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le 09 décembre 2014

Equations de Painlevé et théorie des champs conformes

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Résumé

Ce mémoire d'habilitation présente mes travaux de recherche sur les liens entre la théorie des champs conformes en dimension 2 et les déformations isomonodromiques. Il est organisé en quatre parties. La première se voudrait une introduction à la problématique des équations de Painlevé et déformations isomonodromiques en général, suivie d'une discussion informelle des résultats obtenus dans le cadre de la théorie conforme. Dans une deuxième partie, nous montrons comment la relation entre les blocs conformes de l'algèbre de Virasoro et la fonction tau de l'équation de Painlevé VI peut être obtenue de façon heuristique. Nous y présentons aussi les séries combinatoires donnant les solutions générales de Painlevé VI, V et trois équations inéquivalentes de Painlevé III. La troisième partie est consacrée à la démonstration d'un résultat plus général: nous montrons que la solution du problème de Riemann-Hilbert pour les systèmes Fuchsiens de rang 2 ainsi que la fonction tau isomonodromique associée coïncident avec les transformations de Fourier des blocs conformes de Virasoro avec $c = 1$ par rapport aux impulsions intermédiaires. Nous nous intéressons par la suite au problème de connexion pour la fonction tau de Painlevé VI et calculons la matrice de fusion des blocs conformes avec $c = 1$. Enfin, diverses perspectives sont esquissées.

Abstract

This habilitation thesis presents several results on the relation between the two-dimensional conformal field theory and isomonodromic deformations. It is organized into four chapters. The first one is intended to be an introduction into problematics of Painlevé equations and isomonodromic deformations in general, followed by an informal discussion of the results obtained in the CFT framework. In the second part, we show how the relation between conformal blocks of the Virasoro algebra and the tau function of Painlevé VI equation can be derived heuristically. There we also present the combinatorial series representations of the general solutions of Painlevé VI, V and three inequivalent Painlevé III equations. The third part is devoted to the proof of a more general result: we show that the solution of the Riemann-Hilbert problem for rank 2 Fuchsian systems as well as the associated isomonodromic tau function coincide with Fourier transforms of $c = 1$ Virasoro conformal blocks with respect to intermediate momenta. In the next part we solve the connection problem for Painlevé VI tau function and compute the fusion kernel for $c = 1$ conformal blocks. We conclude with a discussion of various perspectives.

Remerciements

Ce mémoire d'habilitation présente quelques résultats d'un programme de recherche initié autour de 2010. Je voudrais tout d'abord remercier tous mes collaborateurs qui ont pris part à cette aventure: Sasha Gamayun, Pasha Gavrylenko, Kolya Iorgov, Alexander Its, Anton Shchechkin, Joerg Teschner et Yura Tykhyy.

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“Certes, une équation est “plus intégrée” lorsqu’on sait que ses solutions sont des fonctions entières ou méromorphes que si on a simplement appris à lui appliquer la méthode de Cauchy ou les approximations successives de M. Picard... Mais ceci ne doit pas nous faire oublier la différence entre l’existence d’une expression et le fait de la connaître effectivement. Lorsque Jacobi a exprimé par les fonctions theta la fonction inverse de l’intégrale elliptique de première espèce, c’est-à-dire la solution d’une équation différentielle du premier ordre, il ne s’est pas contenté de montrer que les fonctions qu’il introduisait étaient des fonctions entières : il en a donné une expression précise par des séries *dont les termes successifs sont explicitement connus*... Il s’en faut que nous en soyons là en ce qui concerne les transcendantes de Painlevé, et que nous ayons en mains tous les éléments nécessaires pour en trouver aisément les propriétés.”

Jacques Hadamard, *L’oeuvre scientifique de Paul Painlevé*, (1934) [68, p.303]

Table des matières

Document de synthèse **3**

1 Introduction **23**

1.1 Painlevé property 23

1.2 Monodromy preserving deformation 24

1.3 Riemann-Hilbert correspondence 26

1.4 Are they solvable? 27

1.5 Tau functions and CFT 29

1.6 Connection vs fusion 33

1.7 Organization of the manuscript 35

2 Heuristic CFT approach to Painlevé transcendents **36**

2.1 Preliminaries 36

2.2 Field content 38

2.3 Painlevé VI 42

2.4 Conformal blocks from OPEs 47

2.5 $\mathcal{N} = 2$ SUSY gauge theories 50

2.6 Confluence cascade 51

2.7 General solutions of P_{VI}, P_{V} and $\mathrm{P}_{\mathrm{III}}$'s 54

2.8 Some examples 58

3 Isomonodromic tau functions from Virasoro conformal blocks **63**

3.1 The Riemann-Hilbert problem 63

3.2 Trace coordinates 64

3.3 Conformal blocks from chiral vertex operators 66

3.4 Degenerate fields 68

3.5 Monodromy action on spaces of conformal blocks 71

3.6 Solving the Riemann-Hilbert problem 73

3.7 Painlevé VI and Jimbo's formula 78

3.8 Algebro-geometric solutions of the Schlesinger system 80

4 Painlevé VI connection problem and $c = 1$ fusion matrix 84

4.1 Fusion transformations of conformal blocks 84

4.2 Painlevé VI and hyperbolic tetrahedron 89

4.3 Connection coefficient for P_{VI} tau function 93

4.4 Fusion kernel at $c = 1$ 99

5 Conclusion and perspectives 103

Bibliographie 105

Document de synthèse

Propriété de Painlevé

Au tournant du vingtième siècle, il y a eu des tentatives répétées de décrire les équations différentielles ordinaires (EDOs) en termes de singularités de leurs solutions dans le domaine complexe. Pour les équations nonlinéaires, ce problème est assez difficile à aborder, car en général les positions des singularités et même leur type dépendent des conditions initiales: les points singuliers deviennent mobiles. Une version plus raffinée du problème est de classifier les équations différentielles à ramification "prévisible", en autorisant uniquement les pôles mobiles et les singularités essentielles mobiles non-ramifiées ainsi que tout type de points singuliers fixes. Cette dernière condition est connue aujourd'hui sous le nom de *propriété de Painlevé*.

La classification des équations différentielles algébriques du premier ordre ayant la propriété de Painlevé a été achevée par L. Fuchs [57] en 1884, avec quelques développements ultérieurs par H. Poincaré [131]. Il s'avère que de telles équations sont soit équivalentes à des équations linéaires, soit solubles en termes de fonctions elliptiques. Le traitement du problème est considérablement simplifié par le fait que les points singuliers mobiles des équations du premier ordre ne peuvent être que des pôles ou des points de branchement algébriques.

Lorsque l'ordre de l'équation est supérieur ou égal à 2, nous ne pouvons pas exclure *a priori* des singularités mobiles plus complexes, telles que les singularités essentielles ou bords naturels. Cela a conduit E. Picard d'exprimer son opinion sceptique quant à la possibilité d'une classification dans une lettre à G. Mittag-Leffler en 1893. Néanmoins en 1900–1910 P. Painlevé [127, 128] et B. Gambier [63] ont entrepris une remarquable tentative de classification des équations différentielles d'ordre 2 et de degré 1 sans points de ramification mobiles. Ces équations sont de la forme $q'' = F(t, q, q')$, où t note la variable indépendante, F est rationnelle en q, q' et localement analytique en t.

Le résultat de l'étude de Painlevé et Gambier se présente sous forme d'une liste de 50 classes d'équivalence d'EDOs dont la plupart se réduisent à des équations linéaires ou intégrables par quadratures. Cependant il reste six équations nonlinéaires irréductibles — les six équations de Painlevé:

$$\begin{aligned} q'' = {} & \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)(q')^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)q' + \\ & + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{q^2}+\frac{\gamma(t-1)}{(q-1)^2}+\frac{\delta t(t-1)}{(q-t)^2}\right), \end{aligned} \tag{$\mathrm{P_{VI}}$}$$

$$q'' = \left(\frac{1}{2q} + \frac{1}{q-1}\right)(q')^2 - \frac{q'}{t} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1}, \qquad (\mathrm{P_V})$$

$$q'' = \frac{(q')^2}{2q} + \frac{3}{2}q^3 + 4tq^2 + 2\left(t^2 - \alpha\right)q + \frac{\beta}{q}, \qquad (\mathrm{P_{IV}})$$

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} + \frac{\alpha q^2 + \beta}{t} + \gamma q^3 + \frac{\delta}{q}, \qquad (\mathrm{P_{III}})$$

$$q'' = 2q^3 + tq + \alpha, \qquad (\mathrm{P_{II}})$$

$$q'' = 6q^2 + t. \qquad (\mathrm{P_I})$$

Cette liste d'équations différentielles exceptionnelles est restée inchangée après quelques reconsidérations plus soignées de l'approche de Painlevé et Gambier, voir par exemple [72, Chapître 14], [24] et la discussion dans [30].

L'augmentation de l'ordre et/ou du degré de l'équation rend le problème de classification beaucoup plus compliqué. On sait que les singularités essentielles et les bords naturels mobiles apparaissent réellement dans ce cas: ils ont été observés dans les solutions exactes de certains exemples explicites. Les équations différentielles de type Painlevé d'ordre 2 et de degré 2 de la forme $(q'')^2 = F(t, q, q')$ ont été classifiées dans [30]. Quelques résultats partiels sont également disponibles pour les équations d'ordre 3 et de degré 1 [27, 64, 25]. Une des caractéristiques intéressantes de ces développements est leur relation aux équations classiques de Painlevé. Par exemple, les EDOs exceptionnelles de la première classe sont précisément les équations vérifiées par les Hamiltoniens non-autonomes de $(\mathrm{P_I})$–$(\mathrm{P_{VI}})$.

Déformation isomonodromique

La motivation initiale de Painlevé et ses élèves peut sembler quelque peu ésotérique dans une perspective moderne et a été perçue comme telle même par certains de leurs contemporains. Un commentaire célèbre, atribué dans [68] à H. Poincaré, décrit l'oeuvre de Painlevé's comme suit:

> *Les Mathématiques constituent un continent solidement agencé, dont tous les pays sont bien reliés les uns aux autres; l'oeuvre de Paul Painlevé est une île originale et splendide dans l'océan voisin.*

La situation a changé quand il a été réalisé que les équations de Painlevé étaient les exemples les plus élémentaires d'équations décrivant les déformations isomonodromiques d'EDOs linéaires à coefficients rationnels. Le premier pas dans cette direction a été effectué en 1905 [58] par R. Fuchs (fils de L. Fuchs). Il considérait les équations différentielles Fuchsiennes d'ordre 2 à 5 points singuliers sur la sphère de Riemann, une des

singularités étant apparente. On peut mettre ces points à 0, 1, ∞, t et q (la dernière position correspondant à la singularité apparente) en utilisant les transformations homographiques. La découverte principale de Fuchs était que la condition de conservation de la monodromie de l'équation sous variation de t produisait l'équation de Painlevé VI pour q.[1] Dans la situation plus générale de $n+3$ vrais points singuliers et n singularités apparentes la condition d'isomonodromie donne lieu à un système d'EDPs nonlinéaires appelé le système de Garnier $\mathcal{G}_n$ [64, 65]; voir aussi [78] pour une revue.

Nous allons nous placer dans un cadre légèrement différent, développé par L. Schlesinger [141] en 1912. Au lieu de travailler avec une équation scalaire d'ordre supérieur, nous allons nous intéresser aux systèmes Fuchsiens d'équations différentielles linéaires d'ordre 1 à un nombre arbitraire de points singuliers sur $\mathbb{P}^1$. Esquissons brièvement la formulation isomonodromique correspondante de l'équation P_{VI} (qui sera notre exemple illustratif principal dans la suite) en laissant les détails et la discussion de la situation plus générale au texte principal.

Considérons un système Fuchsien de rang 2 à quatre singularités régulières $0, t, 1, \infty$ sur la sphère de Riemann:

$$\partial_z \Phi = \mathcal{A}(z)\,\Phi, \qquad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}. \tag{0.1}$$

Ici les matrices $A_\nu \in \mathfrak{sl}_2(\mathbb{C})$ ($\nu = 0, t, 1$) sont indépendantes de z. Leurs valeurs propres respectives seront notées $\pm\theta_\nu$. Aussi, posons $A_\infty := -A_0 - A_t - A_1 = \mathrm{diag}\,\{\theta_\infty, -\theta_\infty\}$. La forme diagonale de A_∞ n'induit presque pas de perte de généralité et correspond à un choix convenable de jauge.

La matrice fondamentale $\Phi(z)$ est génériquement multivaluée sur $C_{0,4} := \mathbb{P}^1 \backslash \{0, t, 1, \infty\}$ puisque son prolongement analytique le long des chemins fermés non-contractibles peut engendrer une monodromie nontriviale. Le groupe fondamental $\pi_1(C_{0,4})$ est engendré par les lacets χ_ν ($\nu = 0, t, 1, \infty$) indiquées sur la Fig. 1, auxquelles nous allons associer les matrices de monodromie $M_\nu \in \mathrm{SL}(2, \mathbb{C})$. Notons qu'elles vérifient la contrainte $M_\infty M_1 M_t M_0 = \mathbf{1}$.

On obtient ainsi un anti-homomorphisme $\rho : \pi_1(C_{0,4}) \to \mathrm{SL}(2, \mathbb{C})$ appelé la représentation de monodromie. Les différents choix de la base des solutions du système (0.1) conduisent à des représentations équivalentes de $\pi_1(C_{0,4})$. L'idée de la déformation isomonodromique est de construire une famille de systèmes linéaires qui partagent la même classe d'équivalence de la monodromie.

[1] En effet, c'est dans l'article [58] que la forme générale de P_{VI} a été signalée pour la première fois; cette équation manquait dans la classification initiale de Painlevé en raison d'erreurs de calculs.

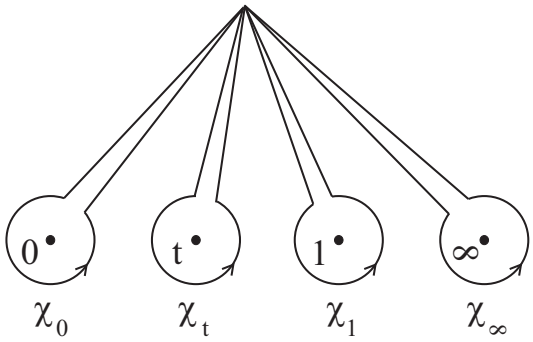


Fig. 1: La base de $\pi_1\left(C_{0,4}\right)$

Les équations de Schlesinger décrivant les déformations isomonodromiques du système Fuchsien (0.1) sont données par

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}. \tag{0.2}$$

On peut décrire l'espace de paramètres $\mathcal{P}$ de ce système avec une paire de fonctions scalaires de t. L'une de ces fonctions peut être définie comme le zéro nontrivial unique de l'élément hors-diagonal $\mathcal{A}_{12}(z)$. Les équations d'isomonodromie s'avèrent équivalentes à l'équation de Painlevé VI vérifiée par cette dernière quantité. Quatre paramètres de P_{VI} s'expriment en fonction des valeurs propres de $A_{0,t,1,\infty}$ comme $(\alpha, \beta, \gamma, \delta) = \left(\left(\theta_\infty + \frac{1}{2}\right)^2, -\theta_0^2, \theta_1^2, \frac{1}{4} - \theta_t^2\right)$. Le lecteur intéressé par les détails calculatoires pourra consulter [29, cours de G. Mahoux].

Les cinq autres équations de Painlevé peuvent être mises dans un contexte analogue en considérant des systèmes linéaires à singularités irrégulières (pôles d'ordre supérieur). La structure de données de la monodromie dans le cas irrégulier est plus intriquée. Elle inclut en particulier les matrices de Stokes. L'extension correspondante de la théorie des déformations isomonodromiques a été développée au début des années 80s par H. Flaschka et A. Newell [52], et par M. Jimbo, T. Miwa et K. Ueno [82, 83].

Nous noterons au passage que, comme il était déjà connu à P. Painlevé, on peut obtenir les équations $\mathrm{P}_{\mathrm{I-V}}$ à partir de P_{VI} dans certaines limites de scaling. Ces limites correspondent à des confluences successives des points singuliers du système Fuchsien initial et donnent un moyen alternatif d'obtenir les problèmes linéaires irréguliers associés.

Correspondance de Riemann-Hilbert

La construction ci-dessus définit l'application de la monodromie

$$\mathfrak{m} : \begin{array}{c}\text{espace de paramètres } \mathcal{P} \\ \text{du problème linéaire}\end{array} \longrightarrow \begin{array}{c}\text{espace de modules } \mathcal{M} \\ \text{de la monodromie}\end{array}$$

parfois dénommée la correspondance de Riemann-Hilbert. Dans un cadre plus général, cette correspondance relie les espaces de connexions plates logarithmiques sur les surfaces de Riemann avec des points marqués et les espaces de représentations de leurs groupes fondamentaux.

Dans le cas de P_{VI}, le triplet (t, q, q') définit un ensemble de coordonnées locales sur $\mathcal{P}$. L'application $\mathfrak{m}$ associe à toute branche de toute solution de Painlevé VI la classe de conjugaison d'un triplet de matrices de monodromie représentant un point de l'espace de représentations

$$\mathcal{M} = \mathrm{Hom}\left(\pi_1\left(C_{0,4}\right), G\right)/G \cong G^3/G, \qquad G = \mathrm{SL}(2, \mathbb{C}). \tag{0.3}$$

Un comptage naïf de paramètres donne $\dim \mathcal{M} = 6$. Quatre exposants de la monodromie locale autour des singularités régulières sont reliés aux quatre paramètres $\theta_{0,t,1,\infty}$ de P_{VI}. Les deux paramètres restants correspondent aux conditions initiales. En d'autres termes, les données de la monodromie forment un ensemble complet de quantités conservées de Painlevé VI.

Un regain d'intérêt aux équations de Painlevé est survenu dans les années 80-90s, plus d'un demi-siècle après leur découverte. Sa première motivation est venue des applications. On s'est rendu compte que les réductions exactes des équations différentielles nonlinéaires intégrables classiques (telles que l'équation de Korteweg-de Vries, sine-Gordon ou Schrödinger nonlinéaire) conduisent à des équations de Painlevé, voir par exemple [1]. D'autres domaines d'applications remarquables sont la théorie des matrices aléatoires [80, 148, 149, 55] et les modèles exactement solubles de la mécanique statistique et la théorie quantique des champs [158, 137, 138, 139, 155, 81, 51, 162, 12], en particulier le modèle d'Ising bidimensionnel.

Une autre raison importante de ce regain d'intérêt était le développement de nouvelles méthodes d'analyse globale des fonctions de Painlevé, basées sur la correspondance de Riemann-Hilbert. L'idée centrale a été inspirée par de puissantes techniques de la diffusion inverse dans la théorie des systèmes intégrables classiques. Nous allons informellement énoncer cette idée comme ceci: les questions sur l'équation *nonlinéaire* doivent être posées en termes de la monodromie du problème *linéaire* associé, et les réponses extraites de l'analyse de ce dernier.

La mise en pratique de cette philosophie dans la théorie des équations de Painlevé a été initiée en 1982 par l'article fondateur de M. Jimbo [79]. Dans ce travail, l'asymptotique de la sixième transcendante de Painlevé (générique) au voisinage des points critiques (c'est-à-dire, lorsque $t \to 0, 1$ ou ∞) a été exprimée en termes de la monodromie du système linéaire (0.1). Cela résout automatiquement le problème de connexion entre les paramètres asymptotiques des différents points critiques. De nombreux cas spéciaux ont

été récemment discutés dans [67]. Les formules de connexion pour les autres transcendantes de Painlevé étaient étudiées dans [84, 85, 90] (pour P_{I}), [74, 73, 37] (pour P_{II}), [109, 119, 120, 86, 87, 95] (pour $\mathrm{P}_{\mathrm{III}}$), [88, 75] (pour P_{IV}) et [107, 108, 5] (pour P_{V}). Pour une exposition moderne de l'approche de Riemann-Hilbert aux équations de Painlevé, on renvoie le lecteur à la monographie [54].

Sont-elles solubles?

Le formalisme de Riemann-Hilbert met sur un pied d'égalité les solutions des équations nonlinéaires de Painlevé et les fonctions spéciales classiques vérifiant les équations différentielles linéaires de type hypergéométrique. Plus précisément, il nous permet de réaliser (au moins en principe) le programme suivant:

- paramétrer convenablement les fonctions de Painlevé de différents types par les données de monodromie des systèmes linéaires auxiliaires,
- exprimer leur asymptotique au voisinage des points critiques en termes de la monodromie.

Nous pourrions concevoir par la suite des codes numériques pour le calcul de solutions des équations de Painlevé en dehors des points critiques en utilisant leur asymptotique comme conditions initiales. Cette suggestion est une des idées derrière le "Projet Painlevé" lancé récemment [31].

Peut-on aller au-delà de l'asymptotique? Il est bien sûr possible (même si la tâche reste techniquement nontrivial en raison de nonlinéarité) de calculer les corrections sous-dominantes à l'asymptotique ordre par ordre à partir des équations de Painlevé. Cependant la vraie question est la suivante: est-il possible d'obtenir une description explicite non-itérative de l'inverse $\mathfrak{m}^{-1}$ de l'application de la monodromie?

Problème A: *Construire les solutions générales des équations de Painlevé en termes des données de monodromie des problèmes linéaires associés.*

A première vue ce problème semble désespérément insoluble. En effet, depuis la découverte même des équations de Painlevé il y avait un débat sur la possibilité de leur intégration en termes de fonctions élémentaires ou fonctions spéciales connues auparavant. Rappelons qu'une fonction d'une variable t s'appelle classique [152] si on peut l'obtenir à partir du corps $\mathbb{C}(t)$ des fonctions rationnelles par une itération finie de certaines opérations admissibles telles que les opérations arithmétiques habituelles, quadratures, solution d'équations différentielles linéaires à coefficients classiques, substitution

dans une fonction abélienne, etc. Les solutions générales des équations de Painlevé sont aujourd'hui connues d'être transcendantes dans ce sens [117, 151, 153, 121, 154, 156, 157, 26]. Les solutions classiques n'apparaissent que pour des valeurs particulières de paramètres et des conditions initiales spéciales, et ont été étudiées en grand détail.

Par exemple, dans le cas de P_{VI} les résultats de H. Watanabe [157] suggèrent que toute solution est soit a) de type Riccati, soit b) une fonction algébrique ou alors c) elle est transcendante[2]. Les solutions de Riccati s'expriment en termes de fonctions hypergéométriques de Gauss et sont connues explicitement. Elles apparaissent lorsque les matrices de monodromie du système linéaire (0.1) peuvent être ramenées simultanément à la forme triangulaire supérieure ou bien si l'une de ces matrices est égale à $\pm\mathbf{1}$.

Les solutions algébriques de Painlevé VI ont été classifiées par l'auteur de ce manuscrit et Yu. Tykhyy en 2008 [102]. L'idée de la classification est basée sur la correspondance de Riemann-Hilbert: le prolongement analytique de toute solution induit une action du groupe de tresses à 3 brins sur l'espace de modules de la monodromie. Les solutions à ramification finie conduisent à des orbites finies de cette action. Leur description se réduit à son tour à la classification des orbites périodiques d'un système dynamique polynomial discret sur $\mathbb{C}^3$. L'existence de certaines contraintes trigonométriques-diophantiennes implique que l'ensemble des candidats possibles aux orbites finies est lui-même fini (mais de cardinalité assez grande, $\sim 10^8$). Nous avons selectionné les véritables orbites finies en parcourant les éléments de cet ensemble un par un à l'aide de l'ordinateur. Le résultat final est le suivant: les solutions algébriques de P_{VI} sont soit a) des cas particuliers des solutions hypergéométriques de Riccati, soit b) des cas particuliers des solutions elliptiques de Picard, soit c) elles appartiennent à une des 3 familles explicites continues, ou bien d) elles sont équivalentes à une des 45 solutions algébriques exceptionnelles, connues des travaux antérieurs de F. V. Andreev et A. V. Kitaev [6], P. Boalch [14, 15, 16, 17], B. A. Dubrovin [42], B. A. Dubrovin et M. Mazzocco [43], et A. V. Kitaev [91, 92]. Notons que pour les autres équations de Painlevé la classification des solutions algébriques se simplifie énormement car dans ces cas le groupe fondamental du complément du lieu critique devient abélien.

Malgré cet aspect de transcendance, les solutions générales des équations de Painlevé partagent un grand nombre de propriétés des fonctions spéciales classiques (telles que la fonction hypergéométrique de Gauss, les fonctions de Whittaker, Hermite, Bessel, ...). En plus des problèmes de connexion solubles, on peut mentionner l'existence de diverses symétries, relations de récurrence, formules de dérivation (transformations de Bäcklund-

[2]Les solutions transcendantes incluent (quelque peu contre-intuitivement) la famille des solutions elliptiques de Picard [130, 105]. A transformations de Bäcklund près, les paramètres correspondants de P_{VI} sont donnés par $\theta_0 = \theta_t = \theta_1 = 0$, $\theta_\infty = \frac{1}{2}$, et il n y a pas de contraintes sur les conditions initiales.

Okamoto [123, 124, 125, 126], voir aussi [55, Chapître 8] pour une revue) et des transformations algébriques de différents ordres [89, 96, 106]. La différence majeure entre les transcendantes de Painlevé et les fonctions spéciales de type hypergéométrique est le manque de connexion transparente à la théorie des représentations des groupes et algèbres de Lie. C'est l'obstacle conceptuel derrière l'absence de représentations explicites des fonctions de Painlevé, qu'il s'agisse de séries ou de représentations intégrales. Cela nous conduit au problème suivant:

Problème B: *Mettre les transcendantes de Painlevé dans le cadre de la théorie des représentations.*

Ce mémoire décrit une tentative de résolution des Problèmes A et B via la théorie des champs conformes (CFT) en dimension 2. Nous allons montrer qu'il existe une relation très simple entre les *fonctions tau* de Painlevé VI, V et trois équations inéquivalentes de Painlevé III, et les *blocs conformes* de l'algèbre de Virasoro de charge centrale $c = 1$ ainsi que leurs versions irrégulières. En particulier, cela permet d'obtenir les solutions explicites des problèmes inverses de la monodromie sous forme de séries combinatoires.

Fonctions tau et CFT

Le premier lien frappant entre les déformations isomonodromiques et la théorie quantique des champs a été mis en évidence par M. Sato, T. Miwa and M. Jimbo (SMJ) dans une série d'articles apparus à la fin des années 70s, voir en particulier [135, 136], et [140] pour une revue. Leurs résultats incluent l'identification des fonctions tau isomonodromiques des systèmes Fuchsiens de rang N à un nombre arbitraire de singularités régulières sur la sphère de Riemann avec certaines fonctions de corrélation dans une théorie quantique des champs décrivant N copies de fermions libres chiraux de masse nulle.

Les fonctions tau sont importantes pour plusieurs raisons:

- Elles sont les fonctions génératrices pour les Hamiltoniens des flots isomonodromiques [82, 83]. En outre, elles coïncident avec les déterminants régularisés des opérateurs de Cauchy-Riemann avec les domaines composés des fonctions multivaluées avec la monodromie appropriée [129]. La définition des fonctions tau a été étendue également au cas irrégulier.

- Ce sont les fonctions tau au lieu de transcendantes de Painlevé elles-mêmes (même si les unes sont directement reliées aux autres) qui se montrent habituellement dans les applications, en représentant les probabilités de "gap" dans la théorie des matrices aléatoires, les fonctions de corrélation du modèle d'Ising et la théorie des champs de sine-Gordon au point de fermions libres, etc.

En plus de fermions, les corrélateurs de la théorie de SMJ incluent des champs locaux d'un autre type (dans la suite nous utiliserons le terme "champs de monodromie" au lieu de "champs quantiques holonomes" de SMJ). Ces derniers peuvent être vus dans le formalisme opératoriel comme des transformations de Bogoliubov de l'algèbre fermionique assurant les propriétés nécessaires de monodromie. La fonction tau a été initialement définie comme étant la fonction de corrélation de ces champs de monodromie. En particulier, pour Painlevé VI elle est représentée par un corrélateur à quatre points:

$$\tau(t) = \langle \mathcal{O}_{M_\infty}(\infty)\mathcal{O}_{M_1}(1)\mathcal{O}_{M_t}(t)\mathcal{O}_{M_0}(0)\rangle.$$

Afin de rendre de telles expressions formelles adaptées au calculs réels, la caractérisation des champs de monodromie devrait être rendue plus précise. Nous allons atteindre cet objectif en utilisant les outils de la théorie des champs conformes [11, 38, 164].

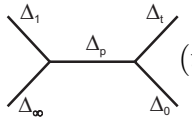
Nous commencerons par donner un argument heuristique montrant que la CFT correspondante est caractérisée par la charge centrale $c = N-1$, c'est-à-dire, pour Painlevé VI nous avons $c = 1$. Les champs de monodromie sont des champs primaires de Virasoro et dans le cas de P_{VI} leurs dimensions conformes sont $\Delta_\nu = \frac{1}{2}\,\mathrm{Tr}\,A_\nu^2 = \theta_\nu^2$, où $\nu = 0, 1, t, \infty$. Cela signifie que le développement de $\tau(t)$, disons autour de $t = 0$, est fortement contraint.

En effet, la symétrie conforme de dimension infinie fixe la structure de fonctions de corrélation et conduit à la notion de blocs conformes: ce sont les parties universelles chirales des corrélateurs qui correspondent aux différents choix de familles conformes intermédiaires dans les développements de produits d'opérateurs (OPE) des champs primaires. Le corrélateur chiral de quatre champs primaires génériques est donné par la somme

$$\langle \mathcal{O}_\infty(\infty)\mathcal{O}_1(1)\mathcal{O}_t(t)\mathcal{O}_0(0)\rangle = \sum_p C_{0t}^p C_{p\infty}^1 \mathcal{F}_c\left(\Delta, \Delta_p; t\right),$$

où l'indice de sommation énumère les familles conformes présentes dans l'OPE de $\mathcal{O}_t$ et $\mathcal{O}_0$, Δ_p note la dimension du champ primaire intermédiaire correspondant $\mathcal{O}_p$ et $\Delta = (\Delta_0, \Delta_t, \Delta_1, \Delta_\infty)$ est l'ensemble des dimensions extérieures. Le bloc conforme

$$\mathcal{F}_c\left(\Delta, \Delta_p; t\right) = \tag{0.4}$$



$$(t) = t^{\Delta_p - \Delta_0 - \Delta_t} \sum_{k=0}^{\infty} \mathcal{F}_{c,k}\left(\Delta, \Delta_p\right) t^k,$$

associé au canal p est une série de puissances de t habituellement normalisée par $\mathcal{F}_{c,0}\left(\Delta, \Delta_p\right) = 1$. Le reste de la série est complètement déterminé par la symétrie de Virasoro [11]. Les constantes de structure C_{0t}^p, $C_{p\infty}^1$ rassemblent les blocs conformes en fonctions de corrélation de théories spécifiques; elles ne sont pas fixées par l'invariance conforme et doivent être obtenues séparément.

Le calcul des coefficients $\mathcal{F}_{c,k}(\Delta, \Delta_p)$ du bloc conforme est un problème purement algébrique. Il devient assez laborieux avec l'augmentation de k. Cependant, très récemment (2009–10) ce problème a été complètement résolu dans le cadre de la conjecture célèbre de Alday-Gaiotto-Tachikawa (AGT) [4] qui relie les blocs conformes de l'algèbre de Virasoro et les fonctions de partition instantoniques des théories de Yang-Mills supersymétriques en quatre dimensions avec $\mathcal{N} = 2$ [21, 53, 114, 115].

Entre autre, la correspondance AGT produit des évaluations combinatoires des blocs conformes, subséquemment démontrées par V. A. Alba, V. A. Fateev, A. V. Litvinov et G. M. Tarnopolsky [2]. De telles représentations combinatoires expriment la contribution des descendants de niveau fixe au bloc conforme à quatre points $\mathcal{F}_c(\Delta, \Delta_p; t)$ en termes de sommes de fonctions simples explicites de Δ, Δ_p et c par rapport aux bipartitions de taille fixe.

Approche conforme heuristique

Par conséquent, pour obtenir le développement complet de la fonction tau de P_{VI} il suffirait de déterminer le spectre des champs primaires qui apparaissent dans les OPEs de champs de monodromie et les constantes de structure associées. A cette fin supposons que l'ensemble des champs de monodromie est fermé par rapport à l'algèbre des OPEs. La conservation de la monodromie alors implique que le spectre des dimensions des primaires présents dans l'OPE de $\mathcal{O}_{M_t}(t)\mathcal{O}_{M_0}(0)$ est discret: il a la forme $(\sigma + \mathbb{Z})^2$, où $2\cos 2\pi\sigma = \operatorname{Tr} M_0 M_t$. Les constantes de structure correspondantes peuvent être calculées à partir de la formule asymptotique de Jimbo [79] interprétée comme une relation de récurrence. Après un changement de normalisation des blocs conformes, le résultat final [61] prend la forme de transformation de Fourier

$$\tau(t) = \mathrm{const} \cdot \sum_{n \in \mathbb{Z}} e^{in\tau} \mathcal{F}_1\left(\Delta, (\sigma + n)^2; t\right). \tag{0.5}$$

Les paramètres complexes (σ, τ) correspondent à deux constantes d'intégration de Painlevé VI. Ils sont explicitement reliés à la monodromie du système Fuchsien (0.1).

Certaines caractéristiques de l'approche que nous venons d'esquisser sont plutôt inhabituelles du point de vue de la CFT. Par exemple, les fonctions de corrélation dans les théories conformes standards contiennent généralement les contributions des deux types de blocs conformes: holomorphes et anti-holomorphes. De plus ces corrélateurs vérifient la condition d'invariance sous l'action du groupe de tresses sur les positions des champs. Les corrélateurs chiraux qui représentent les fonctions tau isomonodromiques ne sont pas invariants par rapport à cette action; néanmoins ils se transforment de façon "covariante" induite par l'action de Hurwitz sur les matrices de monodromie.

Dans tous les cas, de chaque côté de la formule (0.5) nous avons des quantités bien définies qu'on peut calculer indépendamment. Le développement de $\tau(t)$ à courtes distances peut être calculé de façon récursive en utilisant P_{VI} et l'asymptotique dominante. Son homologue à droite peut être obtenu soit à partir des premiers principes de la CFT (ce qui revient à un calcul direct de l'inverse de la matrice de Kac-Shapovalov constituée des produits scalaires des états descendants de Virasoro), soit en utilisant les formules de Nekrasov pour les fonctions de partition instantoniques.

La façon la plus simple et la plus convaincante de tester la conjecture (0.5) est de comparer les deux développements ordre par ordre. Nous avons effectué de telles vérifications en montant jusqu'au niveau 10 en descendants (~ 30 premiers termes dans les développements asymptotiques). En outre, dans quelques cas particuliers où les solutions de P_{VI} sont connues explicitement, l'accord peut être vérifié à tout ordre. Ceci inclut les solutions elliptiques de Picard ainsi que les solutions hypergéométriques de Riccati, qui correspondent respectivement aux blocs conformes du modèle d'Ashkin-Teller [160] et aux corrélateurs contenants des champs dégénérés ou les blocs conformes des champs libres.

Blocs conformes de Virasoro et fonctions tau isomonodromiques

La partie suivante de ce mémoire est consacrée à la démonstration du développement conforme (0.5). En réalité, nous allons démontrer une affirmation plus générale [70].

Résultat principal: *Pour $c = 1$, les transformations de Fourier des blocs conformes de Virasoro à n points par rapport à leurs impulsions intermédiaires coïncident avec les fonctions tau associées aux déformations isomonodromiques de connexions plates logarithmiques de rang 2 sur la sphère de Riemann avec n points marqués $C_{0,n}$.*

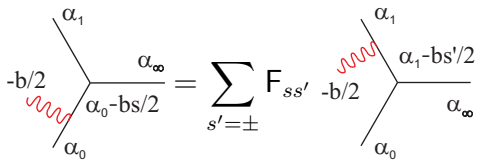
L'idée principale est de montrer d'abord que la transformation de Fourier des mêmes blocs conformes avec des insertions supplémentaires dégénérés au niveau 2 donne une solution constructive du problème de Riemann-Hilbert isomonodromique.

En effet, le prolongement analytique des blocs conformes par rapport aux positions des champs dégénérés peut être vu comme une suite de transformations élémentaires localisées de fusion et braiding, voir par exemple [3, 41]. Cette procédure que nous allons élaborer en détail plus loin peut être informellement expliquée comme ceci. L'OPE d'un champ primaire générique d'impulsion[3] α avec un autre champ d'impulsion $-b/2$

[3]Nous utilisons ici le paramétrage standard du modèle de Liouville: b est relié à la charge centrale de l'algèbre de Virasoro par $c = 1 + 6Q^2$, $Q = b + b^{-1}$ et les dimensions conformes sont données par $\Delta_\alpha = \alpha(Q - \alpha)$.

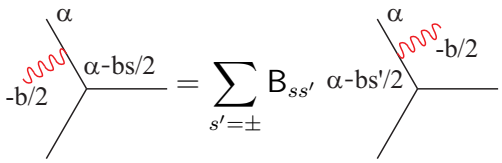
dégénéré au niveau 2 ne contient que deux familles conformes caractérisées par les impulsions $\alpha \pm b/2$. Alors il convient de représenter les transformations élémentaires de façon picturale:

- *Mouvement* S:



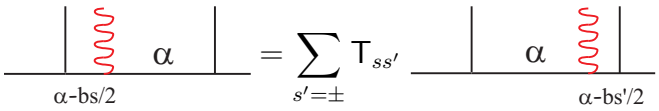
  Ici la ligne rouge ondulée correspond au champ dégénéré. La forme explicite de la matrice de fusion $\mathsf{F}_{ss'}$ (notons qu'elle est de taille 2×2) peut être calculée en utilisant les représentations hypergéométriques des blocs conformes à 4 points contenant un champ dégénéré.

- *Mouvement* B:



  Ce mouvement élémentaire correspond à un demi-tour du champ dégénéré autour d'un champ primaire générique. La matrice de monodromie correspondante $\mathsf{B}_{ss'}$ est diagonale.

- *Mouvement* T:



  Purement notationnelle, cette dernière transformation reste néanmoins extrêmement importante. Les éléments de la matrice $\mathsf{T}_{ss'}$ sont des *opérateurs* de translation d'impulsions intermédiaires de $\pm b/2$.

Par conséquent, le prolongement analytique des blocs conformes en positions des champs dégénérés le long des contours fermés donne toujours des combinaisons linéaires des blocs conformes de la même forme mais avec les impulsions internes éventuellement décalées.

Cela conduit à une monodromie opératorielle qui inclut des translations des impulsions intermédiaires. Pour $c = 1$, la transformation de Fourier diagonalise les opérateurs de translation et produit une monodromie ordinaire à valeurs dans $\mathrm{SL}(2, \mathbb{C})$, en donnant ainsi la solution recherchée du problème de Riemann-Hilbert. Les matrices de monodromie peuvent être calculées explicitement en termes de n impulsions extérieures, $n - 3$ impulsions intermédiaires, et $n - 3$ paramètres conjugués de la transformation de Fourier. Ces

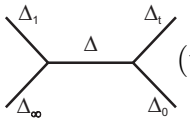
variables définissent un ensemble convenable de coordonnées locales sur l'espace $\mathcal{M}$ de données de la monodromie.

Exprimer les fonctions tau en termes de blocs conformes semble avoir un certain nombre d'avantages. Par exemple, la formule asymptotique de Jimbo pour l'équation de Painlevé VI [79] est une conséquence facile de cette relation. Plus généralement, on peut profiter de nombreux résultats connus sur les blocs conformes de Virasoro (en particulier de leurs représentations combinatoires explicites engendrées par la correspondance AGT) afin d'obtenir les informations détaillées sur les fonctions tau isomonodromiques. Pour donner une autre application intéressante, nous montrerons comment les solutions algébro-géométriques du système de Schlesinger [94, 32] s'obtiennent à partir des blocs conformes du modèle d'Ashkin-Teller critique [161, 164].

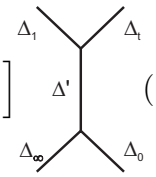
De la connexion à la fusion

Réciproquement, la relation des déformations isomonodromiques à la théorie des champs conformes fournit de nouveaux résultats non-triviaux sur les blocs conformes de l'algèbre de Virasoro. Un exemple intéressant de telles implications concerne les transformations de fusion, ou de la symétrie de croisement [71].

L'équivalence des différentes façons de décomposer la fonction de corrélation donnée en une somme sur les familles conformes intermédiaires suggère l'existence de transformations de dualité des blocs conformes. Celles-ci sont formalisées par le concept de groupoïde de Moore-Seiberg [111]. En particulier, il devrait exister une application linéaire élémentaire inversible reliant les blocs conformes de Virasoro à quatre points sur la sphère de Riemann, calculés dans les canaux s et t. Cela correspond à la dualité couplage fort/couplage faible du côté théorie de jauge de la correspondance AGT. Schématiquement, on pourrait écrire



$$(t) = \int d\Delta' \; K_c\left[\begin{array}{cc;c} \Delta_1 & \Delta_t & \Delta' \\ \Delta_\infty & \Delta_0 & \Delta \end{array}\right]$$



$$(1-t).$$

Le noyau intégral K_c de cette transformation est appelé la matrice de fusion. Sa forme explicite a été obtenue dans [132, 133] par la solution de certaines équations fonctionnelles (qui découlent du formalisme de Moore-Seiberg) à l'aide de la théorie de représentations du double modulaire de $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. Une démonstration alternative, basée sur les représentations "libre-champ" des opérateurs de vertex chiraux, a été proposée un peu plus tard dans [144, 145].

Les résultats de [132, 133, 144, 145] sont valables pour les valeurs complexes génériques de la charge centrale c de Virasoro. Malheureusement il n'est pas clair comment pourrait-on les étendre au segment semi-infini $c \in \mathbb{R}_{\leq 1}$, qui inclut un point particulièrement intéressant $c = 1$ [134, 142, 46] à la frontière entre les modèles minimaux et la théorie de Liouville.

Nous allons aborder ce dernier problème en exploitant la relation entre les déformations isomonodromiques et la théorie conforme. On va montrer que pour $c = 1$ la *matrice de fusion* coïncide, à un facteur élémentaire près, avec le *coefficient de connexion* de la fonction tau de Painlevé VI exprimé en termes de données de la monodromie du système linéaire associé. Les développements conformes de type (0.5) impliquent que ce coefficient vérifie certaines relations de récurrence. D'autre part, l'équivalence des différents points critiques de Painlevé VI peut être interprétée comme une sorte de symétrie de croisement. En reliant les paramètres des développements dans les différents canaux, elle rend les relations de récurrence hautement non-triviales et restrictives. Leur solution s'avère liée à la géométrie de Poisson de l'espace de modules de la monodromie et au volume complexifié du tétraèdre hyperbolique générique.

Problème C: *Obtenir une expression explicite pour la matrice de fusion des blocs conformes de Virasoro dans le cas $c = 1$ et résoudre le problème de connexion pour les fonctions tau des équations de Painlevé.*

Il est à noter que le problème de connexion (calcul des facteurs constants dans l'asymptotique de la fonction tau) à un fort intérêt indépendant. Le problème analytique principal dans les applications des équations de Painlevé à la description des fonctions de corrélation des modèles intégrables de la mécanique statistique et la théorie quantique des champs concerne leur comportement asymptotique à long temps et/ou à longues distances, avec un accent particulier sur les facteurs constants mentionnés ci-dessus.

Depuis les travaux célèbres de L. Onsager et B. Kaufman sur le modèle d'Ising dont les besoins mathématiques ont conduit à la naissance du théorème dit "fort" de Szegö dans la théorie des déterminants de Toeplitz (voir par exemple [35] pour l'histoire de la question), l'évaluation des termes constants dans l'asymptotique de différentes distributions et fonctions de corrélation de la théorie des matrices aléatoires et la théorie des modèles integrables statistiques a toujours été un grand défi du domaine. En plus du théorème de Szegö, on peut mentionner l'article de C. Tracy [147], où le "problème de constante" liée au modèle d'Ising a été résolu. C'était en effet la première solution rigoureuse d'un problème de connexion pour les fonctions tau de Painlevé (un cas particulier de Painlevé III).

Les progrès ultérieurs [98], [45], [36] concernent en particulier le calcul rigoureux de la

constante de Dyson [44] dans l'asymptotique du déterminant du noyau sinus qui décrit la probabilité de "gap" dans la distribution des valeurs propres de matrices aléatoires. Ce déterminant coïncide avec la fonction tau d'une solution particulière de Painlevé V. D'autres "problèmes de constantes" ont été considérés et résolus dans les articles [8], [22], [33], [100] et [7].

Comme nous l'avons déjà mentionné, le cadre naturel pour l'analyse asymptotique globale des équations de Painlevé est donné par la technique de Riemann-Hilbert. Cependant elle aborde directement les transcendantes standards de Painlevé et non pas les fonctions tau. Ces dernières sont des intégrales logarithmiques de certaines fonctions rationnelles de transcendantes de Painlevé — notamment les Hamiltoniens des équations correspondantes. D'où le problème: afin de résoudre le problème de connexion pour les fonctions tau, on doit être capable de calculer les intégrales de certaines combinaisons des transcendantes de Painlevé et leurs dérivées. Jusqu'à présent, ce problème a été traité avec succès seulement pour des solutions très particulières des équations de Painlevé. Les fonctions tau évaluées sur ces solutions admettent des représentations supplémentaires en termes de certains déterminants de Fredholm ou de Toeplitz. C'est cette propriété qui, malgré quelques différences dans les approches, était la raison principale du succès dans tous les travaux mentionnés dans le paragraphe précédent.

La situation avec les "problèmes des constantes" a changé après la découverte des développements conformes des fonctions tau. Leur périodicité manifeste par rapport aux données de la monodromie produit certaines relations fonctionnelles qu'on peut résoudre [71, 77] en utilisant une interprétation hamiltonienne de la paramétrisation des transcendantes de Painlevé par la monodromie.

Par exemple, comme toute autre équation de Painlevé, l'équation P_{VI} est un système hamiltonien non-autonome classique. On peut choisir les données de monodromie de sorte à former une paire de coordonnées canoniquement conjuguées sur l'espace de ses solutions. La correspondance entre les paramètres asymptotiques des différrents points critiques peut être interprétée comme une transformation canonique reliant deux ensembles différents de telles coordonnées. Les coefficients de connexion qui nous intéressent sont étroitement liés aux *fonctions génératrices* de ces transformations. Nous montrerons que cette observation permet, en fin de compte, de les calculer explicitement.

Organisation du mémoire

Le contenu de ce mémoire est essentiellement basé sur quatre articles [61], [62], [70] et [71]. La Section 2 correspond aux deux premières publications. Dans cette section on montre comment la relation entre les blocs conformes de Virasoro et les équations de Painlevé peut être obtenue de façon heuristique. Les résultats principaux — notamment les séries combinatoires donnant les solutions générales de Painlevé VI, V et trois équations inéquivalentes de Painlevé III — sont formulés dans la Section 2.7.

Dans la Section 3, en suivant l'article [70], nous montrons que la solution du problème de Riemann-Hilbert isomonodromique pour les systèmes Fuchsiens de rang 2 sur la sphère de Riemann avec n points marqués $C_{0,n}$ et la fonction tau associée sont données par les transformations de Fourier des blocs conformes avec $c = 1$. Dans le cas $n = 4$ cela fournit une démonstration des conjectures formulées dans la section précédente. La Section 4 est basée sur l'article [71] et est consacrée à la solution du problème de connexion pour la fonction tau de Painlevé VI et à l'obtention de la matrice de fusion des blocs conformes pour $c = 1$.

D'autres résultats de l'auteur sur des sujets reliés (mais qui n'ont pas été traités dans ce mémoire) incluent [99, 100] (l'étude asymptotique d'une famille de solutions de P_{VI} données par des déterminants de Fredholm), [102] (classification des solutions algébriques de P_{VI}), et [77] (solution du problème de connexion pour la fonction tau de l'équation de sine-Gordon radiale, liée à l'équation $\mathrm{P}_{\mathrm{III}}$ la plus dégénérée et les blocs conformes irréguliers).

Conclusion et pistes de recherche

Dans ce mémoire d'habilitation, nous avons présenté quelques résultats de ces dernières années visant à mieux développer la compréhension des liens entre la théorie des champs conformes en dimension 2 et les déformations isomonodromiques. En se référant au commentaire de H. Poincaré ci-dessus, on espère convaincre le lecteur que la théorie conforme dresse un pont entre l'île splendide de Painlevé et le continent de la théorie des représentations.

La solution du problème de Riemann-Hilbert isomonodromique pour les systèmes Fuchsiens de rang 2 sur $\mathbb{P}^1$ et la fonction tau associée ont été reliées aux blocs conformes de l'algèbre de Virasoro de charge centrale $c = 1$. En combinaison avec la correspondance AGT, cette relation permet d'écrire des représentations explicites des transcendantes génériques de Painlevé VI, V et III sous forme de séries combinatoires. Elle permet également de résoudre certains problèmes asymptotiques tels que le calcul du coefficient de connexion pour la fonction tau de P_{VI}, et de décrire les transformations de fusion des blocs conformes avec $c = 1$.

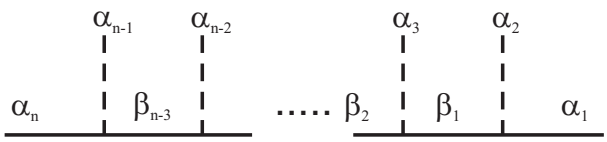
Qu'en-est-il au cas irrégulier? Dans un article récent [77], nous avons étudié un exemple particulier de $\mathrm{P}_{\mathrm{III}_3}$ — l'equation de Painlevé III la plus dégénérée, équivalente à l'équation radiale de sine-Gordon. Les blocs conformes irréguliers figurant dans le développement à courtes distances de la fonction tau de $\mathrm{P}_{\mathrm{III}_3}$ ont la forme la plus simple possible: ils coïncident avec les normes des états de Gaiotto. Ces derniers sont les vecteurs de Whittaker qui annulent les générateurs $L_{n\geq 2}$ de Virasoro et diagonalisent L_1. La surprise expérimentale principale est que le développement de la fonction tau *à longues distances*, que l'on ne peut pas obtenir à partir des développements conformes de Painlevé VI par une limite de scaling, met en évidence une périodicité très similaire par rapport aux données de la monodromie irrégulière.

On peut donc s'attendre à ce que cette périodicité soit un phénomène général dans les problèmes isomonodromiques. Elle porte une certaine analogie avec la conjecture de Fisher-Hartwig généralisée dans la théorie de déterminants de Toeplitz [9, 34]. De plus, elle suggère l'existence de nouvelles bases distinguées des espaces de blocs conformes irréguliers, et il serait très intéressant de comprendre leur origine algébrique. Comme il a été montré dans [77], cette périodicité permet aussi de calculer les coefficients de connexion pour les fonctions tau correspondantes.

Du point de vue général on pourrait s'attendre à ce que les problèmes isomonodromiques dans le rang $N > 2$ soient reliés aux $\mathfrak{sl}_N$-théories de Toda et les blocs conformes des algèbres $\mathcal{W}_N$ [47, 159]. Ces algèbres apparaissent lors de la quantification de la sec-

onde structure de Poisson des hiérarchies N-KdV. Cependant en essayant de rendre la correspondance aussi explicite et précise que dans le cas $N = 2$ on rencontre un certain nombre de difficultés. Du côté de la CFT, les blocs conformes ne sont pas complètement déterminés par l'invariance $\mathcal{W}$. Du côté isomonodromique, les solutions des problèmes "élémentaires" (tels que le problème de Riemann-Hilbert sur $C_{0,3}$ ou le problème asymptotique de Jimbo sur $C_{0,4}$) ne sont pas connues pour le rang supérieur.

Cependant il existe des cas particuliers intéressants où la généralisation de notre formalisme semble être assez abordable. Ils concernent les systèmes Fuchsiens de type spectral spécial, où deux matrices de monodromie restent génériques alors que toutes les autres sont données par des reflections complexes. Par exemple, le problème de Riemann-Hilbert sur $C_{0,3}$ dans ce cas devient soluble en termes de fonctions hypergéométriques généralisées de type ${}_NF_{N-1}$, et l'asymptotique de la fonction tau sur $C_{0,4}$ peut être obtenue en étudiant l'interaction entre deux systèmes Fuchsiens hypergéométriques de ce type. Il semblerait que cette situation spéciale soit liée aux blocs conformes de l'algèbre $\mathcal{W}_N$ de la forme



où les impulsions α, β appartiennent au dual de la sous-algèbre de Cartan de $\mathfrak{sl}_N$. Les lignes pointillées représentent les champs semi-dégénérés dont les impulsions sont proportionnelles à un poids fondamental de $\mathfrak{sl}_N$. Les transformations de fusion et braiding de ces blocs conformes par rapport aux positions d'insertions supplémentaires complètement dégénérées peuvent être décrites en utilisant les résultats de [48, 49], voir aussi la prépublication très récente [23]. Il est alors plausible que l'ensemble du programme de notre Section 3 puisse être réalisé dans ce cas. On notera également que les blocs conformes spéciaux de $\mathcal{W}_N$ représentés sur la figure ci-dessus sont précisément ceux pour lesquels la correspondance AGT fournit des représentations combinatoires (qui, en outre, ont été démontrées dans [50]).

Une approche possible à la démonstration des relations de type (0.5) entre les fonctions tau des équations de Painlevé et les blocs conformes est de montrer que ces derniers vérifient certaines relations bilinéaires différentielles récursives [76]. Cela a été accompli dans un travail récent [13] de M. Bershtein et A. Shchechkin. L'idée de [13] est d'utiliser un plongement $\mathsf{Vir} \oplus \mathsf{Vir} \subset \mathsf{F} \oplus \mathsf{NSR}$ de deux copies de l'algèbre de Virasoro dans l'algèbre de Neveu-Schwarz-Ramond étendue par les fermions libres. Une des caractéristiques intéressantes de cette approche est qu'elle donne une déformation

continue (à un paramètre) des relations bilinéaires nécessaires. Le rôle du paramètre de déformation est joué par la charge centrale de NSR qui détermine à son tour les charges des deux Virasoro (en général, inégales). Le cas de Painlevé correspond au point particulier $c_{\mathsf{NSR}} = c_{\mathsf{Vir}_{1,2}} = 1$. La question intéressante est de comprendre quelle serait la déformation appropriée des équations de Painlevé et fonctions tau.

Il existe d'autres relations entre le problème de déformation isomonodromique et la théorie des champs conformes en deux dimensions. En particulier, la limite semiclassique ($c \to \infty$) des équations de decouplage de vecteurs nuls dans la théorie de Liouville conduit à des équations de type Hamilton-Jacobi qui définissent les Hamiltoniens des flots isomonodromiques [146, 113, 101]. Il semble assez remarquable qu'il existe des liens entre les fonctions tau et les blocs conformes de Virasoro à la fois pour $c = 1$ et $c \to \infty$; néanmoins une bonne explication reste à trouver.

1. Introduction

1.1 Painlevé property

At the turn of the twentieth century there have been repeated attempts to describe ordinary differential equations in terms of singularities of their solutions in the complex domain. For nonlinear ODEs this task is quite difficult to tackle, as in general the positions of singularities and even their type may depend on the initial conditions: the singular points become movable. A more refined version of the problem is to classify the equations with predictable branching by allowing only movable poles, movable single-valued essential singularities and fixed singular points of any type. The latter requirement is nowadays known as the *Painlevé property*.

The classification of algebraic 1st order ODEs of Painlevé type was achieved by L. Fuchs [57] in 1884, with further developments by H. Poincaré [131]. It turns out that such equations are either reducible to linear ones or can be solved in terms of elliptic functions. The treatment of the problem is considerably simplified by the fact that movable singular points of the 1st order equations can only be poles or algebraic branch points.

When the order of an ODE is 2 or higher, one cannot *a priori* exclude more complicated movable singularities, such as essential singular points or natural boundaries. This led E. Picard to express his sceptical opinion about the possibility of classification in a letter to G. Mittag-Leffler in 1893. Nevertheless in 1900–1910 P. Painlevé [127, 128] and B. Gambier [63] undertook a remarkable attempt of classifying the 2nd order, 1st degree ODEs without movable branch points. Such equations have the form

$$q'' = F\left(t, q, q'\right),$$

where t denotes the independent variable, F is rational in q, q' and locally analytic in t. The outcome of Painlevé and Gambier studies is a list of 50 equivalence classes of ODEs, most of which are can be reduced to linear equations or integrated by quadratures. There remain, however, six irreducible nonlinear ODEs — the Painlevé equations:

$$\begin{aligned} q'' &= \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)(q')^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)q' + \\ &\quad + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{q^2}+\frac{\gamma(t-1)}{(q-1)^2}+\frac{\delta t(t-1)}{(q-t)^2}\right), \end{aligned} \tag{$\mathrm{P_{VI}}$}$$

$$q'' = \left(\frac{1}{2q}+\frac{1}{q-1}\right)(q')^2 - \frac{q'}{t} + \frac{(q-1)^2}{t^2}\left(\alpha q+\frac{\beta}{q}\right)+\frac{\gamma q}{t}+\frac{\delta q(q+1)}{q-1}, \tag{$\mathrm{P_V}$}$$

$$q'' = \frac{(q')^2}{2q} + \frac{3}{2}q^3 + 4tq^2 + 2\left(t^2-\alpha\right)q + \frac{\beta}{q}, \tag{$\mathrm{P_{IV}}$}$$

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} + \frac{\alpha q^2 + \beta}{t} + \gamma q^3 + \frac{\delta}{q}, \tag{$\mathrm{P}_{\mathrm{III}}$}$$

$$q'' = 2q^3 + tq + \alpha, \tag{P_{II}}$$

$$q'' = 6q^2 + t. \tag{P_{I}}$$

This list of exceptional differential equations remained unchanged after subsequent more careful reconsiderations of the Painlevé-Gambier approach, see e.g. [72, Chapter 14], [24] and the discussion in [30].

Increasing the order and/or degree makes the classification problem much more involved. Movable essential singularities and natural boundaries are known to be possible in this case: they have been observed in the exact solutions of certain explicit examples. The Painlevé type differential equations of the 2nd order and 2nd degree of the form $(q'')^2 = F(t, q, q')$ have been described in [30]. Partial results are also available for the 3rd order, 1st degree equations [27, 64, 25]. One of the interesting features of these developments is their relation to the classical Painlevé equations. For example, the exceptional ODEs of the former class are precisely the equations satisfied by the non-autonomous Hamiltonians of (P_{I})–(P_{VI}).

1.2 Monodromy preserving deformation

The original motivation of Painlevé and his students may seem somewhat esoteric from a modern perspective and was perceived as such even by some of their contemporaries. A famous quote, attributed in [68] to H. Poincaré, describes Painlevé's work as follows:

> *Les Mathématiques constituent un continent solidement agencé, dont tous les pays sont bien reliés les uns aux autres; l'oeuvre de Paul Painlevé est une ile originale et splendide dans l'océan voisin.*[4]

The situation has changed when it was realized that Painlevé equations are the simplest examples of equations describing monodromy preserving deformations of linear ODEs with rational coefficients. The first step in this direction [58] was made in 1905 by R. Fuchs (son of L. Fuchs). He considered 2nd degree Fuchsian differential equations with 5 singular points on the Riemann sphere, one of the singularities being apparent. These points can be put to 0, 1, ∞, t and q (the last one corresponds to the apparent singularity) using the Möbius transformations. The discovery of Fuchs was that the condition for the monodromy of the equation to be preserved under variation of t leads to Painlevé VI equation

[4] The Mathematics is a tightly arranged continent, with all countries well connected to each other; the work of Paul Painlevé is an original beautiful island in the neighboring ocean.

for q.[5] In a more general situation of $n+3$ genuine and n apparent singular points the isomonodromy requirement gives rise to the so-called Garnier system $\mathcal{G}_n$ of nonlinear PDEs [64, 65]; see also [78] for a nice review.

We are going to use a related framework, developed by L. Schlesinger [141] in 1912, where instead of working with a scalar equation of higher order one deals with a Fuchsian system of linear 1st order ODEs of arbitrary rank and with an arbitrary number of singular points on $\mathbb{P}^1$. Let us briefly outline the corresponding isomonodromy formulation of P_{VI} (which is going to be our main illustrative example) leaving the details and the discussion of a more general situation to the main text.

Consider a rank 2 Fuchsian system with four regular singular points $0, t, 1, \infty$ on the Riemann sphere:

$$\partial_z \Phi = \mathcal{A}(z)\,\Phi, \qquad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}. \tag{1.1}$$

Here the matrices $A_\nu \in \mathfrak{sl}_2(\mathbb{C})$ ($\nu = 0, t, 1$) are independent of z and have eigenvalues denoted by $\pm\theta_\nu$. Also, set $A_\infty := -A_0 - A_t - A_1 = \mathrm{diag}\,\{\theta_\infty, -\theta_\infty\}$. The diagonal form of A_∞ involves almost no loss of generality and corresponds to a convenient choice of the gauge.

The fundamental matrix $\Phi(z)$ is generically multivalued on $C_{0,4} := \mathbb{P}^1 \backslash \{0, t, 1, \infty\}$, as its analytic continuation along non-contractible closed loops may produce nontrivial monodromy. The fundamental group $\pi_1(C_{0,4})$ is generated by the loops χ_ν ($\nu = 0, t, 1, \infty$) shown in Fig. 2, to which we associate monodromy matrices $M_\nu \in \mathrm{SL}(2, \mathbb{C})$. Note that they satisfy the constraint $M_\infty M_1 M_t M_0 = \mathbf{1}$.

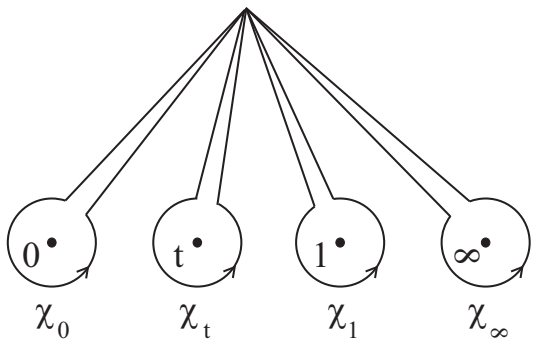


Fig. 2: Basis of $\pi_1(C_{0,4})$

One thereby obtains an anti-homomorphism $\rho : \pi_1(C_{0,4}) \to \mathrm{SL}(2, \mathbb{C})$ called monodromy representation. Different choices of the basis of solutions of (1.1) lead to equivalent repre-

[5] Actually [58] is the paper where the general P_{VI} equation was reported for the first time; it was missing in the initial Painlevé classification because of computational errors.

sentations of $\pi_1\left(C_{0,4}\right)$. The idea of the monodromy preserving deformation is to construct a family of linear systems that share a given equivalence class of monodromy.

Schlesinger equations of the isomonodromic deformation of the Fuchsian system (1.1) are given by

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}. \tag{1.2}$$

The parameter set $\mathcal{P}$ of (1.1) may be described in terms of a pair of scalar functions of t. One of them can be chosen as the unique nontrivial zero of the off-diagonal element $\mathcal{A}_{12}(z)$. The isomonodromy equations turn out to be equivalent to Painlevé VI equation satisfied by the latter quantity. Four P_{VI} parameters are expressed in terms of the eigenvalues of $A_{0,t,1,\infty}$ as $(\alpha,\beta,\gamma,\delta) = \left(\left(\theta_\infty + \frac{1}{2}\right)^2, -\theta_0^2, \theta_1^2, \frac{1}{4} - \theta_t^2\right)$. The reader interested in the computational details is referred to [29, lectures of G. Mahoux].

The other five Painlevé equations may be put into analogous context by considering linear systems with irregular singularities (higher order poles). The irregular monodromy data have more intricate structure and include Stokes matrices. The corresponding extension of the theory of monodromy preserving deformations has been developed in the beginning of 80s by H. Flaschka and A. Newell [52], and by M. Jimbo, T. Miwa and K. Ueno [82, 83].

We note in passing that, as was known already to P. Painlevé, the equations $\mathrm{P}_{\mathrm{I-V}}$ can be obtained from P_{VI} in certain scaling limits. These limits correspond to successive confluences of the singular points of the initial Fuchsian system and provide an alternative way to derive the associated irregular linear problems.

1.3 Riemann-Hilbert correspondence

The above construction defines the monodromy map

$$\mathfrak{m}: \begin{array}{c}\text{parameter set } \mathcal{P}\\ \text{of the linear problem}\end{array} \longrightarrow \begin{array}{c}\text{moduli space } \mathcal{M}\\ \text{of monodromy data}\end{array} \tag{1.3}$$

sometimes referred to as the Riemann-Hilbert correspondence. In a more general setting, it relates the spaces of flat logarithmic connections on the punctured Riemann surfaces and the spaces of representations of their fundamental groups.

In the P_{VI} case, the triple (t, q, q') provides a set of local coordinates on $\mathcal{P}$. To every branch of every Painlevé VI solution the map $\mathfrak{m}$ assigns a conjugacy class of a triple of monodromy matrices representing a point of the monodromy manifold

$$\mathcal{M} = \mathrm{Hom}\left(\pi_1\left(C_{0,4}\right), G\right)/G \cong G^3/G, \qquad G = \mathrm{SL}(2,\mathbb{C}). \tag{1.4}$$

A rough dimension count shows that $\dim \mathcal{M} = 6$. Four local monodromy exponents around the regular singularities are related to four P_{VI} parameters $\theta_{0,t,1,\infty}$. The remaining pair of monodromy parameters describes the initial conditions. In other words, the monodromy data form a complete set of P_{VI} integrals of motion.

A regain of interest to Painlevé equations occured in the 80-90s, more than half of a century after their discovery. Its first motivation came from applications. It was realized that exact ODE reductions of the classical integrable nonlinear partial differential equations (like Korteweg-de Vries, nonlinear Schrödinger or sine-Gordon equation) lead to Painlevé equations, see e.g. [1]. Other remarkable domains of applications are the random matrix theory [80, 148, 149, 55] and exactly solvable models of statistical mechanics and quantum field theory [158, 137, 138, 139, 155, 81, 51, 162, 12], in particular the two-dimensional Ising model.

Another important reason of this surge of interest was the development of new methods of global analysis of Painlevé functions, based on the Riemann-Hilbert correspondence. The central idea was inspired by the powerful inverse scattering techniques for integrable PDEs. We informally state it as follows: the questions about the *nonlinear* equation should be asked in terms of monodromy of the associated *linear* problem, and the answers extracted from the analysis of the latter.

Practical implementation of this philosophy in Painlevé theory was initiated in 1982 by the foundational paper of M. Jimbo [79]. In this work, the asymptotics of the generic P_{VI} transcendent near the critical points (i.e. as $t \to 0, 1$ or ∞) was expressed in terms of monodromy of the linear system (1.1). This automatically solves the connection problem of relating the asymptotic parameters at different critical points. Various non-generic cases have been recently discussed in [67]. Connection formulae for the other Painlevé transcendents have been studied in [84, 85, 90] (for P_{I}), [74, 73, 37] (for P_{II}), [109, 119, 120, 86, 87, 95] (for $\mathrm{P}_{\mathrm{III}}$), [88, 75] (for P_{IV}) and [107, 108, 5] (for P_{V}). For a modern exposition of the Riemann-Hilbert approach to Painlevé equations, see [54].

1.4 Are they solvable?

The Riemann-Hilbert formalism puts on equal footing the solutions of nonlinear Painlevé equations and classical special functions solving linear ordinary differential equations of hypergeometric type. More precisely, it enables us to realize (at least in principle) the following program:

- introduce a convenient labeling of Painlevé functions of different types by the monodromy data of the appropriate auxiliary linear problems,

- express their asymptotics near the critical points in terms of monodromy.

One could further try to design numerical codes for computation of solutions of Painlevé equations outside the critical points using the known asymptotics as initial conditions. This suggestion is one of the ideas behind the recently initiated Painlevé Project [31].

Can we go beyond the asymptotics? It is of course straightforward (though still technically nontrivial because of nonlinearity) to compute subleading corrections to the asymptotics from Painlevé equations order by order. However, the real question is: can we obtain an explicit non-iterative description of the inverse $\mathfrak{m}^{-1}$ of monodromy map?

Problem A: *Construct general solutions of Painlevé equations in terms of monodromy data of the associated linear problems.*

At first sight, this problem looks desperately unsolvable. Indeed, since the very discovery of Painlevé equations there has been a debate on whether they can be integrated in terms of elementary or known special functions. A function of one variable t is called classical [152] if it can be obtained from the field $\mathbb{C}(t)$ of rational functions by a finite iteration of certain permissible operations such as arithmetic operations, quadratures, solving linear ODEs with classical coefficients, substitution into an abelian function, etc. The general solutions of Painlevé equations are nowadays known to be transcendental in this sense [117, 151, 153, 121, 154, 156, 157, 26]. Classical solutions appear only for special values of parameters and constrained initial conditions, and have been fully described.

For instance, in the P_{VI} case the results of H. Watanabe [157] suggest that any solution is either a) of Riccati type or b) algebraic or c) transcendental[6]. The Riccati solutions are expressed in terms of Gauss hypergeometric functions and are known explicitly. They arise when the monodromy matrices of the linear system (1.1) can be simultaneously brought to upper triangular form or when one of these matrices is equal to $\pm\mathbf{1}$.

Algebraic P_{VI} solutions have been classified by the author and Yu. Tykhyy in 2008 [102]. The classification idea was based on the Riemann-Hilbert correspondence: analytic continuation of a solution induces an action of the pure braid group on 3 strands on the space of monodromy data. The solutions with finite branching therefore lead to finite braid group orbits. Their description reduces to the classification of periodic orbits of a discrete polynomial dynamical system on $\mathbb{C}^3$. The existence of certain trigonometric-diophantine constraints implies that the set of possible finite orbit candidates is finite (but has very large cardinality $\sim 10^8$). We have selected the genuine finite orbits by performing a computer-assisted check of the elements of this set one by one. The final result

[6]The transcendental solutions somewhat counter-intuitively include the Picard family of elliptic solutions [130, 105]. Up to Bäcklund transformations, the corresponding P_{VI} parameters are given by $\theta_0 = \theta_t = \theta_1 = 0$, $\theta_\infty = \frac{1}{2}$, and there are no constraints on the initial conditions.

is as follows: algebraic P_{VI} solutions are either a) special cases of the hypergeometric Riccati solutions or b) special cases of the elliptic Picard solutions or c) belong to one of three explicit continuous families or d) are equivalent to one of the 45 exceptional algebraic solutions, known from the earlier works of F. V. Andreev and A. V. Kitaev [6], P. Boalch [14, 15, 16, 17], B. A. Dubrovin [42], B. A. Dubrovin and M. Mazzocco [43], and A. V. Kitaev [91, 92]. Note that for the other Painlevé equations the classification of algebraic solutions is almost trivial since the fundamental group of the complement of the critical locus in that case is abelian.

In spite of the transcendency aspect, general solutions of Painlevé equations share many properties of the classical special functions (such as Gauss hypergeometric function, Whittaker, Hermite, Bessel, ...). In addition to solvable connection problems, one could mention the existence of various symmetries, recurrence relations, differentiation formulas (Bäcklund-Okamoto transformations [123, 124, 125, 126], see also [55, Chapter 8] for a review) and algebraic transformations of different orders [89, 96, 106]. The major difference between Painlevé transcendents and the special functions of hypergeometric type is the lack of a transparent relation to representation theory of Lie groups and algebras. This is the conceptual reason of the absence of explicit series or integral representations of Painlevé functions. It leads us to

Problem B: *Include Painlevé functions into the framework of representation theory*.

The present manuscript describes an attempt to address the Problems A and B via the two-dimensional conformal field theory (CFT). We are going to show that the *tau functions* of Painlevé VI, V and three inequivalent Painlevé III equations are very simply related to *conformal blocks* of the Virasoro algebra with central charge $c = 1$ and their irregular versions. In particular, this provides explicit solutions of the corresponding inverse monodromy problems in the form of combinatorial series.

1.5 Tau functions and CFT

A first striking relation of monodromy preserving deformations and quantum field theory was exhibited by M. Sato, T. Miwa and M. Jimbo (SMJ) in a series of papers that appeared at the end of 70s, see in particular [135, 136], and [140] for a review. Their results include the identification of the isomonodromic tau functions of rank N Fuchsian systems with an arbitrary number of regular singular points on the Riemann sphere with certain correlation functions in a quantum field theory describing N free massless chiral fermion copies.

The tau functions are important for a number of reasons:

- They are the generating functions for the Hamiltonians of the isomonodromic flows

[82, 83]. Furthermore they coincide with regularized determinants of Cauchy-Riemann operators whose domains consist of multivalued functions with appropriate monodromy [129]. Their definition has also been extended to the irregular case.

- It is the tau functions instead of Painlevé transcendents themselves (though both are directly related to each other) that typically show up in the applications of Painlevé equations, representing gap probabilities in random matrix theory, correlation functions of the Ising model and sine-Gordon field theory at the free-fermion point, etc.

Besides fermions, the correlators in SMJ theory involve local fields of another type (below we use the term "monodromy fields" instead of SMJ's "holonomic") which can be seen in the operator formalism as Bogoliubov transformations of the fermion algebra ensuring the required monodromy properties. Isomonodromic tau function was originally defined as the correlation function of monodromy fields. In particular, for Painlevé VI it is given by a four-point correlator

$$\tau(t) = \langle \mathcal{O}_{M_\infty}(\infty)\mathcal{O}_{M_1}(1)\mathcal{O}_{M_t}(t)\mathcal{O}_{M_0}(0)\rangle. \tag{1.5}$$

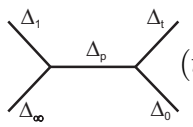
In order to make such formal expressions suitable for actual computations, the characterization of monodromy fields should be made more precise. We are going to achieve this using the tools of conformal field theory [11, 38, 164].

We will start by giving an heuristic argument that the relevant CFT has central charge $c = N - 1$, i.e. for Painlevé VI one has $c = 1$. The monodromy fields are Virasoro primary fields and in the P_{VI} case they have dimensions $\Delta_\nu = \frac{1}{2}\operatorname{Tr} A_\nu^2 = \theta_\nu^2$, where $\nu = 0, 1, t, \infty$. Therefore the expansion of $\tau(t)$, say, near $t = 0$, is strongly constrained.

Indeed, the infinite-dimensional conformal symmetry determines the structure of correlation functions and leads to the notion of conformal blocks: these are universal chiral parts of correlators corresponding to different choices of intermediate conformal families in the successive operator product expansions (OPEs) of primary fields. The chiral correlator of four generic primaries for arbitrary central charge is given by the sum

$$\langle \mathcal{O}_\infty(\infty)\mathcal{O}_1(1)\mathcal{O}_t(t)\mathcal{O}_0(0)\rangle = \sum_p C_{0t}^p C_{p\infty}^1 \mathcal{F}_c\left(\Delta, \Delta_p; t\right), \tag{1.6}$$

where the summation index runs over conformal families appearing in the OPE of $\mathcal{O}_t$ and $\mathcal{O}_0$, Δ_p denotes the dimension of the corresponding intermediate primary field $\mathcal{O}_p$ and $\Delta = (\Delta_0, \Delta_t, \Delta_1, \Delta_\infty)$ is the set of external dimensions. Conformal block

$\mathcal{F}_c\left(\Delta, \Delta_p; t\right) =$ 

$(t) = t^{\Delta_p - \Delta_0 - \Delta_t} \sum_{k=0}^{\infty} \mathcal{F}_{c,k}\left(\Delta, \Delta_p\right) t^k,$ (1.7)

associated to the channel p is a power series usually normalized by $\mathcal{F}_{c,0}(\Delta, \Delta_p) = 1$. The rest of the series is completely fixed by the Virasoro symmetry [11]. The structure constants C^p_{0t}, $C^1_{p\infty}$ combine conformal blocks into correlation functions of specific theories and should be obtained from another source.

The computation of the conformal block coefficients $\mathcal{F}_{c,k}(\Delta, \Delta_p)$ is a purely algebraic problem. It becomes quite laborious with the growth of k. However, very recently (2009–10) this problem was completely solved in the framework of the Alday-Gaiotto-Tachikawa (AGT) conjecture [4] relating Virasoro conformal blocks and instanton partition functions of $\mathcal{N} = 2$ supersymmetric gauge theories [21, 53, 114, 115]. The AGT correspondence produces conjectural combinatorial evaluations of conformal blocks, subsequently proved by V. A. Alba, V. A. Fateev, A. V. Litvinov and G. M. Tarnopolsky [2]. In particular, such combinatorial representations express the contribution of fixed level of descendants to the four-point conformal block $\mathcal{F}_c(\Delta, \Delta_p; t)$ in terms of sums of simple explicit functions of Δ, Δ_p and c over bipartitions of fixed size.

Hence, to obtain full expansion of the P_{VI} tau function, it suffices to determine the dimension spectrum of primaries present in the OPEs of monodromy fields and the associated structure constants. To this end let us assume that the set of monodromy fields closes under the OPE algebra. The conservation of monodromy then implies that the dimension spectrum of primary fields appearing in the OPE of $\mathcal{O}_{M_t}(t)\mathcal{O}_{M_0}(0)$ is discrete and has the form $(\sigma + \mathbb{Z})^2$, where $2\cos 2\pi\sigma = \operatorname{Tr} M_0 M_t$. The corresponding structure constants can be obtained from the Jimbo's asymptotic formula [79] interpreted as a recursion relation. After a suitable change of normalization of conformal blocks, the final result [61] can be written as a Fourier transform

$$\tau(t) = \mathrm{const} \cdot \sum_{n\in\mathbb{Z}} e^{in\tau} \mathcal{F}_1\left(\Delta, (\sigma + n)^2; t\right). \tag{1.8}$$

Complex parameters (σ, τ) correspond to two Painlevé VI integration constants and are explicitly related to the monodromy of the Fuchsian system (1.1).

Some features of the above setup may seem unusual from the CFT point of view. For instance, CFT correlation functions usually contain contributions of both holomorphic and antiholomorphic conformal blocks and are subject to the requirement of invariance under the action of braid group on the positions of fields. The chiral correlators representing the isomonodromic tau functions are not invariant under this action but transform in a natural way, induced by the Hurwitz action on monodromy matrices.

In any case, at both sides of the conjectured formula (1.8) we have well-defined quantities that can be calculated independently. The short-distance expansion of $\tau(t)$ may be recursively computed from P_{VI} starting from the leading asymptotics. Its counterpart on

the right may be found either from the CFT first principles (which amounts to direct inversion of the Kac-Shapovalov matrix of scalar products of Virasoro descendant states) or using Nekrasov formulas for instanton partition functions. The most straightforward and convincing way to test the conjecture (1.8) is to compare both expansions order by order. We performed such checks going up to level 10 in descendants (~ 30 first terms in the asymptotic expansion). Also, in a few special cases where P_{VI} solutions are known explicitly, the check can be carried out to arbitrary order. This includes Picard elliptic solutions as well as solutions of Riccati type, which correspond to Ashkin-Teller conformal blocks [160] and the correlators involving degenerate fields or free-field conformal blocks.

The next part of the manuscript is devoted to the proof of the conformal expansion (1.8). In fact we are going to demonstrate a more general statement [70].

Main result: *The tau functions for monodromy preserving deformations of flat logarithmic* $\mathrm{SL}(2,\mathbb{C})$*-connections on* n*-punctured spheres coincide with Fourier transforms of the* n*-point Virasoro conformal blocks at* $c=1$ *with respect to the intermediate momenta.*

The principal idea is to show that the Fourier transforms of the same conformal blocks with additional level 2 degenerate insertions provide a constructive solution of the appropriate isomonodromic Riemann-Hilbert problem.

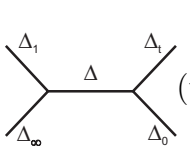
As a matter of fact, the analytic continuation of conformal blocks with respect to the positions of degenerate fields can be viewed as a sequence of elementary fusion and braiding transformations, see e.g. [3, 41]. It leads to an operator-valued monodromy of conformal blocks which involves shifts of the intermediate momenta. The Fourier transform at $c=1$ diagonalizes the shift operators and produces an ordinary $\mathrm{SL}(2,\mathbb{C})$-valued monodromy. The latter can be computed explicitly in terms of n external momenta, $n-3$ intermediate momenta and their $n-3$ Fourier conjugates. These variables define a convenient set of local coordinates on the $(3n-6)$-dimensional space $\mathcal{M}$ of monodromy data.

Expressing the isomonodromic tau functions in terms of Virasoro conformal blocks appears to have a number of advantages. For example, Jimbo's asymptotic formula for Painlevé VI [79] is an easy consequence of this relation. More generally, one may take advantage of the various results known about the Virasoro conformal blocks (in particular, their explicit combinatorial series representations generated by the AGT correspondence) in order to get detailed information on the isomonodromic tau functions. As another interesting application we are going to show how the known algebro-geometric solutions of the Schlesinger system [94, 32] arise from conformal blocks of the Ashkin-Teller critical model [161, 164].

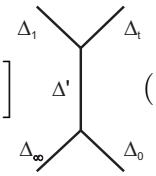
1.6 Connection vs fusion

Conversely, the relation of monodromy preserving deformations and conformal field theory yields nontrivial new results on the Virasoro conformal blocks. An interesting example of such implications concerns the so-called fusion, or crossing symmetry transformations [71].

Equivalence of different ways to decompose a correlation function into a sum over conformal families suggests the existence of duality transformations of conformal blocks, formalized by the concept of Moore-Seiberg groupoid [111]. In particular, there should exist an elementary invertible linear map connecting s- and t-channel four-point Virasoro conformal blocks on the Riemann sphere, AGT-related to weak/strong coupling S-duality on the gauge side. Schematically, one could write



$$(t) = \int d\Delta' \; K_c\left[\begin{array}{cc;c} \Delta_1 & \Delta_t & \Delta' \\ \Delta_\infty & \Delta_0 & \Delta \end{array}\right]$$



$$(1-t). \tag{1.9}$$

The integral kernel K_c of this transformation is called the fusion matrix. Its explicit form was obtained in [132, 133] by solving certain functional equations (which follow from the Moore-Seiberg formalism) with the help of representation theory of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. An alternative derivation, based on the free-field representations of chiral vertex operators, was proposed later in [144, 145].

The results of [132, 133, 144, 145] hold for generic complex values of the Virasoro central charge c. Unfortunately, it is not clear whether/how they can be extended to the half-line $c \in \mathbb{R}_{\leq 1}$, including a particularly interesting point $c = 1$ [134, 142, 46] at the borderline between minimal models and Liouville theory.

We are going to address the last problem by exploiting the relation of isomonodromic deformations and CFT. It will be shown that $c = 1$ *fusion matrix* coincides, up to an elementary dressing prefactor, with the *connection coefficient* for the P_{VI} tau function expressed in terms of monodromy data of the auxiliary linear problem. Conformal expansions of type (1.8) imply that this coefficient satisfies certain recurrence relations. On the other hand, equivalence of different critical points of Painlevé VI may be seen as a kind of crossing symmetry condition. Connecting expansion parameters in different channels, it makes the recurrence relations highly nontrivial and restrictive. Their solution appears to be related to the Poisson geometry of the moduli space of monodromy data and complexified volume of generic hyperbolic tetrahedron.

Problem C: *Obtain explicit expression for the $c = 1$ fusion matrix and solve the connection problem for tau functions of Painlevé equations.*

It is worth mentioning that the connection problem (computation of constant factors in the tau function asymptotics) has a strong independent interest. The main analytic issue in the applications of Painlevé theory to the description of correlation functions of integrable statistical mechanics and quantum field models is the large time and large distance asymptotics with a particular focus on the relevant connection formulae and the evaluation of the above mentioned constant factors. The latter, very often, contain the most important information about the models in question.

Starting from the seminal works of L. Onsager and B. Kaufman on the Ising model whose mathematical needs led to the birth of the Strong Szegö Theorem in the theory of Toeplitz matrices (see e.g. [35] for more on the history of the matter), the evaluation of the constant terms in the asymptotics of different correlation and distribution functions of the random matrix theory and of the theory of solvable statistical mechanics models has always been a great challenge in the field. In addition to the Strong Szegö Theorem we mention the work of C. Tracy [147], where the "constant problem" related to the Ising model was solved. This was the first rigorous solution of a "constant problem" for Painlevé equations (a special Painlevé III transcendent).

Further developments [98], [45], [36] were devoted to the rigorous derivation of Dyson's constant [44] in the asymptotics of the sine-kernel determinant describing the gap probability in large random matrices. This determinant represents the tau function of a particular solution of the fifth Painlevé equation. Other "constant" problems were also considered and solved in the works [8], [22], [33], [100] and [7].

As already mentioned, a natural framework for the global asymptotic analysis of the solutions of Painlevé equations is provided by the Riemann-Hilbert technique. However, it addresses directly Painlevé functions and not the associated tau functions. The latter are logarithmic antiderivatives of certain rational functions of Painlevé transcendents — the Hamiltonians of Painlevé equations. Hence the problem: one should be able to evaluate integrals of certain combinations of Painlevé transcendents and their derivatives. So far, this problem has been successfully handled only for very special solutions of the Painlevé equations. The tau functions evaluated on these solutions admit additional representations in terms of certain Fredholm or Toeplitz determinants. It is this extra property that, in spite of the difference in the approaches, is a principal reason of the success in the solution of the "constant problem" in all of the works mentioned in the previous paragraphs.

The situation with "constant problems" has changed with the discovery of conformal block representations of the tau functions. Their manifest periodicity with respect to

monodromy data yields certain functional relations that can be solved [71, 77] using the hamiltonian interpretation of the monodromy parameterization of Painlevé transcendents.

For instance, as every other Painlevé equation, P_{VI} is a classical non-autonomous hamiltonian system. Appropriately chosen monodromy data form a canonically conjugated pair of coordinates on the space of its solutions. The correspondence between the asymptotic parameters at different critical points can be interpreted as a canonical map between two different sets of such coordinates. The connection coefficients we are interested in are closely related to the *generating functions* of these maps. As we will see, this ultimately allows to evaluate them in explicit form.

1.7 Organization of the manuscript

The content of the manuscript is essentially based on four papers [61], [62], [70] and [71]. Section 2 corresponds to the first two of them. There we show how the relation between Virasoro conformal blocks and Painlevé equations can be derived heuristically. The main results — namely, the combinatorial series representations for the general solutions of Painlevé VI, V and three inequivalent Painlevé III equations — are formulated in Subsection 2.7. In Section 3, following Ref. [70], we show that the solution of the isomonodromic Riemann-Hilbert problem for rank 2 Fuchsian systems on the n-punctured Riemann sphere and the associated tau function are given by Fourier transforms of the $c = 1$ conformal blocks. In the case $n = 4$ this yields a proof of the conjectures of the previous section. Section 4 is based on Ref. [71] and is devoted to the solution of the connection problem for Painlevé VI tau function and the calculation of $c = 1$ fusion matrix.

Other results of the author on related topics that were left outside the scope of the manuscript include [99, 100] (asymptotic study of a family of Fredholm determinant P_{VI} solutions), [102] (classification of algebraic P_{VI} solutions), and [77] (solution of the connection problem for the tau function of the radial sine-Gordon equation, related to the most degenerate $\mathrm{P}_{\mathrm{III}}$ and irregular conformal blocks).

2. Heuristic CFT approach to Painlevé transcendents

2.1 Preliminaries

It is instructive to start with the general case of rank N linear system with n regular singular points $a = \{a_1, \dots, a_n\}$ on $\mathbb{P}^1$. Instead of (1.1) one has

$$\partial_z \Phi = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \sum_{\nu=1}^{n} \frac{A_\nu}{z - a_\nu}. \tag{2.1}$$

The absence of singularity at infinity implies that $N \times N$ constant matrices $A_\nu \in \mathrm{GL}(N, \mathbb{C})$ satisfy the constraint $\sum_{\nu=1}^{n} A_\nu = 0$. They are assumed to be diagonalizable so that $A_\nu = G_\nu T_\nu G_\nu^{-1}$ with some $T_\nu = \mathrm{diag}\,\{\lambda_{\nu,1}, \dots, \lambda_{\nu,N}\}$. The fundamental solution will be normalized by $\Phi(z_0) = \mathbf{1}_N$. It is useful to introduce the matrix

$$\mathcal{J}(z) = \Phi^{-1}\partial_z \Phi = \Phi^{-1}\mathcal{A}(z)\Phi.$$

The coefficients of the Taylor series of $\Phi(z)$ around $z = z_0$ can be expressed in terms of $\mathcal{J}$ and its derivatives. In particular,

$$\Phi(z \to z_0) = \mathbf{1}_N + \mathcal{J}\,(z_0)\,(z - z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right) \frac{(z - z_0)^2}{2} + \dots \tag{2.2}$$

Near the singular points, the fundamental solution has the following expansions (under additional non-resonancy assumption $\lambda_{\nu,j} - \lambda_{\nu,k} \notin \mathbb{Z}$ for $j \neq k$):

$$\Phi(z \to a_\nu) = \mathcal{G}_\nu(z)\,(z - a_\nu)^{T_\nu}\, C_\nu. \tag{2.3}$$

Here $\mathcal{G}_\nu(z)$ is holomorphic and invertible in a neighborhood of $z = a_\nu$ and satisfies $\mathcal{G}_\nu(a_\nu) = G_\nu$. The connection matrix C_ν is independent of z. Counterclockwise continuation of $\Phi(z)$ around a_ν leads to the monodromy matrix $M_\nu = C_\nu^{-1} e^{2\pi i T_\nu} C_\nu$.

Let us now vary the positions of singularities and normalization point, simultaneously evolving A_ν's in such a way that the monodromy is preserved. A classical result of Schlesinger [141] translates this requirement into a system of PDEs

$$\partial_{a_\nu}\Phi = -\,\frac{z_0 - z}{z_0 - a_\nu}\,\frac{A_\nu}{z - a_\nu}\,\Phi, \tag{2.4}$$

$$\partial_{z_0}\Phi = -\,\mathcal{A}\,(z_0)\,\Phi. \tag{2.5}$$

It is important to note that the matrix $\mathcal{J}(z)$ remains invariant under isomonodromic variation of z_0. Schlesinger deformation equations are obtained as the compatibility conditions

of (2.1), (2.4) and (2.5). Explicitly,

$$\partial_{a_{\nu'}} A_\nu = \frac{z_0 - a_\nu}{z_0 - a_{\nu'}} \frac{[A_{\nu'}, A_\nu]}{a_{\nu'} - a_\nu}, \qquad \nu \neq \nu', \tag{2.6a}$$

$$\partial_{a_\nu} A_\nu = -\sum_{\nu' \neq \nu} \frac{[A_{\nu'}, A_\nu]}{a_{\nu'} - a_\nu}, \qquad \partial_{z_0} A_\nu = -\sum_{\nu' \neq \nu} \frac{[A_{\nu'}, A_\nu]}{z_0 - a_{\nu'}}. \tag{2.6b}$$

The Lax form of the Schlesinger system (2.6) implies that the eigenvalues of A_ν's are conserved under deformation. This is of course expected due to the obvious relation between the spectra of A_ν's and monodromy matrices.

The isomonodromic tau function $\tau(a)$ is defined by

$$d \ln \tau = \sum_{\nu < \nu'} \operatorname{Tr} A_\nu A_{\nu'} \, d \ln (a_{\nu'} - a_\nu) \, . \tag{2.7}$$

It is a nontrivial consequence of the deformation equations that the 1-form in the r.h.s. is closed. To show that it does not depend on z_0, one can rewrite (2.7) as

$$\partial_{a_\nu} \ln \tau = \sum_{\nu' \neq \nu} \frac{\operatorname{Tr} A_\nu A_{\nu'}}{a_\nu - a_{\nu'}} = \frac{1}{2} \operatorname{res}_{z = a_\nu} \operatorname{Tr} \mathcal{J}^2(z). \tag{2.8}$$

Here the first equality follows from (2.7) and the second one from the fact that $\mathcal{J}(z)$ is conjugate to $\mathcal{A}(z)$.

Finally, let us decompose all A_ν's into the sum of scalar and traceless part as $A_\nu = \frac{\operatorname{Tr} A_\nu}{N} \mathbf{1}_N + \hat{A}_\nu$. It can then be easily checked that

$$\Phi_{\mathcal{A}}(z) = \prod_\nu \Big(\frac{z - a_\nu}{z_0 - a_\nu} \Big)^{\frac{\operatorname{Tr} A_\nu}{N}} \Phi_{\hat{\mathcal{A}}}(z), \tag{2.9}$$

$$\mathcal{J}_{\mathcal{A}}(z) = \frac{1}{N} \sum_\nu \frac{\operatorname{Tr} A_\nu}{z - a_\nu} \mathbf{1}_N + \mathcal{J}_{\hat{\mathcal{A}}}(z), \tag{2.10}$$

$$\tau_{\mathcal{A}}(a) = \prod_{\nu < \nu'} (a_{\nu'} - a_\nu)^{\frac{\operatorname{Tr} A_\nu \operatorname{Tr} A_{\nu'}}{N}} \tau_{\hat{\mathcal{A}}}(a). \tag{2.11}$$

This allows to assume without any loss of generality that A_ν's are traceless, but we deliberately postpone the imposition of this condition.

Fractional linear maps $z \mapsto f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ form the automorphism group of the Riemann sphere. It is clear that under the action of these transformations on z, z_0 and a the quantities $\Phi(z)$ and $\{A_\nu\}$ transform as functions and $\mathcal{J}(z)$ as a vector field. Let us try to understand the effect of global conformal mappings on the tau function.

To this end we first compute $\tau(a)$ explicitly in the case $n = 3$. Since $A_1 + A_2 + A_3 = 0$, the coefficients $\operatorname{Tr} A_\nu A_{\nu'}$ in (2.7) are conserved quantities. Hence, integrating (2.7), we

find

$$\tau(a_1, a_2, a_3) = \text{const} \cdot (a_1 - a_2)^{\Delta_3 - \Delta_2 - \Delta_1} (a_1 - a_3)^{\Delta_2 - \Delta_1 - \Delta_3} (a_2 - a_3)^{\Delta_1 - \Delta_2 - \Delta_3},$$

where $\Delta_\nu = \frac{1}{2} \operatorname{Tr} A_\nu^2$ with $\nu = 1, 2, 3$. One recognizes here the general expression for the three-point correlation function of (quasi)primary fields of dimensions $\Delta_{1,2,3}$ in the two-dimensional conformal field theory.

Given the above example, it is easy to guess that for arbitrary n the tau function transforms as an n-point correlation function of primaries with appropriate dimensions:

$$\tau(f(a)) = \prod_{\nu=1}^{n} [f'(a_\nu)]^{-\Delta_\nu} \tau(a). \tag{2.12}$$

To prove the last formula, it is sufficient to consider infinitesimal transformations generated by the vector field $Z = (A + Bz + Cz^2)\partial_z$, which amounts to checking three differential constraints:

$$\sum_\nu \partial_{a_\nu} \ln \tau = 0,$$
$$\sum_\nu (a_\nu \partial_{a_\nu} \ln \tau + \Delta_\nu) = 0,$$
$$\sum_\nu \left(a_\nu^2 \partial_{a_\nu} \ln \tau + 2\Delta_\nu a_\nu\right) = 0.$$

These relations can indeed be straightforwardly demonstrated using the first equality in (2.8) and the condition $\sum_\nu A_\nu = 0$.

2.2 Field content

The fundamental solution Φ is completely fixed by its monodromy, normalization and the singular behaviour (2.3). The last point is particularly important: indeed, adding an integer to any diagonal element of T_ν modifies the asymptotics of Φ as $z \to a_\nu$ without changing monodromy matrices. Therefore, in what follows, the notion of monodromy will include not only the set of M_ν's but also the choice of their logarithm branches

$$\mu_\nu = C_\nu^{-1} T_\nu C_\nu. \tag{2.13}$$

Let us now try to construct a formal QFT solution of the isomonodromic deformation problem, extending the ideas of [135, 136] and [110]. The starting point will be the following ansatz for Φ:

$$\Phi_{jk}(z) = (z - z_0)^{2\Delta} \frac{\langle \varphi_k(z) \bar{\varphi}_j(z_0) \mathcal{O}_{\mu_n}(a_n) \dots \mathcal{O}_{\mu_1}(a_1) \rangle}{\langle \mathcal{O}_{\mu_n}(a_n) \dots \mathcal{O}_{\mu_1}(a_1) \rangle}, \tag{2.14}$$

with $j, k = 1, \ldots, N$. Here it is assumed that $\{\mathcal{O}_{\mu_\nu}\}$, $\{\bar{\varphi}_j\}$, $\{\varphi_k\}$ are primary fields in a 2D CFT characterized by some central charge c. Further, we require the OPEs of $\bar{\varphi}$'s with φ's to contain the identity operator. This forces them to have equal dimensions, to be denoted by Δ. Normalization of these fields is fixed by the normalization of Φ; the leading OPE term should be equal to

$$\varphi_k(z)\bar{\varphi}_j(z_0) \sim (z - z_0)^{-2\Delta}\, \delta_{jk}. \tag{2.15}$$

Since $\Phi(z)$ may be represented by an entire series near $z = z_0$, the dimensions of all other primaries appearing in the this OPE should be given by strictly positive integers. Monodromy fields $\{\mathcal{O}_{\mu_\nu}\}$ are defined by the condition that their complete OPEs with auxiliary fields $\{\varphi_k\}$ have the form

$$\varphi_k(z)\mathcal{O}_{\mu_\nu}(a_\nu) = \sum_{j=1}^{n} \left((z-a)^{\mu_\nu}\right)_{jk} \sum_{\ell=0}^{\infty} \mathcal{O}_{\mu_\nu,j,\ell}(a)\,(z-a_\nu)^{\ell}\,,$$

where $\{\mathcal{O}_{\mu_\nu,j,\ell}\}$ are some local fields. In particular, the raw vector $(\varphi_1 \ldots \varphi_n)$ should be multiplied by M_ν when continued around $\mathcal{O}_{\mu_\nu}$. If one succeeds in finding a set of fields with all mentioned properties, the correlator ratio (2.14) will automatically give the solution of the linear system (2.1).

The definition (2.7) of the tau function arises very naturally in the CFT framework. To illustrate this, let us compute two more orders in the OPE (2.15). The identity field has no level 1 descendants, therefore the leading correction is given by a new primary field J_{jk} of dimension 1. The next-to-leading order correction comes from three sources: i) nonvanishing level 2 descendant of the identity operator given by the energy-momentum tensor T, ii) level 1 current descendant ∂J_{jk} and iii) new primaries of dimension 2 which can be combined into a single field S_{jk}. Thus

$$\begin{aligned}\varphi_k(z)\bar{\varphi}_j(z_0) = (z-z_0)^{-2\Delta}\Big[\,\delta_{jk} + J_{jk}(z_0)\,(z-z_0) + \\ +\Big(\frac{4\Delta}{c}\,T(z_0)\delta_{jk} + (\partial J_{jk})(z_0) + S_{jk}(z_0)\Big)\frac{(z-z_0)^2}{2} + O\left((z-z_0)^3\right)\Big]\,.\end{aligned} \tag{2.16}$$

We will make a further assumption of tracelessness of S, which is essentially motivated by the examples considered below. Now, substituting (2.16) into (2.14) and matching the result with (2.2), one finds that

$$\mathcal{J}(z) = \frac{\langle J(z)\mathcal{O}_{\mu_n}(a_n)\ldots\mathcal{O}_{\mu_1}(a_1)\rangle}{\langle \mathcal{O}_{\mu_n}(a_n)\ldots\mathcal{O}_{\mu_1}(a_1)\rangle}, \tag{2.17a}$$

$$\mathrm{Tr}\,\mathcal{J}^2(z) = \frac{\langle T(z)\mathcal{O}_{\mu_n}(a_n)\ldots\mathcal{O}_{\mu_1}(a_1)\rangle}{\langle \mathcal{O}_{\mu_n}(a_n)\ldots\mathcal{O}_{\mu_1}(a_1)\rangle}\frac{4N\Delta}{c}. \tag{2.17b}$$

Standard CFT arguments allow to rewrite the r.h.s. of the last formula as

$$\frac{\langle T(z)\mathcal{O}_{\mu_n}(a_n)\dots\mathcal{O}_{\mu_1}(a_1)\rangle}{\langle\mathcal{O}_{\mu_n}(a_n)\dots\mathcal{O}_{\mu_1}(a_1)\rangle}=\sum_{\nu=1}^{n}\left\{\frac{\tilde{\Delta}_\nu}{(z-a_\nu)^2}+\frac{1}{z-a_\nu}\,\partial_{a_\nu}\ln\langle\mathcal{O}_{\mu_n}(a_n)\dots\mathcal{O}_{\mu_1}(a_1)\rangle\right\},$$

where $\tilde{\Delta}_\nu$ denotes the conformal dimension of $\mathcal{O}_{\mu_\nu}$. Comparison of (2.17b) with (2.8) then shows that the tau function can be identified with a power of the correlator of monodromy fields,

$$\tau(a)=\langle\mathcal{O}_{\mu_n}(a_n)\dots\mathcal{O}_{\mu_1}(a_1)\rangle^{\frac{2N\Delta}{c}}\,. \tag{2.18}$$

In what follows, we will be exclusively interested in the case when

$$c=2N\Delta. \tag{2.19}$$

Such a condition implies, in particular, that the dimensions $\tilde{\Delta}_\nu$ of monodromy fields coincide with the quantities $\Delta_\nu=\frac{1}{2}\,\mathrm{Tr}\,A_\nu^2$ from the previous subsection.

One possible realization of the above conditions is provided by the theory of N free complex fermions. Its central charge $c=N$ agrees with the conformal dimension $\Delta=\frac{1}{2}$ of fermionic fields $\{\bar{\psi}_j\}$, $\{\psi_k\}$ which play the role of $\bar{\varphi}$'s and φ's. The currents are by definition given by $J_{jk}=(\bar{\psi}_j\,\psi_k)$, while the energy-momentum tensor T and the fields $\{S_{jk}\}$ may be expressed as

$$\begin{aligned}T&=\frac{1}{2}\sum_k\left[(\bar{\psi}_k\,\partial\psi_k)-(\partial\bar{\psi}_k\,\psi_k)\right],\\ S_{jk}&=(\bar{\psi}_j\,\partial\psi_k)-(\partial\bar{\psi}_j\,\psi_k)-\frac{2}{N}\,T\delta_{jk}.\end{aligned}$$

To represent monodromy fields, recall the usual bosononization formulas

$$\begin{aligned}&\bar{\psi}_k=:e^{-i\phi_k}:,\qquad \psi_k=:e^{i\phi_k}:,\\ &J_{jk}=\begin{cases}:e^{i(\phi_k-\phi_j)}:, & j\neq k,\\ i\,\partial\phi_k, & j=k,\end{cases}\\ &T=-\frac{1}{2}\sum_k(\partial\phi_k\,\partial\phi_k)\,,\end{aligned}$$

where $\{\phi_k\}_{k=1,\dots,N}$ are free complex bosonic fields with the propagator $\langle\phi_k(w)\phi_k(z)\rangle\sim-\ln(w-z)$. Also note that for $C\in\mathrm{GL}(N,\mathbb{C})$, the monodromy matrices for the linearly transformed fermions

$$\bar{\psi}'_j=\sum_k C_{jk}\bar{\psi}_k,\qquad \psi'_j=\sum_k C^{-1}_{kj}\psi_k,$$

are obtained from M_ν's by conjugation by C. In particular, setting $C = C_\nu$, one obtains fermions $\{\bar\psi_k^{(\nu)}\}$, $\{\psi_k^{(\nu)}\}$ with diagonal monodromy around a_ν. Denote by $\{\phi_k^{(\nu)}\}$ bosonic fields associated to this "diagonal" fermionic basis, then monodromy field $\mathcal{O}_{\mu_\nu}$ can be written as

$$\mathcal{O}_{\mu_\nu} =: e^{i\sum_k \lambda_{\nu,k}\phi_k^{(\nu)}} : .$$

We thus need to deal with n different bosonization schemes of the same theory, each of them being adapted for representing one of the monodromy fields. The corresponding N-tuples of bosons are related by complicated nonlocal transformations.

The formulas (2.9)–(2.11) are a signature of the well-known decomposition of fermionic CFT into the direct sum $\hat{u}(1) \oplus \hat{su}(N)_1$ of two WZW theories. Fermion and monodromy fields are given by products of fields from the two summands:

$$\bar\psi_k =: e^{-i\phi_0/\sqrt{N}} : \otimes \hat{\bar\varphi}_k, \qquad \psi_k =: e^{i\phi_0/\sqrt{N}} : \otimes \hat\varphi_k,$$
$$\mathcal{O}_{\mu_\nu} =: e^{\frac{i\,\mathrm{Tr}\,A_\nu}{\sqrt{N}}\phi_0} : \otimes \mathcal{O}_{\hat\mu_\nu}.$$

Bosonic field ϕ_0 in the $\hat{u}(1)$ factors is expressed in terms of fields introduced before as $\phi_0 = \frac{1}{\sqrt{N}}\sum_{k=1}^N \phi_k$. The fields $\{\hat{\bar\varphi}_k\}$, $\{\hat\varphi_k\}$ and $\{\mathcal{O}_{\hat\mu_\nu}\}$ live in the $\hat{su}(N)_1$ WZW theory and can be formally written as ordered exponentials of integrated linear combinations of $\hat{su}(N)_1$-currents. It should be emphasized that they are Virasoro primaries but not necessarily WZW primaries. The fields $\{\hat{\bar\varphi}_k\}$ and $\{\hat\varphi_k\}$ have the same dimension $\Delta = \frac{N-1}{2N}$, in accordance with the central charge $c_{\hat{su}(N)_1} = N-1$ [97]. The dimension of $\mathcal{O}_{\hat\mu_\nu}$ is equal to $\frac{1}{2}\,\mathrm{tr}\,\hat{A}_\nu^2$, where as above, $\hat{A}_\nu = A_\nu - \frac{\mathrm{Tr}\,A_\nu}{N}\mathbf{1}_N$ stands for the traceless part of A_ν.

Now it becomes clear that imposing the tracelessness of $\mathcal{A}(z)$ corresponds to factoring out the $\hat{u}(1)$ piece from the fermionic theory. This innocently looking procedure is in fact crucial, as it drastically reduces the number of primary fields in the OPEs and thus makes the computation of correlation functions much more efficient as compared to fermionic realization. Therefore, in the rest of this section we set $\mathrm{Tr}\,\mathcal{A}(z) = 0$, remove the hats from $\hat{A}_\nu$'s, $\hat\mu_\nu$'s, $\hat{\bar\varphi}$'s and $\hat\varphi$'s to lighten the notation, and interpret the isomonodromic tau function as a correlation function of primaries with dimensions Δ_ν in a CFT with $c = N-1$.

We close this subsection with an example of application of the field-theoretic machinery in the case $N = 2$. It is somewhat distinguished from CFT point of view, since for $c = 1$ the dimension $\Delta = \frac{1}{4}$ of $\bar\varphi$'s and φ's corresponds to level 2 degenerate states, and the dimension 1 of $\{J_{jk}\}$ is degenerate at level 3. Hence the correlation functions

$$\mathcal{P}_{jk} = \langle \varphi_k(z)\bar\varphi_j(z_0)\mathcal{O}_{\mu_n}(a_n)\dots\mathcal{O}_{\mu_1}(a_1)\rangle,$$
$$\mathcal{Q}_{jk} = \langle J_{jk}(z)\mathcal{O}_{\mu_n}(a_n)\dots\mathcal{O}_{\mu_1}(a_1)\rangle,$$

have to satisfy linear PDEs of order 2 and 3, fixed by Virasoro symmetry. This results into the following statement (cf observations made in [118]):

Proposition 2.1. *Under assumption* $\operatorname{Tr}\mathcal{A}(z)=0$*, the matrices*

$$\mathcal{P}=(z-z_0)^{-\frac{1}{2}}\tau\Phi, \qquad \mathcal{Q}=\tau\Phi^{-1}\partial_z\Phi,$$

satisfy the differential equations

$$\partial_{zz}\mathcal{P}=\left\{\frac{1}{z-z_0}\,\partial_{z_0}+\frac{1}{4\,(z-z_0)^2}+\sum_\nu\left(\frac{1}{z-a_\nu}\,\partial_{a_\nu}+\frac{\Delta_\nu}{(z-a_\nu)^2}\right)\right\}\mathcal{P},$$

$$\partial_{zzz}\mathcal{Q}=\left\{4\sum_\nu\left(\frac{1}{z-a_\nu}\,\partial_{a_\nu z}+\frac{\Delta_\nu}{(z-a_\nu)^2}\,\partial_z\right)+2\sum_\nu\left(\frac{1}{(z-a_\nu)^2}\,\partial_{a_\nu}+\frac{2\Delta_\nu}{(z-a_\nu)^3}\right)\right\}\mathcal{Q}.$$

Proof. Straightforward but tedious calculation using the relations (2.1), (2.4), (2.5), (2.8) and the identity $A^2=\frac{1}{2}\operatorname{tr}A^2\,\mathbf{1}_2$ verified by any traceless 2×2 matrix A. □

2.3 Painlevé VI

Recall that global conformal symmetry allows to fix the positions of three singular points. Painlevé VI equation corresponds to setting $N=2$, $n=4$ and sending these three points to 0, 1 and ∞. The remaining singular point, $t=\dfrac{(a_2-a_1)(a_4-a_3)}{(a_3-a_1)(a_4-a_2)}$, represents the cross-ratio of singularities, which is preserved by Möbius transformations.

For $\nu=0,t,1,\infty$, let us denote by $\pm\theta_\nu$ the eigenvalues of A_ν and consider the logarithmic derivative of the P_{VI} tau function,

$$\sigma(t):=t(t-1)\frac{d}{dt}\ln\tau=(t-1)\ \operatorname{Tr}A_0A_t+t\operatorname{Tr}A_tA_1. \tag{2.20}$$

We first derive (following [69]) a differential equation for this quantity. Differentiating it twice with respect to t and using the deformation equations (1.2), one finds

$$\sigma'(t)=\operatorname{Tr}A_0A_t+\operatorname{Tr}A_tA_1, \tag{2.21}$$

$$\sigma''(t)=\frac{\operatorname{Tr}[A_0,A_t]A_1}{t(1-t)}. \tag{2.22}$$

But since any triple of traceless 2×2 matrices $A_{0,t,1}$ satisfies the identity

$$(\operatorname{Tr}[A_0,A_t]A_1)^2=-2\det\begin{pmatrix}\operatorname{Tr}A_0^2 & \operatorname{Tr}A_0A_t & \operatorname{Tr}A_0A_1\\ \operatorname{Tr}A_tA_0 & \operatorname{Tr}A_t^2 & \operatorname{Tr}A_tA_1\\ \operatorname{Tr}A_1A_0 & \operatorname{Tr}A_1A_t & \operatorname{Tr}A_1^2\end{pmatrix},$$

the relations (2.20)–(2.22) imply that the function $\sigma(t)$ satisfies the ODE

$$\Big(t(t-1)\sigma''\Big)^2 = \tag{2.23}$$
$$= -2\det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}.$$

This is the so called σ-form of Painlevé VI equation.

Our preceding arguments show that the P_{VI} tau function $\tau(t)$ defined by (2.20) is nothing but the four-point correlator of monodromy fields,

$$\tau(t) = \langle \mathcal{O}_{\mu_\infty}(\infty)\mathcal{O}_{\mu_1}(1)\mathcal{O}_{\mu_t}(t)\mathcal{O}_{\mu_0}(0)\rangle, \tag{2.24}$$

and that these fields are Virasoro primaries with dimensions $\Delta_\nu = \theta_\nu^2$ in a $c=1$ conformal field theory. The field at infinity should be understood according to the usual CFT prescription

$$\langle \mathcal{O}(\infty)\ldots\rangle := \lim_{R\to\infty} R^{2\Delta_\mathcal{O}}\langle \mathcal{O}(R)\ldots\rangle \tag{2.25}$$

It is clear that auxiliary fields $\{\varphi_k\}$ should have monodromy M_tM_0 around all fields in the OPE of $\mathcal{O}_{\mu_0}$ and $\mathcal{O}_{\mu_t}$. Let $e^{\pm 2\pi i\sigma_{0t}}$ denote the eigenvalues of the product M_tM_0 and C_{0t} be its diagonalizing transformation. Since σ_{0t} is defined only up to an integer, it is natural to expect that the set of primaries present in the OPE of $\mathcal{O}_{\mu_0}$ and $\mathcal{O}_{\mu_t}$ consists of an infinite number of monodromy fields $\mathcal{O}_{\mu_{0t}^{(n)}}$ with $n\in\mathbb{Z}$ and

$$\mu_{0t}^{(n)} = C_{0t}^{-1}\begin{pmatrix} \sigma_{0t}+n & 0 \\ 0 & -\sigma_{0t}-n \end{pmatrix} C_{0t},$$

i.e. of all possible monodromy fields associated to the monodromy matrix M_tM_0. Taking into account that the conformal dimension of $\mathcal{O}_{\mu_{0t}^{(n)}}$ is equal to $(\sigma_{0t}+n)^2$, we arrive at the following intermediate conjecture:

$$\tau(t) = \sum_{n\in\mathbb{Z}} C_n\,\mathcal{F}(\vec{\theta},\sigma_{0t}+n,t), \tag{2.26}$$

where $\mathcal{F}(\vec{\theta};\sigma;t) := \mathcal{F}_1(\theta_0^2,\theta_t^2,\theta_1^2,\theta_\infty^2,\sigma^2,t)$ denotes the appropriate s-channel $c=1$ conformal block (1.7), and the structure constants C_n are to be determined.

The latter problem can be solved using the Jimbo's asymptotic formula [79]. To give its explicit form, we need to introduce local coordinates on the P_{VI} space $\mathcal{M}$ of monodromy data defined by (1.4). It will be convenient to work with monodromy exponents $\vec{\sigma} = (\sigma_{0t},\sigma_{1t},\sigma_{01})$ and $\vec{\theta} = (\theta_0,\theta_t,\theta_1,\theta_\infty)$ defined by

$$p_\nu := \mathrm{Tr}\,M_\nu = 2\cos 2\pi\theta_\nu, \tag{2.27a}$$

$$p_{\nu\nu'} := \mathrm{Tr}\,M_\nu M_{\nu'} = 2\cos 2\pi\sigma_{\nu\nu'}. \tag{2.27b}$$

Painlevé VI monodromy manifold is described by the Jimbo-Fricke affine cubic surface $W(p_{0t}, p_{1t}, p_{01}) = 0$, where

$$W(p_{0t}, p_{1t}, p_{01}) = p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_{0t}p_{1t}p_{01} - \omega_{0t}p_{0t} - \omega_{1t}p_{1t} - \omega_{01}p_{01} + \omega_4, \quad (2.28)$$

and the parameters ω_{0t}, ω_{1t}, ω_{01}, ω_4 depend only on $\vec{\theta}$:

$$\begin{aligned}
\omega_{0t} &= p_0 p_t + p_1 p_\infty, \\
\omega_{1t} &= p_t p_1 + p_0 p_\infty, \\
\omega_{01} &= p_0 p_1 + p_t p_\infty, \\
\omega_4 &= p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0 p_t p_1 p_\infty - 4.
\end{aligned}$$

This surface defines the $\mathrm{SL}(2,\mathbb{C})$-character variety of the four-punctured sphere $C_{0,4}$. The relation $W(p_{0t}, p_{1t}, p_{01}) = 0$ is of course consistent with $\dim \mathcal{M} = 6$ and allows to interpret the triple $\vec{\sigma}$ as a pair of P_{VI} integration constants. Note that one of them already appeared in (2.26), where it parameterizes the intermediate dimensions of conformal blocks.

Now, in addition to $\Delta_\nu = \theta_\nu^2$ ($\nu = 0, t, 1, \infty$), let us introduce a suggestive notation $\Delta_\sigma = \sigma^2$. The asymptotic behavior of P_{VI} tau function near the critical point $t = 0$ is related to the monodromy data $\vec{\theta}$, $\vec{\sigma}$ by the following

Theorem 2.2 ([79]). *Under assumptions that*

$$\begin{aligned}
&\theta_0, \theta_t, \theta_1, \theta_\infty \notin \mathbb{Z}/2, \\
&|\Re\sigma_{0t}| < \frac{1}{2}, \quad \sigma_{0t} \neq 0, \\
&\theta_t + \epsilon\theta_0 + \epsilon'\sigma_{0t},\ \theta_1 + \epsilon\theta_\infty + \epsilon'\sigma_{0t} \notin \mathbb{Z}, \qquad \epsilon, \epsilon' = \pm 1,
\end{aligned}$$

the tau function $\tau(t)$ has the following asymptotics as $t \to 0$:

$$\begin{aligned}
\tau(t) = {} & C\left[\begin{matrix} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{matrix}; \sigma_{0t}\right] t^{\Delta_{\sigma_{0t}} - \Delta_0 - \Delta_t} \left(1 + \frac{(\Delta_{\sigma_{0t}} - \Delta_0 + \Delta_t)(\Delta_{\sigma_{0t}} - \Delta_\infty + \Delta_1)}{2\Delta_{\sigma_{0t}}} t\right) + \\
& + C\left[\begin{matrix} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{matrix}; \sigma_{0t} - 1\right] s_{0t}^{-1} t^{\Delta_{\sigma_{0t}-1} - \Delta_0 - \Delta_t} + \\
& + C\left[\begin{matrix} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{matrix}; \sigma_{0t} + 1\right] s_{0t}\, t^{\Delta_{\sigma_{0t}+1} - \Delta_0 - \Delta_t} + \\
& + O\Big(\max\{t^{\Delta_{\sigma_{0t}+1} - \Delta_0 - \Delta_t + 1}, t^{\Delta_{\sigma_{0t}-1} - \Delta_0 - \Delta_t + 1}\}\Big),
\end{aligned} \quad (2.29)$$

where

$$s_{0t} = \frac{(\omega_{1t} - 2p_{1t} - p_{0t}p_{01}) - (\omega_{01} - 2p_{01} - p_{0t}p_{1t})\, e^{2\pi i \sigma_{0t}}}{(2\cos 2\pi(\theta_t - \sigma_{0t}) - p_0)(2\cos 2\pi(\theta_1 - \sigma_{0t}) - p_\infty)}, \quad (2.30)$$

$$C\Big[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma\Big]=\frac{\prod_{\epsilon,\epsilon'=\pm}G(1+\theta_t+\epsilon\theta_0+\epsilon'\sigma)\,G(1+\theta_1+\epsilon\theta_\infty+\epsilon'\sigma)}{\prod_{\epsilon=\pm}G(1+2\epsilon\sigma)}, \tag{2.31}$$

and $G(z)$ denotes the Barnes G-function.

A few comments are in order:

- Two Painlevé VI integration constants correspond to the parameters σ_{0t} and s_{0t} in the asymptotic formula. The first one is present already in the leading term $t^{\Delta_{\sigma_{0t}}-\Delta_0-\Delta_t}$ (note that it appears almost trivially within the CFT framework), whereas the second appears in subleading orders (2nd and 3rd line of (2.29)).

- Fixing p_{0t} in the relation $W(p_{0t},p_{1t},p_{01})=0$, one obtains a quadric in p_{1t}, p_{01}. The quantity s_{0t} defined by a somewhat obscure formula (2.30) is in fact a uniformizing parameter on this quadric.

- Barnes G-function may be defined as the infinite product

  $$G\left(1+z\right)=(2\pi)^{\frac{z}{2}}\exp\left(-\frac{z+z^2\left(1+\gamma\right)}{2}\right)\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^k\exp\left(\frac{z^2}{2k}-z\right),$$

  or using analytic continuation of the integral representation

  $$G(1+z)=(2\pi)^{\frac{z}{2}}\exp\int_0^\infty\frac{ds}{s}\left[\frac{1-e^{-zs}}{4\sinh^2\frac{s}{2}}-\frac{z}{s}+\frac{z^2}{2}\,e^{-s}\right],\qquad \Re z>-1.$$

  It is essentially characterized by the recurrence relation $G(z+1)=\Gamma(z)\,G(z)$ and normalization $G(1)=1$.

The tau function is defined only up to a multiplicative constant, cf e.g. (2.20). It can be chosen so that the formula (2.29) will contain only the ratios of the coefficients $C\Big[\ldots\Big]$. These ratios furthermore can be written in terms of gamma functions only; this is the way the result was originally formulated in [79]. However, the normalization using Barnes functions turns out to be more convenient. Indeed, interpreting Jimbo's formula as a recursion relation for C_n's, one immediately finds

$$C_n=\text{const}\cdot C\Big[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma_{0t}+n\Big]\,s_{0t}^n. \tag{2.32}$$

Together with (2.26), this gives what we are really interested in — namely, a complete s-channel conformal expansion of the Painlevé VI tau function:

$$\tau(t)=\chi_0\sum_{n\in\mathbb{Z}}C\Big[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma_{0t}+n\Big]\,s_{0t}^n\,\mathcal{F}\left(\theta_0,\theta_t,\theta_1,\theta_\infty;\sigma_{0t}+n;t\right) \tag{2.33}$$

Remark 2.3. Since the critical points play completely analogous roles in Painlevé VI, one can write similar expansions at $t = 1$ and $t = \infty$.[7] For instance, the series around $t = 1$ is obtained by the exchange [79, 100]

$$t \leftrightarrow 1-t, \qquad \theta_0 \leftrightarrow \theta_1, \qquad \sigma_{0t} \leftrightarrow \sigma_{1t}, \qquad p_{01} \leftrightarrow \omega_{01} - p_{01} - p_{0t}p_{1t}.$$

This yields the t-channel expansion

$$\tau(t) = \chi_1 \sum_{n\in\mathbb{Z}} C\left[\begin{matrix} \theta_0 & \theta_t \\ \theta_\infty & \theta_1 \end{matrix}; \sigma_{1t}+n\right] s_{1t}^n \mathcal{F}\left(\theta_1, \theta_t, \theta_0, \theta_\infty; \sigma_{1t}+n; 1-t\right), \tag{2.34}$$

where s_{1t} is given by a formula analogous to (2.30):

$$s_{1t} = \frac{(\omega_{0t} - 2p_{0t} - p_{1t}p_{01}) - (\omega_{01} - 2p_{01} - p_{0t}p_{1t})\, e^{-2\pi i\sigma_{1t}}}{(2\cos 2\pi(\theta_t - \sigma_{1t}) - p_1)\,(2\cos 2\pi(\theta_0 - \sigma_{1t}) - p_\infty)}. \tag{2.35}$$

Remark 2.4. The structure constants (2.32) cannot be completely factorized into the products of three-point functions due to the presence of the parameter s_{0t}. This is an artefact of a nontrivial braid group action on the correlation functions of monodromy fields.

To illustrate what we have in mind, consider the analytic continuation of $\tau(t)$ along a closed counterclockwise contour around the branch point $t = 0$. In general, such a continuation induces an action of the 3-braid group (more precisely, of the modular group $\Gamma(2)$) on monodromy [43]. In the case at hand, new monodromy matrices are given by

$$M_0' = M_t M_0 M_t^{-1}, \qquad M_t' = (M_t M_0)\, M_t\, (M_t M_0)^{-1}, \qquad M_1' = M_1,$$

so that $\sigma_{0t}' = \sigma_{0t}$ and

$$p_{01}' = p_0 p_1 + p_t p_\infty - p_{01} - p_{0t}p_{1t}, \tag{2.36a}$$

$$p_{1t}' = p_1 p_t + p_0 p_\infty - p_{1t} - p_{0t}p_{01}'. \tag{2.36b}$$

Therefore, the change of the branch of $\tau(t)$ is encoded in the change of the structure constants. On the other hand, one can perform analytic continuation directly in the expansion (2.33). Up to an irrelevant overall factor, this amounts to multiplication of $C_n(\vec{\theta}, \vec{\sigma})$ by the braiding factor $e^{4\pi i n\sigma_{0t}}$. Since both results should coincide, the structure constants have to satisfy the functional relation

$$C_n(\vec{\theta}, \vec{\sigma}') = \varsigma \cdot e^{4\pi i n\sigma_{0t}} C_n(\vec{\theta}, \vec{\sigma}),$$

where ς is independent of n. The factor s_{0t}^n in (2.32) is a minimal solution of this relation, as for $\vec{\sigma}'$ defined by (2.36) one has $s_{0t}(\vec{\theta}, \vec{\sigma}') = e^{4\pi i\sigma_{0t}} s_{0t}(\vec{\theta}, \vec{\sigma})$.

[7] One can construct expansions in any of the six anharmonic ratios t, $1-t$, $\frac{1}{t}$, $\frac{t-1}{t}$, $\frac{t}{t-1}$, $\frac{1}{1-t}$. As the radius of convergence of conformal block series is believed to be 1, the corresponding tau functions series have overlapping regions of validity and glue into a global P_{VI} solution.

2.4 Conformal blocks from OPEs

This subsection is a digression recalling some basics on conformal blocks in 2D CFT [11]. For the sake of brevity, simplicity and relevance for the rest of the manuscript, we will concentrate on conformal blocks for the 4-point correlator on $\mathbb{P}^1$ and the Virasoro algebra

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n+m,0}. \tag{2.37}$$

Only chiral primary fields $\mathcal{O}$ will be considered, i.e. their antiholomorphic conformal dimensions $\bar{\Delta}_{\mathcal{O}} = 0$. We do not require invariance of correlators under the braid group action on the positions of fields to avoid constraints on dimensions.

The three-point correlator of primary fields is fixed by conformal symmetry up to a constant factor,

$$\langle \mathcal{O}_3(z_3)\mathcal{O}_2(z_2)\mathcal{O}_1(z_1)\rangle = C_{\Delta_3,\Delta_2,\Delta_1} z_{21}^{\Delta_3-\Delta_1-\Delta_2} z_{32}^{\Delta_1-\Delta_2-\Delta_3} z_{31}^{\Delta_2-\Delta_1-\Delta_3},$$

where $z_{ij} = z_i - z_j$ and Δ_j stand for holomorphic dimensions. Thanks to conformal invariance, it suffices to consider more special coordinate dependence, namely, we may set $z_1 = 0$, $z_2 = t$, $z_3 = R$ with $R \gg t$. It is also customary to use the convention (2.25) so that, for instance,

$$\langle \mathcal{O}_3(\infty)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle = C_{\Delta_3,\Delta_2,\Delta_1}\, t^{\Delta_3-\Delta_1-\Delta_2}.$$

Besides primary fields, conformal field theory also contains their descendants $L_{-\lambda}\mathcal{O} = L_{-\lambda_N}\ldots L_{-\lambda_1}\mathcal{O}$, naturally labeled by partitions $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N > 0\}$. Partitions can be identified in the obvious way with Young diagrams. As they play an important role below, we take the opportunity to fix some notation for later purposes.

The set of all Young diagrams will be denoted by $\mathbb{Y}$. For $\lambda \in \mathbb{Y}$, λ' denotes the transposed diagram, λ_i and λ'_j the number of boxes in ith row and jth column of λ, and $|\lambda|$ the total number of boxes (it gives the level of the corresponding descendant). Given a box $(i,j) \in \lambda$, its hook length is defined as $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$, see Fig. 3.

Conformal symmetry also allows to compute in explicit form the three-point functions involving one descendant:

$$\langle \mathcal{O}_3(\infty)\, \mathcal{O}_2(t) L_{-\lambda}\mathcal{O}_1(0)\rangle = C_{\Delta_3,\Delta_2,\Delta_1}\, \gamma_\lambda(\Delta_1,\Delta_2,\Delta_3)\, t^{-|\lambda|}, \tag{2.38a}$$

$$\langle L_{-\lambda}\mathcal{O}_3(\infty)\, \mathcal{O}_2(t)\mathcal{O}_1(0)\rangle = C_{\Delta_3,\Delta_2,\Delta_1}\, \gamma_\lambda(\Delta_3,\Delta_2,\Delta_1)\, t^{|\lambda|}, \tag{2.38b}$$

where

$$\gamma_\lambda(\Delta_1,\Delta_2,\Delta_3) = \prod_{j=1}^{N}\Big(\Delta_1 - \Delta_3 + \lambda_j\Delta_2 + \sum_{k=1}^{j-1}\lambda_k\Big). \tag{2.39}$$

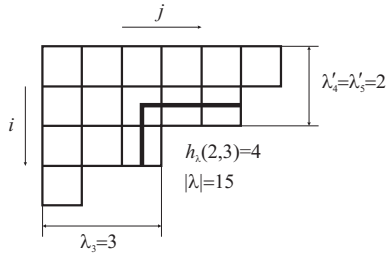


Fig. 3: Young diagram associated to the partition $\lambda = \{6, 5, 3, 1\}$

The action of $L_{-\lambda}$ on the field at infinity in (2.38b) should be understood as a result of successive contour integration with the energy-momentum tensor. This action can be transferred to the fields at 0 and t by deformation of the contour.

The relation (2.38b) is extremely important as it allows to determine the coefficients of the OPE of primary fields

$$\mathcal{O}_2(t)\mathcal{O}_1(0) = \sum_p \sum_{\mu\in\mathbb{Y}} C_{\Delta_p,\Delta_2,\Delta_1}\, \beta_\mu\left(\Delta_p, \Delta_2, \Delta_1\right) t^{\Delta_p-\Delta_1-\Delta_2+|\mu|} L_{-\mu}\mathcal{O}_p\left(0\right). \tag{2.40}$$

Indeed, assuming orthonormality of the basis of primaries, $\langle \mathcal{O}_p\left(\infty\right)\mathcal{O}_{p'}\left(0\right)\rangle = \delta_{pp'}$, and considering the correlator of both sides of the last relation with the descendant $L_{-\lambda}\mathcal{O}_p\left(\infty\right)$, one finds that

$$\beta_\lambda\left(\Delta_p, \Delta_2, \Delta_1\right) = \sum_{\mu\in\mathbb{Y}} \left[Q\left(\Delta_p\right)\right]^{-1}_{\lambda\mu} \gamma_\mu\left(\Delta_p, \Delta_2, \Delta_1\right), \tag{2.41}$$

where $Q_{\lambda\mu}\left(\Delta_p\right) = \langle L_{-\lambda}\mathcal{O}_p\left(\infty\right) L_{-\mu}\mathcal{O}_p\left(0\right)\rangle$ are the elements of the Kac-Shapovalov matrix. They can be computed algebraically as scalar products of descendant states

$$Q_{\lambda\mu}\left(\Delta\right) = \langle\Delta| L_{\lambda_1}\ldots L_{\lambda_N} L_{-\mu_M}\ldots L_{-\mu_1}|\Delta\rangle, \tag{2.42}$$

where $|\Delta\rangle$ and $\langle\Delta|$ denote the highest weight vectors annihilated by all $L_{n>0}$ and, respectively, all $L_{n<0}$, satisfying $L_0|\Delta\rangle = \Delta|\Delta\rangle$, $\langle\Delta|L_0 = \langle\Delta|\Delta$ and normalized as $\langle\Delta|\Delta\rangle = 1$. It is easy to understand that the only non-vanishing scalar products correspond to descendants from the same level, which means that $Q(\Delta)$ has a block-diagonal structure: $Q_{\lambda\mu}\left(\Delta\right) \sim \delta_{|\lambda|,|\mu|}$.

We can now finally calculate the four-point correlator $\langle\mathcal{O}_4(\infty)\mathcal{O}_3(1)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle$. Replace therein the product of fields $\mathcal{O}_2(t)\mathcal{O}_1(0)$ by the OPE (2.40) and then use (2.38a) and (2.41). The result is

$$\langle\mathcal{O}_4(\infty)\mathcal{O}_3(1)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle = \sum_p C_{\Delta_4,\Delta_3,\Delta_p}\, C_{\Delta_p,\Delta_2,\Delta_1}\, \mathcal{F}_c\left(\Delta_{1\ldots4}, \Delta_p, t\right), \tag{2.43}$$

where we have introduced the notation

$$\mathcal{F}_c\left(\Delta_{1\ldots4},\Delta,t\right)=t^{\Delta-\Delta_1-\Delta_2}\sum_{\lambda,\mu\in\mathbb{Y}}\gamma_\lambda\left(\Delta,\Delta_3,\Delta_4\right)\left[Q\left(\Delta\right)\right]^{-1}_{\lambda\mu}\gamma_\mu\left(\Delta,\Delta_2,\Delta_1\right)t^{|\lambda|}. \quad (2.44)$$

The representation (2.43) separates model-dependent information (three-point functions $C_{\Delta_i,\Delta_j,\Delta_k}$) from the universal pieces fixed solely by Virasoro symmetry.

The special function (2.44) is called *four-point conformal block*. This quantity appeared above in formulas such as (1.7), (1.8) or (2.26), (2.33). Sometimes it becomes convenient to use a related function

$$\mathcal{B}_c\left(\Delta_{1\ldots4},\Delta,t\right)=t^{\Delta_1+\Delta_2-\Delta}\mathcal{F}_c\left(\Delta_{1\ldots4},\Delta,t\right) \quad (2.45)$$

It is an entire power series in t with coefficients depending on four external dimensions $\Delta_{1\ldots4}$, one intermediate dimension Δ, and the central charge c which enters via the Kac-Shapovalov matrix. These coefficients can in principle be calculated using (2.39), (2.42) and the Virasoro commutation relations.

Example 2.5. For instance, bringing all $L_{n<0}$ in (2.42) to the left and all $L_{n>0}$ to the right, we easily determine non-zero elements of $Q(\Delta)$ for levels 1 and 2:

$$\begin{cases} Q_{\square\square}\left(\Delta\right)=2\Delta, \\ Q_{\square\square\,\square\square}\left(\Delta\right)=4\Delta+\frac{c}{2}, \\ Q_{\square\square\,\begin{smallmatrix}\square\\\square\end{smallmatrix}}\left(\Delta\right)=Q_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\square\square}\left(\Delta\right)=6\Delta, \\ Q_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\begin{smallmatrix}\square\\\square\end{smallmatrix}}\left(\Delta\right)=4\Delta\left(2\Delta+1\right). \end{cases} \quad (2.46)$$

Also, from (2.39) we deduce that

$$\begin{cases} \gamma_{\square}\left(\Delta,\Delta_2,\Delta_1\right)=\ \Delta-\Delta_1+\Delta_2, \\ \gamma_{\square\square}\left(\Delta,\Delta_2,\Delta_1\right)=\ \Delta-\Delta_1+2\Delta_2, \\ \gamma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\left(\Delta,\Delta_2,\Delta_1\right)=\left(\Delta-\Delta_1+\Delta_2\right)\left(\Delta-\Delta_1+\Delta_2+1\right). \end{cases} \quad (2.47)$$

Substituting (2.46)–(2.47) into the series (2.44) and computing the inverses of the corresponding blocks of $Q(\Delta)$, one arrives at

$$\begin{aligned} \mathcal{B}_c\left(\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta,t\right)&=1+\frac{\left(\Delta-\Delta_1+\Delta_2\right)\left(\Delta-\Delta_4+\Delta_3\right)}{2\Delta}t+ \\ &+\left[\frac{\left(\Delta-\Delta_1+\Delta_2\right)\left(\Delta-\Delta_1+\Delta_2+1\right)\left(\Delta-\Delta_4+\Delta_3\right)\left(\Delta-\Delta_4+\Delta_3+1\right)}{2\Delta\left(1+2\Delta\right)}+\right. \\ &\left.+\frac{\left(1+2\Delta\right)\left(\Delta_1+\Delta_2+\frac{\Delta(\Delta-1)-3(\Delta_1-\Delta_2)^2}{1+2\Delta}\right)\left(\Delta_4+\Delta_3+\frac{\Delta(\Delta-1)-3(\Delta_4-\Delta_3)^2}{1+2\Delta}\right)}{\left(1-4\Delta\right)^2+\left(c-1\right)\left(1+2\Delta\right)}\right]\frac{t^2}{2}+O\left(t^3\right). \end{aligned} \quad (2.48)$$

Direct (i.e. based on (2.44)) computation of conformal block coefficients becomes rather complicated at higher levels. An explicit representation for the arbitrary level was found only recently in a surprisingly different framework.

2.5 $\mathcal{N} = 2$ SUSY gauge theories

The AGT correspondence [4] relates Virasoro conformal blocks to Nekrasov functions [114, 115]. These functions represent the instanton parts of ϵ_1, ϵ_2-regularized partition functions in 4D $\mathcal{N} = 2$ SUSY quiver gauge theories.

The simplest case of AGT correspondence deals with $\mathrm{SU}(2)$ gauge theory with extra $N_f = 2N_c = 4$ fundamental (i.e. transforming in the spin-$\frac{1}{2}$ representation of the gauge group) matter hypermultiplets with masses $m_1, \ldots, m_4$. Parameters of this theory are related to those of 4-point conformal block on the sphere by

$$\begin{aligned} m_1 &= \alpha_3 - \alpha_4 + \frac{\epsilon}{2}, & m_2 &= \alpha_1 - \alpha_2 + \frac{\epsilon}{2}, \\ m_3 &= \alpha_1 + \alpha_2 - \frac{\epsilon}{2}, & m_4 &= \alpha_3 + \alpha_4 - \frac{\epsilon}{2}, \\ c &= 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}, & \epsilon &= \epsilon_1 + \epsilon_2, \\ \Delta_\nu &= \frac{\alpha_\nu\left(\epsilon - \alpha_\nu\right)}{\epsilon_1\epsilon_2}, & \nu &= 1, 2, 3, 4. \end{aligned}$$

The intermediate dimension Δ of conformal block is expressed via the eigenvalues $\pm a$ of the vacuum expectation value of scalar field in the gauge multiplet:

$$\Delta = \frac{\alpha\left(\epsilon - \alpha\right)}{\epsilon_1\epsilon_2}, \qquad \alpha = \frac{\epsilon}{2} + a.$$

The parameter t (anharmonic ratio of four points on the sphere) in the conformal block expansion is related to the bare complex coupling constant τ_{UV} on the gauge side by

$$t = \exp 2\pi i \tau_{UV}, \qquad \tau_{UV} = \frac{4\pi i}{g_{UV}^2} + \frac{\theta_{UV}}{2\pi}.$$

Partition function in the regularized theory is an integral over a compactified moduli space $\mathfrak{M}$ of instantons. The integral is given by a sum of explicitly computable contributions coming from fixed points of a torus action on $\mathfrak{M}$, which are labeled by pairs of partitions. On the CFT side, this is interpreted as existence of a geometrically distinguished basis of states in the highest weight representations of the Virasoro algebra. For more details, generalizations and further references, the reader is referred to [10].

When all four masses $m_{1,2,3,4} \to \infty$, the fundamental hypermultiplets decouple and we get pure gauge theory. Decoupling only some of them yields asymptotically free theories

with $N_f < 2N_c$. From the gauge theory point of view, the parameter t_{N_f} can be considered as a dynamically generated scale. Taking into account the RG dependence of the coupling constant, one finds that in the decoupling process this scale should transform in the appropriate way: $m_{N_f} \to \infty$, $t_{N_f} \to 0$, $t_{N_f-1} = m_{N_f} t_{N_f}$ fixed [39, 143]. The corresponding Nekrasov functions are related to irregular conformal blocks [59, 19, 60, 104].

Recall that conformal blocks relevant to Painlevé VI equation are characterized by the central charge $c = 1$ and external dimensions $\Delta_\nu = \theta_\nu^2$ (with $\nu = 0, t, 1, \infty$). Hence we can set $\epsilon_1 = -\epsilon_2 = 1$, $\alpha_1 = \theta_0$, $\alpha_2 = \theta_t$, $\alpha_3 = \theta_1$, $\alpha_4 = \theta_\infty$ and

$$m_1 = \theta_1 - \theta_\infty, \qquad m_2 = \theta_0 - \theta_t, \qquad m_3 = \theta_0 + \theta_t, \qquad m_4 = \theta_1 + \theta_\infty.$$

We will see in the next subsection that, under such identification of parameters, successive decoupling of the matter hypermultiplets corresponds to the scaling limits leading from P_{VI} to P_{V} and three inequivalent $\mathrm{P}_{\mathrm{III}}$ equations (Fig. 4).

$$\begin{array}{ccccccccc} N_f = 4 & \xrightarrow{m_4 \to \infty} & N_f = 3 & \xrightarrow{m_3 \to \infty} & N_f = 2 & \xrightarrow{m_2 \to \infty} & N_f = 1 & \xrightarrow{m_1 \to \infty} & \text{pure gauge theory} \\ (P_{\mathrm{VI}}) & & (P_{\mathrm{V}}) & & (P_{\mathrm{III}_1}) & & (P_{\mathrm{III}_2}) & & (P_{\mathrm{III}_3}) \end{array}$$

Fig. 4: Decoupling of matter hypermultiplets

2.6 Confluence cascade

This subsection collects the necessary technical material about the lower Painlevé equations, in particular about their confluences. We will need it in the study of scaling limits of the conformal expansion (2.33).

If $\delta \neq 0$ in P_{V}, then one can set $\delta = -\frac{1}{2}$ by rescaling the independent variable. P_{V} with $\delta = 0$ is reducible to $\mathrm{P}_{\mathrm{III}}$ (see e.g. [30, transformations (1.24)–(1.26)]) which will be treated separately. Hence we may set

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{V}} = \left(2\theta_0^2, -2\theta_t^2, 2\theta_* - 1, -\frac{1}{2}\right). \tag{2.49}$$

The case of $\mathrm{P}_{\mathrm{III}}$ is slightly more involved. First of all we note that it is often convenient to use instead of P_{III} an equivalent equation

$$\frac{d^2 q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2(\alpha + \gamma q)}{4t^2} + \frac{\beta}{4t} + \frac{\delta}{4q}, \tag{$\mathrm{P}_{\mathrm{III}'}$}$$

which reduces to $\mathrm{P}_{\mathrm{III}}$ by setting $t_{\mathrm{III}'} = t_{\mathrm{III}}^2$, $q_{\mathrm{III}'} = t_{\mathrm{III}} q_{\mathrm{III}}$. In the generic situation, when $\gamma\delta \neq 0$, one can assume that $\gamma = -\delta = 4$ by rescaling t and q. We will then write

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{III}_1} = (8\theta_\star, 4 - 8\theta_*, 4, -4)\,. \tag{2.50}$$

The variable change $q \to q^{-1}$ maps $\mathrm{P}_{\mathrm{III}}$ with $\delta = 0$ to $\mathrm{P}_{\mathrm{III}}$ with $\gamma = 0$. Assume that $\gamma = 0$ and $\alpha\delta \neq 0$, then the scaling freedom can be used to set

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{III}_2} = (8, 4 - 8\theta_*, 0, -4)\,. \tag{2.51}$$

For $\gamma = \delta = 0$, $\alpha\beta \neq 0$ we can set

$$(\alpha, \beta, \gamma, \delta)_{\mathrm{III}_3} = (8, -8, 0, 0)\,. \tag{2.52}$$

Finally, for $\alpha = \gamma = 0$ (and, similarly, for $\beta = \delta = 0$ by $q \to q^{-1}$), the general (two-parameter) solution of $\mathrm{P}_{\mathrm{III}}$ is known [103]. It reads

$$q(t) = \mu t^{1-\nu} + \frac{\beta}{\nu^2}\, t + \frac{\beta^2 + \nu^2\delta}{4\mu\nu^4}\, t^{1+\nu},$$

where μ, ν are two arbitrary integration constants. Excluding this last solvable case, there remain *three* inequivalent $\mathrm{P}_{\mathrm{III}}$'s with two, one and zero parameters.

The time-dependent Hamiltonians of P_{V} and $\mathrm{P}_{\mathrm{III}_{1,2,3}}$ satisfy the σ-forms of the corresponding Painlevé equations:

$$\mathrm{P}_{\mathrm{V}}: \qquad (t\sigma'')^2 = \left(\sigma - t\sigma' + 2\,(\sigma')^2\right)^2 - \frac{1}{4}\left((2\sigma' - \theta_*)^2 - 4\theta_0^2\right)\left((2\sigma' + \theta_*)^2 - 4\theta_t^2\right), \tag{2.53}$$

$$\mathrm{P}_{\mathrm{III}_1'}: \qquad (t\sigma'')^2 = \left(4\,(\sigma')^2 - 1\right)(\sigma - t\sigma') - 4\theta_*\theta_\star\sigma' + \left(\theta_*^2 + \theta_\star^2\right), \tag{2.54}$$

$$\mathrm{P}_{\mathrm{III}_2'}: \qquad (t\sigma'')^2 = 4\,(\sigma')^2\,(\sigma - t\sigma') - 4\theta_*\sigma' + 1, \tag{2.55}$$

$$\mathrm{P}_{\mathrm{III}_3'}: \qquad (t\sigma'')^2 = 4\,(\sigma')^2\,(\sigma - t\sigma') - 4\sigma', \tag{2.56}$$

The solutions of P_{V}, $\mathrm{P}_{\mathrm{III}_{1,2,3}}$ can thus be mapped to solutions of (2.53)–(2.56). Conversely, one can recover conventional Painlevé functions from the solutions of σ-Painlevé equations using the following formulas:

$$\mathrm{P}_{\mathrm{V}}: \qquad q = \frac{2\left(t\sigma'' + \sigma - t\sigma' + 2\,(\sigma')^2\right)}{(2\sigma' - \theta_*)^2 - 4\theta_0^2}, \tag{2.57}$$

$$\mathrm{P}_{\mathrm{III}_1'}: \qquad q = -\frac{2t\sigma'' + 4\theta_*\sigma' - 2\theta_\star}{4\,(\sigma')^2 - 1}, \tag{2.58}$$

$$\mathrm{P}_{\mathrm{III}_2'}: \qquad q = -\frac{t\sigma'' + 2\theta_*\sigma' - 1}{2\,(\sigma')^2}, \tag{2.59}$$

$$\mathrm{P}_{\mathrm{III}_3'}: \qquad q = -\frac{1}{\sigma'}. \tag{2.60}$$

Finally, we define the tau functions of P_{V} and $\mathrm{P}_{\mathrm{III}}$ by

$$\sigma_{\mathrm{V}}(t) = t\frac{d}{dt}\ln\left(e^{-\frac{\theta_* t}{2}}\, t^{-\theta_0^2 - \theta_t^2 - \frac{\theta_*^2}{2}}\,\tau_{\mathrm{V}}(t)\right), \tag{2.61}$$

$$\sigma_{\mathrm{J}}(t) = t\frac{d}{dt}\ln\tau_{\mathrm{J}}(t), \qquad \mathrm{J} = \mathrm{III}_{1,2,3}'. \tag{2.62}$$

Our solutions below will formulated in terms of combinatorial expansions of these tau functions in powers of t. Expansions of σ's and q's can then be obtained from the relations (2.57)–(2.60) and (2.61)–(2.62).

As is well-known, Painlevé VI produces all the other Painlevé equations in certain scaling limits. The equations considered in this manuscript form the first line of the coalescence cascade

$$\begin{tikzcd}
\mathrm{VI} \arrow[r] & \mathrm{V} \arrow[r] \arrow[d] & \mathrm{III}_1 \arrow[r] \arrow[d] & \mathrm{III}_2 \arrow[r] \arrow[d] & \mathrm{III}_3 \\
& \mathrm{IV} \arrow[r] & \mathrm{II} \arrow[r] & \mathrm{I} &
\end{tikzcd}$$

Fig. 5: Coalescence diagram for Painlevé equations

Every step to the right or to the bottom of the diagram decreases by 1 the number of parameters in the corresponding equation. There exist more limits than shown in Fig. 5; the interested reader may consult [93, 122] for details.

Let us now describe the scaling limits we need below (1st line):

- $\mathrm{P}_{\mathrm{VI}} \to \mathrm{P}_{\mathrm{V}}$: set in P_{VI}

$$\theta_1 = \frac{\Lambda + \theta_*}{2}, \qquad \theta_\infty = \frac{\Lambda - \theta_*}{2}, \tag{2.63a}$$

then solutions of P_{V} can be obtained as the limits

$$1 - q_{\mathrm{V}}(t) = \lim_{\Lambda\to\infty} \frac{t/\Lambda}{q_{\mathrm{VI}}(t/\Lambda)}, \tag{2.63b}$$

$$\sigma_{\mathrm{V}}(t) = \lim_{\Lambda\to\infty} \left(\frac{\Lambda^2 - \Lambda t - \theta_*^2 - 2\theta_0^2 - 2\theta_t^2}{4} - \sigma_{\mathrm{VI}}(t/\Lambda) \right), \tag{2.63c}$$

$$\tau_{\mathrm{V}}(t) = \lim_{\Lambda\to\infty} (t/\Lambda)^{\theta_0^2+\theta_t^2} \tau_{\mathrm{VI}}(t/\Lambda), \tag{2.63d}$$

- $\mathrm{P}_{\mathrm{V}} \to \mathrm{P}_{\mathrm{III}_1'}$: this limiting transition is described by

$$\theta_0 = \frac{\Lambda - \theta_\star}{2}, \qquad \theta_t = \frac{\Lambda + \theta_\star}{2}, \tag{2.64a}$$

$$q_{\mathrm{III}_1'}(t) = \lim_{\Lambda\to\infty} \Lambda\left(1 - q_{\mathrm{V}}(t/\Lambda)\right), \tag{2.64b}$$

$$\sigma_{\mathrm{III}_1'}(t) = \lim_{\Lambda\to\infty} \left(\frac{\Lambda^2 + \theta_*^2 + \theta_\star^2}{2} + \sigma_{\mathrm{V}}(t/\Lambda) \right), \tag{2.64c}$$

$$\tau_{\mathrm{III}_1'}(t) = \lim_{\Lambda\to\infty} \tau_{\mathrm{V}}(t/\Lambda), \tag{2.64d}$$

- $\mathrm{P}_{\mathrm{III}_1'} \to \mathrm{P}_{\mathrm{III}_2'}$: similarly,

$$q_{\mathrm{III}_2'}(t) = \lim_{\theta_\star\to\infty} \theta_\star q_{\mathrm{III}_1'}(t/\theta_\star)\,, \tag{2.65a}$$

$$\sigma_{\mathrm{III}_2'}(t) = \lim_{\theta_\star\to\infty} \sigma_{\mathrm{III}_1'}(t/\theta_\star)\,, \tag{2.65b}$$

$$\tau_{\mathrm{III}_2'}(t) = \lim_{\theta_\star\to\infty} \tau_{\mathrm{III}_1'}(t/\theta_\star)\,. \tag{2.65c}$$

- $\mathrm{P}_{\mathrm{III}_2'} \to \mathrm{P}_{\mathrm{III}_3'}$:

$$q_{\mathrm{III}_3'}(t) = \lim_{\theta_*\to\infty} q_{\mathrm{III}_2'}(t/\theta_*)\,, \tag{2.66a}$$

$$\sigma_{\mathrm{III}_3'}(t) = \lim_{\theta_*\to\infty} \sigma_{\mathrm{III}_2'}(t/\theta_*)\,, \tag{2.66b}$$

$$\tau_{\mathrm{III}_3'}(t) = \lim_{\theta_*\to\infty} \tau_{\mathrm{III}_2'}(t/\theta_*)\,. \tag{2.66c}$$

2.7 General solutions of P_{VI}, P_{V} and $\mathrm{P}_{\mathrm{III}}$'s

We are now finally ready to obtain explicit series representations for the general solutions of P_{VI}, P_{V} and three inequivalent $\mathrm{P}_{\mathrm{III}}$ equations.

Conjecture 2.6. *Generic Painlevé VI tau function can be written in the form of conformal expansion around the critical point $t = 0$:*

$$\tau_{\mathrm{VI}}(t) = \sum_{n\in\mathbb{Z}} C_{\mathrm{VI}}(\vec{\theta}, \sigma+n)\, s_{\mathrm{VI}}^n\, t^{(\sigma+n)^2-\theta_0^2-\theta_t^2} \mathcal{B}_{\mathrm{VI}}(\vec{\theta}, \sigma+n, t). \tag{2.67}$$

Here the structure constants $C_{\mathrm{VI}}(\vec{\theta},\sigma) := C\left[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma\right]$ are determined by (2.31). The parameters $\sigma := \sigma_{0t}$ and $s_{\mathrm{VI}} := s_{0t}$ play the role of two integration constants and are related to the monodromy of the linear system (1.1) by (2.27b) and (2.30). The function $\mathcal{B}_{\mathrm{VI}}(\vec{\theta},\sigma,t)$ coincides with the conformal block $\mathcal{B}_{c=1}\left(\theta_0^2,\theta_t^2,\theta_1^2,\theta_\infty^2,\sigma^2,t\right)$ and is explicitly given by the combinatorial series

$$\mathcal{B}_{\mathrm{VI}}(\vec{\theta},\sigma,t) = (1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{(\mathrm{VI})}_{\lambda,\mu}(\vec{\theta},\sigma)\, t^{|\lambda|+|\mu|}, \tag{2.68}$$

whose coefficients are determined by

$$\mathcal{B}^{(\mathrm{VI})}_{\lambda,\mu}(\vec{\theta},\sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)^2} \times \tag{2.69}$$
$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)^2}\,.$$

This conjecture is in fact a rewrite of (2.33): the formulas (2.68)–(2.69) are nothing but the AGT representation of the $c = 1$ conformal block rewritten in terms of P_{VI} parameters.

As explained in the Introduction, σ-Painlevé VI equation (2.23) allows to reconstruct the tau function expansions recursively, starting from the leading asymptotic terms determined by Jimbo's formula. The result can then be checked against the CFT predictions. Technical details of this procedure are described in [61, Section 3].

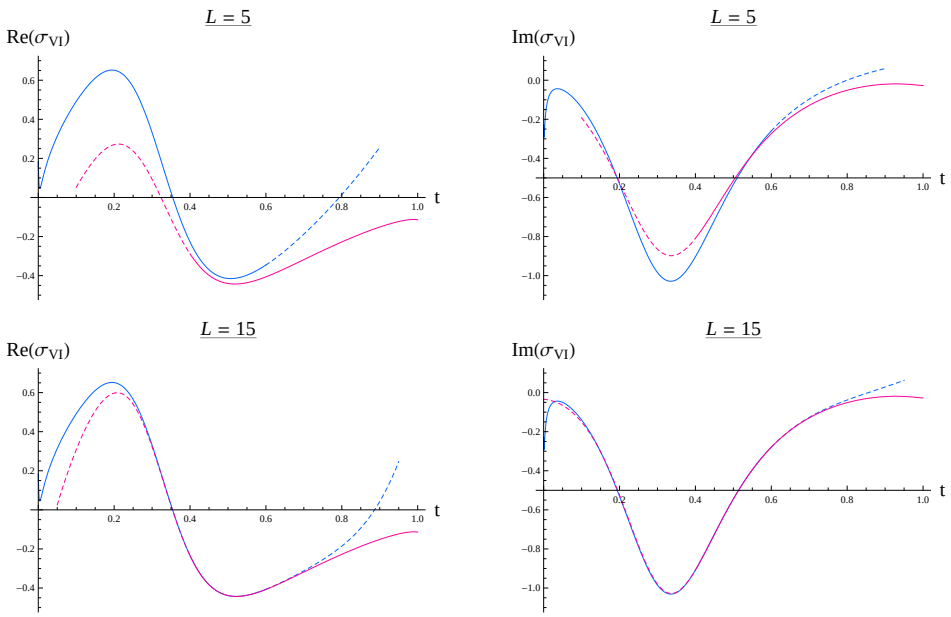


Fig. 6: Truncated P_{VI} series at $t = 0$ (blue curve) and $t = 1$ (red curve) with $\vec{\theta} = (0.1902 + 0.3106i, 0.4182 - 0.2109i, 0.3429 + 0.3314i, 0.0163 + 0.1805i)$, $\vec{\sigma} = (-0.3272 - 0.4811i, 0.0958 + 0.3168i, 0.4762 + 0.1028i)$

Numerical efficiency of the expansions (2.33), (2.34) is illustrated in Fig. 6. For random complex $\vec{\theta}$, $\vec{\sigma}$ we plot on the same graph the series for $\sigma_{\mathrm{VI}}(t)$ around $t = 0$ and $t = 1$ keeping the terms up to $O\left(t^L\right)$ and $O\left((1-t)^L\right)$ with $L = 5, 15$. Zooming near the endpoints $t = 0, 1$ would display oscillations of rapidly increasing frequency and decreasing amplitude due to non-zero imaginary parts of σ_{0t}, σ_{1t}.

Next we consider the scaling limit $\mathrm{P}_{\mathrm{VI}} \to \mathrm{P}_{\mathrm{V}}$ given by (2.63). Conformal block function $\mathcal{B}_{\mathrm{VI}}\left(\theta_0, \theta_t, \frac{\Lambda+\theta_*}{2}, \frac{\Lambda-\theta_*}{2}, \sigma; \frac{t}{\Lambda}\right)$ has a well-defined limit as $\Lambda \to \infty$, which can be calculated termwise in (2.68)–(2.69). The asymptotics of the structure constants $C_{\mathrm{VI}}\left(\theta_0, \theta_t, \frac{\Lambda+\theta_*}{2}, \frac{\Lambda-\theta_*}{2}, \sigma\right)$ ensures consistency of the expansion (2.67) with the limit (2.63d). More precisely, using the known large argument behavior of the Barnes function, it is easy to check that

$$\lim_{\Lambda\to\infty} \frac{\Lambda^{-\sigma^2} C_{\mathrm{VI}}\left(\theta_0, \theta_t, \frac{\Lambda+\theta_*}{2}, \frac{\Lambda-\theta_*}{2}, \sigma\right)}{G^2\left(1+\Lambda\right)} = C_{\mathrm{V}}\left(\theta_0, \theta_t, \theta_*, \sigma\right), \tag{2.70}$$

where

$$C_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma\right)=\prod_{\epsilon=\pm}G\left[\begin{array}{c}1+\theta_*+\epsilon\sigma,1+\theta_t+\theta_0+\epsilon\sigma,1+\theta_t-\theta_0+\epsilon\sigma\\ 1+2\epsilon\sigma\end{array}\right]. \tag{2.71}$$

One could even completely get rid of the denominator in the l.h.s. of (2.70) by modifying the normalization of P_{VI} tau function in (2.67) (e.g. by dividing all structure constants in (2.31) by a σ-independent factor $G^2\left(1+\theta_1+\theta_\infty\right)$).

Altogether, this leads to

Conjecture 2.7. *Painlevé V is solved by the following tau function expansion at $t=0$:*

$$\tau_{\mathrm{V}}(t)=\sum_{n\in\mathbb{Z}}C_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma+n\right)s_{\mathrm{V}}^n\,t^{(\sigma+n)^2}\mathcal{B}_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma+n;t\right). \tag{2.72}$$

Here again σ and s_{V} are arbitrary parameters, the irregular conformal block $\mathcal{B}_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma;t\right)$ is a power series defined by

$$\mathcal{B}_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma;t\right)=e^{-\theta_t t}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}^{(\mathrm{V})}_{\lambda,\mu}\left(\theta_0,\theta_t,\theta_*,\sigma\right)t^{|\lambda|+|\mu|}, \tag{2.73}$$

$$\begin{aligned}\mathcal{B}^{(\mathrm{V})}_{\lambda,\mu}\left(\theta_0,\theta_t,\theta_*,\sigma\right)=&\prod_{(i,j)\in\lambda}\frac{\left(\theta_*+\sigma+i-j\right)\left(\left(\theta_t+\sigma+i-j\right)^2-\theta_0^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j+\mu_i-i-j+1+2\sigma\right)^2}\times\\ &\times\prod_{(i,j)\in\mu}\frac{\left(\theta_*-\sigma+i-j\right)\left(\left(\theta_t-\sigma+i-j\right)^2-\theta_0^2\right)}{h_\mu^2(i,j)\left(\lambda_i+\mu'_j-i-j+1-2\sigma\right)^2},\end{aligned} \tag{2.74}$$

and the structure constants $C_{\mathrm{V}}\left(\theta_0,\theta_t,\theta_*,\sigma\right)$ are given by (2.71).

The second P_{V} critical point $t=\infty$ corresponds to irregular singularity of the associated 2×2 linear system and is obtained by the fusion of two P_{VI} critical points $1,\infty$. The expansion around this point cannot be extracted from P_{VI} series and requires the knowledge of complete irregular OPEs. For the same reason, we are so far unable to treat P_{IV}, P_{II} and P_{I}. However, long-distance expansions of this kind are available in a few special cases where the solutions of P_{V}, $\mathrm{P}_{\mathrm{III}}$ can be expressed in terms of Fredholm determinants.

Because of the presence of irregular singular points, monodromy data for P_{V} involve Stokes multipliers. The expression for the integration constants σ, s_{V} of Conjecture 2.7 in terms of monodromy can be extracted from Jimbo's paper [79].

Repeating the previous arguments almost literally for the scaling limits (2.64), (2.65) and (2.66), one obtains short-distance expansions of the tau functions of three nontrivial $\mathrm{P}_{\mathrm{III}}$ equations:

Conjecture 2.8. *Expansion of $\tau_{\mathrm{III}_1'}(t)$ at $t=0$ can be written as*

$$\tau_{\mathrm{III}_1'}(t)=\sum_{n\in\mathbb{Z}} C_{\mathrm{III}_1'}\left(\theta_*,\theta_\star,\sigma+n\right) s_{\mathrm{III}_1'}^{n}\, t^{(\sigma+n)^2}\mathcal{B}_{\mathrm{III}_1'}\left(\theta_*,\theta_\star,\sigma+n;t\right), \tag{2.75}$$

where the irregular conformal block $\mathcal{B}_{\mathrm{III}_1'}\left(\theta_*,\theta_\star,\sigma;t\right)$ is given by

$$\mathcal{B}_{\mathrm{III}_1'}\left(\theta_*,\theta_\star,\sigma;t\right)=e^{-\frac{t}{2}}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}^{(\mathrm{III}_1')}\left(\theta_*,\theta_\star,\sigma\right)t^{|\lambda|+|\mu|}, \tag{2.76}$$

$$\begin{aligned}\mathcal{B}_{\lambda,\mu}^{(\mathrm{III}_1')}\left(\theta_*,\theta_\star,\sigma\right)=&\prod_{(i,j)\in\lambda}\frac{\left(\theta_*+\sigma+i-j\right)\left(\theta_\star+\sigma+i-j\right)}{h_\lambda^2(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)^2}\times\\ &\times\prod_{(i,j)\in\mu}\frac{\left(\theta_*-\sigma+i-j\right)\left(\theta_\star-\sigma+i-j\right)}{h_\mu^2(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)^2},\end{aligned} \tag{2.77}$$

and the structure constants can be written as

$$C_{\mathrm{III}_1'}\left(\theta_*,\theta_\star,\sigma\right)=\prod_{\epsilon=\pm}\frac{G(1+\theta_*+\epsilon\sigma)G(1+\theta_\star+\epsilon\sigma)}{G(1+2\epsilon\sigma)}. \tag{2.78}$$

Conjecture 2.9. *Expansion of $\tau_{\mathrm{III}_2'}(t)$ at $t=0$ is given by*

$$\tau_{\mathrm{III}_2'}(t)=\sum_{n\in\mathbb{Z}} C_{\mathrm{III}_2'}\left(\theta_*,\sigma+n\right) s_{\mathrm{III}_2'}^{n}\, t^{(\sigma+n)^2}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}^{(\mathrm{III}_2')}\left(\theta_*,\sigma+n\right)t^{|\lambda|+|\mu|}, \tag{2.79}$$

with arbitrary σ, $s_{\mathrm{III}_2'}$ and

$$\begin{aligned}\mathcal{B}_{\lambda,\mu}^{(\mathrm{III}_2')}\left(\theta_*,\sigma\right)=&\prod_{(i,j)\in\lambda}\frac{\theta_*+\sigma+i-j}{h_\lambda^2(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)^2}\times\\ &\times\prod_{(i,j)\in\mu}\frac{\theta_*-\sigma+i-j}{h_\mu^2(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)^2},\end{aligned} \tag{2.80}$$

The $\mathrm{P}_{\mathrm{III}_2'}$ structure constants are given by $C_{\mathrm{III}_2'}\left(\theta_*,\sigma\right)=\prod_{\epsilon=\pm}\frac{G\left(1+\theta_*+\epsilon\sigma\right)}{G\left(1+2\epsilon\sigma\right)}$.

Conjecture 2.10. *Complete expansion of the $\mathrm{P}_{\mathrm{III}_3'}$ tau function at $t=0$ is:*

$$\tau_{\mathrm{III}_3'}(t)=\sum_{n\in\mathbb{Z}} C_{\mathrm{III}_3'}\left(\sigma+n\right) s_{\mathrm{III}_3'}^{n}\, t^{(\sigma+n)^2}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}^{(\mathrm{III}_3')}\left(\sigma+n\right)t^{|\lambda|+|\mu|}, \tag{2.81}$$

where $C_{\mathrm{III}_3'}\left(\sigma\right)=\left[G\left(1+2\sigma\right)G\left(1-2\sigma\right)\right]^{-1}$ and

$$\begin{aligned}\mathcal{B}_{\lambda,\mu}^{(\mathrm{III}_3')}\left(\sigma\right)=&\Bigg[\prod_{(i,j)\in\lambda}h_\lambda(i,j)\left(\lambda_j'+\mu_i-i-j+1+2\sigma\right)\times\\ &\times\prod_{(i,j)\in\mu}h_\mu(i,j)\left(\lambda_i+\mu_j'-i-j+1-2\sigma\right)\Bigg]^{-2}.\end{aligned} \tag{2.82}$$

It is possible to show (see e.g. [77, Proposition 1] for $\mathrm{P}_{\mathrm{III}'_3}$) that for $2\sigma \notin \mathbb{Z}$ combinatorial series for conformal blocks and tau functions have non-zero radius of convergence. As in the P_{VI} case, Conjectures 2.7–2.10 can be verified by iterative reconstruction of the tau function expansions from the leading asymptotic terms using the equations $\mathrm{P}_{\mathrm{V,III}'_{1,2,3}}$.

2.8 Some examples

2.8.1 Picard solutions and Ashkin-Teller conformal blocks

As a first illustration of conformal expansions, consider Painlevé VI solutions of Picard type. Here the four parameters of P_{VI} are chosen as $\vec{\theta}_{\text{Picard}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Up to Bäcklund transformations, this is the only case where the general two-parameter solution of P_{VI} is available. Quite remarkably, for precisely these $\vec{\theta}$ there exists an explicit formula for $c = 1$ conformal block, found by Al. B. Zamolodchikov [160]. We now briefly explain the relation between the two subjects.

Consider a complex torus $\mathbb{C}/\left(\mathbb{Z} + \tau\mathbb{Z}\right)$ and identify its points related by multiplication by -1. This yields a double cover of $\mathbb{P}^1$ with 4 ramification points. Their cross-ratio (our variable t) remains invariant under the natural action of the group $\Gamma(2)$ and is explicitly given by the elliptic lambda function

$$t = \frac{\vartheta_2^4\left(0|\tau\right)}{\vartheta_3^4\left(0|\tau\right)}, \tag{2.83}$$

where $\vartheta_{2,3}\left(z|\tau\right)$ are the usual Jacobi theta functions

$$\vartheta_2\left(z|\tau\right) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{i\pi n^2\tau + 2inz}, \qquad \vartheta_3\left(z|\tau\right) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau + 2inz}.$$

The inverse map can be written as

$$\tau = i\frac{K(1-t)}{K(t)}, \tag{2.84}$$

where $K(t)$ denotes complete elliptic integral of the 1st kind:

$$K(t) = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}.$$

It is clear that in the limit $t \to +0$ one has $\tau \to i\infty$. Even more specifically, one can check that $\lim\limits_{t\to+0} t^{-1}e^{i\pi\tau} = \frac{1}{16}$.

In this parametrization, conformal block with external dimensions corresponding to $\vec{\theta}_{\text{Picard}}$ is given by (see Eq. (2.28) in [160])

$$\mathcal{F}(\vec{\theta}_{\text{Picard}}, \sigma, t) = \frac{2^{4\sigma^2} e^{i\pi\sigma^2\tau}}{t^{\frac{1}{8}}\left(1-t\right)^{\frac{1}{8}}\vartheta_3\left(0|\tau\right)}. \tag{2.85}$$

On the other hand, Barnes function duplication identity reduces the formula (2.31) for the structure constants to

$$C\left[\begin{array}{cc}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{array};\sigma\right]_{\text{Picard}} = \text{const}\cdot(-)^n\, 2^{-4\sigma^2}.$$

The expression for s_{0t} also simplifies drastically. Jimbo-Fricke relation implies that for fixed σ_{0t}, σ_{1t} the two possible values of $\cos 2\pi\sigma_{01}$ are given by $-\cos 2\pi(\sigma_{0t}\mp\sigma_{1t})$ and therefore $s_{0t} = -e^{\pm 2\pi i\sigma_{1t}}$.

Summation over conformal families in (2.33) now gives theta function series so that we get an explicit expression

$$\tau_{\text{Picard}}(t) = \text{const}\cdot\ \frac{e^{i\pi\sigma_{0t}^2\tau}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\frac{\vartheta_3\left(\sigma_{0t}\pi\tau\pm\sigma_{1t}\pi|\tau\right)}{\vartheta_3(0|\tau)}.$$

This function is indeed known to satisfy Painlevé VI with Picard parameters [94], i.e. the conformal expansion (2.33) is correct and complete.

2.8.2 *Riccati solutions and degenerate fields: AGT vs Gessel's theorem*

In 2002, P. J. Forrester and N. S. Witte [56] have proved a remarkable determinant representation for a family of Riccati solutions of Painlevé VI. Their result can be restated as follows. Define a five-parameter family of $N\times N$ Toeplitz determinants

$$D_N^{(\nu,\nu',\eta,\xi)}(t) = \det\left[A_{j-k}^{(\nu,\nu',\eta,\xi)}(t)\right]_{j,k=0}^{N-1}, \tag{2.86}$$

$$\begin{aligned} A_m^{(\nu,\nu',\eta,\xi)}(t) = {} & \frac{\Gamma\left(1+\nu'\right)t^{\frac{\eta-m}{2}}\left(1-t\right)^{\nu}}{\Gamma\left(1+\eta-m\right)\Gamma\left(1-\eta+m+\nu'\right)}\,{}_2F_1\left[\begin{array}{c}-\nu,1+\nu'\\ 1+\eta-m\end{array}\middle|\frac{t}{t-1}\right]+ \\ & +\frac{\xi\Gamma\left(1+\nu\right)t^{\frac{m-\eta}{2}}\left(1-t\right)^{\nu'}}{\Gamma\left(1-\eta+m\right)\Gamma\left(1+\eta-m+\nu\right)}\,{}_2F_1\left[\begin{array}{c}1+\nu,-\nu'\\ 1-\eta+m\end{array}\middle|\frac{t}{t-1}\right]. \end{aligned} \tag{2.87}$$

Then the function

$$\tau_N^{(\nu,\nu',\eta,\xi)}(t) = (1-t)^{-\frac{N(N+\nu+\nu')}{2}}\,D_N^{(\nu,\nu',\eta,\xi)}(t) \tag{2.88}$$

is a P_{VI} tau function with parameters

$$(\theta_0,\theta_t,\theta_1,\theta_\infty) = \frac{1}{2}\left(\eta, N, -N-\nu-\nu', \nu-\nu'+\eta\right).$$

Looking at the asymptotic expansions of $D_N^{(\nu,\nu',\eta,\xi)}(t)$ at 0 and 1, one can also identify the monodromy exponents

$$(\sigma_{0t},\sigma_{1t},\sigma_{01}) = \frac{1}{2}\left(N+\eta, \nu+\nu', N+\nu-\nu'+\eta\right).$$

Almost all structure constants in (2.67) vanish because of the relations $\theta_t = \frac{N}{2}$, $\sigma_{0t} = \theta_0 + \theta_t$ (recall that Barnes G-function has zeros at negative integer values of the argument). The only non-zero contributions correspond to $n = 0, -1, \ldots, -N$, so that there remain only $N+1$ conformal blocks in the expansion (2.33). The parameter $s_{\rm VI}$ in (2.67) is related to ξ in (2.87) by

$$\xi s_{\rm VI} = \frac{\sin \pi\nu \sin \pi(\eta - \nu')}{\sin \pi\nu' \sin \pi(\eta + \nu)}. \tag{2.89}$$

Let us first examine in more detail the case $\xi \to 0$. The relation (2.89) then implies that $s_{\rm VI} \to \infty$, which in turn means that the expansion (2.67) at $t = 0$ contains only one ($n = 0$) conformal block $\mathcal{B}_{\rm VI}\left(\frac{\eta}{2}, \frac{N}{2}, \frac{N+\nu+\nu'}{2}, \frac{\nu-\nu'+\eta}{2}, \frac{N+\eta}{2}, t\right)$. The product over boxes of μ in the combinatorial representation (2.69) contains a factor $i - j$ due to the relation $\sigma_{0t} = \theta_0 + \theta_t$. Since this expression vanishes for the box $(1,1)$, the quantity $\mathcal{B}^{(\rm VI)}_{\lambda,\mu}$ does so for any non-empty μ. Moreover, the factor $i - j + N$ in the product over boxes of λ reduces the summation in (2.68) to Young diagrams with $\lambda_1 \le N$ (i.e. with the length of their first row not exceeding N).

Therefore, Conjecture 2.6 for the above parameters is equivalent to the following identity:

$$D_N^{(\nu,\nu',\eta,0)}(t) = C_N \sum_{\lambda\in\mathbb{Y}|\lambda_1\le N} t^{|\lambda|+\frac{N\eta}{2}} \prod_{(i,j)\in\lambda} \frac{i-j+N}{i-j+N+\eta} \frac{(i-j-\nu)\,(i-j-\nu'+\eta)}{h_\lambda^2(i,j)}, \tag{2.90}$$

where the constant prefactor

$$C_N = G\left[\begin{array}{c} 1+N, 1+\nu'+N, 1+\eta, 1-\eta+\nu' \\ 1+\eta+N, 1-\eta+\nu'+N, 1+\nu' \end{array}\right] \tag{2.91}$$

can be computed using the pure Fisher-Hartwig determinant.

In the limit $\eta \to 0$, the left hand side of (2.90) reduces to $N \times N$ Toeplitz determinant with the symbol

$$A(\zeta) = \left(1 + \sqrt{t}\,\zeta\right)^{\nu} \left(1 + \sqrt{t}\,\zeta^{-1}\right)^{\nu'}. \tag{2.92}$$

Also, $C_N = 1$ and the first factor in the product on the right disappears so that the r.h.s. coincides with the length distribution function of the first row of a random Young diagram distributed according to the so-called z-measure [20]. The equality (2.90) can then be rigorously demonstrated using a dual version of Gessel's theorem [66, 150].

Remark 2.11. We draw the reader's attention to the fact that Toeplitz determinant with the symbol (2.92) with $\nu = -\nu' = \frac{1}{2}$ coincides with the diagonal two-point Ising spin correlation function on the infinite square lattice. Its relation to $\mathrm{P}_{\rm VI}$ is rather well-known [81]. It is intriguing, however, that this lattice correlator is equal to a (particular limit of) conformal block in continuous 2D CFT with $c = 1$.

In the general case $\xi \neq 0$, the function $\tau^{(\nu,\nu',\eta,\xi)}(t)$ is a polynomial of degree N in ξ. The coefficients of $N+1$ different powers of ξ are s-channel conformal blocks with internal dimensions $(\theta_0+\theta_t-k)^2$, where $k=0,\dots,N$. Alternatively, one can first transform hypergeometric functions to make them depend on $1-t$ and then expand the determinant in powers of $\tilde{\xi}$, the analog of parameter ξ. The result has the form (2.34) of a sum of t-channel conformal blocks with internal dimensions $(\theta_1+\theta_t-k)^2$, again with $k=0,\dots,N$. The relations between the expansion parameters are given by (2.89), $\tilde{\xi}\tilde{s}_{\rm VI}=\xi s_{\rm VI}=\mathcal{K}$ and $(1-s_{\rm VI})(1-\tilde{s}_{\rm VI})=1+\mathcal{K}$.

The CFT interpretation of this picture is as follows. The tau function (2.88) is a four-point correlator of primaries which involves a field degenerate at level $N+1$ (here $\mathcal{O}_{\mu_t}(t)$). Its expansions at $t=0$ and $t=1$ incorporate all allowed intermediate dimensions. Determinant representation (2.86) can in fact be used to compute the fusion matrix for the corresponding two sets of conformal blocks. This task simplifies in the case $\xi=0$, where we are left with a single s-channel block tranforming into a linear combination of $N+1$ t-channel ones.

2.8.3 Algebraic solutions

Algebraic Painlevé VI solutions correspond to finite orbits of the braid/modular group action on monodromy of the associated linear system. All such solutions have been classified in [102]. There are 45 exceptional equivalence classes with fixed rational parameters $\vec{\theta}$, $\vec{\sigma}$ and three families continuously depending on some of them.

In the exceptional cases, the contributions coming from different conformal blocks overlap in the tau function expansion, which makes difficult their identification. On the other hand, this task is rather straightforward for continuous families, see examples below.

Example 2.12. For $\vec{\theta}=\left(a,a,b,\frac{1}{2}-b\right)$, $\vec{\sigma}=\left(2a,\frac{1}{4},\frac{1}{4}\right)$ there is a solution

$$\tau(t)=\text{const}\cdot\underbrace{t^{2a^2}(1-t)^{-a^2-b^2+\frac{1}{16}}\left(\frac{1+\sqrt{1-t}}{2}\right)^{-4a^2+\left(2b-\frac{1}{2}\right)^2}}_{\mathcal{F}\left(a,a,b,\frac{1}{2}-b,2a,t\right)},$$

arising from the contribution of a single conformal block (Solution II in [102]).

Example 2.13. For $\vec{\theta}=\left(2a,a,a,\frac{1}{3}\right)$, $\vec{\sigma}=\left(3a,\frac{1}{6},\frac{1}{4}\right)$ there is a solution that can be parametrized as follows (Solution III in [102]):

$$\tau(t(s))=\text{const}\cdot(s-2)^{4a^2}(s-1)^{2a^2-\frac{7}{72}}(s+1)^{-10a^2+\frac{1}{8}}(s+2)^{10a^2-\frac{1}{9}},$$
$$t(s)=\frac{(s+1)^2(s-2)}{(s-1)^2(s+2)}\,.$$

Consider the interval $s \in (2, \infty)$ which maps to $t \in (0, 1)$. Again only one block contributes to the tau function expansion on the corresponding branch at $t = 0$. It is explicitly given by

$$\mathcal{F}\left(2a, a, a, \frac{1}{3}, 3a, t\right) = \left(\frac{9(s_t - 2)}{4}\right)^{4a^2} (s_t - 1)^{2a^2 - \frac{7}{72}} \left(\frac{s_t + 1}{3}\right)^{-10a^2 + \frac{1}{8}} \left(\frac{s_t + 2}{4}\right)^{10a^2 - \frac{1}{9}},$$

with $s_t = \dfrac{(1 + \sqrt{t})^{\frac{2}{3}} + (1 - \sqrt{t})^{\frac{2}{3}}}{(1 - t)^{\frac{1}{3}}}$. Note that $t = 0$ is not really a branch point of s_t.

Example 2.14. Taking $\vec{\theta} = \left(a, a, a, \frac{1}{4}\right)$, $\vec{\sigma} = \left(2a, \frac{1}{6}, \frac{1}{6}\right)$ one has a solution (corresponding to Solution IV in [102])

$$\begin{aligned} \tau(t(s)) &= \text{const} \cdot s^{-6a^2 + \frac{1}{12}} (s - 1)^{-6a^2 + \frac{1}{12}} (s + 1)^{2a^2} (2 - s)^{2a^2} (2s - 1)^{5a^2 - \frac{1}{16}}, \\ t(s) &= \frac{s^3(2 - s)}{2s - 1}. \end{aligned}$$

Under this parametrization, the interval $s \in (1, 2)$ is mapped to $t \in (0, 1)$. Tau function asymptotics near $t = 0$ on the relevant branch yields the conformal block

$$\mathcal{F}\left(a, a, a, \frac{1}{4}, 2a, t\right) = \left(\frac{8(2 - s_t)}{3}\right)^{2a^2} \left(\frac{s_t(s_t - 1)}{2}\right)^{-6a^2 + \frac{1}{12}} \left(\frac{s_t + 1}{3}\right)^{2a^2} \left(\frac{2s_t - 1}{3}\right)^{5a^2 - \frac{1}{16}},$$

where

$$s_t = \frac{1}{2}\left(1 + u_t + \sqrt{3 - u_t^2 + \frac{2 - 4t}{u_t}}\right), \qquad u_t = \sqrt{1 - (4t(1 - t))^{\frac{1}{3}}}.$$

An infinite number of other new explicit examples of algebraic $c = 1$ conformal blocks can be generated by Bäcklund-Okamoto transformations for Painlevé VI.

3. Isomonodromic tau functions from Virasoro conformal blocks

The goal of this section is to show that the isomonodromic Riemann-Hilbert problem to find multivalued analytic functions with $\mathrm{SL}(2,\mathbb{C})$-valued monodromy on Riemann surfaces of genus zero with n punctures can be solved by taking suitable linear combinations of the Virasoro conformal blocks at $c = 1$. This implies a similar representation for the isomonodromic tau function. In the case $n = 4$ we thereby get a proof of the relations between tau functions and conformal blocks outlined in the previous section.

3.1 The Riemann-Hilbert problem

The fundamental group π_1 of $C_{0,n} := \mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}$ has n generators $\chi_1, \ldots, \chi_n$ subject to one relation $\chi_1 \circ \chi_2 \circ \cdots \circ \chi_n = \mathbf{1}$. Representations ρ of $\pi_1(C_{0,n})$ in $\mathrm{SL}(2,\mathbb{C})$ are specified by collections of matrices $M_\nu := \rho(\chi_\nu) \in \mathrm{SL}(2,\mathbb{C})$, $\nu = 1, \ldots, n$ satisfying $M_n \cdot M_{n-1} \cdot \ldots \cdot M_1 = \mathbf{1}$ up to overall conjugation with elements of $\mathrm{SL}(2,\mathbb{C})$. We will be interested in the cases where the matrices M_ν are diagonalizable with fixed eigenvalues $e^{\pm 2\pi i\theta_\nu}$. The space of all such representations of $\pi_1(C_{0,n})$ is then $2(n-3)$-dimensional.

It will be convenient to choose a base-point z_0 on $C_{0,n}$. The dependence on the choice of z_0 will turn out to be inessential. We may then represent the generators χ_ν by closed paths starting and ending at z_0. The Riemann-Hilbert problem is to find a multivalued analytic invertible matrix $\Phi(z)$ on $C_{0,n}$ such that the monodromy along χ_ν is represented as

$$\Phi(\chi_\nu.z) \;=\; \Phi(z) M_\nu \,, \tag{3.1}$$

where $\Phi(\chi_\nu.z)$ denotes the analytic continuation of $\Phi(z)$ along χ_ν.

The solution to this problem is unique up to left multiplication with single valued matrix functions. In order to fix this ambiguity we need to specify the singular behavior of $\Phi(z)$, leading to the following refined version of the Riemann-Hilbert problem: find a matrix function $\Phi(z)$ such that the following conditions are satisfied.

i) $\Phi(z_0) \;=\; \mathbf{1}\,,$

ii) $\Phi(z)$ is multivalued, analytic and invertible on $C_{0,n}$,

iii) There exist neighborhoods of a_ν, $\nu = 1, \ldots, n$ where $\Phi(z)$ can be represented as

$$\Phi(z) \;=\; \hat{\Phi}^{(\nu)}(z)\,(z - a_\nu)^{\mu_\nu}\,, \qquad M_\nu = e^{2\pi \mathrm{i}\mu_\nu}\,, \tag{3.2}$$

with $\hat{\Phi}^{(\nu)}(z)$ being holomorphic and invertible at $z = a_\nu$ and $\mu_1, \ldots, \mu_n \in \mathfrak{sl}(2,\mathbb{C})$.

If such function $\Phi(z)$ exists, it is uniquely determined by the monodromy data $\mu = (\mu_1, \ldots, \mu_n)$.

The refined Riemann-Hilbert problem naturally arises in the study of logarithmic flat $\mathfrak{sl}(2,\mathbb{C})$-connections on $C_{0,n}$. Any such connection is gauge equivalent to a holomorphic connection of the form $\partial_z - \mathcal{A}(z)$, with $\mathcal{A}(z)$ of the form (2.1),

$$\mathcal{A}(z) \;=\; \sum_{\nu=1}^{n} \frac{A_\nu}{z - a_\nu}\,, \tag{3.3}$$

where $A_1, \ldots A_n \in \mathfrak{sl}(2,\mathbb{C})$, $\sum_{\nu=1}^{n} A_\nu = 0$. One may then consider the fundamental matrix solution $\Phi(z)$ of the Fuchsian system

$$\partial_z \Phi(z) \;=\; \mathcal{A}(z)\Phi(z)\,, \tag{3.4}$$

normalized by $\Phi(z_0) = \mathbf{1}$. It will automatically satisfy the conditions ii) and iii) for certain $\mu_1, \ldots, \mu_n \in \mathfrak{sl}(2,\mathbb{C})$, provided that the eigenvalues $\pm\theta_\nu$ of A_ν (they also coincide with those of μ_ν) satisfy the non-resonancy condition $2\theta_\nu \notin \mathbb{Z}$.

Any representation $\rho : \pi_1(C_{0,n}) \to \mathrm{SL}(2,\mathbb{C})$ can be realized as monodromy representation of such a Fuchsian system, which means that a solution to the Riemann-Hilbert problem formulated above will generically exist. The Riemann-Hilbert correspondence between flat connections $\partial_z - \mathcal{A}(z)$ and representations $\rho : \pi_1(C_{0,n}) \to \mathrm{SL}(2,\mathbb{C})$ allows us to identify the moduli space $\mathcal{M}_{\mathrm{flat}}(C_{0,n})$ of flat $\mathfrak{sl}(2,\mathbb{C})$-connections on $C_{0,n}$ with the moduli space $\mathcal{M}(C_{0,n}) := \mathrm{Hom}(\pi_1(C_{0,n}), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$ of monodromy data.

3.2 Trace coordinates

Useful sets of coordinates for $\mathcal{M}(C_{0,n})$ are given by the trace functions $L_\gamma := \mathrm{tr}\,\rho(\gamma)$ associated to any simple closed curve γ on $C_{0,n}$. Minimal sets of trace functions that can be used to parameterize $\mathcal{M}(C_{0,n})$ can be identified using pants decompositions. In order to have uniform notations, let us replace the punctures $a_1, \ldots, a_n$ by little holes obtained by cutting along non-intersecting simple closed curves δ_ν surrounding the punctures a_ν, $\nu = 1, \ldots, n$, respectively. A pants decomposition is defined by cutting $C_{0,n}$ along $n-3$ simple closed curves γ_r, $r = 1, \ldots, n-3$ on $C_{0,n}$. This will decompose $C_{0,n}$ into a disjoint union of $n-2$ trinions (three-holed spheres) $C^t_{0,3}$, $t = 1, \ldots, n-2$. The collection $\mathcal{C} = \{\gamma_1, \ldots, \gamma_{n-3}\}$ of curves will be called the cut system.

To each curve $\gamma_r \in \mathcal{C}$ let us associate the union of the two three-holed spheres that have γ_r in its boundary, a four-holed sphere $C^r_{0,4}$. It will be assumed that the curves γ_r, $r = 1, \ldots, n-3$ are oriented. The orientation of γ_r allows us to introduce a natural numbering of the boundaries of $C^r_{0,4}$. We may then consider the curves $\gamma^r_s \equiv \gamma_r$ and γ^r_t which encircle

the pairs of boundary components of $C^r_{0,4}$ with numbers $(1,2)$ and $(2,3)$, respectively, see Fig. 7. The corresponding trace functions will be denoted as L^r_s and L^r_t. The collection of pairs of trace functions (L^r_s, L^r_t), $r = 1, \ldots, n-3$ can be used to locally parameterize $\mathcal{M}(C_{0,n})$.

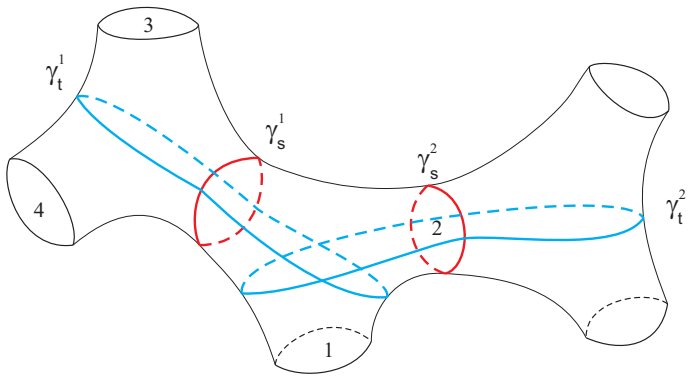


Fig. 7: Five-punctured sphere $C_{0,5}$ decomposed into 3 trinions $C_{0,3}$ by two curves γ^1_s, γ^2_s (red); the curves γ^1_t, γ^2_t are shown in blue; the labeling of boundaries $1 \ldots 4$ is indicated for the 4-punctured sphere $C^1_{0,4}$

A closely related set of coordinates for $\mathcal{M}(C_{0,n})$ is obtained by parameterizing L^r_s and L^r_t in terms of complex numbers (σ_r, τ_r) as

$$L^r_s = 2\cos 2\pi\sigma_r\,, \tag{3.5a}$$

$$(\sin(2\pi\sigma_r))^2\, L^r_t = C_+(\sigma_r)\, e^{i\tau_r} + C_0(\sigma_r) + C_-(\sigma_r)\, e^{-i\tau_r}\,, \tag{3.5b}$$

where

$$C_+(\sigma_r) = -4 \prod_{s=\pm 1} \sin\pi(\sigma_r + s(\sigma^r_1 - \sigma^r_2)) \sin\pi(\sigma_r + s(\sigma^r_3 - \sigma^r_4))\,, \tag{3.6a}$$

$$\begin{aligned} C_0(\sigma_r) = {} & 2\left[\cos 2\pi\sigma^r_2 \cos 2\pi\sigma^r_3 + \cos 2\pi\sigma^r_1 \cos 2\pi\sigma^r_4\right] \\ & - 2\cos 2\pi\sigma_r \left[\cos 2\pi\sigma^r_1 \cos 2\pi\sigma^r_3 + \cos 2\pi\sigma^r_2 \cos 2\pi\sigma^r_4\right]\,, \end{aligned} \tag{3.6b}$$

$$C_-(\sigma_r) = -4 \prod_{s=\pm 1} \sin\pi(\sigma_r + s(\sigma^r_1 + \sigma^r_2)) \sin\pi(\sigma_r + s(\sigma^r_3 + \sigma^r_4))\,. \tag{3.6c}$$

In order to define σ^r_i, $i = 1, \ldots, 4$ in (3.6) let us note that the boundary of $C^r_{0,4}$ with label i may either be a curve $\gamma_{r'} \in \mathcal{C}$, or it must coincide with a curve δ_ν surrounding the puncture a_ν. We will identify $\sigma^r_i \equiv \sigma_{r'}$ in the first case, while σ^r_i will be identified with an eigenvalue θ_ν of $\mu_\nu \in \mathfrak{sl}(2,\mathbb{C})$ otherwise.

The collection of data (σ_r, τ_r), $r = 1, \ldots, n-3$ will be denoted as (σ, τ). For $n = 4$ the coordinates (σ, τ) are related to the parameters used in [79] and our Subsection 2.3 (they are close relatives of σ_{0t} and s_{0t}, defined by (2.27b) and (2.30)). They are also closely related to the coordinates used in [116].

3.3 Conformal blocks from chiral vertex operators

Let us introduce the necessary definitions and results on the representation theory of the Virasoro algebra which has generators L_n, $n \in \mathbb{Z}$ that obey the commutation relations (2.37).

Although we will ultimately be interested in the case $c = 1$, it will be useful to consider more general values of c in some of our arguments. It will be convenient to parameterize the central charge as $c = 1 + 6Q^2$ and conformal dimensions as $\Delta_\alpha = \alpha(Q - \alpha)$. Highest weight representations $\mathcal{V}_\alpha$ are generated from vectors $|\alpha\rangle$ which satisfy

$$L_n \,|\, \alpha \,\rangle \;=\; 0\,, \quad n > 0\,, \quad L_0 \,|\, \alpha \,\rangle \;=\; \Delta_\alpha \,|\, \alpha \,\rangle\,. \tag{3.7}$$

The representations $\mathcal{V}_\alpha$ can be decomposed into the so-called energy-eigenspaces

$$\mathcal{V}_\alpha \;\simeq\; \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{V}_\alpha^{(n)}\,, \tag{3.8}$$

defined by the condition $L_0 v = (\Delta_\alpha + n) v$ for all $v \in \mathcal{V}_\alpha^{(n)}$.

Chiral vertex operators $V^\alpha_{\beta_2\beta_1}(z)$ can be defined as operators that map $\mathcal{V}_{\beta_1} \to \mathcal{V}_{\beta_2}$ such that

$$L_n \, V^\alpha_{\beta_2\beta_1}(z) - V^\alpha_{\beta_2\beta_1}(z) \, L_n \;=\; z^n (z\partial_z + \Delta_\alpha(n+1)) V^\alpha_{\beta_2\beta_1}(z)\,. \tag{3.9}$$

We have in particular

$$V^\alpha_{\beta_2\beta_1}(z) \,|\, \beta_1 \,\rangle \;=\; N(\beta_2, \alpha, \beta_1)\, z^{\Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_\alpha} \left[\,|\, \beta_2 \,\rangle + O(z) \,\right], \tag{3.10}$$

with a normalization factor $N(\beta_2, \alpha, \beta_1)$ that will be specified later. The conditions (3.9) define $z^{\Delta_{\beta_1} + \Delta_\alpha - \Delta_{\beta_2}} V^\alpha_{\beta_2\beta_1}(z)$ uniquely in the sense of formal power series in z,

$$V^\alpha_{\beta_2\beta_1}(z) \;=\; z^{\Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_\alpha} \sum_{n=-\infty}^{\infty} z^n \, W^\alpha_{\beta_2\beta_1}(n)\,, \qquad W^\alpha_{\beta_2\beta_1}(n) : \mathcal{V}^{(k)}_{\beta_1} \to \mathcal{V}^{(k+n)}_{\beta_2}\,. \tag{3.11}$$

It has furthermore been argued in [145] that the composition $V^{\alpha_2}_{\beta_3\beta_2}(z) V^{\alpha_1}_{\beta_2\beta_1}(w)$ of such vertex operators exists for $|w/z| < 1$, and that matrix elements such as

$$\langle \, \alpha_n \,|\, V^{\alpha_{n-1}}_{\alpha_n\beta_{n-3}}(a_{n-1}) V^{\alpha_{n-2}}_{\beta_{n-3}\beta_{n-4}}(a_{n-2}) \cdots V^{\alpha_2}_{\beta_1\alpha_1}(a_2) \,|\, \alpha_1 \,\rangle\,, \tag{3.12}$$

are represented by absolutely convergent power series in $a_\nu / a_{\nu+1}$, $\nu = 2, \ldots, n-2$. These matrix elements define a large class of conformal blocks. For instance, for $n = 4$ the above expression coincides with the 4-point s-channel conformal block considered in Subsection 2.4. It is often convenient to represent (3.12) pictorially as shown in Fig. 8.

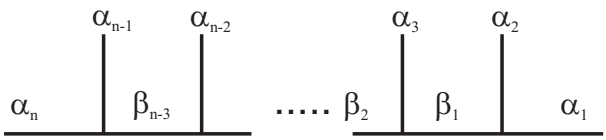


Fig. 8: Pictorial representation of the conformal block (3.12)

From each chiral vertex operator $V^{\alpha}_{\beta_2\beta_1}(z)$ one may generate a family of vertex operators called descendants of $V^{\alpha}_{\beta_2\beta_1}(z)$. The descendants of $V^{\alpha}_{\beta_2\beta_1}(z)$ are labelled by the vectors in $\mathcal{V}_\alpha$, and the descendant corresponding to $v \in \mathcal{V}_\alpha$ will be denoted as $V^{\alpha}_{\beta_2\beta_1}[v](z)$. The descendants may be defined by means of the recursion relations

$$V^{\alpha}_{\beta_2\beta_1}[|\alpha\rangle](z) \equiv V^{\alpha}_{\beta_2\beta_1}(z)\,, \tag{3.13a}$$

$$V^{\alpha}_{\beta_2\beta_1}[L_{-1}v](z) \equiv \partial_z V^{\alpha}_{\beta_2\beta_1}[v](z)\,, \tag{3.13b}$$

$$V^{\alpha}_{\beta_2\beta_1}[L_{-2}v](z) \equiv :T(z)V^{\alpha}_{\beta_2\beta_1}[v](z):\,, \tag{3.13c}$$

where the following notation has been used in (3.13c):

$$:T(z)V^{\alpha}_{\beta_2\beta_1}[v](z): \equiv \sum_{k<-1} z^{-k-2} L_k\, V^{\alpha}_{\beta_2\beta_1}[v](z) + \sum_{k\geq -1} z^{-k-2}\, V^{\alpha}_{\beta_2\beta_1}[v](z)\, L_k\,. \tag{3.14}$$

The recursion relations (3.13) suffice to define $V^{\alpha}_{\beta_2\beta_1}[L_{-n}v](z)$ for all $n > 0$ thanks to the Virasoro algebra commutation relations (2.37).

Using the descendants one may define a trilinear form $\mathcal{C}_{0,3} : \mathcal{V}_{\alpha_3} \otimes \mathcal{V}_{\alpha_2} \otimes \mathcal{V}_{\alpha_1} \to \mathbb{C}$ as

$$\mathcal{C}_{0,3}(v_3 \otimes v_2 \otimes v_1) := \langle\, v_3 \,|\, V^{\alpha_2}_{\alpha_3\alpha_1}[v_2](z) \,|\, v_1 \,\rangle\,. \tag{3.15}$$

This trilinear form can be identified with the conformal block associated to the three-punctured sphere $C_{0,3}$.

The definition of descendants allows us to introduce another way to compose chiral vertex operators. We may e.g. consider

$$V^{\beta_3}_{\beta_2\beta_1}\big[\, V^{\alpha_2}_{\beta_3\alpha_1}[v_2](w-z)v_1 \,\big](z)\,, \tag{3.16}$$

which is defined a priori as a formal power series in $w - z$. Quadrilinear forms such as

$$\mathcal{C}_{0,4}(v_4 \otimes \ldots \otimes v_1) := \langle\, v_4 \,|\, V^{\beta}_{\alpha_4\alpha_1}\big[\, V^{\alpha_3}_{\beta\alpha_2}[v_3](w-z)v_2 \,\big](z) \,|\, v_1 \,\rangle\,, \tag{3.17}$$

will define absolutely convergent series in $w - z$ for all $v_4, \ldots, v_1$ of finite energy. The quadrilinear forms $\mathcal{C}_{0,4}(v_4 \otimes \ldots \otimes v_1)$ can be identified with t-channel conformal blocks associated to the four-punctured sphere $C_{0,4}$.

By using the two types of composition of chiral vertex operators introduced above one may construct conformal blocks associated to *arbitrary* pants decompositions of n-punctured spheres (i.e. to arbitrary sequence of OPEs in the framework of Subsection 2.4).

Descendant vertex operators can also be used to define irregular conformal blocks [59, 19, 60]. The latter involve coherent states (Whittaker vectors) in the highest weight representations $\mathcal{V}_\alpha$ on which the annihilation part of the Virasoro algebra acts diagonally.

3.4 Degenerate fields

Of particular importance for us will be the special case where $\alpha = -b/2$, assuming that Q is represented as $Q = b + b^{-1}$. If furthermore β_2 and β_1 are related as $\beta_2 = \beta_1 \mp b/2$, the vertex operators $\psi_s(z) \equiv \psi_{\beta_1,s}(z) := V^{-b/2}_{\beta_1 - sb/2,\beta_1}(z)$, $s = \pm 1$, are well-known to satisfy a differential equation of the form

$$\partial_z^2 \psi_{\beta_1,s}(z) + b^2 : T(z)\psi_{\beta_1,s}(z) := 0\,, \tag{3.18}$$

with normal ordering defined in (3.14). The chiral vertex operators $\psi_{\beta_1,s}(z)$ are called degenerate fields. It follows from (3.18) that conformal blocks with degenerate insertions such as

$$\begin{aligned} &\mathcal{F}(\,\alpha;\beta\,|\,a\,|\,z_0\,|\,z\,) := \langle\,\alpha_n\,|\,\psi_{s'}(z_0)\psi_s(z)\,|\,\Theta\,\rangle\,, \\ &|\,\Theta\,\rangle := V^{\alpha_{n-1}}_{\alpha_n+(s+s')\frac{b}{2},\beta_{n-3}}(a_{n-1})V^{\alpha_{n-2}}_{\beta_{n-3}\beta_{n-4}}(a_{n-2})\cdots V^{\alpha_2}_{\beta_1\alpha_1}(a_2)V^{\alpha_1}_{\alpha_1,0}(a_1)|\,0\,\rangle\,, \end{aligned} \tag{3.19}$$

will satisfy the partial differential equation $\mathcal{D}_{\mathrm{BPZ}}\mathcal{F} = 0$, with

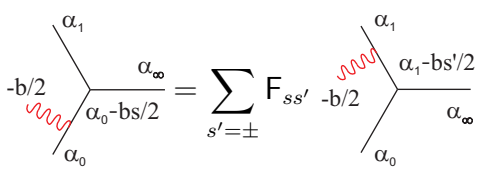
$$\mathcal{D}_{\mathrm{BPZ}} := \frac{1}{b^2}\frac{\partial^2}{\partial z^2} + \frac{\Delta_{-\frac{b}{2}}}{(z-z_0)^2} + \frac{1}{z-z_0}\frac{\partial}{\partial z_0} + \sum_{\nu=1}^{n-1}\left(\frac{\Delta_{\alpha_\nu}}{(z-a_\nu)^2} + \frac{1}{z-a_\nu}\frac{\partial}{\partial a_\nu}\right)\,, \tag{3.20}$$

together with a similar differential equation for z_0. Using this differential equation it may be shown that $\mathcal{F}(\,\alpha;\beta\,|\,a\,|\,z_0\,|\,z\,)$, considered as a function of z, can be analytically continued to a multivalued analytic function on $C_{0,n}$.

3.4.1 Outlook

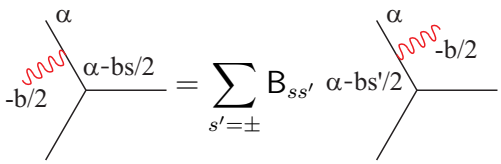
The program that we are going to realize can be informally explained as follows. The OPE of a generic primary field of momentum α with a field degenerate at level 2 of momentum $-b/2$ contains only two conformal families with momenta $\alpha \pm b/2$. The analytic continuation of conformal blocks in the position of degenerate field $\psi_s(z)$ can be viewed as a sequence of localized elementary transformations that are best understood pictorially:

- *S-move*:



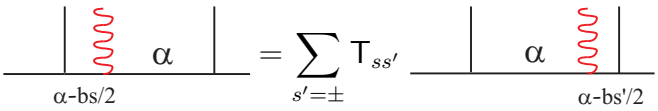
Here the curly red line corresponds to the degenerate leg and the explicit form of the 2×2 fusion matrix $\mathsf{F}_{ss'}$ can be computed using the hypergeometric representations of the 4-point conformal blocks involving one degenerate field.

- *B-move*:



This move corresponds to a half-braid of the degenerate field around the generic one. The corresponding monodromy matrix $\mathsf{B}_{ss'}$ is diagonal.

- *T-move*:



This elementary transformation is purely notational but extremely important. The matrix $\mathsf{T}_{ss'}$ is *operator-valued*: it involves shifts of intermediate momenta by $\pm b/2$.

The result of analytic continuation along closed contours is therefore given by a linear combination of conformal blocks of the same form but with possibly shifted intermediate momenta. A more precise description of the continuation procedure and the corresponding operator-valued monodromy is given below.

3.4.2 Braiding and fusion of degenerate fields

The differential equations (3.18) satisfied by the degenerate fields can be used to get a precise description of the monodromies of the conformal blocks $\mathcal{F}(\,\alpha;\beta\,|\,a\,|\,z_0\,|\,z\,)$ defined in (3.19) w.r.t analytic continuation in z. Let us briefly summarize the relevant results. There are three ways to compose a degenerate field with a generic chiral vertex operator,

$$\begin{aligned} &(1)\quad V^{\alpha_2}_{\alpha_3,\alpha_1-s\frac{b}{2}}[v_2](w)\,\psi_s(z)\,, \\ &(3)\quad \psi_{-s}(z)V^{\alpha_2}_{\alpha_3-s\frac{b}{2},\alpha_1}[v_2](w)\,, \end{aligned} \qquad (2)\quad V^{\alpha_2-s\frac{b}{2}}_{\alpha_3,\alpha_1}\big[\psi_s(z-w)v_2\big](w)\,. \tag{3.21}$$

The three ways (3.21) to compose these vertex operators correspond to having the degenerate field $\psi_s(z)$ located in the vicinity of the boundary components with labels 1, 2 and 3, respectively, referring to Fig. 9 for the notations.

The conformal blocks defined using the three compositions (3.21) are single-valued and analytic in neighborhoods of the black dots marked in Fig. 9 on the boundaries of the three holes of $C_{0,3}$, respectively. We are going to describe their analytic continuation to

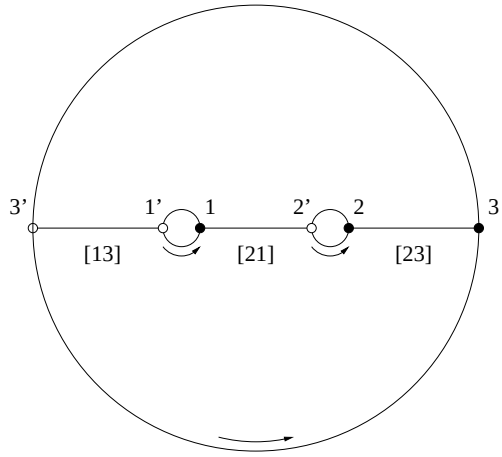


Fig. 9: A sphere with three holes. The arrows indicate our orientation conventions.

the universal cover of $C_{0,3}$. It will be helpful to introduce a separate notation for the vertex operator $\psi_s(z)$ when it is inserted at the antipodal point of the circle $|z| = \mathrm{const}$,

$$\psi'_{\beta,s}(z) \;=\; B_s(\beta)\psi_{\beta,s}(e^{-\pi i}z)\,, \qquad B_s(\alpha) \;=\; e^{\pi i(\Delta_{\alpha-s\frac{b}{2}}-\Delta_\alpha-\Delta_{-\frac{b}{2}})}\,. \tag{3.22}$$

The vertex operators $\psi'_{\beta,s}(z)$ are single-valued in an open neighborhood containing segments of the negative real axis. One may naturally consider compositions (1)′–(3)′ of the form (3.21), but with $\psi_s(z)$ replaced by $\psi'_s(z)$. Regions on $C_{0,3}$ where the compositions (1)′–(3)′ define single-valued analytic conformal blocks are neighborhoods of the small empty circles in Fig. 9.

The main building block for the monodromies will be the following relations,

$$\psi_{-s_1}(z)V^{\alpha_2}_{\alpha_3-s_1\frac{b}{2},\alpha_1}[v_2](w) = \sum_{s_2=\pm1} F^{[23]}_{s_1,s_2}\, V^{\alpha_2-s_2\frac{b}{2}}_{\alpha_3,\alpha_1}\big[\psi_{s_2}(z-w)v_2\big](w), \tag{3.23a}$$

$$V^{\alpha_2}_{\alpha_3,\alpha_1-s_1\frac{b}{2}}[v_2](w)\,\psi_{s_1}(z) = \sum_{s_2=\pm1} F^{[21]}_{s_1,s_2}\, V^{\alpha_2-s_2\frac{b}{2}}_{\alpha_3,\alpha_1}\big[\psi'_{s_2}(z-w)v_2\big](w)\,, \tag{3.23b}$$

$$\psi'_{-s_1}(z)\,V^{\alpha_2}_{\alpha_3-s_1\frac{b}{2},\alpha_1}[v_2](w) = \sum_{s_2=\pm1} F^{[13]}_{s_1,s_2}\, V^{\alpha_2}_{\alpha_3,\alpha_1-s_2\frac{b}{2}}[v_2](w)\,\psi'_{s_2}(z)\,. \tag{3.23c}$$

The relevant transport matrices are given respectively as

$$F^{[ji]}_{s_1,s_2} = \frac{\Gamma(1+s_1b(2\alpha_i-Q))\Gamma(s_2b(Q-2\alpha_j))}{\prod_{s_3=\pm}\Gamma\big(\frac{1}{2}+s_1b(\alpha_i-Q/2)-s_2b(\alpha_j-Q/2)+s_3b(\alpha_k-Q/2)\big)}\,, \tag{3.24}$$

valid if the vertex operators $V^{\alpha}_{\beta_2,\beta_1}(w)$ are normalized via (3.10) with $N(\alpha_3,\alpha_2,\alpha_1)\equiv 1$.

Remark 3.1. Comparing with the Moore-Seiberg formalism let us note that

$$F^{[21]}_{s_1,s_2} \;\equiv\; F\left[\begin{smallmatrix}\alpha_2 & -b/2\\ \alpha_3 & \alpha_1\end{smallmatrix}\right]_{s_1s_2} \;\equiv\; F_{\alpha_1-s_1\frac{b}{2};\alpha_2-s_2\frac{b}{2}}\left[\begin{smallmatrix}\alpha_2 & -b/2\\ \alpha_3 & \alpha_1\end{smallmatrix}\right]\,, \tag{3.25a}$$

$$F^{[23]}_{s_1,s_2} \;\equiv\; F\left[\begin{smallmatrix}-b/2 & \alpha_2\\ \alpha_3 & \alpha_1\end{smallmatrix}\right]_{s_1s_2} \;\equiv\; F_{\alpha_3-s_1\frac{b}{2};\alpha_2-s_2\frac{b}{2}}\left[\begin{smallmatrix}-b/2 & \alpha_2\\ \alpha_3 & \alpha_1\end{smallmatrix}\right]\,, \tag{3.25b}$$

$$F^{[13]}_{s_1,s_2} \;\equiv\; F\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ -b/2 & \alpha_1\end{smallmatrix}\right]_{s_1s_2} \;\equiv\; F_{\alpha_3-s_1\frac{b}{2};\alpha_1-s_2\frac{b}{2}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ -b/2 & \alpha_1\end{smallmatrix}\right]\,, \tag{3.25c}$$

The relevant fusion matrices are related to each other by the symmetries

$$F\left[\begin{smallmatrix}\alpha_2 & -b/2\\ \alpha_3 & \alpha_1\end{smallmatrix}\right]_{s_1 s_2} = F\left[\begin{smallmatrix}-b/2 & \alpha_2\\ \alpha_1 & \alpha_3\end{smallmatrix}\right]_{s_1 s_2} = F\left[\begin{smallmatrix}\alpha_3 & \alpha_1\\ \alpha_2 & -b/2\end{smallmatrix}\right]_{s_1 s_2}, \tag{3.26}$$

together with

$$F\left[\begin{smallmatrix}\alpha_2 & -b/2\\ \alpha_3 & \alpha_1\end{smallmatrix}\right]^{-1} = F\left[\begin{smallmatrix}\alpha_1 & -b/2\\ \alpha_3 & \alpha_2\end{smallmatrix}\right]. \tag{3.27}$$

The definition of the antipodal vertex operators $\psi'_{\beta,s}(z)$ in (3.22) is related to the elementary braid relation

$$[\, V^{\alpha_2}_{\alpha_3,\alpha_1}(z)\, V^{\alpha_1}_{\alpha_1,0}(w)|\, 0\,\rangle\,]_{\circlearrowleft} \;=\; \Omega^{\alpha_3}_{\alpha_2,\alpha_1}\, V^{\alpha_1}_{\alpha_3,\alpha_2}(w)\, V^{\alpha_2}_{\alpha_2,0}(z)|\, 0\,\rangle\,, \tag{3.28}$$

with left hand side defined by means of analytic continuation making z encircle w in the anti-clockwise sense. It is easy to see that the "half-monodromy" used in (3.22) is related to the composition of analytic continuation (3.28) with a suitable translation. It follows that the braiding phase factor $B_s(\alpha)$ is related to the factors $\Omega^{\alpha_3}_{\alpha_2,\alpha_1}$ in (3.28) as $B_s(\alpha) \;=\; \Omega^{\alpha-sb/2}_{-b/2,\alpha}$.

In the normalisation where $N(\alpha_3,\alpha_2,\alpha_1) \equiv 1$ one may observe that the conformal blocks and the fusion matrices $F^{[ji]}$ are perfectly analytic with respect to the central charge c. We may in particular take the limit $c \to 1$ without encountering any problem. This is not the case for the kernel of the integral transformation relating conformal blocks associated to different pants decompositions.

3.5 Monodromy action on spaces of conformal blocks

Using these ingredients it is straightforward to show that the analytic continuation of the conformal blocks $\mathcal{F}(\,\alpha;\beta\,|\,a\,|\,z_0\,|\,z\,)$ in z along the closed paths χ_ν can be expressed as a linear combination of the matrix elements $\mathcal{F}(\,\alpha;\beta'\,|\,a\,|\,z_0\,|\,z\,)$ having parameters β'_r that differ from β_r by integer multiples of the parameter b. In order to have a convenient notation let us define the shift operators $\mathsf{V}^{\frac{1}{2}}_{\beta_r}$ which acts on functions to the left as

$$\mathcal{F}(\,\alpha;\beta\,|\,a\,|\,z_0\,|\,z\,)\cdot \mathsf{V}^{\frac{1}{2}}_{\beta_r} \;=\; \mathcal{F}(\,\alpha;\beta-\tfrac{b}{2}e_r\,|\,a\,|\,z_0\,|\,z\,)\,, \tag{3.29}$$

where e_r is the vector in $\mathbb{C}^{n-3}$ with components δ_{rs}.

3.5.1 Geometric setup

It will be useful for us to refine the pants decompositions as follows. On each curve γ in the extended cut system $\hat{\mathcal{C}} := \{\gamma_1,\ldots,\gamma_{2n-3}\}$, where $\gamma_{n-3+\nu} := \delta_\nu$ for $\nu = 1,\ldots,n$ let

us mark two points, a black one and a white one. On each pair of pants with label t let us introduce a collection of two non-intersecting arcs $[23]_t$, and $[13]_t$ that connect marked points on the boundary components labelled by 1, 2 and 3, respectively. These contours are depicted in Fig. 9.

Let us next note that any generator χ_ν of $\pi_1(C_{0,n})$ may be represented as a concatenation $\eta_1 \circ \eta_2 \circ \cdots \circ \eta_N$ of oriented arcs η_a, each contained within a three-holed sphere $C^t_{0,3}$. It will not cause a loss of generality to assume that each arc η_a is of the following two types:

- An arc $[ji]_t$ on $C^t_{0,3}$ running from the marked point on boundary component i of trinion t to the one on boundary component j as depicted in Fig. 9,
- An arc b^t_i connecting the two marked points on boundary component i of $C^t_{0,3}$ with positive orientation[8].

We will assume that the point z_0 is located on the boundary circle δ_n of $C_{0,n}$. It will be useful to introduce the notation $[ji^k]_t$ for the composite arcs $(b^t_i)^k \circ [ji]_t$, $k \in \mathbb{Z}$.

3.5.2 The algorithm

Using the results from the previous subsection and the definitions from 3.5.1 we may now formulate a simple algorithm for calculating the result of the analytic continuation $\mathcal{F}(\,\alpha;\beta\,|\,a\,|\,z_0\,|\,z\,)$ along χ_ν. We will use the geometric setup introduced in Subsection 3.5.1, in particular the decomposition of the paths χ_ν into a collection of arcs. Note that the basic building blocks are close relatives of the moves introduced in (3.23) such as

$$\begin{aligned}\psi_{s_2}(z)V^{\alpha_2}_{\alpha_3,\alpha_1}[v_2](w) &= \psi_{s_2}(z)V^{\alpha_2}_{\alpha_3+s_2\frac{b}{2},\alpha_1}[v_2](w)\cdot \mathsf{V}^{\frac{1}{2}s_2}_{\alpha_3}\\ &= \sum_{s_1=\pm1} F^{[23]}_{-s_2,s_1}\, V^{\alpha_2-s_1\frac{b}{2}}_{\alpha_3,\alpha_1}\big[\psi_{s_1}(z-w)v_2\big](w)\cdot \mathsf{V}^{\frac{1}{2}s_2}_{\alpha_3}\,. \end{aligned} \tag{3.30}$$

In this way we find that the arcs $[ji^k]_t$ are represented by the matrices

$$\mathsf{S}^t_{[ji]} := \mathsf{F}^t_{[ji]}\cdot \mathsf{T}^t_i\,, \qquad \mathsf{C}^{t,k}_{[ji]} := \mathsf{S}^t_{[ji]}\cdot(\mathsf{B}^t_i)^k\,, \tag{3.31a}$$

where

- $\mathsf{F}^t_{[ji]}$ is obtained from $F^{[ji]}$ by replacing $\alpha_i \to \alpha^t_i$, $i = 1,2,3$ and transposition,

[8]The orientation is indicated in Fig. 9. The starting point is determined either by the base point or by the endpoint of the previous arc.

- T_i^t is defined as

$$(\mathsf{T}_i^t)_{s_1 s_2} \;=\; \delta_{s_1,-s_2}\,(\mathsf{V}_t)^{\frac{1}{2}s_2}\,, \tag{3.31b}$$

  where V_t is the shift operator which shifts the variable $\alpha_t \equiv \alpha_i^t$ as defined in equation (3.29). The operators V_t act to the left in the product of matrices.

- B_i^t is the matrix with elements

$$(\mathsf{B}_i^t)_{s_1 s_2} \;=\; \delta_{s_1 s_2}\,B_{s_1}(\alpha_i)\,. \tag{3.31c}$$

Arcs b_i^t will be represented by the matrix B_i^t. If χ_ν is a simple closed curve on $C_{0,n}$ starting and ending at z_0 represented by the ordered concatenation $\eta_1 \circ \eta_2 \circ \cdots \circ \eta_K$ of the arcs defined above, we will define

$$\mathsf{M}_\nu = \mathsf{N}_K \cdot \mathsf{N}_{K-1} \cdot \cdots \cdot \mathsf{N}_1\,, \tag{3.32}$$

where N_k are the 2×2-matrices associated to the arcs η_k. We may thereby define the sought-for collection of matrices M_ν, $\nu = 1, \ldots, n$ describing the action of monodromies of the degenerate fields on spaces of conformal blocks.

One should not forget that the resulting monodromy matrix is *operator-valued*: it is a matrix which has elements containing the operators V_t shifting the parameters β.

3.6 Solving the Riemann-Hilbert problem

We shall now specialize to $c = 1$. For that case we shall replace the parameters α_ν and β_r by variables θ_ν and σ_r giving the conformal dimensions as $\Delta_{\theta_\nu} = \theta_\nu^2$ and $\Delta_{\sigma_r} = \sigma_r^2$, for $\nu = 1, \ldots, n$ and $r = 1, \ldots, n-3$, respectively.

3.6.1 The construction

Let us now consider,

$$\begin{aligned} &\mathcal{F}_{s's}\big(\,\theta;\sigma\,|\,a\,|\,z_0\,|\,z\,\big) := \big\langle\,\theta_n\,|\,\psi_{-s'}(z_0)\psi_s(z)\,|\,\Theta_{s-s'}\,\big\rangle\,, \\ &|\,\Theta_\epsilon\,\rangle \;=\; V^{\theta_{n-1}}_{\theta_n+\frac{\epsilon}{2}\,,\,\sigma_{n-3}}(a_{n-1})\ldots V^{\theta_3}_{\sigma_2,\sigma_1}(a_3)\,V^{\theta_2}_{\sigma_1,\theta_1}(a_2)\,|\,\theta_1\,\big\rangle\,, \end{aligned} \tag{3.33}$$

where $V^{\theta}_{\sigma_2,\sigma_1}(a)$ maps $\mathcal{V}_{\sigma_1}$ to $\mathcal{V}_{\sigma_2}$ and $\psi_s(z)$ maps $\mathcal{V}_\sigma$ to $\mathcal{V}_{\sigma-s/2}$ for all σ. We will from now on assume that the vertex operators $V^{\theta}_{\sigma_2,\sigma_1}(a)$ are normalized by (3.10) with $N(\sigma_3,\sigma_2,\sigma_1)$ being chosen as

$$\begin{aligned} &N(\sigma_3,\sigma_2,\sigma_1) \;= \\ &= \frac{G(1+\sigma_3-\sigma_2-\sigma_1)G(1+\sigma_1-\sigma_3-\sigma_2)G(1+\sigma_2-\sigma_1-\sigma_3)G(1+\sigma_3+\sigma_2+\sigma_1)}{G(1+2\sigma_3)G(1-2\sigma_2)G(1-2\sigma_1)}\,, \end{aligned} \tag{3.34}$$

where $G(\sigma)$ is the Barnes G-function.

Consider the matrix $\Psi(z;z_0)$ which has elements

$$\Psi_{s's}(z;z_0) := \frac{\pi s'(z_0-z)^{\frac{1}{2}}}{\sin 2\pi\theta_n} \frac{\left\langle\, \theta_n \,|\, \psi_{-s'}(z_0)\psi_s(z) \,|\, \Theta^{\mathrm{D}}_{s-s'} \,\right\rangle}{\left\langle\, \theta_n \,|\, \Theta^{\mathrm{D}}_0 \,\right\rangle}\,, \tag{3.35a}$$

where

$$|\,\Theta^{\mathrm{D}}_\epsilon(\sigma,\tau)\,\rangle := \sum_{\vec{n}\in\mathbb{Z}^N} \prod_{r=1}^{N} e^{in_r\tau_r}\,|\,\Theta_\epsilon(\sigma+\vec{n})\,\rangle\,. \tag{3.35b}$$

We have introduced $N := n-3$, and the summation is over vectors $\vec{n} = (n_1,\dots,n_N)$ in $\mathbb{Z}^N$. We claim that $\Psi_{s's}(z;z_0)$ represents the sought-for solution to the Riemann-Hilbert problem in which we set without loss of generality $a_1 = 0$, $a_n = \infty$. The proof of this statement is given in the following subsections. At this point we only remark that the prefactor in (3.35a) ensures the normalization $\Psi(z_0;z_0) = \mathbf{1}$.

The observations above provide the input needed to apply the reasoning presented in Subsection 2.2 to show that the isomonodromic tau function is nothing but

$$\begin{aligned} \tau(\,a\,) &= \left\langle\, \theta_n \,|\, \Theta^{\mathrm{D}}_0 \,\right\rangle = \\ &= \sum_{\vec{n}\in\mathbb{Z}^N} \prod_{r=1}^{N} e^{in_r\tau_r} \left\langle\, \theta_n \,|\, V^{\theta_{n-1}}_{\theta_n,\sigma_N+n_N}(a_{n-1}) \dots V^{\theta_3}_{\sigma_2+n_2,\sigma_1+n_1}(a_3)\, V^{\theta_2}_{\sigma_1+n_1,\theta_1}(a_2) \,|\, \theta_1 \,\right\rangle. \end{aligned} \tag{3.36}$$

Our results for the case $n = 4$ yield in particular a proof of the relation between the tau function for Painlevé VI and Virasoro conformal blocks (Conjecture 2.6). Its irregular versions (Conjectures 2.7–2.10) are derived by taking suitable scaling limits as explained in Subsection 2.7.

3.6.2 Existence of classical monodromies

We may calculate the monodromies by the algorithm formulated in Subsection 3.5.2 with input data $F^{[ji]}_{s_1,s_2}$ and $B_s(\alpha)$ now given by

$$F^{[ji]}_{s_1,s_2} = s_1 \frac{\cos\pi(\sigma_k + s_2\sigma_j - s_1\sigma_i)}{\sin 2\pi\sigma_j}\,, \tag{3.37a}$$

$$B_s(\sigma) = e^{-\pi \mathrm{i}\, s\, \sigma}\,. \tag{3.37b}$$

Note that the change of normalization of chiral vertex operators simplifies the fusion matrix to *trigonometric* form. The operator V_t may now be represented as $\mathsf{V}_t = e^{\mathrm{i}\mathsf{q}_t}$, where $\mathsf{q}_t = i\frac{\partial}{\partial\sigma_t}$. Let us denote the resulting operator-valued monodromy matrices by M_γ.

We may now make a key observation: the monodromy matrices M_γ have matrix elements that are rational functions of $\mathsf{U}_t = e^{2\pi i\sigma_t}$ and V_t which generate a *commutative* subalgebra of the algebra of all operators[9] acting on the space of conformal blocks.

In order to see that M_γ depends only on V_t rather than $(\mathsf{V}_t)^{\frac{1}{2}}$ let us note that each curve of the cut system traversed on the way must be crossed a second time before one can return to the starting point. In a similar way one may see that M_γ depends on σ_t only via $U_t = e^{2\pi i\sigma_t}$: the elements of the matrices $F^{[ji]}_{s_1,s_2}$ are linear combinations of the form $Ae^{\pi i\sigma_t} + Be^{-\pi i\sigma_t}$. As the product of matrices representing M_γ will always contain an even number of matrices depending on a given variable σ_t, it follows that M_γ depends on σ_t only via $e^{2\pi i\sigma_t}$.

But this means that the algebra generated by the matrix elements of M_γ becomes classical (commutative) in the limit $c \to 1$! This allows us to diagonalize the operator V_t by taking linear combinations of the form (3.35b). The generalized Fourier transformation (3.35b) diagonalizes V_t with eigenvalue $e^{i\tau_t}$, while $e^{2\pi i\sigma_t}$ will act on $\Psi_{s's}$ by multiplication. The matrix obtained from M_γ by means of the transformation (3.35b) will be denoted M_γ.

3.6.3 Calculation of monodromies

In order to formulate the rules for the calculation of the monodromy matrices M_ν, let us assume without loss of generality that the path connecting boundary component δ_n to δ_ν passes through the trinions $t_1, t_2, \ldots, t_L$ in the given order, each trinion being traversed exactly once. We claim that we may then calculate the monodromy matrices M_ν as

$$M_\nu = \sigma_z \cdot \left[C^{t_L,k_L}_{[j_L i_L]} \ldots C^{t_1,k_1}_{[j_1 i_1]}\right]^{-1} \cdot (\mathsf{B}^{t_L}_\nu)^2 \cdot \left[C^{t_L,k_L}_{[j_L i_L]} \ldots C^{t_1,k_1}_{[j_1 i_1]}\right] \cdot \sigma_z, \tag{3.38}$$

where $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $C^{t,k}_{[ji]}$ is defined as

$$C^{t,k}_{[ji]} := F^t_{[ji]} \cdot (TB)^{t,k}_i \,, \tag{3.39a}$$

with matrices $F^t_{[ji]}$, and $(TB)^{t,k}_i$ defined as

$$(F^t_{[ji]})_{s_1,s_2} = s_2 \frac{\sin \pi(\sigma^t_k + s_2\sigma^t_j - s_1\sigma^t_i)}{\sin 2\pi\sigma^t_j} \,, \tag{3.39b}$$

$$[(TB)^{t,k}_i]_{s_1 s_2} = \delta_{s_1,-s_2}\, \mathrm{i}^{s_2 k} e^{s_2 \frac{\mathrm{i}}{2}\tau^t_i} e^{-\pi \mathrm{i}\, k\, s_2\, \sigma^t_i} \,. \tag{3.39c}$$

In order to derive these rules let us note that application of the algorithm formulated in Subsection 3.5.2 will produce monodromy matrices M_ν of the following form:

$$\mathsf{M}_\nu = \left[\mathsf{C}^{t_L,k_L}_{[j_L i_L]} \cdot \ldots \mathsf{C}^{t_1,k_1}_{[j_1 i_1]}\right]^{-1} \cdot (\mathsf{B}^{t_L}_\nu)^2 \cdot \left[\mathsf{C}^{t_L,k_L}_{[j_L i_L]} \cdot \ldots \mathsf{C}^{t_1,k_1}_{[j_1 i_1]}\right] . \tag{3.40}$$

[9] We mean operators acting on the conformal blocks built from σ_t, ∂_{σ_t}.

The matrices $\mathsf{C}^{t,k}_{[ji]}$, $k \in \mathbb{Z}$, represent the contribution of the segments connecting boundary component i to j in trinion t. Recall that $\mathsf{C}^{t,k}_{[ji]} = \mathsf{S}^t_{[ji]} \cdot (\mathsf{B}^t_i)^k$. The matrices $\mathsf{S}^t_{[ji]}$ and $(\mathsf{S}^t_{[ji]})^{-1}$ are explicitly given as

$$\begin{aligned}
\mathsf{S}^t_{[ji]} &= \frac{1}{\sin 2\pi\sigma^t_j} \begin{pmatrix} -\sin\pi(\sigma^t_k + \sigma^t_j - \sigma^t_i - \frac{1}{2}) & \sin\pi(\sigma^t_k - \sigma^t_j - \sigma^t_i + \frac{1}{2}) \\ -\sin\pi(\sigma^t_k + \sigma^t_j + \sigma^t_i - \frac{3}{2}) & \sin\pi(\sigma^t_k - \sigma^t_j + \sigma^t_i - \frac{1}{2}) \end{pmatrix} \begin{pmatrix} 0 & e^{-\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \\ e^{\frac{\mathrm{i}}{2}\mathsf{q}^t_i} & 0 \end{pmatrix} \\
&= \frac{1}{\sin 2\pi\sigma^t_j} \begin{pmatrix} -e^{\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \sin\pi(\sigma^t_k - \sigma^t_j - \sigma^t_i) & e^{-\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \sin\pi(\sigma^t_k + \sigma^t_j - \sigma^t_i) \\ -e^{\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \sin\pi(\sigma^t_k - \sigma^t_j + \sigma^t_i) & e^{-\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \sin\pi(\sigma^t_k + \sigma^t_j + \sigma^t_i) \end{pmatrix}, \\
(\mathsf{S}^t_{[ji]})^{-1} &= \frac{1}{\sin 2\pi\sigma^t_i} \begin{pmatrix} \sin\pi(\sigma^t_k + \sigma^t_j + \sigma^t_i) e^{-\frac{\mathrm{i}}{2}\mathsf{q}^t_i} & -\sin\pi(\sigma^t_k + \sigma^t_j - \sigma^t_i) e^{-\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \\ \sin\pi(\sigma^t_k - \sigma^t_j + \sigma^t_i) e^{+\frac{\mathrm{i}}{2}\mathsf{q}^t_i} & -\sin\pi(\sigma^t_k - \sigma^t_j - \sigma^t_i) e^{+\frac{\mathrm{i}}{2}\mathsf{q}^t_i} \end{pmatrix}.
\end{aligned}$$

In order to calculate the effect of the transformation (3.35b) it is convenient to move the operators $e^{\pm\mathrm{i}\mathsf{q}_t}$ to the left in (3.40). To this aim let us analyze the dependence of M_ν on $\sigma_{i_a} \equiv \sigma^{t_a}_{i_a}$ and the shift operator $e^{\mathrm{i}\mathsf{q}_a}$, where $\mathsf{q}_a \equiv \mathsf{q}^{t_a}_{i_a}$. The dependence on $e^{\mathrm{i}\mathsf{q}_a}$ can be made explicit by writing M_ν as

$$\mathsf{M}_\nu = \left[\mathsf{C}^{t_{a-2}\ldots t_1}\right]^{-1} \cdot \left[\mathsf{C}^{t_{a-1},k_{a-1}}_{[j_{a-1}i_{a-1}]}\right]^{-1} \cdot [\mathsf{B}^{t_a}_{i_a}]^{-k_a} \cdot \mathsf{M}'_{\nu,a} \cdot (\mathsf{B}^{t_a}_{i_a})^{k_a} \cdot \mathsf{C}^{t_{a-1},k_{a-1}}_{[j_{a-1}i_{a-1}]} \cdot \left[\mathsf{C}^{t_{a-2}\ldots t_1}\right], \quad (3.41)$$

where

$$\mathsf{M}'_{\nu,a} := \left[\mathsf{S}^{t_a}_{[j_a i_a]}\right]^{-1} \cdot \left[\mathsf{C}^{t_L,k_L}_{[j_L i_L]} \cdots \mathsf{C}^{t_{a+1},k_{a+1}}_{[j_{a+1}i_{a+1}]}\right]^{-1} \cdot (\mathsf{B}^{t_L}_\nu)^2 \cdot \left[\mathsf{C}^{t_L,k_L}_{[j_L i_L]} \cdots \mathsf{C}^{t_{a+1},k_{a+1}}_{[j_{a+1}i_{a+1}]}\right] \cdot \mathsf{S}^{t_a}_{[j_a i_a]}.$$

It is easy to see that the dependence of the matrix $\mathsf{M}'_{\nu,a}$ on $e^{\mathrm{i}\mathsf{q}_a}$ is of the form

$$\mathsf{M}'_{\nu,a} = \begin{pmatrix} (\mathsf{m}'_{\nu,a})_{++} & -e^{-\mathrm{i}\mathsf{q}_a} (\mathsf{m}'_{\nu,a})_{+-} \\ -e^{\mathrm{i}\mathsf{q}_a} (\mathsf{m}'_{\nu,a})_{-+} & (\mathsf{m}'_{\nu,a})_{--} \end{pmatrix},$$

where $\mathsf{m}'_{\nu,a}$ is the matrix one would obtain by replacing q_a by 0 and $\mathsf{F}^{t_a}_{[j_a i_a]}$ by $F^{t_a}_{[j_a i_a]}$ from the very beginning. The extra minus sign is the result of the application of the exchange relation

$$\sin\pi(\sigma_{k_a} + \sigma_{j_a} + \sigma_{i_a}) e^{-\mathrm{i}\mathsf{q}_a} = -e^{-\mathrm{i}\mathsf{q}_a} \sin\pi(\sigma_{k_a} + \sigma_{j_a} + \sigma_{i_a}).$$

The only matrices in (3.41) to the left of $\mathsf{M}'_{\nu,a}$ containing dependence on the variable σ_{i_a} are $[\mathsf{C}^{t_{a-1},k_{a-1}}_{[j_{a-1}i_{a-1}]}]^{-1}$ and $[\mathsf{B}^{t_a}_{i_a}]^{-k_a}$. The matrix elements of both $\mathsf{S}^{t_{a-1}}_{[j_{a-1}i_{a-1}]}$ and $\mathsf{B}^{t_a}_{i_a}$ are both anti-periodic under shifts $\sigma_a \to \sigma_a + 1$. Moving $e^{\pm\mathrm{i}\mathsf{q}_a}$ through the product $[\mathsf{C}^{t_{a-1},k_{a-1}}_{[j_{a-1}i_{a-1}]}]^{-1} \cdot [\mathsf{B}^{t_a}_{i_a}]^{-k_a}$ will for $a > 1$ produce an extra sign $(-)^{1+k_a}$. This sign is taken into account by means of the factor $\mathrm{i}^{s_2 k_a}$ in (3.39c). The extra sign $(-)^{1+k_a}$ should be replaced by $(-)^{k_a}$ in the case $a = 1$. This is taken into account by means of conjugation with σ_z in (3.38).

3.6.4 Calculation of trace functions

Let us now calculate the trace functions L_s^r and L_t^r associated to each curve γ_r of the cut system using the algorithm developed in Subsection 3.5.2 along with the rules of Subsection 3.6.3. We want to express the result in terms of the parameters $\sigma_{1\ldots4}^r$, σ_r, τ_r. The reader is referred to Fig. 10 for the labeling of pairs of pants and boundary components.

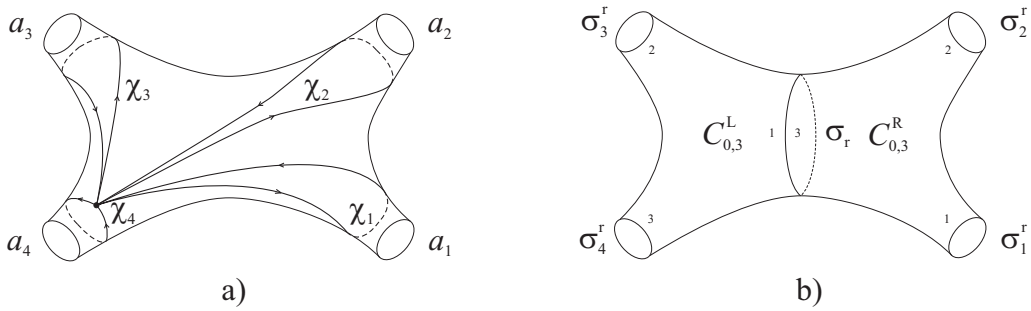


Fig. 10: Basis of loops of $\pi_1(C_{0,4})$ and the decomposition $C_{0,4} = C_{0,3}^L \cup C_{0,3}^R$

The trace functions are determined by the classical monodromies around the punctures a_1, a_2, a_3. To find them explicitly, we first note that the corresponding operator-valued monodromy matrices are given by

$$\mathsf{M}_1 = \left[\mathsf{C}_{[13]}^{R,0}\mathsf{C}_{[13]}^{L,1} \cdot \mathsf{C}\right]^{-1}\left(\mathsf{B}_1^R\right)^2 \mathsf{C}_{[13]}^{R,0}\mathsf{C}_{[13]}^{L,1} \cdot \mathsf{C}, \tag{3.42a}$$

$$\mathsf{M}_2 = \left[\mathsf{C}_{[23]}^{R,-1}\mathsf{C}_{[13]}^{L,1} \cdot \mathsf{C}\right]^{-1}\left(\mathsf{B}_2^R\right)^2 \mathsf{C}_{[23]}^{R,-1}\mathsf{C}_{[13]}^{L,1} \cdot \mathsf{C}, \tag{3.42b}$$

$$\mathsf{M}_3 = \left[\mathsf{C}_{[23]}^{L,0} \cdot \mathsf{C}\right]^{-1}\left(\mathsf{B}_2^L\right)^2 \mathsf{C}_{[23]}^{L,0} \cdot \mathsf{C}. \tag{3.42c}$$

Here the common factor C corresponds to the part of analytic continuation path which relates the base-point z_0 to the boundary component 3 of $C_{0,3}^L$ (the neighborhood of the black dot on the boundary circle in Fig. 9. The factor next to it depends on what one wants to achieve at the subsequent step: the black circle on the boundary 2 or the empty circle on the boundary 1 of $C_{0,3}^L$. In the latter case, for instance, the arc $[13]_L$ should be preceded by the half-turn b_3^L.

The observations of Subsection 3.6.3 allow one to get rid of the shift operators in the computation of classical monodromies by replacing the operator-valued matrices $\mathsf{C}_{[ji]}^{t,k}$ by the ordinary matrices $C_{[ji]}^{t,k}$ defined by (3.39). We may therefore set $\mathsf{C} = 1$ in the calculation of the trace functions. Also note that the resulting expressions are independent of the parameter τ_4 associated to the boundary curve δ_4: this is a consequence of the

factorization

$$(TB)_i^{t,k} = \left(\tilde{B}_i^t\right)^{-k} \begin{pmatrix} 0 & e^{-\frac{\mathrm{i}}{2}\tau_i^t} \\ e^{\frac{\mathrm{i}}{2}\tau_i^t} & 0 \end{pmatrix}, \qquad \tilde{B}_i^t = \mathrm{i}\,\sigma_z B_i^t. \tag{3.43}$$

We can now write L_s^r, L_t^s as the traces

$$\begin{aligned} L_s^r &= \mathrm{Tr}\left(\left[C_{[23]}^{R,-1}\right]^{-1}\left(B_2^R\right)^2 C_{[23]}^{R,-1}\left[C_{[13]}^{R,0}\right]^{-1}\left(B_1^R\right)^2 C_{[13]}^{R,0}\right) = \\ &= \mathrm{Tr}\left(\left(\tilde{B}_3^R\right)^{-1} F_{[32]}^R \left(\tilde{B}_2^R\right)^2 F_{[23]}^R\, \tilde{B}_3^R\, F_{[31]}^R \left(\tilde{B}_1^R\right)^2 F_{[13]}^R\right), \end{aligned} \tag{3.44a}$$

$$L_t^r = \mathrm{Tr}\left(\left[C_{[23]}^{R,-1} C_{[13]}^{L,1}\right]^{-1}\left(B_2^R\right)^2 C_{[23]}^{R,-1} C_{[13]}^{L,1}\left[C_{[23]}^{L,0}\right]^{-1}\left(B_2^L\right)^2 C_{[23]}^{L,0}\right). \tag{3.44b}$$

The first of the equations (3.5) then follows from the easily verified identity

$$F_{[31]}^t \tilde{B}_1^t\, F_{[12]}^t \tilde{B}_2^t\, F_{[23]}^t \tilde{B}_3^t = \mathrm{i}, \tag{3.45}$$

which should be understood as a version of the Moore-Seiberg hexagonal relation. To demonstrate the second equation, observe that (3.44b) may be rewritten as

$$\begin{gathered} L_t^r = G_{+-}^R G_{+-}^L e^{i\tau_r} + \left(G_{++}^R G_{--}^L + G_{--}^R G_{++}^L\right) + G_{-+}^R G_{-+}^L e^{-i\tau_r}, \\ G^R = \left[F_{[23]}^R \tilde{B}_3^R\right]^{-1}\left(\tilde{B}_2^R\right)^2 F_{[23]}^R \tilde{B}_3^R, \qquad G^L = \tilde{B}_1^L F_{[12]}^L \left(\tilde{B}_2^L\right)^2 \left[\tilde{B}_1^L F_{[12]}^L\right]^{-1}. \end{gathered}$$

The rest of the computation is straightforward.

3.7 Painlevé VI and Jimbo's formula

Let us now consider some of the applications of the above general formalism to the theory of monodromy preserving deformations. We start by providing a CFT derivation of the Jimbo's asymptotic formula [79] for the tau function of Painlevé VI equation.

Consider the simplest nontrivial case of four punctures. The fundamental group $\pi_1(C_{0,4})$ is isomorphic to free group of rank 3. Let $\chi_1, \ldots, \chi_4$ be the four loops shown in Fig. 10a, then

$$\pi_1(C_{0,4}) = \langle \chi_1, \chi_2, \chi_3, \chi_4 \,|\, \chi_1 \circ \chi_2 \circ \chi_3 \circ \chi_4 = 1 \rangle. \tag{3.46}$$

We denote by $M_1, \ldots, M_4 \in \mathrm{SL}(2,\mathbb{C})$ the monodromy matrices associated to these loops, satisfying $M_4 M_3 M_2 M_1 = \mathbf{1}$. Conjugacy classes of irreducible representations of $\pi_1(C_{0,4})$ in $\mathrm{SL}(2,\mathbb{C})$ are uniquely specified by seven invariants

$$L_\nu = \mathrm{Tr}\, M_\nu = 2\cos 2\pi\theta_\nu, \qquad \nu = 1, \ldots, 4, \tag{3.47a}$$

$$L_s = \mathrm{Tr}\, M_1 M_2, \qquad L_t = \mathrm{Tr}\, M_2 M_3, \qquad L_u = \mathrm{Tr}\, M_1 M_3, \tag{3.47b}$$

generating the algebra of invariant polynomial functions on $\mathrm{Hom}\,(\pi_1(C_{0,4}), \mathrm{SL}(2,\mathbb{C}))$. These traces satisfy the quartic equation

$$\begin{aligned} L_1L_2L_3L_4 + L_sL_tL_u + L_s^2 + L_t^2 + L_u^2 + L_1^2 + L_2^2 + L_3^2 + L_4^2 = \\ = (L_1L_2 + L_3L_4)\,L_s + (L_2L_3 + L_1L_4)\,L_t + (L_1L_3 + L_2L_4)\,L_u + 4. \end{aligned} \tag{3.48}$$

Remark 3.2. To recover the notation used earlier in Subsection 2.3, we could make the following identifications:

$$(a_1, a_2, a_3, a_4) \mapsto (0, t, 1, \infty), \qquad (\theta_1, \theta_2, \theta_3, \theta_4) \mapsto (\theta_0, \theta_t, \theta_1, \theta_\infty), \tag{3.49a}$$

$$(L_1, L_2, L_3, L_4) \mapsto (p_0, p_t, p_1, p_\infty), \qquad (L_s, L_t, L_u) \mapsto (p_{0t}, p_{1t}, p_{01}). \tag{3.49b}$$

The affine algebraic variety defined by (3.48) is the character variety of $C_{0,4}$. For every choice of $\theta_1, \ldots, \theta_4$, it defines a cubic surface in $\mathbb{C}^3$ in the variables L_s, L_t, L_u. If we further fix the trace function $L_s = 2\cos 2\pi\sigma$, the resulting quadric in L_t, L_u admits rational parameterization

$$\left(L_s^2 - 4\right) L_t = D_{t,+}\mathsf{s} + D_{t,-}\mathsf{s}^{-1} + D_{t,0}, \tag{3.50a}$$

$$\left(L_s^2 - 4\right) L_u = D_{u,+}\mathsf{s} + D_{u,-}\mathsf{s}^{-1} + D_{u,0}. \tag{3.50b}$$

with coefficients given by

$$D_{t,0} = L_s\,(L_1L_3 + L_2L_4) - 2\,(L_1L_4 + L_2L_3)\,, \tag{3.51a}$$

$$D_{u,0} = L_s\,(L_2L_3 + L_1L_4) - 2\,(L_1L_3 + L_2L_4)\,, \tag{3.51b}$$

$$D_{t,\pm} = 16\prod_{\epsilon=\pm} \sin\pi\,(\theta_2 \mp \sigma + \epsilon\theta_1)\sin\pi\,(\theta_3 \mp \sigma + \epsilon\theta_4)\,, \tag{3.51c}$$

$$D_{u,\pm} = -D_{t,\pm}e^{\mp 2\pi i\sigma}. \tag{3.51d}$$

Under identification (3.49), the uniformizing parameter s coincides with the quantity s_{0t} defined by the formula (2.30).

The Riemann surface $C_{0,4}$ is glued from two three-holed spheres $C_{0,3}^L$, $C_{0,3}^R$ as shown in Fig. 10b. The local coordinates (σ, τ) associated to this pants decomposition parameterize trace functions via (3.5)–(3.6) (as well as their counterparts for L_u). Comparing these expressions with (3.50)–(3.51), we find that

$$\mathsf{s} = \frac{\sin\pi\,(\sigma - \theta_1 + \theta_2)\sin\pi\,(\sigma + \theta_3 - \theta_4)}{\sin\pi\,(\sigma - \theta_1 - \theta_2)\sin\pi\,(\sigma - \theta_3 - \theta_4)}\,e^{i\tau}. \tag{3.52}$$

Now let us send (a_1, a_2, a_3, a_4) to $(0, t, 1, \infty)$ by Möbius transformation. The representation (3.36) of $\tau(a)$ as a Fourier transform of the $c = 1$ Virasoro conformal block is then more explicitly written as

$$\tau(t) = \sum_{n\in\mathbb{Z}} \langle\, \theta_4 \,|\, V_{\theta_4,\sigma+n}^{\theta_3}\,(1)\, V_{\sigma+n,\theta_1}^{\theta_2}\,(t) \,|\, \theta_1 \rangle\, e^{in\tau}. \tag{3.53}$$

Assuming without loss of generality that $-\frac{1}{2} < \Re\sigma < \frac{1}{2}$, letting $t \to 0$ in the last formula, and taking into account the normalization (3.34) of the chiral vertex operators, we deduce the asymptotics

$$\tau(z) = \sum_{n=0,\pm 1} N(\theta_4, \theta_3, \sigma + n) N(\sigma + n, \theta_2, \theta_1) e^{in\tau} t^{(\sigma+n)^2 - \theta_1^2 - \theta_2^2} + O\left(t^{\sigma^2 - \theta_1^2 - \theta_2^2 + 1}\right). \tag{3.54}$$

This is equivalent to the famous Jimbo's asymptotic formula (Theorem 2.2 of Subsection 2.3) expressing the critical behavior of the Painlevé VI tau function in terms of monodromy data.

3.8 Algebro-geometric solutions of the Schlesinger system

As another illustration, let us show how the known algebro-geometric solutions of the Schlesinger system on $C_{0,n}$ [94, 32] arise from the conformal blocks of the Ashkin-Teller critical model [161, 164].

Consider the pants decomposition of $C_{0,2g+2}$ schematically depicted in Fig. 11, and denote by $\mathcal{F}(\theta \,|\, p, p' \,|\, z)$ the corresponding $c = 1$ conformal block. Its external legs are combined into $g+1$ pairs. The momenta obtained by fusing different pairs are connected to a "black box". Its internal structure is not essential for the final result. However, to fix the notations, we will choose it in a particular way and parameterize it by $g-2$ internal momenta $p'_1, \ldots, p'_{g-2}$.

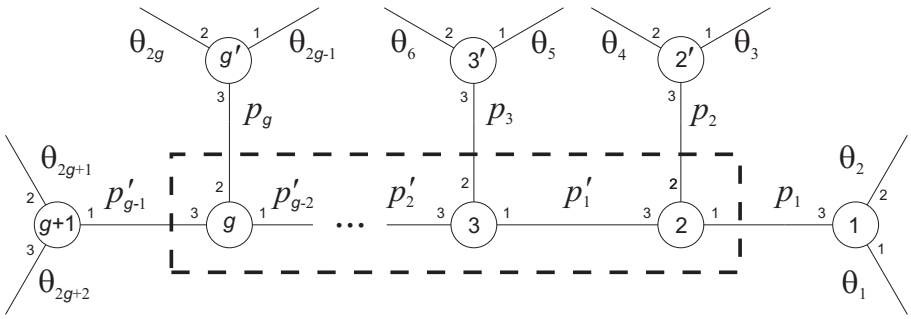


Fig. 11: Labeling of pairs of pants for the conformal block $\mathcal{F}(\theta \,|\, p, p' \,|\, z)$

As explained in Section 3.6, summation of conformal blocks over integer shifts of momenta gives an isomonodromic tau function of the Schlesinger system,

$$\tau(z) = \sum_{n \in \mathbb{Z}^g} \sum_{n' \in \mathbb{Z}^{g-1}} \mathcal{F}(\theta \,|\, p + n, p' + n' \,|\, z)\, e^{in\cdot\tau + in'\cdot\tau'}. \tag{3.55}$$

The variables p, p', τ, τ' provide a set of local coordinates on the $(4g-2)$-dimensional space of monodromy data (with fixed local monodromy exponents).

Let us impose a free-field-like conservation constraint on momenta of the unshifted conformal block at each vertex inside the box. These conditions determine the black box momenta $p' = p'[p]$ in terms of p. Explicitly,

$$p'_k[p] = p'_{k-1}[p] + p_{k+1}, \qquad p'_0[p] \equiv p_1.$$

Also, for $k = 1, \ldots, g-1$ we define

$$\ell_k = n'_k - n'_{k-1} - n_{k+1}, \qquad n'_0 \equiv n_1.$$

Since Barnes G-function vanishes at non-positive integer values of the argument, the form of the normalization coefficient (3.34) restricts the sum (3.55) to the domain $\ell_1, \ldots, \ell_{g-1} \geq 0$. In the limit

$$\tau_j \to -i\infty, \qquad \tau'_k \to i\infty, \tag{3.56a}$$

$$\tau_j + \textstyle\sum_{k=j}^{g} \tau'_{k-1} \to \xi_j, \qquad \tau'_0 \equiv 0, \qquad j = 1, \ldots, g, \tag{3.56b}$$

this sum further reduces to the values $\ell_1 = \ldots = \ell_{g-1} = 0$. We thus get a $2g$-parameter family of tau functions

$$\tau(z) = \sum_{n \in \mathbb{Z}^g} \mathcal{F}\left(\theta \,|\, p+n, p'[p+n] \,|\, z\right) e^{in\cdot\xi}. \tag{3.57}$$

Notice that at each of $g-1$ internal vertices of conformal blocks which appear in (3.57), the corresponding momenta satisfy the same conservation conditions as in the unshifted case.

Conformal blocks of this form with $\theta = \theta_{\text{AT}} \equiv \left(\frac{1}{4}, \ldots, \frac{1}{4}\right)$ describe correlation functions of the Ashkin-Teller critical model [161, 164]. They can be expressed in terms of certain quantities associated to the hyperelliptic curve Σ of genus g defined by

$$\lambda^2 = \prod_{k=1}^{2g+2} \left(\zeta - z_k\right). \tag{3.58}$$

Let us fix the canonical homology basis of a- and b-cycles on Σ as shown in Fig. 12. The g-dimensional space of holomorphic 1-forms on Σ is spanned by

$$d\omega_k = \frac{\zeta^{k-1} d\zeta}{\lambda}, \qquad k = 1, \ldots, g.$$

The $g \times g$ matrices of a- and b-periods

$$a_{jk} = \oint_{a_k} d\omega_j, \qquad b_{jk} = \oint_{b_k} d\omega_j, \tag{3.59}$$

determine the symmetric period matrix $\Omega = a^{-1}b$ of Σ. The hyperelliptic Riemann theta function with characteristics $[\,p,q\,] \in \mathbb{C}^{2g}$ is defined as the following series:

$$\theta[\,p,q\,]\big(x\,|\,\Omega\big) = \sum_{n\in\mathbb{Z}^g} e^{\pi i(n+p)\cdot\Omega\cdot(n+p)+2\pi i(n+p)\cdot(x+q)}. \tag{3.60}$$

Even characteristics $[\,p_S, q_S\,]$ correspond to its non-trivial half-periods and are indexed by partitions $S = \{\{z_{\alpha_1},\ldots,z_{\alpha_{g+1}}\},\{z_{\beta_1},\ldots,z_{\beta_{g+1}}\}\}$ of the set of ramification points into two subsets of equal size.

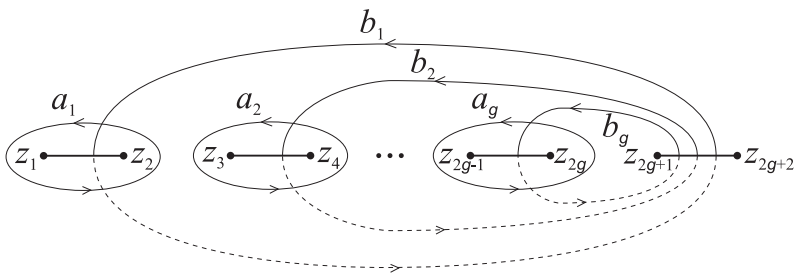


Fig. 12: Canonical homology basis on Σ

In this notation, the Ashkin-Teller conformal block is given by [161, 164]

$$\mathcal{F}\left(\theta_{\mathrm{AT}}\,|\,p,p'(p)\,|\,z\right) = \mathcal{G}(p)\mathcal{K}(z)\,\frac{e^{i\pi p\cdot\Omega\cdot p}}{\theta[\,p_S,q_S\,]\big(0\,|\,\Omega\big)}, \tag{3.61}$$

$$\mathcal{G}(p) = \frac{\cos\pi p'_{g-1}}{\pi^{1-g/2}}\,\frac{\hat{G}^2(p'_{g-1}+\frac{1}{2})}{\prod_{k=1}^{g}\hat{G}^2(p_k)}, \tag{3.62}$$

$$\mathcal{K}(z) = \left(\frac{\prod_{j<k}^{g+1}(z_{\alpha_j}-z_{\alpha_k})\prod_{j<k}^{g+1}(z_{\beta_j}-z_{\beta_k})}{\prod_{j,k}^{g+1}(z_{\alpha_j}-z_{\beta_k})}\right)^{\frac{1}{8}}. \tag{3.63}$$

Here we denote $\hat{G}(p) = \dfrac{G(1+p)}{G(1-p)}$. The prefactor $\mathcal{G}(p)$ comes from our normalization (3.34) of the chiral vertex operators. Taking into account the recurrence relation $\hat{G}\,(p+1) = -\pi\,(\sin\pi p)^{-1}\,\hat{G}\,(p)$, we see that the sum (3.57) reduces to the theta function series (3.60), so that

$$\tau(z) = \mathrm{const}\cdot\mathcal{K}(z)\,\frac{\theta[\,p,q\,]\big(0\,|\,\Omega\big)}{\theta[\,p_S,q_S\,]\big(0\,|\,\Omega\big)}, \tag{3.64}$$

with $e^{2\pi i q_k} := -\dfrac{\sin^2\pi p_k}{\cos^2\pi p'_{g-1}}e^{i\xi_k}$. We thus reproduce the $2g$-parameter family of tau functions found in [94]. The elliptic case $g=1$ corresponds to Picard solutions of Painlevé VI considered in Subsection 2.8.1.

At last let us compute the actual monodromy matrices for $\theta = \theta_{\mathrm{AT}}$ applying the rules formulated in Subsection 3.6.3. Up to overall conjugation, one has

$$M_{2k-1} = \left[C^{k',1}_{[13]} C^{[k,\dots,g]} C^{g+1,1}_{[13]}\right]^{-1} \left(B^{k'}_1\right)^2 C^{k',1}_{[13]} C^{[k,\dots,g]} C^{g+1,1}_{[13]}, \tag{3.65a}$$

$$M_{2k} = \left[C^{k',0}_{[23]} C^{[k,\dots,g]} C^{g+1,1}_{[13]}\right]^{-1} \left(B^{k'}_2\right)^2 C^{k',0}_{[23]} C^{[k,\dots,g]} C^{g+1,1}_{[13]}, \tag{3.65b}$$

with $C^{[k,\dots,g]} = C^{k,-1}_{[23]} C^{k+1,0}_{[13]} \dots C^{g,0}_{[13]}$ and $k = 2, \dots, g$. The conservation of momenta at the vertices $k, \dots, g-1$ implies that all matrices in the product $C^{[k,\dots,g]}$ are lower triangular. This enables one to explicitly calculate the monodromies in the limit (3.56). Again up to conjugation, the result is

$$M_k = \begin{pmatrix} 0 & i m_k^{-1} \\ i m_k & 0 \end{pmatrix}, \qquad k = 1, \dots, 2g+2, \tag{3.66}$$

with $m_{2g+1} = e^{2\pi i p'_{g-1}}$, $m_{2g+2} = 1$ and

$$m_{2k-1} = e^{2\pi i (p'_{k-2} + q_k)}, \qquad m_{2k} = -e^{2\pi i (p'_{k-1} + q_k)}. \tag{3.67}$$

Note in particular that in the chosen basis the products $M_{2k-1} M_{2k}$ and $M_{2k} M_{2k+1}$ are given by diagonal matrices, cf [94, Theorem 3.2].

4. Painlevé VI connection problem and $c = 1$ fusion matrix

4.1 Fusion transformations of conformal blocks

4.1.1 Notation

In this subsection, we again use a Liouville-type parameterization of the central charge

$$c = 1 + 6Q^2, \qquad Q = b + b^{-1}.$$

To cover all possible complex values of c, it suffices to consider b from the first quadrant. The weak-coupling region $c \geq 25$ then corresponds to $b \in \mathbb{R}_{\geq 1}$, the values $c \leq 1$ to $b \in i\mathbb{R}_{\geq 1}$, and $1 \leq c \leq 25$ to a quarter of the unit circle $b = e^{i\varphi}$, $\varphi \in [0, \frac{\pi}{2}]$. Here it will be convenient to represent conformal weights of primary fields in the form

$$\Delta = \frac{c-1}{24} + \theta^2,$$

where the parameters θ will be referred to as momenta. Four-point s-channel Virasoro conformal block on $\mathbb{P}^1$ with external dimensions $\Delta_\nu = \frac{c-1}{24} + \theta_\nu^2$ attached to the points $\nu = 0, t, 1, \infty$ and internal dimension $\Delta_\sigma = \frac{c-1}{24} + \sigma^2$ will be represented as

$$\mathcal{F}_c\left(\{\Delta_\nu\}, \Delta_\sigma; t\right) = \mathcal{F}_c\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \sigma\right](t)\,. \tag{4.1}$$

We will also use the function (2.45)

$$\mathcal{B}_c\left(\{\Delta_\nu\}, \Delta_\sigma; t\right) = t^{\Delta_0 + \Delta_t - \Delta_\sigma} \mathcal{F}_c\left(\{\Delta_\nu\}, \Delta_\sigma; t\right),$$

which is a power series in t normalized as $\mathcal{B}_c\left(\{\Delta_\nu\}, \Delta_\sigma; 0\right) = 1$.

As a function of t, $\mathcal{F}_c$ is believed to be analytically continuable to the universal cover of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. Some intuition about this analytic behavior may be gained by looking at the limit $c \to \infty$, Δ's finite, where conformal block reduces to the Gauss hypergeometric series

$$\mathcal{B}_\infty\left(\{\Delta_\nu\}, \Delta_\sigma; t\right) = {}_2F_1\left(\Delta_t - \Delta_0 + \Delta_\sigma, \Delta_1 - \Delta_\infty + \Delta_\sigma; 2\Delta_\sigma; t\right).$$

The previous sections dealt with another special case $c = 1$, where conformal block turns out to be related by a Fourier transform to the tau function of the sixth Painlevé equation, whose only critical points are 0, 1 and ∞.

4.1.2 Symmetries

As a function of parameters, conformal block enjoys a number of symmetries, analogous to Euler-Pfaff fractional linear transformations of ${}_2F_1\left(a,b;c;z\right)$:

- *Trivial symmetries*. Changing the sign of any of $\theta_{0,t,1,\infty}$ or σ has no effect on conformal block as the latter depends on dimensions only.

- *R-symmetries* allow the exchange of columns and rows of external momenta:

$$\mathcal{B}_c\left[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma\right](t)=(1-t)^{\Delta_0-\Delta_t-\Delta_1+\Delta_\infty}\,\mathcal{B}_c\left[\begin{array}{cc}\theta_\infty & \theta_0\\ \theta_1 & \theta_t\end{array};\sigma\right](t)= \tag{4.2a}$$

$$=\mathcal{B}_c\left[\begin{array}{cc}\theta_t & \theta_1\\ \theta_0 & \theta_\infty\end{array};\sigma\right](t)=(1-t)^{\Delta_0-\Delta_t-\Delta_1+\Delta_\infty}\,\mathcal{B}_c\left[\begin{array}{cc}\theta_0 & \theta_\infty\\ \theta_t & \theta_1\end{array};\sigma\right](t)\,. \tag{4.2b}$$

- *T-symmetry* enables one to exchange the dimensions in one column:

$$\mathcal{B}_c\left[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma\right](t)=(1-t)^{\Delta_0-\Delta_t-\Delta_\sigma}\,\mathcal{B}_c\left[\begin{array}{cc}\theta_\infty & \theta_t\\ \theta_1 & \theta_0\end{array};\sigma\right]\left(\frac{t}{t-1}\right). \tag{4.3}$$

- *Regge-Okamoto symmetry*. There is an identity

$$\mathcal{B}_c\left[\begin{array}{cc}\theta_1-\delta & \theta_t-\delta\\ \theta_\infty-\delta & \theta_0-\delta\end{array};\sigma\right](t)=(1-t)^{\delta_{1t}\delta}\,\mathcal{B}_c\left[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma\right](t)\,, \tag{4.4}$$

  where δ, δ_{1t} are defined by

$$2\delta=\theta_0+\theta_t+\theta_1+\theta_\infty,\qquad \delta_{1t}=\theta_t+\theta_1-\theta_0-\theta_\infty.$$

  This is reminiscent of the "unexpected" Regge symmetry of Racah-Wigner $6j$ symbols and Okamoto symmetry of Painlevé VI [18]. The latter can be actually seen as a $c=1$ specialization of the above. Though it is not easy to derive (4.4) from CFT first principles, this relation becomes almost obvious on the gauge side of the AGT correspondence where it corresponds to a permutation of masses of the matter hypermultiplets. Being combined with trivial symmetries, it relates conformal blocks with three distinct (unordered) sets of external dimensions.

4.1.3 Linear transformations

It is useful to think of the variable t as being the cross-ratio $t=\dfrac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}$ of four points $z_1=0$, $z_2=t$, $z_3=1$, $z_4=\infty$. The mapping class group $\Gamma=\mathrm{PSL}\left(2,\mathbb{Z}\right)$

of the 4-punctured sphere is the quotient of the braid group on 3 strings by its center. It naturally acts on conformal blocks by braiding transformations of $z_{1,2,3,4}$ and appropriate permutations of dimensions. One of the generators of this action is given by the above T-transformation. The second generator acts as $S : \theta_0 \leftrightarrow \theta_1, t \leftrightarrow 1-t$. It can be checked that S and T satisfy the modular group defining relations $S^2 = (ST)^3 = 1$.

It is expected that the linear span of conformal blocks (4.1) with different internal dimensions realizes an infinite-dimensional representation of Γ due to associativity of the operator product expansions. More precisely, there should be a linear "S-duality" relation between the conformal blocks calculated in different channels:

$$\mathcal{F}_c\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array};\sigma\right](t) = \int_{\mathbb{R}^+} F_c\left[\begin{array}{ccc} \theta_1 & \theta_t & \rho \\ \theta_\infty & \theta_0 & \sigma \end{array}\right]\mathcal{F}_c\left[\begin{array}{cc} \theta_0 & \theta_t \\ \theta_\infty & \theta_1 \end{array};\rho\right](1-t)\, d\rho. \tag{4.5}$$

The t-independent kernel F_c is the *fusion matrix*. It may be assumed to be even function of parameters θ_ν, σ, ρ and has a number of symmetries similar to (4.2) and (4.4).

The explicit form of the fusion kernel was found by B. Ponsot and J. Teschner who identified it with the Racah-Wigner matrix for a class of infinite-dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [132, 133]. Their result reads

$$F_c\left[\begin{array}{ccc} \theta_1 & \theta_t & \rho \\ \theta_\infty & \theta_0 & \sigma \end{array}\right] = \prod_{\epsilon,\epsilon'=\pm}\hat{\Gamma}_b\left[\begin{array}{c} \epsilon\theta_1 - \theta_t + \epsilon'\rho, \epsilon\theta_0 + \theta_\infty + \epsilon'\rho \\ \epsilon\theta_0 - \theta_t + \epsilon'\sigma, \epsilon\theta_1 + \theta_\infty + \epsilon'\sigma \end{array}\right]\prod_{\epsilon=\pm}\hat{\Gamma}_b\left[\begin{array}{c} 2\epsilon\sigma - \frac{iQ}{2} \\ 2\epsilon\rho + \frac{iQ}{2} \end{array}\right]\times$$
$$\times\int_{\mathcal{C}} dx \prod_{\epsilon=\pm}\hat{S}_b\left[\begin{array}{c} \frac{iQ}{2} + \epsilon\theta_0 - \theta_t + x, \frac{iQ}{2} + \epsilon\theta_1 + \theta_\infty + x \\ \epsilon\sigma + x, \theta_\infty - \theta_t + \epsilon\rho + x \end{array}\right]. \tag{4.6}$$

where we use the standard convention $f\left[\begin{array}{c} \alpha_1,\ldots,\alpha_n \\ \beta_1,\ldots,\beta_m \end{array}\right] = \frac{\prod_{k=1}^n f(\alpha_k)}{\prod_{k=1}^m f(\beta_k)}$.

The functions $\hat{\Gamma}_b(x)$ and $\hat{S}_b(x) = \hat{\Gamma}_b(x)/\hat{\Gamma}_b(-x)$ are closely related to the Barnes double gamma function and quantum dilogarithm. They can be defined by analytic continuation of the integral representations

$$\ln\hat{\Gamma}_b(x) = \int_0^\infty \frac{dt}{t}\left\{\frac{e^{-2ixt}-1}{4\sinh bt\sinh b^{-1}t} + \frac{1}{2}x^2e^{-2t} + \frac{ix}{2t}\right\},$$
$$\ln\hat{S}_b(x) = \int_0^\infty \frac{dt}{it}\left\{\frac{\sin 2xt}{2\sinh bt\sinh b^{-1}t} - \frac{x}{t}\right\}.$$

The function $\hat{S}_b(x)$ has an infinite number of zeros and poles in the complex x-plane:

- zeros: $x = -ib\left(m+\frac{1}{2}\right) - ib^{-1}\left(n+\frac{1}{2}\right)$ with $m,n \in \mathbb{Z}_{\geq 0}$,
- poles: $x = ib\left(m+\frac{1}{2}\right) + ib^{-1}\left(n+\frac{1}{2}\right)$ with $m,n \in \mathbb{Z}_{\geq 0}$.

This implies that, for instance, for real $b \geq 1$ the integrand in (4.6) has eight infinite half-lines of poles shown in Fig. 13a. As $b = e^{i\varphi}$, $\varphi \in \left(0, \frac{\pi}{2}\right)$, the half-lines open to 2D lattice sectors, see Fig. 13b. Similar picture holds for any b with $\Re b > 0$.

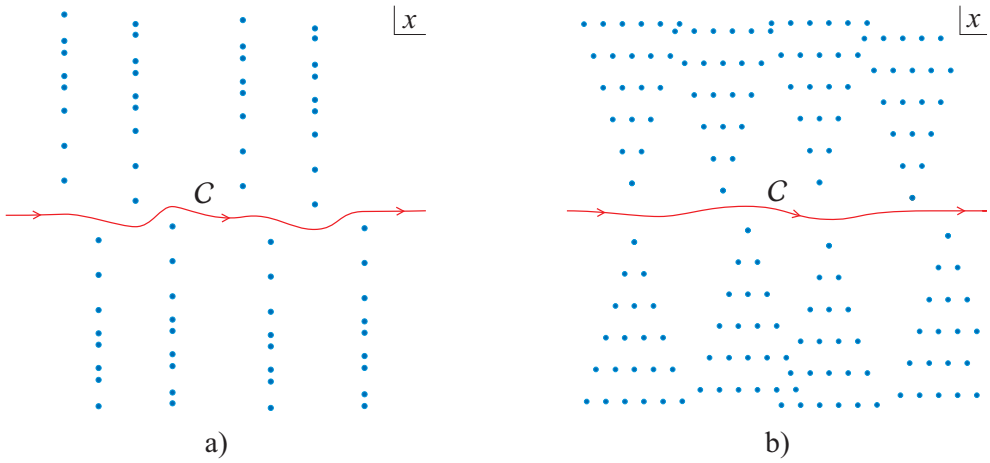


Fig. 13: Integrand singularities and integration contour in Ponsot-Teschner formula for a) $c > 25$ and b) $1 < c < 25$

The integration contour $\mathcal{C}$ in (4.6) runs from $-\infty$ to $+\infty$ passing between the upper and lower pole sectors. With this prescription, Ponsot-Teschner formula gives the fusion kernel for any complex value of the central charge *except* for the half-line $c \in \mathbb{R}_{\leq 1}$ corresponding to purely imaginary b. Our main concern is the edge point $c = 1$ of this excluded region.

4.1.4 Checking Ponsot-Teschner formula: an example with $c = 25$

Let us illustrate the formula (4.6) for the fusion kernel with an explicit example. It will be based on the evaluation of conformal block with $c = 25$, arbitrary internal dimension and all external dimensions equal to $\frac{15}{16}$, found by Al. B. Zamolodchikov [160, footnote (1)]. This corresponds to setting $b = 1$ and $\theta_{0,t,1,\infty} = \frac{i}{4}$ in the above.

The answer appears in [160] in the elliptic parameterization already used in Subsection 2.8.1 for the description of Ashkin-Teller conformal blocks and Picard solutions of P_{VI}. Namely, using the formulas (2.83), (2.84), the result can be stated as

$$\mathcal{F}_{25}\left[\begin{matrix} \frac{i}{4} & \frac{i}{4} \\ \frac{i}{4} & \frac{i}{4} \end{matrix}; \sigma\right](t) = \frac{2^{4\sigma^2} e^{i\pi\sigma^2\tau}}{t^{\frac{7}{8}}\,(1-t)^{\frac{7}{8}}\,\vartheta_3^3\,(0|\tau)}. \tag{4.7}$$

The modular transformation S exchanging t and $1-t$ maps τ to $-\tau^{-1}$. Then, applying Jacobi's imaginary transformation to (4.7), it is straightforward to verify that the S-duality

relation (4.5) is satisfied by Fourier transform conjugated by simple diagonal factors

$$F_{25}\left[\begin{array}{cc} \frac{i}{4} & \frac{i}{4} \\ \frac{i}{4} & \frac{i}{4} \end{array} ; \begin{array}{c} \rho \\ \sigma \end{array}\right] = \left(2^{-4\sigma^2}\sigma\right)^{-1} 2\sin 2\pi\sigma\rho \, \left(2^{-4\rho^2}\rho\right). \tag{4.8}$$

Next let us try to derive this relation from the Ponsot-Teschner formula. The functions $\hat{\Gamma}_b(x)$ and $\hat{S}_b(x)$ in the limit $b \to 1$ are expressed by means of the Barnes G-function:

$$\hat{\Gamma}_{b\to 1}(x) = \frac{(2\pi)^{\frac{ix}{2}}}{G(1+ix)}, \qquad \hat{S}_{b\to 1}(x) = (2\pi)^{ix}\frac{G(1-ix)}{G(1+ix)}.$$

Thanks to the doubling identity

$$G\left(1+2x\right) = 2^{x(2x-1)}\pi^{-x-\frac{1}{2}} G\left[\begin{array}{c} \frac{1}{2}+x, 1+x, 1+x, \frac{3}{2}+x \\ \frac{1}{2}, \frac{1}{2} \end{array}\right]$$

the prefactor in the 1st line of (4.6) reduces to

$$\left(2^{-4\sigma^2}\sigma\right)^{-1} 4\sinh 2\pi\sigma \sinh 2\pi\rho \, \left(2^{-4\rho^2}\rho\right).$$

Similarly simplifying the integrand in the 2nd line, it is possible to show that the fusion kernel (4.8) will follow from (4.6) provided that

$$\int_{\mathcal{C}} \frac{dx}{(2\pi)^{2ix}} \frac{G(1-2ix)}{G(1+2ix)} \prod_{\epsilon=\pm} G\left[\begin{array}{cc} 2+i\epsilon\sigma+ix, & 2+i\epsilon\rho+ix \\ i\epsilon\sigma-ix, & i\epsilon\rho-ix \end{array}\right] = \frac{8\pi^4 \sin 2\pi\sigma\rho}{\sinh 2\pi\sigma \sinh 2\pi\rho}. \tag{4.9}$$

The contour $\mathcal{C}$ passes between the half-lines of zeros of the Barnes functions in the denominator, as discussed above. In particular, for $\sigma, \rho \in \mathbb{R}$ it can be chosen as horizontal line with $0 < \Im x < \frac{1}{2}$.

The intriguing integral identity (4.9) looks rather nontrivial and we will not attempt to rigorously prove it here. Instead, we contented ourselves with its numerical verification for several randomly chosen values of σ and ρ.

4.1.5 Probing $c = 1$ fusion

As c approaches the interval $(-\infty, 1]$, Ponsot-Teschner formula becomes singular. The sectors in Fig. 13b transform into overlapping half-planes containing an infinite number of dense lines of poles. The function $\hat{\Gamma}_b(x)$ has a natural boundary of analyticity at purely imaginary b. It is therefore legitimate to ask whether fusion transformations merely exist. Of course, they do for degenerate dimensions [40]. We suggest that this is also true for arbitrary dimensions at $c = 1$.

As an illustration, consider the Ashkin-Teller conformal block (2.85), characterized by $c = 1$ and all external dimensions equal to $\frac{1}{16}$. It is very easy to check by evaluating Gaussian integrals that (4.5) is again satisfied by a Fourier-type fusion kernel

$$F_1\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}\end{matrix}\,;\,\begin{matrix}\rho \\ \sigma\end{matrix}\right] = 2^{4\sigma^2-4\rho^2+1}\cos 2\pi\sigma\rho. \tag{4.10}$$

4.2 Painlevé VI and hyperbolic tetrahedron

4.2.1 Connection problem

We have seen in the previous sections that the Painlevé VI tau function admits several conformal block representations associated to different channels. In particular, one can write

$$\tau(t) = \chi_0 \sum_{n\in\mathbb{Z}} C\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}\,;\sigma_{0t}+n\right] s_{0t}^n\, \mathcal{F}_1\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}\,;\sigma_{0t}+n\right](t) = \tag{4.11a}$$

$$= \chi_1 \sum_{n\in\mathbb{Z}} C\left[\begin{matrix}\theta_0 & \theta_t \\ \theta_\infty & \theta_1\end{matrix}\,;\sigma_{1t}+n\right] s_{1t}^n\, \mathcal{F}_1\left[\begin{matrix}\theta_0 & \theta_t \\ \theta_\infty & \theta_1\end{matrix}\,;\sigma_{1t}+n\right](1-t)\,. \tag{4.11b}$$

The first representation is particularly suitable in the vicinity of $t = 0$, and the second one gives the expansion of $\tau(t)$ around $t = 1$. The structure constants are given by (2.31). Any of the two pairs of integration constants (σ_{0t}, s_{0t}) and (σ_{1t}, s_{1t}) specifies the initial conditions for P_{VI} in the form of solution asymptotics near a given critical point. The relation between the two pairs is most conveniently formulated in terms of monodromy data for the associated linear problem by the formulas such as (2.27b), (2.30) and (2.35).

The definition (2.20) of the P_{VI} tau function contains an obvious normalization ambiguity, which implies that the coefficients $\chi_{0,1}$ in (4.11a)–(4.11b) are intrinsically indefinite. However, their ratio

$$\chi_{01}(\vec{\theta};\sigma_{0t},\sigma_{1t};p_{01}) = \chi_0^{-1}\chi_1 \tag{4.12}$$

is completely fixed by the σ-Painlevé VI equation (2.23) and initial conditions for $\sigma(t)$. It determines relative normalization of the expansions of $\tau(t)$ near 0 and 1, and will be called *connection coefficient*.

Sometimes it will become convenient to include the structure constants into the definition of relative normalization by introducing

$$\bar{\chi}_{01}(\vec{\theta};\sigma_{0t},\sigma_{1t};p_{01}) = \chi_{01}(\vec{\theta};\sigma_{0t},\sigma_{1t};p_{01})\, C\left[\begin{matrix}\theta_0 & \theta_t \\ \theta_\infty & \theta_1\end{matrix}\,;\sigma_{1t}\right] \Big/ C\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}\,;\sigma_{0t}\right]. \tag{4.13}$$

In particular, for $-\frac{1}{2} < \Re\sigma_{0t}$, $\Re\sigma_{1t} < \frac{1}{2}$ one can write

$$\bar{\chi}_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01}) = \frac{\lim_{t\to 1}(1-t)^{\theta_1^2+\theta_t^2-\sigma_{1t}^2}\tau(t)}{\lim_{t\to 0} t^{\theta_0^2+\theta_t^2-\sigma_{0t}^2}\tau(t)}. \tag{4.14}$$

Finding the explicit form of the connection coefficients (4.12), (4.13) in terms of monodromy data constitutes one of the main goals of this section.

4.2.2 Canonical coordinates

There is a natural Poisson bracket $\{,\}$ on P_{VI} monodromy manifold $\mathcal{M}$ inherited from the Atiyah-Bott symplectic structure on the moduli space of flat logarithmic $\mathrm{SL}(2,\mathbb{C})$-connections on the 4-punctured sphere. This bracket is defined by

$$\{p_{0t}, p_{1t}\} = q_{01}, \qquad \{p_{1t}, p_{01}\} = q_{0t}, \qquad \{p_{01}, p_{0t}\} = q_{1t}, \tag{4.15}$$

where $q_{\nu\nu'} = \dfrac{\partial W}{\partial p_{\nu\nu'}}$ and $W(p_{0t}, p_{1t}, p_{01})$ is the polynomial (2.28) defining the Jimbo-Fricke cubic. Thus, for instance,

$$q_{01} = 2p_{01} + p_{0t}p_{1t} - \omega_{01}. \tag{4.16}$$

It turns out that the cubic itself may be rewritten in terms of these variables in a nice determinantal form, e.g.,

$$q_{01}^2 = \det \mathcal{G}, \qquad \mathcal{G} = \begin{pmatrix} 2 & -p_0 & -p_t & p_{1t} \\ -p_0 & 2 & p_{0t} & -p_\infty \\ -p_t & p_{0t} & 2 & -p_1 \\ p_{1t} & -p_\infty & -p_1 & 2 \end{pmatrix}. \tag{4.17}$$

Parameterizing s_{0t}, s_{1t} from (2.30), (2.35) as

$$s_{0t} = e^{\eta_{0t}/2\pi i}, \qquad s_{1t} = e^{\eta_{1t}/2\pi i}, \tag{4.18}$$

it can be easily verified (an equivalent observation is contained in a recent work [116]) that the local coordinates η_{0t}, η_{1t} are conjugate to monodromy exponents σ_{0t}, σ_{1t} with respect to the above Poisson bracket:

$$\{\sigma_{0t}, \eta_{0t}\} = \{\sigma_{1t}, \eta_{1t}\} = 1.$$

Two pairs of Darboux coordinates (σ_{0t}, η_{0t}) and (σ_{1t}, η_{1t}) on $\mathcal{M}$ are well-adapted for characterizing the expansions of $\tau(t)$ near $t = 0$ and $t = 1$, respectively, cf (4.11a)–(4.11b). Recall that generic $c = 1$ four-point Virasoro conformal blocks, as functions

of t and $1-t$, literally coincide with Fourier expansion coefficients of the appropriately normalized Painlevé VI tau functions with respect to the dual coordinates η_{0t} and η_{1t}.

The pairs (σ_{0t}, η_{0t}) and (σ_{1t}, η_{1t}) are related by a canonical transformation whose generating function $\mathcal{S}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ is defined by the equations

$$\eta_{0t} = \frac{\partial \mathcal{S}}{\partial \sigma_{0t}}, \qquad \eta_{1t} = -\frac{\partial \mathcal{S}}{\partial \sigma_{1t}}. \tag{4.19}$$

Remarkably, $\mathcal{S}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ can be found in explicit form. It essentially coincides [116] with the complexified volume of the hyperbolic tetrahedron $\mathcal{T}$ with dihedral angles $2\pi\vec{\theta}$, $2\pi\sigma_{0t}+\pi$, $2\pi\sigma_{1t}+\pi$, whose mnemonic graphical representation can be obtained by gluing external legs of s- or t-channel conformal blocks (see Fig. 14). A tetrahedral signature shows up already in (4.17): the 4×4 matrix $\mathcal{G}$ is nothing but the Gram matrix of scalar products of length $\sqrt{2}$ vectors normal to faces of $\mathcal{T}$ and oriented outwards.

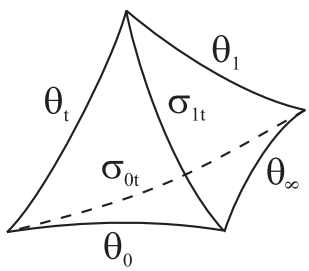


Fig. 14: Tetrahedron $\mathcal{T}$ obtained by gluing external legs of conformal blocks

4.2.3 Volume of $\mathcal{T}$

The explicit formula for the volume is most conveniently written in terms of the Lobachevsky function, which is essentially a half of the imaginary part of the Euler dilogarithm:

$$\Lambda(\sigma) = \frac{1}{4i}\left[\mathrm{Li}_2\left(e^{2\pi i\sigma}\right) - \mathrm{Li}_2\left(e^{-2\pi i\sigma}\right)\right], \qquad \sigma \in \mathbb{R}. \tag{4.20}$$

This definition differs from the standard one by a factor of π in the argument of Λ. The dilogarithms are evaluated on their main sheets, which implies that $\Lambda(\sigma)$ is continuous and periodic.

Define the parameters

$$\begin{aligned}
\nu_1 &= \sigma_{0t} + \theta_0 + \theta_t, & \lambda_1 &= \theta_0 + \theta_t + \theta_1 + \theta_\infty,\\
\nu_2 &= \sigma_{0t} + \theta_1 + \theta_\infty, & \lambda_2 &= \sigma_{0t} + \sigma_{1t} + \theta_0 + \theta_1,\\
\nu_3 &= \sigma_{1t} + \theta_0 + \theta_\infty, & \lambda_3 &= \sigma_{0t} + \sigma_{1t} + \theta_t + \theta_\infty,\\
\nu_4 &= \sigma_{1t} + \theta_t + \theta_1, & \lambda_4 &= 0,\\
2\nu_\Sigma &= \nu_1 + \nu_2 + \nu_3 + \nu_4,
\end{aligned} \tag{4.21}$$

then the volume of $\mathcal{T}$ is given by [28, 112]

$$\mathrm{Vol}\left(\mathcal{T}\right) = \frac{1}{2}\sum_{k=1}^{4}\sum_{\epsilon=\pm}\epsilon\left[\Lambda\left(\omega_\epsilon + \nu_k\right) - \Lambda\left(\omega_\epsilon + \lambda_k\right)\right]. \tag{4.22}$$

Here $\omega_\pm$ represent two nontrivial solutions $z_\pm = e^{2\pi i\omega_\pm}$ of the equation

$$\prod_{k=1}^{4}\left(1 - ze^{2\pi i\nu_k}\right) = \prod_{k=1}^{4}\left(1 - ze^{2\pi i\lambda_k}\right). \tag{4.23}$$

which can be expressed in terms of $\vec{\theta}$, $\vec{\sigma}$ as

$$z_\pm = \frac{4\sin 2\pi\sigma_{0t}\sin 2\pi\sigma_{1t} + 4\sin 2\pi\theta_t\sin 2\pi\theta_\infty + 4\sin 2\pi\theta_0\sin 2\pi\theta_1 \pm q_{01}}{2\sum_{k=1}^{4}\left(e^{2\pi i(\nu_\Sigma-\nu_k)} - e^{2\pi i(\nu_\Sigma-\lambda_k)}\right)}. \tag{4.24}$$

Their product does not contain p_{01} and can be written as

$$z_+z_- = \frac{\sum_{k=1}^{4}\left(e^{2\pi i(\nu_k-\nu_\Sigma)} - e^{2\pi i(\lambda_k-\nu_\Sigma)}\right)}{\sum_{k=1}^{4}\left(e^{2\pi i(\nu_\Sigma-\nu_k)} - e^{2\pi i(\nu_\Sigma-\lambda_k)}\right)}. \tag{4.25}$$

Note that for the genuine hyperbolic tetrahedra θ's, σ_{0t} and σ_{1t} are real. Also, q_{01} is purely imaginary since $\mathcal{G}$ should have the signature $(-,+,+,+)$ of the ambient space $\mathbb{R}^{1,3} \supset \mathbb{H}^3$. The parameters $z_\pm$ then lie on the unit circle, which makes (4.22) compatible with our earlier conventions for $\Lambda(\sigma)$.

The precise relation between $\mathrm{Vol}\left(\mathcal{T}\right)$ and the generating function $\mathcal{S}(\vec{\theta};\sigma_{0t},\sigma_{1t})$ from the previous subsection is provided by

Lemma 4.1. *We have*

$$2\frac{\partial}{\partial\sigma_{0t}}\mathrm{Vol}\left(\mathcal{T}\right) = i\eta_{0t} + \pi\ln\prod_{\epsilon=\pm}\frac{\sin\pi\left(\theta_t + \sigma_{0t} + \epsilon\theta_0\right)\sin\pi\left(\theta_1 + \sigma_{0t} + \epsilon\theta_\infty\right)}{\sin\pi\left(\theta_t - \sigma_{0t} + \epsilon\theta_0\right)\sin\pi\left(\theta_1 - \sigma_{0t} + \epsilon\theta_\infty\right)}, \tag{4.26}$$

$$-2\frac{\partial}{\partial\sigma_{1t}}\mathrm{Vol}\left(\mathcal{T}\right) = i\eta_{1t} + \pi\ln\prod_{\epsilon=\pm}\frac{\sin\pi\left(\theta_t + \sigma_{1t} + \epsilon\theta_1\right)\sin\pi\left(\theta_0 + \sigma_{1t} + \epsilon\theta_\infty\right)}{\sin\pi\left(\theta_t - \sigma_{1t} + \epsilon\theta_1\right)\sin\pi\left(\theta_0 - \sigma_{1t} + \epsilon\theta_\infty\right)}. \tag{4.27}$$

Let us also mention that the formula (4.22) for $\mathrm{Vol}\left(\mathcal{T}\right)$ can be rewritten in terms of ω_+ or ω_- only (instead of using both of these parameters) thanks to the following result, cf [112, Theorem 2]:

Lemma 4.2. *The quantity*

$$\mathcal{V}\left(\mathcal{T}\right) = \frac{1}{2}\sum_{k=1}^{4}\sum_{\epsilon=\pm}\left[\Lambda\left(\omega_\epsilon + \nu_k\right) - \Lambda\left(\omega_\epsilon + \lambda_k\right)\right], \tag{4.28}$$

can be alternatively expressed as

$$\mathcal{V}(\mathcal{T}) = \frac{1}{2}\sum_{\epsilon,\epsilon'=\pm} \epsilon\epsilon' \Big[\Lambda(\theta_0 + \epsilon\sigma_{0t} + \epsilon'\theta_t) + \Lambda(\theta_\infty + \epsilon\sigma_{0t} + \epsilon'\theta_1) + \\ + \Lambda(\theta_t + \epsilon\sigma_{1t} + \epsilon'\theta_1) + \Lambda(\theta_\infty + \epsilon\sigma_{1t} + \epsilon'\theta_0)\Big]. \tag{4.29}$$

In the case of complex angles the function $\mathrm{Vol}(\mathcal{T})$ may be defined via continuation from an open set $U \subset \mathbb{C}^6$. However, in doing this the periodicity with respect to angles will be lost, just as if instead of taking $\sigma \in \mathbb{R}$ and fixing the principal branches of $\mathrm{Li}_2(z)$ in (4.20) we tried to continue $\Lambda(\sigma)$ analytically from a suitable open subset of $\mathbb{C}$.

4.3 Connection coefficient for P_{VI} tau function

4.3.1 Functional relations

In this subsection, we compute the connection coefficient defined by (4.12). The idea is to consider P_{VI} parameters $\vec{\theta}$ as fixed and obtain $\chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01})$ by solving certain difference equations with respect to σ_{0t} and σ_{1t}.

Given $\vec{\theta}$, σ_{0t} and σ_{1t}, the value of p_{01} which enters the tau function expansions (4.11a)–(4.11b) via s_{0t} and s_{1t}, is fixed up to the choice of solution of the Jimbo-Fricke relation $W(p_{0t}, p_{1t}, p_{01}) = 0$. Therefore the space $\mathcal{M}_{\vec{\theta}}$ of triples $(\sigma_{0t}, \sigma_{1t}; p_{01})$ associated to $\vec{\sigma}$ at fixed $\vec{\theta}$ is a double cover of $\mathbb{C}^2 \ni (\sigma_{0t}, \sigma_{1t})$. It may be safely assumed that $\chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01})$ is meromorphic on the complement of the ramification locus of $\mathcal{M}_{\vec{\theta}}$. The structure of conformal expansions (4.11a)–(4.11b) then gives two recurrence relations for χ_{01}:

$$\frac{\chi_{01}(\vec{\theta}; \sigma_{0t}+1, \sigma_{1t}; p_{01})}{\chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01})} = s_{0t}^{-1}, \tag{4.30a}$$

$$\frac{\chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}+1; p_{01})}{\chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01})} = s_{1t}, \tag{4.30b}$$

where s_{0t}, s_{1t} are defined by (2.30), (2.35). The main difficulty in the solution of (4.30a)–(4.30b) is hidden in the dependence of these quantities on p_{01}, as the latter depends on σ_{0t}, σ_{1t} in a rather complicated way.

As a warm-up exercise, let us consider the implications of the difference equations for the symmetrized product of connection coefficients over two sheets of $\mathcal{M}_{\vec{\theta}}$:

$$\kappa_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}) = \chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01})\chi_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p'_{01}). \tag{4.31}$$

Our notation means the following: if p_{01} is one of the two roots of the equation $W(p_{0t}, p_{1t}, p_{01}) = 0$, then $p'_{01} = \omega_{0t} - p_{01} - p_{0t}p_{1t}$ denotes the other root. Also, denote by $q'_{\mu\nu}$, $s'_{\mu\nu}$ the appropriate modifications of $q_{\mu\nu}$, $s_{\mu\nu}$. In particular, one has

$$q'_{0t} = q_{0t} - p_{1t}q_{01}, \qquad q'_{1t} = q_{1t} - p_{0t}q_{01}, \qquad q'_{01} = -q_{01}.$$

Now write s_{0t}, s_{1t}, s'_{0t}, s'_{1t} in terms of q_{01}. It turns out, expectedly, that the products $s_{0t}s'_{0t}$ and $s_{1t}s'_{1t}$ depend only on q_{01}^2. Using (4.17), it can be deduced from (4.30a)–(4.30b) that

$$\frac{\kappa_{01}(\vec{\theta}; \sigma_{0t} + 1, \sigma_{1t})}{\kappa_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})} = \prod_{\epsilon=\pm} \frac{\sin \pi \left(\theta_t - \sigma_{0t} + \epsilon\theta_0\right) \sin \pi \left(\theta_1 - \sigma_{0t} + \epsilon\theta_\infty\right)}{\sin \pi \left(\theta_t + \sigma_{0t} + \epsilon\theta_0\right) \sin \pi \left(\theta_1 + \sigma_{0t} + \epsilon\theta_\infty\right)}, \tag{4.32a}$$

$$\frac{\kappa_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t} + 1)}{\kappa_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})} = \prod_{\epsilon=\pm} \frac{\sin \pi \left(\theta_t + \sigma_{1t} + \epsilon\theta_1\right) \sin \pi \left(\theta_0 + \sigma_{1t} + \epsilon\theta_\infty\right)}{\sin \pi \left(\theta_t - \sigma_{1t} + \epsilon\theta_1\right) \sin \pi \left(\theta_0 - \sigma_{1t} + \epsilon\theta_\infty\right)}. \tag{4.32b}$$

The general solution of (4.32a)–(4.32b) may be constructed in terms of the Barnes G-function. Indeed, since $G\left(1 + \sigma\right) = \Gamma\left(\sigma\right) G\left(\sigma\right)$, the function

$$\hat{G}\left(\sigma\right) = \frac{G\left(1 + \sigma\right)}{G\left(1 - \sigma\right)} \tag{4.33}$$

satisfies

$$\hat{G}\left(\sigma \pm 1\right) = \mp \left(\frac{\sin \pi\sigma}{\pi}\right)^{\mp 1} \hat{G}\left(\sigma\right). \tag{4.34}$$

One then easily derives

Lemma 4.3. *The general solution of (4.32a)–(4.32b) is given by*

$$\kappa_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}) = \kappa_{\rm per}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}) \prod_{\epsilon,\epsilon'=\pm} \hat{G}\left[\begin{array}{c} \theta_t + \epsilon\theta_0 + \epsilon'\sigma_{0t}, \theta_1 + \epsilon\theta_\infty + \epsilon'\sigma_{0t} \\ \theta_t + \epsilon\theta_1 + \epsilon'\sigma_{1t}, \theta_0 + \epsilon\theta_\infty + \epsilon'\sigma_{1t} \end{array}\right], \tag{4.35}$$

where $\kappa_{\rm per}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ is an arbitrary periodic function of both σ_{0t}, σ_{1t} with periods 1.

What about $\kappa_{\rm per}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$? The simplest guess is to assume that this quantity does not depend on σ_{0t}, σ_{1t} (some arguments in favor of this hypothesis will be given in the next subsection). The guess is readily confirmed by numerical experiments, but in fact the numerics reveals much more: $\kappa_{\rm per}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ is simply equal to 1!

The final formula can now be written as

$$\bar{\chi}_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01})\bar{\chi}_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p'_{01}) = \frac{\Phi\left(\theta_t, \theta_1, \sigma_{1t}\right) \Phi\left(\theta_0, \theta_\infty, \sigma_{1t}\right)}{\Phi\left(\theta_0, \theta_t, \sigma_{0t}\right) \Phi\left(\theta_1, \theta_\infty, \sigma_{0t}\right)}, \tag{4.36}$$

where

$$\Phi\left(\theta, \theta', \theta''\right) = \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm} G\left(1 + \epsilon\theta + \epsilon'\theta' + \epsilon''\theta''\right)}{\prod_{\epsilon=\pm} G\left(1 + 2\epsilon\theta\right) G\left(1 + 2\epsilon\theta'\right) G\left(1 + 2\epsilon\theta''\right)}.$$

The right side of (4.36) coincides with a ratio of three-point functions Φ in the time-like $c = 1$ Liouville theory [163]. A conceptual explanation of this intriguing coincidence is yet to be found.

4.3.2 Minimal solution

Let us now come back to the relations (4.30a)–(4.30b). As their solution is much easier to check than to guess, the reader interested only in the final result may jump directly to Lemma 4.4. What follows is an attempt to elucidate the origins of this Lemma.

Taking the logarithmic derivatives of both sides of (4.30a)–(4.30b) and recalling the parameterization (4.18), we obtain

$$\frac{\partial}{\partial\sigma_{0t}}\ln\frac{\chi_{01}(\sigma_{0t}+1,\sigma_{1t})}{\chi_{01}(\sigma_{0t},\sigma_{1t})}=-\frac{1}{2\pi i}\frac{\partial\eta_{0t}}{\partial\sigma_{0t}}, \tag{4.37a}$$

$$\frac{\partial}{\partial\sigma_{1t}}\ln\frac{\chi_{01}(\sigma_{0t}+1,\sigma_{1t})}{\chi_{01}(\sigma_{0t},\sigma_{1t})}=-\frac{1}{2\pi i}\frac{\partial\eta_{0t}}{\partial\sigma_{1t}}, \tag{4.37b}$$

$$\frac{\partial}{\partial\sigma_{0t}}\ln\frac{\chi_{01}(\sigma_{0t},\sigma_{1t}+1)}{\chi_{01}(\sigma_{0t},\sigma_{1t})}=\frac{1}{2\pi i}\frac{\partial\eta_{1t}}{\partial\sigma_{0t}} \tag{4.37c}$$

$$\frac{\partial}{\partial\sigma_{1t}}\ln\frac{\chi_{01}(\sigma_{0t},\sigma_{1t}+1)}{\chi_{01}(\sigma_{0t},\sigma_{1t})}=\frac{1}{2\pi i}\frac{\partial\eta_{1t}}{\partial\sigma_{1t}}, \tag{4.37d}$$

where all other arguments of χ_{01} are temporarily omitted to lighten the notation.

A tentative solution of (4.37a)–(4.37d) can be written in the form

$$\ln\tilde{\chi}_{01}(\sigma_{0t},\sigma_{1t})=\frac{1}{2\pi i}\int_P^{(\sigma_{0t},\sigma_{1t})}\sigma_{1t}d\eta_{1t}-\sigma_{0t}d\eta_{0t}, \tag{4.38}$$

where the integral is calculated along a path starting at some fixed point P on the (infinite-sheeted covering of) Jimbo-Fricke surface. Indeed, the integrand is a closed 1-form; its differential $d\sigma_{1t}\wedge d\eta_{1t}-d\sigma_{0t}\wedge d\eta_{0t}$ vanishes since the transformation $(\sigma_{0t},\eta_{0t})\to(\sigma_{1t},\eta_{1t})$ is canonical, see Subsection 4.2.2. Hence the integral value depends only on the homotopy class of the path on the Jimbo-Fricke surface with excluded one-dimensional subspaces corresponding to the singularities of the integrand. In particular, restricting to $\sigma_{1t}=\mathrm{const}$, one obtains a complex curve with punctures at the poles of the integrand. For $\tilde{\chi}_{01}$ defined by (4.38) one has, e.g.,

$$\ln\frac{\tilde{\chi}_{01}(\sigma_{0t}+1,\sigma_{1t})}{\tilde{\chi}_{01}(\sigma_{0t},\sigma_{1t})}=\frac{1}{2\pi i}\int_{(\sigma_{0t},\sigma_{1t})}^{(\sigma_{0t}+1,\sigma_{1t})}\sigma_{1t}d\eta_{1t}-\sigma_{0t}d\eta_{0t}. \tag{4.39}$$

Differentiating the right side, it is straightforward to check that $\tilde{\chi}_{01}$ satisfies (4.37a)–(4.37b). The other two relations are verified analogously.

Thus we have shown that, up to an additive constant independent of σ_{0t}, σ_{1t} but a priori depending on homotopy class of integration path, $\ln\tilde{\chi}_{01}$ satisfies the same functional relations as $\ln\chi_{01}$. On the other hand, χ_{01} is expected to be a single-valued function of σ_{0t}, σ_{1t}. If it were possible to present it in the form (4.38), the integrals corresponding

to different paths could only differ by integer multiples of $2\pi i$. This appears not to be the case: the residues (e.g. calculated at the poles of the integrand restricted to $\sigma_{1t} = \mathrm{const}$) are not integers. Therefore, one may try to use the freedom in the choice of the additive constant to correct the integrand by a closed 1-form with periodic coefficients which would ensure the necessary analytic properties.

The form of the correction term can be guessed as follows. The integral in (4.38) is obviously related to the generating function

$$\mathcal{S}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}) = \int_P^{(\sigma_{0t},\sigma_{1t})} \eta_{0t} d\sigma_{0t} - \eta_{1t} d\sigma_{1t} \tag{4.40}$$

of the canonical transformation $(\sigma_{0t}, \eta_{0t}) \to (\sigma_{1t}, \eta_{1t})$. This function is in turn related (see Lemma 4.1) to the complexified volume (4.22) of hyperbolic tetrahedron, expressed in terms of Lobachevsky functions. Both $\mathcal{S}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ and $\mathrm{Vol}(\mathcal{T})$ are multivalued functions of σ_{0t}, σ_{1t}, and we take the point of view that this multivaluedness stems from the possibility to consider homotopically inequivalent paths in (4.40). Now using that

$$\Lambda(\sigma) = -\pi\sigma \ln \frac{\sin \pi\sigma}{\pi} - \pi \ln \hat{G}(\sigma), \tag{4.41}$$

one can decompose (4.38) into a Barnes function piece which has the required analytic behaviour, and an elementary function piece responsible for multivaluedness. It is natural to assume that the latter contribution can be compensated by the correction 1-form mentioned above, so that the genuine single-valued connection coefficient χ_{01} comes from (4.22) and Lemmas 4.1–4.2 by keeping only log-Barnes term of (4.41) in each $\Lambda(\sigma)$. This finally leads to

Lemma 4.4. *The general solution of (4.30a)–(4.30b) is given by*

$$\bar{\chi}_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01}) = \chi_{\mathrm{per}}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}) \times \tag{4.42}$$
$$\times \prod_{\epsilon,\epsilon'=\pm} G\left[\begin{matrix} 1+\epsilon\sigma_{1t}+\epsilon'\theta_t-\epsilon\epsilon'\theta_1, 1+\epsilon\sigma_{1t}+\epsilon'\theta_0-\epsilon\epsilon'\theta_\infty \\ 1+\epsilon\sigma_{0t}+\epsilon'\theta_t+\epsilon\epsilon'\theta_0, 1+\epsilon\sigma_{0t}+\epsilon'\theta_1+\epsilon\epsilon'\theta_\infty \end{matrix}\right] \prod_{\epsilon=\pm} \frac{G(1+2\epsilon\sigma_{0t})}{G(1+2\epsilon\sigma_{1t})} \prod_{k=1}^{4} \frac{\hat{G}(\omega_+ + \nu_k)}{\hat{G}(\omega_+ + \lambda_k)},$$

where $\nu_{1\ldots4}$ and $\lambda_{1\ldots4}$ are defined by (4.21), ω_+ by (4.23)–(4.24), and $\chi_{\mathrm{per}}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ is an arbitrary periodic function of σ_{0t}, σ_{1t} with periods 1.

Observe that the right side of (4.42) is a periodic function of ω_+ thanks to (4.23) and (4.34), which enables one to choose the solution of $z_+ = e^{2\pi i\omega_+}$ arbitrarily.

The periodic prefactor $\chi_{\mathrm{per}}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ can be fixed from the following considerations. Let $\sigma_{0t} = \alpha$ be a point where the connection coefficient tends to infinity. Unless this singular behavior is compensated by the coefficients of (4.11a) and (4.11b), all terms in

the $t = 0$ tau function expansion vanish whereas the $t = 1$ series produces a nontrivial solution to Painlevé VI. This contradiction suggests that $\chi_{\mathrm{per}}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ is a nowhere vanishing holomorphic function of both σ_{0t} and σ_{1t}. Making an additional assumption of nice behavior at infinity, one concludes that $\chi_{\mathrm{per}}(\vec{\theta}; \sigma_{0t}, \sigma_{1t})$ is in fact independent of the last two parameters.

The remaining dependence on $\vec{\theta}$ can be strongly constrained using P_{VI} solutions known in closed form and depending on continuous parameters. For instance, such solutions are known for an infinite number of affine hyperplanes in the $\vec{\theta}$-space (Riccati solutions). In CFT language, they correspond to conformal blocks involving degenerate fields or (by Regge-Okamoto symmetry (4.4)) to conformal blocks of free-field exponentials with screening insertions. The simplest nontrivial example of this type is given by

$$\tau(t) = t^{2\theta_0\theta_1}(1-t)^{2\theta_t\theta_1}{}_2F_1\left(1-2\theta_\infty, 2\theta_t; 2\theta_0+2\theta_t; t\right), \tag{4.43}$$

where $\vec{\theta}$ are subject to the constraint $\theta_0+\theta_t+\theta_1+\theta_\infty = 1$ and $\vec{\sigma} = (\theta_0+\theta_t, \theta_1+\theta_t, \theta_0+\theta_1)$. More general formulas can be found in Subsection 2.8.2. Some further examples come from the continuous algebraic families of P_{VI} transcendents living on affine $\vec{\theta}$-subspaces of dimensions 1 and 2.

It turns out that the connection coefficients computed directly in these particular cases are reproduced by the simplest possible ansatz $\chi_{\mathrm{per}}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}) = 1$. It is further supported by numerical computations with random values of $\vec{\theta}$, $\vec{\sigma}$ and analytic checks using exceptional algebraic solutions of P_{VI} that we are going to discuss in the next subsection. This transforms Lemma 4.4 into the following

Conjecture 4.5. *Connection coefficient (4.13) for the generic Painlevé VI tau function has the following expression in terms of monodromy data:*

$$\bar{\chi}_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01}) = \tag{4.44}$$
$$= \prod_{\epsilon,\epsilon'=\pm} G\begin{bmatrix} 1+\epsilon\sigma_{1t}+\epsilon'\theta_t-\epsilon\epsilon'\theta_1, 1+\epsilon\sigma_{1t}+\epsilon'\theta_0-\epsilon\epsilon'\theta_\infty \\ 1+\epsilon\sigma_{0t}+\epsilon'\theta_t+\epsilon\epsilon'\theta_0, 1+\epsilon\sigma_{0t}+\epsilon'\theta_1+\epsilon\epsilon'\theta_\infty \end{bmatrix} \prod_{\epsilon=\pm} \frac{G(1+2\epsilon\sigma_{0t})}{G(1+2\epsilon\sigma_{1t})} \prod_{k=1}^{4} \frac{\hat{G}(\omega_+ + \nu_k)}{\hat{G}(\omega_+ + \lambda_k)}$$

It is worth noting that the formula (4.44) possesses a non-obvious symmetry: its right side remains invariant upon sign reversal of any of the parameters $\theta_{0,t,1,\infty}$, σ_{0t} and σ_{1t}.

4.3.3 Algebraic checks

Algebraic solutions of Painlevé VI provide an instructive way to test the general answer (4.44) for the connection coefficient. For example, P_{VI} with parameters $\vec{\theta} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}\right)$

admits the solution (obtained from the Solution 30 of [102])

$$\tau(t(s)) = \frac{(s^4-1)^{-\frac{1}{8}}(s^4+1)^{-\frac{5}{192}}(i+(1-i)s+s^2)}{s^{\frac{1}{32}}(1+2s-s^2)^{\frac{7}{24}}(s^2+2s-1)^{\frac{1}{24}}(1+6s^2+s^4)^{\frac{1}{6}}}\times$$
$$\times\left[\frac{(s^2+i)(1-2is+s^2)}{(s^2-i)(1+2is+s^2)}\right]^{\frac{1}{8}}, \tag{4.45}$$
$$t(s) = -\frac{(1+s^2)^2(1-6s^2+s^4)^3}{32s^2(1+s^4)^3}. \tag{4.46}$$

This solution has 16 branches. Let us choose one of them, e.g. corresponding to the interval $s \in (1, \sqrt{2}+1)$. It is straightforward to check that the latter interval is bijectively mapped by (4.46) to $t \in (0,1)$. In particular,

$$t\left(s \to \sqrt{2}+1\right) \sim \frac{32}{27}\left(10-7\sqrt{2}\right)\left(\sqrt{2}+1-s\right)^3, \tag{4.47a}$$
$$1-t\left(s\to 1\right) \sim 8\left(s-1\right)^2. \tag{4.47b}$$

Expanding the tau function (4.45) near the endpoints and using (4.47), one also finds that

$$\tau\left(t\to 0\right) = 2^{\frac{197}{576}}\cdot 3^{\frac{1}{64}}\cdot e^{-i\phi}\cdot t^{-\frac{7}{72}}\left[1+O\left(t^{\frac{2}{3}}\right)\right], \tag{4.48a}$$
$$\tau\left(t\to 1\right) = 2^{\frac{5}{64}}\cdot(1-t)^{-\frac{1}{16}}\left[1+\frac{3}{8}e^{-\frac{i\pi}{4}}(1-t)^{\frac{1}{2}}+O\left(1-t\right)\right], \tag{4.48b}$$

where the phase ϕ is a non-rational multiple of π, explicitly given by

$$\phi = \frac{1}{8}\left(\pi - \arctan\frac{7}{4\sqrt{2}}\right).$$

The asymptotics (4.48) corresponds to monodromy exponents $\vec{\sigma} = \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{6}\right)$. The connection coefficient $\bar{\chi}_{01}$ can be computed directly from these formulas and the relation (4.14). To show that the answer obtained in this way coincides with our expression (4.44), it suffices to demonstrate the following identity:

$$G\left[\begin{array}{c}\frac{4}{3},\frac{1}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{1}{8},\frac{9}{8},\frac{11}{8},\frac{11}{8}\\ \frac{1}{2},\frac{3}{2},\frac{5}{3},\frac{5}{6},\frac{5}{6},\frac{13}{24},\frac{17}{24},\frac{23}{24},\frac{43}{24}\end{array}\right]\prod_{k=1}^{4}\frac{\hat{G}(\omega_+ + \nu_k)}{\hat{G}(\omega_+ + \lambda_k)} = 2^{-\frac{19}{72}}\cdot 3^{-\frac{1}{64}}\cdot e^{i\phi}, \tag{4.49}$$

where

$$\omega_+ = \frac{5}{48} + \frac{1}{4\pi i}\ln\left(2-\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}\right),$$
$$(\nu_1,\nu_2,\nu_3,\nu_4) = \left(\frac{2}{3},\frac{19}{24},\frac{7}{8},\frac{3}{4}\right),$$
$$(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \left(\frac{9}{8},\frac{11}{12},\frac{25}{24},0\right).$$

The identity (4.49) is readily confirmed numerically by comparison of the first 500 significant digits at both sides. Similar checks of (4.44) have been done for more than 50 branches of about 20 exceptional algebraic P_{VI} solutions.

4.4 Fusion kernel at $c = 1$

4.4.1 Relation to connection coefficient

It is clear from the form of Painlevé VI tau function expansions (4.11a)–(4.11b) that the connection coefficient (4.12) is a close relative of the fusion matrix (4.5) for $c = 1$ conformal blocks. Let us now try to spell out their relation more explicitly.

First observe that

$$\mathcal{F}_1\left[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma_{0t}\right](t) = \oint_{C_\Lambda}\frac{ds_{0t}}{2\pi i s_{0t}}\frac{\chi_0^{-1}\tau(t)}{C\left[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma_{0t}\right]} =$$

$$= \oint_{C_\Lambda}\frac{ds_{0t}}{2\pi i s_{0t}}\sum_{n\in\mathbb{Z}}\bar{\chi}_{01}(\vec{\theta};\sigma_{0t},\sigma_{1t}+n;p_{01})\,\mathcal{F}_1\left[\begin{array}{cc}\theta_0 & \theta_t\\ \theta_\infty & \theta_1\end{array};\sigma_{1t}+n\right](1-t)\,, \tag{4.51}$$

where C_Λ denotes the circle $|s_{0t}| = e^{-2\pi\Lambda}$ in the complex s_{0t}-plane and the last equality is obtained by combining (4.11a)–(4.11b) with the functional relation (4.30b).

The next step is to transform the integral over s_{0t} into an integral over σ_{1t}. The latter was considered so far as a function of σ_{0t}, s_{0t} implicitly determined by (2.30) — recall that s_{0t} parameterizes the pairs (p_{1t}, p_{01}) at fixed p_{0t}. It is not difficult to show (cf (3.50a)) that

$$p_{1t} = \frac{\alpha_+(\vec{\theta},\sigma_{0t})s_{0t} + \alpha_-(\vec{\theta},\sigma_{0t})s_{0t}^{-1} + (p_{0t}\omega_{01} - 2\omega_{1t})}{p_{0t}^2 - 4}, \tag{4.52}$$

with $\alpha_\pm(\vec{\theta},\sigma_{0t})$ given by

$$\alpha_\pm(\vec{\theta},\sigma_{0t}) = 16\prod_{\epsilon=\pm}\sin\pi\left(\theta_t \mp \sigma_{0t} + \epsilon\theta_0\right)\sin\pi\left(\theta_1 \mp \sigma_{0t} + \epsilon\theta_\infty\right).$$

Observe that, as $\Lambda \to \infty$, p_{1t} becomes very large, which means that either (a) $\Im\sigma_{1t} \sim \Lambda$ so that $e^{2\pi i\sigma_{1t}} \sim s_{0t}$ or (b) $\Im\sigma_{1t} \sim -\Lambda$, in which case $e^{2\pi i\sigma_{1t}} \sim s_{0t}^{-1}$.

Since Λ may indeed be chosen sufficiently large, (4.52) implies that σ_{1t}-integration contour may be chosen as a segment $[\sigma_{1t}^*, \sigma_{1t}^* + 1]$ with $\Im\sigma_{1t}^*$ being sufficiently large for all singularities of the integrand in (4.51) to be located below the line $\mathbb{R} + i\Lambda$. The sum over

n in (4.51) then produces an integral over the whole line so that

$$\mathcal{F}_1\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \sigma_{0t}\right](t) = \int_{\mathbb{R}+i\Lambda} F\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \begin{array}{c} \sigma_{1t} \\ \sigma_{0t} \end{array}\right] \mathcal{F}_1\left[\begin{array}{cc} \theta_0 & \theta_t \\ \theta_\infty & \theta_1 \end{array}; \sigma_{1t}\right](1-t)\, d\sigma_{1t}, \tag{4.53}$$

with

$$F\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \begin{array}{c} \sigma_{1t} \\ \sigma_{0t} \end{array}\right] = \mu(\vec{\theta}, \vec{\sigma}) \cdot \bar{\chi}_{01}(\vec{\theta}; \sigma_{0t}, \sigma_{1t}; p_{01}) \tag{4.54}$$

and $\mu(\vec{\theta}, \vec{\sigma}) = \dfrac{1}{2\pi i} \dfrac{\partial \ln s_{0t}}{\partial \sigma_{1t}}$. The root p_{01} of the Jimbo-Fricke relation is chosen as to reproduce the asymptotics $s_{0t} \to 0$ as $\operatorname{Im} \sigma_{1t} \to +\infty$.

Lemma 4.1 indicates that the prefactor $\mu(\vec{\sigma}, \vec{\theta})$ from the last relation can be rewritten in a symmetric form

$$\mu(\vec{\theta}, \vec{\sigma}) = \frac{i}{2\pi^2} \frac{\partial^2}{\partial \sigma_{0t} \partial \sigma_{1t}} \operatorname{Vol}(\mathcal{T}), \tag{4.55}$$

where $\operatorname{Vol}(\mathcal{T})$ denotes the tetrahedral volume of Section 4.2.3. One can also obtain an explicit trigonometric expression

$$\mu(\vec{\theta}, \vec{\sigma}) = -\frac{4 \sin 2\pi\sigma_{0t} \sin 2\pi\sigma_{1t}}{\sqrt{\det \mathcal{G}}}, \tag{4.56}$$

where $\mathcal{G}$ denotes the Gram matrix defined by (4.17) and the branch of the square root is chosen so that $\sqrt{\det \mathcal{G}} = q_{01}$.

The formula (4.53) is a $c = 1$ analog of the fusion relation (4.5). One apparent difference is that here it becomes more convenient to integrate over a complex contour in the momentum space rather than $\mathbb{R}^+$. The integral kernel $F\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \begin{array}{c} \sigma_{1t} \\ \sigma_{0t} \end{array}\right]$ is the fusion matrix for $c = 1$ conformal blocks. We emphasize that, up to an elementary prefactor, it coincides with Painlevé VI connection coefficient explicitly given by (4.44). In contrast to Ponsot-Teschner result for generic central charge, the formula (4.54) for $c = 1$ fusion kernel does not involve any integration.

4.4.2 Numerics

The expression (4.54) for the fusion kernel can in principle be checked numerically. Fix, for instance,

$$\begin{pmatrix} \theta_0 \\ \theta_t \\ \theta_1 \\ \theta_\infty \end{pmatrix} = \begin{pmatrix} 0.49 + 0.42i \\ 0.64 - 0.31i \\ 0.28 - 0.35i \\ 0.19 + 0.47i \end{pmatrix}, \qquad \sigma_{0t} = 0.26 + 0.45i.$$

To specify the integration contour in (4.53), one needs to analyze the singularities of $\bar{\chi}_{01}$ with respect to σ_{1t}. The poles (and zeros) may only be located at $\epsilon\theta_0 + \epsilon'\theta_\infty + \mathbb{Z}$, $\epsilon\theta_t + \epsilon'\theta_1 + \mathbb{Z}$, $\mathbb{Z}/2$ with $\epsilon, \epsilon' = \pm$. One also has square root branch points corresponding to zeros of q_{01}; in our case they are given by $\pm(0.40-0.37i)+\mathbb{Z}$, $\pm(0.11+0.49i)+\mathbb{Z}$. This results into the singularity structure shown in Fig. 15. Hence in the integration contour $\mathbb{R} + i\Lambda$ on the right of (4.53) one may set $\Lambda = 1$.

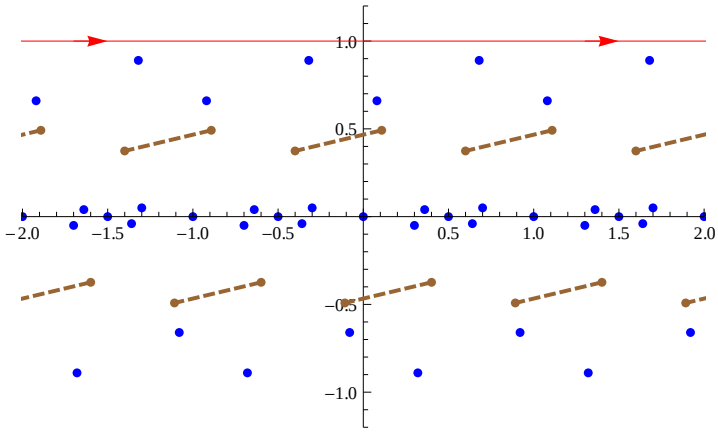


Fig. 15: Singularities of $\bar{\chi}_{01}$ in the complex σ_{1t}-plane: the dots represent zeros and poles, the dashed lines correspond to branch cuts

In practice, the integrand decays very rapidly so that one can approximate the integral by the Riemann sum over uniform partition of the segment $[-1.5 + i, 2.5 + i]$ into 400 subintervals. Numerical values of conformal block functions were obtained by taking several first terms in their series expansions (30 on the left and 15 on the right of (4.53)).

Fig. 16 shows the graphs of the left and right sides of (4.53), as functions of $t \in (0, 1)$, calculated in this way.

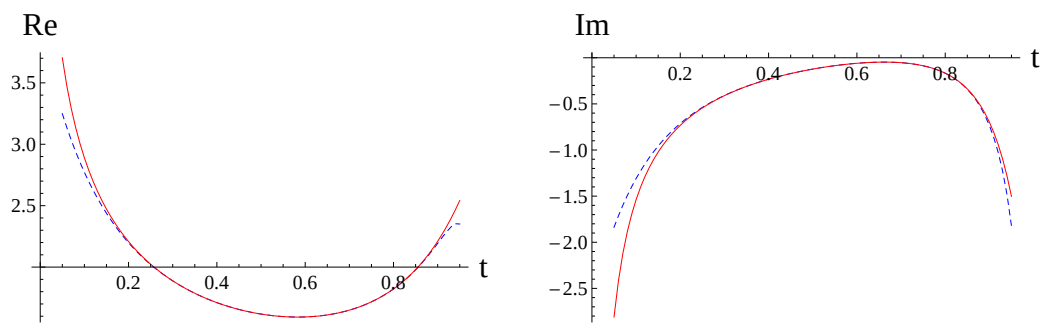


Fig. 16: Real and imaginary parts of the l.h.s. (solid curve) and r.h.s. (dashed curve) of (4.53)

Discrepancies at the endpoints are related to the fact that efficient appoximation of conformal block $\mathcal{F}_1(t)$ by truncated series requires taking into account more and more terms as t approaches 1.

4.4.3 Further checks: Ashkin-Teller conformal block

As $\vec{\theta} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, the Jimbo-Fricke cubic $W(p_{0t}, p_{1t}, p_{01}) = 0$ reduces to

$$p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_{0t}p_{1t}p_{01} = 4.$$

Considering σ_{0t} and σ_{1t} as fixed, one obtains the two possible solutions for p_{01}:

$$p_{01} = -2\cos 2\pi(\sigma_{0t} \pm \sigma_{1t})$$

which implies that $q_{01} = \pm 4 \sin 2\pi\sigma_{0t} \sin 2\pi\sigma_{1t}$ and $s_{0t} = -e^{\mp 2\pi i \sigma_{1t}}$. The parameter ω_+ which appears in the connection formula (4.44) is then determined by

$$e^{2\pi i \omega_+} = -e^{-\pi i(\sigma_{0t}+\sigma_{1t})}\Big(\cos\pi(\sigma_{0t} - \sigma_{1t}) \mp i \sin\pi(\sigma_{0t} + \sigma_{1t})\Big)^{\pm 1}.$$

Upon substitution of this expression into the connection formula (4.44), the latter rather nontrivially simplifies to

$$\bar{\chi}_{01}(\sigma_{0t}, \sigma_{1t}; p_{01}) = 2^{4\sigma_{0t}^2 - 4\sigma_{1t}^2} e^{\pm 2\pi i \sigma_{0t}\sigma_{1t}}.$$

The origin of this simplification is that here $\mu(\vec{\sigma}) = \mp 1$, cf e.g. (4.55) and (4.22).

As explained in the previous subsections, the formulas (4.53)–(4.54) come with a prescription for the choice of p_{01}. We should select from its two possible values the one characterized by vanishing of s_{0t} as $\operatorname{Im}\sigma_{1t} \to \infty$. This corresponds to picking out the lower sign in the above formulas, and finally yields the folowing expression for the fusion matrix:

$$F\left[\begin{matrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{matrix} ; \begin{matrix} \sigma_{1t} \\ \sigma_{0t} \end{matrix}\right] = 2^{4\sigma_{0t}^2 - 4\sigma_{1t}^2} e^{-2\pi i \sigma_{0t}\sigma_{1t}}. \tag{4.57}$$

Since the integral in (4.53) is calculated over the whole line, the last expression is clearly equivalent to the fusion kernel (4.10) for Ashkin-Teller conformal blocks.

5. Conclusion and perspectives

In this habilitation thesis, we reviewed a progress achieved in the last few years in understanding the relation between the two-dimensional conformal field theory and isomonodromic deformations. Referring to the earlier quote of H. Poincaré, the 2D CFT seems to be a bridge connecting Painlevé island to the land of representation theory.

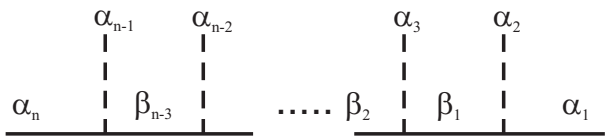
The solution of the isomonodromic Riemann-Hilbert problem for rank 2 Fuchsian systems on $\mathbb{P}^1$ and the associated tau function were related to Virasoro conformal blocks at $c = 1$. Combined with the AGT correspondence, this relation provides, in particular, explicit combinatorial series representations of generic Painlevé VI, V and III transcendents. It also allows to solve asymptotic problems such as calculation of the connection coefficient of the P_{VI} tau function and the description of fusion transformations for $c = 1$ conformal blocks.

What about the irregular case? In [77], we studied a particular example of the most degenerate $\mathrm{P}_{\mathrm{III}_3}$ equivalent to the radial sine-Gordon equation. Irregular conformal blocks which appear in the short-distance expansion of the corresponding tau function have the simplest possible form: they coincide with the norms of Gaiotto states annihilating all $L_{n\geq 2}$ and diagonalizing L_1. The main experimental surprise is that the *long-distance* expansion of the tau function, which cannot be obtained from P_{VI} conformal expansions, exhibits very similar periodicity with respect to monodromy data. We expect this periodicity to be a general phenomenon in isomonodromy problems which is somehow reminiscent of the generalized Fisher-Hartwig conjecture in the theory of Toeplitz determinants [9, 34]. It suggests the existence of new distinguished bases in spaces of irregular conformal blocks, and it would be very interesting to elucidate its algebraic origin. Furthermore, as shown in [77], this periodicity allows to address the connection problems for the corresponding tau functions.

It can be expected on general grounds that the isomonodromy problems for Fuchsian systems of higher rank N are related to $\mathfrak{sl}_N$ Toda CFT and conformal blocks of $\mathcal{W}_N$ algebras [47, 159]. These algebras arise as a quantization of the second Poisson structure of the N-KdV hierarchies. On the way of making the correspondence as precise as for $N = 2$ one meets a number of conceptual obstacles. On the CFT side, conformal blocks are not completely fixed by the $\mathcal{W}$-invariance. On the isomonodromic side, the solutions of the "elementary block" problems, such as the Riemann-Hilbert problem on $C_{0,3}$ or Jimbo asymptotic problem on $C_{0,4}$, are unknown for higher rank.

However, there are interesting special cases where generalization of our formalism seems to be possible and almost straightforward. They concern linear systems of special spectral

type, where two of n matrices A_ν in (2.1) are generic and all the others have rank 1. For instance, the $n = 3$ Riemann-Hilbert problem is then solvable in ${}_NF_{N-1}$ hypergeometric functions and the tau function asymptotics on $C_{0,4}$ could in principle be obtained from an interplay between two Fuchsian systems of this type. It can be expected that this situation corresponds to $\mathcal{W}_N$ conformal blocks of the form



where the momenta α, β now live in the dual of the Cartan subalgebra of $\mathfrak{sl}_N$. The dashed lines represent the so-called semi-degenerate fields whose momenta are proportional to a fundamental weight of $\mathfrak{sl}_N$. Fusion and braiding transformations of such conformal blocks with respect to additional completely degenerate insertions can be described using the results of [48, 49], see also very recent paper [23]. It is therefore plausible that the whole program of our Section 3 can be carried out in this case. Note that the above special $\mathcal{W}_N$ conformal blocks are precisely those for which the AGT correspondence produces combinatorial representations which furthermore have been proved in [50].

A possible way to prove relations of type (2.33) between the Painlevé tau functions and $c = 1$ conformal blocks is to show that the latter satisfy certain bilinear differential-difference identities [76]. This was accomplished in a recent work [13] by M. Bershtein and A. Shchechkin. The idea of [13] is to use an embedding $\mathsf{Vir} \oplus \mathsf{Vir} \subset \mathsf{F} \oplus \mathsf{NSR}$ of two copies of Virasoro algebra into Neveu-Schwarz-Ramond algebra extended by free fermions. One of the interesting features of this setup is that it yields a one-parameter deformation of the necessary bilinear relations. The role of the deformation parameter is played by the NSR central charge which determines the (in general, unequal) charges of two Virasoro algebras. Painlevé case corresponds to a particular point $c_{\mathsf{NSR}} = c_{\mathsf{Vir}_{1,2}} = 1$. An intriguing question is: what is the appropriate deformation of the Painlevé equations and tau functions?

There are further connections between the isomonodromic deformation problem and 2D CFT. In particular, the semiclassical ($c \to \infty$) limit of the null-vector decoupling equations in Liouville theory yields Hamilton-Jacobi-like equations that define the Hamiltonians generating the isomonodromic deformation flows [146, 113, 101]. It seems remarkable that there exist relations between Virasoro conformal blocks and isomonodromic tau functions for both $c = 1$ and $c \to \infty$; a good explanation remains to be found.

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