# Hecke algebras, generalisations and representation theory <br> Maria Chlouveraki 

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Mémoire des travaux en vue de l'obtention d'une HABILITATION À DIRIGER DES RECHERCHES<br>Spécialité : Mathématiques<br>Présentée par<br>MARIA CHLOUVERAKI

# Algèbres de Hecke, généralisations <br> et théorie des représentations 

## Hecke algebras, generalisations and representation theory

Soutenue le 25 novembre 2016 devant le jury composé de
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## Introduction en Français

Les algèbres de Iwahori-Hecke associées aux groupes de Weyl apparaissent naturellement dans l'étude des groupes réductifs finis comme des algèbres d'endomorphismes de la représentation de permutation par rapport à un sous-groupe de Borel. Elles peuvent aussi être définies indépendamment comme déformations des algèbres de groupe des groupes de Coxeter finis. L'objectif de ce mémoire est d'étudier certains aspects de la théorie des représentations des algèbres de Iwahori-Hecke et la façon dont elles se généralisent dans le cas des :

- algèbres de Hecke cyclotomiques, qui sont obtenues comme déformations des algèbres de groupe des groupes de réflexions complexes,
- algèbres de Ariki-Koike, qui sont obtenues comme généralisations des algèbres de Iwahori-Hecke de types $A$ et $B$,
- algèbres de Yokonuma-Hecke, qui sont obtenues lors de l'étude des groupes réductifs finis comme des algèbres d'endomorphismes de la représentation de permutation par rapport à un sous-groupe unipotent maximal.

Au cours de ce mémoire, nous allons aussi étudier une autre famille d'algèbres associées aux groupes de réflexions complexes, les algèbres de Cherednik rationnelles, dont la théorie des représentations a beaucoup de liens avec la théorie des représentations des algèbres de Hecke.

Les aspects de la théorie des représentations de ces algèbres sur lesquelles nous allons nous concentrer seront la paramétrisation et description des représentations irréductibles dans les cas semisimple et nonsemisimple, les blocs, la structure d'algèbre symétrique et la détermination de la matrice de décomposition associée à une spécialisation.

Les groupes de Coxeter finis sont des groupes finis de matrices réelles qui sont engendrés par des réflexions. Ils incluent les groupes de Weyl, qui sont des objets fondamentaux pour la classification des algèbres de Lie simples sur $\mathbb{C}$ et des groupes algébriques simples. Les algèbres de IwahoriHecke associées aux groupes de Weyl apparaissent naturellement comme des algèbres d'endomorphismes des représentations induites dans l'étude des groupes réductifs finis. Elles peuvent aussi être définies indépendamment comme déformations des algèbres de groupes des groupes de Coxeter finis, où la déformation dépend d'un paramètre $q$ et une fonction de poids $L$. Pour $q=1$, nous obtenons l'algèbre de groupe. Pour un groupe de Coxeter fini $W$, nous désignerons par $\mathcal{H}_{q}(W, L)$ l'algèbre de Iwahori-Hecke associée.

Lorsque $q$ est une indéterminée, l'algèbre de Iwahori-Hecke $\mathcal{H}_{q}(W, L)$ est semisimple. Suivant le théorème de déformation de Tits, il existe une bijection entre l'ensemble des représentations irréductibles de $\mathcal{H}_{q}(W, L)$ et l'ensemble $\operatorname{Irr}(W)$ des représentations irréductibles de $W$. Grâce à cette bijection, Lusztig associe à chaque représentation irréductible de $W$ un entier qui dépend de $L$, en définissant ainsi la fameuse fonction $a$. Ici nous définissons la fonction $a$ en utilisant la structure d'algèbre symétrique et les éléments de Schur de $\mathcal{H}_{q}(W, L)$. La fonction $a$ intervient dans la définition de Lusztig des familles de caractères, une partition de $\operatorname{Irr}(W)$ qui joue un rôle central dans l'organisation des familles de caractères unipotents dans le cas des groupes réductifs finis.

La théorie de Kazhdan-Lusztig est une clef pour la compréhension de la théorie des représentations de l'algèbre de Iwahori-Hecke $\mathcal{H}_{q}(W, L)$. Il existe une base spéciale de $\mathcal{H}_{q}(W, L)$, appelé la base de KazhdanLusztig, qui nous permet de définir des cellules de Kazhdan-Lusztig pour $\mathcal{H}_{q}(W, L)$, un certain ensemble de classes d'équivalence sur $W$. Les cellules de Kazhdan-Lusztig aident à construire les représentations
de $\mathcal{H}_{q}(W, L)$. Elles permettent aussi une deuxième définition équivalente, plus combinatoire, des familles de caractères de Lusztig.

Lorsque $q$ se spécialise en un nombre complexe $\eta$ différent de zéro, et plus spécifiquement en une racine de l'unité, l'algèbre de Iwahori-Hecke spécialisée $\mathcal{H}_{\eta}(W, L)$ n'est pas nécessairement semisimple et nous n'avons plus une bijection entre ses représentations irréductibles et $\operatorname{Irr}(W)$. Nous obtenons donc une matrice de décomposition qui décrit comment les représentations irréductibles de l'algèbre semisimple se décomposent après la spécialisation. Un ensemble basique canonique est un sous-ensemble de $\operatorname{Irr}(W)$ en bijection avec les représentations irréductibles de $\mathcal{H}_{\eta}(W, L)$ (et ainsi un ensemble qui paramètre les colonnes de la matrice de décomposition) avec de bonnes propriétés. Ces bonnes propriétés impliquent que la matrice de décomposition a une forme unitriangulaire inférieure alors que la fonction $a$ croît (au sens large) le long des colonnes. Les ensembles basiques canoniques ont été définis par Geck et Rouquier GeRo, qui ont prouvé leur existence dans quelques cas en utilisant la théorie de Kazhdan-Lusztig. Grâce à un travail collectif, il est maintenant prouvé que des ensembles basiques canoniques existent et sont explicitement décrits pour tous les groupes de Coxeter finis et tout choix de $L$ (en caractéristique $0)$.

Les groupes de Coxeter finis sont des cas particuliers de groupes de réflexions complexes, c'est-à-dire des groupes finis de matrices complexes engendrés par des "pseudo-réflexions". La classification des groupes de réflexions complexes est due à Shephard et Todd ShTo : si $W$ est un groupe de réflexions complexe (irréductible), alors soit $W$ appartient à la série infinie $G(l, p, n)$ soit $W$ est un des 34 groupes exceptionnels $G_{4}, \ldots, G_{37}$ (voir Théorème 3.1). D'importants développements durant les deux dernières décennies ont suggéré que les groupes de réflexions complexes jouent un rôle crucial, mais seulement partiellement compris, en théorie de représentations, et pourraient même devenir aussi importants que les groupes de Coxeter finis dans l'étude d'autres structures mathématiques. En effet, ils se comportent de façon tellement analogue aux groupes de réflexions réels que Broué, Malle et Michel [BMM1 ont conjecturé qu'ils pourraient jouer le rôle de groupes de Weyl pour des objets, toujours mystérieux, qui généralisent les groupes réductifs finis. Ces objets s'appellent "Spetses".

Broué, Malle et Rouquier [BMR ont défini des algèbres de Hecke pour les groupes de réflexions complexes comme déformations de leurs algèbres de groupe. Dans le cas des groupes de réflexions complexes de type $G(l, 1, n)$, elle coïncident avec les algèbres de Ariki-Koike, déjà définies par Ariki et Koike ArKo comme généralisations des algèbres de Iwahori-Hecke de types $A$ et $B$. Une théorie des cellules de Kazhdan-Lusztig généralisée pour ces algèbres, connues sous le nom d'algèbres de Hecke cyclotomiques, permettrait de se diriger vers une définition des Spetses. Malheureusement, nous n'avons pas de base de Kazhdan-Lusztig pour les groupes de réflexions complexes. Néanmoins, nous pouvons définir des familles de caractères en utilisant la définition de Rouquier : dans Ro1, Rouquier a donné une définition alternative pour les familles de caractères de Lusztig en prouvant que, dans le cas des groupes de Weyl, elles coïncident avec les blocs de l'algèbre de Iwahori-Hecke sur un certain anneau, qui s'appelle l'anneau de Rouquier. Cette définition se généralise naturellement au cas des groupes de réflexions complexes et leurs algèbres de Hecke cyclotomiques, en produisant les familles de Rouquier. Ces familles sont maintenant déterminées pour toutes les algèbres de Hecke cyclotomiques de tous les groupes de réflexions complexes, voir Ch3, Ch4, Ch5. Un rôle-clef dans cette détermination est joué par la structure d'algèbre symétrique (toujours conjecturée dans certains cas) des algèbres de Hecke cyclotomiques et les éléments de Schur correspondants.

En utilisant les éléments de Schur, nous pouvons aussi définir une fonction $a$ et des ensembles basiques canoniques pour les algèbres de Hecke cyclotomiques. Même s'il n'existe pas de théorie de KazhdanLusztig dans le cas complexe, il est maintenant prouvé que des ensembles basiques existent pour les groupes de la série infinie $G(l, p, n)$ et pour quelques groupes exceptionnels. Afin d'obtenir des ensembles basiques canoniques pour les algèbres de Ariki-Koike, Geck et Jacon ont utilisé le théorème d'Ariki sur la catégorification des représentations des algèbres de Hecke et le travail de Uglov sur les bases canoniques pour des espaces de Fock de plus haut niveau GeJa1, Ja2, Ja3, GeJa2]. Le résultat pour $G(l, p, n)$ est déduit du résultat pour $G(l, 1, n)$ avec l'utilisation de la théorie de Clifford GenJa, ChJa2.

Les algèbres de réflexions symplectiques ont été introduites par Etingof et Ginzburg EtGi] pour l'étude des résolutions symplectiques de l'espace d'orbites $V / G$, où $V$ est un espace vectoriel complexe symplectique et $G$ est un groupe de réflexions symplectique agissant sur $V$, i.e., $G$ est un groupe fini engendré par des réflexions symplectiques. Un groupe de réflexions complexe $W \subset \mathrm{GL}(\mathfrak{h})$, où $\mathfrak{h}$ est un espace vectoriel complexe, peut être vu comme un groupe de réflexions symplectique agissant sur $V=\mathfrak{h} \oplus \mathfrak{h}^{*}$. Les algèbres de réflexions symplectiques associées aux groupes de réflexions complexes
sont connues sous le nom d'algèbres de Cherednik rationnelles. Elles dépendent de façon cruciale d'un paramètre $t$, et leur théorie des représentations varie beaucoup selon si $t$ est nul ou non.

Si $t \neq 0$, il existe une catégorie de représentations de l'algèbre de Cherednik rationnelle importante, la catégorie $\mathcal{O}$, et un foncteur exact, le foncteur de Knizhnik-Zamalodchikov ou simplement KZ, de $\mathcal{O}$ vers la catégorie de représentations d'une certaine algèbre de Hecke cyclotomique spécialisée $\mathcal{H}_{\eta}(W)$ (la spécialisation dépend du choix des paramètres de l'algèbre de Cherednik rationnelle; toute algèbre de Hecke spécialisée peut être obtenue de cette façon). La catégorie $\mathcal{O}$ est une catégorie de plus haut poids, et elle contient un ensemble des modules standards $\{\Delta(E) \mid E \in \operatorname{Irr}(W)\}$, un ensemble des modules simples $\{\mathrm{L}(E) \mid E \in \operatorname{Irr}(W)\}$ et une matrice de décomposition qui décrit le nombre de fois que $\mathrm{L}(E)$ apparaît dans la série de composition de $\Delta\left(E^{\prime}\right)$ pour $E, E^{\prime} \in \operatorname{Irr}(W)$. L'exactitude de KZ nous permet de déduire la matrice de décomposition de $\mathcal{H}_{\eta}(W)$ de la matrice de décomposition de la catégorie $\mathcal{O}$. En utilisant cette propriété, nous avons prouvé dans [GG] l'existence des ensembles basiques canoniques pour tous les groupes de Coxeter finis et pour les groupes de réflexions complexes de type $G(l, 1, n)$ (en caractéristique $0)$. En particulier, nous avons montré que $E$ appartient à l'ensemble basique canonique pour $\mathcal{H}_{\eta}(W)$ si et seulement si $\mathrm{KZ}(\mathrm{L}(E)) \neq 0$. Notre preuve de leur existence est assez générale et n'utilise pas le théorème d'Ariki pour le type $G(l, 1, n)$. Néanmoins la description explicite des ensembles basiques canoniques dans ces cas par des travaux antérieurs permet de déterminer quels modules simples sont envoyés sur zéro par le foncteur KZ. Nous avons aussi démontré que les images des modules standards via le foncteur KZ sont isomorphes aux modules cellulaires des algèbres de Hecke.

En ce qui concerne le cas $t=0$, Gordon GO a introduit et étudié un quotient de dimension finie de l'algèbre de Cherednik rationnelle, l'algèbre de Cherednik rationnelle restreinte, dont les modules simples sont paramétrés par $\operatorname{Irr}(W)$. La décomposition de cette algèbre en blocs induit une partition de $\operatorname{Irr}(W)$, connue sous le nom de partition de Calogero-Moser. Nous pensons qu'il existe un lien entre la partition de Calogero-Moser et les familles de caractères, d'abord suggéré par Gordon et Martino GoMa] pour le type $B$. Dans tous les cas étudiés jusqu'à maintenant, la partition en familles de Rouquier (pour une algèbre de Hecke cyclotomique choisie convenablement) rend la partition de Calogero-Moser plus fine ("la conjecture de Martino"), alors que pour les groupes de Coxeter finis les deux partitions coïncident. Ce lien n'est pas encore expliqué, étant donné qu'il n'y a pas de connexion évidente entre les algèbres de Hecke et les algèbres de Cherednik rationnelles à $t=0$. Motivés par cette idée, et afin de pouvoir construire une théorie des cellules de Kazhdan-Lusztig généralisée, Bonnafé et Rouquier ont utilisé la partition de Calogero-Moser pour développer une théorie des "cellules de Calogero-Moser" qui peut être appliquée à tous les groupes de réflexions complexes BoRo1, BoRo2. Il reste à voir les fruits de cette approche très récente.

Les algèbres de Yokonuma-Hecke ont été introduites par Yokonuma Yo dans le contexte des groupes réductifs finis comme qénéralisations des algèbres de Iwahori-Hecke. Elle sont définies comme des algèbres d'endomorphismes de la représentation de permutation par rapport à un sous-groupe unipotent maximal, et peuvent ainsi être vues comme des cas particuliers des algèbres de Hecke unipotentes. Nous allons nous intéresser ici aux algèbres de Yokonuma-Hecke de type $A$, qui peuvent être aussi définies indépendamment comme déformations des algèbres de groupe des groupes de réflexions complexes de type $G(l, 1, n)$. Nous obtenons ainsi un autre type d'algèbres de Hecke associées à $G(l, 1, n)$, différentes des algèbres de ArikiKoike. Ayant étudié la théorie des représentations des algèbres de Yokonuma-Hecke dans [ChPdA1], nous avons découvert beaucoup de similarités avec la théorie des représentations des algèbres de IwahoriHecke de type $A$; ces similarités sont expliquées par un résultat de Lusztig Lu7, qui a démontré que les algèbres de Yokonuma-Hecke sont isomorphes à des sommes directes des algèbres matricielles sur des produits tensoriels de certaines sous-algèbres des algèbres de Iwahori-Hecke.

Nous avons aussi introduit et étudié beaucoup de nouvelles algèbres intéressantes dans le contexte des algèbres de Yokonuma-Hecke, qui sont obtenues comme généralisations (ou, comme elles sont appelées dans JuLa5, "pondérisations") d'algèbres importantes dans le contexte des algèbres de Iwahori-Hecke. Tout d'abord, nous avons les algèbres de Yokonuma-Hecke affines et cyclotomiques, qui généralisent respectivement les algèbres de Hecke affines de type $A$ et les algèbres de Ariki-Koike [ChPdA2]. En fait, nous avons montré dans [ChSe que les algèbres de Yokonuma-Hecke affines apparaissent aussi naturellement dans l'étude des groupes réductifs $p$-adiques, avec une construction analogue à la construction de Yokonuma, alors que les algèbres de Yokonuma-Hecke cyclotomiques produisent en même temps les algèbres de Ariki-Koike et les algèbres de Yokonuma-Hecke de type $A$, comme des quotients et des cas particuliers. De plus, nous avons trois généralisations possibles de l'algèbre de Temperley-Lieb : l'algèbre de Yokonuma-Temperley-Lieb [GJKL1, l'algèbre de Temperley-Lieb à poids ("Framisation of
the Temperley-Lieb algebra") GJKL2] et l'algèbre de Temperley-Lieb de réflexions complexe GJKL2]. Notre étude de la structure et la théorie des représentations de ces trois algèbres dans [ChPo1, ChPo2] indique que l'algèbre de Temperley-Lieb à poids est l'analogue le plus naturel de l'algèbre de TemperleyLieb dans ce contexte.

Une des raisons pour lesquelles l'intérêt porté aux algèbres de Yokonuma-Hecke s'est accru récemment est leurs applications topologiques. Suivant la méthode de Jones Jo2, Jo3, la trace de Ocneanu sur l'algèbre de Iwahori-Hecke de type $A$ et son quotient, l'algèbre de Temperley-Lieb, peut être utilisée pour la définition des invariants de noeuds classiques, connues sous le nom de polynôme Homflypt (ou polynôme de Jones en 2 variables) et de polynôme de Jones respectivement. De façon similaire, en utilisant la trace de Markov sur l'algèbre de Yokonuma-Hecke de type $A$ définie par Juyumaya Ju2, nous pouvons définir des invariants en 2 variables de noeuds à poids ("framed"), classiques et singuliers (voir JuLa1-JuLa4, CJKL).

Durant les années dernières, nous travaillions sur la comparaison entre les invariants en 2 variables de noeuds classiques obtenus par les algèbres de Yokonuma-Hecke et le polynôme Homflypt, qui apparaît dans cette famille (voir ChLa). Très récemment, nous avons réussi à prouver que nous avons en effet obtenu une nouvelle famille d'invariants, qui ne sont pas topologiquement équivalents au polynôme Homflypt CJKL. En fait, nous avons montré que nos invariants sont topologiquement équivalents au polynôme Homflypt sur les noeuds, mais qu'ils distinguent des entrelacs (c'est à dire, des enchevêtrements de plusieurs noeuds) que le polynôme Homflypt ne peut pas distinguer. En plus des données calculées par ordinateur, nous avons donné une preuve diagrammatique basée sur le résultat suivant : les invariants provenant des algèbres de Yokonuma-Hecke peuvent être définis de façon diagrammatique via une relation d'écheveau ("skein relation") qui est appliquée seulement à des croisements de composantes différentes. Ce fait est très important en lui-même, puisque il existe très peu d'invariants de noeuds qui sont définis via des relations d'écheveau. Enfin, nous avons généralisé cette famille d'invariants à un nouvel invariant d'écheveau en 3 variables qui est plus fort que le polynôme Homflypt.

Dans ChPdA2, nous avons défini des traces de Markov sur les algèbres de Yokonuma-Hecke affines et cyclotomiques. Appliquant la méthode de Jones, nous avons obtenu des invariants de noeuds classiques et à poids dans le tore solide. Étant donnés les résultats de [JJKL, ces invariants ne sont pas topologiquement équivalents aux invariants de noeuds classiques dans le tore solide de type Homflypt obtenus des algèbres de Hecke affines et cyclotomiques dans [La1, GeLa, La2]. De façon similaire, des invariants en 1 variable de noeuds à poids et classiques peuvent être obtenus grâce à l'algèbre de Temperley-Lieb à poids en utilisant la trace de Juyumaya; ces invariants ne sont pas topologiquement équivalents au polynôme de Jones sur les entrelacs classiques GJKL2].

Ce mémoire est organisé de la façon suivante : dans le premier chapitre, nous présentons quelques résultats préliminaires sur les blocs, les formes symétrisantes et les éléments de Schur, les applications et les matrices de décomposition. Dans le deuxième chapitre, nous étudions les algèbres de IwahoriHecke associées aux groupes de Coxeter finis et leur théorie des cellules de Kazhdan-Lusztig. Nous définissons les familles de caractères de trois façons différentes, et les ensembles basiques canoniques en caractéristique 0. Dans le troisième chapitre, nous étudions les algèbres de Hecke associées aux groupes de réflexions complexes et explorons quelles propriétés se généralisent du cas réel au cas complexe. Nous donnons la définition des algèbres de Hecke génériques et cyclotomiques associées aux groupes de réflexions complexes. Puis nous utilisons la définition de Rouquier afin de généraliser la notion de familles de caractères au cas des groupes de réflexions complexes, et nous expliquons comment nous pouvons déterminer les familles de Rouquier dans tous les cas. Un rôle-clef dans cette détermination est joué par la forme des éléments de Schur, qui ont été calculés, sous certaines hypothèses, pour tous les groupes de réflexions complexes. Ces éléments de Schur sont aussi utilisés pour la définition d'une fonction $a$ et des ensembles basiques canoniques dans ce cas, et nous présentons nos résultats dans cette direction, qui concernent plutôt les groupes de la série infinie. Dans le chapitre 4, nous nous concentrons sur les algèbres de Ariki-Koike, i.e., les algèbres de Hecke associées aux groupes de réflexions complexes de type $G(l, 1, n)$. Même si elles sont des cas particuliers d'algèbres de Hecke étudiées dans le chapitre précedent, les algèbres de Ariki-Koike ont été définies indépendamment plus tôt et étudiées combinatoirement par plusieurs personnes. Elles constituent le cas le mieux exploré jusqu'à présent. Nous donnons de jolies formules combinatoires pour les éléments de Schur des algèbres de Ariki-Koike et la fonction $a$. Nous montrons aussi que les familles de caractères dans ce cas peuvent être complètement déterminées par les familles de caractères des groupes de Weyl de type $B$. Enfin, nous discutons les applications de décomposition
pour les algèbres de Ariki-Koike (non-graduées et graduées), et la preuve de Geck et Jacon de l'existence des ensembles basiques canoniques via l'utilisation du théorème d'Ariki. Dans le cinquième chapitre, nous introduisons les algèbres de Cherednik rationnelles associées aux groupes de réflexions complexes, qui dépendent d'un paramètre $t$. Si $t \neq 0$, alors il existe un lien entre la théorie des représentations des algèbres de Cherednik rationnelles et celle des algèbres de Hecke cyclotomiques via le foncteur KZ, et nous montrons comment on peut utiliser des informations sur l'un pour obtenir des informations sur l'autre, en particulier en ce qui concerne les matrices de décomposition et les ensembles basiques canoniques. Nous obtenons l'existence des ensembles basiques canoniques pour plusieurs algèbres de Hecke cyclotomiques, parmi lesquelles les algèbres de Iwahori-Hecke associées aux groupes de Coxeter finis et les algèbres de Ariki-Koike (sans l'utilisation du théorème d'Ariki). Si $t=0$, il existe aussi un lien (qui n'est pas encore bien compris) entre la théorie des représentations des algèbres de Cherednik rationnelles et celle des algèbres de Hecke cyclotomiques, qui pourrait permettre le développement d'une théorie des cellules de Kazhdan-Lusztig pour les groupes de réflexions complexes. Dans le dernier chapitre, nous étudions les algèbres de Yokonuma-Hecke, qui sont obtenues dans le contexte des groupes réductifs finis comme généralisations des algèbres de Iwahori-Hecke, et nous nous concentrons sur l'algèbre de Yokonuma-Hecke de type $A$. Nous étudions sa théorie des représentations, nous construisons une forme symétrisante sur celle-ci et nous déterminons ses éléments de Schur. Puis nous démontrons des résultats similaires pour les algèbres de Yokonuma-Hecke affines et cyclotomiques, que nous introduisons comme généralisations des algèbres de Yokonuma-Hecke de type $A$. Nous étudions également la théorie des représentations et construisons des bases pour les trois quotients de type Temperley-Lieb des algèbres de Yokonuma-Hecke: l'algèbre de Yokonuma-Temperley-Lieb, l'algèbre de Temperley-Lieb à poids et l'algèbre de Temperley-Lieb de réflexions complexe. Nous déduisons que l'algèbre de Temperley-Lieb à poids est l'analogue le plus naturel de l'algèbre de Temperley-Lieb dans ce cas. Enfin, nous parlons des applications topologiques de toutes ces algèbres, qui peuvent être utilisées pour la définition des invariants de noeuds à poids, classiques et singuliers avec l'utilisation des traces de Markov que nous définissons sur celles-ci et la méthode de Jones. Nous montrons que les invariants de noeuds classiques provenant des algèbres de Yokonuma-Hecke sont en effet nouveaux, et ne sont pas topologiquement équivalents aux fameux invariants polynomiaux obtenus dans le contexte classique des algèbres de Iwahori-Hecke de type $A$ (notamment, le polynôme Homflypt, les invariants de noeuds classiques dans le tore solide de type Homflypt et le polynôme de Jones).

## Introduction in English

Iwahori-Hecke algebras associated to Weyl groups appear naturally in the study of finite reductive groups as endomorphism rings of the permutation representation with respect to a Borel subgroup. They can also be defined independently as deformations of group algebras of finite Coxeter groups. The aim of this memoir is to study some aspects of the representation theory of Iwahori-Hecke algebras and the way they generalise in the cases of

- cyclotomic Hecke algebras, which are obtained as deformations of group algebras of complex reflection groups,
- Ariki-Koike algebras, which are obtained as generalisations of Iwahori-Hecke algebras of types $A$ and $B$,
- Yokonuma-Hecke algebras, which are obtained in the study of finite reductive groups as endomorphism rings of the permutation representation with respect to a maximal unipotent subgroup.

In the process, we will also study another family of algebras associated to complex reflection groups, the rational Cherednik algebras, whose representation theory has many connections with the representation theory of Hecke algebras.

The aspects of the representation theory of these algebras on which we will focus will be the parametrisation and description of the irreducible representations in the semisimple and non-semisimple case, the block structure, the symmetric algebra structure and the determination of the decomposition matrix with respect to a specialisation.

Finite Coxeter groups are finite groups of real matrices that are generated by reflections. They include the Weyl groups, which are fundamental in the classification of simple Lie algebras over $\mathbb{C}$ as well as simple algebraic groups. Iwahori-Hecke algebras associated to Weyl groups appear naturally as endomorphism algebras of induced representations in the study of finite reductive groups. They can also be defined independently as deformations of group algebras of finite Coxeter groups, where the deformation depends on a parameter $q$ and a weight function $L$. For $q=1$, we recover the group algebra. For a finite Coxeter group $W$, we will denote by $\mathcal{H}_{q}(W, L)$ the associated Iwahori-Hecke algebra.

When $q$ is an indeterminate, the Iwahori-Hecke algebra $\mathcal{H}_{q}(W, L)$ is semisimple. By Tits's deformation theorem, there exists a bijection between the set of irreducible representations of $\mathcal{H}_{q}(W, L)$ and the set $\operatorname{Irr}(W)$ of irreducible representations of $W$. Through this bijection, Lusztig attaches to every irreducible representation of $W$ an integer depending on $L$, thus defining the famous $a$-function. Here we define the $a$-function using the symmetric algebra structure and the Schur elements of $\mathcal{H}_{q}(W, L)$. The $a$-function is used in Lusztig's definition of families of characters, a partition of $\operatorname{Irr}(W)$ which plays a principal role in the organisation of families of unipotent characters in the case of finite reductive groups.

Kazhdan-Lusztig theory is a key to understanding the representation theory of the Iwahori-Hecke algebra $\mathcal{H}_{q}(W, L)$. There exists a special basis of $\mathcal{H}_{q}(W, L)$, called the Kazhdan-Lusztig basis, which allows us to define the Kazhdan-Lusztig cells for $\mathcal{H}_{q}(W, L)$, a certain set of equivalence classes on $W$. The construction of Kazhdan-Lusztig cells yields the construction of representations for $\mathcal{H}_{q}(W, L)$. It also gives another, more combinatorial, definition for Lusztig's families of characters.

Now, when $q$ specialises to a non-zero complex number $\eta$, and more specifically to a root of unity, the specialised Iwahori-Hecke algebra $\mathcal{H}_{\eta}(W, L)$ is not necessarily semisimple and we no longer have a bijection between its irreducible representations and $\operatorname{Irr}(W)$. We obtain then a decomposition matrix which records how the irreducible representations of the semisimple algebra decompose after the specialisation.

A canonical basic set is a subset of $\operatorname{Irr}(W)$ in bijection with the irreducible representations of $\mathcal{H}_{\eta}(W, L)$ (and thus a labelling set for the columns of the decomposition matrix) with good properties. These good properties ensure that the decomposition matrix has a lower unitriangular form while the $a$-function increases (roughly) down the columns. Canonical basic sets were defined by Geck and Rouquier GeRo, who also proved their existence in certain cases with the use of Kazhdan-Lusztig theory. Thanks to the work of many people, canonical basic sets are now proved to exist and explicitly described for all finite Coxeter groups and for any choice of $L$ (in characteristic 0 ).

Finite Coxeter groups are particular cases of complex reflection groups, that is, finite groups of complex matrices generated by "pseudo-reflections". Their classification is due to Shephard and Todd ShTo: if $W$ is an (irreducible) complex reflection group, then either $W$ belongs to the infinite series $G(l, p, n)$ or $W$ is one of the 34 exceptional groups $G_{4}, \ldots, G_{37}$ (see Theorem 3.1). Important work in the last two decades has suggested that complex reflection groups play a crucial, but not yet understood role in representation theory, and may even become as ubiquitous as finite Coxeter groups in the study of other mathematical structures. In fact, they behave so much like real reflection groups that Broué, Malle and Michel [BMM1] conjectured that they could play the role of Weyl groups for, as yet mysterious, objects generalising finite reductive groups. These objects are called "Spetses".

Broué, Malle and Rouquier BMR defined Hecke algebras for complex reflection groups as deformations of their group algebras. In the case of complex reflection groups of type $G(l, 1, n)$, these coincide with the Ariki-Koike algebras, already defined by Ariki and Koike ArKo as generalisations of Iwahori-Hecke algebras of types $A$ and $B$. A generalised Kazhdan-Lusztig cell theory for these algebras, known as cyclotomic Hecke algebras, is expected to help find Spetses. Unfortunately, we do not have a Kazhdan-Lusztig basis for complex reflection groups. However, we can define families of characters using Rouquier's definition: in Ro1, Rouquier gave an alternative definition for Lusztig's families of characters by proving that, in the case of Weyl groups, they coincide with the blocks of the Iwahori-Hecke algebra over a certain ring, called the Rouquier ring. This definition generalises naturally to the case of complex reflection groups and their cyclotomic Hecke algebras, producing the so-called Rouquier families. These families have now been determined for all cyclotomic Hecke algebras of all complex reflection groups, see Ch3, Ch4, Ch5. A key role in this determination is played by the (still conjectural in some cases) symmetric algebra structure of cyclotomic Hecke algebras and the corresponding Schur elements.

Using the Schur elements, we can also define an $a$-function and canonical basic sets for cyclotomic Hecke algebras. Although there is no Kazhdan-Lusztig theory in the complex case, canonical basic sets are now known to exist for the groups of the infinite series $G(l, p, n)$ and for some exceptional ones. In order to obtain canonical basic sets for Ariki-Koike algebras, Geck and Jacon used Ariki's Theorem on the categorification of Hecke algebra representations and Uglov's work on canonical bases for higher level Fock spaces GeJa1, Ja2, Ja3, GeJa2. The result for $G(l, p, n)$ derives from that for $G(l, 1, n)$ with the use of Clifford Theory GenJa, ChJa2.

Symplectic reflection algebras were introduced by Etingof and Ginzburg EtGi] for the study of symplectic resolutions of the orbit space $V / G$, where $V$ is a symplectic complex vector space and $G$ is a symplectic reflection group acting on $V$, that is, $G$ is a finite group generated by symplectic reflections. A complex reflection group $W \subset G L(\mathfrak{h})$, where $\mathfrak{h}$ is a complex vector space, can be seen as a symplectic reflection group acting on $V=\mathfrak{h} \oplus \mathfrak{h}^{*}$. Symplectic reflection algebras associated with complex reflection groups are known as rational Cherednik algebras. They depend on a parameter $t$, and their representation theory varies a lot according to whether $t$ is zero or not.

If $t \neq 0$, there exists an important category of representations of the rational Cherednik algebra, the category $\mathcal{O}$, and an exact functor, the Knizhnik-Zamalodchikov functor or simply KZ, from $\mathcal{O}$ to the category of representations of a certain specialised cyclotomic Hecke algebra $\mathcal{H}_{\eta}(W)$ (the specialisation depends on the choice of parameters for the rational Cherednik algebra; every specialised Hecke algebra can arise this way). Category $\mathcal{O}$ is a highest weight category, and it comes equipped with a set of standard modules $\{\Delta(E) \mid E \in \operatorname{Irr}(W)\}$, a set of simple modules $\{\mathrm{L}(E) \mid E \in \operatorname{Irr}(W)\}$ and a decomposition matrix that records the number of times that $\mathrm{L}(E)$ appears in the composition series of $\Delta\left(E^{\prime}\right)$ for $E, E^{\prime} \in \operatorname{Irr}(W)$. The exactness of KZ allows us to read off the decomposition matrix of $\mathcal{H}_{\eta}(W)$ from the decomposition matrix of category $\mathcal{O}$. Using this, we proved in CGG the existence of canonical basic sets for all finite Coxeter groups and for complex reflection groups of type $G(l, 1, n)$ (in characteristic 0 ). In particular, we showed that $E$ belongs to the canonical basic set for $\mathcal{H}_{\eta}(W)$ if and only if $\mathrm{KZ}(\mathrm{L}(E)) \neq 0$. Our proof of existence is quite general and it does not make use of Ariki's Theorem for type $G(l, 1, n)$. However, the explicit description of canonical basic sets in these cases by previous works answers simultaneously
the question of which simple modules are killed by the KZ-functor. We also proved that the images of the standard modules via the KZ-functor are isomorphic to the cell modules of Hecke algebras.

As far as the case $t=0$ is concerned, Gordon [G0 introduced and studied extensively a finitedimensional quotient of the rational Cherednik algebra, called the restricted rational Cherednik algebra, whose simple modules are parametrised by $\operatorname{Irr}(W)$. The decomposition of this algebra into blocks induces a partition of $\operatorname{Irr}(W)$, known as Calogero-Moser partition. It is believed that there exists a connection between the Calogero-Moser partition and the families of characters, first suggested by Gordon and Martino GoMa for type $B$. In every case studied so far, the partition into Rouquier families (for a suitably chosen cyclotomic Hecke algebra) refines the Calogero-Moser partition ("Martino's conjecture"), while for finite Coxeter groups the two partitions coincide. The reasons for this connection are still unknown, since there is no apparent connection between Hecke algebras and rational Cherednik algebras at $t=0$. Inspired by this, and in an effort to construct a generalised Kazhdan-Lusztig cell theory, Bonnafé and Rouquier have used the Calogero-Moser partition to develop a "Calogero-Moser cell theory" which can be applied to all complex reflection groups BoRo1, BoRo2. The fruits of this very recent approach remain to be seen.

Yokonuma-Hecke algebras were introduced by Yokonuma Yo in the context of finite reductive groups as generalisations of Iwahori-Hecke algebras. They are defined as endomorphism rings of the permutation representation with respect to a maximal unipotent subgroup, and can thus be regarded as particular cases of unipotent Hecke algebras. We will focus here on Yokonuma-Hecke algebras of type $A$, which can be also defined independently as deformations of group algebras of complex reflection groups of type $G(l, 1, n)$. We thus obtain another type of Hecke algebras associated to $G(l, 1, n)$, different from ArikiKoike algebras. Having studied the representation theory of Yokonuma-Hecke algebras in ChPdA1, we have discovered many similarities with the representation theory of Iwahori-Hecke algebras of type $A$; these similarities are explained by a result of Lusztig Lu7, who has proved that Yokonuma-Hecke algebras are isomorphic to direct sums of matrix algebras over tensor products of certain subalgebras of Iwahori-Hecke algebras.

We have now introduced and studied many interesting new algebras in the Yokonuma-Hecke algebra setting, which are obtained as generalisations (or, as called in JuLa5, "framisations") of important algebras in the Iwahori-Hecke algebra setting. First of all, we have the affine and cyclotomic YokonumaHecke algebras, which generalise respectively the affine Hecke algebras of type $A$ and the Ariki-Koike algebras ChPdA2. In fact, we have shown in ChSe that affine Yokonuma-Hecke algebras appear also naturally in the study of $p$-adic reductive groups, arising from a construction analogous to the one used by Yokonuma, while cyclotomic Yokonuma-Hecke algebras give rise to both Ariki-Koike algebras and classical Yokonuma-Hecke algebras of type $A$, both as quotients and as particular cases. Further, we have three possible generalisations of the Temperley-Lieb algebra: the Yokonuma-Temperley-Lieb algebra GJKL1, the Framisation of the Temperley-Lieb algebra GJKL2 and the Complex Reflection Temperley-Lieb algebra GJKL2. Our study of the structure and the representation theory of these three algebras in ChPo1, ChPo2 indicates that the Framisation of the Temperley-Lieb algebra is the most natural analogue of the Temperley-Lieb algebra in this setting.

One of the reasons that Yokonuma-Hecke algebras have attracted a lot of interest recently is their topological applications. After Jones's method Jo2, Jo3, the Ocneanu trace on the Iwahori-Hecke algebra of type $A$ and its quotient, the Temperley-Lieb algebra, can be used to define isotopy invariants for classical knots and links, known as the Homflypt polynomial (or 2-variable Jones polynomial) and the Jones polynomial respectively. In a similar way, using the Markov trace on the Yokonuma-Hecke algebra of type $A$ defined by Juyumaya Ju2, one can define 2 -variable isotopy invariants for framed, classical and singular knots and links (see JJuLa1- JuLa4, (CJKL).

For the past years, we have been trying to compare the family of 2 -variable isotopy invariants of oriented classical links arising from Yokonuma-Hecke algebras with the Homflypt polynomial, which is included in this family (see ChLa ). Very recently, we were successful in proving that we have indeed obtained a new family of invariants, which are not topologically equivalent to the Homflypt polynomial CJKL. In fact, we have shown that our invariants are topologically equivalent to the Homflypt polynomial on knots, but they distinguish links that the Homflypt polynomial does not. Besides computational data, we have given a diagrammatic proof based on the following result: the invariants from Yokonuma-Hecke algebras can be defined diagrammatically via a special skein relation involving only crossings between different components. This fact is very important in its own right, since there are very few link invariants defined through skein relations. Finally, we have generalised this family of
invariants to a new 3-variable skein link invariant which is stronger than the Homflypt polynomial.
In [ChPdA2], we have defined Markov traces on affine and cyclotomic Yokonuma-Hecke algebras. Applying Jones's method, we have obtained invariants for framed and classical links in the solid torus. In view of the results of CJKL, these invariants are not topologically equivalent to the Homflypt-type invariants for classical links in the solid torus obtained from affine and cyclotomic Hecke algebras in La1, GeLa, La2]. Similarly, 1-variable isotopy invariants of oriented framed and classical links can be obtained from the Framisation of the Temperley-Lieb algebra with the use of Juyumaya's trace; these invariants are not topologically equivalent to the Jones polynomial on classical links GJKL2.

This memoir is organised as follows: In the first chapter, we present some preliminary results on blocks, symmetrising forms and Schur elements, decomposition maps and matrices. In the second chapter, we discuss Iwahori-Hecke algebras associated with finite Coxeter groups and their Kazhdan-Lusztig cell theory. We define families of characters in three different ways, and canonical basic sets in characteristic 0 . In the third chapter, we study the Hecke algebras associated with complex reflection groups and explore which properties generalise from the real case to the complex case. We give the definition of generic and cyclotomic Hecke algebras associated with complex reflection groups. We then use Rouquier's definition in order to generalise the notion of families of characters to the case of complex reflection groups, and we explain how we can determine the Rouquier families in all cases. A key role in this determination is played by the form of the Schur elements, which have been calculated, under certain assumptions, for all complex reflection groups. These Schur elements are also used for the definition of an $a$-function and canonical basic sets in this case, and we present our results in this direction, which mostly concern the groups of the infinite series. In Chapter 4, we focus on Ariki-Koike algebras, that is, Hecke algebras associated with complex reflection groups of type $G(l, 1, n)$. Even though these are included in the Hecke algebras studied in the previous chapter, Ariki-Koike algebras were defined earlier independently and studied combinatorially by many people, so this is the best understood case. We give nice combinatorial formulas for the Schur elements of Ariki-Koike algebras and the $a$-function. We also prove that the families of characters in this case can be completely determined by the families of characters of Weyl groups of tye $B$. Finally, we discuss decomposition maps for Ariki-Koike algebras (ungraded and graded), and Geck and Jacon's proof of existence of canonical basic sets with the use of Ariki's Theorem. In the fifth chapter, we introduce rational Cherednik algebras associated with complex reflection groups, which depend on a parameter $t$. If $t \neq 0$, then there is a connection between the representation theory of rational Cherednik algebras and that of cyclotomic Hecke algebras through the KZ-functor, and we show how we can use information on the former to obtain information on the latter and vice versa, especially in regard to decomposition matrices and canonical basic sets. We obtain the existence of canonical basic sets for many cyclotomic Hecke algebras, including Iwahori-Hecke algebras associated with finite Coxeter groups and Ariki-Koike algebras (without the use of Ariki's Theorem). If $t=0$, there is also a (not yet understood) connection between the representation theory of rational Cherednik algebras and that of cyclotomic Hecke algebras, which could allow the development of a Kazhdan-Lusztig cell theory for complex reflection groups. In the last chapter, we study Yokonuma-Hecke algebras, which are obtained in the context of finite reductive groups as generalisations of Iwahori-Hecke algebras, and we focus on the Yokonuma-Hecke algebra of type $A$. We study its representation theory, construct a symmetrising form on it and determine its Schur elements. We then provide similar results for affine and cyclotomic Yokonuma-Hecke algebras, which we introduce as generalisations of Yokonuma-Hecke algebras of type $A$. We also study the representation theory and construct bases for the three suggested Temperley-Lieb type quotients of Yokonuma-Hecke algebras: the Yokonuma-Temperley-Lieb algebra, the Framisation of the Temperley-Lieb algebra and the Complex Reflection Temperley-Lieb algebra. We deduce that the Framisation of the Temperley-Lieb algebra is the most natural analogue of the Temperley-Lieb algebra in this case. Finally, we discuss the topological applications of all these algebras, which can be used to define invariants for framed, classical and singular knots and links using the Markov traces that we define on them and Jones's recipe. We show that the classical link invariants arising from YokonumaHecke algebras are actually new, and not topologically equivalent to the famous polynomial invariants obtained in the classical setting of Iwahori-Hecke algebras of type $A$ (namely, the Homflypt polynomial, the Homflypt-type invariants of links in the solid torus and the Jones polynomial).

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## Chapter 1

## Algebras of Finite Type

Let $R$ be a commutative integral domain and let $A$ be an $R$-algebra, free and finitely generated as an $R$-module. If $R^{\prime}$ is a commutative integral domain containing $R$, we will write $R^{\prime} A$ for $R^{\prime} \otimes_{R} A$ and we will denote by $\operatorname{Irr}\left(R^{\prime} A\right)$ the set of irreducible representations of $R^{\prime} A$.

### 1.1 Blocks

We define the block-idempotents of $A$ to be the primitive idempotents of the centre $Z(A)$ of $A$. Let $K$ be a field containing $R$ such that the algebra $K A$ is semisimple. Then $K A$ is isomorphic to a direct product of simple algebras,

$$
K A \cong \prod_{V \in \operatorname{Irr}(K A)} M_{V}
$$

where $M_{V}$ is a simple $K$-algebra. For all $V \in \operatorname{Irr}(K A)$, we denote by $\pi_{V}: K A \rightarrow M_{V}$ the projection onto the $V$-factor and by $e_{V}$ the element of $K A$ such that

$$
\pi_{V^{\prime}}\left(e_{V}\right)= \begin{cases}1_{M_{V}} & \text { if } V=V^{\prime} \\ 0 & \text { if } V \neq V^{\prime}\end{cases}
$$

We have that the elements $\left\{e_{V}\right\}_{V \in \in \operatorname{Irr}(K A)}$ are the block-idempotents of $K A$. Further, there exists a unique partition $\mathrm{Bl}(A)$ of $\operatorname{Irr}(K A)$ that is the finest with respect to the property:

$$
\forall B \in \operatorname{Bl}(A), \quad e_{B}:=\sum_{V \in B} e_{V} \in A
$$

The elements $\left\{e_{B}\right\}_{B \in \operatorname{Bl}(A)}$ are the block-idempotents of $A$. We have $A \cong \prod_{B \in \operatorname{Bl}(A)} A e_{B}$. From now on, we will refer to the parts of $\operatorname{Bl}(A)$ as the blocks of the algebra $A$.

Example 1.1 Let $G$ be a finite group. The group algebra $\mathbb{Z}[G]$ has only one block, which contains all the irreducible representations of $\mathbb{C}[G]$.

### 1.2 Symmetrising forms and Schur elements

A symmetrising form on the algebra $A$ is a linear map $\tau: A \rightarrow R$ such that
(a) $\tau(a b)=\tau(b a)$ for all $a, b \in A$, that is, $\tau$ is a trace function, and
(b) the map $\widehat{\tau}: A \rightarrow \operatorname{Hom}_{R}(A, R), a \mapsto(x \mapsto \tau(a x))$ is an isomorphism of $A$-bimodules.

If there exists a symmetrising form on $A$, we say that $A$ is a symmetric algebra.
Example 1.2 Let $G$ be a finite group. The linear map $\tau: \mathbb{Z}[G] \rightarrow \mathbb{Z}$ defined by $\tau(1)=1$ and $\tau(g)=0$ for all $g \in G \backslash\{1\}$ is a symmetrising form on $\mathbb{Z}[G]$; it is called the canonical symmetrising form on $\mathbb{Z}[G]$.

Suppose that there exists a symmetrising form $\tau$ on $A$. Let $\mathcal{B}=\left(b_{i}\right)_{i \in I}$ be an $R$-basis of $A$, and let $\mathcal{B}^{\vee}=\left(b_{i}^{\vee}\right)_{i \in I}$ denote the dual basis of $A$ with respect to $\tau$, that is, $\tau\left(b_{i} b_{j}^{\vee}\right)=\delta_{i j}$. Let $K$ be a field containing $R$ such that the algebra $K A$ is split. The map $\tau$ can be extended to $K A$ by extension of scalars. Let $V \in \operatorname{Irr}(K A)$ with character $\chi_{V}$. We have [GePf, Lemma 7.1.7]:

$$
\begin{equation*}
\widehat{\tau}^{-1}\left(\chi_{V}\right)=\sum_{i \in I} \chi_{V}\left(b_{i}\right) b_{i}^{\vee} \in Z(K A) \tag{1.1}
\end{equation*}
$$

For any $z \in Z(K A)$, Schur's lemma implies that $z$ acts as a scalar on $V$; we denote this scalar by $\omega_{V}(z)$. The $K$-algebra homomorphism $\omega_{V}: Z(K A) \rightarrow K$ is the central character associated with $V$. We define

$$
s_{V}:=\omega_{V}\left(\hat{\tau}^{-1}\left(\chi_{V}\right)\right)
$$

to be the Schur element associated with $V$. We have $s_{V} \in R_{K}$, where $R_{K}$ denotes the integral closure of $R$ in $K$ [GePf, Proposition 7.3.9]. Moreover, note that the element $s_{V}$ satisfies:

$$
s_{V} \chi_{V}(1)=\sum_{i \in I} \chi_{V}\left(b_{i}\right) \chi_{V}\left(b_{i}^{\vee}\right)
$$

Example 1.3 Let $G$ be a finite group and let $\tau$ be the canonical symmetrising form on $A:=\mathbb{Z}[G]$. The set $\{g\}_{g \in G}$ forms a basis of $A$ over $\mathbb{Z}$, with $\left\{g^{-1}\right\}_{g \in G}$ the dual basis of $A$ with respect to $\tau$. If $K$ is an algebraically closed field of characteristic 0 , then $K A$ is a split semisimple algebra and $s_{V}=|G| / \chi_{V}(1) \in \mathbb{Q}$ for all $V \in \operatorname{Irr}(K A)$. Because of the integrality of the Schur elements, we must have $|G| / \chi_{V}(1) \in \mathbb{Z}_{K} \cap \mathbb{Q}=\mathbb{Z}$ for all $V \in \operatorname{Irr}(K A)$. Thus, we have also shown that $\chi_{V}(1)$ divides $|G|$.

Now, the algebra $K A$ is semisimple if and only if $s_{V} \neq 0$ for all $V \in \operatorname{Irr}(K A)$ GePf, Theorem 7.2.6]. If this is the case, we have

$$
\begin{equation*}
\tau=\sum_{V \in \operatorname{Irr}(K A)} \frac{1}{s_{V}} \chi_{V} \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
e_{V}=\frac{1}{s_{V}} \widehat{\tau}^{-1}\left(\chi_{V}\right) \tag{1.3}
\end{equation*}
$$

where $e_{V}$ is the block-idempotent of $K A$ corresponding to $V$. Both results are due to Curtis and Reiner CuRe, but we follow the exposition in [GePf, Theorem 7.2.6] and [GePf, Proposition 7.2.7] respectively. Combining (1.1) with (1.3), and following the characterisation of blocks of the algebra $A$ given in Section 1.1. we obtain the following description for the blocks of a symmetric algebra:

Proposition 1.4 Let $K$ be a field containing $R$ such that the algebra $K A$ is split semisimple. The blocks of the symmetric algebra $A$ are the non-empty subsets $B$ of $\operatorname{Irr}(K A)$ that are minimal with respect to the property:

$$
\sum_{V \in B} \frac{1}{s_{V}} \chi_{V}(a) \in R \quad \forall a \in A
$$

### 1.3 Decomposition maps

Let $K$ be a field containing $R$. Let $R_{0}(K A)$ be the Grothendieck group of finite-dimensional $K A$-modules. The group $R_{0}(K A)$ is generated by expressions [ $V$ ], one for each $K A$-module $V$ (up to isomorphism), with relations $[V]=\left[V^{\prime}\right]+\left[V^{\prime \prime}\right]$ for each exact sequence $0 \rightarrow V^{\prime} \rightarrow V \rightarrow V^{\prime \prime} \rightarrow 0$ of $K A$-modules. Two $K A$ modules $V, V^{\prime}$ give rise to the same element in $R_{0}(K A)$, if $V$ and $V^{\prime}$ have the same composition factors, counting multiplicities. It follows that $R_{0}(K A)$ is free abelian with basis given by the isomorphism classes of simple modules. Finally, let $R_{0}^{+}(K A)$ be the subset of $R_{0}(K A)$ consisting of elements [ $V$ ], where $V$ is a finite-dimensional $K A$-module.

From now on, we assume that $R$ is integrally closed in $K$. Let $\theta: R \rightarrow L$ be a ring homomorphism into a field $L$ such that $L$ is the field of fractions of $\theta(R)$. We call such a ring homomorphism a specialisation of $R$.

There exists a valuation ring $\mathcal{O} \subseteq K$ such that $R \subseteq \mathcal{O}$ and $\operatorname{Ker} \theta$ consists of the elements of $R$ that belong to the maximal ideal $J(\mathcal{O})$ of $\mathcal{O}$. Let $k$ denote the residue field of $\mathcal{O}$. Then the restriction of the
canonical map $\pi: \mathcal{O} \rightarrow k$ to $R$ has kernel $J(\mathcal{O}) \cap R=\operatorname{Ker} \theta$. Since $L$ is the field of fractions of $\theta(R)$, we may regard $L$ as a subfield of $k$. Thus, we obtain a commutative diagram


If now $V$ is a $K A$-module, then there exists an $\mathcal{O} A$-lattice $\tilde{V}$ such that $K \otimes_{\mathcal{O}} \tilde{V}=V$. The $k$-vector space $k \otimes_{\mathcal{O}} \tilde{V}$ is a $k A$-module via $(1 \otimes a)(1 \otimes v)=1 \otimes a v(a \in A, v \in \tilde{V})$, which we call the modular reduction of $\tilde{V}$.

From now on, we will assume that either $L A$ is split, or $L=k$ and $k$ is perfect. One consequence of this assumption is that we can identify the Grothendieck groups $R_{0}(L A)$ and $R_{0}(k A)$. We then have the following [GePf, Theorem 7.4.3]:

Theorem 1.5 The modular reduction induces an additive map $d_{\theta}: R_{0}^{+}(K A) \rightarrow R_{0}^{+}(L A)$ such that $d_{\theta}([K \tilde{V}])=[k \tilde{V}]$, where $\tilde{V}$ is an $\mathcal{O}$ A-lattice and $[k \tilde{V}]$ is regarded as an element of $R_{0}^{+}(L A)$ via the identification of $R_{0}(k A)$ and $R_{0}(L A)$.

The map $d_{\theta}$ is called the decomposition map associated with the specialisation $\theta: R \rightarrow L$. Note that the map $d_{\theta}$ depends only on $\theta$ and not on the choice of the valuation ring $\mathcal{O}$. The matrix $D_{\theta}$ of this map with respect to the bases of $R_{0}(K A)$ and $R_{0}(L A)$ consisting of the classes of the simple modules is called the decomposition matrix associated with $\theta$. The rows of $D_{\theta}$ are labelled by $\operatorname{Irr}(K A)$ and the columns of $D_{\theta}$ are labelled by $\operatorname{Irr}(L A)$.

The following result gives a criterion for $d_{\theta}$ to be trivial. It is known as "Tits's deformation theorem". For its proof, the reader may refer, for example, to [GePf, Theorem 7.4.6].

Theorem 1.6 Assume that $K A$ and $L A$ are split. If $L A$ is semisimple, then $K A$ is also semisimple and the decomposition map $d_{\theta}$ is an isomorphism which preserves isomorphism classes of simple modules. In particular, we have a bijection $\operatorname{Irr}(K A) \leftrightarrow \operatorname{Irr}(L A)$.

Finally, if $A$ is symmetric, we can check whether the assumption of Tits's deformation theorem is satisfied using the following semisimplicity criterion involving the Schur elements [GePf, Theorem 7.4.7].

Theorem 1.7 Assume that $K A$ and $L A$ are split and that $A$ is symmetric with symmetrising form $\tau$. For any simple $K A$-module $V$, let $s_{V} \in R$ be the Schur element with respect to $\tau$. Then $L A$ is semisimple if and only if $\theta\left(s_{V}\right) \neq 0$ for all $V \in \operatorname{Irr}(K A)$.

## Chapter 2

## Iwahori-Hecke Algebras

In this chapter we will focus on real reflection groups, while in the next chapter we will see what happens in the complex case.

Let $(W, S)$ be a finite Coxeter system. By definition, the finite Coxeter group $W$ has a presentation of the form

$$
W=\left\langle S \mid(s t)^{m_{s t}}=1 \quad \forall s, t \in S\right\rangle
$$

with $m_{s s}=1$ and $m_{s t} \geqslant 2$ for $s \neq t$. We have a length function $\ell: W \rightarrow \mathbb{Z}_{\geqslant 0}$ defined by $\ell(w):=$ $\min \left\{r \mid w=s_{i_{1}} \ldots s_{i_{r}}\right.$ with $\left.s_{i_{j}} \in S\right\}$ for all $w \in W$.

Let $L: W \rightarrow \mathbb{Z}_{\geqslant 0}$ be a weight function, that is, a map such that $L\left(w w^{\prime}\right)=L(w)+L\left(w^{\prime}\right)$ whenever $\ell\left(w w^{\prime}\right)=\ell(w)+\ell\left(w^{\prime}\right)$. For $s, t \in S$, we have $L(s)=L(t)$ whenever $s$ and $t$ are conjugate in $W$. Let $q$ be an indeterminate. We define the Iwahori-Hecke algebra of $W$ with parameter $L$, denoted by $\mathcal{H}_{q}(W, L)$, to be the $\mathbb{Z}\left[q, q^{-1}\right]$-algebra generated by elements $\left(T_{s}\right)_{s \in S}$ satisfying the relations:

$$
\left(T_{s}-q^{L(s)}\right)\left(T_{s}+q^{-L(s)}\right)=0 \quad \text { and } \quad \underbrace{T_{s} T_{t} T_{s} T_{t} \ldots}_{m_{s t}}=\underbrace{T_{t} T_{s} T_{t} T_{s} \ldots}_{m_{s t}} \text { for } s \neq t \text {. }
$$

If $L(s)=L(t)$ for all $s, t \in S$, we say that we are in the equal parameter case. Since $L$ is a weight function, unequal parameters can only occur in irreducible types $B_{n}, F_{4}$ and dihedral groups $I_{2}(m)$ for $m$ even.

Example 2.1 Let $W=\mathfrak{S}_{3}$. We have $W=\left\langle s, t \mid s^{2}=t^{2}=(s t)^{3}=1\right\rangle$. Let $L: W \rightarrow \mathbb{Z}_{\geqslant 0}$ be a weight function and set $l:=L(s)=L(t)$. We have

$$
\mathcal{H}_{q}(W, l):=\mathcal{H}_{q}(W, L)=\left\langle T_{s}, T_{t} \mid T_{s} T_{t} T_{s}=T_{t} T_{s} T_{t},\left(T_{s}-q^{l}\right)\left(T_{s}+q^{-l}\right)=\left(T_{t}-q^{l}\right)\left(T_{t}+q^{-l}\right)=0\right\rangle .
$$

Let $w \in W$ and let $w=s_{i_{1}} \ldots s_{i_{r}}$ be a reduced expression for $w$, that is, $r=\ell(w)$. Set $T_{w}:=$ $T_{s_{i_{1}}} \ldots T_{s_{i_{r}}}$. The element $T_{w}$ is well-defined due to Matsumoto's theorem (see, for example, GePf, Theorem 1.2.2]). As a $\mathbb{Z}\left[q, q^{-1}\right]$-module, $\mathcal{H}_{q}(W, L)$ is generated by the elements $\left(T_{w}\right)_{w \in W}$ satisfying the following multiplication formulas:

$$
\begin{cases}T_{s}^{2}=1+\left(q^{L(s)}-q^{-L(s)}\right) T_{s} & \text { for } s \in S \\ T_{w} T_{w^{\prime}}=T_{w w^{\prime}} & \text { if } \ell\left(w w^{\prime}\right)=\ell(w)+\ell\left(w^{\prime}\right)\end{cases}
$$

The elements $\left(T_{w}\right)_{w \in W}$ form a basis of $\mathcal{H}_{q}(W, L)$, the standard basis.

### 2.1 Kazhdan-Lusztig cells

Let $\iota$ be the algebra involution on $\mathcal{H}_{q}(W, L)$ given by $\iota(q)=q^{-1}$ and $\iota\left(T_{s}\right)=T_{s}^{-1}$ for $s \in S$ (as a consequence, $\iota\left(T_{w}\right)=T_{w^{-1}}^{-1}$ for all $w \in W$ ). By KaLu1, Theorem 1.1] (see [Lu3, Proposition 2] for the unequal parameter case), for each $w \in W$, there exists an element $C_{w} \in \mathcal{H}_{q}(W, L)$ uniquely determined by the conditions

$$
\iota\left(C_{w}\right)=C_{w} \quad \text { and } \quad \iota\left(C_{w}\right)=T_{w}+\sum_{x \in W, x<w} P_{x, w} T_{x}
$$

where $<$ stands for the Chevalley-Bruhat order on $W$ and $P_{x, w} \in q^{-1} \mathbb{Z}\left[q^{-1}\right]$. The elements $\left(C_{w}\right)_{w \in W}$ also form a basis of $\mathcal{H}_{q}(W, L)$, the Kazhdan-Lusztig basis.

Example 2.2 We have $C_{1}=T_{1}=1$ and, for all $s \in S, C_{s}=\left\{\begin{array}{ll}T_{s} & \text { if } L(s)=0 \\ T_{s}+q^{-L(s)} T_{1} & \text { if } L(s)>0\end{array}\right.$.
Using the Kazhdan-Lusztig basis, we can define the following three relations on $W$. For $x, y \in W$, we have:

- $x \leftarrow_{\mathcal{L}} y$, if $C_{x}$ appears with non-zero coefficient in $h C_{y}$ for some $h \in \mathcal{H}_{q}(W, L)$.
- $x \leftarrow_{\mathcal{R}} y$, if $C_{x}$ appears with non-zero coefficient in $C_{y} h^{\prime}$ for some $h^{\prime} \in \mathcal{H}_{q}(W, L)$.
- $x \leftarrow_{\mathcal{L R}} y$, if $C_{x}$ appears with non-zero coefficient in $h C_{y} h^{\prime}$ for some $h, h^{\prime} \in \mathcal{H}_{q}(W, L)$.

If now we take the transitive closures of the above relations, we can define the following three preorders on $W: \leqslant_{\mathcal{L}}, \leqslant_{\mathcal{R}}, \leqslant_{\mathcal{L} \mathcal{R}}$.

The preorder $\leqslant_{\mathcal{L}}$ defines an equivalence relation $\sim_{\mathcal{L}}$ on $W$ as follows:

$$
x \sim_{\mathcal{L}} y \Leftrightarrow x \leqslant_{\mathcal{L}} y \text { and } y \leqslant_{\mathcal{L}} x .
$$

The equivalence classes for $\sim_{\mathcal{L}}$ are called left cells. Similarly, one can define equivalence relations $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{L R}}$ on $W$, whose equivalence classes are called, respectively, right cells and two-sided cells.

Example 2.3 Following the notation of Example 2.1. for $W=\mathfrak{S}_{3}=\{1, s, t, s t, t s, s t s=t s t\}$ and $l>0$,

- the left cells are $\{1\},\{s, t s\},\{t, s t\}$ and $\{s t s\}$;
- the right cells are $\{1\},\{s, s t\},\{t, t s\}$ and $\{s t s\}$;
- the two-sided cells are $\{1\},\{s, t, s t, t s\}$ and $\{s t s\}$.

If $l=0$, then all elements of $W$ belong to the same cell (left, right or two-sided).
Let now $\mathfrak{C}$ be a left cell of $W$. The following two $\mathbb{Z}\left[q, q^{-1}\right]$-modules are left ideals of $\mathcal{H}_{q}(W, L)$ :

$$
\mathcal{H}_{\leqslant \mathcal{L} \mathfrak{C}}=\left\langle C_{y} \mid y \leqslant_{\mathcal{L}} w, w \in \mathfrak{C}\right\rangle_{\mathbb{Z}\left[q, q^{-1}\right]} \quad \text { and } \quad \mathcal{H}_{<\mathcal{L} \mathfrak{C}}=\left\langle C_{y} \mid y \leqslant \mathcal{L} w, w \in \mathfrak{C}, y \notin \mathfrak{C}\right\rangle_{\mathbb{Z}\left[q, q^{-1}\right]}
$$

Then

$$
\mathcal{M}_{\mathfrak{C}}:=\mathcal{H}_{\leqslant \mathcal{L}} \mathfrak{C} / \mathcal{H}_{<_{\mathcal{L}} \mathfrak{C}}
$$

is a free left $\mathcal{H}_{q}(W, L)$-module with basis indexed by the elements of $\mathfrak{C}$.
Let $K \subseteq \mathbb{C}$ be a field such that the algebra $K(q) \mathcal{H}_{q}(W, L)$ is split semisimple (for example, we can take $K=\mathbb{C}$, or $K=\mathbb{Q}$ if $W$ is a Weyl group). Now, since the left cells form a partition of $W$, we obtain a corresponding direct sum decomposition of $K(q) \mathcal{H}_{q}(W, L)$ :

$$
\begin{equation*}
K(q) \mathcal{H}_{q}(W, L) \cong \bigoplus_{\mathfrak{C} \text { left cell }} K(q) \mathcal{M}_{\mathfrak{C}} \quad \text { (isomorphism of left } K(q) \mathcal{H}_{q}(W, L) \text {-modules) } \tag{2.1}
\end{equation*}
$$

where $K(q) \mathcal{M}_{\mathfrak{C}}:=K(q) \otimes_{\mathbb{Z}\left[q, q^{-1}\right]} \mathcal{M}_{\mathfrak{C}}$. We obtain analogous decompositions with respect to right and two-sided cells.

### 2.2 Schur elements and the $a$-function

From now on, set $R:=\mathbb{Z}\left[q, q^{-1}\right]$ and let $K \subseteq \mathbb{C}$ be a field such that he algebra $K(q) \mathcal{H}_{q}(W, L)$ is split semisimple.

Using the standard basis of the Iwahori-Hecke algebra, we define the linear map $\tau: \mathcal{H}_{q}(W, L) \rightarrow R$ by setting

$$
\tau\left(T_{w}\right):= \begin{cases}1 & \text { if } w=1 \\ 0 & \text { otherwise }\end{cases}
$$

The map $\tau$ is a symmetrising form on $\mathcal{H}_{q}(W, L)$, and the elements $\left(T_{w^{-1}}\right)_{w \in W}$ form a basis of $\mathcal{H}_{q}(W, L)$ dual to the standard basis with respect to $\tau$ (that is, $\tau\left(T_{w^{-1}} T_{w^{\prime}}\right)=\delta_{w, w^{\prime}}$ ) [GePf, Proposition 8.1.1]. The map $\tau$ is called the canonical symmetrising form on $\mathcal{H}_{q}(W, L)$, because it specialises to the canonical symmetrising form on the group algebra $\mathbb{Z}[W]$ when $q \mapsto 1$.

Now, the map $\tau$ can be extended to $K(q) \mathcal{H}_{q}(W, L)$ by extension of scalars. By Tits's deformation theorem, the specialisation $q \mapsto 1$ induces a bijection between the set of irreducible representations $\operatorname{Irr}\left(K(q) \mathcal{H}_{q}(W, L)\right)$ of $K(q) \mathcal{H}_{q}(W, L)$ and the set of irreducible representations $\operatorname{Irr}(W)$ of $W$. For $E \in$ $\operatorname{Irr}(W)$, we will denote by $V^{E}$ the corresponding irreducible representation of $K(q) \mathcal{H}_{q}(W, L)$ and by $s_{E}$ the Schur element of $\mathcal{H}_{q}(W, L)$ associated with $V^{E}$. We have $s_{E} \in \mathbb{Z}_{K}\left[q, q^{-1}\right]$ for all $E \in \operatorname{Irr}(W)$, where $\mathbb{Z}_{K}$ denotes the integral closure of $\mathbb{Z}$ in $K$.

Example 2.4 The irreducible representations of the symmetric group $\mathfrak{S}_{n}$ are parametrised by the partitions of $n$. For $W=\mathfrak{S}_{3}$, there are three irreducible representations. Let $E^{(3)}, E^{(2,1)}$ and $E^{(1,1,1)}$ denote respectively the trivial, reflection and sign representation of $\mathfrak{S}_{3}$. Using again the notation of Example 2.1, we have

$$
s_{E^{(3)}}=\left(q^{2 l}+1\right)\left(q^{4 l}+q^{2 l}+1\right), s_{E^{(2,1)}}=q^{2 l}+1+q^{-2 l}, s_{E^{(1,1,1)}}=\left(q^{-2 l}+1\right)\left(q^{-4 l}+q^{-2 l}+1\right) .
$$

We can define the functions $a: \operatorname{Irr}(W) \rightarrow \mathbb{Z}$ and $A: \operatorname{Irr}(W) \rightarrow \mathbb{Z}$ by setting

$$
a(E):=-\operatorname{valuation}_{q}\left(s_{E}\right) \quad \text { and } \quad A(E):=-\operatorname{degree}_{q}\left(s_{E}\right) .
$$

Note that both functions depend on $L$. For brevity, we will write $a_{E}$ for $a(E)$ and $A_{E}$ for $A(E)$.
Example 2.5 Following Example 2.4, for $W=\mathfrak{S}_{3}$, we have

$$
a_{E^{(3)}}=0, a_{E^{(2,1)}}=2 l, a_{E^{(1,1,1)}}=6 l \quad \text { and } \quad A_{E^{(3)}}=-6 l, A_{E^{(2,1)}}=-2 l, A_{E^{(1,1,1)}}=0
$$

The Schur elements of $\mathcal{H}_{q}(W, L)$ have been explicitly calculated for all finite Coxeter groups:

- for type $A_{n}$ by Steinberg [St1,
- for type $B_{n}$ by Hoefsmit [H0,
- for type $D_{n}$ by Benson and Gay BeGa (it derives from type $B_{n}$ with the use of Clifford theory),
- for dihedral groups $I_{2}(m)$ by Kilmoyer and Solomon KiSo,
- for $F_{4}$ by Lusztig Lu1,
- for $E_{6}$ and $E_{7}$ by Surowski Su,
- for $E_{8}$ by Benson Be ,
- for $H_{3}$ by Lusztig Lu2,
- for $H_{4}$ by Alvis and Lusztig AlLu].

There have been other subsequent proofs of the above results. For example, Iwahori-Hecke algebras of types $A_{n}$ and $B_{n}$ are special cases of Ariki-Koike algebras, whose Schur elements have been independently obtained by Geck-Iancu-Malle GIM] and Mathas Mat.

A case-by-case analysis shows that the Schur elements of $\mathcal{H}_{q}(W, L)$ can be written in the form

$$
\begin{equation*}
s_{E}=\xi_{E} q^{-a_{E}} \prod_{\Phi \in \mathrm{Cyc}_{E}} \Phi\left(q^{n_{E, \Phi}}\right) \tag{2.2}
\end{equation*}
$$

where $\xi_{E} \in \mathbb{Z}_{K}, n_{E, \Phi} \in \mathbb{Z}_{>0}$ and $\mathrm{Cyc}_{E}$ is a family of $K$-cyclotomic polynomials (see GePf, Chapters 10 \& 11], [Ch4, Theorem 4.2.5 \& Proposition 4.3.5]).

Example 2.6 Following Example 2.4 for $W=\mathfrak{S}_{3}$, we have

$$
s_{E^{(3)}}=\Phi_{2}\left(q^{2 l}\right) \Phi_{3}\left(q^{2 l}\right), s_{E^{(2,1)}}=q^{-2 l} \Phi_{3}\left(q^{2 l}\right), s_{E^{(1,1,1)}}=q^{-6 l} \Phi_{2}\left(q^{2 l}\right) \Phi_{3}\left(q^{2 l}\right),
$$

where $\Phi_{n}$ denotes the $n$-th $\mathbb{Q}$-cyclotomic polynomial.

### 2.3 Families of characters and Rouquier families

The families of characters are a special partition of the set of irreducible representations of $W$. In the case where $W$ is a Weyl group, these families play an essential role in the definition of the families of unipotent characters for the corresponding finite reductive groups. Their original definition is due to Lusztig [Lu4, 4.2] and uses the $a$-function.

Let $I \subseteq S$ and consider the parabolic subgroup $W_{I} \subseteq W$ generated by $I$. Then we have a corresponding parabolic subalgebra $\mathcal{H}\left(W_{I}, L\right) \subseteq \mathcal{H}_{q}(W, L)$. By extension of scalars from $R$ to $K(q)$, we also have a subalgebra $K(q) \mathcal{H}\left(W_{I}, L\right) \subseteq K(q) \mathcal{H}_{q}(W, L)$, and a corresponding $a$-function on the set of irreducible representations of $W_{I}$. Denote by $\operatorname{Ind}_{I}^{S}$ the induction of representations from $W_{I}$ to $W$. Let $E \in \operatorname{Irr}(W)$ and $M \in \operatorname{Irr}\left(W_{I}\right)$. We will write $M \rightsquigarrow_{L} E$ if $E$ is a constituent of $\operatorname{Ind}_{I}^{S}(M)$ and $a_{E}=a_{M}$.

Definition 2.7 The partition of $\operatorname{Irr}(W)$ into families is defined inductively as follows: When $W=\{1\}$, there is only one family; it consists of the unit representation of $W$. Assume now that $W \neq\{1\}$ and that the families have already been defined for all proper parabolic subgroups of $W$. Then $E, E^{\prime} \in \operatorname{Irr}(W)$ are in the same family of $W$ if there exists a finite sequence $E=E_{0}, E_{1}, \ldots, E_{r}=E^{\prime}$ in $\operatorname{Irr}(W)$ such that, for each $i \in\{0,1, \ldots, r-1\}$, the following condition is satisfied: There exist a subset $I_{i} \varsubsetneqq S$ and $M_{i}, M_{i}^{\prime} \in \operatorname{Irr}\left(W_{I_{i}}\right)$ such that $M_{i}, M_{i}^{\prime}$ belong to the same family of $W_{I_{i}}$ and either

$$
M_{i} \rightsquigarrow{ }_{L} E_{i} \quad \text { and } \quad M_{i}^{\prime} \rightsquigarrow_{L} E_{i+1}
$$

or

$$
M_{i} \rightsquigarrow_{L} E_{i} \otimes \varepsilon \quad \text { and } \quad M_{i}^{\prime} \rightsquigarrow{ }_{L} E_{i+1} \otimes \varepsilon,
$$

where $\varepsilon$ denotes the sign representation of $W$. We will also refer to these families as Lusztig families.
Lusztig [Lu5, 3.3 and 3.4] has shown that the functions $a$ and $A$ are both constant on the families of characters, that is, if $E$ and $E^{\prime}$ belong to the same family, then $a_{E}=a_{E^{\prime}}$ and $A_{E}=A_{E^{\prime}}$.

The decomposition of $W$ into two-sided cells can be used to facilitate the description of the partition of $\operatorname{Irr}(W)$ into families of characters. As we saw in the previous section, Tits's deformation theorem yields a bijection between $\operatorname{Irr}\left(K(q) \mathcal{H}_{q}(W, L)\right)$ and $\operatorname{Irr}(W)$. Let $E \in \operatorname{Irr}(W)$ and let $V^{E}$ be the corresponding simple module of $K(q) \mathcal{H}_{q}(W, L)$. Following the direct sum decomposition given by (2.1), there exists a left cell $\mathfrak{C}$ such that $V^{E}$ is a constituent of $K(q) \mathcal{M}_{\mathfrak{C}}$; furthermore, all such left cells are contained in the same two-sided cell. This two-sided cell, therefore, only depends on $E$ and will be denoted by $\mathcal{F}_{E}$. Thus, we obtain a natural surjective map

$$
\operatorname{Irr}(W) \rightarrow\{\text { set of two-sided cells of } W\}, E \mapsto \mathcal{F}_{E}
$$

(see [Lu4, 5.15] for the equal parameter case; the same argument works in general).
Definition 2.8 Let $E, E^{\prime} \in \operatorname{Irr}(W)$. We will say that $E$ and $E^{\prime}$ belong to the same Kazhdan-Lusztig family if $\mathcal{F}_{E}=\mathcal{F}_{E^{\prime}}$.

The following remarkable result, relating Lusztig families and Kazhdan-Lusztig families, has been proved by Barbasch-Vogan and Lusztig for finite Weyl groups in the equal parameter case [Lu4, 5.25]. It was subsequently proved, in Lu6, 23.3] and Ge5, to hold for any finite Coxeter group and any weight function $L$, assuming that Lusztig's conjectures P1-P15 Lu6, 14.2] are satisfied.

Theorem 2.9 Assume that Lusztig's conjectures P1-P15 hold. The Lusztig families and the KazhdanLusztig families coincide.

Lusztig's conjectures P1-P15 concern properties of the Kazhdan-Lusztig basis which should hold for any finite Coxeter group and in the general multiparameter case. For the moment, Conjectures P1-P15 have been proved in the following cases:

- Equal parameter case for finite Weyl groups KaLu2, Lu6, Sp.
- Equal parameter case for $H_{3}, H_{4}$ and dihedral groups $I_{2}(m)$ Al, DuCloux.
- Unequal parameter case for $F_{4}$ and dihedral groups $I_{2}(m)$ Ge4, Ge9.
- Asymptotic case and some other cases for $B_{n}$ BoIa, Bo1, BGIL, GeIa.

Moreover, these are exactly the cases where we have a description of the Kazhdan-Lusztig cells and Kazhdan-Lusztig families. A conjectural combinatorial description of the Kazhdan-Lusztig cells for type $B_{n}$ is given by BGIL.

Example 2.10 The group $\mathfrak{S}_{3}$ has three irreducible representations. Recalling the notation of Example 2.1, for $l>0$, each irreducible representation forms a family on its own. This is true in general for the symmetric group $\mathfrak{S}_{n}$. For $l=0$, all irreducible representations belong to the same family. This is true in general for the group algebra $(L(s)=0$ for all $s \in S)$ of every finite Coxeter group.

In [Ro1] Rouquier gave an alternative definition for Lusztig's families. He showed that, for finite Weyl groups in the equal parameter case, the families of characters coincide with the blocks of the Iwahori-Hecke algebra $\mathcal{H}_{q}(W, L)$ over the Rouquier ring

$$
\mathcal{R}_{K}(q):=\mathbb{Z}_{K}\left[q, q^{-1},\left(q^{n}-1\right)_{n \geqslant 1}^{-1}\right] .
$$

These are the Rouquier families of $\mathcal{H}_{q}(W, L)$. One advantage of this definition, as we will see in the next chapter, is that it can be also applied to complex reflection groups. This is important in the project "Spetses" BMM1, BMM2.

Following the determination of Rouquier families for all complex reflection groups (see Section 3.3 for references), and thus for all finite Coxeter groups, one can check that Rouquier's result holds for all finite Coxeter groups for all choices of parameters (by comparing the Rouquier families with the already known Lusztig families [Lu4, Lu6]); that is, we have the following:

Theorem 2.11 Let $(W, S)$ be a finite Coxeter system and let $\mathcal{H}_{q}(W, L)$ be an Iwahori-Hecke algebra associated to $W$. The Lusztig families and the Rouquier families of $\mathcal{H}_{q}(W, L)$ coincide.

### 2.4 Decomposition maps and canonical basic sets

As we saw in Section 2.2, the specialisation $q \mapsto 1$ yields a bijection between the set of irreducible representations of $K(q) \mathcal{H}_{q}(W, L)$ and $\operatorname{Irr}(W)$. What happens though when $q$ specialises to a complex number? The resulting Iwahori-Hecke algebra is not necessarily semisimple and the first questions that need to be answered are the following: What are the simple modules for the newly obtained algebra? Is there a good way to parametrise them? What are their dimensions? One major approach to answering these questions is through the existence of "canonical basic sets".

Let $\theta: \mathbb{Z}_{K}\left[q, q^{-1}\right] \rightarrow K(\eta), q \mapsto \eta$ be a ring homomorphism such that $\eta$ is a non-zero complex number. Let us denote by $\mathcal{H}_{\eta}(W, L)$ the algebra obtained as a specialisation of $\mathcal{H}_{q}(W, L)$ via $\theta$. Set $\mathbb{K}:=K(\eta)$. Due to the form of the Schur elements of $\mathcal{H}_{q}(W, L)$ given by 2.2 , and following Theorem 1.7. the algebra $\mathbb{K} \mathcal{H}_{\eta}(W, L)$ is semisimple unless $\eta$ is a root of unity.

Example 2.12 Since $\mathfrak{S}_{3}$ is a Weyl group, the algebra $\mathbb{Q}(q) \mathcal{H}_{q}\left(\mathfrak{S}_{3}, l\right)$ is split semisimple. Now, following Example 2.6. the algebra $\mathbb{Q}(\eta) \mathcal{H}_{\eta}\left(\mathfrak{S}_{3}, l\right)$ is semisimple if and only if $\eta^{2 l} \notin\left\{-1, \omega, \omega^{2}\right\}$, where $\omega:=$ $\exp (2 \pi i / 3)$.

If $\mathbb{K} \mathcal{H}_{\eta}(W, L)$ is semisimple, then, by Tits's deformation theorem, the specialisation $\theta$ yields a bijection between $\operatorname{Irr}\left(K(q) \mathcal{H}_{q}(W, L)\right)$ and $\operatorname{Irr}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right)$. Thus, the irreducible representations of $\mathbb{K} \mathcal{H}_{\eta}(W, L)$ are parametrised by $\operatorname{Irr}(W)$. If $\mathbb{K} \mathcal{H}_{\eta}(W, L)$ is not semisimple, then we have a well-defined decomposition map

$$
d_{\theta}: R_{0}\left(K(q) \mathcal{H}_{q}(W, L)\right) \rightarrow R_{0}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right)
$$

such that, for all $E \in \operatorname{Irr}(W)$, we have

$$
d_{\theta}\left(\left[V^{E}\right]\right)=\sum_{M \in \operatorname{Irr}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right)}\left[V^{E}: M\right][M] .
$$

The matrix

$$
D_{\theta}=\left(\left[V^{E}: M\right]\right)_{E \in \operatorname{Irr}(W), M \in \operatorname{Irr}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right)}
$$

is the decomposition matrix with respect to $\theta$. If $\mathbb{K} \mathcal{H}_{\eta}(W, L)$ is semisimple, then $D_{\theta}$ is a permutation matrix.

Definition 2.13 A canonical basic set with respect to $\theta$ is a subset $\mathcal{B}_{\theta}$ of $\operatorname{Irr}(W)$ such that
(a) there exists a bijection $\operatorname{Irr}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right) \leftrightarrow \mathcal{B}_{\theta}, M \mapsto E_{M}$;
(b) $\left[V^{E_{M}}: M\right]=1$ for all $M \in \operatorname{Irr}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right)$;
(c) if $\left[V^{E}: M\right] \neq 0$ for some $E \in \operatorname{Irr}(W), M \in \operatorname{Irr}\left(\mathbb{K} \mathcal{H}_{\eta}(W, L)\right)$, then either $E=E_{M}$ or $a_{E_{M}}<a_{E}$.

If a canonical basic set exists, the decomposition matrix has a lower unitriangular form (with an appropriate ordering of the rows). Thus, we can obtain a lot of information about the simple modules of $\mathbb{K} \mathcal{H}_{\eta}(W, L)$ from what we already know about the simple modules of $K(q) \mathcal{H}_{q}(W, L)$.

A general existence result for canonical basic sets is proved by Geck in [Ge7, Theorem 6.6], following earlier work of Geck Ge2], Geck-Rouquier GeRo and Geck-Jacon GeJa1. Another proof is given in GeJa2. Canonical basic sets are explicitly described for all finite Coxeter groups:

- for type $A_{n}$ by Geck $G \mathrm{Ge} 2$,
- for type $B_{n}$ by Jacon Ja22,
- for type $D_{n}$ by Geck Ge1 and Jacon Ja1,
- for all remaining groups from the explicit tables of decomposition matrices obtained by Geck, Lux and Müller.

For a complete survey on the topic, we refer the reader to GeJa2].
Example 2.14 Let $W$ be the symmetric group $\mathfrak{S}_{n}$. Then $W$ is generated by the transpositions $s_{i}=$ $(i, i+1)$ for all $i=1, \ldots, n-1$, which are all conjugate in $W$. Set $l:=L\left(s_{1}\right)$ and let $\eta^{2 l}$ be a primitive root of unity of order $e>1$. By [DiJa, Theorem 7.6], we have that, in this case, the canonical basic set $\mathcal{B}_{\theta}$ is the set of $e$-regular partitions (a partition is $e$-regular if it does not have $e$ non-zero equal parts). For example, for $n=3$, we have $\mathcal{B}_{\theta}=\left\{E^{(3)}, E^{(2,1)}\right\}$ for $e \in\{2,3\}$, and $\mathcal{B}_{\theta}=\operatorname{Irr}\left(\mathfrak{S}_{3}\right)$ for $e>3$.

## Chapter 3

## Cyclotomic Hecke Algebras

Cyclotomic Hecke algebras generalise the notion of Iwahori-Hecke algebras to the case of complex reflection groups. For any positive integer $e$ we will write $\zeta_{e}$ for $\exp (2 \pi i / e) \in \mathbb{C}$.

Let $\mathfrak{h}$ be a finite dimensional complex vector space. A pseudo-reflection is a non-trivial element $s \in \operatorname{GL}(\mathfrak{h})$ that fixes a hyperplane pointwise, that is, $\operatorname{dim}_{\mathbb{C}} \operatorname{Ker}\left(s-\operatorname{id}_{\mathfrak{h}}\right)=\operatorname{dim}_{\mathbb{C}} \mathfrak{h}-1$. The hyperplane $\operatorname{Ker}\left(s-\mathrm{id}_{\mathfrak{h}}\right)$ is the reflecting hyperplane of $s$. A complex reflection group is a finite subgroup of GL( $\left.\mathfrak{h}\right)$ generated by pseudo-reflections. The classification of (irreducible) complex reflection groups is due to Shephard and Todd [ShTo:

Theorem 3.1 Let $W \subset \mathrm{GL}(\mathfrak{h})$ be an irreducible complex reflection group (i.e., $W$ acts irreducibly on $\mathfrak{h})$. Then one of the following assertions is true:

- There exists a positive integer $n$ such that $(W, \mathfrak{h}) \cong\left(\mathfrak{S}_{n}, \mathbb{C}^{n-1}\right)$.
- There exist positive integers $l, p, n$ with $l / p \in \mathbb{Z}$ and $l>1$ such that $(W, \mathfrak{h}) \cong\left(G(l, p, n), \mathbb{C}^{n}\right)$, where $G(l, p, n)$ is the group of all $n \times n$ monomial matrices whose non-zero entries are l-th roots of unity, while the product of all non-zero entries is an $(l / p)$-th root of unity.
- $(W, \mathfrak{h})$ is isomorphic to one of the 34 exceptional groups $G_{n}(n=4, \ldots, 37)$.

If $W \subset \mathrm{GL}(\mathfrak{h})$ is an irreducible complex reflection group, then the dimension of $\mathfrak{h}$ is called the rank of $W$. We have $\operatorname{rank}\left(\mathfrak{S}_{n}\right)=n-1, \operatorname{rank}(G(l, p, n))=n$ for $l>1$ and $\operatorname{rank}\left(G_{n}\right) \in\{2,3, \ldots, 8\}$ for $n=4, \ldots, 37$.

Remark 3.2 We have $G(l, 1, n) \cong(\mathbb{Z} / l \mathbb{Z}) \backslash \mathfrak{S}_{n} \cong(\mathbb{Z} / l \mathbb{Z})^{n} \rtimes \mathfrak{S}_{n}$.
Remark 3.3 We have $G(1,1, n) \cong \mathfrak{S}_{n}, G(2,1, n) \cong B_{n}, G(2,2, n) \cong D_{n}, G(m, m, 2) \cong I_{2}(m)$, $G_{23} \cong H_{3}, G_{28} \cong F_{4}, G_{30} \cong H_{4}, G_{35} \cong E_{6}, G_{36} \cong E_{7}, G_{37} \cong E_{8}$.

Let $W \subset \mathrm{GL}(\mathfrak{h})$ be a complex reflection group. Benard [Ben] and Bessis [Bes] have proved (using a case-by-case analysis) that the field $K$ generated by the traces on $\mathfrak{h}$ of all the elements of $W$ is a splitting field for $W$. The field $K$ is called the field of definition of $W$. If $K \subseteq \mathbb{R}$, then $W$ is a finite Coxeter group, and if $K=\mathbb{Q}$, then $W$ is a Weyl group.

### 3.1 Hecke algebras for complex reflection groups

Let $\mathcal{A}$ be the set of reflecting hyperplanes of $W$. Let $\mathfrak{h}^{\text {reg }}:=\mathfrak{h} \backslash \bigcup_{H \in \mathcal{A}} H$ and $B_{W}:=\pi_{1}\left(\mathfrak{h}^{\text {reg }} / W, x_{0}\right)$, where $x_{0}$ is some fixed basepoint. The group $B_{W}$ is the braid group of $W$. For every orbit $\mathcal{C}$ of the action of $W$ on $\mathcal{A}$, we set $e_{\mathcal{C}}$ the common order of the subgroups $W_{H}$, where $H$ is any element of $\mathcal{C}$ and $W_{H}$ is the pointwise stabiliser of $H$. Note that $W_{H}$ is cyclic, for all $H \in \mathcal{A}$.

We choose a set of indeterminates $\mathbf{u}=\left(u_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ and we denote by $\mathbb{Z}\left[\mathbf{u}, \mathbf{u}^{-1}\right]$ the Laurent polynomial ring in the indeterminates $\mathbf{u}$. We define the generic Hecke algebra $\mathcal{H}(W)$ of $W$ to be the quotient of the group algebra $\mathbb{Z}\left[\mathbf{u}, \mathbf{u}^{-1}\right]\left[B_{W}\right]$ by the ideal generated by the elements of the form

$$
\left(\mathbf{s}-u_{\mathcal{C}, 0}\right)\left(\mathbf{s}-u_{\mathcal{C}, 1}\right) \cdots\left(\mathbf{s}-u_{\mathcal{C}, e_{\mathcal{C}}-1}\right),
$$

where $\mathcal{C}$ runs over the set $\mathcal{A} / W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in $\mathfrak{h}^{\text {reg }} / W$ of the elements of $\mathcal{C}$ [BMR, §4].

From now on, we will make the following assumptions for $\mathcal{H}(W)$ :

## Assumptions 3.4

(a) (Freeness conjecture) The algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}\left[\mathbf{u}, \mathbf{u}^{-1}\right]$-module of rank $|W|$.
(b) (Trace conjecture) There exists a canonical symmetrising form $\tau$ on $\mathcal{H}(W)$ that satisfies certain canonicality conditions [BMM1, §1 and 2]; the form $\tau$ specialises to the canonical symmetrising form on the group algebra of $W$ when $u_{\mathcal{C}, j} \mapsto \zeta_{e_{\mathcal{C}}}^{j}$.
The above assumptions are known to hold for all finite Coxeter groups [Bou, IV, §2] and for the complex reflection groups of the infinite series $G(l, p, n)$ BMM1, BreMa, MalMat, GIM. As far as the exceptional complex reflection groups are concerned, the freeness conjecture has been proved for all groups of rank at least $3\left(G_{23}, \ldots, G_{37}\right)$ Mar1, Mar2, MarPf] and (almost all) groups belonging to the first two families of groups of rank $2\left(G_{4}, \ldots, G_{15}\right)$ [ha1], plus the groups $G_{16}$ [Cha2] and $G_{22}$ [MarPf]. For the five remaining groups $\left(G_{17}, \ldots, G_{21}\right)$, the freeness conjecture is still open, but seems now to be within reach. In all the cases for which the freeness conjecture is proved, one can actually find a basis originating from the braid group itself, as expected in BMM1, §1.17]. A weaker version of this conjecture states that $\mathcal{H}(W)$ should be at least finitely generated as a $\mathbb{Z}\left[\mathbf{u}, \mathbf{u}^{-1}\right]$-module. The weak version is known for every group EtRa, Mar1, Mar2, MarPf, and it is strong enough to imply some of the results that we will see in this chapter. Finally, the only (non-Coxeter) exceptional groups for which the trace conjecture has been proved are $G_{4}, G_{12}, G_{22}$ and $G_{24}$ MalMi] (the case of $G_{4}$ was later independently checked in MarWa).

Under Assumptions 3.4, we can always find $N_{W} \in \mathbb{Z}_{>0}$ such that if we take

$$
\begin{equation*}
u_{\mathcal{C}, j}=\zeta_{\mathcal{C}_{\mathcal{C}}}^{j} v_{\mathcal{C}, j}^{N_{W}} \tag{3.1}
\end{equation*}
$$

and set $\mathbf{v}:=\left(v_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$, then the $K(\mathbf{v})$-algebra $K(\mathbf{v}) \mathcal{H}(W)$ is split semisimple. Malle [Mal3, 5.2] has shown that taking $N_{W}$ to be the number of roots of unity in $K$ works every time, but sometimes it is enough to take $N_{W}$ to be even as small as 1 (for example, if $W=G(l, 1, n)$ or $W=G_{4}$ ). By Tits's deformation theorem, it follows that the specialisation $v_{\mathcal{C}, j} \mapsto 1$ induces a bijection between $\operatorname{Irr}(K(\mathbf{v}) \mathcal{H}(W))$ and $\operatorname{Irr}(W)$. From now on, we will consider $\mathcal{H}(W)$ as an algebra over $\mathbb{Z}_{K}\left[\mathbf{v}, \mathbf{v}^{-1}\right]$, where $\mathbb{Z}_{K}$ denotes the integral closure of $\mathbb{Z}$ in $K$.

Example 3.5 The group $W=G(l, 1, n)$ is isomorphic to the wreath product $(\mathbb{Z} / l \mathbb{Z}) \imath \mathfrak{S}_{n}$ and its splitting field is $K=\mathbb{Q}\left(\zeta_{l}\right)$. In this particular case, we can take $N_{W}=1$ ArKo. The algebra $K(\mathbf{v}) \mathcal{H}(W)$ is generated by elements $\mathbf{s}, \mathbf{t}_{1}, \ldots, \mathbf{t}_{n-1}$ satisfying the braid relations of type $B_{n}$ :
$\mathbf{s t}_{1} \mathbf{s} \mathbf{t}_{1}=\mathbf{t}_{1} \mathbf{s} \mathbf{t}_{1} \mathbf{s}, \quad \mathbf{s t}_{i}=\mathbf{t}_{i} \mathbf{s}$ and $\mathbf{t}_{i-1} \mathbf{t}_{i} \mathbf{t}_{i-1}=\mathbf{t}_{i} \mathbf{t}_{i-1} \mathbf{t}_{i}$ for $i=2, \ldots, n-1, \quad \mathbf{t}_{i} \mathbf{t}_{j}=\mathbf{t}_{j} \mathbf{t}_{i}$ for $|i-j|>1$, together with the extra relations:

$$
\left(\mathbf{s}-v_{\mathbf{s}, 0}\right)\left(\mathbf{s}-\zeta_{l} v_{\mathbf{s}, 1}\right) \cdots\left(\mathbf{s}-\zeta_{l}^{l-1} v_{\mathbf{s}, l-1}\right)=0 \text { and }\left(\mathbf{t}_{i}-v_{\mathbf{t}, 0}\right)\left(\mathbf{t}_{i}+v_{\mathbf{t}, 1}\right)=0 \text { for all } i=1, \ldots, n-1
$$

The Hecke algebra of $G(l, 1, n)$ is also known as Ariki-Koike algebra, with the last quadratic relation usually looking like this:

$$
\begin{equation*}
\left(\mathbf{t}_{i}-q\right)\left(\mathbf{t}_{i}+1\right)=0, \tag{3.2}
\end{equation*}
$$

where $q$ is an indeterminate. The irreducible representations of $G(l, 1, n)$, and thus the irreducible representations of $K(\mathbf{v}) \mathcal{H}(W)$, are parametrised by the $l$-partitions of $n$. We will talk more about Ariki-Koike algebras in the next chapter.

Let now $q$ be an indeterminate and let $\mathbf{m}=\left(m_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ be a family of integers. The $\mathbb{Z}_{K}$-algebra morphism

$$
\varphi_{\mathbf{m}}: \mathbb{Z}_{K}\left[\mathbf{v}, \mathbf{v}^{-1}\right] \rightarrow \mathbb{Z}_{K}\left[q, q^{-1}\right], v_{\mathcal{C}, j} \mapsto q^{m_{\mathcal{C}, j}}
$$

is called a cyclotomic specialisation. The $\mathbb{Z}_{K}\left[q, q^{-1}\right]$-algebra $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ obtained as the specialisation of $\mathcal{H}(W)$ via $\varphi_{\mathbf{m}}$ is called a cyclotomic Hecke algebra associated with $W$. The Iwahori-Hecke algebras defined in the previous chapter are cyclotomic Hecke algebras associated with real reflection groups.

Following [Ch4, Proposition 4.3.4], the algebra $K(q) \mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ is split semisimple. By Tits's deformation theorem, the specialisation $q \mapsto 1$ yields a bijection between $\operatorname{Irr}\left(K(q) \mathcal{H}_{\varphi_{\mathrm{m}}}(W)\right)$ and $\operatorname{Irr}(W)$.

Example 3.6 We say that $\varphi_{\mathbf{m}}$ is the spetsial cyclotomic specialisation if $m_{\mathcal{C}, 0}=1$ and $m_{\mathcal{C}, j}=0$ for all $1 \leqslant j \leqslant e_{\mathcal{C}}-1$. Fow $W=G(l, 1, n)$, the spetsial cyclotomic Hecke algebra $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ is generated by the elements $\mathbf{s}, \mathbf{t}_{1}, \ldots, \mathbf{t}_{n-1}$ satisfying the braid relations of type $B_{n}$, together with the extra relations:

$$
(\mathbf{s}-q)\left(\mathbf{s}^{l-1}+\mathbf{s}^{l-2}+\cdots+\mathbf{s}+1\right)=0 \text { and }\left(\mathbf{t}_{i}-q\right)\left(\mathbf{t}_{i}+1\right)=0 \text { for all } i=1, \ldots, n-1
$$

### 3.2 Schur elements and the $a$-function

The canonical symmetrising form $\tau$ on $\mathcal{H}(W)$ (see Assumptions 3.4) can be extended to $K(\mathbf{v}) \mathcal{H}(W)$ by extension of scalars, and can be used to define Schur elements $\left(s_{E}\right)_{E \in \operatorname{Irr}(W)}$ for $\mathcal{H}(W)$, as in the real case. The Schur elements of $\mathcal{H}(W)$ have been explicitly calculated for all complex reflection groups:

- for finite Coxeter groups see $\$ 2.2$.
- for complex reflection groups of type $G(l, 1, n)$ by Geck-Iancu-Malle GIM] and Mathas [Mat;
- for complex reflection groups of type $G(l, 2,2)$ by Malle Mal2;
- for the non-Coxeter exceptional complex reflection groups by Malle [Mal2, Mal4].

With the use of Clifford theory, we obtain the Schur elements for type $G(l, p, n)$ from those of type $G(l, 1, n)$ when $n>2$ or $n=2$ and $p$ is odd. The Schur elements for type $G(l, p, 2)$ when $p$ is even derive from those of type $G(l, 2,2)$ (see [Mal1], [Ch4, A.7]).

Using a case-by-case analysis, we have been able to determine that the Schur elements of $\mathcal{H}(W)$ have the following form [Ch4, Theorem 4.2.5].

Theorem 3.7 Let $E \in \operatorname{Irr}(W)$. The Schur element $s_{E}$ is an element of $\mathbb{Z}_{K}\left[\mathbf{v}, \mathbf{v}^{-1}\right]$ of the form

$$
\begin{equation*}
s_{E}=\xi_{E} N_{E} \prod_{i \in I_{E}} \Psi_{E, i}\left(M_{E, i}\right) \tag{3.3}
\end{equation*}
$$

where
(a) $\xi_{E}$ is an element of $\mathbb{Z}_{K}$,
(b) $N_{E}=\prod_{\mathcal{C}, j} v_{\mathcal{C}, j}^{b_{\mathcal{C}, j}}$ is a monomial in $\mathbb{Z}_{K}\left[\mathbf{v}, \mathbf{v}^{-1}\right]$ with $\sum_{j=0}^{e_{\mathcal{C}}-1} b_{\mathcal{C}, j}=0$ for all $\mathcal{C} \in \mathcal{A} / W$,
(c) $I_{E}$ is an index set,
(d) $\left(\Psi_{E, i}\right)_{i \in I_{E}}$ is a family of $K$-cyclotomic polynomials in one variable,
(e) $\left(M_{E, i}\right)_{i \in I_{E}}$ is a family of monomials in $\mathbb{Z}_{K}\left[\mathbf{v}, \mathbf{v}^{-1}\right]$ such that if $M_{E, i}=\prod_{\mathcal{C}, j} v_{\mathcal{C}, j}^{a_{\mathcal{C}, j}}$, then $\operatorname{gcd}\left(a_{\mathcal{C}, j}\right)=1$ and $\sum_{j=0}^{e_{\mathcal{C}}-1} a_{\mathcal{C}, j}=0$ for all $\mathcal{C} \in \mathcal{A} / W$.

Equation (3.3) gives the factorisation of $s_{E}$ into irreducible factors. The monomials $\left(M_{E, i}\right)_{i \in I_{E}}$ are unique up to inversion, and we will call them potentially essential for $W$.
Remark 3.8 Theorem 3.7 was independently obtained by Rouquier [Ro2, Theorem 3.5] using a general argument on rational Cherednik algebras.

Example 3.9 Let us consider the example of $\mathfrak{S}_{3}$, which is isomorphic to $G(1,1,3)$. We have $s_{E^{(3)}}=\Phi_{2}\left(v_{\mathbf{t}, 0} v_{\mathbf{t}, 1}^{-1}\right) \Phi_{3}\left(v_{\mathbf{t}, 0} v_{\mathbf{t}, 1}^{-1}\right), s_{E^{(2,1)}}=v_{\mathbf{t}, 0}^{-1} v_{\mathbf{t}, 1} \Phi_{3}\left(v_{\mathbf{t}, 0} v_{\mathbf{t}, 1}^{-1}\right), s_{E^{(1,1,1)}}=v_{\mathbf{t}, 0}^{-3} v_{\mathbf{t}, 1}^{3} \Phi_{2}\left(v_{\mathbf{t}, 0} v_{\mathbf{t}, 1}^{-1}\right) \Phi_{3}\left(v_{\mathbf{t}, 0} v_{\mathbf{t}, 1}^{-1}\right)$, where $\Phi_{n}$ denotes the $n$-th $\mathbb{Q}$-cyclotomic polynomial.

Let $\varphi_{\mathrm{m}}: v_{\mathcal{C}, j} \mapsto q^{m_{\mathcal{C}, j}}$ be a cyclotomic specialisation. The canonical symmetrising form on $\mathcal{H}(W)$ specialises via $\varphi_{\mathrm{m}}$ to become a canonical symmetrising form $\tau_{\varphi_{\mathrm{m}}}$ on $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$. The Schur elements of $\mathcal{H}_{\varphi_{\mathbf{m}}}(W)$ with respect to $\tau_{\varphi_{\mathrm{m}}}$ are $\left(\varphi_{\mathbf{m}}\left(s_{E}\right)\right)_{E \in \operatorname{Irr}(W)}$, hence they can be written in the form 2.2. We can again define the $a$-function $a^{\mathbf{m}}: \operatorname{Irr}(W) \rightarrow \mathbb{Z}$ and the $A$-function $A^{\mathbf{m}}: \operatorname{Irr}(W) \rightarrow \mathbb{Z}$ by setting

$$
a_{E}^{\mathbf{m}}:=-\operatorname{val}_{q}\left(\varphi_{\mathbf{m}}\left(s_{E}\right)\right) \quad \text { and } \quad A_{E}^{\mathbf{m}}:=-\operatorname{deg}_{q}\left(\varphi_{\mathbf{m}}\left(s_{E}\right)\right)
$$

### 3.3 Families of characters and Rouquier families

Let $\varphi_{\mathbf{m}}: v_{\mathcal{C}, j} \mapsto q^{m_{\mathcal{C}, j}}$ be a cyclotomic specialisation and let $\mathcal{H}_{\varphi_{\mathbf{m}}}(W)$ be the corresponding cyclotomic Hecke algebra associated with $W$. How can we define families of characters for $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ ? We cannot apply Lusztig's original definition, because parabolic subgroups of complex reflection groups $\$^{11}$ do not have a nice presentation as in the real case, and certainly not a "corresponding" parabolic Hecke algebra. On the other hand, we do not have a Kazhdan-Lusztig basis for $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$, so we cannot construct KazhdanLusztig cells and use them to define families of characters for complex reflection groups in the usual way. However, we can define the families of characters to be the Rouquier families of $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$, that is, the blocks of $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ over the Rouquier ring $\mathcal{R}_{K}(q)$, where

$$
\mathcal{R}_{K}(q)=\mathbb{Z}_{K}\left[q, q^{-1},\left(q^{n}-1\right)_{n \geqslant 1}^{-1}\right] .
$$

By Proposition 1.4, the Rouquier families are the non-empty subsets $B$ of $\operatorname{Irr}(W)$ that are minimal with respect to the property:

$$
\sum_{E \in B} \frac{\varphi_{\mathbf{m}}\left(\chi_{E}(h)\right)}{\varphi_{\mathbf{m}}\left(s_{E}\right)} \in \mathcal{R}_{K}(q) \quad \forall h \in \mathcal{H}(W)
$$

where $\chi_{E}$ and $s_{E}$ are respectively the character and the Schur element of $K(\mathbf{v}) \mathcal{H}(W)$ corresponding to $E \in \operatorname{Irr}(W)$.

Broué and Kim BrKi determined the Rouquier families for the complex reflection groups of type $G(l, 1, n)$, but their results are only true when $l$ is a power of a prime number or $\varphi_{\mathrm{m}}$ is a "good" cyclotomic specialisation (for example, we have shown that their results hold when $\varphi_{\mathbf{m}}$ is the spetsial cyclotomic specialisation [Ch3, Proposition 3.20]). The same problem persists, and some new appear, in the determination of the Rouquier families for $G(l, p, n)$ by Kim Kim. Malle and Rouquier MalRo calculated the Rouquier families for some exceptional complex reflection groups and the dihedral groups, for the spetsial cyclotomic specialisation. More recently, we managed to determine the Rouquier families for all cyclotomic Hecke algebras of all complex reflection groups Ch1, Ch3, Ch4, Ch5, thanks to their property of "semicontinuity". In order to explain this property, we will need some definitions.

Let $M=\prod_{\mathcal{C}, j} v_{\mathcal{C}, j}^{a_{\mathcal{C}, j}}$ be a potentially essential monomial for $W$. We say that the family of integers $\mathbf{m}=\left(m_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ belongs to the potentially essential hyperplane $H_{M}\left(\right.$ of $\left.\mathbb{R}^{\sum_{\mathcal{C}} e_{\mathcal{C}}}\right)$ if $\sum_{\mathcal{C}, j} m_{\mathcal{C}, j} a_{\mathcal{C}, j}=0$.

Suppose that $\mathbf{m}$ belongs to no potentially essential hyperplane. Then the Rouquier families of $\mathcal{H}_{\varphi_{\mathbf{m}}}(W)$ are called Rouquier families associated with no essential hyperplane. Now suppose that $\mathbf{m}$ belongs to a unique potentially essential hyperplane $H$. Then the Rouquier families of $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ are unions of the Rouquier families associated with no essential hyperplane and they are called Rouquier families associated with $H$. If they do not coincide with the Rouquier families associated with no essential hyperplane, then $H$ is called an essential hyperplane for $W$. All these notions are well-defined and they do not depend on the choice of $\mathbf{m}$ because of the following theorem [Ch4, §4.4]:

Theorem 3.10 (Semicontinuity property of Rouquier families) Let $\mathbf{m}=\left(m_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ be a family of integers and let $\varphi_{\mathbf{m}}: v_{\mathcal{C}, j} \mapsto q^{m_{\mathcal{C}, j}}$ be the corresponding cyclotomic specialisation. The Rouquier families of $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ are unions of the Rouquier families associated with the essential hyperplanes that $\mathbf{m}$ belongs to and they are minimal with respect to this property.

Thanks to the above result, it is enough to do calculations in a finite number of cases in order to obtain the families of characters for all cyclotomic Hecke algebras, whose number is infinite.

Example 3.11 For $W=\mathfrak{S}_{3}$, the Rouquier families associated with no essential hyperplane are trivial (that is, every block is a singleton). The hyperplane $H_{M}$ corresponding to the monomial $M=v_{\mathbf{t}, 0} v_{\mathbf{t}, 1}^{-1}$ is essential, and it is the unique essential hyperplane for $\mathfrak{S}_{3}$. Let $\varphi_{\mathbf{m}}: v_{\mathbf{t}, j} \mapsto q^{m_{j}}, j=0,1$, be a cyclotomic specialisation. We have that $\mathbf{m}=\left(m_{0}, m_{1}\right)$ belongs to $H_{M}$ if and only if $m_{0}=m_{1}$. There is a single Rouquier family associated with $H_{M}$, which contains all irreducible representations of $\mathfrak{S}_{3}$.

[^0]We have computed the Rouquier families associated with no and each essential hyperplane:

- With the use of GAP3 for all exceptional complex reflection groups [h4; the results can be found on my webpage Ch-web, while very recently, together with Jean Michel, we programmed the algorithm described in Ch4] in the GAP3 package CHEVIE (see chevie/contr/rouquierblockdata.g).
- Combinatorially for the groups of type $G(l, 1, n)$ Ch3; we will see more on these results in the next chapter.
- With the use of Clifford theory for the groups $G(l, p, n)$, for $n>2$ or $n=2$ and $p$ odd, from the Rouquier families for $G(l, 1, n)$ [Ch5].
- Applying the algorithm of Ch4 for $G(l, 2,2)$ Ch5.
- With the use of Clifford theory for the groups $G(l, p, 2)$, for $p$ even, from the Rouquier families for $G(l, 2,2)$ Ch5.

A consequence of the above results is the following:
Proposition 3.12 The Rouquier families associated with no essential hyperplane are trivial if and only if $W=G_{4}$ or $W$ is of type $G(l, 1, n)$.

Note that, by definition, the Rouquier families associated with an essential hyperplane are not trivial for any complex reflection group.

Now, part of the algorithm used for the determination of the Rouquier families is the easy-to-prove fact that the function $a+A$ is constant on the Rouquier families (see, for example, BrKi, Proposition 2.9]). However, we have also managed to show that, similarly to the real case, both functions $a$ and $A$ are constant on the Rouquier families, for all cyclotomic Hecke algebras of all complex reflection groups:

- With the use of GAP3 for all exceptional complex reflection groups Ch2; the programme used, Degval.g, is available on my webpage Ch-web.
- Combinatorially for the groups of type $G(l, 1, n)$ BrKi, Ch3.
- With the use of Clifford theory for the groups $G(l, p, n)$, for $n>2$ or $n=2$ and $p$ odd, from the analogous result for $G(l, 1, n)$ Ch5.
- With explicit calculations for $G(l, 2,2)$ Ch5.
- With the use of Clifford theory for the groups $G(l, p, 2)$, for $p$ even, from the analogous result for $G(l, 2,2)$ Ch5.

Finally, we would like to point out that the semicontinuity property of the Rouquier families is part of the block behaviour of algebras whose Schur elements are of the form (3.3). We have studied the block theory of this type of algebras, which we call essential algebras, in Ch4. Besides generic Hecke algebras of complex reflection groups, another example of essential algebras are the Yokonuma-Hecke algebras which will be studied in the last chapter.

### 3.4 Decomposition maps and canonical basic sets

Given a cyclotomic Hecke algebra $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ and a ring homomorphism $\theta: q \mapsto \eta \in \mathbb{C} \backslash\{0\}$, we obtain a semisimplicity criterion and a decomposition map exactly as in $\$ 2.4$. A canonical basic set with respect to $\theta$ is also defined in the same way.

In [ChJa1, we showed the existence of canonical basic sets with respect to any $\theta$ for all cyclotomic Hecke algebras associated with finite Coxeter groups, that is, when the weight function $L$ in the definition of $\mathcal{H}_{q}(W, L)$ is also allowed to take negative values.

For non-Coxeter complex reflection groups, things become more complicated. For $W=G(l, 1, n)$, consider the specialised Ariki-Koike algebra with relations

$$
\left(\mathbf{s}-\zeta_{e}^{s_{0}}\right)\left(\mathbf{s}-\zeta_{e}^{s_{1}}\right) \cdots\left(\mathbf{s}-\zeta_{e}^{s_{l-1}}\right)=0, \quad\left(\mathbf{t}_{i}-\zeta_{e}\right)\left(\mathbf{t}_{i}+1\right)=0 \quad \text { for } i=1, \ldots, n-1,
$$

where $\left(s_{0}, \ldots, s_{l-1}\right) \in \mathbb{Z}^{l}$ and $e \in \mathbb{Z}_{>0}$. With the use of Ariki's Theorem Ar2 and Uglov's work on canonical bases for higher level Fock spaces Ug , Geck and Jacon GeJa1, Ja2, Ja3, GeJa2] have shown that, for a suitable choice of $\mathbf{m}$, the corresponding function $a^{\mathbf{m}}$ yields a canonical basic set for the above specialised Ariki-Koike algebra. However, this does not work the other way round: not all cyclotomic Ariki-Koike algebras admit canonical basic sets. We will see more on canonical basic sets for type $G(l, 1, n)$ in the next chapter.

In ChJa2, building on work by Genet and Jacon GenJa, we generalised the above result to obtain canonical basic sets for all groups of type $G(l, p, n)$ with $n>2$, or $n=2$ and $p$ odd.

Finally, for the exceptional complex reflection groups of rank $2\left(G_{4}, \ldots, G_{22}\right)$, we have shown the existence of canonical basic sets for the cyclotomic Hecke algebras appearing in BrMa with respect to any specialisation $\theta$ ChMi.

## Chapter 4

## Ariki-Koike Algebras

Ariki-Koike algebras were introduced by Ariki and Koike ArKo as generalisations of Iwahori-Hecke algebras of types $A$ and $B$. They can be viewed as the Hecke algebras associated with the complex reflection groups of type $G(l, 1, n)$. In this chapter, we will switch from the notation for the indeterminates used in Example 3.5 to the notation more traditionally used for Ariki-Koike algebras.

Let $R$ be a commutative integral domain with 1 . Fix elements $q, Q_{0}, \ldots, Q_{l-1}$ of $R$, and assume that $q$ is invertible in $R$. Set $\mathbf{q}:=\left(Q_{0}, \ldots, Q_{l-1} ; q\right)$. The Ariki-Koike algebra $\mathcal{H}_{n}^{\mathbf{q}}$ is the unital associative $R$-algebra with generators $\mathbf{s}, \mathbf{t}_{1}, \ldots, \mathbf{t}_{n-1}$ and relations:

$$
\begin{aligned}
\left(\mathbf{s}-Q_{0}\right)\left(\mathbf{s}-Q_{1}\right) \cdots\left(\mathbf{s}-Q_{l-1}\right)=0 & \\
\left(\mathbf{t}_{i}-q\right)\left(\mathbf{t}_{i}+1\right)=0 & \text { for } 1 \leqslant i \leqslant n-1 \\
\mathbf{s t}_{1} \mathbf{s t}_{1}=\mathbf{t}_{1} \mathbf{s t}_{1} \mathbf{s} & \\
\mathbf{t}_{i} \mathbf{t}_{i+1} \mathbf{t}_{i}=\mathbf{t}_{i+1} \mathbf{t}_{i} \mathbf{t}_{i+1} & \text { for } 1 \leqslant i \leqslant n-2 \\
\mathbf{t}_{i} \mathbf{t}_{j}=\mathbf{t}_{j} \mathbf{t}_{i} & \text { for } 0 \leqslant i<j \leqslant n-1 \text { with } j-i>1 .
\end{aligned}
$$

The Ariki-Koike algebra $\mathcal{H}_{n}^{\mathbf{q}}$ is a deformation of the group algebra of the complex reflection group $G(l, 1, n)$.

For $w \in \mathfrak{S}_{n}$, if $w=s_{i_{1}} \ldots s_{i_{r}}$ is a reduced expression for $w$, with $s_{i}=(i, i+1)$, set $\mathbf{t}_{w}:=\mathbf{t}_{i_{1}} \ldots \mathbf{t}_{i_{r}}$. Further, define elements $X_{m}:=q^{1-m} \mathbf{t}_{m-1} \ldots \mathbf{t}_{1} \mathbf{s} \mathbf{t}_{1} \ldots \mathbf{t}_{m-1}$ for $m=1,2, \ldots, n$. Ariki and Koike ArKo, have proved that the set

$$
\begin{equation*}
\left\{X_{1}^{a_{1}} X_{2}^{a_{2}} \ldots X_{n}^{a_{n}} \mathbf{t}_{w} \mid w \in \mathfrak{S}_{n}, 0 \leqslant a_{m} \leqslant l-1 \text { for all } m=1,2, \ldots, n\right\} \tag{4.1}
\end{equation*}
$$

is a basis of $\mathcal{H}_{n}^{\mathbf{q}}$ as an $R$-module. In particular, $\mathcal{H}_{n}^{\mathbf{q}}$ is a free $R$-module of rank $l^{n} n!=|G(l, 1, n)|$.
In addition, when $R$ is a field, Ariki and Koike have constructed an $\mathcal{H}_{n}^{\text {q }}$-module $V^{\boldsymbol{\lambda}}$ for each $l$-partition $\boldsymbol{\lambda}$ of $n$, which is called the Specht module. These modules form a complete set of non-isomorphic simple modules in the case where $\mathcal{H}_{n}^{\mathbf{q}}$ is semisimple. By Ariki's semisimplicity criterion [Ar1, the algebra $\mathcal{H}_{n}^{\mathbf{q}}$ is (split) semisimple if and only if

$$
\begin{equation*}
\prod_{1 \leqslant i \leqslant n}\left(1+q+\cdots+q^{i-1}\right) \prod_{0 \leqslant s<t \leqslant l-1} \prod_{-n<k<n}\left(q^{k} Q_{s}-Q_{t}\right) \neq 0 \tag{4.2}
\end{equation*}
$$

### 4.1 Combinatorics of partitions and multipartitions

Let $l$ and $n$ be positive integers.

### 4.1.1 Partitions

A partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right)$ is a family of positive integers such that $\lambda_{1} \geqslant \lambda_{2} \geqslant \ldots \geqslant \lambda_{h} \geqslant 1$. We call $h$ the length or height of the partition $\lambda$. We write $|\lambda|:=\sum_{i=1}^{h} \lambda_{i}$ and we say that $\lambda$ is a partition of $n$
if $n=|\lambda|$. We denote by $\mathcal{P}(n)$ the set of partitions of $n$. To each partition $\lambda$ we associate its $\beta$-number, $\beta_{\lambda}=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{h}\right)$, defined by

$$
\beta_{1}:=h+\lambda_{1}-1, \beta_{2}:=h+\lambda_{2}-2, \ldots, \beta_{h}:=h+\lambda_{h}-h=\lambda_{h} .
$$

Note that $\beta_{1}>\beta_{2}>\cdots>\beta_{h} \geqslant 1$.
We define the set of nodes [ $\lambda$ ] of $\lambda$ to be the set

$$
[\lambda]:=\left\{(i, j) \mid 1 \leqslant i \leqslant h, 1 \leqslant j \leqslant \lambda_{i}\right\} .
$$

We identify partitions with their Young diagrams: the Young diagram of $\lambda$ is a left-justified array of $h$ rows such that the $i$-th row contains $\lambda_{i}$ boxes (nodes) for all $i=1, \ldots, h$.

The conjugate partition $\lambda^{\prime}=\left(\lambda_{1}^{\prime}, \ldots, \lambda_{h^{\prime}}^{\prime}\right)$ of $\lambda$ is defined by $\lambda_{j}^{\prime}:=\#\left\{i \mid 1 \leqslant i \leqslant h\right.$ such that $\left.\lambda_{i} \geqslant j\right\}$. We have $h^{\prime}=\lambda_{1}$. The Young diagram of $\lambda^{\prime}$ is the transpose of the Young diagram of $\lambda$. A node $(i, j) \in[\lambda]$ if and only if $(j, i) \in\left[\lambda^{\prime}\right]$.

If $x=(i, j) \in[\lambda]$ and $\mu$ is another partition, we define the generalised hook length of $x$ with respect to $(\lambda, \mu)$ to be the integer:

$$
h_{i, j}^{\lambda, \mu}:=\lambda_{i}-i+\mu_{j}^{\prime}-j+1 .
$$

For $\mu=\lambda$, the above formula becomes the classical hook length formula (giving the length of the hook of $\lambda$ that $x$ belongs to). Moreover, we define the content of $x$ to be the difference $j-i$.

Finally, we set

$$
N(\lambda):=\sum_{i=1}^{h}(i-1) \lambda_{i}=\frac{1}{2} \sum_{i=1}^{h^{\prime}}\left(\lambda_{i}^{\prime}-1\right) \lambda_{i}^{\prime}=\sum_{i=1}^{h^{\prime}}\binom{\lambda_{i}^{\prime}}{2} .
$$

### 4.1.2 Multipartitions

An $l$-partition (or multipartition) of $n$ is an ordered $l$-tuple $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(l-1)}\right)$ of partitions such that $|\boldsymbol{\lambda}|:=\sum_{i=0}^{l-1}\left|\lambda^{(i)}\right|=n$. We denote by $\mathcal{P}(l, n)$ the set of $l$-partitions of $n$. The empty multipartition, denoted by $\emptyset$, is an $l$-tuple of empty partitions. If $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(l-1)}\right) \in \mathcal{P}(l, n)$, we denote by $\boldsymbol{\lambda}^{\prime}$ the $l$-partition $\left(\lambda^{(0)^{\prime}}, \lambda^{(1)^{\prime}}, \ldots, \lambda^{(l-1)^{\prime}}\right)$. For $i=0,1, \ldots, l-1$, we denote by $h^{(i)}$ and $\beta^{(i)}$ the height and $\beta$-number of $\lambda^{(i)}$ respectively. Finally, we denote by $\overline{\boldsymbol{\lambda}}$ the simple partition of $n$ obtained by writing in decreasing order all the entries of $\boldsymbol{\lambda}$ (that is, all the elements of the multiset $\lambda^{(0)} \cup \lambda^{(1)} \cup \ldots \cup \lambda^{(l-1)}$ ).

Example 4.1 Let us take $l=2, n=6$ and $\boldsymbol{\lambda}=((2,1),(3))$. We have $\boldsymbol{\lambda}^{\prime}=((2,1),(1,1,1))$ and $\overline{\boldsymbol{\lambda}}=(3,2,1)$.

If $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{h}\right)$ is a sequence of positive integers such that $\beta_{1}>\beta_{2}>\cdots>\beta_{h}$ and $m$ is a positive rational number, then the $m$-shifted of $\beta$ is the sequence of numbers defined by

$$
\beta[m]=\left(\beta_{1}+m, \beta_{2}+m, \ldots, \beta_{h}+m, m-1, m-2, \ldots, m-[m]+1, m-[m]\right),
$$

where $[m$ ] denotes the integer part of $m$.
From now on, we suppose that we have a given weight system $\mathbf{m}:=\left(m_{0}, m_{1}, \ldots, m_{l-1}\right) \in \mathbb{Q}^{l}$. We define the $(l, \mathbf{m})$-charged height of $\boldsymbol{\lambda}$ to be the family $\left(h c^{(0)}, h c^{(1)}, \ldots, h c^{(l-1)}\right)$, where

$$
h c^{(0)}:=h^{(0)}-m_{0}, h c^{(1)}:=h^{(1)}-m_{1}, \ldots, h c^{(l-1)}:=h^{(l-1)}-m_{l-1}
$$

and the $\mathbf{m}$-charged height of $\boldsymbol{\lambda}$ to be the integer

$$
h c_{\boldsymbol{\lambda}}:=\max _{0 \leqslant i \leqslant l-1} h c^{(i)} .
$$

For $i=0,1, \ldots, l-1$, we set

$$
B c_{\boldsymbol{\lambda}}^{(i)}:=\beta^{(i)}\left[h c_{\boldsymbol{\lambda}}-h c^{(i)}\right] .
$$

We then have

$$
B c_{\boldsymbol{\lambda}}^{(i)}=\left(b c_{j}^{(i)}\right)_{1 \leqslant j \leqslant h c_{\boldsymbol{\lambda}}+\left[m_{i}\right]} \quad \text { where } \quad b c_{j}^{(i)}=\beta_{j}^{(i)}+h c_{\boldsymbol{\lambda}}-h c^{(i)}=\lambda_{j}^{(i)}-j+h c_{\boldsymbol{\lambda}}+m_{i}
$$

$\left(\operatorname{taking} \beta_{j}^{(i)}=\lambda_{j}^{(i)}=0\right.$ for $\left.j>h^{(i)}\right)$.
The $\mathbf{m}$-charged standard symbol of $\boldsymbol{\lambda}$ is the family of numbers defined by

$$
B c_{\boldsymbol{\lambda}}=\left(B c_{\boldsymbol{\lambda}}^{(0)}, B c_{\boldsymbol{\lambda}}^{(1)}, \ldots, B c_{\boldsymbol{\lambda}}^{(l-1)}\right)
$$

It is a tableau of numbers arranged into $l$ rows indexed by the set $\{0,1, \ldots, l-1\}$ such that the $i$-th row has length equal to $h c_{\boldsymbol{\lambda}}+\left[m_{i}\right]$. In general, an $\mathbf{m}$-charged symbol of $\boldsymbol{\lambda}$ is a family of numbers obtained from the $\mathbf{m}$-charged standard symbol of $\boldsymbol{\lambda}$ by shifting all the rows by the same positive rational number.

The $\mathbf{m}$-charged content of $\boldsymbol{\lambda}$ is the multiset

$$
\operatorname{Contc}_{\boldsymbol{\lambda}}=B c_{\boldsymbol{\lambda}}^{(0)} \cup B c_{\boldsymbol{\lambda}}^{(1)} \cup \ldots \cup B c_{\boldsymbol{\lambda}}^{(l-1)}
$$

Example 4.2 Let us take $l=2, n=6, \boldsymbol{\lambda}=((2,1),(3))$ and $\mathbf{m}=(-1,2)$. Then

$$
B c_{\boldsymbol{\lambda}}=\left(\begin{array}{lllll}
3 & 1 & & & \\
7 & 3 & 2 & 1 & 0
\end{array}\right)
$$

We have Contc $_{\boldsymbol{\lambda}}=\{7,3,3,2,1,1,0\}$.
Let $s \in \mathbb{Z}$ with $s \geqslant h c_{\boldsymbol{\lambda}}$. Let $B c_{\boldsymbol{\lambda}}^{s}$ be the $\mathbf{m}$-charged symbol of $\boldsymbol{\lambda}$ obtained from $B c_{\boldsymbol{\lambda}}$ by shifting all the rows by $s-h c_{\boldsymbol{\lambda}}$. Let $\kappa_{1} \geqslant \kappa_{2} \geqslant \cdots \geqslant \kappa_{t}$, with $t=l s+\sum_{i=0}^{l-1}\left[m_{i}\right]$, be the elements of the multiset $B c_{\boldsymbol{\lambda}}^{s}$ written in decreasing order (if $s=h c_{\boldsymbol{\lambda}}$, these are simply the elements of Contc $\boldsymbol{\lambda}_{\boldsymbol{\lambda}}$ ). We set

$$
n_{\mathbf{m}}(\boldsymbol{\lambda}):=\sum_{i=1}^{t}(i-1) \kappa_{i}
$$

Let $r \in \mathbb{Z}_{>0}$, By GeJa2, Proposition 5.5.11], the following element of $\mathbb{Q}$ does not depend on the choice of the integer $s$ :

$$
\begin{equation*}
\mathbf{a}^{(\mathbf{m}, r)}(\boldsymbol{\lambda}):=r\left(n_{\mathbf{m}}(\boldsymbol{\lambda})-n_{\mathbf{m}}(\emptyset)\right) . \tag{4.3}
\end{equation*}
$$

We will see in the next section that $\mathbf{a}^{(\mathbf{m}, r)}$ corresponds to the $a$-function of an Ariki-Koike algebra.

### 4.2 Schur elements and the $a$-function

Let $\tau: \mathcal{H}_{n}^{\mathbf{q}} \rightarrow R$ be the linear map defined by

$$
\tau\left(X_{1}^{a_{1}} X_{2}^{a_{2}} \ldots X_{n}^{a_{n}} \mathbf{t}_{w}\right)= \begin{cases}1 & \text { if } a_{1}=a_{2}=\cdots=a_{n}=0 \text { and } w=1 \\ 0 & \text { otherwise }\end{cases}
$$

where $w \in \mathfrak{S}_{n}$ and $0 \leqslant a_{m} \leqslant l-1$ for all $m=1,2, \ldots, n$. The function $\tau$ was introduced by Bremke and Malle BreMa who showed that $\tau$ is a trace form and that $\tau$ is essentially independent of the choice of basis of $\mathcal{H}_{n}^{q}$. It is not obvious from the definition above that $\tau$ coincides with the form introduced by Bremke and Malle; however, this was proved by Malle and Mathas in MalMat, where they also showed that $\tau$ is a canonical symmetrising form on $\mathcal{H}_{n}^{\mathbf{q}}$ whenever all $Q_{i}$ 's are invertible in $R$. As we will see in Chapter 6, $\tau$ is the Markov trace on $\mathcal{H}_{n}^{\mathbf{q}}$ with all parameters equal to 0 .

From now on, we assume that all $q, Q_{0}, \ldots, Q_{l-1}$ are invertible in $R$. Let $K$ be a field containing $R$ such that $K \mathcal{H}_{n}^{\mathbf{q}}$ is split semisimple. As mentioned in the beginning of this chapter, but also due to Tits's deformation theorem, the simple modules of $K \mathcal{H}_{n}^{\mathbf{q}}$ are parametrised by the $l$-partitions of $n$. For $\boldsymbol{\lambda} \in \mathcal{P}(l, n)$, we denote by $V^{\boldsymbol{\lambda}}$ the corresponding simple module and by $s_{\boldsymbol{\lambda}}(\mathbf{q})$ the Schur element of $V^{\boldsymbol{\lambda}}$ with respect to $\tau$. As we saw in $\S 1.2, s_{\boldsymbol{\lambda}}(\mathbf{q})$ belongs to the integral closure $R_{K}$ of $R$ in $K$.

For example, we can take $q, Q_{0}, \ldots, Q_{l-1}$ to be indeterminates, $R=\mathbb{Z}\left[q^{ \pm 1}, Q_{0}^{ \pm 1}, \ldots, Q_{l-1}^{ \pm 1}\right]$ and $K=$ $\mathbb{Q}\left(q, Q_{0}, \ldots, Q_{l-1}\right)$. In this case, we call $\mathcal{H}_{n}^{\mathbf{q}}$ the generic Ariki-Koike algebra, and we have $s_{\boldsymbol{\lambda}}(\mathbf{q}) \in$ $\mathbb{Z}\left[q^{ \pm 1}, Q_{0}^{ \pm 1}, \ldots, Q_{l-1}^{ \pm 1}\right]$ for all $\boldsymbol{\lambda} \in \mathcal{P}(l, n)$.

The Schur elements of $\mathcal{H}_{n}^{\mathbf{q}}$ have been calculated independently by Geck, Iancu and Malle GIM, and by Mathas Mat. Their papers provide different combinatorial formulas for the calculation of $s_{\boldsymbol{\lambda}}(\mathbf{q})$, which are described in both cases as quotients of elements of $K$. Even though it was theoretically known that these quotients belonged to $R_{K}$, there was no combinatorial formula to describe the terms remaining after the division. In ChJa2, we managed to obtain a cancellation-free formula for the Schur elements of $\mathcal{H}_{n}^{\mathbf{q}}$ :

Theorem 4.3 Let $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(l-1)}\right)$ be an l-partition of $n$. Then

$$
s_{\boldsymbol{\lambda}}(\mathbf{q})=(-1)^{n(l-1)} q^{-N(\overline{\boldsymbol{\lambda}})}(q-1)^{-n} \prod_{0 \leqslant s \leqslant l-1} \prod_{(i, j) \in\left[\lambda^{(s)}\right]} \prod_{0 \leqslant t \leqslant l-1}\left(q^{h_{i, j}^{\lambda^{(s)}, \lambda^{(t)}}} Q_{s} Q_{t}^{-1}-1\right)
$$

Since the total number of nodes in $\boldsymbol{\lambda}$ is equal to $n$, the above formula can be rewritten as follows:

$$
s_{\lambda}(\mathbf{q})=(-1)^{n(l-1)} q^{-N(\overline{\boldsymbol{\lambda}})} \prod_{0 \leqslant s \leqslant l-1} \prod_{(i, j) \in\left[\lambda^{(s)}\right]}\left(\left[h_{i, j}^{\lambda^{(s)}, \lambda^{(s)}}\right]_{q} \prod_{0 \leqslant t \leqslant l-1, t \neq s}\left(q^{h_{i, j}^{\lambda^{(s)}, \lambda^{(t)}}} Q_{s} Q_{t}^{-1}-1\right)\right),
$$

where $[m]_{q}:=\left(q^{m}-1\right) /(q-1)=q^{m-1}+q^{m-2}+\cdots+q+1$, for all $m \in \mathbb{Z}_{>0}$.
Jean Michel and the author programmed the cancellation-free formula for the Schur elements of Ariki-Koike algebras into the GAP3 package CHEVIE. This made the algorithm for the computation of the Schur elements faster than the pre-existing one.

Now, applying Theorem 1.7 with the formulas for the Schur elements given in Theorem 4.3 yields easily Ariki's semisimplicity criterion (4.2). Further, if we take $q=u^{r}$ and $Q_{i}=\zeta_{l}^{i} u^{r_{i}}$ for all $i=0, \ldots, l-1$, where $u$ is an indeterminate, $r \in \mathbb{Z}_{>0}$ and $r_{0}, \ldots, r_{l-1} \in \mathbb{Z}$, then the algebra $\mathcal{H}_{n}^{\mathbf{q}}$ is a cyclotomic ArikiKoike algebra (since it can be obtained via the cyclotomic specialisation $\varphi: q \mapsto u^{r}, Q_{i} \mapsto \zeta_{l}^{i} u^{r_{i}}$ from the generic Ariki-Koike algebra). Set $m_{i}:=r_{i} / r$ for all $i=0, \ldots, l-1$ and $\mathbf{m}:=\left(m_{0}, \ldots, m_{l-1}\right) \in \mathbb{Q}^{l}$. Let $\boldsymbol{\lambda} \in \mathcal{P}(d, n)$. Then the $a$-value of the irreducible representation $E^{\boldsymbol{\lambda}}$ of $G(l, 1, n)$ corresponding to $\boldsymbol{\lambda}$ (that is, the negative of the valuation of $\left.s_{\boldsymbol{\lambda}}(\mathbf{q})\right)$ is equal to $\mathbf{a}^{(\mathbf{m}, r)}$ GeJa2, Proposition 5.5.11]. Theorem 4.3 allows us to give an alternative description of the $a$-value of $E^{\boldsymbol{\lambda}}$ [ChJa2, Proposition 4.5]:

Proposition 4.4 Let $\mathcal{H}_{n}^{\mathbf{m}, r}$ be the cyclotomic Ariki-Koike algebra defined above, and let $\boldsymbol{\lambda} \in \mathcal{P}(d, n)$. The a-value of $E^{\boldsymbol{\lambda}}$ is

$$
\mathbf{a}^{(\mathbf{m}, r)}(\boldsymbol{\lambda})=r\left(N(\overline{\boldsymbol{\lambda}})-\sum_{0 \leqslant s \leqslant l-1} \sum_{(i, j) \in\left[\lambda^{(s)}\right]} \sum_{0 \leqslant t \leqslant l-1, t \neq s} \min \left(h_{i, j}^{\lambda^{(s)}, \lambda^{(t)}}+m_{s}-m_{t}, 0\right)\right) .
$$

### 4.3 Families of characters and Rouquier families

The Rouquier families for cyclotomic Hecke algebras associated with the complex reflection group $G(l, 1, n)$, that is, cyclotomic Ariki-Koike algebras, have been first determined by Broué and Kim BrKi], but it turned out that their results were only true in the case where $l$ is a power of a prime number.

In [Ch3, we gave the correct families for any $l$, using the property of semicontinuity explained in the previous chapter. In fact, we discovered that the Rouquier families for $G(l, 1, n)$ are determined by the families of characters of Weyl groups of type $B$.

Let $r, r_{0}, r_{1}, \ldots, r_{l} \in \mathbb{Z}$. If $r \neq 0$, set $m_{i}:=r_{i} / r$ for all $i=0, \ldots, l-1$ and $\mathbf{m}:=\left(m_{0}, \ldots, m_{l-1}\right) \in \mathbb{Q}^{l}$. Let $u$ be an indeterminate. Let

$$
\varphi: \mathbb{Z}\left[q^{ \pm 1}, Q_{0}^{ \pm 1}, \ldots, Q_{l-1}^{ \pm 1}\right] \rightarrow \mathbb{Z}\left[u^{ \pm 1}\right]
$$

be the cyclotomic specialisation defined by

$$
\varphi(q)=u^{r} \quad \text { and } \quad \varphi\left(Q_{i}\right)=\zeta_{l}^{i} u^{r_{i}} .
$$

Then we have the following result due to Broué and Kim BrKi].
Proposition 4.5 Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}(l, n)$ and assume that $r \neq 0$. If $E^{\boldsymbol{\lambda}}$ and $E^{\boldsymbol{\mu}}$ are in the same Rouquier family, then $\operatorname{Contc}_{\boldsymbol{\lambda}}=\operatorname{Contc}_{\boldsymbol{\mu}}$ with respect to the weight system $\mathbf{m}$. The converse is true when $l$ is a power of a prime number.

Now, using the form of the Schur elements, we obtain that the essential hyperplanes for $G(l, 1, n)$ are:

- $k r+r_{s}-r_{t}=0$ for all $-n<k<n$ if $\zeta_{l}^{s}-\zeta_{l}^{t}$ is not a unit in $\mathbb{Z}\left[\zeta_{l}\right]$.
- $r=0$.

Remark 4.6 Note that, according to Theorem 3.10, the Rouquier families do not depend on the actual values of the $r_{i}$ 's and $r$, but only on the essential hyperplanes to which they belong. Therefore, as long as $r \neq 0$, we can always restrict to the case $r=1$.

It is obvious from the form of the Schur elements that the Rouquier families associated with no essential hyperplane are all trivial. We now have the following two results about the Rouquier families associated with the essential hyperplanes described above [Ch3, Propositions 3.15, 3.16, 3.17]:

Proposition 4.7 Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}(l, n)$. The following are equivalent :
(1) $E^{\boldsymbol{\lambda}}$ and $E^{\boldsymbol{\mu}}$ are in the same Rouquier family associated with the essential hyperplane $k r+r_{s}-r_{t}=0$.
(2) $\lambda^{(i)}=\mu^{(i)}$ for all $i \notin\{s, t\}$ and $\operatorname{Contc}_{\left(\lambda^{(s)}, \lambda^{(t)}\right)}=\operatorname{Contc}_{\left(\mu^{(s)}, \mu^{(t)}\right)}$ with respect to the weight system $(0, k)$.
(3) $\lambda^{(i)}=\mu^{(i)}$ for all $i \notin\{s, t\}$ and $\operatorname{Contc}_{\left(\lambda^{(s)}, \lambda^{(t)}\right)}=\operatorname{Contc}_{\left(\mu^{(s)}, \mu^{(t)}\right)}$ with respect to the weight system $\left(m_{s}, m_{t}\right)$.
(4) $\lambda^{(i)}=\mu^{(i)}$ for all $i \notin\{s, t\}$ and $E^{\left(\lambda^{(s)}, \lambda^{(t)}\right)}$, $E^{\left(\mu^{(s)}, \mu^{(t)}\right)}$ are in the same Rouquier family of the cyclotomic Ariki-Koike algebra of type $B_{\left|\lambda^{(s)}\right|+\left|\lambda^{(t)}\right|}$ obtained via the specialisation :

$$
q \mapsto u^{r}, Q_{0} \mapsto u^{r_{s}}, Q_{1} \mapsto-u^{r_{t}}
$$

Proposition 4.8 Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}(l, n)$. Then $E^{\boldsymbol{\lambda}}$ and $E^{\boldsymbol{\mu}}$ are in the same Rouquier family associated with the essential hyperplane $r=0$ if and only if $\left|\lambda^{(i)}\right|=\left|\mu^{(i)}\right|$ for all $i=0, \ldots, l-1$.

Using the above results, we managed to prove that the functions $a$ and $A$ are constant on Rouquier families (for $r \neq 0$, this was already a known consequence of Proposition 4.5). We also created the program RBAK.g, available on my webpage Ch-web, which allows the computation of the Rouquier families for any cyclotomic Ariki-Koike algebra on GAP3.

Finally, the results above allowed for the correct determination of the Rouquier families of the cyclotomic Hecke algebras associated with the complex reflection groups $G(l, p, n)$, with the use of Clifford theory, in the cases where $n>2$, or $n=2$ and $p$ is odd Ch5.

### 4.4 Decomposition maps and canonical basic sets

A few of the results in this section have been briefly discussed in 3.4 . They are repeated here for reasons of completeness.

### 4.4.1 Classical setting

We now consider the specialised Ariki-Koike algebra with relations

$$
\begin{equation*}
\left(\mathbf{s}-\zeta_{e}^{s_{0}}\right)\left(\mathbf{s}-\zeta_{e}^{s_{1}}\right) \cdots\left(\mathbf{s}-\zeta_{e}^{s_{l}-1}\right)=0, \quad\left(\mathbf{t}_{i}-\zeta_{e}\right)\left(\mathbf{t}_{i}+1\right)=0 \quad \text { for } i=1, \ldots, n-1 \tag{4.4}
\end{equation*}
$$

where $\left(s_{0}, \ldots, s_{l-1}\right) \in \mathbb{Z}^{l}$ and $e \in \mathbb{Z}_{>0}$.
Geck and Jacon GeJa1, Ja2, Ja3, GeJa2 have shown that, for a suitable choice of $\mathbf{m}$, the corresponding function $a^{\mathbf{m}}:=a^{(\mathbf{m}, r)}$, where $r$ can be any positive integer, yields a canonical basic set for the above specialised Ariki-Koike algebra. This canonical basic set consists of the so-called "Uglov $l$-partitions" [Ja3, Definition 3.2]. However, this does not work the other way round: not all cyclotomic Ariki-Koike algebras admit canonical basic sets. For a study about which values of $\mathbf{m}$ yield canonical basic sets, see Ger.

The key result in the determination of the decomposition matrix for Ariki-Koike algebras has been Ariki's proof Ar2, Ar3] of the Lascoux-Leclerc-Thibon Conjecture LLT], which relates this matrix to the canonical basis matrix of the irreducible highest weight module of highest weight determined by $\left(s_{0}, \ldots, s_{l-1}\right)$ as a submodule of the Fock space in affine type $A$. The latter depends on a parameter
$v$ and, according to Ariki's Theorem, the decomposition matrix of the Ariki-Koike algebra equals the canonical basis matrix (for any realisation of the highest weight module) when $v=1$.

In the next chapter, we will see that we can prove the existence of canonical basic sets for Ariki-Koike algebras without the use of Ariki's Theorem, but with the use of the category $\mathcal{O}$ for rational Cherednik algebras.

Generalising the above results, we proved in ChJa2 the existence of canonical basic sets for the specialised Ariki-Koike algebra with relations

$$
\begin{equation*}
\left(\mathbf{s}-\zeta_{e}^{s_{0}}\right)\left(\mathbf{s}-\zeta_{e}^{s_{1}}\right) \cdots\left(\mathbf{s}-\zeta_{e}^{s_{l-1}}\right)=0, \quad\left(\mathbf{t}_{i}-\zeta_{e}^{r}\right)\left(\mathbf{t}_{i}+1\right)=0 \quad \text { for } i=1, \ldots, n-1, \tag{4.5}
\end{equation*}
$$

where $\left(s_{0}, \ldots, s_{l-1}\right) \in \mathbb{Z}^{l}$ and $r, e \in \mathbb{Z}_{>0}$; we called the elements of this canonical basic set "generalised Uglov $l$-partitions".

The basis of our proof is a reduction theorem by Dipper and Mathas DiMa which allows us to restrict to the case $r=1$ : more specifically, the specialised Ariki-Koike algebra given by 4.5) is Morita equivalent to a direct sum of tensor products of Ariki-Koike algebras of the form 4.4) (associated with groups $G\left(l_{i}, 1, n_{i}\right)$ with $l_{i} \leqslant l$ and $\left.n_{i} \leqslant n\right)$. Morita equivalent algebras have the same decomposition matrix.

This result allowed us, with the use of Clifford theory, and building on previous work by Genet and Jacon GenJa, to obtain canonical basic sets for all cyclotomic Hecke algebras associated with groups of type $G(l, p, n)$ with $n>2$, or $n=2$ and $p$ odd ChJa2.

### 4.4.2 Graded setting

Recently, using the works of Khovanov-Lauda KhLa and Rouquier Ro3, Brundan, Kleshchev and Rouquier have shown the existence of a $\mathbb{Z}$-grading on Ariki-Koike algebras defined over any field BrK11, Ro3]. The next natural step is to study the graded representation theory of these algebras. In the characteristic 0 case, using a grading on the Specht modules explicitly constructed in BKW, Brundan and Kleshchev BrKl1, BrKl2 have shown the existence of a certain "graded decomposition matrix" (with coefficients in $\mathbb{N}\left[v, v^{-1}\right]$ for some indeterminate $v$ ). In addition, they have proved that this matrix, which may be viewed as a quantisation of the usual decomposition matrix, corresponds to the canonical basis matrix in one realisation of the irreducible highest weight module as a submodule of the Fock space in affine type $A$. This gives a graded analogue of Ariki's Theorem.

However, there exist several possible realisations of the irreducible highest weight modules as submodules generated by the vacuum vector of the Fock space, and thus several canonical bases matrices, in affine type $A$. Therefore, it is natural to ask whether all these matrices can be expressed as graded decomposition matrices. From this perspective, one problem is to obtain a canonical way to define these graded decomposition matrices; indeed, Brundan and Kleshchev's approach depends on the existence and the choice of the Specht modules for the Ariki-Koike algebras.

In ChJa3 we developped a general theory for graded decomposition matrices, using the concept of specialisation. This pursuit is motivated by the works above, but can be also interesting as a subject in its own right, providing a graded analogue of the usual notion of decomposition matrices associated with specialisation maps. In particular, we obtain a graded analogue of Theorem 1.5 and a factorisation result which is a graded analogue of [Ge3, Proposition 2.6]. Combined with the results obtained in [AJL], our results could indicate the existence of several graded representation theories for Ariki-Koike algebras.

## Chapter 5

## Rational Cherednik Algebras

Let $W \subset \mathrm{GL}(\mathfrak{h})$ be a complex reflection group. There is a natural pairing $():, \mathfrak{h} \times \mathfrak{h}^{*} \rightarrow \mathbb{C}$ given by $(y, x):=x(y)$. Let $\mathcal{S}$ be the set of all pseudo-reflections in $W$ and let $\mathbf{c}: \mathcal{S} \rightarrow \mathbb{C}$ be a conjugacy invariant function, that is, a map such that

$$
\mathbf{c}\left(w s w^{-1}\right)=\mathbf{c}(s) \quad \forall s \in \mathcal{S}, w \in W
$$

Let $V:=\mathfrak{h} \oplus \mathfrak{h}^{*}$ and let $T V^{*}$ denote the tensor algebra on the dual space of $V$. For $s \in \mathcal{S}$, fix $\alpha_{s} \in \mathfrak{h}^{*}$ to be a basis of the one-dimensional vector space $\left.\operatorname{Im}\left(s-\mathrm{id}_{V}\right)\right|_{\mathfrak{h}^{*}}$ and $\alpha_{s}^{\vee} \in \mathfrak{h}$ to be a basis of the onedimensional vector space $\left.\operatorname{Im}\left(s-\mathrm{id}_{V}\right)\right|_{\mathfrak{h}}$. Let $t \in \mathbb{C}$. The rational Cherednik algebra $\mathbf{H}_{t, \mathbf{c}}(W)$ of $W$ is the quotient of $T V^{*} \rtimes W$ by the relations:

$$
\begin{equation*}
\left[x_{1}, x_{2}\right]=0,\left[y_{1}, y_{2}\right]=0,[y, x]=t(y, x)-2 \sum_{s \in \mathcal{S}} \mathbf{c}(s) \frac{\left(y, \alpha_{s}\right)\left(\alpha_{s}^{\vee}, x\right)}{\left(\alpha_{s}^{\vee}, \alpha_{s}\right)} s \tag{5.1}
\end{equation*}
$$

for all $x_{1}, x_{2}, x \in \mathfrak{h}^{*}$ and $y_{1}, y_{2}, y \in \mathfrak{h}$.
Example 5.1 Let $W=\mathfrak{S}_{n}$ and $\mathfrak{h}=\mathbb{C}^{n}$. Choose a basis $x_{1}, \ldots, x_{n}$ of $\mathfrak{h}^{*}$ and a dual basis $y_{1}, \ldots, y_{n}$ of $\mathfrak{h}$ so that

$$
\sigma x_{i}=x_{\sigma(i)} \sigma \quad \text { and } \quad \sigma y_{i}=y_{\sigma(i)} \sigma \quad \forall \sigma \in \mathfrak{S}_{n}, 1 \leqslant i \leqslant n .
$$

The set $\mathcal{S}$ is the set of all transpositions in $\mathfrak{S}_{n}$. We denote by $s_{i j}$ the transposition $(i, j)$. Set

$$
\alpha_{i j}:=x_{i}-x_{j} \quad \text { and } \quad \alpha_{i j}^{\vee}=y_{i}-y_{j} \quad \forall 1 \leqslant i<j \leqslant n .
$$

We have $\left(\alpha_{i j}^{\vee}, \alpha_{i j}\right)=2$. There is a single conjugacy class in $\mathcal{S}$, so take $\mathbf{c} \in \mathbb{C}$. Then $\mathbf{H}_{t, \mathbf{c}}\left(\mathfrak{S}_{n}\right)$ is the quotient of $T V^{*} \rtimes \mathfrak{S}_{n}$ by the relations:

$$
\left[x_{i}, x_{j}\right]=0, \quad\left[y_{i}, y_{j}\right]=0, \quad\left[y_{i}, x_{i}\right]=t-\mathbf{c} \sum_{j \neq i} s_{i j}, \quad\left[y_{i}, x_{j}\right]=\mathbf{c} s_{i j} \text { for } i \neq j .
$$

Remark 5.2 For all $\lambda \in \mathbb{C}^{\times}$, we have $\mathbf{H}_{\lambda t, \lambda \mathbf{c}}(W) \cong \mathbf{H}_{t, \mathbf{c}}(W)$. So we only need to consider the cases $t=1$ and $t=0$.

There is a natural filtration $\mathcal{F}$ on $\mathbf{H}_{t, \mathbf{c}}(W)$ given by putting $V^{*}$ in degree one and $W$ in degree zero. The crucial result by Etingof and Ginzburg is the Poincaré-Birkhoff-Witt (PBW) Theorem EtGi, Theorem 1.3].

Theorem 5.3 Let $\mathbb{C}[V]$ be the set of regular functions on $V$ (that is, the symmetric algebra $\operatorname{Sym}\left(V^{*}\right)$ of the dual space of $V$ ). There is an isomorphism of algebras

$$
\operatorname{gr}_{\mathcal{F}}\left(\mathbf{H}_{t, \mathbf{c}}(W)\right) \cong \mathbb{C}[V] \rtimes W,
$$

given by $\sigma(v) \mapsto v, \sigma(w) \mapsto w$, where $\sigma(h)$ denotes the image of $h \in \mathbf{H}_{t, \mathbf{c}}(W)$ in $\operatorname{gr}_{\mathcal{F}}\left(\mathbf{H}_{t, \mathbf{c}}(W)\right)$. In particular, there is an isomorphism of vector spaces

$$
\mathbf{H}_{t, \mathbf{c}}(W) \cong \mathbb{C}[V] \otimes \mathbb{C}[W] .
$$

### 5.1 Rational Cherednik Algebras at $t=1$

The PBW Theorem implies that the rational Cherednik algebra $\mathbf{H}_{1, \mathbf{c}}(W)$, as a vector space, has a "triangular decomposition"

$$
\mathbf{H}_{1, \mathbf{c}}(W) \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[W] \otimes \mathbb{C}\left[\mathfrak{h}^{*}\right] .
$$

Another famous example of a triangular decomposition is the one of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional semisimple complex Lie algebra $\mathfrak{g}$ (into the enveloping algebras of the Cartan subalgebra, the nilpotent radical of the Borel subalgebra and its opposite). In the representation theory of $\mathfrak{g}$, one of the categories of modules most studied, and best understood, is category $\mathcal{O}$, the abelian category generated by all highest weight modules. Therefore, it makes sense to want to construct and study an analogue of category $\mathcal{O}$ for rational Cherednik algebras.

### 5.1.1 Category $\mathcal{O}$

Let $\mathbf{H}_{1, \mathbf{c}}(W)$-mod be the category of all finitely generated $\mathbf{H}_{1, \mathbf{c}}(W)$-modules. We say that a module $M \in \mathbf{H}_{1, \mathbf{c}}(W)$-mod is locally nilpotent for the action of $\mathfrak{h} \subset \mathbb{C}\left[\mathfrak{h}^{*}\right]$ if for each $m \in M$ there exists $N \gg 0$ such that $\mathfrak{h}^{N} \cdot m=0$.

Definition 5.4 We define $\mathcal{O}$ to be the category of all finitely generated $\mathbf{H}_{1, \mathbf{c}}(W)$-modules that are locally nilpotent for the action of $\mathfrak{h} \subset \mathbb{C}\left[\mathfrak{h}^{*}\right]$.

Remark 5.5 Each module in category $\mathcal{O}$ is finitely generated as a $\mathbb{C}[\mathfrak{h}]$-module.
Category $\mathcal{O}$ has been thoroughly studied in GGOR. Proofs of all its properties presented here can be found in this paper.

For all $E \in \operatorname{Irr}(W)$, we set

$$
\Delta(E):=\mathbf{H}_{1, \mathbf{c}}(W) \otimes_{\mathbb{C}\left[\mathfrak{h}^{*}\right] \rtimes W} E,
$$

where $\mathbb{C}\left[\mathfrak{h}^{*}\right]$ acts trivially on $E$ (that is, the augmentation ideal $\mathbb{C}\left[\mathfrak{h}^{*}\right]_{+}$acts on $E$ as zero) and $W$ acts naturally. The module $\Delta(E)$ belongs to $\mathcal{O}$ and is called a standard module (or Verma module). Each standard module $\Delta(E)$ has a simple head $\mathrm{L}(E)$ and the set

$$
\{\mathrm{L}(E) \mid E \in \operatorname{Irr}(W)\}
$$

is a complete set of pairwise non-isomorphic simple modules of the category $\mathcal{O}$. Every module in $\mathcal{O}$ has finite length, so we obtain a well-defined square decomposition matrix

$$
\mathbf{D}=\left(\left[\Delta(E): \mathrm{L}\left(E^{\prime}\right)\right]\right)_{E, E^{\prime} \in \operatorname{Irr}(W)}
$$

where $\left[\Delta(E): \mathrm{L}\left(E^{\prime}\right)\right]$ equals the multiplicity with which the simple module $\mathrm{L}\left(E^{\prime}\right)$ appears in the composition series of $\Delta(E)$. We have $[\Delta(E): \mathrm{L}(E)]=1$.

Proposition 5.6 The following are equivalent:
) $\mathcal{O}$ is semisimple.
(2) $\Delta(E)=\mathrm{L}(E)$ for all $E \in \operatorname{Irr}(W)$.
(3) $\mathbf{D}$ is the identity matrix.

Now, there exist several orderings on the set of standard modules of $\mathcal{O}$ (and consequently on $\operatorname{Irr}(W)$ ) for which $\mathcal{O}$ is a highest weight category in the sense of [CPS (see also [Ro2, §5.1]). If $<_{\mathcal{O}}$ is such an ordering on $\operatorname{Irr}(W)$, and if $\left[\Delta(E): \mathrm{L}\left(E^{\prime}\right)\right] \neq 0$ for some $E, E^{\prime} \in \operatorname{Irr}(W)$, then either $E=E^{\prime}$ or $E^{\prime}<_{\mathcal{O}} E$. Thus, we can arrange the rows of $\mathbf{D}$ so that the decomposition matrix is lower unitriangular. We will refer to these orderings on $\operatorname{Irr}(W)$ as orderings on the category $\mathcal{O}$. A famous example of such an ordering is the one given by the $c$-function.

### 5.1.2 A change of parameters and the $c$-function

In order to relate rational Cherednik algebras with cyclotomic Hecke algebras via the KZ-functor in the next subsection, we need to change the parametrisation of $\mathbf{H}_{1, \mathbf{c}}(W)$. As in $\$ 3.1$, let $\mathcal{A}$ denote the set of reflecting hyperplanes of $W$. For $H \in \mathcal{A}$, let $W_{H}$ be the pointwise stabiliser of $H$ in $W$. The group $W_{H}$ is cyclic and its order, denoted by $e_{\mathcal{C}}$, only depends on the orbit $\mathcal{C} \in \mathcal{A} / W$ that $H$ belongs to. We have that

$$
\mathcal{S}=\bigcup_{H \in \mathcal{A}} W_{H} \backslash\{1\}
$$

For each $s \in W_{H} \backslash\{1\}$, we have $\operatorname{Ker} \alpha_{s}=H$. Without loss of generality, we may assume that $\alpha_{s}=\alpha_{s^{\prime}}$ and $\alpha_{s}^{\vee}=\alpha_{s^{\prime}}^{\vee}$ for all $s, s^{\prime} \in W_{H} \backslash\{1\}$. Set $\alpha_{H}:=\alpha_{s}$ and $\alpha_{H}^{\vee}:=\alpha_{s}^{\vee}$. Then the third relation in (5.1) becomes

$$
[y, x]=(y, x)-2 \sum_{H \in \mathcal{A}} \frac{\left(y, \alpha_{H}\right)\left(\alpha_{H}^{\vee}, x\right)}{\left(\alpha_{H}^{\vee}, \alpha_{H}\right)} \sum_{s \in W_{H} \backslash\{1\}} \mathbf{c}(s) s \quad \forall x \in \mathfrak{h}^{*}, y \in \mathfrak{h} .
$$

We define a family of complex numbers $\mathbf{k}=\left(k_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ by

$$
-2 \sum_{s \in W_{H} \backslash\{1\}} \mathbf{c}(s) s=\sum_{s \in W_{H} \backslash\{1\}}\left(\sum_{j=0}^{e_{\mathcal{C}}-1} \operatorname{det}(s)^{-j}\left(k_{\mathcal{C}, j}-k_{\mathcal{C}, j-1}\right)\right) s \quad \text { for } H \in \mathcal{C},
$$

with $k_{\mathcal{C},-1}=0$. This implies that

$$
\mathbf{c}(s)=-\frac{1}{2} \sum_{j=0}^{e_{\mathcal{C}}-1} \operatorname{det}(s)^{-j}\left(k_{\mathcal{C}, j}-k_{\mathcal{C}, j-1}\right)
$$

From now on, we will denote by $\mathbf{H}_{\mathbf{k}}(W)$ the quotient of $T V^{*} \rtimes W$ by the relations:

$$
\left[x_{1}, x_{2}\right]=0,\left[y_{1}, y_{2}\right]=0,[y, x]=(y, x)+\sum_{H \in \mathcal{A}} \frac{\left(y, \alpha_{H}\right)\left(\alpha_{H}^{\vee}, x\right)}{\left(\alpha_{H}^{\vee}, \alpha_{H}\right)} \gamma_{H},
$$

where

$$
\gamma_{H}=\sum_{w \in W_{H} \backslash\{1\}}\left(\sum_{j=0}^{e_{\mathcal{C}}-1} \operatorname{det}(w)^{-j}\left(k_{\mathcal{C}, j}-k_{\mathcal{C}, j-1}\right)\right) w
$$

for all $x_{1}, x_{2}, x \in \mathfrak{h}^{*}$ and $y_{1}, y_{2}, y \in \mathfrak{h}$. We have $\mathbf{H}_{\mathbf{k}}(W)=\mathbf{H}_{1, \mathbf{c}}(W)$.
Let $E \in \operatorname{Irr}(W)$. We denote by $c_{E}$ the scalar by which the element

$$
-\sum_{H \in \mathcal{A}} \sum_{j=0}^{e_{\mathcal{C}}-1}\left(\sum_{w \in W_{H}}(\operatorname{det} w)^{-j} w\right) k_{\mathcal{C}, j} \in Z(\mathbb{C}[W])
$$

acts on $E$. We obtain thus a function $c: \operatorname{Irr}(W) \rightarrow \mathbb{C}, E \mapsto c_{E}$. The $c$-function defines an ordering $<_{c}$ on the category $\mathcal{O}$ as follows: For all $E, E^{\prime} \in \operatorname{Irr}(W)$,

$$
E^{\prime}<_{c} E \text { if and only if } c_{E}-c_{E^{\prime}} \in \mathbb{Z}_{>0}
$$

Remark 5.7 If $c_{E}-c_{E^{\prime}} \notin \mathbb{Z} \backslash\{0\}$ for all $E, E^{\prime} \in \operatorname{Irr}(W)$, then $\mathbf{D}$ is the identity matrix, and thus $\mathcal{O}$ is semisimple.

Remark 5.8 In the rational Cherednik algebra literature the function $c$ is usually taken to be the negative of the one defined here. In the context of this chapter the above definition is more natural. In both cases we obtain an ordering on the category $\mathcal{O}$.

### 5.1.3 The KZ-functor

Following GGOR, §5.3], there exists an exact functor, known as the Knizhnik-Zamalodchikov functor or simply KZ, between the category $\mathcal{O}$ of $\mathbf{H}_{\mathbf{k}}(W)$ and the category of representations of a certain specialised Hecke algebra $\mathcal{H}_{\mathbf{k}}(W)$. Using the notation of $\$ 3.1$, this specialised Hecke algebra $\mathcal{H}_{\mathbf{k}}(W)$ is the quotient of the group algebra $\mathbb{C}\left[B_{W}\right]$ by the ideal generated by the elements of the form

$$
\left(\mathbf{s}-\exp \left(2 \pi i k_{\mathcal{C}, 0}\right)\right)\left(\mathbf{s}-\zeta_{e_{\mathcal{C}}} \exp \left(2 \pi i k_{\mathcal{C}, 1}\right)\right) \cdots\left(\mathbf{s}-\zeta_{e_{\mathcal{C}}}^{e_{\mathcal{C}}-1} \exp \left(2 \pi i k_{\mathcal{C}, e_{\mathcal{C}}-1}\right)\right)
$$

where $\mathcal{C}$ runs over the set $\mathcal{A} / W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in $\mathfrak{h}^{\text {reg }} / W$ of the elements of $\mathcal{C}$. The algebra $\mathcal{H}_{\mathbf{k}}(W)$ is obtained from the generic Hecke algebra $\mathbb{C}\left[\mathbf{v}, \mathbf{v}^{-1}\right] \mathcal{H}(W)$ via the specialisation $\Theta: v_{\mathcal{C}, j}^{N_{W}} \mapsto \exp \left(2 \pi i k_{\mathcal{C}, j}\right)$ (recall that $N_{W}$ is a positive power to which the indeterminates $v_{\mathcal{C}, j}$ appear in the defining relations of the generic Hecke algebra so that the algebra $\mathbb{C}(\mathbf{v}) \mathcal{H}(W)$ is split; see (3.1)). We always assume that Assumptions 3.4 hold for $\mathcal{H}(W)$.

The functor KZ is represented by a projective object $P_{\mathrm{KZ}} \in \mathcal{O}$, and $\mathcal{H}_{\mathbf{k}}(W) \cong \operatorname{End}_{\mathbf{H}_{\mathbf{k}}(W)}\left(P_{\mathrm{KZ}}\right)^{o p}$ [GGOR, §5.4]. Based on this, we obtain that the category $\mathcal{O}$ is semisimple if and only if the algebra $\mathcal{H}_{\mathbf{k}}(W)$ is semisimple [Va, Theorem 2.1]. We can thus use the semisimplicity criterion for $\mathcal{H}_{\mathbf{k}}(W)$ given by Theorem 1.7 in order to determine for which values of $\mathbf{k}$ the category $\mathcal{O}$ is semisimple.

Now let $<_{\mathcal{O}}$ be any ordering on the category $\mathcal{O}$ as in Subsection 5.1.1.
Proposition 5.9 Set $\mathbf{B}:=\{E \in \operatorname{Irr}(W) \mid \operatorname{KZ}(\mathrm{L}(E)) \neq 0\}$.
(a) The set $\{\mathrm{KZ}(\mathrm{L}(E)) \mid E \in \mathbf{B}\}$ is a complete set of pairwise non-isomorphic simple $\mathcal{H}_{\mathbf{k}}(W)$-modules.
(b) For all $E \in \operatorname{Irr}(W), E^{\prime} \in \mathbf{B}$, we have $\left[\Delta(E): \mathrm{L}\left(E^{\prime}\right)\right]=\left[\mathrm{KZ}(\Delta(E)): \mathrm{KZ}\left(\mathrm{L}\left(E^{\prime}\right)\right)\right]$.
(c) If $E \in \mathbf{B}$, then $[\mathrm{KZ}(\Delta(E)): \mathrm{KZ}(\mathrm{L}(E))]=1$.
(d) If $\left[\mathrm{KZ}(\Delta(E)): \mathrm{KZ}\left(\mathrm{L}\left(E^{\prime}\right)\right)\right] \neq 0$ for some $E \in \operatorname{Irr}(W)$ and $E^{\prime} \in \mathbf{B}$, then either $E=E^{\prime}$ or $E^{\prime}<_{\mathcal{O}} E$.

Property (a) follows from [GGOR, Theorem 5.14]. For the proof of properties (b), (c) and (d), all of them deriving from the fact that KZ is exact, the reader may refer to [CGG, Proposition 3.1].

The simple modules killed by the KZ-functor are exactly the ones that do not have full support. Their determination, and thus the determination of the set $\mathbf{B}$, is a very difficult problem.

### 5.2 Canonical basic sets for Hecke algebras from rational Cherednik algebras

Let $\Theta$ and $\mathcal{H}_{\mathbf{k}}(W)$ be defined as above. We have a decomposition matrix $D_{\mathbf{k}}$ for the specialised Hecke algebra $\mathcal{H}_{\mathbf{k}}(W)$ with respect to the specialisation $\Theta$. The rows of $D_{\mathbf{k}}$ are indexed by $\operatorname{Irr}(W)$ and its columns by $\operatorname{Irr}\left(\mathcal{H}_{\mathbf{k}}(W)\right)$. Following Proposition $5.9, D_{\mathbf{k}}$ can be obtained from the decomposition matrix $\mathbf{D}$ of the category $\mathcal{O}$ by removing the columns that correspond to the simple modules killed by the KZfunctor, that is, the columns labelled by $\operatorname{Irr}(W) \backslash \mathbf{B}$. This implies that $D_{\mathbf{k}}$ becomes lower unitriangular when its rows are ordered with respect to $<_{\mathcal{O}}$, in the same way that, in the cases where $\Theta$ factors through a cyclotomic Hecke algebra, the existence of a canonical basic set implies that $D_{\mathbf{k}}$ becomes lower unitriangular when its rows are ordered with respect to the $a$-function. If we could show that the $a$-function defines an ordering on the category $\mathcal{O}$, we would automatically obtain the existence of a canonical basic set for $\mathcal{H}_{\mathbf{k}}(W)$. At the same time, we would obtain the determination of $\mathbf{B}$ in the cases where canonical basic sets have already been explicitly described.

### 5.2.1 The $(a+A)$-function

Let $\mathbf{m}=\left(m_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ be a family of integers and let $\varphi_{\mathbf{m}}: v_{\mathcal{C}, j} \mapsto q^{m_{\mathcal{C}, j}}$ be the corresponding cyclotomic specialisation for the Hecke algebra $\mathcal{H}(W)$. Let $\theta: q \mapsto \eta$ be a specialisation such that $\eta$ is a non-zero complex number. If $\eta$ is not a root of unity or $\eta=1$, then, due to Theorem 1.7 and the form of the Schur elements of $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$, the specialised Hecke algebra $\mathcal{H}_{\eta}(W)$ is semisimple. So we may assume from now on that $\eta$ is a root of unity of order $e>1$, namely $\eta=\zeta_{e}^{r}$ for some $r \in \mathbb{Z}_{>0}$ such that $\operatorname{gcd}(e, r)=1$.

Let $\mathbf{k}=\left(k_{\mathcal{C}, j}\right)_{(\mathcal{C} \in \mathcal{A} / W)\left(0 \leqslant j \leqslant e_{\mathcal{C}}-1\right)}$ be the family of rational numbers defined by

$$
k_{\mathcal{C}, j}:=\frac{r N_{W}}{e} m_{\mathcal{C}, j} \quad \text { for all } \mathcal{C}, j
$$

Then $\mathcal{H}_{\mathbf{k}}(W)=\mathcal{H}_{\eta}(W)$. Following [CGG, §3.3], we obtain the following equation which relates the functions $a^{\mathbf{m}}$ and $A^{\mathbf{m}}$ for $\mathcal{H}_{\varphi_{\mathrm{m}}}(W)$ with the $c$-function for $\mathbf{H}_{\mathbf{k}}(W)$ :

$$
\begin{equation*}
a_{E}^{\mathrm{m}}+A_{E}^{\mathrm{m}}=\frac{e}{r N_{W}} c_{E}+\sum_{H \in \mathcal{A}} \sum_{j=0}^{e_{\mathcal{C}}-1} m_{\mathcal{C}, j} \quad \text { for all } E \in \operatorname{Irr}(W) \tag{5.2}
\end{equation*}
$$

where $\mathcal{C}$ denotes the orbit of $H \in \mathcal{A}$ under the action of $W$.
Remark 5.10 The above formula was also obtained in [GGOR, §6.2] for finite Weyl groups in the equal parameter case.

Equation (5.2) implies that $a^{\mathbf{m}}+A^{\mathbf{m}}$ yields the same ordering on $\operatorname{Irr}(W)$ as the $c$-function (note that in this case $c_{E} \in \mathbb{Q}$ for all $\left.E \in \operatorname{Irr}(W)\right)$. Thus, $a^{\mathbf{m}}+A^{\mathbf{m}}$ is also an ordering on the category $\mathcal{O}$, that is, if $\left[\Delta(E): \mathrm{L}\left(E^{\prime}\right)\right] \neq 0$ for some $E, E^{\prime} \in \operatorname{Irr}(W)$, then either $E=E^{\prime}$ or $a_{E^{\prime}}^{\mathrm{m}}+A_{E^{\prime}}^{\mathrm{m}}<a_{E}^{\mathbf{m}}+A_{E}^{\mathrm{m}}$. If now the function $a^{\mathbf{m}}$ is compatible with $a^{\mathbf{m}}+A^{\mathbf{m}}$, that is, for all $E, E^{\prime} \in \operatorname{Irr}(W)$,

$$
\begin{equation*}
a_{E^{\prime}}^{\mathbf{m}}+A_{E^{\prime}}^{\mathbf{m}}<a_{E}^{\mathbf{m}}+A_{E}^{\mathbf{m}} \Rightarrow a_{E^{\prime}}^{\mathbf{m}}<a_{E}^{\mathbf{m}} \tag{5.3}
\end{equation*}
$$

then $a^{\mathbf{m}}$ is an ordering on the category $\mathcal{O}$ and we obtain the existence of a canonical basic set for $\mathcal{H}_{\varphi_{\mathbf{m}}}(W)$ with respect to $\theta$ by Proposition 5.9. This is true in several cases, but unfortunately not true in general. Some exceptional complex reflection groups where 5.3 holds and the above argument works are:

$$
G_{23}=H_{3}, G_{24}, G_{27}, G_{29} \text { and } G_{30}=H_{4}
$$

This yields the existence of canonical basic sets for the groups $G_{24}, G_{27}$ and $G_{29}$, which was not known before. To summarise, we have the following [Ch6, Proposition 5.11]:

Proposition 5.11 Let $W=G_{n}, n \in\{23,24,27,29,30\}$. Let $\mathbf{m}$ and $\mathbf{k}$ be defined as above, and let $E, E^{\prime} \in \operatorname{Irr}(W)$. If $\left[\Delta(E): \mathrm{L}\left(E^{\prime}\right)\right] \neq 0$, then either $E=E^{\prime}$ or $a_{E^{\prime}}^{\mathbf{m}}<a_{E}^{\mathbf{m}}$. In particular, we have $\mathrm{KZ}(\mathrm{L}(E)) \neq 0$ if and only if $E$ belongs to the canonical basic set of $\mathcal{H}_{\varphi_{m}}(W)$ with respect to $\theta: q \mapsto \zeta_{e}^{r}$.

### 5.2.2 Canonical basic sets for Iwahori-Hecke algebras

Equation (5.2) has also allowed us to show that, in the case where $W$ is a finite Coxeter group, and assuming that Lusztig's conjectures P1 - P15 hold, the $c$-function is compatible with the ordering $\leqslant_{\mathcal{L} \mathcal{R}}$ on two-sided cells, since $a$ and $A$ are (see [Ge8, Remark 5.4] for the $a$-function, [u6, Corollary 21.6] and ChJa1, Proposition 2.8] for $A$ ). This in turn was crucial in showing CGG, Corollary 4.7]:

Proposition 5.12 Let $(W, S)$ be a finite Coxeter group and let $\mathcal{H}_{q}(W, L)$ be the Iwahori-Hecke algebra of $W$ with parameter $L$, as defined in Chapter 2. For $H \in \mathcal{A}$, let $s_{H} \in W$ be the reflection with reflecting hyperplane $H$ and let $\mathcal{C}$ be the orbit of $H$ under the action of $W$. If $H^{\prime} \in \mathcal{C}$, then we have $L\left(s_{H}\right)=L\left(s_{H^{\prime}}\right)$ and we can set $L_{\mathcal{C}}:=L\left(s_{H}\right)$. Let $e, r \in \mathbb{Z}_{>0}$ such that $\operatorname{gcd}(e, r)=1$, and take, for all $\mathcal{C} \in \mathcal{A} / W$,

$$
k_{\mathcal{C}, 0}=\frac{r L_{\mathcal{C}}}{e} \quad \text { and } \quad k_{\mathcal{C}, 1}=-\frac{r L_{\mathcal{C}}}{e} .
$$

If $E \in \operatorname{Irr}(W)$, then $\mathrm{KZ}(\mathrm{L}(E)) \neq 0$ if and only if $E$ belongs to the canonical basic set of $\mathcal{H}_{q}(W, L)$ with respect to $\theta: q \mapsto \zeta_{e}^{r}$.

The proof uses a connection, established in [CGG, Proposition 4.6], between category $\mathcal{O}$ and the cellular structure of the Iwahori-Hecke algebra. More specifically, if $E \in \operatorname{Irr}(W)$, then $\mathrm{KZ}(\Delta(E))$ is isomorphic to the cell module $W_{\theta}(E)$ defined in [Ge6, Example 4.4]; we will not go into further details here. Note though that, in Proposition 5.12, we have not included the assumption that Lusztig's conjectures must hold. The reason is that the only case where they are not known to hold, the case of $B_{n}$, is covered by Corollary 5.17 below.

Remark 5.13 The above result can be generalised to the case where $k_{\mathcal{C}, 0}=\lambda L_{\mathcal{C}}$ and $k_{\mathcal{C}, 1}=-\lambda L_{\mathcal{C}}$ for any complex number $\lambda$. If $\lambda \in \mathbb{Z}$ or $\lambda \in \mathbb{C} \backslash \mathbb{Q}$, then both category $\mathcal{O}$ and $\mathcal{H}_{\exp (2 \pi i \lambda)}(W, L)$ are semisimple, so the statement trivially holds. If $\lambda$ is a negative rational number, let us say $\lambda=-r / e$ for some $e, r \in \mathbb{Z}_{>0}$ with $\operatorname{gcd}(e, r)=1$, and $E \in \operatorname{Irr}(W)$, then $\operatorname{KZ}(\mathrm{L}(E)) \neq 0$ if and only if $E$ belongs to the canonical basic set of $\mathcal{H}_{q}(W,-L)$ with respect to $\theta: q \mapsto \zeta_{e}^{r}$. We recall now that the canonical basic sets for finite Coxeter groups where $L$ can take negative values are described in ChJa1. In fact, $E$ belongs to the canonical basic set of $\mathcal{H}_{q}(W,-L)$ with respect to $\theta: q \mapsto \zeta_{e}^{r}$ if and only if $E \otimes \varepsilon$ belongs to the canonical basic set of $\mathcal{H}_{q}(W, L)$ with respect to $\theta$, where $\varepsilon$ denotes the sign representation of $W$.

Proposition 5.12 yields the existence of canonical basic sets for all finite Coxeter groups in a uniform way. At the same time, it yields a description of the simple modules that are not killed by the KZfunctor, since canonical basic sets for finite Coxeter groups are explicitly known (see, for example, GeJa2] ). However, it does not imply that the $a$-function is an ordering on the category $\mathcal{O}$, because we do not know what happens with the simple modules killed by the KZ-functor. We do believe though that, for finite Coxeter groups, the $a$-function is an ordering on the category $\mathcal{O}$.

Example 5.14 Let $W$ be the symmetric group $\mathfrak{S}_{n}$ and let $l:=L(s)$ for every transposition $s \in \mathfrak{S}_{n}$ (there exists only one orbit $\mathcal{C}$ in $\mathcal{A} / W)$. Let $\eta^{2 l}:=\zeta_{e}^{r}$ for some $e, r \in \mathbb{Z}_{>0}$ with $\operatorname{gcd}(e, r)=1$. As we saw in Example 2.14, the canonical basic set $\mathcal{B}_{\theta}$ of $\mathcal{H}_{q}(W, l)$ with respect to $\theta: q \mapsto \eta$ consists of the $e$-regular partitions of $n$. Now take $k_{\mathcal{C}, 0}=r / 2 e$ and $k_{\mathcal{C}, 1}=-r / 2 e$. Let $\lambda$ be a partition of $n$ and let $E^{\lambda}$ be the corresponding irreducible representation of $\mathfrak{S}_{n}$. We have $\operatorname{KZ}\left(\mathrm{L}\left(E^{\lambda}\right)\right) \neq 0$ if and only if $\lambda$ is $e$-regular.

### 5.2.3 Canonical basic sets for Ariki-Koike algebras

As we have said and seen earlier, there exist several orderings on the category $\mathcal{O}$. For $W=G(l, 1, n)$, where the irreducible representations are parametrised by the set $\mathcal{P}(l, n)$ of $l$-partitions of $n$, one combinatorial ordering on the category $\mathcal{O}$ is given by Dunkl and Griffeth in DuGr. More precisely, in this case, there are two hyperplane orbits in $\mathcal{A} / W$; we will denote them by $\mathcal{\mathcal { C } _ { \mathrm { s } }}$ and $\mathcal{C}_{\mathrm{t}}$. We have $e_{\mathcal{C}_{\mathrm{s}}}=l$ and $e_{\mathcal{C}_{\mathbf{t}}}=2$. Let $\left(s_{0}, \ldots, s_{l-1}\right) \in \mathbb{Z}^{l}$ and $e \in \mathbb{Z}_{>0}$. We define $\mathbf{k}=\left(k_{\mathcal{C}_{\mathbf{s}}, 0}, \ldots, k_{\mathcal{C}_{\mathbf{s}}, l-1}, k_{\mathcal{C}_{\mathbf{t}}, 0}, k_{\mathcal{C}_{\mathbf{t}}, 1}\right)$ by

$$
\begin{equation*}
k_{\mathcal{C}_{\mathbf{s}}, j}=\frac{s_{j}}{e}-\frac{j}{l} \quad \text { for } j=0, \ldots, l-1, \quad k_{\mathcal{C}_{\mathbf{t}}, 0}=\frac{1}{e}, \quad k_{\mathcal{C}_{\mathbf{t}}, 1}=0 \tag{5.4}
\end{equation*}
$$

Then the KZ-functor goes from the category $\mathcal{O}$ of $\mathbf{H}_{\mathbf{k}}(W)$ to the category of representations of the specialised Ariki-Koike algebra $\mathcal{H}_{\mathbf{k}}(W)$ with relations

$$
\left(\mathbf{s}-\zeta_{e}^{s_{0}}\right)\left(\mathbf{s}-\zeta_{e}^{s_{1}}\right) \cdots\left(\mathbf{s}-\zeta_{e}^{s_{l-1}}\right)=0, \quad\left(\mathbf{t}_{i}-\zeta_{e}\right)\left(\mathbf{t}_{i}+1\right)=0 \quad \text { for } i=1, \ldots, n-1
$$

as in (4.4).
Let $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \ldots, \lambda^{(l-1)}\right) \in \mathcal{P}(l, n)$. We will denote by $E^{\boldsymbol{\lambda}}$ the corresponding irreducible representation of $G(l, 1, n)$. We define the set of nodes of $\boldsymbol{\lambda}$ to be the set

$$
[\boldsymbol{\lambda}]=\left\{(a, b, c): 0 \leqslant c \leqslant l-1, a \geqslant 1,1 \leqslant b \leqslant \lambda_{a}^{(c)}\right\}
$$

Let $\gamma=(a(\gamma), b(\gamma), c(\gamma)) \in[\boldsymbol{\lambda}]$. We set $\vartheta(\gamma):=b(\gamma)-a(\gamma)+s_{c(\gamma)}$. We then have the following DuGr, Proof of Theorem 4.1]:

Proposition 5.15 Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}^{\prime} \in \mathcal{P}(l, n)$. If $\left[\Delta\left(E^{\boldsymbol{\lambda}}\right): \mathrm{L}\left(E^{\boldsymbol{\lambda}^{\prime}}\right)\right] \neq 0$, then there exist orderings $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n}$ and $\gamma_{1}^{\prime}, \gamma_{2}^{\prime}, \ldots, \gamma_{n}^{\prime}$ of the nodes of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^{\prime}$ respectively, and non-negative integers $\mu_{1}, \mu_{2}, \ldots, \mu_{n}$, such that, for all $1 \leqslant i \leqslant n$,

$$
\mu_{i} \equiv c\left(\gamma_{i}\right)-c\left(\gamma_{i}^{\prime}\right) \bmod l \quad \text { and } \quad \mu_{i}=c\left(\gamma_{i}\right)-c\left(\gamma_{i}^{\prime}\right)+\frac{l}{e}\left(\vartheta\left(\gamma_{i}^{\prime}\right)-\vartheta\left(\gamma_{i}\right)\right)
$$

Now, there are several different cyclotomic Ariki-Koike algebras that produce the specialised ArikiKoike algebra $\mathcal{H}_{\mathbf{k}}(W)$ defined above and they may have distinct $a$-functions attached to them. Let $\mathbf{u}=\left(u_{0}, \ldots, u_{l-1}\right) \in \mathbb{Q}^{l}$ be a list of rational numbers such that $0<u_{j}-u_{i}<e$ whenever $i<j$. Set $t_{j}:=s_{j}-u_{j}$, for all $0 \leqslant j \leqslant l-1$, and $\mathbf{t}:=\left(t_{0}, \ldots, t_{l-1}\right)$. We then use the combinatorial description of the $a$-function for $G(l, 1, n)$ given by 4.3), taking $a_{E^{\boldsymbol{\lambda}}}:=\mathbf{a}^{(\mathbf{t}, 1)}(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \mathcal{P}(l, n)$. Note that this
definition captures all $a$-functions for $G(l, 1, n)$ in the literature: the function $a^{\mathbf{m}}$ for $m_{\mathcal{C}_{\mathbf{s}}, j}=s_{j} l-e j$, $j=0, \ldots, l-1$, given by Jacon Ja3 and studied in the context of Uglov's work on canonical bases for higher level Fock spaces, and also the $a$-function for type $B_{n}(l=2)$ arising from the KazhdanLusztig theory for Iwahori-Hecke algebras with unequal parameters (see [GeJa2, 6.7]). We have shown in [CGG, §5] that this $a$-function is compatible with the ordering on category $\mathcal{O}$ given by Proposition 5.15 Consequently, the $a$-function also defines a highest weight category structure on $\mathcal{O}$, that is, we have the following:

Proposition 5.16 Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}^{\prime} \in \mathcal{P}(l, n)$. If $\left[\Delta\left(E^{\boldsymbol{\lambda}}\right): \mathrm{L}\left(E^{\boldsymbol{\lambda}^{\prime}}\right)\right] \neq 0$, then either $\boldsymbol{\lambda}=\boldsymbol{\lambda}^{\prime}$ or $a_{E^{\boldsymbol{\lambda}^{\prime}}}<a_{E^{\boldsymbol{\lambda}}}$.
The above result, combined with Proposition 5.9, yields the following:
Corollary 5.17 Let $W=G(l, 1, n)$. Let $\left(s_{0}, \ldots, s_{l-1}\right) \in \mathbb{Z}^{l}$ and $e \in \mathbb{Z}_{>0}$. Let $\mathbf{k}$ be defined as in 5.4. If $\boldsymbol{\lambda} \in \mathcal{P}(l, n)$, then $\mathrm{KZ}\left(\mathrm{L}\left(E^{\boldsymbol{\lambda}}\right)\right) \neq 0$ if and only if $E^{\boldsymbol{\lambda}}$ belongs to the canonical basic set of $\mathcal{H}_{\mathbf{k}}(W)$ with respect to the a-function above.

Thus, we obtain the existence of canonical basic sets for Ariki-Koike algebras without the use of Ariki's Theorem. On the other hand, the description of the canonical basic sets for Ariki-Koike algebras by [Ja3, Main Theorem] yields a description of the set $\mathbf{B}=\left\{E^{\boldsymbol{\lambda}} \in \operatorname{Irr}(W) \mid \operatorname{KZ}\left(\mathrm{L}\left(E^{\boldsymbol{\lambda}}\right)\right) \neq 0\right\}$ : we have that $E^{\boldsymbol{\lambda}} \in \mathbf{B}$ if and only if $\boldsymbol{\lambda}$ is an Uglov $l$-partition.

Finally, we expect a result similar to Corollary 5.17 to hold in the case where $W=G(l, p, n)$ for $p>1$.

Remark 5.18 By repeating the proof of CGG, Proposition 4.6] for $W=G(l, 1, n)$, we can obtain that the images of the standard modules via the KZ-functor agree with the Specht modules defined by the cellular structure on the Ariki-Koike algebras in DJM, provided that the parameters $\mathbf{k}$ are chosen to belong to the so-called "asymptotic region" (see Ro2, Proposition 6.4] for the explicit description of this region).

### 5.3 Rational Cherednik Algebras at $t=0$

Let us now consider the rational Cherednik algebra $\mathbf{H}_{0, \mathbf{c}}(W)$. Set $\mathfrak{A}:=\mathbb{C}[\mathfrak{h}]^{W} \otimes \mathbb{C}\left[\mathfrak{h}^{*}\right]^{W}$. We define the restricted rational Cherednik algebra to be

$$
\overline{\mathbf{H}}_{0, \mathbf{c}}(W):=\mathbf{H}_{0, \mathbf{c}}(W) / \mathfrak{A}_{+} \mathbf{H}_{0, \mathbf{c}}(W)
$$

where $\mathfrak{A}_{+}$denotes the ideal of $\mathfrak{A}$ consisting of elements with zero constant term. This algebra was originally introduced, and extensively studied, in G0. The PBW Theorem implies that, as a vector space,

$$
\overline{\mathbf{H}}_{0, \mathbf{c}}(W) \cong \mathbb{C}[\mathfrak{h}]^{c o W} \otimes \mathbb{C}[W] \otimes \mathbb{C}\left[\mathfrak{h}^{*}\right]^{c o W}
$$

where $\mathbb{C}[\mathfrak{h}]^{\text {co } W}=\mathbb{C}[\mathfrak{h}] /\left\langle\mathbb{C}[\mathfrak{h}]_{+}^{W}\right\rangle$ is the coinvariant algebra. Since $W$ is a complex reflection group, $\mathbb{C}[\mathfrak{h}]^{\text {coW }}$ has dimension $|W|$ and is isomorphic to the regular representation as a $\mathbb{C}[W]$-module. Thus, $\operatorname{dim}_{\mathbb{C}} \overline{\mathbf{H}}_{0, \mathbf{c}}(W)=|W|^{3}$.

Let $E \in \operatorname{Irr}(W)$. We set

$$
\bar{\Delta}(E):=\overline{\mathbf{H}}_{0, \mathbf{c}}(W) \otimes_{\mathbb{C}\left[\mathfrak{h}^{*}\right]^{c o W} \rtimes W} E,
$$

where $\mathbb{C}\left[\underline{h}^{*}\right]^{c o W}$ acts trivially on $E$ (that is, $\mathbb{C}\left[\mathfrak{h}^{*}\right]_{+}^{c o W}$ acts on $E$ as zero) and $W$ acts naturally. The module $\bar{\Delta}(E)$ is the baby Verma module of $\overline{\mathbf{H}}_{0, \mathbf{c}}(W)$ associated with $E$. We summarise, as is done in [Go, Proposition 4.3], the results of [HoNa applied to this situation.

Proposition 5.19 Let $E, E^{\prime} \in \operatorname{Irr}(W)$.
(a) The baby Verma module $\bar{\Delta}(E)$ has a simple head, $\overline{\mathrm{L}}(E)$. Hence, $\bar{\Delta}(E)$ is indecomposable.
(b) $\bar{\Delta}(E) \cong \bar{\Delta}\left(E^{\prime}\right)$ if and only if $E \cong E^{\prime}$.
(c) The set $\{\overline{\mathrm{L}}(E) \mid E \in \operatorname{Irr}(W)\}$ is a complete set of pairwise non-isomorphic simple $\overline{\mathbf{H}}_{0, \mathbf{c}}$-modules.

### 5.4 Families of characters for Hecke algebras from rational Cherednik algebras

Since the algebra $\overline{\mathbf{H}}_{0, \mathbf{c}}$ is finite dimensional, we can define its blocks in the usual way. Let $E, E^{\prime} \in \operatorname{Irr}(W)$. Following GoMa, we define the Calogero-Moser partition of $\operatorname{Irr}(W)$ to be the set of equivalence classes of $\operatorname{Irr}(W)$ under the equivalence relation:

$$
E \sim_{C M} E^{\prime} \text { if and only if } \overline{\mathrm{L}}(E) \text { and } \overline{\mathrm{L}}\left(E^{\prime}\right) \text { belong to the same block of } \overline{\mathbf{H}}_{0, \mathbf{c}} .
$$

We will simply write $C M_{\mathbf{c}}$-partition for the Calogero-Moser partition of $\operatorname{Irr}(W)$. Using the classification of irreducible complex reflection groups (see Theorem 3.1), Bellamy has shown the following Bel1, Theorem 1.1]:

Theorem 5.20 Let $W$ be an irreducible complex reflection group. The $C M_{\mathbf{c}}$-partition is trivial for generic values of $\mathbf{c}$ if and only if $W$ is of type $G(l, 1, n)$ or $G_{4}$.

Remark 5.21 The cases where the $C M_{\mathbf{c}}$-partition is trivial for generic values of $\mathbf{c}$ are exactly the cases where the space $\left(\mathfrak{h} \oplus \mathfrak{h}^{*}\right) / W$ admits a symplectic resolution.

### 5.4.1 The Calogero-Moser partition and Rouquier families

It just so happens that the cases where $X_{\mathbf{c}}(W)$ is generically smooth, and the Calogero-Moser partition generically trivial, are exactly the cases where the Rouquier families are generically trivial (that is, the Rouquier families associated with no essential hyperplane are singletons). This, combined with the fact that the Calogero-Moser partition into blocks enjoys some property of semicontinuity, led to the question whether there is a connection between the two partitions.

The question was first asked by Gordon and Martino GoMa in terms of a connection between the Calogero-Moser partition and families of characters for type $B_{n}$. In their paper, they computed the $C M_{\mathbf{c}}$-partition, for all $\mathbf{c}$, for complex reflection groups of type $G(l, 1, n)$ and showed that for $l=2$, using the conjectural combinatorial description of Kazhdan-Lusztig cells for type $B_{n}$ by BGIL, the $C M_{\mathrm{c}}$-partition coincides with the partition into Kazhdan-Lusztig families. After that, Martino Mart, compared the combinatorial description of the $C M_{\mathrm{c}}$-partition for type $G(l, 1, n)$ given in GoMa with the description of the partition into Rouquier families, given by Ch3, for a suitable cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{c}}$ of $G(l, 1, n)$ (different from the one defined in $\$ 5.1 .3$ ). He showed that the two partitions coincide when $l$ is a power of a prime number (which includes the cases of type $A_{n}$ and $B_{n}$ ), but not in general. In fact, he showed that the $C M_{\mathbf{c}}$-partition for $G(l, 1, n)$ is the same as the one obtained by BrKi. He thus obtained the following two connections between the $C M_{\mathbf{c}}$-partition and the partition into Rouquier families for $G(l, 1, n)$, and he conjectured that they hold for every complex reflection group $W$ [Mart, 2.7]:
(a) The $C M_{\mathbf{c}}$-partition for generic $\mathbf{c}$ coincides with the generic partition into Rouquier families (both being trivial for $W=G(l, 1, n)$ );
(b) The partition into Rouquier families refines the $C M_{\mathbf{c}}$-partition, for all choices of $\mathbf{c}$; that is, if $E, E^{\prime} \in$ $\operatorname{Irr}(W)$ belong to the same Rouquier family of $\mathcal{H}_{\mathbf{c}}$, then $E \sim_{C M} E^{\prime}$.

Conditions (a) and (b) are known as "Martino's Conjecture". Using the combinatorics of GoMa] and Mart, Bellamy computed the $C M_{\mathbf{c}}$-partition, for all c, and proved Martino's conjecture in the case where $W$ is of type $G(l, p, n)$ Bel2]; note that when $p>1$ the generic partitions in this case are not trivial. However, a counter-example for (a) was found by Thiel Th] in the case where $W=G_{25}$. Thiel calculated the $C M_{\mathbf{c}}$-partition for generic $\mathbf{c}$ for the exceptional complex reflection groups $G_{4}, G_{5}, G_{6}, G_{8}, G_{10}, G_{23}=$ $H_{3}, G_{24}, G_{25}$ and $G_{26}$. Comparing his results with the generic partition into Rouquier families for these groups, given by Ch4, he showed that Part (a) of Martino's Conjecture holds in every cas $\Phi^{1}$ except for when $W=G_{25}$. In this particular case, the generic partition into Rouquier families simply refines the $C M_{\mathrm{c}}$-partition for generic c. So we will state here as a conjecture only Part (b) of Martino's conjecture, which is still an open problem, and proved in all the above cases.

[^1]Conjecture 5.22 (Martino's Conjecture) Let $W$ be a complex reflection group. The partition into Rouquier families (for a suitably chosen cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{c}}$ of $W$ ) refines the $C M_{\mathbf{c}}$-partition, for all choices of $\mathbf{c}$; that is, if $E, E^{\prime} \in \operatorname{Irr}(W)$ belong to the same Rouquier family of $\mathcal{H}_{\mathbf{c}}$, then $E \sim_{C M} E^{\prime}$.

Remark 5.23 Note that, in all the cases checked so far where $W$ is a finite Coxeter group, the partition into Rouquier families and the $C M_{\mathbf{c}}$-partition coincide. This covers the finite Coxeter groups of types $A_{n}, B_{n}, D_{n}$ and the dihedral groups for all choices of $\mathbf{c}$, and $H_{3}$ for generic $\mathbf{c}$.

### 5.4.2 The Calogero-Moser partition and Kazhdan-Lusztig cells

In an effort to develop a generalised Kazhdan-Lusztig cell theory, Bonnafé and Rouquier [BoRo1, BoRo2] used the Calogero-Moser partition to define Calogero-Moser cells for all complex reflection groups. An advantage of this, quite geometric, approach is that the Calogero-Moser partition exists naturally for all complex reflection groups. It also implies automatically the existence of a semicontinuity property for cells, a property that was conjectured and proved in some cases for Kazhdan-Lusztig cells by Bonnafé Bo2]. However, Calogero-Moser cells are very hard to compute and their construction depends on an "uncontrollable" choice. After very long computations, it has been confirmed that the Calogero-Moser cells coincide with the Kazhdan-Lusztig cells for some finite Coxeter groups of rank 2: the Weyl groups $A_{2}, B_{2}, G_{2}$ BoRo1] and the dihedral groups $I_{2}(m)$ with $m$ odd [Bo3]; there is still a lot of work that needs to be done.

## Chapter 6

## Yokonuma-Hecke Algebras

Yokonuma-Hecke algebras were introduced by Yokonuma [Y0 for the same reason that Ariki-Koike algebras were introduced by Ariki and Koike: as generalisations of Iwahori-Hecke algebras. More precisely, the Iwahori-Hecke algebra associated to a finite reductive group $G$ is the centraliser algebra associated to the permutation representation of $G$ with respect to a Borel subgroup of $G$. The Yokonuma-Hecke algebra is the centraliser algebra associated to the permutation representation of $G$ with respect to a maximal unipotent subgroup of $G$. Thus, Yokonuma-Hecke algebras can be also regarded as particular cases of unipotent Hecke algebras.

In this chapter, we will focus on the Yokonuma-Hecke algebra of type $A$. In recent years, the presentation of this algebra has been transformed in Ju1, JuKa, Ju2, ChPdA1, ChPo2 to the one that we will use here. This new presentation is given by generators and relations, depending on two positive integers, $d$ and $n$, and a parameter $q$. For $q=p^{m}$ and $d=p^{m}-1$, where $p$ is a prime number and $m$ is a positive integer, the Yokonuma-Hecke algebra of type $A$, denoted by $\mathrm{Y}_{d, n}(q)$, is the centraliser algebra associated to the permutation representation of $\mathrm{GL}_{n}\left(\mathbb{F}_{q}\right)$ with respect to a maximal unipotent subgroup.

Let $n \in \mathbb{N}, d \in \mathbb{N}^{*}$. Let $q$ be an indeterminate. The Yokonuma-Hecke algebra (of type $A$ ), denoted by $\mathrm{Y}_{d, n}(q)$, is an associative $\mathbb{C}\left[q, q^{-1}\right]$-algebra generated by the elements

$$
g_{1}, \ldots, g_{n-1}, t_{1}, \ldots, t_{n}
$$

subject to the following relations:

| $\left(\mathrm{b}_{1}\right)$ | $g_{i} g_{j}$ | $=g_{j} g_{i}$ |  |
| ---: | :--- | :--- | :--- |
| $\left(\mathrm{~b}_{2}\right)$ | $g_{i} g_{i+1} g_{i}$ | $=g_{i+1} g_{i} g_{i+1}$ |  |
| for all $i, j=1, \ldots, n-1$ such that $i=1, \ldots, n-2$, |  |  |  |
| $\left(\mathrm{f}_{1}\right)$ | $t_{i} t_{j}$ | $=t_{j} t_{i}$ |  |
| $\left(\mathrm{f}_{2}\right)$ | $t_{j} g_{i}$ | $=g_{i} t_{s_{i}(j)}$ |  |
| $\left(\mathrm{f}_{3}\right)$ |  | for all $i, j=1, \ldots, n$, |  |
| $\left(t_{j}^{d}\right.$ | $=1$ |  | for all $i=1, \ldots, n-1$ and $j=1, \ldots, n$, |
|  |  |  |  |

where $s_{i}$ is the transposition $(i, i+1)$, together with the quadratic relations:

$$
\begin{equation*}
g_{i}^{2}=q+(q-1) e_{i} g_{i} \quad \text { for all } i=1, \ldots, n-1 \tag{6.2}
\end{equation*}
$$

where

$$
\begin{equation*}
e_{i}:=\frac{1}{d} \sum_{s=0}^{d-1} t_{i}^{s} t_{i+1}^{-s} . \tag{6.3}
\end{equation*}
$$

It is easily verified that the elements $e_{i}$ are idempotents in $\mathrm{Y}_{d, n}(q)$. Also, that the elements $g_{i}$ are invertible, with

$$
\begin{equation*}
g_{i}^{-1}=q^{-1} g_{i}+\left(q^{-1}-1\right) e_{i} \quad \text { for all } i=1, \ldots, n-1 \tag{6.4}
\end{equation*}
$$

If we specialise $q$ to 1 , the defining relations (6.1) $\sqrt{6.2}$ become the defining relations for the complex reflection group $G(d, 1, n)$. Thus, the algebra $\mathrm{Y}_{d, n}(q)$ is a deformation of the group algebra over $\mathbb{C}$ of $G(d, 1, n)$, different from the Ariki-Koike algebra. Moreover, for $d=1$, the Yokonuma-Hecke algebra $\mathrm{Y}_{1, n}(q)$ coincides with the Iwahori-Hecke algebra $\mathcal{H}_{n}(q)$ of type $A$, and thus, for $d=1$ and $q$ specialised to 1 , we obtain the group algebra over $\mathbb{C}$ of the symmetric group $\mathfrak{S}_{n}$.

Remark 6.1 Note that in all the papers prior to ChPdA1, the algebra $\mathrm{Y}_{d, n}(q)$ is generated by elements $\bar{g}_{1}, \ldots, \bar{g}_{n-1}, t_{1}, \ldots, t_{n}$ satisfying relations 6.1) and the quadratic relations:

$$
\begin{equation*}
\bar{g}_{i}^{2}=1+(q-1) e_{i}+(q-1) e_{i} \bar{g}_{i} \quad \text { for all } i=1, \ldots, n-1 \tag{6.5}
\end{equation*}
$$

This presentation changed in [ChPdA1, where we considered $\mathrm{Y}_{d, n}(q)$ defined over $\mathbb{C}\left[q^{1 / 2}, q^{-1 / 2}\right]$ and generated by elements $\widetilde{g}_{1}, \ldots, \widetilde{g}_{n-1}, t_{1}, \ldots, t_{n}$ satisfying relations 6.1) and the quadratic relations:

$$
\begin{equation*}
\widetilde{g}_{i}^{2}=1+\left(q^{1 / 2}-q^{-1 / 2}\right) e_{i} \widetilde{g}_{i} \quad \text { for all } i=1, \ldots, n-1 \tag{6.6}
\end{equation*}
$$

By taking $\bar{g}_{i}:=\widetilde{g}_{i}+\left(q^{1 / 2}-1\right) e_{i} \widetilde{g}_{i}$ (and thus, $\widetilde{g}_{i}=\bar{g}_{i}+\left(q^{-1 / 2}-1\right) e_{i} \bar{g}_{i}$ ), we obtain the old presentation of the Yokonuma-Hecke algebra. By taking $g_{i}:=q^{1 / 2} \widetilde{g}_{i}$ we obtain our presentation of the Yokonuma-Hecke algebra.

Now let $w \in \mathfrak{S}_{n}$, and let $w=s_{i_{1}} s_{i_{2}} \ldots s_{i_{r}}$ be a reduced expression for $w$. Since the generators $g_{i}$ of the Yokonuma-Hecke algebra satisfy the same braid relations as the generators of $\mathfrak{S}_{n}$, Matsumoto's theorem implies that the element $g_{w}:=g_{i_{1}} g_{i_{2}} \ldots g_{i_{r}}$ is well-defined, that is, it does not depend on the choice of the reduced expression of $w$.

Juyumaya Ju2 has proved that the following set is a $\mathbb{C}\left[q, q^{-1}\right]$-basis of $\mathrm{Y}_{d, n}(q)$ :

$$
\begin{equation*}
\mathcal{B}_{d, n}^{\mathrm{H}}:=\left\{t_{1}^{r_{1}} \ldots t_{n}^{r_{n}} g_{w} \mid w \in \mathfrak{S}_{n}, 0 \leqslant r_{j} \leqslant d-1 \text { for all } j=1,2, \ldots, n\right\} \tag{6.7}
\end{equation*}
$$

In particular, $\mathrm{Y}_{d, n}(q)$ is a free $\mathbb{C}\left[q, q^{-1}\right]$-module of rank $d^{n} n!$.

### 6.1 Representation theory of Yokonuma-Hecke algebras

The representation theory of Yokonuma-Hecke algebras has been first studied by Thiem Th1, Th2, Th3, in the general context of unipotent Hecke algebras. The generality of his results and the new presentation for $\mathrm{Y}_{d, n}(q)$ has led us to develop a combinatorial approach to the representation theory of the YokonumaHecke algebra of type $A$ in ChPdA1. In this section, we will give an explicit description of the irreducible representations of $\mathrm{Y}_{d, n}(q)$ in terms of $d$-partitions and standard $d$-tableaux.

Recall that we denote by $\mathcal{P}(d, n)$ the set of $d$-partitions of $n$. Let $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \ldots, \lambda^{(d-1)}\right) \in \mathcal{P}(d, n)$. A node $\boldsymbol{\theta}$ of $\boldsymbol{\lambda}$ is a triple $(x, y, i)$, where $0 \leqslant i \leqslant d-1$ and $(x, y)$ is a node of the partition $\lambda^{(i)}$. We define $\mathrm{p}(\boldsymbol{\theta}):=i$ to be the position of $\boldsymbol{\theta}$ and $\mathrm{c}(\boldsymbol{\theta}):=q^{y-x}$ to be the quantum content of $\boldsymbol{\theta}$.

A $d$-tableau of shape $\boldsymbol{\lambda}$ is a bijection between the set $\{1, \ldots, n\}$ and the set of nodes of $\boldsymbol{\lambda}$. In other words, a $d$-tableau of shape $\boldsymbol{\lambda}$ is obtained by placing the numbers $1, \ldots, n$ in the nodes of $\boldsymbol{\lambda}$. The size of a $d$-tableau of shape $\boldsymbol{\lambda}$ is $n$, that is, the size of $\boldsymbol{\lambda}$. A $d$-tableau is standard if its entries increase along any row and down any column of every diagram in $\boldsymbol{\lambda}$. For $d=1$, a standard 1-tableau is a usual standard tableau.

For a $d$-tableau $\mathcal{T}$, we denote respectively by $\mathrm{p}(\mathcal{T} \mid i)$ and $\mathrm{c}(\mathcal{T} \mid i)$ the position and the quantum content of the node with the number $i$ in it. For example, for the standard 3 -tableau $\mathcal{T}=(\boxed{1 \mid 3}, \varnothing, 2)$ of size 3, we have

$$
\mathrm{p}(\mathcal{T} \mid 1)=0, \quad \mathrm{p}(\mathcal{T} \mid 2)=2, \quad \mathrm{p}(\mathcal{T} \mid 3)=0 \quad \text { and } \quad \mathrm{c}(\mathcal{T} \mid 1)=1, \quad \mathrm{c}(\mathcal{T} \mid 2)=1, \quad \mathrm{c}(\mathcal{T} \mid 3)=q
$$

For any $d$-tableau $\mathcal{T}$ of size $n$ and any permutation $\sigma \in \mathfrak{S}_{n}$, we denote by $\mathcal{T}^{\sigma}$ the $d$-tableau obtained from $\mathcal{T}$ by applying the permutation $\sigma$ on the numbers contained in the nodes of $\mathcal{T}$. We have

$$
\mathrm{p}\left(\mathcal{T}^{\sigma} \mid i\right)=\mathrm{p}\left(\mathcal{T} \mid \sigma^{-1}(i)\right) \quad \text { and } \quad \mathrm{c}\left(\mathcal{T}^{\sigma} \mid i\right)=\mathrm{c}\left(\mathcal{T} \mid \sigma^{-1}(i)\right) \quad \text { for all } i=1, \ldots, n
$$

Note that if the $d$-tableau $\mathcal{T}$ is standard, the $d$-tableau $\mathcal{T}^{\sigma}$ is not necessarily standard.
Now, let $\boldsymbol{\lambda} \in \mathcal{P}(d, n)$, and let $V_{\boldsymbol{\lambda}}$ be a $\mathbb{C}(q)$-vector space with a basis $\left\{\mathbf{v}_{\mathcal{T}}\right\}$ indexed by the standard $d$-tableaux of shape $\boldsymbol{\lambda}$. We set $\mathbf{v}_{\mathcal{T}}:=0$ for any non-standard $d$-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. By ChPdA1, Proposition $5 \&$ Theorem 1] and [ChPo2, Theorem 3.7], we have the following description of the irreducible representations of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ :

Theorem 6.2 Let $\left\{\xi_{0}, \xi_{1}, \ldots, \xi_{d-1}\right\}$ be the set of all d-th roots of unity (ordered arbitrarily). Let $\mathcal{T}$ be a standard $d$-tableau of shape $\boldsymbol{\lambda} \in \mathcal{P}(d, n)$. For brevity, we set $\mathrm{p}_{i}:=\mathrm{p}(\mathcal{T} \mid i)$ and $\mathrm{c}_{i}:=\mathrm{c}(\mathcal{T} \mid i)$ for all $i=1, \ldots, n$. The vector space $V_{\boldsymbol{\lambda}}$ is a representation of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ with the action of the generators on the basis element $\mathbf{v}_{\mathcal{T}}$ defined as follows: for $j=1, \ldots, n$,

$$
\begin{equation*}
t_{j}\left(\mathbf{v}_{\mathcal{T}}\right)=\xi_{\mathrm{p}_{j}} \mathbf{v}_{\mathcal{T}} \tag{6.8}
\end{equation*}
$$

for $i=1, \ldots, n-1$, if $\mathrm{p}_{i}>\mathrm{p}_{i+1}$ then

$$
\begin{gather*}
g_{i}\left(\mathbf{v}_{\mathcal{T}}\right)=\mathbf{v}_{\mathcal{T}^{s_{i}}}  \tag{6.9}\\
g_{i}\left(\mathbf{v}_{\mathcal{T}}\right)=q \mathbf{v}_{\mathcal{T}^{s_{i}}} \tag{6.10}
\end{gather*}
$$

if $\mathrm{p}_{i}<\mathrm{p}_{i+1}$ then
and if $\mathrm{p}_{i}=\mathrm{p}_{i+1}$ then

$$
\begin{equation*}
g_{i}\left(\mathbf{v}_{\mathcal{T}}\right)=\frac{q \mathrm{c}_{i+1}-\mathrm{c}_{i+1}}{\mathrm{c}_{i+1}-\mathrm{c}_{i}} \mathbf{v}_{\mathcal{T}}+\frac{q \mathrm{c}_{i+1}-\mathrm{c}_{i}}{\mathrm{c}_{i+1}-\mathrm{c}_{i}} \mathbf{v}_{\mathcal{T}^{s_{i}}} \tag{6.11}
\end{equation*}
$$

where $s_{i}$ is the transposition $(i, i+1)$. Further, the set $\left\{V_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}(d, n)\right\}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$.

The above theorem implies that the algebra $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ is split. As we have already mentioned, when $q \mapsto 1$, the algebra $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ specialises to the group algebra $\mathbb{C}[G(d, 1, n)]$, which is semisimple. By Tits's deformation theorem, we obtain that the algebra $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ is also semisimple.

Let now $\theta: \mathbb{C}\left[q, q^{-1}\right] \rightarrow \mathbb{C}$ be a ring homomorphism such that $\theta(q)=\eta \in \mathbb{C} \backslash\{0\}$. Using the representation theory of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$, we have proved the following semisimplicity criterion for $\mathbb{C Y}_{d, n}(\eta)$ ChPdA1, Proposition 9]:

Proposition 6.3 The specialised Yokonuma-Hecke algebra $\mathbb{C Y}_{d, n}(\eta)$ is (split) semisimple if and only if $\theta(P(q)) \neq 0$, where

$$
P(q)=\prod_{1 \leqslant i \leqslant n}\left(1+q+\cdots+q^{i-1}\right)
$$

Note that following Ariki's semisimplicity criterion 4.2), the algebra $\mathbb{C Y}_{d, n}(\eta)$ is semisimple if and only if the specialised Iwahori-Hecke algebra $\mathbb{C} \mathcal{H}_{n}(\eta)$ is semisimple.

Another way to obtain the above result is through our definition of a canonical symmetrising form $\tau$ on $\mathrm{Y}_{d, n}(q)$ ChPdA1, Proposition 10]. Having calculated the Schur elements of $\mathrm{Y}_{d, n}(q)$ with respect to $\tau$ [hPdA1, Proposition 11], we can deduce the above semisimplicity criterion with the use of Theorem 1.7. More precisely, we have the following:

Theorem 6.4 We define the linear map $\tau: \mathrm{Y}_{d, n}(q) \rightarrow \mathbb{C}\left[q, q^{-1}\right]$ by

$$
\tau\left(t_{1}^{r_{1}} \ldots t_{n}^{r_{n}} g_{w}\right)= \begin{cases}1 & \text { if } w=1 \text { and } r_{j}=0 \text { for all } j=1,2, \ldots, n  \tag{6.12}\\ 0 & \text { otherwise },\end{cases}
$$

where $w \in \mathfrak{S}_{n}$ and $0 \leqslant r_{j} \leqslant d-1$ for all $j=1,2, \ldots, n$. Then $\tau$ is a symmetrising form on $\mathrm{Y}_{d, n}(q)$, called the canonical symmetrising form. If $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \ldots, \lambda^{(d-1)}\right) \in \mathcal{P}(d, n)$, then the Schur element of $V_{\boldsymbol{\lambda}}$ with respect to $\tau$ is

$$
\begin{equation*}
s_{\boldsymbol{\lambda}}=d^{n} s_{\lambda^{(0)}} s_{\lambda^{(1)}} \ldots s_{\lambda^{(d-1)}} \tag{6.13}
\end{equation*}
$$

where $s_{\lambda^{(i)}}$ is the Schur element of the Iwahori-Hecke algebra $\mathcal{H}_{\left|\lambda^{(i)}\right|}(q)$ corresponding to $\lambda^{(i)}$ for all $i=0,1, \ldots, d-1$ (we take $s_{\emptyset}:=1$ ).

A simple formula for the calculation of the Schur elements of Iwahori-Hecke algebras of type $A$ is given by Theorem 4.3 .

The connection between the representation theory of the Yokonuma-Hecke algebra and that of Iwahori-Hecke algebras of type $A$ implied by (6.13) is explained by a result of Lusztig [Lu7, §34], who has proved that Yokonuma-Hecke algebras, in general, are isomorphic to direct sums of matrix algebras over certain subalgebras of classical Iwahori-Hecke algebras. Using the new presentation for $\mathrm{Y}_{d, n}(q)$, Jacon and Poulain d'Andecy [JaPdA] have explicitly described this isomorphism between the

Yokonuma-Hecke algebra of type $A$ and a direct sum of matrix algebras over tensor products of IwahoriHecke algebras of type $A$. Another proof of this result has been given recently in EsRy, where Espinoza and Ryom-Hansen have constructed a concrete isomorphism between $\mathrm{Y}_{d, n}(q)$ and Shoji's modified ArikiKoike algebra. Note that in all cases the result has been obtained over the ring $\mathbb{C}\left[q^{1 / 2}, q^{-1 / 2}\right]$ (using the generators $\widetilde{g}_{i}$ defined in Remark 6.1). We have managed to show that it is still valid over the smaller ring $\mathbb{C}\left[q, q^{-1}\right]$. We have [ChPo2, Theorem 4.3]:

$$
\begin{equation*}
\mathrm{Y}_{d, n}(q) \cong \bigoplus_{\mu \in \operatorname{Comp}_{d}(n)} \operatorname{Mat}_{m_{\mu}}\left(\mathcal{H}_{\mu_{0}}(q) \otimes \mathcal{H}_{\mu_{1}}(q) \otimes \cdots \otimes \mathcal{H}_{\mu_{d-1}}(q)\right) \tag{6.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\operatorname{Comp}_{d}(n)=\left\{\mu=\left(\mu_{0}, \mu_{1}, \ldots, \mu_{d-1}\right) \in \mathbb{N}^{d} \mid \mu_{0}+\mu_{1}+\cdots+\mu_{d-1}=n\right\} \tag{6.15}
\end{equation*}
$$

and

$$
\begin{equation*}
m_{\mu}=\frac{n!}{\mu_{0}!\mu_{1}!\ldots \mu_{d-1}!} \tag{6.16}
\end{equation*}
$$

### 6.2 Affine and Cyclotomic Yokonuma-Hecke algebras

We have now seen two families of algebras that are deformations of the groups algebra of the complex reflection group $G(d, 1, n)$ : the Ariki-Koike algebras and the Yokonuma-Hecke algebras of type $A$. Both families include the Iwahori-Hecke algebras of type $A$. The former preserve the "good" quadratic relations of the Iwahori-Hecke algebras, while the latter preserve the wreath product structure of $G(d, 1, n) \cong$ $(\mathbb{Z} / d \mathbb{Z})$ 〕 $\mathfrak{S}_{n}$.

In ChPdA2, we introduced the affine and cyclotomic Yokonuma-Hecke algebras, which give rise to both Ariki-Koike algebras and Yokonuma-Hecke algebras of type $A$ as quotients and as special cases. Let $n \in \mathbb{N}, d \in \mathbb{N}^{*}$ and $l \in \mathbb{N}^{*} \cup\{\infty\}$. Let $q$ and $\left(Q_{i}\right)_{i \in \mathbb{N}}$ be indeterminates, and set $\mathcal{R}_{l}:=$ $\mathbb{C}\left[q^{ \pm 1}, Q_{0}^{ \pm 1}, Q_{1}^{ \pm 1}, \ldots, Q_{l-1}^{ \pm 1}\right]$ if $l<\infty$, and $\mathcal{R}_{\infty}:=\mathbb{C}\left[q^{ \pm 1}\right]$. We define the algebra $\mathrm{Y}(d, l, n)$ to be the associative $\mathcal{R}_{l}$-algebra generated by the elements

$$
g_{1}, \ldots, g_{n-1}, t_{1}, \ldots, t_{n}, X_{1}, X_{1}^{-1}
$$

subject to the relations $\sqrt{6.1}-\sqrt{6.2}$, together with the following relations concerning the generator $X_{1}$ :

$$
\begin{array}{rlrl}
X_{1} X_{1}^{-1} & =X_{1}^{-1} X_{1}=1 & \\
X_{1} g_{1} X_{1} g_{1} & =g_{1} X_{1} g_{1} X_{1} & & \\
X_{1} g_{i} & =g_{i} X_{1} & & \text { for all } i=2, \ldots, n-1,  \tag{6.17}\\
X_{1} t_{j} & =t_{j} X_{1} & & \text { for all } j=1, \ldots, n,
\end{array}
$$

and if $l<\infty$,

$$
\begin{equation*}
\left(X_{1}-Q_{0}\right)\left(X_{1}-Q_{1}\right) \cdots\left(X_{1}-Q_{l-1}\right)=0 \tag{6.18}
\end{equation*}
$$

The algebra $\mathrm{Y}(d, \infty, n)$ is called the affine Yokonuma-Hecke algebra. For $l<\infty$, the algebra $\mathrm{Y}(d, l, n)$ is called the cyclotomic Yokonuma-Hecke algebra. These algebras are isomorphic to the modular framisations of, respectively, the affine Hecke algebra $(l=\infty)$ and the Ariki-Koike algebra $(l<\infty)$; see definitions in [JuLa5, Section 6] and [ChPdA1, Remark 1].

The cyclotomic Yokonuma-Hecke algebra is a quotient of the affine Yokonuma-Hecke algebra by the relation 6.18). If we map $X_{1} \mapsto Q_{0}$ for $l<\infty$ or $X_{1} \mapsto 1$ for $l=\infty$, we obtain a surjection of $\mathrm{Y}(d, l, n)$ onto $\mathrm{Y}_{d, n}(q)$. If we map $t_{j} \mapsto 1$ for all $j=1, \ldots, n$, then we obtain a surjection of $\mathrm{Y}(d, l, n)$ onto $\mathcal{H}(l, n)$, where $\mathcal{H}(l, n)$ denotes the Ariki-Koike algebra associated to $G(l, 1, n)$ for $l<\infty$ and $\mathcal{H}(\infty, n)$ denotes the affine Hecke algebra of type $A$. Moreover, we have $\mathrm{Y}(d, 1, n) \cong \mathrm{Y}_{d, n}(q)$ and $\mathrm{Y}(1, l, n) \cong \mathcal{H}(l, n)$. In particular, we have $\mathrm{Y}(1,1, n) \cong \mathcal{H}_{n}(q)$. All these relations are depicted in the following commutative diagram (where $l$ is taken to be a positive integer):

Remark 6.5 Let $p$ be a prime number. In a recent series of papers Vi1, Vi2, Vi3, Vignéras introduced and studied a large family of algebras, called pro-p-Iwahori-Hecke algebras. They generalise convolution algebras of compactly supported functions on a $p$-adic connected reductive group that are bi-invariant under the pro- $p$-radical of an Iwahori subgroup, which play an important role in the $p$-modular representation theory of $p$-adic reductive groups. In ChSe we have shown that the affine Yokonuma-Hecke algebra $\mathrm{Y}(d, \infty, n)$ is a pro- $p$-Iwahori-Hecke algebra. Thus, the affine Yokonuma-Hecke algebra generalises the affine Hecke algebra of type $A$ in a similar way that the Yokonuma-Hecke algebra generalises the Iwahori-Hecke algebra of type $A$. In particular, for $q=p^{m}$ and $d=p^{m}-1$, where $m$ is a positive integer, $\mathrm{Y}(d, \infty, n)$ is isomorphic to the convolution algebra of complex valued and compactly supported functions on the group $\mathrm{GL}_{n}(F)$, with $F$ a suitable $p$-adic field, that are bi-invariant under the pro- $p$ radical of an Iwahori subgroup.

Remark 6.6 Following Lusztig's approach in Lu7, Cui Cuil has established an explicit algebra isomorphism between the affine Yokonuma-Hecke algebra $\mathrm{Y}(d, \infty, n)$ and a direct sum of matrix algebras over tensor products of affine Hecke algebras of type $A$, similar to 6.14). More recently, Poulain d'Andecy PdA obtained the same result, as well as an isomorphism between the cyclotomic Yokonuma-Hecke algebra $\mathrm{Y}(d, l, n)$, where $l<\infty$, and a direct sum of matrix algebras over tensor products of Ariki-Koike algebras, using the same approach as in JaPdA. The isomorphism theorem for cyclotomic YokonumaHecke algebras has been subsequently re-obtained by Rostam Ros using his result that cyclotomic Yokonuma-Hecke algebras are cyclotomic quiver Hecke algebras.

In ChPdA2 we have constructed several bases for the algebra $\mathrm{Y}(d, l, n)$. In order to describe them here, we introduce the following notation: Let $Z_{l}:=\{0, \ldots, l-1\}$ for $l<\infty$ and $Z_{\infty}:=\mathbb{Z}$. We define inductively elements $X_{2}, \ldots, X_{n}$ of $\mathrm{Y}(d, l, n)$ by setting

$$
X_{i+1}:=q^{-1} g_{i} X_{i} g_{i} \quad \text { for all } i=1, \ldots, n-1
$$

Let $\mathcal{B}_{d, n}$ be a basis of the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q) \cong \mathrm{Y}(d, 1, n)$ over $\mathcal{R}_{l}$ (we can take, for example, $\mathcal{B}_{d, n}^{\mathrm{H}}$ defined in (6.7)). We denote by $\mathcal{B}_{d, l, n}^{\mathrm{AK}}$ the following set of elements of $\mathrm{Y}(d, l, n)$ :

$$
X_{1}^{a_{1}} \ldots X_{n}^{a_{n}} \cdot \omega, \quad a_{k} \in Z_{l} \text { and } \omega \in \mathcal{B}_{d, n}
$$

Now, for $k=1, \ldots, n$, we set

$$
\begin{aligned}
& W_{J, a, b}^{(k)}:=g_{J}^{-1} \ldots g_{2}^{-1} g_{1}^{-1} X_{1}^{a} t_{1}^{b} g_{1} g_{2} \ldots g_{k-1}, \\
& W_{J, a, b}^{(k)-}:=g_{J} \ldots g_{2} g_{1} X_{1}^{a} t_{1}^{b} g_{1}^{-1} g_{2}^{-1} \ldots g_{k-1}^{-1}, \\
& \widetilde{W}_{J, a, b}^{(k)}:=g_{J} \ldots g_{2} g_{1} X_{1}^{a} t_{1}^{b} g_{1} g_{2} \ldots g_{k-1} \\
& \widetilde{W}_{J, a, b}^{(k)-}:=g_{J}^{-1} \ldots g_{2}^{-1} g_{1}^{-1} X_{1}^{a} t_{1}^{b} g_{1}^{-1} g_{2}^{-1} \ldots g_{k-1}^{-1},
\end{aligned}
$$

where $J \in\{0, \ldots, k-1\}$ and $a, b \in \mathbb{Z}$. We use the following standard conventions: for $\epsilon= \pm 1$, $g_{J}^{\epsilon} \ldots g_{2}^{\epsilon} g_{1}^{\epsilon}:=1$ and $g_{k-J}^{\epsilon} \ldots g_{k-2}^{\epsilon} g_{k-1}^{\epsilon}:=1$ if $J=0$. Then we denote, respectively, by $\mathcal{B}_{d, l, n}^{\text {Ind }}, \mathcal{B}_{d, l, n}^{\text {Ind }-}$, $\widetilde{\mathcal{B}}_{d, l, n}^{\text {Ind }}$ and $\widetilde{\mathcal{B}}_{d, l, n}^{\text {Ind }}$ the following sets of elements of $\mathrm{Y}(d, l, n)$ :

$$
W_{J_{n}, a_{n}, b_{n}}^{(n)} \ldots W_{J_{2}, a_{2}, b_{2}}^{(2)} W_{J_{1}, a_{1}, b_{1}}^{(1)}, \quad J_{k} \in\{0, \ldots, k-1\}, a_{k} \in Z_{l} \text { and } b_{k} \in\{0, \ldots, d-1\} .
$$

$$
\begin{array}{ll}
W_{J_{n}, a_{n}, b_{n}}^{(n)-} \ldots W_{J_{2}, a_{2}, b_{2}}^{(2)-} W_{J_{1}, a_{1}, b_{1}}^{(1)}, & J_{k} \in\{0, \ldots, k-1\}, a_{k} \in Z_{l} \text { and } b_{k} \in\{0, \ldots, d-1\} . \\
\widetilde{W}_{J_{n}, a_{n}, b_{n}}^{(n)} \ldots \widetilde{W}_{J_{2}, a_{2}, b_{2}}^{(2)} \widetilde{W}_{J_{1}, a_{1}, b_{1}}^{(1)}, & J_{k} \in\{0, \ldots, k-1\}, a_{k} \in Z_{l} \text { and } b_{k} \in\{0, \ldots, d-1\} . \\
\widetilde{W}_{J_{n}, a_{n}, b_{n}}^{(n)-} \ldots \widetilde{W}_{J_{2}, a_{2}, b_{2}}^{(2)-} \widetilde{W}_{J_{1}, a_{1}, b_{1}}^{(1)-}, & J_{k} \in\{0, \ldots, k-1\}, a_{k} \in Z_{l} \text { and } b_{k} \in\{0, \ldots, d-1\} .
\end{array}
$$

We then have the following [ChPdA2, Theorem 4.4]:
Theorem 6.7 Each set $\mathcal{B}_{d, l, n}^{\mathrm{AK}}, \mathcal{B}_{d, l, n}^{\mathrm{Ind}}, \mathcal{B}_{d, l, n}^{\text {Ind- }}, \widetilde{\mathcal{B}}_{d, l, n}^{\text {Ind }}$ and $\widetilde{\mathcal{B}}_{d, l, n}^{\mathrm{Ind}}$ is an $\mathcal{R}_{l}$-basis of $\mathrm{Y}(d, l, n)$. In particular, $\mathrm{Y}(d, l, n)$ is a free $\mathcal{R}_{l}$-module and, if $l<\infty$, its rank is equal to $(d l)^{n} n$ !.

Remark 6.8 The set $\mathcal{B}_{d, l, n}^{\mathrm{AK}}$ is the analogue of the Ariki-Koike basis 4.1) of the Ariki-Koike algebra $\mathcal{H}(l, n)$ for $l<\infty$, and the standard Bernstein basis of the affine Hecke algebra of type $A$ for $l=\infty$. The four other sets are inductive sets with respect to $n$, which are analogous to the inductive bases of $\mathcal{H}(l, n)$ studied in La2, OgPo.

Furthermore, in ChPdA2 we have studied the representation theory of the cyclotomic YokonumaHecke algebra $\mathrm{Y}(d, l, n)$, which is quite similar to the representation theory of the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q)$. From now on, we only consider the case $l<\infty$.

Let $\mathcal{K}_{l}$ denote the field of fractions of $\mathcal{R}_{l}$. We will see that the irreducible representations of the algebra $\mathcal{K}_{l} \mathrm{Y}(d, l, n)$ are parametrised by the $d l$-partitions of $n$. Instead of looking though at $d l$-partitions as $d l$-tuples of partitions, we look at them as $d$-tuples of $l$-partitions, and we call them $(d, l)$-partitions when seen as such. We denote by $\mathcal{P}(d, l, n)$ the set of $(d, l)$-partitions of $n$. If $\boldsymbol{\lambda} \in \mathcal{P}(d, l, n)$, then $\boldsymbol{\lambda}=\left(\boldsymbol{\lambda}^{(0)}, \boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(d-1)}\right)$, where $\boldsymbol{\lambda}^{(i)}$ is an $l$-partition for all $i=0,1, \ldots, d-1$, and $\sum_{i=0}^{d-1}\left|\boldsymbol{\lambda}^{(i)}\right|=n$. We thus have $\boldsymbol{\lambda}^{(i)}=\left(\lambda^{(i, 0)}, \lambda^{(i, 1)}, \ldots, \lambda^{(i, l-1)}\right)$, where $\lambda^{(i, j)}$ is a partition for all $i=0,1, \ldots, d-1$ and $j=0,1, \ldots, l-1$, and $\sum_{i=0}^{d-1} \sum_{j=0}^{l-1}\left|\lambda^{(i, j)}\right|=n$.

A node $\boldsymbol{\theta}$ of $\boldsymbol{\lambda}$ is a 4 -tuple $(x, y, i, j)$, where $0 \leqslant i \leqslant d-1,0 \leqslant j \leqslant l-1$ and $(x, y)$ is a node of the partition $\lambda^{(i, j)}$. We define $\mathrm{p}(\boldsymbol{\theta}):=i$ to be the $d$-position of $\boldsymbol{\theta}$ and $\mathrm{c}(\boldsymbol{\theta}):=Q_{j} q^{y-x}$ to be the l-quantum content of $\boldsymbol{\theta}$.

Following the definitions in 6.1, a $(d, l)$-tableau is simply a dl-tableau and a standard $(d, l)$-tableau is simply a standard $d l$-tableau. For a $(d, l)$-tableau $\mathcal{T}$ and for $i=1, \ldots, n$, we denote respectively by $\mathrm{p}(\mathcal{T} \mid i)$ and $\mathrm{c}(\mathcal{T} \mid i)$ the $d$-position and the $l$-quantum content of the node with the number $i$ in it.

Now, let $\boldsymbol{\lambda} \in \mathcal{P}(d, l, n)$, and let $V_{\boldsymbol{\lambda}}$ be a $\mathbb{C}(q)$-vector space with a basis $\left\{\mathbf{v}_{\mathcal{T}}\right\}$ indexed by the standard $(d, l)$-tableaux of shape $\boldsymbol{\lambda}$. We set $\mathbf{v}_{\mathcal{T}}:=0$ for any non-standard $(d, l)$-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. By ChPdA2, Propositions $3.2 \& 3.4]$, the vector space $V_{\boldsymbol{\lambda}}$ is a representation of $\mathcal{K}_{l} \mathrm{Y}(d, l, n)$, with the action of the generators $g_{1}, \ldots, g_{n-1}, t_{1}, \ldots, t_{n}$ on the basis element $\mathbf{v}_{\mathcal{T}}$ defined exactly as in Theorem 6.2, and the action of the generator $X_{1}$ given by:

$$
\begin{equation*}
X_{1}\left(\mathbf{v}_{\mathcal{T}}\right)=\mathrm{c}(\mathcal{T} \mid 1) \mathbf{v}_{\mathcal{T}} \tag{6.19}
\end{equation*}
$$

Further, the set $\left\{V_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}(d, l, n)\right\}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathcal{K}_{l} \mathrm{Y}(d, l, n)$.

Remark 6.9 We can easily show, by induction on $i$, that ChPdA2, Lemma 3.3]:

$$
\begin{equation*}
X_{i}\left(\mathbf{v}_{\mathcal{T}}\right)=\mathrm{c}(\mathcal{T} \mid i) \mathbf{v}_{\mathcal{T}} \quad \text { for all } i=1, \ldots, n \tag{6.20}
\end{equation*}
$$

We also have a semisimplicity criterion for cyclotomic Yokonuma-Hecke algebras, which is exactly the same as Ariki's semisimplicity criterion 4.2) for Ariki-Koike algebras [ChPdA2, Proposition 4.7]:

Proposition 6.10 Let $\theta: \mathcal{R}_{l} \rightarrow \mathbb{C}$ be a ring homomorphism such that $\theta(q) \prod_{j=0}^{l-1} \theta\left(Q_{j}\right) \neq 0$. The specialised cyclotomic Yokonuma-Hecke algebra $\mathbb{C Y}(d, l, n)_{\theta}$, defined via $\theta$, is (split) semisimple if and only if $\theta(P) \neq 0$, where

$$
P=\prod_{1 \leqslant i \leqslant n}\left(1+q+\cdots+q^{i-1}\right) \prod_{0 \leqslant s<t \leqslant l-1} \prod_{-n<k<n}\left(q^{k} Q_{s}-Q_{t}\right) .
$$

We deduce that the algebra $\mathbb{C Y}(d, l, n)_{\theta}$ is semisimple if and only if the specialised Ariki-Koike algebra $\mathbb{C H}(l, n)_{\theta}$ is semisimple.

Finally, we have proved the existence of a "canonical" symmetrising form on $\mathcal{K}_{l} \mathrm{Y}(d, l, n)$ and calculated the Schur elements with respect to it ChPdA2, §7]:

Theorem 6.11 We define the linear map $\tau: \mathrm{Y}(d, l, n) \rightarrow \mathcal{R}_{l}$ by

$$
\tau\left(X_{1}^{a_{1}} \ldots X_{n}^{a_{n}} t_{1}^{b_{1}} \ldots t_{n}^{b_{n}} g_{w}\right)= \begin{cases}1 & \text { if } w=1 \text { and } a_{j}=b_{j}=0 \text { for all } j=1,2, \ldots, n,  \tag{6.21}\\ 0 & \text { otherwise },\end{cases}
$$

where $w \in \mathfrak{S}_{n}, a_{j} \in Z_{l}$ and $0 \leqslant b_{j} \leqslant d-1$ for all $j=1,2, \ldots, n$. Then $\tau$ (extended linearly) is $a$ symmetrising form on $\mathcal{K}_{l} Y(d, l, n)$. If $\boldsymbol{\lambda}=\left(\boldsymbol{\lambda}^{(0)}, \ldots, \boldsymbol{\lambda}^{(d-1)}\right) \in \mathcal{P}(d, l, n)$, then the Schur element of $V_{\boldsymbol{\lambda}}$ with respect to $\tau$ is

$$
\begin{equation*}
s_{\boldsymbol{\lambda}}=d^{n} s_{\boldsymbol{\lambda}^{(0)}} s_{\boldsymbol{\lambda}^{(1)}} \ldots s_{\boldsymbol{\lambda}^{(d-1)}} \tag{6.22}
\end{equation*}
$$

where $s_{\boldsymbol{\lambda}^{(i)}}$ is the Schur element of the Ariki-Koike algebra $\mathcal{H}\left(l,\left|\boldsymbol{\lambda}^{(i)}\right|\right)$ corresponding to $\boldsymbol{\lambda}^{(i)}$ for all $i=0,1, \ldots, d-1$ (we take $s_{\emptyset}:=1$ ).

A simple formula for the calculation of the Schur elements of Ariki-Koike algebras is given by Theorem 4.3.

Remark 6.12 The map $\tau$ is known to be a symmetrising form on $\mathrm{Y}(d, l, n)$ (defined over $\left.\mathcal{R}_{l}\right)$ in cases $d=1$ MalMat and $l=1$ ChPdA1]. In these cases, $\tau$ is called the canonical symmetrising form on $\mathrm{Y}(d, l, n)$.

Remark 6.13 Equation 6.22 hints towards an isomorphism between the cyclotomic Yokonuma-Hecke algebra $\mathrm{Y}(d, l, n)$ and a direct sum of matrix algebras over tensor products of Ariki-Koike algebras; this isomorphism was recently described by Poulain d'Andecy in [PdA and by Rostam in Ros.

### 6.3 Temperley-Lieb quotients of Yokonuma-Hecke algebras

The Temperley-Lieb algebra was introduced by Temperley and Lieb in TeLi] for its applications in statistical mechanics. Jones Jo1, Jo2, Jo3] later showed that it can be obtained as a quotient of the Iwahori-Hecke algebra $\mathcal{H}_{n}(q)$ of type $A$ by a two-sided ideal, and used it for the construction of the knot invariant known as the Jones polynomial.

Since Yokonuma-Hecke algebras can be also used for the definition of knot invariants, it was natural to ask what the analogue of the Temperley-Lieb algebra would be in this case. As it is explained in more detail in JuLa5, where the technique of framisation is thoroughly discussed, three possible candidates arose: the Yokonuma-Temperley-Lieb algebra GJKL1, the Framisation of the Temperley-Lieb algebra GJKL2] and the Complex Reflection Temperley-Lieb algebra GJKL2]. All three are defined as quotients of the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q)$ of type $A$ by a suitable two-sided ideal, and they specialise to the classical Temperley-Lieb algebra for $d=1$.

In this section, we will determine the irreducible representations of the three algebras by showing which representations of $\mathrm{Y}_{d, n}(q)$ pass to each quotient. We will compute their dimensions and construct bases for them. At the end of this section, it will be clear that the most natural analogue of the Temperley-Lieb algebra in this setting is the Framisation of the Temperley-Lieb algebra.

First, let us recall some information about the classical setting. Let $n \geqslant 3$. The Temperley-Lieb algebra $\mathrm{TL}_{n}(q)$ is defined as the quotient of the Iwahori-Hecke algebra $\mathcal{H}_{n}(q) \cong \mathrm{Y}_{1, n}(q)$ by the ideal $I_{n}$ generated by the elements

$$
g_{i, i+1}:=1+g_{i}+g_{i+1}+g_{i} g_{i+1}+g_{i+1} g_{i}+g_{i} g_{i+1} g_{i}=\sum_{w \in\left\langle s_{i}, s_{i+1}\right\rangle} g_{w}
$$

for all $i=1, \ldots, n-2$. It turns out that this ideal is principal, and we have $I_{n}=\left\langle g_{1,2}\right\rangle$.
Since the algebra $\mathbb{C}(q) \mathcal{H}_{n}(q)$ is semisimple, the algebra $\mathbb{C}(q) \mathrm{TL}_{n}(q)$ is also semisimple and its irreducible representations are precisely the irreducible representations of $\mathbb{C}(q) \mathcal{H}_{n}(q)$ that pass to the quotient. That is, for $\lambda \in \mathcal{P}(n), V_{\lambda}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{TL}_{n}(q)$ if and only if $g_{1,2}\left(\mathrm{v}_{\mathcal{T}}\right)=0$ for every standard tableau $\mathcal{T}$ of shape $\lambda$. It is easy to see that the latter is equivalent to the trivial representation not being a direct summand of the restriction $\operatorname{Res}_{\left\langle s_{1}, s_{2}\right\rangle}^{\mathfrak{G}_{n}}\left(E^{\lambda}\right)$, where $E^{\lambda}$ is the irreducible representation of the symmetric group $\mathfrak{S}_{n}$ labelled by $\lambda$. Since the restriction from $\mathfrak{S}_{n}$ to $\mathfrak{S}_{3} \cong\left\langle s_{1}, s_{2}\right\rangle$ corresponds to the simple removal of boxes from the Young diagram of $\lambda$, and the trivial representation of $\mathfrak{S}_{3}$ is labelled by the partition (3), we obtain the following description of the irreducible representations of $\mathbb{C}(q) \mathrm{TL}_{n}(q)$ :

Theorem 6.14 Let $\lambda \in \mathcal{P}(n)$. We have that $V_{\lambda}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{TL}_{n}(q)$ if and only if the Young diagram of $\lambda$ has at most two columns.

Now, let $n \in \mathbb{N}$, and let $\underline{i}=\left(i_{1}, \ldots, i_{p}\right)$ and $\underline{k}=\left(k_{1}, \ldots k_{p}\right)$ be two $p$-tuples of non-negative integers, with $0 \leqslant p \leqslant n-1$. We denote by $\mathfrak{H}_{n}$ the set of pairs $(\underline{i}, \underline{k})$ such that

$$
1 \leqslant i_{1}<i_{2}<\cdots<i_{p} \leqslant n-1 \text { and } i_{j}-k_{j} \geqslant 1 \quad \forall j=1, \ldots, p
$$

For $(\underline{i}, \underline{k}) \in \mathfrak{H}_{n}$, we set

$$
g_{\underline{i}, \underline{k}}:=\left(g_{i_{1}} g_{i_{1}-1} \ldots g_{i_{1}-k_{1}}\right)\left(g_{i_{2}} g_{i_{2}-1} \ldots g_{i_{2}-k_{2}}\right) \ldots\left(g_{i_{p}} g_{i_{p}-1} \ldots g_{i_{p}-k_{p}}\right) \in \mathcal{H}_{n}(q) .
$$

We take $g_{\emptyset, \emptyset}$ to be equal to 1 . We have that the set

$$
\mathcal{B}_{1, n}^{\mathrm{H}}=\left\{g_{w} \mid w \in \mathfrak{S}_{n}\right\}=\left\{g_{\underline{i}, \underline{k}} \mid(\underline{i}, \underline{k}) \in \mathfrak{H}_{n}\right\}
$$

is the standard basis of $\mathcal{H}_{n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module.
Further, let us denote by $\mathfrak{T}_{n}$ the subset of $\mathfrak{H}_{n}$ consisting of the pairs $(\underline{i}, \underline{k})$ such that

$$
1 \leqslant i_{1}<i_{2}<\cdots<i_{p} \leqslant n-1 \quad \text { and } \quad 1 \leqslant i_{1}-k_{1}<i_{2}-k_{2}<\cdots<i_{p}-k_{p} \leqslant n-1
$$

Jones Jo1 has shown that the set

$$
\mathcal{B}_{1, n}^{\mathrm{TL}}:=\left\{g_{\underline{i}, \underline{k}} \mid(\underline{i}, \underline{k}) \in \mathfrak{T}_{n}\right\}
$$

is a basis of $\mathrm{TL}_{n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module. We have $\left|\mathcal{B}_{n}^{\mathrm{TL}}\right|=C_{n}$, where $C_{n}$ is the $n$-th Catalan number, i.e.,

$$
C_{n}=\frac{1}{n+1}\binom{2 n}{n}=\frac{1}{n+1} \sum_{k=0}^{n}\binom{n}{k}^{2}
$$

### 6.3.1 The Yokonuma-Temperley-Lieb algebra

Let $d \in \mathbb{N}^{*}$ and let $n \in \mathbb{N}$ with $n \geqslant 3$. The Yokonuma-Temperley-Lieb algebra $\mathrm{YTL}_{d, n}(q)$ is defined as the quotient of the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q)$ by the ideal $I_{d, n}:=\left\langle g_{1,2}\right\rangle$.

Since the algebra $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ is semisimple, the algebra $\mathbb{C}(q) \mathrm{YTL}_{d, n}(q)$ is also semisimple and its irreducible representations are precisely the irreducible representations of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ that pass to the quotient. That is, for $\boldsymbol{\lambda} \in \mathcal{P}(d, n), V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{YTL}_{d, n}(q)$ if and only if $g_{1,2}\left(\mathbf{v}_{\mathcal{T}}\right)=0$ for every standard $d$-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. It is easy to see that the latter is equivalent to the trivial representation not being a direct summand of the restriction $\operatorname{Res}_{\left\langle s_{1}, s_{2}\right\rangle}^{G(d, 1, n)}\left(E^{\boldsymbol{\lambda}}\right)$, where $E^{\boldsymbol{\lambda}}$ is the irreducible representation of the complex reflection group $G(d, 1, n)$ labelled by $\boldsymbol{\lambda}$. Unfortunately, this restriction for $d>1$ does not correspond to the simple removal of boxes from the Young diagram of $\boldsymbol{\lambda}$ (as in the symmetric group case), but it is controlled by the so-called Littlewood-Richardson coefficients. Using algebraic combinatorics, we obtain the following description of the irreducible representations of $\mathbb{C}(q) \mathrm{YTL}_{d, n}(q)$ ChPo1, Theorem 3]:

Theorem 6.15 Let $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \ldots, \lambda^{(d-1)}\right) \in \mathcal{P}(d, n)$. We have that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{YTL}_{d, n}(q)$ if and only if the Young diagram of $\boldsymbol{\lambda}$ has at most two columns in total, that is, $\sum_{i=0}^{d-1} \lambda_{1}^{(i)} \leqslant 2$.

Using the fact that the algebra $\mathbb{C}(q) \mathrm{YTL}_{d, n}(q)$ is semisimple and the above description of its irreducible representations, we have been able to calculate the dimension of the Yokonuma-Temperley-Lieb algebra ChPo1, Proposition 4]. We have

$$
\operatorname{dim}_{\mathbb{C}(q)}\left(\mathbb{C}(q) \mathrm{YTL}_{d, n}(q)\right)=\frac{n\left(d^{2}-d\right)+d^{2}+d}{2} C_{n}-\left(d^{2}-d\right)
$$

What is more, we have shown in ChPo1 that $\mathrm{YTL}_{d, n}(q)$ is a free $\mathbb{C}\left[q, q^{-1}\right]$-module of rank equal to the dimension above. However, note that, even though the set

$$
\mathcal{B}_{d, n}^{\mathrm{H}}=\left\{t_{1}^{r_{1}} \ldots t_{n}^{r_{n}} g_{\underline{i}, \underline{k}} \mid(\underline{i}, \underline{k}) \in \mathfrak{H}_{n}, 0 \leqslant r_{j} \leqslant d-1 \text { for all } j=1,2, \ldots, n\right\}
$$

is a basis of $\mathrm{Y}_{d, n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module, the set

$$
\mathcal{B}_{d, n}^{\mathrm{TL}}=\left\{t_{1}^{r_{1}} \ldots t_{n}^{r_{n}} g_{\underline{i}, \underline{k}} \mid(\underline{i}, \underline{k}) \in \mathfrak{T}_{n}, 0 \leqslant r_{j} \leqslant d-1 \text { for all } j=1,2, \ldots, n\right\}
$$

is not a basis of $\mathrm{YTL}_{d, n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module, since $\left|\mathcal{B}_{d, n}^{\mathrm{TL}}\right|=d^{n} C_{n}$. The set $\mathcal{B}_{d, n}^{\mathrm{TL}}$ is simply a generating set for $\mathrm{YTL}_{d, n}(q)$, and we have managed to find a subset $\mathcal{B}_{d, n}^{\mathrm{YTL}}$ of $\mathcal{B}_{d, n}^{\mathrm{TL}}$ that is a basis of $\mathrm{YTL}_{d, n}(q)$ by proving the following remarkable property: Let $(\underline{i}, \underline{k}) \in \mathfrak{T}_{n}$. We denote by $\mathcal{I}\left(g_{\underline{i}, \underline{k}}\right)$ the set (without repetition) of all indices of the $g_{j}$ 's appearing in $g_{i-k}$, i.e.,

$$
\mathcal{I}\left(g_{i, \underline{k}}\right)=\left\{i_{1}, i_{1}-1, \ldots, i_{1}-k_{1}, i_{2}, i_{2}-1, \ldots, i_{2}-k_{2}, \ldots, i_{p}, i_{p}-1, \ldots, i_{p}-k_{p}\right\}
$$

We define the weight of $g_{\underline{i}, \underline{k}}$ to be $w_{\underline{i}, \underline{k}}:=\left|\mathcal{I}\left(g_{\underline{i}, \underline{k}}\right)\right|$. We then have ChPo1, Propositions 9, 11, 12]:

$$
\left|\left\{\left(r_{1}, \ldots, r_{n}\right) \in\{0, \ldots, d-1\}^{n} \mid t_{1}^{r_{1}} \ldots t_{n}^{r_{n}} g_{\underline{i}, \underline{k}} \in \mathcal{B}_{d, n}^{\mathrm{YTL}}\right\}\right|=2^{n-w_{i, \underline{k}}-1}\left(d^{2}-d\right)+d-\delta_{w_{i, k}, \underline{0}}\left(d^{2}-d\right)
$$

where $\delta_{i, j}$ stands for Kronecker's delta (note that we have $w_{\underline{i}, \underline{k}}=0$ if and only if $g_{i, \underline{k}}=1$ ). Thanks to this property, an explicit basis for $\mathrm{YTL}_{d, n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module is described in ChPo1.

### 6.3.2 The Framisation of the Temperley-Lieb algebra

Let $d \in \mathbb{N}^{*}$ and let $n \in \mathbb{N}$ with $n \geqslant 3$. The Framisation of the Temperley-Lieb algebra $\mathrm{FTL}_{d, n}(q)$ is defined as the quotient of the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q)$ by the ideal $J_{d, n}:=\left\langle e_{1} e_{2} g_{1,2}\right\rangle$. We remark that $J_{d, n}$ can be also defined as the ideal generated by the element $\sum_{0 \leqslant a, b \leqslant d-1} t_{1}^{a} t_{2}^{b} t_{3}^{-a-b} g_{1,2}$.

Again, since the algebra $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ is semisimple, the algebra $\mathbb{C}(q) \mathrm{FTL}_{d, n}(q)$ is also semisimple and its irreducible representations are precisely the irreducible representations of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ that pass to the quotient. That is, for $\boldsymbol{\lambda} \in \mathcal{P}(d, n), V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{FTL}_{d, n}(q)$ if and only if $e_{1} e_{2} g_{1,2}\left(\mathbf{v}_{\mathcal{T}}\right)=0$ for every standard $d$-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. Using the formulas for the irreducible representations of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ given by Theorem 6.2 , we obtain the following description of the irreducible representations of $\mathbb{C}(q) \mathrm{FTL}_{d, n}(q)$ ChPo2, Theorem 3.10]:

Theorem 6.16 Let $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \ldots, \lambda^{(d-1)}\right) \in \mathcal{P}(d, n)$. We have that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{FTL}_{d, n}(q)$ if and only if the Young diagram of $\lambda^{(i)}$ has at most two columns for each $i=0, \ldots, d-1$.

Following the recipe of JaPdA, §3], we have proved the following isomorphism theorem for $\mathrm{FTL}_{d, n}(q)$ ChPo2, Theorem 4.3]:

Theorem 6.17 There exists $a \mathbb{C}\left[q, q^{-1}\right]$ algebra isomorphism

$$
\psi_{n}: \mathrm{FTL}_{d, n}(q) \rightarrow \bigoplus_{\mu \in \operatorname{Comp}_{d}(n)} \operatorname{Mat}_{m_{\mu}}\left(\mathrm{TL}_{\mu_{0}}(q) \otimes \mathrm{TL}_{\mu_{1}}(q) \otimes \cdots \otimes \mathrm{TL}_{\mu_{d-1}}(q)\right)
$$

where $\operatorname{Comp}_{d}(n)$ and $m_{\mu}$ are as defined in 6.15 and 6.16), and we take $\mathrm{TL}_{n}(q) \cong \mathcal{H}_{n}(q)$ for $n<3$.
We deduce that the following set is a basis of $\mathrm{FTL}_{d, n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module:

$$
\left\{\psi_{n}^{-1}\left(b_{0}^{\mu} b_{1}^{\mu} \ldots b_{d-1}^{\mu} M_{k, l}^{\mu}\right) \mid \mu \in \operatorname{Comp}_{d}(n), b_{i}^{\mu} \in \mathcal{B}_{1, \mu_{i}}^{\mathrm{TL}} \text { for all } i=0, \ldots, d-1,1 \leqslant k, l \leqslant m_{\mu}\right\}
$$

where $M_{k, l}^{\mu}$ denotes the elementary $m_{\mu} \times m_{\mu}$ matrix with 1 in position $(k, l)$. In particular, $\operatorname{FTL}_{d, n}(q)$ is a free $\mathbb{C}\left[q, q^{-1}\right]$-module of rank

$$
\sum_{\mu \in \operatorname{Comp}_{d}(n)} m_{\mu}^{2} C_{\mu_{0}} C_{\mu_{1}} \cdots C_{\mu_{d-1}}
$$

### 6.3.3 The Complex Reflection Temperley-Lieb algebra

Let $d \in \mathbb{N}^{*}$ and let $n \in \mathbb{N}$ with $n \geqslant 3$. The Complex Reflection Temperley-Lieb algebra $\operatorname{CTL}_{d, n}(q)$ is defined as the quotient of the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q)$ by the ideal $K_{d, n}:=\left\langle\sum_{s=0}^{d-1} t_{1}^{s} e_{1} e_{2} g_{1,2}\right\rangle$. We remark that $K_{d, n}$ can be also viewed as the ideal generated by the element $\sum_{0 \leqslant a, b, c \leqslant d-1} t_{1}^{a} t_{2}^{b} t_{3}^{c} g_{1,2}$.

Once more, the algebra $\mathbb{C}(q) \mathrm{CTL}_{d, n}(q)$ is semisimple and, for $\boldsymbol{\lambda} \in \mathcal{P}(d, n), V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{CTL}_{d, n}(q)$ if and only if $\sum_{s=0}^{d-1} t_{1}^{s} e_{1} e_{2} g_{1,2}\left(\mathbf{v}_{\mathcal{T}}\right)=0$ for every standard $d$-tableau $\mathcal{T}$ of shape $\boldsymbol{\lambda}$. Using the formulas for the irreducible representations of $\mathbb{C}(q) \mathrm{Y}_{d, n}(q)$ given by Theorem 6.2, we obtain the following description of the irreducible representations of $\mathbb{C}(q) \mathrm{CTL}_{d, n}(q)$ ChPo2, Theorem 5.3]:

Theorem 6.18 Let $\left\{\xi_{0}, \xi_{1}, \ldots, \xi_{d-1}\right\}$ be the set of all d-th roots of unity (ordered arbitrarily) as in Theorem 6.2. Let $i_{0} \in\{0, \ldots, d-1\}$ be such that $\xi_{i_{0}}=1$, and let $\boldsymbol{\lambda}=\left(\lambda^{(0)}, \ldots, \lambda^{(d-1)}\right) \in \mathcal{P}(d, n)$. We have that $V_{\boldsymbol{\lambda}}$ is an irreducible representation of $\mathbb{C}(q) \mathrm{CTL}_{d, n}(q)$ if and only if the Young diagram of $\lambda^{\left(i_{0}\right)}$ has at most two columns.

Following the recipe of JaPdA, §3], we have proved the following isomorphism theorem for $\mathrm{CTL}_{d, n}(q)$ ChPo2, Theorem 5.8]:

Theorem 6.19 There exists a $\mathbb{C}\left[q, q^{-1}\right]$ algebra isomorphism

$$
\bar{\psi}_{n}: \operatorname{CTL}_{d, n}(q) \rightarrow \bigoplus_{\mu \in \operatorname{Comp}_{d}(n)} \operatorname{Mat}_{m_{\mu}}\left(\operatorname{TL}_{\mu_{0}}(q) \otimes \mathcal{H}_{\mu_{1}}(q) \otimes \mathcal{H}_{\mu_{2}}(q) \otimes \cdots \otimes \mathcal{H}_{\mu_{d-1}}(q)\right),
$$

where $\operatorname{Comp}_{d}(n)$ and $m_{\mu}$ are as defined in 6.15 and 6.16), and we take $\operatorname{TL}_{n}(q) \cong \mathcal{H}_{n}(q)$ for $n<3$.
We deduce that the following set is a basis of $\mathrm{CTL}_{d, n}(q)$ as a $\mathbb{C}\left[q, q^{-1}\right]$-module:
$\left\{\bar{\psi}_{n}^{-1}\left(b_{0}^{\mu} b_{1}^{\mu} \ldots b_{d-1}^{\mu} M_{k, l}^{\mu}\right) \mid \mu \in \operatorname{Comp}_{d}(n), b_{0}^{\mu} \in \mathcal{B}_{1, \mu_{0}}^{\mathrm{TL}}, b_{i}^{\mu} \in \mathcal{B}_{1, \mu_{i}}^{\mathrm{H}}\right.$ for all $\left.i=1, \ldots, d-1,1 \leqslant k, l \leqslant m_{\mu}\right\}$,
where $M_{k, l}^{\mu}$ denotes the elementary $m_{\mu} \times m_{\mu}$ matrix with 1 in position $(k, l)$. In particular, $\operatorname{CTL}_{d, n}(q)$ is a free $\mathbb{C}\left[q, q^{-1}\right]$-module of rank

$$
\sum_{\mu \in \operatorname{Comp}_{d}(n)} m_{\mu}^{2} C_{\mu_{0}} \mu_{1}!\ldots \mu_{d-1}!
$$

### 6.3.4 Relations between the Temperley-Lieb quotients

Let $d \in \mathbb{N}^{*}$ and let $n \in \mathbb{N}$ with $n \geqslant 3$. The inclusions of ideals $K_{d, n} \subseteq J_{d, n} \subseteq I_{d, n}$ yield the following commutative diagram of natural algebra epimorphisms:

where the non-horizontal arrows are defined by $t_{j} \mapsto 1$ for all $j=1, \ldots, n$.

### 6.4 Knot invariants

One of the reasons that the interest in Yokonuma-Hecke algebras was rekindled in the past years was that Juyumaya and Lambropoulou used the Markov trace on them defined by the former Ju2 in order to construct isotopy invariants for framed JuLa1, JuLa2, classical JuLa3 and singular JuLa4 knots and links. Their method is the same as the one applied by Jones, who used the Ocneanu trace on the Iwahori-Hecke algebra of type $A$ to define the Jones polynomial Jo2, an invariant for classical knots and links arising from the Temperley-Lieb algebra. The same technique, when applied directly to the Iwahori-Hecke algebra, yielded the Homflypt polynomial or 2-variable Jones polynomial HOMFLY, PT].

### 6.4.1 Invariants from Yokonuma-Hecke algebras for framed knots and links

From now on, set $\mathcal{R}:=\mathbb{C}\left[q, q^{-1}\right]$. Using the natural inclusions $\mathrm{Y}_{d, n}(q) \subset \mathrm{Y}_{d, n+1}(q)$, we can define a Markov trace on $\bigcup_{n \geqslant 0} \mathrm{Y}_{d, n}(q)$ as follows (cf. [Ju2, Theorem 12]):

Theorem 6.20 Let $z, x_{1}, \ldots, x_{d-1}$ be indeterminates over $\mathbb{C}(q)$. There exists a unique linear Markov trace

$$
\operatorname{tr}_{d}: \bigcup_{n \geqslant 0} \mathrm{Y}_{d, n}(q) \longrightarrow \mathcal{R}\left[z, x_{1}, \ldots, x_{d-1}\right]
$$

defined inductively on $\mathrm{Y}_{d, n}(q)$, for all $n \geqslant 0$, by the following rules:

$$
\begin{align*}
\operatorname{tr}_{d}(a b) & =\operatorname{tr}_{d}(b a) & & a, b \in \mathrm{Y}_{d, n}(q)  \tag{1}\\
\operatorname{tr}_{d}(1) & =1 & & 1 \in \mathrm{Y}_{d, n}(q) \tag{2}
\end{align*}
$$

$$
\text { (3) } \operatorname{tr}_{d}\left(a g_{n}\right)=z \operatorname{tr}_{d}(a) \quad a \in \mathrm{Y}_{d, n}(q) \quad \text { (Markov property) }
$$

$$
\text { (4) } \operatorname{tr}_{d}\left(a t_{n+1}^{k}\right)=x_{k} \operatorname{tr}_{d}(a) \quad a \in \mathrm{Y}_{d, n}(q) \quad(1 \leqslant k \leqslant d-1)
$$

For $d=1$, the trace $\operatorname{tr}_{1}$ is defined by only the first three rules, and it is the Ocneanu trace on the Iwahori-Hecke algebra $\mathcal{H}_{n}(q) \cong \mathrm{Y}_{1, n}(q)$.

Remark 6.21 The canonical symmetrising form on $\mathrm{Y}_{d, n}(q)$ defined by $\sqrt{6.12}$ coincides with the Markov trace with parameters $z=x_{1}=\cdots=x_{d-1}=0$.

The relations $\left(b_{1}\right),\left(b_{2}\right),\left(f_{1}\right)$ and $\left(f_{2}\right)$ in the definition of the Yokonuma-Hecke algebra (6.1) are defining relations for the framed braid group $\mathcal{F}_{n} \cong \mathbb{Z} \backslash B_{n}$, where $B_{n}$ is the classical braid group on $n$ strands (its generators usually denoted by $\sigma_{1}, \ldots, \sigma_{n-1}$ ), with the $t_{j}$ 's being interpreted as the "elementary framings" (framing 1 on the $j$-th strand). The relations $t_{j}^{d}=1$ mean that the framing of each braid strand is regarded modulo $d$. Thus, the algebra $\mathrm{Y}_{d, n}(q)$ arises naturally as a quotient of the framed braid group algebra over the modular relations $\left(\mathrm{f}_{3}\right)$ and the quadratic relations (6.2). Moreover, relations (6.1) are defining relations for the modular framed braid group $\mathcal{F}_{d, n} \cong(\mathbb{Z} / d \mathbb{Z})$ 々 $\overline{B_{n}}$, so the algebra $\mathrm{Y}_{d, n}(q)$ can be also seen as a quotient of the modular framed braid group algebra over the quadratic relations (6.2).

For any $\alpha \in \mathcal{F}_{n}$, we denote by $\widehat{\alpha}$ the link obtained as the closure of $\alpha$. By the Alexander Theorem, we have that $\mathcal{L}_{f}:=\cup_{n}\left\{\widehat{\alpha} \mid \alpha \in \mathcal{F}_{n}\right\}$ is the set of all oriented framed links. Further, by the Markov Theorem (see KoSm), isotopy of framed links is generated by conjugation in $\mathcal{F}_{n}$ and by positive and negative stabilisation and destabilisation $\left(\alpha \sim \alpha \sigma_{n}^{ \pm 1}\right)$, for any $n \in \mathbb{N}$. Using now the natural $\mathcal{R}$-algebra epimorphism from $\mathcal{R}\left[\mathcal{F}_{n}\right]$ onto $\mathrm{Y}_{d, n}(q)$ given by $\sigma_{i} \mapsto g_{i}$ and $t_{j} \mapsto t_{j}$, and abusing notation, we can define the $\operatorname{trace} \operatorname{tr}_{d}$ on the elements of $\mathbb{C} \mathcal{F}_{n}$, and thus, in particular, on the elements of $\mathcal{F}_{n}$. In order to use $\operatorname{tr}_{d}$ for constructing invariants for framed knots and links after Jones's method, the trace $\operatorname{tr}_{d}$ has to be normalised, so that the closures of the framed braids $\alpha$ and $\alpha \sigma_{n}\left(\alpha \in \mathcal{F}_{n}\right)$ be assigned the same value of the invariant, and re-scaled, so that the closures of the framed braids $\alpha \sigma_{n}^{-1}$ and $\alpha \sigma_{n}\left(\alpha \in \mathcal{F}_{n}\right)$ be assigned the same value of the invariant. However, $\operatorname{tr}_{d}\left(\alpha \sigma_{n}^{-1}\right)$ does not factor through $\operatorname{tr}_{d}(\alpha)$ as in the classical case, that is,

$$
\operatorname{tr}_{d}\left(\alpha \sigma_{n}^{-1}\right) \stackrel{\sqrt{6.4}}{=} q^{-1} \operatorname{tr}_{d}\left(\alpha \sigma_{n}\right)+\left(q^{-1}-1\right) \operatorname{tr}_{d}\left(\alpha e_{n}\right) \neq \operatorname{tr}_{d}(\alpha) \operatorname{tr}_{d}\left(\sigma_{n}^{-1}\right) .
$$

The reason is that, although $\operatorname{tr}_{d}\left(\alpha \sigma_{n}\right)=z \operatorname{tr}_{d}(\alpha)=\operatorname{tr}_{d}(\alpha) \operatorname{tr}_{d}\left(\sigma_{n}\right), \operatorname{tr}_{d}\left(\alpha e_{n}\right)$ does not always factor through $\operatorname{tr}_{d}(\alpha)$, that is,

$$
\operatorname{tr}_{d}\left(\alpha e_{n}\right) \neq \operatorname{tr}_{d}(\alpha) \operatorname{tr}_{d}\left(e_{n}\right)
$$

Forcing the so-called E-condition

$$
\operatorname{tr}_{d}\left(a e_{n}\right)=\operatorname{tr}_{d}(a) \operatorname{tr}_{d}\left(e_{n}\right) \quad \text { for any } n \geqslant 1 \text { and all } a \in \mathrm{Y}_{d, n}(q)
$$

yields that the trace parameters $x_{1}, \ldots, x_{d-1}$ have to satisfy the following non-linear system of equations, called the E-system:

$$
\sum_{s=0}^{d-1} x_{k+s} x_{d-s}=x_{k} \sum_{s=0}^{d-1} x_{s} x_{d-s} \quad(1 \leqslant k \leqslant d-1),
$$

where the sub-indices on the $x_{j}$ 's are regarded modulo $d$ and $x_{0}:=1$. Note that

$$
\frac{1}{d} \sum_{s=0}^{d-1} x_{s} x_{d-s}=\operatorname{tr}_{d}\left(e_{i}\right) \quad \text { for all } i=1, \ldots, n-1
$$

It has been shown by Gérardin JuLa2, Appendix] that the solutions of the E-system are parametrised by the non-empty subsets of $\mathbb{Z} / d \mathbb{Z}$. More specifically, if $D$ is a non-empty subset of $\mathbb{Z} / d \mathbb{Z}$, then the corresponding solution of the E-system is $X_{D}:=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{d-1}\right)$, where

$$
\begin{equation*}
\mathrm{x}_{k}:=\frac{1}{|D|} \sum_{j \in D} \zeta_{d}^{k j} \quad(1 \leqslant k \leqslant d-1) \tag{6.23}
\end{equation*}
$$

Let now $D$ be a non-empty subset of $\mathbb{Z} / d \mathbb{Z}$ and let $X_{D}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{d-1}\right)$ be the corresponding solution of the E-system. We shall call specialised trace and denote by $\operatorname{tr}_{d, D}$, the trace $\operatorname{tr}_{d}$ where the parameter $x_{j}$ is specialised to $\mathrm{x}_{j}$ for all $j=1, \ldots, d-1$ [ChLa, Definition 3]. We then have

$$
E_{D}:=\operatorname{tr}_{d, D}\left(e_{i}\right)=\frac{1}{|D|} \quad \text { for all } i=1, \ldots, n-1
$$

and (cf. JuLa2, Theorem 7 \& Lemma 8]):

$$
\begin{equation*}
\operatorname{tr}_{d, D}\left(a e_{n}\right)=E_{D} \operatorname{tr}_{d, D}(a) \quad \text { and } \quad \operatorname{tr}_{d, D}\left(a e_{n} g_{n}\right)=z \operatorname{tr}_{d, D}(a) \quad \text { for all } a \in \mathrm{Y}_{d, n}(q) \tag{6.24}
\end{equation*}
$$

Note that, for $d=1$, the specialised trace $\operatorname{tr}_{1,\{0\}}$ coincides with $\operatorname{tr}_{1}$, which in turn coincides with the Ocneanu trace on $\mathcal{H}_{n}(q) \cong \mathrm{Y}_{1, n}(q)$.

Remark 6.22 Following (6.23), we have

$$
E_{D}=1 \Leftrightarrow|D|=1 \Leftrightarrow \mathrm{x}_{1}^{d}=1 \text { and } \mathrm{x}_{k}=\mathrm{x}_{1}^{k} \quad(1 \leqslant k \leqslant d-1)
$$

and

$$
E_{D}=\frac{1}{d} \Leftrightarrow D=\mathbb{Z} / d \mathbb{Z} \Leftrightarrow \mathrm{x}_{k}=0(1 \leqslant k \leqslant d-1)
$$

The latter is the "trivial" solution of the E-system.
We now set

$$
\begin{equation*}
\lambda_{D}:=\frac{z+(1-q) E_{D}}{q z} \quad \text { and } \quad \Lambda_{D}:=\frac{1}{z \sqrt{\lambda_{D}}} \tag{6.25}
\end{equation*}
$$

We also set $\mathcal{R}_{D}:=\mathcal{R}\left[z^{ \pm 1},{\sqrt{\lambda_{D}}}^{ \pm 1}, \sqrt{q}^{ \pm 1}\right]$. Normalising and re-scaling the specialised trace $\operatorname{tr}_{d, D}$ (which is possible due to the E -condition) yields the following [CJKL, Theorem 3.1]:

Theorem 6.23 For any framed braid $\alpha \in \mathcal{F}_{n}$, we define

$$
\Phi_{d, D}(\widehat{\alpha}):=\Lambda_{D}^{n-1}\left(\sqrt{\lambda_{D}}\right)^{\epsilon(\alpha)} \operatorname{tr}_{d, D}(\alpha),
$$

where $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators $\sigma_{i}$ in the word $\alpha$. Then the map

$$
\Phi_{d, D}(q, z): \mathcal{L}_{f} \rightarrow \mathcal{R}_{D}, L \mapsto \Phi_{d, D}(L)
$$

is a 2-variable isotopy invariant of oriented framed links.
Remark 6.24 Note that, for every $d \in \mathbb{N}^{*}$, we have $2^{d}-1$ distinct solutions of the E-system, so the above construction yields $2^{d}-1$ seemingly distinct isotopy invariants for framed links.

Remark 6.25 Using (6.25, we can obtain defining equations for $z$ and $\Lambda_{D}$ with respect to $\lambda_{D}$, namely,

$$
z:=\frac{(1-q) E_{D}}{q \lambda_{D}-1} \quad \text { and } \quad \Lambda_{D}:=\frac{1}{z \sqrt{\lambda_{D}}}
$$

Accordingly, we can use the notation $\Phi_{d, D}\left(q, \lambda_{D}\right)$ instead of $\Phi_{d, D}(q, z)$.

Remark 6.26 In JuLa2, the old quadratic relation 6.5 was used. Using the natural surjection of $\mathcal{R}\left[\mathcal{F}_{n}\right]$ onto $\mathrm{Y}_{d, n}(q)$ given by $\sigma_{i} \mapsto \bar{g}_{i}$ and $t_{j} \mapsto t_{j}$, and normalising and re-scaling the specialised trace, invariants $\Gamma_{d, D}(q, z)$ for oriented framed links were defined in JuLa2, Theorem 8].

In [JuLa2, Proposition 7] a skein relation is found for the invariant $\Gamma_{d, D}$, involving the braiding and the framing generators. It reads:

$$
\begin{equation*}
\frac{1}{\sqrt{\lambda_{D}}} \Gamma_{d, D}\left(L_{+}\right)-\sqrt{\lambda_{D}} \Gamma_{d, D}\left(L_{-}\right)=\frac{1-q^{-1}}{d} \sum_{s=0}^{d-1} \Gamma_{d, D}\left(L_{s}\right)+\frac{1-q^{-1}}{d \sqrt{\lambda_{D}}} \sum_{s=0}^{d-1} \Gamma_{d, D}\left(L_{s \times}\right) \tag{6.26}
\end{equation*}
$$

where the links $L_{+}, L_{-}, L_{s}$ and $L_{s \times}$ are illustrated in Figure 6.1.

$L_{-}=\widehat{\beta \sigma_{1}^{-1}}$
$L_{s}=\beta \widehat{t_{1}^{s} t_{2}^{d-s}}$
$L_{s \times}=\beta t_{1} \widehat{t_{2}^{d-s}}{ }_{\sigma}$
Figure 6.1: The framed links in the skein relation in open braid form.

Analogously, with the new quadratic relations, the invariant $\Phi_{d, D}$ satisfies the following skein relation:

$$
\begin{equation*}
\frac{1}{\sqrt{q \lambda_{D}}} \Phi_{d, D}\left(L_{+}\right)-\sqrt{q \lambda_{D}} \Phi_{d, D}\left(L_{-}\right)=\frac{q^{1 / 2}-q^{-1 / 2}}{d} \sum_{s=0}^{d-1} \Phi_{d, D}\left(L_{s}\right) \tag{6.27}
\end{equation*}
$$

where the links $L_{+}, L_{-}$and $L_{s}$ are illustrated in Figure 6.1. The fact that the skein relation for $\Phi_{d, D}$ is very different from the skein relation for $\Gamma_{d, D}$ is an indication that the two invariants may not be topologically equivalent. We also have computational data that seem to support this claim Ai].

### 6.4.2 Invariants from Yokonuma-Hecke algebras for classical knots and links

Let $\mathcal{L}$ denote the set of oriented classical links. By the Alexander Theorem, we have $\mathcal{L}=\cup_{n}\left\{\widehat{\alpha} \mid \alpha \in B_{n}\right\}$. The classical braid group $B_{n}$ injects into the framed braid group $\mathcal{F}_{n} \cong \mathbb{Z}^{n} \rtimes B_{n}$, whereby elements of $B_{n}$ are viewed as framed braids with all framings equal to zero. So, by the classical Markov braid equivalence, comprising conjugation in the groups $B_{n}$ and positive and negative stabilisations and destabilisations, and by treating the $t_{j}$ 's as formal generators, $\Phi_{d, D}(q, z)$ becomes an isotopy invariant of oriented classical links when restricted to $\mathcal{L}$ (see also [JuLa3]). This invariant of classical links will be denoted by $\Theta_{d, D}(q, z)$. Accordingly, for any classical braid $\alpha \in B_{n}$, we have

$$
\Theta_{d, D}(\widehat{\alpha}):=\Lambda_{D}^{n-1}\left(\sqrt{\lambda_{D}}\right)^{\epsilon(\alpha)} \operatorname{tr}_{d, D}(\alpha)
$$

where $\lambda_{D}$ and $\Lambda_{D}$ are given by 6.25 , and $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators $\sigma_{i}$ in the word $\alpha$.

Remark 6.27 Following Remark 6.24 the above construction yields $2^{d}-1$ seemingly distinct isotopy invariants for classical links. However, we shall see soon that, for classical links, we only obtain one invariant for every $d \in \mathbb{N}^{*}$.

The invariants $\Theta_{d, D}(q, z)$ need to be compared with known invariants of classical links, especially with the Homflypt polynomial. The Homflypt (or 2-variable Jones) polynomial $P(q, z)$ is a 2 -variable isotopy invariant of oriented classical links that was constructed from the Iwahori-Hecke algebras $\mathcal{H}_{n}(q)$ after normalising and re-scaling the Ocneanu trace Jo3. Here we define $P(q, z)$ via the invariants $\Theta_{d, D}(q, z)$,
since, for $d=1$, the algebras $\mathcal{H}_{n}(q)$ and $\mathrm{Y}_{1, n}(q)$ coincide, while the Ocneanu trace coincides with $\operatorname{tr}_{1}$ and $\operatorname{tr}_{1,\{0\}}$. For any classical braid $\alpha \in B_{n}$, we define

$$
P(\widehat{\alpha}):=\Theta_{1,\{0\}}(\widehat{\alpha})=\left(\frac{1}{z \sqrt{\lambda_{\mathrm{H}}}}\right)^{n-1}\left(\sqrt{\lambda_{\mathrm{H}}}\right)^{\epsilon(\alpha)} \operatorname{tr}_{1,\{0\}}(\alpha),
$$

where

$$
\lambda_{\mathrm{H}}:=\frac{z+(1-q)}{q z}=\lambda_{\{0\}},
$$

and $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators $\sigma_{i}$ in the word $\alpha$. Further, the Homflypt polynomial satisfies the following skein relation Jo3]:

$$
\begin{equation*}
\frac{1}{\sqrt{q \lambda_{\mathrm{H}}}} P\left(L_{+}\right)-\sqrt{q \lambda_{\mathrm{H}}} P\left(L_{-}\right)=\left(q^{1 / 2}-q^{-1 / 2}\right) P\left(L_{0}\right) \tag{6.28}
\end{equation*}
$$

where $L_{+}, L_{-}, L_{0}$ is a Conway triple.
Contrary to the case of framed links, the skein relation of the invariant $\Phi_{d, D}(q, z)$ has no topological interpretation in the case of classical links, since it introduces framings. This makes it very difficult to compare the invariants $\Theta_{d, D}(q, z)$ with the Homflypt polynomial using diagrammatic methods. On the algebraic level, there are no algebra homomorphisms connecting the algebras and the traces (see ChLa). Further, for generic values of the parameters $q, z$ the invariants $\Theta_{d, D}(q, z)$ do not coincide with the Homflypt polynomial; they only coincide when $q=1$ or when $E_{D}=1$ (cf. [ChLa, Theorem 5]).

Remark 6.28 Similarly to $\Phi_{d, D}(q, z)$, the invariant $\Gamma_{d, D}(q, z)$ becomes an isotopy invariant of oriented classical links when restricted to $\mathcal{L}$. This invariant of classical links is denoted by $\Delta_{d, D}(q, z)$ and it is the one studied in JuLa3 and ChLa. Again, there is no reason that the invariants $\Delta_{d, D}(q, z)$ and $\Theta_{d, D}(q, z)$ are topologically equivalent. In fact, we have computational data that seem to indicate that they are not [Ai]. Note though that, for $d=1, \Delta_{1,\{0\}}(q, z)=\Theta_{1,\{0\}}(q, z)$ (this can be easily seen by writing down the skein relation for $\Delta_{1,\{0\}}$ and for $\left.\Theta_{1,\{0\}}\right)$, so both families of invariants include the Homflypt polynomial as a special case.

We will now see that the specialised trace $\operatorname{tr}_{d, D}$, when computed on elements of the classical braid group $B_{n}$, can be computed via rules involving only the traces of the elements $g_{i}$ and $e_{i}$. This in turn will imply that the invariants $\Theta_{d, D}$ of oriented classical links depend only on the cardinality of $D$.

Let $\delta: \mathcal{R}\left[B_{n}\right] \rightarrow \mathrm{Y}_{d, n}(q)$ be the natural $\mathcal{R}$-algebra homomorphism given by $\sigma_{i} \mapsto g_{i}$. Then, for any $\alpha \in B_{n}, \delta(\alpha)$ involves only the braiding generators $g_{1}, \ldots, g_{n-1}$. In fact, we have that the image of $\delta$ is the subalgebra $\mathrm{Y}_{d, n}(q)^{(\mathrm{br})}$ of $\mathrm{Y}_{d, n}(q)$ generated by $g_{1}, \ldots, g_{n-1}$ [CJKL, Proposition 4.1].

Remark 6.29 If $q \neq 1$, then

$$
e_{i}=\frac{1}{q-1}\left(q^{-1} g_{i}^{3}-g_{i}\right)-\left(q^{-1} g_{i}^{2}-1\right)
$$

for all $i=1, \ldots, n-1$. So, when defined over $\mathcal{R}\left[(q-1)^{-1}\right], \mathrm{Y}_{d, n}(q)^{(b r)}$ coincides with the subalgebra of $\mathrm{Y}_{d, n}(q)$ generated by the elements $g_{1}, \ldots, g_{n-1}, e_{1}, \ldots, e_{n-1}$, which in turn should be isomorphic, when $d \geqslant n$, to the so-called algebra of braids and ties (see EsRy).

Now, during the calculation of the specialised trace $\operatorname{tr}_{d, D}$ on $\alpha \in B_{n}$, the framing generators appear only when the quadratic relation (6.2) and the inverse relation (6.4) are applied, and then only in the form of the idempotents $e_{i}$. So, it would make sense in this setting to substitute rule (4) of the definition of $\operatorname{tr}_{d, D}$ by rules involving only the $e_{i}$ 's, such as (6.24). Indeed, we have the following CJKL, Theorem 4.3]:

Theorem 6.30 Let $m \in\{1, \ldots, d\}$ and set $E_{m}:=1 / m$. Let $z$ be an indeterminate over $\mathbb{C}$. There exists a unique linear Markov trace

$$
\operatorname{tr}_{d, m}: \bigcup_{n \geqslant 0} \mathrm{Y}_{d, n}(q)^{(\mathrm{br})} \longrightarrow \mathcal{R}[z]
$$

defined inductively on $\mathrm{Y}_{d, n}(q)^{(\mathrm{br})}$, for all $n \geqslant 0$, by the following rules:
(i) $\quad \operatorname{tr}_{d, m}(a b)=\operatorname{tr}_{d, m}(b a)$
$a, b \in \mathrm{Y}_{d, n}(q)^{(\mathrm{br})}$
(ii) $\quad \operatorname{tr}_{d, m}(1)=1$
(iii) $\quad \operatorname{tr}_{d, m}\left(a g_{n}\right)=z \operatorname{tr}_{d, m}(a) \quad a \in \mathrm{Y}_{d, n}(q)^{(\mathrm{br})} \quad$ (Markov property)
(iv) $\operatorname{tr}_{d, m}\left(a e_{n}\right)=E_{m} \operatorname{tr}_{d, m}(a)$
(v) $\operatorname{tr}_{d, m}\left(a e_{n} g_{n}\right)=z \operatorname{tr}_{d, m}(a)$
$a \in \mathrm{Y}_{d, n}(q)^{(\mathrm{br})}$
$a \in \mathrm{Y}_{d, n}(q)^{(\mathrm{br})}$.
For all $a \in \bigcup_{n \geqslant 0} \mathrm{Y}_{d, n}(q)^{(\mathrm{br})}$, we have $\operatorname{tr}_{d, m}(a)=\operatorname{tr}_{d, D}(a)$ where $D$ is any subset of $\mathbb{Z} / d \mathbb{Z}$ such that $|D|=m$. Note that, in this case, $E_{m}=E_{D}$.

For $\alpha \in B_{n}$, we set $\operatorname{tr}_{d, m}(\alpha):=\operatorname{tr}_{d, m}(\delta(\alpha))$. Theorem 6.30 implies that the specialised trace $\operatorname{tr}_{d, D}$ on classical knots and links depends only on $|D|$ and not on the solution $X_{D}$ of the E-system. It does not even depend on $d$, since we have the following [CJKL, Corollary 4.4]:

Corollary 6.31 Let $d, d^{\prime}$ be positive integers with $d \leqslant d^{\prime}$. For all $\alpha \in B_{n}$, we have

$$
\operatorname{tr}_{d^{\prime}, d}(\alpha)=\operatorname{tr}_{d, d}(\alpha)=\operatorname{tr}_{d, \mathbb{Z} / d \mathbb{Z}}(\alpha)
$$

Remark 6.32 Recall that $\mathbb{Z} / d \mathbb{Z}$ parametrises the trivial solution of the E-system, that is, the one given by $\mathrm{x}_{1}=\mathrm{x}_{2}=\cdots=\mathrm{x}_{d-1}=0$.

The above way of calculating the trace $\operatorname{tr}_{d, D}$ allows us to prove new results for the invariants $\Theta_{d, D}$ of classical knots and links and to compare them with the Homflypt polynomial. Here is the first observation. The construction of the invariants $\Theta_{d, D}$ yielded seemingly $2^{d}-1$ invariants for every choice of $d$. However, an immediate consequence of Theorem 6.30 and Corollary 6.31 is the following CJKL, Proposition 4.6]:

Proposition 6.33 The values of the isotopy invariants $\Theta_{d, D}$ for classical links depend only on the cardinality $|D|$ of $D$. Hence, for a fixed d, we only obtain d invariants. Further, for $d, d^{\prime}$ positive integers with $d \leqslant d^{\prime}$, we have $\Theta_{d, D}=\Theta_{d^{\prime}, D^{\prime}}$ as long as $|D|=\left|D^{\prime}\right|$. We deduce that, if $\left|D^{\prime}\right|=d$, then $\Theta_{d^{\prime}, D^{\prime}}=\Theta_{d, \mathbb{Z} / d \mathbb{Z}}$. Therefore, the invariants $\Theta_{d, D}$ can be parametrised by the positive natural numbers, setting $\Theta_{d}:=\Theta_{d, \mathbb{Z} / d \mathbb{Z}}$ for all $d \in \mathbb{N}^{*}$.

Theorem 6.30 also enabled the development of a program for computing the invariants $\Theta_{d}$ with much lower computational complexity than the one used for the computation of the invariants $\Gamma_{d, D}$ (see ChmJKL). Such a program has been developed by Karvounis Kar and it is available on La-web.

Now, the main result of this section concerns the comparison of the invariants $\Theta_{d}$ with the Homflypt polynomial. In CJKL we have obtained the following remarkable result:

Theorem 6.34 Let $d \in \mathbb{N}, d>1$. The invariants $\Theta_{d}$ are topologically equivalent to the Homflypt polynomial on knots, and more generally on links that are obtained as disjoint unions of knots. However, the invariants $\Theta_{d}$ are NOT topologically equivalent to the Homflypt polynomial on links.

For the first part, using the properties of the specialised trace, we have proved that, for all $\alpha \in B_{n}$ such that $\widehat{\alpha}$ is a knot, we have [CJKL, Theorem 5.8]:

$$
\Theta_{d}(q, z)(\widehat{\alpha})=\Theta_{1}(q, d z)(\widehat{\alpha})=P(q, d z)(\widehat{\alpha}) .
$$

More generally, for all $\alpha \in B_{n}$ such that $\widehat{\alpha}$ is a disjoint union of $k$ knots, we have [CJKL, Theorem 6.2]:

$$
\Theta_{d}(q, z)(\widehat{\alpha})=d^{k-1} \Theta_{1}(q, d z)(\widehat{\alpha})=d^{k-1} P(q, d z)(\widehat{\alpha})
$$

Note that, following Remark 6.25, the above equality can be rewritten as

$$
\Theta_{d}\left(q, \lambda_{D}\right)(\widehat{\alpha})=E_{D}^{1-k} \Theta_{1}\left(q, \lambda_{D}\right)(\widehat{\alpha})=E_{D}^{1-k} P\left(q, \lambda_{D}\right)(\widehat{\alpha})
$$

where $D=\mathbb{Z} / d \mathbb{Z}$. We will simply write

$$
\begin{equation*}
\Theta_{d}(\widehat{\alpha})=E_{D}^{1-k} P(\widehat{\alpha}) \tag{6.29}
\end{equation*}
$$

For the second part, as mentioned earlier, a computer program has been developed on Mathematica, which calculates the specialised trace $\operatorname{tr}_{d, D}$ (and thus the invariants $\Theta_{d}$ ) by using only the five rules of Theorem 6.30 and the new quadratic relation. Now, out of 4.188 links (with up to 11 crossings), there are 89 pairs of $P$-equivalent links which do not differ only by orientation, that is, they are different links if considered as unoriented links. Using the data from LinkInfo ChaLi, we computed the invariants $\Theta_{d}$ on all of them. Out of these $89 P$-equivalent pairs of links, 83 are still $\Theta_{d}$-equivalent for generic $d$, yet we found that the following six pairs of 3 -component $P$-equivalent links are not $\Theta_{d}$-equivalent for every $d>1$ :

| $L 11 n 358\{0,1\}$ | $L 11 n 418\{0,0\}$ |
| :---: | :---: |
| $L 11 a 467\{0,1\}$ | $L 11 a 527\{0,0\}$ |
| $L 11 n 325\{1,1\}$ | $L 11 n 424\{0,0\}$ |
| $L 10 n 79\{1,1\}$ | $L 10 n 95\{1,0\}$ |
| $L 11 a 404\{1,1\}$ | $L 11 a 428\{0,1\}$ |
| $L 10 n 76\{1,1\}$ | $L 11 n 425\{1,0\}$ |

Table 6.1: Six $P$-equivalent pairs of 3 -component links which are not $\Theta_{d}$-equivalent.
Further, in CJKL, Theorem 7.3], we have proved diagrammatically that the invariants $\Theta_{d}$, for $d>1$, distinguish the pair of $P$-equivalent links $\operatorname{L11} n 358\{0,1\}$ and $\operatorname{L11n418\{ 0,0\} }{ }^{1}$, using another remarkable property of the invariants $\Theta_{d}$, namely [JJKL, Theorem 6.19]:

Theorem 6.35 The invariants $\Theta_{d}$ can be completely defined via the Homflypt skein relation.
The above result derives from the fact that the invariants $\Theta_{d}$ satisfy a special skein relation CJKL, Proposition 6.8], which can only be applied on crossings involving two different components (and thus, it cannot be applied, for example, on knots). Consider the standard Conway triple $L_{+}, L_{-}, L_{0}$, as in the following figure, where different colours represent different components:


Figure 6.2: The links in the special skein relation in open braid form.

Then the skein relation 6.27 for the framed link invariants $\Phi_{d, D}$ becomes:

$$
\frac{1}{\sqrt{q \lambda_{D}}} \Phi_{d, D}\left(L_{+}\right)-\sqrt{q \lambda_{D}} \Phi_{d, D}\left(L_{-}\right)=\left(q^{1 / 2}-q^{-1 / 2}\right) \Phi_{d, D}\left(L_{0}\right)
$$

which in turn yields the following skein relation for the invariants $\Theta_{d}$ :

$$
\begin{equation*}
\frac{1}{\sqrt{q \lambda_{D}}} \Theta_{d}\left(L_{+}\right)-\sqrt{q \lambda_{D}} \Theta_{d}\left(L_{-}\right)=\left(q^{1 / 2}-q^{-1 / 2}\right) \Theta_{d}\left(L_{0}\right), \tag{6.30}
\end{equation*}
$$

where $D=\mathbb{Z} / d \mathbb{Z}$. Note that this is exactly the skein relation 6.28) of the Homflypt polynomial $P\left(q, \lambda_{D}\right)$.
Remark 6.36 With the old quadratic relations for the Yokonuma-Hecke algebra, we would not have been able to derive the special skein relation (6.30). The reason is that the skein relation (6.26) for the invariants $\Gamma_{d, D}$ contains also diagrams $L_{s \times}$ of $L_{+}$with framings which cannot be collected together (to form $L_{0}$ ) since they belong to different components.

[^2]Let now $L$ be an oriented link with $l$ components. The value of $\Theta_{d}$ on $L$ at variables $\left(q, \lambda_{D}\right)$ can be computed diagrammatically by applying the following procedure:

Step 1. Apply the skein relation 6.30 on crossings linking different components until the link $L$ is decomposed into disjoint unions of knots. More specifically, we obtain that $\Theta_{d}(L)$ is written as a linear combination of values of $\Theta_{d}$ on disjoint unions of knots with up to $l$ components. For $k=1, \ldots, l$, let $\mathcal{N}(L)_{k}$ denote the set of all disjoint unions of $k$ knots appearing in this linear combination. We thus have

$$
\Theta_{d}(L)=\sum_{k=1}^{l} \sum_{\widehat{\alpha} \in \mathcal{N}(L)_{k}} c(\widehat{\alpha}) \Theta_{d}(\widehat{\alpha})
$$

for some $c(\widehat{\alpha}) \in \mathbb{Q}\left[q^{ \pm 1},{\sqrt{\lambda_{D}}}^{ \pm 1}\right]$.
Step 2. Apply 6.29, which yields

$$
\Theta_{d}(L)=\sum_{k=1}^{\ell} E_{D}^{1-k} \sum_{\widehat{\alpha} \in \mathcal{N}(L)_{k}} c(\widehat{\alpha}) P(\widehat{\alpha}),
$$

where $D=\mathbb{Z} / d \mathbb{Z}$. We thus obtain that $\Theta_{d}(L)$ is a linear combination of Homflypt polynomials of disjoint unions of knots obtained by the special skein relation [JJKL, Theorem 6.16].

Step 3. Apply the skein relation 6.28 of the Homflypt polynomial to obtain the value of $P$ on $\widehat{\alpha}$ at variables $\left(q, \lambda_{D}\right)$, for all disjoint unions of knots $\widehat{\alpha} \in \mathcal{N}(L)_{k}, k=1, \ldots, l$.

Finally, observing that in the above procedure, the rational number $E_{D}=1 / d$ could be taken to be a parameter, we have constructed a new 3 -variable skein link invariant $\Theta$ generalising both the invariants $\Theta_{d}$ and the Homflypt polynomial [JJKL, Theorems 8.1]:

Theorem 6.37 Let $q, \lambda, E$ be indeterminates over $\mathbb{C}$. There exists a unique isotopy invariant of classical oriented links $\Theta: \mathcal{L} \rightarrow \mathbb{C}\left[q^{ \pm 1 / 2}, \lambda^{ \pm 1}, E^{ \pm 1}\right]$ defined by the following rules:

1. For a disjoint union $L$ of $k$ knots, with $k \geqslant 1$, it holds that:

$$
\Theta(L)=E^{1-k} P(L)
$$

2. On crossings involving different components the following skein relation holds:

$$
\frac{1}{\sqrt{q \lambda}} \Theta\left(L_{+}\right)-\sqrt{q \lambda} \Theta\left(L_{-}\right)=\left(q^{1 / 2}-q^{-1 / 2}\right) \Theta\left(L_{0}\right)
$$

where $L_{+}, L_{-}, L_{0}$ is a Conway triple.
The invariant $\Theta$ specialises to the invariant $\Theta_{d}$ for $E=E_{D}$ and $\lambda=\lambda_{D}$, where $D=\mathbb{Z} / d \mathbb{Z}$ (recall that $\Theta_{1}=P$ ). It also distinguishes the six pairs of links of Table 6.1. Hence, we conclude the following CJKL Theorem 8.2]:

Theorem 6.38 The invariant $\Theta(q, \lambda, E)$ is stronger than the Homflypt polynomial.
Remark 6.39 The invariants $\left\{\Theta_{d}\right\}_{d \in \mathbb{N}^{*}}$ and $\Theta$ are topologically equivalent to the Homflypt polynomial on knots. Since there is at least one pair of knots distinguished by the Homflypt polynomial but not by the Kauffman polynomial, the invariants $\left\{\Theta_{d}\right\}_{d \in \mathbb{N}^{*}}$ and $\Theta$ are not topologically equivalent to the Kauffman polynomial.

Remark 6.40 An appendix was recently added to CJKL; it contains a closed formula, obtained by Lickorish, for the 3 -variable invariant $\Theta$. For a given link $L$, this formula expresses $\Theta(L)$ as a nontrivial mixture of linking numbers and Homflypt polynomials of all sublinks of $L$. The same result was independently proved in [PdAWa.

### 6.4.3 Invariants from affine and cyclotomic Yokonuma-Hecke algebras

Recall now the notation of Section 6.2 Let $n \in \mathbb{N}, d \in \mathbb{N}^{*}$ and $l \in \mathbb{N}^{*} \cup\{\infty\}$. We have $\mathcal{R}_{l}=$ $\mathbb{C}\left[q^{ \pm 1}, Q_{0}^{ \pm 1}, Q_{1}^{ \pm 1}, \ldots, Q_{l-1}^{ \pm 1}\right]$ if $l<\infty$, and $\mathcal{R}_{\infty}:=\mathbb{C}\left[q^{ \pm 1}\right]$. We define inductively elements $\widetilde{X}_{1}, \widetilde{X}_{2}, \ldots, \widetilde{X}_{n}$ of $\mathrm{Y}(d, l, n)$ by setting

$$
\widetilde{X}_{1}:=X_{1} \quad \text { and } \quad \widetilde{X}_{i+1}:=g_{i}^{-1} \widetilde{X}_{i} g_{i} \quad \text { for } i=1, \ldots, n-1 .
$$

Note that $\widetilde{X}_{i}^{a}=W_{i-1, a, 0}^{(i)}$ for all $i=1, \ldots, n$ and $a \in \mathbb{Z}$. We have $t_{i}^{b} \widetilde{X}_{i}^{a}=W_{i-1, a, b}^{(i)}=\widetilde{X}_{i}^{a} t_{i}^{b}$ for all $b \in \mathbb{Z}$.
Using now the natural inclusions $\mathrm{Y}(d, l, n) \subset \mathrm{Y}(d, l, n+1)$, we can define a Markov trace on $\bigcup_{n \geqslant 0} \mathrm{Y}(d, l, n)$ as follows (cf. [ChPdA2, §5]):

Theorem 6.41 Let $z$ and $\left(x_{a, b}\right)_{a \in Z_{l}, b \in\{0, \ldots, d-1\}}$ be indeterminates over $\mathcal{R}_{l}$. There exists a unique linear Markov trace

$$
\operatorname{tr}_{d, l}: \bigcup_{n \geqslant 0} \mathrm{Y}(d, l, n) \longrightarrow \mathcal{R}_{l}\left[z,\left(x_{a, b}\right)_{a \in Z_{l}, b \in\{0, \ldots, d-1\}}\right]
$$

defined inductively on $\mathrm{Y}(d, l, n)$, for all $n \geqslant 0$, by the following rules:
(4) $\operatorname{tr}_{d, l}\left(u \widetilde{X}_{d, l}^{a}\left(u g_{n}\right)=z \operatorname{tr}_{d, l}(u)\right.$

$$
\begin{align*}
& u, v \in \mathrm{Y}(d, l, n)  \tag{1}\\
& 1 \in \mathrm{Y}(d, l, n)  \tag{2}\\
& u \in \mathrm{Y}(d, l, n) \quad(\text { Markov property })  \tag{3}\\
& u \in \mathrm{Y}(d, l, n) \quad\left(a \in Z_{l}, 0 \leqslant b \leqslant d-1\right) .
\end{align*}
$$

Remark 6.42 For $d=1$, the Markov trace $\operatorname{tr}_{1, l}$ on the algebra $\mathcal{H}(l, n) \cong \mathrm{Y}(1, l, n)$ was introduced

- by Ocneanu/Jones Jo2, Jo3 for $l=1$,
- by Lambropoulou and Geck La1, GeLa for $l=2$, and
- by Lambropoulou [La2] for $l \in \mathbb{Z}_{\geqslant 3} \cup\{\infty\}$.

For $l=1, \operatorname{tr}_{d, 1}$ is the Markov trace $\operatorname{tr}_{d}$ on the Yokonuma-Hecke algebra $\mathrm{Y}_{d, n}(q)$.
Remark 6.43 The canonical symmetrising form on $\mathcal{K}_{l} \mathrm{Y}(d, l, n)$ defined by 6.21) coincides with the Markov trace $\operatorname{tr}_{d, l}$ with parameters $z=0$ and $x_{a, b}=\delta_{a, 0} \delta_{b, 0}$.

The relations $\left(b_{1}\right),\left(b_{2}\right),\left(f_{1}\right)$ and $\left(f_{2}\right)$ in (6.1) together with relations 6.17) are defining relations for the framed affine braid group $\mathcal{F}_{n}^{\text {aff }} \cong \mathbb{Z} \imath B_{n}^{\text {aff }}$, where $B_{n}^{\text {aff }}$ is the affine braid group (its generators usually denoted by $\sigma_{0}, \sigma_{1}, \ldots, \sigma_{n-1}$ ), with the $t_{j}$ 's being interpreted as the "elementary framings" (framing 1 on the $j$-th strand). Moreover, relations (6.1) together relations (6.17) are defining relations for the modular framed affine braid group $\mathcal{F}_{d, n}^{\text {aff }} \cong(\mathbb{Z} / d \mathbb{Z})$ \} $B_{n}^{\text {aff }}$. Thus, the algebra $\mathrm{Y}(d, l, n)$ can be seen as a quotient of both framed affine braid group algebra and modular framed affine braid group algebra

For any $\alpha \in \mathcal{F}_{n}^{\text {aff }}$, we denote by $\widehat{\alpha}$ the link obtained as the closure of $\alpha$. By the Alexander Theorem, we have that $\mathcal{L}_{f}^{\text {aff }}:=\cup_{n}\left\{\widehat{\alpha} \mid \alpha \in \mathcal{F}_{n}^{\text {aff }}\right\}$ is the set of all oriented framed links in the solid torus. Further, in [ChPdA2, Theorem 6.6], we have shown that isotopy of framed links in the solid torus is generated by conjugation in $\mathcal{F}_{n}^{\text {aff }}$ and by positive and negative stabilisation and destabilisation ( $\alpha \sim \alpha \sigma_{n}^{ \pm 1}$ ), for any $n \in \mathbb{N}$. Using now the natural $\mathcal{R}_{l}$-algebra epimorphism from $\mathcal{R}_{l}\left[\mathcal{F}_{n}^{\text {aff }}\right]$ onto $\mathrm{Y}(d, l, n)$ given by $\sigma_{0} \mapsto X_{1}$, $\sigma_{i} \mapsto g_{i}$ and $t_{j} \mapsto t_{j}$, and abusing notation, we can define the $\operatorname{trace}^{\operatorname{tr}} \mathrm{tr}_{d, l}$ on the elements of $\mathbb{C} \mathcal{F}_{n}^{\text {aff }}$, and thus, in particular, on the elements of $\mathcal{F}_{n}^{\text {aff }}$. In order to use $\operatorname{tr}_{d, l}$ for constructing invariants for framed knots and links after Jones's method, the trace $\operatorname{tr}_{d, l}$ has to be normalised, so that the closures of the framed affine braids $\alpha$ and $\alpha \sigma_{n}\left(\alpha \in \mathcal{F}_{n}\right)$ be assigned the same value of the invariant, and re-scaled, so that the closures of the framed affine braids $\alpha \sigma_{n}^{-1}$ and $\alpha \sigma_{n}\left(\alpha \in \mathcal{F}_{n}\right)$ be assigned the same value of the invariant. As in the non-affine case, we need to impose the affine E -condition:

$$
\operatorname{tr}_{d, l}\left(u e_{n}\right)=\operatorname{tr}_{d, l}(u) \operatorname{tr}_{d, l}\left(e_{n}\right) \quad \text { for any } n \geqslant 1 \text { and all } u \in \mathrm{Y}(d, l, n)
$$

This condition yields in turn that the trace parameters $\left(x_{a, b}\right)_{a \in Z_{l}, b \in\{0, \ldots, d-1\}}$ have to satisfy the following non-linear system of equations, called the affine E -system [ChPdA2, Proposition 6.13]:

$$
\sum_{s=0}^{d-1} x_{a, b+s} x_{0, d-s}=x_{a, b} \sum_{s=0}^{d-1} x_{0, s} x_{0, d-s} \quad\left(a \in Z_{l}, 0 \leqslant b \leqslant d-1\right) .
$$

Note that

$$
\frac{1}{d} \sum_{s=0}^{d-1} x_{0, s} x_{0, d-s}=\operatorname{tr}_{d, l}\left(e_{i}\right) \quad \text { for all } i=1, \ldots, n-1
$$

We have solved the affine E-system and obtained the following [ChPdA2, Proposition 6.15]:
Proposition 6.44 The affine E-condition holds if and only if there exists a non-empty subset $D$ of $\mathbb{Z} / d \mathbb{Z}$ such that

$$
x_{0, b}=\frac{1}{|D|} \sum_{j \in D} \zeta_{d}^{b j} \quad(0 \leqslant b \leqslant d-1),
$$

that is, $\left(x_{0,1}, \ldots, x_{0, d-1}\right)$ is solution of the classical $\mathrm{E}-$ system, and, for $a \neq 0$,

$$
\left(x_{a, 0}, x_{a, 1}, \ldots, x_{a, d-1}\right) \in \operatorname{Span}_{\mathcal{R}_{l}}\left(\left\{\left(1, \zeta_{d}^{j}, \zeta_{d}^{2 j}, \ldots, \zeta^{(d-1) j}\right) \mid j \in D\right\}\right)
$$

In this case, we have

$$
\operatorname{tr}_{d, l}\left(e_{i}\right)=\frac{1}{|D|} \quad \text { for all } i=1, \ldots, n-1
$$

Assume now that the trace parameters $\left(x_{a, b}\right)_{a \in Z_{l}, b \in\{0, \ldots, d-1\}}$ are a solution of the affine E-system corresponding to a non-empty subset $D$ of $\mathbb{Z} / d \mathbb{Z}$. We set

$$
E_{D}:=\frac{1}{|D|}, \quad \lambda_{D}:=\frac{z+(1-q) E_{D}}{q z} \quad \text { and } \quad \Lambda_{D}:=\frac{1}{z \sqrt{\lambda_{D}}} .
$$

We also set $\widetilde{\mathcal{R}}_{l}:=\mathcal{R}_{l}\left[z^{ \pm 1},{\sqrt{\lambda_{D}}}^{ \pm 1}, \sqrt{q}^{ \pm 1}\right]$. We then have the following ChPdA2, Theorem 6.8]:
Theorem 6.45 For any framed affine braid $\alpha \in \mathcal{F}_{n}^{\text {aff }}$, we define

$$
\Psi_{d, l, D}(\widehat{\alpha}):=\Lambda_{D}^{n-1}\left(\sqrt{\lambda_{D}}\right)^{\epsilon(\alpha)} \operatorname{tr}_{d, l}(\alpha),
$$

where $\epsilon(\alpha)$ is the sum of the exponents of the braiding generators $\sigma_{1}, \ldots, \sigma_{n-1}$ in the word $\alpha$. The map

$$
\Psi_{d, l, D}: \mathcal{L}_{f}^{\mathrm{aff}} \rightarrow \widetilde{\mathcal{R}}_{l}, L \mapsto \Psi_{d, l, D}(L)
$$

is an isotopy invariant of oriented framed links in the solid torus.
The invariant $\Psi_{d, l, D}$ when restricted to the set $\mathcal{L}_{f}$ of framed links coincides with the invariant $\Phi_{d, D}$ of oriented framed links from the Yokonuma-Hecke algebra defined in Theorem 6.23

Let now $\mathcal{L}^{\text {aff }}$ denote the set of oriented (non-framed) links in the solid torus. The invariant $\Psi_{d, l, D}$ when restricted to $\mathcal{L}^{\text {aff }}$ becomes an invariant of oriented links in the solid torus, which we will denote by $\Omega_{d, l, D}$. For $d=1$, the invariants $\Omega_{1, l,\{0\}}$ coincide with the Homflypt-type invariants of oriented links in the solid torus obtained from the cyclotomic and affine Hecke algebras in La1, GeLa, La2. When further restricted to the set of $\mathcal{L}$ of oriented classical links, the invariants $\Omega_{d, l, D}$ and $\Omega_{1, l,\{0\}}$ coincide respectively with the invariants $\Theta_{d, D}$ and the Homflypt polynomial $\Theta_{1,\{0\}}$, defined in the previous subsection. Following Theorem 6.34 we deduce that the six pairs of links of Table 6.1 are distinguished by $\Omega_{d, l, D}$ when $|D|>1$, but not by $\Omega_{1, l,\{0\}}$. We conclude the following [ChPdA2, Proposition 6.10]:

Proposition 6.46 The invariants $\Omega_{d, l, D}$ with $|D|>1$ are not topologically equivalent to the Homflypttype invariants of oriented links in the solid torus obtained from the cyclotomic and affine Hecke algebras.

### 6.4.4 Invariants from Temperley-Lieb quotients of Yokonuma-Hecke algebras

The Ocneanu trace on the Iwahori-Hecke algebra $\mathcal{H}_{n}(q)$ passes to the Temperley-Lieb algebra $\mathrm{TL}_{n}(q)$ if and only if $z=-1 /(q+1)$ or $z=-1$. Using the natural surjection of $\mathcal{R}\left[B_{n}\right]$ onto $\mathrm{TL}_{n}(q)$ given by $\sigma_{i} \mapsto g_{i}$, and normalising and re-scaling the Ocneanu trace for $z=-1 /(q+1)$ (the case $z=-1$ not being interesting topologically), Jones Jo2 defined an 1-variable isotopy invariant of oriented classical links $V(q)$, known as the Jones polynomial. We have $V(q)=P(q,-1 /(q+1))$.

Trying to repeat the same procedure for the construction of 1-variable isotopy invariants of oriented framed and classical links from the Temperley-Lieb quotients of $\mathrm{Y}_{d, n}(q)$, Goundaroulis, Juyumaya, Kontogeorgis and Lambropoulou have determined the values of the parameters $z, x_{1}, \ldots, x_{d-1}$ for which the trace $\operatorname{tr}_{d}$ passes to the Yokonuma-Temperley-Lieb algebra GJKL1, the Framisation of the TemperleyLieb algebra GJKL2 and the Complex Reflection Temperley-Lieb algebra GJKL2. In particular, they have obtained that $\operatorname{tr}_{d}$ passes to:

- $\mathrm{YTL}_{d, n}(q)$ only if $x_{1}$ is a $d$-th root of unity and $x_{k}=x_{1}^{k}$ for all $k=1, \ldots, d-1$.
- $\mathrm{FTL}_{d, n}(q)$ only if $\left(x_{1}, \ldots, x_{d-1}\right)$ is a solution of the E-system.
- $\mathrm{CTL}_{d, n}(q)$ for many values of $\left(x_{1}, \ldots, x_{d-1}\right)$, including all solutions of the E-system.

Subsequently, in order to define knot invariants following Jones's method, they have had to impose the E-condition on the parameters $x_{1}, \ldots, x_{d-1}$. If now $x_{1}$ is a $d$-th root of unity and $x_{k}=x_{1}^{k}$ for all $k=1, \ldots, d-1$, then $\left(x_{1}, \ldots, x_{d-1}\right)$ is a solution of the E -system parametrised by a singleton subset of $\mathbb{Z} / d \mathbb{Z}$ (case $E_{D}=1$ ). In this case, there is a loss of the framing information, while if we restrict to the case of classical links, the invariants from the Yokonuma-Temperley algebra coincide with the Jones polynomial (cf. ChLa, Theorem 5]). Thus, the Yokonuma-Temperley-Lieb algebra is not very interesting topologically.

For the Framisation of the Temperley-Lieb algebra the conditions are not so restrictive. If $D$ is a non-empty subset of $\mathbb{Z} / d \mathbb{Z}$ and $X_{D}$ is the corresponding solution of the E-system, then the specialised trace $\operatorname{tr}_{d, D}$ passes to $\mathrm{FTL}_{d, n}(q)$ if and only if $z=-E_{D} /(q+1)$ or $z=-E_{D}$ [GJKL2, Proposition 8]. Using the natural surjection of $\mathcal{R}\left[\mathcal{F}_{n}\right]$ onto $\mathrm{FTL}_{d, n}(q)$ given by $\sigma_{i} \mapsto g_{i}$ and $t_{j} \mapsto t_{j}$, and normalising and re-scaling $\operatorname{tr}_{d, D}$ for $z=-E_{D} /(q+1)$ (the case $z=-E_{D}$ not being interesting topologically), one can construct 1-variable invariants $\phi_{d, D}(q)$ of oriented framed links [GJKL2, Definition 7]. We have $\phi_{d, D}(q)=\Phi_{d, D}\left(q,-E_{D} /(q+1)\right)$. Similarly to the Yokonuma-Hecke algebra case, these invariants, when restricted to classical links, are parametrised by the positive natural numbers; we denote them by $\theta_{d}$, where $d \in \mathbb{N}^{*}$. We then have $\theta_{d}(q)=\Theta_{d}(q,-1 / d(q+1))$, and so $V(q)=\theta_{1}(q)$. The invariants $\Theta_{d}(q,-1 / d(q+1))$ still distinguish the six $P$-equivalent pairs of Table 6.1, whence, following Theorem 6.34 we conclude the following (see also [GJKL2, Proposition 11 \& Theorem 9]):

Proposition 6.47 Let $d \in \mathbb{N}, d>1$. The invariants $\theta_{d}$ are topologically equivalent to the Jones polynomial on knots, and more generally on links that are obtained as disjoint unions of knots. However, the invariants $\theta_{d}$ are not topologically equivalent to the Jones polynomial on links.

Remark 6.48 A 2-variable generalisation of $\theta_{d}$ and the Jones polynomial (obtained as a specialiasation of the invariant $\Theta$ defined in Theorem 6.37) was recently introduced and studied in GouLa.

Finally, as far as the Complex Reflection Temperley-Lieb algebra is concerned, we can take again $\left(x_{1}, \ldots, x_{d-1}\right)$ to be any solution of the E-system. Let $D$ be a non-empty subset of $\mathbb{Z} / d \mathbb{Z}$ and let $X_{D}$ be the corresponding solution of the E-system. By GJKL2, Proposition 9], if $0 \notin D$, then $\operatorname{tr}_{d, D}$ passes to $\mathrm{CTL}_{d, n}(q)$ for any value of $z$; if $0 \in D$, then $\operatorname{tr}_{d, D}$ passes to $\mathrm{CTL}_{d, n}(q)$ if and only if $z=-E_{D} /(q+1)$ or $z=-E_{D}$. We conclude the following about the invariants arising from the Complex Reflection Temperley-Lieb algebra GJKL2, Proposition 10]:

Proposition 6.49 Let $D$ be a non-empty subset of $\mathbb{Z} / d \mathbb{Z}$ and let $X_{D}$ be the corresponding solution of the E -system. If $0 \notin D$, then the invariants obtained from $\mathrm{CTL}_{d, n}(q)$ coincide with the invariants obtained from $\mathrm{Y}_{d, n}(q)$. If $0 \in D$, then the invariants obtained from $\mathrm{CTL}_{d, n}(q)$ coincide witht the invariants obtained from $\mathrm{FTL}_{d, n}(q)$.

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[^0]:    ${ }^{1}$ The parabolic subgroups of a complex reflection group $W \subset G L(\mathfrak{h})$ are the pointwise stabilisers of the subsets of $\mathfrak{h}$. It is a remarkable theorem by Steinberg [t2 Theorem 1.5] that all parabolic subgroups of $W$ are again complex reflection groups.

[^1]:    ${ }^{1}$ for $G_{4}$ this was already known by [Bel1].

[^2]:    ${ }^{1}$ Similar diagrammatic proofs can be given for the remaining five pairs.

