group identifies with its geometrical boundary. The map $(\gamma, \xi) \mapsto K_z(\gamma^{-1}, \xi)$ is a multiplicative cocycle.

It is well known (and not too difficult to check) that the Martin Kernel is $1/z$ -harmonic:

$$\int_{\Gamma} K_z(\gamma, \xi) d\mu(\gamma) = \frac{1}{z}.$$

IV.3. The real functions k, k^+, k_η

IV.3.1. Definition and first properties. Put

$$k^+(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sup_{\xi \in X} \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma) \right),$$

$$k(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int_G \|\gamma\|^s d\mu^n(\gamma).$$

where $\|\cdot\|$ is the operator norm associated to the euclidean norm on \mathbf{R}^2 . This choice is practical because of the equality $2 \log \|\gamma\|$ equal the hyperbolic distance between o and γo .

Let $X = \mathbb{P}^1$. For $\nu \in \mathcal{M}_1(X)$ a probability measure on X , let

$$k_\nu(s) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\int_{\mathbb{P}^1} \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma) d\nu(\xi) \right) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu(P_s^n(1)).$$

Let us first mention that, for $s \geq 0$, this three functions are equal, analytic, strictly increasing, convex, the operator P_s has a spectral gap, a unique positive

eigenfunction, and P_s^* has a unique probability eigenmeasure of eigenvalue $e^{k(s)}$. Moreover, the derivative of k at zero is equal to the Lyapunov exponent of the random walk; all these results are proved in [22]. From now on, we will limit our study to $s < 0$.

We first prove some properties of k^+ and relate it with the spectral radius of P_s and P_s^* . We endow the space of continuous functions of X with the supremum norm $|\cdot|_\infty$, and the space $\mathcal{M}(X)$ of signed measures on X with the total mass $\|\cdot\|_{\mathcal{M}(X)}$.

PROPOSITION IV.2. *Let $\eta \in \mathcal{M}_1(X)$, and $s \leq 0$. Then*

- (1) *The limit in the definition of $k^+(s)$ exists and is finite.*
- (2) *The spectral radius of P_s and P_s^* are equal to $e^{k^+(s)}$.*
- (3) *The maps $s \mapsto k(s)$, $s \mapsto k_\eta(s)$, $s \mapsto k^+(s)$ are convex.*
- (4) *The limit in the definition of $k(s)$ exists and is finite.*
- (5) *The map k is non-decreasing.*
- (6) *We have*

$$k(s) \leq k_\eta(s) \leq k^+(s).$$

PROOF. Let

$$a_n(s) = \sup_{\xi \in X} \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma) = |P_s^n 1|_\infty \leq \|P_s^n\|.$$

For any function f ,

$$|P_s^n f|_\infty = \sup_{\xi \in X} \int_G \sigma(\gamma, \xi)^s f(\gamma\xi) d\mu^n(\gamma) \leq |f|_\infty \sup_{\xi \in X} \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma) = a_n(s),$$

so we conclude that $a_n(s) = \|P_s^n\| = \|(P_s^*)^n\|$, since the total mass norm is the dual norm of the sup norm. The definition of the spectral radius then implies (1) and (2). Concerning the convexity of k^+ the maps

$$a_{n,\xi}(s) = \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma),$$

are probability integrals of log-convex functions, so $s \mapsto \log a_{n,\xi}(s)$ is convex. Then $\log a_n(s) = \sup_\xi \log a_{n,\xi}(s)$ is the supremum of convex functions, so is also convex. Since $k^+(s)$ is a limit of convex functions, it is also convex.

Let $b_n(s) = \log \int_G \|\gamma\|^s d\mu^n(\gamma)$, this is a convex function. For $s \leq 0$, we have

$$b_{n+m}(s) = \log \int_{G \times G} \|\gamma_1 \gamma_2\|^s d\mu^n(\gamma_1) d\mu^m(\gamma_2) \geq b_n(s) + b_m(s),$$

so the sequence $b_n(s)/n$ converges because of superadditivity and

$$k(s) = \lim_{n \rightarrow +\infty} \frac{b_n(s)}{n} = \sup \frac{b_n(s)}{n} \in \mathbf{R} \cup \{+\infty\}.$$

For $\xi \in X$, we have

$$\sigma(\gamma, \xi)^s \geq \|\gamma\|^s,$$

so integrating over γ with respect to μ^n , we obtain

$$\int_G \|\gamma\|^s d\mu^n(\gamma) \leq \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma) \leq \sup_{x \in X} \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma),$$

integrating ξ over η , taking the logarithm and limsup proves the point (6). This implies that $k(s)$ is finite for all $s \leq 0$, point (4).

The statement (3) follows from the fact that the limsup of convex functions is still convex. The map $s \mapsto \|\gamma\|^s$ are non-decreasing, so are the functions $b_n(s)$. This implies (5). □

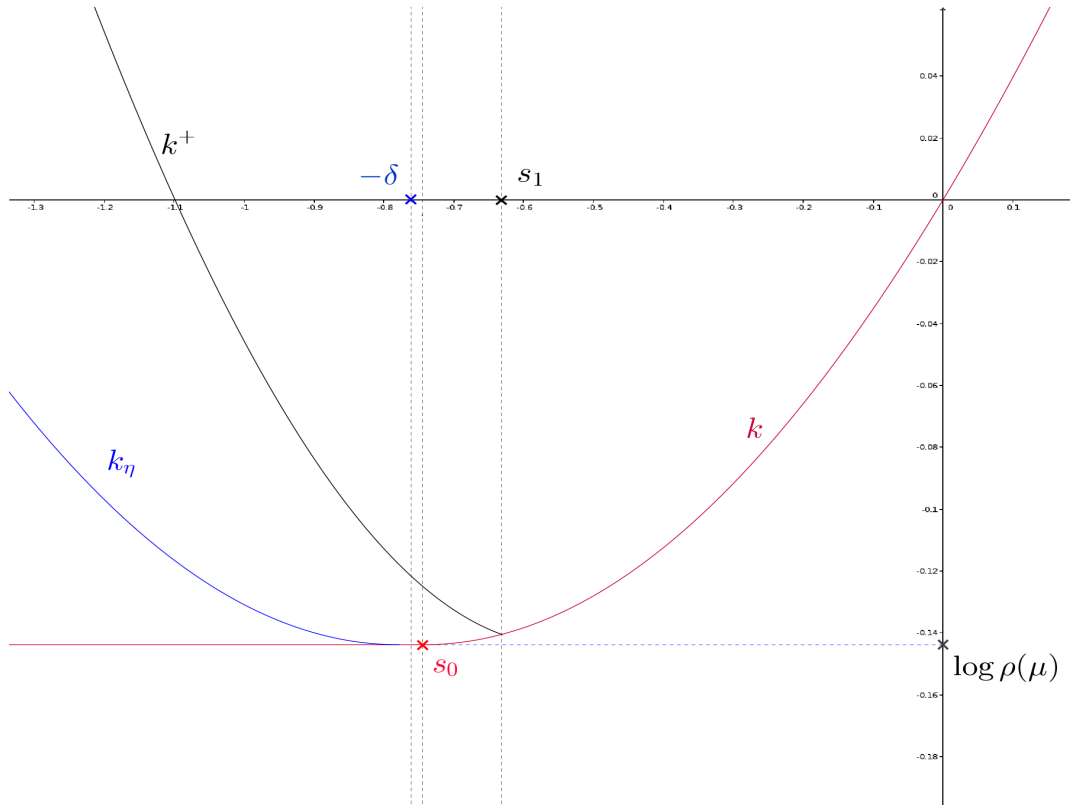


FIGURE 1. The functions k (red), k^+ (black then red), k_η for the Patterson Measure (blue then red), for the parameter $\ell = 2$ in the free group example

IV.3.2. The Green-Poincaré Series. We define the *Green-Poincaré series* for $s \in \mathbf{R}$ and $z \in [0, 1/\rho(\mu)]$, by

$$GP(s, z) = \sum_{\gamma \in \Gamma} \|\gamma\|^s G_z(e, \gamma) = \sum_{\gamma \in \Gamma} e^{\frac{s}{2}d(o, \gamma o)} G_z(e, \gamma).$$

It can be written also as a power series of z :

$$GP(s, z) = \sum_{n \geq 0} \left(\int_G \|\gamma\|^s d\mu^n(\gamma) \right) z^n.$$

The Cauchy-Hadamard formula for the radius of convergence implies that $GP(s, z)$ converges if $z < e^{-k(s)}$ and diverges if $z > e^{-k(s)}$.

IV.3.3. The function k for small s . The function k turns out to be constant on an interval $(-\infty, s_0]$, where $s_0 \in [-\delta, 0]$ is a priori unknown.

PROPOSITION IV.3. *For all $s < -\delta$, where δ is the critical exponent of Γ , we have $k(s) = \log \rho(\mu)$.*

PROOF. Notice first that

$$k(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int_G \|\gamma\|^s d\mu^n(\gamma) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu^n(e) \geq \log \rho(\mu).$$

Since the Green-Poincaré series diverges for $z > e^{-k(s)}$, it is sufficient to show that it converges for $z < 1/\rho(\mu)$. Indeed, it implies that $k(s) \leq \log \rho(\mu)$, and is therefore equal to $\log \rho(\mu)$.

By the Cauchy-Schwarz inequality,

$$GP(s, z) \leq \left(\sum_{\gamma \in \Gamma} \|\gamma\|^{2s} \right)^{1/2} \left(\sum_{\gamma \in \Gamma} G_z(e, \gamma)^2 \right)^{1/2}.$$

Recall that the relationship between the norm and the hyperbolic distance is given by the formula $\|\gamma\| = e^{\frac{1}{2}d(o, \gamma o)}$. The first expression in the right-hand side of the previous equation is thus the classical Poincaré series, which converges because $s < -\delta$, by definition of the critical exponent. The second expression is also finite, because we have shown that $G_z(e, \cdot) \in \ell^2(\Gamma)$. □

IV.3.4. Symmetry formula for k_η . Recall that η is said to be a Γ -conformal density of dimension κ if for any $\gamma \in \Gamma$ and $\xi \in \mathbb{P}^1$,

$$\frac{d\gamma_*\eta}{d\eta}(\xi) = \sigma(\gamma^{-1}, \xi)^{-2\kappa}.$$

Two examples are the $SO(2)$ -invariant probability \mathcal{L} on \mathbb{P}^1 , of dimension 1, and Patterson measures ν_o for the origin o , which is of dimension δ .

PROPOSITION IV.4. *Assume that η is a conformal density of dimension κ . Then*

$$k_\eta(s) = k_\eta(-s - 2\kappa),$$

that is, the graph of k_η is symmetric with respect to $x = -\kappa$.

PROOF. We have

$$\eta(P_s^n(1)) = \int_{\mathbb{P}^1} \int_G \sigma(\gamma, \xi)^s d\mu^n(\gamma) d\eta(\xi).$$

Noting that $\sigma(\gamma, \xi)\sigma(\gamma^{-1}, \gamma\xi) = 1$, the change of variables $\xi' = \gamma\xi$ transforms this equation into:

$$\eta(P_s^n(1)) = \int_G \int_{\mathbb{P}^1} \sigma(\gamma^{-1}, \xi')^{-s} d\gamma_*\eta(\xi') d\mu^n(\gamma).$$

Using the hypothesis on the Radon-Nikodym derivative of $\gamma_*\eta$, and the symmetry of μ , that implies:

$$\eta(P_s^n(1)) = \eta(P_{-s-2\kappa}^n(1)).$$

Taking n -th root and the limit as n tends to infinity proves the claim. \square

IV.3.5. The equality $k_\eta = k$ for large enough s . If one has some estimate on the lower dimension of the measure η , one can deduce k_η from the knowledge of k for s larger than this dimension. More precisely, define the property (ULD_ϵ) (for uniform lower dimension) by the condition

$$\begin{aligned} & \exists C > 0, \text{ such that} \\ (ULD_\epsilon) \quad & \forall \xi \in \mathbb{P}^1, \forall r \in (0, 1], \\ & \eta(B(\xi, r)) \leq Cr^\epsilon \end{aligned}$$

where the distance considered on \mathbb{P}^1 is the sinus of the angle distance, denoted by $|\xi \wedge \xi'|$, since it's equal to the norm of the wedge product between unitary representatives of ξ, ξ' . Recall that if ν has (ULD_ϵ) then for all ϵ' such that $0 \leq \epsilon' < \epsilon$, there exists $M(\epsilon')$ such that for all $\xi \in \mathbb{P}^1$,

$$\int_{\mathbb{P}^1} |\xi \wedge x|^{-\epsilon'} d\eta(x) \leq M(\epsilon').$$

PROPOSITION IV.5. *If η satisfies (ULD_ϵ) , then*

$$\forall s \geq -\epsilon, k_\eta(s) = k(s).$$

Examples include the Lebesgue measure on \mathbb{P}^1 , which satisfies (ULD_1) , and the Patterson density, which satisfies (ULD_δ) . In particular, thanks to the symmetry formula, one can deduce the function k_η for all s from the knowledge of k , when η is one of these two measures.

PROOF. The proposition follows immediately from the following inequality: for all $s > -\epsilon$, there exist $C_1, C_2 > 0$ such that for any $\gamma \in G$,

$$(7) \quad C_1 \|\gamma\|^s \leq \int_{\mathbb{P}^1} \sigma(\gamma, \xi)^s d\eta(\xi) \leq C_2 \|\gamma\|^s,$$

The left inequality is a straightforward consequence of $\sigma(\gamma, \xi) \leq \|\gamma\|$. Let us prove the right one. Since ${}^t\gamma\gamma$ is symmetric, up to a change of orthonormal basis, we have

$${}^t\gamma\gamma = \begin{pmatrix} \|\gamma\|^2 & 0 \\ 0 & \|\gamma\|^{-2} \end{pmatrix},$$

Note that the change of orthonormal basis does not change the distance on \mathbb{P}^1 , and thus the constants given by the inequality (ULD_ϵ). We parametrize points x in \mathbb{P}^1 by an angle $\theta(x) \in [-\pi/2, \pi/2[$, more precisely

$$x(\theta) = \pm \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$

Let

$$A = \{x \in \mathbb{P}^1, |\theta(x)| \leq \|\gamma\|^{-2}\},$$

$$B = \{x \in \mathbb{P}^1, |\theta(x)| > \|\gamma\|^{-2}\}.$$

We shall find upper bounds for the integrals over A and B . First notice that

$$\sigma(\gamma, x)^2 = \langle \gamma x, \gamma x \rangle = \langle {}^t\gamma\gamma x, x \rangle = \|\gamma\|^2 \sin^2 \theta + \|\gamma\|^{-2} \cos^2 \theta,$$

and if $y_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\sigma(\gamma, x)^2 \geq \|\gamma\|^2 |x \wedge y_0|^2,$$

so

$$\int_B \sigma(\gamma, x)^s d\nu(x) \leq \|\gamma\|^s \int_B |x \wedge y_0|^s d\eta(x) \leq M(-s) \|\gamma\|^s.$$

Now, note that since $\|\gamma\| \geq 1$, on A one has $\cos(\theta) \geq \cos(1)$, so for all $x \in A$,

$$\sigma(\gamma, x)^2 \geq \|\gamma\|^{-2} \cos(1)^2,$$

thus

$$\int_A |\gamma x|^s d\nu(x) \leq \cos(1)^s \|\gamma\|^{-s} \eta(A) \leq \cos(1)^s \|\gamma\|^{-s} C \|\gamma\|^{-2\epsilon}.$$

Recall that $s > -\epsilon$, so $-s - 2\epsilon \leq s$ and

$$\int_A \sigma(\gamma, x)^s d\eta(x) \leq C \cos(1)^s \|\gamma\|^s. \quad \square$$

IV.4. Construction of eigenmeasures

IV.4.1. An abstract result on the existence of eigenmeasures. By the Schauder fixed point Theorem, the operator P_s^* has a least one eigenvalue. We can actually say a little more.

The following Theorem implies that for any probability measure η , both $e^{k^+(s)}$ and $e^{k_\eta(s)}$ are eigenvalues of P_s^* . I was unable to find a reference about this result, although I doubt it was unknown. It is not really a generalisation of the Perron-Frobenius Theorem, because no contraction is required - it applies even to the identity map.

We write $\mathcal{M}(X)$ the Banach Space of finite signed Radon measures on a compact space X , endowed with the total mass norm: if $\nu = \nu^+ - \nu^-$, where ν^+, ν^- are non-negative measures, singular to each other, $\|\nu\| = \nu^+(X) + \nu^-(X)$. We write ν^+ and ν^- the unique measures in the above Jordan decomposition. Call a continuous operator A on $\mathcal{M}(X)$ non-negative if for all non-negative measure ν , $A\nu$ is also non-negative. This implies that $(A\nu)^+ \leq A(\nu^+)$ for any ν .

THEOREM IV.6. *Let X be a compact space, and $A : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ a continuous (for both weak-* and strong topologies) non-negative operator on signed measures. Then*

- (1) *The spectral radius of A is an eigenvalue of A , with eigenvector a probability.*
- (2) *For ν any probability then $\limsup \|A^n \nu\|^{1/n}$ is a eigenvalue of A , with eigenvector a probability.*

The following Lemma, whose proof is a simple exercise, will be useful.

LEMMA IV.7. *Let $(b_n)_{n \geq 0}$ be a sequence of positive numbers such that $\limsup b_n^{1/n} = 1$. Then*

$$\liminf_{n \rightarrow +\infty} \frac{b_n}{\sum_{k=0}^{n-1} b_k} = 0,$$

and if $\sum_n b_n$ is finite, then

$$\liminf_{n \rightarrow +\infty} \frac{b_n}{\sum_{k \geq n+1} b_k} = 0.$$

PROOF. We first prove (1). Without loss of generality, we can assume that the spectral radius of A is equal to 1. Let $\Sigma_N = \sum_{k=0}^{N-1} A^k$. Note that

$$A\Sigma_N = \Sigma_N + A^N - Id,$$

If one can find $(\nu_n) \in \mathcal{M}(X)$ probabilities satisfying for some subsequence $(n_m)_{m \geq 0}$ both

- $\lim_{m \rightarrow +\infty} \|\Sigma_{n_m}(\nu_{n_m})\| = +\infty,$

$$\bullet \lim_{m \rightarrow +\infty} \frac{\|A^{nm}(\nu_{nm})\|}{\|\Sigma_{nm}(\nu_{nm})\|} = 0$$

Then any weak limit ν of $\Sigma_{n_m}(\nu_{n_m})/\|\Sigma_{n_m}(\nu_{n_m})\|$ satisfies $A\nu = \nu$, thus proving (1).

We will show that the preceding criterion is satisfied in two separate cases, whether the family $(A^n)_n$ is bounded, or not.

(*) First assume that (A^n) is uniformly bounded, that is $\|A^n\| \leq K$ for some $K > 0$ independent of n .

Let ν_n be signed, finite measures of total mass 1 such that

$$\|A^n(\nu_n)\| \geq \frac{1}{2}\|A^n\| \geq \frac{1}{2},$$

this last inequality being justified by the fact that the spectral radius is the infimum of $\|A^n\|^{1/n}$.

We can and will assume that ν_n is non-negative, ie a probability. Indeed, the probability $\nu_n^+ + \nu_n^-$ satisfies the same inequality, because A is non-negative.

Then

$$1/2 \leq \|A^n(\nu_n)\| \leq \|A^k\| \cdot \|A^{n-k}(\nu_n)\| \leq K \|A^{n-k}(\nu_n)\|,$$

so

$$\|\Sigma_n(\nu_n)\| = \sum_{k=0}^{n-1} \|A^{n-k}(\nu_n)\| \geq \frac{n}{2K}.$$

The second point is as follows

$$\frac{\|A^n(\nu_n)\|}{\|\Sigma_n(\nu_n)\|} \leq \frac{2K^2}{n}.$$

(*) Now we assume that (A^n) is not bounded, that is $\sup_n \|A^n\| = +\infty$. By the Banach-Steinhaus Theorem, there exists some signed finite measure ν such that

$$\limsup_n \|A^n(\nu)\| = +\infty.$$

We can again reduce to the case where ν is a probability, using the positivity of A .

Since $\|\Sigma(\nu)\| = \sum_{k=0}^{n-1} \|A^k(\nu)\|$, one has $\lim_{n \rightarrow +\infty} \|\Sigma(\nu)\| = +\infty$. Let $b_n = \|A^n(\nu)\|$. Then $\limsup b_n^{1/n} = 1$, so by Lemma IV.7, one has

$$\liminf \frac{b_n}{\sum_{k=0}^{n-1} b_k} = 0.$$

So, for some subsequence n_k , one has $\lim_{k \rightarrow +\infty} \frac{\|A^{n_k}(\nu)\|}{\|\Sigma_{n_k}(\nu)\|} = 0$.

This finishes the proof of (1). Now, let ν_0 be any probability measure, and define $\lambda = \limsup \|A^n \nu_0\|^{1/n}$. First assume that $\lambda = 0$. If there is a $n \geq 0$ such that $A^n \nu_0 = 0$, then there is nothing to prove. Otherwise, $\liminf \frac{A(A^n \nu_0)(1)}{A^n \nu_0(1)} = 0$, so on the set $\mathcal{M}_1(X)$ of probabilities, the infimum of the continuous map $\nu \mapsto A\nu(1)$ is zero. By weak compactness of $\mathcal{M}_1(X)$, there is a probability ν such that $A\nu = 0$.

We now consider the case $\lambda > 0$. Put $B = A/\lambda$, and define now $S_N = \sum_{k=0}^{N-1} B^k$. Write $b_k = \|B^k \nu_0\|$. Then

$$BS_N = S_N + B^N - I.$$

First case: $\|S_N \nu_0\| = \sum_{k=0}^{N-1} b_k$ goes to infinity with N . By the previous lemma, there exists a subsequence n_k such that $b_{n_k}/\|S_{n_k} \nu_0\|$ goes to zero as k goes to infinity. Then any weak limit ν of $\frac{S_{n_k} \nu_0}{\|S_{n_k} \nu_0\|}$ satisfies $B\nu = \nu$, as required.

Second case: $\|S_N \nu_0\|$ is bounded. Put $R_N = \sum_{k \geq N+1} B^k$, then $R_N \nu_0$ is a finite measure. By Lemma IV.7, there is a subsequence n_k such that $b_{n_k}/\|R_{n_k-1} \nu_0\|$ goes to zero as k goes to infinity. Since

$$B \frac{R_{N-1} \nu_0}{\|R_{N-1} \nu_0\|} = \frac{R_{N-1} \nu_0}{\|R_{N-1} \nu_0\|} - \frac{B^N \nu_0}{\|R_{N-1} \nu_0\|},$$

any weak limit ν of the probabilities $\frac{R_{n_k-1} \nu_0}{\|R_{n_k-1} \nu_0\|}$ satisfies $A\nu = \lambda\nu$. \square

IV.4.2. Characterisation of eigenmeasures in terms of harmonicity.

One can translate the condition to be an eigenmeasure into a condition on Radon-Nikodym derivatives.

PROPOSITION IV.8. *Let ν be a probability on X . Then ν is an eigenmeasure for P_s^* of eigenvalue λ if and only if for every $\gamma \in \Gamma$, $\gamma_* \nu$ is absolutely continuous with respect to ν , and for ν -almost every ξ ,*

$$F_\xi : \gamma \mapsto \sigma(\gamma, \xi)^{-s} \frac{d\gamma_*^{-1} \nu}{d\nu}(\xi)$$

is λ -harmonic.

PROOF. Let ν a probability such that $P_s^* \nu = \lambda\nu$, and $\gamma \in \Gamma$. Let $n \geq 1$ be such that $\mu^n(\gamma) > 0$, then for any non-negative $f \in C^0(\mathbb{P}^1)$,

$$\lambda^n \nu(f) = (P_s^*)(\nu)(f) \mu(g) \geq \mu^n(\gamma) \int_{\mathbb{P}^1} f(\gamma\xi) \sigma(\gamma, \xi)^s d\nu(\xi),$$

and therefore $\gamma_* \nu$ is absolutely continuous with respect to ν .

Now let ν such that $\gamma_*\nu$ is absolutely continuous with respect to ν for μ -almost every γ . Then, for any f continuous on X ,

$$\begin{aligned} P_s^*\nu(f) &= \int_{\mathbb{P}^1} \int_G f(\gamma\xi) \sigma(\gamma, \xi)^s d\mu(\gamma) d\nu(\xi) \\ &= \int_G \left(\int_{\mathbb{P}^1} f(\xi') \sigma(\gamma, \gamma^{-1}\xi')^s \frac{d\gamma_*\nu}{d\nu}(\xi') d\nu(\xi') \right) d\mu(\gamma) \\ &= \int_{\mathbb{P}^1} f(\xi') \left(\int_G \sigma(\gamma^{-1}, \xi')^{-s} \frac{d\gamma_*\nu}{d\nu}(\xi') d\mu(\gamma) \right) d\nu(\xi'). \end{aligned}$$

This proves that ν is an eigenmeasure of P_s^* of eigenvalue λ if and only if, for ν -almost every ξ ,

$$\int_G F_\xi(\gamma^{-1}) d\mu(\gamma) = \lambda = \lambda F_\xi(e),$$

that is, λ -harmonicity at the point e .

As noted before, this is equivalent to the λ -harmonicity for a cocycle. \square

IV.5. Green-Poincaré Series and (Γ, μ) -conformal measures

The criterion of Proposition IV.8 gives us another way to find eigenmeasures for the operators P_s^* . Indeed, we already know λ -harmonic functions, namely the Martin kernel. This motivates the following definition.

DEFINITION IV.9. *Let ν be a finite measure on X . ν is said to be (Γ, μ) -conformal of parameters (s, z) if for any $\gamma \in \Gamma$, $\gamma_*\nu$ is absolutely continuous with respect to ν and*

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = \sigma(\gamma^{-1}, \xi)^s K_z(\gamma, \xi).$$

Such measures exist, by the following Theorem.

THEOREM IV.10. *Let $z \in (1, 1/\rho(\mu))$, and s be the unique solution to $e^{-k(s)} = z$. Then there exists a (Γ, μ) -conformal measure of parameter (s, z) , and it is an eigenvector for P_s^* of eigenvalue $e^{k(s)}$.*

The proof follows closely the construction of Patterson [36]. Such an approach is also present in the work of Yue [44], where he constructs the stationary measure this way, ie the case $s = 0$. In this case, we have a priori no clue if the Green-Poincaré series diverges or not on the critical curve $z = e^{-k(s)}$, so Patterson's weight function will be useful.

IV.5.1. Patterson's weight function. Let $1 < z < 1/\rho(\mu)$ be fixed, and $s(z)$ be the unique s such that $e^{-k(s)} = z$. The Green-Poincaré series converges for $s < s(z)$ and diverges for $s > s(z)$.

For $0 \leq t_1 < t_2$, we consider the annulus

$$A_{t_1, t_2} = \{\gamma \in \Gamma; t_1 \leq d(o, \gamma o) < t_2\}.$$

Like in [36], we define inductively an increasing map $\tau : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, by defining it on a sequence of intervals $[t_{n-1}, t_n]$, such that its logarithm has slow variations. Let $t_0 = 0$, and $\tau(0) = 0$. Assume τ has been defined up to time t_{n-1} , as an increasing function.

We define t_n as the infimum of the times $t \geq t_{n-1} + 1$ such that

$$\sum_{\gamma \in A_{t_{n-1}, t}} e^{(s(z)/2+1/n)d(o, \gamma o)} G_z(e, \gamma) \geq e^{t_{n-1}/n} / \tau(t_{n-1}).$$

Such a t exists because the Green-Poincaré Series diverges for $s > s(z)$. On the interval $(t_{n-1}, t_n]$, define

$$\tau(t) = \tau(t_{n-1}) e^{(t-t_{n-1})/n}.$$

Obviously, τ is still increasing on $[t_{n-1}, t_n]$. Since $t_n \geq t_{n-1} + 1$, this defines correctly τ on the whole real line.

Note that for all $\epsilon > 0$, $\tau(t) \leq e^{\epsilon t}$ for all sufficiently large t .

The weight function τ satisfies:

(1) The series

$$\sum_{\gamma \in \Gamma} e^{(s/2)d(o, \gamma o)} G_z(e, \gamma) \tau(d(o, \gamma o))$$

converges for all $s < s(z)$.

(2) The above series diverges for $s = s(z)$.

(3) $\tau(t+c)/\tau(t) \rightarrow 1$ as $t \rightarrow +\infty$.

The proof is elementary.

IV.5.2. Weighted measures. For $s < s(z)$, define

$$\nu_{s,z} = \sum_{\gamma \in \Gamma} e^{(s/2)d(o, \gamma o)} G_z(e, \gamma) \tau(d(o, \gamma o)) \delta_{\gamma o},$$

$$p_{s,z} = \frac{\nu_{s,z}}{\|\nu_{s,z}\|}.$$

For $y \in \Gamma o$, $y = go$, for some $g \in \Gamma$, the derivative of an element $\gamma \in \Gamma$ is given by:

$$\frac{d\gamma_* p_{s,z}}{dp_{s,z}}(y) = \frac{d\gamma_* \nu_{s,z}}{d\nu_{s,z}}(y) = \frac{e^{(s/2)d(o, \gamma^{-1}y)} G_z(e, \gamma^{-1}g) \tau(d(o, \gamma^{-1}y))}{e^{(s/2)d(o, y)} G_z(e, g) \tau(d(o, y))}.$$

Since $d(o, \gamma^{-1}go) \leq d(o, \gamma^{-1}o) + d(o, go)$, this Radon-Nikodym derivative has a limit when y tends to some $\xi \in \mathbb{P}^1$:

$$\lim_{y \rightarrow \xi} \frac{d\gamma_* p_{s,z}}{dp_{s,z}}(y) = e^{(s/2)\beta_\xi(\gamma o, o)} K_z(\gamma, \xi),$$

where β is the Busemann function.

Because of the divergence at $s(z)$ of the series considered here, any weak limit ν of $p_{s,z}$ as $s \rightarrow s(z)^-$, is a probability measure with support on the limit set. Moreover, the previous computation imply that

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{(s(z)/2)\beta_\xi(\gamma^o, o)} K_z(\gamma, \xi).$$

This proves the existence of a (Γ, μ) -conformal measure of parameters (s, z) . As noted before, Proposition IV.8 and the properties of the Martin Kernel imply that it is necessarily an eigenmeasure supported on the limit set of eigenvalue $1/z = e^{k(s)}$. This concludes the proof of Theorem IV.10.

IV.6. Examples

IV.6.1. Absolutely continuous measure. Although outside the discrete case considered here, it is instructive to see what happens when μ is absolutely continuous with respect to the Haar measure.

IV.6.1.1. *The Jentzsch-Birkhoff Theorem.* If μ is absolutely continuous, with continuous derivative, and symmetric, the operator P_s and its iterates can be interpreted as an integral operator, where \mathcal{L} is the $SO(2)$ -invariant measure on \mathbb{P}^1 :

$$P_s^n f(\xi) = \int_{\mathbb{P}^1} h_n(\xi, \xi') f(\xi') d\mathcal{L}(\xi'),$$

where $\xi' \mapsto h_n(\xi, \xi')$ is obtained as the mass of the leaf $\{g \in G : g\xi = \xi'\}$ in the decomposition of $\sigma(g, g^{-1}\xi)^s d\mu^n(g)$ with respect to the foliation $\{g \in G : g\xi = \xi'\}_{\xi' \in \mathbb{P}^1}$ and the transverse measure \mathcal{L} . For some iterate n , one can show that $\inf h > 0$. Therefore, the Jentzsch-Birkhoff Theorem [5] can be applied, and P_s^n has a unique positive continuous eigenfunction (up to multiplication) and a spectral gap. In this case, P_s^* also has a unique probability eigenmeasure (see [5, Lemma 3]).

IV.6.1.2. *Left $SO(2)$ -invariant measure.* Although not necessarily absolutely continuous, this case is even simpler. Assume that μ is invariant under left multiplication by $SO(2)$, and compactly supported. Then, by invariance, if r_θ is the rotation of angle θ ,

$$P_s f(\xi) = \int_{SO(2)} \int_G f(r_\theta g \xi) \sigma(g, \xi)^s d\theta d\mu(g),$$

so

$$P_s f(\xi) = \left(\int_{\mathbb{P}^1} f d\mathcal{L} \right) \left(\int_G \sigma(g, \xi)^s d\mu(g) \right).$$

Therefore, P_s is a rank one projector, with eigenfunction

$$F_s(\xi) = \int_G \sigma(g, \xi)^s d\mu(g).$$

where $|x|$ stands for the word distance from e to x . Note that this expression has a limit when $z \rightarrow 1/\rho(\mu)$, namely

$$G_{1/\rho(\mu)}(e, x) = \frac{3}{2} \left(\frac{1}{\sqrt{3}} \right)^{|x|}.$$

IV.6.2.2. *The critical exponent.* Things would not get very interesting in this example if the distance considered was the word metric.

Endow the Cayley tree X of $(\Gamma, \{a, b\})$ with a distance d such that edges labelled by a have length 1, and edges labelled by b have length ℓ . For $\xi \in \partial X$, and $\gamma \in \Gamma$, the cocycle $\sigma(\gamma, \xi)$ has to be understood as $e^{\frac{1}{2}\beta_\xi(e, \gamma)}$, where β is the Busemann function defined using the above distance. The norm $\|\gamma\|$ is then $e^{\frac{1}{2}d(e, \gamma)}$.

The critical exponent of the group Γ can be computed more or less explicitly. Let $a_n(s)$ be the sum of $e^{sd(e, \gamma)}$ over all reduced words of length n beginning by a or a^{-1} , and $b_n(s)$ similarly for words of length n beginning with b or b^{-1} . Then

$$\begin{pmatrix} a_{n+1}(s) \\ b_{n+1}(s) \end{pmatrix} = \begin{pmatrix} e^s & 2e^s \\ 2e^{s\ell} & e^{s\ell} \end{pmatrix} \begin{pmatrix} a_n(s) \\ b_n(s) \end{pmatrix}.$$

Thus the growth of $a_n(s)$ and $b_n(s)$ are determined by the spectral radius of the positive matrix

$$M_s = \begin{pmatrix} e^s & 2e^s \\ 2e^{s\ell} & e^{s\ell} \end{pmatrix},$$

by the Perron-Frobenius Theorem. This implies that the critical exponent δ of the Poincaré series is the only $-s$ such that the above matrix has 1 as eigenvalue. It is given by the following implicit equation:

$$3e^{-\delta(\ell+1)} + e^{-\delta} + e^{-\delta\ell} = 1.$$

After a little bit of analysis, one can obtain an equivalent as $\ell \rightarrow +\infty$:

$$\delta \sim \frac{\log \ell}{\ell}.$$

IV.6.2.3. *The function k .* Let (s, z) be such that $0 < z < 1/\rho(\mu)$, $s < 0$. The function k can be obtained via the study of the Green-Poincaré series,

$$GP(s, z) = \sum_{\gamma \in \Gamma} e^{\frac{s}{2}d(o, \gamma o)} G_z(e, \gamma).$$

Here, it can be computed rather explicitly, by regrouping the terms corresponding to reduced words of the same length, because $G_z(e, \gamma)$ depends only on the word length:

$$GP(s, z) = \frac{3}{2 + \sqrt{4 - 3z^2}} \left[\sum_{n \geq 0} (a_n(s/2) + b_n(s/2)) \left(\frac{2 - \sqrt{4 - 3z^2}}{3z} \right)^n \right].$$

Since $a_n(s/2) + b_n(s/2)$ grows like powers of the spectral radius of $M_{s/2}$, the convergence or divergence of this series changes on the curve

$$\frac{3z}{2 - \sqrt{4 - 3z^2}} = \rho(M_{s/2}),$$

which imply that ²

$$z = \frac{4\rho(M_{s/2})}{3 + \rho(M_{s/2})^2},$$

provided this value is $< 1/\rho(\mu)$ (otherwise, the Green series is even not defined). The positive eigenvalue of $M_{s/2}$ is given, after a short calculation of the characteristic polynomial, by the formula

$$\rho(M_{s/2}) = \frac{1}{2} \left(e^{s/2} + e^{s\ell/2} + \sqrt{e^s + e^{s\ell} + 14e^{s(\ell+1)/2}} \right).$$

Putting $z = e^{-k(s)}$ inside these formulas gives an explicit (not very not enlightening, but useful for the pictures) expression for $k(s)$.

More interesting is to try to understand if the criterion that says that k is constant on some interval $(-\infty, s_0]$ for $s_0 \geq -\delta$ is tight, or not, i.e. whether s_0 is equal or strictly larger than $-\delta$. Because $G_z(e, \cdot)$ has a limit when $z = 1/\log \rho(\mu)$ of exponential decay given by $\frac{1}{\sqrt{3}}$, that tells us that

$$\rho(M_{s_0/2}) = \sqrt{3}.$$

Again, this is expressed in the implicit equation $\det(M_{s_0/2} - \sqrt{3}I) = 0$, that is:

$$(e^{s_0/2} - \sqrt{3})(e^{s_0\ell/2} - \sqrt{3}) - 4e^{s_0(\ell+1)/2} = 0.$$

A bit of analysis gives the equivalent, when $\ell \rightarrow +\infty$:

$$(8) \quad s_0 \sim - \left(2 \log \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) \frac{1}{\ell}.$$

Therefore, $s_0 > -\delta$ for large ℓ .

IV.6.2.4. *Computation of a special k_η .* Let $\xi = (b)_\infty = \lim_{k \rightarrow +\infty} b^k$ be the point at infinity in $\partial\Gamma$ obtained by following the directed edges b from the origin. Let \mathcal{D}_ξ be the Dirac measure at ξ , we will obtain a lower bound for k^+ by computing $k_{\mathcal{D}_\xi}$. By definition, the series

$$S(s, z) = \sum_{\gamma \in \Gamma} e^{\frac{s}{2}\beta_\xi(e, \gamma)} G_z(e, \gamma) = \sum_{m \geq 0} \left(\sum_{\gamma \in \Gamma} \sigma(\gamma, \xi)^s \mu^m(\gamma) \right) z^m,$$

converges if $z < e^{-k_{\mathcal{D}_\xi}(s)}$ and diverges if $z > e^{-k_{\mathcal{D}_\xi}(s)}$. Notice that since $s < 0$ the series S is greater than the Green-Poincaré series; Let π be the projection on

²It is actually equivalent only if $s \geq s_0$, for the s_0 defined below.

the geodesic between $(b)_\infty$ and $(b^{-1})_\infty$, which is simply $(b^n)_{n \in \mathbf{Z}}$. Decompose the series along this geodesic:

$$S(s, z) = \sum_{n \in \mathbf{Z}} \sum_{\gamma \in \pi^{-1}(b^n)} e^{\frac{s}{2}\beta_\xi(e, \gamma)} G_z(e, \gamma).$$

Note that the geodesic from e to γ passes through $\pi(\gamma)$, and this is also true for the geodesic from γ to $(b)_\infty$. Therefore, up to some bounded multiplicative constant,

$$G_z(e, \gamma) \simeq G_z(e, \pi(\gamma))G_z(\pi(\gamma), \gamma).$$

This very general fact follows from Ancona's work, but is rather straightforward here as we have an explicit formula for G_z . The same is true (with an equality) for the Busemann cocycle

$$e^{\frac{s}{2}\beta_\xi(e, \gamma)} = e^{\frac{s}{2}\beta_\xi(e, \pi(\gamma))} e^{\frac{s}{2}\beta_\xi(\pi(\gamma), \gamma)}.$$

Once can thus estimate (up to bounded multiplicative constants)

$$S(s, z) \simeq \sum_{n \in \mathbf{Z}} e^{\frac{s}{2}\beta_\xi(e, b^n)} G_z(e, b^n) \left(\sum_{\gamma \in \pi^{-1}(b^n)} e^{\frac{s}{2}\beta_\xi(\pi(\gamma), \gamma)} G_z(\pi(\gamma), \gamma) \right).$$

Note that the factored term does not depend on b^n , as all fibers are isometric. Therefore, $S(s, z)$ is essentially a product:

$$S(s, z) \simeq \left(\sum_{n \in \mathbf{Z}} e^{-sln/2} u_z^{|n|} \right) \left(\sum_{\gamma \in \pi^{-1}(e)} e^{\frac{s}{2}d(e, \gamma)} u_z^{|\gamma|} \right),$$

where $u_z = \frac{2 - \sqrt{4 - 3z^2}}{3z}$. The second sum is certainly smaller than the Green-Poincaré series, so converges if $z < e^{-k(s)}$, and will not produce any new condition. The first sum is a two sided sum: the $n < 0$ part always converge because $u_z < 1$ and $s < 0$. The second side ($n > 0$) has a critical curve defined by

$$e^{-sl/2} \frac{2 - \sqrt{4 - 3z^2}}{3z} = 1.$$

This curve is the graph a increasing function $z_0(s)$, explicitly:

$$z_0(s) = \frac{4e^{sl/2}}{3e^{sl} + 1},$$

defined for $s \leq \frac{-\log 3}{\ell} = s_1$, where it takes the critical value $1/\rho(\mu) = 2/\sqrt{3}$. The expression $k_{\mathcal{D}_\xi}$ is given as a supremum

$$k_{\mathcal{D}_\xi}(s) = \max\{k(s), -\log z_0(s)\}.$$

Since k is strictly increasing on $(s_0, 0)$, and $s_0 < s_1$ for large ℓ because of the equivalent (8), the graph of the two functions k and $-\log z_0(s)$ do intersect, taking a value $> \log \rho(\mu)$, thus proving the existence of at least two eigenvalues for

the operator P_s^* for $s \in (s_0, s_1)$.

One can show in this case that in fact $k^+ = k_{\mathcal{D}_\xi}$, using the above methods. Indeed, decomposing the sum along a geodesic is quite general, and the convergence of the series on the fiber is always implied by the convergence of the Green-Poincaré series. The geodesic considered here is clearly 'the fastest one' as soon as $\ell > 1$.

IV.6.2.5. *Geometrical approximation in $\mathrm{SL}(2, \mathbf{R})$.* The above considerations on the free group imply that such examples where k is distinct from k^+ and $s_0 > -\delta$ exist also in $\mathrm{SL}(2, \mathbf{R})$. Indeed, consider the distance td , where $t > 0$ is very large. Its related functions k, k^+ only change by multiplication of the variable by t , and so δ is divided by t . On the other hand, there is a representation of the free group on $\mathrm{SL}(2, \mathbf{R})$, for which the distance on Γ induced by the hyperbolic metric is quasi-isometric with arbitrarily pinched constants (depending on t) to the initial distance td . Therefore, the new functions k, k^+ obtained by the hyperbolic metric will have the same properties.

IV.7. Further comments

We have also a proof (not detailed here) of the fact that, for $s > s_1$, the operator P_s has a spectral gap on the space of Hölder functions of sufficiently small exponent.

Some of the measures constructed here as eigenvalues may be atomic, and irrelevant from a dynamical perspective. It would be interesting to obtain a lower bound on the dimension for some specific measures.

I believe it is probably not very hard to adapt the arguments present in the study of the example to show that k^+ is the maximum of k and a decreasing function. This function can be interpreted as asymptotic quasi-isometry constants between the Green distances of various parameter z , and the hyperbolic distance. This implies in particular that if $s_1 > s_0$, then the left and right derivatives of k at s_1 are different.

And of course, it remains to show a large deviation Theorem for the norm, using the Legendre-Fenchel dual of k .

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