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# Optimal control in discrete-time framework and in infinite horizon

Thoi-Nhan Ngo

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UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

LABORATOIRE SAMM EA 4543

**École Doctorale de Sciences Mathématiques de Paris Centre**

# Thèse de doctorat

Discipline : Mathématiques

présentée par

**Thoi-Nhan Ngo**

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# Contrôle optimal en temps discret et en horizon infini

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dirigée par Joël Blot

Soutenue le 21 novembre 2016 devant le jury composé de :

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# Résumé


## Résumé

Une première théorie consacrée à l'optimisation des problèmes dynamiques est le Calcul des Variations. Les mathématiciens qui ont créé cette théorie sont: Jean BERNOULLI (1667-1748), Leonhard EULER (1707-1793), Joseph Louis LAGRANGE (1736-1813), Adrien LEGENDRE (1752-1833), Carl JACOBI (1804-1851), William HAMILTON (1805-1865), Karl WEIERSTRASS (1815-1897), Adolph MAYER (1819-1904) et Oscar BOLZA (1857-1942). Dans les problèmes de Calcul des Variations, l'observateur n'intervient pas sur le problème. Aujourd'hui, le Calcul des Variations est encore un champ de recherche très actif.

Pour répondre à des questions technologiques, issues de diverses industries, une seconde théorie, consacrée à l'optimisation des problèmes dynamiques, naît au milieu du vingtième siècle: la théorie du Contrôle Optimal. Dans cette théorie, l'observateur agit sur le problème. Outre la "variable d'état" qui décrit le comportement du système dynamique, il y a une "variable de contrôle" qui est pilotée par l'observateur.

Historiquement, il y a deux grands points de vue en théorie du Contrôle Optimal: le point de vue de Lev Pontryagin (Principe du Maximum) et le point de vue de Richard Bellman (Programmation Dynamique).

Le premier cadre qui fut utilisé en Contrôle Optimal est celui du temps continu en horizon fini. Plus tard le cadre du temps discret fut aussi étudié. Le développement de l'utilisation des ordinateurs pour faire des calculs approchés ou des simulations constitue une motivation supplémentaire pour étudier le cadre du temps discret en Contrôle Optimal.

À propos de l'horizon infini en Calcul des Variations et en Contrôle Optimal, des motivations importantes et historiques viennent de la théorie économique: les travaux de F. P. Ramsey (1928), de H. Hotelling (1931), de C. C. von Weizsacker (1965). Plus tard, des modèles de développement durable, par exemple la gestion optimale des forêts et des pêcheries, ont été traités par la théorie du Contrôle Optimal en horizon infini.

Cette thèse contient des contributions originales à la théorie du Contrôle Optimal en temps discret et en horizon infini du point de vue de Pontryagin. Le point de vue de Pontryagin fournit des conditions nécessaires d'optimalité. De telles conditions ont du sens dans les problèmes considérés; ce sont des lois de comportement. De plus, l'utilisation de telles conditions nécessaires d'optimalité peut permettre d'améliorer la modélisation du phénomène étudié, par exemple en montrant que les seuls candidats possibles à l'optimalité sont inadaptés au problème. Dans certain cas, il est possible d'établir des théorèmes de condition suffisante dans le point de vue de Pontryagin.

Nous décrivons maintenant le contenu de cette thèse.

Dans le chapitre 1, nous rappelons des résultats sur les espaces de suites à valeur dans $\mathbb{R}^k$. Nous rappelons aussi des résultats de Calcul Différentiel: sur la dérivabilité directionnelle, sur la Gâteaux différentiabilité, sur la Fréchet-différentiabilité et sur la stricte différentiabilité.

Dans le chapitre 2, nous étudions le problème

$$\left.\begin{array}{rl} \text{Maximiser} & K(\underline{y},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \psi(y_t,u_t) \\ \text{quand} & \underline{y} = (y_t)_{t\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}\ ,\ \underline{u} = (u_t)_{t\in\mathbb{N}} \in U^{\mathbb{N}} \\ & y_0 = \eta,\ \lim_{t\to+\infty} y_t = y_\infty, \\ & \underline{u} \text{ est bornée}, \\ & \forall t \in \mathbb{N},\ y_{t+1} = g(y_t,u_t), \end{array}\right\}$$

où $y_\infty$ est donné. En utilisant la structure d'espace affine de Banach de l'ensemble des suites convergentes vers $y_\infty$, et la structure d'espace vectoriel de Banach de l'ensemble des suites bornées, nous traduisons ce problème en un problème d'optimisation statique dans des espaces de Banach. Après avoir établi des résultats originaux sur les opérateurs de Nemytskii sur les espaces de suites et après avoir adapté à notre problème un théorème d'existence de multiplicateurs (au sens de Fritz John et au sens de Karush-Kuhn-Tucker), nous établissons un nouveau principe de Pontryagin faible pour notre problème.

Dans le chapitre 3, nous établissons un principe de Pontryagin fort pour les problèmes considérés au chapitre 2 en utilisant un résultat de Ioffe-Tihomirov. Nous établissons aussi un théorème de conditions suffisantes qui est nouveau, sous des conditions adaptées de concavité.

Le chapitre 4 est consacré aux problèmes de Contrôle Optimal, en temps discret et en horizon infini, generaux avec plusieurs critères différents, sans condition de borne ou de comportement asymptotique sur la variable d'état et la variable de contrôle. La méthode utilisée est celle de la réduction à l'horizon fini, initiée par J. Blot et H. Chebbi en 2000. Les problèmes considérés sont gouvernés par des équations aux différences ou des inéquations aux différences. Un nouveau principe de Pontryagin faible est établi en utilisant un résultat récent de J. Blot sur les multiplicateurs à la Fritz John.

Le chapitre 5 est consacré aux problèmes multicritères de Contrôle Optimal en temps discret et en horizon infini. De nouveaux principes de Pontryagin faibles et forts sont établis, là-aussi en utilisant des résultats récents d'optimisation, sous des hypothèses plus faibles que celles des résultats existants. En corollaires de nouveaux résultats sur les problèmes multicritères, on obtient de nouveaux résultats sur les problèmes avec un seul critère.

## Mots-clefs

Contrôle Optimal, temps discret, horizon infini, principe de Pontryagin, système dynamique

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# Introduction

A first theory which is devoted to the optimization of dynamical problems is the theory of Calculus of Variations, a field of mathematical analysis that deals with maximizing or minimizing functionals. The first mathematicians who gave contribution to the theory of Calculus of Variations are Johann Bernoulli (1667-1748), Leonhard Euler (1707-1793), Joseph-Louis Lagrange (1736-1813), Andrien Legendre (1752-1833), Carl Jacobi (1804-1851), William Hamilton (1805-1865), Karl Weierstrass (1815-1897), Adolph Mayer (1839-1907), and Oskar Bolza (1857-1942). Today it is ever an important field of research. In theory of Calculus of Variations, the researcher does not take action on the considered problem but to play the role of an observer. He only acknowledges the behaviour of system and understands it through the observations.

Along with the development of technology, there was a need to answer the technological problem of finding a control law for a given system such that a certain optimality criterion is achieved. Then a second theory devoted to the optimization of dynamical problems was born in the middle of twentieth century: the Optimal Control Theory, a mathematical optimization method for deriving control policies. Beside the so-called "state variable" which represents the behaviour of the dynamical system, there is also a so-called "control variable", which is chosen by the researcher over the time. Now not only that the researcher observes, but he also takes action on the dynamical system over the time and plays the role of a controller. By this theory, studying the optimization of a dynamical problems becomes more interactive.

Historically, there exist two great methods in Optimal Control Theory, which are due to the work of Lev Pontryagin (1908-1988) and Richard Bellman (1920-1984) in the 1950s: The minimum (maximum) principle of Pontryagin and the dynamic programming of Bellman.

Optimal Control Theory has found applications in many different fields of science, including aerospace, process control, robotics, bioengineering, economics, finance, and management science, and it continues to be an active research area within control theory. The first framework which was used in Optimal Control Theory is the continuous-time framework. Later, the discrete-time framework was also studied. The reason is that while theory of differential equations in continuous-time models is not well known by all the scientists, except for mathematicians and physicists, the equations of a discrete-time dynamical system do not require sophisticated mathematical tools. Thus, discrete-time models can simplify the communication between mathematicians and the researchers of other scientific fields. Besides, studying the same phenomenon using both discrete-time model and continuous-time model can lead to a comparison between their respective results and can provide interesting consequences. Moreover, the development of the use of electronic computers to calculate approximations or to realize simulations of the optimal solutions is an additional motivation to study the discrete-time framework in Optimal Control Theory. In fact, contemporary control theory is now primarily concerned with discrete-time systems and their solutions.

A historical motivation for infinite-horizon variational problems and infinite-horizon optimal control problems is found in the macroeconomic optimal growth theory in the works of Ramsey [50], Hotelling [35], von Weizsäcker [62] and Brock [20]. In such a theory, an agent represents itself and all its progeny, and the infinite horizon avoids to deal with the problems of the end of the world. Another important field of knowledge which uses the infinite-horizon optimal control is the management of natural resources as forests and fisheries, which are introduced in [22]. More generally, the study of some aspect of sustainable development naturally leads to a framework where a final time does not exist.

Those are the reasons for the author to choose infinite-horizon discrete-time optimal control problems to be the studied object of the thesis.

In finite-horizon continuous-time optimal control theory, there exists two main historical approaches: Pontryagin's approach and Bellman's approach. In infinite-horizon discrete-time optimal control problems, the dynamic programming of Bellman is currently used. In this thesis, we follow the other approach: the viewpoint of Pontryagin.

Pontryagin's viewpoint provides necessary conditions of optimality which are principles that the optimal solutions ought to satisfy, and these principles possess a meaning in the considered phenomenon. Moreover, the role of necessary conditions of optimality is to narrow the set of all processes which are candidates to be solutions of the problem, and this can also improve the modeling. In some cases, it is also possible to formulate sufficient conditions of optimality in the spirit of the conditions initiated by Seierstad and Sydsaeter for the continuous-time problems. During the process of establishing Pontryagin principles for discrete-time optimal control problems in infinite horizon, in some cases, we can also establish those for such problems in finite horizon.

The structure of this thesis is as follows. In Chapter 1, firstly, we recall preliminary basis of sequence spaces. In this part, we introduce some classical sequence spaces in $(\mathbb{R}^k)^{\mathbb{N}}$. Then we study their norms, their dualities and their completeness. In second part, we recall some basic results on differential calculus in normed linear space, particularly, the various types of differential in normed linear space, their properties, the Mean Value Theorem and the differential in product space. Those are the fundamental mathematical tools that we use throughout the thesis.

Chapters 2-5 are the main chapters of this thesis which contain new results on Pontryagin principles for infinite-horizon discrete-time optimal control problems. There are several ways to establish Pontryagin principles for our considered problems. The first method is to translate the original problem into an optimization problem that defined in Banach spaces; then use an appropriate multiplier rule in Banach spaces to obtain Pontryagin princiles. This method is direct and it requires the considered problems to have the capability of being translated into an optimization problem in Banach spaces. It is used in Chapter 2 and Chapter 3. The second method, which is first proposed by Blot and Chebbi in 2000, is to reduce the infinite-horizon problems into families of finite-horizon problems; then use an appropriate multiplier rule to obtain Pontryagin principle for the finite-horizon problem; and finally, extend that result to the infinite-horizon case by using some additional assumptions. Chapter 4 and Chapter 5 follow the second method.

In Chapter 2, we study the following problem:

$$\begin{cases} \text{Maximize} & K(\underline{y},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \psi(y_t,u_t) \\ \text{when} & \underline{y} = (y_t)_{t\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}\ ,\ \underline{u} = (u_t)_{t\in\mathbb{N}} \in U^{\mathbb{N}}, \\ & y_0 = \eta,\ \lim_{t\to\infty} y_t = y_\infty, \\ & \underline{u} \text{ is bounded}, \\ & \forall t \in \mathbb{N},\ y_{t+1} = g(y_t,u_t). \end{cases}$$

This problem is a special case of single-objective optimal control problem with bounded processes which was studied in by Blot and Hayek in [14] and [15]. The difference is that

now the problem contains an asymptotic constraint at infinity on the state variable. We will use an approach of functional analytic for this problem after translating it into the form of an optimization problem in Banach (sequence) spaces. Then a weak Pontyagin principle is established for this problem by using a classical multiplier rule in Banach spaces. Some new properties of Nemytskii operators are also studied in this chapter.

In Chapter 3, we establish a strong Pontryagin principle and a sufficient condition for the considered problems in Chapter 2. To obtain the strong principle, we use the non linear functional analytic approach as in Chapter 2 and apply a multiplier rule of Ioffe and Tihomirov in which a convexity condition is necessary. Sufficient condition is obtained by using the weak Pontryagin principle's conclusions as assumptions and an assumption of concavity of the Hamiltonian.

In Chapter 4, we study the infinite-horizon discrete-time single-objective optimal control problem, which is more general than the problem considered in Chapter 2 and Chapter 3. For such problems, we consider the dynamical system with difference equation and with difference inequation and in the presence of the constraints on optimal control at each period of time. There exists several results on Pontryagin principles for such problems which are established in Blot and Hayek [15]. However, some require the Lipschitzian conditions to use Clarke's calculus, while others require the smoothness or at least, the Fréchet differentiability and the continuity on a neighborhood of the optimal solution of the functions, which are present in the problem. The aim of this chapter is to establish Pontryagin maximum principle under the weak form for such problems using lighter assumptions than the usual ones by applying recent result of Blot on the multiplier rule in [10].

In Chapter 5, necessary conditions of Pareto optimality under the form of Pontryagin principles for finite-horizon and infinite-horizon multiobjective optimal control problems in discrete-time framework are studied. The considered problems in this chapter are similar to the ones in Chapter 4 but with multicriteria objective function. The aim of this chapter is to establish weak and strong maximum principles of Pontryagin for problems in the presence of constraints and under assumptions which are weaker than the usual ones. In this way, this chapter generalizes existing results for single-objective optimal control problems and for multiobjective optimal control problems with or without constraints. To establish weak principles of Pontryagin, we provide new multiplier rules for static multiobjective optimization problems, which are in the spirit of the multiplier rule for static single-objective optimization problems of Blot in [10]. The strong principles of Pontryagin are established relied on the multiplier rule of Khanh and Nuong in [39]. Sufficient conditions of optimality for the considered problems are also provided in the end of this chapter by using the weighting method.

# Chapter 1

# Preliminary on Sequence Spaces and Differential Calculus in Normed Spaces

## 1.1 Sequence Spaces

In this section we provide elements on sequence spaces, essentially on the space $\ell^\infty(\mathbb{N}, \mathbb{R}^k)$ of the bounded sequences in $\mathbb{R}^k$ and the space $c_0(\mathbb{N}, \mathbb{R}^k)$ of bounded sequences in $\mathbb{R}^k$ which converge to 0. Firstly, we define our notation and we recall some basic facts. Then we provide some basic analyses on sequence spaces in $\mathbb{R}^k$ and recall the results on the dual spaces of $c_0(\mathbb{N}, \mathbb{R}^k)$ and $\ell^\infty(\mathbb{N}, \mathbb{R}^k)$ which are useful in the establishment of Pontryagin principles for optimal control problems with processes from these spaces. Finally we prove the completeness of the classical sequence spaces in $\mathbb{R}^k$.

The main references that we use on the sequence spaces are Chapter 16 in Aliprantis and Border [2], Section 31 in Köthe [40] and Appendix A in Blot and Hayek [15].

### 1.1.1 Notation and Recall

**Notation 1.1. (Basic sets and spaces)**

- $\mathbb{N}$ is the set of all nonnegative integers. $\mathbb{N}_* = \mathbb{N}\backslash\{0\} = \{1, 2, \ldots\}$.
- $\mathbb{R}$ is the set of all real numbers.
- When $k \in \mathbb{N}_*$, $\mathbb{R}^k$ is the space of all $k$-dimensional real vectors. If $v \in \mathbb{R}^k$ then $v = (v^1, v^2, \ldots, v^k)$ where $v^i \in \mathbb{R},\ i \in \{1, \ldots, k\}$. The canonical basis of $\mathbb{R}^k$ is denoted by $(e_j)_{1\leq j\leq k}$ where $e^i_j := \delta^i_j$ for all $i,\ j \in \{1, \ldots, k\}$. ($\delta^i_j$ is the Kronecker symbol).
- $\mathbb{R}^{k*}$ denotes the dual space of $\mathbb{R}^k$ which is the space of all linear functionals from $\mathbb{R}^k$ into $\mathbb{R}$.
- Let $E$ be a set. Then $E^\mathbb{N}$ denotes the set of all sequences in $E$. For $E^\mathbb{N}$, $\underline{x} = (x_t)_{t\in\mathbb{N}}$ denotes its element. Here, $x_t \in E$ for each $t \in \mathbb{N}$.
- Let $M$ be a finite set. $|M|$ denotes the number of elements of $M$.

**Definition 1.2. (Norm and normed space)**

Given a vector space $E$ over the field $\mathbb{R}$. A norm on $E$ is a map $\|\cdot\| : E \to \mathbb{R}$ that satisfies the following conditions:

**(N1)** $\|\lambda x\| = |\lambda|\,\|x\|$ for all $\lambda \in \mathbb{R},\ x \in E$;

**(N2)** $\|x\| = 0$ if and only if $x = 0$;

**(N3)** $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E,\ \|\cdot\|)$ is then called a normed space.

**Definition 1.3. (Banach space)**
A Banach space is a vector space $E$ over the field $\mathbb{R}$ which is equipped with a norm $\|\cdot\|$ and which is complete with respect to that norm. That is to say, for every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $E$, there exists an element $x$ in $E$ such that

$$\lim_{n\to+\infty} x_n = x,$$

or equivalently:

$$\lim_{n\to+\infty} \|x_n - x\| = 0.$$

**Notation 1.4. (norm and norm dual on $\mathbb{R}^k$)**
- When $k \in \mathbb{N}_*$, on $\mathbb{R}^k$ we consider the norm

$$\|v\| := \max\left\{\left|v^j\right| : j \in \{1,\dots,k\}\right\},$$

  where $v = (v^1,\dots,v^k) \in \mathbb{R}^k$. For simplicity, from now on we denote the norm on $\mathbb{R}^k$ by $|\cdot|$.
- The canonical norm on the dual $\mathbb{R}^{k*}$ is

$$|p|_* := \sup\{|\langle p, v\rangle| : v \in \mathbb{R}^k,\ |v| \le 1\}$$

  where $\langle p, v\rangle := p(v)$ is the duality bracket. Note that when $\mathbb{R}^k$ is endowed with the norm $|\cdot|$, $|p|_* = \sum_{j=1}^k |p_j|$, where $p_j := \langle p, e_j\rangle$.

When $E = \mathbb{R}^k$, $(E, |\cdot|)$ is a finite-dimensional normed real vector space, we consider $E^{\mathbb{N}}$, the set of all sequences in $E$. This set can be turned into a vector space by defining vector addition as follows

$$(x_t)_{t\in\mathbb{N}} + (y_t)_{t\in\mathbb{N}} := (x_t + y_t)_{t\in\mathbb{N}} \text{ for all } \underline{x}, \underline{y} \in E^{\mathbb{N}},$$

and the scalar multiplication as follows

$$\alpha(x_t)_{t\in\mathbb{N}} := (\alpha x_t)_{t\in\mathbb{N}} \text{ for all } \underline{x} \in E^{\mathbb{N}}.$$

There are some basic sequence spaces which are subspaces of $E^{\mathbb{N}}$ and are defined as follows:

**Definition 1.5. (Basic sequence spaces)**
- **Space** $\ell^p(\mathbb{N}, E)$ :
  For all $p \in [1, +\infty)$, $\ell^p(\mathbb{N}, E) := \{\underline{x} \in E^{\mathbb{N}} : \sum_{t=0}^{+\infty} |x_t|^p < +\infty\}$. Endowed with the norm $\|\underline{x}\|_p := \left(\sum_{t=0}^{+\infty} |x_t|^p\right)^{1/p}$, it is a Banach space.
- **Space** $\ell^\infty(\mathbb{N}, E)$ :
  $\ell^\infty(\mathbb{N}, E) := \{\underline{x} \in E^{\mathbb{N}} : \sup_{t\in\mathbb{N}} |x_t| < +\infty\}$. Endowed with the norm $\|\underline{x}\|_\infty := \sup_{t\in\mathbb{N}} |x_t|$, it is Banach space.
- **Space** $c(\mathbb{N}, E)$ :
  $c(\mathbb{N}, E) := \{\underline{x} \in E^{\mathbb{N}} : \lim_{t\to+\infty} x_t \text{ exists in } E\}$. Endowed with the norm $\|\underline{x}\|_\infty$, it is a Banach subspace of $\ell^\infty(\mathbb{N}, E)$.
- **Space** $c_0(\mathbb{N}, E)$ :
  $c_0(\mathbb{N}, E) := \{\underline{x} \in E^{\mathbb{N}} : \lim_{t\to+\infty} x_t = 0\}$. Endowed with the norm $\|\underline{x}\|_\infty$, it is a Banach subspace of $\ell^\infty(\mathbb{N}, E)$.

– **Space** $c_{00}(\mathbb{N}, E)$ :
$c_{00}(\mathbb{N}, E) := \{\underline{x} \in E^{\mathbb{N}} : x_t = 0$ except for finitely many indexes $t\}$. It is a subspace of $\ell^p(\mathbb{N}, E)$ for all $p \geq 1$.

By direct verification, it is evident that $\|\cdot\|_\infty$ satisfies all conditions (N1), (N2), (N3) of the definition of norm. Hence, $\ell^\infty(\mathbb{N}, E)$, $c(\mathbb{N}, E)$ and $c_0(\mathbb{N}, E)$ are normed spaces. We will show that $\|\cdot\|_p$ also satisfies the definition of a norm in next subsection. The completeness of space $\ell^p(\mathbb{N}, E)$, $\ell^\infty(\mathbb{N}, E)$, $c(\mathbb{N}, E)$ and $c_0(\mathbb{N}, E)$ will be proven in the end of this section.

For the above-mentioned sequence spaces, we have the following theorem:

**Theorem 1.6.** *For all $p, q \in [1, +\infty)$ such that $p \leq q$, the following inclusions hold:*

$$c_{00}(\mathbb{N}, E) \subset \ell^p(\mathbb{N}, E) \subset \ell^q(\mathbb{N}, E) \subset c_0(\mathbb{N}, E) \subset c(\mathbb{N}, E) \subset \ell^\infty(\mathbb{N}, E).$$

*Proof.* We will prove the above-mentioned inclusions by the inverse order.

– Prove $c(\mathbb{N}, E) \subset \ell^\infty(\mathbb{N}, E)$ : let $\underline{x} = (x_t)_{t\in\mathbb{N}} \in c(\mathbb{N}, E)$ then $(x_t)_{t\in\mathbb{N}}$ is convergent. A convergent sequence is clearly bounded. So $\underline{x} \in \ell^\infty(\mathbb{N}, E)$.
– Prove $c_0(\mathbb{N}, E) \subset c(\mathbb{N}, E)$ : it is clear from the definition of these two spaces.
– Prove $\ell^q(\mathbb{N}, E) \subset c_0(\mathbb{N}, E)$ with $q \geq 1$ : let $\underline{x} = (x_t)_{t\in\mathbb{N}} \in \ell^q(\mathbb{N}, E)$ then $\sum_{t\in\mathbb{N}} |x_t|^q < +\infty$. From this we deduce that $\lim_{t\to+\infty} |x_t|^q = 0$ and hence $\lim_{t\to+\infty} x_t = 0$. And so, $\underline{x} \in c_0(\mathbb{N}, E)$.
– Prove $\ell^p(\mathbb{N}, E) \subset \ell^q(\mathbb{N}, E)$ for all $p, q \in [1, +\infty)$ such that $p \leq q$:
Let $\underline{x} = (x_t)_{t\in\mathbb{N}} \in \ell^p(\mathbb{N}, E)$. From the above-mentioned inclusion, we know that $\lim_{t\to+\infty} x_t = 0$. Then there exists $T \in \mathbb{N}_*$ big enough such that when $t > T$ we have $|x_t| < 1$. Now we consider the series

$$\begin{aligned}\sum_{t\in\mathbb{N}} |x_t|^q &= \sum_{t=0}^{T} |x_t|^q + \sum_{t=T+1}^{\infty} |x_t|^q \\ &\leq \sum_{t=0}^{T} |x_t|^q + \sum_{t=T+1}^{\infty} |x_t|^p \\ &\leq \sum_{t=0}^{T} |x_t|^q + \sum_{t\in\mathbb{N}} |x_t|^p .\end{aligned}$$

It is obvious that $\sum_{t=0}^{T} |x_t|^q < +\infty$ as it is a finite sum of positive numbers. Besides, $\sum_{t\in\mathbb{N}} |x_t|^p < +\infty$ since $\underline{x} = (x_t)_{t\in\mathbb{N}} \in \ell^p(\mathbb{N}, E)$. Hence, $\sum_{t\in\mathbb{N}} |x_t|^q < +\infty$ which means that $\underline{x} \in \ell^q(\mathbb{N}, E)$.
– Prove $c_{00}(\mathbb{N}, E) \subset \ell^p(\mathbb{N}, E)$ : Let $\underline{x} = (x_t)_{t\in\mathbb{N}} \in c_{00}(\mathbb{N}, E)$. We denote $NZ = \{t \in \mathbb{N} : x_t \neq 0\}$. From the definition of space $c_{00}(\mathbb{N}, E)$ we know that $|NZ| < +\infty$. Now, consider the series

$$\sum_{t\in\mathbb{N}} |x_t|^p = \sum_{t\in NZ} |x_t|^p < +\infty.$$

Then $\underline{x} \in \ell^p(\mathbb{N}, E)$.

□

### 1.1.2 Some Basic Analyses on Sequence Spaces

In this subsection, we will provide some basic analyses on sequence spaces. For simplicity, we set by default $E = \mathbb{R}^k$ where $k \in \mathbb{N}_*$.

**Definition 1.7.** (**real convex function**)
A map $f : \mathbb{R}^m \to \mathbb{R}$ is called convex if for all $x,\ y \in \mathbb{R}^m$ and for all $\alpha \in [0,1]$ we have:

$$f(x + \alpha(y - x)) \leq f(x) + \alpha(f(y) - f(x)).$$

For the convexity of real function we have the following theorem:

**Theorem 1.8.** *Let $f$ be a real function which is differentiable on the open interval $(a,b)$. Then $f$ is convex on $(a,b)$ if and only if its derivative $f'$ is increasing on $(a,b)$.*

**Example 1.9.** After the previous theorem, the exponential function $e^x$ is convex since $(e^x)' = e^x > 0$ for all $x \in \mathbb{R}$. From the definition of convex function we have the following inequality

$$e^{B+\alpha(A-B)} \leq e^B + \alpha(e^A - e^B)$$

for all $A, B \in \mathbb{R}$ and for all $\alpha \in [0,1]$.

**Proposition 1.10.** ***(Hölder inequality)***
*Let $p,\ q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\underline{x} \in \ell^p(\mathbb{N}, E)$ and $\underline{y} \in \ell^q(\mathbb{N}, E^*)$ one has*

$$\textstyle\sum_{t\in\mathbb{N}} |\langle y_t, x_t\rangle| \ \leq \|\underline{x}\|_p \|\underline{y}\|_q.$$

*Proof.* For $a, b > 0$, set $A := p \ln a,\ B := q \ln b$. The exponential function $e^x$ is convex, thus:

$$\begin{aligned} e^{\frac{A}{p}+\frac{B}{q}} &= e^{B+\frac{1}{p}(A-B)} \\ &\leq e^B + \frac{1}{p}(e^A - e^B) \\ &= \frac{1}{p}e^A + \frac{1}{q}e^B. \end{aligned}$$

Then after substituting $A = p \ln a$ and $B = q \ln b$, we have

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

It is obvious that when $\underline{x} = 0$ or $\underline{y} = 0$, the Hölder inequality holds.
Now, for $\underline{x} = (x_t)_{t\in\mathbb{N}} \in \ell^p(\mathbb{N}, E)$ and $\underline{y} = (y_t)_{t\in\mathbb{N}} \in \ell^q(\mathbb{N}, E^*)$ such that $\|\underline{x}\|_p = 1 = \|\underline{y}\|_q$, from the above-mentioned result one has

$$\begin{aligned} &\forall t \in \mathbb{N}, \quad |\langle y_t, x_t\rangle| \leq |x_t|\,|y_t|_* \leq \tfrac{1}{p}|x_t|^p + \tfrac{1}{q}|y_t|_*^q \\ &\Rightarrow \quad \textstyle\sum_{t\in\mathbb{N}} |\langle y_t, x_t\rangle| \ \leq \ \tfrac{1}{p}\sum_{t\in\mathbb{N}} |x_t|^p + \tfrac{1}{q}\sum_{t\in\mathbb{N}} |y_t|_*^q = \tfrac{1}{p} + \tfrac{1}{q} = 1. \end{aligned}$$

For $\underline{x} \in \ell^p(\mathbb{N}, E),\ \underline{x} \neq 0$ and $\underline{y} \in \ell^q(\mathbb{N}, E^*),\ \underline{y} \neq 0$, using the above-mentioned result for $\underline{x}' := \frac{\underline{x}}{\|\underline{x}\|_p}$ and $\underline{y}' := \frac{\underline{y}}{\|\underline{y}\|_q}$ we come to the following inequality:

$$\begin{aligned} \textstyle\sum_{t\in\mathbb{N}} |\langle y_t', x_t'\rangle| \ &\leq \ 1 \Leftrightarrow \textstyle\sum_{t\in\mathbb{N}} \left|\left\langle \frac{y_t}{\|\underline{y}\|_q}, \frac{x_t}{\|\underline{x}\|_p}\right\rangle\right| \ \leq \ 1 \\ &\Leftrightarrow \frac{1}{\|\underline{x}\|_p \|\underline{y}\|_q} \textstyle\sum_{t\in\mathbb{N}} |\langle y_t, x_t\rangle| \leq 1. \end{aligned}$$

Multiply both sides of the last inequality with $\|\underline{x}\|_p \|\underline{y}\|_q$ we obtain the result of this proposition. $\square$

**Lemma 1.11.** ***(Supremum formula)***
*Let $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $\underline{x} \in E^{\mathbb{N}}$ we have*

$$\|\underline{x}\|_p = \sup\left\{|\textstyle\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle| : \ \underline{y} \in c_{00}(\mathbb{N}, E^*), \ \|\underline{y}\|_q \leq 1\right\},$$

*whereas, equality is meant in $[0, +\infty]$.*

*Proof.* Let $\underline{x} \in E^{\mathbb{N}}$ and let $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We denote

$$C = \sup\left\{|\textstyle\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle| : \ \underline{y} \in c_{00}(\mathbb{N}, E^*), \ \|\underline{y}\|_q \leq 1\right\}.$$

- Prove that $\|\underline{x}\|_p \geq C$ :
  If $\|\underline{x}\|_p = +\infty$ then $\|\underline{x}\|_p \geq C$ is true. Now we assume that $\|\underline{x}\|_p < +\infty$. Following Hölder inequality:

  $$|\textstyle\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle| \leq \sum_{t\in\mathbb{N}} |\langle y_t, x_t\rangle| \ \leq \|\underline{x}\|_p$$

  for all $\underline{y} \in \ell_q(\mathbb{N}, E^*)$ (including $\underline{y} \in c_{00}(\mathbb{N}, E^*)$ ) with $\|\underline{y}\|_q \leq 1$. And so, $C \leq \|\underline{x}\|_p$.
- Prove that $\|\underline{x}\|_p \leq C$ :
  It is obviously that when $\|\underline{x}\|_p = 0$ we have $C = 0$. So the inequality is satisfied for this case. If $\|\underline{x}\|_p \neq 0$ then $\underline{x} \neq 0$. Choose $\underline{\lambda} = (\lambda_t)_{t\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$ such that $|\lambda_t|_* = 1$ and $\langle \lambda_t, x_t\rangle = |x_t|$ for all $t \in \mathbb{N}$. For sufficient big $N \in \mathbb{N}$, we know that $A := \left(\sum_{t=0}^{N} |x_t|^p\right)^{-1/q}$ exists and $A \geq 0$. Define $\underline{y} = (y_t)_{t\in\mathbb{N}} \in c_{00}(\mathbb{N}, E^*)$ as follows: $y_t := A\,|x_t|^{\frac{p}{q}}\,\lambda_t$ for $0 \leq t \leq N$ and $y_t := 0$ for $t > N$. Then we have:

  $$\begin{aligned}\|\underline{y}\|_q &= \left(\textstyle\sum_{t=0}^{N} \left|A\lambda_t\,|x_t|^{\frac{p}{q}}\right|_*^q\right)^{\frac{1}{q}} = A\left(\sum_{t=0}^{N} |x_t|^p\,|\lambda_t|_*^q\right)^{\frac{1}{q}} \\ &\overset{|\lambda_t|_*=1}{=} A\left(\textstyle\sum_{t=0}^{N} |x_t|^p\right)^{\frac{1}{q}} = 1.\end{aligned}$$

  Now with this $\underline{y} \in c_{00}(\mathbb{N}, E^*)$ we have

  $$\begin{aligned}C &\geq \left|\textstyle\sum_{t=0}^{N} \langle y_t, x_t\rangle\right| = \left|\sum_{t=0}^{N} \left\langle A\,|x_t|^{\frac{p}{q}}\,\lambda_t, x_t\right\rangle\right| \\ &= A\left|\textstyle\sum_{t=0}^{N} |x_t|^{\frac{p}{q}}\,\langle \lambda_t, x_t\rangle\right| \overset{\langle\lambda_t, x_t\rangle = |x_t|}{=} A\sum_{t=0}^{N} |x_t|^{1+\frac{p}{q}} \\ &= A\sum_{t=0}^{N} |x_t|^p = A\sum_{t=0}^{N} |x_t|^p = \left(\textstyle\sum_{t=0}^{N} |x_t|^p\right)^{1-\frac{1}{q}} = \left(\sum_{t=0}^{N} |x_t|^p\right)^{\frac{1}{p}}.\end{aligned}$$

  The last inequality: $C \geq \left(\sum_{t=0}^{N} |x_t|^p\right)^{\frac{1}{p}}$ is satisfied for all $N \in \mathbb{N}$, hence $C \geq \|\underline{x}\|_p$.

From the above-mentioned arguments, we have $\|\underline{x}\|_p = C$. □

**Proposition 1.12.** *For $1 \leq p < +\infty$, $\ell^p(\mathbb{N}, E)$ is a normed sequence space.*

*Proof.* We will prove this statement for the case $p = 1$ and for the case $p \in (1, +\infty)$ individually.

- Case $p = 1$:
  We have $\ell^1(\mathbb{N}, E) := \{\underline{x} \in E^{\mathbb{N}} : \|\underline{x}\|_1 := \sum_{t\in\mathbb{N}} |x_t|_\infty < +\infty\}$. It is clear that $\|\cdot\|_1$ satisfies all the properties (N1), (N2) and (N3) of a norm, hence $\ell^1(\mathbb{N}, E)$ is a normed space.

– Case $1 < p < +\infty$:
Obviously, $\|\cdot\|_p$ satisfies properties (N1) and (N2). Now let $q = \frac{p}{p-1}$, thus $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\underline{x},\ \underline{y} \in \ell^p(\mathbb{N}, E)$ and for all $\underline{z} \in E^{\mathbb{N}}$ (including all $\underline{z} \in c_{00}(\mathbb{N}, E^*)$) such that $\|\underline{z}\|_q \leq 1$ the following triangle inequality takes place:

$$\forall N \in \mathbb{N},\ \left|\sum_{t=0}^{N} \langle z_t, x_t + y_t\rangle\right| \leq \left|\sum_{t=0}^{N} \langle z_t, x_t\rangle\right| + \left|\sum_{t=0}^{N} \langle z_t, y_t\rangle\right|.$$

Let $N \to +\infty$ and using supremum formula, we have:

$$\left|\sum_{t\in\mathbb{N}} \langle z_t, x_t + y_t\rangle\right| \leq \left|\sum_{t\in\mathbb{N}} \langle z_t, x_t\rangle\right| + \left|\sum_{t\in\mathbb{N}} \langle z_t, y_t\rangle\right| \leq \|\underline{x}\|_p + \|\underline{y}\|_p.$$
$$\Rightarrow \left|\sum_{t\in\mathbb{N}} \langle z_t, x_t + y_t\rangle\right| \leq \|\underline{x}\|_p + \|\underline{y}\|_p$$

Take supremum both sides of the last inequality and using the supremum formula we obtain the property (N2) of $\|\cdot\|_p$:

$$\|x + y\|_p \leq \|\underline{x}\|_p + \|\underline{y}\|_p.$$

Hence $\ell^p(\mathbb{N}, E)$ is a normed space.

□

**Lemma 1.13.** ***(Convergence of norm)***
*Let $\underline{x} \in \ell^p(\mathbb{N}, E)$ where $p \in [1, +\infty)$ . Then $\|\underline{x}\|_p \overset{p\to+\infty}{\longrightarrow} \|\underline{x}\|_\infty$.*

*Proof.* Since

$$|x_k| \ \leq \left(\sum_{t\in\mathbb{N}} |x_t|^p\right)^{\frac{1}{p}}$$

for all $k \in \mathbb{N}$ and $p \geq 1$. So we have $\|\underline{x}\|_\infty \leq \|\underline{x}\|_p$. Thus, in particular

$$\|\underline{x}\|_\infty \leq \liminf_{p\to+\infty} \|\underline{x}\|_p.$$

On the other hand, we know that

$$\|\underline{x}\|_p = \left(\sum_{t\in\mathbb{N}} |x_t|^{p-q} \cdot |x_t|^q\right)^{\frac{1}{p}} \leq \|\underline{x}\|_\infty^{\frac{p-q}{p}} \cdot \left(\sum_{t\in\mathbb{N}} |x_t|^q\right)^{\frac{1}{p}} = \|\underline{x}\|_\infty^{1-\frac{q}{p}} \cdot \|\underline{x}\|_q^{\frac{q}{p}},$$

for all $q < p$ where we used $|x_t| \leq \|\underline{x}\|_\infty$ for all $t \in \mathbb{N}$. Therefore, we arrive at

$$\limsup_{p\to+\infty} \|\underline{x}\|_p \leq \limsup_{p\to+\infty} \left(\|\underline{x}\|_\infty^{1-\frac{q}{p}} \cdot \|\underline{x}\|_q^{\frac{q}{p}}\right) = \|\underline{x}\|_\infty \cdot 1.$$

We conclude

$$\limsup_{p\to+\infty} \|\underline{x}\|_p \leq \|\underline{x}\|_\infty \leq \liminf_{p\to+\infty} \|\underline{x}\|_p.$$

This shows that $\lim_{p\to+\infty} \|\underline{x}\|_p$ exists and equals $\|\underline{x}\|_\infty$. □

**Corollary 1.14.** ***(Extended supremum formula)***
*The supremum formula is also true for $p = \infty$ and $p = 1$:*

$$\|\underline{x}\|_\infty = \sup\left\{\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| :\ \underline{y} \in c_{00}(\mathbb{N}, E^*),\ \|\underline{y}\|_1 \leq 1\right\}$$

*and*

$$\|\underline{x}\|_1 = \sup\left\{\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| :\ \underline{y} \in c_{00}(\mathbb{N}, E^*),\ \|\underline{y}\|_\infty \leq 1\right\}.$$

*Proof.* In supremum formula we let $p \to +\infty$ with $q = \frac{p}{p-1}$ and $q \to +\infty$ with $p = \frac{q}{q-1}$, respectively. □

### 1.1.3 Duality of Sequence Spaces

**Definition 1.15. (Dual space of a sequence space)**
When $P$ and $Q$ are sequence spaces, $Q$ is called the dual space of $P$ if

**(D1)** For each $\underline{y} \in Q$ the series $\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle =: y(\underline{x})$ is convergent for all $\underline{x} \in P$ and defines an element $y(.)$ in $P^*$, with $\|y(.)\|_{P^*} = \|\underline{y}\|_Q$;

**(D2)** For each $\eta(.) \in P^*$, an element $\underline{y} \in Q$ exists with $y(.) = \eta(.)$.

and we can write $P^* = Q$.

So the notation $P^* = Q$ means that the map $\underline{y} \longmapsto y(.)$ from $Q$ into $P^*$ is an isometric isomorphism between $Q$ and $P^*$.

We also recall that norm of a bounded linear operator is defined as follows.

$$\|y(.)\|_{P^*} = \sup\{|y(\underline{x})| : \underline{x} \in P,\ \|\underline{x}\|_P \leq 1\}.$$

**Proposition 1.16.** *Let $p, q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell^p(\mathbb{N}, E)^* = \ell^q(\mathbb{N}, E^*)$. Furthermore, $c_0(\mathbb{N}, E)^* = \ell^1(\mathbb{N}, E^*)$ and $\ell^1(\mathbb{N}, E)^* = \ell^\infty(\mathbb{N}, E^*)$.*

*Proof.* Let $p, q \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. For simplicity, in this proof we denote the norm on dual space by $\|\cdot\|_*$.

- Prove $\ell^p(\mathbb{N}, E)^* = \ell^q(\mathbb{N}, E^*)$:
  - (D1) For each $\underline{y} \in \ell^q(\mathbb{N}, E^*)$ the series $\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle =: y(\underline{x})$ is absolutely convergent (from Hölder inequality) for all $\underline{x} \in \ell^p(\mathbb{N}, E)$ and hence it defines an element $y(.)$ in $\ell^p(\mathbb{N}, E)^*$. Using Hölder inequality we have:

    $$|y(\underline{x})| = |\textstyle\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle| \leq \|\underline{x}\|_p \|\underline{y}\|_q.$$

    Therefore, from the definition of norm of linear operator, we obtain $\|y(.)\|_* \leq \|\underline{y}\|_q$. Now, using supremum formula with interchange $\underline{x}$ and $\underline{y}$, $p$ and $q$, we have:

    $$\begin{aligned}
    \|\underline{y}\|_q &= \sup\left\{|\textstyle\sum_{t\in\mathbb{N}} \langle x_t, y_t\rangle| :\ \underline{x} \in c_{00}(\mathbb{N}, (E^*)^*),\ \|\underline{x}\|_p \leq 1\right\} \\
    &\overset{E=\mathbb{R}^k}{=} \sup\left\{|\textstyle\sum_{t\in\mathbb{N}} \langle x_t, y_t\rangle| :\ \underline{x} \in c_{00}(\mathbb{N}, E),\ \|\underline{x}\|_p \leq 1\right\} \\
    &\leq \sup\left\{|\textstyle\sum_{t\in\mathbb{N}} \langle x_t, y_t\rangle| :\ \underline{x} \in \ell^p(\mathbb{N}, E),\ \|\underline{x}\|_p \leq 1\right\} \\
    &= \sup\left\{|y(\underline{x})| :\ \underline{x} \in \ell^p(\mathbb{N}, E),\ \|\underline{x}\|_p \leq 1\right\} \\
    &= \|y(.)\|_*.
    \end{aligned}$$

    Then, we have proven $\|y(.)\|_* = \|\underline{y}\|_q$.
  - (D2) Let $\eta(.) \in \ell^p(\mathbb{N}, E)^*$. If $y_t : E \to \mathbb{R}$ defined by $y_t(x_t) = \eta(0, 0, \dots, x_t, 0, 0, \dots)$ for all $t \in \mathbb{N}$ then $y_t \in E^*$ and $\underline{y} = (y_t)_{t\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$. Now, for all $\underline{x} \in \ell^p(\mathbb{N}, E)$ we have

    $$\begin{aligned}
    \eta(x) = \eta(x_1, x_2, \dots) &= \sum_{t\in\mathbb{N}} \eta(0, 0, \dots, x_t, 0, 0, \dots) \\
    &= \sum_{t\in\mathbb{N}} y_t(x_t) = \textstyle\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle.
    \end{aligned}$$

    From here, using supremum formula as before we obtain $\|\underline{y}\|_q \leq \|\eta(.)\|_* < +\infty$ hence $\underline{y} \in \ell^q(\mathbb{N}, E^*)$. Therefore, if from $\underline{y}$ we define $y(.) \in \ell^p(\mathbb{N}, E)^*$ by setting $y(\underline{x}) := \sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle$ then $\eta(.) = y(.)$. And so (D2) is satisfied.

  From the above-mentioned arguments, we can conclude that $\ell^p(\mathbb{N}, E)^* = \ell^q(\mathbb{N}, E)$.
- Prove $c_0(\mathbb{N}, E)^* = \ell^1(\mathbb{N}, E^*)$:

– (D1) For each $\underline{y} \in \ell^1(\mathbb{N}, E^*)$, we consider series $\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle$ where $\underline{x} = (x_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N}, E)$ arbitrary. We have

$$\begin{aligned}
\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| &\leq \sum_{t\in\mathbb{N}} |\langle y_t, x_t\rangle| \\
&= \lim_{N\to+\infty} \sum_{t=0}^{N} |\langle y_t, x_t\rangle| \\
&\leq \lim_{N\to+\infty} \sum_{t=0}^{N} |y_t|_* |x_t| \\
&\leq \sup_{t\in\mathbb{N}} |x_t| \lim_{N\to+\infty} \sum_{t=0}^{N} |y_t|_* \\
&= \|\underline{x}\|_\infty . \sum_{t\in\mathbb{N}} |y_t|_* \\
&= \|\underline{x}\|_\infty . \|\underline{y}\|_1 < +\infty.
\end{aligned}$$

And so, for each $\underline{y} \in \ell^1(\mathbb{N}, E^*)$ the series $\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle =: y(\underline{x})$ is absolutely convergent for all $\underline{x} \in c_0(\mathbb{N}, E)$ and hence, we can define an element $y(.) \in c_0(\mathbb{N}, E)^*$. Moreover, from the previous inequality we have $\|y(.)\|_* \leq \|\underline{y}\|_1$. Now, using the extended supremum formula

$$\begin{aligned}
\|\underline{y}\|_1 &= \sup\left\{\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| : \ \underline{x} \in c_{00}(\mathbb{N}, E), \ \|\underline{x}\|_\infty \leq 1\right\} \\
&\leq \ \sup\left\{\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| : \ \underline{x} \in c_0(\mathbb{N}, E), \ \|\underline{x}\|_\infty \leq 1\right\} \\
&= \sup\left\{|y(\underline{x})| : \ \underline{x} \in c_0(\mathbb{N}, \mathbb{R}), \ \|\underline{x}\|_\infty \leq 1\right\} \\
&= \|y(.)\|_* .
\end{aligned}$$

Then, we have proven $\|y(.)\|_* = \|\underline{y}\|_1$. And so condition (D1) is satisfied.

– (D2) Let $\eta(.) \in c_0(\mathbb{N}, E)^*$. If $y_t : E \to \mathbb{R}$ defined by $y_t(x_t) = \eta(0, 0, \dots, x_t, 0, 0, \dots)$ for all $t \in \mathbb{N}$ then $y_t \in E^*$ and $\underline{y} = (y_t)_{t\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$. Now, for all $\underline{x} \in c_0(\mathbb{N}, E)$ we have

$$\begin{aligned}
\eta(x) = \eta(x_1, x_2, \dots) &= \sum_{t\in\mathbb{N}} \eta(0, 0, \dots, x_t, 0, 0, \dots) \\
&= \sum_{t\in\mathbb{N}} y_t(x_t) = \sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle .
\end{aligned}$$

From here, using supremum formula as before we obtain $\|\underline{y}\|_1 \leq \|\eta(.)\|_* < +\infty$ hence $\underline{y} \in \ell^1(\mathbb{N}, E^*)$. Therefore, if from $\underline{y}$ we define $y(.) \in \ell^1(\mathbb{N}, E)^*$ by setting $y(\underline{x}) := \sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle$ then $\eta(.) = y(.)$. And so (D2) is satisfied.

From the above-mentioned arguments, we can conclude that $c_0(\mathbb{N}, E)^* = \ell^1(\mathbb{N}, E^*)$.

– Prove $\ell^1(\mathbb{N}, E)^* = \ell^\infty(\mathbb{N}, E^*)$:

– (D1) For each $\underline{y} \in \ell^\infty(\mathbb{N}, E^*)$, we consider series $\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle$ where $\underline{x} = (x_t)_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}, E)$ arbitrary. We have

$$\begin{aligned}
\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| &\leq \sum_{t\in\mathbb{N}} |\langle y_t, x_t\rangle| \\
&= \lim_{N\to+\infty} \sum_{t=0}^{N} |\langle y_t, x_t\rangle| \\
&\leq \lim_{N\to+\infty} \sum_{t=0}^{N} |y_t|_* |x_t| \\
&\leq \sup_{k\in\mathbb{N}} |y_k|_* \lim_{N\to+\infty} \sum_{t=0}^{N} |x_t| \\
&= \|\underline{y}\|_\infty . \sum_{t\in\mathbb{N}} |x_t| \\
&= \|\underline{y}\|_\infty \|\underline{x}\|_1 < +\infty.
\end{aligned}$$

And so, for each $\underline{y} \in \ell^\infty(\mathbb{N}, E^*)$ the series $\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle =: y(\underline{x})$ is absolutely convergent for all $\underline{x} \in \ell^1(\mathbb{N}, E)$ and hence, we can define an element $y(.) \in c_0(\mathbb{N}, E)^*$. Moreover, from the previous inequality we have $\|y(.)\|_* \leq \|\underline{y}\|_\infty$. Now, using the extended supremum formula, we have

$$\begin{aligned}
\|\underline{y}\|_\infty &= \sup\left\{\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| : \ \underline{x} \in c_{00}(\mathbb{N}, E), \ \|\underline{x}\|_1 \leq 1\right\} \\
&\leq \ \sup\left\{\left|\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle\right| : \ \underline{x} \in \ell^1(\mathbb{N}, E), \ \|\underline{x}\|_1 \leq 1\right\} \\
&= \sup\left\{|y(\underline{x})| : \underline{x} \in \ell^1(\mathbb{N}, E), \ \|\underline{x}\|_1 \leq 1\right\} \\
&= \|y(.)\|_* .
\end{aligned}$$

Then, we have proven $\|y(.)\|_* = \|\underline{y}\|_1$. And so condition (D1) is satisfied.

– (D2) Let $\eta(.) \in \ell^1(\mathbb{N}, E)^*$. If $y_t : E \to \mathbb{R}$ defined by $y_t(x_t) = \eta(0, 0, \dots, x_t, 0, 0, \dots)$ for all $t \in \mathbb{N}$ then $y_t \in E^*$ and hence, $\underline{y} = (y_t)_{t\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$. Now, for all $\underline{x} \in \ell^1(\mathbb{N}, E)$ we have

$$\begin{aligned}\eta(\underline{x}) = \eta(x_0, x_1, \dots) &= \sum_{t\in\mathbb{N}} \eta(0, 0, \dots, x_t, 0, 0, \dots)\\ &= \sum_{t\in\mathbb{N}} y_t(x_t) = \textstyle\sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle .\end{aligned}$$

From here, using supremum formula as before we obtain $\|\underline{y}\|_\infty \le \|\eta(.)\|_* < +\infty$ hence $\underline{y} \in \ell^1(\mathbb{N}, E^*)$. Therefore, if from $\underline{y}$ we define $y(.) \in \ell^1(\mathbb{N}, E)^*$ by setting $y(\underline{x}) := \sum_{t\in\mathbb{N}} \langle y_t, x_t\rangle$ then $\eta(.) = y(.)$. And so (D2) is satisfied.

From the above-mentioned arguments, we can conclude that $\ell^1(\mathbb{N}, E)^* = \ell^\infty(\mathbb{N}, E^*)$. □

**The dual space of $\ell^\infty(\mathbb{N}, \mathbb{R}^k)$:**

In [2] the following space is defined

**Definition 1.17.** $\ell^1_d(\mathbb{N}, \mathbb{R})$ is the set of all linear functionals $\theta \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ such that there exists $\xi \in \mathbb{R}$ satisfying $\langle \theta, \underline{x}\rangle = \xi \lim_{t\to+\infty} x_t$ for all $\underline{x} \in c(\mathbb{N}, \mathbb{R})$. Its elements are called the singular functionals of $\ell^\infty(\mathbb{N}, \mathbb{R})^*$.

In [2] the following result is established.

**Theorem 1.18.** $\ell^\infty(\mathbb{N}, \mathbb{R})^* = \ell^1(\mathbb{N}, \mathbb{R}) \oplus \ell^1_d(\mathbb{N}, \mathbb{R})$.

The meaning of this equality is the following: for all $\Lambda \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ there exists a unique $(\underline{q}, \theta) \in \ell^1(\mathbb{N}, \mathbb{R}) \times \ell^1_d(\mathbb{N}, \mathbb{R})$ such that $\langle \Lambda, \underline{x}\rangle = \left\langle \underline{x}, \underline{q}\right\rangle + \langle \theta, \underline{x}\rangle$ for all $\underline{x} \in \ell^\infty(\mathbb{N}, \mathbb{R})$.

Now we extend this space and the previous description to sequences in $\mathbb{R}^k$.

**Definition 1.19.** $\ell^1_d(\mathbb{N}, \mathbb{R}^k)$ is the set of all linear functionals $\theta \in \ell^\infty(\mathbb{N}, \mathbb{R}^{k*})$ such that there exists $\zeta \in \mathbb{R}^{k*}$ satisfying $\langle \theta, \underline{x}\rangle = \langle \zeta, \lim_{t\to+\infty} x_t\rangle$ for all $\underline{x} \in c(\mathbb{N}, \mathbb{R}^k)$. Its elements are call the singular functionals of $\ell^\infty(\mathbb{N}, \mathbb{R}^k)^*$.

**Proposition 1.20.** $\ell^\infty(\mathbb{N}, \mathbb{R}^k)^* = \ell^1(\mathbb{N}, \mathbb{R}^{k*}) \oplus \ell^1_d(\mathbb{N}, \mathbb{R}^k)$.

*Proof.* Let $\Lambda \in \ell^\infty(\mathbb{N}, \mathbb{R}^k)^*$. When $\underline{x} \in \ell^\infty(\mathbb{N}, \mathbb{R}^k)$, we can identify it with $(\underline{x}^1, \underline{x}^2, \dots, \underline{x}^k) \in (\ell^\infty(\mathbb{N}, \mathbb{R}))^k$ . And we can write $\langle \Lambda, \underline{x}\rangle = \sum_{i=1}^k \langle \Lambda_i, \underline{x}^i\rangle$ where

$$\left\langle \Lambda_i, \underline{x}^i\right\rangle = \left\langle \Lambda, (\underline{0}, \dots, \underline{0}, \underline{x}^i, \underline{0}, \dots, \underline{0})\right\rangle .$$

Note that $\Lambda_i \in \ell^\infty(\mathbb{N}, \mathbb{R})^*$ and then, using Theorem 1.18, we know that there exist $\underline{q}^i \in \ell^1(\mathbb{N}, \mathbb{R})$ and $\theta^i \in \ell^1_d(\mathbb{N}, \mathbb{R})$ such that $\langle \Lambda_i, \underline{r}\rangle = \left\langle \underline{r}, \underline{q}^i\right\rangle + \langle \theta^i, \underline{r}\rangle$ for all $\underline{r} \in \ell^\infty(\mathbb{N}, \mathbb{R})$.

Denoting by $(e_i^*)_{1\le i\le k}$ the dual basis of the canonical basis of $\mathbb{R}^k$, we set $q_t := \sum_{i=1}^k q_t^i e_i^*$. Since $|q_t|_* = \sum_{i=1}^k |q_t^i|$ we obtain that $\underline{q} = (q_t)_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}, \mathbb{R}^{k*})$.

We set $\langle \theta, \underline{x}\rangle := \sum_{i=1}^k \langle \theta^i, \underline{x}^i\rangle$ . We see that $\theta$ is a linear functional from $\ell^\infty(\mathbb{N}, \mathbb{R}^k)$ into $\mathbb{R}$. Since the projection $\pi_i : \ell^\infty(\mathbb{N}, \mathbb{R}^k) \to \ell^\infty(\mathbb{N}, \mathbb{R})$, $\pi_i(\underline{x}) := \underline{x}^i$, are continuous, $\theta = \sum_{i=1}^k \theta^i \circ \pi_i$ is continuous as a finite sum of compositions of continuous functions. And so we obtain $\theta \in \ell^\infty(\mathbb{N}, \mathbb{R}^k)^*$. When $\underline{x} \in c(\mathbb{N}, \mathbb{R}^k)$ we have $\underline{x}^i \in c(\mathbb{N}, \mathbb{R})$ and since $\theta^i \in \ell^1_d(\mathbb{N}, \mathbb{R})$ there exists $\xi_i \in \mathbb{R}$ such that $\langle \theta^i, \underline{x}^i\rangle = \xi_i \lim_{t\to+\infty} x_t^i$. We set $\xi := \sum_{i=1}^k \xi_i e_i^* \in \mathbb{R}^{k*}$,and then we have $\langle \theta, \underline{x}\rangle = \sum_{i=1}^k \langle \theta^i, \underline{x}^i\rangle = \sum_{i=1}^k \xi_i \lim_{t\to+\infty} x_t^i = \langle \xi, \lim_{t\to+\infty} x_t\rangle$ . Hence, $\theta \in \ell^1_d(\mathbb{N}, \mathbb{R}^k)$. The existence is proven.

To check the uniqueness we assume that there exist $\underline{p} \in \ell^1(\mathbb{N}, \mathbb{R}^{k*})$ and $\underline{\rho} \in \ell^1_d(\mathbb{N}, \mathbb{R}^k)$ such that $\left\langle \underline{x}, \underline{q} \right\rangle + \langle \theta, \underline{x} \rangle = \left\langle \underline{x}, \underline{p} \right\rangle + \left\langle \underline{\rho}, \underline{x} \right\rangle$ for all $\underline{x} \in \ell^\infty(\mathbb{N}, \mathbb{R}^k)$. When $x \in c_0(\mathbb{N}, \mathbb{R}^k)$, this equality becomes $\left\langle \underline{x}, \underline{q} \right\rangle = \left\langle \underline{x}, \underline{p} \right\rangle$, and since $\ell^1(\mathbb{N}, \mathbb{R}^{k*})$ is the dual space of $c_0(\mathbb{N}, \mathbb{R}^k)$ we obtain $\underline{q} = \underline{p}$, from which we deduce $\theta = \underline{\rho}$. □

### 1.1.4 Completeness of Sequence Spaces

In this subsection, we will show that $\ell^p(\mathbb{N}, E)$ with $p \in [1, +\infty)$, $\ell^\infty(\mathbb{N}, E)$, $c(\mathbb{N}, E)$ and $c_0(\mathbb{N}, E)$ are Banach spaces.

**Theorem 1.21.** *$\ell^p(\mathbb{N}, E)$ where $p \in [1, +\infty]$ is a Banach space.*

*Proof.* We will prove this theorem for the cases $p \in (1, +\infty)$, $p = 1$ and $p = +\infty$ individually.

- Case $p \in (1, +\infty)$:
  Let $(\underline{x}^m)^{m\in\mathbb{N}} = (\underline{x}^0, \underline{x}^1, \dots, \underline{x}^n, \dots)$ be a Cauchy sequence in $\ell^p(\mathbb{N}, E)$. Here, for each $n \in \mathbb{N}$, $\underline{x}^n = (x^n_t)_{t\in\mathbb{N}} \in \ell^p(\mathbb{N}, E)$. Now, since $(\underline{x}^m)^{m\in\mathbb{N}}$ is a Cauchy sequence then for all $\varepsilon > 0$ small enough, there exists $M \in \mathbb{N}_*$ such that when $m, n > M$ we have

$$\|\underline{x}^m - \underline{x}^n\|_p < \varepsilon \Leftrightarrow \left( \sum_{t\in\mathbb{N}} |x^m_t - x^n_t|^p \right)^{1/p} < \varepsilon$$
$$\Leftrightarrow \sum_{t\in\mathbb{N}} |x^m_t - x^n_t|^p < \varepsilon^p < \varepsilon.$$

  Then for every $t \in \mathbb{N}$ we have $|x^m_t - x^n_t|^p < \varepsilon$ and it means that for each $t \in \mathbb{N}$ sequence $(x^m_t)^{m\in\mathbb{N}}$ is a Cauchy sequence in space $E = \mathbb{R}^k$. Since $\mathbb{R}^k$ is a Banach space we deduce that $\lim_{m\to+\infty} x^m_t = x_t$ for each $t \in \mathbb{N}$. We set $\underline{x} = (x_t)_{t\in\mathbb{N}}$. For any $N > 1$ and $n, m > M$ we have:

$$\sum_{t=0}^{N} |x^m_t - x^n_t|^p \leq \sum_{t\in\mathbb{N}} |x^m_t - x^n_t|^p < \varepsilon^p < \varepsilon.$$

  Let $n \to +\infty$ in the previous inequality, we obtain the following:

$$\sum_{t=0}^{N} |x^m_t - x_t|^p < \varepsilon^p < \varepsilon$$

  This inequality holds for all $N > 1$. Then we can take the limit as $N \to +\infty$ :

$$\sum_{t=0}^{+\infty} |x^m_t - x_t|^p \leq \varepsilon^p < \varepsilon \Leftrightarrow \|\underline{x}^m - \underline{x}\|_p < \varepsilon.$$

  And so, $\underline{x}^m - \underline{x} \in \ell^p(\mathbb{N}, E)$ hence $\underline{x} \in \ell^p(\mathbb{N}, E)$ since $\underline{x}^m \in \ell^p(\mathbb{N}, E)$. Moreover, $\|\underline{x}^m - \underline{x}\|_p < \varepsilon$ for all $m > M$ then we deduce that $\lim_{m\to+\infty} \underline{x}^m = \underline{x} \in \ell^p(\mathbb{N}, E)$. Hence, $\ell^p(\mathbb{N}, E)$ is a Banach space.
- Case $p = 1$ :
  Let $(\underline{x}^m)^{m\in\mathbb{N}} = (\underline{x}^0, \underline{x}^1, \dots, \underline{x}^n, \dots)$ be a Cauchy sequence in $\ell^1(\mathbb{N}, E)$. Here, for each $n \in \mathbb{N}$, $\underline{x}^n = (x^n_t)_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}, E)$. Now, since $(\underline{x}^m)^{m\in\mathbb{N}}$ is a Cauchy sequence then for all $\varepsilon > 0$, there exists $M \in \mathbb{N}_*$ such that when $m, n > M$ we have

$$\|\underline{x}^m - \underline{x}^n\|_1 < \varepsilon \Leftrightarrow \sum_{t\in\mathbb{N}} |x^m_t - x^n_t| < \varepsilon.$$

Then for every $t \in \mathbb{N}$ we have $|x_t^m - x_t^n| < \varepsilon$ and it means that for each $t \in \mathbb{N}$ sequence $(x_t^m)^{m\in\mathbb{N}}$ is a Cauchy sequence in space $E = \mathbb{R}^k$. Since $\mathbb{R}^k$ is a Banach space we deduce that $\lim_{m\to+\infty} x_t^m = x_t$ for each $t \in \mathbb{N}$. We set $\underline{x} = (x_t)_{t\in\mathbb{N}}$. For any $N > 1$ and $n, m > M$ we have:

$$\sum_{t=0}^{N} |x_t^m - x_t^n| \leq \sum_{t\in\mathbb{N}} |x_t^m - x_t^n| < \varepsilon.$$

Let $n \to +\infty$ in the previous inequality, we obtain the following:

$$\sum_{t=0}^{N} |x_t^m - x_t|^p < \varepsilon.$$

This inequality holds for all $N > 1$. Then we can take the limit as $N \to +\infty$ :

$$\sum_{t=0}^{+\infty} |x_t^m - x_t|^p < \varepsilon \Leftrightarrow \|\underline{x}^m - \underline{x}\|_1 < \varepsilon.$$

And so, $\underline{x}^m - \underline{x} \in \ell^1(\mathbb{N}, E)$ hence $\underline{x} \in \ell^1(\mathbb{N}, E)$ since $\underline{x}^m \in \ell^1(\mathbb{N}, E)$. Moreover, $\|\underline{x}^m - \underline{x}\|_1 < \varepsilon$ for all $m > M$ then we deduce that $\lim_{m\to+\infty} \underline{x}^m = \underline{x} \in \ell^1(\mathbb{N}, E)$. Hence, $\ell^1(\mathbb{N}, E)$ is a Banach space.

– Case $p = +\infty$ :
Let $(\underline{x}^m)^{m\in\mathbb{N}} = (\underline{x}^0, \underline{x}^1, \dots, \underline{x}^n, \dots)$ be a Cauchy sequence in $\ell^\infty(\mathbb{N}, E)$. Here, for each $n \in \mathbb{N}$, $\underline{x}^n = (x_t^n)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, E)$. Now, since $(\underline{x}^m)^{m\in\mathbb{N}}$ is a Cauchy sequence then for all $\varepsilon > 0$, there exists $M \in \mathbb{N}_*$ such that when $m, n > M$ we have

$$\|\underline{x}^m - \underline{x}^n\|_\infty < \varepsilon \Leftrightarrow \sup_{t\in\mathbb{N}} |x_t^m - x_t^n| < \varepsilon.$$

Then for every $t \in \mathbb{N}$ we have $|x_t^m - x_t^n| < \varepsilon$ and it means that for each $t \in \mathbb{N}$ sequence $(x_t^m)^{m\in\mathbb{N}}$ is a Cauchy sequence in space $E = \mathbb{R}^k$. Since $\mathbb{R}^k$ is a Banach space we deduce that $\lim_{m\to+\infty} x_t^m = x_t$ for each $t \in \mathbb{N}$. We set $\underline{x} = (x_t)_{t\in\mathbb{N}}$. For any $N > 1$ and $n, m > M$ we have:

$$\max_{0\leq t\leq N} |x_t^m - x_t^n| \leq \sup_{t\in\mathbb{N}} |x_t^m - x_t^n| < \varepsilon.$$

Let $n \to +\infty$ in the previous inequality, we obtain the following:

$$\max_{0\leq t\leq N} |x_t^m - x_t| < \varepsilon.$$

This inequality holds for all $N > 1$. Then we can take the limit as $N \to +\infty$ :

$$\sup_{t\in\mathbb{N}} |x_t^m - x_t| < \varepsilon \Leftrightarrow \|\underline{x}^m - \underline{x}\|_\infty < \varepsilon.$$

And so, $\underline{x}^m - \underline{x} \in \ell^\infty(\mathbb{N}, E)$ hence $\underline{x} \in \ell^\infty(\mathbb{N}, E)$ since $\underline{x}^m \in \ell^\infty(\mathbb{N}, E)$. Moreover, $\|\underline{x}^m - \underline{x}\|_\infty < \varepsilon$ for all $m > M$ then we deduce that $\lim_{m\to+\infty} \underline{x}^m = \underline{x} \in \ell^\infty(\mathbb{N}, E)$. Hence, $\ell^\infty(\mathbb{N}, E)$ is a Banach space.

□

**Lemma 1.22.** *$c(\mathbb{N}, E)$ and $c_0(\mathbb{N}, E)$ are closed subspace of $\ell^\infty(\mathbb{N}, E)$.*

*Proof.* – Prove $c_0(\mathbb{N}, E)$ is a closed subspace of $\ell^\infty(\mathbb{N}, E)$ :
Let $(\underline{x}^m)^{m\in\mathbb{N}} = (\underline{x}^0, \underline{x}^1, \dots, \underline{x}^n, \dots)$ be a convergent sequence of elements from $c_0(\mathbb{N}, E)$ and let $\lim_{m\to+\infty} \underline{x}^m = \underline{x}$. Since $\ell^\infty(\mathbb{N}, E)$ is a Banach space, we know that $\underline{x} \in \ell^\infty(\mathbb{N}, E)$.Now we need to prove $\underline{x} \in c_0(\mathbb{N}, E)$. From $\lim_{m\to+\infty} \underline{x}^m = \underline{x}$, for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}_*$ such that when $m > N_1$ we have:

$$\|\underline{x}^m - \underline{x}\|_\infty < \frac{\varepsilon}{2} \Leftrightarrow \sup_{t\in\mathbb{N}} |x_t^m - x_t| < \frac{\varepsilon}{2}.$$

From this we deduce that $|x_t^m - x_t| < \frac{\varepsilon}{2}$ for each $t \in \mathbb{N}$ when $m > N_1$.
Now, $\underline{x}^m \in c_0(\mathbb{N}, E)$ for all $m \in \mathbb{N}$, hence $\lim_{t\to+\infty} x_t^m = 0$. It means that for all $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}_*$ such that when $t > N_2$ we have $|x_t^m| < \frac{\varepsilon}{2}$ for all $m \in \mathbb{N}$. And so, when $m > N_1$ and $t > N_2$ we have

$$\begin{aligned} |x_t| &= |x_t - x_t^m + x_t^m| \leq |x_t - x_t^m| + |x_t^m| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The last inequality holds for all $t > N_2$, hence we can conclude that $\lim_{t\to+\infty} x_t = 0$ and $\underline{x} \in c_0(\mathbb{N}, E)$. Then we obtain $c_0(\mathbb{N}, E)$ is closed subspace of $\ell^\infty(\mathbb{N}, E)$.

– Prove $c(\mathbb{N}, E)$ is a closed subspace of $\ell^\infty(\mathbb{N}, E)$ :
Let $(\underline{x}^m)^{m\in\mathbb{N}} = (\underline{x}^0, \underline{x}^1, \dots, \underline{x}^n, \dots)$ be a convergent sequence of elements from $c(\mathbb{N}, E)$ and let $\lim_{m\to+\infty} \underline{x}^m = \underline{x}$. We know that $\underline{x} \in \ell^\infty(\mathbb{N}, E)$ and we need to prove $\underline{x} \in c(\mathbb{N}, E)$. From $\lim_{m\to+\infty} \underline{x}^m = \underline{x}$, for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}_*$ such that when $m > N_1$ we have:

$$\|\underline{x}^m - \underline{x}\|_\infty < \frac{\varepsilon}{3} \Leftrightarrow \sup_{t\in\mathbb{N}} |x_t^m - x_t| < \frac{\varepsilon}{3}.$$

From this we deduce that $|x_t^m - x_t| < \frac{\varepsilon}{3}$ for each $t \in \mathbb{N}$ when $m > N_1$.
Now, $\underline{x}^m \in c(\mathbb{N}, E)$ for all $m \in \mathbb{N}$, hence $\lim_{t\to+\infty} x_t^m$ exists for all $m \in \mathbb{N}$ and from this we obtain $(x_t^m)_{t\in\mathbb{N}}$ is a Cauchy sequence. Therefore, for all $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}_*$ such that when $t, t' > N_2$ we have:

$$|x_t^m - x_{t'}^m| \leq \frac{\varepsilon}{3}.$$

Notice that the previous inequality holds for each $m \in \mathbb{N}$.
Now consider

$$\begin{aligned} |x_t - x_{t'}| &= |x_t - x_t^m + x_t^m - x_{t'}^m + x_{t'}^m - x_{t'}| \\ &\leq |x_t - x_t^m| + |x_t^m - x_{t'}^m| + |x_{t'}^m - x_{t'}| \end{aligned}$$

In the previous inequality, we take $t, t' > N_2$ and $m > N_1$ then:

$$|x_t - x_{t'}| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

It means that $\underline{x} = (x_t)_{t\in\mathbb{N}}$ is a Cauchy sequence of element from $E = \mathbb{R}^k$. Since $\mathbb{R}^k$ is complete, we deduce that $\underline{x} = (x_t)_{t\in\mathbb{N}}$ is convergent. Hence, $\underline{x} \in c(\mathbb{N}, E)$ and therefore, $c(\mathbb{N}, E)$ is closed subspace of $\ell^\infty(\mathbb{N}, E)$.

□

**Lemma 1.23.** *A closed subspace of a Banach space is also a Banach space.*

*Proof.* Let $P$ be a Banach space and $Q$ is a closed subspace of $P$. Consider an arbitrary Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $Q$ then $(x_n)_n$ is a Cauchy sequence in $P$ as well since $Q \subset P$. Now $P$ is Banach, hence, $(x_n)_n$ is convergent and $\lim_{n\to+\infty} x_n = x \in P$. Since $Q$ is closed then if $\lim_{n\to+\infty} x_n$ exists, it must belongs to $Q$. Hence, $x \in Q$. And so, every Cauchy sequence in $Q$ is convergent within $Q$. It means that $Q$ is a Banach space. □

**Corollary 1.24.** *$c(\mathbb{N}, E)$ and $c_0(\mathbb{N}, E)$ are Banach spaces.*

*Proof.* $c(\mathbb{N}, E)$ and $c_0(\mathbb{N}, E)$ are closed subspaces of $\ell^\infty(\mathbb{N}, E)$; hence, they are also Banach spaces since $\ell^\infty(\mathbb{N}, E)$ is a Banach space. □

## 1.2 Recall of Differential Calculus in Normed Linear Spaces

In this section, we recall the basis of differential calculus in normed linear spaces, especially on diverse kinds of differentials and their properties which will be useful for the proofs in later chapters. The main references for this section are Chapter XIII in Lang [41] and Section 2.2 in Alekseev-Tihomirov-Fomin [1].

### 1.2.1 Directional Derivative, Gâteaux and Fréchet Differentials and Strict Differentiability

Let $X$ and $Y$ be normed linear spaces, let $U$ be a neighborhood of a point $\hat{x}$ in $X$, and let $F$ be a mapping from $U$ into $Y$. The directional derivative is usually defined as follows

**Definition 1.25.** Let $h \in X$. If the limit

$$\lim_{\lambda\to 0^+} \frac{F(\hat{x} + \lambda h) - F(\hat{x})}{\lambda} \tag{1.1}$$

exists, it is called the directional derivative of $F$ at the point $\hat{x}$ in the direction $h$ and it is denoted by $\vec{D}F(\hat{x}; h)$.

**Definition 1.26.** Let us suppose that for any $h \in X$ there exists the directional derivative $\vec{D}F(\hat{x}; h)$ and there exists a continuous linear operator $\Lambda \in \mathcal{L}(X, Y)$ such that $\vec{D}F(\hat{x}; h) \equiv \Lambda h$. Then the operator $\Lambda$ is called the Gâteaux differential of the mapping $F$ at the point $\hat{x}$ and is denoted by $D_G F(\hat{x})$.

Thus, $D_G F(\hat{x})$ is an element of $\mathcal{L}(X, Y)$ such that, given any $h \in X$, the relation

$$F(\hat{x} + \lambda h) = F(\hat{x}) + \lambda D_G F(\hat{x}) \cdot h + \lambda \cdot \rho_h(\lambda) \tag{1.2}$$

holds when $\lambda$ is positive and small enough. Here $\rho_h : \mathbb{R} \to Y$ is a mapping satisfying $\lim_{\lambda\to 0^+} \rho_h(\lambda) = 0$. It readily follows that the Gâteaux differential is determined uniquely since the directional derivatives are determined uniquely.

**Definition 1.27.** Let it be possible to represent a mapping $F$ in a neighborhood of a point $\hat{x}$ in the form

$$F(\hat{x} + h) = F(\hat{x}) + \Lambda.h + \alpha(h). \|h\|, \tag{1.3}$$

where $\Lambda \in \mathcal{L}(X, Y)$ and $\alpha : X \to Y$ is a mapping which is defined for all sufficient small $h$ in $X$ and such that

$$\lim_{h\to 0} \alpha(h) = 0. \tag{1.4}$$

Then the mapping $F(\cdot)$ is said to be Fréchet differentiable at the point $\hat{x}$. The operator $\Lambda$ is called the Fréchet differential of the mapping $F$ at the point $\hat{x}$ and is denoted by $DF(\hat{x})$.

From this definition, $DF(\hat{x}) = \Lambda$ is a continuous linear function which belongs to $\mathcal{L}(X, Y)$. Relations (1.3) and (1.4) can also be written thus:

$$F(\hat{x} + h) = F(\hat{x}) + DF(\hat{x}).h + o(\|h\|), \tag{1.5}$$

It readily follows that the Fréchet differential is determined uniquely because if $\Lambda_1$ and $\Lambda_2$ from $\mathcal{L}(X, Y)$ simultaneously satisfy relation (1.5) then $\|\Lambda_1 h - \Lambda_2 h\| = o(\|h\|)$. It is only possible when $\Lambda_1 = \Lambda_2$. Moreover, if a mapping is Fréchet differentiable at a point then it is also Gâteaux differentiable at that point and $DF(\hat{x}) \equiv D_G F(\hat{x})$. This assertion is easily verified by setting $h = \lambda v$ in the definition of Fréchet differential. Finally, in the term of the $\varepsilon$, $\delta$ formalism the relations Relations (1.3) and (1.4) are stated thus: given an arbitrary $\varepsilon > 0$, there is $\delta > 0$ for which the inequality

$$\|F(\hat{x} + h) - F(\hat{x}) - \Lambda.h\| \leq \varepsilon \|h\| \tag{1.6}$$

holds for all $h$ such that $\|h\| < \delta$. This naturally leads to a further strengthening:

**Definition 1.28.** A mapping F is said to be strictly differentiable at a point $\hat{x}$ if there is an operator $\Lambda \in \mathcal{L}(X, Y)$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that for all $y$ and $z$ satisfying the inequalities $\|y - \hat{x}\| < \delta$ and $\|z - \hat{x}\| < \delta$ the inequality

$$\|F(y) - F(z) - \Lambda.(y - z)\| \leq \varepsilon \|y - z\| \tag{1.7}$$

holds.

Putting $z = \hat{x}$ and $y = \hat{x} + h$ in (1.7) we obtain (1.6), and hence a strictly differentiable mapping is Fréchet differentiable and $\Lambda = DF(\hat{x})$ .

**Example 1.29.** A mapping $A : X \to Y$ of one linear space into another is said to be affine if there exists a linear mapping $\Lambda : X \to Y$ and a constant $\alpha \in Y$ such that $A(x) = \Lambda.x + \alpha$. If $X$ and $Y$ are normed spaces and $\Lambda \in \mathcal{L}(X, Y)$, then the mapping $A$ is strictly differentiable at any point $x$, and, moreover, $DA(x) = \Lambda$.

This assertion can be verified directly. Indeed, we have:

$$\begin{aligned} \|A(y) - A(z) - \Lambda.(y - z)\| &= \|\Lambda.y - \Lambda.z - \Lambda.(y - z)\| \\ &= 0 \leq \varepsilon \|y - z\| \end{aligned}$$

for any $\varepsilon > 0$ and for all $y,\ z \in X$ (in particular, including $y, z \in X$ such that $\|y - x\| < \delta$ and $\|z - x\| < \delta$ where $\delta$ is some sufficient small positive number). Hence, $A$ is strictly differentiable at any point $x \in X$.

When $\alpha = 0$ the mapping $A$ degrades into a continuous linear mapping $\Lambda \in \mathcal{L}(X, Y)$. And from the above-mentioned argument, we can deduce that a mapping $\Lambda \in \mathcal{L}(X, Y)$ is also a strictly differentiable at any point $x \in X$ and $D\Lambda(x) = \Lambda$.

For Fréchet differentiable mapping, we have the following well-known property:

**Theorem 1.30.** *If $F$ is Fréchet differentiable at the point $x$ then $F$ is continuous at $x$.*

Now with strictly differentiable mapping, we have a similar but stronger result:

**Theorem 1.31.** *If $F$ is strictly differentiable at the point $x$ then $F$ is continuous on a neighborhood of $x$.*

*Proof.* Let $F$ be a mapping which is strictly differentiable at the point $x$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_1$ and $x_2$ satisfying the inequalities $\|x_1 - x\| < \delta$ and $\|x_2 - x\| < \delta$ we have the inequality

$$\|F(x_1) - F(x_2) - \Lambda.(x_1 - x_2)\| \leq \varepsilon \|x_1 - x_2\|.$$

Let $z$ be a point in a $\frac{\delta}{2}$ - neighborhood of $x$.We will show that $F$ is continuous at $z$. We set $y = z + h$ where $h \in X$ and $\|h\| < \frac{\delta}{2}$. Then

$$\|y - x\| = \|z + h - x\| \leq \|z - x\| + \|h\| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

From this we deduce that both $y$ and $z$ belongs to $\delta$ - neighborhood of $x$, hence:

$$\begin{aligned} & \|F(y) - F(z) - \Lambda.(y - z)\| \leq \varepsilon \|y - z\| \\ \Leftrightarrow \quad & \|F(z + h) - F(z) - \Lambda.h\| \leq \varepsilon \|h\|. \end{aligned}$$

The last inequality shows that $F$ is Fréchet differentiable at $z$ and hence, continuous at that point. Since taking $z$ is arbitrary in $\frac{\delta}{2}$ - neighborhood of $x$, then one can conclude that $F$ is continuous on this neighborhood of $x$. □

**Definition 1.32.** If $F$ is (Gâteaux or Fréchet or strictly) differentiable at every point $x$ of $U$, then we say that $F$ is ( Gâteaux or Fréchet or strictly) differentiable on $U$. In that case, the differential $\mathcal{D}F$ ($\mathcal{D}F$ can be $DF$ or $D_G F$) is a mapping

$$\mathcal{D}F : U \to \mathcal{L}(X, Y)$$

from $U$ into the space of continuous linear maps $\mathcal{L}(X, Y)$, and thus to each $x \in U$, we have associated the linear map $\mathcal{D}F(x) \in \mathcal{L}(X, Y)$. If $\mathcal{D}F$ is continuous, we say that $F$ is continuously differentiable or simply, $F$ is of class $C^1$.

Since $\mathcal{D}F$ maps $U$ into the Banach space $\mathcal{L}(X, Y)$, we can define inductively $F$ to be of class $C^p$ if all the differentials $\mathcal{D}^k F$ exist and are continuous for $1 \leq k \leq p$.

When a mapping $F$ is Fréchet differentiable on an open set, the following theorem shows the necessary and sufficient condition for $F$ to be strictly differentiable:

**Theorem 1.33.** *If $F$ is Fréchet differentiable on an open set, then $F$ is continuously differentiable if and only if $F$ is strictly differentiable on the same set.*

The proof of this theorem can be found in [55], page 682.

### 1.2.2 Subgradient and subdifferential

Let $X$ be a normed linear space, $F : X \to \mathbb{R}$ is a functional and $x \in X$.

**Definition 1.34.** An element $\zeta$ of $X^*$ is called a subgradient of $F$ at $x$ (in the sense of convex analysis) if it satisfies the following subgradient inequality:

$$F(y) - F(x) \geq \langle \zeta, y - x \rangle, y \in X.$$

The set of all subgradients of $F$ at $x$ is called the subdifferential of $F$ at $x$ and is denoted by $\partial F(x)$. When $X = X_1 \times X_2$, $x = (x_1, x_2)$, the partial subdifferential of $F$ with respect to $x_1$ (respectively, $x_2$) at $(x_1, x_2)$ is denoted by $\partial_1 F(x_1, x_2)$ (respectively, $\partial_2 F(x_1, x_2)$).

The following properties of subdifferential are taken from [1] and [24].

**(i)** The subdifferential $\partial F(x)$ is a closed convex set (possibly empty).

**(ii)** If $F$ is convex then

$$\partial F(x) = \{\zeta \in X^* : \vec{D}F(x;h) \geq \langle \zeta, h\rangle, \forall h \in X\}.$$

**(iii)** If $F$ is convex and it is Gâteaux differentiable at $x$ then $\partial F(x) = \{D_G F(x)\}$.

**(iv)** If $F$ is convex and continuous at $x$ then $\partial F(x)$ is nonempty closed convex. When $X = \mathbb{R}^n$, then for any $x \in \mathbb{R}^n, \partial F(x)$ is a non empty convex compact set.

### 1.2.3 Properties of Differentials

We recall some basic properties of differentials.

**Sum:** Let $X$, $Y$ be normed vector spaces, and let $U$ be open in $X$. Let $f$, $g : U \to Y$ be maps which are differentiable (in the sense of Gâteaux or Fréchet or strict differentiability) at $x \in U$. Then $f + g$ is differentiable at $x$ and

$$\mathcal{D}(f + g)(x) = \mathcal{D}f(x) + \mathcal{D}g(x).$$

If $c$ is a real number, then

$$\mathcal{D}(cf)(x) = c\mathcal{D}f(x).$$

**Product:** Let $X$ be a normed vector space. Let $Y$, $Z$ and $W$ be complete normed vector spaces, and let $Y \times Z \to W$ be a continuous bilinear map. Let $U$ be open in $X$ and let $f : U \to Y$, and $g : U \to Z$ be maps differentiable at $x \in U$. Then the product map $fg$ is differentiable at $x$ and for all $h \in X$, one has

$$\mathcal{D}(fg)(x) \cdot h = (\mathcal{D}f(x) \cdot h)g(x) + f(x)(\mathcal{D}g(x) \cdot h).$$

**Chain rule:** Let $X$, $Y$, $Z$ be normed vector spaces, let $U$ be open in $X$ and let $V$ be open in $Y$, let $f : U \to V$ and $g : V \to Z$ be mappings and let $h = g \circ f : U \to Z$ be the composition of the mapping $f$ and $g$. Let $\hat{x} \in U$ and $\hat{y} = f(\hat{x}) \in V$.

Assume that $f$ is differentiable (in Gâteaux or Fréchet's sense) at $\hat{x}$ and $g$ is Fréchet differentiable at $\hat{y}$. Then $h$ is differentiable (in the same sense of $f$) at $\hat{x}$ and for all $h \in X$,

$$\mathcal{D}(g \circ f)(\hat{x}) \cdot h = Dg(\hat{y})(\mathcal{D}f(\hat{x}) \cdot h). \tag{1.8}$$

If $f$ is strictly differentiable at $\hat{x}$ and $g$ is strictly differentiable at $\hat{y}$ then $h$ is strictly differentiable at $\hat{x}$.

Those are well-known properties of differentiable mapping in normed space. The proofs for them can be found for instance in [1] or [41].

**Example 1.35.** Let $f : U \to Y$ be a (Gâteaux or Fréchet or strictly) differentiable map, and let $\lambda : Y \to Z$ be a continuous linear map. Then for each $x \in U$, $\lambda \circ f$ is differentiable in the same sense of $f$ at $x$, and for every $v \in U$ we have

$$\mathcal{D}(\lambda \circ f)(x) \cdot v = \lambda(\mathcal{D}f(x) \cdot v).$$

This result follows directly from Example 1.29 and chain rule.

### 1.2.4 Mean Value Theorem

Let $X$ be normed space and let $a, b \in X$. We denote closed line segment which connects $a$ and $b$ as follows:

$$[a, b] = \{x : x = a + t(a - b),\ 0 \le t \le 1\},$$

and the open line segment $(a, b)$ is defined as folows:

$$(a, b) = \{x : x = a + t(a - b),\ 0 < t < 1\}.$$

**Theorem 1.36.** ***(Mean Value Theorem)*** *Let $X$ and $Y$ be normed linear spaces, and let an open set $U \subset X$ contains a closed line segment $[a, b]$. If $f : U \to Y$ is a Gâteaux differentiable function at each point $x \in [a, b]$, then*

$$\|f(b) - f(a)\| \le \sup_{c \in [a,b]} \|D_G f(c)\| \, \|b - a\| .$$

*Proof.* Let us take an arbitrary $y^* \in Y^*$ and consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(t) := \langle y^*,\ f(a + t(b - a))\rangle .$$

This function possesses derivatives at each point of the closed interval $[0, 1]$:

$$\begin{aligned}
\Phi'(t) &= \lim_{\lambda \to 0} \frac{\Phi(t + \lambda) - \Phi(t)}{\lambda} = \left\langle y^*,\ \lim_{\lambda \to 0} \frac{f(a + (t + \lambda)(b - a)) - f(a + t(b - a))}{\lambda} \right\rangle \\
&= \left\langle y^*,\ \lim_{\lambda \to 0} \frac{f(a + t(b - a) + \lambda(b - a)) - f(a + t(b - a))}{\lambda} \right\rangle \\
&= \langle y^*,\ D_G f(a + t(b - a)).(b - a)\rangle .
\end{aligned}$$

Hence, function $\Phi$ is differentiable (in the ordinary sense) on the closed interval $[0, 1]$ and therefore it is continuous on this interval. By Lagrange's formula, there exists $\theta \in (0, 1)$ such that

$$\langle y^*,\ f(b) - f(a)\rangle = \Phi(1) - \Phi(0) = \Phi'(\theta) = \langle y^*,\ D_G f(a + \theta(b - a)) \cdot (b - a)\rangle .$$

Now we shall make use of Corollary 1 of the Hahn-Banach Theorem in [1] (page 76), according to which, for any element $y \in Y$, there is a linear functional $y^* \in Y^*$ such that $\|y^*\| = 1$ and $\langle y^*, y\rangle = \|y\|$. Taking this functional $y^*$ for the element $y = f(b) - f(a)$, we obtain

$$\begin{aligned}
\|f(b) - f(a)\| &= \langle y^*,\ f(b) - f(a)\rangle = \langle y^*,\ D_G f(a + \theta(b - a)).(b - a)\rangle \\
&\le \|y^*\| \, \|D_G f(a + \theta(b - a))\| \, \|(b - a)\| \\
&= \|D_G f(a + \theta(b - a))\| \, \|(b - a)\| \\
&\le \sup_{c \in [a,b]} \|D_G f(c)\| \, \|b - a\| .
\end{aligned}$$

□

**Corollary 1.37.** *Let all the conditions of the Mean Value Theorem be fulfilled, and let $\Lambda \in \mathcal{L}(X, Y)$. Then*

$$\|f(b) - f(a) - \Lambda(b - a)\| \le \sup_{c \in [a,b]} \|D_G f(c) - \Lambda\| \, \|b - a\| .$$

*Proof.* The proof reduces to the application of the Mean Value Theorem to the mapping $g(x) = f(x) - \Lambda \cdot x$. □

**Corollary 1.38.** *Let $X$ and $Y$ be normed spaces, let $U$ be a neighborhood of a point $\hat{x}$ in $X$, and let a mapping $f : U \to Y$ be Gâteaux differentiable at each point $x \in U$. If the mapping $x \mapsto D_G f(x)$ is continuous at the point $\hat{x}$, then the mapping $f$ is strictly differentiable at $\hat{x}$ (and consequently it is Fréchet differentiable at that point).*

*Proof.* Given $\varepsilon > 0$, there is $\delta > 0$ such that the relation

$$\|x - \hat{x}\| < \delta \implies \|D_G f(x) - D_G f(\hat{x})\| < \varepsilon \tag{1.9}$$

holds. If $\|x_1 - \hat{x}\| < \delta$ and $\|x_2 - \hat{x}\| < \delta$, then for any $x = x_1 + t(x_2 - x_1) \in [x_1,\ x_2]$, $0 \le t \le 1$, we have

$$\begin{aligned} \|x - \hat{x}\| &= \|x_1 + t(x_2 - x_1) - \hat{x}\| = \|t(x_2 - \hat{x}) + (1-t)(x_1 - \hat{x})\| \\ &\le t\,\|x_2 - \hat{x}\| + (1-t)\,\|x_1 - \hat{x}\| < t\delta + (1-t)\delta = \delta, \end{aligned}$$

and therefore, by virtue of (1.9), we have $\|D_G f(x) - D_G f(\hat{x})\| < \varepsilon$.

Applying Corollary 1.37 to $\Lambda = D_G f(\hat{x})$, we obtain

$$\begin{aligned} \|f(x_1) - f(x_2) - D_G f(\hat{x}) \cdot (x_1 - x_2)\| &\le \sup_{x \in [x_1, x_2]} \|D_G f(x) - D_G f(\hat{x})\|\,\|x_1 - x_2\| \\ &\le \varepsilon\,\|x_1 - x_2\|, \end{aligned}$$

which implies the strict differentiability of $f$ at $\hat{x}$. □

### 1.2.5 Differentiation in a Product Space. Partial Derivatives. Theorem on The Total Differential.

In this subsection $X, Y$ and $Z$ are normed spaces. We shall begin with the case of a mapping whose values belong to the product space $Y \times Z$, i.e., $F : U \to Y \times Z$, $U \subset X$. Since a point of $Y \times Z$ is a pair $(y, z)$, the mapping $F$ also consists of two components: $F(x) = (G(x), H(x))$, where $G : U \to Y$ and $H : U \to Z$. The corresponding definitions immediately imply:

**Proposition 1.39.** *Let $X, Y$ and $Z$ be normed spaces, let $U$ be a neighborhood of a point $x$ in $X$, and let $G : U \to Y$ and $H : U \to Z$.*

*For the mapping $F = (G, H) : U \to Y \times Z$ to be differentiable at the point $x$ in the sense of Gâteaux or Fréchet or strict differentiability, it is necessary and sufficient that $G$ and $H$ possess the same property. Moreover, in this case*

$$\mathcal{D}F(x) = (\mathcal{D}G(x),\ \mathcal{D}H(x)),$$

*or for all $h \in X$, we have*

$$\mathcal{D}F(x) \cdot h = (\mathcal{D}G(x) \cdot h,\ \mathcal{D}H(x) \cdot h).$$

Now we turn to the case when the domain of the mapping $F : U \to Z$ belongs to the product space: $U \subset X \times Y$.

**Definition 1.40.** Let $X, Y$ and $Z$ be normed spaces, let $U$ be a neighborhood of a point $(\hat{x}, \hat{y})$ in $X \times Y$, and let $F : U \to Z$. If the mapping $x \mapsto F(x, \hat{y})$ is (Gâteaux or Fréchet or strictly) differentiable at the point $\hat{x}$, its differential is called the partial differantial of the mapping $F$ with respect to $x$ at the point $(\hat{x}, \hat{y})$ and is denoted $\mathcal{D}_1 F(\hat{x}, \hat{y})$. The partial differential $\mathcal{D}_2 F(\hat{x}, \hat{y})$ with respect to y is defined in an analogous manner.

**Theorem 1.41.** ***(Theorem on the Total Differential)*** *Let $X, Y$ and $Z$ be normed spaces, let $U$ be a neighborhood of a point $(\hat{x}, \hat{y})$ in $X \times Y$ and let $F : U \to Z$ be a mapping possessing the partial derivatives $\mathcal{D}_1 F(x, y)$ and $\mathcal{D}_2 F(x, y)$ in Gâteaux' sense at each point $(x, y) \in U$.*

*If the mappings $(x, y) \mapsto \mathcal{D}_1 F(x, y)$ and $(x, y) \mapsto \mathcal{D}_2 F(x, y)$ are continuous at a point $(\hat{x}, \hat{y}) \in U$ in the uniform topology of operators, then $F$ is strictly differentiable at that point, and moreover,*

$$\mathcal{D}F(\hat{x}, \hat{y}).(\xi, \eta) = \mathcal{D}_1 F(\hat{x}, \hat{y}).\xi + \mathcal{D}_2 F(\hat{x}, \hat{y}).\eta$$

*for all $(\xi, \eta) \in X \times Y$.*

*Proof.* Given an arbitrary $\varepsilon > 0$, let us choose $\delta > 0$ small enough so that the neighborhood

$$V = \{(x, y) : \|x - \hat{x}\| < \delta,\ \|y - \hat{y}\| < \delta\}$$

of the point $(\hat{x}, \hat{y})$ is contained in $U$ and the inequalities

$$\|\mathcal{D}_1 F(x, y) - \mathcal{D}_1 F(\hat{x}, \hat{y})\| < \varepsilon,\ \|\mathcal{D}_2 F(x, y) - \mathcal{D}_2 F(\hat{x}, \hat{y})\| < \varepsilon \tag{1.10}$$

hold in that neighborhood. Now we have

$$\begin{aligned}
\Delta &= F(x_1, y_1) - F(x_2, y_2) - \mathcal{D}_1 F(\hat{x}, \hat{y})(x_1 - x_2) - \mathcal{D}_2 F(\hat{x}, \hat{y})(y_1 - y_2) \\
&= (F(x_1, y_1) - F(x_1, y_2) - \mathcal{D}_1 F(\hat{x}, \hat{y})(x_1 - x_2)) \\
&\quad + (F(x_1, y_2) - F(x_2, y_2) - \mathcal{D}_2 F(\hat{x}, \hat{y})(y_1 - y_2)).
\end{aligned}$$

It can readily be seen that if the points $(x_1, y_1), (x_2, y_2)$ belong to $V$, then $(x_2,\ y_1) \in V$, and moreover the line segments $[(x_1, y_1), (x_2, y_1)]$ and $[(x_2, y_1),\ (x_2, y_2)]$ are contained in $V \subset U$ . Therefore, the functions $x \mapsto F(x, y_1)$ and $y \mapsto F(x_2, y)$ are Gâteaux differentiable: the first of them possesses the differential $\mathcal{D}_1 F$ on $[x_1, x_2]$ and the other possesses the differential $\mathcal{D}_2 F$ on $[y_1, y_2]$. Applying the Mean Value Theorem to these functions we obtain, by virtue of (1.10), the relations

$$(x_1, y_1), (x_2, y_2) \in V \implies$$

$$\begin{aligned}
\|\Delta\| &\le \sup_{z \in [x_1,\ x_2]} \|\mathcal{D}_1 F(z, y_1) - \mathcal{D}_1 F(\hat{x}, \hat{y})\| \, \|x_1 - x_2\| \\
&\quad + \sup_{w \in [y_1,\ y_2]} \|\mathcal{D}_2 F(x_2, w) - \mathcal{D}_2 F(\hat{x}, \hat{y})\| \, \|y_1 - y_2\| \\
&\le \varepsilon(\|x_1 - x_2\| + \|y_1 - y_2\|).
\end{aligned}$$

And so, $F$ is strictly differentiable at $(\hat{x}, \hat{y})$. □

# Chapter 2

# Infinite-Horizon Optimal Control Problem in Presence of Asymptotical Constraint and a Weak Pontryagin Principle

## 2.1 Introduction

The aim of this chapter is to introduce the infinite-horizon optimal control problem in discrete time framework with asymptotical constraint and to establish for it necessary condition of optimality in the form of weak Pontryagin principle. We consider the following optimal control problem

$$\left.\begin{array}{rl} \text{Maximize} & K(\underline{y},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \psi(y_t,u_t) \\ \text{when} & \underline{y} = (y_t)_{t\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}\ ,\ \underline{u} = (u_t)_{t\in\mathbb{N}} \in U^{\mathbb{N}}, \\ & y_0 = \eta,\ \lim_{t\to+\infty} y_t = y_\infty, \\ & \underline{u} \text{ is bounded}, \\ & \forall t \in \mathbb{N},\ y_{t+1} = g(y_t,u_t), \end{array}\right\} \qquad (P_m)$$

where $U \subset \mathbb{R}^d$ is nonempty; $\beta \in (0,1)$; $\eta,\ y_\infty \in \mathbb{R}^n$ fixed; function $\psi : \mathbb{R}^n \times U \to \mathbb{R}$ and function $g : \mathbb{R}^n \times U \to \mathbb{R}^n$, and $(\mathbb{R}^n)^{\mathbb{N}}$ (respectively $U^{\mathbb{N}}$) denotes the set of the sequences in $\mathbb{R}^n$ (respectively $U$). In comparison with existing results on bounded processes like [14] and [16], the specificness of the present work is the presence of the asymptotical constraint on the state variable $\lim_{t\to+\infty} y_t = y_\infty$. Its meaning is that the optimal state of the problem stays near a "good" or "expected" value on the long run.

The approach to this problem is functional analytic; we translate our problem into static form of optimization in suitable Banach sequence spaces. We describe the content of this chapter as follows.

- In Section 2.2 we introduce a problem of optimal control which is equivalent to the initial problem and is defined in the following classical sequence spaces: $c_0(\mathbb{N},\mathbb{R}^n)$ the space of all sequences in $(\mathbb{R}^n)^{\mathbb{N}}$ which converge to zero at infinity, and $\ell^\infty(\mathbb{N},U)$ the space of all sequences in $U^{\mathbb{N}}$ which are bounded.

- In Section 2.3 we study the properties of operators and functionals on sequence spaces. A first novelty is a characterization of the operators which send $c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ into $c_0(\mathbb{N},\mathbb{R}^n)$ (Theorem 2.2). The other results use this characterization and existing results on Nemytskii operators from $\ell^\infty(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ into $\ell^\infty(\mathbb{N},\mathbb{R}^m)$.

- Section 2.4 is devoted to the solutions which converge toward zero of a linear difference equation. These results are useful to establish regularity properties of the differential of operators formalizing the nonlinear difference equation which governs the dynamical system.

- In Section 2.5 we establish a variation of a Karush-Kuhn-Tucker theorem which is useful for weak Pontryagin principles in later subsections.

- In Section 2.6 and Section 2.7 we establish weak Pontryagin principles for the considered problems.

## 2.2 The Supporting Problem

In this section we introduce a supporting problem which can be translated equivalently into the form of Problem $(P_m)$. The purpose of this is that with the supporting problem, we can work in classical Banach sequence spaces. We consider the following optimal control problem:

$$\left.\begin{array}{rl} \text{Maximize} & J(\underline{x},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \phi(x_t,u_t) \\ \text{when} & \underline{x} = (x_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n), \\ & \underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},U), \\ & x_0 = \eta, \\ & \forall t \in \mathbb{N},\ x_{t+1} = f(x_t,u_t). \end{array}\right\} \qquad (P_s)$$

Here $U$, $\beta$ and $\eta$ are exactly the same like those in the main problem. Functions $\phi$ and $f$ are defined like functions $\psi$ and $g$, respectively. Notice that Problem $(P_m)$ can be translated into the form of Problem $(P_s)$ by using the following transformation

- Let for all $t \in \mathbb{N}$, $z_t := y_t - y_\infty$. Then $\underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n)$ since $\lim_{t\to+\infty} z_t = \lim_{t\to+\infty}(y_t - y_\infty) = 0$.

- Now $K(\underline{y},\underline{u}) = \sum_{t=0}^{+\infty} \beta^t \psi(y_t,u_t) = \sum_{t=0}^{+\infty} \beta^t \psi(z_t + y_\infty, u_t)$. We set $N(\underline{z},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \varphi(z_t,u_t)$ where $\varphi(z,u) := \psi(z + y_\infty, u)$ for all $(z,u) \in \mathbb{R}^n \times U$ then $K(\underline{y},\underline{u}) = N(\underline{z},\underline{u})$.

- Finally, we set $\ell(z,u) := g(z + y_\infty, u) - y_\infty$ for all $(z,u) \in \mathbb{R}^n \times U$ then for all $t \in \mathbb{N}$, from $y_{t+1} = g(y_t,u_t)$ we obtain the equivalent equation $z_{t+1} = \ell(z_t,u_t)$.

After these settings, Problem $(P_m)$ becomes:

$$\left.\begin{array}{rl} \text{Maximize} & N(\underline{z},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \varphi(z_t,u_t) \\ \text{when} & \underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n), \\ & \underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},U), \\ & z_0 = \eta - y_\infty, \\ & \forall t \in \mathbb{N},\ z_{t+1} = \ell(z_t,u_t). \end{array}\right\} \qquad (P1)$$

This problem has precisely the form of Problem $(P_s)$.

## 2.3 Some Useful Properties of Nemytskii Operators

This section is devoted to the study of several operators between sequence spaces; notably the Nemytskii operators, and to the study of functionals and the mappings which define the criterion and the dynamical system of our maximization problems. We also establish results of continuity and Fréchet differentiability for those operators.

**Lemma 2.1.** *Let $\zeta$ be a mapping from $X \times V$ into $Y$ where $X$, $V$ and $Y$ are normed spaces and $U$ be a nonempty subset of $V$. Then from*

**(i)** *For all $B$ bounded, nonempty in $U$:* $\lim_{x\to 0}(\sup_{u\in B} \|\zeta(x,u)\|) = 0$,

*we obtain*

**(ii)** *For all $\underline{x} \in c_0(\mathbb{N}, X)$, for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$ : $(\zeta(x_t, u_t))_{t\in\mathbb{N}} \in c_0(\mathbb{N}, Y)$ or equivalently,* $\lim_{t\to+\infty} \zeta(x_t, u_t) = 0$.

*Proof.* Let $\underline{x} \in c_0(\mathbb{N}, X)$ and $\underline{u} \in \ell^\infty(\mathbb{N}, U)$. Take $B = \{u_t,\ t \in \mathbb{N}\}$, then $B$ is bounded since $(u_t)_t \in \ell^\infty(\mathbb{N}, U)$. Using the hypothesis (i) of this lemma, we know that for this set $B$, $\lim_{x\to 0}(\sup_{u\in B} \|\zeta(x,u)\|) = 0$. It means that with $\varepsilon > 0$ arbitrarily, there exists $\delta_\varepsilon > 0$ such that for all $x \in X$, $\sup_{u\in B} \|\zeta(x,u)\| < \varepsilon$ when $\|x\| < \delta_\varepsilon$ which also means that for all $t \in \mathbb{N}$, $\|\zeta(x, u_t)\| < \varepsilon$ when $\|x\| < \delta_\varepsilon$. From $(x_t)_t \in c_0(\mathbb{N}, X)$ we obtain $\lim_{t\to+\infty} x_t = 0$. It means that with $\delta_\varepsilon > 0$ above, there exists $T \in \mathbb{R}_+$ such that for all $t \in \mathbb{N}$, when $t > T$ we have $\|x_t\| < \delta_\varepsilon$.
From these arguments, we derive that with $\varepsilon > 0$ arbitrarily, there exists $T \in \mathbb{R}_+$ such that for all $t \in \mathbb{N}$, when $t > T$ we have $\|x_t\| < \delta_\varepsilon$ and hence, $\|\zeta(x_t, u_t)\| < \varepsilon$. And so, we have proven $\lim_{t\to\infty} \zeta(x_t, u_t) = 0$ and we obtain $(\zeta(x_t, u_t))_{t\in\mathbb{N}} \in c_0(\mathbb{N}, Y)$. □

**Theorem 2.2.** *Let $X$, $V$, $Y$ be three normed spaces, $U$ be a nonempty subset of $V$ and $\zeta : X \times U \to Y$ be a mapping such that, for all $x \in X$, the partial mapping $u \mapsto \zeta(x, u)$ transforms the bounded subsets of $U$ into bounded subsets of $Y$. Then the assertions (i) and (ii) of Lemma 2.1 are equivalent.*

*Proof.* Using Lemma 2.1 we have $(i) \implies (ii)$. Now we prove the converse implication $(ii) \implies (i)$. Let $B$ be nonempty bounded subset of $U$. Let $\underline{x} \in c_0(\mathbb{N}, X)$. From the assumption on $\zeta$, we know that, for all $t \in \mathbb{N}$, we have $\sup_{u\in B} \|\zeta(x_t, u)\| < +\infty$. Therefore, for all $t \in \mathbb{N}$, there exists $u_t \in B$ such that

$$0 \le \sup_{u\in B} \|\zeta(x_t, u)\| \le \|\zeta(x_t, u_t)\| + \frac{1}{t+1}.$$

Note that for all $t \in \mathbb{N}$, $u_t \in B$, then $\underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, U)$. Using (ii), we obtain $\lim_{t\to+\infty} \|\zeta(x_t, u_t)\| = 0$ and from previous inequality we obtain $\lim_{t\to+\infty} \sup_{u\in B} \|\zeta(x_t, u)\| = 0$, and since we work in normed spaces we can use the sequential characterization of the limit (which will be proven in Remark 2.6) and assert that we obtain (i). □

**Remark 2.3.** Assertion (ii) of Lemma 2.1 permits to define the Nemytskii operator

$$N_\zeta : c_0(\mathbb{N}, X) \times \ell^\infty(\mathbb{N}, U) \to c_0(\mathbb{N}, Y),\ N_\zeta(\underline{x}, \underline{u}) := (\zeta(x_t, u_t))_{t\in\mathbb{N}}.$$

*Proof.* Let $\underline{x} \in c_0(\mathbb{N}, X)$ and $\underline{u} \in \ell^\infty(\mathbb{N}, U)$. From assertion (ii) of Lemma 2.1 we have $\lim_{t\to+\infty} \zeta(x_t, u_t) = 0$. It means that $N_\zeta(\underline{x}, \underline{u}) = (\zeta(x_t, u_t))_{t\in\mathbb{N}} \in c_0(\mathbb{N}, Y)$. Hence, $N_\zeta$ is well defined. □

**Remark 2.4.** Let $B_R := \{v \in V :\ \|v\| \le R\}$ when $R \in (0, +\infty)$. We have the following statements:

1. In the setting of Theorem 2.2, the assumption on $\zeta$ is equivalent to the following condition:
$$\forall x \in X,\ \forall R \in (0, +\infty),\quad \sup_{u\in U\cap B_R} \|\zeta(x, u)\| < +\infty.$$

2. The assertion (i) of Lemma 2.1 is equivalent to

$$\forall R \in (0,+\infty),\ \lim_{x\to 0}\left(\sup_{u\in U\cap B_R}\|\zeta(x,u)\|\right)=0.$$

3. It is also noticed that assertion (i) of Lemma 2.1 and the continuity of $[x \mapsto \zeta(x,u)]$ for all $u \in U$ imply $\zeta(0,u)=0$ for all $u \in U$.

*Proof.* We will prove these assertions one after another.

1. Let $\zeta$ is a function that satisfies the assumption of Theorem 2.2. It is obvious that $B_R := \{v \in V :\ \|v\| \le R\}$ is bounded for all $R \in (0,+\infty)$. Then $B := U \cap B_R$ is a bounded subset of $U$ and hence, for all $x \in X$, $\zeta(x,.)$ transforms $B$ into bounded subset of $Y$ which means that for all $x \in X$, $\sup_{u\in U\cap B_R}\|\zeta(x,u)\| < +\infty$. On the contrary, let $B$ be a bounded subset of $U$ then there exists $R \in (0,+\infty)$ such that $B \subset (B_R \cap U)$. Hence, from $\left(\forall x \in X,\ \sup_{u\in U\cap B_R}\|\zeta(x,u)\| < +\infty\right)$ we obtain $(\forall x \in X,\ \sup_{u\in B}\|\zeta(x,u)\| < +\infty)$. Therefore, for all $x \in X$, the partial mapping $u \to \zeta(x,u)$ transforms the bounded subsets of $U$ into bounded subsets of $Y$.
2. The reasoning is similar to 1.
3. Since $B = \{u\}$ is a bounded set for all $u \in U$, we obtain $\|\zeta(0,u)\| = \lim_{x\to 0}\|\zeta(x,u)\| = \lim_{x\to 0}(\sup_{u\in\{u\}}\|\zeta(x,u)\|) = 0$. Hence, for all $u \in U$, $\zeta(0,u)=0$.

□

**Remark 2.5.** In the setting of Theorem 2.2, if in addition, we assume that $\dim V < +\infty$ and $U$ is closed then if $\zeta(x,.) \in C^0(U,Y)$, $\zeta(x,.)$ transforms the bounded subsets of $U$ into bounded subsets of $Y$.

*Proof.* Let $B$ be a bounded subset of $U$. Since $U$ is closed, $\overline{B}$, the closure of $B$, is also a subset of $U$ and since $B$ is bounded, $\overline{B}$ is compact. Now if $\zeta(x,.) \in C^0(U,Y)$, then $\zeta(x,.)$ is continuous on $\overline{B}$. Using Theorem 4.14 on page 89 of [53], we obtain $\{\zeta(x,u),\ u \in \overline{B}\}$ is also compact in $Y$. Therefore, its subset $\{\zeta(x,u),\ u \in B\}$ is bounded. □

**Remark 2.6.** Let $\zeta \in C^0(X \times V, Y)$ where $X$, $V$ and $Y$ are finite-dimensional Banach spaces and $U$ be a nonempty closed subset of $V$. Then the following statement are equivalent:

**(i)** For all $B$ bounded, nonempty in $U$, $\lim_{x\to 0}(\sup_{u\in B}\|\zeta(x,u)\|) = 0$.

**(ii)** For all $(u_t)_t \in \ell^\infty(\mathbb{N},U)$, For all $(x_t)_t \in c_0(\mathbb{N},X)$, $\lim_{t\to+\infty}\zeta(x_t,u_t)=0$.

**(iii)** For all $B$ bounded, nonempty in $U$, for all $(x_t)_t \in c_0(\mathbb{N},X)$,

$$\lim_{t\to+\infty}(\sup_{u\in B}\|\zeta(x_t,u)\|)=0.$$

**(iv)** For all $K$ compact in $U$, $\lim_{x\to 0}(\sup_{u\in K}\|\zeta(x,u)\|)=0$.

*Proof.* We will prove the equivalent of (i) and (ii), then we will prove that (i) $\iff$ (iii) and finally, (i) $\iff$ (iv).

- Prove (i) $\iff$ (ii): it is a direct consequence of Remark 2.5 and Theorem 2.2.
- Prove (i) $\iff$ (iii)
  - (i) $\implies$ (iii): suppose that we have (i). Let $B$ be some bounded nonempty set in $U$ and sequence $(x_t)_t \in c_0(\mathbb{N},X)$. Now $\lim_{x\to 0}(\sup_{u\in B}\|\zeta(x,u)\|)=0$ means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{u\in B}\|\zeta(x,u)\| < \varepsilon$ when $\|x\| < \delta$. Now since $(x_t)_t \in c_0(\mathbb{N},X)$ then with $\delta$ above there exists $T \in \mathbb{R}_+$ such that when $t > T$, $t \in \mathbb{N}$ we have $\|x_t\| < \delta$ and hence, $\sup_{u\in B}\|\zeta(x_t,u)\| < \varepsilon$ from the above-mentioned argument. And so $\lim_{t\to+\infty}(\sup_{u\in B}\|\zeta(x_t,u)\|)=0$.

- (iii) $\Longrightarrow$ (i): Let $B$ be a bounded nonempty set in space $U$. For all $x \in X$, we set $S(x) := \sup_{u\in B} \|\zeta(x,u)\|$. Obviously, $S(x) \geq 0$ on $X$. Now we can rewrite (iii) $\Rightarrow$ (i) as follows

$$\forall (x_t)_t \in c_0(\mathbb{N}, X), \lim_{t\to+\infty} S(x_t) = 0 \ \Rightarrow \lim_{x\to 0} S(x) = 0.$$

We will prove this implication by contradiction. Suppose that we have (ii) and not (i). We have

$$\begin{aligned} \text{not (i)} &\Leftrightarrow \text{not}[\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0,\ \forall x \in X,\ \|x\| \leq \delta_\varepsilon \Rightarrow S(x) \leq \varepsilon] \\ &\Leftrightarrow [\exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x_\delta \in X,\ \|x_\delta\| \leq \delta \text{ and } S(x_\delta) > \varepsilon]. \end{aligned}$$

We take $\delta = \frac{1}{n}$ when $n \in \mathbb{N}_*$. Then from the meaning of not (i), the following statement holds

$$[\exists \varepsilon > 0,\ \forall n \in \mathbb{N}_*,\ \exists z_n := x_{\frac{1}{n}} \in X,\ \|z_n\| \leq \frac{1}{n} \text{ and } S(z_n) > \varepsilon].$$

Then there exists $(z_n)_{n\in\mathbb{N}_*} \in c_0(\mathbb{N}, X)$ such that $S(z_n) > \varepsilon$ for all $n \in \mathbb{N}_*$. It means that $\lim_{n\to+\infty} S(z_n) \geq \varepsilon > 0$. Using (ii), since $(z_n)_{n\in\mathbb{N}_*} \in c_0(\mathbb{N}, X)$, we have $\lim_{n\to+\infty} S(z_n) = 0$. Hence, we obtain the contradiction and so the implication (iii) $\Rightarrow$ (i) is proven.

- Prove (i) $\Longleftrightarrow$ (iv):
  - First, we prove that (i) $\Longrightarrow$ (iv). Let $K$ be some compact set in $U$. Then $K$ is bounded. From (i) we immediately obtain the result of (iv).
  - Now on the other hand, suppose that, for all $K$ compact in $U$, $\lim_{x\to 0}(\sup_{u\in K} \|\zeta(x,u)\|) = 0$. Let $B$ be some bounded nonempty subset of $U$. Then its closure $\bar{B}$ is compact and $\bar{B} \subset U$ since $\dim V < +\infty$ and $U$ is closed. From (iv) we know that $\lim_{x\to 0}(\sup_{u\in\bar{B}} \|\zeta(x,u)\|) = 0$. Hence, we only need to prove that for all $x \in X$, $\sup_{u\in\bar{B}} \|\zeta(x,u)\| = \sup_{u\in B} \|\zeta(x,u)\|$.
    When $x \in X$, we set $\psi(u) := \|\zeta(x,u)\|$ then $\psi$ is continuous on $U$ since $\|\cdot\|$ and $\zeta(x,.)$ are continuous. It is known that $\sup \psi(B) = \sup_{u\in B} \|\zeta(x,u)\| \leq \sup_{u\in\bar{B}} \|\zeta(x,u)\| = \sup \psi(\bar{B})$.
    Now $\bar{B}$ is compact and then from Weierstrass theorem there exists $\hat{u} \in \bar{B}$ such that $\psi(\hat{u}) = \sup \psi(\bar{B})$. Since $\hat{u} \in \bar{B}$, there exists a sequence $(u_n)_n \in B^{\mathbb{N}}$ such that $\lim_{n\to+\infty} u_n = \hat{u}$. Since $\psi$ is continuous, we deduce that $\lim_{n\to+\infty} \psi(u_n) = \psi(\hat{u})$. Hence, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\psi(\hat{u}) \leq \psi(u_n) + \varepsilon \leq \sup \psi(B) + \varepsilon$. Then for all $\varepsilon > 0$, $\sup \psi(\bar{B}) \leq \sup \psi(B) + \varepsilon$. Let $\varepsilon \to 0$, we have $\sup \psi(\bar{B}) \leq \sup \psi(B)$.
    From the above arguments, we can conclude that $\sup \psi(B) = \sup \psi(\bar{B})$ which means that for all $x \in X$, $\sup_{u\in\bar{B}} \|\zeta(x,u)\| = \sup_{u\in B} \|\zeta(x,u)\|$. Hence, from $\lim_{x\to 0}(\sup_{u\in\bar{B}} \|\zeta(x,u)\|) = 0$ we have $\lim_{x\to 0}(\sup_{u\in B} \|\zeta(x,u)\|) = 0$.

□

Let $U$ be a nonempty closed subset of $\mathbb{R}^d$. Let us call

($\alpha$.1) $\zeta \in C^o(\mathbb{R}^n \times U, \mathbb{R}^m)$ ;

($\alpha$.2) For all $B$ bounded, nonempty in $U$: $\lim_{x\to 0}(\sup_{u\in B} \|\zeta(x,u)\|) = 0$.

Now we introduce some important facts before proving the continuity of $N_\zeta$. Let us fix an arbitrary point $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Since $\underline{x} \in c_0(\mathbb{N}, \mathbb{R}^n) \subset \ell^\infty(\mathbb{N}, \mathbb{R}^n)$ and $\underline{u} \in \ell^\infty(\mathbb{N}, U)$ we can assert that $\{(x_t, u_t) : t \in \mathbb{N}\}$ is bounded. Moreover, since $\dim(\mathbb{R}^n \times U) < +\infty$ then $K := \overline{\{(x_t, u_t) : t \in \mathbb{N}\}}$, the closure of $\{(x_t, u_t) : t \in \mathbb{N}\}$, is

compact and $K \subset \mathbb{R}^n \times U$ since $U$ is closed. Using a property of locally compact metric spaces, we know that since $\dim \mathbb{R}^n \times U < +\infty$, there exists $\rho \in (0, +\infty)$ such that $L := \overline{\{\nu = (x,u) \in \mathbb{R}^n \times U : d(\nu, K) < \rho\}}$, the closure of $\{\nu = (x,u) \in \mathbb{R}^n \times U : d(\nu, K) < \rho\}$, is also a compact set (see [26], page 65) and it is contained in $\mathbb{R}^n \times U$ since $U$ is closed. Here $d(\nu, K)$ is the distance from a point $\nu$ to set $K$ which is defined by $d(\nu, K) := \inf_{\nu' \in K} \|\nu - \nu'\|$.

**Remark 2.7.** Let $U$ be a nonempty closed subset of $\mathbb{R}^d$. Under condition $(\alpha.1)$, $\zeta$ is uniformly continuous from $L$ into $\mathbb{R}^m$ where $L$ is defined as above.

*Proof.* We recall Heine - Cantor's theorem: if $\zeta : M \to N$ is a continuous mapping between two metric spaces and $M$ is compact then $\zeta$ is uniformly continuous on $M$ (see Theorem 4.19, page 91 of [53]). Under $(\alpha.1)$ we have $\zeta : L \to \mathbb{R}^m$ is a continuous mapping. Besides, $L$ is compact as already proven above. Hence, apply Theorem Heine - Cantor for our case, we deduce that that $\zeta$ is uniformly continuous from $L$ into $\mathbb{R}^m$. It means that

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0,\ \forall \nu, \nu' \in L,\ \|\nu - \nu'\| < \delta_\varepsilon \Rightarrow \|\zeta(\nu) - \zeta(\nu')\| < \varepsilon.$$

□

We will use these facts to prove the continuity of operator $N_\zeta$ in the following lemma.

**Theorem 2.8.** *Let $U$ be a nonempty closed subset of $\mathbb{R}^d$ and let $\zeta : \mathbb{R}^n \times U \to \mathbb{R}^m$ be a mapping that satisfies conditions $(\alpha.1)$ and $(\alpha.2)$. Then the Nemytskii operator $N_\zeta$ is well defined and is continuous on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, i.e. $N_\zeta \in C^0(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U); c_0(\mathbb{N}, \mathbb{R}^m))$.*

*Proof.* It can be seen that condition $(\alpha.1)$ is the condition on $\zeta$ and condition $(\alpha.2)$ is the assertion (i) in Remark 2.6. Then using Remark 2.6 we can deduce that $N_\zeta$ is well defined. Now we prove its continuity. Take $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ arbitrarily. Then $K = \overline{\{(x_t, u_t) : t \in \mathbb{N}\}}$ is compact and there exists $\rho \in (0, +\infty)$ such that $L = \overline{\{\nu = (x,u) \in \mathbb{R}^n \times U : d(\nu, K) < \rho\}}$ is also compact. From Remark 2.7 we know that $\zeta$ is uniformly continuous on $L$. Then for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $\nu, \nu' \in L$, if $\|\nu - \nu'\| < \delta_\varepsilon$ then $\|\zeta(\nu) - \zeta(\nu')\| < \varepsilon$.

Let $(\underline{x}', \underline{u}') \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ such that $\|\underline{x} - \underline{x}'\| + \|\underline{u} - \underline{u}'\| < \delta := \min\{\delta_\varepsilon, \rho\}$. From this we obtain for all $t \in \mathbb{N}$, $\|x_t - x'_t\| + \|u_t - u'_t\| < \rho$ . Hence, for all $t \in \mathbb{N}$, if we set $\nu_t := (x_t, u_t)$ and $\nu'_t := (x'_t, u'_t)$ then from the definition of set $L$ we see that for all $t \in \mathbb{N}$, $\nu_t$ and $\nu'_t$ belong to $L$ (because $d(\nu_t, K) = 0$ and $d(\nu'_t, K) \le d(\nu_t, \nu'_t) = \|x_t - x'_t\| + \|u_t - u'_t\| < \rho$).

On the other hand, since for all $t \in \mathbb{N}$, $\nu_t$ and $\nu'_t$ belong to $L$ and $\|\nu_t - \nu'_t\| = \|x_t - x'_t\| + \|u_t - u'_t\| < \delta_\varepsilon$ then $\|\zeta(\nu_t) - \zeta(\nu'_t)\| < \varepsilon$. This inequality is satisfied for all $t \in \mathbb{N}$, then $\sup_{t \in \mathbb{N}} \|\zeta(x_t, u_t) - \zeta(x'_t, u'_t)\| \le \varepsilon$ or equivalently, $\|N_\zeta(\underline{x}, \underline{u}) - N_\zeta(\underline{x}', \underline{u}')\| \le \varepsilon$. The last inequality holds for all $(\underline{x}', \underline{u}') \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ satisfying $\|\underline{x} - \underline{x}'\| + \|\underline{u} - \underline{u}'\| < \delta$ and it means that operator $N_\zeta$ is continuous at $(\underline{x}, \underline{u})$. From the arbitrary choice of $(\underline{x}, \underline{u})$ in $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ we obtain $N_\zeta \in C^0(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U); c_0(\mathbb{N}, \mathbb{R}^m))$. □

**Remark 2.9.** Let $X$, $V$, $W$ be real Banach spaces, and $U$ be a nonempty subset of $V$. Let $F : X \times U \to W$ be a mapping. We say that $F$ is of class $C^1$ at $(x, u) \in X \times U$ when there exist an open neighborhood $N_{x,u}$ of $(x, u)$ in $X \times V$ and a mapping $F_1 \in C^1(N_{x,u}, W)$ such that $F_{1|_{N_{x,u} \cap X \times U}} = F_{|_{N_{x,u} \cap X \times U}}$. In finite dimension, it is showed that if $F$ is of class $C^1$ at each point of $X \times U$, then there exist an open subset $G$ of $V$ and a mapping $\tilde{F} \in C^1(X \times G, W)$ such that $U \subset G$ and $\tilde{F}_{|_{X \times U}} = F$ (see [46], page 7).

**Remark 2.10.** Recall that $U$ is star-shaped with respect to $u^0$ means that, for all $u \in U$, the segment $[u^0, u]$ is included in $U$ (see page 93 of [58]). It is also noticed that if $U$ is convex then $U$ is star-shaped with respect to all of its elements.

**Remark 2.11.** When $F_1,\ F_2 \in C^1(X \times G, W)$ such that $F_{1|_{X\times U}} = F_{2|_{X\times U}} = F$, when $U$ is star-shaped with respect to $u^0$, when $u,\ u^0 \in U$ and $x^0 \in X$, note that, for all $\theta \in (0,1)$, we have $F_1(x^0, (1-\theta)u^0 + \theta u) = F_2(x^0, (1-\theta)u^0 + \theta u) = F(x^0, (1-\theta)u^0 + \theta u)$. Therefore we have $D_2F_1(x^0, u^0)(u-u^0) = \frac{d}{d\theta}_{|\theta=0} F_1(x^0, (1-\theta)u^0+\theta u) = \frac{d}{d\theta}_{|\theta=0} F_2(x^0, (1-\theta)u^0+\theta u) = D_2F_2(x^0, u^0)(u-u^0)$, and so $D_2F(x^0, u^0)(u-u^0)$ does not depend of extension of $F$.

To treat the differentiability of Nemytskii operator $N_\zeta$, we introduce the two following conditions:

($\alpha$.3) For all $x \in \mathbb{R}^n$ and for all $u \in U$, the Fréchet differential $D\zeta(x,u)$ exists and $D\zeta$ is of class $C^0$ on $\mathbb{R}^n \times U$;

($\alpha$.4) For all $B$ bounded, nonempty in $U$: $\lim_{x\to 0}(\sup_{u\in B} \|D\zeta(x,u)\|) = 0$.

Under ($\alpha$.1), ($\alpha$.3) and ($\alpha$.4), condition ($\alpha$.2) can be lightened into a weaker one which is shown in the following remark.

**Remark 2.12.** Let $U$ be a nonempty closed subset of $\mathbb{R}^d$ and let $\zeta : \mathbb{R}^n \times U \to \mathbb{R}^m$ be a mapping which belongs to class $C^1(\mathbb{R}^n \times U,\ \mathbb{R}^m)$. Consider the following assumptions.

**(i)** $\lim_{x\to 0}(\sup_{u\in B} \|D\zeta(x,u)\|) = 0$ for all $B$ bounded, nonempty in $U$.

**(ii)** There exists $u^0 \in U$ such that $\zeta(0, u^0) = 0$ and $U$ is star-shaped with respect to $u^0$.

If (i) and (ii) hold, then we obtain $\lim_{x\to 0}(\sup_{u\in B} \|\zeta(x,u)\|) = 0$ for all $B$ bounded, nonempty in $U$.

*Proof.* Let $\zeta$ be a mapping which satisfies all the assumptions of this remark. It is noticed that in this remark, assumption $\zeta \in C^1(\mathbb{R}^n \times U,\ \mathbb{R}^m)$ is equivalent to $\zeta$ satisfies ($\alpha$.1) and ($\alpha$.3); assumption (i) is ($\alpha$.4); and the conclusion is ($\alpha$.2). Let $B$ be a nonempty bounded set in $U$ then there exists $R \in (0, +\infty)$ such that $\left(B \cup \{u^0\}\right) \subset B_R$ where $B_R = \{u \in \mathbb{R}^d : \|u\| \le R\}$. Using Mean Value Theorem, for all $u \in B$ and for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\|\zeta(x,u)\| - \left\|\zeta(x,u^0)\right\| &\le \left\|\zeta(x,u) - \zeta(x,u^0)\right\| \\ &\le \sup_{w\in[u^0,u]} \|D_2\zeta(x,w)\| \left\|u - u^0\right\| \le 2R. \sup_{w\in B_R\cap U} \|D\zeta(x,w)\|\end{aligned}$$

since $\sup_{w\in[u,u^0]} \|D_2\zeta(x,w)\| \le \sup_{w\in[u,u^0]} \|D\zeta(x,w)\| \le \sup_{w\in B_R\cap U} \|D\zeta(x,w)\|$. Here we notice that $U$ contains segment $[u^0, u]$ since $U$ is star-shaped with respect to $u^0$ and hence, for all $x \in \mathbb{R}^n$, $D\zeta(x,.)$ exists at any $w \in [u^0, u]$. Besides, $B_R \cap U$ is nonempty bounded subset of $U$ since it contains $u^0$ and $B_R \cap U$ is a subset of $B_R$. The last inequality is equivalent to the following

$$\|\zeta(x,u)\| \le \left\|\zeta(x,u^0)\right\| + 2R. \sup_{w\in B_R\cap U} \|D\zeta(x,w)\|.$$

This inequality holds for all $u \in B$, hence

$$\sup_{u\in B} \|\zeta(x,u)\| \le \left\|\zeta(x,u^0)\right\| + 2R. \sup_{w\in B_R\cap U} \|D\zeta(x,w)\|. \tag{2.1}$$

Using continuity of $\zeta$ and assumption (ii) of this remark, we obtain

$$\lim_{x\to 0} \zeta(x, u^0) = \zeta(0, u^0) = 0,$$

and using assumption (i), since $(B_R \cap U)$ is a nonempty bounded subset of $U$, we obtain $\lim_{x\to 0}(\sup_{w\in B_R\cap U} \|D\zeta(x,w)\|) = 0$. Hence, if we take limit when $x \to 0$ on both sides of (2.1), we obtain

$$\lim_{x\to 0}\left(\sup_{u\in B}\|\zeta(x,u)\|\right) = 0.$$

□

We will call the weaker condition as follows.

($\alpha.2'$) There exists $u^0 \in U$ such that $\zeta(0, u^0) = 0$ and $U$ is star-shaped with respect to $u^0$.

From these conditions, we have following theorem on the differentiability of Nemytskii operator.

**Theorem 2.13.** *Let $U$ be a nonempty closed subset of $\mathbb{R}^d$ and let $\zeta : \mathbb{R}^n \times U \to \mathbb{R}^m$ be a mapping which satisfies conditions ($\alpha.1$), ($\alpha.2'$), ($\alpha.3$) and ($\alpha.4$). Then $N_\zeta \in C^1(c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U); c_0(\mathbb{N},\mathbb{R}^m))$. Moreover, for all $(\underline{x},\underline{u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ and for all $(\underline{\delta x},\underline{\delta u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$, we have*

$$\begin{aligned} DN_\zeta(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u}) &= (D\zeta(x_t,u_t).(\delta x_t,\delta u_t))_{t\in\mathbb{N}} \\ &= (D_1\zeta(x_t,u_t).\delta x_t + D_2\zeta(x_t,u_t).\delta u_t)_{t\in\mathbb{N}} \end{aligned}$$

*where $D_1$ and $D_2$ denote the partial Fréchet differentiations.*

*Proof.* Under ($\alpha.1$), ($\alpha.2'$), ($\alpha.3$) and ($\alpha.4$), after Remark 2.12, we can assert that $\zeta$ satisfies condition ($\alpha.2$). Under conditions ($\alpha.1$) and ($\alpha.2$), using Theorem 2.8, we obtain $N_\zeta \in C^0(c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U); c_0(\mathbb{N},\mathbb{R}^m))$. Now we define a new Nemytskii operator $N_{D\zeta}$ on $c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ by setting $N_{D\zeta}(\underline{x},\underline{u}) := (D\zeta(x_t,u_t))_{t\in\mathbb{N}}$ for all $(\underline{x},\underline{u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$. Under conditions ($\alpha.3$) and ($\alpha.4$), by realizing a similar interpretation as in the proof of Theorem 2.8, we obtain $N_{D\zeta}$ is of class $C^0$ on $c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$.
Take $(\underline{x},\underline{u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ and $\varepsilon > 0$ arbitrarily. Since $N_{D\zeta}$ is continuous at $(\underline{x},\underline{u})$, there exists $\delta_\epsilon > 0$ such that for all $(\underline{x}',\underline{u}') \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ satisfying $\|\underline{x}'-\underline{x}\| + \|\underline{u}'-\underline{u}\| < \delta_\epsilon$, the following inequality holds $\|N_{D\zeta}(\underline{x},\underline{u}) - N_{D\zeta}(\underline{x}',\underline{u}')\| < \epsilon$. That means for all $t \in \mathbb{N}$, $\|D\zeta(x_t,u_t) - D\zeta(x_t',u_t')\| < \epsilon$.

Now we consider the following expression

$$G_t = \left\|\zeta(x_t',u_t') - \zeta(x_t,u_t) - D\zeta(x_t,u_t).(x_t'-x_t,u_t'-u_t)\right\|.$$

For all $t \in \mathbb{N}$, using Corollary 1.37 in Chapter 1, we know that

$$G_t \le \sup_{(z_t,w_t)\in[(x_t,u_t),(x_t',u_t')]} \|D\zeta(z_t,w_t) - D\zeta(x_t,u_t)\| \, \|(\delta x_t,\delta u_t)\|,$$

where $\delta x_t := x_t' - x_t \in \mathbb{R}^n$; $\delta u_t := u_t' - u_t \in \mathbb{R}^d$. For all $(z_t,w_t) \in [(x_t,u_t),(x_t',u_t')]$, we have $\|z_t - x_t\| + \|w_t - u_t\| \le \|x_t - x_t'\| + \|u_t - u_t'\| < \delta$, hence $\|D\zeta(z_t,w_t) - D\zeta(x_t,u_t)\| < \varepsilon$. Therefore, we obtain

$$\sup_{(z_t,w_t)\in[(x_t,u_t),(x_t',u_t')]} \|D\zeta(z_t,w_t) - D\zeta(x_t,u_t)\| < \varepsilon.$$

Thus, we have

$$\forall t \in \mathbb{N},\ G_t \le \varepsilon\,\|(\delta x_t,\delta u_t)\| = \varepsilon(\|x_t - x_t'\| + \|u_t - u_t'\|),$$

which can be rewritten as follows

$$\begin{aligned}
&\sup_{t\in\mathbb{N}} G_t \le \varepsilon(\|\underline{x}' - \underline{x}\| + \|\underline{u}' - \underline{u}\|) = \varepsilon.(\|\underline{\delta x}\| + \|\underline{\delta u}\|)\\
\iff &\left\|(\zeta(x_t', u_t'))_{t\in\mathbb{N}} - (\zeta(x_t, u_t))_{t\in\mathbb{N}} - (D\zeta(x_t, u_t).(\delta x_t, \delta u_t))_{t\in\mathbb{N}}\right\| \le \varepsilon.(\|\underline{\delta x}\| + \|\underline{\delta u}\|)\\
\iff &\|N_\zeta(\underline{x}', \underline{u}') - N_\zeta(\underline{x}, \underline{u}) - N_{D\zeta}(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u})\| \le \varepsilon(\|\underline{\delta x}\| + \|\underline{\delta u}\|).
\end{aligned}$$

The last inequality means that $N_\zeta$ is Fréchet differentiable at the point $(\underline{x}, \underline{u})$. Since the choice of $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ is arbitrary, $N_\zeta$ is Fréchet differentiable on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and moreover, for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and for all $(\underline{\delta x}, \underline{\delta u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, \mathbb{R}^d)$ we have

$$\begin{aligned}
DN_\zeta(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) &= N_{D\zeta}(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) = (D\zeta(x_t, u_t).(\delta x_t, \delta u_t))_{t\in\mathbb{N}}\\
&= (D_1\zeta(x_t, u_t).\delta x_t + D_2\zeta(x_t, u_t).\delta u_t)_{t\in\mathbb{N}}.
\end{aligned}$$

It is noticed that $DN_\zeta(\underline{x}, \underline{u})$ and $N_{D\zeta}(\underline{x}, \underline{u})$ are equal so $DN_\zeta(\underline{x}, \underline{u})$ is continuous on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ from the continuity of $N_{D\zeta}(\underline{x}, \underline{u})$. Hence, $N_\zeta \in C^1(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U); c_0(\mathbb{N}, \mathbb{R}^m))$. □

Finally, we introduce a theorem which relates to the functionals in the criterion of Problem $(P_s)$.

**Theorem 2.14.** *Let $U$ be a nonempty closed subset of $\mathbb{R}^d$ and let $\phi : \mathbb{R}^n \times U \to \mathbb{R}$ be a mapping. If $\phi \in C^1(\mathbb{R}^n \times U, \mathbb{R})$ then the following assertions hold*

**(i)** *$N_\phi : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to \ell^\infty(\mathbb{N}, \mathbb{R})$ is well defined where $N_\phi(\underline{x}, \underline{u}) := (\phi(x_t, u_t))_{t\in\mathbb{N}}$.*

**(ii)** *$N_\phi$ is continuous on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$.*

**(iii)** *For all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, $DN_\phi(\underline{x}, \underline{u})$ exists and $DN_\phi$ is continuous on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Moreover, for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and for all $(\underline{\delta x}, \underline{\delta u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, \mathbb{R}^d)$ we have*

$$DN_\phi(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) = (D_1\phi(x_t, u_t).\delta x_t + D_2\phi(x_t, u_t).\delta u_t)_{t\in\mathbb{N}}.$$

*Proof.* **(i)** Let $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Let $V = \{\xi_t = (x_t,\ u_t) : t \in \mathbb{N}\}$. Then $V \subset \mathbb{R}^n \times U$ and $V$ is bounded since

$$\begin{aligned}
\sup_{t\in\mathbb{N}} \|\xi_t\|_\infty = \sup_{t\in\mathbb{N}} \|(x_t,\ u_t)\|_\infty &\le \sup_{t\in\mathbb{N}} \|x_t\|_\infty + \sup_{t\in\mathbb{N}} \|u_t\|_\infty\\
&= \|\underline{x}\| + \|\underline{u}\| < +\infty.
\end{aligned}$$

Since $\phi$ is continuous on $V$, $\phi(V)$ is bounded in $\mathbb{R}$, i.e. $\sup_{t\in\mathbb{N}} |\phi(\xi_t)| = \sup_{t\in\mathbb{N}} |\phi(x_t,\ u_t)| < +\infty \implies (\phi(x_t, u_t))_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbb{R})$. Hence, $N_\phi$ is well defined and (i) holds.

**(ii)** After (i) we know that $N_\phi : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to \ell^\infty(\mathbb{N}, \mathbb{R})$ is well defined. By a similar proceeding like in the proof of Theorem 2.8 we obtain the continuity of $N_\phi$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$.

**(iii)** We know that $\phi \in C^1(\mathbb{R}^n \times U,\ \mathbb{R})$ then $D\phi$ is of class $C^0$ on $\mathbb{R}^n \times U$. We define new Nemytskii operator $D\Phi : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to \ell^\infty(\mathbb{N},\ \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^d,\ \mathbb{R}))$, where $D\Phi(\underline{x}, \underline{u}) := (D\phi(x_t, u_t))_{t\in\mathbb{N}}$. Proceeding like in (i) and (ii), we obtain $D\Phi$ is well defined and is continuous on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Then using a similar interpretation like the proof of Theorem 2.13, we obtain assertion (iii), in which $DN_\phi$ coincides with $D\Phi$.

After (i), (ii) and (iii) we can assert that $N_\phi \in C^1(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U);\ \ell^\infty(\mathbb{N}, \mathbb{R}))$. □

## 2.4 Linear Difference Equations

We will establish a result on the existence of a solution of a nonhomogeneous linear equation which belongs to $c_0(\mathbb{N},\mathbb{R}^n)$ when the second member belongs to $c_0(\mathbb{N},\mathbb{R}^n)$. $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ denotes the space of linear mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$, and $\|A\|_{\mathcal{L}} := \sup\{\|Ax\| : x \in \mathbb{R}^n,\ \|x\| \leq 1\}$ is the norm on $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$.

**Proposition 2.15.** *Let $(A_t)_{t\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $\underline{z} \in c_0(\mathbb{N},\mathbb{R}^n)$ and $w \in \mathbb{R}^n$. We consider the following Cauchy problem*

$$\left.\begin{array}{r} \ell_{t+1} = A_t\ell_t + z_t \\ \ell_0 = w. \end{array}\right\} \qquad (DE)$$

*We assume that $\sup_{t\in\mathbb{N}} \|A_t\|_{\mathcal{L}} < 1$. Then the solution of (DE) belongs to $c_0(\mathbb{N},\mathbb{R}^n)$.*

*Proof.* We denotes by $\underline{\ell} = (\ell_t)_{t\in\mathbb{N}}$ the solution of $(DE)$. We need to prove that $\underline{\ell}$ exists and $\underline{\ell} \in c_0(\mathbb{N},\mathbb{R}^n)$. Obviously, for any $\underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n)$ this system of difference equations has an unique solution which can be generalized by induction as follows

$$\begin{aligned}
&\ell_0 = w;\\
&\ell_1 = A_0\ell_0 + z_0 = A_0w + z_0;\\
&\ell_2 = A_1\ell_1 + z_1 = A_1A_0w + A_1z_0 + z_1;\\
&\ell_3 = A_2\ell_2 + z_2 = A_2A_1A_0w + A_2A_1z_0 + A_2z_1 + z_2;\\
&\quad\vdots\\
&\ell_{t+1} = (\textstyle\prod_{i=0}^t A_i)w + (\prod_{i=1}^t A_i)z_0\\
&\qquad\quad + (\textstyle\prod_{i=2}^t A_i)z_1 + \cdots + A_tz_{t-1} + z_t.
\end{aligned}$$

Now we prove that $\underline{\ell} = (\ell_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{R}^n)$. From the last equality, we have:

$$\begin{aligned}
\|\ell_{t+1}\| \leq\ & (\textstyle\prod_{i=0}^t \|A_i\|_{\mathcal{L}})\,\|w\| + (\prod_{i=1}^t \|A_i\|_{\mathcal{L}})\,\|z_0\|\\
&+(\textstyle\prod_{i=2}^t \|A_i\|_{\mathcal{L}})\,\|z_1\| + \cdots + \|A_t\|_{\mathcal{L}}\,\|z_{t-1}\| + \|z_t\|.
\end{aligned}$$

Let $r = \max\{\|\underline{z}\|_\infty, \|w\|\} < +\infty$ and $M = \sup_{t\in\mathbb{N}} \|A_t\| < 1$. Then we have

$$\begin{aligned}
\|\ell_{t+1}\| &\leq r\left(\begin{array}{c}(\prod_{i=0}^t \|A_i\|_{\mathcal{L}}) + \prod_{i=1}^t \|A_i\|_{\mathcal{L}}\\ +\prod_{i=2}^t \|A_i\|_{\mathcal{L}} + \cdots + \|A_t\|_{\mathcal{L}} + 1\end{array}\right)\\
&< r(1 + M + M^2 + \cdots + M^{t+1})\\
\Rightarrow\ \sup_{t\in\mathbb{N}} \|\ell_{t+1}\| &\leq r\textstyle\sum_{i=0}^{+\infty} M^i = r\frac{1}{1-M} < +\infty.
\end{aligned}$$

And so, $(\ell_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{R}^n)$.

Finally, we prove that $\underline{\ell} = (\ell_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n)$. We have:

$$\begin{aligned}
&\forall t \in \mathbb{N},\ \ell_{t+1} = A_t\ell_t + z_t\\
\Rightarrow\ &\forall t \in \mathbb{N},\ \|\ell_{t+1}\| \leq \|A_t\|\,\|\ell_t\| + \|z_t\| \leq M\,\|\ell_t\| + \|z_t\|.
\end{aligned}$$

Since $(\ell_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{R}^n)$, $\limsup_{t\to+\infty} \|\ell_t\|$ exists and it is finite. Take $\limsup_{t\to+\infty}$ of both sides of last inequation and notice that $\limsup_{t\to+\infty} \|\ell_{t+1}\| = \limsup_{t\to+\infty} \|\ell_t\|$ and $\limsup_{t\to+\infty} \|z_t\| = 0$ (since $(z_t)_t \in c_0(\mathbb{N},\mathbb{R}^n)$) then we obtain the following inequality

$$\limsup_{t\to+\infty} \|\ell_t\| \leq M \limsup_{t\to+\infty} \|\ell_t\| \Leftrightarrow (1-M) \limsup_{t\to+\infty} \|\ell_t\| \leq 0.$$

From this we obtain $\limsup_{t\to+\infty} \|\ell_t\| = 0$ because $M < 1$. Now since $\limsup_{t\to+\infty} \|\ell_t\| = 0$ we obtain $\lim_{t\to+\infty} \|\ell_t\| = 0$. Therefore, $(\ell_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n)$. $\square$

**Corollary 2.16.** *Let $(B_t)_{t\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $\underline{d}\in c_0(\mathbb{N},\mathbb{R}^n)$ and $e\in\mathbb{R}^n$. We consider the following Cauchy problem*

$$\left.\begin{array}{l} k_{t+1}=B_tk_t+d_t,\\ k_0=e.\end{array}\right\} \qquad (DE1)$$

*We assume that there exists $t_*\in\mathbb{N}$ such that $\sup_{t\geq t_*}\|B_t\|_{\mathcal{L}}<1$. Then the solution of (DE1) belongs to $c_0(\mathbb{N},\mathbb{R}^n)$.*

*Proof.* For all $t\in\mathbb{N}$, we set $A_t:=B_{t+t_*}\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ and $z_t:=d_{t+t_*}$. Then we have $\sup_{t\in\mathbb{N}}\|A_t\|_{\mathcal{L}}<1$ and $\underline{z}\in c_0(\mathbb{N},\mathbb{R}^n)$. We denote by $\underline{k}$ the solution of $(DE1)$. We set $\ell_t:=k_{t+t_*}$ for all $t\in\mathbb{N}$. Then we have $\ell_{t+1}=k_{t+t_*+1}=B_{t+t_*}k_{t+t_*}+d_{t+t_*}=A_t\ell_t+z_t$ for all $t\in\mathbb{N}$ and $\ell_0=k_{t_*}\in\mathbb{R}^n$. Using Proposition 2.15 we obtain $\lim_{t\to+\infty}\ell_t=0$, i.e. $\lim_{t\to+\infty}k_{t+t_*}=0$ which implies $\lim_{t\to+\infty}k_t=0$. □

## 2.5 Static Optimization - a Result in Abstract Banach Spaces

In this section we establish a result in the form of Karush-Kuhn-Tucker theorem in abstract Banach spaces. It will be useful for the proof of weak Pontryagin principles in next sections.

**Lemma 2.17.** *Let $\mathcal{X}$, $\mathcal{V}$, $\mathcal{W}$ be real Banach spaces, and $\mathcal{U}$ be a nonempty subset of $\mathcal{V}$. Let $\mathcal{J}\in C^1(\mathcal{X}\times\mathcal{U},\mathbb{R})$ and $\Gamma\in C^1(\mathcal{X}\times\mathcal{U},\mathcal{W})$. Let $(\hat{x},\hat{u})$ be a solution of the following optimization problem*

$$\left\{\begin{array}{rl} \textit{Maximize} & \mathcal{J}(x,u)\\ \textit{when} & x\in\mathcal{X},\ u\in\mathcal{U},\ \Gamma(x,u)=0.\end{array}\right.$$

*We assume that $D_1\Gamma(\hat{x},\hat{u})$ is invertible and that $\mathcal{U}$ is star-shaped with respect to $\hat{u}$. Then there exists $M\in\mathcal{W}^*$ which satisfies the following conditions.*

**(i)** $D_1\mathcal{J}(\hat{x},\hat{u})+M\circ D_1\Gamma(\hat{x},\hat{u})=0.$

**(ii)** $\forall u\in\mathcal{U},\ \langle D_2\mathcal{J}(\hat{x},\hat{u})+M\circ D_2\Gamma(\hat{x},\hat{u}),u-\hat{u}\rangle\leq 0.$

*Proof.* Let $\mathcal{U}_1$ be an open subset of $\mathcal{V}$ such that $\mathcal{U}\subset\mathcal{U}_1$ and such that there exists $\Gamma_1\in C^1(\mathcal{X}\times\mathcal{U}_1,\mathcal{W})$ such that $\Gamma_{1|_{\mathcal{X}\times\mathcal{U}}}=\Gamma$. Since $D_1\Gamma_1(\hat{x},\hat{u})=D_1\Gamma(\hat{x},\hat{u})$ is invertible, we can use Implicit Function Theorem and assert that there exist $\mathcal{N}_{\hat{x}}$ an open neighborhood of $\hat{x}$ in $\mathcal{X}$, $\mathcal{N}_{\hat{u}}$ an open convex neighborhood of $\hat{u}$ in $\mathcal{U}_1$, and a mapping $\pi\in C^1(\mathcal{N}_{\hat{u}},\mathcal{N}_{\hat{x}})$ such that

$$\{(x,u)\in\mathcal{N}_{\hat{x}}\times\mathcal{N}_{\hat{u}}:\Gamma_1(x,u)=0\}=\{(\pi(u),u):u\in\mathcal{N}_{\hat{u}}\}.$$

Differentiating $\Gamma_1(\pi(u),u)=0$ at $\hat{u}$ we obtain $D_1\Gamma_1(\hat{x},\hat{u})\circ D\pi(\hat{u})+D_2\Gamma_1(\hat{x},\hat{u})=0$ which implies

$$D\pi(\hat{u})=-(D_1\Gamma(\hat{x},\hat{u}))^{-1}\circ D_2\Gamma(\hat{x},\hat{u}). \qquad (2.2)$$

Since $(\hat{x},\hat{u})$ is a solution of the initial problem, $\hat{u}$ is a solution of the following problem

$$\left\{\begin{array}{rl} \text{Maximize} & \mathcal{B}(u)\\ \text{when} & u\in\mathcal{N}_{\hat{u}}\cap\mathcal{U}\end{array}\right.$$

where $\mathcal{B}(u) = \mathcal{J}(\pi(u), u)$. Since $\mathcal{B}$ is differentiable (as a composition of differentiable mappings) and $\mathcal{N}_{\hat{u}} \cap \mathcal{U}$ is also star-shaped with respect to $\hat{u}$, a necessary condition of optimality for the last problem is

$$\forall u \in \mathcal{N}_{\hat{u}} \cap \mathcal{U}, \langle D\mathcal{B}(\hat{u}), u - \hat{u}\rangle \leq 0, \tag{2.3}$$

since $0 \geq \lim_{\theta \to 0+} \frac{1}{\theta}(\mathcal{B}(\hat{u} + \theta(u - \hat{u})) - \mathcal{B}(\hat{u})) = \langle D\mathcal{B}(\hat{u}), u - \hat{u}\rangle$. When $u \in \mathcal{U}$, there exists $\theta_u \in (0, 1)$ such that $(1 - \theta_u)\hat{u} + \theta_u u \in \mathcal{N}_{\hat{u}} \cap \mathcal{U}$. Using (2.3) we obtain

$$\theta_u \cdot \langle D\mathcal{B}(\hat{u}), u - \hat{u}\rangle) = \langle D\mathcal{B}(\hat{u}), \theta_u(u - \hat{u})\rangle = \langle D\mathcal{B}(\hat{u}), [(1 - \theta_u)\hat{u} + \theta_u u] - \hat{u}\rangle \leq 0,$$

and so we obtain

$$\forall u \in \mathcal{U}, \langle D\mathcal{B}(\hat{u}), u - \hat{u}\rangle \leq 0. \tag{2.4}$$

Using the chain rule we obtain

$$D\mathcal{B}(\hat{u}) = D_1\mathcal{J}(\hat{x}, \hat{u}) \circ D\pi(\hat{u}) + D_2\mathcal{J}(\hat{x}, \hat{u}). \tag{2.5}$$

We define

$$M := -D_1\mathcal{J}(\hat{x}, \hat{u}) \circ (D_1\Gamma(\hat{x}, \hat{u}))^{-1} \in \mathcal{W}^*. \tag{2.6}$$

From (2.6) we obtain

$$D_1\mathcal{J}(\hat{x}, \hat{u}) + M \circ D_1\Gamma(\hat{x}, \hat{u}) = 0. \tag{2.7}$$

Using (2.5) and (2.2) we have

$$\begin{aligned} D\mathcal{B}(\hat{u}) &= -D_1\mathcal{J}(\hat{x}, \hat{u}) \circ (D_1\Gamma(\hat{x}, \hat{u}))^{-1} \circ D_2\Gamma(\hat{x}, \hat{u}) + D_2\mathcal{J}(\hat{x}, \hat{u}) \\ &= M \circ D_2\Gamma(\hat{x}, \hat{u}) + D_2\mathcal{J}(\hat{x}, \hat{u}) \end{aligned}$$

and therefore, from (2.4) we obtain

$$\forall u \in \mathcal{U}, \langle D_2\mathcal{J}(\hat{x}, \hat{u}) + M \circ D_2\Gamma(\hat{x}, \hat{u}), u - \hat{u}\rangle \leq 0. \tag{2.8}$$

□

**Remark 2.18.** There exist several results like this one in the books [25] and [64] which use the convexity of $U$. In the necessary conditions of optimality we prefer to avoid the convexity of the sets; it is why we have established this lemma.

## 2.6 Weak Pontryagin Principle for Problem (Ps)

We start by a translation of Problem (Ps) into a simpler abstract optimization problem in Banach spaces. Let $\mathcal{T} : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n$ where $\mathcal{T}(\underline{x}, \underline{u}) = (F(\underline{x}, \underline{u}),\ h(\underline{x}, \underline{u}))$. Here $F(\underline{x}, \underline{u}) := (f(x_t, u_t) - x_{t+1})_{t\in\mathbb{N}}$ and $h(\underline{x}, \underline{u}) := x_0 - \eta$. Then we can translate Problem $(P_s)$ into the following problem.

$$(P2) \begin{cases} \text{Maximize} & J(\underline{x}, \underline{u}) \\ \text{when} & \underline{x} \in c_0(\mathbb{N}, \mathbb{R}^n), \underline{u} \in \ell^\infty(\mathbb{N}, U) \\ & \mathcal{T}(\underline{x}, \underline{u}) = \underline{0}. \end{cases}$$

We consider the following list of assumptions:

(A1) $U$ is a nonempty closed subset of $\mathbb{R}^d$.

(A2) $\phi \in C^1(\mathbb{R}^n \times U, \mathbb{R})$ and $f \in C^1(\mathbb{R}^n \times U, \mathbb{R}^n)$.

(A3) There exists $u^0 \in U$ such that $f(0, u^0) = 0$ and $U$ is star-shaped with respect to $u^0$.

(A4) $\lim_{x\to 0}(\sup_{u\in B}\|Df(x,u)\|_{\mathfrak{L}})=0$ for all nonempty bounded subset $B\subset U$.

**Lemma 2.19.** *Under condition (A2), the functional $J$ is of class $C^1$ on $c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ and moreover, the following formula holds for all $(\underline{x},\underline{u})$, $(\underline{\delta x},\underline{\delta u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$*

$$DJ(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})=\sum_{t=0}^{+\infty}\beta^t D_1\phi(x_t,u_t)\delta x_t+\sum_{t=0}^{+\infty}\beta^t D_2\phi(x_t,u_t)\delta u_t.$$

*Proof.* Under condition (A2), the Nemytskii operator $N_\phi : c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)\to\ell^\infty(\mathbb{N},\mathbb{R})$ defined by $N_\phi(\underline{x},\underline{u}):=(\phi(x_t,u_t))_{t\in\mathbb{N}}$ is of class $C^1$ on $c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ using Theorem 2.14. Moreover, for all $(\underline{x},\underline{u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ and for all $(\underline{\delta x},\underline{\delta u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},\mathbb{R}^d)$, we have

$$DN_\phi(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})=(D_1\phi(x_t,u_t)\delta x_t)_{t\in\mathbb{N}}+(D_2\phi(x_t,u_t)\delta u_t)_{t\in\mathbb{N}}.$$

Consider the functional $S:\ell^\infty(\mathbb{N},\mathbb{R})\to\mathbb{R}$ defined by $S(\underline{r}):=\sum_{t=0}^{+\infty}\beta^t r^t$. It is easy to verify that $S(\alpha_1\underline{r_1}+\alpha_2\underline{r_2})=\alpha_1 S(\underline{r_1})+\alpha_2 S(\underline{r_2})$ for all $\alpha_1,\ \alpha_2\in\mathbb{R}$ and for all $\underline{r_1},\ \underline{r_2}\in\ell^\infty(\mathbb{N},\mathbb{R})$. Hence, $S$ is linear. Besides, $|S(\underline{r})|\le(\sum_{t=0}^{+\infty}\beta^t)\|\underline{r}\|_\infty=\frac{1}{1-\beta}\|\underline{r}\|_\infty<+\infty$ so $S$ is bounded and consequently, continuous. From Example 1.29 in Chapter 2, we obtain $S$ is of class $C^1$ and moreover, for all $\underline{r},\ \delta\underline{r}\in\ell^\infty(\mathbb{N},\mathbb{R})$, we have $DS(\underline{r}).\underline{\delta r}=S(\underline{\delta r})=\sum_{t=0}^{+\infty}\beta^t\delta r^t$. We see that $J=S\circ N_\phi$ is of class $C^1$ as a composition of two $C^1$ -mappings. Using the chain rule we obtain for all $(\underline{x},\underline{u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ and for all $(\underline{\delta x},\underline{\delta u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},\mathbb{R}^d)$ we have

$$\begin{aligned}DJ(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})&=DS(N_\phi(\underline{x},\underline{u}))\circ DN_\phi(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})\\&=S(DN_\phi(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u}))\\&=\sum_{t=0}^{+\infty}\beta^t D_1\phi(x_t,u_t)\delta x_t+\sum_{t=0}^{+\infty}\beta^t D_2\phi(x_t,u_t)\delta u_t.\end{aligned}$$

□

**Lemma 2.20.** *Under conditions (A1-A4), the Nemytskii operator $N_f : c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)\to c_0(\mathbb{N},\mathbb{R}^n)$ defined by $N_f(\underline{x},\underline{u}):=(f(x_t,u_t))_{t\in\mathbb{N}}$ and the operator $F$ defined before ($F(\underline{x},\underline{u}):=(f(x_t,u_t)-x_{t+1})_{t\in\mathbb{N}}$) are of class $C^1$ on $c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ and moreover, the following formulas hold for all $(\underline{x},\underline{u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ and for all $(\underline{\delta x},\underline{\delta u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},\mathbb{R}^d)$:*

1. $DN_f(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})=(D_1f(x_t,u_t)\delta x_t)_t+(D_2f(x_t,u_t)\delta u_t)_t$;
2. $DF(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})=(D_1f(x_t,u_t)\delta x_t-\delta x_{t+1})_t+(D_2f(x_t,u_t)\delta u_t)_t$.

*Proof.* Under conditions (A2-A4), we can assert that the mapping $f$ satisfies all the conditions $(\alpha.1)$, $(\alpha.2')$, $(\alpha.3)$ and $(\alpha.4)$ (by replacing $\zeta$ by $f$). Then using condition (A1) and Theorem 2.13, we know that $N_f\in C^1(c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U);c_0(\mathbb{N},\mathbb{R}^n)$. Moreover, for all $(\underline{x},\underline{u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ and for all $(\underline{\delta x},\underline{\delta u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},\mathbb{R}^d)$, we have

$$DN_f(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})=(D_1f(x_t,u_t)\delta x_t)_{t\in\mathbb{N}}+(D_2f(x_t,u_t)\delta u_t)_{t\in\mathbb{N}}.$$

So conclusion 1 is proven.

Now we consider operator $A(\underline{x},\underline{u}):=(-x_{t+1})_{t\in\mathbb{N}}$ from $c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ into $c_0(\mathbb{N},\mathbb{R}^n)$. Obviously, $A$ is linear and from $\|A(\underline{x},\underline{u})\|\le\|\underline{x}\|_\infty\le\|\underline{x}\|_\infty+\|\underline{u}\|_\infty<+\infty$, we obtain $A$ is bounded and hence, continuous. So $A$ is of class $C^1$ and for all $(\underline{x},\underline{u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$, for all $(\underline{\delta x},\underline{\delta u})\in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},\mathbb{R}^d)$ we have

$$DA(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u})=A(\underline{\delta x},\underline{\delta u})=(-\delta x_{t+1})_{t\in\mathbb{N}}.$$

We see that $F = N_f + A$ then $F$ is of class $C^1$ as a sum of two $C^1$- mappings. Moreover, for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and for all $(\underline{\delta x}, \underline{\delta u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, \mathbb{R}^d)$ we have

$$\begin{aligned} DF(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) &= DN_f(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) + DA(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) \\ &= (D_1 f(x_t, u_t)\delta x_t)_t + (D_2 f(x_t, u_t)\delta u_t)_t + (-\delta x_{t+1})_t \\ &= (D_1 f(x_t, u_t)\delta x_t - \delta x_{t+1})_t + (D_2 f(x_t, u_t)\delta u_t)_t. \end{aligned}$$

Hence, conclusion 2 is proven. □

Under conditions (A1-A4), using Lemma 2.19 and Lemma 2.20 we obtain $J$ and $F$ are continuously differentiable on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Now we consider $\mathcal{T}(\underline{x}, \underline{u}) = (F(\underline{x}, \underline{u}),\ h(\underline{x}, \underline{u}))$ where $F(\underline{x}, \underline{u}) = (f(x_t, u_t) - x_{t+1})_{t\in\mathbb{N}}$ and $h(\underline{x}, \underline{u}) = x_0 - \eta$ as they were defined before. The operator $h$ is obviously an affine continuous mapping since we can consider $h$ as a sum of linear continuous mapping $\mathrm{Pr}_0$ (projection with respect to the 0-coordinate of $\underline{x}$) and a constant $-\eta$. Here $\mathrm{Pr}_0 : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to \mathbb{R}^n$ defined by $\mathrm{Pr}_0(\underline{x}, \underline{u}) := x_0$. We know that mapping $\mathrm{Pr}_0$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, for all $(\underline{\delta x}, \underline{\delta u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, \mathbb{R}^d)$, we have

$$D\,\mathrm{Pr}_0(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) = \mathrm{Pr}_0(\underline{\delta x}, \underline{\delta u}) = \delta x_0.$$

In addition, a constant $-\eta$ is continuously differentiable on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and its differential is null function. Therefore, $h$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, for all $(\underline{\delta x}, \underline{\delta u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, \mathbb{R}^d)$ we have

$$\begin{aligned} Dh(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) &= (D\,\mathrm{Pr}_0 + 0)(\underline{x}, \underline{u}) \cdot (\underline{\delta x}, \underline{\delta u}) \\ &= \mathrm{Pr}_0(\underline{\delta x}, \underline{\delta u}) = \delta x_0. \end{aligned}$$

More specifically, $Dh(\underline{x}, \underline{u}).(\underline{\delta x}, \underline{\delta u}) = D_1 h(\underline{x}, \underline{u}).\underline{\delta x} + D_2 h(\underline{x}, \underline{u}).\underline{\delta u}$ and from the above-mentioned arguments, we obtain $D_1 h(\underline{x}, \underline{u}).\underline{\delta x} = \delta x_0$ and $D_2 h(\underline{x}, \underline{u}).\underline{\delta u} = 0$. From this assertion, we obtain $\mathcal{T}$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ since each of its element mappings $F$ and $h$ are of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$.

**Lemma 2.21.** *We assume assumptions (A1-A4) fulfilled. Let $(\underline{\hat{x}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of (P2) and assume that for all $t \in \mathbb{N}$, $U$ is star-shaped with respect to $\hat{u}_t$. Then there exists $(\underline{q}, \mu) \in \ell^1(\mathbb{N}, \mathbb{R}^{n*}) \times \mathbb{R}^{n*}$ which satisfies the two following conditions.*

**(i)** $D_1 J(\underline{\hat{x}}, \underline{\hat{u}}) + (\underline{q}, \mu) \circ D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) = \underline{0}$.

**(ii)** *For all* $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, $\left\langle D_2 J(\underline{\hat{x}}, \underline{\hat{u}}) + (\underline{q}, \mu) \circ D_2 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}), \underline{u} - \underline{\hat{u}} \right\rangle \leq 0$.

*Proof.* Under conditions (A1-A4), after Lemma 2.19 and Lemma 2.20 we know that the mappings $J$ and $\mathcal{T}$ are of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Hence, the partial Fréchet differential with respect to $\underline{x}$ of mapping $\mathcal{T}$ exists at the optimal solution $(\underline{\hat{x}}, \underline{\hat{u}})$. We will prove that $D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}})$ is invertible. Let $\underline{d} = (d_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N}, \mathbb{R}^n)$ and $e \in \mathbb{R}^n$ arbitrarily. We need to show that there exists an unique $\underline{k} = (k_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N}, \mathbb{R}^n)$ such that $D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}).\underline{k} = (-\underline{d}, e)$. Since

$$D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}).\underline{k} = (D_1 F(\underline{\hat{x}}, \underline{\hat{u}}).\underline{k},\ D_1 h(\underline{\hat{x}}, \underline{\hat{u}}).\underline{k}),$$

the problem is equivalent to the following one

$$\begin{cases} D_1F(\underline{\hat{x}},\underline{\hat{u}}).\underline{k} = -\underline{d}, \\ D_1h(\underline{\hat{x}},\underline{\hat{u}}).\underline{k} = e \end{cases}$$
$$\Leftrightarrow \begin{cases} D_1F(\underline{\hat{x}},\underline{\hat{u}}).\underline{k} = -\underline{d}, \\ k_0 = e. \end{cases}$$

Now using Lemma 2.20, we know that $D_1F(\underline{\hat{x}},\underline{\hat{u}}).\underline{k} = (D_1f(\hat{x}_t,\hat{u}_t)k_t - k_{t+1})_{t\in\mathbb{N}}$ then system above can be translated into the following problem of solutions in space $c_0(\mathbb{N},\mathbb{R}^n)$ of difference equations

$$\begin{cases} k_{t+1} = B_tk_t + d_t, \ t \in \mathbb{N}, \\ k_0 = e, \end{cases}$$

where $B_t = D_1f(\hat{x}_t,\hat{u}_t) \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$. We will prove that sequence $(B_t)_{t\in\mathbb{N}}$ satisfies the property mentioned in Corollary 2.16. Set $\hat{B} := \{\hat{u}_t : \ t \in \mathbb{N}\}$ then $\hat{B}$ is nonempty bounded set in $U$ since $\underline{\hat{u}} \in \ell^\infty(\mathbb{N},U)$. For all $t \in \mathbb{N}$, we have

$$0 \le \|B_t\|_{\mathcal{L}} = \|D_1f(\hat{x}_t,\hat{u}_t)\|_{\mathcal{L}} \le \|Df(\hat{x}_t,\hat{u}_t)\|_{\mathcal{L}} \le \sup_{u\in\hat{B}} \|Df(\hat{x}_t,u)\|$$

and therefore, using (A4) and Remark 2.6 we obtain $\lim_{t\to+\infty}\|B_t\|_{\mathcal{L}} = 0$. Hence, there exists $t_* \in \mathbb{N}$ such that when $\sup_{t\ge t_*}\|B_t\|_{\mathcal{L}} < 1$. Finally, using Corollary 2.16 we obtain the unique solution $\underline{k} = (k_t)_{t\in\mathbb{N}}$ of the above system belongs to $c_0(\mathbb{N},\mathbb{R}^n)$. And so, we have proven that for any $\underline{d} = (d_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n)$ and $e \in \mathbb{R}^n$ the equation $D_1\mathcal{T}(\underline{\hat{x}},\underline{\hat{u}}).\underline{k} = (-\underline{d},e)$ always has an unique solution $(k_t)_{t\in\mathbb{N}}$ that belongs to $c_0(\mathbb{N},\mathbb{R}^n)$. It means that operator $D_1\mathcal{T}(\underline{\hat{x}},\underline{\hat{u}})$ is bijective, hence $D_1\mathcal{T}(\underline{\hat{x}},\underline{\hat{u}})$ is invertible.

Now take $\underline{u}$ in $\ell^\infty(\mathbb{N},U)$ and take $\alpha \in (0,1)$ arbitrarily. From the boudedness of $\underline{\hat{u}}$ and $\underline{u}$ we obtain $\alpha\underline{\hat{u}}+(1-\alpha)\underline{u}$ is bounded. Moreover, for all $t \in \mathbb{N}$, since $U$ is star-shaped with respect to $\hat{u}_t$ we have $\alpha\hat{u}_t+(1-\alpha)u_t \in U$. Hence, we deduce that $\underline{\hat{u}}+(1-\alpha)\underline{u} \in \ell^\infty(\mathbb{N},U)$. It means that $\ell^\infty(\mathbb{N},U)$ is star-shaped with respect to $\underline{\hat{u}}$.

Recall that in Chapter 1, we have proven that $\ell^1(\mathbb{N},\mathbb{R}^{n*})$ can be assimilated to the dual topological space of $c_0(\mathbb{N},\mathbb{R}^n)$, i.e. an element of $\ell^1(\mathbb{N},\mathbb{R}^{n*})$ can be considered as a continuous linear functional on $c_0(\mathbb{N},\mathbb{R}^n)$. Now all the assumptions of Lemma 2.17 are fulfilled and we can use Lemma 2.17 and assert that there exists Lagrange multiplier $(\underline{q},\mu) \in (c_0(\mathbb{N},\mathbb{R}^n)\times\mathbb{R}^n)^* = \ell^1(\mathbb{N},\mathbb{R}^{n*}) \times\mathbb{R}^{n*}$ which satisfies the announced conclusions. □

**Theorem 2.22.** *We assume assumptions (A1-A4) fulfilled. Let $(\underline{\hat{x}},\underline{\hat{u}}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ be a solution of Problem $(P_s)$ and assume that for all $t \in \mathbb{N}$, $U$ is star-shaped with respect to $\hat{u}_t$. Then there exists $\underline{p} \in \ell^1(\mathbb{N}_*,\mathbb{R}^{n*})$ such that*

**(AE1)** $p_t = p_{t+1}\circ D_1f(\hat{x}_t,\hat{u}_t) + \beta^t.D_1\phi(\hat{x}_t,\hat{u}_t)$ *for all* $t \in \mathbb{N}_*$;

**(WM1)** $\langle p_{t+1}\circ D_2f(\hat{x}_t,\hat{u}_t) + \beta^t.D_2\phi(\hat{x}_t,\hat{u}_t), u-\hat{u}_t\rangle \le 0$ *for all* $t \in \mathbb{N}$, *for all* $u \in U$.

*Proof.* Since $(\underline{\hat{x}},\underline{\hat{u}}) \in c_0(\mathbb{N},\mathbb{R}^n)\times\ell^\infty(\mathbb{N},U)$ is a solution of Problem $(P_s)$, it is also a solution of Problem (P2). Then Lemma 2.21 provides $(\underline{q},\mu) \in \ell^1(\mathbb{N},\mathbb{R}^{n*}) \times\mathbb{R}^{n*}$ such that

$$\left.\begin{aligned} D_1J(\underline{\hat{x}},\underline{\hat{u}}) + (\underline{q},\mu)\circ D_1\mathcal{T}(\underline{\hat{x}},\underline{\hat{u}}) &= \underline{0}, \\ \left\langle D_2J(\underline{\hat{x}},\underline{\hat{u}}) + (\underline{q},\mu)\circ D_2\mathcal{T}(\underline{\hat{x}},\underline{\hat{u}}), \underline{u}-\underline{\hat{u}}\right\rangle &\le 0 \end{aligned}\right\} \tag{2.9}$$

for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$. In (2.9), the equation can be rewritten equivalently as follows

$$D_1J(\underline{\hat{x}},\underline{\hat{u}}) + \underline{q}\circ D_1F(\underline{\hat{x}},\underline{\hat{u}}) + \mu\circ D_1h(\underline{\hat{x}},\underline{\hat{u}}) = 0.$$

Then for all $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ such that $\delta x_0 = 0$ we have

$$\Big(D_1 J(\underline{\hat{x}}, \underline{\hat{u}}) + \underline{q} \circ D_1 F(\underline{\hat{x}}, \underline{\hat{u}}) + \mu \circ D_1 h(\underline{\hat{x}}, \underline{\hat{u}})\Big).\underline{\delta x} = 0. \tag{2.10}$$

Using results of Lemma 2.19 and Lemma 2.20, we know that $D_1 J(\underline{\hat{x}}, \underline{\hat{u}}).\underline{\delta x} = \sum_{t=0}^{+\infty} \beta^t D_1 \phi(\hat{x}_t, \hat{u}_t)\delta x_t$, $D_1 F(\underline{\hat{x}}, \underline{\hat{u}}).\underline{\delta x} = (D_1 f(x_t, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}}$ and $D_1 h(\underline{\hat{x}}, \underline{\hat{u}}).\underline{\delta x} = \delta x_0 = 0$. Hence, equation (2.10) becomes

$$\sum_{t=0}^{+\infty} \beta^t D_1 \phi(\hat{x}_t, \hat{u}_t)\delta x_t + \sum_{t=0}^{+\infty} \langle q_t, D_1 f(\hat{x}_t, \hat{u}_t)\delta x_t - \delta x_{t+1}\rangle = 0$$

$$\iff \sum_{t=1}^{+\infty} \beta^t D_1 \phi(\hat{x}_t, \hat{u}_t)\delta x_t + \sum_{t=1}^{+\infty} \langle q_t, D_1 f(\hat{x}_t, \hat{u}_t)\delta x_t - \delta x_{t+1}\rangle - \langle q_0, \delta x_1\rangle = 0. \tag{2.11}$$

We fix $t \in \mathbb{N}_*$ arbitrarily. Take $\underline{\delta x} = (\delta x_s)_{s\in\mathbb{N}^*}$ such that when $s \neq t$, $\delta x_s = 0$ and $\delta x_t$ varies in $\mathbb{R}^n$. Then the last equation becomes

$$\begin{aligned} & \beta^t D_1 \phi(\hat{x}_t, \hat{u}_t)\delta x_t + (q_t \circ D_1 f(\hat{x}_t, \hat{u}_t)) \cdot \delta x_t - q_{t-1} \cdot \delta x_t = 0 \\ \Leftrightarrow \quad & \langle \beta^t D_1 \phi(\hat{x}_t, \hat{u}_t) + q_t \circ D_1 f(\hat{x}_t, \hat{u}_t) - q_{t-1}, \delta x_t\rangle = 0. \end{aligned}$$

This equality is verified for all $\delta x_t \in \mathbb{R}^n$ and for all $t \in \mathbb{N}_*$. And so we have

$$\forall t \in \mathbb{N}_*,\ q_{t-1} = q_t \circ D_1 f(\hat{x}_t, \hat{u}_t) + \beta^t D_1 \phi(\hat{x}_t, \hat{u}_t). \tag{2.12}$$

Now in (2.9), the inequation can be rewritten equivalently as follows

$$\Big\langle D_2 J(\underline{\hat{x}}, \underline{\hat{u}}) + \underline{q} \circ D_2 F(\underline{\hat{x}}, \underline{\hat{u}}) + \mu \circ D_2 h(\underline{\hat{x}}, \underline{\hat{u}}), \underline{u} - \underline{\hat{u}}\Big\rangle \leq 0 \text{ for all } \underline{u} \in \ell^\infty(\mathbb{N}, U).$$

Using results of Lemma 2.19 and Lemma 2.20, we know that $D_2 J(\underline{\hat{x}}, \underline{\hat{u}}).(\underline{u} - \underline{\hat{u}}) = \sum_{t=0}^{+\infty} \beta^t D_2 \phi(\hat{x}_t, \hat{u}_t)(u_t - \hat{u}_t)$, $D_2 F(\underline{\hat{x}}, \underline{\hat{u}}).(\underline{u} - \underline{\hat{u}}) = (D_2 f(x_t, u_t).(u_t - \hat{u}_t))_{t\in\mathbb{N}}$ and $D_2 h(\underline{\hat{x}}, \underline{\hat{u}}).(\underline{u} - \underline{\hat{u}}) = 0$. Hence, the last inequation becomes

$$\sum_{t=0}^{+\infty} \beta^t D_2 \phi(\hat{x}_t, \hat{u}_t)(u_t - \hat{u}_t) + \sum_{t=0}^{+\infty} \langle q_t \circ D_2 f(\hat{x}_t, \hat{u}_t), (u_t - \hat{u}_t)\rangle \leq 0 \text{ for all } (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, U). \tag{2.13}$$

Fix $t \in \mathbb{N}$ arbitrarily. Take $\underline{u} = (u_s)_{s\in\mathbb{N}}$ such that when $s \neq t$, $u_s = \hat{u}_s$ and $u_t$ varies in $U$. Then from equation (2.12) we have

$$\beta^t D_2 \phi(\hat{x}_t, \hat{u}_t)(u_t - \hat{u}_t) + \langle q_t \circ D_2 f(\hat{x}_t, \hat{u}_t), (u_t - \hat{u}_t)\rangle \leq 0$$

$$\iff \Big\langle q_t \circ D_2 f(\hat{x}_t, \hat{u}_t) + \beta^t.D_2 \phi(\hat{x}_t, \hat{u}_t), u_t - \hat{u}_t\Big\rangle \leq 0.$$

This inequation is verified for all $u_t \in U$ and for all $t \in \mathbb{N}$. Hence, we have

$$\forall t \in \mathbb{N},\ \forall u \in U,\ \Big\langle q_t \circ D_2 f(\hat{x}_t, \hat{u}_t) + \beta^t.D_2 \phi(\hat{x}_t, \hat{u}_t), u_t - \hat{u}_t\Big\rangle \leq 0. \tag{2.14}$$

In (2.12) and (2.14), by setting $p_{t+1} := q_t$ for all $t \in \mathbb{N}$, we obtain $\underline{p} = (p_{t+1})_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ such that the following statements hold

**(i)** $p_t = p_{t+1} \circ D_1 f(\hat{x}_t, \hat{u}_t) + \beta^t.D_1 \phi(\hat{x}_t, \hat{u}_t)$ for all $t \in \mathbb{N}_*$.

**(ii)** $\langle p_{t+1} \circ D_2 f(\hat{x}_t, \hat{u}_t) + \beta^t.D_2 \phi(\hat{x}_t, \hat{u}_t), u - \hat{u}_t\rangle \leq 0$ for all $t \in \mathbb{N}$, for all $u \in U$.

These statements are conclusions (AE1) and (WM1). The proof is complete. □

**Remark 2.23.** In Theorem 2.22, (AE1) means Adjoint Equation, (WM1) means weak Maximum Principle. Since $\underline{p} = (p_{t+1})_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$, note that the transversality condition at infinity for Problem $(P_s)$, $\lim_{t\to+\infty} p_t = 0$, is satisfied.

## 2.7 Weak Pontryagin Principle for Main Problem

So far in Section 1 we have introduced the main problem as follows:

$$\left.\begin{array}{rl} \text{Maximize} & K(\underline{y},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \psi(y_t,u_t) \\ \text{when} & \underline{y} = (y_t)_{t\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}\ ,\ \underline{u} = (u_t)_{t\in\mathbb{N}} \in U^{\mathbb{N}} \\ & y_0 = \eta,\ \lim_{t\to+\infty} y_t = y_\infty, \\ & \underline{u} \text{ is bounded}, \\ & \forall t\in\mathbb{N},\ y_{t+1} = g(y_t,u_t). \end{array}\right\} \qquad (P_m)$$

By setting $z_t := y_t - y_\infty$ for all $t \in \mathbb{N}$, $N(\underline{z},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \varphi(z_t,u_t)$ where $\varphi(z,u) := \psi(z+y_\infty,u)$ for all $(z,u)\in\mathbb{R}^n\times U$ and $\ell(z,u) := g(z+y_\infty,u)-y_\infty$ for all $(z,u)\in\mathbb{R}^n\times U$, we can translate Problem $(P_m)$ into Problem $(P1)$ which has the form of Problem $(P_s)$

$$\left.\begin{array}{rl} \text{Maximize} & N(\underline{z},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \varphi(z_t,u_t) \\ \text{when} & \underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n), \\ & \underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},U), \\ & z_0 = \eta - y_\infty, \\ & \forall t\in\mathbb{N},\ z_{t+1} = \ell(z_t,u_t). \end{array}\right\} \qquad (P1)$$

Let $(\underline{\hat{z}},\underline{\hat{u}})$ be a solution of Problem $(P1)$ and assume that for all $t\in\mathbb{N}$, $U$ is star-shaped with respect to $\hat{u}_t$. Then by the inverse transformation, $(\underline{\hat{y}},\underline{\hat{u}})$ is a solution of Problem $(P_m)$. Apply Theorem 2.22 to Problem $(P1)$ we know that if in Problem $(P1)$ the following conditions

**(1)** $U$ is a nonempty closed subset of $\mathbb{R}^d$.

**(2)** $\varphi\ \in C^1(\mathbb{R}^n\times U,\ \mathbb{R}),\ \ell\ \in C^1(\mathbb{R}^n\times U,\ \mathbb{R}^n)$,

**(3)** There exists $u^0\in U$ such that $\ell\ (0,u^0) = 0$ and $U$ is star-shaped with respect to $u^0$,

**(4)** $\lim_{x\to 0}(\sup_{u\in B}\|D\ell(z,u)\|) = 0$ for all nonempty bounded subset $B\subset U$

are fulfilled then there exists $(p_{t+1})_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}_*,\mathbb{R}^{n*})$ such that

**(i)** $p_t = p_{t+1}\circ D_1\ell(\hat{z}_t,\hat{u}_t) + \beta^t.D_1\varphi(\hat{z}_t,\hat{u}_t)$ for all $t\in\mathbb{N}_*$.

**(ii)** $\langle p_{t+1}\circ D_2\ell(\hat{z}_t,\hat{u}_t) + \beta^t.D_2\varphi(\hat{z}_t,\hat{u}_t), u-\hat{u}_t\rangle \le 0$ for all $t\in\mathbb{N}$, for all $u\in U$.

Now we will study the statement of weak Pontryagin principle for our main problem based on that for Problem $(P1)$.

– **Study the assumptions**

- The assumption on $U$ and assumption (1) remain unchanged.

- Because of the definitions of functions $\psi$, $g$, $\varphi$ and $\ell$, it is evident that the assumption (2) above is equivalent to following condition.

$$(2) \iff \psi\in C^1(\mathbb{R}^n\times U,\ \mathbb{R}) \text{ and } g\in C^1(\mathbb{R}^n\times U,\ \mathbb{R}^n).$$

- From assumption (3) above, we make the equivalent changes as follows:

$$\ell(0,u^0) = 0 \iff g(0+y_\infty,u^0) - y_\infty = 0 \iff g(y_\infty,u^0) = y_\infty.$$

And finally, we translate assumption (4) as follows.

$$\begin{array}{cl} & \lim_{z\to 0}(\sup_{u\in B}\|D\ell(z,u)\|) = 0 \\ \Leftrightarrow & \lim_{z\to 0}(\sup_{u\in B}\|D(g(z+y_\infty,u)-y_\infty)\|) = 0 \\ \overset{y:=z+y_\infty}{\Leftrightarrow} & \lim_{y\to y_\infty}(\sup_{u\in B}\|D(g(y,u)-y_\infty)\|) = 0 \\ \Leftrightarrow & \lim_{y\to y_\infty}(\sup_{u\in B}\|Dg(y,u)\|) = 0. \end{array}$$

– **Study the conclusions**

We can easily rewrite conclusions (i) and (ii) for Problem $(P_m)$ as follows

**(i)** $p_t = p_{t+1} \circ D_1 g(\hat{y}_t, \hat{u}_t) + \beta^t . D_1 \psi(\hat{y}_t, \hat{u}_t)$ for all $t \in \mathbb{N}_*$,

**(ii)** $\langle p_{t+1} \circ D_2 g(\hat{y}_t, \hat{u}_t) + \beta^t . D_2 \psi(\hat{y}_t, \hat{u}_t), u - \hat{u}_t \rangle \leq 0$ for all $t \in \mathbb{N}$, for all $u \in U$.

Here we notice that

$$D_1 g(\hat{y}_t, \hat{u}_t) = D_1 \ell(\hat{z}_t, \hat{u}_t);\ D_1 \psi(\hat{y}_t, \hat{u}_t) = D_1 \varphi(\hat{z}_t, \hat{u}_t);$$
$$D_2 g(\hat{y}_t, \hat{u}_t) = D_2 \ell(\hat{z}_t, \hat{u}_t) \text{ and } D_2 \psi(\hat{y}_t, \hat{u}_t) = D_2 \varphi(\hat{z}_t, \hat{u}_t).$$

The reasoning is like what we did when studying assumption (4).
Finally, we can state the weak Pontryagin principle for main problem.

**Theorem 2.24.** *Assume that the following conditions are fulfilled*

**(B1)** *$U$ is a nonempty closed subset of $\mathbb{R}^d$.*

**(B2)** *$\psi \in C^1(\mathbb{R}^n \times U, \mathbb{R})$, $g \in C^1(\mathbb{R}^n \times U, \mathbb{R}^n)$.*

**(B3)** *There exists $u^0 \in U$ such that $g(y_\infty, u^0) = y_\infty$ and $U$ is star-shaped with respect to $u^0$.*

**(B4)** $\lim_{y \to y_\infty} (\sup_{u \in B} \|Dg(y, u)\|) = 0$.

*Let $(\underline{\hat{y}}, \underline{\hat{u}})$ in $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of Problem $(P_m)$ and assume that for all $t \in \mathbb{N}$, $U$ is star-shaped with respect to $\hat{u}_t$. Then there exists $(p_{t+1})_{t \in \mathbb{N}} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ such that*

**(AE)** *$p_t = p_{t+1} \circ D_1 g(\hat{y}_t, \hat{u}_t) + \beta^t . D_1 \psi(\hat{y}_t, \hat{u}_t)$ for all $t \in \mathbb{N}_*$.*

**(WM)** *$\langle p_{t+1} \circ D_2 g(\hat{y}_t, \hat{u}_t) + \beta^t . D_2 \psi(\hat{y}_t, \hat{u}_t), u - \hat{u}_t \rangle \leq 0$ for all $t \in \mathbb{N}$, for all $u \in U$.*

# Chapter 3

# Strong Pontryagin Principle for Infinite-Horizon Optimal Control Problem in Presence of Asymptotical Constraint and a Sufficient Condition of Optimality

## 3.1 Introduction

This chapter is devoted to establish strong Pontryagin principles and sufficient conditions of optimality for the problems introduced in Chapter 2. The content of this chapter is as follows:

- In Section 3.2, we recall the main problem and the supporting problem which were introduced in Chapter 2.

- In Section 3.3, by fixing the sequence of control variable, we study the properties of Nemytskii operator from $c_0(\mathbb{N}, \mathbb{R}^n)$ into $c_0(\mathbb{N}, \mathbb{R}^m)$ and of Nemystskii operator from $c_0(\mathbb{N}, \mathbb{R}^n)$ into $\ell^\infty(\mathbb{N}, \mathbb{R})$.

- In Section 3.4, we recall a result on static optimization in Banach spaces which is useful for establishing strong Pontryagin principles.

- In Section 3.5 and 3.6, we establish strong Pontryagin principles for the supporting problem and the main problem.

- In Section 3.7 and 3.8, we establish results on sufficient condition of optimality for them.

## 3.2 Recall of the Main Problem and the Supporting Problem

Let $U$ be nonempty subset of $\mathbb{R}^d$. We consider the same problem like in Chapter 2.

$$\left.\begin{array}{rl} \text{Maximize} & K(\underline{y}, \underline{u}) := \sum_{t=0}^{+\infty} \beta^t \psi(y_t, u_t) \\ \text{when} & \underline{y} = (y_t)_{t\in\mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}\ ,\ \underline{u} = (u_t)_{t\in\mathbb{N}} \in U^{\mathbb{N}} \\ & y_0 = \eta,\ \lim_{t\to+\infty} y_t = y_\infty, \\ & \underline{u} \text{ is bounded}, \\ & \forall t \in \mathbb{N},\ y_{t+1} = g(y_t, u_t). \end{array}\right\} \qquad (P_m)$$

Here functional $K$, functions $\psi$, $g$, real number $\beta$ and elements $\eta$, $y_\infty$ were already defined in Chapter 2. For this problem, we have introduced its supporting problem in previous chapter as follows

$$\left.\begin{array}{rl} \text{Maximize} & J(\underline{x},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \phi(x_t,u_t) \\ \text{when} & \underline{x} = (x_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n), \\ & \underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},U), \\ & x_0 = \eta, \\ & \forall t \in \mathbb{N},\ x_{t+1} = f(x_t,u_t). \end{array}\right\} \qquad (P_s)$$

Functional $J$, functions $\phi$, $f$, real number $\beta$ and element $\eta$ were defined in Chapter 2. We knew that Problem $(P_m)$ can be translated into the form of Problem $(P_s)$ by the following transformation:

- For all $t \in \mathbb{N}$, $z_t := y_t - y_\infty$. Then $\underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n)$.
- Set $N(\underline{z},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \varphi(z_t,u_t)$ where $\varphi(z,u) := \psi(z+y_\infty,u)$ for all $(z,u) \in \mathbb{R}^n \times U$ then $K(\underline{y},\underline{u}) = N(\underline{z},\underline{u})$.
- Set $\ell(z,u) := g(z+y_\infty,u) - y_\infty$ for all $(z,u) \in \mathbb{R}^n \times U$ then from $y_{t+1} = g(y_t,u_t)$ we get the equivalent equation $z_{t+1} = \ell(z_t,u_t)$.

Now the main problem becomes the following one

$$\left.\begin{array}{rl} \text{Maximize} & N(\underline{z},\underline{u}) := \sum_{t=0}^{+\infty} \beta^t \varphi(z_t,u_t) \\ \text{when} & \underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N},\mathbb{R}^n), \\ & \underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N},U), \\ & z_0 = \eta - y_\infty, \\ & \forall t \in \mathbb{N},\ z_{t+1} = \ell(z_t,u_t), \end{array}\right\} \qquad (P1)$$

which has the form of supporting problem.

## 3.3 Some Useful Properties of Nemytskii Operators

Let $U$ be nonempty subset of $\mathbb{R}^d$. We recall the conditions $(\alpha.1)$ and $(\alpha.2)$ in Chapter 2. For the sake of consistence, in this chapter, we will call those conditions by $(\beta.1)$ and $(\beta.2)$.

$(\beta.1)$ $\zeta \in C^o(\mathbb{R}^n \times U, \mathbb{R}^m)$ .

$(\beta.2)$ For all $B$ bounded, nonempty in $U$, $\lim_{x\to 0}(\sup_{u\in B} \|\zeta(x,u)\|) = 0$.

Now to deal with partial Fréchet differential of Nemytskii operator, we introduce the following conditions.

$(\beta.3)$ For all $(x,u) \in \mathbb{R}^n \times U$, $D_1\zeta(x,u)$ exists and, for all $u \in U$, $D_1\zeta(.,u) \in C^0(\mathbb{R}^n,\ \mathcal{L}(\mathbb{R}^n,\mathbb{R}^m))$.

$(\beta.4)$ For all $B$ bounded nonempty in $U$, $\lim_{x\to 0}(\sup_{u\in B} \|D_1\zeta(x,u)\|) = 0$.

$(\beta.5)$ $D_1\zeta$ transforms the nonempty bounded subsets of $\mathbb{R}^n \times U$ into bounded subsets of $\mathbb{R}^m$.

$(\beta.6)$ For all $u \in U$, $\zeta(0,u) = 0$.

**Remark 3.1.** Under conditions $(\beta.3)$, $(\beta.5)$ and $(\beta.6)$, condition $(\beta.2)$ holds.

*Proof.* Let $B$ be a nonempty bounded subset of $U$. We fix $R \in (0, +\infty)$. For all $x \in \mathbb{R}^n$ such that $\|x\| \leq R$, using ($\beta$.3) and the Mean Value Theorem we have

$$\begin{aligned}\|\zeta(x,u)\| - \|\zeta(0,u)\| &\leq \|\zeta(x,u) - \zeta(0,u)\| \\ &\leq \sup_{z\in[0,x]} \|D_1\zeta(z,u)\| \, \|x\| \leq \|x\| \, . \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| \, .\end{aligned}$$

Then using ($\beta$.6) we have

$$\begin{aligned}\|\zeta(x,u)\| &\leq \|\zeta(0,u)\| + \|x\| \, . \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| \\ &\leq \|x\| \, . \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| \, .\end{aligned}$$

Take supremum when $u \in B$ on both sides of the last inequality, we obtain

$$\sup_{u\in B} \|\zeta(x,u)\| \leq \|x\| \, . \sup_{u\in B} \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| \, . \tag{3.1}$$

Let $V = \{z \in \mathbb{R}^n : \|z\| \leq R\} \times B$. Then $V$ is nonempty bounded subset of $\mathbb{R}^n \times U$. Using ($\beta$.5) we obtain $\sup_{u\in B} \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| = \sup_{(z,u)\in V} \|D_1\zeta(z,u)\| < +\infty$. Finally, take limit when $x \to 0$ on both sides of (3.1) we have

$$\begin{aligned}\lim_{x\to 0} \sup_{u\in B} \|\zeta(x,u)\| &\leq \lim_{x\to 0} \left( \|x\| \, . \sup_{u\in B} \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| \right) \\ &= \sup_{u\in B} \sup_{\|z\|\leq R} \|D_1\zeta(z,u)\| \lim_{x\to 0} \|x\| \\ &= 0.\end{aligned}$$

The final expression indicates that condition ($\beta$.2) holds. □

**Theorem 3.2.** *Let $U$ be nonempty subset of $\mathbb{R}^d$ and let $\zeta : \mathbb{R}^n \times U \to \mathbb{R}^m$ be a mapping. Under conditions ($\beta$.1), ($\beta$.3 - $\beta$.6), the following assertions hold.*

1. *$N_\zeta \in C^0(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U);\ c_0(\mathbb{N}, \mathbb{R}^m))$.*
2. *For all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, $D_1 N_\zeta\ (\underline{x}, \underline{u})$ exists.*
3. *For all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, the partial differential $D_1 N_\zeta(., \underline{u})$ belongs to $C^0(c_0(\mathbb{N}, \mathbb{R}^n);\ \mathcal{L}(c_0(\mathbb{N}, \mathbb{R}^n), c_0(\mathbb{N}, \mathbb{R}^m)))$. Moreover, for all $\underline{x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ and for all $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ we have*

$$D_1 N_\zeta(\underline{x}, \underline{u}).\underline{\delta x} = (D_1\zeta(x_t, u_t).\delta x_t)_{t\in\mathbb{N}}.$$

*Proof.* Under conditions ($\beta$.1), ($\beta$.3), ($\beta$.5) and ($\beta$.6), using Remark 3.1 we obtain $\zeta$ satisfies property ($\beta$.2). Then under ($\beta$.1) and ($\beta$.2), using Theorem 2.8 we deduce that $N_\zeta \in C^0(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U);\ c_0(\mathbb{N}, \mathbb{R}^m))$. Let $\mathcal{D}_1\mathcal{N} : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to c_0(\mathbb{N},\ \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ defined by $\mathcal{D}_1\mathcal{N}(\underline{x}, \underline{u}) := (D_1\zeta(x_t, u_t))_{t\in\mathbb{N}}$. Under ($\beta$.4) and ($\beta$.5), after Theorem 2.2, $\mathcal{D}_1\mathcal{N}$ is well defined.

Now we fix $\underline{u} \in \ell^\infty(\mathbb{N}, U)$. We take $\underline{x} = (x_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N}, \mathbb{R}^n)$ arbitrarily. Let $K = \overline{\{(x_t) : t \in \mathbb{N}\}}$ - the closure of the bounded set $\{(x_t) : t \in \mathbb{N}\}$ then $K$ is compact since $\{(x_t) : t \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}^n$. Let $L = \overline{\{x \in \mathbb{R}^n : d(x, K) < \rho\}}$. By the analogous argument as in Chapter 2, we obtain that $L$ is also compact. Under ($\beta$.3), by fixing $\underline{u}$ and proceeding as in the proof of Theorem 2.8 we deduce that for all $t \in \mathbb{N}$, $D_1\zeta(., u_t)$ is uniformly continuous on $L$ and hence, $\mathcal{D}_1\mathcal{N}(.,\ \underline{u})$ is of class $C^0$ on $c_0(\mathbb{N}, \mathbb{R}^n)$. That means for all $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, \rho)$ such that

$\|\mathcal{D}_1\mathcal{N}(\underline{x},\ \underline{u}) - \mathcal{D}_1\mathcal{N}(\underline{x}',\ \underline{u})\| < \varepsilon$ for all $\underline{x}' \in c_0(\mathbb{N},\mathbb{R}^n)$ satisfying $\|\underline{x} - \underline{x}'\| < \delta_\varepsilon$. Then we deduce that, for all $t \in \mathbb{N}$, $\|D_1\zeta(x_t,u_t) - D_1\zeta(x'_t,u_t)\| < \varepsilon$ whenever $\|x_t - x'_t\| < \delta_\varepsilon$. Let us consider the following expression

$$G_t = \left\|\zeta(x'_t,u_t) - \zeta(x_t,u_t) - D_1\zeta(x_t,u_t).(x'_t - x_t)\right\| \text{ where } t \in \mathbb{N}.$$

For all $t \in \mathbb{N}$, using Corollary 1.37 in Chapter 1, we have

$$G_t \leq \sup_{z_t\ \in[x_t,x'_t]} \|D_1\zeta(z_t,u_t) - D_1\zeta(x_t,u_t)\| \, \|\delta x_t\| \,,$$

where $\delta x_t := x'_t - x_t$. Now for all $t \in \mathbb{N}$, for all $z_t \in [x_t, x'_t]$ , we have$\|z_t - x_t\| \leq \|x_t - x'_t\| < \delta_\varepsilon$ and hence $\|D_1\zeta(z_t,u_t) - D_1\zeta(x_t,u_t)\| < \varepsilon$, which implies that $\sup_{z_t\ \in[x_t,x'_t]} \|D_1\zeta(z_t,u_t) - D_1\zeta(x_t,u_t)\| \leq \varepsilon$. So we have

$$\forall t \in \mathbb{N},\ G_t \leq \varepsilon \, \|\delta x_t\| \,.$$

Hence,

$$\begin{aligned}
&\sup_{t\in\mathbb{N}} G_t \leq \varepsilon.\, \|\underline{\delta x}\| \text{ where } \underline{\delta x} = (\delta x_t)_{t\in\mathbb{N}} \\
\iff &\left\|(\zeta(x'_t,u_t))_{t\in\mathbb{N}} - (\zeta(x_t,u_t))_{t\in\mathbb{N}} - (D_1\zeta(x_t,u_t))_{t\in\mathbb{N}}.(\delta x_t)_{t\in\mathbb{N}}\right\| \leq \varepsilon \, \|\underline{\delta x}\| \\
\iff &\left\|N_\zeta(\underline{x}',\underline{u}) - N_\zeta(\underline{x},\underline{u}) - \mathcal{D}_1\mathcal{N}(\underline{x},\underline{u}).\underline{\delta x}\right\| \leq \varepsilon \, \|\underline{\delta x}\| \,.
\end{aligned}$$

The last inequality means that $N_\zeta$ is partial Fréchet differentiable with respect to $\underline{x}$ at point $(\underline{x},\underline{u})$. Since the choice of $\underline{x} \in c_0(\mathbb{N},\mathbb{R}^n)$ is arbitrary, $N_\zeta$ is partially Fréchet differentiable with respect to $\underline{x}$ on $c_0(\mathbb{N},\mathbb{R}^n)$. Moreover, for all $\underline{x} \in c_0(\mathbb{N},\mathbb{R}^n)$ and for all $\underline{\delta x} \in c_0(\mathbb{N},\mathbb{R}^n)$ we have

$$D_1N_\zeta(\underline{x},\underline{u}).\underline{\delta x} = \mathcal{D}_1\mathcal{N}(\underline{x},\underline{u}).\underline{\delta x} = (D_1\zeta(x_t,u_t).\delta x_t)_{t\in\mathbb{N}}.$$

Therefore, $D_1N_\zeta(.,\underline{u})$, and $\mathcal{D}_1\mathcal{N}(.,\underline{u})$ are equal. And so, $D_1N_\zeta(.,\underline{u})$ is continuous on $c_0(\mathbb{N},\mathbb{R}^n)$ from the continuity of $\mathcal{D}_1\mathcal{N}(.,\underline{u})$. Hence, for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$, $D_1N_\zeta(.,\underline{u}) \in C^0(c_0(\mathbb{N},\mathbb{R}^n);\ \mathcal{L}(c_0(\mathbb{N},\mathbb{R}^n), c_0(\mathbb{N},\mathbb{R}^m)))$. □

**Theorem 3.3.** *Let $U$ be nonempty subset of $\mathbb{R}^d$ and let $\phi : \mathbb{R}^n \times U \to \mathbb{R}$ be a mapping which satisfies conditions ($\beta$.1) and ($\beta$.3) with $m = 1$. Then the following assertions hold*

**(i)** *$N_\phi : c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U) \to \ell^\infty(\mathbb{N},\mathbb{R})$ is well defined.*

**(ii)** *$N_\phi$ is continuous on $c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$.*

**(iii)** *For all $(\underline{x},\underline{u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$, $D_1N_\phi(\underline{x},\underline{u})$ exists and for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$, $D_1N_\phi(.,\underline{u})$ is continuous on $c_0(\mathbb{N},\mathbb{R}^n)$. Moreover, for all $\underline{x} \in c_0(\mathbb{N},\mathbb{R}^n)$ and for all $\underline{\delta x} \in c_0(\mathbb{N},\mathbb{R}^n)$ we have*

$$D_1N_\phi(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u}) = (D_1\phi(x_t,u_t).\delta x_t)_{t\in\mathbb{N}}.$$

*Proof.* The assertions (i) and (ii) are directly derived from (i) and (ii) of Theorem 2.14. Now we prove the assertion (iii). Under ($\beta$.3) we know that for all $(x,u) \in \mathbb{R}^n \times U$, $D_1\phi(x,u)$ exists and, for all $u \in U$, $D_1\phi(.,u) \in C^0(\mathbb{R}^n,\ \mathcal{L}(\mathbb{R}^n,\mathbb{R}))$. We fix $\underline{u} \in \ell^\infty(\mathbb{N},U)$ and we set $\mathcal{D}_1\Phi(.,\underline{u}) : c_0(\mathbb{N},\mathbb{R}^n) \to \ell^\infty(\mathbb{N},\mathcal{L}(\mathbb{R}^n,\mathbb{R}))$ where $\mathcal{D}_1\Phi(\underline{x},\underline{u}) := (D_1\phi(x_t,u_t))_{t\in\mathbb{N}}$. Let $\underline{x} \in c_0(\mathbb{N},\mathbb{R}^n)$ arbitrarily and let $V = \{x_t : t \in \mathbb{N}\}$. Then $V \subset \mathbb{R}^n$ and $V$ is bounded since $\sup_{t\in\mathbb{N}} \|x_t\|_\infty = \|\underline{x}\| < +\infty$. Since for all $t \in \mathbb{N}$, $D_1\phi(.,u_t)$ is continuous on $V$ then for all $t \in \mathbb{N}$, the image of $V$ under the mapping $D_1\phi(.,u_t)$ is bounded in $\mathcal{L}(\mathbb{R}^n,\mathbb{R})$. Then $\sup_{t\in\mathbb{N}} \|D_1\phi(x_t,u_t)\| < +\infty \implies \mathcal{D}_1\Phi(\underline{x},\underline{u}) \in \ell^\infty(\mathbb{N},\mathcal{L}(\mathbb{R}^n,\mathbb{R}))$. Hence, $\mathcal{D}_1\Phi(.,\underline{u}) : c_0(\mathbb{N},\mathbb{R}^n) \to \ell^\infty(\mathbb{N},\mathcal{L}(\mathbb{R}^n,\mathbb{R}))$ is well-defined. By an analogous argument

like that in Theorem 3.2, we obtain $\mathcal{D}_1\Phi(.,\underline{u})$ is of class $C^0$ on $c_0(\mathbb{N},\mathbb{R}^n)$ and assertion (iii) holds with $D_1N_\phi(.,\underline{u})$ coincides with $\mathcal{D}_1\Phi(.,\underline{u})$. Moreover, for all $\underline{x}\in c_0(\mathbb{N},\mathbb{R}^n)$ and for all $\underline{\delta x}\in c_0(\mathbb{N},\mathbb{R}^n)$ we have

$$D_1N_\phi(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u}) = (D_1\phi(x_t,u_t).\delta x_t)_{t\in\mathbb{N}}.$$

□

## 3.4 Static Optimization - a Result in Abstract Banach Spaces

In this section we recall a result issued from the book of Ioffe and Tihomirov [36]. As a corollary of the extremal principle in mixed problems (Theorem 3, page 71 in [36]), we obtain the following lemma.

**Lemma 3.4.** *Let $\mathcal{X}$, $\mathcal{V},\mathcal{W}$ be real Banach spaces and $\mathcal{U}$ be a nonempty set in $\mathcal{V}$. Let $\mathcal{J}:\mathcal{X}\times\mathcal{U}\to\mathbb{R}$ - a functional and $\Gamma:\mathcal{X}\times\mathcal{U}\to\mathcal{W}$ - a mapping. Let $(\hat{x},\hat{u})\in\mathcal{X}\times\mathcal{U}$ be a solution of the following optimization problem*

$$\left.\begin{array}{rl} \textit{Maximize} & \mathcal{J}(x,u) \\ \textit{when} & (\underline{x},\underline{u})\in\mathcal{X}\times\mathcal{U},\ \Gamma(x,u)=0. \end{array}\right\} \qquad (P)$$

*We assume that the following conditions are fulfilled.*

**(a)** *For all $u\in\mathcal{U}$, $[x\to\Gamma(x,u)]$ and $[x\to\mathcal{J}(x,u)]$ are of class $C^1$ at $\hat{x}$;*

**(b)** *There exists a neighborhood $\mathcal{N}$ of $\hat{x}$ in $\mathcal{X}$ such that for all $x\in\mathcal{N}$, for all $u',u''\in\mathcal{U}$ and for all $\theta\in[0,1]$ there exists $u\in\mathcal{U}$ satisfying*

$$\left\{\begin{array}{l} \Gamma(x,u)=(1-\theta)\Gamma(x,u')+\theta\Gamma(x,u''), \\ \mathcal{J}(x,u)\geq(1-\theta)J(x,u')+\theta J(x,u''). \end{array}\right.$$

**(c)** *The codimension of $D_1\Gamma(\hat{x},\hat{u})$ is finite.*

**(d)** *The set $\{D_1\Gamma(\hat{x},\hat{u}).x+\Gamma(\hat{x},u):x\in\mathcal{X},\ u\in\mathcal{U}\}$ contains a neighborhood of the origin of $\mathcal{W}$.*

*Then there exists $M\in\mathcal{W}^*$ such that the following assertions hold.*

**(i)** $D_1\mathcal{J}(\hat{x},\hat{u})+M\circ D_1\Gamma(\hat{x},\hat{u})=0.$

**(ii)** *For all $u\in\mathcal{U}$, $\mathcal{J}(\hat{x},\hat{u})+\langle M,\Gamma(\hat{x},\hat{u})\rangle\geq\mathcal{J}(\hat{x},u)+\langle M,\Gamma(\hat{x},u)\rangle$.*

## 3.5 Strong Pontryagin Principle for Problem (Ps)

First we introduce the Hamiltonian of Pontryagin which is defined, for all $t\in\mathbb{N}$, as follows

$$H_t:\mathbb{R}^n\times U\times\mathbb{R}^{n*}\to\mathbb{R},\ H_t(x,u,p):=\beta^t\phi(x,u)+\langle p,f(x,u)\rangle. \qquad (3.2)$$

Note that the condition (WM1) of Theorem 2.22 in Chapter 2 is equivalent to the following condition

$$\forall u\in U,\forall t\in\mathbb{N},\langle D_2H_t(\hat{x}_t,\hat{u}_t,p_{t+1}),u-\hat{u}_t\rangle\leq 0.$$

In this section, we will replace (WM1) by the strengthened condition $H_t(\hat{x}_t,\hat{u}_t,p_{t+1})=\max_{u\in U}H_t(\hat{x}_t,u,p_{t+1})$ for all $t\in\mathbb{N}$. Note that (WM1) can be considered as a first-order necessary condition of the optimality of $H_t(\hat{x}_t,.,p_{t+1})$ at $\hat{u}_t$ on $U$.

We consider the following conditions:

**(C1)** $U$ is a nonempty compact subset of $\mathbb{R}^d$.

**(C2)** $\phi \in C^0(\mathbb{R}^n \times U,\ \mathbb{R})$ and $f \in C^0(\mathbb{R}^n \times U,\ \mathbb{R}^n)$.

**(C3)** For all $u \in U,\ f(0,u) = 0$.

**(C4)** For all $u \in U,\ D_1\phi(x,u)$ and $D_1 f(x,u)$ exist for all $x \in \mathbb{R}^n$, and $D_1\phi(.,u) \in C^0(\mathbb{R}^n,\ \mathbb{R}^{n*})$, and $D_1 f(.,u) \in C^0(\mathbb{R}^n,\ \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))$.

**(C5)** $D_1 f$ transforms the nonempty bounded subsets of $\mathbb{R}^n \times U$ into bounded subsets of $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$.

**(C6)** For all $B$ bounded, nonempty set in $U$ we have

$$\lim_{x\to 0}(\sup_{u\in B}\|D_1 f(x,u)\|) = 0.$$

**(C7)** For all $t \in \mathbb{N}$, for all $x_t \in \mathbb{R}^n$, for all $u'_t,\ u''_t\ \in U$ and for all $\theta \in (0,1)$ there exists $u_t \in U$ such that

$$\begin{cases} \phi(x_t,u_t) \geq (1-\theta)\phi(x_t,u'_t) + \theta\phi(x_t,u''_t), \\ f(x_t,u_t) = (1-\theta)f(x_t,u'_t) + \theta f(x_t,u''_t). \end{cases}$$

Using these conditions, we will prove the smoothness properties of the criterion $J$ and the Nemytskii operator $F$ in the following lemmas.

**Lemma 3.5.** *Under the conditions (C1), (C2) and (C4), for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$ the functional $\underline{x} \mapsto J(\underline{x},\underline{u})$ is of class $C^1$ on $c_0(\mathbb{N},\mathbb{R}^n)$ and moreover, for all $(\underline{x},\underline{u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ and for all $\underline{\delta x} \in c_0(\mathbb{N},\mathbb{R}^n)$ the following formula holds*

$$D_1 J(\underline{x},\underline{u}).\underline{\delta x} = \sum_{t=0}^{+\infty}\beta^t D_1\phi(x_t,u_t)\delta x_t.$$

*Proof.* Under the condition (C1), $U$ is closed. Under conditions (C2) and (C4), the Nemytskii operator $N_\phi : c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U) \to \ell^\infty(\mathbb{N},\mathbb{R})$ defined by $N_\phi(\underline{x},\underline{u}) := (\phi(x_t,u_t))_{t\in\mathbb{N}}$ is of class $C^0$ on $c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$ using assertions (i) and (ii) of Theorem 3.3. Besides, from assertion (iii) of Theorem 3.3, we know that for all $(\underline{x},\underline{u}) \in c_0(\mathbb{N},\mathbb{R}^n) \times \ell^\infty(\mathbb{N},U)$, $D_1 N_\phi(\underline{x},\underline{u})$ exists; for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$, $D_1 N_\phi(.,\underline{u})$ is continuous on $c_0(\mathbb{N},\mathbb{R}^n)$ and moreover, for all $\underline{x} \in c_0(\mathbb{N},\mathbb{R}^n)$ and for all $\underline{\delta x} \in c_0(\mathbb{N},\mathbb{R}^n)$ we have

$$D_1 N_\zeta(\underline{x},\underline{u}).(\underline{\delta x},\underline{\delta u}) = (D_1\zeta(x_t,u_t).\delta x_t)_{t\in\mathbb{N}}.$$

Consider the functional $S : \ell^\infty(\mathbb{N},\mathbb{R}) \to \mathbb{R}$ defined by $S(\underline{r}) := \sum_{t=0}^{+\infty}\beta^t r^t$. In Chapter 2, we have proven that $S$ is of class $C^1$ and for all $\underline{r},\ \underline{\delta r} \in \ell^\infty(\mathbb{N},\mathbb{R})$, we have $DS(\underline{r}).\underline{\delta r} = S(\underline{\delta r}) = \sum_{t=0}^{+\infty}\beta^t\delta r^t$. We knew that $J = S \circ N_\phi$. Since $S$ is of class $C^1$ and for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$, $N_\phi(.,\underline{u})$ is of class $C^1$ then for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$, $[\ \underline{x} \mapsto J(\underline{x},\underline{u})]$ is of class $C^1$on $c_0(\mathbb{N},\mathbb{R}^n)$ as a composition of two $C^1$ - mappings. Using the chain rule we obtain that for all $\underline{u} \in \ell^\infty(\mathbb{N},U)$, for all $\underline{x}$, $\underline{\delta x} \in c_0(\mathbb{N},\mathbb{R}^n)$ we have

$$\begin{aligned} D_1 J(\underline{x},\underline{u}).\underline{\delta x} &= DS(N_\phi(\underline{x},\underline{u})) \circ D_1 N_\phi(\underline{x},\underline{u}).\underline{\delta x} \\ &= S(D_1 N_\phi(\underline{x},\underline{u})).\underline{\delta x} \\ &= \sum_{t=0}^{+\infty}\beta^t D_1\phi(x_t,u_t)\delta x_t. \end{aligned}$$

□

**Lemma 3.6.** *Under the conditions (C1-C6), for all* $\underline{u} \in \ell^\infty(\mathbb{N}, U)$ *the mapping* $[\underline{x} \mapsto F(\underline{x}, \underline{u})]$ *are of class* $C^1$*on* $c_0(\mathbb{N}, \mathbb{R}^n)$ *and moreover, for all* $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ *and for all* $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ *the following formula holds*

$$D_1F(\underline{x}, \underline{u}).\underline{\delta x} = (D_1f(x_t, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}}. \tag{3.3}$$

*Proof.* Under conditions (C1-C6), we can assert that $U$ is closed and that the mapping $f$ satisfies all the conditions $(\beta.1)$, $(\beta.3 - \beta.6)$ (by replacing $\zeta$ by $f$). Using Theorem 3.2, we obtain $N_f \in C^0(c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U);\ c_0(\mathbb{N}, \mathbb{R}^n))$; for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, $D_1N_f\ (\underline{x}, \underline{u})$ exists; and for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, $D_1N_f\ (., \underline{u}) \in C^0(c_0(\mathbb{N}, \mathbb{R}^n);\ \mathcal{L}(c_0(\mathbb{N}, \mathbb{R}^n), c_0(\mathbb{N}, \mathbb{R}^n)))$. Moreover, for all $\underline{x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ and for all $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ we have

$$D_1N_f(\underline{x}, \underline{u}).\underline{\delta x} = (D_1f(x_t, u_t).\delta x_t)_{t\in\mathbb{N}}.$$

Now we set $A(\underline{x}, \underline{u}) := (-x_{t+1})_t$. As we knew before, $A$ is a bounded linear operator from $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ into $c_0(\mathbb{N}, \mathbb{R}^n)$; operator $A$ is Fréchet differentiable and its partial differential with respect to $\underline{x}$ is $D_1A(\underline{x}, \underline{u}) \cdot \underline{\delta x} = A(\underline{\delta x}, \underline{u}) = (-\delta x_{t+1})_{t\in\mathbb{N}}$. Obviously, for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, $D_1A(., \underline{u})$ is continuous on $c_0(\mathbb{N}, \mathbb{R}^n)$. In Chapter 2, we knew that $F(\underline{x}, \underline{u}) = N_f(\underline{x}, \underline{u}) + A(\underline{x}, \underline{u})$ is a mapping from $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ into $c_0(\mathbb{N}, \mathbb{R}^n)$. Using this fact we can easily obtain these following results when we fix $\underline{u} \in \ell^\infty(\mathbb{N}, U)$.

- $F(., \underline{u})$ is a mapping from $c_0(\mathbb{N}, \mathbb{R}^n)$ into itself.
- $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is of class $C^0$ on $c_0(\mathbb{N}, \mathbb{R}^n)$ because $F = N_f + A$ is of class $C^0$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ as a sum of two $C^0$- mappings on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$.
- $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is Fréchet differentiable on $c_0(\mathbb{N}, \mathbb{R}^n)$. This is a consequence of the existence of $D_1N_f$ and $D_1A$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Moreover, for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and for all $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ we have

$$\begin{aligned} D_1F(\underline{x}, \underline{u}).\underline{\delta x} &= D_1N_f(\underline{x}, \underline{u}).\underline{\delta x} + D_1A(\underline{x}, \underline{u}).\underline{\delta x} \\ &= (D_1f(x_t, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}}. \end{aligned}$$

- $\underline{x} \mapsto D_1F(\underline{x}, \underline{u})$ is continuous on $c_0(\mathbb{N}, \mathbb{R}^n)$ as a sum of two continuous mappings on $c_0(\mathbb{N}, \mathbb{R}^n)$ $(D_1F(., \underline{u}), = D_1N_f(., \underline{u}) + D_1A(., \underline{u}))$.

And so, for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$ the mapping $\underline{x} \mapsto F(\underline{x}, \underline{u})$ are of class $C^1$on $c_0(\mathbb{N}, \mathbb{R}^n)$ and formula (3.3) holds. □

Now for $\mathcal{T}(\underline{x}, \underline{u}) = (F(\underline{x}, \underline{u}),\ h(\underline{x}, \underline{u}))$, from above-mentioned argument we already know that for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$ the mapping $\underline{x} \mapsto F(\underline{x}, \underline{u})$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n)$. In Chapter 2, we have proven that $h$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$. Hence, it is trivial that for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, the mapping $\underline{x} \mapsto h(\underline{x}, \underline{u})$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n)$ and moreover, for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, for all $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$ we have $D_1h(\underline{x}, \underline{u}) \cdot \underline{\delta x} = \delta x_0$. Then for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, the mapping $\underline{x} \mapsto \mathcal{T}(\underline{x}, \underline{u})$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n)$ and moreover, for all $(\underline{x}, \underline{u}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$, for all $\underline{\delta x} \in c_0(\mathbb{N}, \mathbb{R}^n)$, the following formula holds.

$$\begin{aligned} D_1\mathcal{T}(\underline{x}, \underline{u}).\underline{\delta x} &= (D_1F(\underline{x}, \underline{u}).\underline{\delta x},\ D_1h(\underline{x}, \underline{u}).\underline{\delta x}) \\ &= ((D_1f(x_t, u_t)\delta x_t - \delta x_{t+1})_{t\in\mathbb{N}},\ \delta x_0). \end{aligned}$$

In Chapter 2, we have translated Problem $(P_s)$ into the following form

$$(P2)\begin{cases} \text{Maximize} & J(\underline{x}, \underline{u}) \\ \quad\text{when} & \underline{x} \in c_0(\mathbb{N}, \mathbb{R}^n), \underline{u} \in \ell^\infty(\mathbb{N}, U) \\ & \mathcal{T}(\underline{x}, \underline{u}) = \underline{0}, \end{cases}$$

where $\mathcal{T} : c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U) \to c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n$ and $\mathcal{T}(\underline{x}, \underline{u}) = (F(\underline{x}, \underline{u}),\ h(\underline{x}, \underline{u}))$ . Recall that $F(\underline{x}, \underline{u}) := (f(x_t, u_t) - x_{t+1})_{t\in\mathbb{N}}$ and $h(\underline{x}, \underline{u}) := x_0 - \eta$. We will make use of Lemma 3.4 and these above-mentioned lemmas to prove the following result.

**Lemma 3.7.** *Under assumptions (C1-C7), let $(\underline{\hat{x}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of Problem (P2). Then there exists $(\underline{q}, \mu) \in \ell^1(\mathbb{N}, \mathbb{R}^{n*}) \times \mathbb{R}^{n*}$ which satisfies the following properties.*

1. $D_1 J(\underline{\hat{x}}, \underline{\hat{u}}) + (\underline{q}, \mu) \circ D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) = 0.$
2. $J(\underline{\hat{x}}, \underline{\hat{u}}) + \left\langle (\underline{q}, \mu), \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) \right\rangle = \max_{\underline{u}\in\ell^\infty(\mathbb{N},U)} \left( J(\underline{\hat{x}}, \underline{u}) + \left\langle (\underline{q}, \mu), \mathcal{T}(\underline{\hat{x}}, \underline{u}) \right\rangle \right).$

*Proof.* We want to apply Lemma 3.4 with $\mathcal{J} = J$ and $\Gamma = \mathcal{T}$. Under assumptions (C1-C6), after Lemma 3.5 and Lemma 3.6 we know that for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, the mappings $\underline{x} \mapsto \mathcal{T}(\underline{x}, \underline{u})$ and $\underline{x} \mapsto J(\underline{x}, \underline{u})$ are of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n)$. Hence, assumption (a) of Lemma 3.4 is fulfilled.

Let $\underline{x} = (x_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N}, \mathbb{R}^n)$, $\underline{u}' = (u'_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, U)$, $\underline{u}'' = (u''_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, U)$ and $\theta \in [0, 1]$ arbitrarily. Using condition (C7), we know that for each $t \in \mathbb{N}$, there exists an element $u_t \in U$ such that

$$\phi(x_t, u_t) \geq (1 - \theta)\phi(x_t, u'_t) + \theta\phi(x_t, u''_t), \tag{3.4}$$

and

$$f(x_t, u_t) = (1 - \theta)f(x_t, u'_t) + \theta f(x_t, u''_t). \tag{3.5}$$

We set $\underline{u} = (u_t)_{t\in\mathbb{N}}$ then $\underline{u} \in \ell^\infty(\mathbb{N}, U)$ since $U$ is compact (condition (C1)). From equation (3.4), for any $\beta \in [0, 1]$ we have:

$$\beta^t\phi(x_t, u_t) \geq (1 - \theta)\beta^t\phi(x_t, u'_t) + \theta\beta^t\phi(x_t, u''_t), \tag{3.6}$$

Take sum both sides of equation (3.6) for all $t$ in $\mathbb{N}$ we have:

$$\sum_{t\in\mathbb{N}} \beta^t\phi(x_t, u_t) \geq (1 - \theta)\sum_{t\in\mathbb{N}} \beta^t\phi(x_t, u'_t) + \theta\sum_{t\in\mathbb{N}} \beta^t\phi(x_t, u''_t),$$

or equivalently,

$$J(\underline{x}, \underline{u}) \geq (1 - \theta)J(\underline{x}, \underline{u}') + \theta J(\underline{x}, \underline{u}'').$$

Now, equation (3.5) holds for all $t \in \mathbb{N}$, hence

$$(f(x_t, u_t))_{t\in\mathbb{N}} = (1 - \theta)(f(x_t, u'_t))_{t\in\mathbb{N}} + \theta(f(x_t, u''_t))_{t\in\mathbb{N},}$$

or equivalently,

$$F(\underline{x}, \underline{u}) = (1 - \theta)F(\underline{x}, \underline{u}') + \theta F(\underline{x}, \underline{u}'').$$

Besides, by the virtue of the definition of $h$, we always have

$$h(\underline{x}, \underline{u}) = (1 - \theta)h(\underline{x}, \underline{u}') + \theta h(\underline{x}, \underline{u}'').$$

From the last two equations, we obtain

$$\mathcal{T}(\underline{x}, \underline{u}) = (1 - \theta)\mathcal{T}(\underline{x}, \underline{u}') + \theta\mathcal{T}(\underline{x}, \underline{u}'').$$

And so assumption (b) of Lemma 3.4 is fulfilled.

To verify assumption (c) of Lemma 3.4, we see that under conditions (C1-C6), after Lemma 3.6 we know that the for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, the mapping $\underline{x} \mapsto \mathcal{T}(\underline{x}, \underline{u})$ is of class $C^1$ on $c_0(\mathbb{N}, \mathbb{R}^n)$. Hence, the partial Fréchet differential with respect to $\underline{x}$

of mapping $\mathcal{T}$ exists at the optimal solution $(\underline{\hat{x}}, \underline{\hat{u}})$. We set $\hat{B} := \{\hat{u}_t : t \in \mathbb{N}\}$ then $\hat{B}$ is nonempty bounded set in $U$ since $\underline{\hat{u}} \in \ell^\infty(\mathbb{N}, U)$. For all $t \in \mathbb{N}$, we have

$$0 \leq \|D_1 f(\hat{x}_t, \hat{u}_t)\|_{\mathcal{L}} \leq \sup_{u \in \hat{B}} \|D_1 f(\hat{x}_t, u)\|_{\mathcal{L}},$$

and therefore, using (C6) and Remark 2.6 we obtain $\lim_{t \to +\infty} \|D_1 f(\hat{x}_t, \hat{u}_t)\|_{\mathcal{L}} = 0$. Hence, there exists $t_* \in \mathbb{N}$ such that $\sup_{t \geq t_*} \|D_1 f(\hat{x}_t, \hat{u}_t)\|_{\mathcal{L}} < 1$. Then proceeding as in Corollary 2.16, and using Lemma 2.21, we deduce that $\operatorname{Im} D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) = c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n$. That means the codimension of $D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}})$ is finite. Hence, assumption (c) of Lemma 3.4 is fulfilled.

Finally, after assumption (c), $\operatorname{Im} D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) = c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n$ then it is clear that the set $\{D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}).x + \mathcal{T}(\underline{\hat{x}}, u) : \underline{x} \in c_0(\mathbb{N}, \mathbb{R}^n),\ \underline{u} \in \ell^\infty(\mathbb{N}, U)\} = \operatorname{Im} D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) + \mathcal{T}(\{\underline{\hat{x}}\} \times \ell^\infty(\mathbb{N}, U))$ contains $\operatorname{Im} D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}})$, and consequently, contains a neighborhood of the origin of $c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n$. Hence, assumption (d) of Lemma 3.4 is fulfilled.

Now we can use Lemma 3.4 and we obtain $(\underline{q}, \mu) \in (c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n)^* = \ell^1(\mathbb{N}, \mathbb{R}^{n*}) \times \mathbb{R}^{n*}$ that satisfied the announced conclusions. □

We will prove the following theorem which is called strong Pontryagin principle for Problem $(P_s)$.

**Theorem 3.8.** *Assume that conditions (C1)-(C7) are fulfilled. Let $(\underline{\hat{x}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of Problem $(P_s)$. Then there exists $(p_t)_{t \in \mathbb{N}_*} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ such that the following statements hold:*

**(AE1)** $p_t = p_{t+1} \circ D_1 f(\hat{x}_t, \hat{u}_t) + \beta^t . D_1 \phi(\hat{x}_t, \hat{u}_t)$ *for all* $t \in \mathbb{N}_*$.

**(MP1)** $\beta^t \phi(\hat{x}_t, \hat{u}_t) + p_{t+1}, f(\hat{x}_t, \hat{u}_t) = \max_{u \in U}(\beta^t \phi(\hat{x}_t, u) + \langle p_{t+1}, f(\hat{x}_t, u)\rangle)$ *for all* $t \in \mathbb{N}$.

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of Problem $(P_s)$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is also a solution of Problem (P2). Under conditions (C1-C7), after Lemma 3.7 we obtain the existence of $(\underline{q}, \mu) \in (c_0(\mathbb{N}, \mathbb{R}^n) \times \mathbb{R}^n)^* = \ell^1(\mathbb{N}, \mathbb{R}^{n*}) \times \mathbb{R}^{n*}$ such that

$$\left.\begin{array}{l} D_1 J(\underline{\hat{x}}, \underline{\hat{u}}) + (\underline{q}, \mu) \circ D_1 \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) = 0, \\ J(\underline{\hat{x}}, \underline{\hat{u}}) + \left\langle (\underline{q}, \mu), \mathcal{T}(\underline{\hat{x}}, \underline{\hat{u}}) \right\rangle = \max_{\underline{u} \in \ell^\infty(\mathbb{N}, U)} \left( J(\underline{\hat{x}}, \underline{u}) + \left\langle (\underline{q}, \mu), \mathcal{T}(\underline{\hat{x}}, \underline{u}) \right\rangle \right). \end{array}\right\} \tag{3.7}$$

Then from first equality of (3.7), proceeding like in the proof of Theorem 2.22 in Chapter 2, we obtain (AE1). We rewrite the second equality of (3.7) as follows

$$\begin{array}{l} J(\underline{\hat{x}}, \underline{\hat{u}}) + \left\langle \underline{q}, F(\underline{\hat{x}}, \underline{\hat{u}}) \right\rangle + \langle \mu, h(\underline{\hat{x}}, \underline{\hat{u}}) \rangle \\ = \max\limits_{\underline{u} \in \ell^\infty(\mathbb{N}, U)} (J(\underline{\hat{x}}, \underline{u}) + \left\langle \underline{q}, F(\underline{\hat{x}}, \underline{u}) \right\rangle + \langle \mu, h(\underline{\hat{x}}, \underline{u}) \rangle). \end{array} \tag{3.8}$$

Since for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, $h(\underline{\hat{x}}, \underline{u}) = \hat{x}_0 - \eta = h(\underline{\hat{x}}, \underline{\hat{u}})$ then we have

$$(3.8) \Leftrightarrow J(\underline{\hat{x}}, \underline{\hat{u}}) + \left\langle \underline{q}, F(\underline{\hat{x}}, \underline{\hat{u}}) \right\rangle = \max_{\underline{u} \in \ell^\infty(\mathbb{N}, U)} (J(\underline{\hat{x}}, \underline{u}) + \left\langle \underline{q}, F(\underline{\hat{x}}, \underline{u}) \right\rangle).$$

That means for all $\underline{u} \in \ell^\infty(\mathbb{N}, U)$,

$$\begin{array}{rl} & \sum\limits_{t=0}^{+\infty} \beta^t \phi(\hat{x}_t, \hat{u}_t) + \sum_{t=0}^{+\infty} \langle q_t, f(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle \\ & \quad \geq \sum\limits_{t=0}^{+\infty} \beta^t \phi(\hat{x}_t, u_t) + \sum_{t=0}^{+\infty} \langle q_t, f(\hat{x}_t, u_t) - \hat{x}_{t+1} \rangle \\ \Leftrightarrow & \sum\limits_{t=0}^{+\infty} \beta^t \phi(\hat{x}_t, \hat{u}_t) + \sum_{t=0}^{+\infty} \langle q_t, f(\hat{x}_t, \hat{u}_t) \rangle \\ & \quad \geq \sum\limits_{t=0}^{+\infty} \beta^t \phi(\hat{x}_t, u_t) + \sum_{t=0}^{+\infty} \langle q_t, f(\hat{x}_t, u_t) \rangle . \end{array}$$

Now we arbitrarily fix $t \in \mathbb{N}$. Take $\underline{u} = (u_s)_{s\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, U)$ such that when $s \neq t,\ u_s = \hat{u}_s$ and $u_t$ varies in $U$. Then from the last inequality we have

$$\begin{aligned} & \forall u_t \in U,\ \beta^t\phi(\hat{x}_t, \hat{u}_t) + \langle p_t, f(\hat{x}_t, \hat{u}_t)\rangle \geq \beta^t\phi(\hat{x}_t, u_t) + \langle q_t, f(\hat{x}_t, u_t)\rangle \\ \Leftrightarrow \quad & \beta^t\phi(\hat{x}_t, \hat{u}_t) + \langle q_t, f(\hat{x}_t, \hat{u}_t)\rangle = \max_{u_t\in U}\left(\beta^t\phi(\hat{x}_t, u_t) + \langle q_t, f(\hat{x}_t, u_t)\rangle\right). \end{aligned}$$

This expression is satisfied for all $t \in \mathbb{N}$. So we have

$$\forall t \in \mathbb{N},\ \beta^t\phi(\hat{x}_t, \hat{u}_t) + \langle q_t, f(\hat{x}_t, \hat{u}_t)\rangle = \max_{u\in U}\Big(\beta^t\phi(\hat{x}_t, u) + \langle q_t, f(\hat{x}_t, u)\rangle\Big). \tag{3.9}$$

From (3.9), by setting $p_{t+1} := q_t$ for all $t \in \mathbb{N}$, we obtain $\underline{p} = (p_{t+1})_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ such that (SM1) holds. The proof is complete. □

**Remark 3.9.** In Theorem 3.8, (SM1) means Strong Maximum principle.

## 3.6 Strong Pontryagin Principle for Main Problem

In previous section, we already proven the strong Pontryagin principle for supporting problem. Now we study that for main problem. Notice that main problem can be translated into a problem in the form of supporting problem and we named it by $(P1)$ as follows

$$\left.\begin{array}{rl} \text{Maximize} & N(\underline{z}, \underline{u}) := \sum_{t=0}^{+\infty}\beta^t\varphi(z_t, u_t) \\ \text{when} & \underline{z} = (z_t)_{t\in\mathbb{N}} \in c_0(\mathbb{N}, \mathbb{R}^n), \\ & \underline{u} = (u_t)_{t\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, U), \\ & z_0 = \eta - y_\infty, \\ & \forall t \in \mathbb{N},\ z_{t+1} = \ell(z_t, u_t). \end{array}\right\} \tag{$P1$}$$

Apply Theorem 3.8 to Problem $(P1)$, we obtain the following statement.
Let $(\underline{\hat{z}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of Problem $(P_m)$. Assume that these following conditions hold.

**(1)** $U$ is a nonempty compact subset of $\mathbb{R}^d$.

**(2)** $\varphi \in C^0(\mathbb{R}^n \times U,\ \mathbb{R})$ and $\ell \in C^0(\mathbb{R}^n \times U,\ \mathbb{R}^n)$.

**(3)** For all $u \in U,\ \ell(0, u) = 0$.

**(4)** For all $u \in U,\ D_1\varphi(z, u)$ and $D_1\ell(z, u)$ exist for all $z \in \mathbb{R}^n$, and $D_1\varphi(., u) \in C^0(\mathbb{R}^n,\ \mathbb{R}^{n*})$, and $D_1\ell(., u) \in C^0(\mathbb{R}^n,\ \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$.

**(5)** $D_1\ell$ transforms the nonempty bounded subsets of $\mathbb{R}^n \times U$ into bounded subsets of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

**(6)** For all $B$ bounded, nonempty set in $U$ we have

$$\lim_{z\to 0}(\sup_{u\in B}\|D_1\ell(z, u)\|) = 0.$$

**(7)** For all $t \in \mathbb{N}$, for all $z_t \in \mathbb{R}^n$, for all $u'_t,\ u''_t \in U$ and for all $\theta \in [0, 1]$ there exists $u_t \in U$ such that

$$\begin{cases} \varphi(z_t, u_t) \geq (1-\theta)\varphi(z_t, u'_t) + \theta\varphi(z_t, u''_t), \\ \ell(z_t, u_t) = (1-\theta)\ell(z_t, u'_t) + \theta l(z_t, u''_t). \end{cases}$$

Then there exists $(p_t)_{t\in\mathbb{N}_*} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ such that

**(i)** For all $t \in \mathbb{N}_*,\ p_t = p_{t+1} \circ D_1\ell(\hat{z}_t, \hat{u}_t) + \beta^t.D_1\varphi(\hat{z}_t, \hat{u}_t)$;

**(ii)** For all $t \in \mathbb{N}$, for all $u \in U$,

$$\beta^t \varphi(\hat{z}_t, \hat{u}_t) + \langle p_{t+1}, \ell(\hat{z}_t, \hat{u}_t)\rangle \geq \beta^t \varphi(\hat{z}_t, u) + \langle p_{t+1}, \ell(\hat{z}_t, u)\rangle .$$

We study the statement of strong Pontryagin principle for main problem based on the statement above. We know that if $(\underline{\hat{z}}, \underline{\hat{u}})$ is a solution of Problem $(P1)$ then $(\underline{\hat{y}}, \underline{\hat{u}})$ is a solution of Problem $(P_m)$ where $\hat{y}_t = \hat{z}_t + y_\infty$, for all $t \in \mathbb{N}$.

**Study the assumptions:** similar to what we did in Chapter 2, the assumptions (1-7) for Problem $(P1)$ can be rewritten equivalently for Problem $(P_m)$ as follows.

**(D1)** $U$ is a nonempty compact subset of $\mathbb{R}^d$.

**(D2)** $\psi \in C^0(\mathbb{R}^n \times U,\ \mathbb{R})$ and $g \in C^0(\mathbb{R}^n \times U,\ \mathbb{R}^n)$.

**(D3)** For all $u \in U,\ g(y_\infty, u) = 0$.

**(D4)** For all $u \in U,\ D_1\psi(y, u)$ and $D_1 g(y, u)$ exist for all $y \in \mathbb{R}^n$, and $D_1\psi(., u) \in C^0(\mathbb{R}^n,\ \mathbb{R}^{n*})$, and $D_1 g(., u) \in C^0(\mathbb{R}^n,\ \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$.

**(D5)** $D_1 g$ transforms the nonempty bounded subsets of $\mathbb{R}^n \times U$ into bounded subsets of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

**(D6)** For all $B$ bounded, nonempty set in $U$ we have

$$\lim_{y \to y_\infty} (\sup_{u \in B} \|D_1 g(y, u)\|) = 0.$$

For assumption (7), using the definition of mappings $\varphi$ and $\ell$ (as we know before, $\varphi(z, u) = \psi(z + y_\infty, u)$; $\ell(z, u) = g(z + y_\infty, u) - y_\infty$ for all $(z, u) \in \mathbb{R}^n \times U$) and recall that $z = y - y_\infty$ for all $z \in \mathbb{R}^n$, we have the following equivalent transformations

$$\begin{aligned} & \varphi(z_t, u_t) \geq (1-\theta)\varphi(z_t, u_t') + \theta\varphi(z_t, u_t'') \\ \Leftrightarrow\ & \psi(z_t + y_\infty, u_t) \geq (1-\theta)\psi(z_t + y_\infty, u_t') + \theta\psi(z_t + y_\infty, u_t'') \\ \Leftrightarrow\ & \psi(y_t, u_t) \geq (1-\theta)\psi(y_t, u_t') + \theta\psi(y_t, u_t''), \end{aligned}$$

and

$$\begin{aligned} & \ell(z_t, u_t) = (1-\theta)\ell(z_t, u_t') + \theta h(z_t, u_t'') \\ \Leftrightarrow\ & g(z_t + y_\infty, u_t) - y_\infty = \ (1-\theta)\left(g(z_t + y_\infty, u_t') - y_\infty\right) \\ & \qquad\qquad + \theta\left(g(z + y_\infty, u_t'') - y_\infty\right) \\ \Leftrightarrow\ & g(z_t + y_\infty, u_t) = (1-\theta)g(z_t + y_\infty, u_t') + \theta g(z + y_\infty, u_t'') \\ \Leftrightarrow\ & g(y_t, u_t) = (1-\theta)g(y_t, u_t') + \theta g(y_t, u_t''). \end{aligned}$$

Then we obtain assumption (D7) as follows

**(D7)** For all $t \in \mathbb{N}$, for all $y_t \in \mathbb{R}^n$, for all $u_t',\ u_t'' \in U$ and for all $\theta \in [0, 1]$ there exists $u_t \in U$ such that

$$\begin{cases} \psi(y_t, u_t) \geq (1-\theta)\psi(y_t, u_t') + \theta\psi(y_t, u_t''), \\ g(y_t, u_t) = (1-\theta)g(y_t, u_t') + \theta g(y_t, u_t''). \end{cases}$$

**Study the conclusions:** By the same reasoning as in Chapter 2, conclusion (i) can be rewritten as follows.

**(i)** For all $t \in \mathbb{N}_*,\ p_t = p_{t+1} \circ D_1 g(\hat{y}_t, \hat{u}_t) + \beta^t . D_1\psi(\hat{y}_t, \hat{u}_t)$.

For conclusion (ii), we make the following equivalent transformations:

$$\begin{aligned} & \beta^t \varphi(\hat{z}_t, \hat{u}_t) + \langle p_{t+1}, \ell(\hat{z}_t, \hat{u}_t)\rangle \geq \beta^t \varphi(\hat{z}_t, u) + \langle p_{t+1}, \ell(\hat{z}_t, u)\rangle \\ \Leftrightarrow\ & \beta^t \psi(\hat{y}_t, \hat{u}_t) + \langle p_{t+1}, g(\hat{y}_t, \hat{u}_t) - y_\infty\rangle \geq \beta^t \psi(\hat{y}_t, u) + \langle p_{t+1}, g(\hat{y}_t, u) - y_\infty\rangle \\ \Leftrightarrow\ & \beta^t \psi(\hat{y}_t, \hat{u}_t) + \langle p_{t+1}, g(\hat{y}_t, \hat{u}_t)\rangle \geq \beta^t \psi(\hat{y}_t, u) + \langle p_{t+1}, g(\hat{y}_t, u)\rangle . \end{aligned}$$

Then we can rewrite (ii) as follows

**(ii)** For all $t \in \mathbb{N}$, for all $u \in U$: $\beta^t \psi(\hat{y}_t, \hat{u}_t) + \langle p_{t+1}, g(\hat{y}_t, \hat{u}_t)\rangle \geq \beta^t \psi(\hat{y}_t, u) + \langle p_{t+1}, g(\hat{y}_t, u)\rangle$.

Finally, we state the strong Pontryagin principle for Problem $(P_m)$:

**Theorem 3.10.** *Let $(\underline{\hat{y}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ be a solution of Problem $(P_m)$. Assume that the assumptions (D1-D7) are fulfilled then there exists $(p_{t+1})_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ such that the following assertions hold.*

**(AE)** $p_t = p_{t+1} \circ D_1 g(\hat{y}_t, \hat{u}_t) + \beta^t . D_1 \psi(\hat{y}_t, \hat{u}_t)$ *for all* $t \in \mathbb{N}_*$.

**(SM)** $\beta^t \psi(\hat{y}_t, \hat{u}_t) + \langle p_{t+1}, g(\hat{y}_t, \hat{u}_t)\rangle = \max_{u\in U}(\beta^t \psi(\hat{y}_t, u) + \langle p_{t+1}, g(\hat{y}_t, u)\rangle)$ *for all* $t \in \mathbb{N}$.

## 3.7 Sufficient Condition for Problem (P1)

In this section we establish a result of sufficient condition of optimality which uses the adjoint equation, the weak maximum principle and the concavity of the Hamiltonian with respect to the state variable and the control variable.

**Theorem 3.11.** *Let $U$ be a nonempty convex subset of $\mathbb{R}^d$, $\beta \in (0,1)$, $\eta$, $y_\infty \in \mathbb{R}^n$ and two mappings $\varphi : \mathbb{R}^n \times U \to \mathbb{R}$ and $\ell : \mathbb{R}^n \times U \to \mathbb{R}^n$. Let $(\underline{\hat{z}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and $\underline{p} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$. Assume that the following conditions hold.*

**(i)** $\hat{z}_{t+1} = \ell(\hat{z}_t, \hat{u}_t)$ *for all* $t \in \mathbb{N}$, *and* $\hat{z}_0 = \eta - y_\infty$.

**(ii)** $\varphi \in C^1(\mathbb{R}^n \times U, \mathbb{R})$ *and* $\ell \in C^1(\mathbb{R}^n \times U, \mathbb{R}^n)$.

**(iii)** $\varphi$ *transforms bounded subsets of* $\mathbb{R}^n \times U$ *into bounded subsets of* $\mathbb{R}$.

**(iv)** $p_t = p_{t+1} \circ D_1 \ell(\hat{z}_t, \hat{u}_t) + \beta^t D_1 \varphi(\hat{z}_t, \hat{u}_t)$ *for all* $t \in \mathbb{N}_*$.

**(v)** $\langle p_{t+1} \circ D_2 \ell(\hat{z}_t, \hat{u}_t) + D_2 \varphi(\hat{z}_t, \hat{u}_t), u - \hat{u}_t\rangle \leq 0$ *for all* $u \in U$, *for all* $t \in \mathbb{N}$.

**(vi)** *The function* $[(z,u) \mapsto \langle p_{t+1}, \ell(z,u)\rangle + \beta^t \varphi(z,u))]$ *is concave on* $\mathbb{R}^n \times U$ *for all* $t \in \mathbb{N}$.

*Then $(\underline{\hat{z}}, \underline{\hat{u}})$ is a solution of (P1).*

*Proof.* Let $(\underline{z}, \underline{u})$ be an admissible process for (P1), i.e. $\underline{z} \in c_0(\mathbb{N}, \mathbb{R}^n)$, $\underline{u} \in \ell^\infty(\mathbb{N}, U)$, $z_{t+1} = \ell(z_t, u_t)$ for all $t \in \mathbb{N}$, and $z_0 = \eta - y_\infty$. From (iii), since $\{\varphi(z_t, u_t) : t \in \mathbb{N}\}$ is bounded, $N(\underline{z}, \underline{u}) = \sum_{t=0}^{+\infty} \beta^t \varphi(z_t, u_t)$ exists in $\mathbb{R}$. From (ii) and (iv) and from the definition of Hamiltonian of Pontryagin (3.2) we obtain

$$D_1 H_t(\hat{z}_t, \hat{u}_t, p_{t+1}) = p_t. \tag{3.10}$$

From (vi) we obtain, for all $t \in \mathbb{N}$,

$$\begin{aligned} H_t(\hat{z}_t, \hat{u}_t, p_{t+1}) - H_t(z_t, u_t, p_{t+1}) - \langle D_1 H_t(\hat{z}_t, \hat{u}_t, p_{t+1}), \hat{z}_t - z_t\rangle \\ -\langle D_2 H_t(\hat{z}_t, \hat{u}_t, p_{t+1}), \hat{u}_t - u_t\rangle \quad \geq 0. \end{aligned} \tag{3.11}$$

From (v) the following relation holds for all $t \in \mathbb{N}$

$$\langle D_2 H_t(\hat{z}_t, \hat{u}_t, p_{t+1}), \hat{u}_t - u_t\rangle \geq 0. \tag{3.12}$$

For all $t \in \mathbb{N}$ we have

$$\begin{aligned} \beta^t \varphi(\hat{z}_t, \hat{u}_t) - \beta^t \varphi(z_t, u_t) &= H_t(\hat{z}_t, \hat{u}_t, p_{t+1}) - \langle p_{t+1}, \ell(\hat{z}_t, \hat{u}_t)\rangle \\ &\quad -H_t(z_t, u_t, p_{t+1}) + \langle p_{t+1}, \ell(z_t, u_t) \\ &= H_t(\hat{z}_t, \hat{u}_t, p_{t+1}) - H_t(z_t, u_t, p_{t+1}) \\ &\quad -\langle p_{t+1}, \hat{z}_{t+1} - z_{t+1}\rangle. \end{aligned}$$

Then, using (3.10) and (3.12) we obtain

$$\begin{array}{rcl}\beta^t\varphi(\hat{z}_t,\hat{u}_t)-\beta^t\varphi(z_t,u_t) & \geq & H_t(\hat{z}_t,\hat{u}_t,p_{t+1})-H_t(z_t,u_t,p_{t+1})\\ & & -\langle D_2H_t(\hat{z}_t,\hat{u}_t,p_{t+1}),\hat{u}_t-u_t\rangle\\ & & -\langle D_1H_{t+1}(\hat{z}_{t+1},\hat{u}_{t+1},p_{t+2}),\hat{z}_{t+1}-z_{t+1}\rangle,\end{array}$$

which implies

$$\begin{array}{rcl}\beta^t\varphi(\hat{z}_t,\hat{u}_t)-\beta^t\varphi(z_t,u_t) & \geq & [H_t(\hat{z}_t,\hat{u}_t,p_{t+1})-H_t(z_t,u_t,p_{t+1})\\ & & -\langle D_1H_t(\hat{z}_t,\hat{u}_t,p_{t+1}),\hat{z}_t-z_t\rangle\\ & & -\langle D_2H_t(\hat{z}_t,\hat{u}_t,p_{t+1}),\hat{u}_t-u_t\rangle]\\ & & +[\langle D_1H_t(\hat{z}_t,\hat{u}_t,p_{t+1}),\hat{z}_t-z_t\rangle\\ & & -\langle D_1H_{t+1}(\hat{z}_{t+1},\hat{u}_{t+1},p_{t+2}),\hat{z}_{t+1}-z_{t+1}\rangle],\end{array}$$

and using (3.11) we obtain

$$\begin{array}{rcl}\beta^t\varphi(\hat{z}_t,\hat{u}_t)-\beta^t\varphi(z_t,u_t) & \geq & [\langle D_1H_t(\hat{z}_t,\hat{u}_t,p_{t+1}),\hat{z}_t-z_t\rangle\\ & & -\langle D_1H_{t+1}(\hat{z}_{t+1},\hat{u}_{t+1},p_{t+2}),\hat{z}_{t+1}-z_{t+1}\rangle].\end{array}$$

Therefore, using (3.10), we obtain, for all $T\in\mathbb{N}_*$,

$$\begin{array}{rcl}\sum_{t=0}^T\beta^t\varphi(\hat{z}_t,\hat{u}_t)-\sum_{t=0}^T\beta^t\varphi(z_t,u_t) & \geq & \langle D_1H_0(\eta-y_\infty,\hat{u}_0,p_1),\eta-y_\infty-(\eta-y_\infty)\rangle\\ & & -\langle p_{T+1},\hat{z}_{T+1}-z_{T+1}\rangle\end{array}$$

$$\Longrightarrow\sum_{t=0}^T\beta^t\varphi(\hat{z}_t,\hat{u}_t)-\sum_{t=0}^T\beta^t\varphi(z_t,u_t)\geq-\langle p_{T+1},\hat{z}_{T+1}-z_{T+1}\rangle\,. \tag{3.13}$$

Since $\underline{p}\in\ell^1(\mathbb{N}_*,\mathbb{R}^{n*})$, we have $\lim_{T\to+\infty}p_{T+1}=0$, and since $\underline{\hat{z}},\underline{z}\in c_0(\mathbb{N},\mathbb{R}^n)$ we have $\lim_{T\to+\infty}(\hat{z}_{T+1}-z_{T+1})=0$ which implies $\lim_{T\to+\infty}(-\langle p_{T+1},\hat{z}_{T+1}-z_{T+1}\rangle)=0$, and then, from (3.13), doing $T\to+\infty$ we obtain $N(\underline{\hat{z}},\underline{\hat{u}})-N(\underline{z},\underline{u})\geq 0$. And so we have proven that $(\underline{\hat{z}},\underline{\hat{u}})$ is a solution of (P1). □

**Remark 3.12.** The structure of the previous proof is inspired by the proof of Theorem 5.1 in [15]. Note that our assumption (iii) permits to avoid to assume that $U$ is compact. Moreover, note that we can replace the assumption (iii) by the condition: $U$ is closed.

**Remark 3.13.** Note that under our assumptions, the process $(\underline{\hat{z}},\underline{\hat{u}})$ is also solution of the following problem

$$\left\{\begin{array}{cl}\text{Maximize} & \sum_{t=0}^{+\infty}\beta^t\varphi(x_t,u_t)\\ \text{when} & \underline{z}\in\ell^\infty(\mathbb{N},\mathbb{R}^n),\underline{u}\in\ell^\infty(\mathbb{N},U),\\ \text{and} & \forall t\in\mathbb{N},z_{t+1}=\ell(z_t,u_t),\ z_0=\eta-y_\infty,\end{array}\right.$$

since, in the previous proof, when we obtain (3.13), having $\underline{\hat{z}}$ and $\underline{z}$ bounded is sufficient to obtain $\lim_{T\to+\infty}(-\langle p_{T+1},\hat{z}_{T+1}-z_{T+1}\rangle)=0$ and consequently to have the optimality of $(\underline{\hat{z}},\underline{\hat{u}})$ for the last problem.

## 3.8 Sufficient Condition for Main Problem

This section is devoted to the translation of the result of sufficient condition of optimality for $(P1)$ into an analogous result for $(P_m)$. When $y_\infty\in\mathbb{R}^n$, we denote by $c_{y_\infty}(\mathbb{N},\mathbb{R}^n)$, the set of the sequences $\underline{y}$ in $\mathbb{R}^n$ such that $\lim_{t\to+\infty}y_t=y_\infty$. It is a complete affine subset of $\ell^\infty(\mathbb{N},\mathbb{R}^n)$.

**Theorem 3.14.** *Let $U$ be a nonempty convex subset of $\mathbb{R}^d$, $\beta \in (0,1)$, $\eta, y_\infty \in \mathbb{R}^n$, and two mappings $\psi : \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R}^n \times U \to \mathbb{R}^n$. Let $(\underline{\hat{y}}, \underline{\hat{u}}) \in c_{y_\infty}(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ and $\underline{p} \in \ell^1(\mathbb{N}_*, \mathbb{R}^{n*})$ which satisfy the following conditions.*

**(i)** *For all $t \in \mathbb{N}$, $\hat{y}_{t+1} = g(\hat{y}_t, \hat{u}_t)$, and $\hat{y}_0 = \eta$.*

**(ii)** *$\psi \in C^1(\mathbb{R}^n \times U, \mathbb{R})$ and $g \in C^1(\mathbb{R}^n \times U, \mathbb{R}^n)$.*

**(iii)** *$\psi$ transforms bounded subsets of $\mathbb{R}^n \times U$ into bounded subsets of $\mathbb{R}$.*

**(iv)** *$p_t = p_{t+1} \circ D_1 g(\hat{y}_t, \hat{u}_t) + \beta^t D_1 \psi(\hat{y}_t, \hat{u}_t)$ for all $t \in \mathbb{N}_*$.*

**(v)** *$\langle p_{t+1} \circ D_2 g(\hat{y}_t, \hat{u}_t) + \beta^t D_2 \psi(\hat{y}_t, \hat{u}_t), u - \hat{u} \rangle \leq 0$ for all $u \in U$, for all $t \in \mathbb{N}$.*

**(vi)** *The function $[(y,u) \mapsto \langle p_{t+1}, g(y,u) \rangle + \beta^t \psi(y,u)]$ is concave on $\mathbb{R}^n \times U$ for all $t \in \mathbb{N}$.*

*Then $(\underline{\hat{y}}, \underline{\hat{u}})$ is a solution of $(P_m)$.*

*Proof.* We set $\hat{z}_t = \hat{y}_t - y_\infty$ for all $t \in \mathbb{N}$. We see that $(\underline{\hat{z}}, \underline{\hat{u}}) \in c_0(\mathbb{N}, \mathbb{R}^n) \times \ell^\infty(\mathbb{N}, U)$ satisfies all the assumptions of Theorem 3.11. And so $(\underline{\hat{z}}, \underline{\hat{u}})$ is a solution of $(P1)$ which implies that $(\underline{\hat{y}}, \underline{\hat{u}})$ is a solution of $(P_m)$. $\square$

# Chapter 4

# Lightenings of Assumptions for Pontryagin Principles in Finite Horizon and Discrete Time

## 4.1 Introduction

In this chapter, we will establish maximum principles of Pontryagin under assumptions which are weaker than those of existing results for the infinite horizon optimal control problems in discrete time framework. The considered problems is stated as follows.

The discrete time is denoted by the letter $t \in \mathbb{N}$. For all $t \in \mathbb{N}$, $X_t$ is a nonempty open subset of $\mathbb{R}^n$, $U_t$ is a nonempty subset of $\mathbb{R}^d$, and $f_t : X_t \times U_t \to X_{t+1}$ is a mapping where $n$ and $d$ are fixed positive integers. The usual order of $\mathbb{R}^n$ is $x = (x^1, x^2, \ldots, x^n) \leq (y^1, y^2, \ldots, y^n) = y$ defined by $x^i \leq y^i$ for all $i \in \{1, \ldots, n\}$. And $x < y$ means that $x \leq y$ and $x \neq y$. To abridge the writing we use the notation $\underline{x} := (x_t)_{t\in\mathbb{N}} \in \prod_{t\in\mathbb{N}} X_t$ and also $\underline{u} := (u_t)_{t\in\mathbb{N}} \in \prod_{t\in\mathbb{N}} U_t$ where $\prod_{t\in\mathbb{N}} X_t$ and $\prod_{t\in\mathbb{N}} U_t$ are the Cartesian products. We work with two families of controlled dynamical systems: difference equations and difference inequations. They are

**(De)** $\quad x_{t+1} = f_t(x_t, u_t)$

and

**(Di)** $\quad x_{t+1} \leq f_t(x_t, u_t)$.

The variable $x_t$ is called the *state variable* and the variable $u_t$ is called the *control variable*. When we fix an initial state $\eta \in X_0$, we denote by $\mathrm{Adm}_e$ the set of all processes $(\underline{x}, \underline{u}) \in \prod_{t\in\mathbb{N}} X_t \times \prod_{t\in\mathbb{N}} U_t$ which satisfy (De) at each time $t \in \mathbb{N}$ and such that $x_0 = \eta$. These processes are called the *admissible* for (De) and $\eta$. The letter $e$ as lower index means equation. Similarly, we denote by $\mathrm{Adm}_i$ the set of all processes $(\underline{x}, \underline{u}) \in \prod_{t\in\mathbb{N}} X_t \times \prod_{t\in\mathbb{N}} U_t$ which satisfy (Di) at each time $t \in \mathbb{N}$ and such that $x_0 = \eta$. The letter $i$ as lower index means inequation.

For all $t \in \mathbb{N}$, we consider a function $\phi_t : X_t \times U_t \to \mathbb{R}$. When $k \in \{i, e\}$, we define $\mathrm{Dom}_k$ as the set of the $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$ such that the series $\sum_{t=0}^{+\infty} \phi_t(x_t, u_t)$ is convergent in $\mathbb{R}$. We define the functional $J : \mathrm{Dom}_k \to \mathbb{R}$ by setting $J(\underline{x}, \underline{u}) := \sum_{t=0}^{+\infty} \phi_t(x_t, u_t)$.

When $k \in \{i, e\}$, we consider the following list of problems.

$(\mathbf{P^1_k})$ Maximize $J(\underline{x}, \underline{u})$ when $(\underline{x}, \underline{u}) \in \mathrm{Dom}_k$.

$(\mathbf{P_k^2})$ Find $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Adm}_k$ such that, for all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$

$$\limsup_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t(\hat{x}_t, \hat{u}_t) - \sum_{t=0}^{h} \phi_t(x_t, u_t)\Big) \geq 0.$$

$(\mathbf{P_k^3})$ Find $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{Adm}_k$ such that, for all $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$

$$\liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t(\hat{x}_t, \hat{u}_t) - \sum_{t=0}^{h} \phi_t(x_t, u_t)\Big) \geq 0.$$

Now we describe the content of this chapter.

- In Section 4.2, we recall the method of reduction to finite horizon (Theorem 4.1).

- In Section 4.3, firstly, we recall the Multiplier Rule of Halkin. Then we introduce the New Multiplier Rules of Blot for maximization static problems with only inequality constraints and with both equality and inequality constraints. After that, we apply the New Multiplier Rules to obtain weak Pontryagin principles for the reduced problems in Section 4.2.

- In Section 4.4, at first, we establish weak maximum principles where the values of the optimal control belong to the interior of the sets of controls for systems which governed by difference inequations (Theorem 4.14 and Theorem 4.16). These results are new and only use the Gâteaux differentiability of the criterion, of the vector field and of the inequality constraints. Neither continuity on a neighborhood of optimal solution nor Fréchet differentiability is necessary. In the end of this section, we state a a similar result for problem in which the system is governed by difference equations.

- In Section 4.5, firstly, we establish a weak maximum principle when the sets of controls are defined by inequalities (Theorem 4.21) and when the system is governed by a difference inequations. This result also only uses the Gâteaux differentiability of the criterion, of the vector field and of the inequality constraints and a condition of separation of the origin and of the convex hull of the Gâteaux differentials of the inequelities constraints in the spirit of the Mangarasian-Fromowitz condition. Secondly, we establish a weak maximum principle when the sets of controls are defined by equalities and inequalities (Theorem 4.25) when the system is governed by a difference inequation. Such a case is treated in [7] (Theorem 3.1 and Theorem 3.2). In comparison with the result of [7], the improvements are the following ones: we avoid a condition of continuity for the saturated inequality constraints and for the vector field, we avoid a condition of linear independence of all the differentials of the constraints. A similar result is Theorem 4.26 for which the system is governed by a difference equation.

## 4.2 Reduction to Finite Horizon

The **general principle** is the following one: when a process is optimal on $\mathbb{N}$ (until infinity), then for all $T \in \mathbb{N}^*$ its restriction to $[0, T] \cap \mathbb{N}$ is optimal by fixing the final condition at $T$.

**Theorem 4.1.** *The two assertions hold.*

**(a)** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_e^j)$ and let $T \in \mathbb{N}^*$ when $j \in \{1, 2, 3\}$. Then the restriction $((\hat{x}_0, \ldots, \hat{x}_{T+1}), (\hat{u}_0, \ldots, \hat{u}_T))$ is an optimal solution of the following finite-horizon*

*problem:*

$$\left.\begin{array}{rl} \textit{Maximize} & F_T((x_t)_{0\le t\le T+1},(u_t)_{0\le t\le T}) := \sum_{t=0}^{T}\phi_t(x_t,u_t) \\ \textit{when} & \forall t\in\{0,\dots,T+1\},\ x_t\in X_t \\ & \forall t\in\{0,\dots,T\},\ u_t\in U_t \\ & \forall t\in\{0,\dots,T\},\ x_{t+1}=f_t(x_t,u_t) \\ & x_0=\eta,\ x_{T+1}=\hat{x}_{T+1}. \end{array}\right\} \qquad (F_e^T)$$

**(b)** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_i^j)$ let $T\in\mathbb{N}^*$ when $j\in\{1,2,3\}$. Then the restriction $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is an optimal solution of the following finite-horizon problem:*

$$\left.\begin{array}{rl} \textit{Maximize} & F_T((x_t)_{0\le t\le T+1},(u_t)_{0\le t\le T}) := \sum_{t=0}^{T}\phi_t(x_t,u_t) \\ \textit{when} & \forall t\in\{0,\dots,T+1\},\ x_t\in X_t \\ & \forall t\in\{0,\dots,T\},\ u_t\in U_t \\ & \forall t\in\{0,\dots,T\},\ x_{t+1}\le f_t(x_t,u_t) \\ & x_0=\eta,\ x_{T+1}=\hat{x}_{T+1}. \end{array}\right\} \qquad (F_i^T)$$

*Proof.* **For the (De) case:** We will prove for each case of $(P_e^j)$ where $j\in\{1,2,3\}$.

- **For** $(P_e^1)$: Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_e^1)$. We proceed by contradiction. Assume that $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is not optimal for $(F_e^T)$. Then there exists $((z_0,\dots,z_{T+1}),(w_0,\dots,w_T))$ which is admissible for $(F_e^T)$ such that

$$\sum_{t=0}^{T}\phi_t(z_t,w_t) > \sum_{t=0}^{T}\phi_t(\hat{x}_t,\hat{u}_t).$$

When $t>T+1$, we set $z_t:=\hat{x}_t$ and when $t\ge T+1$, we set $w_t:=\hat{u}_t$. From the admissibility and this setting, it is obvious that $\underline{z}\in\prod_{t\in\mathbb{N}}X_t$ and $\underline{w}\in\prod_{t\in\mathbb{N}}U_t$. Also, $z_{t+1}=\hat{x}_{t+1}=f_t(\hat{x}_t,\hat{u}_t)\ =f_t(z_t,w_t)$ when $t\ge T+1$. It implies that $(\underline{z},\underline{w})$ belongs to $\mathrm{Adm}_e$, the admissible set of $(P_e^1)$. Now, we have

$$\sum_{t=T+1}^{+\infty}\phi_t(z_t,w_t)=\sum_{t=T+1}^{+\infty}\phi_t(\hat{x}_t,\hat{u}_t)<+\infty \text{ then } \sum_{t=0}^{+\infty}\phi_t(z_t,w_t)<+\infty.$$

It implies that $(\underline{z},\underline{w})\in\mathrm{Dom}_e$. Then

$$\begin{aligned}\sum_{t=0}^{+\infty}\phi_t(z_t,w_t)&=\sum_{t=0}^{T}\phi_t(z_t,w_t)+\sum_{t=T+1}^{+\infty}\phi_t(z_t,w_t)=\sum_{t=0}^{T}\phi_t(z_t,w_t)+\sum_{t=T+1}^{+\infty}\phi_t(\hat{x}_t,\hat{u}_t)\\ &>\sum_{t=0}^{T}\phi_t(\hat{x}_t,\hat{u}_t)+\sum_{t=T+1}^{+\infty}\phi_t(\hat{x}_t,\hat{u}_t)=\sum_{t=0}^{+\infty}\phi_t(\hat{x}_t,\hat{u}_t)\end{aligned}$$

This is contradiction since $(\underline{\hat{x}},\underline{\hat{u}})$ is the optimal solution for Problem $(P_e^1)$. Hence, $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ must be optimal for $(F_e^T)$.

- **For** $(P_e^3)$: Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_e^3)$. By an analogous proceed like in the previous case, we obtain $(\underline{z},\underline{w})\in\mathrm{Adm}_e$ such that $\sum_{t=0}^{T}\phi_t(z_t,w_t)>\sum_{t=0}^{T}\phi_t(\hat{x}_t,\hat{u}_t)$. Then we have when $h\ge T$,

$$\liminf_{h\to+\infty}\Big(\sum_{t=0}^{h}\phi_t(z_t,w_t)-\sum_{t=0}^{h}\phi_t(\hat{x}_t,\hat{u}_t)\Big)=\sum_{t=0}^{T}\phi_t(z_t,w_t)-\sum_{t=0}^{T}\phi_t(\hat{x}_t,\hat{u}_t)>0,$$

which is a contradiction.

- **For** $(P_e^2)$: Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_e^2)$. It is clear that it is also a solution of $(P_e^3)$ and hence, its restriction $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is an optimal solution for $(F_e^T)$.

**For the (Di) case**: the proof is completely similar to the (DE) case. □

## 4.3 The New Multiplier Rule

### 4.3.1 Recall of the Multiplier Rule of Halkin

We consider two nonnegative integer numbers $n_I$ and $n_E$, a nonempty open subset $\Omega$ in $\mathbb{R}^n$, and functions $g^0,\ g^1,\dots,\ g^{n_I},\ h^1,\ h^2,\dots,\ h^{n_E}$ from $\Omega$ into $\mathbb{R}$. With these elements we formulate the following maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & g^0(z) \\ \text{when} & \forall\alpha\in\{1,2,\dots,n_I\},\ g^\alpha(z)\geq 0 \\ & \forall\beta\in\{1,2,\dots,n_E\},\ h^\beta(z)=0. \end{array}\right\} \qquad (M)$$

The conditions $g^\alpha(z)\geq 0$ are called the inequality constraints, the conditions $h^\beta(z)=0$ are called the equality constraints, and $g^0(z)$ is called the criterion. A point $z\in\Omega$ which satisfies all the inequality constraints and all the equality constraints is called *admissible* for $(M)$.

**Definition 4.2.** The function $\mathfrak{L}:\Omega\times\mathbb{R}^{n_I}\times\mathbb{R}^{n_E}\to\mathbb{R}$ defined by

$$\mathfrak{L}(z,\lambda_1,\dots,\lambda_{n_I},\mu_1,\dots,\mu_{n_E}):=g^0(z)+\sum_{\alpha=1}^{n_I}\lambda_\alpha g^\alpha(z)+\sum_{\beta=1}^{n_E}\mu_\beta h^\beta(z)$$

is called the Lagrangian of $(M)$.

The function $\mathfrak{G}:\Omega\times\mathbb{R}\times\mathbb{R}^{n_I}\times\mathbb{R}^{n_E}\to\mathbb{R}$ defined by

$$\begin{aligned}\mathfrak{G}(z,\lambda_0,\lambda_1,\dots,\lambda_{n_I},\mu_1,\dots,\mu_{n_E}):&=\lambda_0 g^0(z)+\sum_{\alpha=1}^{n_I}\lambda_\alpha g^\alpha(z)+\sum_{\beta=1}^{n_E}\mu_\beta h^\beta(z)\\ &=\sum_{\alpha=0}^{n_I}\lambda_\alpha g^\alpha(z)+\sum_{\beta=1}^{n_E}\mu_\beta h^\beta(z)\end{aligned}$$

is called the generalized Lagrangian of $(M)$. Note that the difference between $\mathfrak{G}$ and $\mathfrak{L}$ is the presence of a scalar $\lambda_0$ associated to the criterion.

The following theorem is established in the paper of Halkin [31]. In [45], Michel provides a proof which is different from this one of Halkin.

**Theorem 4.3.** *Let $z_*$ be a solution of $(M)$. We assume that the functions $g^0,\ g^1,\dots,g^{nI}$ and $h^1,\ h^2,\dots,\ h^{nE}$ are continuous on a neighborhood of $z_*$ and that they are Fréchet differentiable at $z_*$. Then there exist real numbers $\lambda_0,\dots,\lambda_{n_I},\mu_1,\dots,\mu_{n_E}$ which satisfy the following conditions:*

**(a)** *$\lambda_0,\lambda_1,\dots,\lambda_{n_I},\mu_1,\dots,\mu_{n_E}$ are not simultaneously equal to zero.*

**(b)** *For all $\forall\alpha\in\{0,1,\dots,n_I\},\ \lambda_\alpha\geq 0$.*

**(c)** *For all $\forall\alpha\in\{1,\dots,n_I\},\ \lambda_\alpha g^\alpha(z_*)=0$.*

**(d)** *$D_1\mathfrak{G}(z_*,\lambda_0,\lambda_1,\dots,\lambda_{n_I},\mu_1,\dots,\mu_{n_E})=0$ where $D_1$ denotes the partial differential with respect to the first variable $z$.*

The real numbers of the conclusion of the theorem are called the multipliers associated to $z_*$. $\lambda_0$ is called the multiplier associated to the criterion; when $\alpha\in\{1,\dots,n_I\},\ \lambda_\alpha$ is called the multiplier associated to the inequality constraint $g^\alpha(z)\geq 0$; and when $\beta\in\{1,2,\dots,n_E\},\ \mu_\beta$ is called the multiplier associated to the equality constraint $h^\beta(z)=0$. About the conclusion (a), it is easy to see that when all the multipliers are zero then all the conclusions hold even if $z_*$ is not a solution of the problem. The conclusion (c) is called

the slackness condition; it means that when $g^\alpha(z_*) > 0$ then the associated multiplier is zero and consequently we can delete it. Conclusion (d) can be translated as follows.

$$\sum_{\alpha=0}^{n_I} \lambda_\alpha Dg^\alpha(z_*) + \sum_{\beta=1}^{n_E} \mu_\beta Dh^\beta(z_*) = 0$$

Note that when $(\lambda_0, \lambda_1, \ldots, \lambda_{n_I}, \mu_1, \ldots, \mu_{n_E})$ satisfies (a–d), for all real number $r > 0$, the new list $(r\lambda_0, r\lambda_1, \ldots, r\lambda_{n_I}, r\mu_1, \ldots, r\mu_{n_E})$ also satisfies (a–d) (this is a property of cones). Consequently it is possible to normalize a list $(\lambda_0, \lambda_1, \ldots, \lambda_{n_I}, \mu_1, \ldots, \mu_{n_E})$ which satisfies (a–d): choosing a norm $\|.\|$ on $\mathbb{R} \times \mathbb{R}^{n_I} \times \mathbb{R}^{n_E}$, we can choose a suitable list such that $\|\lambda_0, \ldots, \lambda_{n_I}, \mu_1, \ldots, \mu_{n_E}\| = 1$. Also note that the set of all lists $(\lambda_0, \ldots, \lambda_{n_I}, \mu_1, \ldots, \mu_{n_E})$ which satisfy (a–d) is a convex subset of $\mathbb{R}^{1+n_I+n_E}$.

### 4.3.2 New Multiplier Rules

Let $\Omega$ be a nonempty open subset of $\mathbb{R}^n$, let $f_i : \Omega \to \mathbb{R}$ (when $i \in \{0, \ldots, m\}$) be functions, let $g_i : \Omega \to \mathbb{R}$ (when $i \in \{0, \ldots, p\}$) and $h_i : \Omega \to \mathbb{R}$ (when $i \in \{1, \ldots, q\}$) be functions. With these elements, we consider the two following problems:

$$\left.\begin{array}{rl} \text{Maximize} & f_0(x) \\ \text{when} & x \in \Omega \\ \text{and when} & \forall i \in \{1, \ldots, m\},\ f_i(x) \geq 0, \end{array}\right\} \qquad (\mathcal{I})$$

and

$$\left.\begin{array}{rl} \text{Maximize} & g_0(x) \\ \text{when} & x \in \Omega \\ \text{when} & \forall i \in \{1, \ldots, p\},\ g_i(x) \geq 0 \\ \text{and when} & \forall j \in \{1, \ldots, q\},\ h_j(x) = 0. \end{array}\right\} \qquad (\mathcal{M})$$

For Problem $(\mathcal{I})$, the conditions $f_i(x) \geq 0$ are called the inequality constraints and $f_0(x)$ is called the criterion. A point $x \in \Omega$ which satisfies all the inequality constraints is called *admissible* for $(\mathcal{I})$.

Similarly, for Problem $(\mathcal{M})$, the conditions $g_i(x) \geq 0$ are called the inequality constraints, the conditions $h_j(x) = 0$ are called the equality constraints, and $g_0(x)$ is called the criterion. A point $x \in \Omega$ which satisfies all the inequality constraints and all the equality constraints is called *admissible* for $(\mathcal{M})$.

Recently, Blot has improved the Multiplier Rule of Halkin by the following theorems. Their statements and proofs are taken from [10].

**Theorem 4.4.** *Let $\hat{x}$ be a solution of $(\mathcal{I})$. We assume that the following assumptions are fulfilled.*

**(i)** *For all $i \in \{0, \ldots, m\}$, $f_i$ is Gâteaux differentiable at $\hat{x}$.*

**(ii)** *For all $i \in \{1, \ldots, m\}$, $f_i$ is lower semicontinuous at $\hat{x}$ when $f_i(\hat{x}) > 0$.*

*Then there exist $\lambda^0, \ldots, \lambda^m \in \mathbb{R}_+$ such that the following conditions hold.*

**(a)** $(\lambda^0, \ldots, \lambda^m) \neq (0, \ldots, 0)$.

**(b)** *For all $i \in \{1, \ldots, m\}$, $\lambda^i f_i(\hat{x}) = 0$.*

**(c)** $\sum_{0 \leq i \leq m} \lambda^i D_G f_i(\hat{x}) = 0$.

*If, in addition, we assume that the following assumption is fulfilled,*

**(iii)** *There exists $w \in \mathbb{R}^n$ such that, for all $i \in \{1, \ldots, m\}$, $D_G f_i(\hat{x}).w > 0$ when $f_i(\hat{x}) = 0$,*

*then we can take*

**(d)** $\lambda^0 = 1$.

**Theorem 4.5.** ***(New Multiplier Rule)*** *Let $\hat{x}$ be a solution of $(\mathcal{M})$. We assume that the following assumptions are fulfilled.*

**(i)** *$g_0$ is Fréchet differentiable at $\hat{x}$.*

**(ii)** *For all $i \in \{1, \dots, p\}$, $g_i$ is Fréchet differentiable at $\hat{x}$ when $g_i(\hat{x}) = 0$.*

**(iii)** *For all $i \in \{1, \dots, p\}$, $g_i$ is Gâteaux differentiable at $\hat{x}$ and lower semicontinuous at $\hat{x}$ when $g_i(\hat{x}) > 0$.*

**(iv)** *For all $i \in \{1, \dots, q\}$, $h_i$ is continuous on a neighborhood of $\hat{x}$ and Fréchet differentiable at $\hat{x}$.*

*Then there exist $\lambda^0, \dots, \lambda^p \in \mathbb{R}_+$ and $\mu^1, \dots, \mu^q \in \mathbb{R}$ such that the following conditions are satisfied.*

**(a)** $(\lambda^0, \dots, \lambda^p, \mu^1, \dots, \mu^q) \neq (0, \dots, 0)$.

**(b)** *For all $i \in \{1, \dots, p\}$, $\lambda^i g_i(\hat{x}) = 0$.*

**(c)** $\lambda^0 D_G\phi(\hat{x}) + \sum_{1 \leq i \leq p} \lambda^i D_G g_i(\hat{x}) + \sum_{1 \leq i \leq q} \mu^i D_G h_i(\hat{x}) = 0$.

*Moreover, under the additional assumption*

**(v)** *$Dh_1(\hat{x}), \dots, Dh_q(\hat{x})$ are linearly independent,*

*we can take*

**(d)** $(\lambda^0, \dots, \lambda^p) \neq (0, \dots, 0)$.

*Furthermore, under (v) and under the additional assumption*

**(vi)** *There exists $w \in \bigcap_{1 \leq j \leq q} Ker Dh_j(\hat{x})$ such that, for all $i \in \{1, \dots, p\}$, $Dg_i(\hat{x}).w > 0$ when $g_i(\hat{x}) = 0$,*

*we can take*

**(e)** $\lambda^0 = 1$.

In comparison with the Halkin's Multiplier Rule, for Problem $(\mathcal{I})$, the assumptions of local continuity on a neighborhood of $\hat{x}$ of the $f_i$, $i \in \{0, \dots, m\}$ have been deleted and their Fréchet differentiability has been replaced by their Gâteaux differentiability, and for Problem $(\mathcal{M})$, the assumptions of local continuity on $g_0$ and on the $g_i$, $i \in \{1, \dots, p\}$ have been deleted. In comparison with the result of [49] for Problem $(\mathcal{I})$, the Fréchet differentiability of the $f_i$ has been replaced by their Gâteaux differentiability. Note that the Gâteaux differentiability of a mapping at a point does not imply the continuity of this mapping at this point.

### 4.3.3 Proof of Theorem 4.4

#### Some fundamental tools

Before proving Theorem 4.4, we recall the following well-known results.

**Theorem 4.6.** *Let $m$, $n \in \mathbb{N}_*$, $\varphi_1, \dots, \varphi_m \in \mathbb{R}^{n*}$, and $a \in \mathbb{R}^{n*}$. The two following assertions are equivalent.*

**(i)** *For all $x \in \mathbb{R}^n$, $(\forall i \in \{1, \dots, m\},\ \varphi_i.x \geq 0) \implies (a.x \geq 0)$.*

**(ii)** *There exists $\lambda^1, \dots, \lambda^m \in \mathbb{R}_+$ such that $a = \sum_{1 \leq i \leq m} \lambda^i \varphi_i$.*

A complete proof of this result is given in [61] (Chapter 4, Sections 4.14 - 4.19) and in [38] (Chapter 2, Sections 2.5, 2.6). This result is presented in many books (for example, in [4] page 164 and in [60] page 176 ).

A second fundamental tool that we recall is the Implicit Function Theorem of Halkin for the Fréchet differentiable mappings which are not necessarily continuously Fréchet differentiable.

**Theorem 4.7.** *Let $X$, $Y$, $Z$ be three real finite-dimensional normed vector spaces. Let $A \subset X \times Y$ be a nonempty open subset, let $f : A \to Z$ be a mapping, and let $(\bar{x}, \bar{y}) \in A$. We assume that the following conditions are fulfilled.*

**(i)** $f(\bar{x}, \bar{y}) = 0$.

**(ii)** *$f$ is continuous on a neighborhood of $(\bar{x}, \bar{y})$.*

**(iii)** *$f$ is Fréchet differentiable at $(\bar{x}, \bar{y})$ and the partial Fréchet differential $D_2 f(\bar{x}, \bar{y})$ is bijective.*

*Then there exist a neighborhood $U$ of $\bar{x}$ in $X$, a neighborhood $V$ of $\bar{y}$ in $Y$ such that $U \times V \subset A$, and a mapping $\psi : U \to V$ which satisfy the following conditions.*

**(a)** $\psi(\bar{x}) = y$.

**(b)** *For all $x \in U$, $f(x, \psi(x)) = 0$.*

**(c)** *$\psi$ is Fréchet differentiable at $\bar{x}$ and $D\psi(\bar{x}) = -D_2 f(\bar{x}, \bar{y})^{-1} \circ D_1 f(\bar{x}, \bar{y})$.*

This result is proven in [31]. Its proof uses the Fixed Point Theorem of Brouwer. The assumptions of this theorem are well explained by the electronic paper of Border [19]. Halkin does not use an open subset $A$. His function is defined on $X \times Y$ but it is easy to adapt his result. Since $\psi$ is Fréchet differentiable at $\bar{x}$, $\psi$ is continuous at $\bar{x}$ and then we can consider a neighborhood $V$ of $\bar{y}$ and a neighborhood $U$ of $\bar{x}$ such that $\psi(U) \subset V$ and such that $U \times V \subset A$.

Now we will prove Theorem 4.4. Let $\hat{x}$ be a solution of Problem $(\mathcal{I})$. Doing a change of index, we can assume that $I := \{1, \ldots, e\} = \{i \in \{1, \ldots, m\} : f_i(\hat{x}) = 0\}$. If $f_i(\hat{x}) > 0$ for all $i \in \{1, \ldots, m\}$ then $I = \emptyset$ (or equivalently, $e = 0$). Using the lower semicontinuity of (ii), there exists an open neighborhood $\Theta$ of $\hat{x}$ on which $f_i(\hat{x}) > 0$ for all $i \in \{1, \ldots, m\}$. Hence, $\hat{x}$ maximizes $f_0$ (without constraints) on open set $\Theta$. Then using (i) and the definition of Gâteaux differential, we obtain $D_G f_0(\hat{x}) = 0$, and we conclude by taking $\lambda^0 := 1$ and $\lambda^i := 0$ for all $i \in \{1, \ldots, m\}$. And so, for sequel of the proof we assume that $1 \leq e \leq m$.

### Proof of (a), (b), (c)

Ever using (ii), when $e < m$ we can assert that there exists an open neighborhood $\Omega_1 \subset \Omega$ of $\hat{x}$ such that, for all $x \in \Omega_1$ and for all $i \in \{e+1, \ldots, m\}$, $f_i(x) > 0$. When $e = m$ we simply take $\Omega_1 := \Omega$. Then for all case of $e$, $\hat{x}$ is a solution of the following problem.

$$\left.\begin{array}{rl} \text{Maximize} & f_0(x) \\ \text{when} & x \in \Omega_1 \\ \text{and when} & \forall i \in \{1, \ldots, e\},\ f_i(x) \geq 0. \end{array}\right\} \qquad (\mathcal{P})$$

For all $k \in \{0, \ldots, e\}$, we introduce the set

$$A_k := \{v \in \mathbb{R}^n : \forall i \in \{k, \ldots, e\},\ D_G f_i(\hat{x}).v > 0\} \qquad (4.1)$$

We will prove that $A_0 = \emptyset$. To realize that, we proceed by contradiction; we assume that $A_0 \neq \emptyset$, and so there exists $w \in \mathbb{R}^n$ such that $D_G f_i(\hat{x}).w > 0$ for all $i \in \{0, \dots, e\}$. Since $\Omega_1$ is open, there exists $\theta_* \in (0, +\infty)$ such that $\hat{x} + \theta w \in \Omega_1$ for all $\theta \in [0, \theta_*]$. After (i), for all $i \in \{0, \dots, e\}$, the function $\sigma_i : [0, \theta_*] \to \mathbb{R}$, defined by $\sigma_i(\theta) := f_i(\hat{x} + \theta w)$, is differentiable at 0, and its derivative is $\sigma_i'(0) = D_G f_i(\hat{x}).w$ as follows

$$\begin{aligned}\sigma_i'(0) &= \lim_{t \to 0} \frac{\sigma_i(t) - \sigma_i(0)}{t} = \lim_{t \to 0} \frac{f_i(\hat{x} + tw) - f_i(\hat{x})}{t} \\ &= \vec{D} f(\hat{x}, w) = D_G f_i(\hat{x}).w\end{aligned}$$

The differentiability of at 0 implies the existence of a function $\rho_i : [0, \theta_*] \to \mathbb{R}$ such that $\lim_{\theta \to 0} \rho_i(\theta) = 0$ and such that $\sigma_i(\theta) = \sigma_i(0) + \sigma_i'(0)\theta + \rho_i(\theta)\theta$ for all $\theta \in [0, \theta_*]$ and for all $i \in \{0, \dots, e\}$. Translating this last equality we obtain for all $\theta \in [0, \theta_*]$ and for all $i \in \{0, \dots, e\}$,

$$f_i(\hat{x} + \theta w) = f_i(\hat{x}) + \theta(D_G f_i(\hat{x}).w + \rho_i(\theta)).$$

Since $D_G f_i(\hat{x}).w > 0$ and since $\lim_{\theta \to 0} \rho_i(\theta) = 0$, we obtain the existence of $\theta_i \in (0, \theta)$ such that $D_G f_i(\hat{x}).w + \rho_i(\theta) > 0$ for all $\theta \in (0, \theta_i]$. Setting $\hat{\theta} := \min\{\theta_i : i \in \{0, \dots, e\}\}$ we obtain that $f_i(\hat{x} + \theta w) < f_i(\hat{x})$ for all $\theta \in (0, \hat{\theta}]$ and for all $i \in \{0, \dots, e\}$. Then using $i \in \{1, \dots, e\}$, this last relation ensures that $\hat{x} + \theta w$ is admissible for $(\mathcal{P})$ when $\theta \in (0, \hat{\theta}]$, and using this last relation when $i = 0$ we obtain $f_0(\hat{x} + \theta w) > f_0(\hat{x})$ when $\theta \in (0, \hat{\theta}]$, that is impossible since $\hat{x}$ is a solution of $(\mathcal{P})$. And so the reasoning by contradiction is complete, and we have proven

$$A_0 = \emptyset. \tag{4.2}$$

When $A_e = \emptyset$ there is not any $v \in \mathbb{R}^n$ such that $D_G f_e(\hat{x}).v > 0$. This implies that $D_G f_e(\hat{x}) = 0$ since it belongs to $\mathbb{R}^{n*}$. Then taking $\lambda^e := 1$ and $\lambda^i := 0$ when $i \in \{0, \cdots, \}\backslash\{e\}$, we obtain the conclusions (a), (b), (c). And so we have proven

$$A_e = \emptyset \implies ((a), (b), (c) \text{ hold}). \tag{4.3}$$

Now we assume that $A_e \neq \emptyset$. Since we have $A_0 = \emptyset$ after (4.2) and $A_i \subset A_{i+1}$ we can define

$$k := \min\{i \in \{1, \dots, e\} : A_i \neq \emptyset\}. \tag{4.4}$$

Note that $A_k \neq \emptyset$ and that $A_{k-1} = \emptyset$. We consider the following problem

$$\left.\begin{array}{rl} \text{Maximize} & D_G f_{k-1}(\hat{x}).v \\ \text{when} & v \in \mathbb{R}^n \\ \text{and when} & \forall i \in \{k, \dots, e\},\ D_G f_i(\hat{x}).v \geq 0. \end{array}\right\} \tag{$\mathcal{Q}$}$$

We want to prove that 0 is a solution of $(\mathcal{Q})$. To do that, we proceed by contradiction. We assume that there exists $y \in \mathbb{R}^n$ such that $(\forall i \in \{k, \dots, e\}, D_G f_i(\hat{x}).y \geq 0)$ and $D_G f_{k-1}(\hat{x}).y > 0 = D_G f_{k-1}(\hat{x}).0$. Since $A_k \neq \emptyset$, there exists $z \in \mathbb{R}^n$ such that $D_G f_i(\hat{x}).z > 0$ when $i \in \{k, \dots, e\}$. We can not have $D_G f_{k-1}(\hat{x}).z > 0$ since $A_{k-1} = \emptyset$. Therefore we have $D_G f_{k-1}(\hat{x}).z \leq 0$. If $D_G f_{k-1}(\hat{x}).z < 0$ we can choose $\varepsilon > 0$ such that $D_G f_{k-1}(\hat{x}).y + \varepsilon D_G f_{k-1}(\hat{x}).z > 0$ (i.e $0 < \varepsilon < \frac{-D_G f_{k-1}(\hat{x}).y}{D_G f_{k-1}(\hat{x}).z}$). If $D_G f_{k-1}(\hat{x}).z = 0$ we arbitrarily choose $\varepsilon \in (0, +\infty)$ and we also have $D_G f_{k-1}(\hat{x}).y + \varepsilon D_G f_{k-1}(\hat{x}).z > 0$. We set $u_\varepsilon := y + \varepsilon z$, and we note that $D_G f_{k-1}(\hat{x}).u_\varepsilon = D_G f_{k-1}(\hat{x}).y + \varepsilon D_G f_{k-1}(\hat{x}).z > 0$. Furthermore, when $i \in \{k, \dots, e\}$, we have $D_G f_i(\hat{x}).u_\varepsilon = D_G f_i(\hat{x}).y + \varepsilon D_G f_i(\hat{x}).z > 0$

since the three terms are positive. Therefore we have $u_\varepsilon \in A_{k-1}$ that is impossible since $A_{k-1} = \emptyset$ . And so the reasoning by contradiction is complete, and we have proven

$$A_e \neq \emptyset \implies (0 \text{ solves } (\mathcal{Q})). \tag{4.5}$$

Since 0 solves $(\mathcal{Q})$, we have, for all $v \in \mathbb{R}^n$,

$$\begin{aligned}&(\forall i \in \{k, \dots, e\},\ D_G f_i(\hat{x}).v \geq 0) \implies (D_G f_{k-1}(\hat{x}).v \leq 0)\\ &\quad \iff (\forall i \in \{k, \dots, e\},\ D_G f_i(\hat{x}).v \geq 0) \implies (-D_G f_{k-1}(\hat{x}).v \geq 0)\,.\end{aligned}$$

Then we use Theorem 4.6 that ensures the existence of $\alpha^k, \dots,\ \alpha^e\ \in \mathbb{R}_+$ such that $\alpha^k D_G f_k(\hat{x}) + \dots + \alpha^e D_G f_e(\hat{x}) = -D_G f_{k-1}(\hat{x})$ or equivalently, $D_G f_{k-1}(\hat{x}) + \alpha^k D_G f_k(\hat{x}) + \dots + \alpha^e D_G f_e(\hat{x}) = 0$. We set

$$\lambda^i := \begin{cases} 0 & \text{if } i < k-1,\\ 1 & \text{if } i = k-1,\\ \alpha^i & \text{if } i \in \{k, \dots, e\},\\ 0 & \text{if } i \in \{e+1, \dots, m\},\end{cases}$$

and we obtain

$$A_e \neq \emptyset \implies ((a), (b), (c) \text{ hold}). \tag{4.6}$$

Then with (4.3) and (4.6) the conclusions (a), (b), (c) are proven.

**Proof of (d)**

The assumption (iii) means that $A_1 \neq \emptyset$ and by (4.2) we know that $A_0 = \emptyset$. Proceeding like in the proof of (4.5) we can prove that 0 is a solution of the following problem

$$\left.\begin{array}{rl}\text{Maximize} & D_G f_0(\hat{x}).v\\ \text{when} & v \in \mathbb{R}^n\\ \text{and when} & \forall i \in \{1, \dots, e\},\ D_G f_i(\hat{x}).v \geq 0.\end{array}\right\}$$

Then using Theorem 4.6, there exist $\alpha^1, \dots,\ \alpha^e\ \in \mathbb{R}_+$ such that

$$D_G f_0(\hat{x}) + \alpha^1 D_G f_1(\hat{x}) + \dots + \alpha^e D_G f_e(\hat{x}) = 0.$$

Then we conclude by setting

$$\lambda^i := \begin{cases} 1 & \text{if } i = 0,\\ \alpha^i & \text{if } i \in \{1, \dots, e\},\\ 0 & \text{if } i \in \{e+1, \dots, m\}.\end{cases}$$

And so the proof of Theorem is complete.

**Remark 4.8.** The use of the sets $A_k$ comes from the book of Alekseev-Tihomirov-Fomin [1], and the proof of formula (4.6) is similar to their proof (p. 247-248). The use of the set $A_0$ is yet done in [31].

### 4.3.4 Proof of Theorem 4.5

**First step - a simple case**

For a simple case, if $Dh_1(\hat{x}), \dots,\ Dh_q(\hat{x})$ are linearly dependent, there exist $\mu_1, \dots, \mu_q$ such that $(\mu_1, \dots, \mu_q) \neq (0, \dots, 0)$ and such that $\sum_{1 \leq i \leq q} \mu_i Dh_i(\hat{x}) = 0$. Then it suffices to take $\lambda^i = 0$ for all $i \in \{0, \dots, p\}$ to obtain the conclusions (a), (b), (c).

Now in the remaining of the proof we assume that the assumption (v) is fulfilled.

**Second step - deleting the non satured inequality constraints**

We will delete the non satured inequality constraints. Doing a change of index, we can assume that $\{1,\dots,e\} = \{i \in \{1,\dots,p\} : g_i(\hat{x}) = 0\}$. Using the lower semicontinuity at $\hat{x}$ of the $g_i$ when $i \in \{e+1,\dots,p\}$, we can say that there exists an open neighborhood $\Omega_1$ of $\hat{x}$ in $\Omega$ such that $g_i(x) > 0\}$ when $x \in \Omega_1$ and when $i \in \{e+1,\dots,p\}$. And so $\hat{x}$ is a solution of the following problem

$$\left.\begin{array}{rl} \text{Maximize} & g_0(x) \\ \text{when} & x \in \Omega_1 \\ \text{when} & \forall i \in \{1,\dots,e\},\ g_i(x) \geq 0 \\ \text{and when} & \forall i \in \{1,\dots,q\},\ h_i(x) = 0. \end{array}\right\} \qquad (\mathcal{M}_1)$$

**Third step - deleting the equality constraints**

We consider the mapping $h : \Omega_1 \to \mathbb{R}^q$ defined by $h(x) := (h_1(x),\dots,\ h_q(x))$. Under (iv) and (v), $h$ is continuous on a neighborhood of $\hat{x}$, and it is Fréchet differentiable at $\hat{x}$ with $Dh(\hat{x})$ onto.

We set $E_1 :=\mathrm{Ker}Dh(\hat{x})$ and we take a vector subspace $E_2$ of $\mathbb{R}^n$ such that $E_1 \oplus E_2 = \mathbb{R}^n$. Since $Dh(\hat{x}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$ and it is onto hence, $\dim E_1 = n - q$ and $\dim E_2 = q$. And we can do the assimilation $\mathbb{R}^n = E_1 \times E_2$. We set $(\hat{x}_1,\ \hat{x}_2) := \hat{x}\ \in E_1 \times E_2$. Then the partial differential $D_2h(\hat{x})$ is an isomorphism from $E_2$ onto $\mathbb{R}^q$. Now we can use Theorem 4.7 and assert that there exist a neighborhood $U_1$ of $\hat{x}_1$ in $E_1$, a neighborhood $U_2$ of $\hat{x}_2$ in $E_2$, and a mapping $\psi : U_1 \to U_2$ such that $\psi(\hat{x}_1) = \hat{x}_2$, $h(x_1, \psi(x_1)) = 0$ for all $x_1 \in U_1$, and such that $\psi$ is Fréchet differentiable at $\hat{x}_1$ with $D\psi(\hat{x}_1) = -\left(D_2h(\hat{x})\ \right)^{-1} \circ D_1h(\hat{x}) = 0$ since $D_1h(\hat{x}) = Dh(\hat{x})|_{E_1} = 0$.

We define $f_i : U_1 \to \mathbb{R}$ by setting $f_i(x_1) := g_i(x_1, \psi(x_1))$ for all $i \in \{0,\dots,\ e\}$. Since $\hat{x}$ is a solution of $(\mathcal{M}_1)$, $\hat{x}_1$ is a solution of the following problem without equality constraints

$$\left.\begin{array}{rl} \text{Maximize} & f_0(x_1) \\ \text{when} & x_1 \in U_1 \\ \text{when} & \forall i \in \{1,\dots,e\},\ f_i(x_1) \geq 0. \end{array}\right\} \qquad (\mathcal{R})$$

**Fourth step: using Theorem 4.4**

Since $\psi$ is Fréchet differentiable at $\hat{x}_1$, the mapping $(x_1 \mapsto (x_1, \psi(x_1))$ is Fréchet differentiable at $\hat{x}_1$, and using (i) and (ii), we obtain that $f_i$ is Fréchet differentiable (and therefore Gâteaux differentiable) at $\hat{x}_1$, for all $i \in \{0,\dots,\ e\}$. Note that $f_i(\hat{x}_1) = 0$ for all $i \in \{1,\dots,\ e\}$. Consequently we can use Theorem 4.4 on $(\mathcal{R})$ that permits us to ensure the existence of $\lambda^0,\ \lambda^1,\dots,\ \lambda^e \in \mathbb{R}_+$ such that

$$(\lambda^0,\ \lambda^1,\dots,\ \lambda^e) \neq (0,0,\dots,0), \tag{4.7}$$

$$\forall i \in \{1,\dots,e\},\ \lambda^i f_i(\hat{x}_1) = 0, \tag{4.8}$$

$$\sum_{0 \leq i \leq e} \lambda^i D_G f_i(\hat{x}_1) = 0. \tag{4.9}$$

**The proof of (a), (b), (c)**

We have for all $i \in \{0, \dots, e\}$, $D_G f_i(\hat{x}_1) = Df_i(\hat{x}_1) = Dg_i(\hat{x}_1, \psi(\hat{x}_1)) = D_1 g_i(\hat{x}_1, \psi(\hat{x}_1)) + D_2 g_i(\hat{x}_1, \psi(\hat{x}_1)) \circ D_1\psi(\hat{x}_1) = D_1 g_i(\hat{x})$ since $D_1\psi(\hat{x}_1) = 0$. The formula (4.9) implies

$$\sum_{0 \le i \le e} \lambda^i D_1 g_i(\hat{x}) = 0. \tag{4.10}$$

We will find operator $M \in \mathbb{R}^{q*}$ such that

$$\sum_{0 \le i \le e} \lambda^i D_2 g_i(\hat{x}) + M \circ D_2 h(\hat{x}) = 0.$$

Since $D_2 h(\hat{x})$ is invertible, $M$ is easily found as follows

$$M = \left( - \sum_{0 \le i \le e} \lambda^i D_2 g_i(\hat{x}) \right) \circ (D_2 h(\hat{x}))^{-1}. \tag{4.11}$$

Denoting by $\mu^1, \dots, \mu^q$ the coordinates of $M$ in the canonical basis of $\mathbb{R}^{q*}$, we obtain

$$\sum_{0 \le i \le e} \lambda^i D_2 g_i(\hat{x}) + \sum_{1 \le j \le q} \mu^j D_2 h_j(\hat{x}) = 0. \tag{4.12}$$

Since $E_1 := \mathrm{Ker} Dh(\hat{x}) = \bigcap_{1 \le j \le q} \mathrm{Ker} Dh_j(\hat{x})$, we have $D_1 h_j(\hat{x}) = Dh_j(\hat{x})|_{E_1} = 0$ for all $j \in \{1, \dots, q\}$, from (4.10) we obtain

$$\sum_{0 \le i \le e} \lambda^i D_1 g_i(\hat{x}) + \sum_{1 \le j \le q} \mu^j D_1 h_j(\hat{x}) = 0. \tag{4.13}$$

From (4.12) and (4.13) we obtain

$$\sum_{0 \le i \le e} \lambda^i D g_i(\hat{x}) + \sum_{1 \le j \le q} \mu^j D h_j(\hat{x}) = 0. \tag{4.14}$$

We set $\lambda^i := 0$ when $i \in \{e+1, \dots, p\}$, and so (4.14) implies (c). With (4.7) we obtain (a) and with (4.8) we obtain (b). And so the proof of (a), (b), (c) is complete.

**The proof of (d)**

The relation (4.7) provides the conclusion (d).

**The proof of (e)**

When $i \in \{1, \dots, e\}$, we have yet seen that $Df_i(\hat{x}_1) = D_1 g_i(\hat{x}) = Dg_i(\hat{x})|_{E_1}$. And so the translation of the assumption (vi) gives

$$\exists w \in E_1 \text{ such that } \forall i \in \{1, \dots, e\},\ Df_i(\hat{x}_1).w > 0.$$

That permits us to use the last assertion of Theorem 4.4 on $(\mathcal{R})$ to ensure that we can choose $\lambda^0 = 1$.

Then the proof of Theorem 4.5 is complete.

**Remark 4.9.** We see that in this proof, the assumption of Fréchet differentiability of the $h_j$ is used in order to apply the Implicit Function Theorem of Halkin. The assumption of Fréchet differentiability of $g_0$ and of the $g_i$ for which the associated constraint is satured is used to obtain the differentiability when we compose them with $h_j$ (to obtain the differentiability of the $f_j$). The Hadamard differentiability is sufficient to do that, but in finite-dimensional spaces, the Hadamard differentiability coincides with the Fréchet differentiability ([23], page 266).

### 4.3.5 Weak Pontryagin Principles for The Reduced Problems

We will apply New Multiplier Rule for the reduced to finite-horizon problems $(F_e^T)$ and $(F_i^T)$. To do it, we will translate these problems into static optimization problems. Note that, in the reduced problems, $x_0$ and $x_T$ are fixed and so they are not unknown variables. Like in [15], page 11, we will assume that for all $t \in \mathbb{N}$, $X_t$ is open and the sets of admissible controls $U_t$, with $t \in \mathbb{N}$ are defined by equalities and inequalities

$$U_t = \left(\bigcap_{i=1}^{m_i}\{u \in \mathbb{R}^d : g_t^i(u) \geq 0\}\right) \cap \left(\bigcap_{k=1}^{m_e}\{u \in \mathbb{R}^d : e_t^k(u) = 0\}\right), \tag{4.15}$$

where for all $i \in \{1, \ldots, m_i\}$, for all $k \in \{1, \ldots, m_e\}$, for all $t \in \mathbb{N}$, $g_t^i : \mathbb{R}^d \to \mathbb{R}$ and $e_t^k : \mathbb{R}^d \to \mathbb{R}$. We also assume that $U_t \neq \emptyset$ for all $t \in \mathbb{N}$.

**For Problem $(F_e^T)$**

We arbitrarily fix $T \in \mathbb{N}_*$. Let $\eta$ and $\hat{x}_{T+1}$ be given. We rewrite Problem $(F_e^T)$ under model of $(\mathcal{M})$.

Let $\Omega = \prod_{t=1}^{T} X_t \times \left(\mathbb{R}^d\right)^T$, then $\Omega$ is open as a finite product topology of open sets and the element $z$ from $\Omega$ has the form $z = (x_1, x_2, \ldots, x_T, u_0, u_1, \ldots, u_T)$. We set:

$$\tilde{g}^0(x_1, \ldots, x_T, u_0, \ldots, u_T) := \phi_0(\eta, u_0) + \textstyle\sum_{t=1}^{T} \phi_t(x_t, u_t);$$

and for all $i \in \{1, \ldots, m_i\}$, for all $t \in \{0, \ldots, T\}$, we set

$$\tilde{g}_t^i(x_1, \ldots, x_T, u_0, \ldots, u_T) := g_t^i(u_t);$$

and for all $k \in \{1, \ldots, m_e\}$, for all $t \in \{0, \ldots, T\}$, we set

$$\tilde{e}_t^k(x_1, \ldots, x_T, u_0, \ldots, u_T) := e_t^k(u_t);$$

and for all $\alpha \in \{1, \ldots, n\}$,

$$\psi_0^\alpha(x_1, \ldots, x_T, u_0, \ldots, u_T) := f_0^\alpha(\eta, u_0) - x_1^\alpha;$$

and for all $\alpha \in \{1, \ldots, n\}$, for all $t \in \{1, \ldots, T-1\}$

$$\psi_t^\alpha(x_1, \ldots, x_T, u_0, \ldots, u_T) := f_t^\alpha(x_t, u_t) - \hat{x}_{t+1}^\alpha;$$

and for all $\alpha \in \{1, \ldots, n\}$,

$$\psi_T^\alpha(x_1, \ldots, x_T, u_0, \ldots, u_T) := f_T^\alpha(x_T, u_T) - \hat{x}_{T+1}^\alpha.$$

So we have translated Problem $(F_e^T)$ into this form:

$$\left.\begin{array}{rl}\text{Maximize} & \tilde{g}^0(x_1,\dots,x_T,u_0,\dots,u_T);\\ \text{when} & \forall t\in\{1,\dots,T\},\ x_t\in X_t,\\ & \forall t\in\{0,\dots,T\},\ u_t\in\mathbb{R}^d,\\ & \forall i\in\{1,\dots,m_i\},\forall t\in\{0,\dots,T\},\ \tilde{g}_t^i(x_1,\dots,x_T,u_0,\dots,u_T)\geq 0,\\ & \forall k\in\{1,\dots,m_e\},\forall t\in\{0,\dots,T\},\ \tilde{e}_t^k(x_1,\dots,x_T,u_0,\dots,u_T)=0,\\ & \forall\alpha\in\{1,\dots,n\},\ \forall t\in\{0,\dots,T\},\ \psi_t^\alpha(x_1,\dots,x_T,u_0,\dots,u_T)=0.\end{array}\right\}\quad(R_e^T)$$

This is equivalent to:

$$\left\{\begin{array}{rl}\text{Maximize} & \tilde{g}^0(z);\\ \text{when} & z\in\Omega;\\ & \forall i\in\{1,\dots,m_i\},\forall t\in\{0,\dots,T\},\tilde{g}_t^i(z)\geq 0,\\ & \forall k\in\{1,\dots,m_e\},\forall t\in\{0,\dots,T\},\tilde{e}_t^k(z)=0,\\ & \forall\alpha\in\{1,\dots,n\},\ \forall t\in\{0,\dots,T\},\ \psi_t^\alpha(z)=0.\end{array}\right.$$

Now we see that the Problem $(R_e^T)$ has the form of Problem $(\mathcal{M})$. Let $\hat{z}=(\hat{x}_1,\dots,\hat{x}_T,\hat{u}_0,\dots,\hat{u}_T)$ be a solution of the problem above. If all the above-mentioned functions (functions $\tilde{g}^0$, $\tilde{g}_t^i$ ,$\tilde{e}_t^k$ and $\psi_t^\alpha$) satisfy all the assumptions of the New Multiplier Rule then we can apply Theorem 4.5 to obtain the multiplier $\lambda_o^T\in\mathbb{R}_+$ for the criterion function $\tilde{g}^0(z)$, multipliers $\lambda_{i,t}^T\in\mathbb{R}_+$ for inequality constraints $\tilde{g}_t^i(z)\geq 0$, multipliers $\mu_{k,t}^T\in\mathbb{R}$ for equality constraints $\tilde{e}_t^i(z)=0$ and multipliers $p_{t+1,\alpha}^T\in\mathbb{R}$ for the equality constraints $\psi_t^\alpha(z)=0$ where $t\in\{0,\dots,T\}$, $i\in\{1,\dots,m_i\}$, $\alpha\in\{1,\dots,n\}$ and $k\in\{1,\dots,m_e\}$ such that all the conclusions of Theorem 4.5 are satisfied.

From conclusion (a) of Theorem 4.5, we know that the multipliers are not simultaneously equal to zero.

From conclusion (b) of Theorem 4.5, we have

$$\forall i\in\{1,\dots,m_i\},\forall t\in\{0,\dots,T\},\ \lambda_{i,t}^T\tilde{g}_t^i(\hat{z})=0$$

i.e.

$$\forall i\in\{1,\dots,m_i\},\forall t\in\{0,\dots,T\},\ \lambda_{i,t}^T g_t^i(\hat{u}_t)=0.$$

We set $p_{t+1}^T:=\sum\limits_{\alpha=1}^n p_{t+1,\alpha}^T e_\alpha^*\in\mathbb{R}^{n*}$ where $(e_\alpha^*)_{1\leq\alpha\leq n}$ is the dual basis of the canonical basis of $\mathbb{R}^n$. Then the generalized Lagrangian of this problem is:

$$\begin{aligned}&\mathcal{L}(z,\lambda_o^T,\lambda_{1,0}^T,\dots,\lambda_{m_i,T}^T,\mu_{1,0}^T,\dots,\mu_{m_e,T}^T,p_{1,1}^T,\dots,p_{T+1,n}^T)\\ &=\lambda_o^T.\tilde{g}^0(z)+\sum_{t=0}^T\sum_{i=1}^{m_i}\lambda_{i,t}^T\tilde{g}_t^i(z)+\sum_{t=0}^T\sum_{k=1}^{m_e}\mu_{k,t}^T\tilde{e}_t^i(z)+\sum_{t=0}^T\sum_{\alpha=1}^n p_{t+1,\alpha}^T\psi_t^\alpha(z)\\ &=\lambda_o^T(\phi_0(\eta,u_0)+\textstyle\sum_{t=1}^T\phi_t(x_t,u_t))+\sum_{t=0}^T\sum_{i=1}^{m_i}\lambda_{i,t}^T g_t^i(u_t)+\sum_{t=0}^T\sum_{k=1}^{m_e}\mu_{k,t}^T e_t^i(u_t)\\ &+\sum_{t=0}^T\left\langle p_{t+1}^T,f_t(x_t,u_t)-x_{t+1}\right\rangle.\end{aligned}$$

As preliminary calculations, since all the functions in $\mathcal{L}$ are Gâteaux differentiable, the differential of the generalized Lagrangian with respect to $x_t$ (in the Gâteaux' sense) is

$$\begin{aligned}
&\tfrac{\partial \mathcal{L}}{\partial x_t}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T)\\
&= D_{G,x_t}\mathcal{L}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T)\\
&= \lambda_o^T.D_{G,1}\phi_t(x_t, u_t) + p_{t+1}^T \circ D_{G,1}f_t(x_t, u_t) - p_t^T,\\
&\qquad\qquad \text{for all } t \in \{1, \dots, T\}.
\end{aligned}$$

and the differential of the generalized Lagrangian with respect to $u_t$ is

$$\begin{aligned}
&\tfrac{\partial \mathcal{L}}{\partial u_t}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T)\\
&= D_{G,u_t}\mathcal{L}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T)\\
&= \lambda_o^T.D_{G,2}\phi_t(x_t, u_t) + p_{t+1}^T \circ D_{G,2}f_t(x_t, u_t) + \sum_{i=1}^{m_i} \lambda_{i,t}^T D_G g_t^i(u_t) + \sum_{k=1}^{m_e} \mu_{k,t}^T D_G e_t^i(u_t),\\
&\qquad\qquad \text{for all } t \in \{0, \dots, T\}.
\end{aligned}$$

From the conclusion (c) of Theorem of New Multiplier Rule, the Gâteaux differential of $\mathcal{L}$ with respect to $z$ vanishes at $\hat{z}$. It means that

$$\begin{cases}
\forall t \in \{1, \dots, T\},\ D_{G,x_t}\mathcal{L}(\hat{z}, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) = 0,\\
\forall t \in \{0, \dots, T\},\ D_{G,u_t}\mathcal{L}(\hat{z}, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) = 0
\end{cases}$$

$$\Longleftrightarrow$$

$$\left.\begin{aligned}
&\forall t \in \{1, \dots, T\},\ p_t^T = \lambda_o^T.D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_{G,1_t}f_t(\hat{x}_t, \hat{u}_t),\\
&\forall t \in \{0, \dots, T\},\ \ \lambda_o^T.D_{G,2}\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_{G,2}f_t(\hat{x}_t, \hat{u}_t)\\
&\qquad\qquad + \sum_{i=1}^{m_i} \lambda_{i,t}^T D_G g_t^i(\hat{u}_t) + \sum_{k=1}^{m_e} \mu_{k,t}^T D_G e_t^i(\hat{u}_t) = 0
\end{aligned}\right\} \tag{4.16}$$

And so, we can state the following result which is called weak Pontryagin principle for the finite-horizon (reduced) problem $(F_e^T)$.

**Proposition 4.10.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_e^j)$ where $j \in \{1, 2, 3\}$. We assume that $X_t$ are open for all $t \in \mathbb{N}$ and $U_t$ is defined by (4.15). We also assume that the following assumptions are fulfilled:*

**(i)** $\forall t \in \mathbb{N},\ \phi_t$ *is Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(ii)** *For all $i \in \{1, \dots, m_i\}$, for all $t \in \mathbb{N}$, $g_t^i$ is Fréchet differentiable at $\hat{u}_t$ when $g_t^i(\hat{u}_t) = 0$.*

**(iii)** *For all $i \in \{1, \dots, m_i\}$, for all $t \in \mathbb{N}$, $g_t^i$ is Gâteaux differentiable at $\hat{u}_t$ and lower semicontinuous at $\hat{u}_t$ when $g_t^i(\hat{u}_t) > 0$.*

**(iv)** *For all $k \in \{1, \dots, m_e\}$, for all $t \in \mathbb{N}$, $e_t^i$ is continuous on a neighborhood of $\hat{u}_t$ and Fréchet differentiable at $\hat{u}_t$.*

*For all $t \in \mathbb{N}$, $f_t$ is continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

*Then for all $T \in \mathbb{N}_*$ there exist $\lambda_o^T$, $\lambda_{i,t}^T \in \mathbb{R}_+$, $\mu_{k,t}^T \in \mathbb{R}$ and $p_{t+1}^T \in \mathbb{R}^{n*}$ where $t \in \{0, \dots, T\}$, $i \in \{1, \dots, m_i\}$ and $k \in \{1, \dots, m_e\}$ such that the following conditions are satisfied.*

**(a)** *The multipliers are not simultaneously equal to zero.*

**(b)** $\forall i \in \{1, \dots, m_i\}, \forall t \in \{0, \dots, T\},\ \lambda_{i,t}^T g_t^i(\hat{u}_t) = 0.$

**(c)** $\forall t \in \{1, \dots, T\},\ p_t^T = \lambda_o^T.D_1\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t).$

**(d)** $\forall t \in \{0, \dots, T\},$

$$\lambda_o^T.D_2\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t) + \sum_{i=1}^{m_i} \lambda_{i,t}^T D_G g_t^i(\hat{u}_t) + \sum_{k=1}^{m_e} \mu_{k,t}^T D h_t^i(\hat{u}_t) = 0.$$

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_e^j)$ where $j \in \{1,2,3\}$. Using Theorem 4.1, the restriction $\hat{z} = ((\hat{x}_0, \ldots, \hat{x}_T), (\hat{u}_0, \ldots, \hat{u}_{T-1}))$ is an optimal solution of Problem $(F_e^T)$, and then it is a solution of Problem $(R_e^T)$. Under (i), (ii), (iii) and (iv) we obtain the following assertions in turn:

- Function $\tilde{g}^0$ is Fréchet differentiable at $\hat{z}$ as a sum of $T$ Fréchet differentiable functions.
- For all $i \in \{1, \ldots, m_i\}$, for all $t \in \{0, \ldots, T\}$, functions $\tilde{g}_t^i$ is Fréchet differentiable at $\hat{z}$ when $\tilde{g}_t^i(\hat{z}) = 0$.
- For all $i \in \{1, \ldots, m_i\}$, for all $t \in \{0, \ldots, T\}$, functions $\tilde{g}_t^i$ is Gâteaux differentiable at $\hat{z}$ and lower semicontinuous at $\hat{z}$ when $\tilde{g}_t^i(\hat{z}) > 0$.
- For all $k \in \{1, \ldots, m_e\}$, for all $t \in \{0, \ldots, T\}$, $\tilde{e}_t^i$ is continuous on a neighborhood of $\hat{z}$ and Fréchet differentiable at $\hat{z}$. For all $\alpha \in \{1, \ldots, n\}$, for all $t \in \{0, \ldots, T\}$, $\psi_t^\alpha$ is continuous on a neighborhood of $\hat{z}$ and Fréchet differentiable at $\hat{z}$.

Then using Theorem of New Multiplier Rule for Problem $(R_e^T)$ as we did before the statement of this proposition, we easily obtain the multipliers $\lambda_o^T$, $\lambda_{i,t}^T \in \mathbb{R}_+$, $\mu_{k,t}^T \in \mathbb{R}$ and $p_{t+1}^T \in \mathbb{R}^{n*}$ where $t \in \{0, \ldots, T\}$, $i \in \{1, \ldots, m_i\}$ and $k \in \{1, \ldots, m_e\}$ that satisfy results (a) and (b). Results (c) and (d) are obtained after identifying Gâteaux differential with Fréchet differential of Fréchet differentiable functions in (4.16). □

In the special case when for all $t \in \mathbb{N}$, $U_t$ is an arbitrary subset of $\mathbb{R}^d$ and $\hat{u}_t$ belongs the interior of $U_t$, Problem $(R_e^T)$ now contains only phase constraints an can be rewritten as follows.

$$\left.\begin{array}{rl} \text{Maximize} & \tilde{g}^0(z); \\ \text{when} & z \in \Omega; \\ & \forall \alpha \in \{1, \ldots, n\},\ \forall t \in \{0, \ldots, T\},\ \psi_t^\alpha(z) = 0. \end{array}\right\} \qquad (int_e^T)$$

Then applying Theorem of New Multiplier Rule, we obtain the following simpler statement.

**Corollary 4.11.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_e^j)$ where $j \in \{1,2,3\}$. We assume that $X_t$ is open and $\hat{u}_t \in int(U_t)$ for all $t \in \mathbb{N}$ . We also assume that the following assumptions are fulfilled:*

**(i)** $\forall t \in \mathbb{N}$, *$\phi_t$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

**(ii)** *For all $t \in \mathbb{N}$, $f_t$ are continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

*Then for all $T \in \mathbb{N}_*$ there exist $\lambda_o^T \in \mathbb{R}_+$ and $p_{t+1}^T \in \mathbb{R}^{n*}$ where $t \in \{0, \ldots, T\}$ such that the following conditions are satisfied.*

**(a)** $\lambda_o^T$, $p_{t+1}^T$, $t \in \{0, \ldots, T\}$ *are not all zeros.*

**(b)** $\forall t \in \{1, \ldots, T\}$, $p_t^T = \lambda_o^T.D_1\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t)$.

**(c)** $\forall t \in \{0, \ldots, T\}$, $0 = \lambda_o^T.D_2\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t)$.

**For Problem $(F_i^T)$:** the way we treat it is almost similar to what we did with the (De) case. The different point is just the following: In $(R_e^T)$ the equal sign in $\psi_t^\alpha(x_1, \ldots, x_T, u_0, \ldots, u_T) = 0$ will be replaced by the greater or equal sign $(\geq)$ because here the phase constraint has the form $x_{t+1} \leq f_t(x_t, u_t)$. And then, we obtain following problem

$$\left.\begin{array}{rl}\text{Maximize} & \tilde{g}^0(x_1,\dots,x_T,u_0,\dots,u_T);\\ \text{when} & \forall t\in\{1,\dots,T\},\ x_t\in X_t,\\ & \forall t\in\{0,\dots,T\},\ u_t\in\mathbb{R}^d,\\ & \forall i\in\{1,\dots,m_i\},\forall t\in\{0,\dots,T\},\tilde{g}_t^i(x_1,\dots,x_T,u_0,\dots,u_T)\geq 0,\\ & \forall k\in\{1,\dots,m_e\},\forall t\in\{0,\dots,T\},\tilde{e}_t^k(x_1,\dots,x_T,u_0,\dots,u_T)=0,\\ & \forall\alpha\in\{1,\dots,n\}\forall t\in\{0,\dots,T\},\psi_t^\alpha(x_1,\dots,x_T,u_0,\dots,u_T)\geq 0,\end{array}\right\}\qquad (R_i^T)$$

Apply the Theorem 4.5 for this problem with a notice that now the inequality constraints group includes functions $\tilde{g}_t^i$ and $\psi_t^\alpha$ and the equality constraints group includes only functions $e_t^k$, we receive the weak Pontryagin principle like before but with a slight difference for the inequalities $\psi_t^\alpha(z)\geq 0$. The whole new proposition is presented here:

**Proposition 4.12.** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_i^j)$ when $j\in\{1,2,3\}$. We assume that $X_t$ are open for all $t\in\mathbb{N}$ and $U_t$ is defined by (4.15). We also assume that the following assumptions are fulfilled:*

**(i)** *$\forall t\in\mathbb{N},\ \phi_t$ is Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$.*

**(ii)** *For all $i\in\{1,\dots,m_i\}$, for all $t\in\mathbb{N},\ g_t^i$ is Fréchet differentiable at $\hat{u}_t$ when $g_t^i(\hat{u}_t)=0$.*

*For all $\alpha\in\{1,\dots,n\}$, for all $t\in\mathbb{N},\ f_t^\alpha$ is Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$ when $f_t^\alpha(\hat{x}_t,\hat{u}_t)=\hat{x}_{t+1}^\alpha$.*

**(iii)** *For all $i\in\{1,\dots,m_i\}$, for all $t\in\mathbb{N},\ g_t^i$ is Gâteaux differentiable at $\hat{u}_t$ and lower semicontinuous at $\hat{u}_t$ when $g_t^i(\hat{u}_t)>0$.*

*For all $\alpha\in\{1,\dots,n\}$, for all $t\in\mathbb{N},\ f_t^\alpha$ is Gâteaux differentiable and and lower semicontinuous at $(\hat{x}_t,\hat{u}_t)$ when $f_t^\alpha(\hat{x}_t,\hat{u}_t)=\hat{x}_{t+1}^\alpha$.*

**(iv)** *For all $k\in\{1,\dots,m_e\}$, for all $t\in\mathbb{N},\ e_t^i$ is continuous on a neighborhood of $\hat{u}_t$ and Fréchet differentiable at $\hat{u}_t$.*

*Then for all $T\in\mathbb{N}_*$ there exist $\lambda_o^T,\ \lambda_{i,t}^T\in\mathbb{R}_+,\ \mu_{k,t}^T\in\mathbb{R}$ and $p_{t+1}^T\in\mathbb{R}_+^{n*}$ where $t\in\{0,\dots,T\},\ i\in\{1,\dots,m_i\}$ and $k\in\{1,\dots,m_e\}$ such that the following conditions are satisfied.*

**(a)** *The multipliers are not simultaneously equal to zero.*

**(b)** *For all $i\in\{1,\dots,m_i\}$, for all $t\in\{0,\dots,T\},\ \lambda_{i,t}^T g_t^i(\hat{u}_t)=0$.*

*For all $t\in\{0,\dots,T\},\ \left\langle p_{t+1}^T,\ f_t(\hat{x}_t,\hat{u}_t)-\hat{x}_{t+1}\right\rangle=0$.*

**(c)** *$\forall t\in\{1,\dots,T\},\ p_t^T=\ \lambda_o^T.D_1\phi_t(\hat{x}_t,\hat{u}_t)+p_{t+1}^T\circ D_1f_t(\hat{x}_t,\hat{u}_t)$.*

**(d)** *$\forall t\in\{0,\dots,T\}$,*

$$\lambda_o^T.D_2\phi_t(\hat{x}_t,\hat{u}_t)+p_{t+1}^T\circ D_2f_t(\hat{x}_t,\hat{u}_t)+\sum_{i=1}^{m_i}\lambda_{i,t}^T D_G g_t^i(\hat{u}_t)+\sum_{k=1}^{m_e}\mu_{k,t}^T De_t^i(\hat{u}_t)=0.$$

In the special case when for all $t\in\mathbb{N}$, $U_t$ is an arbitrary subset of $\mathbb{R}^d$ and $\hat{u}_t$ belongs the interior of $U_t$, Problem $(R_i^T)$ now contains only phase constraints an can be rewritten as follows.

$$\left\{\begin{array}{rl}\text{Maximize} & \tilde{g}^0(z);\\ \text{when} & z\in\Omega;\\ & \forall\alpha\in\{1,\dots,n\},\ \forall t\in\{0,\dots,T\},\ \psi_t^\alpha(z)\geq 0.\end{array}\right.\qquad (int_i^T)$$

This has the form of Problem $(\mathcal{I})$. Then we obtain the following corollary after applying Theorem 4.4.

**Corollary 4.13.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_i^j)$ when $j \in \{1,2,3\}$. We assume that for all $t \in \mathbb{N}$, $X_t$ are open and $\hat{u}_t \in int(U_t)$. We also assume that the following assumptions are fulfilled:*

**(i)** $\forall t \in \mathbb{N},\ \phi_t,\ f_t$ *is Gâteaux differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(ii)** *For all* $\alpha \in \{1,\dots,n\}$, *for all* $t \in \mathbb{N}$, $f_t^\alpha$ *is lower semicontinuous at* $(\hat{x}_t, \hat{u}_t)$ *when* $f_t^\alpha(\hat{x}_t, \hat{u}_t) > \hat{x}_{t+1}^\alpha$.

*Then for all $T \in \mathbb{N}_*$ there exist $\lambda_o^T \in \mathbb{R}_+$ and $p_{t+1}^T \in \mathbb{R}_+^{n*}$ where $t \in \{0,\dots,T\}$ such that the following conditions are satisfied.*

**(a)** $\lambda_o^T,\ \ p_{t+1}^T,\ t \in \{0,\dots,T\}$ *are not all zeros.*

**(b)** *For all* $t \in \{0,\dots,T\}$, $\left\langle p_{t+1}^T,\ f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \right\rangle = 0$.

**(c)** $\forall t \in \{1,\dots,T\},\ p_t^T = \lambda_o^T.D_1\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t)$.

**(d)** $\forall t \in \{0,\dots,T\},\ 0 = \lambda_o^T.D_2\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t)$.

## 4.4 Weak Pontryagin Principles for Infinite-Horizon Problems with Interior Optimal Controls

In this section we consider the case where values of the optimal control sequence belong to the topological interior of the set $U_t$ of the considered controls at each time $t$, and where the system is governed by the difference inequation (Di).

**Theorem 4.14.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_i^j)$ when $j \in \{1,2,3\}$. We assume that the following assumptions are fulfilled.*

**(i)** *For all* $t \in \mathbb{N}$, $\hat{u}_t \in \mathrm{int}U_t$.

**(ii)** *For all* $t \in \mathbb{N}$, $\phi_t$ *and* $f_t$ *are Gâteaux differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(iii)** *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1,\dots,n\}$, $f_t^\alpha$ *is lower semicontinuous at* $(\hat{x}_t, \hat{u}_t)$ *when* $f_t^\alpha(\hat{x}_t, \hat{u}_t) > \hat{x}_{t+1}^\alpha$.

**(iv)** *For all* $t \in \mathbb{N}_*$, $D_{G,1}f_t(\hat{x}_t, \hat{u}_t)$ *is invertible.*

*Then there exist $\lambda_0 \in \mathbb{R}$ and $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfy the following properties.*

**(NN)** $(\lambda_0, p_1) \neq (0,0)$.

**(Si)** $\lambda_0 \geq 0$ *and, for all* $t \in \mathbb{N}_*$, $p_t \geq 0$.

**(S$\ell$)** *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1,\dots,n\}$, $p_{t+1}^\alpha \cdot (f_t^\alpha(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}^\alpha) = 0$.

**(AE)** *For all* $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_{G,1}f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t)$.

**(WM)** *For all* $t \in \mathbb{N}$, $p_{t+1} \circ D_{G,2}f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_{G,2}\phi_t(\hat{x}_t, \hat{u}_t) = 0$.

*Proof.* Our assumptions (i, ii, iii) imply that the assumptions of Corollary 4.13 in previous section are fulfilled and so we know that, for all $T \in \mathbb{N}_*$, there exists $(\lambda_0^T, p_1^T, \dots, p_{T+1}^T) \in \mathbb{R} \times (\mathbb{R}_+^{n*})^{T+1}$ which satisfies the following conditions.

$$(\lambda_o^T,\ \ p_1^T, \dots, p_{T+1}^T) \neq (0,0,\dots,0). \tag{4.17}$$

$$\lambda_o^T \geq 0 \text{ and } \forall t \in \{1,\dots,T+1\},\ p_t^T \geq 0. \tag{4.18}$$

$$\forall t \in \{0,\dots,T\},\ \left\langle p_{t+1}^T,\ f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \right\rangle = 0. \tag{4.19}$$

$$\forall t \in \{1,\dots,T\},\ p_t^T = \lambda_o^T.D_1\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t). \tag{4.20}$$

$$\forall t \in \{0,\dots,T\},\ 0 = \lambda_o^T.D_2\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t). \tag{4.21}$$

Using assumption (iv) we can formulate (4.20) as follows.

$$\forall t \in \{1,\dots,T\},\ p_{t+1}^T = p_t^T \circ [D_1 f_t(\hat{x}_t,\hat{u}_t)]^{-1} - \lambda_o^T.D_1\phi_t(\hat{x}_t,\hat{u}_t) \circ [D_1 f_t(\hat{x}_t,\hat{u}_t)]^{-1}. \quad (4.22)$$

From this last equation we easily see that $(\lambda_0^T, p_1^T) = (0,0) \implies (\lambda_0^T, p_1^T, \dots, p_{T+1}^T) = (0,0,\dots,0)$ and then from (4.17) we have a contradiction. Therefore we can assert that

$$(\lambda_0^T, p_1^T) \neq (0,0). \quad (4.23)$$

Since the set of the lists of multipliers of Problem $(int_i^T)$ is a cone, we can normalize the multipliers by setting

$$\left\|(\lambda_0^T, p_1^T)\right\| = \left|\lambda_0^T\right| + \left\|p_1^T\right\| = 1. \quad (4.24)$$

Since the values of the sequence $(\lambda_0^T, p_1^T)_{T\in\mathbb{N}_*}$ belong to the unit sphere of $\mathbb{R}\times\mathbb{R}^{n*}$ which is compact, using the Bolzano-Weierstrass theorem we can say that there exist an increasing function $\varphi : \mathbb{N}_* \to \mathbb{N}_*$ and $(\lambda_0, p_1) \in \mathbb{R}\times\mathbb{R}^{n*}$ such that $|\lambda_0| + \|p_1\| = 1$, $\lim_{T\to+\infty} \lambda_0^{\varphi(T)} = \lambda_0$ and $\lim_{T\to+\infty} p_1^{\varphi(T)} = p_1$.

Note that $p_2^{\varphi(T)} = (p_1^{\varphi(T)} - \lambda_0^{\varphi(T)} D_{G,1}\phi_1(\hat{x}_1,\hat{u}_1)) \circ D_{G,1} f_1(\hat{x}_1,\hat{u}_1)^{-1}$ for all $T \geq t-1$, which implies that

$$p_2 = \lim_{T\to+\infty} p_2^{\varphi(T)} = (p_1 - \lambda_0.D_{G,1}\phi_1(\hat{x}_1,\hat{u}_1)) \circ D_{G,1} f_1(\hat{x}_1,\hat{u}_1)^{-1}$$

Proceeding recursively we define, for all $t \in \mathbb{N}_*$,

$$\begin{aligned} p_{t+1} &:= \lim_{T\to+\infty} p_{t+1}^{\varphi(T)} = \lim_{T\to+\infty} (p_t^{\varphi(T)} - \lambda_0^{\varphi(T)} D_{G,1}\phi_t(\hat{x}_t,\hat{u}_t)) \circ D_{G,1} f_t(\hat{x}_t,\hat{u}_t)^{-1} \\ &= (p_t - \lambda_0 D_{G,1}\phi_t(\hat{x}_t,\hat{u}_t)) \circ D_{G,1} f_t(\hat{x}_t,\hat{u}_t)^{-1}. \end{aligned}$$

And so we have built $\lambda_0 \in \mathbb{R}$ and a sequence $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ which satisfies (AE). We have yet seen that (NN) is satisfied. From (4.18) we obtain (Si). From (4.19) we obtain (S$\ell$). From (4.21) we obtain (WM). □

**Notation 4.15.** Note that, for all $t \in \mathbb{N}_*$ , $D_1 f_t(\hat{x}_t,\hat{u}_t)$ belongs to $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, the space of all continuous linear operators from $\mathbb{R}^n$ into $\mathbb{R}^n$. It can be represented by a $n\times n$ Jacobian matrix. We will denote the element at position $(\alpha,\beta)$ of this matrix by $\dfrac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x_t^\beta}$ where $\alpha,\beta \in \{1,\dots,n\}$.

In the next theorem, we will replace the assumption of invertibility (assumption (iv)) in previous theorem by another one, which is called positivity assumption. The theorem is stated as follows.

**Theorem 4.16.** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_i^j)$ when $j \in \{1,2,3\}$. We assume that the following assumptions (i,ii,iii) of Theorem 4.14 are fulfilled. Moreover we assume that the following assumption is fulfilled.*

**(v)** *For all $t \in \mathbb{N}_*$, for all $\alpha,\beta \in \{1,\dots,n\}$, $\dfrac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x_t^\beta} \geq 0$ and $\dfrac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x_t^\alpha} > 0$.*

*Then the conclusions of Theorem 4.14 hold.*

Before proving this theorem, we recall the following useful lemma which related to condition (v).

**Lemma 4.17.** *Under assumption (v) of Theorem 4.16, setting $\rho_t := \min\limits_{1\leq\alpha\leq n} \dfrac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x_t^\alpha}$ - a real positive number. Then the following assertions hold:*

**(i)** *For all* $y \in \mathbb{R}^n_+$, $D_1 f_t(\hat{x}_t, \hat{u}_t).y \geq \rho_t.y$.

**(ii)** *For all* $\pi \in \mathbb{R}^{n*}_+$, $\pi \circ D_1 f_t(\hat{x}_t, \hat{u}_t) \geq \rho_t.\pi$.

*Proof.* **(i)** Let $y \in \mathbb{R}^n_+$ arbitrary. For all $\alpha \in \{1, \ldots, n\}$ we have

$$\sum_{\beta=1}^{n} \frac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x_t^\beta}.y^\beta \geq \frac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x_t^\alpha}.y^\alpha \geq \rho_t.y^\alpha.$$

That means $D_1 f_t(\hat{x}_t, \hat{u}_t).y \geq \rho_t.y$.

**(ii)** Let $\pi \in \mathbb{R}^{n*}_+$. Using (i) we know for all $y \in \mathbb{R}^n_+$, $D_1 f_t(\hat{x}_t, \hat{u}_t).y \geq \rho_t.y$. Then we have

$$\begin{aligned}\pi \circ D_1 f_t(\hat{x}_t, \hat{u}_t).y &= \pi(D_1 f_t(\hat{x}_t, \hat{u}_t).y) = \pi(z + \rho_t.y) \text{ where } z \in \mathbb{R}^n_+ \\ &= \pi(z) + \pi(\rho_t.y) \\ &\geq \pi(\rho_t.y) = \rho_t.\pi(y).\end{aligned}$$

We have the change above because $\pi$ is linear. The last inequality means $\pi \circ D_1 f_t(\hat{x}_t, \hat{u}_t) \geq \rho_t.\pi$.

□

Now we move to the proof of Theorem 4.16.

*Proof.* Proceeding as in the proof of Theorem 4.14we obtain for all $T \in \mathbb{N}_*$, there exists $(\lambda_0^T, p_1^T, \ldots, p_{T+1}^T) \in \mathbb{R} \times (\mathbb{R}^{n*}_+)^{T+1}$ which satisfies the conditions (4.17-4.21). We have

$$\forall t \in \{1, \ldots, T\},\ p_t^T = \lambda_o^T.D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$$

$$\Rightarrow \forall t \in \{1, \ldots, T\},\ p_{t+1}^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) = p_t^T - \lambda_o^T.D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t).$$

Since $p_{t+1}^T \geq 0$, using Lemma 4.17 we have:

$$\forall t \in \{1, \ldots, T\},\ p_{t+1}^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) \geq \rho_t.p_{t+1}^T$$

where $\rho_t := \min_{1 \leq \alpha \leq n} \dfrac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x_t^\alpha}$. Then

$$\forall t \in \{1, \ldots, T\},\ 0 \leq \rho_t.p_{t+1}^T \leq p_t^T - \lambda_o^T.D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t)$$

$$\Rightarrow \rho_t \left\|p_{t+1}^T\right\| = \left\|\rho_t.p_{t+1}^T\right\| \leq \left\|p_t^T\right\| + \lambda_o^T.\left\|D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t)\right\|,\ (\text{since } \lambda_o^T \geq 0)$$

$$\Rightarrow\ \forall t \in \{1, \ldots, T\},\ \left\|p_{t+1}^T\right\| \leq \frac{1}{\rho_t}\left(\left\|p_t^T\right\| + \lambda_o^T.\left\|D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t)\right\|\right),$$

which implies that if $(\lambda_0^T, p_1^T) = (0, 0)$ then $(\lambda_0^T, p_1^T, \ldots, p_{T+1}^T) = (0, 0, \ldots, 0)$ which is a contradiction with condition (4.17). Then condition (4.23) holds and by normalizing we obtain (4.24). Then $0 \leq \lambda_o^T \leq 1$ and the following relation holds for all $t \in \mathbb{N}_*$ and for all $T \geq t - 1$.

$$\left\|p_{t+1}^T\right\| \leq \frac{1}{\rho_t}\left(\left\|p_t^T\right\| + \left\|D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t)\right\|\right) \tag{4.25}$$

Now we want to prove the following assertion:

$$\forall t \in \mathbb{N}_*,\ \exists \zeta_t \in (0, +\infty), \forall T \geq t - 1,\ \left\|p_t^T\right\| \leq \zeta_t. \tag{4.26}$$

We proceed by induction. When $t = 1$, from (4.24), we know that $\left\|p_1^T\right\| \leq 1$. And so it suffices to take $\zeta_1 := 1$. Assume that (4.26) holds for $t$, then for $t+1$, from (4.25) we obtain

$$\left\|p_{t+1}^T\right\| \leq \frac{1}{\rho_t}\left(\zeta_t + \|D_{G,1}\phi_t(\hat{x}_t, \hat{u}_t)\|\right) =: \zeta_{t+1}.$$

Hence, assertion (4.26) is proven.

Using (4.26) and the diagonal process of Cantor as it is formulated in [15] (Theorem A.1, p.94), we can assert that there exist an increasing function $\varphi : \mathbb{N}_* \to \mathbb{N}_*$, a real number $\lambda_o \geq 0$ and a sequence $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}_+^{n*})^{\mathbb{N}_*}$ such that $\lim_{T\to+\infty} \lambda_0^{\varphi(T)} = \lambda_0$ and $\lim_{T\to+\infty} p_t^{\varphi(T)} = p_t$ for all $t \in \mathbb{N}_*$ . Now we conclude as in the proof of Theorem 4.14. □

## 4.5 Weak Pontryagin Principles for Infinite-Horizon Problems with Constrained Controls

In this section we first consider the case where the sets of controls are defined by inequalities for each $t \in \mathbb{N}$.

$$U_t = \bigcap_{1\leq k\leq m} \{u \in \mathbb{R}^d : g_t^k(u) \geq 0\} \tag{4.27}$$

where $g_t^k : \mathbb{R}^d \to \mathbb{R}$ and we assume that $U_t \neq \emptyset$ for all $t \in \mathbb{N}$.

In this subsection, we use the following notations for linear span and convex hull of a finite set of vectors in real normed vector space.

**Notation 4.18.** Let $S$ be a finite subset of a normed vector space $X$. We denote the linear span of $S$ by $span(S)$ and the convex hull of $S$ by $conv(S)$.

**Lemma 4.19.** *Let $E$ be a finite-dimensional real normed vector space and $I$ be a nonempty finite set. Let $(\varphi_i)_{i\in I} \in (E^*)^I$. The three following assertions are equivalent.*

**(i)** $0 \notin conv\{\varphi_i : i \in I\}$.

**(ii)** *For all* $(\lambda_i)_{i\in I} \in (\mathbb{R}_+)^I$, $\sum_{i\in I} \lambda_i\varphi_i = 0 \Longrightarrow \lambda_i = 0$ *for all* $i \in I$.

**(iii)** *There exists* $w \in E$ *such that,* $\langle\varphi_i, w\rangle > 0$ *for all* $i \in I$.

*Proof.* Firstly, we prove that non(ii) implies non(i). From non(ii) we deduce that there exists $(\lambda_i)^{i\in I} \in (\mathbb{R}_+)^I$ such that $(\lambda_i)^{i\in I} \neq 0$ and $\sum_{i\in I}(\frac{\lambda_i}{\sum_{j\in I}\lambda_j})\varphi_i = 0$ which implies non(i). Secondly, we prove that non(i) implies non(ii). From non(i) there exists $(\alpha_i)_{i\in I} \in (\mathbb{R}_+)^I$ such that $\sum_{i\in I} \alpha_i = 1$ and $0 = \sum_{i\in I} \alpha_i\varphi_i$, and since $(\alpha_i)_{i\in I}$ is non zero, non(ii) is fulfilled. And so we have proven that non(i) and non(ii) are equivalent.

To prove that (i) implies (iii), note that $0 \notin conv\{\varphi_i : i \in I\} =: K$, and $K$ is a nonempty convex compact set. Using the theorem of separation of Hahn-Banach, we can assert that there exist $\xi \in \mathbb{R}^{n**}$ and $a \in (0, +\infty)$ such that $\langle\xi, \varphi\rangle \geq a$ for all $\varphi \in K$, and $\langle\xi, 0\rangle = 0 < a$. Since $\mathbb{R}^n$ is reflexive, there exists $w \in \mathbb{R}^n$ such $\langle\xi, \varphi\rangle = \langle\varphi, w\rangle$ for all $\varphi \in \mathbb{R}^{n*}$. Therefore for all $i \in I$, we have $\langle\varphi_i, w\rangle \geq a > 0$ that is (iii).

To prove that (iii) implies (i) we set $\gamma := \min_{i\in I}\langle\varphi_i, w\rangle > 0$. When $\varphi \in conv\{\varphi_i : i \in I\}$, there exists $(\alpha_i)_{i\in I} \in (\mathbb{R}_+)^I$ such that $\sum_{i\in I} \alpha_i = 1$ and $\varphi = \sum_{i\in I} \alpha_i\varphi_i$. Then we have $\langle\varphi, w\rangle = \sum_{i\in I} \alpha_i\langle\varphi_i, w\rangle \geq \sum_{i\in I} \alpha_i\gamma = \gamma > 0$ which implies $\varphi \neq 0$, and so (i) is satisfied. □

**Lemma 4.20.** *Let $E$ be a finite-dimensional real normed vector space and $I$ be a nonempty finite set. Let $(\varphi_i)_{i\in I} \in (E^*)^I$ such that $0 \notin conv\{\varphi_i : i \in I\}$. For all $i \in I$, let $(r_i^h)_{h\in\mathbb{N}_*} \in R_+^{\mathbb{N}_*}$. We assume that the sequence $(\psi_h)_{h\in\mathbb{N}_*} := (\sum_{i\in I} r_i^h \varphi_i)_{h\in\mathbb{N}_*}$ is bounded in $E^*$. Then there exists an increasing function $\rho : \mathbb{N}_* \to \mathbb{N}_*$ such that, for all $i \in I$, the sequence $(r_i^{\rho(h)})_{h\in\mathbb{N}_*}$ is convergent in $R_+$.*

*Proof.* Firstly, we prove that $\liminf_{h\to+\infty} \sum_{i\in I} r_i^h < +\infty$. We proceed by contradiction: we assume that $\liminf_{h\to+\infty} \sum_{i\in I} r_i^h = +\infty$. Therefore we have $\lim_{h\to+\infty} \sum_{i\in I} r_i^h = +\infty$. We set $s_i^h := \frac{r_i^h}{\sum_{j\in I} r_j^h} \in \mathbb{R}_+$. We have $\sum_{i\in I} s_i^h = 1$ and so $\sum_{i\in I} s_i^h \varphi_i \in conv\{\varphi_i : i \in I\}$. Note that $\|\sum_{i\in I} s_i^h \varphi_i\| = \frac{1}{\sum_{j\in I} r_j^h} \|\psi_h\|$ converges to 0 when $h \to +\infty$ since $(\psi_h)_{h\in\mathbb{N}_*}$ is bounded. Thus, we have $\lim_{h\to+\infty} \sum_{i\in I} s_i^h \varphi_i = 0$ which implies that $0 \in conv\{\varphi_i : i \in I\}$ and that is a contradiction with one assumption. And so we have proven that $s := \liminf_{h\to+\infty} \sum_{i\in I} r_i^h < +\infty$.

Now we can assert that there exists an increasing function $\tau : \mathbb{N}_* \to \mathbb{N}_*$ such that $\lim_{h\to+\infty} \sum_{i\in I} r_i^{\tau(h)} = s$. Therefore there exists $M \in \mathbb{R}_+$ such that $0 \le \sum_{i\in I} r_i^{\tau(h)} \le M$ for all $h \in \mathbb{N}_*$. Since for all $i \in I$, we have $0 \le r_i^{\tau(h)} \le \sum_{j\in I} r_j^{\tau(h)} \le M$, i.e. the sequence $(r_i^{\tau(h)})_{h\in\mathbb{N}_*}$ is bounded in $R_+$. Using several times the Bolzano-Weierstrass theorem we can assert that there exist an increasing function $\tau_1 : \mathbb{N}_* \to \mathbb{N}_*$ and $r_i^* \in \mathbb{R}_+$ for all $i \in I$, such that $\lim_{h\to+\infty} r_i^{\tau\circ\tau_1(h)} = r_i^*$. It suffices to take $\rho := \tau \circ \tau_1$. □

**Theorem 4.21.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(P_i^j)$ where $j \in \{1, 2, 3\}$ and where the sets $U_t$ are defined by (4.27). We assume that the following assumptions are fulfilled.*

**(i)** *For all $t \in \mathbb{N}$, $\phi_t$ and $f_t$ are Gâteaux differentiable at $(\hat{x}_t, \hat{u}_t)$.*

**(ii)** *For all $t \in \mathbb{N}$, for all $k \in \{1, \dots, m\}$, $g_t^k$ is Gâteaux differentiable at $\hat{u}_t$.*

**(iii)** *For all $t \in \mathbb{N}$, for all $\alpha \in \{1, \dots, n\}$, $f_t^\alpha$ is lower semicontinuous at $(\hat{x}_t, \hat{u}_t)$ when $f_t^\alpha(\hat{x}_t, \hat{u}_t) > \hat{x}_{t+1}^\alpha$.*

**(iv)** *For all $t \in \mathbb{N}$, for all $k \in \{1, \dots, m\}$, $g_t^k$ is lower semicontinuous at $\hat{u}_t$ when $g_t^k(\hat{u}_t) > 0$.*

**(v)** *For all $t \in \mathbb{N}$, $0 \notin conv\{D_G g_t^k(\hat{u}_t) : k \in I_t^s\}$ where $I_t^s := \{k \in \{1, \dots, m\} : g_t^k(\hat{u}_t) = 0\}$.*

**(vi)** *For all $t \in \mathbb{N}_*$, $D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$ is invertible.*

**(vii)** *For all $t \in \mathbb{N}_*$, for all $\alpha, \beta \in \{1, \dots, n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x^\beta} \ge 0$ and for all $\alpha \in \{1, \dots, n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x^\alpha} > 0$.*

*Then, under (i-vi) or under (i-v) and (vii), there exist $\lambda_0 \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, $(\lambda_t^1)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, ..., and $(\lambda_t^m)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ which satisfy the following conditions.*

**(NN)** $(\lambda_0, p_1) \neq (0, 0)$.

**(Si)** $\lambda_0 \ge 0$, , $p_t \ge 0$ *for all* $t \in \mathbb{N}_*$, *and* $\lambda_t^k \ge 0$ *for all* $t \in \mathbb{N}$ *and for all* $k \in \{1, \dots, m\}$.

**(S$\ell$)** *For all $t \in \mathbb{N}$, for all $\alpha \in \{1, \dots, n\}$, $p_{t+1}^\alpha \cdot (f_t^\alpha(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}^\alpha) = 0$, and for all $k \in \{1, \dots, m\}$, $\lambda_t^k g_t^k(\hat{u}_t) = 0$.*

**(AE)** *For all $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_{G,1} \phi_t(\hat{x}_t, \hat{u}_t)$.*

**(WM)** *For all $t \in \mathbb{N}$, $p_{t+1} \circ D_{G,2} f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_{G,2} \phi_t(\hat{x}_t, \hat{u}_t) + \sum_{k=1}^m \lambda_t^k D_G g_t^k(\hat{u}) = 0$.*

*Proof.* From Theorem 4.1 (b) we know that, for all $T \in \mathbb{N}_*$, $(\hat{x}_0, \dots, \hat{x}_{T+1}, \hat{u}_0, \dots, \hat{u}_T)$ is a solution of the following finite-horizon problem.

$$\begin{cases} \text{Maximize} & J(x_0, \dots, x_{T+1}, u_0, \dots, u_T) = \sum_{t=0}^{T} \phi_t(x_t, u_t) \\ \text{when} & \forall t \in \{0, \dots, T\},\ f_t(x_t, u_t) - x_{t+1} \geq 0 \\ & \forall t \in \{0, \dots, T+1\}, x_t \in X_t \\ & x_0 = \eta,\ x_{T+1} = \hat{x}_{T+1} \\ & \forall t \in \{0, \dots, T\}, \forall k \in \{1, \dots, m\},\ g_t^k(u_t) \geq 0. \end{cases}$$

From Theorem 4.4 we can assert that there exists $(\lambda_0^T, p_1^T, \dots, p_{T+1}^T, \lambda_1^{1,T}, \dots, \lambda_T^{m,T}) \in \mathbb{R} \times (\mathbb{R}^{n*})^T + 1 \times \mathbb{R}^{mT}$ which satisfies the following assertions.

$$(\lambda_0^T, p_1^T, \dots, p_{T+1}^T, \lambda_1^{1,T}, \dots, \lambda_m^{T,T}) \neq 0. \tag{4.28}$$

$$\left.\begin{array}{l} \lambda_0^T \geq 0, \forall t \in \{1, \dots, T+1\}, p_t \geq 0, \\ \text{and } \forall t \in \{0, \dots, T\}, \forall k \in \{1, \dots, m\}, \lambda_t^{k,t} \geq 0. \end{array}\right\} \tag{4.29}$$

$$\left.\begin{array}{l} \forall t \in \{0, \dots, T\}, \forall \alpha \in \{1, \dots, n\}, p_{t+1}^\alpha \cdot (f_t^\alpha(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}^\alpha) = 0, \\ \text{and } \forall t \in \{0, \dots, T\}, \forall k \in \{1, \dots, m\}, \lambda_t^{k,T} \cdot g_t^k(\hat{u}_t) = 0. \end{array}\right\} \tag{4.30}$$

$$\left.\begin{array}{l} \forall t \in \{1, \dots, T\}, \\ p_t^T = p_{t+1}^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^T D_{G,1} \phi_t(\hat{x}_t, \hat{u}_t). \end{array}\right\} \tag{4.31}$$

$$\left.\begin{array}{l} \forall t \in \{0, \dots, T\}, \\ p_{t+1}^T \circ D_{G,2} f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^T D_{G,2} \phi_t(\hat{x}_t, \hat{u}_t) + \sum_{k=1}^m \lambda_t^{k,T} D_G g_t^k(\hat{u}_t) = 0. \end{array}\right\} \tag{4.32}$$

Using (4.31) under (vi) or (vii) and working as in the proof of Theorem 4.14 or Theorem 4.16, we obtain

$$(\lambda_0^T, p_1^T) = (0, 0) \Longrightarrow (\lambda_0^T, p_1^T, \dots, p_{T+1}^T) = (0, 0, \dots, 0).$$

Proceeding by contradiction, assuming that $(\lambda_0^T, p_1^T) = (0, 0)$, from the previous implication and (4.32) we obtain $\sum_{k=1}^m \lambda_t^{k,T} D_G g_t^k(\hat{u}_t) = 0$, and then using Lemma 4.19 we obtain the $\lambda_t^{k,T} = 0$. Therefore, we obtain a contradiction with (4.28). And so we have proven that $(\lambda_0^T, p_1^T) \neq (0, 0)$. Under (vi), proceeding as in the proof of Theorem 4.14 and under (vii), proceeding as in the proof of Theorem 4.16 we obtain the existence of an increasing function $\rho : \mathbb{N}_* \to \mathbb{N}_*$ and of $\lambda_0 \in \mathbb{R}_+$ and of $(p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}_+^{n*})^{\mathbb{N}_*}$ such that $\lambda_0 = \lim_{T \to +\infty} \lambda_0^{\rho(T)}$, $p_t = \lim_{T \to +\infty} p_t^{\rho(T)}$, $(\lambda_0, p_1) \neq (0, 0)$, and $p_t^\alpha \cdot (f_t^\alpha(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}^\alpha) = 0$ for all $t \in \mathbb{N}_*$ and for all $\alpha \in \{1, \dots, n\}$.

We fix $t \in \mathbb{N}$ and we consider, for all $T \in \mathbb{N}_*$,

$$\begin{array}{rcl} \varphi_T & := & \sum_{k \in I_t^s} \lambda_t^{k,\rho(T)} D_G g_t^k(\hat{u}_t) = \sum_{k=1}^m \lambda_t^{k,\rho(T)} D_G g_t^k(\hat{u}_t) \\ & = & -(p_{t+1}^{\rho(T)} \circ D_{G,2} f_t(\hat{x}_t, \hat{u}_t) + \lambda_0^{\rho(T)} D_{G,2} \phi_t(\hat{x}_t, \hat{u}_t)). \end{array}$$

Therefore, we have $\lim_{T \to +\infty} \varphi_T = -(p_{t+1} \circ D_{G,2} f_t(\hat{x}_t, \hat{u}_t) + \lambda_0 D_{G,2} \phi_t(\hat{x}_t, \hat{u}_t))$, and consequently the sequence $(\varphi_T)_{T \in \mathbb{N}_*}$ is bounded in $\mathbb{R}^{n*}$. Using Lemma 4.20 we can assert that there exist an increasing function $\rho_1 : \mathbb{N}_* \to \mathbb{N}_*$ and $\lambda_t^1, \dots, \lambda_t^m \in \mathbb{R}_+$ such that $\lim_{T \to +\infty} \lambda_t^{k,\rho \circ \rho_1(T)} = \lambda_t^k \in \mathbb{R}_+$. And then the assertions (NN), (Si), (S$\ell$), (AE) and (WM) are satisfied. □

Now we consider the case where the sets of controls are defined by equalities and inequalities for each $t \in \mathbb{N}$ as in (4.15),

$$U_t = \left( \bigcap_{1 \leq k \leq m_i} \{u \in \mathbb{R}^d : g_t^k(u) \geq 0\} \right) \cap \left( \bigcap_{1 \leq k \leq m_e} \{u \in \mathbb{R}^d : e_t^k(u) = 0\} \right). \tag{4.33}$$

The following lemmas are useful for the proofs of the weak Pontryagin principles for problems with constraints on control sets defined as above.

**Lemma 4.22.** *Let $E$ be a real finite-dimensional normed vector space; let $J$ and $K$ be two nonempty finite sets, and let $(\psi^j)_{j\in J}$ and $(\varphi^k)_{k\in K}$ be two families of elements of the dual $E^*$. Then the two following assertions are equivalent.*

**(i)** *$span\{\psi^j : j \in J\} \cap conv\{\varphi^k : k \in K\} = \emptyset$.*

**(ii)** *There exists $w \in E$ such that $\langle \psi^j, w\rangle = 0$ for all $j \in J$ and $\langle \varphi^k, w\rangle > 0$ for all $k \in K$.*

*Proof.* We set $S := span\{\psi^j : j \in J\}$ and $C := conv\{\varphi^k : k \in K\}$.
$[i \Longrightarrow ii]$ Under (i) using the theorem of separation of Hahn-Banach, there exist $\xi \in E^{**}$ and $a \in (0, +\infty)$ such that $\langle \xi, \psi\rangle \le a$ for all $\psi \in S$, and $\langle \xi, \varphi\rangle > a$ for all $\varphi \in C$. When $\psi \in S$ is non zero, we have $|\langle \xi, \psi\rangle| \le a$ since $-\psi \in S$, and therefore, for all $\lambda \in \mathbb{R}$, we have $|\lambda| \cdot |\langle \xi, \psi\rangle| \le a$ which impossible if $|\langle \xi, \psi\rangle| \ne 0$, therefore we have $\langle \xi, \psi\rangle = 0$ for all $\psi \in S$. Since $E^{**}$ is isomorphic to $E$ there exists $w \in E$ such $\langle \xi, \chi\rangle = \langle \chi, w\rangle$ for all $\chi \in E^*$, and then we obtain (ii).
$[ii \Longrightarrow i]$ Under (ii) we define $a := \min_{k\in K}\langle \varphi^k, w\rangle > 0$. When $\varphi \in C$ there exists $(\theta_k)_{k\in K} \in \mathbb{R}^K_+$ such that $\sum_{k\in K} \theta_k = 1$ and $\sum_{k\in K} \theta_k\varphi^k = \varphi$. Then $\langle \varphi, w\rangle = \sum_{k\in K} \theta_k\langle \varphi^k, w\rangle \ge \sum_{k\in K} \theta_k \cdot a > 0$. When $\psi \in S$ there exists $(\zeta_j)_{j\in J} \in \mathbb{R}^J$ such that $\sum_{j\in J} \zeta_j\psi^j = \psi$. Therefore we have $\langle \psi, w\rangle = \sum_{j\in J} \zeta_j\langle \psi^j, w\rangle = 0$. We have proven that $\langle \psi, w\rangle = 0$ for all $\psi \in S$ and $\langle \varphi, w\rangle > 0$ for all $\varphi \in C$, which implies (i). □

**Lemma 4.23.** *In the framework of Lemma 4.22, under condition (i) of Lemma 4.22, when $(\mu_j)_{j\in J} \in \mathbb{R}^J$ and $(\lambda_k)_{k\in K} \in \mathbb{R}^K_+$, we have*

$$\sum_{j\in J} \mu_j\psi^j + \sum_{k\in K} \lambda_k\varphi^k = 0 \Longrightarrow (\forall k \in K,\ \lambda_k = 0).$$

*Proof.* We proceed by contraposition, we assume that there exists $k \in K$ such that $\lambda_k \ne 0$. Then $\overline{\lambda} := \sum_{k\in K} \lambda_k > 0$ and so $\sum_{k\in K} \frac{\lambda_k}{\overline{\lambda}}\varphi^k \in conv\{\varphi^k : k \in K\}$ and $\sum_{k\in K} \frac{\lambda_k}{\overline{\lambda}}\varphi^k = -\sum_{j\in J} \frac{\mu_j}{\overline{\lambda}}\psi^j \in span\{\psi^j : j \in J\}$ which provides a contradiction with condition (i). □

**Lemma 4.24.** *Let $E$ be a real finite-dimensional normed vector space; let $J$ and $K$ be two nonempty finite sets, and let $(\psi^j)_{j\in J}$ and $(\varphi^k)_{k\in K}$ be two families of elements of the dual $E^*$. We assume that the following assumptions are fulfilled.*

**(a)** *The family $(\psi^j)_{j\in J}$ is linearly independent.*

**(b)** *$span\{\psi^j : j \in J\} \cap conv\{\varphi^k : k \in K\} = \emptyset$.*

*Let $(\mu^h_j)_{j\in J} \in \mathbb{R}^J$ and $(\lambda^h_k)_{k\in K} \in \mathbb{R}^K_+$ for all $h \in \mathbb{N}_*$ such that the sequence $(\chi^h)_{h\in\mathbb{N}_*} := (\sum_{j\in J} \mu^h_j\psi^j + \sum_{k\in K} \lambda^h_k\varphi^k)_{h\in\mathbb{N}_*}$ is bounded in $E^*$. Then there exists an increasing function $\rho : \mathbb{N}_* \to \mathbb{N}_*$ such that the sequences $(\mu^{\rho(h)}_j)_{h\in\mathbb{N}_*}$ are convergent in $\mathbb{R}$ for all $j \in J$ and the sequences and $(\lambda^{\rho(h)}_k)_{h\in\mathbb{N}_*}$ are convergent in $\mathbb{R}_+$ for all $k \in K$.*

*Proof.* We set $S := span\{\psi^j : j \in J\}$ and $C := conv\{\varphi^k : k \in K\}$. Firstly, we prove that $\liminf_{h\to+\infty} \sum_{k\in K} \lambda^h_k < +\infty$. We proceed by contradiction, we assume that $\liminf_{h\to+\infty} \sum_{k\in K} \lambda^h_k = +\infty$. Therefore we have $s := \lim_{h\to+\infty} \sum_{k\in K} \lambda^h_k = +\infty$. We set $\pi^h_k := \frac{\lambda^h_k}{\sum_{k'\in K} \lambda^h_{k'}} \in \mathbb{R}_+$. We have $\sum_{k\in K} \pi^h_k = 1$, and therefore $\sum_{k\in K} \pi^h_k\varphi^k \in C$. Note that

$$\|\sum_{j\in J} \frac{\mu^h_j}{\sum_{k'\in K} \lambda^h_{k'}}\psi^j + \sum_{k\in K} \pi^h_k\varphi^k\| = \frac{1}{\sum_{k'\in K} \lambda^h_{k'}}\|\chi^h\| \to 0$$

when $h \to +\infty$, therefore

$$\lim_{h\to+\infty}\left(\sum_{j\in J}\frac{\mu_j^h}{\sum_{k'\in K}\lambda_{k'}^h}\psi^j+\sum_{k\in K}\pi_k^h\varphi^k\right)=0.$$

Since $C$ is compact there exists an increasing function $\tau:\mathbb{N}_*\to\mathbb{N}_*$ and $\varphi_*\in C$ such that $\lim_{h\to+\infty}\sum_{k\in K}\pi_k^{\tau(h)}\varphi^k=\varphi_*$. Consequently $\lim_{h\to+\infty}\sum_{j\in J}\frac{-\mu_j^{\tau(h)}}{\sum_{k'\in K}\lambda_{k'}^{\tau(h)}}\psi^j=\varphi_*$. Since a finite-dimensional normed vector space is complete, $S$ is closed in $E^*$, and consequently we have $\varphi_*\in S$, and then $\varphi_*\in S\cap C$ which is a contradiction with assumption (b). And so we have proven that $\liminf_{h\to+\infty}\sum_{k\in K}\lambda_k^h<+\infty$.

Now, the previous result implies that there exists an increasing function $r:\mathbb{N}_*\to\mathbb{N}_*$ such that $\lim_{h\to+\infty}\sum_{k\in K}\lambda_k^{r(h)}=\liminf_{h\to+\infty}\sum_{k\in K}\lambda_k^h$. Thus, the sequence $(\sum_{k\in K}\lambda_k^{r(h)})_{h\in\mathbb{N}_*})_{h\in\mathbb{N}_*}$ is bounded in $\mathbb{R}_+$. Since $0\le\lambda_k^{r(h)}\le\sum_{k\in K}\lambda_k^{r(h)}$, we obtain $(\lambda_k^{r(h)})_{h\in\mathbb{N}_*}$ is bounded in $\mathbb{R}_+$ for all $k\in K$. Therefore $(\sum_{k\in K}\lambda_k^{r(h)}\varphi^k)_{h\in\mathbb{N}_*}$ is bounded in $E^*$. Consequently, $(\sum_{j\in J}\mu_j^{r(h)}\psi^j)_{h\in\mathbb{N}_*}=(\chi^{r(h)}-\sum_{k\in K}\lambda_k^{r(h)}\varphi^k)_{h\in\mathbb{N}_*}$ is bounded as a difference of two bounded sequences. Under assumption (a) we can use Lemma 5.5 in [7] and assert that there exists an increasing function $r_1:\mathbb{N}_*\to\mathbb{N}_*$ such $(\mu_j^{r\circ r_1(h)})_{h\in\mathbb{N}_*}$ is convergent in $\mathbb{R}$ for all $j\in J$. Using $|K|$ times the Bolzano-Weierstrass theorem, there exists an increasing function $r_2:\mathbb{N}_*\to\mathbb{N}_*$ such that $\lambda_k^{r\circ r_1\circ r_2(h)}{}_{h\in\mathbb{N}_*}$ is convergent in $\mathbb{R}_+$. Taking $\rho:=r\circ r_1\circ r_2$ we have proven the lemma. □

**Theorem 4.25.** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_i^j)$ where $j\in\{1,2,3\}$ and where the sets $U_t$ are defined in (4.33). We assume that the following assumptions are fulfilled for all $t\in\mathbb{N}$.*

**(i)** *$\phi_t$ is Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$.*

**(ii)** *For all $\alpha\in\{1,\dots,n\}$, $f_t^\alpha$ is Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$ when $f_t^\alpha(\hat{x}_t,\hat{u}_t)=x_{t+1}^\alpha$.*

**(iii)** *For all $\alpha\in\{1,\dots,n\}$, $f_t^\alpha$ is lower semicontinuous and Gâteaux differentiable at $(\hat{x}_t,\hat{u}_t)$ when $f_t^\alpha(\hat{x}_t,\hat{u}_t)>x_{t+1}^\alpha$.*

**(iv)** *For all $k\in\{1,\dots,m_i\}$, $g_t^k$ is Fréchet differentiable at $\hat{u}_t$ when $g_t^k(\hat{u}_t)=0$.*

**(v)** *For all $k\in\{1,\dots,m_i\}$, $g_t^k$ is lower semicontinuous and Gâteaux differentiable at $\hat{u}_t$ when $g_t^k(\hat{u}_t)>0$.*

**(vi)** *For all $j\in\{1,\dots,m_e\}$, $e_t^j$ is continuous on a neighborhood of $\hat{u}_t$ and Fréchet differentiable at $\hat{u}_t$.*

**(vii)** *$span\{De_t^j(\hat{u}_t):j\in\{1,\dots,m_e\}\}\cap conv\{D_Gg_t^k(\hat{u}_t):k\in I_t^s\}=\emptyset$, where $I_t^s:=\{k\in\{1,\dots,m_i\}:g_t^k(\hat{u}_t)=0\}$.*

**(viii)** *$De_t^1(\hat{u}_t)$, …, $De^{m_e}(\hat{u}_t)$ are linearly independent.*

**(ix)** *For all $t\in\mathbb{N}_*$, $D_{G,1}f_t(\hat{x}_t,\hat{u}_t)$ is invertible.*

**(x)** *For all $t\in\mathbb{N}_*$, for all $\alpha,\beta\in\{1,\dots,n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x^\beta}\ge0$ and for all $\alpha\in\{1,\dots,n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x^\alpha}>0$.*

*Then under (i-ix) or under (i-viii) and (x) there exist $\lambda_0\in\mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*}\in(R^{n*})^{\mathbb{N}_*}$, $(\mu_{1,t})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, …, $(\mu_{m_e,t})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, $(\lambda_{1,t})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, …, $(\lambda_{m_i,t})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ which satisfy the following conditions.*

**(NN)** $(\lambda_0,p_1)\neq(0,0)$.

**(Si)** *$\lambda_0\ge0$, $p_t\ge0$ for all $t\in\mathbb{N}_*$, $\lambda_{k,t}\ge0$ for all $t\in\mathbb{N}$ and for all $k\in\{1,\dots,m_i\}$.*

**(S$\ell$)** *For all $t \in \mathbb{N}$, for all $\alpha \in \{1,\dots,n\}$, $p^\alpha_{t+1} \cdot (f^\alpha_t(\hat{x}_t,\hat{u}_t) - x^\alpha_{t+1}) = 0$, and for all $k \in \{1,\dots,m_i\}$, $\lambda_{k,t} \cdot g^k_t(\hat{u}_t) = 0$.*

**(AE)** *For all $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_{G,1} f_t(\hat{x}_t,\hat{u}_t) + \lambda_0 D_1\phi_t(\hat{x}_t,\hat{u}_t)$.*

**(WM)** *For all $t \in \mathbb{N}$,*

$$p_{t+1} \circ D_{G,2} f_t(\hat{x}_t,\hat{u}_t) + \lambda_0 D_2\phi_t(\hat{x}_t,\hat{u}_t) + \sum_{k=1}^{m_i} \lambda_{k,t} D_G g^k_t(\hat{u}_t) + \sum_{j=1}^{m_e} \mu_{j,t} De^j(\hat{u}_t) = 0.$$

*Proof.* Under assumptions (i-vi), using Proposition 4.12 we obtain the existence of real numbers $\lambda^T_0$, $p^T_{t,\alpha}$ (for $t \in \{1,\dots,T+1\}$ and $\alpha \in \{1,\dots,n\}$), $\mu^T_{t,j}$ (for $t \in \{0,\dots,T\}$ and $j \in \{1,\dots,m_e\}$), $\lambda^T_{t,k}$ (for $t \in \{0,\dots,T\}$ and $k \in \{1,\dots,m_i\}$) which satisfy the following conditions:

$$(\lambda^T_0, p^T_{1,1},\dots,p^T_{T+1,n},\mu^T_{1,0},\dots,\mu^T_{m_e,T},\lambda^T_{1,0},\dots,\lambda^T_{m_i,T}) \neq 0 \tag{4.34}$$

$$\left.\begin{array}{r} \lambda^T_0 \geq 0, (\forall t \in \{1,\dots,T+1\}, \forall \alpha \in \{1,\dots,n\}, p^T_{t,\alpha} \geq 0) \\ (\forall t \in \{0,\dots,T\}, \forall k \in \{1,\dots,m_i\}, \lambda^T_{t,j} \geq 0) \end{array}\right\} \tag{4.35}$$

$$\forall t \in \{0,\dots,T\}, \forall \alpha \in \{1,\dots,n\},\ p^T_{t+1,\alpha}.(f^\alpha_t(\hat{x}_t,\hat{u}_t) - \hat{x}^\alpha_t) = 0 \tag{4.36}$$

$$\forall t \in \{0,\dots,T\}, \forall k \in \{1,\dots,m_i\}, \lambda^T_{t,k}.g^k_t(\hat{u}_t) = 0 \tag{4.37}$$

$$\forall t \in \{1,\dots,T\},\ \lambda^T_0 D_1\phi_t(\hat{x}_t,\hat{u}_t) + p^T_{t+1} \circ D_{G,1} f_t(\hat{x}_t,\hat{u}_t) = p^T_t. \tag{4.38}$$

$$\left.\begin{array}{r} \lambda^T_0 D_2\phi_t(\hat{x}_t,\hat{u}_t) + p^T_{t+1} \circ D_{G,2} f_t(\hat{x}_t,\hat{u}_t) \\ + \sum_{j=1}^{m_e} \mu^T_{t,j} De^j_t(\hat{u}_t) + \sum_{k=1}^{m_i} \lambda^T_{t,k} D_G g^k_t(\hat{u}_t) = 0. \end{array}\right\} \tag{4.39}$$

Using (ix) and working as in the proof of Theorem 4.14 or using (x) and working as in the proof of Theorem 4.16 , from (4.38) we obtain the following condition.

$$(\lambda^T_0, p^T_1) = (0,0) \Longrightarrow (\lambda^T_0, p^T_1,\dots,p^T_{T+1}) = (0,0,\dots,0). \tag{4.40}$$

If $(\lambda^T_0, p^T_1) = (0,0)$, using (4.39), (4.40) implies

$$\sum_{j=1}^{m_e} \mu^T_{t,j} De^j_t(\hat{u}_t) + \sum_{k=1}^{m_i} \lambda^T_{t,k} D_G g^k_t(\hat{u}_t) = 0,$$

and using (4.37) we obtain $\lambda^T_{t,k} = 0$ if $k \notin I^s_t$, and so we obtain the following relation $\sum_{j=1}^{m_e} \mu^T_{t,j} De^j_t(\hat{u}_t) + \sum_{k \in I^s_t} \lambda^T_{t,k} D_G g^k_t(\hat{u}_t) = 0$. Then using (vii) and Lemma 4.23 we obtain $\lambda^T_{t,k} = 0$ for all $k \in I^s_t$, and consequently we have $\lambda^T_{t,k} = 0$ for all $k \in \{1,\dots,m_i\}$. Therefore we have $\sum_{j=1}^{m_e} \mu^T_{t,j} De^j_t(\hat{u}_t) = 0$. Using (viii) we obtain $\mu^T_{t,j} = 0$ for all $j \in \{1,\dots,m_e\}$. And so we have proven that $(\lambda^T_0, p^T_1) = (0,0)$ implies $(\lambda^T_0, p^T_{1,1},\dots,p^T_{T+1,n},\mu^T_{1,0},\dots,\mu^T_{m_e,T},\lambda^T_{1,0},\dots,\lambda^T_{m_i,T}) = (0,\dots,0)$ which is a contradiction with (4.34). And so we have proven the following condition.

$$(\lambda^T_0, p^T_1) \neq (0,0). \tag{4.41}$$

From (4.41) under (ix) proceeding as in the proof of Theorem 4.14 or, under (x) proceeding as in the proof of Theorem 4.16 we obtain the existence of an increasing function $r : \mathbb{N}_* \to \mathbb{N}_*$, of $\lambda_0 \in \mathbb{R}$ and of $(p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$ such that

$$\lim_{T \to +\infty} \lambda_0^{r(T)} = \lambda_0, (\forall t \in \mathbb{N}_*, \lim_{T \to +\infty} p_t^{r(T)} = p_t), (\lambda_0, p_1) \neq (0,0). \tag{4.42}$$

From (4.42) we see that the sequences $(\lambda_0^{r(T)})_{T\in\mathbb{N}_*}$ and $(p_t^{r(T)})_{T\in\mathbb{N}_*}$ are bounded and then, using (4.39), we deduce that the sequence

$$(\sum_{j=1}^{m_e}\mu_{t,j}^{r(T)}De_t^j(\hat{u}_t)+\sum_{k=1}^{m_i}\lambda_{t,k}^{r(T)}D_Gg_t^k(\hat{u}_t))_{T\in\mathbb{N}_*}=$$
$$(\sum_{j=1}^{m_e}\mu_{t,j}^{r(T)}De_t^j(\hat{u}_t)+\sum_{k\in I_t^s}\lambda_{t,k}^{r(T)}D_Gg_t^k(\hat{u}_t))_{T\in\mathbb{N}_*}$$

is bounded for all $t\in\mathbb{N}$. Using (vii), (viii) and Lemma 4.24 we can assert that there exist an increasing function $r_1:\mathbb{N}_*\to\mathbb{N}_*$, $\mu_{t,j}\in\mathbb{R}$ (for all $t\in\mathbb{N}$ and for all $j\in\{1,\dots,m_e\}$), $\lambda_{t,k}\in\mathbb{R}$ (for all $t\in\mathbb{N}$ and for all $k\in\{1,\dots,m_i\}$) such that

$$\lim_{T\to+\infty}\mu_{t,j}^{r\circ r_1(T)}=\mu_{t,j},\lim_{T\to+\infty}\lambda_{t,k}^{r\circ r_1(T)}=\lambda_{t,k}. \tag{4.43}$$

Finally (4.42) implies (NN), (4.42), (4.43) and (4.35) imply (Si),(4.42), (4.43), (4.36) and (4.37) imply (S$\ell$), (4.42), (4.38) imply (AE), and (4.42), (4.43) and (4.39) imply (WM). □

**Theorem 4.26.** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a solution of $(P_e^j)$ where $j\in\{1,2,3\}$ and where the sets $U_t$ are defined in (4.27). We assume that the following assumptions are fulfilled for all $t\in\mathbb{N}$.*

**(i)** *$\phi_t$ is Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$.*

**(ii)** *$f_t$ is continuous on a neighborhood of $(\hat{x}_t,\hat{u}_t)$ and Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$.*

**(iii)** *For all $k\in\{1,\dots,m_i\}$, $g_t^k$ is Fréchet differentiable at $\hat{u}_t$ when $g_t^k(\hat{u}_t)=0$.*

**(iv)** *For all $k\in\{1,\dots,m_i\}$, $g_t^k$ is lower semicontinuous and Gâteaux differentiable at $\hat{u}_t$ when $g_t^k(\hat{u}_t)>0$.*

**(v)** *For all $j\in\{1,\dots,m_e\}$, $e_t^j$ is continuous on a neighborhood of $\hat{u}_t$ and Fréchet differentiable at $\hat{u}_t$.*

**(vi)** *$span\{De_t^j(\hat{u}_t):j\in\{1,\dots,m_e\}\}\cap conv\{D_Gg_t^k(\hat{u}_t):k\in I_t^s\}=\emptyset$, where $I_t^s:=\{k\in\{1,\dots,m_i\}:g_t^k(\hat{u}_t)=0\}$.*

**(vii)** *$De_t^1(\hat{u}_t)$, …, $De_t^{m_e}(\hat{u}_t)$ are linearly independent.*

**(viii)** *For all $t\in\mathbb{N}_*$, $D_{G,1}f_t(\hat{x}_t,\hat{u}_t)$ is invertible.*

*Then under (i-viii) there exist $\lambda_0\in\mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*}\in(R^{n*})^{\mathbb{N}_*}$, $(\mu_{t,1})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, …, $(\mu_{t,m_e})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, $(\lambda_{t,1})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, …, $(\lambda_{t,m_i})_{t\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ which satisfy the following conditions.*

**(NN)** $(\lambda_0,p_1)\neq(0,0)$.

**(Si)** *$\lambda_0\geq 0$, $\lambda_{k,t}\geq 0$ for all $t\in\mathbb{N}$ and for all $k\in\{1,\dots,m_i\}$.*

**(S$\ell$)** *For all $t\in\mathbb{N}$, for all $k\in\{1,\dots,m_i\}$, $\lambda_{k,t}\cdot g_t^k(\hat{u}_t)=0$.*

**(AE)** *For all $t\in\mathbb{N}_*$, $p_t=p_{t+1}\circ D_{G,1}f_t(\hat{x}_t,\hat{u}_t)+\lambda_0D_1\phi_t(\hat{x}_t,\hat{u}_t)$.*

**(WM)** *For all $t\in\mathbb{N}$,*

$$p_{t+1}\circ D_{G,2}f_t(\hat{x}_t,\hat{u}_t)+\lambda_0D_2\phi_t(\hat{x}_t,\hat{u}_t)+\sum_{k=1}^{m_i}\lambda_{t,k}D_Gg_t^k(\hat{u}_t)+\sum_{j=1}^{m_e}\mu_{t,j}De^j(\hat{u}_t)=0.$$

The proof of this theorem is similar to the one of Theorem 4.25. The difference is the replacement of inequality constraints by equality constraints in the problem issued from the reduction to finite horizon. The consequence of this difference is the lost of the sign of the adjoint variable $p_t$.

# Chapter 5

# Pontryagin Principles for Infinite-Horizon Discrete-Time Multiobjective Optimal Control Problems

## 5.1 Introduction

Multiobjective optimal control is an important branch of Optimal Control Theory. Multiobjective optimal control problems naturally arise, for example, in economics ([27, 29, 57] and references therein), in aerospace, mechanical and chemical engineering ([5, 6] and references therein), in multiobjective control design ([63] and references therein), in environmental studies ([28] and references therein)...

Multiobjective optimal control was first studied by Zadeh [67]. Some works followed like Salukvadze [54], Yu and Leitmann [66], Toivonen [59], Ishizuka and Shimizu [37], Khanh and Nuong [39], Yang and Teo [65], Giannessi et al. [30] and references therein who developed necessary and sufficient conditions as well as various methods for multiobjective optimal control. Multiple linear quadratic control problems can be found in Li [42], Liao and Li [43], Liu [44].

The first works on infinite-horizon single-objective optimal control problems are due to Pontryagin and his school [48] and Halkin [32]. Other works followed as Carlson et al. [21], Zaslavski [68, 70, 69], Blot and Chebbi [11], Blot and Hayek [13, 14, 15], Blot [8, 9], Blot et al. [16], Blot and Ngo [17, 18].

Infinite-horizon multiobjective optimal control problems in the continuous-time framework can be found in Bellaassali and Jourani [3], in Zhu [71] and in Reddy and Engwerda [51] and references therein.

Infinite-horizon multiobjective optimal control problems in the discrete-time framework can be found in Hayek [34, 33] and in Blot and Hayek [15] and in references therein.

In this chapter, necessary conditions of Pareto optimaltity under the form of Pontryagin principles for finite horizon and infinite-horizon multiobjective optimal control problems in discrete-time framework are studied. The aim of this chapter is to establish weak and strong maximum principles of Pontryagin for problems in the presence of constraints and under assumptions which are weaker than the usual ones. In this way, this chapter generalizes existing results for single-objective optimal control problems and for multiobjective optimal control problems with or without constraints. The general method we follow is to reduce the infinite-horizon problems into finite-horizon problems and

then to translate the finite-horizon multiobjective optimal control problems into static multiobjective optimization problems, then use an appropriate multiplier rule for such problems. However, many existing multiplier rules require the smoothness or at least, the Fréchet differentiability at the optimal solution and the continuity on a neighborhood of the optimal solution of the functions issued in the problem. To establish weak maximum principles, we provide new multiplier rules for static multiobjective optimization problems, which are built in the spirit of the single-objective static optimization rules that Blot presents in [10]. In these rules, in some places, instead of the usual Fréchet differentiability at the optimal solution we use Gâteaux differentiability at it and we replace the continuity on a neighborhood of the optimal solution by lower semicontinuity at the optimal solution. To establish strong maximum principles, we rely on a multiplier rule of Khanh and Nuong [39]. So in some places, instead of the usual $C^1$ differentiability with respect to the optimal state variable, we use the directional differentiability with respect to the optimal state variable with a concavity property, and the continuity at the optimal state variable. Since we study multiobjective optimal control problems in the presence of constraints, this chapter generalizes to the multiobjective case some results of Blot [9] and Blot and Hayek [13, 15], where infinite-horizon single-objective optimal control problems under constraints were studied. It also generalizes some results of strong Pontryagin principles and results of weak Pontryagin principles when replacing weak Pareto optimality by Pareto optimality from Hayek [34], where infinite-horizon multiobjective optimal control problems in the discrete time framework are studied. In this chapter, we provide weaker smoothness assumptions and moreover, we study problems under constraints. Sufficient conditions of optimality for considered problems are also studied in this chapter, in which, similar to the one in Chapter 3, concavity assumption on the Hamiltonian is required.

The structure of this chapter is as follows: After the introduction, the infinite-horizon multiobjective optimal control problems are presented in Section 5.2 and Theorem of Reduction to finite horizon is provided in Section 5.3. In Section 5.4, New Multiplier Rules for multiobjective static optimization problems are established. Then, in Section 5.5, weak and strong Pontryagin principles for the multiobjective optimal control problems in the finite-horizon setting are given where the weak ones rely on the New Multiplier Rules for static multiobjective optimization problems. In Section 5.6, weak and strong Pontryagin principles for multiobjective optimal control problems in infinite horizon are provided. Moreover, using weak principles, by adding additional condition, a transversality condition is achieved. Finally, in Section 5.7, we provide sufficient conditions of optimality.

## 5.2 The Multiobjective Optimal Control Problems

In previous chapter, we have established weak Pontryagin principles for the single-objective optimal control problems $(P_k^j)$ where $k \in \{i, e\}$ and $j \in \{1, 2, 3\}$. It is worth to recall that in these problems, there are two families of controlled dynamical systems that governed by difference equations (De) and by difference inequations (Di).

In this chapter, we will study Pontryagin principles and sufficient condition of optimality for the very same problems but with multiobjective criterion. All the settings remain the same like those of single-objective problems, except for the criterion and their domain. Recall that when $k \in \{i, e\}$, $\mathrm{Adm}_k$ is the set of all processes $(\underline{x}, \underline{u}) \in \prod_{t\in\mathbb{N}} X_t \times \prod_{t\in\mathbb{N}} U_t$ which satisfy (D$k$) at each time $t \in \mathbb{N}$ and such that $x_0 = \eta$. These processes are called admissible for (D$k$) and $\eta$. For all $t \in \mathbb{N}$, for all $j \in \{1, \ldots, \ell\}$, we consider a functional $\phi_t^j : X_t \times U_t \to \mathbb{R}$. For each $j \in \{1, \ldots, \ell\}$, we set $J_j(\underline{x}, \underline{u}) := \sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t)$ and we also denote by $\mathrm{Dom}_k(J_j)$ as the set of the $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$ such that the series $\sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t)$ is convergent in $\mathbb{R}$. The optimality criterion that we consider here is

defined by using the vector-function (multiobjective) $J(\underline{x},\underline{u}) := (J_1,\dots,J_\ell)$. The order for criterion is the usual order in $\mathbb{R}^\ell$. Now, we introduce the domain for the multiobjective optimal control problems with criterion $J$, denoted by $\mathrm{DOM}_k(J) := \left(\bigcap_{j=1}^{\ell}\mathrm{Dom}_k J_j\right)$ where $k \in \{e,i\}$. We define the following multiobjective problem when $k \in \{e,i\}$ :

$(\mathbf{PM_k^1})$ Maximize $J(\underline{x},\underline{u})$ when $(\underline{x},\underline{u}) \in \mathrm{DOM}_k(J)$.

**Definition 5.1.** A process $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{DOM}_k(J)$ is called a Pareto optimal solution of Problem $(PM_k^1)$, if there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{DOM}_k(J)$ such that for all $j \in \{1,\dots,\ell\},\ J_j(\underline{x},\underline{u}) \geq J_j(\underline{\hat{x}},\underline{\hat{u}})$ and for some $i \in \{1,\dots,\ell\},\ J_i(\underline{x},\underline{u}) > J_i(\underline{\hat{x}},\underline{\hat{u}})$.

A process $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{DOM}_k(J)$ is called a weak Pareto optimal solution of Problem $(PM_k^1)$, if there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{DOM}_k(J)$ such that for all $j \in \{1,\dots,\ell\},\ J_j(\underline{x},\underline{u}) > J_j(\underline{\hat{x}},\underline{\hat{u}})$.

It is obvious that a Pareto optimal solution of Problem $(PM_k^1)$ is a weak Pareto optimal solution of Problem $(PM_k^1)$.

Consider now the following problems for the cases where the infinite series does not necessarily converge:

$(\mathbf{PM_k^2})$ Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}_k$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}_k$ such that for all $j \in \{1,\dots,\ell\}, \limsup\limits_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) \geq 0$ and for some $i \in \{1,\dots,\ell\}$, $\limsup_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^i(x_t,u_t) - \sum_{t=0}^{h}\phi_t^i(\hat{x}_t,\hat{u}_t)) > 0$.

$(\mathbf{PM_k^{2'}})$ Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}_k$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}_k$ such that for all $j \in \{1,\dots,\ell\}, \limsup\limits_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) > 0$.

$(\mathbf{PM_k^3})$ Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}_k$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}_k$ such that for all $j \in \{1,\dots,\ell\}, \liminf\limits_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) \geq 0$ and for some $i \in \{1,\dots,\ell\}$, $\liminf_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^i(x_t,u_t) - \sum_{t=0}^{h}\phi_t^i(\hat{x}_t,\hat{u}_t)) > 0$.

$(\mathbf{PM_k^{3'}})$ Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}_k$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}_k$ such that for all $j \in \{1,\dots,\ell\}, \liminf\limits_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) > 0$.

Since for all $j \in \{1,\dots,\ell\}$, $\limsup\limits_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) \geq \liminf\limits_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t))$, then a solution of Problem $(PM_k^3)$ is also a solution of Problem $(PM_k^2)$. Besides, it is obvious that a solution of Problem $(PM_k^{j'})$ is also a solution of Problem $(PM_k^j)$ when $j \in \{2,3\}$.

## 5.3 Reduction to Finite Horizon

Let $T$ be a fixed number in $\mathbb{N}_*$. We set $J_j^T((x_0,\dots,x_{T+1}),(u_0,\dots,u_T)) := \sum\limits_{t=0}^{T}\phi_t^j(x_t,u_t)$ and $J^T := (J_1^T,\dots,J_\ell^T)$. Consider the following reduced problems when $k \in \{e,i\}$

$$\left.\begin{array}{rl} \text{Maximize} & J^T((x_t)_{0\leq t\leq T+1},(u_t)_{0\leq t\leq T}) \\ \text{when} & \forall t \in \{0,\dots,T+1\},\ x_t \in X_t \\ & \forall t \in \{0,\dots,T\},\ u_t \in U_t \\ & \forall t \in \{0,\dots,T\},\ (D_k) \text{ holds} \\ & x_0 = \eta,\ x_{T+1} = \hat{x}_{T+1}. \end{array}\right\} \qquad (FM_k^T)$$

**Definition 5.2.** $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is called a Pareto optimal solution of Problem $(FM_k^T)$ where $k \in \{e,i\}$, if there does not exist any $((x_0,\dots,x_{T+1}),(u_0,\dots,u_T))$ admissible for Problem $(FM_k^T)$ such that for all $j \in \{1,\dots,\ell\}$, $J_j^T((x_0,\dots,x_{T+1}),(u_0,\dots,u_T)) \geq J_j^T((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ and for some $i \in \{1,\dots,\ell\}, J_i^T((x_0,\dots,x_{T+1}),(u_0,\dots,u_T)) > J_i^T((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$.

$((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is called a weak Pareto optimal solution of Problem $(FM_k^T)$ where $k \in \{e,i\}$, if there does not exist any $((x_0,\dots,x_{T+1}),(u_0,\dots,u_T))$ admissible for Problem $(FM_k^T)$ such that for all $j \in \{1,\dots,\ell\}$, $J_j^T((x_0,\dots,x_{T+1}),(u_0,\dots,u_T)) > J_j^T((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$.

Here admissibility means that all the constraints, including the dynamical system, the initial and final conditions, are satisfied. Then we have the following theorem.

**Theorem 5.3.** *The two following assertions hold.*

**(i)** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a Pareto optimal solution of Problem $(PM_k^1)$ (respectively, solution of $(PM_k^2)$, $(PM_k^3)$) where $k \in \{e,i\}$ and let $T \in \mathbb{N}^*$. Then the restriction $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is a Pareto optimal solution of the finite-horizon problem $(FM_k^T)$.*

**(ii)** *Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_k^1)$ (respectively, solution of $(PM_k^{2'})$, $(PM_k^{3'})$) where $k \in \{e,i\}$ and let $T \in \mathbb{N}^*$. Then the restriction $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is a weak Pareto optimal solution of the finite-horizon problem $(FM_k^T)$.*

*Proof.* We will prove assertion (i) first.
**(De) case:** We will prove for each case of $(PM_e^j)$ where $j \in \{1,2,3\}$.

- **For $(\mathbf{PM_e^1})$:** We proceed by contradiction. Assume that $((\hat{x}_0,\dots,\hat{x}_{T+1}),(\hat{u}_0,\dots,\hat{u}_T))$ is not Pareto optimal for $(FM_k^T)$. Then there exists $((x_0,\dots,x_{T+1}),(u_0,\dots,u_T))$ which is admissible for $(FM_k^T)$ such that

$$J^T((x_t)_{0\leq t\leq T+1},(u_t)_{0\leq t\leq T}) > J^T((\hat{x}_t)_{0\leq t\leq T+1},(\hat{u}_t)_{0\leq t\leq T})$$

This inequality means that for all $j \in \{1,\dots,\ell\}$, $J_j^T((x_t)_{0\leq t\leq T+1},(u_t)_{0\leq t\leq T}) \geq J_j^T((\hat{x}_t)_{0\leq t\leq T+1},(\hat{u}_t)_{0\leq t\leq T})$ and for some $k \in \{1,\dots,\ell\}$, $J_k^T((x_t)_{0\leq t\leq T+1},(u_t)_{0\leq t\leq T}) > J_k^T((\hat{x}_t)_{0\leq t\leq T},(\hat{u}_t)_{0\leq t\leq T-1})$.

When $t > T+1$, we set $x_t := \hat{x}_t$ and when $t \geq T+1$, we set $u_t := \hat{u}_t$. From the admissibility and this setting, it is obvious that $\underline{x} \in \prod_{t\in\mathbb{N}} X_t$ and $\underline{u} \in \prod_{t\in\mathbb{N}} U_t$. Also, $x_{t+1} = \hat{x}_{t+1} = f_t(\hat{x}_t,\hat{u}_t) = f_t(x_t,u_t)$ when $t \geq T+1$. It implies that $(\underline{x},\underline{u})$ belongs to the admissible set of $(PM_e^1)$ i.e. $(\underline{x},\underline{u}) \in \mathrm{Adm}_e$. Now, we have for all $j \in \{1,\dots,\ell\}$, $\sum_{t=T+1}^{+\infty} \phi_t^j(x_t,u_t) = \sum_{t=T+1}^{+\infty} \phi_t^j(\hat{x}_t,\hat{u}_t) < +\infty$ then $\sum_{t=0}^{+\infty} \phi_t^j(x_t,u_t) < +\infty$ or $J_j(\underline{x},\underline{u}) < +\infty$. And so, $(\underline{x},\underline{u}) \in \mathrm{Dom}_e(J_j)$ for all $j \in \{1,\dots,\ell\}$ which implies that $(\underline{x},\underline{u}) \in \left(\bigcap_{j=1}^{\ell} \mathrm{Dom}_e J_j\right) = \mathrm{DOM}_e(J)$.

Now, for all $j \in \{1, \dots, \ell\}$, we have

$$\begin{aligned} J_j(\underline{x}, \underline{u}) &= \sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t) = \sum_{t=0}^{T} \phi_t^j(x_t, u_t) + \sum_{t=T+1}^{+\infty} \phi_t^j(x_t, u_t) \\ &= J_j^T((x_t)_{0 \le t \le T+1}, (u_t)_{0 \le t \le T}) + \sum_{t=T+1}^{+\infty} \phi_t^j(\hat{x}_t, \hat{u}_t) \\ &\ge J_j^T((\hat{x}_t)_{0 \le t \le T+1}, (\hat{u}_t)_{0 \le t \le T}) + \sum_{t=T+1}^{+\infty} \phi_t^j(\hat{x}_t, \hat{u}_t) \\ &= \sum_{t=0}^{T} \phi_t^j(x_t, u_t) + \sum_{t=T+1}^{+\infty} \phi_t^j(\hat{x}_t, \hat{u}_t) = J_j(\underline{\hat{x}}, \underline{\hat{u}}). \end{aligned}$$

Besides, for some $k \in \{1, \dots, \ell\}$, we have

$$\begin{aligned} J_k(\underline{x}, \underline{u}) &= \sum_{t=0}^{+\infty} \phi_t^k(x_t, u_t) = \sum_{t=0}^{T} \phi_t^k(x_t, u_t) + \sum_{t=T+1}^{+\infty} \phi_t^k(x_t, u_t) \\ &= J_k^T((x_t)_{0 \le t \le T+1}, (u_t)_{0 \le t \le T}) + \sum_{t=T+1}^{+\infty} \phi_t^k(\hat{x}_t, \hat{u}_t) \\ &> J_k^T((\hat{x}_t)_{0 \le t \le T+1}, (\hat{u}_t)_{0 \le t \le T}) + \sum_{t=T+1}^{+\infty} \phi_t^k(\hat{x}_t, \hat{u}_t) \\ &= \sum_{t=0}^{T} \phi_t^k(\hat{x}_t, \hat{u}_t) + \sum_{t=T+1}^{+\infty} \phi_t^k(\hat{x}_t, \hat{u}_t) = J_k(\underline{\hat{x}}, \underline{\hat{u}}). \end{aligned}$$

And so, $J(\underline{x}, \underline{u}) > J(\underline{\hat{x}}, \underline{\hat{u}})$. This is contradiction since $(\underline{\hat{x}}, \underline{\hat{u}})$ is the Pareto optimal solution for Problem $(PM_e^1)$. Hence, $((\hat{x}_0, \dots, \hat{x}_{T+1}), (\hat{u}_0, \dots, \hat{u}_T))$ must be Pareto optimal for $(FM_k^T)$.

- **For** $(\mathbf{PM_e^3})$: Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(PM_e^3)$. Assume that $((\hat{x}_0, \dots, \hat{x}_{T+1}), (\hat{u}_0, \dots, \hat{u}_T))$ is not Pareto optimal for $(FM_k^T)$. By realizing analogous proceedings like in the previous case, we can build a process $(\underline{x}, \underline{u}) \in \mathrm{Adm}_e$ such that for all $j \in \{1, \dots, \ell\}$, $\sum_{t=0}^{T} \phi_t^j(x_t, u_t) \ge \sum_{t=0}^{T} \phi_t^j(\hat{x}_t, \hat{u}_t)$ and for some $k \in \{1, \dots, \ell\}$, $\sum_{t=0}^{T} \phi_t^k(x_t, u_t) > \sum_{t=0}^{T} \phi_t^k(\hat{x}_t, \hat{u}_t)$. Then we have when $h \ge T+1$,

$$\liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t^j(x_t, u_t) - \sum_{t=0}^{h} \phi_t^j(\hat{x}_t, \hat{u}_t)\Big) = \sum_{t=0}^{T} \phi_t^j(x_t, u_t) - \sum_{t=0}^{T} \phi_t^j(\hat{x}_t, \hat{u}_t) \ge 0,$$

for all $j \in \{1, \dots, \ell\}$ and

$$\liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t^k(x_t, u_t) - \sum_{t=0}^{h} \phi_t^k(\hat{x}_t, \hat{u}_t)\Big) = \sum_{t=0}^{T} \phi_t^k(x_t, u_t) - \sum_{t=0}^{T} \phi_t^k(\hat{x}_t, \hat{u}_t) > 0,$$

for some $k \in \{1, \dots, \ell\}$ which is a contradiction since $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of $(PM_e^3)$.

- **For** $(\mathbf{PM_e^2})$: Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a solution of $(PM_e^2)$. It is clear that $(\underline{\hat{x}}, \underline{\hat{u}})$ is also a solution of $(PM_e^3)$ and hence, its restriction $((\hat{x}_0, \dots, \hat{x}_{T+1}), (\hat{u}_0, \dots, \hat{u}_T))$ is a Pareto optimal solution for $(FM_k^T)$.

**(Di) case:** the proof is completely similar. Assertion (i) is proven.
The proof of assertion (ii) is analogous to the one for assertion (i). □

## 5.4 New Multiplier Rules for Multiobjective Problem

Let $\Omega$ be a nonempty open subset of $\mathbb{R}^n$, let $\phi : \Omega \to \mathbb{R}^\ell$ , $\vartheta : \Omega \to \mathbb{R}^\ell$ be mappings, let $f_i : \Omega \to \mathbb{R}$ (when $i \in \{1, \dots, m\}$) be functions, let $g_i : \Omega \to \mathbb{R}$ (when $i \in \{0, \dots, p\}$) and $h_i : \Omega \to \mathbb{R}$ (when $i \in \{1, \dots, q\}$) be functions. With these elements, we consider the two following problems:

$$\left.\begin{array}{rl} \text{Maximize} & \phi(x) = (\phi_1(x), \dots, \phi_\ell(x)) \\ \text{when} & x \in \Omega \\ \text{and when} & \forall i \in \{1, \dots, m\},\ f_i(x) \geq 0, \end{array}\right\} \qquad (\mathcal{I}_m)$$

and

$$\left.\begin{array}{rl} \text{Maximize} & \vartheta(x) = (\vartheta_1(x), \dots, \vartheta_\ell(x)) \\ \text{when} & x \in \Omega \\ \text{when} & \forall i \in \{1, \dots, p\},\ g_i(x) \geq 0 \\ \text{and when} & \forall i \in \{1, \dots, q\},\ h_i(x) = 0. \end{array}\right\} \qquad (\mathcal{M}_m)$$

Notice that Problem $(\mathcal{I}_m)$ is a special case of Problem $(\mathcal{M}_m)$ when the equality constraints are omitted. The Pareto optimal solutions the above-mentioned problems are understood under the same meaning like those in the previous sections.

Before stating New Multiplier Rules for those multiobjective static optimization problems, we introduce the following lemmas:

**Lemma 5.4.** *If $\hat{x}$ is a Pareto optimal solution of Problem $(\mathcal{I}_m)$ then it is also a solution of the following problem*

$$\left.\begin{array}{rl} \textit{Maximize} & \phi_k(x) \\ \textit{when} & x \in \Omega \\ \textit{when} & \forall i \in \{1, \dots, \ell\},\ i \neq k,\ \phi_i(x) \geq \phi_i(\hat{x}) \\ \textit{and when} & \forall i \in \{1, \dots, m\},\ f_i(x) \geq 0, \end{array}\right\} \qquad (5.1)$$

*for any given $k \in \{1, \dots, \ell\}$.*

*Proof.* Let $\hat{x}$ be a Pareto optimal solution of Problem $(\mathcal{I}_m)$ and let $k \in \{1, \dots, \ell\}$ be given. We will prove that $\hat{x}$ is also a solution of Problem (5.1) by contradiction.

If $\hat{x}$ is not a solution of Problem (5.1) then there exists $\bar{x}$ which is admissible for Problem (5.1) and $\phi_k(\bar{x}) > \phi_k(\hat{x})$. From the admissibility of $\bar{x}$ we know that $\forall i \in \{1, \dots, m\}$, $f_i(\bar{x}) \geq 0$. Hence, $\bar{x}$ is admissible for Problem $(\mathcal{I}_m)$. Besides, for all $i \in \{1, \dots, \ell\}$, $i \neq k$, we have $\phi_i(\bar{x}) \geq \phi_i(\hat{x})$ (from the admissibility of $\bar{x}$) and $\phi_k(\bar{x}) > \phi_k(\hat{x})$. Therefore, we can conclude that $\bar{x}$ is admissible for Problem $(\mathcal{I}_m)$ and $\phi(\bar{x}) > \phi(\hat{x})$ which is a contradiction. And so, $\hat{x}$ is an optimal solution of (5.1) for any given $k \in \{1, \dots, \ell\}$. □

By a similar argument, we obtain the following lemma.

**Lemma 5.5.** *If $\hat{x}$ is a Pareto optimal solution of Problem $(\mathcal{M}_m)$ then it is also a solution of the following problem*

$$\left.\begin{array}{rl} \textit{Maximize} & \vartheta_k(x) \\ \textit{when} & x \in \Omega \\ \textit{when} & \forall i \in \{1, \dots, \ell\},\ i \neq k,\ \vartheta_i(x) \geq \vartheta_i(\hat{x}) \\ \textit{when} & \forall i \in \{1, \dots, p\},\ g_i(x) \geq 0, \\ \textit{and when} & \forall i \in \{1, \dots, q\},\ h_i(x) = 0. \end{array}\right\} \qquad (5.2)$$

*for any given $k \in \{1, \dots, \ell\}$.*

Now we introduce the New Multiplier Rules based on the ones for single-objective static optimization problems of Blot that were introduced in previous chapter.

**Theorem 5.6.** *Let $\hat{x}$ be a Pareto optimal solution of $(\mathcal{I}_m)$. We assume that the following assumptions are fulfilled.*

**(i)** *For all $i \in \{1, \ldots, \ell\}$, $\phi_i$ is Gâteaux differentiable at $\hat{x}$.*

*For all $i \in \{1, \ldots, m\}$, $f_i$ is Gâteaux differentiable at $\hat{x}$.*

**(ii)** *For all $i \in \{1, \ldots, m\}$, $f_i$ is lower semicontinuous at $\hat{x}$ when $f_i(\hat{x}) > 0$.*

*Then there exist $\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^m \in \mathbb{R}_+$ such that the following conditions hold.*

**(a)** $(\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^m) \neq (0, \ldots, 0)$.

**(b)** *For all $i \in \{1, \ldots, m\}$, $\lambda^i f_i(\hat{x}) = 0$.*

**(c)** $\sum_{i=1}^{\ell} \theta^i D_G \phi_i(\hat{x}) + \sum_{i=1}^{m} \lambda^i D_G f_i(\hat{x}) = 0$.

*Proof.* Let $\hat{x}$ be a Pareto optimal solution of $(\mathcal{I}_m)$. Using Lemma 5.4, $\hat{x}$ is also an optimal solution of Problem (5.1) for any given $k \in \{1, \ldots, \ell\}$. Let $k = 1$ then $\hat{x}$ solves following problem.

$$\left.\begin{array}{rl} \text{Maximize} & \phi_1(x) \\ \text{when} & x \in \Omega \\ \text{when} & \forall i \in \{2, \ldots, \ell\},\ \phi_i(x) \geq \phi_i(\hat{x}) \\ \text{and when} & \forall i \in \{1, \ldots, m\},\ f_i(x) \geq 0. \end{array}\right\} \tag{5.3}$$

We set $\alpha_i(x) := \phi_i(x) - \phi_i(\hat{x})$ for all $i \in \{2, \ldots, \ell\}$. Then the above-mentioned problem is rewritten as follows.

$$\left.\begin{array}{rl} \text{Maximize} & \phi_1(x) \\ \text{when} & x \in \Omega \\ \text{when} & \forall i \in \{2, \ldots, \ell\},\ \alpha_i(x) \geq 0 \\ \text{and when} & \forall i \in \{1, \ldots, m\},\ f_i(x) \geq 0 \end{array}\right\} \tag{5.4}$$

which has the same form of Problem $(\mathcal{I})$ in Chapter 4. The assumptions of this theorem can be rewritten as follows.

**(i)** $\phi_1$ is Gâteaux differentiable at $\hat{x}$.

For all $i \in \{2, \ldots, \ell\}$, $\alpha_i$ is Gâteaux differentiable at $\hat{x}$.

For all $i \in \{1, \ldots, m\}$, $f_i$ is Gâteaux differentiable at $\hat{x}$.

**(ii)** For all $i \in \{1, \ldots, m\}$, $f_i$ is lower semicontinuous at $\hat{x}$ when $f_i(\hat{x}) > 0$.

Now, all the assumptions of Theorem 4.4 are fulfilled (Here, notice that for $i \in \{2, \ldots, \ell\}$ we do not care about the lower semicontinuity of $\alpha_i$ at $\hat{x}$ because $\alpha_i(\hat{x}) = \phi_i(\hat{x}) - \phi_i(\hat{x}) = 0$). Then we can apply Theorem 4.4 to obtain the multipliers $\theta^1 \in \mathbb{R}_+$ for the objective function $\phi_1(x)$ and $\theta^2, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^m \in \mathbb{R}_+$ for the inequality constraints which satisfy the following results.

**(a)** $(\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^m) \neq (0, \ldots, 0)$.

**(b)** For all $i \in \{1, \ldots, m\}$, $\lambda^i f_i(\hat{x}) = 0$.

For all $i \in \{2, \ldots, \ell\}$, $\theta^i \alpha_i(\hat{x}) = 0$ (this is obvious since $\alpha_i(\hat{x}) = 0$).

**(c)** $\sum_{i=1}^{\ell} \theta^i D_G \phi_i(\hat{x}) + \sum_{i=1}^{m} \lambda^i D_G f_i(\hat{x}) = 0$.

Hence, we have obtained the desired results. □

**Theorem 5.7.** *Let $\hat{x}$ be a solution of $(\mathcal{M}_m)$. We assume that the following assumptions are fulfilled.*

**(i)** *For all $i \in \{1, \ldots, \ell\}$, $\vartheta_i$ is Fréchet differentiable at $\hat{x}$.*

**(ii)** *For all $i \in \{1, \ldots, p\}$, $g_i$ is Fréchet differentiable at $\hat{x}$ when $g_i(\hat{x}) = 0$.*

**(iii)** *For all $i \in \{1, \ldots, p\}$, $g_i$ is Gâteaux differentiable at $\hat{x}$ and lower semicontinuous at $\hat{x}$ when $g_i(\hat{x}) > 0$.*

**(iv)** *For all $i \in \{1, \ldots, q\}$, $h_i$ is continuous on a neighborhood of $\hat{x}$ and Fréchet differentiable at $\hat{x}$.*

*Then there exist $\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^p \in \mathbb{R}_+$ and $\mu^1, \ldots, \mu^q \in \mathbb{R}$ such that the following conditions are satisfied.*

**(a)** $(\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^p, \mu^1, \ldots, \mu^q) \neq (0, \ldots, 0)$.

**(b)** *For all $i \in \{1, \ldots, p\}$, $\lambda^i g_i(\hat{x}) = 0$.*

**(c)** $\sum_{i=1}^{\ell} \theta^i D_G \vartheta_i(\hat{x}) + \sum_{i=1}^{p} \lambda^i D_G g_i(\hat{x}) + \sum_{i=1}^{q} \mu^i D_G h_i(\hat{x}) = 0$.

*Moreover, under the additional assumption*

**(v)** *$Dh_1(\hat{x}), \ldots, Dh_q(\hat{x})$ are linearly independent,*

*we can take*

**(d)** $(\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^p) \neq (0, \ldots, 0)$.

*Proof.* Let $\hat{x}$ be a solution of $(\mathcal{M}_m)$. Using Lemma 5.5, $\hat{x}$ is also an optimal solution of Problem (5.2) for any given $k \in \{1, \ldots, \ell\}$. Let $k = 1$ then $\hat{x}$ solves following problem.

$$\left.\begin{array}{rl} \text{Maximize} & \vartheta_1(x) \\ \text{when} & x \in \Omega \\ \text{when} & \forall i \in \{2, \ldots, \ell\},\ \vartheta_i(x) \geq \vartheta_i(\hat{x}) \\ \text{when} & \forall i \in \{1, \ldots, p\},\ g_i(x) \geq 0, \\ \text{and when} & \forall i \in \{1, \ldots, q\},\ h_i(x) = 0. \end{array}\right\} \tag{5.5}$$

We set $\beta_i(x) := \vartheta_i(x) - \vartheta_i(\hat{x})$ for all $i \in \{2, \ldots, \ell\}$. Then the above-mentioned problem is rewritten as follows.

$$\left.\begin{array}{rl} \text{Maximize} & \vartheta_1(x) \\ \text{when} & x \in \Omega \\ \text{when} & \forall i \in \{2, \ldots, \ell\},\ \beta_i(x) \geq 0 \\ \text{when} & \forall i \in \{1, \ldots, p\},\ g_i(x) \geq 0, \\ \text{and when} & \forall i \in \{1, \ldots, q\},\ h_i(x) = 0. \end{array}\right\} \tag{5.6}$$

which has the same form of Problem $(\mathcal{M})$. Under the assumptions of this theorem, we will verify that all the assumptions of Theorem 4.5 are fulfilled.

**(i)** Under assumption (i), $\vartheta_1$ is Fréchet differentiable at $\hat{x}$.

**(ii)** Under assumption (i) and (ii), for all $i \in \{2, \ldots, \ell\}$, $\beta_i$ is Fréchet differentiable at $\hat{x}$ when $\beta_i(\hat{x}) = 0$ and for all $i \in \{1, \ldots, p\}$, $g_i$ is Fréchet differentiable at $\hat{x}$ when $g_i(\hat{x}) = 0$.

**(iii)** Under assumption (iii), for all $i \in \{1, \ldots, p\}$, $g_i$ is Gâteaux differentiable at $\hat{x}$ and lower semicontinuous at $\hat{x}$ when $g_i(\hat{x}) > 0$. (Here, notice that for $i \in \{2, \ldots, \ell\}$ we do not care about the Gâteaux differentiability and lower semicontinuity of $\beta_i$ at $\hat{x}$ because $\beta_i(\hat{x}) = \vartheta_i(\hat{x}) - \vartheta_i(\hat{x}) = 0$).

**(iv)** Under assumption (iv), for all $i \in \{1, \ldots, q\}$, $h_i$ is continuous on a neighborhood of $\hat{x}$ and Fréchet differentiable at $\hat{x}$.

Then we can apply Theorem of New Multiplier Rule in previous chapter (Theorem 4.5) to obtain the existence of $\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^p \in \mathbb{R}_+$ and $\mu^1, \ldots, \mu^q \in \mathbb{R}$ such that the following conditions are satisfied.

**(a)** $(\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^p, \mu^1, \ldots, \mu^q) \neq (0, \ldots, 0)$.

**(b)** For all $i \in \{1, \ldots, p\}$, $\lambda^i g_i(\hat{x}) = 0$.

For all $i \in \{2, \ldots, \ell\}$, $\theta^i \beta_i(\hat{x}) = 0$ (this is obvious since $\beta_i(\hat{x}) = 0$).

**(c)** $\sum_{i=1}^{\ell} \theta^i D_G\, \vartheta_i(\hat{x}) + \sum_{i=1}^{p} \lambda^i D_G g_i(\hat{x}) + \sum_{i=1}^{q} \mu^i D_G h_i(\hat{x}) = 0$.

Then conclusions (a), (b) and (c) of this theorem hold. Now, under the additional assumption (v) which assumes that $Dh_1(\hat{x}), \ldots, Dh_q(\hat{x})$ are linearly independent, we can choose $(\theta^1, \ldots, \theta^\ell, \lambda^1, \ldots, \lambda^p) \neq (0, \ldots, 0)$. This is easily verified using results (a), (c) and by contradiction and we obtain conclusion (d). The proof is complete. □

## 5.5 New Pontryagin Principles for Multiobjective Optimal Control Problems in Finite-Horizon Setting

In this section, we present Pontryagin principles in both weak and strong forms for multiobjective optimal control problems in a finite horizon framework, namely problems $(FM_k^T)$ when $k \in \{e, i\}$. They differ from the existing results since they use lighter smoothness assumptions.

### 5.5.1 Weak Pontryagin Principles

To obtain weak Pontryagin principles for multiobjective optimal control problems in finite-horizon framework $(FM_k^T)$ when $k \in \{e, i\}$, we will rely on New multiplier Rules established in Section 5.3. To do this, we will translate these problems into static optimization problems and we will apply New Multiplier Rules. Note that, in these problems, $x_0$ and $x_{T+1}$ are fixed and so they are not unknown variables. Assume that for all $t \in \mathbb{N}$, $X_t$ is open and the sets of admissible controls $U_t$, with $t \in \mathbb{N}$ are defined by equalities and inequalities as follows

$$U_t(x_t) = \left( \bigcap_{i=1}^{m_i} \{u \in \mathbb{R}^d : g_t^i(x_t, u) \geq 0 \right) \cap \left( \bigcap_{k=1}^{m_e} \{u \in \mathbb{R}^d : h_t^k(x_t, u) = 0 \right) \tag{5.7}$$

and we also assume that for all $t \in \mathbb{N}$, $U_t(x_t) \neq 0$.

**For Problem $(FM_i^T)$**, we have the following theorem:

**Theorem 5.8.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_T)$ be a Pareto optimal solution of Problem $(FM_i^T)$ when $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. We assume that for all $t \in \{0, \ldots, T\}$, $X_t$ is open and $U_t(x_t)$ is defined by (5.7). We also assume that the following conditions are fulfilled.*

**(i)** *For all $j \in \{1, \ldots, \ell\}$, for all $t \in \{0, \ldots, T\}$, $\phi_t^j$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

**(ii)** *For all $i \in \{1, \ldots, m_i\}$, for all $t \in \{0, \ldots, T\}$, $g_t^i$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$ when $g_t^i(\hat{x}_t, \hat{u}_t) = 0$.*

*For all $j \in \{1, \ldots, n\}$, for all $t \in \{0, \ldots, T\}$, $f_t^j$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$ when $f_t^j(\hat{x}_t, \hat{u}_t) = \hat{x}_{t+1}^j$.*

**(iii)** *For all $i \in \{1,\dots,m_i\}$, for all $t \in \{0,\dots,T\}$, $g_t^i$ is Gâteaux differentiable at $(\hat{x}_t,\hat{u}_t)$ and lower semicontinuous at $(\hat{x}_t,\hat{u}_t)$ when $g_t^i(\hat{x}_t,\hat{u}_t) > 0$.*

*For all $j \in \{1,\dots,n\}$, for all $t \in \{0,\dots,T\}$, $f_t^j$ is Gâteaux differentiable and and lower semicontinuous at $(\hat{x}_t,\hat{u}_t)$ when $f_t^j(\hat{x}_t,\hat{u}_t) > \hat{x}_{t+1}^j$.*

**(iv)** *For all $k \in \{1,\dots,m_e\}$, for all $t \in \{0,\dots,T\}$, $h_t^k$ is continuous on a neighborhood of $(\hat{x}_t,\hat{u}_t)$ and Fréchet differentiable at $(\hat{x}_t,\hat{u}_t)$.*

*Then there exist $\theta_1^T,\dots,\theta_\ell^T \in \mathbb{R}_+$, $\lambda_{i,t}^T \in \mathbb{R}_+$, $\mu_{k,t}^T \in \mathbb{R}$ and $p_{t+1}^T \in \mathbb{R}_+^{n*}$ where $t \in \{0,\dots,T\}$, $i \in \{1,\dots,m_i\}$ and $k \in \{1,\dots,m_e\}$ such that the following conditions are satisfied.*

**(a)** *The multipliers are not simultaneously equal to zero.*

**(b)** *For all $i \in \{1,\dots,m_i\}$, for all $t \in \{0,\dots,T\}$, $\lambda_{i,t}^T g_t^i(\hat{x}_t,\hat{u}_t) = 0$.*

*For all $t \in \{0,\dots,T\}$, $\left\langle p_{t+1}^T,\ f_t(\hat{x}_t,\hat{u}_t) - \hat{x}_{t+1}\right\rangle = 0$.*

**(c)** $\forall t \in \{1,\dots,T\}$, $p_t^T = \sum_{j=1}^{\ell}\left(\theta_j^T.D_1\phi_t^j(\hat{x}_t,\hat{u}_t)\right) + p_{t+1}^T \circ D_{G,1}f_t(\hat{x}_t,\hat{u}_t) + \sum\limits_{i=1}^{m_i}\lambda_{i,t}^T D_{G,1}g_t^i(\hat{x}_t,\hat{u}_t) + \sum\limits_{k=1}^{m_e}\mu_{k,t}^T D_1 h_t^k(\hat{x}_t,\hat{u}_t)$.

**(d)** $\forall t \in \{0,\dots,T\}$, $\sum_{j=1}^{\ell}\left(\theta_j^T.D_2\phi_t^j(\hat{x}_t,\hat{u}_t)\right) + p_{t+1}^T \circ D_{G,2}f_t(\hat{x}_t,\hat{u}_t) + \sum\limits_{i=1}^{m_i}\lambda_{i,t}^T D_{G,2}g_t^i(\hat{x}_t,\hat{u}_t) + \sum\limits_{k=1}^{m_e}\mu_{k,t}^T D_2 h_t^k(\hat{x}_t,\hat{u}_t) = 0$.

*Proof.* We arbitrarily fix $T \in \mathbb{N}_*$. Let $(\hat{x}_0,\dots,\hat{x}_{T+1},\hat{u}_0,\dots,\hat{u}_T)$ be a solution of Problem $(FM_i^T)$ where $\eta$ and $\hat{x}_{T+1}$ be given.

**Step 1:** We rewrite Problem $(FM_i^T)$ under model of $(\mathcal{M}_m)$. Let $\Omega = \prod\limits_{t=1}^{T} X_t \times \left(\mathbb{R}^d\right)^{T+1}$, then $\Omega$ is open. Let $z = (x_1,\dots,x_T,u_0,\dots,u_T) \in \Omega$. We set $\phi(z) = (\phi_1(z),\dots,\phi_\ell(z))$ where for all $j \in \{1,\dots,\ell\}$, $\phi_j(z) := \phi_0^j(\eta,u_0) + \sum_{t=1}^{T}\phi_t^j(x_t,u_t)$; and for all $i \in \{1,\dots,m_i\}$, for all $t \in \{0,\dots,T\}, \tilde{g}_t^i(z) := g_t^i(x_t,u_t)$; and for all $k \in \{1,\dots,m_e\}$, for all $t \in \{0,\dots,T\}, \tilde{h}_t^k(z) := h_t^k(x_t,u_t)$; and for all $j \in \{1,\dots,n\}, \psi_0^j(z) =:= f_0^j(\eta,u_0) - x_1^j$; and for all $j \in \{1,\dots,n\}$, for all $t \in \{1,\dots,T-1\}, \psi_t^j(z) := f_t^j(x_t,u_t) - x_{t+1}^j$; and, finally, for all $j \in \{1,\dots,n\}, \psi_T^j(z) := f_T^j(x_T,u_T) - \hat{x}_{T+1}^j$. So we have translated Problem $(FM_i^T)$ into this form

$$\left.\begin{array}{rl}
\text{Maximize} & \phi(z) = \phi(x_1,\dots,x_T,u_0,\dots,u_T);\\
\text{when} & z \in \Omega,\\
& \forall i \in \{1,\dots,m_i\}, \forall t \in \{0,\dots,T\},\ \tilde{g}_t^i(z) \geq 0,\\
& \forall k \in \{1,\dots,m_e\}, \forall t \in \{0,\dots,T\},\ \tilde{h}_t^k(z) = 0,\\
& \forall j \in \{1,\dots,n\},\ \forall t \in \{0,\dots,T\},\ \psi_t^j(z) \geq 0,
\end{array}\right\} \qquad (FM^T)$$

which has the same form of Problem $(\mathcal{M}_m)$.

**Step 2:** We will prove that with this problem, all conditions of Theorem 5.7 are fulfilled. It is obvious that $\hat{z} = (\hat{x}_1,\dots,\hat{x}_T,\hat{u}_0,\dots,\hat{u}_T)$ is a Pareto optimal solution of the above-mentioned problem. Under (i), (ii), (iii) and (iv) we obtain the following statements: For all $j \in \{1,\dots,\ell\}$ function $\phi_j$ is Fréchet differentiable at $\hat{z}$ as a sum of $T$ Fréchet differentiable functions; For all $i \in \{1,\dots,m_i\}$, for all $t \in \{0,\dots,T\}$, functions $\tilde{g}_t^i$ is Fréchet differentiable at $\hat{z}$ when $\tilde{g}_t^i(\hat{z}) = 0$; For all $i \in \{1,\dots,m_i\}$, for all $t \in \{0,\dots,T\}$, functions $\tilde{g}_t^i$ is Gâteaux differentiable at $\hat{z}$ and lower semicontinuous at $\hat{z}$ when $\tilde{g}_t^i(\hat{z}) > 0$; For all $k \in \{1,\dots,m_e\}$, for all $t \in \{0,\dots,T\}$, $\tilde{h}_t^i$ is continuous on a neighborhood of $\hat{z}$ and Fréchet differentiable at $\hat{z}$; For all $j \in \{1,\dots,n\}$, for all

$t \in \{0, \dots, T\}$, $\psi_t^j$ is Fréchet differentiable at $\hat{z}$ when $\psi_t^j(\hat{z}) = 0$; For all $j \in \{1, \dots, n\}$, for all $t \in \{0, \dots, T\}$, functions $\psi_t^j$ is Gâteaux differentiable at $\hat{z}$ and lower semicontinuous at $\hat{z}$ when $\psi_t^j(\hat{z}) > 0$.

**Step 3:** Application of Theorem 5.7. Now for Problem $(FM_1^i)$, all the conditions of Theorem 5.7 are satisfied, by applying it, we obtain the multiplier $\theta^T = (\theta_1^T, \dots, \theta_\ell^T) \in \mathbb{R}_+^\ell$ for the criterion function $\phi(z)$, multipliers $\lambda_{i,t}^T \in \mathbb{R}_+$ for inequality constraints $\tilde{g}_t^i(z) \geq 0$, multipliers $\mu_{k,t}^T \in \mathbb{R}$ for equality constraints $\tilde{h}_t^i(z) = 0$ and multipliers $p_{t+1,j}^T \in \mathbb{R}_+$ for the equality constraints $\psi_t^j(z) \geq 0$ where $t \in \{0, \dots, T\}$, $i \in \{1, \dots, m_i\}$, $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_e\}$ such that all the conclusions of Theorem 5.7 hold. From conclusion (a) of Theorem 5.7, we know that the multipliers are not simultaneously equal to zero. We set $p_{t+1}^T := \sum_{j=1}^{n} p_{t+1,j}^T e_j^* \in \mathbb{R}^{n*}$. From conclusion (b) of Theorem 5.7, we have: $\forall i \in \{1, \dots, m_i\}$, $\forall t \in \{0, \dots, T\}$, $\lambda_{i,t}^T \tilde{g}_t^i(\hat{z}) = 0$ i.e. $\forall i \in \{1, \dots, m_i\}, \forall t \in \{0, \dots, T\}$, $\lambda_{i,t}^T g_t^i(\hat{x}_t, \hat{u}_t) = 0$; and $\forall t \in \{0, \dots, T\}, \forall j \in \{1, \dots, n\}$, $p_{t+1,j}^T \psi_t^j(\hat{z}) = p_{t+1,j}^T (f_t^j(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}^j) = 0$. i.e. $\forall t \in \{0, \dots, T\}$, $\left\langle p_{t+1}^T, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \right\rangle = 0$. Then the generalized Lagrangian of this problem is

$$\begin{aligned}
&\mathcal{L}(z, \theta_1^T, \dots, \theta_\ell^T, \lambda_{1,0}^T, \dots, \lambda_{m_i,T}^T, \mu_{1,0}^T, \dots, \mu_{m_e,T}^T, p_{1,1}^T, \dots, p_{T+1,n}^T) \\
&= \textstyle\sum_{j=1}^{\ell} \theta_j^T . \phi_j(z) + \sum_{t=0}^{T} \sum_{i=1}^{m_i} \lambda_{i,t}^T \tilde{g}_t^i(z) + \sum_{t=0}^{T} \sum_{k=1}^{m_e} \mu_{k,t}^T \tilde{h}_t^i(z) + \sum_{t=0}^{T} \sum_{j=1}^{n} p_{t+1,j}^T \psi_t^j(z) \\
&= \textstyle\sum_{j=1}^{\ell} \left( \theta_j^T (\phi_0^j(\eta, u_0) + \sum_{t=1}^{T} \phi_t^j(x_t, u_t)) \right) + \sum_{t=0}^{T} \sum_{i=1}^{m_i} \lambda_{i,t}^T g_t^i(x_t, u_t) \\
&+ \textstyle\sum_{t=0}^{T} \sum_{k=1}^{m_e} \mu_{k,t}^T h_t^k(x_t, u_t) + \sum_{t=0}^{T} \left\langle p_{t+1}^T, f_t(x_t, u_t) - x_{t+1} \right\rangle .
\end{aligned}$$

As preliminary calculations, since all the functions in $\mathcal{L}$ are Gâteaux differentiable, then for all $t \in \{1, \dots, T\}$, the partial Gâteaux differential of the generalized Lagrangian with respect to $x_t$ is

$$\begin{aligned}
&\tfrac{\partial \mathcal{L}}{\partial x_t}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) \\
&= D_{G,x_t} \mathcal{L}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) \\
&= \textstyle\sum_{j=1}^{\ell} \left( \theta_j^T . D_{G,1} \phi_t^j(x_t, u_t) \right) + \sum_{i=1}^{m_i} \lambda_{i,t}^T D_{G,1} g_t^i(x_t, u_t) + \sum_{k=1}^{m_e} \mu_{k,t}^T D_{G,1} h_t^k(x_t, u_t) \\
&\quad + p_{t+1}^T \circ D_{G,1} f_t(x_t, u_t) - p_t^T ,
\end{aligned}$$

and for all $t \in \{0, \dots, T\}$ the partial Gâteaux differential of the generalized Lagrangian with respect to $u_t$ is

$$\begin{aligned}
&\tfrac{\partial \mathcal{L}}{\partial u_t}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) \\
&= D_{G,u_t} \mathcal{L}(z, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) \\
&= \textstyle\sum_{j=1}^{\ell} \left( \theta_j^T . D_{G,2} \phi_t^j(x_t, u_t) \right) + p_{t+1}^T \circ D_{G,2} f_t(x_t, u_t) \\
&\quad + \textstyle\sum_{i=1}^{m_i} \lambda_{i,t}^T D_{G,2} g_t^i(x_t, u_t) + \sum_{k=1}^{m_e} \mu_{k,t}^T D_{G,2} h_t^k(x_t, u_t).
\end{aligned}$$

From the conclusion (c) of Theorem 5.7, the partial Gâteaux differential of $\mathcal{L}$ with respect to $z$ vanishes at $\hat{z}$. That means

$$\begin{cases}
\forall t \in \{1, \dots, T\}, \ D_{G,x_t} \mathcal{L}(\hat{z}, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) = 0, \\
\forall t \in \{0, \dots, T\}, \ D_{G,u_t} \mathcal{L}(\hat{z}, \lambda_o^T, p_{1,1}^T, \dots, p_{1,n}^T, \dots, p_{T+1,1}^T, \dots, p_{T+1,n}^T) = 0.
\end{cases}$$

$\Longleftrightarrow$

$$\left.\begin{array}{l}\forall t\in\{1,\dots,T\},\ p_t^T = \sum_{j=1}^{\ell}\left(\theta_j^T.D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right)+p_{t+1}^T\circ D_{G,1_t}f_t(\hat{x}_t,\hat{u}_t)\\ \qquad\qquad +\sum\limits_{i=1}^{m_i}\lambda_{i,t}^T D_{G,1}g_t^i(\hat{x}_t,\hat{u}_t)+\sum\limits_{k=1}^{m_e}\mu_{k,t}^T D_{G,1}h_t^k(\hat{x}_t,\hat{u}_t),\\ \forall t\in\{0,\dots,T-1\},\ \sum_{j=1}^{\ell}\left(\theta_j^T.D_{G,2}\phi_t(\hat{x}_t,\hat{u}_t)\right)+p_{t+1}^T\circ D_{G,2}f_t(\hat{x}_t,\hat{u}_t),\\ \qquad\qquad +\sum\limits_{i=1}^{m_i}\lambda_{i,t}^T D_{G,2}g_t^i(\hat{x}_t,\hat{u}_t)+\sum\limits_{k=1}^{m_e}\mu_{k,t}^T D_{G,2}h_t^k(\hat{x}_t,\hat{u}_t)=0.\end{array}\right\}\tag{5.8}$$

We have yet seen that conclusions (a) and (b) are satisfied. Conclusions (c) and (d) are obtained after identifying Gâteaux differential with Fréchet differential of Fréchet differentiable functions in (5.8). □

In the special case where for every $t\in\mathbb{N}$, $U_t$ is an arbitrary subset of $\mathbb{R}^d$ and $u_t$ belongs to the interior of $U_t$, Problem $(FM^T)$ now contains only inequality constraints as follows

$$\left\{\begin{array}{rl}\text{Maximize} & \phi(z);\\ \text{when} & z\in\Omega;\\ & \forall j\in\{1,\dots,n\},\ \forall t\in\{0,\dots,T\},\ \psi_t^j(z)\geq 0.\end{array}\right.$$

This problem has the form of Problem $(\mathcal{I}_m)$. Then we obtain the following corollary after applying Theorem 5.6.

**Corollary 5.9.** *Let $T\in\mathbb{N}_*$ be given and $(\hat{x}_0,\dots,\hat{x}_{T+1},\hat{u}_0,\dots,\hat{u}_T)$ be a Pareto optimal solution of Problem $(FM_i^T)$ when $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. We assume that for all $t\in\{0,\dots,T\}$, $X_t$ are open and $\hat{u}_t\in int(U_t)$. We also assume that the following assumptions are fulfilled*

**(i)** *For all $j\in\{1,\dots,\ell\}$, for all $t\in\{0,\dots,T\}$, $\phi_t^j$ is Gâteaux differentiable at $(\hat{x}_t,\hat{u}_t)$. For all $t\in\{0,\dots,T\}$, $f_t$ is Gâteaux differentiable at $(\hat{x}_t,\hat{u}_t)$.*

**(ii)** *For all $j\in\{1,\dots,n\}$, for all $t\in\{0,\dots,T\}$, $f_t^j$ is lower semicontinuous at $(\hat{x}_t,\hat{u}_t)$ when $f_t^j(\hat{x}_t,\hat{u}_t)>\hat{x}_{t+1}^j$.*

*Then there exist $\theta_1^T,\dots,\theta_\ell^T\in\mathbb{R}_+$ and $p_{t+1}^T\in\mathbb{R}_+^{n*}$ where $t\in\{0,\dots,T\}$ such that the following conditions are satisfied*

**(a)** $\theta_1^T,\dots,\theta_\ell^T,\ p_{t+1}^T,\ t\in\{0,\dots,T\}$ *are not all zeros.*

**(b)** *For all $t\in\{0,\dots,T\}$, $\left\langle p_{t+1}^T,\ f_t(\hat{x}_t,\hat{u}_t)-\hat{x}_{t+1}\right\rangle=0$.*

**(c)** $\forall t\in\{1,\dots,T\},\ p_t^T=\sum_{j=1}^{\ell}\left(\theta_j^T.D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right)+p_{t+1}^T\circ D_{G,1}f_t(\hat{x}_t,\hat{u}_t)$.

**(d)** $\forall t\in\{0,\dots,T\},\ 0=\sum_{j=1}^{\ell}\left(\theta_j^T.D_{G,2}\phi_t^j(\hat{x}_t,\hat{u}_t)\right)+p_{t+1}^T\circ D_{G,2}f_t(\hat{x}_t,\hat{u}_t)$.

**For Problem $(FM_e^T)$:** the way we treat it is almost similar to what we did with Problem $(FM_i^T)$ case. The difference is that the greater or equal sign in $\psi_t^j(x_1,\dots,x_T,u_0,\dots,u_T)\geq 0$ will be replaced by the equal sign since now the phase constraint has the form of (De). Apply Theorem 5.7 for this problem with a notice that now the inequality constraints group includes only functions $\tilde{g}_t^i$ and the equality constraints group includes functions $\tilde{h}_t^k$ and $\psi_t^j$, we obtain the weak Pontryagin principle for finite horizon problem with (De) as follows.

**Theorem 5.10.** *Let $T\in\mathbb{N}_*$ be given and $(\hat{x}_0,\dots,\hat{x}_{T+1},\hat{u}_0,\dots,\hat{u}_T)$ be a Pareto optimal solution of Problem $(FM_e^T)$ when $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. We assume that for all $t\in\{0,\dots,T\}$, $X_t$ is open and $U_t(x_t)$ is defined by (5.7). We also assume that the following conditions are fulfilled.*

**(i)** *For all* $j \in \{1, \dots, \ell\}$, *for all* $t \in \{0, \dots, T\}$, $\phi_t^j$ *is Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(ii)** *For all* $i \in \{1, \dots, m_i\}$, *for all* $t \in \{0, \dots, T\}$, $g_t^i$ *is Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$ *when* $g_t^i(\hat{x}_t, \hat{u}_t) = 0$.

**(iii)** *For all* $i \in \{1, \dots, m_i\}$, *for all* $t \in \{0, \dots, T\}$, $g_t^i$ *is Gâteaux differentiable at* $(\hat{x}_t, \hat{u}_t)$ *and lower semicontinuous at* $\hat{u}_t$ *when* $g_t^i(\hat{x}_t, \hat{u}_t) > 0$.

**(iv)** *For all* $k \in \{1, \dots, m_e\}$, *for all* $t \in \{0, \dots, T\}$, $h_t^k$ *is continuous on a neighborhood of* $(\hat{x}_t, \hat{u}_t)$ *and Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

*For all* $t \in \{0, \dots, T\}$, $f_t$ *is continuous on a neighborhood of* $(\hat{x}_t, \hat{u}_t)$ *and Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

*Then there exist* $\theta_1^T, \dots, \theta_\ell^T \in \mathbb{R}_+$, $\lambda_{i,t}^T \in \mathbb{R}_+$, $\mu_{k,t}^T \in \mathbb{R}$ *and* $p_{t+1}^T \in \mathbb{R}^{n*}$ *where* $t \in \{0, \dots, T\}$, $i \in \{1, \dots, m_i\}$ *and* $k \in \{1, \dots, m_e\}$ *such that conclusions (a), (c), (d) of Theorem 5.8 are satisfied (in which,* $D_{G,\alpha} f_t(\hat{x}_t, \hat{u}_t)$ *is replaced by* $D_\alpha f_t(\hat{x}_t, \hat{u}_t)$ *where* $\alpha \in \{1, 2\}$*) together with the following one:*

**(b)** $\forall i \in \{1, \dots, m_i\}, \forall t \in \{0, \dots, T\}$, $\lambda_{i,t}^T g_t^i(\hat{x}_t, \hat{u}_t) = 0$.

In the special case where for every $t \in \mathbb{N}$, $U_t$ is an arbitrary subset of $\mathbb{R}^d$ and $u_t$ belongs to the interior of $U_t$, Problem $(FM^T)$ is reduced to the following simpler form of Problem $(\mathcal{M}_m)$:

$$\begin{cases} \text{Maximize} & \phi(z); \\ \text{when} & z \in \Omega; \\ & \forall j \in \{1, \dots, n\},\ \forall t \in \{0, \dots, T\},\ \psi_t^j(z) = 0. \end{cases}$$

Then applying Theorem 5.7, we obtain the following simpler statement

**Corollary 5.11.** *Let* $T \in \mathbb{N}_*$ *be given and* $(\hat{x}_0, \dots, \hat{x}_{T+1}, \hat{u}_0, \dots, \hat{u}_T)$ *be a Pareto optimal solution of Problem* $(FM_e^T)$ *when* $\eta$ *and* $\hat{x}_{T+1}$ *are fixed vectors in* $\mathbb{R}^n$. *We assume that for all* $t \in \{0, \dots, T\}, X_t$ *are open and* $\hat{u}_t \in int(U_t)$. *We also assume that the following conditions are fulfilled:*

**(i)** *For all* $j \in \{1, \dots, \ell\}$, *for all* $t \in \{0, \dots, T\}$, $\phi_t^j$ *is Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(ii)** *For all* $t \in \{0, \dots, T\}$, $f_t$ *are continuous on a neighborhood of* $(\hat{x}_t, \hat{u}_t)$ *and Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

*Then there exist* $\theta_1^T, \dots, \theta_\ell^T \in \mathbb{R}_+$ *and* $p_{t+1}^T \in \mathbb{R}^{n*}$ *where* $t \in \{0, \dots, T\}$ *such that the conclusions (a), (c), (d) of Corollary 5.9 are satisfied in which* $D_{G,\alpha} f_t(\hat{x}_t, \hat{u}_t)$ *is replaced by* $D_\alpha f_t(\hat{x}_t, \hat{u}_t)$ *where* $\alpha \in \{1, 2\}$.

**Remark 5.12.** Notice that in the absence of constraints, one can weaken the assumptions on objective functions $\phi_t^j$ and on $f_t$ as can be seen in the above-mentioned corollaries.

**Remark 5.13.** When all the sets of state and control variables are convex and when all the functions are convex (or concave), Gâteaux differentiability and Fréchet differentiability are identified.

**Remark 5.14.** One can obtain analogous results with the assumption of weak Pareto optimality by applying other multiplier rules (for instance, multiplier rule of Novo and Jimenez (see Theorem 3.10 in [47])). However, the required assumptions for the multiobjective optimal control problems will become very difficult to write due to the complication of qualification constraints needed in such rules to obtain a Fritz John like multiplier rule when the functions are not Fréchet differentiable.

### 5.5.2 Strong Pontryagin Principles

In this section, firstly, we recall a theorem which provides a necessary condition of optimality in a strong form for multiobjective static optimization problems. Let $X,\ Y,\ Z,\ W$ be Banach spaces. Let $Y$ and $Z$ be ordered by cones $K$ and $M$, respectively. Let $U$ be a set equipped with the trivial topology (containing only $U$ and $\varnothing$), so that $X \times U$ is a topological space. Let mappings $J : X \times U \to Y,\ F : X \times U \to Z,\ H : X \times U \to W$ be given. Consider the following problem of vector optimization:

$$\left.\begin{array}{c} \text{Maximize } J(x,u) \\ F(x,u) \geq 0, \\ H(x,u) = 0, \\ x \in X,\ u \in U. \end{array}\right\} \tag{5.9}$$

Let us define the following generalized Lagrangian of Problem (5.9):

$$\mathcal{L}(x,u,\lambda,\mu,\upsilon) := \langle \lambda, J(x,u)\rangle + \langle \mu, F(x,u)\rangle + \langle \upsilon, H(x,u)\rangle .$$

We introduce the multiplier rule for this problem which is proven by Khanh and Nuong in [39] and is described in the following theorem

**Theorem 5.15.** *Assume that Problem (5.9) satisfies the following conditions:*

**(i)** $\mathrm{int}K \neq 0$ *and* $\mathrm{int}M \neq 0$.

**(ii)** *For each* $u \in U$, $H(.,u)$ *is continuously differentiable at* $\hat{x}$.

**(iii)** $J(.,\hat{u})$ *and* $F(.,\hat{u})$ *have directional derivatives at* $\hat{x}$ *which are concave.*

**(iv)** *For each* $u \in U$, $J(.,u)$ *and* $F(.,u)$ *are continuous at* $\hat{x}$ *in any direction* $h$, *in the sense that* $J(\hat{x}+\lambda h,u)$ *and* $F(\hat{x}+\lambda h,u)$ *tend to* $J(\hat{x},u)$ *and* $F(\hat{x},u)$, *respectively, as* $\lambda \to 0^+$.

**(v)** *For each* $x$ *in a neighborhood* $V$ *of* $\hat{x}$, *the following convexity condition is satisfied: if* $u^1 \in U,\ u^2 \in U,\ 0 \leq \alpha \leq 1$, *one can find* $u \in U$ *such that*

$$\begin{aligned} J(x,u) &\geq \alpha J(x,u^1) + (1-\alpha)J(x,u^2), \\ F(x,u) &\geq \alpha F(x,u^1) + (1-\alpha)F(x,u^2), \\ H(x,u) &= \alpha H(x,u^1) + (1-\alpha)H(x,u^2). \end{aligned}$$

**(vi)** $\mathrm{codim}D_1H(\hat{x},\hat{u})$ *is finite.*

*Then, if* $(\hat{x},\hat{u})$ *is a weak Pareto optimal solution of Problem (5.9), there exist* $\lambda \in K^*,\ \mu \in M^*,\ \upsilon \in W^*$, *not all zero, such that*

**(a)** $\left\langle \lambda, \vec{D}_1 J(\hat{x},\hat{u};h)\right\rangle + \left\langle \mu, \vec{D}_1 F(\hat{x},\hat{u};h)\right\rangle + \langle \upsilon, D_1H(\hat{x},\hat{u}).h\rangle \geq 0$, *for all* $h \in X$, *i.e.*

$$0 \in \partial_1 \mathcal{L}(\hat{x},\hat{u},\lambda,\mu,\upsilon).$$

**(b)** $\mathcal{L}(\hat{x},\hat{u},\lambda,\mu,\upsilon) = \max_{u\in U} \mathcal{L}(\hat{x},u,\lambda,\mu,\upsilon)$.

**(c)** $\langle \mu, F(\hat{x},\hat{u})\rangle = 0$.

We will use this theorem to establish a strong Pontryagin principles for finite-horizon multiobjective optimal control problems $(FM_k^T)$.

To obtain strong Pontryagin principles for multiobjective optimal control problems in finite horizon framework $(FM_k^T)$ when $k \in \{e,i\}$, we will rely on the above-mentioned multiplier rule for multiobjective static optimization problems. To do this,

we will translate the multiobjective optimal control problems into multiobjective static optimization problems as we did for the weak principles. In this section we will consider Problems $(FM_k^T)$ when $k \in \{e, i\}$ in the case where there exist inequality constraints on the optimal solution. Assume that for each $t \in \mathbb{N}$, the sets of controls are defined by inequalities as follows

$$W_t(x_t) = \bigcap_{1 \leq k \leq m} \{u \in \mathbb{R}^d : g_t^k(x_t, u) \geq 0\} \tag{5.10}$$

where $g_t^k : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$.
Let $T \in \mathbb{N}_*$ be a fixed number. Assume that $(\hat{x}_0, \dots \hat{x}_{T+1}, \hat{u}_0, \dots \hat{u}_T)$ is a weak Pareto optimal solution of Problem $(FM_k^T)$ when $k \in \{e, i\}$. In this problem, $x_0$ and $x_{T+1}$ are given so we can rewrite $(FM_k^T)$ as follows

$$\left.\begin{array}{c} \text{Maximize } J^T(x^T, u^T) \\ F^T(x^T, u^T) \geq 0 \text{ when } k = i, \\ (\text{or } F^T(x^T, u^T) = 0 \text{ when } k = e) \\ G^T(x^T, u^T) \geq 0, \\ x^T \in X(T),\ u^T \in U(T) = (\mathbb{R}^d)^{T+1}, \end{array}\right\} \tag{5.11}$$

where:

- $X(T) = X_1 \times \cdots \times X_T,\ x^T = (x_1, \dots, x_T),\ u^T = (u_0, \dots, u_T)$;
- $J^T(x^T, u^T) = (J_1^T(x^T, u^T), \dots, J_\ell^T(x^T, u^T))$ where for all $j \in \{1, \dots, \ell\},\ J_j^T(x^T, u^T) = \phi_0^j(\eta, u_0) + \sum_{t=1}^T \phi_t^j(x_t, u_t)$;
- $F^T(x^T, u^T) = (F_0^T(x^T, u^T), \dots, F_T^T(x^T, u^T))$ where $F_0^T(x^T, u^T) = f_0(\eta, u_0) - x_1$, for all $t \in \{1, \dots, T-1\},\ F_t^T(x^T, u^T) = f_t(x_t, u_t) - x_{t+1},\ F_T^T(x^T, u^T) = f_T(x_T, u_T) - \hat{x}_{T+1}$;
- $G^T(x^T, u^T) = (G_0^T(x^T, u^T), \dots, G_T^T(x^T, u^T))$ where for all $t \in \{1, \dots, T\},\ G_t^T(x^T, u^T) = (g_t^1(x_t, u_t), \dots, g_t^m(x_t, u_t))$ and $G_0^T(x^T, u^T) = (g_0^1(\eta, u_0), \dots, g_0^m(\eta, u_0))$.

We can see that $X(T)$ is an open subset of $(\mathbb{R}^n)^T$, $U(T)$ is the whole space $(\mathbb{R}^d)^{T+1}$. The vector function $J^T : X(T) \times (\mathbb{R}^d)^{T+1} \to \mathbb{R}^\ell$ is ordered by the cone $\mathbb{R}_+^\ell$; the mapping $F^T : X(T) \times (\mathbb{R}^d)^{T+1} \to (\mathbb{R}^n)^{T+1}$ is ordered by the cone $(\mathbb{R}_+^n)^{T+1}$; and the mapping $G^T : X(T) \times (\mathbb{R}^d)^{T+1} \to (\mathbb{R}^m)^{T+1}$ is ordered by the cone $(\mathbb{R}_+^m)^{T+1}$. Obviously, $\text{int}\mathbb{R}_+^\ell \neq \varnothing$, $\text{int}(\mathbb{R}_+^n)^{T+1} \neq \varnothing$ and $\text{int}(\mathbb{R}_+^m)^{T+1} \neq \varnothing$. The generalized Lagrangian of this problem has this form

$$\mathcal{L}(x^T, u^T, \theta^T, p^T, \mu^T) := \left\langle \theta^T, J^T(x^T, u^T) \right\rangle + \left\langle p^T, F^T(x^T, u^T) \right\rangle + \left\langle \mu^T, G^T(x^T, u^T) \right\rangle,$$

where $\theta^T = (\theta_1^T, \dots, \theta_\ell^T) \in \mathbb{R}^\ell$, $p^T = (p_1^T, \dots, p_{T+1}^T) \in (\mathbb{R}^{n*})^{T+1}$ and $\mu^T = (\mu_0^T, \dots, \mu_T^T) \in (\mathbb{R}^{m*})^{T+1}$. Now, all the elements depend on $T$. The augmented Hamiltonian is now defined for all $t \in \{0, \dots, T\}$ as follows

$$H_t^T(x_t, u_t, \theta^T, p_{t+1}^T, \mu_t^T) = \sum_{j=1}^\ell \theta_j^T \phi_t^j(x_t, u_t) + \left\langle p_{t+1}^T, f_t(x_t, u_t) \right\rangle + \sum_{k=1}^m \mu_t^{k,T}.g_t^k(x_t, u_t),$$

where $\mu_t^{1,T}, \dots, \mu_t^{m,T}$ are the coordinates of vector $\mu_t^T$ in $\mathbb{R}^{m*}$.

### (Di) case

We present a strong Pontryagin principle for the multiobjective optimal control problems with (Di) in finite horizon as follows.

**Theorem 5.16.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_i^T)$ where $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. Assume that for all $t \in \{0, \ldots, T\}$, the control sets are defined as in (5.10) and that the following conditions are fulfilled*

**(i)** *For all $t \in \{1, \ldots, T\}$, for all $j \in \{1, \ldots, \ell\}$ and for all $k \in \{1, \ldots, m\}$, $\phi_t^j(., \hat{u}_t)$, $f_t(., \hat{u}_t)$ and $g_t^k(., \hat{u}_t)$ have directional derivatives at $\hat{x}_t$ which are concave.*

**(ii)** *For all $t \in \{1, \ldots, T\}$, for all $j \in \{1, \ldots, \ell\}$, for all $k \in \{1, \ldots, m\}$, for all $u \in W_t(\hat{x}_t)$, $\phi_t^j(., u)$ , $f_t(., u)$ and $g_t^k(., u)$ are continuous at $\hat{x}_t$ in any direction $h$.*

**(iii)** *For all $t \in \{0, \ldots, T\}$, for each $x_t$ in a neighborhood $V_t$ of $\hat{x}_t$ such that $x_0 = \eta$, the following convexity condition is satisfied: if $u' \in W_t(x_t)$, $u'' \in W_t(x_t)$, $0 \leq \alpha \leq 1$, one can find $u \in W_t(x_t)$ such that*

$$\begin{aligned} \forall j \in \{1, \ldots, \ell\}, \quad & \phi_t^j(x_t, u) \geq \alpha \phi_t^j(x_t, u') + (1-\alpha)\phi_t^j(x_t, u''), \\ & f_t(x_t, u) \geq \alpha f_t(x_t, u') + (1-\alpha) f_t(x_t, u''), \\ \forall k \in \{1, \ldots, m\}, \quad & g_t^k(x_t, u) \geq \alpha g_t^k(x_t, u') + (1-\alpha) g_t^k(x_t, u''). \end{aligned}$$

*Then there exist $\theta^T = (\theta_1^T, \ldots, \theta_\ell^T) \in \mathbb{R}_+^\ell$, $p^T = (p_1^T, \ldots, p_{T+1}^T) \in (\mathbb{R}_+^{n*})^{T+1}$ and $\mu^T = (\mu_0^T, \ldots, \mu_T^T) \in (\mathbb{R}_+^{m*})^{T+1}$ not all zero which satisfy the following conclusions:*

**(a)** *For all $t \in \{0, \ldots, T\}$, for all $k \in \{1, \ldots, m\}$, $\left\langle p_{t+1}^T, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \right\rangle = 0$ and $\mu_t^k g_t^k(\hat{x}_t, \hat{u}_t) = 0$.*

**(b)** *For all $t \in \{1, \ldots, T\}$, $p_t^T \in \partial_1 H_t^T(\theta^T, \hat{x}_t, \hat{u}_t, p_{t+1}^T, \mu_t^T)$.*

**(c)** *For all $t \in \{0, \ldots, T\}$, $H_t^T(\hat{x}_t, \hat{u}_t, \theta^T, p_{t+1}^T, \mu_t^T) = \max_{u \in \mathbb{R}^d} H_t^T(\hat{x}_t, u, \theta^T, p_{t+1}^T, \mu_t^T)$.*

Firstly, we introduce the following lemmas:

**Lemma 5.17.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_k^T)$ where $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. Assume that condition (i) in Theorem 5.16 is fulfilled. Then, for all $h^T \in (\mathbb{R}^n)^T$, $J^T(., \hat{u}^T)$, $F^T(., \hat{u}^T)$ and $G^T(., \hat{u}^T)$ have directional derivatives at $\hat{x}^T$ in the direction $h^T$, which are concave.*

*Proof.* We will prove the existence of $\vec{D}_1 J^T(\hat{x}^T, \hat{u}^T; h^T)$, i.e. the existence of the following limit

$$\lim_{\lambda \to 0^+} \frac{J^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - J^T(\hat{x}^T, \hat{u}^T)}{\lambda},$$

where $h^T = (h_1, \ldots, h_T)$ is an arbitrary vector from $(\mathbb{R}^n)^T$. Since $J^T$ is a vector mapping, it is equivalent to prove the existence of each element of the above-mentioned limit. For each $j \in \{1, \ldots, \ell\}$, we consider the following relation:

$$\begin{aligned} & \frac{J_j^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - J_j^T(\hat{x}^T, \hat{u}^T)}{\lambda} \\ &= \frac{\phi_0^j(\eta, \hat{u}_0) + \sum_{t=1}^T \phi_t^j(\hat{x}_t + \lambda h_t, \hat{u}_t) - (\phi_0^j(\eta, \hat{u}_0) + \sum_{t=1}^T \phi_t^j(\hat{x}_t, \hat{u}_t))}{\lambda} \\ &= \sum_{t=1}^T \frac{\phi_t^j(\hat{x}_t + \lambda h_t, \hat{u}_t) - \phi_t^j(\hat{x}_t, \hat{u}_t)}{\lambda}. \end{aligned}$$

Using the hypothesis of this lemma, it is obvious that

$$\begin{aligned} \lim_{\lambda \to 0^+} \left( \sum_{t=1}^T \frac{\phi_t^j(\hat{x}_t + \lambda h_t, \hat{u}_t) - \phi_t^j(\hat{x}_t, \hat{u}_t)}{\lambda} \right) &= \sum_{t=1}^T \left( \lim_{\lambda \to 0^+} \frac{\phi_t^j(\hat{x}_t + \lambda h_t, \hat{u}_t) - \phi_t^j(\hat{x}_t, \hat{u}_t)}{\lambda} \right) \\ &= \sum_{t=1}^T \vec{D}_1 \phi_t^j(\hat{x}_t, \hat{u}_t; h_t). \end{aligned}$$

And so, for all $j \in \{1, \ldots, \ell\}$, $J_j^T(., \hat{u}^T)$ has directional derivative at $\hat{x}^T$ and it is concave since it is a sum of $T$ concave mappings. Thus, $\vec{D}_1 J^T(\hat{x}^T, \hat{u}^T; h^T)$ exists for all $h^T \in (\mathbb{R}^n)^T$ and it is concave.

Now, we do the similar process for $F^T(., \hat{u}^T) = (F_0^T(., \hat{u}^T), \ldots, F_T^T(., \hat{u}^T))$, i.e. we will prove the existence of $\vec{D}_1 F_t^T(\hat{x}^T, \hat{u}^T; h^T)$ where $h^T$ is an arbitrary vector in $(\mathbb{R}^n)^T$ and where $t \in \{0, \ldots, T\}$. When $t = 0$, consider the following relation:

$$\begin{aligned}\frac{F_0^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - F_0^T(\hat{x}^T, \hat{u}^T)}{\lambda} &= \frac{f_0(\eta, \hat{u}_0) - (\hat{x}_1 + \lambda h_1) - (f_0(\eta, \hat{u}_0) - \hat{x}_1)}{\lambda} \\ &= \frac{-\lambda h_1}{\lambda} = -h_1.\end{aligned}$$

Since the result does not depend on $\lambda$, we obtain

$$\lim_{\lambda \to 0^+} \frac{F_0^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - F_0^T(\hat{x}^T, \hat{u}^T)}{\lambda} = -h_1$$

Therefore, $F_0^T(., \hat{u}^T)$ has directional derivatives at $\hat{x}^T$ in the direction $h^T$ and obviously, it is linear and thus, concave. Now for $t \in \{1, \ldots, T-1\}$, we have

$$\begin{aligned}&\frac{F_t^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - F_t^T(\hat{x}^T, \hat{u}^T)}{\lambda} \\ &= \frac{f_t(\hat{x}_t + \lambda h_t, \hat{u}_t) - (\hat{x}_{t+1} + \lambda h_{t+1}) - (f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1})}{\lambda} \\ &= -h_{t+1} + \frac{f_t(\hat{x}_t + \lambda h_t, \hat{u}_t) - f_t(\hat{x}_t, \hat{u}_t)}{\lambda}.\end{aligned}$$

Using hypothesis of the lemma, we know that

$$\lim_{\lambda \to 0^+} \frac{f_t(\hat{x}_t + \lambda h_t, \hat{u}_t) - f_t(\hat{x}_t, \hat{u}_t)}{\lambda} = \vec{D}_1 f_t(\hat{x}_t, \hat{u}_t; h_t).$$

Therefore, we can assert that

$$\lim_{\lambda \to 0^+} \frac{F_t^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - F_t^T(\hat{x}^T, \hat{u}^T)}{\lambda} = -h_{t+1} + \vec{D}_1 f_t(\hat{x}_t, \hat{u}_t; h_t).$$

Then for $t \in \{1, \ldots, T-1\}$, $F_t^T(., \hat{u}^T)$ has directional derivatives at $\hat{x}^T$ in the direction $h^T$ and it is concave since it is a sum of a linear mapping and a concave mapping. Finally, when $t = T$, we have

$$\begin{aligned}&\frac{F_T^T(\hat{x}^T + \lambda h^T, \hat{u}^T) - F_T^T(\hat{x}^T, \hat{u}^T)}{\lambda} \\ &= \frac{f_T(\hat{x}_T + \lambda h_T, \hat{u}_T) - \hat{x}_{T+1} - (f_T(\hat{x}_T, \hat{u}_T) - \hat{x}_{T+1})}{\lambda} \\ &= \frac{f_T(\hat{x}_T + \lambda h_T, \hat{u}_T) - f_T(\hat{x}_T, \hat{u}_T)}{\lambda}.\end{aligned}$$

Since

$$\lim_{\lambda \to 0^+} \frac{f_T(\hat{x}_T + \lambda h_T, \hat{u}_T) - f_T(\hat{x}_T, \hat{u}_T)}{\lambda} = \vec{D}_1 f_T(\hat{x}_T, \hat{u}_T; h_T),$$

we can assert that $F_t^T(., \hat{u}^T)$ has directional derivatives at $\hat{x}^T$ in the direction $h^T$ and it is concave since $\vec{D}_1 f_T(\hat{x}_T, \hat{u}_T; h_T)$ is concave. And so, all the element mappings of $F^T(., \hat{u}^T)$ have directional derivatives at $\hat{x}^T$ in the direction $h^T$ which are concave. Therefore, $F^T(., \hat{u}^T)$ has directional derivative at $\hat{x}^T$ in the direction $h^T$ and it is concave.

Finally, we prove the existence of $\vec{D}_1 G^T(\hat{x}^T, \hat{u}^T; h^T)$, where $h^T = (h_1, \ldots, h_T)$ is an arbitrary vector from $(\mathbb{R}^n)^T$. Since $G^T(., u^T)$ is a vector mapping which contains $T$ elements, we will prove the existence of $\vec{D}_1 G_t^T(\hat{x}^T, \hat{u}^T; h^T)$ where $t \in \{0, \ldots, T\}$. Now, each element $G_t^T$ is a $m$-dimensional vector mappings and each of its element $g_t^k$, where $k \in \{1, \ldots, m\}$, has directional derivative and it is concave. Therefore, $\vec{D}_1 G_t^T(\hat{x}^T, \hat{u}^T; h^T)$ exists for all $t \in \{0, \ldots, T\}$ and it is concave. Thus, $\vec{D}_1 G^T(\hat{x}^T, \hat{u}^T; h^T)$ exists for all $h^T \in (\mathbb{R}^n)^T$ and it is concave. The lemma is proven. □

**Lemma 5.18.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_k^T)$ where $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. Assume that condition (ii) in Theorem 5.16 is fulfilled. Then, for each $u^T \in U(T)$, $J^T(., u^T)$, $F(., u^T)$ and $G(., u^T)$ are continuous at $\hat{x}^T$ in any direction $h$.*

*Proof.* Let $T \in \mathbb{N}_*$ and $u^T \in U(T)$. We know that $J^T(., u^T)$, $F(., u^T)$ and $G(., u^T)$ are vector mappings. Their elements are elementary expressions that relate to $\phi_t^j(., u)$ , $f_t(., u)$ and $g_t^k(., u)$ where $t \in \{0, \ldots, T\}$, $j \in \{1, \ldots, \ell\}$ and $k \in \{1, \ldots, m\}$ which were defined before in Problem (5.11). Using the hypothesis of this lemma the following facts: (1) The composition of continuous mappings are continuous; (2) Linear mappings are continuous; (3) Bilinear mappings are continuous and (4) Classical functions of one variable are continuous, we can assert that $J^T(., u^T)$, $F^T(., u^T)$ and $G^T(., u^T)$ are continuous at $\hat{x}^T$ in any direction $h$. □

**Lemma 5.19.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_T)$ be a weak Pareto solution of Problem $(FM_k^T)$ where $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. Assume that condition (iii) in Theorem 5.16 is fulfilled when $k = i$ or condition (iii) in Theorem 5.20 is fulfilled when $k = e$. Then, for each $x^T$ in a neighborhood $V^T$ of $\hat{x}^T$, the following convexity condition is satisfied: if $u^{1,T} \in U(T)$, $u^{2,T} \in U(T)$, $0 \le \alpha \le 1$, one can find $u^T \in U(T)$ such that*

$$\begin{aligned}
J^T(x^T, u^T) &\ge \alpha J^T(x^T, u^{1,T}) + (1-\alpha) J^T(x^T, u^{2,T}),\\
G^T(x^T, u^T) &\ge \alpha G^T(x^T, u^{1,T}) + (1-\alpha) G^T(x^T, u^{2,T}),\\
F^T(x^T, u^T) &\ge \alpha F^T(x^T, u^{1,T}) + (1-\alpha) F^T(x^T, u^{2,T}) \text{ when } k = i\\
\text{or } F^T(x^T, u^T) &= \alpha F^T(x^T, u^{1,T}) + (1-\alpha) F^T(x^T, u^{2,T}) \text{ when } k = e.
\end{aligned}$$

*Proof.* Let $T \in \mathbb{N}_*$, $u^{1,T} = (u_0^1, \ldots, u_T^1) \in U(T)$, $u^{2,T} = (u_0^2, \ldots, u_T^2) \in U(T)$ and $0 \le \alpha \le 1$. Let $x^T$ be in a neighborhood $V^T$ of $\hat{x}^T$. Then using (C3), for all $t \in \{0, \ldots, T\}$, one can find $u_t \in U_t$ such that the following convexity condition is satisfied.

$$\begin{aligned}
\forall j \in \{1, \ldots, \ell\}, \quad & \phi_t^j(x_t, u_t) \le \alpha \phi_t^j(x_t, u_t^1) + (1-\alpha)\phi_t^j(x_t, u_t^2),\\
& f_t(x_t, u_t) \le \alpha f_t(x_t, u_t^1) + (1-\alpha) f_t(x, u_t^2),\\
\forall k \in \{1, \ldots, m\}, \quad & g_t^k(x_t, u) \le \alpha g_t^k(x_t, u') + (1-\alpha) g_t^k(x_t, u'').
\end{aligned}$$

We set $u^T := (u_0, \ldots, u_T) \in U(T)$. The first inequality implies that for all $j \in \{1, \ldots, \ell\}$,

$$\begin{aligned}
J_j^T(x^T, u^T) &= \phi_0^j(\eta, u_0) + \textstyle\sum_{t=1}^T \phi_t^j(x_t, u_t)\\
&\le \alpha(\phi_0^j(\eta, u_0^1) + \textstyle\sum_{t=1}^T \phi_t^j(x_t, u_t^1))\\
&\quad + (1-\alpha)(\phi_0^j(\eta, u_0^2) + \textstyle\sum_{t=1}^T \phi_t^j(x_t, u_t^2),\\
&= \alpha J_j^T(x^T, u^{1,T}) + (1-\alpha) J_j^T(x^T, u^{2,T}).
\end{aligned}$$

Thus, $J^T(x^T, u^T) \le \alpha J^T(x^T, u^{1,T}) + (1-\alpha) J^T(x^T, u^{2,T})$.

When $k = i$, the second inequality implies

$$\begin{aligned} F_0^T(x^T, u^T) &= f_0(\eta, u_0) - x_1 \\ &\leq \alpha(f_0(\eta, u_0^1) - x_1) + (1-\alpha)(f_0(\eta, u_0^2) - x_1) \\ &= \alpha F_0^T(x^T, u^{1,T}) + (1-\alpha) F_0^T(x^T, u^{2,T}); \end{aligned}$$

and when $t \in \{1, \dots, T-1\}$,

$$\begin{aligned} F_t^T(x^T, u^T) &= f_t(x_t, u_0) - x_{t+1} \\ &\leq \alpha(f_t(x_t, u_t^1) - x_{t+1}) + (1-\alpha)(f_t(x_t, u_t^2) - x_{t+1}) \\ &= \alpha F_t^T(x^T, u^{1,T}) + (1-\alpha) F_t^T(x^T, u^{2,T}); \end{aligned}$$

and,

$$\begin{aligned} F_T^T(x^T, u^T) &= f_T(x_T, u_T) - \hat{x}_{T+1} \\ &\leq \alpha(f_T(x_T, u_T^1) - \hat{x}_{T+1}) + (1-\alpha)(f_T(x_T, u_T^2) - \hat{x}_{T+1}) \\ &= \alpha F_T^T(x^T, u^{1,T}) + (1-\alpha) F_T^T(x^T, u^{2,T}). \end{aligned}$$

Thus, $F^T(x^T, u^T) \leq \alpha F^T(x^T, u^{1,T}) + (1-\alpha) F^T(x^T, u^{2,T})$.

When $k = e$ by doing similarly, we obtain

$$F^T(x^T, u^T) = \alpha F^T(x^T, u^{1,T}) + (1-\alpha) F^T(x^T, u^{2,T}).$$

Finally, the third inequality implies that for all $t \in \{1, \dots, T\}$,

$$\begin{aligned} G_t^T(x^T, u^T) &= (g_t^1(x_t, u_t), \dots, g_t^m(x_t, u_t)) \\ &\leq (\alpha g_t^1(x_t, u_t^1) + (1-\alpha) g_t^1(x_t, u_t^2), \dots, \alpha g_t^m(x_t, u_t^1) + (1-\alpha) g_t^m(x_t, u_t^2)) \\ &= \alpha(g_t^1(x_t, u_t^1), \dots, g_t^m(x_t, u_t^1)) + (1-\alpha)(g_t^1(x_t, u_t^2), \dots, g_t^m(x_t, u_t^2)) \\ &= \alpha G_t^T(x^T, u^{1,T}) + (1-\alpha) G_t^T(x^T, u^{2,T}). \end{aligned}$$

This inequality holds for each $t \in \{1, \dots, T\}$, thus,

$$G^T(x^T, u^T) \leq \alpha G^T(x^T, u^{1,T}) + (1-\alpha) G^T(x^T, u^{2,T}).$$

The lemma is proven. □

Now we move to the proof of Theorem 5.16.

*Proof.* Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \dots, \hat{x}_{T+1}, \hat{u}_0, \dots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_i^T)$, then $(\hat{x}^T, \hat{u}^T) = (\hat{x}_1, \dots, \hat{x}_T, \hat{u}_0, \dots, \hat{u}_T)$ is a weak Pareto optimal solution of Problem (5.11) with $k = i$. Now, since conditions (i-iii) are fulfilled, after Lemma 5.17, Lemma 5.18 and Lemma 5.19, all the conditions of Theorem 5.15 are satisfied. Then, there exist $\theta^T = (\theta_1^T, \dots, \theta_\ell^T) \in \mathbb{R}_+^\ell$, $p^T = (p_1^T, \dots, p_{T+1}^T) \in (\mathbb{R}_+^{n*})^{T+1}$ and $\mu^T = (\mu_0^T, \dots, \mu_T^T) \in (\mathbb{R}_+^{m*})^{T+1}$ not all zero, such that the conclusions of Theorem 5.15 hold and thus, we obtain the following statements

$$\left.\begin{aligned} &\forall h^T \in (\mathbb{R}^n)^T,\ \left\langle \theta^T, \vec{D}_1 J^T(\hat{x}^T, \hat{u}^T; h^T) \right\rangle + \left\langle p^T, \vec{D}_1 F^T(\hat{x}^T, \hat{u}^T; h^T) \right\rangle \\ &+ \left\langle \mu^T, \vec{D}_1 G^T(\hat{x}^T, \hat{u}^T; h^T) \right\rangle \geq 0 \text{ i.e. } 0 \in \partial_1 \mathcal{L}(\hat{x}^T, \hat{u}^T, \theta^T, p^T, \mu^T). \end{aligned}\right\} \tag{5.12}$$

$$\mathcal{L}(\hat{x}^T, \hat{u}^T, \theta^T, p^T, \mu^T) = \max_{u^T \in U(T)} \mathcal{L}(\hat{x}^T, u^T, \theta^T, p^T, \mu^T). \tag{5.13}$$

$$\left\langle p^T, F^T(\hat{x}^T, \hat{u}^T)\right\rangle = 0 \text{ and } \left\langle \mu^T, G^T(\hat{x}^T, \hat{u}^T)\right\rangle = 0 \tag{5.14}$$

Using the result on directional derivatives of $J^T(., \hat{u}^T)$, $F^T(., \hat{u}^T)$ and $G^T(., \hat{u}^T)$ at $\hat{x}^T$ which are proven in Lemma 5.17, we can rewrite (a) as follows

$$\begin{gathered}\forall h^T \in (\mathbb{R}^n)^T,\\ \textstyle\sum_{j=1}^{\ell} \theta_j \sum_{t=1}^{T} \vec{D}_1\phi_t^j(\hat{x}_t, \hat{u}_t; h_t) - \left\langle p_1^T, h_1\right\rangle + \sum_{t=1}^{T-1} \left\langle p_{t+1}^T, \vec{D}_1 f_t(\hat{x}_t, \hat{u}_t; h_t) - h_{t+1}\right\rangle\\ + \left\langle p_{T+1}, \vec{D}_1 f_T(\hat{x}_T, \hat{u}_T; h_T)\right\rangle + \sum_{t=1}^{T}\sum_{k=1}^{m} \mu_t^{k,T}.\vec{D}_1 g_t^k(\hat{x}_t, \hat{u}_t; h_t) \geq 0\end{gathered}$$

Now, we consecutively choose $h^T = (0, \ldots, h_t, \ldots, 0)$, in which all the elements are zero except for the $t$-th position to obtain the following:

- When $t = 1$, $h^T = (h_1, 0, \ldots, 0)$, $\sum_{j=1}^{\ell} \theta_j^T \vec{D}_1\phi_1^j(\hat{x}_1, \hat{u}_1; h_1) - \left\langle p_1^T, h_1\right\rangle + \left\langle p_2^T, \vec{D}_1 f_1(\hat{x}_1, \hat{u}_1; h_1)\right\rangle + \sum_{k=1}^{m} \mu_1^k.\vec{D}_1 g_1^k(\hat{x}_1, \hat{u}_1; h_1) \geq 0$.

- For $t \in \{2, \ldots, T-1\}$, $h^T = (0, ., 0, h_t, 0, \ldots, 0)$, i.e. $h_t \neq 0$ at the $t$-th position, $\sum_{j=1}^{\ell} \theta_j^T \vec{D}_1\phi_t^j(\hat{x}_t, \hat{u}_t; h_t) + \left\langle p_{t+1}^T, \vec{D}_1 f_t(\hat{x}_t, \hat{u}_t; h_t)\right\rangle - \left\langle p_t^T, h_t\right\rangle + \sum_{k=1}^{m} \mu_t^k.\vec{D}_1 g_t^k(\hat{x}_t, \hat{u}_t; h_t) \geq 0$.

- When $t = T$, $h^T = (0, \ldots, 0, h_T)$, $\sum_{j=1}^{\ell} \theta_j^T \vec{D}_1\phi_T^j(\hat{x}_T, \hat{u}_T; h_T) - \left\langle p_T^T, h_T\right\rangle + \left\langle p_{T+1}^T, \vec{D}_1 f_T(\hat{x}_T, \hat{u}_T; h_T)\right\rangle + \sum_{k=1}^{m} \mu_T^k.\vec{D}_1 g_T^k(\hat{x}_T, \hat{u}_T; h_T) \geq 0$.

And so, for all $t \in \{1, \ldots, T\}$ and for all $h_t \in \mathbb{R}^n$, the following condition holds:

$$\begin{aligned}&\textstyle\sum_{j=1}^{\ell} \theta_j^T \vec{D}_1\phi_t^j(\hat{x}_t, \hat{u}_t; h_t) + \left\langle p_{t+1}^T, \vec{D}_1 f_t(\hat{x}_t, \hat{u}_t; h_t)\right\rangle\\ &\qquad + \sum_{k=1}^{m} \mu_t^k.\vec{D}_1 g_t^k(\hat{x}_t, \hat{u}_t; h_t) \geq \left\langle p_t^T, h_t\right\rangle\\ \iff &\ \vec{D}_1 H_t^T(\theta^T, \hat{x}_t, \hat{u}_t, p_{t+1}^T, \mu_t^T; h_t) \geq \left\langle p_t^T, h_t\right\rangle,\end{aligned}$$

i.e. $p_t^T \in \partial_1 H_t^T(\theta^T, \hat{x}_t, \hat{u}_t, p_{t+1}^T, \mu_t^T)$ and (b) holds. Now we expand the formula of $\mathcal{L}(\hat{x}^T, u^T, \theta^T, p^T, \mu^T)$ as follows:

$$\begin{aligned}\mathcal{L}(\hat{x}^T, u^T, \theta^T, p^T) &= \left\langle \theta^T, J^T(\hat{x}^T, u^T)\right\rangle + \left\langle p^T, F^T(\hat{x}^T, u^T)\right\rangle + \left\langle \mu^T, G^T(\hat{x}^T, u^T)\right\rangle\\ &= \textstyle\sum_{j=1}^{\ell} \theta_j^T \left(\phi_0^j(\eta, u_0) + \sum_{t=1}^{T} \phi_t^j(\hat{x}_t, u_t)\right) + \left\langle p_1^T, f_0(\eta, u_0) - \hat{x}_1\right\rangle +\\ &\quad + \textstyle\sum_{t=1}^{T} \left\langle p_{t+1}^T, f_t(\hat{x}_t, u_t) - \hat{x}_{t+1}\right\rangle + \sum_{t=0}^{T}\sum_{k=1}^{m} \mu_t^{k,T}.g_t^k(\hat{x}_t, u_t)\\ &= \textstyle\sum_{t=0}^{T} \left(H_t^T(\hat{x}_t, u_t, \theta^T, p_{t+1}^T, \mu_t^T) - \left\langle p_{t+1}^T, \hat{x}_{t+1}\right\rangle\right).\end{aligned}$$

From (5.13), we have

$$\sum_{t=0}^{T} H_t^T(\hat{x}_t, \hat{u}_t, \theta^T, p_{t+1}^T \mu_t^T) = \max_{u^T \in U(T)} \left(\sum_{t=0}^{T} H_t^T(\hat{x}_t, u_t, \theta^T, p_{t+1}^T \mu_t^T)\right).$$

For this equality, for each $t \in \{0, \ldots, T\}$, we consecutively choose $u^T = (\hat{u}_0, \ldots, \hat{u}_{t-1}, u_t, \hat{u}_{t+1}, \ldots, \hat{u}_T)$, i.e. only at $t$-th position, $u_t \neq \hat{u}_t$, and we obtain $H_t^T(\hat{x}_t, \hat{u}_t, \theta^T, p_{t+1}^T, \mu_t^T) = \max_{u_t \in \mathbb{R}^d} H_t^T(\hat{x}_t, u_t, \theta^T, p_{t+1}^T, \mu_t^T)$. So we have proven (c).

Finally, from (5.14), we imply that $\left\langle p_{t+1}^T, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\right\rangle = 0$ and $\mu_t^k.g_t^k(\hat{x}_t, \hat{u}_t) = 0$ for all $t \in \{0, \ldots, T\}$ and for all $k \in \{1, \ldots, m\}$ which gives (a). Thus, the theorem is proven. □

### (De) case

Now we state a strong Pontryagin principle for the multiobjective optimal control problems with (De) in finite horizon as follows

**Theorem 5.20.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_e^T)$ where $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. Assume*

*that for all $t \in \{0, \dots, T\}$, the control sets are defined as in (5.10). We also assume that conditions (i,ii) in Theorem 5.16 hold for $\phi_t^j$ and $g_t^k$ and moreover, assume that the following conditions holds:*

**(iii')** *For all $t \in \{0, \dots, T\}$, for each $x_t$ in a neighborhood $V_t$ of $\hat{x}_t$ such that $x_0 = \eta$, the following convexity condition is satisfied: if $u' \in W_t(x_t)$, $u'' \in W_t(x_t)$, $0 \le \alpha \le 1$, one can find $u \in W_t(x_t)$ such that*

$$\begin{aligned} \forall j \in \{1, \dots, \ell\}, \quad & \phi_t^j(x_t, u) \ge \alpha \phi_t^j(x_t, u') + (1-\alpha)\phi_t^j(x_t, u''), \\ & f_t(x_t, u) = \alpha f_t(x_t, u') + (1-\alpha) f_t(x_t, u''), \\ \forall k \in \{1, \dots, m\}, \quad & g_t^k(x_t, u) \ge \alpha g_t^k(x_t, u') + (1-\alpha) g_t^k(x_t, u''). \end{aligned}$$

**(iv)** *For all $t \in \{1, \dots, T\}$, for all $u_t \in U_t$, $f_t(., u_t)$ is continuously differentiable at $\hat{x}_t$.*

*Then for all $T \in \mathbb{N}_*$, there exist $\theta^T = (\theta_1^T, \dots, \theta_\ell^T) \in \mathbb{R}_+^\ell$ and $p^T = (p_1^T, \dots, p_{T+1}^T) \in (\mathbb{R}^{n*})^{T+1}$, not all zero which satisfy the conclusions (b,c) in Theorem 5.16 and the following one*

**(a')** *For all $t \in \{0, \dots, T\}$, for all $k \in \{1, \dots, m\}$, $\mu_t^k . g_t^k(\hat{x}_t, \hat{u}_t) = 0$.*

Before proving this theorem, we introduce the following lemma.

**Lemma 5.21.** *Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \dots, \hat{x}_{T+1}, \hat{u}_0, \dots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_e^T)$ where $\eta$ and $\hat{x}_{T+1}$ are fixed vectors in $\mathbb{R}^n$. Assume that condition (iv) in Theorem 5.20 is fulfilled. Then, for each $u^T \in U(T)$, $F^T(., u^T)$ is continuously differentiable at $\hat{x}^T$.*

*Proof.* Let $T \in \mathbb{N}_*$ and $u^T \in U(T)$. We know that $F^T(., u^T) = (F_0^T(., u^T), ..., , F_T^T(., u^T))$. And so, to prove that $F^T(., u^T)$ is of class $C^1$ at $\hat{x}^T$, it is equivalent to prove that for all $t = 0, ..., T$, $F_t^T(., u^T)$ is of class $C^1$ at $\hat{x}^T$. Following the definition of $F^T(., u^T)$ in Problem (5.11), we have that for all $t = 1, ..., T-1$, $F_t^T(., u^T) = [x^T \mapsto (x_t, x_{t+1}) \mapsto f_t(x_t, u_t) - x_{t+1}]$. Thus, for all $t = 1, ..., T-1$, $F_t^T(., u^T)$ is continuously differentiable at $\hat{x}^T$ since it is a composition of $C^1$ mappings under hypothesis (iv) of Theorem 4.9. A similar argument is used for $F_0^T(., u^T) = [x^T \mapsto x_1 \mapsto f_0(\eta, u_0) - x_1]$ and $F_T^T(., u^T) = [x^T \mapsto x_T \mapsto f_T(x_T, u_T) - x_{T+1}]$. Thus, $F^T(., u^T)$ is continuously differentiable at $\hat{x}^T$.

□

Now we prove the theorem.

*Proof.* Let $T \in \mathbb{N}_*$ be given and $(\hat{x}_0, \dots, \hat{x}_{T+1}, \hat{u}_0, \dots, \hat{u}_T)$ be a weak Pareto optimal solution of Problem $(FM_e^T)$, then $(\hat{x}^T, \hat{u}^T) = (\hat{x}_1, \dots, \hat{x}_T, \hat{u}_0, \dots, \hat{u}_T)$ is a weak Pareto optimal solution of Problem (5.11) with $k = e$. Now, under the assumptions of this theorem, all the conditions of Theorem 5.15 are fulfilled. Therefore, we can apply it to obtain $\theta^T = (\theta_1^T, \dots, \theta_\ell^T) \in \mathbb{R}_+^\ell$, $p^T = (p_1^T, \dots, p_{T+1}^T) \in (\mathbb{R}^{n*})^{T+1}$ and $\mu^T = (\mu_0^T, \dots, \mu_T^T) \in (\mathbb{R}_+^{m*})^{T+1}$ not all zero, such that conclusions (1,2) in the proof of Theorem 5.16 and the following one are fulfilled

$$\left\langle \mu^T, G^T(\hat{x}^T, \hat{u}^T) \right\rangle = 0. \tag{5.15}$$

The rest of the proof goes like in proof of Theorem 5.16 with only a small change that now, $\vec{D}_1 f_t(\hat{x}_t, \hat{u}_t; h_t)$ is replaced by $D_1 f_t(\hat{x}_t, \hat{u}_t) \cdot h_t$. □

## 5.6 New Pontryagin Principles for Multiobjective Optimal Control Problems in Infinite-Horizon Setting

### 5.6.1 Weak Pontryagin Principles

In this section, we give weak Pontryagin principles for the considered problems with an infinite-horizon setting. The difficulty is in the extraction of subsequences of multipliers

having non-zero limit. We will use some particular assumptions like an invertibility assumption or a positivity assumption used by Blot for single-objective optimal control to overcome this difficulty. The control sets are considered in some specific cases: the ones with both inequality and equality constraints, the special case when control sets are independent with state variables and the case of interior optimal controls.

### Control sets with both inequality and equality constraints

Consider Problem $(PM_k^j)$ where $k \in \{i, e\}$ and where the control sets are defined as in (5.7), i.e. the control sets are defined by inequalities and equalities and for each $t \in \mathbb{N}$, $U_t$ depends on $x_t$. We will establish weak Pontryagin principles for such problems. Notice that in the next theorems and corollaries, we will use the notations *span* for linear span and *conv* for convex hull of a set of vectors, which were introduced in previous chapter.

**Theorem 5.22.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a Pareto optimal solution of Problem $(PM_i^1)$ (respectively, solution of $(PM_i^2)$, $(PM_i^3)$) where the control sets are defined in (5.7). For all $t \in \mathbb{N}$, we assume that the following conditions are fulfilled:*

**(i)** *For all $j \in \{1, \dots, \ell\}$, $\phi_t^j$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

**(ii)** *For all $\alpha \in \{1, \dots, n\}$, $f_t^\alpha$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$ when $f_t^\alpha(\hat{x}_t, \hat{u}_t) = \hat{x}_{t+1}^\alpha$.*

**(iii)** *For all $\alpha \in \{1, \dots, n\}$, $f_t^\alpha$ is lower semicontinuous and Gâteaux differentiable at $(\hat{x}_t, \hat{u}_t)$ when $f_t^\alpha(\hat{x}_t, \hat{u}_t) > \hat{x}_{t+1}^\alpha$.*

**(iv)** *For all $k \in \{1, \dots, m_i\}$, $g_t^k$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$ when $g_t^k(\hat{x}_t, \hat{u}_t) = 0$.*

**(v)** *For all $k \in \{1, \dots, m_i\}$, $g_t^k$ is lower semicontinuous and Gâteaux differentiable at $(\hat{x}_t, \hat{u}_t)$ when $g_t^k(\hat{x}_t, \hat{u}_t) > 0$.*

**(vi)** *For all $i \in \{1, \dots, m_e\}$, $h_t^i$ is continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

*We set $D_{G,\alpha}\hat{g}_t^k := D_{G,\alpha}g_t^k(\hat{x}_t, \hat{u}_t)$, $D_{G,\alpha}\hat{f}_t := D_{G,\alpha}f_t(\hat{x}_t, \hat{u}_t)$, $D_\alpha\hat{h}_t^i := D_\alpha h_t^i(\hat{x}_t, \hat{u}_t)$ for each $\alpha \in \{1, 2\}$. Assume that the following conditions are fulfilled for all $t \in \mathbb{N}_*$:*

**(vii)** *$D_{G,1}\hat{f}_t$ is invertible.*

**(viii)** *Let $v_t^k := D_{G,1}\hat{g}_t^k \circ D_{G,1}\hat{f}_t^{-1} \circ D_{G,2}\hat{f}_t - D_{G,2}\hat{g}_t^k$ for all $k \in \{1, \dots, m_i\}$ and $w_t^i := D_1\hat{h}_t^k \circ D_{G,1}\hat{f}_t^{-1} \circ D_{G,2}\hat{f}_t - D_2\hat{h}_t^k$ for all $i \in \{1, \dots, m_e\}$. Then $span\{w_t^i : i \in \{1, \dots, m_i\}\} \cap conv\{v_t^k : k \in I_t^s\} = \emptyset$, where $I_t^s := \{k \in \{1, \dots, m_i\} : v_t^k = 0\}$.*

**(ix)** *The family $(w_t^i)_{1 \le i \le m_e}$ defined in (viii) is linearly independent.*

*Then there exist $\theta_1, \dots, \theta_\ell \in \mathbb{R}$, $(p_t)_{t \in \mathbb{N}_*} \in (R^{n*})^{\mathbb{N}_*}$, $(\mu_{1,t})_{t \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots, (\mu_{m_e,t})_{t \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $(\lambda_{1,t})_{t \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots, (\lambda_{m_i,t})_{t \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ which satisfy the following conditions.*

**(a)** $(\theta_1, \dots, \theta_\ell, p_1, \lambda_0^1, \dots \lambda_0^{m_i}, \mu_0^1, \dots, \mu_0^{m_e}) \neq (0, \dots, 0)$.

**(b)** *$\theta_j \ge 0$ for all $j \in \{1, \dots, \ell\}$, $p_t \ge 0$ for all $t \in \mathbb{N}_*$, $\lambda_{k,t} \ge 0$ for all $t \in \mathbb{N}$ and for all $k \in \{1, \dots, m_i\}$.*

**(c)** *For all $t \in \mathbb{N}$, for all $\alpha \in \{1, \dots, n\}$, $p_{t+1}^\alpha \cdot (f_t^\alpha(\hat{x}_t, \hat{u}_t) - x_{t+1}^\alpha) = 0$, and for all $k \in \{1, \dots, m_i\}$, $\lambda_{k,t} \cdot g_t^k(\hat{x}_t, \hat{u}_t) = 0$.*

**(d)** *For all $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_{G,1}\hat{f}_t + \sum_{j=1}^\ell \theta_j D_1\hat{\phi}_t^j + \sum_{k=1}^{m_i} \lambda_{k,t} D_{G,1}\hat{g}_t^k + \sum_{i=1}^{m_e} \mu_{i,t} D_1\hat{h}_t^i$, where $D_1\hat{\phi}_t^j := D_1\phi_t^j(\hat{x}_t, \hat{u}_t)$.*

**(e)** *For all* $t \in \mathbb{N}$,

$$p_{t+1} \circ D_{G,2}\hat{f}_t + \sum_{j=1}^{\ell} \theta_j D_2\hat{\phi}_t^j + \sum_{k=1}^{m_i} \lambda_{k,t} D_{G,2}\hat{g}_t^k + \sum_{i=1}^{m_e} \mu_{i,t} D_2\hat{h}_t^i = 0, \text{ where } D_2\hat{\phi}_t^j := D_2\phi_t^j(\hat{x}_t, \hat{u}_t).$$

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a Pareto optimal solution of Problem $(PM_i^j)$ when $j \in \{1,2,3\}$. Using Theorem 5.3, the restriction $((\hat{x}_0, \ldots, \hat{x}_{T+1}), (\hat{u}_0, \ldots, \hat{u}_T))$ is a Pareto optimal solution of the finite-horizon problem $(FM_i^T)$. Our assumptions (i-vi) imply that the assumptions of Theorem 5.8 in Section 5.4 are fulfilled and so we know that, for all $T \in \mathbb{N}_*$, there exist $\theta_1^T, \ldots, \theta_\ell^T \in \mathbb{R}$, $\lambda_{i,t}^T \in \mathbb{R}$, $\mu_{k,t}^T \in \mathbb{R}$ and $p_{t+1}^T \in \mathbb{R}^{n*}$ where $t \in \{0, \ldots, T\}$, $k \in \{1, \ldots, m_i\}$ and $i \in \{1, \ldots, m_e\}$ which satisfy the following conditions.

$$\text{All the multipliers are not simultaneously equal to zero.} \tag{5.16}$$

$$\forall j \in \{1, \ldots, \ell\},\ \theta_j^T \geq 0,\ \forall t \in \{0, \ldots, T\},\ \forall i \in \{1, \ldots, m_i\},\ p_{t+1}^T \geq 0 \text{ and } \lambda_{i,t}^T \geq 0. \tag{5.17}$$

$$\forall t \in \{0, \ldots, T\},\ \forall k \in \{1, \ldots, m_i\},\ \left\langle p_{t+1}^T,\ f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \right\rangle = 0,\ \lambda_{i,t}^T g_t^k(\hat{x}_t, \hat{u}_t) = 0. \tag{5.18}$$

$$\forall t \in \{1, \ldots, T\},\ p_t^T = \sum_{j=1}^{\ell} \theta_j^T . D_1\hat{\phi}_t^j + p_{t+1}^T \circ D_{G,1}\hat{f}_t + \sum_{k=1}^{m_i} \lambda_{k,t}^T D_{G,1}\hat{g}_t^k + \sum_{i=1}^{m_e} \mu_{i,t}^T D_1\hat{h}_t^i. \tag{5.19}$$

$$\forall t \in \{0, \ldots, T\},\ 0 = \sum_{j=1}^{\ell} \theta_j^T . D_2\hat{\phi}_t^j + p_{t+1}^T \circ D_{G,2}\hat{f}_t + \sum_{k=1}^{m_i} \lambda_{k,t}^T D_{G,2}\hat{g}_t^k + \sum_{i=1}^{m_e} \mu_{i,t}^T D_2\hat{h}_t^i. \tag{5.20}$$

We will prove that $(\theta_1^T, \ldots, \theta_\ell^T, p_1^T, \lambda_0^{1,T}, \ldots \lambda_0^{m_i,T}, \mu_0^{1,T}, \ldots, \mu_0^{m_e,T}) \neq (0, \ldots, 0)$ by contradiction. Notice that here, for the multipliers associated with the constraints, only the 0-indexed ones appear. Assume that $(\theta_1^T, \ldots, \theta_\ell^T, p_1^T, \lambda_0^{1,T}, \ldots \lambda_0^{m_i,T}, \mu_0^{1,T}, \ldots, \mu_0^{m_e,T}) = (0, \ldots, 0)$. When $t = 1$, using assumption (vii), we can formulate (5.19) as follows

$$\left.\begin{aligned} -p_2^T &= \left(-p_1^T + \textstyle\sum_{j=1}^{\ell} \theta_j^T . D_1\hat{\phi}_1^j + \sum_{k=1}^{m_i} \lambda_{k,1}^T D_{G,1}\hat{g}_1^k + \sum_{i=1}^{m_e} \mu_{i,1}^T D_1\hat{h}_1^i\right) \circ D_{G,1}\hat{f}_1{}^{-1} \\ &= \left(\textstyle\sum_{k=1}^{m_i} \lambda_{k,1}^T D_{G,1}\hat{g}_1^k + \sum_{i=1}^{m_e} \mu_{i,1}^T D_1\hat{h}_1^i\right) \circ D_{G,1}\hat{f}_1{}^{-1}, \end{aligned}\right\} \tag{5.21}$$

and using (5.20), we have

$$\left.\begin{aligned} -p_2^T \circ D_{G,2}\hat{f}_1 &= \textstyle\sum_{j=1}^{\ell} \theta_j^T . D_2\hat{\phi}_1^j + \sum_{k=1}^{m_i} \lambda_{k,1}^T D_{G,2}\hat{g}_1^k + \sum_{i=1}^{m_e} \mu_{i,1}^T D_2\hat{h}_1^i \\ &= \textstyle\sum_{k=1}^{m_i} \lambda_{k,1}^T D_{G,2}\hat{g}_1^k + \sum_{i=1}^{m_e} \mu_{i,1}^T D_2\hat{h}_1^i. \end{aligned}\right\} \tag{5.22}$$

Using (5.21) and (5.22), we have
$\left(\sum_{k=1}^{m_i} \lambda_{k,1}^T D_{G,1}\hat{g}_1^k + \sum_{i=1}^{m_e} \mu_{i,1}^T D_1\hat{h}_1^i\right) \circ D_{G,1}\hat{f}_1{}^{-1} \circ D_{G,2}\hat{f}_1 - \sum_{k=1}^{m_i} \lambda_{k,1}^T D_{G,2}\hat{g}_1^k - \sum_{i=1}^{m_e} \mu_{i,1}^T D_2\hat{h}_1^i = 0,$
which can be rewritten as follows

$$\left.\begin{aligned} &\textstyle\sum_{k=1}^{m_i} \lambda_{k,1}^T \left(D_{G,1}\hat{g}_1^k \circ D_{G,1}\hat{f}_1{}^{-1} \circ D_{G,2}\hat{f}_1 - D_{G,2}\hat{g}_1^k\right) \\ &+ \textstyle\sum_{i=1}^{m_e} \mu_{i,1}^T \left(D_1\hat{h}_1^i \circ D_{G,1}\hat{f}_1{}^{-1} \circ D_{G,2}\hat{f}_1 - D_2\hat{h}_1^i\right) = 0 \end{aligned}\right\}$$

$$\iff \sum_{k=1}^{m_i} \lambda_{k,1}^T v_1^k + \sum_{i=1}^{m_e} \mu_{i,1}^T w_1^i = 0.$$

Using assumption (viii) and (ix), after Lemma 4.23, we obtain that $\lambda_{k,1}^T = 0$ for all $k \in I_t^s$, and consequently we have $\lambda_{k,1}^T = 0$ for all $k \in \{1, \dots, m_i\}$. Then, using assumption (ix), we obtain $\mu_{i,1}^T = 0$ for all $i \in \{1, \dots, m_e\}$ and thus, using (5.21) again, we have $p_2^T = 0$. Repeat this procedure for each $t \in \{2, \dots, T\}$, we obtain $p_{t+1}^T = 0$, $\lambda_{k,t}^T = 0$ and $\mu_{i,t}^T = 0$ for all $t \in \{0 \dots, T\}$, $k \in \{1, \dots, m_i\}$ and $i \in \{1, \dots, m_e\}$. Thus, all the multipliers are zero which is a contradiction. And so, we have proven that $(\theta_1^T, \dots, \theta_\ell^T, p_1^T, \lambda_0^{1,T}, \dots \lambda_0^{m_i,T}, \mu_0^{1,T}, \dots, \mu_0^{m_e,T}) \neq (0, \dots, 0)$. Since the set of the lists of multipliers of Problem is a cone, we can normalize the multipliers by setting

$$\begin{aligned} &\left\|(\theta_1^T, \dots, \theta_\ell^T, p_1^T, \lambda_{1,0}^T, \dots \lambda_{m_i,0}^T, \mu_{1,0}^T, \dots, \mu_{m_e,0}^T)\right\| \\ &= \textstyle\sum_{j=1}^{\ell} \theta_j^T + \left\|p_1^T\right\| + \sum_{k=1}^{m_i} \lambda_{k,0}^T + \sum_{i=1}^{m_e} \left|\mu_{i,0}^T\right| = 1. \end{aligned} \tag{5.23}$$

Since the values of the sequence $(\theta_1^T, \dots, \theta_\ell^T, p_1^T, \lambda_{1,0}^T, \dots \lambda_{m_i,0}^T, \mu_{1,0}^T, \dots, \mu_{m_e,0}^T)_{T \in \mathbb{N}_*}$ belong to the unit sphere of $\mathbb{R}^\ell \times \mathbb{R}^{n*}$ which is compact, using the Bolzano-Weierstrass theorem we can say that there exist an increasing function $\varphi : \mathbb{N}_* \to \mathbb{N}_*$ and $(\theta_1, \dots, \theta_\ell, p_1, \lambda_{1,0}, \dots \lambda_{m_i,0}, \mu_{1,0}, \dots, \mu_{m_e,0}) \in \mathbb{R}^\ell \times \mathbb{R}^{n*} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_e}$ such that $\sum_{j=1}^{\ell} \theta_j + \|p_1\| + \sum_{k=1}^{m_i} \lambda_{k,0} + \sum_{i=1}^{m_e} |\mu_{i,0}| = 1$, $\lim_{T \to +\infty} \theta_j^{\varphi(T)} = \theta_j$ for all $j \in \{1, \dots, \ell\}$, $\lim_{T \to +\infty} p_1^{\varphi(T)} = p_1$, $\lim_{T \to +\infty} \lambda_{k,0}^{\varphi(T)} = \lambda_{k,0}$ for all $k \in \{1, \dots, m_i\}$ and $\lim_{T \to +\infty} \mu_{i,0}^{\varphi(T)} = \mu_{i,0}$ for all $i \in \{1, \dots, m_e\}$.

Now we will prove that for all $t \in \mathbb{N}_*$, the sequences $T \to p_{t+1}^T$, $T \to \lambda_{k,t}^T$ and $T \to \mu_{i,t}^T$ are bounded for all $k \in \{1, \dots, m_i\}$ and for all $i \in \{1, \dots, m_e\}$. After (5.23), it is clear that the sequences $(\theta_j^T)_T$, $(p_1^T)_T$, $(\lambda_{k,0}^T)_T$ and $(\mu_{i,0}^T)_T$ are bounded for all $j \in \{1, \dots, \ell\}, k \in \{1, \dots, m_i\}$ and $i \in \{1, \dots, m_e\}$. When $t = 1$, using (5.21) and (5.22), we have

$$\begin{aligned} &(-p_1^T + \textstyle\sum_{j=1}^{\ell} \theta_j^T D_1 \hat{\phi}_1^j) \circ D_{G,1} \hat{f}_1{}^{-1} \circ D_{G,2} \hat{f}_1 - \sum_{j=1}^{\ell} \theta_j^T D_2 \hat{\phi}_1^j \\ &+ \textstyle\sum_{k=1}^{m_i} \lambda_{k,1}^T \left( D_{G,1} \hat{g}_1^k \circ D_{G,1} \hat{f}_1{}^{-1} \circ D_{G,2} \hat{f}_1 - D_{G,2} \hat{g}_1^k \right) \\ &+ \textstyle\sum_{i=1}^{m_e} \mu_{i,1}^T \left( D_1 \hat{h}_1^i \circ D_{G,1} \hat{f}_1{}^{-1} \circ D_{G,2} \hat{f}_1 - D_2 \hat{h}_1^i \right) = 0. \end{aligned}$$

Since all the elements in the first line of the last equation are bounded, we can assert that

$$\begin{aligned} &\textstyle\sum_{k=1}^{m_i} \lambda_{k,1}^T \left( D_{G,1} \hat{g}_1^k \circ D_{G,1} \hat{f}_1{}^{-1} \circ D_{G,2} \hat{f}_1 - D_{G,2} \hat{g}_1^k \right) \\ &+ \textstyle\sum_{i=1}^{m_e} \mu_{i,1}^T \left( D_1 \hat{h}_1^i \circ D_{G,1} \hat{f}_1{}^{-1} \circ D_{G,2} \hat{f}_1 - D_2 \hat{h}_1^i \right) \end{aligned}$$

is also bounded. Using Lemma 4.24, we obtain $(\lambda_{k,1}^T)_T$ is bounded in $\mathbb{R}_+$ for all $k \in \{1, \dots, m_i\}$ and $(\mu_{i,1}^T)_T$ is bounded in $\mathbb{R}$ for all $i \in \{1, \dots, m_e\}$. Using (5.20) with $t = 1$, we obtain that $(p_2^T)_T$ is bounded in $\mathbb{R}^{n*}$. Repeat this procedure, for each $t = 2, 3, \dots,$ by induction, we obtain that for all $t \in \mathbb{N}_*$, $(\lambda_{k,t}^T)_{T \geq t}$ is bounded in $\mathbb{R}_+$ for all $k \in \{1, \dots, m_i\}$, $(\mu_{i,t}^T)_{T \geq t}$ is bounded in $\mathbb{R}$ for all $i \in \{1, \dots, m_e\}$ and $(p_{t+1}^T)_{T \geq t+1}$ is bounded in $\mathbb{R}^{n*}$. Then using diagonal process of Cantor which is formulated in [15] (Theorem A.1, p.94), we know that there exist an increasing function $\delta : \mathbb{N}_* \to \mathbb{N}_*$ and sequences $(p_{t+1})_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}, (\lambda_{1,t})_{t \in \mathbb{N}_*}, \dots (\lambda_{m_i,t})_{t \in \mathbb{N}_*} \in \mathbb{R}_+^{\mathbb{N}_*}, (\mu_{1,t})_{t \in \mathbb{N}_*}, \dots, (\mu_{m_e,t})_{t \in \mathbb{N}_*} \in \mathbb{R}^{\mathbb{N}*}$ such that for all $t \in \mathbb{N}_*$, $\lim_{T \to +\infty} p_{t+1}^{\delta(T)} = p_{t+1}$, $\lim_{T \to +\infty} \lambda_{k,t}^{\delta(T)} = \lambda_{k,t}$ for all $k \in \{1, \dots, m_i\}$ and $\lim_{T \to +\infty} \mu_{i,t}^{\delta(T)} = \mu_{i,t}$ for all $i \in \{1, \dots, m_e\}$.

And so we have built $(\theta_1, \dots, \theta_\ell) \in \mathbb{R}^\ell$, sequences $(p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, $(\lambda_{k,t})_{t \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ for all $k \in \{1, \dots, m_i\}$ and $(\mu_{i,t})_{t \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ for all $i \in \{1, \dots, m_e\}$ such that when $T \to +\infty$, from (5.23), (5.17), (5.18), (5.19) and (5.20) we obtain conclusions (a), (b), (c), (d) and (e), respectively (notice that for all the multipliers in these relations, we replace their upper index $T$ by $\varphi \circ \delta(T)$). $\square$

By a similar realization, we can propose a weak Pontryagin principle for Problem $(PM_e^j)$ when $j \in \{1,2,3\}$. The differences in assumptions and conclusions compared to the previous theorem are showed in the following theorem.

**Theorem 5.23.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a Pareto optimal solution of Problem $(PM_e^1)$ (respectively, solution of $(PM_e^2)$, $(PM_e^3)$) where the control sets are defined in (5.7). We assume that assumptions (i,iv-ix) of Theorem 5.22 are satisfied for all $t \in \mathbb{N}$ together with the following one.*

**(ii')** *$f_t$ is continuous on a neighborhood of $(\hat{x}_t, \hat{u}_t)$ and Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

*Then there exist $\theta_1, \ldots, \theta_\ell \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (R^{n*})^{\mathbb{N}_*}$, $(\mu_{t,1})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$,..., $(\mu_{t,m_e})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $(\lambda_{t,1})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$,..., $(\lambda_{t,m_i})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ which satisfy conclusions (a,d,e) of Theorem 5.22 and the following conditions*

**(b)** *$\theta_j \geq 0$ for all $j \in \{1, \ldots, \ell\}$, $\lambda_{k,t} \geq 0$ for all $t \in \mathbb{N}$ and for all $k \in \{1, \ldots, m_i\}$.*

**(c)** *For all $t \in \mathbb{N}$, for all $k \in \{1, \ldots, m_i\}$, $\lambda_{k,t} \cdot g_t^k(\hat{x}_t, \hat{u}_t) = 0$.*

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a Pareto optimal solution of Problem $(PM_e^j)$ when $j \in \{1,2,3\}$. Using Theorem 5.3, the restriction $((\hat{x}_0, \ldots, \hat{x}_{T+1}), (\hat{u}_0, \ldots, \hat{u}_T))$ is a Pareto optimal solution of Problem $(FM_e^T)$. Our assumptions imply that the assumptions of Theorem 5.10 in Section 4 are fulfilled and so we know that, for all $T \in \mathbb{N}_*$, there exist $\theta_1^T, \ldots, \theta_\ell^T \in \mathbb{R}$, $\lambda_{i,t}^T \in \mathbb{R}$, $\mu_{k,t}^T \in \mathbb{R}$ and $p_{t+1}^T \in \mathbb{R}^{n*}$ where $t \in \{0, \ldots, T\}$, $k \in \{1, \ldots, m_i\}$ and $i \in \{1, \ldots, m_e\}$ which satisfy conditions (5.16,5.19,5.20) and the following ones:

$$\forall j \in \{1, \ldots, \ell\},\ \theta_j^T \geq 0,\ \forall t \in \{0, \ldots, T\},\ \forall i \in \{1, \ldots, m_i\},\ \lambda_{i,t}^T \geq 0. \tag{5.24}$$

$$\forall t \in \{0, \ldots, T\},\ \forall k \in \{1, \ldots, m_i\},\ \lambda_{i,t}^T g_t^k(\hat{x}_t, \hat{u}_t) = 0. \tag{5.25}$$

The rest of this proof goes completely like that of the previous one in which relations (5.17) and (5.18) are replaced by relations (5.24) and (5.25), respectively. □

### Control sets that are independent of state variables

In this subsection, we will provide weak Pontryagin principles for the problems in which the control sets are independent of the state variable. Although these results are the corollaries of the theorems in Section 5.6.1, we give here simpler proofs for some of them thanks to the similar results for single-objective optimal control problems in infinite horizon setting in [17].

Firstly, we consider the case when both inequality and equality constraints appear in the problem as follows.

$$U_t = \bigcap_{1\leq k\leq m_i} \{u \in \mathbb{R}^d : g_t^k(u) \geq 0\} \cap (\bigcap_{1\leq k\leq m_e} \{u \in \mathbb{R}^d : h_t^k(u) = 0\}) \tag{5.26}$$

Then we have the following corollaries for multiobjective optimal control problems with (Di).

**Corollary 5.24.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a Pareto optimal solution of Problem $(PM_i^1)$ (respectively, solution of $(PM_i^2)$, $(PM_i^3)$) where the sets $U_t$ are defined in (5.26). For all $t \in \mathbb{N}$, we assume that the following assumptions are fulfilled*

**(i)** *For all $j \in \{1, \ldots, \ell\}$, $\phi_t^j$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$.*

**(ii)** *For all $\alpha \in \{1, \ldots, n\}$, $f_t^\alpha$ is Fréchet differentiable at $(\hat{x}_t, \hat{u}_t)$ when $f_t^\alpha(\hat{x}_t, \hat{u}_t) = \hat{x}_{t+1}^\alpha$.*

**(iii)** *For all* $\alpha \in \{1,\dots,n\}$, $f_t^\alpha$ *is lower semicontinuous and Gâteaux differentiable at* $(\hat{x}_t,\hat{u}_t)$ *when* $f_t^\alpha(\hat{x}_t,\hat{u}_t) > \hat{x}_{t+1}^\alpha$.

**(iv)** *For all* $k \in \{1,\dots,m_i\}$, $g_t^k$ *is Fréchet differentiable at* $\hat{u}_t$ *when* $g_t^k(\hat{u}_t) = 0$.

**(v)** *For all* $k \in \{1,\dots,m_i\}$, $g_t^k$ *is lower semicontinuous and Gâteaux differentiable at* $\hat{u}_t$ *when* $g_t^k(\hat{u}_t) > 0$.

**(vi)** *For all* $i \in \{1,\dots,m_e\}$, $h_t^i$ *is continuous on a neighborhood of* $\hat{u}_t$ *and Fréchet differentiable at* $\hat{u}_t$.

**(vii)** $span\{Dh_t^i(\hat{u}_t) : i \in \{1,\dots,m_e\}\} \cap conv\{D_G g_t^k(\hat{u}_t) : k \in I_t^s\} = \emptyset$, *where* $I_t^s := \{k \in \{1,\dots,m_i\} : g_t^k(\hat{u}_t) = 0\}$.

**(viii)** $Dh_t^1(\hat{u}_t),\dots,Dh^{m_e}(\hat{u}_t)$ *are linearly independent.*

**(ix)** *For all* $t \in \mathbb{N}_*$, $D_{G,1}f_t(\hat{x}_t,\hat{u}_t)$ *is invertible.*

*Then there exist* $\theta_1,\dots,\theta_\ell \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (R^{n*})^{\mathbb{N}_*}$, $(\mu_{1,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}},\dots,(\mu_{m_e,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $(\lambda_{1,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}},\dots,(\lambda_{m_i,t})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ *which satisfy the following conditions.*

**(a)** $(\theta_1,\dots,\theta_\ell,p_1) \neq (0,\dots,0,0)$.

**(b)** $\theta_j \geq 0$ *for all* $j \in \{1,\dots,\ell\}$,, $p_t \geq 0$ *for all* $t \in \mathbb{N}_*$, $\lambda_{k,t} \geq 0$ *for all* $t \in \mathbb{N}$ *and for all* $k \in \{1,\dots,m_i\}$.

**(c)** *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1,\dots,n\}$, $p_{t+1}^\alpha \cdot (f_t^\alpha(\hat{x}_t,\hat{u}_t) - x_{t+1}^\alpha) = 0$, *and for all* $k \in \{1,\dots,m_i\}$, $\lambda_{k,t} \cdot g_t^k(\hat{u}_t) = 0$.

**(d)** *For all* $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_{G,1}f_t(\hat{x}_t,\hat{u}_t) + \sum_{j=1}^\ell \theta_j D_1\phi_t^j(\hat{x}_t,\hat{u}_t)$.

**(e)** *For all* $t \in \mathbb{N}$,

$$p_{t+1} \circ D_{G,2}f_t(\hat{x}_t,\hat{u}_t) + \sum_{j=1}^\ell \theta_j D_2\phi_t^j(\hat{x}_t,\hat{u}_t) + \sum_{k=1}^{m_i} \lambda_{k,t} D_G g_t^k(\hat{u}_t) + \sum_{i=1}^{m_e} \mu_{i,t} Dh^i(\hat{u}_t) = 0.$$

**Corollary 5.25.** *In the setting of Corollary 5.24, assume that conditions (i-viii) of Corollary 5.24 and the following one are fulfilled*

**(ix')** *For all* $t \in \mathbb{N}_*$, *for all* $\alpha,\beta \in \{1,\dots,n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x^\beta} \geq 0$ *and for all* $\alpha \in \{1,\dots,n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x^\alpha} > 0$.

*Then all the conclusions of Corollary 5.24 hold.*

We will prove Corollaries 5.24 and 5.25 simultaneously as follows.

*Proof.* Proceeding as in the proof of Theorem 4.25, with a notice that now there are $\ell$ multipliers for multiobjective criterion instead of one multiplier in that theorem, we obtain the proof for these corollaries. □

Now, we state a weak maximum principle for multiobjective optimal control problems with (De) and with the control sets defined by both inequality and equality constraints.

**Corollary 5.26.** *Let* $(\underline{\hat{x}},\underline{\hat{u}})$ *be a Pareto optimal solution of Problem* $(PM_e^1)$ *(respectively, solution of* $(PM_e^2)$, $(PM_e^3)$*) where the sets* $U_t$ *are defined in (5.26). We assume that the following assumptions are fulfilled for all* $t \in \mathbb{N}$.

**(i)** *For all* $j \in \{1,\dots,\ell\}$, $\phi_t^j$ *is Fréchet differentiable at* $(\hat{x}_t,\hat{u}_t)$.

**(ii)** $f_t$ *is continuous on a neighborhood of* $(\hat{x}_t,\hat{u}_t)$ *and Fréchet differentiable at* $(\hat{x}_t,\hat{u}_t)$.

**(iii)** *For all* $k \in \{1,\dots,m_i\}$, $g_t^k$ *is Fréchet differentiable at* $\hat{u}_t$ *when* $g_t^k(\hat{u}_t) = 0$.

**(iv)** *For all* $k \in \{1,\dots,m_i\}$, $g_t^k$ *is lower semicontinuous and Gâteaux differentiable at* $\hat{u}_t$ *when* $g_t^k(\hat{u}_t) > 0$.

**(v)** *For all* $i \in \{1, \dots, m_e\}$, $h_t^i$ *is continuous on a neighborhood of* $\hat{u}_t$ *and Fréchet differentiable at* $\hat{u}_t$.

**(vi)** $span\{Dh_t^i(\hat{u}_t) : i \in \{1, \dots, m_e\}\} \cap conv\{D_G g_t^k(\hat{u}_t) : k \in I_t^s\} = \emptyset$, *where* $I_t^s := \{k \in \{1, \dots, m_i\} : g_t^k(\hat{u}_t) = 0\}$.

**(vii)** $Dh_t^1(\hat{u}_t), \dots, Dh_t^{m_e}(\hat{u}_t)$ *are linearly independent.*

**(viii)** *For all* $t \in \mathbb{N}_*$, $D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$ *is invertible.*

*Then there exist* $\theta_1, \dots, \theta_\ell \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (R^{n*})^{\mathbb{N}_*}$, $(\mu_{t,1})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots, (\mu_{t,m_e})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $(\lambda_{t,1})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots, (\lambda_{t,m_i})_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ *which satisfy conclusions (a,d,e) of Theorem 5.24 and the following conditions*

**(b)** $\theta_j \geq 0$ *for all* $j \in \{1, \dots, \ell\}$, $\lambda_{k,t} \geq 0$ *for all* $t \in \mathbb{N}$ *and for all* $k \in \{1, \dots, m_i\}$.

**(c)** *For all* $t \in \mathbb{N}$, *for all* $k \in \{1, \dots, m_i\}$, $\lambda_{k,t} \cdot g_t^k(\hat{u}_t) = 0$.

The difference is the replacement of inequality constraints by equality constraints in the problem issued from the reduction to finite horizon, the consequence of this difference is the lost of the sign of the adjoint variable $p_t$.

**Remark 5.27.** In Theorems 5.22, if control sets are independent of state variables then its assumptions (viii,ix) are reduced to assumptions (vii,viii) of Corollary 5.24.

Finally, we consider the case when the control sets are described for each $t \in \mathbb{N}$ as follows

$$U_t = \bigcap_{1\leq k\leq m_i} \{u \in \mathbb{R}^d : g_t^k(u) \geq 0\}) \tag{5.27}$$

Then, for multiobjective optimal control problems with (Di), we have the following corollaries in which, the assumptions are only related to Gâteaux differentiability and lower semicontinuity.

**Corollary 5.28.** *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a Pareto optimal solution of Problem* $(PM_i^1)$ *(respectively, solution of* $(PM_i^2)$, $(PM_i^3)$*) where the sets* $U_t$ *are defined by (5.27). We assume that the following assumptions are fulfilled*

**(i)** *For all* $t \in \mathbb{N}$, *for all* $j \in \{1, \dots, \ell\}$, $\phi_t^j$ *and* $f_t$ *are Gâteaux differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(ii)** *For all* $t \in \mathbb{N}$, *for all* $k \in \{1, \dots, m_i\}$, $g_t^k$ *is Gâteaux differentiable at* $\hat{u}_t$.

**(iii)** *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \dots, n\}$, $f_t^\alpha$ *is lower semicontinuous at* $(\hat{x}_t, \hat{u}_t)$ *when* $f_t^\alpha(\hat{x}_t, \hat{u}_t) > \hat{x}_{t+1}^\alpha$.

**(iv)** *For all* $t \in \mathbb{N}$, *for all* $k \in \{1, \dots, m_i\}$, $g_t^k$ *is lower semicontinuous at* $\hat{u}_t$ *when* $g_t^k(\hat{u}_t) > 0$.

**(v)** *For all* $t \in \mathbb{N}$, $0 \notin conv\{D_G g_t^k(\hat{u}_t) : k \in I_t^s\}$ *where* $I_t^s := \{k \in \{1, \dots, m_i\} : g_t^k(\hat{u}_t) = 0\}$.

**(vi)** *For all* $t \in \mathbb{N}_*$, $D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$ *is invertible.*

*Then, under (i-vi) there exist* $\theta_1, \dots, \theta_\ell \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, $(\lambda_t^1)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \dots,$ *and* $(\lambda_t^m)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ *which satisfy the following conditions.*

**(a)** $(\theta_1, \dots, \theta_\ell, p_1) \neq (0, 0)$.

**(b)** $\theta_j \geq 0$ *for all* $j \in \{1, \dots, \ell\}$ , $p_t \geq 0$ *for all* $t \in \mathbb{N}_*$, *and* $\lambda_t^k \geq 0$ *for all* $t \in \mathbb{N}$ *and for all* $k \in \{1, \dots, m_i\}$.

**(c)** *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \dots, n\}$, $p_{t+1}^\alpha \cdot (f_t^\alpha(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}^\alpha) = 0$, *and for all* $k \in \{1, \dots, m_i\}$, $\lambda_t^k \cdot g_t^k(\hat{u}_t) = 0$.

**(d)** *For all* $t \in \mathbb{N}_*$, $p_t = p_{t+1} \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) + \sum_{j=1}^\ell \theta_j . D_{G,1} \phi_t^j(\hat{x}_t, \hat{u}_t)$.

**(e)** *For all* $t \in \mathbb{N}$, $p_{t+1} \circ D_{G,2} f_t(\hat{x}_t, \hat{u}_t) + \sum_{j=1}^{\ell} \theta_j . D_{G,2} \phi_t^j(\hat{x}_t, \hat{u}_t) + \sum_{k=1}^{m_i} \lambda_t^k D_G g_t^k(\hat{u}) = 0$.

**Corollary 5.29.** *In the setting of Corollary 5.28, assume that assumptions (i-v) of Corollary 5.28 and the following assumption are fulfilled*

**(vi')** *For all* $t \in \mathbb{N}_*$, *for all* $\alpha, \beta \in \{1, \ldots, n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x^\beta} \geq 0$ *and for all* $\alpha \in \{1, \ldots, n\}$, $\frac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x^\alpha} > 0$.

*Then, all the conclusions of Corollary 5.28 hold.*

The proof for Corollary 5.28 can be realized as in Theorem 5.22. However, in this case, by using an existing result in Chapter 4, we can prove Corollaries 5.28 and 5.29 as follows

*Proof.* Proceeding as in the proof of Theorem 4.21, with the same notice as in previous corollaries' proofs, we obtain the proof for Corollaries 5.28 and 5.29. □

### The case of interior optimal controls

In this subsection, we consider the case where for every $t \in \mathbb{N}$, $U_t$ is an arbitrary subset of $\mathbb{R}^d$ and $u_t$ belongs to the interior of $U_t$ and where the system is governed by the difference inequation (Di). For such problems, we can lighten the assumption of smoothness and continuity of the objective functions $\phi_t^j$ and functions $f_t$. This will be shown in the following corollaries:

**Corollary 5.30.** *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a Pareto optimal solution of Problem* $(PM_i^1)$ *(respectively, solution of* $(PM_i^2)$, $(PM_i^3)$*). We also assume that the following assumptions are fulfilled:*

**(i)** *For all* $t \in \mathbb{N}$, $\hat{u}_t \in \mathrm{int} U_t$.

**(ii)** *For all* $j \in \{1, \ldots, \ell\}$, *for all* $t \in \mathbb{N}$, $\phi_t^j$ *and* $f_t$ *are Gâteaux differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(iii)** *For all* $t \in \mathbb{N}$, *for all* $\alpha \in \{1, \ldots, n\}$, $f_t^\alpha$ *is lower semicontinuous at* $(\hat{x}_t, \hat{u}_t)$ *when* $f_t^\alpha(\hat{x}_t, \hat{u}_t) > \hat{x}_{t+1}^\alpha$.

**(iv)** *For all* $t \in \mathbb{N}_*$, $D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$ *is invertible.*

*Then there exist* $\theta_1, \ldots, \theta_\ell \in \mathbb{R}_+$ *and* $p_{t+1} \in \mathbb{R}_+^{n*}$ *where* $t \in \mathbb{N}$ *which satisfy the following conclusions:*

**(a)** $(\theta_1, \ldots, \theta_\ell, p_1) \neq (0, \ldots, 0)$.

**(b)** *For all* $t \in \mathbb{N}$, $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1} \rangle = 0$.

**(c)** *For all* $t \in \mathbb{N}_*$, $p_t = \sum_{j=1}^{\ell} \left( \theta_j . D_{G,1} \phi_t^j(\hat{x}_t, \hat{u}_t) \right) + p_{t+1} \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$.

**(d)** *For all* $t \in \mathbb{N}$, $\sum_{j=1}^{\ell} (\theta_j . D_{G,2} \phi_t(\hat{x}_t, \hat{u}_t)) + p_{t+1} \circ D_{G,2} f_t(\hat{x}_t, \hat{u}_t) = 0$.

*Proof.* Although this is a corollary of Theorem 5.22, however, in this case we can realize a much simpler proof. By proceeding as in the proof of Theorem 4.14, with a notice that now there are $\ell$ multipliers for multiobjective criterion instead of one multiplier in that theorem, we obtain the proof for this corollary. □

**Corollary 5.31.** *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a Pareto optimal solution of Problem* $(PM_i^1)$ *(respectively, solution of* $(PM_i^2)$, $(PM_i^3)$*). We assume that the assumptions (i,ii,iii) of Corollary 5.30 are fulfilled. Moreover, we assume that the following assumption is fulfilled*

**(iv')** *For all* $t \in \mathbb{N}_*$, *for all* $\alpha, \beta \in \{1, \ldots, n\}$, $\dfrac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x_t^\beta} \geq 0$ *and* $\dfrac{\partial f_t^\alpha(\hat{x}_t, \hat{u}_t)}{\partial x_t^\alpha} > 0$.

*Then the conclusions of Corollary 5.30 hold.*

*Proof.* The proof is obtained by proceeding as in the proof of Theorem 4.16, with the same notice as in previous corollary's proof. □

**Remark 5.32.** All the results in Section 5.1 can be considered as generalizations of Theorem 3.1, Theorem 3.2 and Theorem 3.3 in [34] if we replace weak Pareto optimality in those theorems by Pareto optimality. Generalization here can be understood under two meanings: using lighter smoothness assumptions and considering multiobjective optimal control problems in the presence of constraints.

### 5.6.2 Strong Pontryagin Principles

In this section, we will establish strong Pontryagin principles for the considered problems with infinite-horizon settings. To do that, we will use the existing results in finite horizon case, which are presented in Section 4, and add some particular assumptions beside the invertibility assumption to assure that all the multipliers are not simultaneously equal to zero. We recall the definition of control sets:

$$W_t(x_t) = \bigcap_{1\leq k\leq m} \{u \in \mathbb{R}^d : g_t^k(x_t, u) \geq 0\} \tag{5.28}$$

**(Di) with inequality constraints**

Firstly, we state the theorem for multiobjective optimal control problems with (Di) and with control sets defined by inequality constraints.

**Theorem 5.33.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_i^1)$ (respectively, solution of $(PM_i^{2'})$, $(PM_i^{3'})$) where the control sets are defined as in (5.28). We assume that the following conditions are fulfilled*

**(i)** *For all $t \in \mathbb{N}_*$, for all $j \in \{1, \dots, \ell\}$ and for all $k \in \{1, \dots, m\}$, $\phi_t^j(., \hat{u}_t)$ and $g_t^k(., \hat{u}_t)$ have directional derivatives at $\hat{x}_t$ which are concave.*

**(ii)** *For all $t \in \mathbb{N}_*$, for all $j \in \{1, \dots, \ell\}$, for all $k \in \{1, \dots, m\}$, for all $u \in W_t(\hat{x}_t)$, $\phi_t^j(., u)$ , $f_t(., u)$ and $g_t^k(., u)$ are continuous at $\hat{x}_t$ in any direction $h$.*

**(iii)** *For all $t \in \mathbb{N}$, for each $x_t$ in a neighborhood $V_t$ of $\hat{x}_t$ such that $x_0 = \eta$, the following convexity condition is satisfied: if $u' \in W_t(x_t)$, $u'' \in W_t(x_t)$, $0 \leq \alpha \leq 1$, one can find $u \in W_t(x_t)$ such that*

$$\begin{aligned} \forall j \in \{1, \dots, \ell\}, \quad & \phi_t^j(x_t, u) \geq \alpha\phi_t^j(x_t, u') + (1-\alpha)\phi_t^j(x_t, u''), \\ & f_t(x_t, u) \geq \alpha f_t(x_t, u') + (1-\alpha) f_t(x_t, u''), \\ \forall k \in \{1, \dots, m\}, \quad & g_t^k(x_t, u) \geq \alpha g_t^k(x_t, u') + (1-\alpha) g_t^k(x_t, u''). \end{aligned}$$

**(iv)** *For all $t \in \mathbb{N}_*$, $f_t(., \hat{u}_t)$ is Gâteaux differentiable at $\hat{x}_t$ and $D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$ is invertible.*

**(v)** *For all $t \in \mathbb{N}$, there exists $\check{u}_t \in W_t(x_t)$ such that $f_t(\hat{x}_t, \check{u}_t) = f_t(\hat{x}_t, \hat{u}_t)$ and $g_t^k(\hat{x}_t, \check{u}_t) > 0$ for all $k \in \{1, \dots, m\}$.*

*Then there exist $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{R}_+^\ell$, $p_{t+1} \in \mathbb{R}_+^{n*}$ and $\mu_t \in \mathbb{R}_+^{m*}$ where $t \in \mathbb{N}$ which satisfy the following conclusions:*

**(a)** $(\theta_1, \dots, \theta_\ell, p_1) \neq (0, \dots, 0)$.

**(b)** *For all $t \in \mathbb{N}$, $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$ and $\langle \mu_t, g_t(\hat{x}_t, \hat{u}_t)\rangle = 0$.*

**(c)** *For all $t \in \mathbb{N}_*$, $p_t \in \partial_1 H_t(\hat{x}_t, \hat{u}_t, \theta, p_{t+1}, \mu)$.*

**(d)** *For all $t \in \mathbb{N}$, $H_t(\hat{x}_t, \hat{u}_t, \theta, p_{t+1}, \mu) = \max_{u\in\mathbb{R}^d} H_t(\hat{x}_t, u, \theta, p_{t+1}, \mu)$.*

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_i^1)$ (respectively, solution of $(PM_i^{2'})$, $(PM_i^{3'})$). Using Theorem 5.3, for all $T \in \mathbb{N}_*$, we know that $(\hat{x}_0, \ldots, \hat{x}_{T+1}, \hat{u}_0, \ldots, \hat{u}_{T+1})$ is a weak Pareto optimal solution of Problem $(FM_i^T)$. Under conditions (i-iv), after Theorem 5.16, for each $T \in \mathbb{N}_*$, there exist $\theta^T = (\theta_1^T, \ldots, \theta_\ell^T) \in \mathbb{R}_+^\ell$, $p^T = (p_1^T, \ldots, p_{T+1}^T) \in (\mathbb{R}_+^{n*})^{T+1}$ and $\mu^T = (\mu_0^T, \ldots, \mu_T^T) \in (\mathbb{R}_+^{m*})^{T+1}$, not all zero which satisfy the conclusions (a-c) of Theorem 5.16. From (b) of Theorem 5.16, using the assumption (iv), we know that for all $t \in \{1, \ldots, T\}$, for all $j \in \{1, \ldots, \ell\}$ and for all $k \in \{1, \ldots, m\}$, there exist $\varphi_t^{j,T} \in \partial_1 \phi_t^j(\hat{x}_t, \hat{u}_t)$ and $\psi_t^{k,T} \in \partial_1 g_t^k(\hat{x}_t, \hat{u}_t)$ such that

$$\begin{aligned} & p_t^T = \textstyle\sum_{j=1}^\ell \theta_j^T \varphi_t^{j,T} + p_{t+1}^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) + \sum_{k=1}^m \mu_t^{k,T}.\psi_t^{k,T} \\ \implies & p_{t+1}^T = (p_t^T - \textstyle\sum_{j=1}^\ell \theta_j^T \varphi_t^{j,T} - \sum_{k=1}^m \mu_t^{k,T}.\psi_t^{k,T}) \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t)^{-1}. \end{aligned} \tag{5.29}$$

Assume that $(\theta_1^T, \ldots, \theta_\ell^T, p_1^T) = (0, \ldots, 0, 0)$ then from conclusion (c) of Theorem 5.16, for all $t \in \{0, \ldots, T\}$, we have:

$$\left\langle p_{t+1}^T, f_t(\hat{x}_t, \hat{u}_t) \right\rangle + \sum\nolimits_{k=1}^m \mu_t^{k,T} g_t^k(\hat{x}_t, \hat{u}_t) \geq \left\langle p_{t+1}^T, f_t(\hat{x}_t, u) \right\rangle + \sum\nolimits_{k=1}^m \mu_t^{k,T} g_t^k(\hat{x}_t, u)$$

for all $u \in U_t$. Now for each $t \in \{0, \ldots, T\}$, take $u = \check{u}_t$ satisfying (v) then we have $\sum_{k=1}^m \mu_t^{k,T} g_t^k(\hat{x}_t, \hat{u}_t) \geq \sum_{k=1}^m \mu_t^{k,T} g_t^k(\hat{x}_t, \check{u}_t)$. Using (a) of Theorem 5.16, we can assert that for all $t \in \{0, \ldots, T\}$, $0 \geq \sum_{k=1}^m \mu_t^{k,T} g_t^k(\hat{x}_t, \check{u}_t)$ but from (ii), since $g_t^k(\hat{x}_t, \check{u}_t) > 0$, we must have $\mu_t^{k,T} = 0$ for all $t \in \{0, \ldots, T\}$ and for all $k \in \{1, \ldots, m\}$. Now apply those into the backward recursive equation (5.29), we have $p_{t+1}^T = p_t^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t)^{-1}$ for all $t \in \{1, \ldots, T\}$ with the initial condition $p_1^T = 0$. Therefore, it is obvious that $p_t^T = 0$ for all $t \in \{1, \ldots, T+1\}$ and thus all the multipliers are zero which is a contradiction. Hence, $(\theta_1^T, \ldots, \theta_\ell^T, p_1^T) \neq (0, \ldots, 0, 0)$. Since the set of the lists of multipliers is a cone, we can normalize the multipliers by setting

$$\left\| (\theta_1^T, \ldots, \theta_\ell^T, p_1^T) \right\| = \sum_{j=1}^\ell \left| \theta_j^T \right| + \left\| p_1^T \right\| = 1 \tag{5.30}$$

Since the values of the sequence $(\theta_1^T, \ldots, \theta_\ell^T, p_1^T)_{T \in \mathbb{N}_*}$ belong to the unit sphere of $\mathbb{R}^\ell \times \mathbb{R}^{n*}$ which is compact, using the Bolzano-Weierstrass theorem we can say that there exist an increasing function $r^1 : \mathbb{N}_* \to \mathbb{N}_*$ and $(\theta_1, \ldots, \theta_\ell, p_1) \in \mathbb{R}^\ell \times \mathbb{R}^{n*}$ such that $\sum_{j=1}^\ell |\theta_j| + \|p_1\| = 1$, $\lim_{T \to +\infty} \theta_j^{r^1(T)} = \theta_j$ for all $j \in \{1, \ldots, \ell\}$ and $\lim_{T \to +\infty} p_1^{r^1(T)} = p_1$. Since for all $t \in \{1, \ldots, T\}$, for all $j \in \{1, \ldots, \ell\}$ and for all $k \in \{1, \ldots, m\}$, $\varphi_t^{j,T} \in \partial_1 \phi_t^j(\hat{x}_t, \hat{u}_t)$ and $\psi_t^{k,T} \in \partial_1 g_t^k(\hat{x}_t, \hat{u}_t)$, then they are compact. Using Theorem A.1 in the appendix A of [15], there exist an increasing function $r^2 : \mathbb{N}_* \to \mathbb{N}_*$, mappings $\varphi_t^j \in \partial_1 \phi_t^j(\hat{x}_t, \hat{u}_t)$ and $\psi_t^k \in \partial_1 g_t^k(\hat{x}_t, \hat{u}_t)$ such that $\lim_{T \to +\infty} \varphi_t^{j,r^2(T)} = \varphi_t^j$ for all $j \in \{1, \ldots, \ell\}$ and $\lim_{T \to +\infty} \psi_t^{k,r^2(T)} = \psi_t$ for all $k \in \{1, \ldots, m\}$. We set $r = r^2 \circ r^1$. We have

$$p_2^{r(T)} = (p_1^{r(T)} - \sum_{j=1}^\ell \theta_j^{r(T)} \varphi_1^{j,r(T)} - \sum_{k=1}^m \mu_1^{k,r(T)}.\psi_1^{k,r(T)}) \circ D_{G,1} f_1(\hat{x}_1, \hat{u}_1)^{-1},$$

which implies that

$$p_2 = \lim_{T \to +\infty} p_2^{r(T)} = (p_1 - \sum_{j=1}^\ell \theta_j \varphi_1^j - \sum_{k=1}^m \mu_1^k.\psi_1^k) \circ D_{G,1} f_1(\hat{x}_1, \hat{u}_1)^{-1}.$$

Proceeding recursively we define, for all $t \in \mathbb{N}_*$,

$$\begin{aligned} p_{t+1} &:= \lim_{T\to+\infty} p_{t+1}^{r(T)} \\ &= \lim_{T\to+\infty}(p_t^{r(T)} - \textstyle\sum_{j=1}^{\ell} \theta_j^{r(T)}\varphi_t^{j,r(T)} - \sum_{k=1}^{m} \mu_t^{k,r(T)}.\psi_t^{k,r(T)}) \circ D_1 f_t(\hat{x}_t,\hat{u}_t)^{-1} \\ &= (p_t - \textstyle\sum_{j=1}^{\ell} \theta_j\varphi_t^{j} - \sum_{k=1}^{m} \mu_t^{k}.\psi_t^{k}) \circ D_{G,1} f_t(\hat{x}_t,\hat{u}_t)^{-1}. \end{aligned}$$

Then, for all $t \in \mathbb{N}_*$, we have proven that there exist $\varphi_t^j \in \partial_1\phi_t^j(\hat{x}_t,\hat{u}_t)$ for all $j \in \{1,\dots,\ell\}$ and $\psi_t^k \in \partial_1 g_t^k(\hat{x}_t,\hat{u}_t)$ for all $k \in \{1,\dots,m\}$ such that

$$p_t = p_{t+1} \circ D_{G,1} f_t(\hat{x}_t,\hat{u}_t) + \textstyle\sum_{j=1}^{\ell} \theta_j\varphi_t^j + \sum_{k=1}^{m} \mu_t^k.\psi_t^k,$$

i.e. for all $t \in \mathbb{N}_*$, $p_t \in \partial_1 H_t(\hat{x}_t,\hat{u}_t,\theta,p_{t+1},\mu_t)$ which gives (c). We have yet seen that (a) is satisfied. It is obvious that $\theta \in \mathbb{R}_+^\ell$, $(p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}_+^{n*})^{\mathbb{N}_*}$ and $\mu_t \in \mathbb{R}_+^{m*}$ since they are the limits of $\theta^T$, $p_t^T$ and $\mu_t^T$, respectively, when $T \to \infty$. From conclusion (a) of Theorem 5.16 we obtain (b). From conclusion (c) of Theorem 5.16 we obtain (d). □

**Remark 5.34.** Condition (v) is taken from Theorem 4.1 in [33] and it is useful to prove that all the multipliers are not simultaneously equal to zero.

In the previous result, we use the additional condition (iv) on $f_t$ in order to obtain the backward recursive equation which expresses $p_{t+1}$ through $p_t$. In fact, we can lighten the assumption on $f_t$ from Gâteaux differentiable down to directional differentiable. However, by doing that, it will be necessary to have the invertibility of all the mappings belonging to the partial subdifferential of $f_t$ with respect to $x_t$. The next theorem will clarify this point.

**Theorem 5.35.** *Under the settings of Theorem 5.33, we assume that conditions (ii,iii) and (v) of Theorem 5.33 hold together with the following assumption*

**(i')** *For all $t \in \mathbb{N}_*$, for all $j \in \{1,\dots,\ell\}$ and for all $k \in \{1,\dots,m\}$, $\phi_t^j(.,\hat{u}_t)$, $g_t^k(.,\hat{u}_t)$ and $f_t(.,\hat{u}_t)$ have directional derivatives at $\hat{x}_t$ which are concave.*

**(iv')** *For all $t \in \mathbb{N}_*$, if $\zeta \in \partial_1 f_t(\hat{x}_t,\hat{u}_t)$ then it is invertible.*

*Then there exist $\theta = (\theta_1,\dots,\theta_\ell) \in \mathbb{R}_+^\ell$, $p_{t+1} \in \mathbb{R}_+^{n*}$ and $\mu_t \in \mathbb{R}_+^{m*}$ where $t \in \mathbb{N}$ such that all the conclusions of Theorem 5.33 hold.*

*Proof.* Let $(\underline{\hat{x}},\underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_i^1)$ (respectively, solution of $(PM_i^{2'})$, $(PM_i^{3'})$). Using an analogous argument like in previous theorem, for each $T \in \mathbb{N}_*$, there exist $\theta^T = (\theta_1^T,\dots,\theta_\ell^T) \in \mathbb{R}_+^\ell$, $p^T = (p_1^T,\dots,p_{T+1}^T) \in (\mathbb{R}_+^{n*})^{T+1}$ and $\mu^T = (\mu_0^T,\dots,\mu_T^T) \in (\mathbb{R}_+^{m*})^{T+1}$, not all zero which satisfy the conclusions (a-c) of Theorem 5.16.

From conclusion (b) of Theorem 5.16 we know that for all $t \in \{1,\dots,T\}$, for all $j \in \{1,\dots,\ell\}$ and for all $k \in \{1,\dots,m\}$, there exist $\varphi_t^{j,T} \in \partial_1\phi_t^j(\hat{x}_t,\hat{u}_t)$, $\psi_t^{k,T} \in \partial_1 g_t^k(\hat{x}_t,\hat{u}_t)$ and $\zeta_t^T \in \partial_1 f_t(\hat{x}_t,\hat{u}_t)$ such that

$$\begin{aligned} & p_t^T = \textstyle\sum_{j=1}^{\ell} \theta_j^T\varphi_t^{j,T} + p_{t+1}^T \circ \zeta_t^T + \sum_{k=1}^{m} \mu_t^{k,T}.\psi_t^{k,T} \\ \implies\ & p_{t+1}^T = (p_t^T - \textstyle\sum_{j=1}^{\ell} \theta_j^T\varphi_t^{j,T} - \sum_{k=1}^{m} \mu_t^{k,T}.\psi_t^{k,T}) \circ (\zeta_t^T)^{-1}. \end{aligned}$$

By doing an similar procedure like the previous proof, we obtain $(\theta_1^T,\dots,\theta_\ell^T,p_1^T) \neq (0,\dots,0,0)$ and we can normalize it as in (5.30). Now using the same argument like before, there exist an increasing function $r : \mathbb{N}_* \to \mathbb{N}_*$, vector $(\theta_1,\dots,\theta_\ell,p_1) \in \mathbb{R}^\ell \times \mathbb{R}^{n*}$, mappings $\varphi_t^j \in \partial_1\phi_t^j(\hat{x}_t,\hat{u}_t)$, $\psi_t^k \in \partial_1 g_t^k(\hat{x}_t,\hat{u}_t)$ and $\zeta_t \in \partial_1 f_t(\hat{x}_t,\hat{u}_t)$ such that $\sum_{j=1}^{\ell} |\theta_j| + \|p_1\| = 1$, $\lim_{T\to+\infty} \theta_j^{r(T)} = \theta_j$ for all $j \in \{1,\dots,\ell\}$, $\lim_{T\to+\infty} p_1^{r(T)} = p_1$, $\lim_{T\to+\infty} \varphi_t^{j,r(T)} = \varphi_t^j$ for all $j \in \{1,\dots,\ell\}$, $\lim_{T\to+\infty} \psi_t^{k,r(T)} = \psi_t$ for all $k \in \{1,\dots,m\}$ and $\lim_{T\to+\infty} \zeta_t^{r(T)} = \zeta_t$. Then by recursively using the backward equation and by taking the limit when $T \to +\infty$, we have

$$p_{t+1} = (p_t - \sum_{j=1}^{\ell} \theta_j \varphi_t^j - \sum_{k=1}^{m} \mu_t^k.\psi_t^k) \circ (\zeta_t)^{-1}$$

for all $t \in \mathbb{N}_*$ or equivalently,

$$\begin{array}{rc} & p_t = \sum_{j=1}^{\ell} \theta_j \varphi_t^j + p_{t+1} \circ \zeta_t + \sum_{k=1}^{m} \mu_t^k.\psi_t^k \\ \Longrightarrow & p_t \in \partial_1 H_t(\hat{x}_t, \hat{u}_t, \theta, p_{t+1}, \mu_t). \end{array}$$

The rest of the proof goes like that of the previous one. □

**Remark 5.36.** Notice that when $n = 1$, i.e. the space of state variables is identified with $\mathbb{R}$, then condition (iv') in the previous theorem can be rewritten as follows: For all $t \in \mathbb{N}_*, 0 \notin \partial_1 f_t(\hat{x}_t, \hat{u}_t)$. This implies for all $t \in \mathbb{N}_*$, $\hat{x}_t$ is not an extremum of $f_t(., \hat{u}_t)$.

In the special case when the sets of controls are defined by inequalities but only depend on control variables: $W_t = \bigcap_{1 \le k \le m} \{u \in \mathbb{R}^d : g_t^k(u) \ge 0\}$, we can take advantage of special assumption of Corollary 5.30 to state a strong principle for multiobjective optimal control problems with (Di) as in the following corollary.

**Corollary 5.37.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_i^1)$ (respectively, solution of $(PM_i^{2'})$, $(PM_i^{3'})$) when the control sets are defined as above. We assume that conditions (ii,iii) of Theorem 5.33 are satisfied. Moreover, assume that the following additional assumptions are fulfilled*

**(i")** *For all $t \in \mathbb{N}_*$, for all $j \in \{1, \dots, \ell\}$, $\phi_t^j$ has directional derivative at $(\hat{x}_t, \hat{u}_t)$ and it is concave.*

**(iv")** *For all $t \in \mathbb{N}_*$, $f_t$ is Gâteaux differentiable at $(\hat{x}_t, \hat{u}_t)$ and $D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$ is invertible.*

**(v')** *For all $t \in \mathbb{N}$, for all $k \in \{1, \dots, m\}$, $g_t^k$ is Gâteaux differentiable at $\hat{u}_t$ and moreover, $0 \notin conv\{D_G g_t^k(\hat{u}_t) : k \in I_t^s\}$ where $I_t^s := \{k \in \{1, \dots, m\} : g_t^k(\hat{u}_t) = 0\}$.*

*Then there exist $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{R}_+^\ell$, $p_{t+1} \in \mathbb{R}_+^{n*}$ and $\mu_t \in \mathbb{R}_+^{m*}$, where $t \in \mathbb{N}$, such that all the conclusions of Theorem 5.33 hold.*

*Proof.* The proof of this corollary is almost the same like that of Theorem 5.33, except for the step of proving $(\theta_1^T, \dots, \theta_\ell^T, p_1^T) \neq (0, \dots, 0, 0)$. By contradiction, assume that $(\theta_1^T, \dots, \theta_\ell^T, p_1^T) = (0, \dots, 0, 0)$ then from conclusion (b) of Theorem 5.16, using assumption (iv"), we know that for all $t \in \{1, \dots, T\}$, for all $j \in \{1, \dots, \ell\}$, there exists $\varphi_t^{j,T} \in \partial_1 \phi_t^j(\hat{x}_t, \hat{u}_t)$ such that

$$\begin{array}{rc} & p_t^T = \sum_{j=1}^{\ell} \theta_j^T \varphi_t^{j,T} + p_{t+1}^T \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t) \\ \Longrightarrow & p_{t+1}^T = (p_t^T - \sum_{j=1}^{\ell} \theta_j^T \varphi_t^{j,T}) \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t)^{-1}. \end{array} \tag{5.31}$$

Then, since $(\theta_1^T, \dots, \theta_\ell^T, p_1^T) = (0, \dots, 0, 0)$, $p_t^T = 0$ for all $t \in \{1, \dots, T+1\}$. From conclusion (c) of Theorem 5.16, we have $0 \in \partial_2 H_t(\hat{x}_t, \hat{u}_t, \theta, p_{t+1}, \mu_t)$ which means that for all $t \in \{0, \dots, T\}$, for all $j \in \{1, \dots, \ell\}$, there exists $\alpha_t^{j,T} \in \partial_2 \phi_t^j(\hat{x}_t, \hat{u}_t)$ such that

$$0 = \sum_{j=1}^{\ell} \theta_j^T \alpha_t^{j,T} + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t) + \sum_{k=1}^{m} \mu_t^{k,T}.D_G g_t^k(\hat{u}_t). \tag{5.32}$$

From this equation we obtain $\sum_{k=1}^{m} \mu_t^{k,T}.D_G g_t^k(\hat{u}_t) = 0$. By the same argument as in the proof of Corollary 5.28, we have $\mu_t^{k,T} = 0$ for all $t \in \{0, \dots, T\}$, for all $k \in \{1, \dots, m\}$. Thus, all the multipliers are zero which is a contradiction. Then $(\theta_1^T, \dots, \theta_\ell^T, p_1^T) \neq (0, \dots, 0, 0)$. □

**Remark 5.38.** Theorem 5.33, Theorem 5.35 and Corollary 5.37 can be considered as generalizations of Theorem 4.3 in [34] since now there exist the inequality constraints in the control sets. Besides, in these results, we use weaker assumptions compared to the condition of partial continuous differentiability in Theorem 4.3 in [34].

**(De) with inequality constraints**

Now we consider the multiobjective optimal control problems with (De) and with control sets defined by inequality constraints. For these problems, we have the following strong Pontryagin principles which provide a necessary condition of optimality in strong form.

**Theorem 5.39.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_e^1)$ (respectively, solution of $(PM_e^{2'})$, $(PM_e^{3'})$) when the control sets are defined as in (5.28). We assume that conditions (i,ii,v) in Theorem 5.33 hold and the following conditions are fulfilled*

**(iii)** *For all $t \in \mathbb{N}$, for each $x_t$ in a neighborhood $V_t$ of $\hat{x}_t$ such that $x_0 = \eta$, the following convexity condition is satisfied: if $u' \in W_t(x_t)$, $u'' \in W_t(x_t)$, $0 \leq \alpha \leq 1$, one can find $u \in W_t(x_t)$ such that*

$$\begin{aligned} \forall j \in \{1, \dots, \ell\}, \quad & \phi_t^j(x_t, u) \geq \alpha \phi_t^j(x_t, u') + (1-\alpha)\phi_t^j(x_t, u''), \\ & f_t(x_t, u) = \alpha f_t(x_t, u') + (1-\alpha) f_t(x_t, u''), \\ \forall k \in \{1, \dots, m\}, \quad & g_t^k(x_t, u) \geq \alpha g_t^k(x_t, u') + (1-\alpha) g_t^k(x_t, u''). \end{aligned}$$

**(iv)** *For all $t \in \mathbb{N}_*$, for all $u_t \in W_t(\hat{x}_t)$, $f_t(., u_t)$ is continuously differentiable at $\hat{x}_t$ and moreover, $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.*

*Then there exist $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{R}_+^\ell$ and $p_{t+1} \in \mathbb{R}^{n*}$, $t \in \mathbb{N}$ which satisfy conclusions (a,c,d) of Theorem 5.33 and the following one*

**(b)** *For all $t \in \mathbb{N}$, for all $k \in \{1, \dots, m\}$, $\mu_t^k . g_t^k(\hat{x}_t, \hat{u}_t) = 0$.*

The proof of this theorem is similar to the proof of Theorem 5.33. The difference is the lost of sign of $p_{t+1}$ since now the dynamical system is defined by difference equations.

In the special case when the sets of controls are defined by inequalities but only depend on control variable, we can state a corollary as follows.

**Corollary 5.40.** *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem $(PM_e^1)$ (respectively, solution of $(PM_e^{2'})$, $(PM_e^{3'})$) when the control sets are defined as in Corollary 5.37. We assume that condition (ii) in Theorem 5.33 and conditions (iii,iv) in Theorem 5.39 are fulfilled. Moreover, assume that the additional assumptions (v',vi) in Corollary 5.37 are fulfilled. Then there exist $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{R}_+^\ell$, $p_{t+1} \in \mathbb{R}^{n*}$ and $\mu_t \in \mathbb{R}_+^{m*}$ where $t \in \mathbb{N}$ such that all the conclusions of Theorem 5.39 hold.*

The proof for this theorem is similar to that of Corollary 5.37.

**Remark 5.41.** Theorem 5.39 and Corollary 5.40 can be considered as generalizations of Theorem 4.1 in [34] since now there exist the inequality constraints in the control sets. Besides, in these results, we use weaker assumptions compared to the condition of partially continuous differentiability of Theorem 4.1 in [34].

**Remark 5.42.** All the results for multiobjective problems in this chapter become Pontryagin principles for single-objective optimal control problems in Chapter 4 when $\ell = 1$. Therefore, we also obtain strong Pontryagin principles and more general weak Pontryagin principles for single-objective optimal control problems in finite or infinite horizon.

### 5.6.3 Transversality Condition as a Necessary Condition of Optimality

Using Corollary 5.30 and Corollary 5.31, we analyse the transversality condition in the form

$$\lim_{t\to+\infty} p_t = 0$$

in two cases: with invertibility condition and with positivity condition. In each case, we will add more assumptions in order to obtain the transversaltity condition.

#### The transversality condition for the problem with invertibility assumption

From Corollary 5.30, we know that there exist $\theta_1, \dots, \theta_\ell \in \mathbb{R}_+$ and $p_{t+1} \in \mathbb{R}_+^{n*}$ where $t \in \mathbb{N}$ which satisfy conclusions (a-d) of this corollary. Using conclusion (c) of Corollary 5.30, we have:

$$\forall t \in \mathbb{N}^*,\ p_t = \sum_{j=1}^{\ell} \left(\theta_j . D_{G,1}\phi_t^j(\hat{x}_t, \hat{u}_t)\right) + p_{t+1} \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t).$$

$$\Rightarrow \forall t \in \mathbb{N}^*,\ p_{t+1} = (p_t - \sum_{j=1}^{\ell} \theta_j D_{G,1}\phi_t^j(\hat{x}_t, \hat{u}_t)) \circ D_{G,1} f_t(\hat{x}_t, \hat{u}_t)^{-1}.$$

From this, we can assert that, for all $t \in \mathbb{N}^*$,

$$\|p_{t+1}\| \leq \|p_t\| \left\|D_{G,1} f_t(\hat{x}_t, \hat{u}_t)^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j . \left\|D_{G,1}\phi_t^j(\hat{x}_t, \hat{u}_t)\right\| \left\|D_{G,1} f_t(\hat{x}_t, \hat{u}_t)^{-1}\right\|. \quad (5.33)$$

We set for all $t \in \mathbb{N}$, $A_t := D_{G,1} f_t(\hat{x}_t, \hat{u}_t)$, $A_t^{-1} := (D_{G,1} f_t(\hat{x}_t, \hat{u}_t))^{-1}$ and $b_t^j := D_{G,1}\phi_t^j(\hat{x}_t, \hat{u}_t)$. Then, we have for all $t \in \mathbb{N}$, $A_t$, $b_t^j$ are bounded linear operators. Since $A_t^{-1}$ exists for all $t \in \mathbb{N}_*$ (from the invertibility assumption), we can assert that, for all $t \in \mathbb{N}$, $A_t^{-1}$ is also a bounded linear operator (because $A_t . A_t^{-1} = E_n$, the unit matrix in $\mathbb{R}^{n\times n}$). We will find an expression that describes $p_{t+1}$ by $p_1$ and $\theta_1, \dots, \theta_\ell$ (notice that $(\theta_1, \dots, \theta_\ell, p_1) \neq (0, \dots, 0, 0)$).

$$\forall t \in \mathbb{N}^*,\ \|p_{t+1}\| \leq \|p_t\| \left\|A_t^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j . \left\|b_t^j\right\| \left\|A_t^{-1}\right\| \quad (5.34)$$

While $t > 1$ we have:

$$\|p_{t+1}\| \leq \left( \|p_{t-1}\| \left\|A_{t-1}^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j . \left\|b_{t-1}^j\right\| \left\|A_{t-1}^{-1}\right\| \right) \left\|A_t^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j . \left\|b_t^j\right\| \left\|A_t^{-1}\right\|$$

$$\Leftrightarrow \|p_{t+1}\| \leq \|p_{t-1}\| \left\|A_{t-1}^{-1}\right\| \left\|A_t^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j \left( \left\|b_{t-1}^j\right\| \left\|A_{t-1}^{-1}\right\| \left\|A_t^{-1}\right\| + \left\|b_t^j\right\| \left\|A_t^{-1}\right\| \right).$$

By induction we obtain

$$\|p_{t+1}\| \leq \|p_1\| \left\|A_1^{-1}\right\| \dots \left\|A_t^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j \left( \left\|b_1^j\right\| \left\|A_1^{-1}\right\| \dots \left\|A_t^{-1}\right\| + \left\|b_2^j\right\| \left\|A_2^{-1}\right\| \dots \left\|A_t^{-1}\right\| + \dots + \left\|b_t^j\right\| \left\|A_t^{-1}\right\| \right)$$

$$\Leftrightarrow \|p_{t+1}\| \leq \|p_1\| \prod_{s=1}^{t} \left\|A_s^{-1}\right\| + \sum_{j=1}^{\ell} \theta_j \sum_{s=1}^{t} (\left\|b_s^j\right\| \prod_{k=s}^{t} \left\|A_s^{-1}\right\|).$$

Recall that $\sum_{j=1}^{\ell} \theta_j + \|p_1\| = 1$, then

$$\forall t \in \mathbb{N}^*,\ \|p_{t+1}\| \leq \prod_{s=1}^{t} \left\|A_s^{-1}\right\| + \sum_{s=1}^{t} (\left\|b_s^j\right\| \prod_{k=s}^{t} \left\|A_s^{-1}\right\|).$$

Let $\alpha_t = \prod_{s=1}^{t} \|A_s^{-1}\| \geq 0$ and $\beta_t = \sum_{s=1}^{t} (\|b_s^j\| \prod_{k=s}^{t} \|A_s^{-1}\|) \geq 0$. We make an assumption that $\sup_{t\in\mathbb{N}^*} \left\|A_t^{-1}\right\| = M < 1$ and for all $t \in \mathbb{N}$, for all $j \in \{1, \ldots, \ell\}$, $\left\|b_t^j\right\| \leq \gamma_t$ where $(\gamma_t)_{t\in\mathbb{N}} = \underline{\gamma} \in c_0(\mathbb{N}, \mathbb{R})$. From this, we have

$$0 \leq \lim_{t\to+\infty} \alpha_t = \lim_{t\to+\infty} \prod_{s=1}^{t} \left\|A_s^{-1}\right\| \leq \lim_{t\to+\infty} \prod_{s=1}^{t} M = \lim_{t\to+\infty} M^t = 0 \text{ (since } 0 \leq M < 1).$$

That means $\lim_{t\to+\infty} \alpha_t = 0$. Thus, $(\alpha_t)_{t\in\mathbb{N}}$ is bounded. For $\beta_t$, we have

$$\begin{aligned}
\beta_t &= \sum_{s=1}^{t} (\left\|b_s^j\right\| \prod_{k=s}^{t} \left\|A_s^{-1}\right\|) \\
&\leq \sum_{s=1}^{t} \gamma_s M^{t-s+1} \leq \left\|\underline{\gamma}\right\| \sum_{s=1}^{t} M^{t-s+1} \\
&= M^{t+1} \left\|\underline{\gamma}\right\| \sum_{s=1}^{t} M^{-s} = \left\|\underline{\gamma}\right\| M \frac{1-M^t}{1-M} \\
&\leq \left\|\underline{\gamma}\right\| \frac{M}{1-M} < +\infty.
\end{aligned}$$

Then, $(\beta_t)_{t\in\mathbb{N}}$ is also bounded. Therefore, $(p_t)_{t\in\mathbb{N}_*}$ is bounded i.e. $\limsup_{t\to+\infty} \|p_t\| < +\infty$. Take $\limsup$ both side of (5.34) with a notice that $\limsup_{t\to+\infty} \|p_t\| = \limsup_{t\to\infty} \|p_{t+1}\|$ and for all $j \in \{1, \ldots, \ell\}$, $\limsup_{t\to+\infty} \left\|b_t^j\right\| \leq \limsup_{t\to+\infty} |\gamma_t| = 0$ to obtain

$$\begin{aligned}
& \limsup_{t\to+\infty} \|p_t\| \leq \limsup_{t\to+\infty} \|p_t\| \left\|A_t^{-1}\right\| \\
\Longrightarrow\quad & \limsup_{t\to+\infty} \|p_t\| \leq M \limsup_{t\to+\infty} \|p_t\| \\
\Longrightarrow\quad & (1-M) \limsup_{t\to+\infty} \|p_t\| \leq 0.
\end{aligned}$$

From this, we can assert that $\limsup_{t\to+\infty} \|p_t\| = 0$ since $M < 1$. Thus, $\lim_{t\to+\infty} \|p_t\| = 0$ i.e. $\lim_{t\to+\infty} p_t = 0$.

And so, we have proven the following corollary

**Corollary 5.43.** *If $(\underline{\hat{x}}, \underline{\hat{u}})$ satisfies all hypotheses of Corollary 5.30 and the following condition*

**(vi)** $\sup_{t\in\mathbb{N}_*} \left\|A_t^{-1}\right\| = M < 1$ *and* $\left\|D_{G,1}\phi_t^j(\hat{x}_t, \hat{u}_t)\right\| \leq |\gamma_t|$ *for all* $j \in \{1, \ldots, \ell\}$, *for all* $t \in \mathbb{N}$ *where* $(\gamma_t)_{t\in\mathbb{N}} = \underline{\gamma} \in c_0(\mathbb{N}, \mathbb{R})$.

*Then, all the conclusions of Corollary 5.30 hold and moreover,* $\lim_{t\to+\infty} p_t = 0$.

Now we consider the special case when $(\gamma_t)_{t\in\mathbb{N}} = \underline{\gamma} \in \ell^1(\mathbb{N},\mathbb{R})$, i.e. $\sum_{t=0}^{+\infty} |\gamma_t| < +\infty$. This case is usual in Macroeconomic Theory. For example, $\phi_t^j(x,u) = \beta^t \psi_t^j(x,u)$ for all $t \in \mathbb{N}$ and for all $j \in \{1,\dots,\ell\}$ where $\beta \in (0,1)$ and $\psi_t^j$ is a mapping satisfying $\sup_{t\in\mathbb{N}} \left\| D_{G,1}\psi_t^j(\hat{x}_t,\hat{u}_t)\right\| := K < +\infty$ for all $j \in \{1,\dots,\ell\}$. Here we can define $\gamma_t := \beta^t K$ for all $t \in \mathbb{N}$ and then, $\sum_{t=0}^{+\infty} |\gamma_t| = K \sum_{t=0}^{+\infty} \beta^t = \frac{K}{1-\beta^t}$. Another example is $\phi_t^j(x_t,u_t) = ae^{-ct}\psi_t^j(x_t,u_t)$ for all $t \in \mathbb{N}$ and for all $j \in \{1,\dots,\ell\}$ where $a,c > 0$ and $\psi_t^j$ is a same mapping in the previous example. Here we can define $\gamma_t := acKe^{-ct}$ for all $t \in \mathbb{N}$ and then, $\sum_{t=0}^{+\infty} |\gamma_t| = acK \sum_{t=0}^{+\infty} e^{-ct} = \frac{acK}{1-e^{-c}}$. For this case, we will provide another condition which assures the validation of transversality condition. We have the following corollary

**Corollary 5.44.** *If $(\underline{\hat{x}},\underline{\hat{u}})$ satisfies all hypotheses of Corollary 5.30 together with the following condition*

**(vii)** $\sup_{t\in\mathbb{N}_*} \|D_{G,1}f_t(\hat{x}_t,\hat{u}_t)\| = M < 1$ *and* $\left\|D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right\| \le |\gamma_t|$ *for all* $j \in \{1,\dots,\ell\}$, *for all* $t \in \mathbb{N}$ *where* $(\gamma_t)_{t\in\mathbb{N}} = \underline{\gamma} \in \ell^1(\mathbb{N},\mathbb{R})$.

*Then, all the conclusions of Corollary 5.30 hold and moreover,* $\lim_{t\to+\infty} p_t = 0$.

*Proof.* From Corollary 5.30, we know that there exist $\theta_1,\dots,\theta_\ell \in \mathbb{R}_+$ and $p_{t+1} \in \mathbb{R}_+^{n*}$ where $t \in \mathbb{N}$ which satisfy conclusions (a-d) of this corollary. Using conclusion (c) of Corollary 5.30, we have:

$$\forall t \in \mathbb{N}_*,\ p_t = \sum_{j=1}^{\ell} \left(\theta_j . D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right) + p_{t+1} \circ D_{G,1}f_t(\hat{x}_t,\hat{u}_t).$$

Then, we can assert that for all $t \in \mathbb{N}_*$,

$$\begin{aligned} & \|p_t\| \le \textstyle\sum_{j=1}^{\ell} \left(\theta_j . \left\|D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right\|\right) + \|p_{t+1}\| . \|D_{G,1}f_t(\hat{x}_t,\hat{u}_t)\| \\ \Longrightarrow\ & \|p_t\| \le |\gamma_t| \textstyle\sum_{j=1}^{\ell} \theta_j + M \|p_{t+1}\| \\ \Longrightarrow\ & \|p_t\| \le |\gamma_t| + M \|p_{t+1}\| \end{aligned}$$

since $\sum_{j=1}^{\ell} \theta_j \le 1$. Thus,

$$\begin{aligned} & \textstyle\sum_{t=1}^{+\infty} \|p_t\| \le \sum_{t=1}^{+\infty} |\gamma_t| + M \sum_{t=1}^{+\infty} \|p_{t+1}\| \le \sum_{t=1}^{+\infty} |\gamma_t| + M \sum_{t=1}^{+\infty} \|p_t\| \\ \Longrightarrow\ & (1-M) \textstyle\sum_{t=1}^{+\infty} \|p_t\| \le \sum_{t=1}^{+\infty} |\gamma_t| \\ \Longrightarrow\ & \textstyle\sum_{t=1}^{+\infty} \|p_t\| \le \frac{\sum_{t=1}^{+\infty} |\gamma_t|}{1-M} < +\infty. \end{aligned}$$

From the last inequality, we obtain $(p_t)_{t\in\mathbb{N}_*} \in \ell^1(\mathbb{N}_*,\mathbb{R}^{n*})$ which assures that $\lim_{t\to+\infty} p_t = 0$. □

### The transversality condition for the problem with positivity assumption

From the proof of Corollary 5.31, the following condition holds for all $t \in \mathbb{N}_*$ and for all $T \ge t$

$$\left\|p_{t+1}^T\right\| \le \frac{1}{\rho_t}\left(\left\|p_t^T\right\| + \sum_{j=1}^{\ell} \left\|D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right\|\right),$$

where $\rho_t = \min\limits_{1\le\alpha\le n} \dfrac{\partial f_t^\alpha(\hat{x}_t,\hat{u}_t)}{\partial x_t^\alpha} > 0$. In this inequality, we let $T \to +\infty$ and obtain

$$\|p_{t+1}\| \le \frac{1}{\rho_t}\left(\|p_t\| + \sum_{j=1}^{\ell} \left\|D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right\|\right). \tag{5.35}$$

By induction, we have

$$\begin{aligned}\|p_{t+1}\| &\leq \frac{1}{\prod_{s=1}^{t}\rho_t}\|p_1\| + \left(\sum\nolimits_{s=1}^{t}\left(\frac{1}{\prod_{k=s}^{t}\rho_t}\sum_{j=1}^{\ell}\left\|D_{G,1}\phi_s^j(\hat{x}_t,\hat{u}_t)\right\|\right)\right)\\ &\leq \frac{1}{\prod_{s=1}^{t}\rho_s} + \left(\sum\nolimits_{s=1}^{t}\left(\frac{1}{\prod_{k=s}^{t}\rho_k}\sum_{j=1}^{\ell}\left\|D_{G,1}\phi_s^j(\hat{x}_t,\hat{u}_t)\right\|\right)\right)\\ &= \alpha_t + \beta_t,\end{aligned}$$

where $\alpha_t := \frac{1}{\prod_{s=1}^{t}\rho_s} > 0$ and $\beta_t := \sum_{s=1}^{t}\frac{1}{\prod_{k=s}^{t}\rho_k}\sum_{j=1}^{\ell}\left\|D_{G,1}\phi_s^j(\hat{x}_t,\hat{u}_t)\right\| \geq 0$.

Now we make an assumption that $\delta := \inf_{t\in\mathbb{N}_*}\rho_t > 1$ and for all $t \in \mathbb{N}$, for all $j \in \{1,\ldots,\ell\}$, $\left\|D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right\| = \left\|b_t^j\right\| \leq \gamma_t$ where $(\gamma_t)_{t\in\mathbb{N}} = \underline{\gamma} \in c_0(\mathbb{N},\mathbb{R})$. From here, we have

$$\begin{aligned}0 \leq \lim_{t\to+\infty}\alpha_t &= \lim_{t\to+\infty}\frac{1}{\prod_{s=1}^{t}\rho_s} < \lim_{t\to+\infty}\frac{1}{\prod_{s=1}^{t}\delta} = \lim_{t\to+\infty}\frac{1}{\delta^t} = 0\\ &\Rightarrow \lim_{t\to+\infty}\alpha_t = 0.\end{aligned}$$

Then $(\alpha_t)_{t\in\mathbb{N}}$ is bounded. For $\beta_t$, we have

$$\begin{aligned}\beta_t &= \sum_{s=1}^{t}\frac{1}{\prod_{k=s}^{t}\rho_k}\sum_{j=1}^{\ell}\left\|D_{G,1}\phi_s^j(\hat{x}_t,\hat{u}_t)\right\| = \sum_{s=1}^{t}\frac{1}{\prod_{k=s}^{t}\rho_k}\sum_{j=1}^{\ell}\left\|b_s^j\right\|\\ &\leq \ell\left\|\underline{\gamma}\right\|\sum_{s=1}^{t}\frac{1}{\prod_{k=s}^{t}\rho_k} \leq \ell\left\|\underline{\gamma}\right\|\sum_{s=1}^{t}\frac{1}{\delta^{t-s+1}} = \frac{\ell\left\|\underline{\gamma}\right\|}{\delta^{t+1}}\sum_{s=1}^{t}\delta^s\\ &\leq \ell\left\|\underline{\gamma}\right\|\frac{1}{\delta-1} < +\infty.\end{aligned}$$

Then, $(\beta_t)_{t\in\mathbb{N}}$ is also bounded. Thus, $(p_t)_{t\in\mathbb{N}_*}$ is bounded i.e. $\limsup_{t\to+\infty}\|p_t\| < +\infty$. Take lim sup both side of (5.35) with a notice that $\limsup_{t\to+\infty}\|p_t\| = \limsup_{t\to\infty}\|p_{t+1}\|$ and for all $j \in \{1,\ldots,\ell\}$, $\limsup_{t\to+\infty}\left\|b_t^j\right\| \leq \limsup_{t\to+\infty}\gamma_t = 0$ to obtain

$$\begin{aligned}&\limsup_{t\to+\infty}\|p_t\| \leq \limsup_{t\to+\infty}\tfrac{1}{\rho_t}\|p_t\|\\ \Longrightarrow\ &\limsup_{t\to+\infty}\|p_t\| \leq \tfrac{1}{\delta}\limsup_{t\to+\infty}\|p_t\|\\ \Longrightarrow\ &(1-\tfrac{1}{\delta})\limsup_{t\to+\infty}\|p_t\| \leq 0.\end{aligned}$$

Since $\delta > 1$ then $(1-\frac{1}{\delta}) > 0$ and from the last inequality, we can assert that $\limsup_{t\to+\infty}\|p_t\| = 0$ since $M < 1$. Thus, $\lim_{t\to+\infty}\|p_t\| = 0$ i.e. $\lim_{t\to+\infty}p_t = 0$. We have proven the following corollary:

**Corollary 5.45.** *If $(\underline{\hat{x}},\underline{\hat{u}})$ satisfies all hypotheses of Corollary 5.31 and the following condition*

**(viii)** $\delta := \inf_{t\in\mathbb{N}_*}\rho_t > 1$ *and* $\left\|D_{G,1}\phi_t^j(\hat{x}_t,\hat{u}_t)\right\| \leq \gamma_t$ *for all* $j \in \{1,\ldots,\ell\}$, *for all* $t \in \mathbb{N}$ *where* $(\gamma_t)_{t\in\mathbb{N}} = \underline{\gamma} \in c_0(\mathbb{N},\mathbb{R})$.

*Then, all the conclusions of Corollary 5.31 hold and moreover,* $\lim_{t\to+\infty}p_t = 0$.

## 5.7 Sufficient Condition for Multiobjective Optimal Control Problem

In this section we establish results of sufficient condition of optimality which uses the adjoint equation and the maximum principles. All the results will use the concavity assumption on the Hamiltonian. Some results need its concavity with respect to both state variable and control variable. Others need its concavity with respect to only state variable.

Firstly, we introduce the scalarization technique, namely the weighting method. For $k \in \{e, i\}$, consider the following single-objective control problems

$(\mathbf{Q_k^1})$ Maximize $\sum_{j=1}^{\ell} \theta_j J_j(\underline{x}, \underline{u})$ when $(\underline{x}, \underline{u}) \in Dom_k(J)$.

$(\mathbf{Q_k^2})$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_k$ such that, there does not exist a process $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$, satisfying

$$\limsup_{h \to +\infty} \sum_{t=0}^{h} \sum_{j=1}^{\ell} \theta_j (\phi_t^j(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) > 0.$$

$(\mathbf{Q_k^3})$ Find $(\hat{\underline{x}}, \hat{\underline{u}}) \in \mathrm{Adm}_k$ such that, there does not exist a process $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$ satisfying

$$\liminf_{h \to +\infty} \sum_{t=0}^{h} \sum_{j=1}^{\ell} \theta_j (\phi_t^j(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) > 0.$$

We have the following lemma

**Lemma 5.46.** *If $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^2)$ then it is also a solution of Problem $(Q_k^3)$.*

*Proof.* We prove this lemma by contradiction. Suppose that $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^2)$ but not a solution of Problem $(Q_k^3)$. Then there exists $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$ such that $\liminf_{h \to +\infty} \sum_{t=0}^{h} \sum_{j=1}^{\ell} \theta_j (\phi_t^j(x_t, u_t) - \phi_{j,t}(\hat{x}_t, \hat{u}_t)) > 0$. This inequality implies that $\limsup_{h \to +\infty} \sum_{t=0}^{h} \sum_{j=1}^{\ell} \theta_j (\phi_t^j(x_t, u_t) - \phi_t^j(\hat{x}_t, \hat{u}_t)) > 0$ and this contradicts to the fact that $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^2)$. The lemma is proven. □

Now we introduce the lemmas that show the relationship between the solutions of the single-objective weighted problems $\left(Q_k^j\right)$and multiobjective optimal control problems $\left(PM_k^j\right)$ where $k \in \{e, i\}$ and $j \in \{1, 3\}$. The following lemmas are taken from [34].

**Lemma 5.47.** *Assume that $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^1)$. The two following assertions hold:*

- *If $\theta_j > 0$ for all $j \in \{1, \ldots, \ell\}$ then $(\hat{\underline{x}}, \hat{\underline{u}})$ is a Pareto optimal solution of Problem $(PM_k^1)$.*
- *If $\theta_j \geq 0$ for all $j \in \{1, \ldots, \ell\}$ and $\theta = (\theta_1, \ldots, \theta_\ell) \neq 0$ then $(\hat{\underline{x}}, \hat{\underline{u}})$ is a weak Pareto optimal solution of Problem $(PM_k^1)$.*

*Proof.* If $\theta_j > 0$ for all $j \in \{1, \ldots, \ell\}$ and $(\hat{\underline{x}}, \hat{\underline{u}})$ is not a Pareto optimal solution of Problem $(PM_k^1)$; thus there exists a process $(\underline{x}, \underline{u}) \in Dom_k(J)$ such that for all $j \in \{1, \ldots, \ell\}$, $J_j(\underline{x}, \underline{u}) \geq J_j(\hat{\underline{x}}, \hat{\underline{u}})$ and for some $i \in \{1, \ldots, \ell\}$, $J_i(\underline{x}, \underline{u}) > J_i(\hat{\underline{x}}, \hat{\underline{u}})$. And so

$\sum_{j=1}^{\ell} \theta_j J_j(\underline{x}, \underline{u}) > \sum_{j=1}^{\ell} \theta_j J_j(\hat{\underline{x}}, \hat{\underline{u}})$ which is a contradiction since $(\hat{\underline{x}}, \hat{\underline{u}})$ is the solution of $(Q_k^1)$. Hence, $(\hat{\underline{x}}, \hat{\underline{u}})$ is a Pareto optimal solution of Problem $(PM_k^1)$.

Now if $\theta_j \geq 0$ for all $j \in \{1, \ldots, \ell\}$ and $\theta \neq 0$. Assume that $(\hat{\underline{x}}, \hat{\underline{u}})$ is not a weak Pareto optimal solution of Problem $(PM_k^1)$. Thus there exists a process $(\underline{x}, \underline{u}) \in Dom_k(J)$ such that for all $j \in \{1, \ldots, \ell\}$, $J_j(\underline{x}, \underline{u}) > J_j(\hat{\underline{x}}, \hat{\underline{u}})$. And so, since $\theta \neq 0$, $\sum_{j=1}^{\ell} \theta_j J_j(\underline{x}, \underline{u}) > \sum_{j=1}^{\ell} \theta_j J_j(\hat{\underline{x}}, \hat{\underline{u}})$ which is a contradiction since $(\hat{\underline{x}}, \hat{\underline{u}})$ is the solution of $(Q_k^1)$. Hence, $(\hat{\underline{x}}, \hat{\underline{u}})$ is a weak Pareto optimal solution of Problem $(PM_k^1)$. □

**Lemma 5.48.** *Assume that $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^3)$. The two following assertions hold.*

- *If $\theta_j > 0$ for all $j \in \{1, \ldots, \ell\}$ then $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(PM_k^3)$.*
- *If $\theta_j \geq 0$ for all $j \in \{1, \ldots, \ell\}$ and $\theta \neq 0$ then $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(PM_k^{3'})$.*

*Proof.* If $\theta_j > 0$ for all $j \in \{1, \ldots, \ell\}$ and $(\hat{\underline{x}}, \hat{\underline{u}})$ is not a solution of Problem $(PM_k^3)$; thus there exists a process $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$ such that for all $j \in \{1, \ldots, \ell\}$,

$$\liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t^j(x_t, u_t) - \sum_{t=0}^{h} \phi_t^j(\hat{x}_t, \hat{u}_t)\Big) \geq 0$$

and for some $i \in \{1, \ldots, \ell\}$,

$$\liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_{i,t}(x_t, u_t) - \sum_{t=0}^{h} \phi_{i,t}(\hat{x}_t, \hat{u}_t)\Big) > 0.$$

So we have

$$\liminf_{h \to +\infty} \sum_{t=0}^{h} \sum_{j=1}^{\ell} \theta_j (\phi_t^j(x_t, u_t) - \phi_t^j(\hat{x}_t, \hat{u}_t)) \geq \sum_{j=0}^{\ell} \theta_j \liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t^j(x_t, u_t) - \sum_{t=0}^{h} \phi_t^j(\hat{x}_t, \hat{u}_t)\Big) > 0,$$

which contradicts the assumption that $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^3)$. Hence, $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(PM_k^3)$.

If $\theta_j \geq 0$ for all $j \in \{1, \ldots, \ell\}$ and $\theta \neq 0$. Assume that $(\hat{\underline{x}}, \hat{\underline{u}})$ is not a solution of Problem $(PM_k^{3'})$; thus there exists a process $(\underline{x}, \underline{u}) \in \mathrm{Adm}_k$ such that for all $j \in \{1, \ldots, \ell\}$, $\liminf_{h \to +\infty} (\sum_{t=0}^{h} \phi_t^j(x_t, u_t) - \sum_{t=0}^{h} \phi_t^j(\hat{x}_t, \hat{u}_t)) > 0$ . So we have

$$\liminf_{h \to +\infty} \sum_{t=0}^{h} \sum_{j=1}^{\ell} \theta_j (\phi_t^j(x_t, u_t) - \phi_t^j(\hat{x}_t, \hat{u}_t)) \geq \sum_{j=0}^{\ell} \theta_j \liminf_{h \to +\infty} \Big(\sum_{t=0}^{h} \phi_t^j(x_t, u_t) - \sum_{t=0}^{h} \phi_t^j(\hat{x}_t, \hat{u}_t)\Big) > 0,$$

which contradicts the assumption that $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(Q_k^3)$. Hence, $(\hat{\underline{x}}, \hat{\underline{u}})$ is a solution of Problem $(PM_k^{3'})$. □

Now we provide a theorem which uses the concavity assumption to establish sufficient condition for multiobjective problems.

**Theorem 5.49.** *Let $(\hat{\underline{x}}, \hat{\underline{u}}) \in Dom_e(J)$. Assume that there exist $\theta = (\theta_1, \ldots, \theta_\ell) \in (\mathbb{R}^\ell)$, $\underline{p} = (p_t)_{t \in \mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, not all zero such that the following conditions hold.*

**(i)** *For all $t \in \mathbb{N}$, $X_t \times U_t$ is convex.*

**(ii)** $\forall j = 1, \ldots, \ell$, $\forall t \in \mathbb{N}$, $(x,u) \mapsto \phi_t^j(x,u)$ *and* $(x,u) \mapsto f_t(x,u)$ *are Fréchet differentiable at* $(\hat{x}_t, \hat{u}_t)$.

**(iii)** $\lim_{t\to+\infty} \langle p_t, x_t - \hat{x}_t\rangle = 0$.

**(iv)** $p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \sum\limits_{j=1}^{\ell} \theta_j . D_1\phi_t^j(\hat{x}_t, \hat{u}_t)$ *for all* $t \in \mathbb{N}_*$.

**(v)** $\langle p_{t+1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t) + \sum\limits_{j=1}^{\ell} \theta_j D_2\phi_t^j(\hat{x}_t, \hat{u}_t)), u - \hat{u}_t\rangle \leq 0$ *for all* $u \in U$, *for all* $t \in \mathbb{N}$.

**(vi)** *The function* $[(x,u) \mapsto \langle p_{t+1}, f_t(x,u)\rangle + \sum\limits_{j=1}^{\ell} \theta_j . \phi_t^j(x,u))]$ *is concave on* $X_t \times U_t$ *for all* $t \in \mathbb{N}$.

*Then*

- *If* $\theta_j > 0$ *for all* $j \in \{1, \ldots, \ell\}$ *then* $(\underline{\hat{x}}, \underline{\hat{u}})$ *is a Pareto optimal solution of Problem* $(PM_e^1)$.
- *If* $\theta_j \geq 0$ *for all* $j \in \{1, \ldots, \ell\}$ *and* $\theta = (\theta_1, \ldots, \theta_\ell) \neq 0$ *then* $(\underline{\hat{x}}, \underline{\hat{u}})$ *is a weak Pareto optimal solution of Problem* $(PM_e^1)$.

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}})$, $(\underline{x}, \underline{u}) \in \mathrm{Dom}_e(J)$, i.e. $\hat{x}_t$, $x_t \in X_t$, $\hat{u}_t$, $u_t \in U_t$, $\hat{x}_{t+1} = f_t(\hat{x}_t, \hat{u}_t)$, $x_{t+1} = f_t(x_t, u_t)$ for all $t \in \mathbb{N}$, $\hat{x}_0 = x_0 = \eta$ and $J_j(\underline{\hat{x}}, \underline{\hat{u}}) = \sum_{t=0}^{+\infty} \phi_t^j(\hat{x}_t, \hat{u}_t)$, $J_j(\underline{x}, \underline{u}) = \sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t)$ exist in $\mathbb{R}$ for all $j \in \{1, \ldots, \ell\}$. From (ii) and (iv) and from the definition of Hamiltonian of Pontryagin we obtain

$$D_1 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) = p_t. \tag{5.36}$$

From (vi) we obtain, for all $(x_t, u_t) \in X_t \times U_t$ and for all $t \in \mathbb{N}$,

$$\begin{aligned} H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - H_t(\theta, x_t, u_t, p_{t+1}) - \langle D_1 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{x}_t - x_t\rangle \\ -\langle D_2 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{u}_t - u_t\rangle \quad \geq 0. \end{aligned} \tag{5.37}$$

From (v) the relation

$$\langle D_2 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{u}_t - u_t\rangle \geq 0 \tag{5.38}$$

holds for all $u_t \in U_t$ and for all $t \in \mathbb{N}$. For all $t \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{j=1}^{\ell} \theta_j . \left(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\right) &= H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t)\rangle \\ &\quad -H_t(\theta, x_t, u_t, p_{t+1}) + \langle p_{t+1}, f_t(x_t, u_t) \\ &= H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - H_t(\theta, x_t, u_t, p_{t+1}) \\ &\quad -\langle p_{t+1}, \hat{x}_{t+1} - x_{t+1}\rangle. \end{aligned}$$

Then, using (5.36) and (5.38) we obtain

$$\begin{aligned} \sum_{j=1}^{\ell} \theta_j . \left(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\right) &\geq H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - H_t(\theta, x_t, u_t, p_{t+1}) \\ &\quad -\langle D_2 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{u}_t - u_t\rangle \\ &\quad -\langle D_1 H_{t+1}(\hat{x}_{t+1}, \hat{u}_{t+1}, p_{t+2}), \hat{x}_{t+1} - x_{t+1}\rangle, \end{aligned}$$

which implies

$$\begin{aligned} \sum_{j=1}^{\ell} \theta_j . \left(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\right) &\geq [H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - H_t(\theta, x_t, u_t, p_{t+1}) \\ &\quad -\langle D_1 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{x}_t - x_t\rangle \\ &\quad -\langle D_2 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{u}_t - u_t\rangle] \\ &\quad +[\langle D_1 H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{x}_t - x_t\rangle \\ &\quad -\langle D_1 H_{t+1}(\hat{x}_{t+1}, \hat{u}_{t+1}, p_{t+2}), \hat{x}_{t+1} - x_{t+1}\rangle], \end{aligned}$$

and using (5.37) we obtain

$$\begin{aligned}\sum_{j=1}^{\ell}\theta_j.\left(\phi_t^j(\hat{x}_t,\hat{u}_t)-\phi_t^j(x_t,u_t)\right) &\geq [\langle D_1H_t(\theta,\hat{x}_t,\hat{u}_t,p_{t+1}),\hat{x}_t-x_t\rangle\\ &\quad -\langle D_1H_{t+1}(\hat{x}_{t+1},\hat{u}_{t+1},p_{t+2}),\hat{x}_{t+1}-x_{t+1}\rangle].\end{aligned}$$

Therefore, using (5.36), we obtain, for all $T\in\mathbb{N}_*$,

$$\begin{aligned}\sum_{t=0}^{T}\sum_{j=1}^{\ell}\theta_j.\left(\phi_{j,t}(\hat{x}_t,\hat{u}_t)-\phi_t^j(x_t,u_t)\right) &\geq \langle D_1H_0(\eta,\hat{u}_0,p_1),\eta-\eta)\rangle\\ &\quad -\langle p_{T+1},\hat{x}_{T+1}-x_{T+1}\rangle\end{aligned}$$

$$\Longrightarrow \sum_{t=0}^{T}\sum_{j=1}^{\ell}\theta_j.\phi_{j,t}(\hat{x}_t,\hat{u}_t)-\sum_{t=0}^{T}\sum_{j=1}^{\ell}\theta_j.\phi_t^j(x_t,u_t)\geq -\langle p_{T+1},\hat{x}_{T+1}-x_{T+1}\rangle. \tag{5.39}$$

Using (iii), we have $\lim_{T\to+\infty}(-\langle p_{T+1},\hat{x}_{T+1}-x_{T+1}\rangle)=0$, and then, from (5.39), doing $T\to+\infty$ we obtain $\sum_{j=1}^{\ell}\theta_jJ_j(\underline{\hat{x}},\underline{\hat{u}})-\sum_{j=1}^{\ell}\theta_jJ_j(\underline{x},\underline{u}))\geq 0$ which implies $(\underline{\hat{x}},\underline{\hat{u}})$ is a solution of Problem $(Q_e^1)$. Finally, using Lemma 5.47 obtain the conclusion of this theorem. □

One can weaken the hypothesis of concavity of $H_t$ with respect to $x_t$ and $u_t$ and replace it by the concavity of $H_t^*$ with respect to $x_t$ as the following theorem shows. Let $U_t$ be compact for all $t\in\mathbb{N}$ and let $H_t^*(\theta,x_t,p_{t+1})=\max_{u_t\in U_t}H_t(\theta,x_t,u_t,p_{t+1})$. The maximum is atteined since $U_t$ is compact. The following lemma is useful before stating the sufficient condition with the concavity of $H_t^*$.

**Lemma 5.50.** *Let $A$ be a convex subset of $\mathbb{R}^n$ and $\gamma$ a real concave function defined on $A$. Let $\hat{(z)}$ be an interior point of $A$. Let $\phi$ be a real function defined on a ball $B(\hat{z},\delta)$ such that $\phi$ is differentiable at $\hat{z}$, $\phi(\hat{z})=\gamma(\hat{z})$ and $\phi(z)\leq\gamma(z)$ for all $z\in B(\hat{z},\delta)$.*

*Then for all $z\in B(\hat{z},\delta)$, $\gamma(\hat{z})-\gamma(z)\geq D\phi(\hat{z})(\hat{z}-z)$.*

The proof of this lemma can be found in Chapter 5, §23 of [52].

**Theorem 5.51.** *Let $(\underline{\hat{x}},\underline{\hat{u}})\in Dom_e(J)$. Assume that there exist $\theta=(\theta_1,\ldots,\theta_\ell)\in(\mathbb{R}^\ell)$, $\underline{p}=(p_t)_{t\in\mathbb{N}_*}\in(\mathbb{R}^{n*})^{\mathbb{N}_*}$, not all zero such that the conditions (iii) and (iv) of Theorem 5.49 hold together with the following hypotheses*

**(i')** *$\forall t\in\mathbb{N}$, $X_t$ is convex and $U_t$ is compact.*

**(ii')** *$\forall j=1,\ldots,\ell$, $\forall t\in\mathbb{N}$, $\forall u\in U_t$, $x\mapsto\phi_t^j(x,u)$ and $x\mapsto f_t(x,u)$ are Fréchet differentiable at $\hat{x}_t$.*

**(v')** *$\forall t\in\mathbb{N}$, $H_t(\theta,\hat{x}_t,\hat{u}_t,p_{t+1})=\max_{u_t\in U_t}H_t(\theta,x_t,u_t,p_{t+1})$.*

**(vi')** *The function $H_t^*$ is concave with respect to $x_t$ on $X_t$ for all $t\in\mathbb{N}$.*

*Then the conclusion of Theorem 5.49 holds.*

*Proof.* Let $(\underline{\hat{x}},\underline{\hat{u}})$, $(\underline{x},\underline{u})\in\mathrm{Dom}_e(J)$, i.e. $\hat{x}_t$, $x_t\in X_t$, $\hat{u}_t$, $u_t\in U_t$, $\hat{x}_{t+1}=f_t(\hat{x}_t,\hat{u}_t)$, $x_{t+1}=f_t(x_t,u_t)$ for all $t\in\mathbb{N}$, $\hat{x}_0=x_0=\eta$ and $J_j(\underline{\hat{x}},\underline{\hat{u}})=\sum_{t=0}^{+\infty}\phi_t^j(\hat{x}_t,\hat{u}_t)$, $J_j(\underline{x},\underline{u})=\sum_{t=0}^{+\infty}\phi_t^j(x_t,u_t)$ exist in $\mathbb{R}$ for all $j\in\{1,\ldots,\ell\}$. For all $t\in\mathbb{N}$, we have

$$\left.\begin{aligned}\sum_{j=1}^{\ell}\theta_j.\left(\phi_t^j(\hat{x}_t,\hat{u}_t)-\phi_t^j(x_t,u_t)\right) &= H_t(\theta,\hat{x}_t,\hat{u}_t,p_{t+1})-\langle p_{t+1},f_t(\hat{x}_t,\hat{u}_t)\rangle\\ &\quad -H_t(\theta,x_t,u_t,p_{t+1})+\langle p_{t+1},f_t(x_t,u_t)\\ &\geq H_t^*(\theta,\hat{x}_t,p_{t+1})-H_t^*(\theta,x_t,p_{t+1})\\ &\quad -\langle p_{t+1},\hat{x}_{t+1}-x_{t+1}\rangle\end{aligned}\right\} \tag{5.40}$$

by the definition of $H_t^*$ and noticing that

$$H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) = H_t^*(\theta, \hat{x}_t, p_{t+1}),$$

then we have

$$-H_t(\theta, x_t, u_t, p_{t+1}) \geq -H_t(\theta, x_t, \hat{u}_t, p_{t+1}) = -H_t^*(\theta, x_t, p_{t+1}).$$

Take $T \in \mathbb{N}_*$. From (5.40), one has

$$\begin{aligned}&\sum_{t=0}^{T}\sum_{j=1}^{\ell}\theta_j.\left(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\right)\\ \geq &\sum_{t=0}^{T}\left(H_t^*(\theta, \hat{x}_t, p_{t+1}) - H_t^*(\theta, x_t, p_{t+1}) - \langle p_{t+1}, \hat{x}_{t+1} - x_{t+1}\rangle\right).\end{aligned}$$

The right hand side of this inequation can be written as

$$\begin{aligned}\text{RHS} = &H_0^*(\hat{x}_0, p_1) - H_0^*(x_0, p_1) + \sum_{t=1}^{T}\left(H_t^*(\theta, \hat{x}_t, p_{t+1}) - H_t^*(\theta, x_t, p_{t+1}) - \langle p_t, \hat{x}_t - x_t\rangle\right)\\ &- \langle p_{T+1}, \hat{x}_{T+1} - x_{T+1}\rangle\\ = &\sum_{t=1}^{T}\left(H_t^*(\theta, \hat{x}_t, p_{t+1}) - H_t^*(\theta, x_t, p_{t+1}) - \langle D_1H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{x}_t - x_t\rangle\right)\\ &- \langle p_{T+1}, \hat{x}_{T+1} - x_{T+1}\rangle.\end{aligned}$$

Now using the assumptions on $H_t^*$ and Lemma 5.50, for all $t \in \{1, \ldots, T\}$, one can obtain

$$H_t^*(\theta, \hat{x}_t, p_{t+1}) - H_t^*(\theta, x_t, p_{t+1}) - \langle D_1H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}), \hat{x}_t - x_t\rangle \geq 0.$$

Thus, we have

$$\sum_{t=0}^{T}\sum_{j=1}^{\ell}\theta_j.\left(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\right) \geq \langle p_{T+1}, \hat{x}_{T+1} - x_{T+1}\rangle.$$

The end of this proof goes like that of the previous theorem. □

**Corollary 5.52.** *If $(\underline{\hat{x}}, \underline{\hat{u}})$ satisfies all hypotheses of Theorem 5.49 or Theorem 5.51 except for hypothesis (iii), which is replaced by (vii) $\limsup_{t\to+\infty}\langle p_t, x_t - \hat{x}_t\rangle = 0$ or (viii) $\liminf_{h\to+\infty}\langle p_t, x_t - \hat{x}_t\rangle = 0$, then*

- *If $\theta_j > 0$ for all $j \in \{1, \ldots, \ell\}$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of Problem $(PM_e^3)$.*
- *If $\theta_j \geq 0$ for all $j \in \{1, \ldots, \ell\}$ and $\theta \neq 0$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of Problem $(PM_e^{3'})$.*

*Proof.* It is analogous to that of Theorem 5.49 or Theorem 5.51 until we obtain

$$\limsup_{t\to+\infty}\sum_{t=0}^{T}\sum\nolimits_{j=1}^{\ell}(\theta_j\phi_t^j(\hat{x}_t, \hat{u}_t) - \theta_j\phi_t^j(x_t, u_t)) \geq 0$$

using (vii), or equivalently,

$$\liminf_{t\to+\infty}\sum_{t=0}^{T}\sum\nolimits_{j=1}^{\ell}(\theta_j\phi_t^j(x_t, u_t) - \theta_j\phi_t^j(\hat{x}_t, \hat{u}_t)) \leq 0.$$

And so, $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of $(Q_e^3)$ then using Lemma 5.48, we obtain the announced conclusion.

Now under (viii), by analogous realization, we obtain $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of $(Q_e^2)$ and therefore, of $(Q_e^3)$ also. Then using Lemma 5.48 again to obtain the conclusion of this corollary. □

**Theorem 5.53.** *Let $(\underline{\hat{x}}, \underline{\hat{u}}) \in Dom_i(J)$. Assume that there exist $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{R}^\ell$, $\underline{p} = (p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, not all zero such that the conditions (i-vi) of Theorem 5.49 and moreover, the following conditions hold.*

**(ix)** *For all $t \in \mathbb{N}_*$, $p_t \geq 0$.*

**(x)** *For all $t \in \mathbb{N}$, $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$.*

*Then*

- *If $\theta_j > 0$ for all $j \in \{1, \dots, \ell\}$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a Pareto optimal solution of Problem $(PM_i^1)$.*
- *If $\theta_j \geq 0$ for all $j \in \{1, \dots, \ell\}$ and $\theta = (\theta_1, \dots, \theta_\ell) \neq 0$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a weak Pareto optimal solution of Problem $(PM_i^1)$.*

*Proof.* Let $(\underline{\hat{x}}, \underline{\hat{u}}), (\underline{x}, \underline{u}) \in \mathrm{Dom}_i(J)$, i.e. $\hat{x}_t,\ x_t \in X_t,\ \hat{u}_t,\ u_t \in U_t,\ \hat{x}_{t+1} \leq f_t(\hat{x}_t, \hat{u}_t),\ x_{t+1} \leq f_t(x_t, u_t)$ for all $t \in \mathbb{N}$, $\hat{x}_0 = x_0 = \eta$ and $J_j(\underline{\hat{x}}, \underline{\hat{u}}) = \sum_{t=0}^{+\infty} \phi_t^j(\hat{x}_t, \hat{u}_t)$, $J_j(\underline{x}, \underline{u}) = \sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t)$ converge in $\mathbb{R}$ for all $j \in \{1, \dots, \ell\}$. By doing like in Theorem 5.49, we obtain (5.36), (5.37) and (5.38) hold.

For all $t \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{j=1}^{\ell} \theta_j . \Big(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\Big) &= H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t)\rangle \\
&\quad - H_t(\theta, x_t, u_t, p_{t+1}) + \langle p_{t+1}, f_t(x_t, u_t) \\
&= H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - H_t(\theta, x_t, u_t, p_{t+1}) \\
&\quad - \langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle - \langle p_{t+1}, \hat{x}_{t+1}\rangle \\
&\quad + \langle p_{t+1}, f_t(x_t, u_t)\rangle.
\end{aligned}$$

Notice that $\langle p_{t+1}, f_t(\hat{x}_t, \hat{u}_t) - \hat{x}_{t+1}\rangle = 0$ from (x). Moreover, since $f_t(x_t, u_t) - x_{t+1} \geq 0$, from (ix) we obtain $\langle p_{t+1}, f_t(x_t, u_t) - x_{t+1}\rangle \geq 0$, or equivalently, $\langle p_{t+1}, f_t(x_t, u_t) \geq \langle p_{t+1}, x_{t+1}\rangle$. Hence, we have

$$\begin{aligned}
\sum_{j=1}^{\ell} \theta_j . \Big(\phi_t^j(\hat{x}_t, \hat{u}_t) - \phi_t^j(x_t, u_t)\Big) &\geq H_t(\theta, \hat{x}_t, \hat{u}_t, p_{t+1}) - H_t(\theta, x_t, u_t, p_{t+1}) \\
&\quad - \langle p_{t+1}, \hat{x}_{t+1} - x_{t+1}\rangle.
\end{aligned}$$

Now by a similar interpretation as in Theorem 5.49 and using Lemma 5.47, we obtain the announced conclusion. □

**Theorem 5.54.** *Let $(\underline{\hat{x}}, \underline{\hat{u}}) \in Dom_i(J)$. Assume that there exist $\theta = (\theta_1, \dots, \theta_\ell) \in (\mathbb{R}^\ell)$, $\underline{p} = (p_t)_{t\in\mathbb{N}_*} \in (\mathbb{R}^{n*})^{\mathbb{N}_*}$, not all zero and such that all the hypotheses of Theorem 5.51 together with hypotheses (ix) and (x). Then the conclusion of Theorem 5.53 hold.*

*Proof.* The proof is similar to that of Theorem 5.51. □

**Corollary 5.55.** *If $(\underline{\hat{x}}, \underline{\hat{u}})$ satisfies all hypotheses of Theorem 5.53 or Theorem 5.54 except for hypothesis (iii), which is replaced by (vii) or (viii), then*

- *If $\theta_j > 0$ for all $j \in \{1, \dots, \ell\}$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of Problem $(PM_i^3)$.*
- *If $\theta_j \geq 0$ for all $j \in \{1, \dots, \ell\}$ and $\theta = (\theta_1, \dots, \theta_\ell) \neq 0$ then $(\underline{\hat{x}}, \underline{\hat{u}})$ is a solution of Problem $(PM_i^{3'})$.*

*Proof.* The proof is similar to the proof of Corollary 5.52. □

**Remark 5.56.** In case of bounded sequences, i.e. $\underline{x} \in \ell^\infty(\mathbb{N}, \mathbb{R}^n)$ and $\underline{u} \in \ell^\infty(\mathbb{N}, \mathbb{R}^d)$, it is clear that $(x_t - \hat{x}_t)_{t\in\mathbb{N}}$ is a bounded sequence. Moreover, under the setting and hypotheses showed in [15],[14], $(p_t)_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}, \mathbb{R}^{n*})$ and then the condition $\lim_{t\to+\infty} p_t = 0$ automatically holds.

# Notations

$\forall$ universal quantifier, "for every"
$\exists$ existential quantifier, "there exists"
$\Rightarrow$ sign of implication, "... implies ..."
$\Leftrightarrow$ sign of equivalence
$:=$ by definition, is equal to
$x \in X$ the element $x$ belongs to the set $X$
$x \notin X$ the element $x$ does not belong to the set $X$
$|X|$ the cardinality of set X when it is a finite set, i.e. the number its elements
$\emptyset$ empty set
$A \cup B$ the union of the sets A and B
$A \cap B$ the intersection of the sets A and B
$A \setminus B$ the difference of the sets A and B, i.e., the set of elements that belong to the set A but do not belong to the set B
$A \subset B$ the set A is contained in the set B
$A \times B$ the Cartesian product of the sets A and B
$\{x : P(x)\}$ the set of those elements $x$ that possess the property $P(.)$
$\{x_1, ..., x_n, ...\}$ the set which consists of elements $x_1, ..., x_n, ...$
$F : X \to Y$ the mapping $F$ of the set $X$ into the set $Y$; the function $F$ with domain $X$ whose values belong to the set $Y$
$x \mapsto F(x)$ the mapping (function) $F$ assigns the element $F(x)$ to an element $x$; the notation of the mapping (function) $F$ in the case when it is desirable to indicate the notation of its argument
$F(.)$ the notation which stresses that $F$ is a mapping (function)
$F(A)$ the image of the set $A$ under the mapping $F$
$imF = \{y : y = F(x), x \in X\}$ the image of the mapping $F : X \to Y$
$F^{-1}(A)$ the inverse image of the set $A$ under the mapping $F$
$F_{|A}$ the restriction of the mapping $F$ to the set $A$
$F \circ G$ the composition of the mappings $G$ and $F$: $(F \circ G)(x) = F(G(x))$
$\mathbb{R}$ the set of all real numbers
$\mathbb{N}$ the set of all nonnegative integers; $\mathbb{N}_* = \mathbb{N} \backslash \{0\}$
$\inf A$ ($\sup A$) the infimum (supremum) of the numbers which belong to the set $A \in \mathbb{R}$
$\mathbb{R}^n$ the arithmetical n-dimensional space endowed with the standard Euclidean structure
$\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in R^n : x_i \geq 0\}$ the nonnegative orthant of $\mathbb{R}^n$
$e_1, ..., e_n$ the vectors of the standard orthonormal basis in $\mathbb{R}^n$; $e_1 = (1, 0, ..., 0), ..., e_n = (0, ..., 0, 1)$
$[x, y] = \{z : z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\}$ the line segment joining the points x and y
$\mathbb{R}^{n*}$ the arithmetical n-dimensional space conjugate to $\mathbb{R}^n$
$px = \langle p, x \rangle = \sum_{i=1}^n p_i x_i$ for all $p \in \mathbb{R}^{n*}$ and for all $x \in \mathbb{R}^n$
$\|x\|$ the norm of element $x$ in normed space

$\underline{x}$ the sequence $(x_t)_{t\in\mathbb{N}}$

$\ell^p(\mathbb{N}, U), p \geq 1$ space of all sequences $\underline{x}$ from $U^{\mathbb{N}}$ satisfying $\sum_{t=0}^{+\infty} |x_t|^p < +\infty$

$\ell^\infty(\mathbb{N}, U)$ space of all sequences $\underline{x}$ from $U^{\mathbb{N}}$ satisfying $\sum_{t=0}^{+\infty} |x_t| < +\infty$

$c(\mathbb{N}, U)$ space of all sequences $\underline{x}$ from $U^{\mathbb{N}}$ which converge in $U$

$c_0(\mathbb{N}, U)$ space of all sequences $\underline{x}$ from $U^{\mathbb{N}}$ which converge to 0

$c_{00}(\mathbb{N}, U)$ space of all sequences $\underline{x}$ from $U^{\mathbb{N}}$ satisfying $x_t = 0$ except for finitely many indexes $t$

$\overline{A}$ the closure of the set $A$

$intA$ the interior of the set $A$

$spanA$ the linear span of the vectors of set $A$

$convA$ the convex hull of the set $A$

$X^*$ the conjugate space of $X$

$x^*$ an element of the conjugate $X^*$

$\langle x^*, x\rangle$ the value of the linear functional $x^* \in X^*$ on the element $x \in X$

$dimL$ the dimension of the space $L$

$\mathcal{L}(X, Y)$ the space of continuous linear mappings of the space $X$ into the space $Y$. When $X = \mathbb{R}^n, Y = \mathbb{R}^m$ the mappings belonging to the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ can be identified with their matrices relative to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

$Ker\Lambda = \{x : \Lambda x = 0\}$ the kernel of the linear operator $\Lambda$

$Im\Lambda = \{y : y = \Lambda x\}$ the image of the linear operator $\Lambda$

$\overrightarrow{D}F(x; h)$ the directional derivative of the function $F$ at the point $x$ in the direction of the vector $h$

$D_G F(x)$ the Gâteaux differential of the mapping $F$ at the point $x$

$DF(x)$ the Fréchet differential of the mapping $F$ at the point $x$

$\partial F(x)$ the subdifferential of the function $F$ at the point $x$

$\vec{D}_1 F(x_1, x_2; h)$ ($\vec{D}_2 F(x_1, x_2; h)$) the partial directional derivative of $F$ w.r.t $x_1$ ($x_2$) at $(x_1, x_2)$ in the direction of $h$

$D_{G,1} F(x_1, x_2)$ ($D_{G,2} F(x_1, x_2)$) the partial Gâteaux differential of $F$ w.r.t $x_1$ ($x_2$) at $(x_1, x_2)$

$D_1 F(x_1, x_2)$ ($D_2 F(x_1, x_2)$) the partial Fréchet differential of $F$ w.r.t $x_1$ ($x_2$) at $(x_1, x_2)$

$\partial_1 F(x_1, x_2)$ ($\partial_2 F(x_1, x_2)$) the partial subdifferential of $F$ w.r.t $x_1$ ($x_2$) at $(x_1, x_2)$

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