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UNIVERSITÉ PIERRE ET MARIE CURIE

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Unité de recherche **Laboratoire Jacques-Louis Lions, UMR 7598**

Thèse présentée par **Clément MIRSUD**

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Discipline **Mathématiques**

**Méthodes variationnelles et
hyperboliques appliquées aux systèmes
mécaniques sous contrainte**

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MÉTHODES VARIATIONNELLES ET HYPERBOLIQUES APPLIQUÉES AUX SYSTÈMES MÉCANIQUES SOUS CONTRAINTE**Résumé**

Dans cette thèse, nous nous intéressons aux équations aux dérivées partielles hyperboliques sous contraintes ; plus particulièrement aux problèmes provenant de la mécanique de la plasticité parfaite. L'objectif de cette thèse est d'étudier ces problèmes *via* différentes approches, d'analyser les interactions entre ces points de vues distincts et de tirer profit de ces analyses différentes pour obtenir de nouveaux résultats. Un bref historique de l'origine mécanique des problèmes de la plasticité parfaite ainsi que des résultats précédemment obtenus sont décrits dans le Chapitre 1.

Dans le Chapitre 2, nous concentrons notre attention sur les systèmes hyperboliques avec conditions de bord. Dans un premier temps, nous développons une théorie faible pour ces problèmes et expliquons dans un cas simplifié le caractère bien posé de cette théorie. Puis, nous introduisons de manière similaire la notion de solution faible pour des systèmes hyperboliques avec condition de bord soumis à une contrainte.

Le Chapitre 3 est dédié à l'étude d'un modèle simplifié de la dynamique de la plasticité parfaite. Nous confrontons l'approche introduite au chapitre précédent avec celle, plus classique, provenant du calcul des variations qui permet d'obtenir l'existence et l'unicité des solutions pour ce modèle. Cela nous permet de mettre en évidence une nouvelle interaction entre les conditions de bord et les contraintes ainsi que d'aboutir à un théorème de régularité des solutions en temps courts.

Enfin, dans le Chapitre 4, nous nous intéressons à l'approximation numérique des systèmes hyperboliques sous contraintes grâce à des schémas de type volumes finis. Ce travail nous permet d'obtenir un résultat de convergence pour les problèmes sans bord et d'illustrer numériquement les interactions entre les conditions de bord et les contraintes sur l'exemple du chapitre précédent.

Mots clés : systèmes hyperboliques, plasticité parfaite, condition de bord, volumes finis

VARIATIONAL AND HYPERBOLIC METHODS APPLIED TO CONSTRAINED MECHANICAL SYSTEMS**Abstract**

In this thesis, we consider constrained hyperbolic partial differential equations and more precisely mechanical problems coming from perfect plasticity. The goal of this thesis is to study these problems thanks to different approaches, to analyze the interactions between these different points of view and to confront these various analyzes to get new results. A brief review of the mechanical origin of perfect plasticity problems and also of the previous results on these topics are described in Chapter 1.

In Chapter 2, we focus our attention on hyperbolic systems with boundary conditions. First, we develop a weak theory for these problems and explain, in a simplified case, why this theory is well-posed. Then, we introduce similarly a notion of weak solutions for constrained hyperbolic systems with boundary conditions.

Chapter 3 is devoted to the study of the simplified model of dynamical perfect plasticity. We confront the approach introduced in the previous chapter with the one, more standard, coming from calculus of variations that allows us to obtain existence and uniqueness of the solutions for this model. It allows us to bring to light a new interaction between the boundary conditions and the constraints and to get a short-time regularity theorem.

Lastly, in Chapter 4, we are interested in the numerical approximation of constrained hyperbolic systems thanks to finite volume schemes. This work allows us to get a convergence result for problems without boundary condition and to show numerically the link between boundary conditions and constraints on the example of the previous chapter.

Keywords: hyperbolic systems, perfect plasticity, boundary condition, finite volume

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Introduction générale

Le but de cette thèse est d'étudier des problèmes soumis à une contrainte issue de la mécanique. Pour cela, nous avons choisi d'envisager deux approches. La première utilise majoritairement des méthodes provenant du calcul des variations et la seconde est basée sur des outils de la théorie des équations aux dérivées partielles hyperboliques.

Pour illustrer ce que nous entendons par problème mécanique sous contrainte, commençons par présenter un modèle très simple : le modèle patin – ressort (nous renvoyons le lecteur à [72, Section 3.1.1] pour plus de détails). Un patin est amené à se déplacer le long d'une droite. De plus, un ressort est attaché à ce patin et une force de traction peut être appliquée à ce ressort. Illustrons la situation avec la figure suivante.



FIGURE 1.1 – Le patin (trait plein) et le ressort (pointillé) au repos.

Le ressort peut subir une traction dont l'intensité ne peut dépasser un certain seuil, noté $\sigma_c > 0$. Deux comportements mécaniques sont alors observés. Lorsque la traction a une intensité (en norme) strictement inférieure à σ_c , nous sommes alors dans un régime dit élastique (cf Figure 1.2), le ressort subit une élongation.

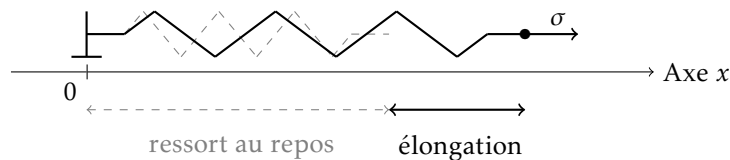


FIGURE 1.2 – Élongation du ressort en régime élastique (traits pleins noirs). La configuration patin – ressort au repos est rappelée en pointillés grisés.

Lorsque la traction atteint le seuil du ressort, nous sommes alors en régime dit plastique. Le patin subit un déplacement illustré en Figure 1.3.

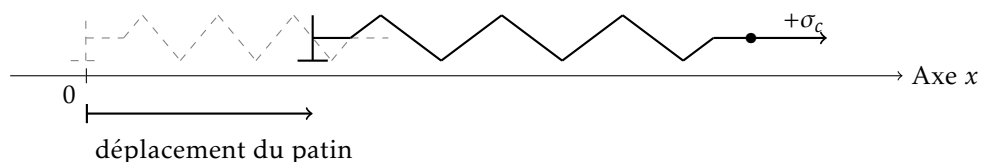


FIGURE 1.3 – Déplacement du patin en régime plastique (traits pleins noirs). La configuration patin – ressort au repos est rappelée en pointillés grisés.

L'unique information sur le déplacement du patin, dont nous disposons en régime plastique, est que si la force de traction vaut $+\sigma_c$ le déplacement se fait vers la droite et si la force de traction vaut $-\sigma_c$ le déplacement se fait vers la gauche.

Ce modèle patin – ressort est une illustration simple (car unidimensionnelle) d’une théorie mécanique plus générale, dite de la plasticité parfaite. L’étude mathématique des problèmes de la plasticité parfaite a permis d’illustrer l’intérêt et la complémentarité des deux approches mathématiques que nous avons évoquées précédemment.

Dans cette introduction, nous expliquons tout d’abord, dans la Section 1.1.1, l’origine mécanique des équations, de la plasticité parfaite, que nous considérons. Puis, nous rappelons dans la Section 1.2 les différents résultats mathématiques obtenus grâce aux outils du calcul des variations. Dans la Section 1.3.1, nous montrons sur un exemple pourquoi ces problèmes peuvent être aussi considérés comme des systèmes hyperboliques sous contraintes. Ensuite, nous décrivons rapidement, dans la Section 1.4, les résultats disponibles dans la littérature pour étudier de tels systèmes hyperboliques. Un autre aspect de notre travail porte sur l’approximation numérique des systèmes hyperboliques sous contraintes, nous expliquons, dans la Section 1.5, les méthodes employées dans cette thèse. Enfin, nous effectuons, dans la Section 1.6 un résumé succinct des différents résultats que le lecteur pourra retrouver dans cette thèse.

1.1 Les équations de la plasticité parfaite

1.1.1 Le modèle mécanique

L’objet de cette section est de décrire l’origine physique des équations de la dynamique de plasticité parfaite. Le lecteur pourra consulter la référence [51] pour un développement plus détaillé de la mécanique des milieux continus et la référence [75] pour une analyse précise des phénomènes plastiques.

La mécanique des matériaux plastiques a été étudié par de nombreux scientifiques (COULOMB (1773), CAUCHY (1828), TRESKA (1864), SAINT-VENANT (1870), LÉVY (1870)...). Le but de cette section n’est pas d’effectuer un historique précis des travaux portant sur la plasticité parfaite mais d’exposer succinctement le cadre mécanique et les hypothèses qui régissent le mouvement d’un matériau plastique parfaite. Pour une description plus complète des différentes contributions à l’étude mécanique des matériaux plastiques, nous renvoyons le lecteur à [53].

Un matériau occupe une certaine région de l’espace \mathbb{R}^3 . Partant d’une configuration de référence, noté $\Omega \subset \mathbb{R}^3$, la configuration du matériau va évoluer au cours du temps. Si x_{ref} désigne la position d’un point du matériau dans la configuration de référence, alors sa position, au temps $t > 0$, sera notée

$$x = \chi_t(x_{\text{ref}}).$$

Regarder l’évolution du problème en utilisant la variable x_{ref} correspond à avoir une description eulérienne, alors qu’en utilisant la variable x , il s’agit d’une description lagrangienne. Une hypothèse classique impose alors à la transformation χ_t d’être bijective. En effet, deux points du matériau ne peuvent venir coïncider à un instant t et à tout point du matériau à l’instant t doit pouvoir correspondre un point du matériau à l’instant initial. De plus, l’orientation doit être préservée. Nous introduisons alors le déplacement du matériau $u(t, x_{\text{ref}}) = \chi_t(x_{\text{ref}}) - x_{\text{ref}}$ et le tenseur des déformations

$$Eu = \frac{\nabla \chi_t^t(x_{\text{ref}}) \nabla \chi_t(x_{\text{ref}}) - \text{Id}}{2} = \frac{\nabla u(x_{\text{ref}}) + \nabla u^t(x_{\text{ref}}) + \nabla u^t(x_{\text{ref}}) \nabla u(x_{\text{ref}})}{2}, \quad (1.1.1)$$

où

$$\nabla \chi_t(x_{\text{ref}}) = \left(\frac{\partial (\chi_t(x_{\text{ref}}))_i}{\partial x_{\text{ref}j}} \right)_{1 \leq i, j \leq 3}.$$

Nous allons simplifier ces hypothèses très générales en supposant que le matériau subit de “petites déformations”.

Hypothèse des petites déformations L’hypothèse dite des “petites déformations” signifie que le déplacement u du matériau reste petit et de plus, le gradient $\nabla u = \left(\frac{\partial (u(t, x_{\text{ref}}))_i}{\partial x_{\text{ref}j}} \right)_{1 \leq i, j \leq 3}$ reste lui aussi petit.

Cette hypothèse a pour conséquences :

- Le domaine $\chi_t(\Omega)$ peut être confondu avec Ω . Le déplacement étant petit, la position $\chi_t(x_{\text{ref}})$ peut être confondue avec la position de référence x_{ref} . En particulier, les conditions de bord seront prescrites sur $\partial\Omega$. Pour simplifier les notations, nous noterons simplement x la variable x_{ref} .
- Les descriptions eulérienne et lagrangienne peuvent être identifiées.
- Le tenseur des déformations s’écrit

$$Eu = \frac{\nabla u(x_{\text{ref}}) + \nabla u^t(x_{\text{ref}})}{2}. \quad (1.1.2)$$

En effet, le terme $\nabla u^t(x_{\text{ref}})\nabla u(x_{\text{ref}})$ peut être négligé dans l'équation (1.1.1). L'expression (1.1.2) porte le nom de tenseur des déformations linéarisé.

- La masse volumique du matériau est indépendante du temps. Dans la suite, nous considérons un matériau homogène et nous supposons ainsi que $\rho(t, x_{\text{ref}}) = \rho(0, x_{\text{ref}}) = 1$ pour simplifier.

Effets thermiques Les phénomènes plastiques sont souvent associés à des phénomènes thermiques. Pour une étude détaillée de ces effets thermiques en plasticité, nous renvoyons le lecteur à [51, 75]. Dans cette thèse, nous nous intéressons à des phénomènes isothermes par conséquent, dans la suite, nous négligeons tous les effets thermiques.

Muni des hypothèses précédentes, nous allons chercher à obtenir l'équation régissant le mouvement du matériau.

Équations du mouvement D'après le principe fondamental de la dynamique (ou deuxième loi de NEWTON), la quantité de mouvement est conservée au cours du temps. Ainsi, pour toute sous-région ω de la configuration de référence Ω , nous avons

$$\frac{d}{dt} \int_{\omega} v(t, x) dx = F_{\text{vol}} + F_{\text{surf}},$$

où v est la vitesse du matériau, *i.e.* la dérivée partielle par rapport au temps du déplacement u , F_{vol} désigne les forces de volume (sur ω) qui s'appliquent sur le matériau et F_{surf} les forces de surface (sur $\partial\omega$). Nous rappelons ici que nous avons supposé que la masse volumique était constante égale à 1. Le théorème de CAUCHY suppose que les forces de surface peuvent s'écrire de la manière suivante

$$F_{\text{surf}} = \int_{\partial\omega} \sigma(t, x) \nu(x) d\mathcal{H}^2(x),$$

où le tenseur des contraintes σ est une matrice symétrique de taille 3×3 dépendant du temps et de l'espace, $\nu(x)$ correspond à la normale extérieure au point $x \in \partial\omega$ et \mathcal{H}^2 est la mesure de HAUSSDORFF sur $\partial\omega$. Les forces de volume peuvent être représentées sous la forme suivante

$$F_{\text{vol}} = \int_{\omega} f(t, x) dx,$$

où f est une fonction dépendant de t et x et à valeurs dans \mathbb{R}^3 . Par conséquent, en intégrant par parties, l'équation du mouvement devient

$$\int_{\omega} \partial_t v(t, x) dx = \int_{\omega} f(t, x) dx + \int_{\omega} \text{div} \sigma(t, x) dx,$$

soit en localisant,

$$\partial_t v - \text{div} \sigma = f, \quad \text{sur } [0, T] \times \Omega,$$

où nous avons noté $\partial_t v$ la dérivée de v par rapport au temps.

Il est alors nécessaire de prescrire des conditions au bord du domaine Ω .

Condition au bord du domaine Ω L'hypothèse la plus classique concernant l'évolution de matériaux plastiques est la suivante : les déplacements u sont prescrits sur une partie Γ_u du bord $\partial\Omega$ de Ω et les forces surfaciques $\langle \sigma; \nu \rangle$ sont prescrites sur une partie Γ_σ du bord $\partial\Omega$. Nous supposons de plus que l'intersection de Γ_u et de Γ_σ est vide et leur union est égale à $\partial\Omega$. Ainsi, la condition au bord s'écrit

$$\begin{cases} u = u_b & \text{sur }]0, T[\times \Gamma_u, \\ \sigma \nu = \Sigma_b & \text{sur }]0, T[\times \Gamma_\sigma. \end{cases}$$

Remarque 1.1.1. Lorsque $\Gamma_u = \partial\Omega$ (et par conséquent, $\Gamma_\sigma = \emptyset$), nous sommes en présence d'une condition de bord dite de DIRICHLET. Lorsque $\Gamma_\sigma = \partial\Omega$ (et par conséquent, $\Gamma_u = \emptyset$), nous sommes en présence d'une condition de bord dite de NEUMANN. Dans les autres cas, nous parlerons de condition mixte.

Hypothèses de la plasticité parfaite Les phénomènes de la plasticité parfaite qui nous intéressent sont basés sur les hypothèses mécaniques suivantes :

- Le tenseur (linéarisé) des déformations Eu se décompose de manière additive sous la forme suivante

$$Eu = e + p,$$

où la variable e encode la déformation élastique et la variable p la déformation plastique.

- Il existe une énergie, dite de déformations, noté W , telle que le tenseur σ est lié au tenseur des déformations élastiques e de la manière suivante :

$$\sigma = \frac{\partial}{\partial e} W(e).$$

- Il existe un domaine convexe fixé \mathcal{C} dans lequel σ est astreint à rester. Ce domaine peut s'écrire sous la forme suivante :

$$\mathcal{C} = \left\{ \sigma \in \mathbb{M}_{\text{sym}}^{3 \times 3} \text{ telles que } \mathcal{F}(\sigma) \leq 0 \right\},$$

où \mathcal{F} est une fonction s'appliquant aux matrices 3×3 symétriques et à coefficients réels, dont l'ensemble est noté $\mathbb{M}_{\text{sym}}^{3 \times 3}$. Les phénomènes dus à la plasticité apparaissent lorsque le tenseur σ vérifie $\mathcal{F}(\sigma) = 0$. De plus, l'état $\sigma = 0$ appartient au convexe \mathcal{C} et que l'ensemble \mathcal{C} est fermé.

- L'évolution du tenseur des déformations plastiques suit la règle dite de normalité qui s'énonce sous la forme suivante :

$$\partial_t p \in N_{\mathcal{C}}(\sigma).$$

où $N_{\mathcal{C}}$ est le cône normal à \mathcal{C} au point σ , défini par (pour plus de détails, voir [15, 90])

$$N_{\mathcal{C}}(\sigma) = \left\{ \tau \in \mathbb{M}_{\text{sym}}^{3 \times 3} \text{ telles que } \sup_{\bar{\sigma} \in \mathcal{C}} ((\bar{\sigma} - \sigma) : \tau) \leq 0 \right\},$$

où la notation $\sigma : \tau$ désigne le produit scalaire usuel (*i.e.* termes à termes) de deux matrices. Cette règle de normalité peut se réécrire sous la forme suivante, appelée principe du travail plastique maximal de HILL,

$$\forall \bar{\sigma} \in \mathcal{C}, \quad (\sigma - \bar{\sigma}) : \partial_t p \geq 0.$$

Il est possible de décrire cette évolution de manière équivalente en utilisant l'analyse convexe. Pour cela, observons tout d'abord que le cône normal à \mathcal{C} au point σ est égal au sous-différentiel de la fonction indicatrice de \mathcal{C} , notée $I_{\mathcal{C}}$, au point σ . Pour rappel, la fonction indicatrice de \mathcal{C} au point τ vaut 0 si τ appartient à \mathcal{C} et $+\infty$. Ainsi la loi d'évolution peut se réécrire (formellement dans un premier temps) sous la forme suivante

$$\partial_t p \in \partial I_{\mathcal{C}}(\sigma) = \left\{ \tau \in \mathbb{M}_{\text{sym}}^{3 \times 3} \text{ telles que pour tout } \bar{\sigma} \in \mathbb{M}_{\text{sym}}^{3 \times 3}, ((\bar{\sigma} - \sigma) : \tau) + I_{\mathcal{C}}(\sigma) \leq I_{\mathcal{C}}(\bar{\sigma}) \right\}.$$

Puis, nous savons que ceci est équivalent (voir [15, Proposition 16.9]) à

$$\sigma : \partial_t p = I_{\mathcal{C}}(\sigma) + I_{\mathcal{C}}^*(\partial_t p),$$

où, la fonction $I_{\mathcal{C}}^*$ est la fonction convexe conjuguée de $I_{\mathcal{C}}$ défini par

$$I_{\mathcal{C}}^*(q) = \sup_{\bar{\sigma} \in \mathbb{M}_{\text{sym}}^{3 \times 3}} (\bar{\sigma} : q - I_{\mathcal{C}}(\bar{\sigma})).$$

Puisque $\sigma \in \mathcal{C}$, nous savons que $I_{\mathcal{C}}(\sigma) = 0$. De plus, la conjuguée convexe de la fonction indicatrice de \mathcal{C} , notée $I_{\mathcal{C}}^*$ ci-dessus, est appelée fonction d'appui de \mathcal{C} (nous la noterons $H_{\mathcal{C}}$ dans la suite). Ainsi, l'évolution de $\partial_t p$ suit la loi, dite d'écoulement, suivante

$$\sigma : \partial_t p = H_{\mathcal{C}}(\partial_t p)$$

Les hypothèses de la plasticité parfaite impliquent, en particulier, que le domaine des contraintes \mathcal{C} est indépendant de l'évolution temporelle du matériau. Dans le cas contraire, nous serions en présence de phénomènes de plasticité avec écrouissage.

Ces hypothèses assurent en particulier que l'évolution du matériau plastique obéit au second principe de la thermodynamique, qui demande la non-négativité de la dissipation mécanique. En effet, dans le cadre que nous nous sommes fixés, ce principe peut se formuler de la façon suivante :

$$\sigma : \partial_t p \geq 0.$$

Remarque 1.1.2. *Le fait que le domaine des contraintes \mathcal{C} soit convexe est en fait imposé par le principe du travail plastique maximal de HILL (qui peut s'énoncer pour un domaine des contraintes \mathcal{C} quelconque). Pour plus de détails, voir [72, Section 3.3.5].*

Enfin, pour décrire complètement ces phénomènes de la plasticité parfaite, il nous reste à fixer la fonction d'énergie de déformations W ainsi que le convexe des contraintes \mathcal{C} .

Dans la suite, nous considérons uniquement des matériaux obéissant à la loi de Hooke, c'est-à-dire satisfaisant une relation linéaire entre σ et e . Cela signifie qu'il existe un tenseur d'ordre 4, noté T , tel que

$$\sigma = T(e).$$

Remarque 1.1.3. *L'énergie de dissipation W au point e vaut alors $1/2T(e) : e$.*

De plus, nous supposons que les matériaux étudiés sont isotropes. Dans ce cas, la loi de Hooke s'écrit simplement

$$\sigma = 2\mu e + \lambda \text{tr}(e)\text{id},$$

où id est la matrice identité de taille 3×3 , μ et λ sont les coefficients de LAMÉ vérifiant les inégalités suivantes

$$\mu > 0, \quad \text{et} \quad 2\mu + 3\lambda > 0.$$

Domaine des contraintes Basée sur des expériences pour tester les seuils de plasticité des matériaux, nous retrouvons dans la littérature différents modèles de domaine des contraintes. Nous ne mentionnerons ici que deux modèles : le modèle de VON MISES et le modèle de TRESCA. Dans le cas du modèle de VON MISES, la fonction \mathcal{F} , qui décrit l'ensemble des contraintes $\mathcal{C} = \{\sigma \text{ tels que } \mathcal{F}(\sigma) \leq 0\}$, prend la forme suivante

$$\mathcal{F}(\sigma) = |\sigma^D|^2 - k^2, \quad (1.1.3)$$

où $\sigma^D = \sigma - \frac{\text{tr}\sigma}{3}\text{id}$ est la partie déviatorique du tenseur σ et k est une constante.

Le modèle de TRESCA fait intervenir les valeurs propres du tenseur σ , que nous noterons $\lambda_1(\sigma)$, $\lambda_2(\sigma)$ et $\lambda_3(\sigma)$. Pour ce modèle, la fonction \mathcal{F} est la suivante

$$\mathcal{F}(\sigma) = \max_{i,j} |\lambda_i(\sigma) - \lambda_j(\sigma)| - k, \quad (1.1.4)$$

où, là encore, k est une constante.

Le modèle dynamique de la plasticité parfaite En résumé, nous recherchons un triplet $(u, e, p) : [0, T] \times \Omega \mapsto \mathbb{R}^3 \times \mathcal{S}_3(\mathbb{R}) \times \mathcal{S}_3(\mathbb{R})$ vérifiant le système d'équations suivant

$$\begin{cases} Eu = (\nabla u + \nabla u^t)/2 = e + p, & \text{sur }]0, T[\times \Omega, \\ \sigma = 2\mu e + \lambda \text{tr}(e)\text{id}, & \text{sur }]0, T[\times \Omega, \\ \sigma \in \mathcal{C}, \quad \partial_t p \in N_{\mathcal{C}}(\sigma), & \text{sur }]0, T[\times \Omega, \\ (u, \partial_t u, e, p)(t=0) = (u_0, v_0, e_0, p_0), & \text{sur } \Omega, \\ \partial_{tt} u - \text{div} \sigma = f, & \text{sur }]0, T[\times \Omega, \\ u = u_b, \quad \text{et} \quad \langle \sigma; \nu \rangle = \Sigma_b, & \text{sur }]0, T[\times \partial\Omega. \end{cases} \quad (1.1.5)$$

La configuration initiale est prescrite *via* une position initiale du matériau décrite par la fonction u_0 , une vitesse initiale v_0 , une déformation élastique e_0 initiale et une déformation plastique p_0 initiale.

Remarque 1.1.4. *Nous parlerons du problème quasi-statique de la plasticité parfaite lorsque l'équation du mouvement est remplacée par l'équation d'équilibre $-\text{div} \sigma = f$.*

1.1.2 Le modèle simplifié

Dans la suite, nous allons concentrer nos efforts sur un modèle simplifié de la dynamique de la plasticité parfaite : le phénomène de cisaillement antiplan. Pour cela, nous allons faire de nouvelles hypothèses. Premièrement, nous allons considérer que la configuration de référence est invariante dans une direction. De manière plus précise, en notant (e_1, e_2, e_3) la base canonique de \mathbb{R}^3 , nous supposons que le domaine Ω est invariant dans la direction e_3 . Par conséquent, la configuration de référence Ω est égale à $\omega \times \mathbb{R}$ où ω est un ouvert borné de \mathbb{R}^2 .

Deuxièmement, nous supposons que le déplacement est antiplan. Cela signifie qu'il s'effectue uniquement selon la direction e_3 et qu'il ne dépend que des deux premières variables d'espace. Ainsi, il existe $u : [0, T] \times \omega \rightarrow \mathbb{R}$ tel que

$$u(x_1, x_2, x_3) = u(x_1, x_2)e_3.$$

Par conséquent, le tenseur des déformations s'écrit

$$Eu = \frac{1}{2} \begin{pmatrix} 0 & 0 & \partial_1 u \\ 0 & 0 & \partial_2 u \\ \partial_1 u & \partial_2 u & 0 \end{pmatrix}.$$

Nous admettons que les tenseur des déformations élastiques et plastiques s'expriment sous une forme similaire *i.e.*

$$e = \frac{1}{2} \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & \varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{pmatrix},$$

et

$$p = \frac{1}{2} \begin{pmatrix} 0 & 0 & \rho_1 \\ 0 & 0 & \rho_2 \\ \rho_1 & \rho_2 & 0 \end{pmatrix},$$

avec $\varepsilon_1, \varepsilon_2, \rho_1$ et ρ_2 des fonctions allant de $[0, T] \times \omega$ dans \mathbb{R} . La décomposition additive du tenseur des déformations peut alors s'écrire simplement comme

$$\nabla u = \varepsilon + \rho,$$

où ε (respectivement ρ) est le vecteur colonne de première composante ε_1 (respectivement ρ_1) et de second composante ε_2 (respectivement ρ_2).

La loi de Hooke implique alors que $\sigma = 2\mu e$. Vu la définition de e , le tenseur σ n'a que deux composantes non nulles σ_{13} et σ_{23} . Nous posons ainsi \mathfrak{S} le vecteur colonne de première composante σ_{13} et de second composante σ_{23} . Par suite, la loi de Hooke se réécrit sous la forme suivante

$$\mathfrak{S} = \mu \varepsilon.$$

Encore une fois, nous supposons que les forces volumiques suivent une évolution antiplane. Cela signifie que la fonction f est égale à $f e_3$ pour une fonction $f : [0, T] \times \omega \rightarrow \mathbb{R}$. L'équation du mouvement se réécrit alors sous la forme suivante

$$\partial_{tt} u - \operatorname{div} \mathfrak{S} = f.$$

Pour ce modèle, nous utilisons le modèle de VON MISES qui se formule alors ainsi

$$|\mathfrak{S}| \leq k/\sqrt{2}.$$

Pour simplifier les calculs, nous supposerons dans la suite que $k = \sqrt{2}$ et $\mu = 1$ de sorte que \mathfrak{S} est contrainte à rester dans la boule unité fermée de \mathbb{R}^2 et de plus, $\mathfrak{S} = \varepsilon$. La fonction d'appui associée à cette boule unité est la norme euclidienne classique. Par conséquent, la loi d'écoulement s'écrit sous la forme suivante

$$\langle \mathfrak{S}; \partial_t \rho \rangle = |\partial_t \rho|.$$

Nous ne détaillons pas le choix de condition de bord pour ce modèle dans cette section. Ce choix va être fait par analogie avec les systèmes hyperboliques. Dans un premier temps, rappelons les résultats mathématiques récents concernant la dynamique de la plasticité parfaite.

1.2 L'approche variationnelle des phénomènes de la plasticité parfaite

L'étude mathématique des problèmes issus de la mécanique de la plasticité s'est développée considérablement durant les quarante dernières années. Cette évolution a profité d'avancées importantes dans les domaines du calcul des variations, de l'analyse convexe, de l'analyse fonctionnelle et de la théorie de la mesure. Nous présentons ici un bref historique des travaux mathématiques récents portant sur la dynamique de la plasticité parfaite (ainsi que sur des modèles relatifs à ce problème).

Dans un premier temps, une étude du problème dynamique décrit dans la Section 1.1.1 (ainsi que l'étude du

problème quasi-statique associée) a été faite par DUVAUT et LIONS dans le livre [39]. La démarche proposée consistait à régulariser le problème de deux manières. Dans un premier temps, la contrainte sur le tenseur σ a été relaxée et fait maintenant intervenir un paramètre ε , dit de viscoplasticité. L'équation d'évolution du tenseur des déformations plastiques s'écrit alors sous la forme

$$\partial_t p = \frac{1}{2\varepsilon}(\sigma - P_{\mathcal{C}}(\sigma)),$$

où $P_{\mathcal{C}}$ est l'opérateur de projection sur le convexe fermé des contraintes \mathcal{C} . Pour pouvoir étudier ce problème (relaxé en contraintes), les auteurs introduisent un second problème régularisé. En plus de la régularisation en contrainte, un second terme de régularisation, grâce à un second paramètre η , est introduit dans la décomposition additive du tenseur des déformations et dans l'équation du mouvement. En particulier, la décomposition additive du tenseur des déformations s'écrit

$$E\partial_t u = \partial_t e + \frac{1}{2\varepsilon}(\sigma - P_{\mathcal{C}}(\sigma)) - \eta \mathcal{D}\sigma,$$

où \mathcal{D} est un opérateur différentiel du second ordre défini par

$$\mathcal{D}\sigma = \left(\frac{\partial_{ii} + \partial_{jj}}{2} \sigma_{ij} \right)_{1 \leq i, j \leq 3}$$

Leur démonstration s'effectue en deux étapes. Dans un premier temps, ils étudient le problème pour des paramètres ε et η strictement positifs et fixés. Puis, ils font tendre le paramètre η vers 0 et ensuite, ils font tendre le paramètre ε vers 0. Cette approche leur permet de prouver l'existence et l'unicité du tenseur des contraintes σ pour le problème de la dynamique de la plasticité parfaite (ils ont éliminé la variable u grâce à l'équation du mouvement). Néanmoins, cela ne permet pas d'avoir des renseignements concernant le tenseur des déformations à la limite $\varepsilon = 0$.

Remarque 1.2.1. *Remarquons que cette approche par relaxation (des contraintes et de l'équation du mouvement) est très proche (voir le problème (1.4.2)) de la dérivation formelle de la définition de systèmes de FRIEDRICHS contraints de [37] (cf Section 1.4) ainsi que de l'approche que nous utilisons pour construire les solutions du modèle simplifié de la plasticité parfaite (cf Section 3.4).*

Notons que MOREAU a aussi démontré dans l'article [83] l'existence et l'unicité du tenseur des contraintes pour le problème quasi-statique totalement unidimensionnel (*i.e.* le domaine Ω est l'intervalle $[0, 1]$ et le déplacement u est un réel) par une approche différente qui fait intervenir un ensemble des contraintes qui dépend du temps. Par ailleurs, KUKSIN a étudiée dans les articles [59] et [60] l'existence et l'unicité du tenseur des contraintes pour le phénomène du cisaillement antiplan (lorsque la condition de bord est une condition mixte). La méthode employée dans les travaux de KUKSIN utilise la théorie des opérateurs monotones. De plus, l'auteur cherche à décrire l'évolution du matériau au niveau de la surface de transition entre le régime élastique (*i.e.* la contrainte \mathfrak{S} appartient à la boule unité ouverte) et le régime plastique (*i.e.* \mathfrak{S} est de norme 1).

L'introduction d'un cadre fonctionnel adapté à ce type de problème dans [93] et [74] a permis une avancée significative concernant l'étude de l'existence et l'unicité des déplacements u pour les problèmes dynamiques et quasi-statiques. Ces auteurs ont introduit l'espace $BD(\Omega)$ des fonctions à déformations bornées : il s'agit des fonctions u appartenant à l'espace $L^1(\Omega)$ dont le tenseur linéarisé (symétrique) des déformations Eu (qui est *a priori* une distribution) est une mesure de RADON. Nous renvoyons le lecteur au livre [95] de TEMAM pour plus de détails concernant cet espace. Une fois ce cadre posé, l'étude du modèle quasi-statique a été faite par SUQUET dans [91] et [92] grâce, là encore, à une méthode de relaxation de la contrainte. Le problème dynamique a quant à lui été résolu par ANZELLOTTI et LUCKHAUS dans l'article [7]. La méthode employée dans ce cas ne relaxe pas les contraintes mais introduit un terme visqueux dans l'équation du mouvement. En régime visqueux, cette dernière s'écrit

$$\partial_{tt} u - \operatorname{div}(\sigma + \varepsilon E \partial_t u) = f.$$

Remarque 1.2.2. *Là encore, cette introduction d'un terme de viscosité dans l'équation du mouvement est similaire au terme visqueux que nous introduisons dans la Section 3.4.*

Dans les deux cas (le problème dynamique ou le problème quasi-statique), la régularité pour les solutions nous permet d'affirmer (en particulier) que $\partial_t u(t) \in BD(\Omega)$ et $\sigma(t) \in L^2(\Omega)$ lorsque $t \in [0, T]$. Par conséquent, la dérivée en temps du tenseur des déformations plastiques au temps t , $\partial_t p(t)$, est une mesure et ainsi le produit $\sigma(t) : \partial_t p(t)$ (intervenant dans le principe du travail plastique maximale de HILL) n'a *a priori* pas de sens. Là encore, des travaux de KOHN et TEMAM (notamment dans l'article [56]) ont permis de donner un sens à ce produit.

Par ailleurs, il nous faut donner un sens à la loi d'évolution ($\partial_t p \in \mathcal{N}_{\mathcal{C}}(\sigma)$). Nous avons vu qu'il était possible de donner un sens au produit $\sigma : \partial_t p$. Il est par ailleurs possible, en suivant les travaux [50, 35, 34] de GOFFMAN, SERRIN,

DEMENGEL et TEMAM, de donner un sens à la notion de fonction convexe d'une mesure. Par conséquent, l'égalité

$$\sigma : \partial_t p = H_{\mathcal{C}}(\partial_t p),$$

qui exprime l'évolution du tenseur des déformations plastiques, peut être comprise au sens des mesures.

Enfin, les travaux récents (en particulier l'article [30]) de DAL MASO, DE SIMONE et MORA ont permis, sur le problème quasi-statique, de montrer que l'égalité $\sigma : \partial_t p = H_{\mathcal{C}}(\partial_t p)$, dite loi d'écoulement, est équivalente à une identité d'énergie pour le système mécanique étudié.

Remarque 1.2.3. *Nous verrons sur le cas du modèle simplifié de la dynamique de la plasticité parfaite (cf Chapitre 3 et plus particulièrement la Section 3.5.3) que l'approche de DAL MASO, DE SIMONE et MORA nous a guidé pour pouvoir obtenir la loi d'écoulement associée à notre modèle.*

1.3 L'analogie avec les systèmes hyperboliques

Les problèmes que nous avons décrits précédemment peuvent se formuler sous la forme de systèmes hyperboliques avec contraintes. Dans un premier temps, nous expliquerons, en détail sur le modèle simplifié décrit dans la Section 1.1.2, cette analogie lorsque le matériau n'est pas contraint à rester dans un domaine prescrit. Puis, toujours sur l'exemple du modèle simplifié de la dynamique de la plasticité parfaite, nous dériverons des inégalités qui nous permettront de voir le lien avec l'analyse des systèmes hyperboliques sous contraintes qui a motivé ce travail.

1.3.1 Le modèle dynamique de l'élasticité linéaire

Lorsque le matériau n'est pas contraint à rester dans un domaine \mathcal{C} (et par conséquent, n'est pas soumis aux différentes hypothèses de la plasticité parfaite), son évolution obéit aux lois de l'élasticité linéaire. En effet, nous supposons que les autres hypothèses (petites déformations, relation linéaire entre σ et e , matériau isotrope) sont toujours valides. Les équations de l'élasticité linéaire en 3D peuvent s'écrire de la manière suivante

$$\begin{cases} Eu = (\nabla u + \nabla u^t)/2 = e, & \text{sur }]0, T[\times \Omega, \\ \sigma = 2\mu e + \lambda \text{tr}(e)\text{id}, & \text{sur }]0, T[\times \Omega, \\ (u, \partial_t u, e)(t=0) = (u_0, v_0, e_0), & \text{sur } \Omega, \\ \partial_{tt} u - \text{div} \sigma = f, & \text{sur }]0, T[\times \Omega, \\ u = u_b, \quad \text{et} \quad \langle \sigma; \nu \rangle = \Sigma_b, & \text{sur }]0, T[\times \partial\Omega. \end{cases} \quad (1.3.1)$$

Ce système peut alors s'écrire sous la forme d'un système hyperbolique (dont l'inconnue est le vecteur $U = (\partial_t u, \sigma) \in \mathbb{R}^9$). Nous ne détaillerons pas cette analogie dans le cadre du modèle 3D complet. Le lecteur pourra trouver les détails de cette analogie dans les travaux de HUGHES et MARSDEN (cf [54]). Il est cependant intéressant de noter que l'analyse de HUGHES et MARSDEN du problème (1.3.1), vu comme un système hyperbolique, est faite en utilisant les conditions de bord provenant de l'approche mécanique (et non de celle des systèmes hyperboliques) : les conditions de DIRICHLET ou de NEUMANN. Dans le cadre du modèle simplifié, les équations de la dynamique de l'élasticité linéaire peuvent se formuler sous la forme suivante

$$\partial_{tt} u - \text{div} \mathfrak{S} = \mathfrak{f}, \quad \nabla u = \mathfrak{S},$$

auxquelles il faut ajouter des conditions initiales et de bord. Nous posons alors le vecteur colonne $U = (\partial_t u, \mathfrak{S}) \in \mathbb{R}^3$. En dérivant par rapport au temps l'égalité $\nabla u = \mathfrak{S}$, nous obtenons alors le système d'équations

$$\begin{cases} \partial_{tt} u - \text{div} \mathfrak{S} = \mathfrak{f}, \\ \partial_t \mathfrak{S} - \nabla \partial_t u = 0. \end{cases}$$

En utilisant alors la variable U , ce système peut se réécrire sous la forme suivante

$$\partial_t U + \sum_{j=1}^2 A_j \partial_j U = F, \quad (1.3.2)$$

où les matrices A_1 et A_2 et le vecteur F sont définis de la manière suivante

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}.$$

Il s'agit ici d'une équation des ondes que nous aurions formulée en utilisant la variable vitesse et mis sous la forme d'un système hyperbolique (dit système de FRIEDRICHS, voir Section 1.4). Il est intéressant de noter que l'étude des bonnes conditions de bord d'un point de vue hyperbolique (cf Section 1.4.1) a été faite (voir notamment [16]) et que les conditions de bord les plus courantes lors de l'analyse variationnelle de ce problème, qui sont les conditions de bord mixtes, ne font pas partie des principales théories hyperboliques sur les conditions de bord. Un des apports de notre travail a été d'utiliser des conditions de bord hyperboliques pour le problème simplifié (et avec contrainte) et d'analyser les liens entre l'approche variationnelle et l'étude hyperbolique des problèmes avec contraintes (que nous détaillons ci-après).

1.3.2 Le modèle simplifié dynamique de la plasticité parfaite

L'introduction d'une contrainte du type $U \in \mathbb{R} \times \mathcal{C}$ dans des systèmes hyperboliques similaires à (1.3.2), et donc l'étude du modèle simplifié de la plasticité parfaite, n'a été étudié très récemment par DESPRÉS, LAGOUTIÈRE et SEGUIN dans [37]. Partant du modèle simplifié et toujours en introduisant la variable $U = (\partial_t u, \mathfrak{S}) \in \mathbb{R}^3$, nous allons expliquer formellement comment dériver une famille d'inégalités (assez proches des inégalités proposés par KRUŽKOV pour les problèmes hyperboliques scalaires non contraint dans [58]). Pour cela, commençons par réécrire le système d'équations aux dérivées partielles (en oubliant pour le moment les conditions initiales et de bord) qui nous concernent :

$$\begin{cases} \partial_{tt} u - \operatorname{div} \mathfrak{S} = f, \\ \partial_t \mathfrak{S} - \nabla \partial_t u = -\partial_t \rho. \end{cases} \quad (1.3.3)$$

Nous allons chercher à nous débarrasser de la variable ρ . Pour cela, nous utilisons le principe du travail plastique maximal de HILL qui nous indique que pour tout vecteur constant \mathfrak{T} dans la boule unité (de \mathbb{R}^2) fermée, nous avons

$$\langle \mathfrak{S} - \mathfrak{T}, \partial_t \rho \rangle \geq 0,$$

où $\langle \cdot; \cdot \rangle$ désigne le produit scalaire euclidienne canonique de \mathbb{R}^2 . Ainsi, nous allons effectuer le produit scalaire de la seconde équation du système (1.3.3) par $\mathfrak{S} - \mathfrak{T}$ où \mathfrak{T} est un vecteur constant dans le convexe des contraintes. Nous obtenons alors que

$$\langle \partial_t \mathfrak{S} - \nabla \partial_t u; \mathfrak{S} - \mathfrak{T} \rangle = -\langle \mathfrak{S} - \mathfrak{T}; \partial_t \rho \rangle \leq 0.$$

De plus, puisque le vecteur \mathfrak{T} est indépendant du temps, ceci implique que

$$\langle \partial_t (\mathfrak{S} - \mathfrak{T}) - \nabla \partial_t u; \mathfrak{S} - \mathfrak{T} \rangle \leq 0. \quad (1.3.4)$$

En introduisant, le vecteur \mathfrak{T} dans la première équation du système (1.3.3), nous avons aussi

$$\partial_{tt} u - \operatorname{div} (\mathfrak{S} - \mathfrak{T}) = f. \quad (1.3.5)$$

Nous avons déjà évoqué, en rappelant brièvement l'historique de l'étude mathématique des problèmes mécaniques liés à la plasticité parfaite, qu'un soucis majeur dans l'approche mathématique de ce type de phénomènes est le manque de régularité du champ de vitesse. Il apparaît ainsi important de faire porter les dérivées spatiales sur une fonction test et non sur les fonctions $\partial_t u$ et \mathfrak{S} . Or, nous savons que

$$\operatorname{div}(\partial_t u (\mathfrak{S} - \mathfrak{T})) = \langle \nabla \partial_t u; \mathfrak{S} - \mathfrak{T} \rangle + \partial_t u \operatorname{div} (\mathfrak{S} - \mathfrak{T}).$$

Nous sommes donc incités à multiplier l'équation (1.3.5) par $\partial_t u$ et à additionner le résultat obtenu avec l'inéquation (1.3.4). Nous arrivons alors à l'inégalité suivante

$$\langle \partial_t (\mathfrak{S} - \mathfrak{T}); \mathfrak{S} - \mathfrak{T} \rangle + \partial_{tt} u \partial_t u - \operatorname{div}(\partial_t u (\mathfrak{S} - \mathfrak{T})) \leq f \partial_t u.$$

Comme prévu, nous prenons avec une fonction test régulière ϕ dépendant des variables de temps et d'espace et à valeurs positives, nous pouvons ainsi multiplier l'inégalité précédente par ϕ et intégrer par parties le terme contenant

les dérivées spatiales pour obtenir

$$\int_0^T \int_\omega \langle \partial_t(\mathfrak{S} - \mathfrak{T}); \mathfrak{S} - \mathfrak{T} \rangle + \partial_{tt} u \partial_t u \phi + \partial_t u \langle \mathfrak{S} - \mathfrak{T}; \nabla \phi \rangle dx dt \leq \int_0^T \int_\omega F \partial_t u \phi dx dt.$$

Lors de l'intégration par parties précédentes, nous avons supposé que les termes de bord étaient nuls, ceci est le cas si, par exemple, la fonction ϕ a un support compact. En réutilisant les notations hyperboliques (U , A_1 , A_2 et F) de la section précédente, cette inégalité se réécrit

$$\int_0^T \int_\omega \langle \partial_t(U - \kappa); U - \kappa \rangle \phi - \frac{1}{2} \sum_{j=1}^2 \langle A_j(U - \kappa); U - \kappa \rangle \partial_j \phi dx dt \leq \int_0^T \int_\omega F U \phi dx dt,$$

où $\kappa = (0, \mathfrak{T})$ et $\langle \cdot; \cdot \rangle$ est maintenant le produit scalaire euclidienne canonique de \mathbb{R}^3 .

Remarque 1.3.1. Dans la suite de cette introduction, nous définissons, par abus de notation, $\langle \cdot; \cdot \rangle$ comme le produit scalaire euclidien entre deux vecteurs et $|\cdot|$ la norme associée quelque soit la dimension du vecteur.

De la même manière, nous pouvons faire porter la dérivée en temps sur la fonction test ϕ (là encore, nous supposons que cette fonction est nulle au bord du domaine temporel) et nous obtenons alors

$$-\frac{1}{2} \int_0^T \int_\omega |U - \kappa|^2 \partial_t \phi - \frac{1}{2} \sum_{j=1}^2 \langle A_j(U - \kappa); U - \kappa \rangle \partial_j \phi dx dt \leq \int_0^T \int_\omega F U \phi dx dt,$$

où de manière équivalente

$$\int_0^T \int_\omega |U - \kappa|^2 \partial_t \phi dx dt + \sum_{j=1}^2 \langle A_j(U - \kappa); U - \kappa \rangle \partial_j \phi + 2F U \phi dx dt \geq 0. \quad (1.3.6)$$

Cette inégalité présente l'intérêt d'être valide dès que U et F appartiennent à $L^2(]0, T[\times \Omega; \mathbb{R}^3)$. La dérivation de ce type d'inégalités, en partant d'une approche visqueuse (similaire à celles évoquées dans la Section 1.2), est à l'origine des travaux sur les systèmes hyperboliques sous contraintes (voir section ci-après). Un lien apparaît alors entre deux approches mathématiques différentes pour étudier le même phénomène mécanique. L'étude de ce lien et de ses conséquences mathématiques fait l'objet d'une partie importante de cette thèse et est développée dans le Chapitre 3.5.

1.4 Les systèmes de FRIEDRICHS

Comme nous l'avons vu dans la section précédente, les problèmes de la plasticité parfaite peuvent se mettre sous la forme d'un système d'équations hyperboliques sous contraintes. Or, depuis les travaux de DESPRÉS, LAGOUTIÈRE et SEGUIN (dans [37]), nous avons à notre disposition une définition de solution pour ce type de problème ainsi qu'un théorème d'existence et d'unicité des solutions lorsque que le problème est posé dans le domaine spatial complet. Rappelons ici brièvement les résultats obtenus dans [37]. Le but principal de cet article est d'étudier le problème de CAUCHY suivant : Trouver une fonction $W : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ telle que

$$\begin{cases} \partial_t W + \sum_{j=1}^n A_j \partial_j W = 0 & \text{sur }]0, T[\times \mathbb{R}^n, \\ W(t, x) \in \mathcal{C} & \text{si } (t, x) \in [0, T] \times \mathbb{R}^n, \\ W(0, x) = W^0(x) & \text{si } x \in \mathbb{R}^n, \end{cases} \quad (1.4.1)$$

avec \mathcal{C} un convexe fixé (*i.e.* indépendant des variables de temps et d'espace) fermé, contenant 0 dans son intérieur. Les matrices A_j sont des matrices symétriques de taille $m \times m$ (indépendantes des variables de temps et d'espace) et $T > 0$.

La notion de solution pour le problème n'est pas évidente *a priori*. En effet, la non-linéarité du problème (provenant du fait que la solution du problème doit rester dans un convexe) pourrait entraîner une perte de régularité pour les solutions (comme dans le cas de la plasticité parfaite), il est donc nécessaire de définir une notion faible de solution. Pour cela, l'approche formelle de [37] a été inspiré des relaxations visqueuses et des méthodes de pénalisations des contraintes que nous avons déjà évoquées pour les différents modèles de la plasticité parfaite. Le problème (1.4.1)

est ainsi remplacé par le problème suivant

$$\begin{cases} \partial_t W_{\epsilon,\eta} - \eta \Delta W_{\epsilon,\eta} + \sum_{j=1}^n A_j \partial_j W_{\epsilon,\eta} = \frac{P_{\mathcal{C}}(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon} & \text{sur }]0, T] \times \mathbb{R}^n, \\ W(0, x) = W^0(x) & \text{si } x \in \mathbb{R}^n. \end{cases} \quad (1.4.2)$$

Si maintenant nous choisissons un vecteur $\kappa \in \mathcal{C}$ constant (*i.e.* indépendant des variables de temps et d'espace), il apparaît que

$$\left\langle \partial_t (W_{\epsilon,\eta} - \kappa) - \eta \Delta (W_{\epsilon,\eta} - \kappa) + \sum_{j=1}^n A_j \partial_j (W_{\epsilon,\eta} - \kappa); W_{\epsilon,\eta} - \kappa \right\rangle = \left\langle \frac{P_{\mathcal{C}}(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}; (W_{\epsilon,\eta} - \kappa) \right\rangle \leq 0.$$

En intégrant (formellement) par parties contre une fonction test ϕ (à valeurs dans \mathbb{R}^+ et nulle au temps T), nous en déduisons que

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} \left(|W_{\epsilon,\eta} - \kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle W_{\epsilon,\eta} - \kappa; A_j (W_{\epsilon,\eta} - \kappa) \rangle \partial_j \phi \right) dx dt + \int_{\mathbb{R}^n} |W^0(x) - \kappa|^2 \phi(0, x) dx \\ & \geq 2\eta \int_0^T \int_{\mathbb{R}^n} |\nabla (W_{\epsilon,\eta} - \kappa)|^2 \phi dx dt - \eta \int_0^T \int_{\mathbb{R}^n} |W_{\epsilon,\eta} - \kappa|^2 \Delta \phi dx dt \\ & \geq -\eta \int_0^T \int_{\mathbb{R}^n} |W_{\epsilon,\eta} - \kappa|^2 \Delta \phi dx dt, \end{aligned} \quad (1.4.3)$$

En supposant que la suite $(W_{\epsilon,\eta})_{\epsilon,\eta}$ converge dans $L^2(]0, T[\times \mathbb{R}^n; \mathbb{R}^m)$ vers une certaine fonction W (qui serait une solution de (1.4.1)), il est alors possible de passer (formellement) à la limite $(\epsilon, \eta) \rightarrow (0, 0)$ dans l'inégalité (1.4.3). La fonction W devrait alors satisfaire l'inégalité suivante :

$$\int_0^T \int_{\mathbb{R}^n} \left(|W - \kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle W - \kappa; A_j (W - \kappa) \rangle \partial_j \phi \right) dx dt + \int_{\mathbb{R}^n} |W^0(x) - \kappa|^2 \phi(0, x) dx \geq 0. \quad (1.4.4)$$

Cette approche formelle a motivé la définition de solutions faibles contraintes pour les problèmes de FRIEDRICHS dans [37]. Nous verrons dans le chapitre 4 que cette approche peut être rendue rigoureuse.

Définition 1.4.1. Soient $W^0 \in L^2(\mathbb{R}^n; \mathcal{C})$ et $T > 0$. Une fonction W est appelée solution faible contrainte du problème (1.4.1) si elle appartient à l'espace $L^2(]0, T[\times \mathbb{R}^n; \mathcal{C})$ et de plus, elle vérifie l'inégalité (1.4.4) pour tout $\kappa \in \mathcal{C}$ et toute fonction $\phi \in C_c^\infty([0, T[\times \mathbb{R}^n)$ avec $\phi(t, x) \geq 0$ pour tout $(t, x) \in [0, T[\times \mathbb{R}^n$.

Nous rappelons ici le résultat majeur de [37] qui indique que la définition 1.4.1 conduit à une théorie d'existence et d'unicité des solutions faibles contraintes du problème (1.4.1).

Théorème 1.4.2. Soit $W^0 \in L^2(\mathbb{R}^n; \mathcal{C})$. Il existe une unique solution faible contrainte $W \in L^2(]0, T[\times \mathbb{R}^n; \mathcal{C})$ au problème (1.4.1) au sens de la Définition 1.4.1. De plus, cette solution appartient à l'espace $C([0, T]; L^2(\mathbb{R}^n; \mathcal{C}))$, et si nous supposons que $W^0 \in H^1(\mathbb{R}^n; \mathcal{C})$, alors $W \in L^\infty([0, T]; H^1(\mathbb{R}^n; \mathcal{C}))$. Enfin, nous disposons de l'inégalité, dite de KATO, suivante : si W et \tilde{W} sont deux solutions (au sens de la Définition 1.4.1) associées aux données initiales W^0 et \tilde{W}^0 alors pour toute fonction $\phi \in C_c^\infty([0, T[\times \mathbb{R}^n; \mathbb{R}^+)$, nous avons

$$\int_0^T \int_{\mathbb{R}^n} |W - \tilde{W}|^2 \partial_t \phi + \sum_{j=1}^n \langle W - \tilde{W}; A_j (W - \tilde{W}) \rangle \partial_j \phi dx dt + \int_{\mathbb{R}^n} |W_0 - \tilde{W}_0|^2 \phi(0) \geq 0. \quad (1.4.5)$$

Ces résultats ont été le point de départ de cette thèse. En effet, les inégalités (1.3.6) et (1.4.4) indiquent une similarité entre l'approche hyperbolique des systèmes de FRIEDRICHS sous contraintes et le modèle simplifié de la plasticité parfaite. Mais, les problèmes de la plasticité parfaite étant posés en domaine borné, la théorie développée dans [37] ne s'applique pas directement à ces problèmes. Il a donc été nécessaire de la modifier pour pouvoir comparer les deux approches pour étudier les phénomènes de la plasticité parfaite : l'approche provenant du calcul des variations (déjà décrites dans la section 1.2) et l'approche "hyperbolique sous contraintes" basée sur l'article [37]. Cette modification s'est faite en deux temps.

Premièrement, il est nécessaire de trouver une formule similaire à (1.4.4) pour les problèmes de FRIEDRICHS *sans* contrainte mais posés dans un domaine avec bord. L'objectif principal est d'obtenir un résultat d'existence et d'unicité pour cette nouvelle formulation. De plus, les problèmes de la plasticité parfaite pouvant développer des singularités, il est nécessaire que cette définition soit valide dans l'espace $L^2(]0, T[\times \Omega; \mathbb{R}^m)$ où Ω est un ouvert (borné ou non) de frontière $\partial\Omega$. Ce type de formulation peut être considérée comme faible car posée dans un espace dans lequel la condition de bord n'a *a priori* pas de sens. Cependant, dans le cas des problèmes scalaires, il existe, depuis les travaux d'OTTO (dans [87]), des formulations, valables dans l'espace $L^\infty(]0, T[\times \Omega)$, pour des problèmes avec condition de bord.

Il est ensuite nécessaire d'introduire dans ces problèmes de FRIEDRICHS avec condition de bord une contrainte similaire à celle du problème posé dans l'espace tout entier (1.4.1). L'étude des problèmes de FRIEDRICHS *sans* contrainte et *avec* condition de bord est le but du Chapitre 2 et est décrite succinctement dans la Section 1.4.1. La formulation adapté aux problèmes de FRIEDRICHS *avec* contrainte et *avec* condition de bord, décrite dans la section 1.4.2, est aussi énoncé dans le Chapitre 2. Son étude détaillée est effectuée sur l'exemple particulier du modèle simplifié de la plasticité parfaite (décrite dans la Section 1.1.2) dans la Section 3.7 du Chapitre 3.

Enfin, notons que d'autres problèmes physiques ou mécaniques peuvent se mettre sous la forme d'équations hyperboliques sous contraintes et ont fait l'objet d'études dans la littérature. Dans les travaux de BERTHELIN et BOUCHUT (voir [17] et [18]), les auteurs s'intéressent au mouvement d'un fluide à deux phases soumis à une contrainte sur la masse de ce fluide. Un autre exemple de système hyperbolique sous contraintes a été étudié par LIONS et MASMOUDI dans [69]. Le problème qu'ils étudient provient là encore de la mécanique des fluides, plus précisément ils concentrent leurs attentions à analyser le comportement d'un fluide compressible barotrope soumis à une contrainte de masse. Une étude plus générale des problèmes scalaires sous contraintes a déjà été faite : dans le cas où la solution w est soumis à la contrainte unilatérale $w \geq 0$ par Barthélemy dans [14] et par Lévi dans [65]; lorsque la solution w satisfait des inégalités de type $\Upsilon_1 \leq w \leq \Upsilon_2$ sur $[0, T] \times \Omega$, où Υ_1 et Υ_2 sont des fonctions données, dans [66]. Une définition faible, valide dans l'espace $L^2(]0, T[\times \Omega; \mathcal{C})$ et assez proche de celles que nous développons pour les systèmes hyperboliques dans cette thèse, a été formulé dans les travaux de MIGNOT et PUEL (cf [80]) pour l'analyse des équations hyperboliques scalaires soumis à la contrainte $w \in \mathcal{C}$.

1.4.1 Présentation de quelques outils possibles pour l'étude des systèmes de FRIEDRICHS en domaine avec bord

L'étude des conditions de bord possibles pour les systèmes de FRIEDRICHS de la forme

$$\partial_t W + \sum_{j=1}^n A_j \partial_j W = f, \quad \text{sur }]0, T[\times \Omega, \quad (1.4.6)$$

munis de la condition initiale suivante

$$W(0, x) = W^0(x), \quad \text{si } x \in \Omega, \quad (1.4.7)$$

est un problème difficile (notamment dans le cas où Ω est un domaine borné régulier quelconque). En effet, le nombre de condition de bord à imposer est égal au nombre de caractéristiques entrantes dans le domaine (cf [16, Section 4.1.1]). Les caractéristiques entrantes sont les valeurs propres strictement négatives de la matrice

$$A_\nu = \sum_{j=1}^n A_j \nu_j, \quad (1.4.8)$$

où $\nu = \nu(x) = (\nu_1, \nu_2, \dots, \nu_n)$ est la normale unitaire sortante au point x du bord $\partial\Omega$ de Ω . Or, la normale sortante variant au bord du domaine, le nombre de conditions à imposer peut évoluer le long du bord du domaine. Dans ce cas, la situation est dite à "multiplicité variable", sinon nous parlons d'un bord à "multiplicité constante". L'étude de tels problèmes fait encore l'objet à l'heure actuelle de travaux (cf [85] par exemple) et de nombreuses questions sont encore ouvertes.

De plus, il apparaît lorsque qu'une caractéristique n'est ni entrante ni sortante (autrement dit une valeur propre de A_ν est nulle) qu'il n'est pas nécessaire d'imposer de condition de bord et ce cas (peu traité dans la littérature, voir [70] et [89]) est appelé "caractéristique". Dans les autres situations, le bord est dit "non caractéristique".

Remarque 1.4.3. *Notons que dans le cas du système hyperbolique provenant du modèle simplifié de la plasticité parfaite (décrit dans la Section 1.3.2) le bord est caractéristique à multiplicité constante (quelque soit le domaine spatial borné Ω).*

En effet, la matrice A_ν vaut alors

$$A_\nu = \begin{pmatrix} 0 & -\nu_1 & -\nu_2 \\ -\nu_1 & 0 & 0 \\ -\nu_2 & 0 & 0 \end{pmatrix},$$

et admet toujours 0 comme valeur propre.

Le problème (1.4.6) et (1.4.7) muni d'une condition de bord a été étudié par de nombreux auteurs : l'article précurseur [47] de FRIEDRICHS a permis de déterminer un premier choix de condition de bord. Deux autres choix (assez proches) de conditions de bord sont alors apparus dans la littérature : les conditions de bord maximales dissipatives dans l'article [62] de LAX et PHILLIPS et les conditions de bord strictement dissipatives, introduites par RAUCH, au séminaire Équations aux dérivées partielles de l'École Polytechnique [88]. Une autre théorie s'est développée, lorsque DUBOIS et LE FLOCH ont écrit l'article [38] pour chercher à déterminer les bonnes conditions de bord pour l'étude de problèmes scalaires (*i.e.* $n = 1$) non-linéaires. En effet, cette théorie a été appliquée (cf [38, Section 1.2]) au cas plus simple des problèmes scalaires linéaires.

Pour aboutir à une formulation valide dans l'espace $L^2(]0, T[\times \Omega; \mathbb{R}^m)$, nous considérons ici une nouvelle approche légèrement différente des précédentes. Dans la suite de cette section, nous exposons les différentes approches pour les systèmes de FRIEDRICHS avec condition de bord. Une comparaison de ces approches a été faite sur un exemple simple (cf Section 2.4).

1.4.1.1 L'approche de FRIEDRICHS

Dans l'article [47], FRIEDRICHS étudie les problèmes (statiques) de la forme suivante

$$\sum_{j=1}^n A_j \partial_j W + B_0 W = f, \quad \text{sur } \Omega, \quad (1.4.9)$$

où W un vecteur de taille m et B_0, A_1, \dots, A_n sont des matrices réelles de taille $m \times m$ (pouvant éventuellement dépendre des variables d'espace) vérifiant les deux conditions suivantes :

1. Les matrices A_1, \dots, A_n sont symétriques.
2. La partie symétrique de la matrice $\mathbb{B} = B_0 - \frac{1}{2} \sum_{j=1}^n \partial_j A_j$ est définie positive.

Remarque 1.4.4. Bien que l'article [47] soit consacré à des problèmes dans lesquels le temps n n'intervient pas, cette technique s'adapte aux problèmes dépendant du temps.

Un des objectifs de cet article est de trouver des conditions de bord permettant d'obtenir l'existence et l'unicité de solutions pour de tels problèmes. Premièrement, FRIEDRICHS introduit la notion de condition de bord "semi-admissible". Il s'agit des conditions de bord (homogènes) s'écrivant de la manière suivante

$$(A_\nu - M_\nu)W = 0, \quad (1.4.10)$$

où M_ν est une matrice dépendant de la variable $\nu(x)$ telle que la partie symétrique de M_ν est positive. Ces conditions permettent de démontrer que l'unicité des solutions régulières du problème (1.4.9) muni de la condition de bord (1.4.10). En effet, pour de telles solutions, nous avons, grâce à une intégration par parties,

$$2 \int_{\Omega} \left\langle W; \sum_{j=1}^n A_j \partial_j W + B_0 W \right\rangle dx + \int_{\partial\Omega} \langle W; (M_\nu - A_\nu)W \rangle d\mathcal{H}^{n-1} = 2 \int_{\Omega} \langle W; \mathbb{B}W \rangle dx + \int_{\partial\Omega} \langle W; M_\nu W \rangle d\mathcal{H}^{n-1}.$$

Les hypothèses sur les matrices \mathbb{B} et M_ν permettent alors de dire qu'il existe une constante $c > 0$ telle que

$$2 \int_{\Omega} \left\langle W; \sum_{j=1}^n A_j \partial_j W + B_0 W \right\rangle dx + \int_{\partial\Omega} \langle W; (M_\nu - A_\nu)W \rangle d\mathcal{H}^{n-1} \geq c \int_{\Omega} |W|^2.$$

Puisque la solution W vérifie la condition de bord, nous arrivons à l'inégalité

$$c \|W\|_{L^2(\Omega; \mathbb{R}^m)}^2 \leq 2 \|W\|_{L^2(\Omega; \mathbb{R}^m)} \|f\|_{L^2(\Omega; \mathbb{R}^m)},$$

l'unicité des solutions régulières s'en déduit facilement. Pour obtenir l'existence de solutions pour les systèmes (1.4.9) muni de (1.4.10), la notion de condition de bord semi-admissible est trop faible. FRIEDRICHS définit alors la notion de condition de bord admissible. Elles s'écrivent toujours sous la forme

$$(A_\nu - M_\nu)W = 0,$$

mais les matrices A_ν et M_ν doivent maintenant vérifier les deux conditions suivantes :

1. La partie symétrique de M_ν est positive (semi-admissibilité).
2. L'espace \mathbb{R}^m peut s'écrire sous la forme de la somme directe suivante

$$\ker(A_\nu - M_\nu) \oplus \ker(A_\nu + M_\nu) = \mathbb{R}^m.$$

Nous ne détaillerons pas comment FRIEDRICHS démontre l'existence de solution une fois ces conditions satisfaites (cette question occupe toute la deuxième partie de l'article [47]), il est cependant important de remarquer que ces conditions doivent être modifiées pour pouvoir prendre en compte notamment le cas caractéristique. En effet, dans le cas simpliste où les matrices A_1, A_2, \dots, A_n sont nulles et où B_0 est une matrice symétrique définie positive fixée, la seconde hypothèse d'admissibilité ne peut être satisfaite (la somme ne peut pas être directe).

1.4.1.2 L'approche de DUBOIS et LE FLOCH

Dans l'article [38], DUBOIS et LE FLOCH s'intéressent aux systèmes hyperboliques non-linéaires de lois de conservation de la forme

$$\partial_t W + \partial_x f(W) = 0, \quad \text{pour } x > 0 \text{ et } t > 0, \quad (1.4.11)$$

où W est toujours un vecteur de taille m et f est une fonction de \mathbb{R}^m dans \mathbb{R}^m . Le but principal de cet article est de proposer et d'étudier deux types de conditions de bord adaptées à (1.4.11). Les auteurs appliquent leurs théories pour les problèmes non-linéaires au cas du problème linéaire (*i.e.* $f(W) = A_1 W$ avec A_1 une matrice constante admettant m valeurs propres réelles distinctes). La condition au bord (en $x = 0$) peut alors s'écrire de la manière suivante

$$W(t, 0) \in \text{Vect}(v_1, \dots, v_p),$$

où $\lambda_1 < \lambda_2 < \dots < \lambda_p \leq 0 < \lambda_{p+1} < \dots < \lambda_m$ sont les valeurs propres de la matrice A_1 et pour tout entier i compris entre 1 et m , v_i est un vecteur propre associé à la valeur propre λ_i . Les valeurs propres $\lambda_1 < \lambda_2 < \dots < \lambda_p \leq 0$ correspondent aux caractéristiques entrantes dans le domaine \mathbb{R}^+ .

1.4.1.3 La théorie dissipative maximale

Au moment où FRIEDRICHS commence à développer sa théorie pour les systèmes hyperboliques avec condition de bord, une théorie, dont la dénomination provient de la théorie des opérateurs, dite "dissipative maximale" apparaît dans les travaux de PHILLIPS et LAX. Nous présenterons cette théorie avec les notations provenant du livre [16] de BENZONI-GAVAGE et SERRE.

Nous avons déjà évoqué que le nombre de conditions de bord à imposer ici est égal au nombre de valeurs propres strictement négatives de A_ν (que nous noterons p dans la suite). Nous supposons dans la suite de cette section que ce nombre n'évolue pas lorsque nous nous déplaçons le long du bord, *i.e.* le problème est à multiplicité constante. De plus, pour simplifier le propos, nous considérons ici un système de FRIEDRICHS de la forme suivante

$$\partial_t W + \sum_{j=1}^n A_j \partial_j W = f, \quad \text{sur }]0, T] \times \Omega, \quad (1.4.12)$$

avec des matrices A_j indépendantes du temps et de l'espace et muni de la condition initiale

$$W(t = 0) = W_0, \quad \text{sur } \Omega. \quad (1.4.13)$$

La condition de bord "dissipative maximale" s'exprime alors de la manière suivante

$$\mathcal{B}W = 0, \quad (1.4.14)$$

où \mathcal{B} est une matrice de taille $p \times m$. La matrice doit vérifier deux propriétés :

1. (Dissipativité) Pour tout vecteur $W \in \ker \mathcal{B}$, l'inégalité suivante est vérifiée $\langle A_\nu W; W \rangle \geq 0$.

2. (Maximalité) L'espace vectoriel $\ker \mathcal{B}$ n'est pas un sous-espace strict d'un espace vectoriel E tel que pour tout vecteur $W \in E$, l'inégalité suivante est vérifiée $\langle A_\nu W; W \rangle \geq 0$. Autrement dit,

$$\nexists E \text{ espace vectoriel, } \ker \mathcal{B} \subsetneq E, \quad \forall W \in E, \quad \langle A_\nu W; W \rangle \geq 0.$$

La condition de dissipativité assure que la norme $L^2(\Omega)$ décroît au cours du temps (en l'absence de terme source). En effet, soit W une solution régulière du problème (1.4.12), (1.4.13) et (1.4.14), nous avons, par des calculs similaires à ceux effectués dans le paragraphe précédent, pour tout $t \in]0, T[$

$$0 = 2 \left(\int_0^t \int_\Omega \langle \partial_t W; W \rangle + \sum_{j=1}^n \langle A_j \partial_j W; W \rangle dx dt \right) = \int_\Omega |W(t)|^2 dx - \int_\Omega |W_0|^2 dx + \int_0^t \int_{\partial\Omega} \langle A_\nu W; W \rangle d\mathcal{H}^{n-1} dt.$$

Par conséquent, la dissipativité de la condition de bord implique que

$$\int_\Omega |W(t)|^2 dx \leq \int_\Omega |W_0|^2 dx.$$

Ce type de condition de bord permet d'aboutir à une théorie d'existence et d'unicité (cf [16, Théorème 3.2]) dans le cas de matrices constantes.

1.4.1.4 La théorie strictement dissipative

La théorie, dite "strictement dissipative", est extrêmement proche de la précédente. Nous nous replaçons dans le même cadre que dans le paragraphe précédent et là encore, nous renvoyons le lecteur au livre [16] pour plus de détails. La condition de bord pour le problème (1.4.12) et (1.4.13) s'exprime toujours sous la forme

$$\mathcal{B}W = 0, \tag{1.4.15}$$

où \mathcal{B} est une matrice de taille $p \times m$. La matrice doit vérifier trois propriétés

1. Pour tout vecteur non nul $W \in \ker \mathcal{B}$, l'inégalité suivante est vérifiée $\langle A_\nu W; W \rangle > 0$.
2. L'espace vectoriel $\ker \mathcal{B}$ n'est pas un sous-espace strict d'un espace vectoriel E tel que pour tout vecteur non nul $W \in E$, l'inégalité suivante est vérifiée $\langle A_\nu W; W \rangle > 0$. C'est-à-dire,

$$\nexists E \text{ espace vectoriel, } \ker \mathcal{B} \subsetneq E, \quad \forall W \in E \text{ avec } W \neq 0, \quad \langle A_\nu W; W \rangle > 0.$$

3. La matrice \mathcal{B} est surjective.

Cette théorie, plus restrictive que la précédente, présente l'intérêt majeur de pouvoir s'adapter aux problèmes à frontière libre de manière plus simple que la théorie maximale dissipative (cf [16]).

1.4.1.5 La théorie dissipative développée dans cette thèse

Dans le cadre de notre approche, nous utilisons un formalisme proche de la théorie développée par FRIEDRICHS. La condition de bord (homogène) s'écrit toujours sous la forme

$$(A_\nu - M_\nu)W = 0, \tag{1.4.16}$$

avec A_ν la matrice définie par (1.4.7) et les matrices A_ν et M_ν doivent vérifier les conditions suivantes :

1. La matrice M_ν est symétrique et positive.
2. L'inclusion suivante est vérifiée

$$\ker A_\nu \subset \ker M_\nu. \tag{1.4.17}$$

3. L'espace \mathbb{R}^m peut se décomposer sous la forme suivante

$$\mathbb{R}^m = \ker(A_\nu - M_\nu) + \ker(A_\nu + M_\nu). \tag{1.4.18}$$

Ce choix de condition de bord pour le système (1.4.6) nous permet d'obtenir une formulation similaire à celle proposée dans la Définition 1.4.1. En effet, supposons que la fonction W soit une solution régulière du problème (1.4.6), (1.4.7)

et (1.4.16) et prenons un vecteur constant $\kappa \in \mathbb{R}^m$. Puisque les dérivées de κ sont nulles, l'équation (1.4.6) implique en particulier que

$$\partial_t(W - \kappa) + \sum_{j=1}^n A_j \partial_j(W - \kappa) = f, \quad \text{sur }]0, T] \times \Omega. \quad (1.4.19)$$

Soit ϕ est une fonction positive régulière à support compact (dans $\mathbb{R} \times \mathbb{R}^n$). En intégrant cette dernière équation contre la fonction $(W - \kappa)\phi$, nous obtenons que

$$\int_0^T \int_{\Omega} \langle \partial_t(W - \kappa); (W - \kappa)\phi \rangle + \sum_{j=1}^n \langle A_j \partial_j(W - \kappa); (W - \kappa)\phi \rangle dx dt = \int_0^T \int_{\Omega} \langle f; (W - \kappa)\phi \rangle dx dt. \quad (1.4.20)$$

Intégrons alors par parties l'équation (1.4.20), nous obtenons alors (puisque la fonction ϕ ne s'annule pas aux bords du domaine $]0, T[\times \Omega$)

$$\begin{aligned} & \int_0^T \int_{\Omega} |W - \kappa|^2 \partial_t \phi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{j=1}^n \langle W - \kappa; A_j(W - \kappa) \rangle \partial_j \phi(t, x) dx dt + \int_0^T \int_{\Omega} 2 \langle f; W - \kappa \rangle \phi dx dt \\ & - \int_0^T \int_{\partial\Omega} \langle W - \kappa; A_\nu(W - \kappa) \rangle \phi d\mathcal{H}^{n-1} dt + \int_{\Omega} |W_0 - \kappa|^2 \phi(0, x) dx - \int_{\Omega} |W(T, x) - \kappa|^2 \phi(T, x) dx = 0, \end{aligned} \quad (1.4.21)$$

Dans le cas non caractéristique (ces calculs sont aussi valables dans le cas caractéristique, pour plus de détails voir Lemme 2.2.3), la somme (1.4.18) est en fait une somme directe. De plus, si nous notons κ_- (respectivement κ_+) la projection de κ sur $\ker(A_\nu - M_\nu)$ (respectivement sur $\ker(A_\nu + M_\nu)$) parallèlement à $\ker(A_\nu + M_\nu)$ (respectivement à $\ker(A_\nu - M_\nu)$), alors l'égalité suivante est vérifiée

$$\langle W - \kappa; A_\nu(W - \kappa) \rangle = \langle (W - \kappa)_-; A_\nu(W - \kappa)_- \rangle + \langle (W - \kappa)_+; A_\nu(W - \kappa)_+ \rangle. \quad (1.4.22)$$

Remarquons alors la condition de bord (1.4.16) peut alors se formuler (en utilisant les projections précédentes) $W_+ = 0$. En outre, en utilisant la matrice M_ν , l'égalité (1.4.22) devient

$$\langle W - \kappa; A_\nu(W - \kappa) \rangle = \langle (W - \kappa)_-; A_\nu(W - \kappa)_- \rangle - \langle \kappa_+; M_\nu \kappa_+ \rangle.$$

Ainsi, le terme de bord dans (1.4.21) se réécrit

$$\int_0^T \int_{\partial\Omega} \langle W - \kappa; A_\nu(W - \kappa) \rangle \phi d\mathcal{H}^{n-1} dt = \int_0^T \int_{\partial\Omega} \langle (W - \kappa)_-; A_\nu(W - \kappa)_- \rangle \phi d\mathcal{H}^{n-1} dt - \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \phi d\mathcal{H}^{n-1} dt.$$

Par suite, nous obtenons (puisque $\phi \geq 0$)

$$\begin{aligned} & \int_0^T \int_{\Omega} |W - \kappa|^2 \partial_t \phi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{j=1}^n \langle W - \kappa; A_j(W - \kappa) \rangle \partial_j \phi(t, x) dx dt + \int_0^T \int_{\Omega} 2 \langle f; W - \kappa \rangle \phi dx dt \\ & + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \phi d\mathcal{H}^{n-1} dt + \int_{\Omega} |W_0 - \kappa|^2 \phi(0, x) dx \\ & = \int_{\Omega} |W(T, x) - \kappa|^2 \phi(T, x) dx + \int_0^T \int_{\partial\Omega} \langle (W - \kappa)_-; A_\nu(W - \kappa)_- \rangle \phi d\mathcal{H}^{n-1} dt \geq 0. \end{aligned} \quad (1.4.23)$$

Finalement, une solution régulière W du problème (1.4.6), (1.4.7) et (1.4.16) vérifie l'inégalité suivante, pour tout $\kappa \in \mathbb{R}^m$ et toute fonction ϕ régulière à support compact (dans $\mathbb{R} \times \mathbb{R}^n$),

$$\begin{aligned} & \int_0^T \int_{\Omega} |W - \kappa|^2 \partial_t \phi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{j=1}^n \langle W - \kappa; A_j(W - \kappa) \rangle \partial_j \phi(t, x) dx dt + \int_0^T \int_{\Omega} 2 \langle f; W - \kappa \rangle \phi dx dt \\ & + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \phi d\mathcal{H}^{n-1} dt + \int_{\Omega} |W_0 - \kappa|^2 \phi(0, x) dx \geq 0. \end{aligned} \quad (1.4.24)$$

Cette inégalité est similaire à l'inégalité (1.4.26) (qui avait servi dans [37] à définir la notion de solution faible contrainte, dans le cas sans bord, pour un système de FRIEDRICHS). Comme (1.4.26), l'inégalité (1.4.24) a un sens dès que $W \in L^2(]0, T[\times \Omega; \mathbb{R}^m)$. Elle n'utilise aucune trace de la solution W (que ce soit au temps initial, comme pour le problème (1.4.1), ou au bord du domaine). Ainsi, nous proposons la définition suivante

Définition 1.4.5. Une fonction $W \in L^2(]0, T[\times \Omega; \mathbb{R}^m)$ est appelée solution dissipative du problème avec condition de bord (1.4.6), (1.4.7) et (1.4.16) si pour toute fonction $\phi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, l'espace des fonctions $W^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$ à valeurs positives et à support compact, et pour tout $\kappa \in \mathbb{R}^m$ l'inégalité suivante est vérifiée

$$\begin{aligned} \int_0^T \int_{\Omega} |W - \kappa|^2 \partial_t \phi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{j=1}^n \langle W - \kappa; A_j(W - \kappa) \rangle \partial_j \phi(t, x) dx dt + \int_0^T \int_{\Omega} 2 \langle f; W - \kappa \rangle \phi dx dt \\ + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \phi d\mathcal{H}^{n-1} dt + \int_{\Omega} |W_0 - \kappa|^2 \phi(0, x) dx \geq 0. \end{aligned}$$

Mais, alors que nous étudions un problème linéaire (contrairement au problème (1.4.1)), la Définition 1.4.5 est non linéaire. En effet, d'après les termes quadratiques de l'inégalité précédente, il n'est pas immédiat que la combinaison linéaire de deux solutions est une solution au sens de la Définition 1.4.5. Il est alors intéressant de se demander si cette définition contient toutes les informations du problème de départ : l'équation aux dérivées partielles (1.4.6), la condition initiale (1.4.7) et la condition de bord (1.4.16). Avons-nous perdu de l'information ? Avons-nous équivalence entre la définition de solution forte et la définition de solution dissipative (*i.e.* au sens de la Définition 1.4.5) pour des solutions régulières ? Puisque une solution dissipative W appartient *a priori* à l'espace $L^2(]0, T[\times \Omega)$, quel sens pouvons-nous donner à la condition initiale et à la condition de bord ? Nous répondons à ces questions (pour certaines, uniquement dans le cas où Ω est le demi-espace) dans le Chapitre 2 et les principaux résultats concernant ce problème sont résumés dans la Section 1.6.1. De plus, nous présentons dans cette section différents choix de condition de bord pour le même problème (1.4.6) muni de la condition initiale (1.4.16). Il est aussi intéressant de comparer ces différents choix. Sont-ils tous équivalents ? Notre approche est-elle la plus générale ? Nous examinons ces questions sur un exemple simple (voir Section 2.4).

1.4.2 Problème contraint posé dans un domaine avec bord

Lorsque le domaine spatial est l'espace tout entier, une solution du problème

$$\begin{cases} \partial_t W + \sum_{j=1}^n A_j \partial_j W = 0 & \text{sur }]0, T[\times \mathbb{R}^n, \\ W(0, x) = W^0(x) & \text{si } x \in \mathbb{R}^n, \end{cases} \quad (1.4.25)$$

peut être définie comme l'unique solution $W \in L^2(]0, T[\times \mathbb{R}^n; \mathbb{R}^m)$, associée à la donnée initiale $W^0 \in L^2(\mathbb{R}^n; \mathbb{R}^m)$ vérifiant l'inégalité

$$\int_0^T \int_{\mathbb{R}^n} \left(|W - \kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle W - \kappa; A_j(W - \kappa) \rangle \partial_j \phi \right) dx dt + \int_{\mathbb{R}^n} |W^0(x) - \kappa|^2 \phi(0, x) dx \geq 0. \quad (1.4.26)$$

pour tout $\kappa \in \mathbb{R}^m$ et toute fonction $\phi \in C_c^\infty([0, T[\times \mathbb{R}^n)$ avec $\phi(t, x) \geq 0$ pour tout $(t, x) \in [0, T[\times \mathbb{R}^n$. Cela signifie que l'introduction des contraintes s'est faite de manière assez simple dans la définition de solution pour le problème (1.3.6). Il suffit d'indiquer que les fonctions W et W_0 sont à valeurs dans le convexe \mathcal{C} des contraintes et de n'utiliser que des vecteurs tests κ qui appartiennent à \mathcal{C} dans la définition de solution. La première idée qui vient à l'esprit lorsque que nous étudions un système de Friedrichs avec contrainte et condition de bord est d'introduire la notion de contrainte de manière similaire. Ainsi, si nous cherchons à étudier le problème (1.4.6), (1.4.7) et (1.4.16) muni de la contrainte $W(t, x) \in \mathcal{C}$ pour $(t, x) \in]0, T[\times \Omega$, il nous suffit de changer la Définition 1.4.5 de la façon suivante

Définition 1.4.6. Une fonction $W \in L^2(]0, T[\times \Omega; \mathcal{C})$ est appelée solution dissipative du problème (1.4.6), (1.4.7) et (1.4.16) sous la contrainte $W(t, x) \in \mathcal{C}$ pour $(t, x) \in]0, T[\times \Omega$ si pour toute fonction $\phi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, l'espace des fonctions

$W^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$ à valeurs positives et à support compact, et pour tout $\kappa \in \mathcal{C}$ l'inégalité suivante est vérifiée

$$\begin{aligned} \int_0^T \int_{\Omega} |W - \kappa|^2 \partial_t \phi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{j=1}^n \langle W - \kappa; A_j(W - \kappa) \rangle \partial_j \phi(t, x) dx dt + \int_0^T \int_{\Omega} 2 \langle f; W - \kappa \rangle \phi dx dt \\ + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \phi d\mathcal{H}^{n-1} dt + \int_{\Omega} |W_0 - \kappa|^2 \phi(0, x) dx \geq 0. \end{aligned}$$

Remarque 1.4.7. Pour des raisons de comptabilité avec les données initiales, l'étude de problèmes contraints (avec ou sans condition de bord) suppose que la condition initiale, W_0 , est à valeurs dans le convexe des contraintes.

Des questions naturelles se posent : existe-t-il des solutions pour ce type de problèmes ? Si oui, sont-elles uniques ? Y-a-t-il une interaction entre la condition de bord et la contrainte ? Nous développerons ces questions sur l'exemple particulier du modèle décrit dans les Sections 1.1.2 et 1.3.2 dans le Chapitre 3.5.

1.5 L'approximation numérique des systèmes de FRIEDRICHS sous contraintes

L'étude théorique des équations aux dérivées partielles hyperboliques permet de mettre en évidence des situations dans lesquelles la solution présente des discontinuités (ondes de chocs par exemple) au cours du temps bien que la solution initiale soit très régulière. Ce type de comportement implique de devoir adopter une approche particulière pour traiter ces situations lors de l'approximation numérique de telles équations. Un moyen de pallier de telles difficultés est de considérer la version intégrale des équations plutôt que de chercher à discrétiser ponctuellement les opérateurs de dérivation. Ainsi le domaine de calcul est divisé en cellules et nous allons chercher à approcher la valeur moyenne de la solution sur chaque cellule (plutôt que de chercher la valeur de la solution en certain(s) point(s) de cette cellule). Il s'agit là du principe de la méthode des volumes finis (pour une description détaillée de cette méthode, nous renvoyons le lecteur aux ouvrages [43, 64]). L'analyse mathématique de la méthode des volumes finis a permis d'obtenir de nombreux résultats concernant la stabilité, la convergence et la précision de tels schémas lorsque le problème est *scalaire* (*i.e.* la solution ne dépend que d'une variable d'espace) et le maillage est quelconque. Dans le cadre de cette thèse, les problèmes qui nous intéressent sont multidimensionnels (*i.e.* la solution est à valeurs vectorielles et dépend de plusieurs variables d'espace). L'analyse générale de schémas, basés sur la méthode des volumes finis, sur maillage quelconque permettant d'obtenir la convergence (ainsi qu'une vitesse de convergence) pour des problèmes linéaires multidimensionnels est assez récente (cf [98, 55, 28]). Notre but principal est de développer des schémas similaires (valides sur maillage non structuré) pour des problèmes sous contrainte.

Il existe déjà des résultats pour des systèmes hyperboliques sous contraintes lorsque la maillage est régulier. En effet, dans l'article [37], les auteurs démontrent l'existence de solutions faibles contraintes pour le problème (1.4.1) grâce à un schéma numérique, de type volumes finis, sur maillage cartésien. Pour cela, ils utilisent un schéma composé de deux étapes : une première étape, utilisant le schéma de RUSANOV (cf [64, Section 12.5]), laisse évoluer l'équation aux dérivées partielles sans se préoccuper des contraintes. Dans un second temps, la solution est projetée dans chaque cellule sur l'espace des contraintes.

Ainsi, ils prouvent que la solution construite par ce schéma converge vers une solution du problème (1.4.1) au sens de la Définition (1.4.1). Cette démonstration se fait en deux étapes. Premièrement, un argument de compacité permet d'extraire une sous-suite convergente de la suite des solutions construites *via* ce schéma. Puis, il est démontré que la limite de cette sous-suite est bien une solution du problème. Il est important de noter que l'étude de la vitesse de convergence des solutions numériques n'a pas été nécessaire dans cet argument.

L'un des objectifs de cette thèse est, dans un premier temps, d'effectuer une étude de la vitesse de convergence sur maillage non structuré dans le cadre des systèmes de FRIEDRICHS contraints et sans bord. Pour cela, nous nous sommes appuyés sur les résultats de VILA et VILLEDIEU (cf [98]). En effet, ils obtiennent, dans cet article, une vitesse de convergence pour les solutions construites par le schéma de type volumes finis décentré amont explicite en temps sur maillage non structuré pour des problèmes de FRIEDRICHS sans bord. Ce travail est très générique : il est valable en toute dimension, les matrices mises en jeu peuvent dépendre du temps et de l'espace, l'équation présente un second membre. VILA et VILLEDIEU démontrent ainsi que la solution construite grâce à leur schéma (que nous noterons W_h) tend vers la solution W du système de FRIEDRICHS (sans contrainte) dans $L^2_{\text{loc}}([0, T] \times \mathbb{R}^n; \mathbb{R}^m)$ et ils obtiennent une estimation de l'erreur sous la forme suivante

$$\|W - W_h\|_{L^2([0, T] \times \mathcal{K}; \mathbb{R}^m)} \leq Ch^{1/2},$$

où \mathcal{K} est un compact de \mathbb{R}^n , C une constante dépendant des données du problème, de \mathcal{K} et de T et h est le plus grand diamètre des cellules du maillage utilisé.

La dérivation de cette estimation dans cet article de VILA et VILLEDIEU peut être rapprochée du cadre des solutions faibles contraintes décrites dans [37]. Cette estimation repose sur l'inégalité suivante (voir [98, Équation (4.7)])

$$\int_0^T \int_{\mathbb{R}^n} |W - W_h|^2 \partial_t \phi + \sum_{j=1}^n \langle W - W_h; A_j(W - W_h) \rangle \partial_j \phi \, dx \, dt \geq \nu_h(\phi) - 2\mu_h(W\phi), \quad (1.5.1)$$

où ϕ est une fonction test à valeurs positives, ν_h et μ_h sont des mesures d'erreurs provenant des approximations numériques. Cette inégalité est très similaire à l'inégalité (1.4.5) de KATO décrite dans le Théorème 1.4.2. Le second membre n'est plus égal à zéro, comme dans (1.4.5), mais fait intervenir des mesures d'erreurs numériques. Cette similarité laisse penser qu'une adaptation des travaux de VILA et VILLEDIEU dans le cadre des systèmes de FRIEDRICHS contraints est possible. Cet étude a été faite dans le Chapitre 4 (plus particulièrement dans la section 4.3) et les résultats obtenus sont rapidement décrits dans la Section 1.6.3.

Ces résultats portant uniquement sur des problèmes posés dans l'espace tout entier (et ne faisant pas intervenir de condition de bord), nous avons aussi cherché (en utilisant la note [28] de COUDIÈRE, VILA et VILLEDIEU) à étudier des schémas pouvant prendre en compte la contrainte et la condition de bord. Ces schémas reposent là encore sur deux étapes : nous laissons évoluer le problème sur un pas de temps sans se préoccuper des contraintes puis nous projetons sur l'espace des contraintes dans chaque cellule. Nous illustrons ces schémas sur l'exemple type de cette interaction entre contraintes et bord : le modèle simplifié de la dynamique de la plasticité parfaite.

1.6 Les principaux résultats de la thèse

Nous présentons ici succinctement les résultats obtenus dans le cadre de cette thèse.

1.6.1 Systèmes de FRIEDRICHS avec condition de bord

Dans le Chapitre 2, nous concentrons notre travail à l'étude des solutions dissipatives (*i.e.* au sens de la Définition 1.4.5) du problème (1.4.6), (1.4.7) et (1.4.16). Dans un premier temps, nous cherchons à obtenir le plus d'informations possible en sachant que les inégalités (1.4.24) sont vérifiées pour tous vecteurs κ et toutes fonctions φ .

Nous parvenons à démontrer des lemmes préliminaires sur les solutions dissipatives. Elles vérifient l'équation aux dérivées partielles (1.4.6) au sens des distributions (Lemme 2.2.13). De plus, il est possible de donner un sens (faible) à la condition initiale (Lemme 2.2.14), qui lorsque les solutions dissipatives sont plus régulières en temps nous permet de retrouver la condition initiale au sens classique. Ensuite, nous cherchons à obtenir des informations sur la condition de bord. Nous examinons pour cela le cas particulier du demi-plan $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^-$. La condition de bord au sens fort peut s'exprimer sous la forme suivante $W_+ = 0$. Nous démontrons alors que dans le cas du demi-plan, une solution dissipative W vérifie que la moyenne de W_+ multipliée par une fonction test quelconque sur la tranche $]0, T[\times]\times \mathbb{R}^{n-1} \times]-\epsilon, 0[$ (avec $\epsilon > 0$) tend vers 0 lorsque ϵ tend vers 0 (Lemme 2.2.15). Encore une fois, si nous savons que la solution dissipative est régulière (suivant les variables d'espace) alors la condition de bord est vérifiée au sens fort.

Dans un second temps, nous démontrons le théorème principal du Chapitre 2 d'existence et d'unicité pour les solutions dissipatives lorsque $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^-$ (Théorème 2.3.1). Ce théorème indique que dès que le second membre de l'équation et la condition initiale sont de carrés intégrables, il existe une solution dissipative pour le problème (1.4.6), (1.4.7) et (1.4.16) et si en plus la donnée initiale appartient à l'espace $H^1(\mathbb{R}^{n-1} \times \mathbb{R}^-; \mathbb{R}^m)$ et vérifie la condition de bord, la solution dissipative est unique.

Puis, nous comparons les différentes approches des conditions de bord pour les systèmes hyperboliques, exposées dans la Section 1.4.1, sur un exemple simple. Et enfin, nous introduisons la notion de solution dissipatives contraintes qui nous permet de considérer des systèmes hyperboliques posés dans un domaine avec bord et soumis à une contrainte. L'étude de ces solutions dissipatives contraintes est faite sur un exemple et fait l'objet du Chapitre 3, dont les résultats sont décrits dans la section suivante.

1.6.2 Le modèle simplifié de la dynamique de la plasticité parfaite

Le Chapitre 3 se consacre à l'étude du modèle simplifié de la dynamique de la plasticité parfaite (décrit dans la Section 1.1.2) et son lien avec le problème hyperbolique sous contraintes qui lui est associé.

Tout d'abord, nous cherchons à déterminer les conditions de bord adaptées, au sens de la théorie développée dans le Chapitre 2, pour ce problème. En oubliant les contraintes, les conditions de bord permettant de définir des solutions dissipatives (au sens de la Définition 1.4.5) sont de la forme $\langle \mathcal{S}; \nu \rangle + \lambda^{-1} \partial_t u = 0$ où λ est un paramètre strictement positif et ν la normale au bord de ω . Nous utilisons l'approche classique en calcul des variations consistant à relaxer

la contrainte ainsi qu'à ajouter un terme de viscosité dans l'équation du mouvement et notre premier résultat est alors un théorème d'existence et d'unicité (Théorème 3.4.1) pour ce problème visqueux.

Il est alors nécessaire d'effectuer une analyse asymptotique pour obtenir le modèle simplifié à partir des solutions du modèle visqueux. Il apparaît alors un phénomène très intéressant : la condition de bord se retrouve modifier par la contrainte. La condition de bord "hyperbolique" pour ce problème est ainsi une condition non linéaire qui peut s'écrire sous la forme

$$\langle \mathfrak{S}; \nu \rangle + \lambda^{-1} T_\lambda(\partial_t \mathbf{u}) = 0,$$

où T_λ est la fonction seuil valant l'identité sur $[-\lambda, \lambda]$, λ sur $]\lambda, +\infty[$ et $-\lambda$ sur $]-\infty, -\lambda[$. Muni de cette condition de bord non linéaire, le modèle simplifié de la plasticité parfaite admet une unique solution (Théorème 3.5.4). Ici le terme de solution réfère à une solution, dite "variationnelle" (car construite à partir de méthodes classiques en calcul des variations) et indique que le triplet $(\mathbf{u}, \mathfrak{S}, \rho)$ vérifie l'équation du mouvement, la décomposition additive du gradient, la contrainte, la loi d'écoulement, la condition de bord, les conditions initiales et une identité d'énergie (par opposition aux solutions dissipatives qui vérifient une série d'inégalités).

Puis, nous démontrons dans le Théorème 3.5.14 que nous retrouvons en passant à la limite $\lambda \rightarrow 0^+$ (respectivement $\lambda \rightarrow +\infty$) les solutions (plus classiques dans la littérature) associées aux conditions de bord de DIRICHLET en vitesse (respectivement NEUMANN).

Ensuite, nous tirons parti de cette analogie avec les systèmes hyperboliques pour obtenir un résultat de régularité en temps courts (Théorème 3.6.4) pour les solutions du modèle simplifié (avec condition de bord non linéaire, de DIRICHLET en vitesse ou de NEUMANN) grâce à un argument, classique pour des problèmes hyperboliques, de propagation à vitesse finie.

Finalement, nous comparons les deux approches : hyperbolique et variationnelle. Nous démontrons, dans la Proposition 3.7.3, d'abord qu'une solution "variationnelle" est une solution dissipative (au sens de la définition de la Section 2.5). Puis nous montrons dans la Proposition 3.7.4 que la notion de solution dissipative (régulière en temps) contient toutes les informations sur le modèle simplifié, c'est-à-dire qu'elle est aussi une solution au sens variationnel.

1.6.3 Étude numérique des systèmes de FRIEDRICHS sous contraintes

Dans un premier temps, nous démontrons dans le Chapitre 4 un résultat théorique préliminaire à notre étude numérique des systèmes de FRIEDRICHS sous contraintes. Plus précisément, ce résultat permet de justifier l'approche formelle (le passage à la limite $(\eta, \epsilon) \rightarrow (0, 0)$ dans les équations (1.4.2)), développée dans [37] et rappelée dans la Section 1.4. Nous démontrons ainsi que l'introduction d'un terme visqueux dans l'équation ainsi que la relaxation des contraintes nous permet de construire des solutions faibles contraintes au sens de [37] (Théorème 4.2.10). Ce travail concerne les problèmes posés dans l'espace tout entier et nous permet de démontrer qu'une solution faible contrainte appartient à $H^1((0, T) \times \mathbb{R}^n; \mathcal{E})$ dès que la donnée initiale est dans $H^1(\mathbb{R}^n; \mathcal{E})$.

Muni de ce résultat théorique, nous présentons un schéma pour les problèmes hyperboliques contraints dans tout l'espace. La stratégie en deux étapes permet d'obtenir un schéma stable ayant la même vitesse de convergence que dans le cas non contraint (obtenue dans [98]) (Théorème 4.3.17). La démonstration d'un tel résultat repose sur l'analogie entre l'approche théorique de [37] et l'estimation de la vitesse de convergence de [98] : ces deux articles utilisent un résultat de comparaison très proche qui nous permet d'étendre le résultat de [98] au cas contraint (cf (1.4.5) et (1.5.1)).

Dans un troisième temps, nous étudions l'approximation des systèmes de FRIEDRICHS contraints en domaine borné. Là encore notre étude se base sur le schéma décrit dans la note [28] et la stratégie en deux étapes de [37] (évolution puis projection), nous démontrons que cette stratégie permet d'obtenir deux schémas (très proches) stables et nous illustrons dans la Section 4.4, sur l'exemple étudié dans le Chapitre 3, que cette méthode permet d'obtenir une solution conforme aux résultats théoriques. En particulier, nous concentrons notre attention sur la condition de bord non linéaire mise en évidence dans l'étude du modèle simplifié de la dynamique de la plasticité parfaite (cf Section 3.5).

Initial boundary value problems for Friedrichs' systems

This chapter is an extended version of the article : Clément MIFSUD, Bruno DESPRÉS et Nicolas SEGUIN. « Dissipative formulation of initial boundary value problems for Friedrichs' systems ». In : *Comm. Partial Differential Equations* 41.1 (2016), p. 51–78.

2.1 Introduction

The aim of this Chapter is to present a notion of solution for initial boundary value problems in the case of linear hyperbolic systems. This problem has been studied by many different authors [47, 57, 73, 70, 13, 89, 40, 29] and there are still a lot of open questions for this kind of problems (in particular in the case of variable multiplicities : see [85]). In addition, many physical systems can be recast in a Friedrichs' framework : linear elasticity theory [82], isotropic Maxwell equations [45], linearized Euler equations, MHD [85].

Having in mind dissipative problems such as viscoplastic problems for which boundary conditions play a important mechanical role and their associated limit plastic problems (for more details see Section 4.2), which can develop singularities and discontinuities, our initial goal was to give a new definition of solutions that requires minimal regularity and in particular, the solution does not need a trace on the boundary. In [37], a new formulation for Friedrichs' systems on the whole space is proposed, very similar to the Kružkov's definition of entropy solutions of scalar quasi-linear equations [58] and to dissipative solutions introduced by Lions for incompressible Euler equations [68]. In [87], Otto extends the concept of entropy solutions to bounded domains with Dirichlet conditions, without requiring any assumptions on the existence of traces of the solutions (contrary to the theory developed in [12]). We aim to do the same for linear systems. In order to verify that this original formalism contains enough information, we prove in the classical case of constant coefficients in the half-plane that this formulation leads to a well-posed theory in L^2 and that our formulation is equivalent to the usual strong formulation for sufficiently smooth solutions. Since different theories of boundary conditions are available in the literature, we compare our formulation to other formulations on a simple example. Finally, we show how to extend the notion of dissipative solutions when convex constraints are added.

This work is organized as follows. In the first section, we explain formally where the definition comes from, how this definition can be extended in order to account for non-homogeneous boundary conditions and variable coefficients. Then we show that a H^1 solution of our weak formulation is a solution of the problem in the strong sense in the case of constant coefficients when the domain is a half-plane. Next we prove that there exists a unique solution (in the sense of our definition) in the non-characteristic setting for H^1 initial data, that satisfies the boundary condition, and L^2 source term, again for constant coefficients and if the spatial domain is a half-plane. The regularity of the initial data is necessary for our proof, but in this case, the classical theory is valid for L^2 initial data and L^2 source term. We also compare our approach to other formulations in the literature (namely to [47],[16] and [38]) and we examine these boundary conditions on the simple example of the wave equation in 1D with Dirichlet condition. Finally, using the same type of boundary condition, we give a definition of a solution to initial boundary value problems for Friedrichs' systems with convex constraints.

2.2 Definition and basic properties

2.2.1 Motivations and definitions

We consider the following problem

$$\begin{cases} \partial_t U + \sum_{i=1}^n A_i \partial_i U = f, & \text{on } (0, T) \times \Omega, \\ U(0, x) = U_0(x), & \text{on } \Omega, \\ (A(\nu(x)) - M(\nu(x)))U(t, x) = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (2.2.1)$$

where Ω is an open domain (not necessarily bounded) of \mathbb{R}^n with (smooth) boundary, denoted by $\partial\Omega$, $U : (0, T) \times \Omega \rightarrow \mathbb{R}^m$, $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$, $U_0 \in L^2(\Omega; \mathbb{R}^m)$, the matrices A_i are $m \times m$ symmetric, $A(\nu(x)) = \sum_{i=1}^n \nu_i A_i$ and $\nu(x) = (\nu_1, \nu_2, \dots, \nu_n)$ is the exterior unit normal of Ω at the point $x \in \partial\Omega$, $M(\nu(x))$ is a $m \times m$ symmetric non-negative matrix such that for all $x \in \partial\Omega$,

$$\ker A(\nu(x)) \subset \ker M(\nu(x)) \quad (2.2.2)$$

and

$$\mathbb{R}^m = \ker(A(\nu(x)) - M(\nu(x))) + \ker(A(\nu(x)) + M(\nu(x))). \quad (2.2.3)$$

We also assume that the matrix-valued function $x \in \partial\Omega \mapsto M(\nu(x))$ belongs to L^∞ . The symmetry of the matrix M is an important hypothesis in this approach. In the work of Friedrichs [47] the matrix M does not need to be symmetric, but its symmetric part has to be non-negative (see Section 2.4). In his work, Friedrichs only considers matrices M such that the previous sum is direct (this hypothesis is too restrictive, especially in the characteristic case where $\det(A(\nu(x))) = 0$), but one can see (using Lemma 2.2.3) that in the non-characteristic case (*i.e.* $\det(A(\nu(x))) \neq 0$) the sum is always direct. In the following, the matrices A_i are independent of the space and time variables but our formulation can be extended to matrices with variables coefficients (see Remark 2.2.9).

Remark 2.2.1. To construct a matrix $M(\nu(x))$, we could consider an orthogonal matrix $P(\nu(x))$ such that

$${}^t P(\nu(x)) A(\nu(x)) P(\nu(x)),$$

is a diagonal matrix and its diagonal terms are $\lambda_{1,1} \leq \dots \leq \lambda_{p,p} < \lambda_{p+1,p+1} = 0 = \dots = \lambda_{q,q} < \dots \leq \lambda_{m,m}$, then define

$$M(\nu(x)) = P(\nu(x)) \text{diag}(|\lambda_{1,1}|, \dots, |\lambda_{p,p}|, 0, \dots, 0, \lambda_{q+1,q+1}, \dots, \lambda_{m,m}) P(\nu(x))^t.$$

It means there is no information coming from the incoming characteristic (since we consider here an homogeneous boundary condition). The hypothesis that M is symmetric still allows us to use a large class of boundary conditions coming from the theory of characteristics.

To shorten notation, we write A_ν instead of $A(\nu(x))$ and M_ν instead of $M(\nu(x))$.

Remark 2.2.2. Here we denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product on \mathbb{R}^m .

Lemma 2.2.3. Let A_ν and M_ν be symmetric matrices verifying the properties (2.2.2) and (2.2.3). The following statements are true

1. We have the decomposition

$$\mathbb{R}^m = \ker A_\nu \oplus ((\ker(A_\nu - M_\nu)) \cap \text{Im} A_\nu) \oplus ((\ker(A_\nu + M_\nu)) \cap \text{Im} A_\nu),$$

and we denote $\kappa = \kappa^0 + \kappa_- + \kappa_+$ the decomposition of $\kappa \in \mathbb{R}^m$ according to this sum.

2. M_ν is positive on $(\ker(A_\nu + M_\nu)) \cap \text{Im} A_\nu$.

3. For all $\kappa_1, \kappa_2 \in \mathbb{R}^m$, $\langle \kappa_1; A_\nu \kappa_2 \rangle = \langle \kappa_{1-}; A_\nu \kappa_{2-} \rangle + \langle \kappa_{1+}; A_\nu \kappa_{2+} \rangle$.

4. For all $\kappa \in \mathbb{R}^m$, $0 \leq \langle \kappa_-; M_\nu \kappa_- \rangle \leq \langle \kappa; M_\nu \kappa \rangle$ and $0 \leq \langle \kappa_+; M_\nu \kappa_+ \rangle \leq \langle \kappa; M_\nu \kappa \rangle$.

Proof. 1. Since A_ν is a diagonalizable matrix, we know that

$$\mathbb{R}^m = \ker A_\nu \oplus \text{Im} A_\nu.$$

Using the fact that we suppose that $\mathbb{R}^m = \ker(A_\nu - M_\nu) + \ker(A_\nu + M_\nu)$, we have

$$\text{Im}A_\nu = ((\ker(A_\nu - M_\nu)) \cap \text{Im}A_\nu) + ((\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu).$$

Suppose that $\kappa \in ((\ker(A_\nu - M_\nu)) \cap \text{Im}A_\nu) \cap ((\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu)$, then $(A_\nu \pm M_\nu)\kappa = 0$ thus $\kappa \in \ker A_\nu$. But by definition, $\kappa \in \text{Im}A_\nu$ and $\ker A_\nu \cap \text{Im}A_\nu = \{0\}$ so $\kappa = 0$. Therefore,

$$\text{Im}A_\nu = ((\ker(A_\nu - M_\nu)) \cap \text{Im}A_\nu) \oplus ((\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu),$$

and thus, we obtain

$$\mathbb{R}^m = \ker A_\nu \oplus ((\ker(A_\nu - M_\nu)) \cap \text{Im}A_\nu) \oplus ((\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu).$$

2. Since M_ν is non-negative matrix, it suffices to show that $\ker M_\nu$ is reduced to $\{0\}$ on $(\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu$. If $M_\nu v = 0$ for some $v \in (\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu$, then $A_\nu v = 0$. It implies, since $\ker A \cap \text{Im}A = \{0\}$, that $v = 0$ and the result follows.
3. We use the decomposition of κ_1 (resp. κ_2) according to the sum

$$\mathbb{R}^m = \ker A_\nu \oplus ((\ker(A_\nu - M_\nu)) \cap \text{Im}A_\nu) \oplus ((\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu),$$

denoted $\kappa_1^0 + \kappa_{1-} + \kappa_{1+}$ (resp. $\kappa_2^0 + \kappa_{2-} + \kappa_{2+}$). First, we notice that

$$\langle \kappa_1; A_\nu \kappa_2 \rangle = \langle \kappa_1; A_\nu (\kappa_{2-} + \kappa_{2+}) \rangle,$$

using the fact that A_ν is symmetric, we see that

$$\langle \kappa_1; A_\nu \kappa_2 \rangle = \langle \kappa_{1-} + \kappa_{1+}; A_\nu (\kappa_{2-} + \kappa_{2+}) \rangle.$$

We now observe that

$$\langle M_\nu \kappa_{1-}; \kappa_{2+} \rangle = \langle A_\nu \kappa_{1-}; \kappa_{2+} \rangle = \langle \kappa_{1-}; A_\nu \kappa_{2+} \rangle = -\langle \kappa_{1-}; M_\nu \kappa_{2+} \rangle,$$

which implies that $\langle \kappa_{1-}; A_\nu \kappa_{2+} \rangle = 0$, from which we get the claim.

4. Since $\ker A_\nu \subset \ker M_\nu$, the same reasoning applies to the matrix M_ν and we obtain

$$\forall \kappa_1, \kappa_2 \in \mathbb{R}^n, \quad \langle \kappa_1; M_\nu \kappa_2 \rangle = \langle \kappa_{1-}; M_\nu \kappa_{2-} \rangle + \langle \kappa_{1+}; M_\nu \kappa_{2+} \rangle.$$

Taking $\kappa_1 = \kappa_2$ and using the fact that M is non-negative leads to the result. □

This lemma allows us to make some algebraic manipulations (see below) to derive the following definition

Definition 2.2.4. We say that $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ is a dissipative solution of System (2.2.1) if for all $\varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, the space of $W^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$ functions with non-negative values and compact supports, and for all $\kappa \in \mathbb{R}^m$ the following inequality holds

$$\begin{aligned} & \int_0^T \int_\Omega |U(t, x) - \kappa|^2 \partial_t \varphi(t, x) dx dt + \int_0^T \int_\Omega \sum_{i=1}^n \langle U(t, x) - \kappa; A_i (U(t, x) - \kappa) \rangle \partial_i \varphi(t, x) dx dt \\ & + \int_\Omega |U_0(x) - \kappa|^2 \varphi(0, x) dx + \int_0^T \int_\Omega 2 \langle f(t, x); U(t, x) - \kappa \rangle \varphi(t, x) dx dt + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt \geq 0, \end{aligned} \quad (2.2.4)$$

where κ_+ stands for the projection onto $(\ker(A_\nu + M_\nu)) \cap \text{Im}A_\nu$.

Remark 2.2.5. One should notice that the test functions do not need to vanish at time $t = T$ (see, for example, the proof of Theorem 2.3.5). One could actually obtain the same results using test functions that belong to $\varphi \in W_{c,+}^{1,\infty}((-\infty, T) \times \mathbb{R}^n)$.

Remark 2.2.6. As A_ν and M_ν depend on $x \in \partial\Omega$, κ_+ also depends on $\partial\Omega$. In the case where A_ν and M_ν do not depend on x , κ_+ is independent of $x \in \partial\Omega$ (see Remark 2.2.16 for more details).

It is worth pointing out that the formulation does not require much regularity on the solution, in particular we do not use any traces of the solution either at the initial time or at the boundary. It is also important to note that the problem (2.2.1) is a linear problem and here we give a nonlinear formulation of a solution to System (2.2.1) due to the quadratic terms of (2.2.4). In fact, if U (resp. \tilde{U}) is a dissipative solution associated with the data (U_0, f) (resp. (\tilde{U}_0, \tilde{f})), one can not deduce easily that for $\lambda \in \mathbb{R}$, $\lambda U + \tilde{U}$ is a dissipative solution. This formulation is inspired by the one exposed in [37] for Friedrichs' systems under constraints in the whole space. Furthermore, this definition can be used in the case of variable coefficients (see Remark 2.2.9) and when the number of non-negative eigenvalues of A_ν is not constant along $\partial\Omega$, this is the case of variable multiplicities (as long as we can find a matrix M_ν that satisfies the hypotheses (2.2.2) and (2.2.3) and belongs to L^∞). These cases are of deep interest for applications (the condition of constant multiplicities is difficult to ensure for a general domain Ω , in particular for a bounded domain $\partial\Omega$) and that is still an active domain of research (see for example [85]).

We call this formulation dissipative since taking $\kappa = 0$ and a function $\varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, such that $\varphi(t, x) = T - t$ on $(0, T) \times \Omega$, in (2.2.4), we get the L^2 -norm of a solution is non-increasing in the case where there is no source term and the domain Ω is bounded,

$$\int_0^T \int_\Omega |U|^2(t, x) dt dx \leq T \int_\Omega |U_0|^2(x) dx.$$

This possible loss of energy of the solution suggests one to call the solution dissipative (as P-L. Lions did for Euler equations in [68]). The word dissipative also refers in the literature to special kind of boundary conditions, our boundary condition can be called dissipative in that sense (see Section 2.4).

Let us briefly explain where this formulation comes from. If we assume that the solution U is smooth, we go from (2.2.1) to

$$\begin{aligned} & \int_0^T \int_\Omega |U(t, x) - \kappa|^2 \partial_t \varphi(t, x) dx dt + \sum_{i=1}^n \langle U(t, x) - \kappa; A_i(U(t, x) - \kappa) \rangle \partial_i \varphi(t, x) + 2 \langle f(t, x); U(t, x) - \kappa \rangle \varphi(t, x) dx dt \\ & - \int_0^T \int_{\partial\Omega} \langle U(t, x) - \kappa; A_\nu(U(t, x) - \kappa) \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt + \int_\Omega |U_0(x) - \kappa|^2 \varphi(0, x) dx - \int_\Omega |U(T, x) - \kappa|^2 \varphi(T, x) dx = 0, \end{aligned}$$

by taking the scalar product with $U - \kappa$ and then integrating by parts (where κ is a constant vector). Then we write the decomposition of $\kappa = \kappa^0 + \kappa_- + \kappa_+$ and observe that since $U \in \ker(A_\nu - M_\nu)$ at the boundary we have $U_+ = 0$ and it follows that

$$\begin{aligned} & \int_0^T \int_{\partial\Omega} \langle U(t, x) - \kappa; A_\nu(U(t, x) - \kappa) \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt \\ & = \int_0^T \int_{\partial\Omega} \langle U(t, x) - \kappa_-; A_\nu(U(t, x) - \kappa_-) \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt + \int_0^T \int_{\partial\Omega} \langle \kappa_+; A_\nu \kappa_+ \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt \\ & = \int_0^T \int_{\partial\Omega} \langle U(t, x) - \kappa_-; M_\nu(U(t, x) - \kappa_-) \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt - \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt, \end{aligned}$$

thanks to Lemma 2.2.3. Consequently,

$$\begin{aligned} & \int_0^T \int_\Omega \left(|U(t, x) - \kappa|^2 \partial_t \varphi(t, x) + \sum_{i=1}^n \langle U(t, x) - \kappa; A_i(U(t, x) - \kappa) \rangle \partial_i \varphi(t, x) dx dt + 2 \langle f(t, x); U(t, x) - \kappa \rangle \varphi(t, x) \right) dx dt \\ & + \int_\Omega |U_0(x) - \kappa|^2 \varphi(0, x) dx + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt \\ & = \int_0^T \int_{\partial\Omega} \langle U(t, x) - \kappa_-; M_\nu(U(t, x) - \kappa_-) \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) + \int_\Omega |U(T, x) - \kappa|^2 \varphi(T, x) dx dt \geq 0. \end{aligned}$$

Remark 2.2.7. We restrict ourselves to the case where the matrices A_i are symmetric for several reasons. First, even if the inequality (2.2.4) still makes sense when the matrices A_i are no longer symmetric, it will be equivalent to the one where we replace the matrices A_i by their symmetric parts. This means that the inequality (2.2.4) will only take into account the symmetric parts of the matrices A_i . In addition, one can see that the results of Lemma 2.2.3 are false in the general case where A_ν is not symmetric (even if we suppose that M_ν is symmetric) and these results are useful to get a formulation that is valid in the general setting of functions in $L^2((0, T) \times \Omega; \mathbb{R}^m)$.

Remark 2.2.8. Here we explain how to take into account of an in-homogeneous boundary condition of the form

$$(A_\nu - M_\nu)(U(t, x) - U_b(t, x)) = 0, \quad \text{on } (0, T) \times \partial\Omega,$$

where $U_b \in L^2((0, T) \times \partial\Omega)^m$. From the previous formal calculations, we want to eliminate the trace of U in the term

$$\int_0^T \int_{\partial\Omega} \langle U(t, x) - \kappa; A_\nu(U(t, x) - \kappa) \rangle \varphi(t, x) d\mathcal{H}^{n-1}(x) dt.$$

Since we know that, formally and using Lemma 2.2.3,

$$\begin{aligned} \langle U - \kappa; A_\nu(U - \kappa) \rangle &= \langle U_b - \kappa; A_\nu(U_b - \kappa) \rangle + \langle U - U_b; A_\nu(U - U_b) \rangle + 2\langle U - U_b; A_\nu(U_b - \kappa) \rangle \\ &= \langle (U_b - \kappa)_-; M_\nu(U_b - \kappa)_- \rangle - \langle (U_b - \kappa)_+; M_\nu(U_b - \kappa)_+ \rangle + \langle U - U_b; M_\nu(U - U_b) \rangle + 2\langle (U - U_b); M_\nu(U_b - \kappa)_- \rangle \\ &\geq -\langle (U_b - \kappa)_+; M_\nu(U_b - \kappa)_+ \rangle, \end{aligned}$$

we just replace the term

$$\int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \varphi(t, x) dt d\mathcal{H}^{n-1}(x),$$

by

$$\int_0^T \int_{\partial\Omega} \langle (\kappa - U_b)_+; M_\nu(\kappa - U_b)_+ \rangle \varphi(t, x) dt d\mathcal{H}^{n-1}(x),$$

in the formulation.

Remark 2.2.9. We also can write a definition of solution for symmetric matrices A_i with variable coefficients. If the functions $(t, x) \mapsto A_i(t, x)$ are $m \times m$ matrix-valued functions belonging to $W^{1,\infty}$ (assuming that one can find a matrix-valued function $(t, x) \in (0, T) \times \partial\Omega \mapsto M(t, \nu(x))$ that satisfies the conditions (2.2.2) and (2.2.3) and belongs to L^∞). One just need to add the term

$$\int_0^T \int_{\Omega} \sum_{i=1}^n \langle U(t, x) - \kappa; \partial_i [A_i](U(t, x) - \kappa) \rangle \varphi(t, x) dx dt,$$

to the left-hand side of the inequality (2.2.4).

Remark 2.2.10. One can see in the functions parameterized by $\kappa \in \mathbb{R}^m$

$$F(U) = |U - \kappa|^2 \tag{2.2.5}$$

the analogue of the entropy functions of Kruřkov [58] in the scalar case of quasi-linear initial value problems.

Example 2.2.11. In the scalar case ($m = 1$), the theory of initial boundary value problems is well known ([12],[87]). Using our theory, we see that if $A_\nu \geq 0$ then $M_\nu = A_\nu$ is the only choice for the boundary matrix and it leads to the fact that there is no condition at that point (this corresponds exactly to the case when the Bardos - Le Roux - Nédélec condition [12] is automatically verified whatever the trace is at that point). In the case $A_\nu < 0$, the choice for M_ν is $-A_\nu$, it implies (at least formally) that $U = 0$ at this point of the boundary (again this corresponds to the Bardos - Le Roux - Nédélec condition).

2.2.2 What is the information contained in the dissipative formulation ?

Our goal is here to show that dissipative solutions (which satisfy inequalities) satisfy the original equation, the initial condition and the boundary condition in a weak sense (that are equalities). For technical reasons (see Remark 2.2.16) we assume that the spatial domain is a half-plane for the study of the boundary condition.

Remark 2.2.12. The Lemmas 2.2.13 and 2.2.14 are also true in the case of variable coefficients and when Ω is a general domain with smooth boundary using the same methods.

We first show that if $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ is a solution of (2.2.1) in the sense of Definition 2.2.4 then U is a solution of the equation

$$\partial_t U + \sum_{i=1}^n A_i \partial_i U = f,$$

in the sense of distributions in $(0, T) \times \Omega$.

Lemma 2.2.13. *A solution U in the sense of Definition 2.2.4 verifies*

$$\partial_t U + \sum_{i=1}^n A_i \partial_i U = f, \quad \text{in } \mathcal{D}'((0, T) \times \Omega; \mathbb{R}^m).$$

Proof. Let φ be a function of $C_c^\infty((0, T) \times \Omega)$ with $\varphi \geq 0$. Then, from Definition 2.2.4 and since a function of $C_c^\infty((0, T) \times \Omega)$ can be extended by zero in order to define a function of $W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, we know that

$$\begin{aligned} \int_0^T \int_\Omega |U(t, x) - \kappa|^2 \partial_t \varphi(t, x) dx dt + \int_0^T \int_\Omega \sum_{i=1}^n \langle U(t, x) - \kappa; A_i (U(t, x) - \kappa) \rangle \partial_i \varphi(t, x) dx dt \\ + 2 \int_0^T \int_\Omega \langle f(t, x); U(t, x) - \kappa \rangle \varphi(t, x) dx dt \geq 0. \end{aligned}$$

Expanding the terms $|U(t, x) - \kappa|^2$ and $\langle U(t, x) - \kappa; A_i (U(t, x) - \kappa) \rangle$ and using the fact that the partial derivatives of κ vanish leads us to the following inequality

$$\begin{aligned} \int_0^T \int_\Omega (|U(t, x)|^2 - 2 \langle \kappa; U(t, x) \rangle) \partial_t \varphi(t, x) dx dt \\ + \int_0^T \int_\Omega \sum_{i=1}^n (\langle U(t, x); A_i U(t, x) \rangle - 2 \langle U(t, x); A_i \kappa \rangle) \partial_i \varphi(t, x) dx dt \\ + 2 \int_0^T \int_\Omega \langle f(t, x); U(t, x) - \kappa \rangle \varphi(t, x) dx dt \geq 0. \end{aligned}$$

Since κ is independent of t and x , we rewrite the previous inequality as

$$\begin{aligned} \int_0^T \int_\Omega \left[|U(t, x)|^2 \partial_t \varphi(t, x) + \sum_{i=1}^n \langle U(t, x); A_i U(t, x) \rangle \partial_i \varphi(t, x) + \langle f(t, x); U(t, x) \rangle \varphi(t, x) \right] dx dt \geq \\ 2 \left\langle \kappa; \int_0^T \int_\Omega U(t, x) \partial_t \varphi(t, x) + \sum_{i=1}^n A_i U(t, x) \partial_i \varphi(t, x) + f(t, x) \varphi(t, x) dx dt \right\rangle. \end{aligned}$$

The left-hand side of the inequality is independent of κ , thus the previous inequality implies that

$$\int_0^T \int_\Omega U(t, x) \partial_t \varphi(t, x) + \sum_{i=1}^n A_i U(t, x) \partial_i \varphi(t, x) dx dt + \int_0^T \int_\Omega f(t, x) \varphi(t, x) dx dt = 0. \quad (2.2.6)$$

It remains to show that Equation (2.2.6) is true for all $\varphi \in C_c^\infty((0, T) \times \Omega)$. Let φ be a function in $C_c^\infty((0, T) \times \Omega)$, then $\varphi = \varphi^+ - \varphi^-$ where φ^+ (resp. φ^-) is the positive part (resp. negative part) of φ and φ^+ (resp. φ^-) can be extended by zero in to order to define an element of $W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$. By density, Equation (2.2.6) is true for φ^+ (resp. φ^-) and thus the lemma is proven. \square

Now, we focus our attention on the initial condition and the boundary condition.

Lemma 2.2.14. *Let $\xi \in W_{c,+}^{1,\infty}(\mathbb{R}^n)$. We have*

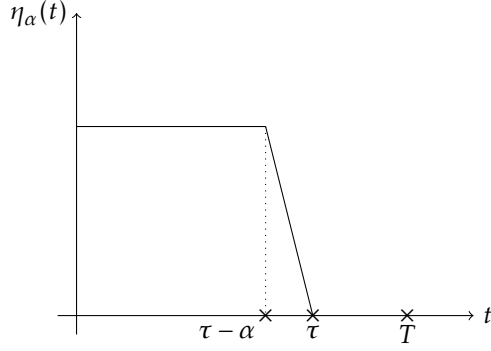
$$\lim_{\tau \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_{\tau-\alpha}^\tau \int_\Omega |U(t, x) - U_0(x)|^2 \xi(x) dx dt = 0.$$

Proof. For convenience, we take $f = 0$. The same conclusion can be drawn for a general $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$. Let $\xi \in W_{c,+}^{1,\infty}(\mathbb{R}^n)$. Let ρ_n^ϵ be a positive n -dimensional mollifier with $\epsilon > 0$ such that

$$\rho_n^\epsilon(x) = \epsilon^{-n} \rho(x/\epsilon),$$

where $\rho \in C_c^\infty(\mathbb{R}^n)$, $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho(x) dx = 1$ and $\text{supp } \rho \subset \overline{B(0, 1)}$. Let $\alpha > 0$ and $\tau \in (\alpha, T)$. We define the function $\varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ by

$$\varphi(t, x, y) = \xi(x) \eta_\alpha(t) \rho_n^\epsilon(x - y),$$

FIGURE 2.1 – Graph of the function η_α on $(0, T)$.

where $\eta_\alpha(t) = \max(0, \min(1, (\tau - t)/\alpha))$ if $t \geq 0$ and $\eta \in W_{c,+}^{1,\infty}(\mathbb{R})$ (see Figure 2.1). Observe that $\eta'_\alpha(t) \leq 0$ on $(0, T)$ and $\eta'_\alpha = -\frac{1}{\alpha} \mathbb{1}_{[\tau-\alpha, \tau]} \xrightarrow{\alpha \rightarrow 0^+} -\delta_\tau$ in the space $\mathcal{D}'((0, T))$.

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |U(t, x) - U_0(y)|^2 &= |U(t, x) - U_0(x)|^2 + |U_0(x) - U_0(y)|^2 + 2 \langle U(t, x) - U_0(x); U_0(x) - U_0(y) \rangle \\ &\geq |U(t, x) - U_0(x)|^2 + |U_0(x) - U_0(y)|^2 - 2|U(t, x) - U_0(x)||U_0(x) - U_0(y)|. \end{aligned}$$

From the inequality $\partial_t \varphi \leq 0$, we get for almost every (t, x, y) ,

$$\begin{aligned} |U(t, x) - U_0(y)|^2 \partial_t \varphi(t, x, y) &\leq |U(t, x) - U_0(x)|^2 \partial_t \varphi(t, x, y) + |U_0(x) - U_0(y)|^2 \partial_t \varphi(t, x, y) \\ &\quad - 2|U(t, x) - U_0(x)||U_0(x) - U_0(y)| \partial_t \varphi(t, x, y). \end{aligned}$$

We integrate this inequality over $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ (we extend U and U_0 by zero outside of Ω) to get, thanks to Fubini's theorem,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t, x) - U_0(y)|^2 \partial_t \varphi(t, x, y) \, dx \, dy \, dt \\ &\quad \leq \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t, x) - U_0(x)|^2 \partial_t \varphi(t, x, y) \, dx \, dy \, dt - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)|^2 \xi(x) \rho_n^\epsilon(x - y) \, dx \, dy \\ &\quad \quad - 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)| \xi(x) \rho_n^\epsilon(x - y) \int_0^T |U(t, x) - U_0(x)| \eta'_\alpha(t) \, dt \, dx \, dy. \end{aligned}$$

We take $\kappa = U_0(y)$ and use the above-defined test function $(t, x) \mapsto \varphi(t, x, y)$ (for a fixed $y \in \mathbb{R}^n$, this function belongs to $W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$) in the definition of a dissipative solution and integrate with respect to y to obtain

$$\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t, x) - U_0(y)|^2 \partial_t \varphi(t, x, y) \, dx \, dy \, dt \geq - \int_0^T \mathcal{R}_{\alpha,\epsilon}(t) \, dt - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)|^2 \xi(x) \rho_n^\epsilon(x - y) \, dx \, dy,$$

where $\mathcal{R}_{\alpha,\epsilon}$ belongs to $L^1(0, T)$ and is for almost every time t defined by

$$\begin{aligned} \mathcal{R}_{\alpha,\epsilon}(t) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n \langle U(t, x) - U_0(y); A_i(U(t, x) - U_0(y)) \rangle \eta_\alpha(t) \partial_i (\xi(x) \rho_n^\epsilon(x - y)) \, dx \, dy \\ &\quad + \int_{\mathbb{R}^n} \int_{\partial\Omega} \langle U_0(y)_+; M_\nu U_0(y)_+ \rangle \xi(x) \eta_\alpha(t) \rho_n^\epsilon(x - y) \, d\mathcal{H}^{n-1}(x) \, dy. \quad (2.2.7) \end{aligned}$$

Using these two inequalities, we get

$$\begin{aligned} - \int_0^T \mathcal{R}_{\alpha,\epsilon}(t) dt &\leq \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t,x) - U_0(x)|^2 \partial_t \varphi(t,x,y) dx dy dt \\ &\quad - 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)| \xi(x) \rho_n^\epsilon(x-y) \int_0^T |U(t,x) - U_0(x)| \eta'_\alpha(t) dt dx dy. \end{aligned}$$

Using the fact that for $(a,b) \in \mathbb{R}^2$, $ab \leq b^2 + \frac{a^2}{4}$ and, since $\partial_t \varphi \leq 0$,

$$\begin{aligned} -|U_0(x) - U_0(y)| |U(t,x) - U_0(x)| \xi(x) \rho_n^\epsilon(x-y) \eta'_\alpha(t) &= |U_0(x) - U_0(y)| |U(t,x) - U_0(x)| |\partial_t \varphi(t,x,y)| \\ &\leq |U_0(x) - U_0(y)|^2 \xi(x) \rho_n^\epsilon(x-y) |\eta'_\alpha(t)| - \frac{|U(t,x) - U_0(x)|^2}{4} \partial_t \varphi(t,x,y). \end{aligned}$$

By integration of this inequality over $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, it follows

$$- \int_0^T \mathcal{R}_{\alpha,\epsilon}(t) dt \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t,x) - U_0(x)|^2 \partial_t \varphi(t,x,y) dx dy dt + 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)|^2 \xi(x) \rho_n^\epsilon(x-y) dx dy. \quad (2.2.8)$$

We are going to send α to 0. By the definition of η_α (and since $\eta_0 = \mathbb{1}_{[0,\tau]}$) and equation (2.2.7), we have

$$\begin{aligned} &\left| \int_0^T [\mathcal{R}_{\alpha,\epsilon}(t) - \mathcal{R}_{0,\epsilon}(t)] dt \right| \\ &= \left| \int_0^T \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n \langle U(t,x) - U_0(y); A_i(U(t,x) - U_0(y)) \rangle (\eta_\alpha(t) - \eta_0(t)) \partial_i (\xi(x) \rho_n^\epsilon(x-y)) dx dy \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{R}^n} \int_{\partial\Omega} \langle U_0(y)_+; M_\nu U_0(y)_+ \rangle \xi(x) (\eta_\alpha(t) - \eta_0(t)) \rho_n^\epsilon(x-y) d\mathcal{H}^{n-1}(x) dy \right] dt \right| \\ &\leq \int_{\tau-\alpha}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n |\langle U(t,x) - U_0(y); A_i(U(t,x) - U_0(y)) \rangle \partial_i \varphi(t,x,y)| dx dy dt \\ &\quad + \int_{\tau-\alpha}^\tau \int_{\mathbb{R}^n} \int_{\partial\Omega} \langle U_0(y)_+; M_\nu U_0(y)_+ \rangle \varphi(t,x,y) d\mathcal{H}^{n-1}(x) dy dt. \end{aligned}$$

We now use the fact that, if $\kappa = \kappa^0 + \kappa_- + \kappa_+$ with $\kappa_\pm \in \ker(A_\nu \pm M_\nu) \cap \text{Im} A_\nu$ and $\kappa^0 \in \ker A_\nu$, then $\langle \kappa_-; M_\nu \kappa_+ \rangle = 0$ and consequently $0 \leq \langle \kappa_+; M_\nu \kappa_+ \rangle \leq \langle \kappa; M_\nu \kappa \rangle$. Therefore the function

$$\begin{aligned} t \mapsto \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n |\langle U(t,x) - U_0(y); A_i(U(t,x) - U_0(y)) \rangle \partial_i \varphi(t,x,y)| dx dy \\ + \int_{\mathbb{R}^n} \int_{\partial\Omega} \langle U_0(y)_+; M_\nu U_0(y)_+ \rangle \varphi(t,x,y) d\mathcal{H}^{n-1}(x) dy \end{aligned}$$

belongs to $L^1(0, T)$, so the dominated convergence theorem implies that

$$\lim_{\alpha \rightarrow 0} \int_0^T \mathcal{R}_{\alpha,\epsilon}(t) dt = \int_0^T \mathcal{R}_{0,\epsilon}(t) dt.$$

From (2.2.8), it follows that,

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_{\tau-\alpha}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t,x) - U_0(x)|^2 \xi(x) \rho_n^\epsilon(x-y) dx dy dt \leq 2 \int_0^T \mathcal{R}_{0,\epsilon}(t) dt + 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)|^2 \xi(x) \rho_n^\epsilon(x-y) dx dy, \quad (2.2.9)$$

while, by Fubini's theorem, we remark that for every $\epsilon > 0$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U(t,x) - U_0(x)|^2 \xi(x) \rho_n^\epsilon(x-y) dx dy = \int_{\mathbb{R}^n} |U(t,x) - U_0(x)|^2 \xi(x) dx.$$

Then, we let τ tend to 0 in (2.2.9), and we use that for all ϵ , we see from (2.2.7) that

$$\begin{aligned} \left| \int_0^T \mathcal{R}_{0,\epsilon}(t) dt \right| &\leq \int_0^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n C_i (|U_0|^2(x) + |U_0|^2(y)) \partial_i [\xi(x) \rho_n^\epsilon(x-y)] dx dy dt \\ &\quad + \int_0^\tau \int_{\mathbb{R}^n} \int_{\partial\Omega} C |U_0|^2(y) \|\xi\|_\infty d\mathcal{H}^{n-1}(x) dy dt, \end{aligned}$$

where C_i and C respectively depend on A_i and M_ν . Consequently, the function

$$t \mapsto \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^n C_i (|U_0|^2(x) + |U_0|^2(y)) \partial_i [\xi(x) \rho_n^\epsilon(x-y)] dx dy + \int_{\mathbb{R}^n} \int_{\partial\Omega} C |U_0|^2(y) \|\xi\|_\infty d\mathcal{H}^{n-1}(x) dy$$

belongs to $L^1(0, T)$ and by the dominated convergence theorem, we get

$$\lim_{\tau \rightarrow 0} \int_0^T \mathcal{R}_{0,\epsilon}(t) dt = 0.$$

Finally, using (2.2.9), we obtain

$$\begin{aligned} \lim_{\tau \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_{\tau-\alpha}^\tau \int_{\Omega} |U(t, x) - U_0(x)|^2 \xi(x) dx dt &= \lim_{\tau \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_{\tau-\alpha}^\tau \int_{\mathbb{R}^n} |U(t, x) - U_0(x)|^2 \xi(x) dx dt \\ &\leq 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |U_0(x) - U_0(y)|^2 \xi(x) \rho_n^\epsilon(x-y) dx dy. \end{aligned}$$

The left-hand side is independent of ϵ and the right-hand side goes to 0 as ϵ goes to 0. \square

We now discuss the boundary condition in the special case of the n -dimensional half-space

$$\Omega = \mathbb{R}_-^n = \{(x_1, x_2, \dots, x_n) = (x', x_n), x_n < 0\}. \quad (2.2.10)$$

In the following, we assume that $\Omega = \mathbb{R}_-^n$ and that the matrices A_i and M_ν are independent of the space and time variables (see Remark 2.2.16 for more details).

Lemma 2.2.15. *Assume Ω is a half-space (2.2.10). If $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ is a solution of (2.2.1) in $(0, T) \times \mathbb{R}_-^n$ in the sense of Definition 2.2.4, then U satisfies the following condition at the boundary*

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\mathbb{R}^{n-1}} \int_{-\epsilon}^0 \int_0^T U(t, x)_+ \psi(t) \lambda(x') dt dx_n dx' = 0,$$

for every $\psi \in W^{1,\infty}(\mathbb{R})$ and $\lambda \in W^{1,\infty}(\mathbb{R}^{n-1})$ with compact supports (resp. in \mathbb{R} and in \mathbb{R}^{n-1}).

Proof. Again, we take $f = 0$ for simplicity. Using the fact that any function of $W^{1,\infty}$ with compact support is the difference of its positive part and its negative part (that belong to $W_{c,+}^{1,\infty}$), we may suppose that $\psi \in W_{c,+}^{1,\infty}(\mathbb{R})$ and $\lambda \in W_{c,+}^{1,\infty}(\mathbb{R}^{n-1})$. We take in the formulation of a solution in Definition 2.2.4 the function $\phi(x, t) = a_\epsilon(x_n) \psi(t) \lambda(x')$ where $a_\epsilon(x_n) = \max(0, (x_n + \epsilon)/\epsilon)$ if $x_n \leq 0$ and $a_\epsilon \in W_{c,+}^{1,\infty}(\mathbb{R})$. We have

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \int_{-\epsilon}^0 \int_0^T |U(t, x) - \kappa|^2 a_\epsilon(x_n) \lambda(x') \psi'(t) dt dx_n dx' + \frac{1}{\epsilon} \int_{\mathbb{R}^{n-1}} \int_{-\epsilon}^0 \int_0^T \langle U - \kappa; A_n(U - \kappa) \rangle \psi(t) \lambda(x') dt dx_n dx' \\ &\quad + \int_{\mathbb{R}^{n-1}} \int_{-\epsilon}^0 \int_0^T \sum_{i=1}^{n-1} \langle U - \kappa; A_i(U - \kappa) \rangle \psi(t) a_\epsilon(x_n) \partial_i \lambda(x') dt dx_n dx' + \int_0^T \int_{\mathbb{R}^{n-1}} \langle \kappa_+; M_\nu \kappa_+ \rangle \psi(t) \lambda(x') dt dx' \geq 0. \end{aligned}$$

Then we use the fact that for all $v \in \mathbb{R}^n$

$$\langle v; A_\nu v \rangle = \langle v; A_n v \rangle = \langle v_-; M_\nu v_- \rangle - \langle v_+; M_\nu v_+ \rangle,$$

and we apply this identity in the second integral to get

$$\begin{aligned} & \int_{\Omega_\epsilon} \int_0^T |U(t, x) - \kappa|^2 a_\epsilon(x) \psi'(t) \lambda(x') dt dx_n dx' + \int_{\Omega_\epsilon} \int_0^T \sum_{i=1}^{n-1} \langle U - \kappa; A_i(U - \kappa) \rangle \psi(t) a_\epsilon(x_n) \partial_i \lambda(x') dt dx_n dx' \\ & + \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle (U(t, x) - \kappa)_-; M_\nu(U(t, x) - \kappa)_- \rangle \psi(t) \lambda(x') dt dx_n dx' \\ & - \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle (U(t, x) - \kappa)_+; M_\nu(U(t, x) - \kappa)_+ \rangle \psi(t) \lambda(x') dt dx_n dx' + \int_0^T \int_{\mathbb{R}^{n-1}} \langle \kappa_+; M_\nu \kappa_+ \rangle \psi(t) \lambda(x') dt dx' \geq 0, \end{aligned}$$

where $\Omega_\epsilon = \mathbb{R}^{n-1} \times (-\epsilon, 0)$. The linearity of the projector and the fact that κ_+ and M_ν are independent of x leads us to

$$\begin{aligned} & \int_{\Omega_\epsilon} \int_0^T |U(t, x) - \kappa|^2 a_\epsilon(x) \psi'(t) \lambda(x') dt dx_n dx' + \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle (U(t, x) - \kappa)_-; M_\nu(U(t, x) - \kappa)_- \rangle \psi(t) \lambda(x') dt dx_n dx' \\ & + \int_{\Omega_\epsilon} \int_0^T \sum_{i=1}^{n-1} \langle U - \kappa; A_i(U - \kappa) \rangle a_\epsilon(x) \psi(t) \partial_i \lambda(x') dt dx_n dx' \geq \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle U(t, x)_+; M_\nu U(t, x)_+ \rangle \psi(t) \lambda(x') dt dx_n dx' \\ & - \frac{2}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle U(t, x)_+; M_\nu \kappa_+ \rangle \psi(t) \lambda(x') dt dx_n dx'. \end{aligned}$$

Now, we set $\kappa = -\alpha \kappa_0$ where $\kappa_0 = \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T U(t, x)_+ \psi(t) \lambda(x') dt dx_n dx'$ and $\alpha \in \mathbb{R}$ is to be chosen later. Observe that $\kappa_0 \in \ker(A_\nu + M_\nu) \cap \text{Im} A_\nu$ and consequently $\kappa_- = 0$. Using that κ in the previous inequality we obtain

$$\begin{aligned} 2\alpha \langle \kappa_0; M_\nu \kappa_0 \rangle & \leq \alpha^2 C_1 \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T |\kappa_0|^2 a_\epsilon(x) dt dx_n dx' + C_2 \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T |U(t, x)|^2 a_\epsilon(x) dt dx_n dx' \\ & + \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle U(t, x)_-; M_\nu U(t, x)_- \rangle \psi(t) \lambda(x') dt dx_n dx', \end{aligned}$$

where C_1 and C_2 are positive numbers depending on A_i , λ and ψ . We now choose α to be equal to

$$\alpha = \frac{\langle \kappa_0; M_\nu \kappa_0 \rangle}{C_1 \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T |\kappa_0|^2 a_\epsilon(x) dt dx_n dx'}, \quad (2.2.11)$$

and consequently we get

$$2\alpha \langle \kappa_0; M_\nu \kappa_0 \rangle - \alpha^2 C_1 \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T |\kappa_0|^2 a_\epsilon(x) dt dx_n dx' = \alpha \langle \kappa_0; M_\nu \kappa_0 \rangle,$$

and we obtain

$$\alpha \langle \kappa_0; M_\nu \kappa_0 \rangle \leq C_2 \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T |U(t, x)|^2 a_\epsilon(x) dt dx_n dx' + \frac{1}{\epsilon} \int_{\Omega_\epsilon} \int_0^T \langle U(t, x)_-; M_\nu U(t, x)_- \rangle \psi(t) \lambda(x') dt dx_n dx'.$$

Since M_ν is coercive on $\ker(A_\nu + M_\nu) \cap \text{Im} A_\nu$ (we denote by γ the constant of coercivity of M_ν), Equation (2.2.11) implies

$$\alpha \geq \frac{\gamma |\kappa_0|^2}{|\kappa_0|^2 C_1 \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T a_\epsilon(x) dt dx_n dx'}.$$

We see that α and ϵ are related by the following inequality

$$\alpha \geq \frac{C}{\epsilon},$$

where $C = C(\psi, M_\nu, A_i, \lambda)$. Finally,

$$\langle \kappa_0; M_\nu \kappa_0 \rangle \leq \frac{C_2 \epsilon}{C} \int_{-\epsilon}^0 \int_{\text{supp } \lambda} \int_0^T |U(t, x)|^2 a_\epsilon(x) dt dx_n dx' + \frac{1}{C} \int_{\Omega_\epsilon} \int_0^T \langle U(t, x)_-; M_\nu U(t, x)_- \rangle \psi(t) \lambda(x') dt dx_n dx'.$$

Since $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$, the first term of the right hand side of the inequality tends to 0 as $\epsilon \rightarrow 0$. The dominated convergence and the fact that $\langle U(t, x)_-; M_\nu U(t, x)_- \rangle \leq \langle U(t, x); M_\nu U(t, x) \rangle \leq C|U(t, x)|^2$ show us that the second term also tends to 0 as $\epsilon \rightarrow 0$. Observe that M_ν is coercive on $\ker(A_\nu + M_\nu) \cap \text{Im}A_\nu$, so by the definition of κ_0 we get the claim. \square

Remark 2.2.16. *In the proof of the previous lemma, we crucially used the fact that we were in the case of constant coefficients to get some information on the boundary conditions when we say that $\kappa_0 \in \ker(A_\nu + M_\nu) \cap \text{Im}A_\nu$. For the initial conditions, we always have some information since the boundary at the time $t = 0$ is flat and the boundary matrix for $t = 0$ is the identity matrix. On the contrary, recovering some information in the case when the matrices A_ν and M_ν depend on x and t seems trickier and is left for further studies.*

Using the previous lemmas, we will show that a regular solution in the sense of Definition 2.2.4 is a solution of Problem 2.2.1 :

Theorem 2.2.17. *Assume that Ω is a half-plane (2.2.10). Assume that U is a dissipative solution and $U \in H^1((0, T) \times \Omega; \mathbb{R}^m)$. Then U is a solution of 2.2.1 in the classical sense, i.e.*

1. $\partial_t U + \sum_{i=1}^n A_i \partial_i U = f$ in $L^2((0, T) \times \Omega; \mathbb{R}^m)$,
2. $U(0, \cdot) = U_0$ in $L^2(\Omega; \mathbb{R}^m)$,
3. $(A_\nu - M_\nu)U = 0$ in $L^2((0, T) \times \partial\Omega)$.

The proof of this theorem follows from the discussion below. First, observe that Lemmas 2.2.13 and 2.2.14 imply respectively the first and the second point of this theorem. Secondly, we need two lemmas to prove that the third point of the previous theorem is true.

Lemma 2.2.18. *Assume Ω is a half-space (2.2.10). Let $U \in W^{1,p}((0, T) \times \Omega; \mathbb{R}^m)$ ($1 < p < \infty$), $\lambda \in C_c^\infty((0, T))$ and $\beta \in C_c^\infty(\mathbb{R}^{n-1})$. Then*

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\epsilon}^0 \int_0^T \int_{\mathbb{R}^{n-1}} |(U(t, x', x_n) - U(t, x', 0)) \lambda(t) \beta(x')| dx' dt dx = 0.$$

Proof. First, observe that

$$\begin{aligned} & \frac{1}{\epsilon} \int_{-\epsilon}^0 \int_0^T \int_{\mathbb{R}^{n-1}} |(U(t, x', x_n) - U(t, x', 0)) \lambda(t) \beta(x')| dx' dt dx_n \\ & \leq \int_{-\epsilon}^0 \int_0^T \int_{\mathbb{R}^{n-1}} \left| \frac{U(t, x', x_n) - U(t, x', 0)}{x_n} \right| |\lambda(t) \beta(x')| dx' dt dx_n \\ & \leq \|\lambda\|_\infty \|\beta\|_\infty \int_0^T \left\| \frac{U(t, x', x_n) - U(t, x', 0)}{x_n} \right\|_{L^2((-\epsilon, 0) \times \mathbb{R}^{n-1})} dt, \end{aligned}$$

Using Hardy's inequality (proved for instance in [94]), we know that for all $t \in (0, T)$

$$\left\| \frac{U(t) - U(t, x_n = 0)}{x_n} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{p-1} \|\partial_n U(t)\|_{L^p(\mathbb{R}^n)},$$

and consequently $(t, x', x_n) \mapsto \frac{U(t, x', x_n) - U(t, x', 0)}{x_n}$ belongs to $L^2((0, T) \times \Omega; \mathbb{R}^m)$. The lemma follows from the dominated convergence theorem. \square

To apply the previous result to U_+ , we need the following lemma

Lemma 2.2.19. *Assume Ω is a half-space (2.2.10). Suppose that $A_\nu U \in H^1((0, T) \times \Omega; \mathbb{R}^m)$. Then $U_+ \in H^1((0, T) \times \Omega; \mathbb{R}^m)$.*

Proof. Since $\ker A_\nu \subset \ker M_\nu$, there exists a matrix B such that $M_\nu = BA_\nu$ and consequently $M_\nu U$ also belongs to $H^1((0, T) \times \Omega; \mathbb{R}^m)$. But

$$(A_\nu - M_\nu)U = (A_\nu - M_\nu)(U^0 + U_- + U_+) = (A_\nu - M_\nu)U_+ = -2M_\nu U_+.$$

We know that M_ν is non-singular on $\ker(A_\nu + M_\nu) \cap \text{Im} A_\nu$, so U_+ belongs to $H^1((0, T) \times \Omega; \mathbb{R}^m)$. \square

Since $U \in H^1((0, T) \times \Omega; \mathbb{R}^m)$, $A_\nu U$ also belongs to $H^1((0, T) \times \Omega; \mathbb{R}^m)$ and thanks to the previous lemma, $U_+ \in H^1((0, T) \times \Omega; \mathbb{R}^m)$. Now using Lemmas 2.2.18 and 2.2.15, we deduce that

$$\int_0^T \int_{\mathbb{R}^{n-1}} U(t, x', 0)_+ \lambda(t) \beta(x') dx' dt = 0,$$

for every λ and β test-functions. Consequently,

$$U_+ = 0, \quad \text{a.e on } (0, T) \times \{x_n = 0\} \times \mathbb{R}^{n-1},$$

and the boundary condition is satisfied in the strong sense. Similar arguments apply to the case of the initial condition thanks to Lemma 2.2.14.

2.3 Existence and uniqueness result in the half-plane

This section is devoted to prove the following theorem

Theorem 2.3.1. *Suppose that $\Omega = \mathbb{R}_+^n$, the matrices A_i and M_ν have constant coefficients and A_n is non singular. For every $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ and $U_0 \in L^2(\Omega; \mathbb{R}^m)$, there exists a dissipative solution of (2.2.1). If $U_0 \in H^1(\Omega; \mathbb{R}^m)$ and satisfies the boundary condition in the following sense : $(A_\nu - M_\nu)U_0 = 0$ in $L^2(\partial\Omega)$, the dissipative solution is unique.*

Remark 2.3.2. *In the previous theorem, we need the H^1 regularity for the initial condition to satisfy the hypothesis of Theorem 2.3.12 but also to know that the function g , defined by (2.3.8), is L^2 . If one could obtain a comparison principle for L^2 data, as in the case developed in [37] for the whole space, the uniqueness would be an easy consequence. For now, the H^1 regularity and the fact that the initial data satisfies the boundary condition allow us to avoid technical problems about traces on the boundary.*

The existence is proven thanks to the classical theory of Friedrichs' systems for regular data and the uniqueness result makes use of mollification in the tangential directions (*i.e* in the time direction and in the x' -direction). In order to use the regularity theory (developed in [73]), we assume that

$$A_\nu = A_n \text{ is non singular.} \quad (2.3.1)$$

In order to prove the existence result, one can use, for example, the theory developed in [16] and [73] for Friedrichs' systems. To do so, we need a matrix $B \in \mathbb{M}^{p \times m}(\mathbb{R})$, the space of $p \times m$ matrices with real coefficients, where p is the number of negative eigenvalues of A_ν and examine the following initial boundary value problem

$$\begin{cases} \partial_t U + \sum_{i=1}^n A_i \partial_i U = f, & \text{on } (0, T) \times \Omega, \\ U(0, x) = U_0(x), & \text{on } \Omega, \\ BU(t, x) = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (2.3.2)$$

where $f \in H^1([0, T] \times \Omega)$ and $U_0 \in H^1(\Omega; \mathbb{R}^m)$. We suppose that U_0 obeys the following compatibility condition

$$\forall x \in \partial\Omega, \quad BU_0(x) = 0. \quad (2.3.3)$$

To connect our boundary condition and the boundary condition using a matrix $B \in \mathbb{M}^{p \times m}(\mathbb{R})$, we need that $\ker B = \ker(A_\nu - M_\nu)$.

For example, since $\ker(A_\nu - M_\nu) \oplus \ker(A_\nu + M_\nu) = \mathbb{R}^m$, one can take the matrix B that gives the coordinates in a basis of $\ker(A_\nu + M_\nu)$ of the projection into the space $\ker(A_\nu + M_\nu)$. In others words, if $(e_1, \dots, e_p, e_{p+1}, \dots, e_m)$ (with p the number of negative eigenvalues of A_ν) is a basis of \mathbb{R}^m such that $e_1, \dots, e_p \in \ker(A_\nu + M_\nu)$ and $e_{p+1}, \dots, e_m \in \ker(A_\nu - M_\nu)$

and if we denote by $\mathcal{V}_1, \dots, \mathcal{V}_m$ the component of a vector \mathcal{V} with respect to this basis, *i.e.*

$$\mathcal{V} = \sum_{i=1}^m \mathcal{V}_i e_i,$$

then we define b to be the linear map $\mathcal{V} \in \mathbb{R}^m \mapsto (\mathcal{V}_1, \dots, \mathcal{V}_p) \in \mathbb{R}^p$ and B is a matrix associated to this map. Since here we are dealing with constant matrices (A_ν and M_ν), B is also constant.

Under all these assumptions, we can say that there exists a unique solution to the problem (2.3.2) and this solution belongs to the space $\mathcal{C}^1([0, T]; L^2(\Omega; \mathbb{R}^m)) \cap \mathcal{C}^0([0, T]; H^1(\Omega; \mathbb{R}^m))$. Consequently, the initial condition is satisfied in a strong sense and the partial differential equation in (2.3.2) has to be understood as an equality in $\mathcal{C}^0([0, T]; L^2(\Omega; \mathbb{R}^m))$. The boundary condition is satisfied in $\mathcal{C}^0([0, T]; L^2(\partial\Omega; \mathbb{R}^m))$.

This result can be found in [73, Theorem 3.1] (see also [16, Theorem 9.16]) where the theorem is stated with general hypotheses (variable coefficients, general domain Ω). To simplify our address, we decide to state here a simpler version of this theorem in the case we are interested in.

Theorem 2.3.3. *Suppose that $\Omega = \mathbb{R}_+^n$ and that the following assumptions are satisfied :*

1. *The matrices A_i have constant coefficients and are symmetric,*
2. *The boundary of Ω is non-characteristic,*
3. *The matrix $B \in \mathbb{M}^{p \times m}(\mathbb{R})$ has constant coefficients and satisfies the following hypothesis*
 - (a) *$U \in \ker B \Rightarrow \langle A_\nu U; U \rangle > 0$, if $U \neq 0$.*
 - (b) *$\ker B$ is maximal for the previous property *i.e.* one can not find a vectorial space $E \subset \mathbb{R}^m$ such that $\ker B \subsetneq E$ and $U \in E \Rightarrow \langle A_\nu U; U \rangle > 0$, if $U \neq 0$.*
 - (c) *B is onto.*

Suppose also that the data are regular in the sense that $U_0 \in H^1(\Omega)$, $f \in H^1((0, T) \times \Omega)$ and that U_0 satisfies the compatibility condition :

$$\forall x \in \partial\Omega, \quad BU_0(x) = 0.$$

Then the solution belongs to the space $\mathcal{C}^1([0, T]; L^2(\Omega; \mathbb{R}^m)) \cap \mathcal{C}^0([0, T]; H^1(\Omega; \mathbb{R}^m))$

Remark 2.3.4. *The fact that the boundary of Ω is non-characteristic is necessary to get the H^1 regularity in space (see [89]). The conditions on the boundary matrix B and the hypotheses we made on the matrices A_i are sufficient to obtain a priori estimates for the problem (2.3.2) and its adjoint (see [16, Proposition 3.1]). These estimates are necessary to obtain this regularity theorem (see [73, Hypotheses (1.2) and (1.3)]).*

The following theorem tells us that the regular solution of the problem (2.3.2) is in fact a regular dissipative solution of (2.2.1).

Theorem 2.3.5. *Assuming that $f \in H^1([0, T] \times \Omega; \mathbb{R}^m)$, $U_0 \in H^1(\Omega; \mathbb{R}^m)$ and condition (2.3.3), there exists a solution to problem (2.2.1) in the sense of Definition 2.2.4.*

Proof. As we said before, we have

$$\partial_t U + \sum_{i=1}^n A_i \partial_i U = f, \quad \text{in } \mathcal{C}^0((0, T); L^2(\Omega; \mathbb{R}^m)), \quad (2.3.4)$$

$$BU(t, x) = 0, \quad \text{in } \mathcal{C}^0((0, T); L^2(\Omega; \mathbb{R}^m)). \quad (2.3.5)$$

If κ is a vector of \mathbb{R}^n independent of t and x , then we also have the following equality

$$\partial_t(U - \kappa) + \sum_{i=1}^n A_i \partial_i(U - \kappa) = f, \quad \text{in } \mathcal{C}^0((0, T); L^2(\Omega; \mathbb{R}^m)). \quad (2.3.6)$$

Consequently, if $\varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$,

$$\begin{aligned} & \int_0^T \int_{\Omega} \langle \partial_t(U - \kappa); U - \kappa \rangle(t, x) \varphi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{i=1}^n \langle U - \kappa; A_i \partial_i(U - \kappa) \rangle(t, x) \varphi(t, x) dx dt \\ &= \int_0^T \int_{\Omega} \langle f; U - \kappa \rangle(t, x) \varphi(t, x) dx dt. \end{aligned}$$

Since for almost every $x \in \Omega$, the function $t \mapsto \langle \partial_t(U - \kappa); U - \kappa \rangle(t, x)$ belongs to $\mathcal{C}^0(0, T)$, the Green formula and Fubini's theorem give us

$$\int_0^T \int_{\Omega} \langle \partial_t(U - \kappa); U - \kappa \rangle(t, x) \varphi(t, x) dx dt = \frac{1}{2} \left(\int_{\Omega} |U(T, x) - \kappa|^2 \varphi(T, x) dx - \int_{\Omega} |U_0(x) - \kappa|^2 \varphi(0, x) dx - \int_0^T \int_{\Omega} |U - \kappa|^2(t, x) \partial_t \varphi(t, x) dx dt \right).$$

Since the function $x \mapsto U(t, x)$ belongs to $H^1(\Omega; \mathbb{R}^m)$ for every $t \in [0, T]$, it implies that the function $x \mapsto \langle U; A_i U \rangle$ belongs to $W^{1,1}(\Omega, \mathbb{R})$ (with $\partial_i \langle U; A_i U \rangle = 2 \langle U; A_i \partial_i U \rangle$) and consequently admits a trace on $\partial\Omega$ (in $L^1(\partial\Omega)$). The Green formula (applied to a $W^{1,1}(\Omega, \mathbb{R})$ function and a $W^{1,\infty}(\Omega, \mathbb{R}_+)$ function) yields

$$\begin{aligned} & \int_{\Omega} \langle U - \kappa; A_i \partial_i(U - \kappa) \rangle(t, x) \varphi(t, x) dx \\ &= -\frac{1}{2} \int_{\Omega} \langle U - \kappa; A_i(U - \kappa) \rangle(t, x) \partial_i \varphi(t, x) dx + \frac{1}{2} \int_{\partial\Omega} \nu_i \langle U - \kappa; A_i(U - \kappa) \rangle(t, x) \varphi(t, x) d\mathcal{H}^{n-1}(x). \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{i=1}^n \int_0^T \int_{\Omega} \langle U - \kappa; A_i \partial_i(U - \kappa) \rangle(t, x) \varphi(t, x) dx dt &= -\frac{1}{2} \sum_{i=1}^n \int_0^T \int_{\Omega} \langle U - \kappa; A_i(U - \kappa) \rangle(t, x) \partial_i \varphi(t, x) dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\partial\Omega} \langle U - \kappa; A_\nu(U - \kappa) \rangle(t, x) \varphi(t, x) d\mathcal{H}^{n-1}(x) dt. \end{aligned}$$

Since we know that $\ker B = \ker(A_\nu - M_\nu)$, we can conclude that $U(x, t) \in \ker(A_\nu - M_\nu)$ almost everywhere on $(0, T) \times \partial\Omega$. Lemma 2.2.3 gives

$$\begin{aligned} & \int_0^T \int_{\partial\Omega} \langle U - \kappa; A_\nu(U - \kappa) \rangle(t, x) \varphi(t, x) d\mathcal{H}^{n-1}(x) dt \\ &= \int_0^T \int_{\partial\Omega} \langle U - \kappa_-; M_\nu(U - \kappa_-) \rangle(t, x) d\mathcal{H}^{n-1}(x) dt - \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle(t, x) d\mathcal{H}^{n-1}(x) dt. \end{aligned}$$

Finally we deduce from the facts that

$$\int_0^T \int_{\partial\Omega} \langle U - \kappa_-; M_\nu(U - \kappa_-) \rangle(t, x) d\mathcal{H}^{n-1}(x) dt \geq 0,$$

and

$$\int_{\Omega} |U(T, x) - \kappa|^2 \varphi(T, x) dx \geq 0,$$

that we have a dissipative solution to the problem (2.2.1) i.e. $\forall \kappa \in \mathbb{R}^n, \forall \varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$ we have the following inequality

$$\begin{aligned} \int_0^T \int_{\Omega} |U(t, x) - \kappa|^2 \partial_t \varphi(t, x) + \sum_{i=1}^n \int_{\Omega} \langle U - \kappa; A_i(U - \kappa) \rangle(t, x) \partial_i \varphi(t, x) dx dt &+ \int_0^T \int_{\Omega} 2 \langle f; U - \kappa \rangle(t, x) \varphi(t, x) dx dt \\ &+ \int_{\Omega} |U_0(x) - \kappa|^2 \varphi(0, x) dx + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle(t, x) d\mathcal{H}^{n-1}(x) dt \geq 0. \end{aligned}$$

□

Using this result, we can compare regular solutions.

Theorem 2.3.6. *Let U and \tilde{U} be two solutions (in the sense of Definition 2.2.4) associated with $(f, U_0) \in L^2((0, T) \times \Omega; \mathbb{R}^m) \times L^2(\Omega; \mathbb{R}^m)$ (resp. $(\tilde{f}, \tilde{U}_0) \in L^2((0, T) \times \Omega; \mathbb{R}^m) \times L^2(\Omega; \mathbb{R}^m)$). We suppose that U and \tilde{U} are $H^1((0, T) \times \Omega; \mathbb{R}^m)$. Then there exists a constant $C > 0$ (depending only on T) such that*

$$\|U - \tilde{U}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 \leq C \left(\|f - \tilde{f}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 + \|U_0 - \tilde{U}_0\|_{L^2(\Omega; \mathbb{R}^m)}^2 \right).$$

Proof. Using the results of Section 2.2, we know that U and \tilde{U} are solutions of the problem (2.2.1) in the classical sense. Consequently, one has

$$\begin{aligned} & \int_0^T \int_{\Omega} \langle \partial_t(U - \tilde{U}); U - \tilde{U} \rangle(t, x) \varphi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{i=1}^n \langle A_i \partial_i(U - \tilde{U}); U - \tilde{U} \rangle(t, x) \varphi(t, x) dx dt \\ &= \int_0^T \int_{\Omega} \langle f - \tilde{f}; U - \tilde{U} \rangle(t, x) \varphi(t, x) dx dt, \end{aligned}$$

Taking the function $\varphi : (t, x) \mapsto (T - t)/T$, we get

$$\begin{aligned} & \frac{1}{2T} \int_0^T \int_{\Omega} |U - \tilde{U}|^2(t, x) dx dt + \frac{1}{2} \int_0^T \langle U - \tilde{U}; A_\nu(U - \tilde{U}) \rangle(t, 0) \varphi(t, 0) dt \\ &= \int_0^T \int_{\Omega} \langle f - \tilde{f}; U - \tilde{U} \rangle(t, x) \varphi(t, x) dx dt + \frac{1}{2} \int_{\Omega} |U_0 - \tilde{U}_0|(x) dx, \end{aligned}$$

Using the fact that $A_\nu(U - \tilde{U}) = M_\nu(U - \tilde{U})$ on the boundary $x_n = 0$, we obtain thanks to the fact that M_ν is non-negative

$$2 \int_0^T \int_{\Omega} \langle f - \tilde{f}; U - \tilde{U} \rangle(t, x) \varphi(t, x) dx dt + \int_{\Omega} |U_0 - \tilde{U}_0|^2(x) dx \geq \frac{1}{T} \int_0^T \int_{\Omega} |U - \tilde{U}|^2(t, x) dx dt.$$

The Cauchy-Schwarz inequality leads us to

$$\frac{1}{T} \|U - \tilde{U}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 \leq 2 \|f - \tilde{f}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)} \|U - \tilde{U}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)} + \|U_0 - \tilde{U}_0\|_{L^2(\Omega; \mathbb{R}^m)}^2.$$

Finally,

$$\frac{1}{2T} \|U - \tilde{U}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 \leq 2T \|f - \tilde{f}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 + \|U_0 - \tilde{U}_0\|_{L^2(\Omega; \mathbb{R}^m)}^2.$$

The assertion of the lemma follows easily. \square

This comparison principle leads to the existence theorem.

Theorem 2.3.7. *If $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ and $U_0 \in L^2(\Omega; \mathbb{R}^m)$, then there exists a solution of System 2.2.1 in the sense of Definition 2.2.4.*

Proof. We regularize the problem by taking $f_p \in C_c^\infty((0, T) \times \Omega; \mathbb{R}^m)$ and $U_{0,p} \in C_c^\infty(\Omega; \mathbb{R}^m)$ (remark that automatically we have $BU_{0,p} = 0$ on $\partial\Omega$) such that f_p (resp. $U_{0,p}$) tends to f in $L^2((0, T) \times \Omega; \mathbb{R}^m)$ (resp. to U_0 in $L^2(\Omega; \mathbb{R}^m)$), we denote by U_p the regular dissipative solution given by Theorem 2.3.5. Then the previous lemma ensures that $(U_p)_{p \in \mathbb{N}}$ is a Cauchy sequence in $L^2((0, T) \times \Omega; \mathbb{R}^m)$ hence it converges to a function $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$. Passing to the limit when p goes to infinity in the inequality (2.2.4) leads to the existence of a solution. \square

Remark 2.3.8. *The comparison principle stated in Theorem 2.3.6 does not lead to the uniqueness of dissipative solutions. It only shows that dissipative solutions can be constructed as limits of classical regular solutions in an unique way.*

Now, we are going to show a uniqueness result in the case when $U_0 \in H^1(\Omega; \mathbb{R}^m)$ and $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$. Here, we take a solution $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ a solution of (2.2.1) associated with the function $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ and $U_0 \in H^1(\Omega; \mathbb{R}^m)$. We are going to regularize $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ in the tangential variables (*i.e.* the variable (t, x')), so that we will be able to integrate by parts as in the proof of Theorem 2.3.6.

Now we use convolution in time (with support in $[-\epsilon, -\epsilon/2]$ and $0 < \epsilon < 1$) and classical convolution in the $x' = (x_1, \dots, x_{n-1})$ variables. We denote by $\rho_{\epsilon, l} = \rho_1^{\epsilon/4}(\cdot + 3\epsilon/4)$ (see the proof of Lemma 2.2.15 for the definition of the mollifier ρ_n^ϵ for $n \in \mathbb{N}^*$ and $\epsilon > 0$) the time convolution kernel, $\rho_{\epsilon, x'} = \rho_{n-1}^\epsilon$ the x' -convolution kernel. First we extend U for negative times and we denote by V the function $U \mathbb{1}_{(0, T) \times \Omega} + U_0 \mathbb{1}_{(-\infty, 0) \times \Omega}$, then we define the function V_ϵ as

$$V_\epsilon(t, x', x_n) = V \star_{t, x'} \check{\rho}_\epsilon(t, x) = V \star_t \rho_{\epsilon, l} \star_{x'} \check{\rho}_\epsilon(t, x) = V \star_t \rho_{\epsilon, l} \star_x \rho_\epsilon(t, x) = \int_{\mathbb{R}^n} V(s, y, x_n) \rho_{\epsilon, l}(s - t) \rho_\epsilon(y - x') ds dy,$$

where \check{f} is the function defined by $\check{f}(x) = f(-x)$. Due to compatibility conditions, we assume also that U_0 satisfies the boundary condition in the strong sense : $(A_\nu - M_\nu)U_0 = 0$ in $L^2(\partial\Omega)$. To avoid technical difficulties due to the boundary terms depending on U_0 , one can suppose first that U_0 is null on $\partial\Omega$.

Lemma 2.3.9. *The function V_ϵ verifies the following properties*

1. V_ϵ goes to U in $L^2((0, T) \times \Omega; \mathbb{R}^m)$ when $\epsilon \rightarrow 0$.
2. V_ϵ verifies

$$\partial_t V_\epsilon + \sum_{i=1}^n A_i \partial_i V_\epsilon = f_\epsilon \quad \text{in } \mathcal{D}'((0, T) \times \Omega).$$

3. for all ϵ , $V_\epsilon \in H^1((0, T) \times \Omega; \mathbb{R}^m)$.
4. for all ϵ , $V_\epsilon(t=0) = U_{0,\epsilon} := U_0 \star_{x'} \rho_\epsilon$ (straightforward with the support of $\rho_{\epsilon,1}$).
5. $V_{\epsilon+} = 0$ on $(0, T) \times \partial\Omega$.

Proof. The fact that V_ϵ goes to U in $L^2((0, T) \times \Omega; \mathbb{R}^m)$ when $\epsilon \rightarrow 0$ follows from the definition of V_ϵ and classical results about mollifiers (see for example [63] Theorem C.19). Let φ be in $W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$. The key observation is the following one : if U is a dissipative solution of (2.2.1) with data f and U_0 then the function V verifies the following inequality

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\Omega} (|V|^2 - 2\langle V; \kappa \rangle)(t, x) \partial_t \varphi(t, x) + \sum_{i=1}^n (\langle V; A_i V \rangle - 2\langle V; A_i k \rangle)(t, x) \partial_i \varphi(t, x) dx dt + 2\langle g; V - \kappa \rangle(t, x) \varphi(t, x) dx dt \\ & - \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \langle U_{0-}; M_\nu(U_{0-} - 2k_-) \rangle(t, x', 0) \varphi(t, x', 0) dx' dt + \int_0^T \int_{\mathbb{R}^{n-1}} \langle \kappa_-; M_\nu \kappa_- \rangle(t, x', 0) \varphi(t, x', 0) dx' dt \geq 0, \end{aligned} \quad (2.3.7)$$

where

$$g = f \mathbb{1}_{(0,T) \times \Omega} + \sum_{i=1}^n A_i \partial_i U_0 \mathbb{1}_{(-\infty, 0) \times \Omega}. \quad (2.3.8)$$

Indeed, we know that U is a solution and consequently, the following inequality is true by definition :

$$\begin{aligned} & \int_0^T \int_{\Omega} (|U|^2 - 2\langle U; \kappa \rangle)(t, x) \partial_t \varphi(t, x) + \sum_{i=1}^n (\langle U; A_i U \rangle - 2\langle U; A_i k \rangle)(t, x) \partial_i \varphi(t, x) + 2\langle f; U - \kappa \rangle(t, x) \varphi(t, x) dx dt \\ & + \int_{\Omega} (|U_0|^2 - 2\langle U_0; \kappa \rangle)(0, x) \varphi(0, x) dx + \int_0^T \int_{\mathbb{R}^{n-1}} \langle \kappa_-; M_\nu \kappa_- \rangle(t, x', 0) \varphi(t, x', 0) dx' dt \geq 0. \end{aligned} \quad (2.3.9)$$

But since U_0 is regular, independent of t , we have

$$\begin{aligned} & \int_{-\infty}^0 \int_{\Omega} (|U_0|^2 - 2\langle U_0; \kappa \rangle)(t, x) \partial_t \varphi(t, x) + \sum_{i=1}^n (\langle U_0; A_i U_0 \rangle - 2\langle U_0; A_i k \rangle)(t, x) \partial_i \varphi(t, x) dt dx \\ & = \int_{\Omega} (|U_0|^2 - 2\langle U_0; \kappa \rangle)(0, x) \varphi(0, x) dx - 2 \int_{-\infty}^0 \int_{\Omega} \sum_{i=1}^n (\langle A_i \partial_i U_0; U_0 - \kappa \rangle)(t, x) \varphi(t, x) dx dt \\ & + \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \langle A_n U_0; U_0 - 2k \rangle(t, x', 0) \varphi(t, x', 0) dx' dt. \end{aligned}$$

Now using the fact that U_0 satisfies the boundary condition and Lemma 2.2.3, we obtain

$$\int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \langle A_n U_0; U_0 - 2k \rangle(t, x', 0) \varphi(t, x', 0) dx' dt = \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \langle U_{0-}; M_\nu(U_{0-} - 2k_-) \rangle(t, x', 0) \varphi(t, x', 0) dx' dt,$$

and

$$\begin{aligned} & \int_{-\infty}^0 \int_{\Omega} (|U_0|^2 - 2\langle U_0; \kappa \rangle)(t, x) \partial_t \varphi(t, x) + \sum_{i=1}^n (\langle U_0; A_i U_0 \rangle - 2\langle U_0; A_i k \rangle)(t, x) \partial_i \varphi(t, x) dt dx \\ & = \int_{\Omega} (|U_0|^2 - 2\langle U_0; \kappa \rangle)(0, x) \varphi(0, x) dx - 2 \int_{-\infty}^0 \int_{\Omega} \sum_{i=1}^n (\langle A_i \partial_i U_0; U_0 - \kappa \rangle)(t, x) \varphi(t, x) dx dt \\ & + \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \langle U_{0-}; M_\nu(U_{0-} - 2k_-) \rangle(t, x', 0) \varphi(t, x', 0) dx' dt. \end{aligned} \quad (2.3.10)$$

Adding this two inequalities, using the definition of V and g and the fact that g and V vanish in $(T, +\infty)$, we obtain (2.3.7). Using the test function $\varphi = \phi \star_t \rho_{\epsilon,l} \star_x \rho_\epsilon$, where ϕ is an arbitrary function of $W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, in the inequality (2.3.7), we obtain that the function V_ϵ satisfies the following inequality, where $\phi_\epsilon = \phi \star_t \rho_{\epsilon,l}$,

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\Omega} (|V_\epsilon|^2 - 2\langle V_\epsilon; \kappa \rangle)(t, x) \partial_t \phi(t, x) dt dx + \int_{\mathbb{R}} \int_{\Omega} \sum_{i=1}^n (\langle V_\epsilon; A_i V_\epsilon \rangle - 2\langle V_\epsilon; A_i k \rangle)(t, x) \partial_i \phi(t, x) dt dx \\
& + \int_{\mathbb{R}} \int_{\Omega} 2\langle g_\epsilon; V_\epsilon - \kappa \rangle(t, x) \phi(t, x) dt dx - \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \langle M_v(U_{0-})_\epsilon; (U_{0-})_\epsilon - 2k_- \rangle(t, x', 0) \phi_\epsilon(t, x', 0) dx' dt \\
& + \int_0^T \int_{\mathbb{R}^{n-1}} \langle \kappa_-; M_v \kappa_- \rangle(t, x', 0) \phi_\epsilon(t, x', 0) dx' dt \\
& \geq \int_{\mathbb{R}} \int_{\Omega} (|V_\epsilon|^2 - |V|^2 \star_{t,x'} \check{\rho}_\epsilon)(t, x) \partial_t \phi(t, x) dt dx \\
& + \int_{\mathbb{R}} \int_{\Omega} \sum_{i=1}^n (\langle V_\epsilon; A_i V_\epsilon \rangle - \langle V; A_i V \rangle \star_{t,x'} \check{\rho}_\epsilon)(t, x) \partial_i \phi(t, x) dx dt \\
& + \int_{\mathbb{R}} \int_{\Omega} 2(\langle g_\epsilon; V_\epsilon \rangle - \langle g; V \rangle \star_{t,x'} \check{\rho}_\epsilon)(t, x) \phi(t, x) dt dx \\
& + \int_{-\infty}^0 \int_{\partial\Omega} [\langle A_n U_{0,\epsilon}; U_{0,\epsilon} \rangle - \langle A_n U_0; U_0 \rangle \star_{x'} \check{\rho}_\epsilon](t, x', 0) \phi_\epsilon(t, x', 0) dx' dt.
\end{aligned}$$

The following lemma and its proof ensure that points 2 and 3 are true. As in the proof of Lemma 2.2.15, we obtain that $V_{\epsilon+} = 0$ on $(0, T) \times \partial\Omega$. \square

The following lemma explains why we only need to regularize in the (t, x') -directions to get the full H^1 -regularity.

Lemma 2.3.10. *Assume that A_n is non singular. Let U be in $L^2((0, T) \times \Omega; \mathbb{R}^m)$ such that $\partial_t U, \partial_1 U, \dots, \partial_{n-1} U$ belong to $L^2((0, T) \times \Omega; \mathbb{R}^m)$ and we suppose that the following inequality holds for every $\kappa \in \mathbb{R}^m$ and $\varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$,*

$$\begin{aligned}
& \int_0^T \int_{\Omega} |U(t, x) - \kappa|^2 \partial_t \varphi(t, x) dx dt + \int_0^T \int_{\Omega} \sum_{i=1}^n \langle U(t, x) - \kappa; A_i (U(t, x) - \kappa) \rangle \partial_i \varphi(t, x) dx dt \\
& + 2 \int_0^T \int_{\Omega} \langle f; U - \kappa \rangle \varphi(t, x) dx dt + \int_{\Omega} |U_0(x) - \kappa|^2 \varphi(0, x) dx \\
& + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_v \kappa_+ \rangle \varphi(t, x) dx dt \\
& \geq \int_0^T \int_{\Omega} g^t(U) \partial_t \varphi(t, x) + \sum_{i=1}^n g^i(U) \partial_i \varphi(t, x) dx dt + \int_{\Omega} g^0(U) \varphi(0, x) dx.
\end{aligned}$$

where $g^t(U)$, $g^i(U)$ and $g^0(U)$ belong to $L^2((0, T) \times \Omega; \mathbb{R}^m)$ and are independent of κ . Then $U \in H^1((0, T) \times \Omega; \mathbb{R}^m)$.

Proof. Following the proof of the fact that a solution in the sense of Definition 2.2.4 is a weak solution, we get that U is a weak solution to the partial differential equation

$$\partial_t U + \sum_{i=1}^n A_i \partial_i U = f.$$

Consequently, since A_n is non-singular by hypothesis, we get

$$\partial_n U = A_n^{-1} \left(f - \partial_t U + \sum_{i=1}^{n-1} A_i \partial_i U \right) \in L^2((0, T) \times \Omega; \mathbb{R}^m).$$

This implies that $U \in H^1((0, T) \times \Omega; \mathbb{R}^m)$. \square

Remark 2.3.11. *The hypothesis that A_v is invertible is important. In fact in the singular case, there is a loss of regularity for the solution (see [89]).*

Theorem 2.3.12. *Let U and \tilde{U} be two $L^2((0, T) \times \Omega; \mathbb{R}^m)$ solutions (in the sense of Definition 2.2.4) associated with $(f, U_0) \in L^2((0, T) \times \Omega; \mathbb{R}^m) \times H^1(\Omega; \mathbb{R}^m)$ and $(\tilde{f}, \tilde{U}_0) \in L^2((0, T) \times \Omega; \mathbb{R}^m) \times H^1(\Omega; \mathbb{R}^m)$ respectively. We suppose that U_0 (resp. \tilde{U}_0) satisfy the boundary condition in the strong sense $(A_\nu - M_\nu)U_0 = 0$ in $L^2(\partial\Omega)$ (resp. $(A_\nu - M_\nu)\tilde{U}_0 = 0$). Then there exists a constant $C > 0$ (depending only on T) such that*

$$\|U - \tilde{U}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 \leq C \left(\|f - \tilde{f}\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 + \|U_0 - \tilde{U}_0\|_{L^2(\Omega; \mathbb{R}^m)}^2 \right).$$

Remark 2.3.13. *From the previous theorem, we deduce that there exists a unique solution in the sense of Definition 2.2.4 of Problem (2.2.1) for $f \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ and $U_0 \in H^1(\Omega; \mathbb{R}^m)$ that satisfies the boundary condition.*

Proof. From U and \tilde{U} , we construct their approximations V_ϵ and \tilde{V}_ϵ , Lemma 2.3.9 allows us to mimic the proof of Theorem 2.3.6 to get

$$\|V_\epsilon - \tilde{V}_\epsilon\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 \leq C \left(\|g_\epsilon - \tilde{g}_\epsilon\|_{L^2((0, T) \times \Omega; \mathbb{R}^m)}^2 + \|U_{0, \epsilon} - \tilde{U}_{0, \epsilon}\|_{L^2(\Omega; \mathbb{R}^m)}^2 \right).$$

Passing to the limit, since V and g are in $L^2_{\text{loc}}((-\infty, T) \times \Omega; \mathbb{R}^m)$ then V_ϵ and g_ϵ tends to V and g in $L^2((0, T) \times \Omega; \mathbb{R}^m)$ (see for example [63]), we get the result. \square

2.4 Comparison with other formulations

In the literature, there are several ways for taking into account the boundary condition for linear hyperbolic systems, for example the initial approach of Friedrichs [47], the formulation developed in [38] by Dubois and Le Floch and the maximal and strictly dissipative theories (summarized in the book of Benzoni-Gavage and Serre [16]). We first describe these different approaches, which are slightly different from the algebraic point of view, and then compare these formulations on a simple example : the wave equation in 1D. We will see on this example that, except for the maximal dissipative theory and Friedrichs' approach, the theory must be modified to take into account the classical Dirichlet boundary condition. This modification can be performed at least formally with the use of a small parameter such that the homogeneous Dirichlet condition is recovered at the limit of a family of dissipative solutions in the sense of definition 1.

2.4.1 Friedrichs' approach

In the seminal work of Friedrichs [47], the boundary condition takes the following form : first he assumes a decomposition of $A_\nu = \beta_- + \beta_+$, M_ν is now the matrix $\beta_+ - \beta_-$ and the boundary condition is $(A_\nu - M_\nu)U = 0$. Friedrichs also assumes that three conditions (denoted by III₀, III₁ and III₂ in [47]) are fulfilled by the matrices β_- and β_+

1. $\ker(\beta_-) \oplus \ker(\beta_+) = \mathbb{R}^m$.
2. $\text{Im}(\beta_-) \cap \text{Im}(\beta_+) = \{0\}$.
3. $M_\nu^t + M_\nu \geq 0$.

From Lemma 2.2.3 and the hypothesis that we made on the matrix M_ν , we see that all Friedrichs' hypotheses are satisfied in the non-characteristic case (we have already said that the first hypothesis was too restrictive in the characteristic case ; for example, take all the matrices A_i equal to zero). As far as assumption 3 is concerned, we have made a restrictive hypothesis : we suppose that M_ν is a symmetric matrix.

2.4.2 Dubois-Le Floch's formulation

Now we describe Dubois-Le Floch's formulation. In order to define admissible boundary condition for nonlinear hyperbolic systems, Dubois and Le Floch (in [38]) also define admissible boundary condition for linear strictly hyperbolic systems. To take account of inhomogeneous boundary condition, they say that the space of admissible condition (for the problem in $(0, T) \times \mathbb{R}^n$) is

$$U_b + \text{Vect} \{r_k, k = p + 1, \dots, m\}, \quad (2.4.1)$$

where r_k is an eigenvector associated with the eigenvalue λ_k (for $k \geq p + 1$, $\lambda_k \geq 0$ and for $k < p + 1$, $\lambda_k < 0$) of the boundary matrix A_ν . Using our formalism, we can rewrite this space of admissible boundary condition as

$$U - U_b \in \ker(A_\nu - M_\nu),$$

where M_ν is the matrix constructed in Remark 2.2.1 with the matrix P associated with the eigenvectors chosen in (2.4.1) (recall that these eigenvectors are orthogonal and we may assume without loss of generality that they are orthonormal and consequently that the matrix P is orthogonal). Therefore the type of boundary condition discussed by Dubois and Le Floch is contained in our formalism (in fact strictly contained, see the example of the wave equation).

2.4.3 Maximal dissipative and strictly dissipative theories

We have already used the fact that our formalism can be transformed to use strictly dissipative theory. In these two theories, the boundary condition is $BV = 0$, where $B \in \mathbb{M}^{p \times n}(\mathbb{R})$ and p is the number of negative eigenvalues of A_ν . We say that the matrix B is maximal dissipative (in the non-characteristic case) if

$$U \in \ker B \quad \Rightarrow \quad \langle A_\nu U; U \rangle \geq 0, \quad (2.4.2)$$

and B is maximal for that property, *i.e.* $\ker B$ is not a proper subspace of vector space V such that

$$U \in V \quad \Rightarrow \quad \langle A_\nu U; U \rangle \geq 0,$$

to get L^2 -well-posedness (Theorem 3.2 of [16]).

This boundary condition (or equivalently the boundary condition used in the work of Rauch, for example in [89]) contains more possibilities than the one we use in this work (in fact, we will see in the next example that the formalism of [16] allows more boundary conditions than Friedrichs' approach).

There is also a stronger condition (see for example [16]): the notion of strictly dissipative boundary condition. A matrix B is strictly dissipative (in the non-characteristic case) if

1. $U \in \ker B \Rightarrow \langle A_\nu U; U \rangle > 0$, if $U \neq 0$.
2. $\ker B$ is maximal for the previous property.
3. B is onto.

2.4.4 A simple example : the wave equation in 1D

We consider here the wave equation problem

$$\partial_{tt}w - \partial_{xx}w = 0, \quad \text{on } (0, T) \times \mathbb{R}_-,$$

that we rewrite as an hyperbolic problem in dimension $m = 2$ as

$$\partial_t \begin{pmatrix} u \\ p \end{pmatrix} - A \partial_x \begin{pmatrix} u \\ p \end{pmatrix} = 0, \quad (2.4.3)$$

where $u = \partial_t w$, $p = \partial_x w$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is classical (see for example [22] section 10.3) that the wave equation is well-posed with the natural following conditions

$$\begin{aligned} w(0, x) &= f(x), & \text{on } \{t = 0\} \times \mathbb{R}_-, \\ \partial_t w(0, x) &= g(x), & \text{on } \{t = 0\} \times \mathbb{R}_-, \\ w(t, 0) &= 0, & \text{on } (0, T) \times \{x = 0\}. \end{aligned}$$

Now that the unknowns in (2.4.3) are $u = \partial_t w$ and $p = \partial_x w$, it seems natural to know if one can obtain the boundary condition

$$u(t, 0) = 0, \quad \text{on } (0, T) \times \{x = 0\} \quad (2.4.4)$$

We are going to show that this kind of boundary condition is not contained in our formalism or Dubois-Le Floch's formalism but it can be obtained with the use of a maximal dissipative boundary condition or in Friedrichs' formalism

First, we are going to show that one can not find two matrices β_- and β_+ such that $\ker \beta_- = \left\{ \begin{pmatrix} 0 \\ p \end{pmatrix}, p \in \mathbb{R} \right\}$ and the conditions 1,2,3 are satisfied. Let β_- be the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In fact, the condition $\ker \beta_- = \left\{ \begin{pmatrix} 0 \\ p \end{pmatrix}, p \in \mathbb{R} \right\}$ implies that $b = d = 0$. The condition 3 rewrites as

$$M_\nu^t + M_\nu = \beta_+ - \beta_- + {}^t \beta_+ - {}^t \beta_- = \begin{pmatrix} -4a & 2c-2 \\ 2c-2 & 0 \end{pmatrix} \geq 0,$$

and consequently, it imposes that $c = 1$ and $a \leq 0$, and then

$$\beta_- = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \beta_+ = \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix}.$$

It implies that the boundary condition (2.4.4) can be directly used in the Friedrichs' framework.

In this particular example, one can see that our framework contains more possibility for the boundary condition than the theory of characteristics of Dubois-Le Floch. Indeed, if we take the example of a homogeneous boundary condition, the only possibility from Dubois-Le Floch is that $u + p = 0$ on the boundary $x = 0$.

Let us determine for the wave equation in 1D all the possible matrices $M_\nu = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ that fulfill the conditions of Section 2.2. In fact, Condition (2.2.3) implies that $\dim \ker(A_\nu \pm M_\nu) \geq 1$. Suppose that $\dim \ker(A_\nu - M_\nu) = 2$ or $\dim \ker(A_\nu + M_\nu) = 2$ (which means that $A_\nu - M_\nu = 0$ or $A_\nu + M_\nu = 0$), these two equalities can not happen since A_ν is neither non-positive nor non-negative. Since we know that $\dim \ker(A_\nu \pm M_\nu) \leq 2$, we obtain that $\dim \ker(A_\nu \pm M_\nu) = 1$. In particular, $\det(A_\nu \pm M_\nu) = 0 = ac - (-1 \pm b)^2$. Subtracting these two equalities implies that $b = 0$. If $b = 0$, to insure that $\det(A_\nu \pm M_\nu) = 0$, we get that $a = \frac{1}{c}$. In our formalism, all the matrices M_ν possible are

$$M_c = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix}, \quad \text{with } c > 0.$$

The matrix M_c is associated with the following boundary condition

$$u + \frac{p}{c} = 0, \quad \text{on } (0, T) \times \{x = 0\}.$$

It allows an infinite number of boundary condition for the wave equation in one dimension, parameterized by c .

Finally, we examine the maximal and strictly dissipative boundary conditions in the case of the wave equation. First, to use a maximal dissipative boundary condition, one need to find a matrix $B \in \mathbb{M}^{1 \times 2}(\mathbb{R})$ satisfying Condition (2.4.2). In fact, any matrix B of the form $\begin{pmatrix} a & 0 \end{pmatrix}$ is maximal dissipative and gives the boundary condition (2.4.4). One can easily see that the boundary condition $u = 0$ can not be strictly dissipative and that in fact the boundary condition (in the general case) that we are considering is such that the space $\ker(A_\nu - M_\nu) \cap \text{Im} A_\nu$ is strictly dissipative in the sense that the first two previous conditions associated with the strictly dissipative theory are verified.

Nevertheless, one way to get a formulation that is valid for a $L^2((0, T) \times \Omega; \mathbb{R}^m)$ solution in this particular example would be to send c to $+\infty$ in the dissipative formulation to get a non-dissipative weak formulation. This is very similar to penalization procedure.

One could also modify the Definition 2.2.4 by restricting the test vectors κ to the ones satisfying $\kappa_+ = 0$ if the support of φ touches $\partial\Omega$. This formulation seems nevertheless less general than the one studied in this work.

2.5 Perspectives

We want to modify the definition of dissipative solutions to Problem (2.2.1) so that it models an additional constraint

$$U(t, x) \in \mathcal{C}, \quad \text{on } (0, T) \times \Omega, \quad (2.5.1)$$

where \mathcal{C} is closed convex subset of \mathbb{R}^m and again Ω is a open domain (not necessarily bounded) of \mathbb{R}^n with (smooth) boundary, denoted by $\partial\Omega$, U is a vector of size m , $f \in L^2((0, T) \times \Omega)^m$, $U_0 \in L^2(\Omega; \mathcal{C})$, the matrices A_i are symmetric,

$A_\nu = \sum_{i=1}^n \nu_i A_i$ and $\nu(x) = (\nu_1, \nu_2, \dots, \nu_n)$ is the exterior unit normal of Ω at the point $x \in \partial\Omega$, M_ν still verifies the

hypotheses (2.2.2) and (2.2.3). Whatever is the way we model the constraint (2.5.1), the problem is now fully nonlinear in the general case. Thanks to our choice of boundary condition, we can define, as in [37] where the problem was posed in the whole space $\Omega = \mathbb{R}^n$, what is a weak solution to the nonlinear Problem (2.2.1) and (2.5.1).

Definition 2.5.1. We say that $U \in L^2((0, T) \times \Omega, \mathcal{C})$ is a dissipative constrained solution of Problem (2.2.1) and (2.5.1) if for all $\varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$, the space of $W^{1,\infty}(\mathbb{R} \times \mathbb{R}^n)$ functions with non-negative values and compact supports, and for all $\kappa \in \mathcal{C}$ the following inequality holds

$$\begin{aligned} & \int_0^T \int_{\Omega} |U(t, x) - \kappa|^2 \partial_t \varphi(t, x) \, dx \, dt + \int_0^T \int_{\Omega} \sum_{i=1}^n \langle U(t, x) - \kappa; A_i(U(t, x) - \kappa) \rangle \partial_i \varphi(t, x) \, dx \, dt \\ & + \int_{\Omega} |U_0(x) - \kappa|^2 \varphi(0, x) \, dx + \int_0^T \int_{\Omega} 2 \langle f(t, x); U(t, x) - \kappa \rangle \varphi(t, x) \, dx \, dt + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_\nu \kappa_+ \rangle \varphi(t, x) \, d\mathcal{H}^{n-1}(x) \, dt \geq 0, \end{aligned} \quad (2.5.2)$$

where κ_+ stands for the projection onto $(\ker(A_\nu + M_\nu)) \cap \text{Im} A_\nu$.

Remark 2.5.2. There are two differences between the definition of a dissipative solution and a dissipative constrained solution. First, we assume in the definition that U and the initial condition U_0 take their values in \mathcal{C} and secondly, the constant vector κ now belongs only to the convex set \mathcal{C} .

Remark 2.5.3. One should observe that most of the work that we have done in the previous section does not extend to the case where the solution follows the constraint (2.5.1). Nevertheless, Lemma 2.2.14 is still true in the constrained case by similar arguments.

We mentioned at the beginning that this work was motivated by initial boundary value problems for nonlinear viscoplastic problems and their associated limit problems. In fact, the system of dynamical linear elasticity can be recast thanks to a change of variable (see Section 4.2) in the Friedrichs' framework. In the Friedrichs' variables, the convex constraint of perfect plasticity can be written, at least formally, in the form (2.5.1). The Definition 2.5.1 allows us to consider the system of perfect plasticity from a new point of view and might lead to new results.

In addition, the dissipative inequality, in the case where $\Omega = \mathbb{R}^n$, in [37] has been introduced thanks to a formal relaxation of the constraint (2.5.1) (see also Section 4.2 for the justification of this formal step) using a source term of the form $(P_{\mathcal{C}}(U) - U)/\epsilon$ (where ϵ is a small parameter). This kind of source term is also used in viscoplastic models to relax the plasticity constraint and approximate the perfect plasticity system (see [86] and [91]). The fact that these approximations are closely related strengthens the link between dissipative constrained solutions and mechanical problems coming from the elastoplastic theory.

This formulation allows us to consider initial boundary value problems under constraints, for which, to our knowledge, there exists no general theory whatever the boundary conditions of Section 2.4 are considered. The solutions to these constrained problems may develop singularities and consequently, it is necessary to propose a weak formulation to the problem made of (2.2.1) and (2.5.1). The study of solutions in the sense of Definition 2.5.1 is still an open problem and is the purpose of future works.

A simplified model of perfect plasticity

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3.1 Introduction

Friedrichs' systems are linear symmetric hyperbolic systems of the form

$$\begin{cases} \partial_t U + \sum_{i=1}^n A_i \partial_i U = 0 & \text{in } (0, T) \times \mathbb{R}^n, \\ U(t=0) = U_0, \end{cases}$$

where $U : (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the unknown of the problem, A_1, \dots, A_n are symmetric $m \times m$ matrices, and $U_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a given initial data. They appear in a number of physical systems such as the wave equation or systems of conservation laws. In particular, the system of three-dimensional linearized elasto-dynamics can be put within this framework (see [54]) where U is a vector of size 9 (with three components for the velocity, and six for the symmetric 3×3 Cauchy stress), and A_1, A_2, A_3 are explicit 9×9 matrices depending on the Lamé coefficients of the material.

Problems of continuum mechanics are usually settled in a bounded domain $\Omega \subset \mathbb{R}^n$, which requires to impose a boundary condition. Of course, it is a difficult issue in hyperbolic equations since the initial condition is transported through the characteristics up to the boundary, where the value of the solution might thus be incompatible with the prescribed boundary data. In other words, one has to impose boundary conditions only on a part of the boundary which is not reached by the characteristics (see *e.g.* [12, 87, 71] in the case of scalar conservation laws, or [38] for one-dimensional nonlinear systems). In Friedrichs' seminal work [47], the following type of boundary conditions are considered

$$(A_\nu - M_\nu)U = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (3.1.1)$$

where $A_\nu = A_\nu(x) := \sum_{i=1}^n A_i \nu_i(x)$ ($\nu(x)$ is the outer unit normal to Ω at the point $x \in \partial\Omega$), and $M_\nu = M(\nu(x))$ is a $m \times m$ matrix, for $x \in \partial\Omega$, satisfying the following algebraic conditions (in the non-characteristic case, *i.e.*, when A_ν is non-singular, see also Section 2.2 for more details)

$$\begin{cases} M_\nu + M_\nu^T \text{ is non-negative,} \\ \text{Im}(A_\nu - M_\nu) \cap \text{Im}(A_\nu + M_\nu) = \{0\}, \\ \mathbb{R}^m = \text{Ker}(A_\nu - M_\nu) \oplus \text{Ker}(A_\nu + M_\nu). \end{cases} \quad (3.1.2)$$

The fact that the symmetric part of M_ν is supposed to be non-negative is a way to ensure that the $L^2(\Omega)$ -norm of the solution decreases in time, and thus this hypothesis is related to the uniqueness of the solution and it is connected with the dissipativity of the equation. The two other assumptions are related to the existence of a solution.

Unfortunately, the previous formulation necessitates to define properly the trace of U on the boundary, which might not be desirable if one is interested in weak solutions in Lebesgue-type spaces (in the spirit of *e.g.* [87, 71])

for a L^∞ -theory of boundary value scalar conservation laws). In Chapter 2, a general L^2 -theory for such boundary value Friedrichs' systems has been introduced. The so-called *dissipative solutions* are defined as functions $U \in L^2((0, T) \times \Omega; \mathbb{R}^m)$ satisfying, for all constant vector $\kappa \in \mathbb{R}^m$ and all $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$,

$$\begin{aligned} \int_0^T \int_\Omega |U - \kappa|^2 \partial_t \varphi \, dx \, dt + \sum_{i=1}^n \int_0^T \int_\Omega \langle A_i(U - \kappa); (U - \kappa) \rangle \partial_i \varphi \, dx \, dt \\ + \int_\Omega |U_0 - \kappa|^2 \varphi(0) \, dx + \int_0^T \int_{\partial\Omega} \langle M_\nu \kappa^+; \kappa^+ \rangle \varphi \, d\mathcal{H}^{n-1} \, dt \geq 0, \end{aligned} \quad (3.1.3)$$

where $\kappa^+ = \kappa^+(x)$ is the orthogonal projection of κ onto the linear space $\text{Ker}(A_\nu + M_\nu) \cap \text{Im}A_\nu$. As discussed in Chapter 2, this formulation naturally discriminates the admissible boundary conditions. In particular, in this case, the matrix M_ν has to be a non-negative symmetric matrix (see Subsection 3.3.3). Note that this kind of dissipative formulation is reminiscent in hyperbolic equations (see *e.g.* [58] in the case of scalar conservation laws). Moreover, the family of functions $U \mapsto |U - \kappa|^2$, where $\kappa \in \mathbb{R}^m$, can be thought of as the analogue of the Kruřkov entropy functions in [58]. The term *dissipative* refers to the decreasing character of the L^2 -norm of the solution which prevents conservation of the energy, and to a special class of boundary conditions for hyperbolic systems (see [16] and Section 2.4).

A number of mechanical problems, such as in elasto-plasticity or generalized non-Newtonian fluids, involve a convex constraint. For that reason, it becomes relevant to ask whether one can incorporate convex constraints within a general theory of Friedrichs' systems. In [37], this problem has been addressed in the full space $\Omega = \mathbb{R}^n$. The authors define a notion of dissipative solutions (from which the previous formulation (3.1.3) in Chapter 2 has been inspired) which are shown in Section 4.2 to be the (unique) limit of a sequence of viscosity solutions for a regularized diffusive model where the constraint is penalized. The formulation of general constrained Friedrichs' systems in bounded domains becomes therefore a natural extension. However, there might be non trivial interactions between the constraint and the boundary condition (see (3.1.8) below), which makes the problem difficult to address in its full generality. This is the reason why, in this Chapter, we focus our attention to the meaningful particular case of dynamical perfect plasticity.

To be more precise, we consider a simplified two dimensional problem of anti-plane shear elasto-plasticity (see Subsection 3.3.2 for a formal derivation from three-dimensional small strain elasto-plasticity), where the displacement field $u : [0, T] \times \Omega \rightarrow \mathbb{R}$ is scalar valued and the stress $\sigma : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ is vector-valued. General considerations of continuum mechanics state that the equation of motion

$$\partial_{tt}u - \text{div} \sigma = 0 \quad \text{in } (0, T) \times \Omega$$

must be satisfied. Then, following standard models of perfect plasticity, the stress is constrained to remain inside a fixed closed and convex set of \mathbb{R}^2 . For simplicity, we assume that

$$|\sigma| \leq 1. \quad (3.1.4)$$

Moreover, the displacement gradient decomposes additively as

$$\nabla u = e + p,$$

where e and $p : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ stand for the elastic and plastic strains, respectively. The elastic strain is related to the stress by means of a linear relation, and, again for simplicity, we set

$$\sigma = e. \quad (3.1.5)$$

Finally, the plastic variable evolves through the so-called flow rule, which stipulates that

$$\begin{cases} \partial_t p = 0 & \text{if } |\sigma| < 1, \\ \frac{\partial_t p}{|\partial_t p|} = \sigma & \text{if } |\sigma| = 1. \end{cases} \quad (3.1.6)$$

This system must be supplemented by initial conditions on $(u, \partial_t u, \sigma, p)$ and boundary conditions. The mathematical analysis of dynamical elasto-plastic models has been performed in [7, 10] (see also [92, 96, 6, 30] in the static and quasi-static cases).

From the hyperbolic point of view, this problem can be interpreted as a constrained Friedrichs' system. Formally it can be put within a hyperbolic formulation of the type (3.1.3) (see Subsection 3.3.3) where $U = (\partial_t u, \sigma) \in \mathbb{R}^3$, and

the 3×3 symmetric matrices A_1, A_2 are given by

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The dissipative formulation is exactly given by (3.1.3), except that the constant vector κ must belong to the constraint set $\mathcal{C} = \mathbb{R} \times \bar{B}$, where \bar{B} is the closed unit ball of \mathbb{R}^2 (see (3.1.4) above). The hyperbolic vision of this problem motivates our choice of boundary conditions. It turns out that, in the unconstrained case (*i.e.* the wave equation), admissible dissipative boundary conditions in the sense of (3.1.1)–(3.1.2) are all of the form

$$\langle \sigma; \nu \rangle + \lambda^{-1} \partial_t u = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (3.1.7)$$

for some $\lambda > 0$ (see Lemma 3.3.1). This choice will be *a posteriori* justified by the fact that the variational and dissipative formulations are essentially equivalent. Note that this type of boundary condition, is quite unusual in solid mechanics. These are not of Robin type since it involves the velocity $\partial_t u$, and not the displacement u . It is closer to Navier's no-slip boundary condition rather found in fluid mechanics problems.

The goal of this Chapter consists thus in studying this particular model (in any space dimension and with a source term) related to dynamical perfect plasticity from both variational and hyperbolic points of view. First of all, using variational methods we establish a well-posedness result for this model. To this aim, we regularize the problem by considering an elasto-visco-plastic model where the constitutive law (3.1.5) is replaced by a Kelvin-Voigt visco-elastic law

$$\sigma = \tilde{\sigma} + \varepsilon \nabla \partial_t u,$$

where $\varepsilon > 0$ is a viscosity parameter, and stress constraint (3.1.4) together with the flow rule (3.1.6) are replaced by a Perzyna visco-plastic law

$$\partial_t p = \frac{\tilde{\sigma} - P_{\bar{B}}(\tilde{\sigma})}{\varepsilon},$$

where $P_{\bar{B}}$ denotes the orthogonal projection operator onto \bar{B} . The equation of motion and the boundary condition are thus to be modified into

$$\begin{cases} \partial_{tt} u - \operatorname{div}(\tilde{\sigma} + \varepsilon \nabla \partial_t u) = 0 & \text{in } (0, T) \times \Omega, \\ \langle \tilde{\sigma} + \varepsilon \nabla \partial_t u; \nu \rangle + \lambda^{-1} \partial_t u = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases}$$

The well-posedness of this regularized model is presented in Section 3.4.

In Section 3.5, we prove the existence and uniqueness of a (variational) solution for the original model by means of a vanishing viscosity analysis as $\varepsilon \rightarrow 0$. However, since in the limit, the stress satisfies the constraint $|\sigma| \leq 1$, the original boundary condition (3.1.7) cannot be satisfied at points of the boundary where $|\partial_t u| > \lambda$. Therefore, a relaxation phenomenon occurs (see Proposition 3.5.10) which implies that the boundary condition (3.1.7) relaxes as

$$\langle \sigma; \nu \rangle + \lambda^{-1} T_\lambda(\partial_t u) = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (3.1.8)$$

where $T_\lambda(z) = \min(-\lambda, \max(z, \lambda))$ is the truncation of $z \in \mathbb{R}$ by the values $\pm\lambda$. It shows an interesting interaction which imposes the boundary condition to accommodate the constraint. Note that since $\lambda \in (0, +\infty)$, the important cases of Dirichlet and Neumann boundary conditions are prohibited by this formulation. In Subsection 3.5.5, we show by means of asymptotic analysis that the Dirichlet (resp. Neumann) boundary condition can be recovered by letting $\lambda \rightarrow 0$ (resp. $\lambda \rightarrow +\infty$).

As usual in plasticity, the solution happens to concentrate, leading to a bounded variation solution for the displacement, and a measure solution for the plastic strain. In Section 3.6, using the property of finite speed propagation, we prove a new regularity result in plasticity which states that, provided the data are smooth and compactly supported in space, the solution is smooth as well in short time. The argument rests on a Kato inequality (Proposition 3.5.12) which states a comparison principle between two solutions associated with different data. The fact that the data is compactly supported in Ω together with the finite speed propagation property ensures that, in short time, the boundary is not reached by the solution so that the boundary condition can be ignored, and one can argue as in the full space. To our knowledge, it seems to be the first regularity result in dynamical perfect-plasticity, and its generalization to more general vectorial models will be the object of a forthcoming work.

Finally, in Section 3.7, we establish rigorous links between the variational and hyperbolic formulations. We show that any variational solutions generate dissipative solutions. Conversely, provided the solution of the hyperbolic problem are smoother in time, variational solutions can be deduced from the dissipative formulation.

3.2 Mathematical preliminaries

3.2.1 General notation

If a and $b \in \mathbb{R}^n$, we write $\langle a; b \rangle$ for the Euclidean scalar product, and we denote by $|a| = \sqrt{\langle a; a \rangle}$ the associated norm. Let $\bar{B} := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed unit ball in \mathbb{R}^n , and $P_{\bar{B}}$ be the orthonormal projection onto \bar{B} , i.e., $P_{\bar{B}}(\sigma) = \sigma/|\sigma|$ if $\sigma \neq 0$, and $P_{\bar{B}}(\sigma) = 0$ if $\sigma = 0$. It is a standard fact of convex analysis that the function

$$\sigma \mapsto \frac{|\sigma - P_{\bar{B}}(\sigma)|^2}{2\varepsilon}$$

is convex, of class \mathcal{C}^1 , and that its differential is given by $\sigma \mapsto (\sigma - P_{\bar{B}}(\sigma))/\varepsilon$. In addition, its convex conjugate is $p \mapsto |p| + \varepsilon|p|^2/2$, and in particular,

$$p = \frac{\sigma - P_{\bar{B}}(\sigma)}{\varepsilon} \iff \langle \sigma; p \rangle = |p| + \varepsilon|p|^2. \quad (3.2.1)$$

We write $\mathbb{M}^{n \times n}$ for the set of real $n \times n$ matrices, and $\mathbb{M}_{\text{sym}}^{n \times n}$ for that of all real symmetric $n \times n$ matrices. Given a matrix $A \in \mathbb{M}^{n \times n}$, we let $|A| := \sqrt{\text{tr}(AA^T)}$ (A^T is the transpose of A , and $\text{tr}A$ is its trace) which defines the usual Euclidean norm over $\mathbb{M}^{n \times n}$. We recall that for any two vectors a and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{n \times n}$ stands for the tensor product, i.e., $(a \otimes b)_{ij} = a_i b_j$ for all $1 \leq i, j \leq n$, and $a \odot b := (a \otimes b + b \otimes a)/2 \in \mathbb{M}_{\text{sym}}^{n \times n}$ denotes the symmetric tensor product.

3.2.2 Functional spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set. We use standard notation for Lebesgue and Sobolev spaces.

We write $\mathcal{M}(\Omega; \mathbb{R}^m)$ (or simply $\mathcal{M}(\Omega)$ if $m = 1$) for the space of bounded Radon measures in Ω with values in \mathbb{R}^m , endowed with the norm $|\mu|(\Omega)$, where $|\mu| \in \mathcal{M}(\Omega)$ is the total variation of the measure μ . The Lebesgue measure in \mathbb{R}^n is denoted by \mathcal{L}^n , and the $(n-1)$ -dimensional Hausdorff measure by \mathcal{H}^{n-1} .

We denote by $H(\text{div}, \Omega)$ the Hilbert space of all $\sigma \in L^2(\Omega; \mathbb{R}^n)$ such that $\text{div} \sigma \in L^2(\Omega)$. We recall that if Ω is bounded with Lipschitz boundary and $\sigma \in H(\text{div}, \Omega)$, its normal trace, denoted by $\langle \sigma; \nu \rangle$, is well defined as an element of $H^{-1/2}(\partial\Omega)$. If further $\sigma \in H(\text{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$, it turns out that $\langle \sigma; \nu \rangle \in L^\infty(\partial\Omega)$ with $\|\langle \sigma; \nu \rangle\|_{L^\infty(\partial\Omega)} \leq \|\sigma\|_{L^\infty(\Omega; \mathbb{R}^n)}$ (see [4, Theorem 1.2]). Moreover, according to [25, Theorem 2.2], if Ω is of class \mathcal{C}^2 , then for all $\varphi \in L^1(\partial\Omega)$,

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 \int_{\partial\Omega} (\langle \sigma(y - \varepsilon s \nu(y)); \nu(y) \rangle - \langle \sigma; \nu \rangle(y)) \varphi(y) d\mathcal{H}^{n-1}(y) ds = 0, \quad (3.2.2)$$

where ν denotes the outer unit normal to $\partial\Omega$.

The space $BV(\Omega)$ of functions of bounded variation in Ω is made of all functions $u \in L^1(\Omega)$ such that the distributional gradient $Du \in \mathcal{M}(\Omega; \mathbb{R}^n)$. We refer to [3, 42, 48] for a detailed presentation of this space. We just recall here few facts. If Ω has Lipschitz boundary, any function $u \in BV(\Omega)$ admits a trace, still denoted by $u \in L^1(\partial\Omega)$, such that Green's formula holds (see e.g. Theorem 1, Section 5.3 in [42]). Moreover, according to [5, Theorem 4], if Ω is further of class \mathcal{C}^1 , for every $\varepsilon > 0$, there exists a constant $c_\varepsilon(\Omega) > 0$ such that for every $u \in BV(\Omega)$,

$$\|u\|_{L^1(\partial\Omega)} \leq (1 + \varepsilon) |Du|(\Omega) + c_\varepsilon(\Omega) \|u\|_{L^1(\Omega)}. \quad (3.2.3)$$

Moreover, if Ω is of class \mathcal{C}^2 , one has

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 \int_{\partial\Omega} |u(y - \varepsilon s \nu(y)) - u(y)| d\mathcal{H}^{n-1}(y) ds = 0. \quad (3.2.4)$$

Conversely, Gagliardo's extension result (see [48, Theorem 2.16 & Remark 2.17]) states that if $\Omega \subset \mathbb{R}^n$ is a bounded open set of class \mathcal{C}^1 , and $g \in L^1(\partial\Omega)$, for each $\varepsilon > 0$, there exists a function $u_\varepsilon \in W^{1,1}(\Omega)$ such that

$$\begin{cases} u_\varepsilon = g \text{ on } \partial\Omega, \\ \|u_\varepsilon\|_{L^1(\Omega)} \leq \varepsilon \|g\|_{L^1(\partial\Omega)}, \\ \|\nabla u_\varepsilon\|_{L^1(\Omega)} \leq (1 + \varepsilon) \|g\|_{L^1(\partial\Omega)}. \end{cases} \quad (3.2.5)$$

Let us finally mention a variant of the usual approximation result for BV functions [4, Lemma 5.2].

Proposition 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $u \in BV(\Omega)$ and $\sigma \in L^2(\Omega; \mathbb{R}^n)$. There exists a sequence $(u_j) \subset W^{1,1}(\Omega)$ such that $u_j \rightharpoonup u$ weakly* in $BV(\Omega)$, $|Du_j - \sigma|(\Omega) \rightarrow |Du - \sigma|(\Omega)$ and $u_j = u$ on $\partial\Omega$ for all $j \in \mathbb{N}$.*

3.2.3 Generalized stress/strain duality

According to [4, Definition 1.4], we define a duality pairing between stresses and plastic strains as follows.

Definition 3.2.2. *Let $u \in BV(\Omega) \cap L^2(\Omega)$ be such that $Du = e + p$ for some $e \in L^2(\Omega; \mathbb{R}^n)$ and $p \in \mathcal{M}(\Omega; \mathbb{R}^n)$, and let $\sigma \in H(\text{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$. We define the distribution $[\sigma \cdot p] \in \mathcal{D}'(\Omega)$ by*

$$\langle [\sigma \cdot p] | \varphi \rangle = \langle [\sigma \cdot Du] | \varphi \rangle - \int_{\Omega} \langle \sigma; e \rangle \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega),$$

where $[\sigma \cdot Du]$ is given by

$$\langle [\sigma \cdot Du] | \varphi \rangle = - \int_{\Omega} u(\text{div} \sigma) \varphi \, dx - \int_{\Omega} u \langle \sigma; \nabla \varphi \rangle \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Proposition 3.2.3. *Let u, e, p, σ be as in Definition 3.2.2. Then for all open subset A of Ω and all $\varphi \in C_c^\infty(A)$, we have*

$$|\langle [\sigma \cdot p] | \varphi \rangle| \leq \|\sigma\|_{L^\infty(\Omega; \mathbb{R}^n)} \|\varphi\|_{L^\infty(\Omega)} |p|(A).$$

In particular, the distribution $[\sigma \cdot p]$ is actually a bounded Radon measure in Ω satisfying

$$\|[\sigma \cdot p]\| \leq \|\sigma\|_{L^\infty(\Omega; \mathbb{R}^n)} |p| \quad \text{in } \mathcal{M}(\Omega).$$

Proof. The proof closely follows the lines of [4, Theorem 1.5]. According to Proposition 3.2.1 and an additional approximation of $W^{1,1}(\Omega)$ functions by $H^1(\Omega)$ ones, there exists a sequence $(u_j)_{j \in \mathbb{N}} \in H^1(\Omega)^{\mathbb{N}}$ that converges to u in the following sense

$$u_j \rightharpoonup u \text{ weakly* in } BV(\Omega) \quad \text{and} \quad |Du_j - \sigma|(\Omega) \rightarrow |Du - \sigma|(\Omega),$$

and define p_j as $p_j = \nabla u_j - e$. Now, if $\varphi \in C_c^\infty(A)$, we consider an open set A' such that $\text{supp} \varphi \subset A'$ and A' is compactly embedded in A . Consequently, we have

$$\langle [\sigma \cdot p_j] | \varphi \rangle = - \int_{\Omega} u_j(\text{div} \sigma) \varphi \, dx - \int_{\Omega} u_j \langle \sigma; \nabla \varphi \rangle \, dx - \int_{\Omega} \langle \sigma; e \rangle \varphi \, dx = - \int_{\Omega} u_j \text{div}(\varphi \sigma) \, dx - \int_{\Omega} \langle \sigma; e \rangle \varphi \, dx$$

The integration by parts formula in $H^1(\Omega)$ gives

$$\langle [\sigma \cdot p_j] | \varphi \rangle = \int_{\Omega} \varphi \langle \sigma; \nabla u_j \rangle \, dx - \int_{\Omega} \langle \sigma; e \rangle \varphi \, dx = \int_{\Omega} \varphi \langle \sigma; \nabla u_j - e \rangle \, dx = \int_{\Omega} \varphi \langle \sigma; p_j \rangle \, dx$$

Hence, we obtain

$$\left| \langle [\sigma \cdot p_j] | \varphi \rangle \right| \leq \|\sigma\|_{L^\infty(\Omega; \mathbb{R}^n)} \|\varphi\|_{\infty} |p_j|(A),$$

and passing to the limit as $j \rightarrow +\infty$

$$|\langle [\sigma \cdot p] | \varphi \rangle| \leq \|\sigma\|_{L^\infty(\Omega; \mathbb{R}^n)} \|\varphi\|_{\infty} |p|(A).$$

□

A slight adaptation of [4, Theorem 1.9] shows the following integration by parts formula.

Proposition 3.2.4. *Let $u \in BV(\Omega) \cap L^2(\Omega)$, $\sigma \in H(\text{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$ and $\varphi \in W^{1,\infty}(\Omega)$. Then,*

$$\int_{\Omega} \varphi \, d[\sigma \cdot Du] + \int_{\Omega} u \varphi(\text{div} \sigma) \, dx + \int_{\Omega} u \langle \sigma; \nabla \varphi \rangle \, dx = \int_{\partial\Omega} \langle \sigma; \nu \rangle u \varphi \, d\mathcal{H}^{n-1}.$$

Proof. First, observe that since $\sigma \in H(\text{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$ and $\varphi \in W^{1,\infty}(\Omega)$, the function $\varphi \sigma$ also belongs to $H(\text{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$ and we have

$$\text{div}(\varphi \sigma) = \varphi \text{div} \sigma + \langle \sigma; \nabla \varphi \rangle.$$

By definition of duality and of the normal trace, we infer that $[\varphi \sigma \cdot Du] = \varphi[\sigma \cdot Du]$ in $\mathcal{M}(\Omega)$, and $\langle \varphi \sigma; \nu \rangle = \varphi \langle \sigma; \nu \rangle$ in

$L^\infty(\partial\Omega)$. Consequently, the integration by parts formula established in [4, Theorem 1.9] yields

$$\begin{aligned} \int_{\partial\Omega} \langle \sigma; \nu \rangle u \varphi \, d\mathcal{H}^{n-1} &= \int_{\partial\Omega} (\varphi \langle \sigma; \nu \rangle) u \, d\mathcal{H}^{n-1} = \int_{\Omega} u \operatorname{div}(\varphi \sigma) \, dx + \int_{\Omega} d[\varphi \sigma \cdot Du] \\ &= \int_{\Omega} u (\operatorname{div} \sigma) \varphi \, dx + \int_{\Omega} u \langle \sigma; \nabla \varphi \rangle \, dx + \int_{\Omega} \varphi \, d[\sigma \cdot Du], \end{aligned}$$

which completes the proof of the Proposition. \square

3.3 Description of the model

3.3.1 Small strain elasto-plasticity

To simplify the presentation of the model, we consider the physical three-dimensional case. We assume that the reference configuration of the elasto-plastic body under consideration occupies the volume $\mathcal{B} \subset \mathbb{R}^3$. In the framework of small strain elasto-plasticity, the natural kinematic variable is the displacement field $\mathbf{u} : [0, T] \times \mathcal{B} \rightarrow \mathbb{R}^3$ (or the velocity $\mathbf{v} := \partial_t \mathbf{u}$). Denoting by $E\mathbf{u} := (D\mathbf{u} + D\mathbf{u}^T)/2 : [0, T] \times \mathcal{B} \rightarrow \mathbb{M}_{\text{sym}}^{3 \times 3}$ the linearized strain tensor, small strain elasto-plasticity assumes that $E\mathbf{u}$ decomposes additively as

$$E\mathbf{u} = \mathbf{e} + \mathbf{p}, \quad (3.3.1)$$

where \mathbf{e} and $\mathbf{p} : [0, T] \times \mathcal{B} \rightarrow \mathbb{M}_{\text{sym}}^{3 \times 3}$ stand for the elastic and plastic strains, respectively. The elastic strain is related to the Cauchy stress tensor $\sigma : [0, T] \times \mathcal{B} \rightarrow \mathbb{M}_{\text{sym}}^{3 \times 3}$ by means of Hooke's law $\sigma = \mathbb{C}\mathbf{e}$, where \mathbb{C} is the symmetric fourth order elasticity tensor. For example, in the isotropic case, one has

$$\sigma = \lambda(\operatorname{tr} \mathbf{e})I + 2\mu \mathbf{e}, \quad (3.3.2)$$

where λ and μ are the Lamé coefficients satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$. In a dynamical framework and in the presence of external body loads $\mathbf{f} : [0, T] \times \mathcal{B} \rightarrow \mathbb{R}^3$, the equations of motion are a system of partial differential equations which writes as

$$\partial_{tt} \mathbf{u} - \operatorname{div} \sigma = \mathbf{f} \quad \text{in } (0, T) \times \mathcal{B}. \quad (3.3.3)$$

Plasticity is characterized by the existence of a yield zone in the stress space beyond which the Cauchy stress is not permitted to live. The stress tensor is indeed constrained to belong to a fixed nonempty, closed and convex subset of $\mathbb{M}_{\text{sym}}^{3 \times 3}$. In the case of Von Mises plasticity, the constraint only acts on the (trace free) deviatoric stress $\sigma_D := \sigma - (\operatorname{tr} \sigma)I/3$, and reads as

$$|\sigma_D| \leq k, \quad (3.3.4)$$

where $k > 0$ is a critical stress value. The evolution of the plastic strain is described by means of the flow rule and is expressed with the Prandtl-Reuss law

$$\begin{cases} \partial_t \mathbf{p} = 0 & \text{if } |\sigma_D| < k, \\ \frac{\partial_t \mathbf{p}}{|\partial_t \mathbf{p}|} = \frac{\sigma_D}{k} & \text{if } |\sigma_D| = k. \end{cases} \quad (3.3.5)$$

The system (3.3.1)–(3.3.5) is supplemented by initial and boundary conditions which will be discussed later.

3.3.2 Anti-plane shear

Denoting by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the canonical basis of \mathbb{R}^3 , we assume that \mathcal{B} is invariant in the \mathbf{e}_3 direction so that $\mathcal{B} = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a bounded open set. We also suppose that the displacement is anti-plane $\mathbf{u}(x_1, x_2, x_3) = u(x_1, x_2)\mathbf{e}_3$ for some scalar function $u : [0, T] \times \Omega \rightarrow \mathbb{R}$, so that computing the linearized strain yields

$$E\mathbf{u} = (\partial_1 u)\mathbf{e}_1 \odot \mathbf{e}_3 + (\partial_2 u)\mathbf{e}_2 \odot \mathbf{e}_3$$

corresponding to pure shear strain. We thus assume that the elastic and plastic strains conserve this special structure so that

$$\mathbf{p} = p_1 \mathbf{e}_1 \odot \mathbf{e}_3 + p_2 \mathbf{e}_2 \odot \mathbf{e}_3, \quad \mathbf{e} = e_1 \mathbf{e}_1 \odot \mathbf{e}_3 + e_2 \mathbf{e}_2 \odot \mathbf{e}_3,$$

for some functions e_1, e_2, p_1 and $p_2 : [0, T] \times \Omega \rightarrow \mathbb{R}$. Denoting by $e = (e_1, e_2)$ and $p = (p_1, p_2)$, the additive decomposition (3.3.1) now reads as

$$\nabla u = e + p.$$

Computing the Cauchy stress according to (3.3.2) yields a pure shear stress $\sigma = 2\mu e$, so that denoting by $\sigma := (\sigma_{13}, \sigma_{23})$ its only nonzero components, we have

$$\sigma = \mu e.$$

We also assume that the body load is compatible with the anti-plane assumption, *i.e.*, $f = f e_3$, for some $f : [0, T] \times \Omega \rightarrow \mathbb{R}$, so that the equations of motion (3.3.3) becomes a scalar equation

$$\partial_{tt} u - \operatorname{div} \sigma = f \quad \text{in } (0, T) \times \Omega.$$

Finally, the stress constraint (3.3.4) now reads as $|\sigma| \leq k/\sqrt{2}$, and the flow rule (3.3.5) is given by

$$\begin{cases} \partial_t p = 0 & \text{if } |\sigma| < k/\sqrt{2}, \\ \frac{\partial_t p}{|\partial_t p|} = \frac{\sqrt{2}\sigma}{k} & \text{if } |\sigma| = k/\sqrt{2}. \end{cases}$$

In order to simplify notation, we assume henceforth that $\mu = 1$ (so that $\sigma = e$) and $k = \sqrt{2}$. The simplified model of plasticity thus consists in looking for functions $u : [0, T] \times \Omega \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ and $p : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ such that the following system holds in $(0, T) \times \Omega$:

$$\begin{cases} \partial_{tt} u - \operatorname{div} \sigma = f, & \nabla u = \sigma + p, \\ |\sigma| \leq 1, & \text{if } |\sigma| < 1, \quad \partial_t p = 0, & \text{if } |\sigma| = 1, \quad \frac{\partial_t p}{|\partial_t p|} = \sigma. \end{cases} \quad (3.3.6)$$

Note that the flow rule can be equivalently be written as

$$\sigma \cdot \partial_t p = |\partial_t p|. \quad (3.3.7)$$

which expresses Hill's principle of maximal plastic work.

We supplement the system with initial conditions on the displacement, the velocity, the stress and the plastic strain

$$(u, \partial_t u, \sigma, p)(0) = (u_0, v_0, \sigma_0, p_0).$$

The precise mathematical formulation of this model will be the object of Section 3.5. In particular, the flow rule (3.3.7) will have to be interpreted in a suitable measure theoretic sense according to the generalized stress/strain duality introduced in Definition 3.2.2.

The discussion of admissible boundary conditions is the object of the following paragraphs, once the hyperbolic structure of the system will be described.

3.3.3 Dissipative formulation of the model

In this section, we perform formal manipulations on the system (3.3.6) in order to write it in a different form, more appropriate to describe hyperbolicity. To do that, we denote by $U := (\partial_t u, \sigma)$, and observe that the first two equations of (3.3.6) can be written as

$$\partial_t U + A_1 \partial_1 U + A_2 \partial_2 U + P = F, \quad (3.3.8)$$

where $F = (f, 0, 0)$, $P = (0, \partial_t p_1, \partial_t p_2)$ and

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (3.3.9)$$

Taking the scalar product of (3.3.8) with U , yields

$$\frac{1}{2} \partial_t |U|^2 + \frac{1}{2} \partial_1 \langle A_1 U; U \rangle + \frac{1}{2} \partial_2 \langle A_2 U; U \rangle + \langle P; U \rangle = \langle F; U \rangle,$$

while, for every constant vector $\kappa = (k, \tau) \in \mathcal{C} := \mathbb{R} \times \bar{B}$, taking the scalar product of (3.3.8) with κ leads to

$$\partial_t \langle U; \kappa \rangle + \partial_1 \langle A_1 U; \kappa \rangle + \partial_2 \langle A_2 U; \kappa \rangle + \langle P; \kappa \rangle = \langle F; \kappa \rangle.$$

Subtracting both previous equalities, and using that $\langle P; (U - \kappa) \rangle = \langle \partial_t p; (\sigma - \tau) \rangle \geq 0$ according to the flow rule written as (3.3.7), we infer that

$$\partial_t |U - \kappa|^2 + \sum_{i=1}^2 \partial_i \langle A_i (U - \kappa); U - \kappa \rangle \leq 2 \langle F; U - \kappa \rangle. \quad (3.3.10)$$

We then multiply the previous inequality by a test function $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$ with $\varphi \geq 0$, and integrate by parts to obtain

$$\int_0^T \int_\Omega |U - \kappa|^2 \partial_t \varphi \, dx \, dt + \sum_{i=1}^2 \int_0^T \int_\Omega \langle A_i (U - \kappa); U - \kappa \rangle \partial_i \varphi \, dx \, dt + \int_\Omega |U_0 - \kappa|^2 \varphi(0) \, dx + 2 \int_0^T \int_\Omega \langle F; U - \kappa \rangle \varphi \, dx \, dt \geq 0,$$

which is precisely the formulation of constrained Friedrichs' systems as defined in [37] without taking care of boundary conditions since φ vanishes in a neighborhood of $(0, T) \times \partial\Omega$.

In order to account for the boundary condition, we follow an approach introduced in Chapter 2. Following the pioneering work [47], we are formally interested in dissipative boundary conditions of the form

$$(A_\nu - M_\nu)U = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (3.3.11)$$

where $A_\nu = A_1 \nu_1 + A_2 \nu_2$ ($\nu = (\nu_1, \nu_2)$ is the outer normal to $\partial\Omega$), and $M_\nu \in \mathbb{M}^{3 \times 3}$ is a boundary matrix satisfying the following algebraic conditions

$$\begin{cases} M_\nu = M_\nu^T, \\ M_\nu \text{ is non-negative,} \\ \text{Ker } A_\nu \subset \text{Ker } M_\nu, \\ \mathbb{R}^3 = \text{Ker}(A_\nu - M_\nu) + \text{Ker}(A_\nu + M_\nu). \end{cases} \quad (3.3.12)$$

Note that in the non-characteristic case (*i.e.* when A_ν is non-singular), conditions (3.3.12) imply those (3.1.2) introduced by Friedrichs (see Chapter 2 for a detailed discussion). Thus, multiplying inequality (3.3.10) by a test function $\varphi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$ with $\varphi \geq 0$, and integrating by parts, we get

$$\begin{aligned} \int_0^T \int_\Omega |U - \kappa|^2 \partial_t \varphi \, dx \, dt + \sum_{i=1}^2 \int_0^T \int_\Omega \langle A_i (U - \kappa); U - \kappa \rangle \partial_i \varphi \, dx \, dt + \int_\Omega |U_0 - \kappa|^2 \varphi(0) \, dx \\ + 2 \int_0^T \int_\Omega \langle F; U - \kappa \rangle \varphi \, dx \, dt - \int_0^T \int_{\partial\Omega} \langle A_\nu (U - \kappa); U - \kappa \rangle \varphi \, dx \, dt \geq 0. \end{aligned} \quad (3.3.13)$$

According to Lemma 2.2.3, we have

$$\mathbb{R}^3 = \text{Ker } A_\nu \oplus \left(\text{Ker}(A_\nu - M_\nu) \cap \text{Im } A_\nu \right) \oplus \left(\text{Ker}(A_\nu + M_\nu) \cap \text{Im } A_\nu \right).$$

For each $\kappa \in \mathbb{R}^3$, we denote by κ^\pm the projection of κ onto $\text{Ker}(A_\nu \pm M_\nu) \cap \text{Im } A_\nu$. Using the (strong) boundary condition (3.3.11), we have $U \in \text{Ker}(A_\nu - M_\nu)$, or still $U^+ = 0$. The algebraic conditions (3.3.12) together with Lemma 2.2.3 thus yield

$$\begin{aligned} \langle A_\nu (U - \kappa); U - \kappa \rangle &= -\langle M_\nu (U - \kappa)^+; (U - \kappa)^+ \rangle + \langle M_\nu (U - \kappa)^-; (U - \kappa)^- \rangle \\ &= -\langle M_\nu \kappa^+; \kappa^+ \rangle + \langle M_\nu (U - \kappa)^-; (U - \kappa)^- \rangle \geq -\langle M_\nu \kappa^+; \kappa^+ \rangle. \end{aligned}$$

Inserting in (3.3.13), we get that for all constant vector $\kappa \in \mathcal{C}$ and all $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$,

$$\begin{aligned} \int_0^T \int_\Omega |U - \kappa|^2 \partial_t \varphi \, dx \, dt + \sum_{i=1}^2 \int_0^T \int_\Omega \langle A_i (U - \kappa); U - \kappa \rangle \partial_i \varphi \, dx \, dt \\ + \int_\Omega |U_0 - \kappa|^2 \varphi(0) \, dx + 2 \int_0^T \int_\Omega \langle F; U - \kappa \rangle \varphi \, dx \, dt + \int_0^T \int_{\partial\Omega} \langle M_\nu \kappa^+; \kappa^+ \rangle \varphi \, d\mathcal{H}^1 \, dt \geq 0. \end{aligned} \quad (3.3.14)$$

The previous family of inequalities defines a notion dissipative solutions $U \in L^2((0, T) \times \Omega; \mathcal{E})$ to the dynamical elasto-plastic problem. Note that it is meaningful within a L^2 theory of Friedrichs' systems (as suggested by (3.3.14)) since the trace of U on the boundary $(0, T) \times \partial\Omega$, which is not well defined, is not involved in this definition (see also [87, 71] for an L^∞ -theory of initial/boundary value conservation laws).

3.3.4 Derivation of the boundary conditions

The well-posedness of this kind of dissipative formulations in the full space $\Omega = \mathbb{R}^2$ has been established in [37]. On the other hand, among the results of Chapter 2, it is shown the existence and uniqueness of a solution to this problem in the unconstrained case ($\mathcal{E} = \mathbb{R}^3$), and the dissipative boundary condition (3.3.11) is proved to be satisfied in a suitable weak sense.

In order to formulate more precisely the admissible boundary conditions in our particular situation, we need to characterize all boundary matrices satisfying the required algebraic conditions (3.3.12).

Lemma 3.3.1. *Assume that A_1 and A_2 are given by (3.3.9) and $v \in \mathbb{R}^2$ satisfies $|v| = 1$. The following assertions are equivalent :*

1. A matrix $M_v \in \mathbb{M}^{3 \times 3}$ satisfies (3.3.12);
2. There exists $\lambda \in (0, +\infty)$ such that

$$M_v = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda v_1^2 & \lambda v_1 v_2 \\ 0 & \lambda v_1 v_2 & \lambda v_2^2 \end{pmatrix}. \quad (3.3.15)$$

Proof. It is immediate to check that any matrix M_v of the form (3.3.15) with $\lambda > 0$ fulfills all conditions (3.3.12). Conversely, assume that

$$M_v = \begin{pmatrix} d_1 & a & b \\ a & d_2 & c \\ b & c & d_3 \end{pmatrix} \in \mathbb{M}_{\text{sym}}^{3 \times 3}$$

satisfies (3.3.12). Since

$$A_v = \begin{pmatrix} 0 & -v_1 & -v_2 \\ -v_1 & 0 & 0 \\ -v_2 & 0 & 0 \end{pmatrix},$$

we get

$$\text{Ker} A_v = \{(v, \sigma) = (v, \sigma_1, \sigma_2) \in \mathbb{R}^3 : v = 0 \text{ and } \langle \sigma; v \rangle = \sigma_1 v_1 + \sigma_2 v_2 = 0\}.$$

Denoting by $v^\perp := \{\sigma \in \mathbb{R}^2 : \langle \sigma; v \rangle = 0\}$ the linear space of dimension 1 in \mathbb{R}^2 , condition $\text{Ker} A_v \subset \text{Ker} M_v$ reads as

$$\begin{cases} v^\perp \subset \{\sigma \in \mathbb{R}^2 : a\sigma_1 + b\sigma_2 = 0\} & =: E_1, \\ v^\perp \subset \{\sigma \in \mathbb{R}^2 : d_2\sigma_1 + c\sigma_2 = 0\} & =: E_2, \\ v^\perp \subset \{\sigma \in \mathbb{R}^2 : c\sigma_1 + d_3\sigma_2 = 0\} & =: E_3. \end{cases}$$

Consequently, we obtain that the dimension of the linear spaces E_1, E_2 and E_3 is larger than or equal to 1. If $\dim E_1 = 1$, then v is orthogonal to E_1 , while if $\dim E_1 = 2$, then $a = b = 0$. In both cases, one can find $\mu_1 \in \mathbb{R}$ such that

$$(a, b) = \mu_1 v.$$

Arguing similarly for E_2 and E_3 , there exist μ_2 and $\mu_3 \in \mathbb{R}$ such that

$$(d_2, c) = \mu_2 v, \quad (c, d_3) = \mu_3 v,$$

so that

$$M_v = \begin{pmatrix} d_1 & \mu_1 v_1 & \mu_1 v_2 \\ \mu_1 v_1 & \mu_2 v_1 & \mu_2 v_2 \\ \mu_1 v_2 & \mu_3 v_1 & \mu_3 v_2 \end{pmatrix}.$$

Using that M_v is symmetric, we must have $\mu_2 v_2 = \mu_3 v_1$. Since $|v| = 1$, then either $v_1 \neq 0$ or $v_2 \neq 0$. Suppose without

loss of generality that $\nu_1 \neq 0$, and define $\lambda = \mu_2/\nu_1$, then

$$M_\nu = \begin{pmatrix} d_1 & \mu_1 \nu_1 & \mu_1 \nu_2 \\ \mu_1 \nu_1 & \lambda \nu_1^2 & \lambda \nu_1 \nu_2 \\ \mu_1 \nu_2 & \lambda \nu_1 \nu_2 & \lambda \nu_2^2 \end{pmatrix}.$$

Using next that M_ν is non-negative, it follows that for all $(v, \sigma) \in \mathbb{R}^3$,

$$\left\langle M_\nu \begin{pmatrix} v \\ \sigma \end{pmatrix}; \begin{pmatrix} v \\ \sigma \end{pmatrix} \right\rangle = d_1 v^2 + 2\mu_1 v \langle \sigma; \nu \rangle + \lambda (\langle \sigma; \nu \rangle)^2 \geq 0,$$

which ensures that $d_1 \geq 0$, $\lambda \geq 0$. In fact, if $d_1 = \lambda = 0$, the previous expression can easily be made negative so that either $d_1 > 0$ or $\lambda > 0$ (since the case $\mu_1 = 0$ is impossible).

From the conditions $\text{Ker} A_\nu \subset \text{Ker} M_\nu$ and $\dim \text{Ker} A_\nu = 1$, we obtain that $\dim \text{Ker}(A_\nu \pm M_\nu) \geq 1$ and $\dim(\text{Ker}(A_\nu - M_\nu) \cap \text{Ker}(A_\nu + M_\nu)) \geq 1$. The last condition $\mathbb{R}^3 = \text{Ker}(A_\nu - M_\nu) + \text{Ker}(A_\nu + M_\nu)$ then implies that $\dim \text{Ker}(A_\nu \pm M_\nu) = 2$ (since A_ν is neither non-negative, nor non-positive). Computing

$$A_\nu \pm M_\nu = \begin{pmatrix} \pm d_1 & (\pm \mu_1 - 1)\nu_1 & (\pm \mu_1 - 1)\nu_2 \\ (\pm \mu_1 - 1)\nu_1 & \pm \lambda \nu_1^2 & \pm \lambda \nu_1 \nu_2 \\ (\pm \mu_1 - 1)\nu_2 & \pm \lambda \nu_1 \nu_2 & \pm \lambda \nu_2^2 \end{pmatrix},$$

we infer that

$$(A_\nu - M_\nu) \begin{pmatrix} v \\ \sigma \end{pmatrix} = 0 \iff \begin{cases} d_1 v + (\mu_1 + 1)\langle \sigma; \nu \rangle = 0, \\ (\mu_1 + 1)v + \lambda \langle \sigma; \nu \rangle = 0. \end{cases}$$

Observe that, since $\dim \text{Ker}(A_\nu - M_\nu) = 2$, we obtain

$$\begin{aligned} d_1 v + (\mu_1 + 1)\langle \sigma; \nu \rangle = 0 &\iff (\mu_1 + 1)v + \lambda \langle \sigma; \nu \rangle = 0 \\ &\iff d_1 \lambda - (\mu_1 + 1)^2 = 0, \end{aligned}$$

and similarly, since $\dim \text{Ker}(A_\nu + M_\nu) = 2$,

$$\begin{aligned} d_1 v + (\mu_1 - 1)\langle \sigma; \nu \rangle = 0 &\iff (\mu_1 - 1)v + \lambda \langle \sigma; \nu \rangle = 0 \\ &\iff d_1 \lambda - (\mu_1 - 1)^2 = 0, \end{aligned}$$

which implies that $\mu_1 = 0$ and $d_1 \lambda = 1$, hence $\lambda > 0$. □

Remark 3.3.2. A similar characterization result can be proved in any dimension $n \geq 2$, i.e., when the matrices M_ν belong to $\mathbb{M}^{(n+1) \times (n+1)}$.

As a consequence of Lemma 3.3.1, it follows that all admissible boundary conditions (3.3.11) for the dissipative formulation are of the form

$$\langle \sigma; \nu \rangle + \lambda^{-1} \partial_t u = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (3.3.16)$$

where $\lambda : \partial\Omega \rightarrow (0, +\infty)$. In the sequel, we will assume for simplicity that $\lambda > 0$ is independent of the space variable.

Remark 3.3.3. Note that, strictly speaking, (homogeneous) Dirichlet and Neumann conditions are not contained within this framework. However, they can be recovered by means of an asymptotic analysis as $\lambda \rightarrow 0^+$ and $\lambda \rightarrow +\infty$, respectively (see Section 3.5.5).

Moreover, since λ is actually a (Borel) function of the space variable, letting $\lambda \rightarrow 0$ in some subset $\Gamma_D \subset \Omega$, and $\lambda \rightarrow +\infty$ on its complementary $\Gamma_N := \partial\Omega \setminus \Gamma_D$ would lead to mixed boundary conditions of Dirichlet type on Γ_D and Neumann type on Γ_N . This problem will not be addressed in the present work.

3.4 The dynamic elasto-visco-plastic model

In order to establish the existence and uniqueness of solution to (3.3.6) and (3.3.16), we consider an elasto-visco-plastic approximation model which consists in regularizing the constitutive law by means of a Kelvin-Voigt type visco-elasticity, and the flow rule thanks to a Perzyna type visco-plasticity. Except for our choice of boundary conditions (3.3.16), the model described below is very similar to that studied in [31] (see also [91, 10]).

This choice of regularization is motivated by the approximation employed in [37] (see also Section 4.2) in order to show the well-posedness of constrained Friedrichs' systems in the whole space. It consists in penalizing the constraint (which is described by Perzyna visco-plasticity), and adding up a diffusive term (which corresponds to Kelvin-Voigt visco-elasticity).

Note also that since the space dimension does not really matter in the subsequent arguments, we perform the analysis in any space dimension.

The main result of this section is the following existence and uniqueness result.

Theorem 3.4.1. *Let Ω be a bounded open set of \mathbb{R}^n with Lipschitz boundary and $\lambda > 0$. Consider a source term $f \in H^1([0, T]; L^2(\Omega))$ and an initial data $(u_0, v_0, \sigma_0, p_0) \in H^1(\Omega) \times H^2(\Omega) \times H(\operatorname{div}, \Omega) \times L^2(\Omega; \mathbb{R}^n)$ such that*

$$\begin{cases} \nabla u_0 = \sigma_0 + p_0 \text{ in } L^2(\Omega; \mathbb{R}^n), \\ \langle \sigma_0; \nu \rangle + \lambda^{-1} v_0 = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ |\sigma_0| \leq 1 \text{ a.e. in } \Omega. \end{cases}$$

For each $\varepsilon > 0$, we define $g_\varepsilon := \varepsilon \langle \nabla v_0; \nu \rangle \in L^2(\partial\Omega)$. Then, there exist a unique triple $(u_\varepsilon, \sigma_\varepsilon, p_\varepsilon)$ with the regularity

$$\begin{cases} u_\varepsilon \in W^{2,\infty}([0, T]; L^2(\Omega)) \cap H^2([0, T]; H^1(\Omega)), \\ \sigma_\varepsilon \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ p_\varepsilon \in H^1([0, T]; L^2(\Omega; \mathbb{R}^n)), \end{cases}$$

which satisfies the following properties :

1. The initial conditions :

$$u_\varepsilon(0) = u_0, \quad \partial_t u_\varepsilon(0) = v_0, \quad \sigma_\varepsilon(0) = \sigma_0, \quad p_\varepsilon(0) = p_0;$$

2. The additive decomposition : for all $t \in [0, T]$

$$\nabla u_\varepsilon(t) = \sigma_\varepsilon(t) + p_\varepsilon(t) \quad \text{in } L^2(\Omega; \mathbb{R}^n); \quad (3.4.1)$$

3. The equation of motion :

$$\partial_{tt} u_\varepsilon - \operatorname{div}(\sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon) = f \quad \text{in } L^2(0, T; L^2(\Omega));$$

4. The dissipative boundary condition :

$$\langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle + \lambda^{-1} \partial_t u_\varepsilon = g_\varepsilon \quad \text{in } L^2(0, T; L^2(\partial\Omega)); \quad (3.4.2)$$

5. The visco-plastic flow rule :

$$\partial_t p_\varepsilon = \frac{\sigma_\varepsilon - P_B(\sigma_\varepsilon)}{\varepsilon} \quad \text{in } L^2(0, T; L^2(\Omega; \mathbb{R}^n)). \quad (3.4.3)$$

In addition, we have the following energy balance : for all $t \in [0, T]$,

$$\begin{aligned} & \frac{1}{2} \|\partial_t u_\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \varepsilon \left(\int_0^t \int_\Omega |\nabla \partial_t u_\varepsilon|^2 \, dx \, ds + \int_0^t \int_\Omega |\partial_t p_\varepsilon|^2 \, dx \, ds \right) + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |\partial_t u_\varepsilon|^2 \, d\mathcal{H}^{n-1} \, ds \\ & + \int_0^t \int_\Omega |\partial_t p_\varepsilon| \, dx \, ds = \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_0^t \int_\Omega f \partial_t u_\varepsilon \, dx \, ds + \int_0^t \int_{\partial\Omega} g_\varepsilon \partial_t u_\varepsilon \, d\mathcal{H}^{n-1} \, ds, \end{aligned} \quad (3.4.4)$$

and the estimate

$$\begin{aligned} & \sup_{t \in [0, T]} \|\partial_{tt} u_\varepsilon(t)\|_{L^2(\Omega)}^2 + \sup_{t \in [0, T]} \|\partial_t \sigma_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \varepsilon \int_0^T \|\nabla \partial_{tt} u_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \, dt + \frac{1}{\lambda} \int_0^T \|\partial_{tt} u_\varepsilon(t)\|_{L^2(\partial\Omega)}^2 \, dt \\ & \leq C \left(\|\operatorname{div}(\sigma_0 + \varepsilon \nabla v_0) + f(0)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \left(\int_0^T \|\partial_t f(t)\|_{L^2(\Omega; \mathbb{R}^n)} \, dt \right)^2 \right), \end{aligned} \quad (3.4.5)$$

for some constant $C > 0$ independent of ε and λ .

The proof of Theorem 3.4.1 is standard and follows the lines of e.g. [31] for the existence and uniqueness in the energy space, and of [10] for the additional regularity results (3.4.5). The main difference with [10, 31] is concerned

with the boundary condition. The proofs relies on a time discretization procedure, described in Subsection 3.4.1.2. We derive, in Subsection 3.4.1.3, some *a priori* estimates that allow us, thanks to compactness results, to obtain some weak convergence when the time discretization parameter goes to 0 in Subsection 3.4.1.4. In particular, we obtain the equation of motion in $L^2((0, T); (H^1(\Omega))')$ (thanks to the definition of a divergence operator given in Subsection 3.4.1.1). In Subsection 3.4.1.5, using calculus in Banach-valued Sobolev space, we obtain the initial condition for the velocity. In order to get the flow rule, we derive a strong convergence result in Subsection 3.4.1.6. The uniqueness of the solution is obtained in Subsection 3.4.1.7 and the energy balance law in Subsection 3.4.1.8. Finally, in Subsection 3.4.1.9, we obtain, thanks to *a posteriori* estimates, the equation of the motion in a strong sense (in $L^2((0, T); H^1(\Omega))$) and the boundary condition.

3.4.1 The proof of Theorem 3.4.1

3.4.1.1 Divergence operator

We will first give a weak sense (in $L^2((0, T), (H^1(\Omega))')$) to the equation of motion. To do so, we will use the following definition.

Definition 3.4.2. Let $v_0 \in H^2(\Omega)$ and $\epsilon > 0$. For all $(v, \Sigma) \in H^1(\Omega) \times L^2(\Omega; \mathbb{R}^n)$, we define the operator $\text{div}_\lambda(v, \Sigma) : H^1(\Omega) \times L^2(\Omega; \mathbb{R}^n) \rightarrow (H^1(\Omega))'$ as

$$\forall \varphi \in H^1(\Omega), \quad \langle \text{div}_\lambda(v, \Sigma) | \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} = -\langle \epsilon \nabla v | \nabla \varphi \rangle_{L^2(\Omega; \mathbb{R}^n)} - \langle \Sigma | \nabla \varphi \rangle_{L^2(\Omega; \mathbb{R}^n)} - \frac{1}{\lambda} \langle v - \epsilon \langle \nabla v_0; v \rangle | \varphi \rangle_{L^2(\partial\Omega)}.$$

Using the continuity of the trace (from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$), we get that there exists a constant C such that

$$\|\text{div}_\lambda(v, \Sigma)\|_{(H^1(\Omega))'} \leq C \left(\|v\|_{H^1(\Omega)} + \|\Sigma\|_{L^2(\Omega; \mathbb{R}^n)} + \|\epsilon \langle \nabla v_0; v \rangle\|_{L^2(\partial\Omega)} \right).$$

3.4.1.2 Time-scheme

We divide the interval $[0, T]$ into N sub-intervals of equal length $\delta = T/N$:

$$0 = t_0 < t_1 < \dots < t_N = T, \quad \text{with } t_{i+1} - t_i = \delta.$$

We define the discrete term source $f_i = f(t^i)$ for all $i \in \{0, \partial_t s, N\}$. Since the boundary term g_ϵ does not depend on the time variable, there is no need to discretize it. Then we define inductively the approximation

$$(u_0, \sigma_0, p_0) = (u_0, \sigma_0, p_0), \quad (u_1, \sigma_1, p_1) = (u_0, \sigma_0, p_0) + \delta(v_0, 0, \nabla v_0),$$

and for all $i \geq 2$, (u_i, σ_i, p_i) is the unique minimum, on the space

$$X = \{(v, \tau, q) \in H^1(\Omega) \times L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n), \quad \nabla v = \tau + q\},$$

of the functional E , which strictly convex, coercive and sequentially weakly lower semi-continuous in X

$$\begin{aligned} E(v, \tau, q) = & \frac{1}{2\delta^2} \int_{\Omega} (v - 2u_{i-1} + u_{i-2})^2 dx + \frac{\epsilon}{2\delta} \int_{\Omega} |\nabla v - \nabla u_{i-1}|^2 dx + \frac{1}{2} \int_{\Omega} |\tau|^2 dx + \int_{\Omega} |q - p_{i-1}| dx \\ & + \frac{\epsilon}{2\delta} \int_{\Omega} |q - p_{i-1}|^2 dx + \frac{1}{2\delta\lambda} \int_{\partial\Omega} |v - u_{i-1}|^2 d\mathcal{H}^{n-1} - \int_{\Omega} f_{i-1} v dx - \int_{\partial\Omega} g_\epsilon v d\mathcal{H}^{n-1}. \end{aligned}$$

Since (u_i, σ_i, p_i) is the unique minimum, we obtain that for all $(v, \tau, q) \in X$ and for all $l \in (0, 1)$,

$$E((u_i, \sigma_i, p_i)) \leq E((u_i, \sigma_i, p_i) + l(v, \tau, q)).$$

Sending l to 0^+ , we have for all $(v, \tau, q) \in X$

$$\begin{aligned} \frac{1}{\delta^2} \int_{\Omega} (u_i - 2u_{i-1} + u_{i-2})v dx + \frac{\epsilon}{\delta} \int_{\Omega} \langle \nabla u_i - \nabla u_{i-1}; \nabla v \rangle dx + \int_{\Omega} \langle \sigma_i; \tau \rangle dx + \int_{\Omega} |p_i - p_{i-1} + q| dx - \int_{\Omega} |p_i - p_{i-1}| dx \\ + \frac{\epsilon}{\delta} \int_{\Omega} \langle p_i - p_{i-1}; q \rangle dx + \frac{1}{\delta\lambda} \int_{\partial\Omega} (u_i - u_{i-1})v d\mathcal{H}^{n-1} - \int_{\Omega} f_{i-1} v dx - \int_{\partial\Omega} g_\epsilon v d\mathcal{H}^{n-1} \geq 0. \end{aligned}$$

Testing this inequality with $\pm(v, \nabla v, 0)$ for all $v \in H^1(\Omega)$, we get

$$\begin{aligned} \int_{\Omega} \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta^2} v \, dx + \epsilon \int_{\Omega} \left\langle \frac{\nabla u_i - \nabla u_{i-1}}{\delta}; \nabla v \right\rangle \, dx + \int_{\Omega} \langle \sigma_i; \nabla v \rangle \, dx + \frac{1}{\lambda} \int_{\partial\Omega} \frac{u_i - u_{i-1}}{\delta} v \, d\mathcal{H}^{n-1} \\ = \int_{\Omega} f_{i-1} v \, dx + \int_{\partial\Omega} g_{\epsilon} v \, d\mathcal{H}^{n-1}. \end{aligned} \quad (3.4.6)$$

In particular, we obtain the discrete version of the equation of the motion,

$$\frac{u_i - 2u_{i-1} + u_{i-2}}{\delta^2} - \operatorname{div} \left(\epsilon \frac{\nabla u_i - \nabla u_{i-1}}{\delta} + \sigma_i \right) = f_{i-1} \quad \text{in } \mathcal{D}'(\Omega), \quad (3.4.7)$$

and consequently, $\epsilon \frac{\nabla u_i - \nabla u_{i-1}}{\delta} + \sigma_i \in H(\operatorname{div}, \Omega)$. Now integrating by parts

$$\begin{aligned} \int_{\Omega} \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta^2} v \, dx - \int_{\Omega} \left(\operatorname{div} \left(\epsilon \frac{\nabla u_i - \nabla u_{i-1}}{\delta} + \sigma_i \right) \right) v \, dx - \int_{\Omega} f_{i-1} v \, dx \\ = \int_{\partial\Omega} g_{\epsilon} v \, d\mathcal{H}^{n-1} - \left\langle \left\langle \epsilon \frac{\nabla u_i - \nabla u_{i-1}}{\delta} + \sigma_i; v \right\rangle \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} - \frac{1}{\lambda} \int_{\partial\Omega} \frac{u_i - u_{i-1}}{\delta} v \, d\mathcal{H}^{n-1}. \end{aligned}$$

Thanks to the equation of motion (3.4.7), we obtain the discrete boundary condition

$$\frac{u_i - u_{i-1}}{\delta} + \lambda \left\langle \epsilon \frac{\nabla u_i - \nabla u_{i-1}}{\delta} + \sigma_i; v \right\rangle = g_{\epsilon}, \quad \text{in } H^{-1/2}(\partial\Omega). \quad (3.4.8)$$

For the viscoplastic equation, we take $v = 0$, $q = \hat{q} - (p_i - p_{i-1})$ and $\tau = -q$ (where \hat{q} is an arbitrary function of $L^2(\Omega; \mathbb{R}^n)$) we obtain

$$\int_{\Omega} \langle \sigma_i; (-\hat{q} + (p_i - p_{i-1})) \rangle \, dx + \frac{\epsilon}{\delta} \int_{\Omega} \langle p_i - p_{i-1}; (\hat{q} - (p_i - p_{i-1})) \rangle \, dx + \int_{\Omega} |\hat{q}| \, dx - \int_{\Omega} |p_i - p_{i-1}| \, dx \geq 0. \quad (3.4.9)$$

Localizing (3.4.9), we get

$$\forall \hat{q} \in \mathbb{R}^n, \quad |\hat{q}| \geq |p_i - p_{i-1}| + \left\langle \sigma_i - \frac{\epsilon}{\delta} (p_i - p_{i-1}); \hat{q} - (p_i - p_{i-1}) \right\rangle, \quad \text{a.e. in } \Omega.$$

which means that $\sigma_i - \epsilon \left(\frac{p_i - p_{i-1}}{\delta} \right)$ belongs to the sub-differential of $\hat{q} \mapsto |\hat{q}|$ at the point $p_i - p_{i-1}$. Equivalently

$$\sigma_i \in \partial \left(|\cdot| + \frac{\epsilon}{2} |\cdot|^2 \right) \left(\frac{p_i - p_{i-1}}{\delta} \right).$$

By duality and denoting $d_{\bar{B}}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ the distance function to the set \bar{B} ,

$$\frac{p_i - p_{i-1}}{\delta} \in \partial \left(\frac{1}{2\epsilon} d_{\bar{B}}^2 \right) (\sigma_i) = \left\{ -\frac{1}{\epsilon} (P_{\bar{B}}(\sigma_i) - \sigma_i) \right\} \quad \text{a.e. in } \Omega,$$

hence,

$$\frac{p_i - p_{i-1}}{\delta} = -\frac{1}{\epsilon} (P_{\bar{B}}(\sigma_i) - \sigma_i) \quad \text{a.e. in } \Omega. \quad (3.4.10)$$

Interpolation We use here three different types of interpolations. Firstly, we consider the piecewise constant interpolation :

$$\bar{u}(t = 0) = u_0, \quad \bar{\sigma}(t = 0) = \sigma_0, \quad \bar{p}(t = 0) = p_0, \quad \bar{f}(t = 0) = f(t = 0),$$

and for all $1 \leq i \leq N$, for all $t \in (t_{i-1}, t_i]$:

$$\bar{u}(t) = u_i, \quad \bar{\sigma}(t) = \sigma_i, \quad \bar{p}(t) = p_i, \quad \bar{f}(t) = f_{i-1} = f(t_{i-1}).$$

We will also need the following piecewise affine interpolation :

$$\hat{u}(t = 0) = u_0, \quad \hat{\sigma}(t = 0) = \sigma_0, \quad \hat{p}(t = 0) = p_0,$$

and for all $1 \leq i \leq N$, for all $t \in (t_{i-1}, t_i]$:

$$\hat{u}(t) = u_{i-1} + \frac{t-t_{i-1}}{\delta}(u_i - u_{i-1}), \quad \hat{\sigma}(t) = \sigma_{i-1} + \frac{t-t_{i-1}}{\delta}(\sigma_i - \sigma_{i-1}), \quad \hat{p}(t) = p_{i-1} + \frac{t-t_{i-1}}{\delta}(p_i - p_{i-1}),$$

and

$$\hat{f}(t) = f_{i-1} + \frac{t-t_{i-1}}{\delta}(f_i - f_{i-1}).$$

Finally, we define the quadratic interpolation \tilde{u} on $[t_{i-1}, t_i]$ (for all $1 \leq i \leq N$) by,

$$\tilde{u}(t) = \frac{u_i - 2u_{i-1} + u_{i-2}}{2\delta^2}(t - t_{i-1})^2 + \frac{u_i - u_{i-2}}{2\delta}(t - t_{i-1}) + u_{i-1}, \quad (3.4.11)$$

with the convention $u_{-1} = u_0$. Observe that for every $t \in [\delta, T]$ (thanks to the definition of div_λ), the equation (3.4.6) rewrites

$$\partial_{tt}\tilde{u}(t) - \text{div}_\lambda(\partial_t\hat{u}(t), \bar{\sigma}(t)) = \bar{f}(t) \quad \text{in } (H^1(\Omega))', \quad (3.4.12)$$

Equation (3.4.10) tells us that for a.e. $t \in [\delta, T]$

$$\partial_t\hat{p} = -\frac{1}{\epsilon}(P_{\mathbb{B}}(\bar{\sigma}) - \bar{\sigma}), \quad \text{in } L^2(\Omega; \mathbb{R}^n), \quad (3.4.13)$$

and finally with (3.4.8), for a.e. $t \in [\delta, T]$

$$\partial_t\hat{u}(t) + \lambda \langle \epsilon \nabla(\partial_t\hat{u}(t) + \bar{\sigma}(t)); \nu \rangle = g_\epsilon, \quad \text{in } H^{-1/2}(\partial\Omega). \quad (3.4.14)$$

3.4.1.3 A priori estimates

Taking $v = u_i - u_{i-1}$ in (3.4.6), we get that for all $i \geq 2$

$$\begin{aligned} \frac{1}{2} \left(\left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\Omega)}^2 - \left\| \frac{u_{i-1} - u_{i-2}}{\delta} \right\|_{L^2(\Omega)}^2 + \left\| \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta} \right\|_{L^2(\Omega)}^2 \right) + \epsilon \delta \left\| \frac{\nabla u_i - \nabla u_{i-1}}{\delta} \right\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\delta}{\lambda} \left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\partial\Omega)}^2 \\ + \int_{\Omega} \langle \sigma_i; \nabla u_i - \nabla u_{i-1} \rangle \, dx = \int_{\Omega} f_{i-1}(u_i - u_{i-1}) \, dx + \int_{\partial\Omega} g_\epsilon(u_i - u_{i-1}) \, d\mathcal{H}^{n-1}. \end{aligned}$$

Using the fact that $\nabla u_i - \nabla u_{i-1} = \sigma_i - \sigma_{i-1} + p_i - p_{i-1}$, we obtain

$$\begin{aligned} \frac{1}{2} \left(\left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\Omega)}^2 - \left\| \frac{u_{i-1} - u_{i-2}}{\delta} \right\|_{L^2(\Omega)}^2 + \left\| \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta} \right\|_{L^2(\Omega)}^2 \right) + \epsilon \delta \left\| \frac{\nabla u_i - \nabla u_{i-1}}{\delta} \right\|_{L^2(\Omega; \mathbb{R}^n)}^2 \\ + \frac{\delta}{\lambda} \left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\partial\Omega)}^2 + \frac{1}{2} \left(\|\sigma_i\|_{L^2(\Omega; \mathbb{R}^n)}^2 - \|\sigma_{i-1}\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\sigma_i - \sigma_{i-1}\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right) \\ + \int_{\Omega} \langle \sigma_i; p_i - p_{i-1} \rangle \, dx \leq \int_{\Omega} f_{i-1}(u_i - u_{i-1}) \, dx + \int_{\partial\Omega} g_\epsilon(u_i - u_{i-1}) \, d\mathcal{H}^{n-1}. \end{aligned}$$

Using (3.4.9) with $\hat{q} = 0$,

$$\int_{\Omega} \langle \sigma_i; p_i - p_{i-1} \rangle \, dx \geq \epsilon \delta \left\| \frac{p_i - p_{i-1}}{\delta} \right\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \delta \left\| \frac{p_i - p_{i-1}}{\delta} \right\|_{L^1(\Omega; \mathbb{R}^n)}.$$

Summing up from $i = 2$ to j gives us (using the initial conditions for u_1 and σ_1)

$$\begin{aligned}
& \frac{1}{2} \left(\left\| \frac{u_j - u_{j-1}}{\delta} \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^j \left\| \frac{u_i - 2u_{i-1} + u_{i-2}}{\delta} \right\|_{L^2(\Omega)}^2 + \|\sigma_j\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \sum_{i=1}^j \|\sigma_i - \sigma_{i-1}\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right) \\
& + \sum_{i=1}^j \delta \left(\epsilon \left\| \frac{\nabla u_i - \nabla u_{i-1}}{\delta} \right\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{1}{\lambda} \left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\partial\Omega)}^2 + \epsilon \left\| \frac{p_i - p_{i-1}}{\delta} \right\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \left\| \frac{p_i - p_{i-1}}{\delta} \right\|_{L^1(\Omega; \mathbb{R}^n)} \right) \\
& \leq \sum_{i=2}^j \delta \left(\|f_{i-1}\|_{L^2(\Omega)} \left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\Omega)} + \|g_\epsilon\|_{L^2(\partial\Omega)} \left\| \frac{u_i - u_{i-1}}{\delta} \right\|_{L^2(\partial\Omega)} \right) + 2\epsilon\delta \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\delta}{\lambda} \|v_0\|_{L^2(\partial\Omega)}^2 \\
& \quad + \frac{1}{2} \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \delta \|\nabla v_0\|_{L^1(\Omega; \mathbb{R}^n)} + \|v_0\|_{L^2(\Omega)}^2, \quad (3.4.15)
\end{aligned}$$

which means that for $t \in (t_{j-1}, t_j]$, we have

$$\begin{aligned}
& \frac{1}{2} \left(\|\partial_t \hat{u}\|_{L^2(\Omega)}^2(t) + \|\bar{\sigma}\|_{L^2(\Omega; \mathbb{R}^n)}^2(t) \right) + \epsilon \int_0^{t_j} \|\nabla(\partial_t \hat{u})\|_{L^2(\Omega; \mathbb{R}^n)}^2(s) ds + \frac{1}{\lambda} \int_0^{t_j} \|\partial_t \hat{u}\|_{L^2(\partial\Omega)}^2(s) ds + \epsilon \int_0^{t_j} \|\partial_t \hat{p}\|_{L^2(\Omega; \mathbb{R}^n)}^2(s) ds \\
& + \int_0^{t_j} \|\partial_t \hat{p}\|_{L^1(\Omega; \mathbb{R}^n)}^2(s) ds \leq \|\bar{f}\|_{L^1((0, T); L^2(\Omega))} \|\partial_t \hat{u}\|_{L^\infty((0, T); L^2(\Omega))} + \sqrt{T} \|g_\epsilon\|_{L^2(\partial\Omega)} \|\partial_t \hat{u}\|_{L^2((0, T) \times \partial\Omega)} + \|v_0\|_{L^2(\Omega)}^2 \\
& + 2\epsilon\delta \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\delta}{\lambda} \|v_0\|_{L^2(\partial\Omega)}^2 + \frac{1}{2} \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \delta \|\nabla v_0\|_{L^1(\Omega; \mathbb{R}^n)}.
\end{aligned}$$

It follows that, in particular, there exists a constant (independent of ϵ , δ and λ) such that

$$\begin{aligned}
& \|\partial_t \hat{u}\|_{L^\infty((0, T); L^2(\Omega))} + \|\bar{\sigma}\|_{L^\infty((0, T); L^2(\Omega; \mathbb{R}^n))} + \sqrt{\epsilon} \|\nabla \partial_t \hat{u}\|_{L^2((0, T) \times \Omega; \mathbb{R}^n)} + \frac{1}{\sqrt{\lambda}} \|\partial_t \hat{u}\|_{L^2((0, T); L^2(\partial\Omega))} + \sqrt{\epsilon} \|\partial_t \hat{p}\|_{L^2((0, T) \times \Omega; \mathbb{R}^n)} \\
& \leq C \left(\|v_0\|_{L^2(\Omega)} + \sqrt{\epsilon\delta} \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)} + \sqrt{\frac{\delta}{\lambda}} \|v_0\|_{L^2(\partial\Omega)} + \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)} + \|\bar{f}\|_{L^1((0, T); L^2(\Omega))} + \delta \|\nabla v_0\|_{L^1(\Omega; \mathbb{R}^n)} + \sqrt{T} \|g_\epsilon\|_{L^2(\partial\Omega)} \right). \quad (3.4.16)
\end{aligned}$$

Using the two identities

$$\hat{u}(t, x) = u_0(x) + \int_0^t \partial_t \hat{u}(s, x) ds,$$

and

$$\nabla \hat{u}(t, x) = \nabla u_0(x) + \int_0^t \nabla \partial_t \hat{u}(s, x) ds,$$

we obtain from the previous bound that there exists a constant C_ϵ (depending on ϵ) such that

$$\|\hat{u}\|_{H^1((0, T); H^1(\Omega))} \leq C_\epsilon. \quad (3.4.17)$$

Using the fact that $\nabla \partial_t \hat{u} = \partial_t \hat{\sigma} + \partial_t \hat{p}$, there exists a constant C_ϵ (depending on ϵ) such that

$$\|\partial_t \hat{\sigma}\|_{L^2((0, T) \times \Omega; \mathbb{R}^n)} \leq C_\epsilon.$$

Finally, we know that for a.e. $t \in [0, T]$

$$\|\partial_{tt} \hat{u}(t)\|_{(H^1(\Omega))'} = \|\operatorname{div}_\lambda(\partial_t \hat{u}(t), \bar{\sigma}(t)) + \bar{f}(t)\|_{(H^1(\Omega))'} \leq C_\epsilon \left(\|\partial_t \hat{u}(t)\|_{H^1(\Omega)} + \|\bar{\sigma}(t)\|_{L^2(\Omega; \mathbb{R}^n)} + \|\bar{f}(t)\|_{L^2(\Omega)} \right). \quad (3.4.18)$$

3.4.1.4 Weak convergence

In this subsection, we deduce from the *a priori* estimates that for $\epsilon > 0$ fixed, the following weak convergences for the three interpolations.

Piece-wise affine interpolation One can find subsequences of \hat{u} , $\hat{\sigma}$ and \hat{p} (not relabeled) and functions

$$u_\epsilon \in H^1((0, T); H^1(\Omega)) \cap W^{1,\infty}((0, T); L^2(\Omega)), \quad \sigma_\epsilon \in H^1((0, T); L^2(\Omega; \mathbb{R}^n)) \quad \text{and} \quad p_\epsilon \in H^1((0, T); L^2(\Omega; \mathbb{R}^n))$$

such that

$$\hat{u} \rightharpoonup_N u_\epsilon \text{ in } H^1((0, T); H^1(\Omega)), \quad \partial_t \hat{u} \overset{*}{\rightharpoonup} \partial_t u_\epsilon \text{ in } L^\infty((0, T); L^2(\Omega)) \quad \text{and} \quad \hat{u} \rightharpoonup_N u_\epsilon \text{ in } H^1((0, T); H^{1/2}(\partial\Omega)).$$

We also have

$$\hat{\sigma} \rightharpoonup_N \sigma_\epsilon \text{ in } H^1((0, T); L^2(\Omega; \mathbb{R}^n)) \quad \text{and} \quad \hat{p} \rightharpoonup_N p_\epsilon \text{ in } H^1((0, T); L^2(\Omega; \mathbb{R}^n)).$$

From the discrete boundary condition, we get that $\langle \bar{\sigma}; \nu \rangle$ weakly converges in $H^{-1/2}(\partial\Omega)$. Now, we want to emphasize that, using for example [31, Lemma 2], we have for every $t \in [0, T]$

$$\hat{u}(t) \rightharpoonup_N u_\epsilon(t) \text{ in } H^1(\Omega), \quad \hat{\sigma}(t) \rightharpoonup_N \sigma_\epsilon(t) \text{ in } L^2(\Omega; \mathbb{R}^n), \quad \hat{p}(t) \rightharpoonup_N p_\epsilon(t) \text{ in } L^2(\Omega; \mathbb{R}^n),$$

and in particular since $(\hat{u}, \hat{\sigma}, \hat{p})(t=0) = (u_0, \sigma_0, p_0)$ for every N , we get that $(u_\epsilon(0), \sigma_\epsilon(0), p_\epsilon(0)) = (u_0, \sigma_0, p_0)$ in $H^1(\Omega) \times L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)$. Also since $\nabla \hat{u}(t) = \hat{\sigma}(t) + \hat{p}(t)$ for every t , we get

$$\nabla u_\epsilon(t) = \sigma_\epsilon(t) + p_\epsilon(t), \quad \text{in } L^2(\Omega; \mathbb{R}^n). \quad (3.4.19)$$

Piecewise constant approximations From the following inequalities,

$$\|\hat{u}(t) - \bar{u}(t)\|_{H^1(\Omega)} \leq 2\delta \|\partial_t \hat{u}\|_{H^1(\Omega)}, \quad \|\hat{\sigma}(t) - \bar{\sigma}(t)\|_{L^2(\Omega; \mathbb{R}^n)} \leq 2\delta \|\partial_t \hat{\sigma}\|_{L^2(\Omega; \mathbb{R}^n)}, \quad \|\hat{p}(t) - \bar{p}(t)\|_{L^2(\Omega; \mathbb{R}^n)} \leq 2\delta \|\partial_t \hat{p}\|_{L^2(\Omega; \mathbb{R}^n)},$$

we deduce that

$$\bar{u} \rightharpoonup_N u_\epsilon \text{ in } L^2((0, T); H^1(\Omega)), \quad \bar{\sigma} \overset{*}{\rightharpoonup} \sigma_\epsilon \text{ in } L^\infty((0, T); L^2(\Omega; \mathbb{R}^n)) \quad \text{and} \quad \bar{p} \rightharpoonup_N p_\epsilon \text{ in } L^2((0, T) \times \Omega; \mathbb{R}^n).$$

Quadratic approximation Using the definition of the quadratic approximation (3.4.11), one can see that

$$\sup_{t \in [0, T]} \|\bar{u}(t) - \hat{u}(t)\|_{L^2(\Omega)} \leq 4\delta \|\partial_t \hat{u}\|_{L^\infty((0, T); L^2(\Omega))} \quad (3.4.20)$$

Therefore, $\bar{u} \overset{*}{\rightharpoonup} u_\epsilon$ in $L^\infty((0, T); L^2(\Omega))$. Thanks to the estimates (3.4.16) and (3.4.18) and the fact that f is in $H^1((0, T); L^2(\Omega)) \subset AC((0, T); L^2(\Omega))$ (so that $\bar{f} \rightarrow f$ in $L^\infty((0, T); L^2(\Omega))$) we get

$$\partial_{tt} \bar{u} \rightharpoonup_N \partial_{tt} u_\epsilon, \quad \text{in } L^2((0, T); (H^1(\Omega))').$$

By definition of div_λ and the weak convergences $\partial_t \hat{u} \rightharpoonup_N \partial_t u_\epsilon$ in $L^2((0, T); H^1(\Omega))$ and $\bar{\sigma} \overset{*}{\rightharpoonup} \sigma_\epsilon$ in $L^\infty((0, T); L^2(\Omega; \mathbb{R}^n))$, one gets the equation of the motion in $L^2((0, T); (H^1(\Omega))')$

$$\partial_{tt} u_\epsilon - \text{div}_\lambda(\partial_t u_\epsilon, \sigma_\epsilon) = f, \quad \text{in } L^2((0, T); (H^1(\Omega))'). \quad (3.4.21)$$

We should also observe that for a.e. $t \in [0, T]$ and a.e. $x \in \Omega$

$$\partial_t \bar{u}(t, x) = v_0(x) + \int_0^t \partial_{tt} \bar{u}(s, x) dx.$$

Consequently, the estimate (3.4.18) implies that there exists a constant $C_\epsilon > 0$ such that

$$\|\partial_t \bar{u}\|_{L^2((0, T); (H^1(\Omega))')} \leq C_\epsilon.$$

Hence,

$$\partial_t \bar{u} \rightharpoonup_N \partial_t u_\epsilon, \quad \text{in } L^2((0, T); (H^1(\Omega))'). \quad (3.4.22)$$

Remark 3.4.3. Thanks to classical results about Banach-valued Sobolev space (see [23, Appendix]) and since $u_\epsilon \in$

$H^1((0, T); H^1(\Omega))$ and $\partial_t u_\epsilon \in H^1((0, T); (H^1(\Omega))')$, we obtain

$$u_\epsilon \in C^{0,1/2}([0, T]; H^1(\Omega)) \quad \text{and} \quad \partial_t u_\epsilon \in C^{0,1/2}([0, T]; (H^1(\Omega))').$$

We also know that

$$\forall w \in (H^1(\Omega))', \quad t \mapsto \langle w | u_\epsilon(t) \rangle_{(H^1(\Omega))', H^1(\Omega)} \text{ is continuous,}$$

and

$$\forall w \in H^1(\Omega), \quad t \mapsto \langle \partial_t u_\epsilon(t) | w \rangle_{(H^1(\Omega))', H^1(\Omega)} \text{ is continuous.}$$

3.4.1.5 Initial condition for the velocity

In order to give a sense to the initial condition for the velocity, we first prove that the equation of the motion is true in a weak sense.

Proposition 3.4.4. For all $\varphi \in H^1((0, T); H^1(\Omega))$ such that $\varphi(T, \cdot) = 0$, we have

$$\begin{aligned} - \int_0^T \int_\Omega \partial_t u_\epsilon \partial_t \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla(\partial_t u_\epsilon); \nabla \varphi \rangle \, dx \, dt + \frac{1}{\lambda} \int_0^T \int_{\partial\Omega} \partial_t u_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt \\ = \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_\Omega v_0 \varphi(0) \, dx + \int_0^T \int_{\partial\Omega} g_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt. \end{aligned}$$

Proof. For $\varphi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n)$, we define the interpolation $\bar{\varphi}$ (which converges strongly in $L^2((0, T); H^1(\Omega))$ to φ) and $\hat{\varphi}$ (such that $\partial_t \hat{\varphi} \rightarrow \partial_t \varphi$ in $L^2((0, T) \times \Omega)$). Using the equality (3.4.6) for $v = \varphi(t_{i-1})$ and summing from $i = 2$ to N and using discrete integration by parts for the first term we get

$$\begin{aligned} - \sum_{i=1}^N \delta \int_\Omega \frac{u_i - u_{i-1}}{\delta} \frac{\varphi(t_i) - \varphi(t_{i-1})}{\delta} \, dx + \sum_{i=2}^N \delta \int_\Omega \left\langle \sigma_i + \epsilon \left(\frac{\nabla u_i - \nabla u_{i-1}}{\delta} \right); \nabla \varphi(t_{i-1}) \right\rangle \, dx + \frac{1}{\lambda} \sum_{i=2}^N \delta \int_\Omega \frac{u_i - u_{i-1}}{\delta} \varphi(t_{i-1}) \, dx \\ = \sum_{i=2}^N \delta \int_\Omega f_{i-1} \varphi(t_{i-1}) \, dx + \int_\Omega v_0 \varphi(0) \, dx + \sum_{i=2}^N \delta \int_{\partial\Omega} g_\epsilon \varphi(t_{i-1}) \, d\mathcal{H}^{n-1}, \end{aligned}$$

which can be rewrite as

$$\begin{aligned} - \int_0^T \int_\Omega \partial_t \hat{u} \partial_t \hat{\varphi} \, dx \, dt + \int_\delta^T \int_\Omega \langle \bar{\sigma} + \epsilon \nabla \hat{u}; \nabla \bar{\varphi} \rangle \, dx \, dt + \frac{1}{\lambda} \int_\delta^T \int_{\partial\Omega} \partial_t \hat{u} \bar{\varphi} \, d\mathcal{H}^{n-1} \, dt \\ = \int_\delta^T \int_\Omega \bar{f} \bar{\varphi} \, dx \, dt + \int_\Omega v_0 \varphi(0) \, dx + \int_\delta^T \int_{\partial\Omega} g_\epsilon \bar{\varphi} \, d\mathcal{H}^{n-1} \, dt. \end{aligned}$$

Passing to the limit $N \rightarrow +\infty$ we get

$$\begin{aligned} - \int_0^T \int_\Omega \partial_t u_\epsilon \partial_t \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla u_\epsilon; \nabla \varphi \rangle \, dx \, dt + \frac{1}{\lambda} \int_0^T \int_{\partial\Omega} \partial_t u_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt \\ = \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_\Omega v_0 \varphi(0) \, dx + \int_0^T \int_{\partial\Omega} g_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt. \end{aligned}$$

By density, the previous equality is still true if $\varphi \in H^1((0, T); H^1(\Omega))$ such that $\varphi(T, \cdot) = 0$. \square

We know that

$$\partial_t u_\epsilon \in L^2(0, T, H^1(\Omega)) \quad \text{and} \quad \partial_{tt} u_\epsilon \in L^2(0, T, (H^1(\Omega))').$$

Thanks to a result [11, Theorem 1.19] on the regularity in time of Banach-valued Sobolev spaces (see also [67]), we obtain that $\partial_t u_\epsilon$ is equal almost everywhere (in time) to a function that belongs to $C^0(0, T; L^2(\Omega))$ and furthermore the function $t \mapsto \langle \partial_t u_\epsilon(t) | v(t) \rangle_{L^2(\Omega)}$ is absolutely continuous (for every v such that $v \in L^2(0, T, H^1(\Omega))$) and $\partial_t v \in L^2(0, T, (H^1(\Omega))')$ and we have, a.e in time,

$$\frac{d}{dt} \langle \partial_t u_\epsilon | v \rangle_{L^2(\Omega)}(t) = \langle \partial_{tt} u_\epsilon | v \rangle_{(H^1(\Omega))', H^1(\Omega)}(t) + \langle \partial_t u_\epsilon | \partial_t v \rangle_{(H^1(\Omega))', H^1(\Omega)}(t),$$

Now using (3.4.21) and the last property, we can get the initial condition for the velocity. Take $\varphi = \psi(t)\chi(x)$ where $\psi(t) = \frac{1}{T}(T-t)$ and $\chi \in \mathcal{C}_c^\infty(\Omega)$ as a test function in Proposition (3.4.4), it yields

$$-\int_0^T \int_\Omega \partial_t u_\epsilon \partial_t \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla(\partial_t u_\epsilon); \nabla \varphi \rangle \, dx \, dt = \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_\Omega v_0 \varphi(0) \, dx. \quad (3.4.23)$$

since $\varphi \in L^2(0, T, H_0^1(\Omega))$. Now observe that $\partial_t \varphi \in L^2(0, T, H^1(\Omega))$, [11, Theorem 1.19] implies that

$$-\int_0^T \int_\Omega \partial_t u_\epsilon \partial_t \varphi \, dx \, dt = -\int_0^T \langle \partial_t u_\epsilon | \partial_t \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)}(t) \, dt = \int_0^T \langle \partial_{tt} u_\epsilon | \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)}(t) \, dt + \int_\Omega \partial_t u_\epsilon(0) \varphi(0) \, dx.$$

Then, (3.4.23) rewrites as

$$\int_0^T \langle \partial_{tt} u_\epsilon | \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)}(t) \, dt + \int_\Omega \partial_t u_\epsilon(0) \varphi(0) \, dx + \int_0^T \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla(\partial_t u_\epsilon); \nabla \varphi \rangle \, dx \, dt = \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_\Omega v_0 \varphi(0) \, dx.$$

Then (3.4.21) and the definition of φ provide

$$\int_\Omega \partial_t u_\epsilon(0) \chi \, dx = \int_\Omega v_0 \chi \, dx,$$

which leads us to the initial condition

$$\partial_t u_\epsilon(t=0) = v_0, \quad \text{in } H^2(\Omega).$$

3.4.1.6 Strong convergences and flow rule

We are going to prove that one can pass to the limit in (3.4.13) to obtain the flow rule. In order to be able to do so, we will use the following lemma.

Lemma 3.4.5. *The following strong convergences hold : $\bar{\sigma}$ and $\hat{\sigma}$ strongly converge in $L^2((0, T) \times \Omega; \mathbb{R}^n)$ to σ_ϵ .*

Proof. For every $t \in (0, T]$, we define $[t]_i = t_i$ where $i \in \{1, 2, \dots, N\}$ is such that $t \in (t_{i-1}, t_i]$. Integrating the equation (3.4.12) minus the equation (3.4.21) against $\mathbb{1}_{[\delta, [t]_j]} \partial_t \hat{u} \in L^2((0, T); (H^1(\Omega))')$, we get

$$\begin{aligned} \mathcal{E}_N(t) &:= \int_\delta^{[t]_j} \langle \partial_{tt} \tilde{u} - \partial_{tt} u_\epsilon | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} \, ds - \int_\delta^{[t]_j} \langle \operatorname{div}_\lambda(\partial_t \hat{u}, \bar{\sigma}) - \operatorname{div}_\lambda(\partial_t u_\epsilon, \sigma_\epsilon) | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} \, ds \\ &= \int_\delta^{[t]_j} (\bar{f} - f) \partial_t \hat{u} \, ds. \end{aligned} \quad (3.4.24)$$

The right hand side goes to zero as N goes to infinity since \bar{f} strongly converges to f in $L^1((0, T); L^2(\Omega))$ and $\partial_t \hat{u}$ is bounded in $L^\infty((0, T); L^2(\Omega))$. The quantity $\mathcal{E}_N(t)$ is bounded uniformly (with respect to N) thanks to the estimates (3.4.18) and (3.4.16) and weak convergences. Consequently, the dominated convergence theorem yields

$$\lim_{N \rightarrow +\infty} \int_0^T \mathcal{E}_N(t) \, dt = 0.$$

By definition of $\operatorname{div}_\lambda$ we have

$$\begin{aligned} & - \int_\delta^{[t]_j} \langle \operatorname{div}_\lambda(\partial_t \hat{u}, \bar{\sigma}) - \operatorname{div}_\lambda(\partial_t u_\epsilon, \sigma_\epsilon) | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} \, ds \\ &= \int_\delta^{[t]_j} \int_\Omega \langle \bar{\sigma} - \sigma_\epsilon; \nabla \partial_t \hat{u} \rangle + \epsilon \langle \nabla \partial_t \hat{u} - \nabla \partial_t u_\epsilon; \nabla \partial_t \hat{u} \rangle \, dx \, ds + \int_\delta^{[t]_j} \int_{\partial\Omega} \frac{1}{\lambda} (\partial_t \hat{u} - \partial_t u_\epsilon) \partial_t \hat{u} + g_\epsilon (\partial_t \hat{u} - \partial_t u_\epsilon) \, d\mathcal{H}^{n-1} \, ds. \end{aligned}$$

Remark that, by weak convergences, the quantities

$$\epsilon \int_\delta^{[t]_j} \int_\Omega \langle \nabla \partial_t \hat{u} - \nabla \partial_t u_\epsilon; \nabla \partial_t u_\epsilon \rangle \, dx \, ds + \frac{1}{\lambda} \int_\delta^{[t]_j} \int_{\partial\Omega} (\partial_t \hat{u} - \partial_t u_\epsilon) \partial_t u_\epsilon \, d\mathcal{H}^{n-1} \, ds,$$

and

$$\int_{\delta}^{[t]_j} \int_{\partial\Omega} g_{\epsilon}(\partial_t \hat{u} - \partial_t u_{\epsilon}) d\mathcal{H}^{n-1} ds$$

go to 0 as N goes to $+\infty$. By the estimate (3.4.16), the two previous quantity are uniformly bounded with respect to N . Consequently, the quantity

$$\int_0^T \int_{\delta}^{[t]_j} \left(\langle \partial_{tt} \tilde{u} - \partial_{tt} u_{\epsilon} | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} + \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \nabla \partial_t \hat{u} \rangle + \epsilon |\nabla \partial_t \hat{u} - \nabla \partial_t u_{\epsilon}|^2 dx + \frac{1}{\lambda} \int_{\partial\Omega} |\partial_t \hat{u} - \partial_t u_{\epsilon}|^2 d\mathcal{H}^{n-1} \right) ds dt$$

goes to 0 as N goes to $+\infty$. Hence, we have

$$\begin{aligned} & \liminf_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \langle \partial_{tt} \tilde{u} - \partial_{tt} u_{\epsilon} | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt + \limsup_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \nabla \partial_t \hat{u} \rangle dx ds dt \\ & \leq \limsup_{N \rightarrow +\infty} \int_0^T \left(\int_{\delta}^{[t]_j} \langle \partial_{tt} \tilde{u} - \partial_{tt} u_{\epsilon} | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds + \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \nabla \partial_t \hat{u} \rangle dx ds \right) dt \leq 0. \end{aligned} \quad (3.4.25)$$

First, we are going to show that

$$\limsup_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \nabla \partial_t \hat{u} \rangle dx ds dt \geq \limsup_{N \rightarrow +\infty} \int_0^T \|\bar{\sigma}(t) - \sigma_{\epsilon}(t)\|_{L^2(\Omega)} dt \quad (3.4.26)$$

Since $\nabla \partial_t \hat{u} = \partial_t \hat{p} + \partial_t \hat{\sigma}$, the left-hand side is equal to :

$$\int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \nabla \partial_t \hat{u} \rangle dx ds dt = \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \partial_t \hat{p} \rangle dx ds dt + \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} \rangle dx ds dt.$$

Equation (3.4.13) and classical properties of the projection onto a closed convex set implies that :

$$\int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \partial_t \hat{p} \rangle dx ds dt \geq 0.$$

Consequently, we have

$$\limsup_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \nabla \partial_t \hat{u} \rangle dx ds dt \geq \limsup_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} \rangle dx ds dt.$$

Rewrite

$$\int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} \rangle dx ds,$$

as

$$\int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \hat{\sigma}; \partial_t \hat{\sigma} \rangle + \langle \hat{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} - \sigma_{\epsilon} \rangle + \langle \hat{\sigma} - \sigma_{\epsilon}; \sigma_{\epsilon} \rangle dx ds,$$

then the weak convergences (and the strong convergences of $\bar{\sigma} - \hat{\sigma}$ to 0) implies that

$$\limsup_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \bar{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} \rangle dx ds dt = \limsup_{N \rightarrow +\infty} \int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \hat{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} - \sigma_{\epsilon} \rangle dx ds dt.$$

Now integrating by parts yields

$$\int_0^T \int_{\delta}^{[t]_j} \int_{\Omega} \langle \hat{\sigma} - \sigma_{\epsilon}; \partial_t \hat{\sigma} - \partial_t \sigma_{\epsilon} \rangle dx ds dt = \frac{1}{2} \left(\int_0^T \|\hat{\sigma} - \sigma_{\epsilon}\|_{L^2(\Omega; \mathbb{R}^n)}^2([t]_j) dt - \int_0^T \|\hat{\sigma} - \sigma_{\epsilon}\|_{L^2(\Omega; \mathbb{R}^n)}^2(\delta) dt \right).$$

Thanks to the definition of $\hat{\sigma}$ and since $\sigma_{\epsilon} \in C^{0,1/2}([0, T]; L^2(\Omega; \mathbb{R}^n))$,

$$\int_0^T \|\hat{\sigma} - \sigma_{\epsilon}\|_{L^2(\Omega; \mathbb{R}^n)}^2(\delta) dt = \int_0^T \|\sigma_0 - \sigma_{\epsilon}(\delta)\|_{L^2(\Omega; \mathbb{R}^n)}^2 dt = \int_0^T \|\sigma_{\epsilon}(0) - \sigma_{\epsilon}(\delta)\|_{L^2(\Omega; \mathbb{R}^n)}^2 dt \xrightarrow{N \rightarrow +\infty} 0.$$

and also that

$$\limsup_{N \rightarrow +\infty} \int_0^T \|\hat{\sigma} - \sigma_\epsilon\|_{L^2(\Omega; \mathbb{R}^n)}([t]_j) dt = \limsup_{N \rightarrow +\infty} \int_0^T \|\bar{\sigma}(t) - \sigma_\epsilon([t]_j)\|_{L^2(\Omega; \mathbb{R}^n)} dt = \limsup_{N \rightarrow +\infty} \int_0^T \|\bar{\sigma}(t) - \sigma_\epsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)} dt.$$

The same reasoning leads to

$$\begin{aligned} & \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} \tilde{u} - \partial_{tt} u_\epsilon | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt \\ &= \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} \tilde{u} | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt - \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} u_\epsilon | \partial_t u_\epsilon \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt \\ & \quad + \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} u_\epsilon | \partial_t u_\epsilon - \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt, \end{aligned}$$

and that

$$\lim_{N \rightarrow +\infty} \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} u_\epsilon | \partial_t u_\epsilon - \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt = 0.$$

Again, thanks to a discrete integration by parts, we have

$$\int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} \tilde{u} | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt \geq \frac{1}{2} \int_0^T (\|\partial_t \hat{u}\|_{L^2(\Omega)}^2(t) - \|v_0\|_{L^2(\Omega)}^2) dt.$$

Taking the lower limit we get (thanks to the weak convergence of $\partial_t \hat{u}$ to $\partial_t u_\epsilon$) that

$$\begin{aligned} \liminf_{N \rightarrow +\infty} \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} \tilde{u} | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt &\geq \frac{1}{2} \liminf_{N \rightarrow +\infty} \int_0^T (\|\partial_t \hat{u}\|_{L^2(\Omega)}^2(t) - \|v_0\|_{L^2(\Omega)}^2) dt \\ &\geq \frac{1}{2} \int_0^T (\|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(t) - \|v_0\|_{L^2(\Omega)}^2) dt. \end{aligned} \quad (3.4.27)$$

Thanks to Theorem 1.19 in [11],

$$\int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} u_\epsilon | \partial_t u_\epsilon \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt = \frac{1}{2} \int_0^T (\|\partial_t u_\epsilon\|_{L^2(\Omega)}^2([t]_j) - \|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(\delta)) dt.$$

Using the fact that $\partial_t u_\epsilon \in C^0([0, T]; L^2(\Omega))$, we have (in particular)

$$\begin{aligned} \lim_{N \rightarrow +\infty} \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} u_\epsilon | \partial_t u_\epsilon \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt &= \lim_{N \rightarrow +\infty} \frac{1}{2} \int_0^T (\|\partial_t u_\epsilon\|_{L^2(\Omega)}^2([t]_j) - \|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(\delta)) dt \\ &= \frac{1}{2} \int_0^T (\|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(t) - \|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(0)) dt. \end{aligned} \quad (3.4.28)$$

It implies that

$$\liminf_{N \rightarrow +\infty} \int_0^T \int_\delta^{[t]_j} \langle \partial_{tt} \tilde{u} - \partial_{tt} u_\epsilon | \partial_t \hat{u} \rangle_{(H^1(\Omega))', H^1(\Omega)} ds dt \geq 0. \quad (3.4.29)$$

Gathering (3.4.25), (3.4.26) and (3.4.29), we obtain that $\bar{\sigma}$ (and then $\hat{\sigma}$ thanks to the fact that $\bar{\sigma} - \hat{\sigma} \rightarrow 0$ in $L^2((0, T) \times \Omega; \mathbb{R}^n)$) strongly converges in $L^2((0, T) \times \Omega; \mathbb{R}^n)$ to σ_ϵ . \square

Thanks to the previous lemma, we also have

$$\partial_t p_\epsilon = -\frac{P_B(\sigma_\epsilon) - \sigma_\epsilon}{\epsilon}, \quad \text{in } L^2((0, T) \times \Omega; \mathbb{R}^n), \quad (3.4.30)$$

using the equation (3.4.13) and the fact that $\partial_t \hat{p}$ weakly converges to $\partial_t p_\epsilon$ in $L^2((0, T) \times \Omega; \mathbb{R}^n)$.

3.4.1.7 Uniqueness

Until now, we have established the existence of weak solutions for the dynamical visco-plastic model. Even if we have proved yet neither the full regularity of the solution nor the energy balance law, let us examine the uniqueness

of the solution. Consider two solutions $(u_\epsilon, \sigma_\epsilon, p_\epsilon)$ and $(v_\epsilon, \tau_\epsilon, q_\epsilon)$ associated with the same initial condition and source term. Subtracting the equations of motion (3.4.21), we get

$$\partial_{tt}u_\epsilon - \partial_{tt}v_\epsilon - (\operatorname{div}_\lambda(\partial_t u_\epsilon, \sigma_\epsilon) - \operatorname{div}_\lambda(\partial_t v_\epsilon, \tau_\epsilon)) = 0, \quad \text{in } L^2((0, T); (H^1(\Omega))'),$$

since $\partial_t u_\epsilon - \partial_t v_\epsilon \in L^2((0, T); H^1(\Omega))$, we have in particular that for every $s \in [0, T]$

$$\int_0^s \langle \partial_{tt}u_\epsilon - \partial_{tt}v_\epsilon | \partial_t u_\epsilon - \partial_t v_\epsilon \rangle_{(H^1(\Omega))', H^1(\Omega)} dt - \int_0^s \langle \operatorname{div}_\lambda(\partial_t u_\epsilon | \sigma_\epsilon) - \operatorname{div}_\lambda(\partial_t v_\epsilon, \tau_\epsilon), \partial_t u_\epsilon - \partial_t v_\epsilon \rangle_{(H^1(\Omega))', H^1(\Omega)} dt = 0.$$

Using the definition of $\operatorname{div}_\lambda$, the flow rule (3.4.30) and integration by parts, it implies that

$$\begin{aligned} & \frac{1}{2} \left(\|\partial_t u_\epsilon - \partial_t v_\epsilon\|_{L^2(\Omega)}^2(s) + \|\sigma_\epsilon - \tau_\epsilon\|_{L^2(\Omega; \mathbb{R}^n)}^2(s) \right) + \epsilon \int_0^s \int_\Omega |\nabla(\partial_t u_\epsilon - \partial_t v_\epsilon)|^2 dx dt \\ & + \frac{1}{\lambda} \int_0^s \int_{\partial\Omega} |\partial_t u_\epsilon - \partial_t v_\epsilon|^2 d\mathcal{H}^{n-1} dt + \frac{1}{\epsilon} \int_0^s \int_\Omega \langle \sigma_\epsilon - \tau_\epsilon; \sigma_\epsilon - \tau_\epsilon - (P_{\mathbb{B}}(\sigma_\epsilon) - P_{\mathbb{B}}(\tau_\epsilon)) \rangle dx dt = 0. \end{aligned}$$

Observing that the last term is non-negative, we deduce that $\partial_t u_\epsilon = \partial_t v_\epsilon$ and $\sigma_\epsilon = \tau_\epsilon$. By hypothesis $u_\epsilon(t=0) = v_\epsilon(t=0)$, it implies that $u_\epsilon = v_\epsilon$ and also $p_\epsilon = q_\epsilon$.

Remark 3.4.6. *The uniqueness of the solution shows that there is no need to extract sub-sequences in all weak and strong convergences obtained before.*

3.4.1.8 Energy balance

We integrate the equation (3.4.21) against the function $(s, x) \mapsto \mathbb{1}_{[0, t]}(s) \partial_t u_\epsilon(s, x)$ and it yields

$$\begin{aligned} & \int_0^t \langle \partial_{tt}u_\epsilon | \partial_t u_\epsilon \rangle_{(H^1(\Omega))', H^1(\Omega)}(t) ds + \int_0^t \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon; \nabla \partial_t u_\epsilon \rangle dx ds \\ & + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |\partial_t u_\epsilon|^2 d\mathcal{H}^{n-1} ds = \int_0^t \int_\Omega f \partial_t u_\epsilon dx ds + \int_0^t \int_{\partial\Omega} g_\epsilon \partial_t u_\epsilon d\mathcal{H}^{n-1} ds. \end{aligned}$$

Again integration by parts and the relation (3.4.19), we have

$$\begin{aligned} & \frac{1}{2} \left(\|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(t) - \|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(0) \right) + \frac{1}{2} \left(\|\sigma_\epsilon\|_{L^2(\Omega; \mathbb{R}^n)}^2(t) - \|\sigma_\epsilon\|_{L^2(\Omega; \mathbb{R}^n)}^2(0) \right) + \epsilon \int_0^t \int_\Omega |\nabla \partial_t u_\epsilon|^2 dx ds \\ & + \int_0^t \int_\Omega \langle \partial_t p_\epsilon; \sigma_\epsilon \rangle dx ds + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |\partial_t u_\epsilon|^2 d\mathcal{H}^{n-1} ds = \int_0^t \int_\Omega f \partial_t u_\epsilon dx ds + \int_0^t \int_{\partial\Omega} g_\epsilon \partial_t u_\epsilon d\mathcal{H}^{n-1} ds. \end{aligned}$$

Observe that, thanks to (3.2.1) and (3.4.30),

$$\int_0^t \int_\Omega \langle \partial_t p_\epsilon; \sigma_\epsilon \rangle dx ds = \int_0^t \int_\Omega |\partial_t p_\epsilon| dx ds + \epsilon \int_0^t \int_\Omega |\partial_t p_\epsilon|^2 dx ds$$

This implies

$$\begin{aligned} & \frac{1}{2} \left(\|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(t) - \|\partial_t u_\epsilon\|_{L^2(\Omega)}^2(0) \right) + \frac{1}{2} \left(\|\sigma_\epsilon\|_{L^2(\Omega; \mathbb{R}^n)}^2(t) - \|\sigma_\epsilon\|_{L^2(\Omega; \mathbb{R}^n)}^2(0) \right) \\ & + \epsilon \int_0^t \int_\Omega |\nabla \partial_t u_\epsilon|^2 dx ds + \int_0^t \int_\Omega |\partial_t p_\epsilon| dx ds + \epsilon \int_0^t \int_\Omega |\partial_t p_\epsilon|^2 dx ds \\ & + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |\partial_t u_\epsilon|^2 d\mathcal{H}^{n-1} ds = \int_0^t \int_\Omega f \partial_t u_\epsilon dx ds + \int_0^t \int_{\partial\Omega} g_\epsilon \partial_t u_\epsilon d\mathcal{H}^{n-1} ds, \quad (3.4.31) \end{aligned}$$

which is the energy balance law.

3.4.1.9 A posteriori estimates

First we extend, for negative times, the functions u_ϵ , σ_ϵ , p_ϵ and f by

$$\forall s < 0, \quad u_\epsilon(s) = u_0 + sv_0, \quad \sigma_\epsilon(s) = \sigma_0, \quad p_\epsilon(s) = p_0, \quad g_\epsilon(s) = \epsilon \langle \nabla v_0; v \rangle, \quad f(s) = f(0).$$

Let t be an element of $[0, T]$ and $h < t$, we now take $\varphi \in L^2(0, T+h, H^1(\Omega))$ and use the equation (3.4.21)

$$\begin{aligned} & \int_0^T \langle \partial_{tt} u_\epsilon(s) | \varphi(s) \rangle_{(H^1(\Omega))', H^1(\Omega)} ds + \int_0^T \int_\Omega \langle \sigma_\epsilon(s) + \epsilon \nabla \partial_t u_\epsilon(s); \nabla \varphi(s) \rangle dx ds \\ & \quad + \frac{1}{\lambda} \int_0^T \int_{\partial\Omega} \partial_t u_\epsilon(s) \varphi(s) d\mathcal{H}^{n-1} ds = \int_0^T \int_\Omega f(s) \varphi(s) dx ds + \int_0^T \int_{\partial\Omega} g_\epsilon \varphi(s) d\mathcal{H}^{n-1} ds, \end{aligned}$$

and at the time $s-h$

$$\begin{aligned} & \int_h^{T+h} \langle \partial_{tt} u_\epsilon(s-h) | \varphi(s) \rangle_{(H^1(\Omega))', H^1(\Omega)} ds + \int_h^{T+h} \int_\Omega \langle \sigma_\epsilon(s-h) + \epsilon \nabla \partial_t u_\epsilon(s-h); \nabla \varphi(s) \rangle dx ds \\ & \quad + \frac{1}{\lambda} \int_h^{T+h} \int_{\partial\Omega} \partial_t u_\epsilon(s-h) \varphi(s) d\mathcal{H}^{n-1} ds = \int_h^{T+h} \int_\Omega f(s-h) \varphi(s) dx ds + \int_h^{T+h} \int_{\partial\Omega} g_\epsilon \varphi(s) d\mathcal{H}^{n-1} ds. \end{aligned}$$

Now we take $\varphi(s, x) = \mathbb{1}_{[0, t]}(s) \frac{\partial_t u_\epsilon(s) - \partial_t u_\epsilon(s-h)}{h} =: \mathbb{1}_{[0, t]}(s) s^h \partial_t u_\epsilon(s)$ and we make the difference between the previous equations to obtain, using the extension for the negative times, that

$$\begin{aligned} & \int_0^t \langle s^h \partial_{tt} u_\epsilon(s) | s^h \partial_t u_\epsilon(s) \rangle_{(H^1(\Omega))', H^1(\Omega)} ds + \int_0^t \int_\Omega \langle s^h (\sigma_\epsilon + \epsilon \nabla (\partial_t u_\epsilon))(s); \nabla s^h \partial_t u_\epsilon(s) \rangle dx ds \\ & \quad + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |s^h \partial_t u_\epsilon(s)|^2 d\mathcal{H}^{n-1} ds + \frac{1}{h} \int_0^h \int_{\partial\Omega} \left(\langle \sigma_0 + \epsilon \nabla v_0; v \rangle + \frac{1}{\lambda} v_0 - g_\epsilon \right) s^h \partial_t u_\epsilon(s) d\mathcal{H}^{n-1} ds \\ & \quad = \frac{1}{h} \int_0^h \int_\Omega (\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)) s^h \partial_t u_\epsilon(s) dx ds + \int_0^t \int_\Omega s^h f(s) s^h \partial_t u_\epsilon(s) dx ds. \end{aligned}$$

Using the definition of g_ϵ and the initial condition, we have

$$\langle \sigma_0 + \epsilon \nabla v_0; v \rangle + \frac{1}{\lambda} v_0 - g_\epsilon = 0.$$

Remark 3.4.7. Here we make a crucial use of the term g_ϵ . Indeed, if the boundary condition was

$$\langle \sigma + \epsilon \nabla v; v \rangle + \frac{1}{\lambda} v = 0,$$

the boundary condition hypothesis

$$\langle \sigma_0; v \rangle + \frac{1}{\lambda} v_0 = 0$$

would have led us to the equality

$$\begin{aligned} & \int_0^t \langle s^h \partial_{tt} u_\epsilon(s) | s^h \partial_t u_\epsilon(s) \rangle_{(H^1(\Omega))', H^1(\Omega)} ds + \int_0^t \int_\Omega \langle s^h (\sigma_\epsilon + \epsilon \nabla (\partial_t u_\epsilon))(s); \nabla s^h \partial_t u_\epsilon(s) \rangle dx ds \\ & \quad + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |s^h \partial_t u_\epsilon(s)|^2 d\mathcal{H}^{n-1} ds + \frac{1}{h} \int_0^h \int_{\partial\Omega} (\epsilon \langle \nabla v_0; v \rangle) s^h \partial_t u_\epsilon(s) d\mathcal{H}^{n-1} ds \\ & \quad = \frac{1}{h} \int_0^h \int_\Omega (\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)) s^h \partial_t u_\epsilon(s) dx ds + \int_0^t \int_\Omega s^h f(s) s^h \partial_t u_\epsilon(s) dx ds. \end{aligned}$$

Then, to get an a posteriori estimates of the boundary term, we would have needed to control the $L^2((0, t); L^2(\partial\Omega))$ -norm of $s^h \partial_t u_\epsilon$ thanks to the term

$$\frac{1}{h} \int_0^h \int_{\partial\Omega} (\epsilon \nabla \langle v_0; v \rangle) s^h \partial_t u_\epsilon(s) d\mathcal{H}^{n-1} ds.$$

But the factor $1/h$ forces us to bound the previous term thanks to the $L^\infty((0, t); L^2(\partial\Omega))$ -norm of $s^h \partial_t u_\epsilon$ and thus, we can not get an estimate of $s^h \partial_t u_\epsilon$ in terms of $L^2((0, t); L^2(\partial\Omega))$ -norm.

We rewrite the previous equation as

$$\begin{aligned} & \int_0^t \left\langle s^h \partial_{tt} u_\epsilon(s) \mid s^h \partial_t u_\epsilon(s) \right\rangle_{(H^1(\Omega))', H^1(\Omega)} ds + \int_0^t \int_\Omega \left\langle s^h (\sigma_\epsilon + \epsilon \nabla(\partial_t u_\epsilon))(s); \nabla s^h \partial_t u_\epsilon(s) \right\rangle dx ds \\ & + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |s^h \partial_t u_\epsilon(s)|^2 d\mathcal{H}^{n-1} ds = \frac{1}{h} \int_0^t \int_\Omega (\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)) s^h \partial_t u_\epsilon(s) dx ds + \int_0^t \int_\Omega s^h f(s) s^h \partial_t u_\epsilon(s) dx ds. \end{aligned} \quad (3.4.32)$$

Using the relation $\nabla \partial_t u_\epsilon = \partial_t \sigma_\epsilon + \partial_t p_\epsilon$ and the extension for negative times

$$\begin{aligned} \int_0^t \int_\Omega \left\langle \nabla s^h \partial_t u_\epsilon(s); s^h \sigma_\epsilon(s) \right\rangle dx ds &= \int_0^t \int_\Omega \left\langle s^h \partial_t \sigma_\epsilon(s); s^h \sigma_\epsilon(s) \right\rangle dx ds + \int_0^t \int_\Omega s^h \left\langle \partial_t p_\epsilon(s); s^h \sigma_\epsilon(s) \right\rangle dx ds \\ &\quad - \frac{1}{h} \int_0^h \int_\Omega \left\langle \nabla v_0; s^h \sigma_\epsilon(s) \right\rangle dx ds. \end{aligned}$$

Using the previous equation and integration by parts, the right-hand side of Equation (3.4.32) satisfies the following inequality :

$$\begin{aligned} & \frac{1}{2} \left(\|s^h \partial_t u_\epsilon(t)\|_{L^2(\Omega)}^2 - \|s^h \partial_t u_\epsilon(0)\|_{L^2(\Omega)}^2 + \|s^h \sigma_\epsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 - \|s^h \sigma_\epsilon(0)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right) + \epsilon \int_0^t \int_\Omega |\nabla s^h \partial_t u_\epsilon(s)|^2 dx ds \\ & + \int_0^t \int_\Omega \left\langle s^h \sigma_\epsilon(s); s^h \partial_t p_\epsilon(s) \right\rangle dx ds - \frac{1}{h} \int_0^h \int_\Omega \left\langle \nabla v_0; s^h \sigma_\epsilon(s) \right\rangle dx ds + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |s^h \partial_t u_\epsilon(s)|^2 d\mathcal{H}^{n-1} ds \\ & \geq \frac{1}{2} \left(\|s^h \partial_t u_\epsilon(t)\|_{L^2(\Omega)}^2 + \|s^h \sigma_\epsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right) + \epsilon \int_0^t \int_\Omega |\nabla s^h \partial_t u_\epsilon(s)|^2 dx ds \\ & \quad - \frac{1}{h} \int_0^h \int_\Omega \left\langle \nabla v_0; s^h \sigma_\epsilon(s) \right\rangle dx ds + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |s^h \partial_t u_\epsilon(s)|^2 d\mathcal{H}^{n-1} ds. \end{aligned}$$

Consequently, the equation (3.4.32) becomes

$$\begin{aligned} & \frac{1}{2} \left(\|s^h \partial_t u_\epsilon(t)\|_{L^2(\Omega)}^2 + \|s^h \sigma_\epsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right) + \epsilon \int_0^t \int_\Omega |\nabla s^h \partial_t u_\epsilon(s)|^2 dx ds + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} |s^h \partial_t u_\epsilon(s)|^2 d\mathcal{H}^{n-1} ds \\ & \leq \frac{1}{h} \int_0^h \int_\Omega (\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)) s^h \partial_t u_\epsilon(s) dx ds + \frac{1}{h} \int_0^h \int_\Omega \left\langle \nabla v_0; s^h \sigma_\epsilon(s) \right\rangle dx ds + \int_0^t \int_\Omega s^h f(s) s^h \partial_t u_\epsilon(s) dx ds \\ & \leq \|\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)\|_{L^2(\Omega)} \sup_{s \in [0, h]} \|s^h \partial_t u_\epsilon(s)\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)} \sup_{s \in [0, h]} \|s^h \sigma_\epsilon(s)\|_{L^2(\Omega; \mathbb{R}^n)} \\ & \quad + \|s_h f\|_{L^1((0, t); L^2(\Omega))} \sup_{s \in [0, t]} \|s^h \partial_t u_\epsilon(s)\|_{L^2(\Omega)}. \end{aligned}$$

Then, there exist a constant C (independent of ϵ and λ)

$$\begin{aligned} & \sup_{s \in [0, T]} \|s^h \partial_t u_\epsilon(s)\|_{L^2(\Omega)} + \sup_{s \in [0, T]} \|s^h \sigma_\epsilon(s)\|_{L^2(\Omega; \mathbb{R}^n)} + \sqrt{\epsilon} \|\nabla s^h \partial_t u_\epsilon\|_{L^2((0, T) \times \Omega; \mathbb{R}^n)} + \frac{1}{\sqrt{\lambda}} \|s^h \partial_t u_\epsilon\|_{L^2((0, T); L^2(\partial\Omega))} \\ & \leq C \left(\|\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)} + \|s_h f\|_{L^1((0, T); L^2(\Omega))} \right). \end{aligned}$$

Letting $h \rightarrow 0$, we get

$$\begin{aligned} & \sup_{s \in [0, T]} \|\partial_{tt} u_\epsilon(s)\|_{L^2(\Omega)} + \sup_{s \in [0, T]} \|\partial_t \sigma_\epsilon(s)\|_{L^2(\Omega; \mathbb{R}^n)} + \sqrt{\epsilon} \|\nabla \partial_{tt} u_\epsilon\|_{L^2((0, T) \times \Omega; \mathbb{R}^n)} + \frac{1}{\lambda} \|\partial_{tt} u_\epsilon\|_{L^2((0, T); L^2(\partial\Omega))} \\ & \leq C \left(\|\operatorname{div}(\sigma_0 + \epsilon \nabla v_0) + f(0)\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)} + \|\partial_t f\|_{L^1((0, T); L^2(\Omega))} \right). \end{aligned}$$

Since $\partial_{tt} u_\epsilon \in L^\infty((0, T); L^2(\Omega))$ and using the fact that $\partial_{tt} u_\epsilon - \operatorname{div}(\sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon) = f$ in $\mathcal{D}'((0, T) \times \Omega)$ (this equality comes

from the equation (3.4.21)), we obtain

$$\operatorname{div}(\sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon) \in L^2((0, T) \times \Omega).$$

Now using the proposition and since $\sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon \in L^2((0, T); H(\operatorname{div}, \Omega))$, we get that, for every $\varphi \in C_c^\infty((0, T); H^1(\Omega))$,

$$\begin{aligned} - \int_0^T \int_\Omega \partial_t u_\epsilon \partial_t \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla(\partial_t u_\epsilon); \nabla \varphi \rangle \, dx \, dt + \frac{1}{\lambda} \int_0^T \int_{\partial\Omega} \partial_t u_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt \\ = \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_0^T \int_{\partial\Omega} g_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt. \end{aligned} \quad (3.4.33)$$

Integration by parts in the last equation gives us

$$\begin{aligned} \int_0^T \int_\Omega \partial_{tt} u_\epsilon \varphi \, dx \, dt - \int_0^T \int_\Omega \operatorname{div}(\sigma_\epsilon + \epsilon \nabla(\partial_t u_\epsilon)) \varphi \, dx \, dt - \int_0^T \int_\Omega f \varphi \, dx \, dt \\ = \int_0^T \left\langle g_\epsilon - \langle \sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon; \nu \rangle - \frac{1}{\lambda} \partial_t u_\epsilon \Big| \varphi \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} \, dt. \end{aligned}$$

Consequently, since $\partial_{tt} u_\epsilon - \operatorname{div}(\sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon) = f$ a.e. in $(0, T) \times \Omega$, we obtain

$$\int_0^T \left\langle g_\epsilon - \langle \sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon; \nu \rangle - \frac{1}{\lambda} \partial_t u_\epsilon \Big| \varphi \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} \, dt = 0.$$

which gives (since $\varphi \in C_c^\infty((0, T); H^1(\Omega))$) by density that

$$\langle \sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon; \nu \rangle + \frac{1}{\lambda} \partial_t u_\epsilon = g_\epsilon, \quad \text{in } L^2((0, T); H^{-1/2}(\partial\Omega)). \quad (3.4.34)$$

Remark 3.4.8. Now that we know that $\partial_{tt} u_\epsilon \in L^\infty((0, T); L^2(\Omega))$, the equation of motion (3.4.33) implies that for every $\varphi \in C_c^\infty((0, T); H^1(\Omega))$ we have

$$\begin{aligned} \int_0^T \int_\Omega \partial_{tt} u_\epsilon \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\epsilon + \epsilon \nabla \partial_t u_\epsilon; \nabla \varphi \rangle \, dx \, dt + \frac{1}{\lambda} \int_0^T \int_{\partial\Omega} \partial_t u_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt \\ = \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_0^T \int_{\partial\Omega} g_\epsilon \varphi \, d\mathcal{H}^{n-1} \, dt. \end{aligned} \quad (3.4.35)$$

3.5 The dynamic elasto-plastic model

The object of this section is to show the well-posedness of the model (3.3.6) and (3.3.16) by letting the viscosity parameter ϵ tend to zero. It turns out that a relaxation phenomenon occurs, and it leads to a modification of the boundary condition which has to accommodate to the stress constraint. Indeed, since it is expected that $|\sigma| \leq 1$, the boundary condition $\langle \sigma; \nu \rangle + \lambda^{-1} \partial_t u = 0$ can only be satisfied at the points of the boundary where $|\partial_t u| \leq \lambda$, while on the part of the boundary where $|\partial_t u| > \lambda$, the velocity has to be truncated by the values $\pm \lambda$. This phenomena is easily explained by looking at the energy balance (3.4.4). In order to pass to the limit in this equality, we must (at least) ensure the sequential lower semi-continuity of the mapping

$$(u, \sigma) \mapsto \int_\Omega |\nabla u - \sigma| \, dx + \frac{1}{2\lambda} \int_{\partial\Omega} |u|^2 \, d\mathcal{H}^{n-1}$$

with respect to a reasonable topology provided by the energy estimates. Unfortunately, this property fails according to the following result (see also [76, 81]).

Proposition 3.5.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Let us define the functional $F : W^{1,1}(\Omega) \times L^2(\Omega; \mathbb{R}^n) \rightarrow [0, +\infty]$ by

$$F(u, \sigma) = \int_\Omega |\nabla u - \sigma| \, dx + \frac{1}{2\lambda} \int_{\partial\Omega} |u|^2 \, d\mathcal{H}^{n-1},$$

with the convention that $F(u, \sigma) = +\infty$ if $u \notin L^2(\partial\Omega)$. Then, the lower semi-continuous envelope of F with respect to the weak* convergence in $BV(\Omega)$ and the strong convergence in $L^2(\Omega; \mathbb{R}^n)$ is given by

$$\bar{F}(u, \sigma) = |Du - \sigma|(\Omega) + \int_{\partial\Omega} \psi_\lambda(u) d\mathcal{H}^{n-1},$$

where $\psi_\lambda : \mathbb{R} \rightarrow [0, +\infty)$ is defined by

$$\psi_\lambda(z) = \begin{cases} \frac{z^2}{2\lambda} & \text{if } |z| \leq \lambda, \\ |z| - \frac{\lambda}{2} & \text{if } |z| \geq \lambda. \end{cases} \quad (3.5.1)$$

Proof. Let us fix $(u, \sigma) \in BV(\Omega) \times L^2(\Omega; \mathbb{R}^n)$.

Step 1 : Lower bound. We must show that for every sequences $(u_k) \subset W^{1,1}(\Omega)$ and $(\sigma_k) \subset L^2(\Omega; \mathbb{R}^n)$ with $u_k \rightharpoonup u$ weakly* in $BV(\Omega)$ and $\sigma_k \rightarrow \sigma$ strongly in $L^2(\Omega; \mathbb{R}^n)$, then

$$\bar{F}(u, \sigma) \leq \liminf_{k \rightarrow \infty} F(u_k, \sigma_k).$$

Since $\psi_\lambda \leq |\cdot|^2/(2\lambda)$, we first observe that

$$\liminf_{k \rightarrow \infty} F(u_k, \sigma_k) \geq \liminf_{k \rightarrow \infty} \bar{F}(u_k, \sigma_k). \quad (3.5.2)$$

Possibly extracting a subsequence, we can assume without loss of generality that

$$\liminf_{k \rightarrow \infty} \bar{F}(u_k, \sigma_k) = \lim_{k \rightarrow \infty} \bar{F}(u_k, \sigma_k) < +\infty,$$

and, up to another subsequence, we can also suppose that $|Du_k - \sigma_k| \rightharpoonup \mu$ weakly* in $\mathcal{M}(\Omega)$ for some non-negative measure $\mu \in \mathcal{M}(\Omega)$.

The argument presented below is very close to that of [81, Proposition 1.2]. Let $\delta > 0$, and $\zeta_\delta \in C_c^\infty(\Omega; [0, 1])$ be a cut-off function such that $\zeta_\delta = 1$ on $A_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\}$ and $|\nabla \zeta_\delta| \leq 2/\delta$ in Ω . We consider the function $w_{\delta,k} := (1 - \zeta_\delta)(u - u_k) \in BV(\Omega)$ which satisfies $w_{\delta,k} = u - u_k$ in a neighborhood of $\partial\Omega$, $w_{\delta,k} = 0$ in A_δ , and $Dw_{\delta,k} = -(u - u_k)\nabla\zeta_\delta + (1 - \zeta_\delta)(Du - Du_k)$ in Ω . According to the trace inequality (3.2.3), we infer that

$$\begin{aligned} \int_{\partial\Omega} |u - u_k| d\mathcal{H}^{n-1} &\leq (1 + \varepsilon)|Dw_{\delta,k}|(\Omega) + c_\varepsilon \int_{\Omega} |w_{\delta,k}| dx \\ &\leq \left(\frac{2(1 + \varepsilon)}{\delta} + c_\varepsilon \right) \int_{\Omega \setminus A_\delta} |u - u_k| dx + (1 + \varepsilon)|Du - Du_k|(\Omega \setminus A_\delta). \end{aligned}$$

Since the function ψ_λ is 1-Lipschitz, we have

$$\int_{\partial\Omega} |\psi_\lambda(u) - \psi_\lambda(u_k)| d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} |u - u_k| d\mathcal{H}^{n-1},$$

while

$$|Du - Du_k|(\Omega \setminus A_\delta) \leq |Du - \sigma|(\Omega \setminus A_\delta) + \int_{\Omega \setminus A_\delta} |\nabla u_k - \sigma_k| dx + \int_{\Omega} |\sigma_k - \sigma| dx.$$

As a consequence,

$$\begin{aligned} \bar{F}(u, \sigma) - \bar{F}(u_k, \sigma_k) &\leq C_{\varepsilon, \delta} \int_{\Omega \setminus A_\delta} |u - u_k| dx + (1 + \varepsilon) \int_{\Omega} |\sigma - \sigma_k| dx \\ &\quad + (1 + \varepsilon)|Du - \sigma|(\Omega \setminus A_\delta) + (1 + \varepsilon)|Du_k - \sigma_k|(\Omega \setminus A_\delta) + |Du - \sigma|(\Omega) - |Du_k - \sigma_k|(\Omega). \end{aligned}$$

Choosing a sequence $(\delta_j)_{j \in \mathbb{N}}$ with $\delta_j \searrow 0^+$ and $\mu(\partial A_{\delta_j}) = 0$ for all $j \in \mathbb{N}$, we get that $|Du_k - \sigma_k|(\Omega \setminus A_{\delta_j}) \rightarrow |Du - \sigma|(\Omega \setminus A_{\delta_j})$ as $k \rightarrow \infty$, and thus

$$\bar{F}(u, \sigma) - \liminf_{k \rightarrow \infty} \bar{F}(u_k, \sigma_k) \leq 2(1 + \varepsilon)|Du - \sigma|(\Omega \setminus A_{\delta_j}).$$

Finally, since A_{δ_j} is increasing to Ω as $j \rightarrow \infty$, we get

$$\bar{F}(u, \sigma) \leq \liminf_{k \rightarrow \infty} \bar{F}(u_k, \sigma_k),$$

which together with (3.5.2) completes the proof of the lower bound.

Step 2 : Upper bound. We show the existence of sequences $(u_k) \subset W^{1,1}(\Omega)$ and $(\sigma_k) \subset L^2(\Omega; \mathbb{R}^n)$ such that $u_k \rightharpoonup u$ weakly* in $BV(\Omega)$, $\sigma_k \rightarrow \sigma$ strongly in $L^2(\Omega; \mathbb{R}^n)$, and

$$\limsup_{k \rightarrow \infty} F(u_k, \sigma_k) \leq \bar{F}(u, \sigma).$$

This proof follows the lines of [20, Lemma 2.1]. Let us denote by $\theta = \max(-\lambda, \min(u, \lambda))$ the truncation of the trace of u on $\partial\Omega$ by the values $\pm\lambda$. Then $\theta \in L^\infty(\partial\Omega)$, and using Gagliardo's extension theorem (3.2.5), for each $k \in \mathbb{N}^*$, one can find a function $w_k \in W^{1,1}(\Omega)$ such that $w_k = \theta - u$ on $\partial\Omega$,

$$\int_{\Omega} |w_k| dx \leq \frac{1}{k} \int_{\partial\Omega} |\theta - u| d\mathcal{H}^{n-1},$$

and

$$\int_{\Omega} |\nabla w_k| dx \leq \left(1 + \frac{1}{k}\right) \int_{\partial\Omega} |\theta - u| d\mathcal{H}^{n-1}.$$

Applying next Proposition 3.2.1, there exists a sequence $(z_k) \subset W^{1,1}(\Omega)$ such that $z_k \rightharpoonup u$ weakly* in $BV(\Omega)$, $|Dz_k - \sigma|(\Omega) \rightarrow |Du - \sigma|(\Omega)$ and $z_k = u$ on $\partial\Omega$ for each k . Setting $\sigma_k \equiv \sigma$ and $u_k := w_k + z_k \in W^{1,1}(\Omega)$, then $u_k = \theta$ on $\partial\Omega$, $u_k \rightharpoonup u$ weakly* in $BV(\Omega)$, and

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k - \sigma| dx \leq |Du - \sigma|(\Omega) + \int_{\partial\Omega} |u - \theta| d\mathcal{H}^{n-1}.$$

Consequently,

$$\begin{aligned} \limsup_{k \rightarrow \infty} F(u_k, \sigma_k) &\leq |Du - \sigma|(\Omega) + \int_{\partial\Omega} |u - \theta| d\mathcal{H}^{n-1} + \frac{1}{2\lambda} \int_{\partial\Omega} |\theta|^2 d\mathcal{H}^{n-1} \\ &= |Du - \sigma|(\Omega) + \int_{\partial\Omega \cap \{u \leq -\lambda\}} (-u - \lambda) d\mathcal{H}^{n-1} + \int_{\partial\Omega \cap \{u \geq \lambda\}} (u - \lambda) d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{2\lambda} \int_{\partial\Omega \cap \{|u| \leq \lambda\}} |u|^2 d\mathcal{H}^{n-1} + \frac{\lambda}{2} \mathcal{H}^{n-1}(\partial\Omega \cap \{|u| > \lambda\}) = \bar{F}(u, \sigma), \end{aligned}$$

which concludes the proof of the upper bound. \square

Remark 3.5.2. In the proof of the lower bound we strongly used the C^1 regularity of the boundary. Indeed [81, Remark 1.3] shows that energy functionals of the form

$$BV(\Omega) \ni u \mapsto |Du|(\Omega) + \int_{\Omega} \psi(u) d\mathcal{H}^{n-1},$$

with $\Omega \subset \mathbb{R}^n$ only Lipschitz, and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ 1-Lipschitz, might fail to be sequentially weakly* lower semi-continuous in $BV(\Omega)$. On the other hand, the use of the C^1 character of the boundary does not seem to be necessary in the proof of the upper bound. Indeed, as observed in [20, Lemma 2.1], Gagliardo's extension result with estimates as in (3.2.5) holds for Lipschitz boundaries as well.

Remark 3.5.3. Let us observe for future use that the function ψ_λ defined in (3.5.1) is convex, 1-Lipschitz, and of class C^1 , with

$$\psi'_\lambda(z) = \begin{cases} -1 & \text{if } z \leq -\lambda, \\ \frac{z}{\lambda} & \text{if } |z| < \lambda, \\ 1 & \text{if } z \geq \lambda. \end{cases}$$

Moreover, its convex conjugate is given, for all $z \in \mathbb{R}$, by

$$\psi_\lambda^*(z) = \frac{\lambda}{2}|z|^2 + I_{[-1,1]}(z), \quad (3.5.3)$$

where $I_{[-1,1]}$ is the indicator function of the interval $[-1, 1]$ which is equal to 0 in $[-1, 1]$, and $+\infty$ outside.

We now state the main result of this section.

Theorem 3.5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary and $\lambda > 0$. Consider a source term $f \in H^1([0, T]; L^2(\Omega))$ and an initial data $(u_0, v_0, \sigma_0, p_0) \in H^1(\Omega) \times H^2(\Omega) \times H(\operatorname{div}, \Omega) \times L^2(\Omega; \mathbb{R}^n)$ such that*

$$\begin{cases} \nabla u_0 = \sigma_0 + p_0 \text{ in } L^2(\Omega; \mathbb{R}^n), \\ \langle \sigma_0; \nu \rangle + \lambda^{-1} v_0 = 0 \quad \mathcal{H}^{n-1} \text{ on } \partial\Omega, \\ |\sigma_0| \leq 1 \text{ a.e. in } \Omega. \end{cases} \quad (3.5.4)$$

Then there exist a unique triple (u, σ, p) with the regularity

$$\begin{cases} u \in W^{2,\infty}([0, T]; L^2(\Omega)) \cap C^{0,1}([0, T]; BV(\Omega)), \\ \sigma \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ p \in C^{0,1}([0, T]; \mathcal{M}(\Omega; \mathbb{R}^n)), \end{cases}$$

which satisfies the following properties :

1. The initial conditions :

$$u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \sigma(0) = \sigma_0, \quad p(0) = p_0;$$

2. The additive decomposition : for all $t \in [0, T]$

$$Du(t) = \sigma(t) + p(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^n);$$

3. The equation of motion :

$$\partial_{tt} u - \operatorname{div} \sigma = f \quad \text{in } L^2(0, T; L^2(\Omega));$$

4. The relaxed boundary condition :

$$\langle \sigma; \nu \rangle + \psi'_\lambda(\partial_t u) = 0 \quad \text{in } L^2(0, T; L^2(\partial\Omega));$$

5. The stress constraint : for all $t \in [0, T]$,

$$|\sigma(t)| \leq 1 \quad \text{a.e. in } \Omega;$$

6. The flow rule : for a.e. $t \in [0, T]$,

$$|\partial_t p(t)| = [\sigma(t) \cdot \partial_t p(t)] \quad \text{in } \mathcal{M}(\Omega).$$

In addition, we have the following energy balance : for all $t \in [0, T]$,

$$\begin{aligned} \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_0^t |\partial_t p(s)|(\Omega) ds + \frac{\lambda}{2} \int_0^t \int_{\partial\Omega} |\langle \sigma; \nu \rangle|^2 d\mathcal{H}^{n-1} ds + \int_0^t \int_{\partial\Omega} \psi_\lambda(\partial_t u) d\mathcal{H}^{n-1} ds \\ = \frac{1}{2} \|v_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_0^t \int_{\Omega} f \partial_t u dx ds. \end{aligned} \quad (3.5.5)$$

Remark 3.5.5. *In the sequel we will refer to the solution given by Theorem 3.5.4 as the variational solution to the elasto-plastic problem associated with the initial data $(u_0, v_0, \sigma_0, p_0)$ and the source term f . Unless otherwise specified, we always assume in the sequel that $f \in H^1([0, T]; L^2(\Omega))$ and $(u_0, v_0, \sigma_0, p_0) \in H^1(\Omega) \times H^2(\Omega) \times H(\operatorname{div}, \Omega) \times L^2(\Omega; \mathbb{R}^n)$ satisfy (3.5.4).*

The rest of this section is devoted to prove Theorem 3.5.4. We consider the unique solution $(u_\varepsilon, \sigma_\varepsilon, p_\varepsilon)$ to the elasto-visco-plastic problem (given by Theorem 3.4.1) associated with the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source terms f and $g_\varepsilon = \varepsilon \langle \nabla v_0; \nu \rangle$. Using the estimates obtained in Theorem 3.4.1, we derive weak convergences which enable one to get the initial conditions, the equation of motion and the stress constraint. We then obtain in Subsection 3.5.2 some strong convergence results despite we did not yet identify the correct boundary condition. Together with the relaxation result Proposition 3.5.1, it allows one to derive a first energy inequality between two

arbitrary times. In Subsection 3.5.3 we show that this inequality is actually an equality which leads to the flow rule in a measure theoretic sense, and the relaxed boundary condition. Eventually, uniqueness of the solution is established in Subsection 3.5.4 as a consequence of a Kato inequality which states a comparison principle between two solutions.

3.5.1 Weak convergences

In the proof of Theorem 3.4.1, we have established the following estimate (see (3.4.5))

$$\begin{aligned} & \sup_{t \in [0, T]} \|\partial_{tt} u_\varepsilon(t)\|_{L^2(\Omega)}^2 + \sup_{t \in [0, T]} \|\partial_t \sigma_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{1}{\lambda} \int_0^T \|\partial_{tt} u_\varepsilon(t)\|_{L^2(\partial\Omega)}^2 dt \\ & \leq C \left(\|\operatorname{div}(\sigma_0 + \varepsilon \nabla v_0) + f(0)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \left(\int_0^T \|\partial_t f(t)\|_{L^2(\Omega; \mathbb{R}^n)} dt \right)^2 \right), \end{aligned} \quad (3.5.6)$$

while the energy balance (3.4.4) gives

$$\begin{aligned} & \sup_{t \in [0, T]} \|\partial_t u_\varepsilon(t)\|_{L^2(\Omega)}^2 + \sup_{t \in [0, T]} \|\sigma_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{1}{\lambda} \int_0^T \|\partial_t u_\varepsilon(t)\|_{L^2(\partial\Omega)}^2 dt \\ & + \varepsilon \int_0^T \|\nabla \partial_t u_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 dt + \varepsilon \int_0^T \|\partial_t p_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 dt + \int_0^T \int_\Omega |\partial_t p_\varepsilon| dx dt \\ & \leq C \left(\|v_0\|_{L^2(\Omega)}^2 + \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \left(\int_0^T \|f(t)\|_{L^2(\Omega; \mathbb{R}^n)} dt \right)^2 + \lambda \varepsilon \|\langle \nabla v_0; \nu \rangle\|_{L^2(\partial\Omega)}^2 \right), \end{aligned} \quad (3.5.7)$$

where the constants $C > 0$ occurring in (3.5.6) and (3.5.7) are independent of ε and λ . Using the fact that the function u_ε belongs to $W^{2,\infty}([0, T]; L^2(\Omega))$ and $u_0 \in H^1(\Omega)$, then

$$u_\varepsilon(t) = u_0 + \int_0^t \partial_t u_\varepsilon(s) ds \quad \text{for all } t \in [0, T],$$

where the integral is intended as a Bochner integral in $L^2(\Omega)$, and we get

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{W^{2,\infty}([0, T]; L^2(\Omega))} < \infty. \quad (3.5.8)$$

Arguing similarly yields

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{H^2([0, T]; L^2(\partial\Omega))} < \infty, \quad (3.5.9)$$

and

$$\sup_{\varepsilon > 0} \|\sigma_\varepsilon\|_{W^{1,\infty}([0, T]; L^2(\Omega))} < \infty. \quad (3.5.10)$$

Thanks to the estimates (3.5.8), (3.5.9) and (3.5.10) we can extract a subsequence (not relabeled), and find functions $u \in W^{2,\infty}([0, T]; L^2(\Omega))$, $\sigma \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$ and $w \in H^2([0, T]; L^2(\partial\Omega))$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly* in } W^{2,\infty}([0, T]; L^2(\Omega)), \\ \sigma_\varepsilon \rightharpoonup \sigma & \text{weakly* in } W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ u_\varepsilon \rightharpoonup w & \text{weakly in } H^2([0, T]; L^2(\partial\Omega)). \end{cases} \quad (3.5.11)$$

Moreover, according to [31, Lemma 2.7], for every $t \in [0, T]$,

$$\begin{cases} u_\varepsilon(t) \rightharpoonup u(t) & \text{weakly in } L^2(\Omega), \\ \partial_t u_\varepsilon(t) \rightharpoonup \partial_t u(t) & \text{weakly in } L^2(\Omega), \\ \sigma_\varepsilon(t) \rightharpoonup \sigma(t) & \text{weakly in } L^2(\Omega; \mathbb{R}^n), \\ u_\varepsilon(t) \rightharpoonup w(t) & \text{weakly in } L^2(\partial\Omega), \\ \partial_t u_\varepsilon(t) \rightharpoonup \partial_t w(t) & \text{weakly in } L^2(\partial\Omega). \end{cases} \quad (3.5.12)$$

In order to derive weak compactness of the sequence of plastic strains, we use the energy balance (3.4.4) between

two arbitrary times $0 \leq t_1 < t_2 \leq T$, which leads to

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\Omega} |\partial_t p_\varepsilon| dx ds &\leq \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u_\varepsilon dx ds + \int_{t_1}^{t_2} \int_{\partial\Omega} g_\varepsilon \partial_t u_\varepsilon d\mathcal{H}^{n-1} ds \\ &\quad + \frac{1}{2} \left(\|\partial_t u_\varepsilon(t_1)\|_{L^2(\Omega)}^2 - \|\partial_t u_\varepsilon(t_2)\|_{L^2(\Omega)}^2 \right) + \frac{1}{2} \left(\|\sigma_\varepsilon(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 - \|\sigma_\varepsilon(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right). \end{aligned} \quad (3.5.13)$$

Using the fact that $f \in L^\infty(0, T; L^2(\Omega))$ and $(\partial_t u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ by (3.5.8), we infer that

$$\left| \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u_\varepsilon dx ds \right| \leq C(t_2 - t_1), \quad (3.5.14)$$

for some constant $C > 0$ independent of ε . Then, using that $g_\varepsilon = \varepsilon \langle \nabla v_0; \nu \rangle \in L^2(\partial\Omega)$ and $(\partial_t u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0, T; L^2(\partial\Omega))$ by (3.5.9),

$$\left| \int_{t_1}^{t_2} \int_{\partial\Omega} g_\varepsilon \partial_t u_\varepsilon d\mathcal{H}^{n-1} ds \right| \leq C(t_2 - t_1), \quad (3.5.15)$$

for some constant $C > 0$ independent of ε . Next, using again (3.5.8) yields

$$\begin{aligned} \left| \|\partial_t u_\varepsilon(t_1)\|_{L^2(\Omega)}^2 - \|\partial_t u_\varepsilon(t_2)\|_{L^2(\Omega)}^2 \right| &= \left| \int_{\Omega} (\partial_t u_\varepsilon(t_1) - \partial_t u_\varepsilon(t_2)) (\partial_t u_\varepsilon(t_1) + \partial_t u_\varepsilon(t_2)) dx \right| \\ &\leq C \|\partial_t u_\varepsilon(t_2) - \partial_t u_\varepsilon(t_1)\|_{L^2(\Omega)} \leq C(t_2 - t_1), \end{aligned} \quad (3.5.16)$$

where again, $C > 0$ is independent of ε . Similarly, using (3.5.10) leads to

$$\left| \|\sigma_\varepsilon(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 - \|\sigma_\varepsilon(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right| \leq C \|\sigma_\varepsilon(t_2) - \sigma_\varepsilon(t_1)\|_{L^2(\Omega; \mathbb{R}^n)} \leq C(t_2 - t_1), \quad (3.5.17)$$

Gathering (3.5.13)–(3.5.17) and using Jensen's inequality yields

$$\int_{\Omega} |p_\varepsilon(t_2) - p_\varepsilon(t_1)| dx \leq C(t_2 - t_1).$$

It is thus possible to apply Ascoli-Arzelà Theorem to get, up to another subsequence independent of time, the existence of $p \in C^{0,1}([0, T]; \mathcal{M}(\Omega; \mathbb{R}^n))$ such that for all $t \in [0, T]$,

$$p_\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega; \mathbb{R}^n). \quad (3.5.18)$$

Using next the decomposition $\nabla u_\varepsilon = p_\varepsilon + \sigma_\varepsilon$ the convergences (3.5.12) and (3.5.18) and the already established regularity properties for (u, σ, p) , we obtain that $u \in C^{0,1}([0, T]; BV(\Omega))$, and for all $t \in [0, T]$,

$$u_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly}^* \text{ in } BV(\Omega). \quad (3.5.19)$$

Remark 3.5.6. Let us stress that, as $u(t) \in BV(\Omega)$ for all $t \in [0, T]$, its trace is well defined as an element of $L^1(\partial\Omega)$. However, the trace mapping from $BV(\Omega)$ to $L^1(\partial\Omega)$ is not sequentially weakly* continuous, and therefore we cannot ensure that $w(t)$ is the trace of $u(t)$.

The initial condition. Since $u_\varepsilon(0) = u_0$, $\partial_t u_\varepsilon(0) = v_0$, $\sigma_\varepsilon(0) = \sigma_0$, $p_\varepsilon(0) = p_0$ for all $\varepsilon > 0$, we obtain that $u(0) = u_0$, $\partial_t u(0) = v_0$, $\sigma(0) = \sigma_0$ and $p(0) = p_0$.

The additive decomposition. Using the additive decomposition (3.4.1) and the weak convergences (3.5.12), (3.5.18) and (3.5.19), we infer that for all $t \in [0, T]$,

$$Du(t) = \sigma(t) + p(t) \text{ in } \mathcal{M}(\Omega; \mathbb{R}^n).$$

The stress constraint. Let $\tau_\varepsilon := P_{\mathbb{B}}(\sigma_\varepsilon)$. Since $\|\tau_\varepsilon\|_{L^\infty((0, T) \times \Omega)} \leq 1$, we can assume, up to another subsequence, that $\tau_\varepsilon \rightharpoonup \tau$ weakly* in $L^\infty((0, T) \times \Omega; \mathbb{R}^n)$ with $\|\tau\|_{L^\infty((0, T) \times \Omega)} \leq 1$. According to estimate (3.5.7) and the flow rule (3.4.3), we get

$$\int_0^T \|\sigma_\varepsilon(t) - \tau_\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 dt \leq C\varepsilon \rightarrow 0,$$

so that $\sigma = \tau$. Consequently, for all $t \in [0, T]$, we have

$$|\sigma(t)| \leq 1 \quad \text{a.e. in } \Omega.$$

The equation of motion. According to (3.5.7), we get that $\varepsilon \nabla \partial_t u_\varepsilon \rightarrow 0$ strongly in $L^2(0, T; L^2(\Omega; \mathbb{R}^n))$. We thus deduce that $\sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon \rightarrow \sigma$ strongly in $L^2(0, T; L^2(\Omega; \mathbb{R}^n))$, and since from the equation of motion at fixed ε , one has $\operatorname{div}(\sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon) = f - \partial_{tt} u_\varepsilon$, we obtain thanks to the estimate (3.5.8) that

$$\sup_{\varepsilon > 0} \|\sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon\|_{L^2(0, T; H(\operatorname{div}, \Omega))} < +\infty.$$

As a consequence $\operatorname{div}(\sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon) \rightarrow \operatorname{div} \sigma$ weakly in $L^2(0, T; L^2(\Omega))$, and

$$\partial_{tt} u - \operatorname{div} \sigma = f \quad \text{in } L^2(0, T; L^2(\Omega)).$$

Note also that $\langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle \rightarrow \langle \sigma; \nu \rangle$ weakly in $L^2(0, T; H^{-1/2}(\partial\Omega))$. Using the boundary condition at fixed ε , we also have $\langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle = g_\varepsilon - \lambda^{-1} \partial_t u_\varepsilon$ which is bounded in $L^2(0, T; L^2(\partial\Omega))$ according to (3.5.9), and consequently,

$$\langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle \rightarrow \langle \sigma; \nu \rangle \quad \text{in } L^2(0, T; L^2(\partial\Omega)). \quad (3.5.20)$$

Remark 3.5.7. *Passing to the limit in (3.4.35), we get for all $\varphi \in L^2(0, T; H^1(\Omega))$,*

$$\int_0^T \int_\Omega \partial_{tt} u \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma; \nabla \varphi \rangle \, dx \, dt + \frac{1}{\lambda} \int_0^T \int_{\partial\Omega} \partial_t w \varphi \, d\mathcal{H}^{n-1} \, dt = \int_0^T \int_\Omega f \varphi \, dx \, dt. \quad (3.5.21)$$

3.5.2 Strong convergences

At this stage, it still remains to prove the boundary condition and the flow rule. To do that, we need to improve some of the weak convergences established above into strong convergences. This is the object of the following result.

Proposition 3.5.8. *The following strong convergences hold :*

$$\begin{cases} \partial_t u_\varepsilon \rightarrow \partial_t u & \text{strongly in } C^0([0, T]; L^2(\Omega)), \\ \sigma_\varepsilon \rightarrow \sigma & \text{strongly in } C^0([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ \sqrt{\varepsilon} \nabla \partial_t u_\varepsilon \rightarrow 0 & \text{strongly in } L^2(0, T; L^2(\Omega; \mathbb{R}^n)), \\ \partial_t u_\varepsilon \rightarrow \partial_t w & \text{strongly in } L^2(0, T; L^2(\partial\Omega)). \end{cases}$$

Proof. Subtracting equations (3.4.35) to (3.5.21), and taking $\varphi := \mathbb{1}_{[0, t]} \partial_t u_\varepsilon \in L^2(0, T; H^1(\Omega))$ where $t \in [0, T]$ as test function, we get

$$\begin{aligned} & \int_0^t \int_\Omega (\partial_{tt} u_\varepsilon - \partial_{tt} u) \partial_t u_\varepsilon \, dx \, ds + \int_0^t \int_\Omega \langle \sigma_\varepsilon - \sigma; \nabla \partial_t u_\varepsilon \rangle \, dx \, ds \\ & + \varepsilon \int_0^t \int_\Omega |\nabla \partial_t u_\varepsilon|^2 \, dx \, ds + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} (\partial_t u_\varepsilon - \partial_t w) \partial_t u_\varepsilon \, d\mathcal{H}^{n-1} \, ds = \int_0^t \int_{\partial\Omega} g_\varepsilon \partial_t u_\varepsilon \, d\mathcal{H}^{n-1} \, ds. \end{aligned}$$

Thanks to the additive decomposition (3.4.1), we have

$$\int_0^t \int_\Omega \langle \sigma_\varepsilon - \sigma; \nabla \partial_t u_\varepsilon \rangle \, dx \, ds = \int_0^t \int_\Omega \langle \sigma_\varepsilon - \sigma; \partial_t p_\varepsilon \rangle \, dx \, ds + \int_0^t \int_\Omega \langle \sigma_\varepsilon - \sigma; \partial_t \sigma_\varepsilon \rangle \, dx \, ds.$$

According to the flow rule (3.4.3) and the fact that for all $t \in [0, T]$, $\|\sigma(t)\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1$, we deduce that

$$\int_0^t \int_\Omega \langle \sigma_\varepsilon - \sigma; \partial_t p_\varepsilon \rangle \, dx \, ds \geq 0,$$

and thus,

$$\begin{aligned} & \int_0^t \int_{\Omega} (\partial_{tt} u_{\varepsilon} - \partial_{tt} u)(\partial_t u_{\varepsilon} - \partial_t u) dx ds + \varepsilon \int_0^t \int_{\Omega} |\nabla \partial_t u_{\varepsilon}|^2 dx ds + \int_0^t \int_{\Omega} \langle \sigma_{\varepsilon} - \sigma; \partial_t \sigma_{\varepsilon} - \partial_t \sigma \rangle dx ds \\ & + \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} (\partial_t u_{\varepsilon} - \partial_t w)^2 d\mathcal{H}^{n-1} ds \leq - \int_0^t \int_{\Omega} (\partial_{tt} u_{\varepsilon} - \partial_{tt} u) \partial_t u dx ds - \int_0^t \int_{\Omega} \langle \sigma_{\varepsilon} - \sigma; \partial_t \sigma \rangle dx ds \\ & - \frac{1}{\lambda} \int_0^t \int_{\partial\Omega} (\partial_t u_{\varepsilon} - \partial_t w) \partial_t w d\mathcal{H}^{n-1} ds + \int_0^t \int_{\partial\Omega} g_{\varepsilon} \partial_t u_{\varepsilon} d\mathcal{H}^{n-1} ds. \end{aligned}$$

The weak convergences (3.5.11) imply that the right hand side of the previous inequality tends to 0 as $\varepsilon \rightarrow 0$. Thus, noticing that

$$\int_0^t \int_{\Omega} (\partial_{tt} u_{\varepsilon} - \partial_{tt} u)(\partial_t u_{\varepsilon} - \partial_t u) dx ds = \frac{1}{2} \|\partial_t u_{\varepsilon}(t) - \partial_t u(t)\|_{L^2(\Omega)}^2,$$

and

$$\int_0^t \int_{\Omega} \langle \sigma_{\varepsilon} - \sigma; \partial_t \sigma_{\varepsilon} - \partial_t \sigma \rangle dx ds = \frac{1}{2} \|\sigma_{\varepsilon}(t) - \sigma(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2,$$

we get the desired strong convergences. \square

An important consequence of the strong convergences is the derivation of an energy inequality between two arbitrary times. The following result makes use of the lower bound inequality established in Proposition 3.5.1.

Proposition 3.5.9. *For every $0 \leq t_1 \leq t_2 \leq T$,*

$$\begin{aligned} & \frac{1}{2} \|\partial_t u(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\lambda}{2} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \sigma; \nu \rangle^2 d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} \int_{\partial\Omega} \psi_{\lambda}(\partial_t u) d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} |\partial_t p(s)|(\Omega) ds \\ & \leq \frac{1}{2} \|\partial_t u(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u dx ds. \end{aligned}$$

In addition, for a.e. $t \in [0, T]$,

$$\frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} + \int_{\partial\Omega} \psi_{\lambda}(\partial_t u(t)) d\mathcal{H}^{n-1} + |\partial_t p(t)|(\Omega) \leq [\sigma(t) \cdot \partial_t p(t)](\Omega) - \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \partial_t u(t) d\mathcal{H}^{n-1}.$$

Proof. Using the energy balance (3.4.4) and the boundary condition, we have for all $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned} & \frac{1}{2} \|\partial_t u_{\varepsilon}(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma_{\varepsilon}(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \\ & + \frac{\lambda}{2} \int_{t_1}^{t_2} \int_{\partial\Omega} |\langle \sigma_{\varepsilon} + \varepsilon \nabla \partial_t u_{\varepsilon}; \nu \rangle - g_{\varepsilon}|^2 d\mathcal{H}^{n-1} ds + \frac{1}{2\lambda} \int_{t_1}^{t_2} \int_{\partial\Omega} |\partial_t u_{\varepsilon}|^2 d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} \int_{\Omega} |\partial_t p_{\varepsilon}| dx ds \\ & \leq \frac{1}{2} \|\partial_t u_{\varepsilon}(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma_{\varepsilon}(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u_{\varepsilon} dx ds + \int_{t_1}^{t_2} \int_{\partial\Omega} g_{\varepsilon} \partial_t u_{\varepsilon} d\mathcal{H}^{n-1} ds. \quad (3.5.22) \end{aligned}$$

By Jensen's inequality, we have

$$\frac{1}{t_2 - t_1} \left(\frac{1}{2\lambda} \int_{t_1}^{t_2} \int_{\partial\Omega} |\partial_t u_{\varepsilon}|^2 d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} \int_{\Omega} |\partial_t p_{\varepsilon}| dx ds \right) \geq \frac{1}{2\lambda} \int_{\partial\Omega} \left(\frac{u_{\varepsilon}(t_2) - u_{\varepsilon}(t_1)}{t_2 - t_1} \right)^2 d\mathcal{H}^{n-1} + \int_{\Omega} \left| \frac{p_{\varepsilon}(t_2) - p_{\varepsilon}(t_1)}{t_2 - t_1} \right| dx,$$

where we used that, as Bochner integrals,

$$\int_{t_1}^{t_2} \partial_t u_{\varepsilon}(s) ds = u_{\varepsilon}(t_2) - u_{\varepsilon}(t_1) \quad \text{in } L^2(\partial\Omega),$$

and

$$\int_{t_1}^{t_2} \partial_t p_{\varepsilon}(s) ds = p_{\varepsilon}(t_2) - p_{\varepsilon}(t_1) \quad \text{in } L^2(\Omega; \mathbb{R}^n).$$

Using (3.5.19) and Proposition 3.5.8, we know that

$$\frac{u_\varepsilon(t_2) - u_\varepsilon(t_1)}{t_2 - t_1} \rightharpoonup \frac{u(t_2) - u(t_1)}{t_2 - t_1} \quad \text{weakly* in } BV(\Omega),$$

and

$$\frac{\sigma_\varepsilon(t_2) - \sigma_\varepsilon(t_1)}{t_2 - t_1} \rightarrow \frac{\sigma(t_2) - \sigma(t_1)}{t_2 - t_1} \quad \text{strongly in } L^2(\Omega; \mathbb{R}^n),$$

from which we deduce, according to Proposition 3.5.1, that

$$\int_{\partial\Omega} \psi_\lambda \left(\frac{u(t_2) - u(t_1)}{t_2 - t_1} \right) d\mathcal{H}^{n-1} + \left| \frac{p(t_2) - p(t_1)}{t_2 - t_1} \right|(\Omega) \leq \liminf_{\varepsilon \rightarrow 0} \left\{ \frac{1}{2\lambda} \int_{\partial\Omega} \left(\frac{u_\varepsilon(t_2) - u_\varepsilon(t_1)}{t_2 - t_1} \right)^2 d\mathcal{H}^{n-1} + \int_{\Omega} \left| \frac{p_\varepsilon(t_2) - p_\varepsilon(t_1)}{t_2 - t_1} \right| dx \right\}.$$

Therefore, thanks to the strong convergences results established in Proposition 3.5.8 together with the weak convergence (3.5.20) of the normal trace, it is possible to pass to the (lower) limit in (3.5.22) to get

$$\begin{aligned} & \frac{1}{2} \frac{\|\partial_t u(t_2)\|_{L^2(\Omega)}^2 - \|\partial_t u(t_1)\|_{L^2(\Omega)}^2}{t_2 - t_1} + \frac{1}{2} \frac{\|\sigma(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 - \|\sigma(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2}{t_2 - t_1} + \frac{\lambda}{2(t_2 - t_1)} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \sigma; \nu \rangle^2 d\mathcal{H}^{n-1} ds \\ & + \int_{\partial\Omega} \psi_\lambda \left(\frac{u(t_2) - u(t_1)}{t_2 - t_1} \right) d\mathcal{H}^{n-1} + \left| \frac{p(t_2) - p(t_1)}{t_2 - t_1} \right|(\Omega) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u dx ds. \end{aligned} \quad (3.5.23)$$

Since $\partial_t u \in W^{1,\infty}([0, T]; L^2(\Omega))$ and $\sigma \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$, using [11, Theorem 1.19], we obtain that functions $t \mapsto \|\partial_t u(t)\|_{L^2(\Omega)}^2$ and $t \mapsto \|\sigma(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2$ are absolutely continuous, and for a.e. $t \in [0, T]$

$$\frac{d}{dt} \|\partial_t u(t)\|_{L^2(\Omega)}^2 = 2 \int_{\Omega} \partial_t u(t) \partial_{tt} u(t) dx, \quad \frac{d}{dt} \|\sigma(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 = 2 \int_{\Omega} \langle \sigma(t); \partial_t \sigma(t) \rangle dx. \quad (3.5.24)$$

In addition, since $u \in C^{0,1}([0, T]; BV(\Omega))$ then according to [30, Theorem 7.1], for a.e. $t \in [0, T]$,

$$\frac{u(s) - u(t)}{s - t} \rightharpoonup \partial_t u(t) \quad \text{weakly* in } BV(\Omega) \text{ as } s \rightarrow t,$$

while the fact that $\sigma \in W^{1,\infty}([0, T], L^2(\Omega; \mathbb{R}^n))$ ensures, according to [23, Corollary A.2], that for a.e. $t \in [0, T]$,

$$\frac{\sigma(s) - \sigma(t)}{s - t} \rightarrow \partial_t \sigma(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^n) \text{ as } s \rightarrow t.$$

Consequently, we can pass to the limit in (3.5.23) as $t_2 \rightarrow t_1 = t$. Applying again Proposition 3.5.1 yields, for a.e. $t \in [0, T]$,

$$\int_{\Omega} \partial_t u(t) \partial_{tt} u(t) dx + \int_{\Omega} \langle \sigma(t); \partial_t \sigma(t) \rangle dx + \frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} + \int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + |\partial_t p(t)|(\Omega) \leq \int_{\Omega} f(t) \partial_t u(t) dx. \quad (3.5.25)$$

On the one hand, using the equation of motion together with the Definition 3.2.2 of duality and the integration by parts formula stated in Proposition 3.2.4 (with $\varphi = 1$), infer that for a.e. $t \in [0, T]$,

$$\frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} + \int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + |\partial_t p(t)|(\Omega) \leq [\sigma(t) \cdot \partial_t p(t)](\Omega) - \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \partial_t u(t) d\mathcal{H}^{n-1}.$$

On the other hand, integrating (3.5.25) between two arbitrary times t_1 and t_2 , and using (3.5.24) yields

$$\begin{aligned} & \frac{1}{2} \|\partial_t u(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\lambda}{2} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \sigma; \nu \rangle^2 d\mathcal{H}^{n-1} ds \\ & + \int_{t_1}^{t_2} \int_{\partial\Omega} \psi_\lambda(\partial_t u) d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} |\partial_t p(s)|(\Omega) ds \leq \frac{1}{2} \|\partial_t u(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u dx ds, \end{aligned}$$

which is the announced energy inequality. \square

3.5.3 Flow rule and boundary condition

We now show that the previous energy inequality is actually an equality, and as a byproduct, we obtain the flow rule together with the boundary condition.

Proposition 3.5.10. *For every $0 \leq t_1 \leq t_2 \leq T$,*

$$\begin{aligned} \frac{1}{2} \|\partial_t u(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\lambda}{2} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \sigma; \nu \rangle^2 d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} \int_{\partial\Omega} \psi_\lambda(\partial_t u) d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} |\partial_t p(s)|(\Omega) ds \\ = \frac{1}{2} \|\partial_t u(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u dx ds. \end{aligned} \quad (3.5.26)$$

In addition,

$$\langle \sigma; \nu \rangle + \psi'_\lambda(\partial_t u) = 0 \quad \text{in } L^2(0, T; L^2(\partial\Omega)), \quad (3.5.27)$$

and, for a.e. $t \in [0, T]$,

$$|\partial_t p(t)| = [\sigma(t) \cdot \partial_t p(t)] \quad \text{in } \mathcal{M}(\Omega).$$

Proof. Deriving the additive decomposition with respect to time yields, for a.e. $t \in [0, T]$,

$$D\partial_t u(t) = \partial_t \sigma(t) + \partial_t p(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^n),$$

where $\partial_t u(t) \in BV(\Omega) \cap L^2(\Omega)$, $\partial_t \sigma(t) \in L^2(\Omega; \mathbb{R}^n)$ and $\partial_t p(t) \in \mathcal{M}(\Omega; \mathbb{R}^n)$. Moreover, since for all $t \in [0, T]$, $\sigma(t) \in H(\text{div}, \Omega)$ and $\|\sigma(t)\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1$, we get from Proposition 3.2.3 that for a.e. $t \in [0, T]$,

$$|\partial_t p(t)| \geq [\sigma(t) \cdot \partial_t p(t)] \quad \text{in } \mathcal{M}(\Omega), \quad (3.5.28)$$

and in particular

$$|\partial_t p(t)|(\Omega) \geq [\sigma(t) \cdot \partial_t p(t)](\Omega). \quad (3.5.29)$$

On the other hand, since $\sigma(t) \in H(\text{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$ with $\|\sigma(t)\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1$, we infer that $\langle \sigma(t); \nu \rangle \in L^\infty(\partial\Omega)$ with $\|\langle \sigma(t); \nu \rangle\|_{L^\infty(\partial\Omega)} \leq 1$. As a consequence, for all $t \in [0, T]$

$$\psi_\lambda(\partial_t u(t)) + \frac{\lambda}{2} \langle \sigma(t); \nu \rangle^2 \geq -\langle \sigma(t); \nu \rangle \partial_t u(t) \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \quad (3.5.30)$$

or still

$$\int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + \frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} \geq - \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \partial_t u(t) d\mathcal{H}^{n-1}. \quad (3.5.31)$$

Summing up (3.5.29) and (3.5.31), using Definition 3.2.2 of duality together with the integration by parts formula given by Proposition 3.2.4 and the equation of motion yields, for a.e. $t \in [0, T]$,

$$|\partial_t p(t)|(\Omega) + \int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + \frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} \geq \int_{\Omega} f(t) \partial_t u(t) dx - \int_{\Omega} \partial_{tt} u(t) \partial_t u(t) dx - \int_{\Omega} \langle \partial_t \sigma(t); \sigma(t) \rangle dx.$$

Thanks to (3.5.24) together with an integration between two arbitrary times $0 \leq t_1 \leq t_2 \leq T$ gives the converse energy inequality

$$\begin{aligned} \frac{1}{2} \|\partial_t u(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_2)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \frac{\lambda}{2} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \sigma; \nu \rangle^2 d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} \int_{\partial\Omega} \psi_\lambda(\partial_t u) d\mathcal{H}^{n-1} ds + \int_{t_1}^{t_2} |\partial_t p(s)|(\Omega) ds \\ \geq \frac{1}{2} \|\partial_t u(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma(t_1)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u dx ds \end{aligned}$$

which gives the energy equality (3.5.26) according to Proposition 3.5.9.

We now derive the energy balance (3.5.26) with respect to time. It follows that for a.e. $t \in [0, T]$,

$$|\partial_t p(t)|(\Omega) + \int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + \frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} = \int_{\Omega} f(t) \partial_t u(t) dx - \int_{\Omega} \partial_{tt} u(t) \partial_t u(t) dx - \int_{\Omega} \langle \partial_t \sigma(t); \sigma(t) \rangle dx,$$

or still, thanks to the equation of motion, Definition 3.2.2 of duality and the integration by parts formula given by

Proposition 3.2.4,

$$|\partial_t p(t)|(\Omega) + \int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + \frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} = [\sigma(t) \cdot \partial_t p(t)](\Omega) - \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \partial_t u(t) d\mathcal{H}^{n-1}.$$

Remembering (3.5.29) and (3.5.31) implies that

$$|\partial_t p(t)|(\Omega) = [\sigma(t) \cdot \partial_t p(t)](\Omega),$$

and

$$\int_{\partial\Omega} \psi_\lambda(\partial_t u(t)) d\mathcal{H}^{n-1} + \frac{\lambda}{2} \int_{\partial\Omega} \langle \sigma(t); \nu \rangle^2 d\mathcal{H}^{n-1} = - \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \partial_t u(t) d\mathcal{H}^{n-1},$$

and by (3.5.28) and (3.5.30),

$$|\partial_t p(t)| = [\sigma(t) \cdot \partial_t p(t)] \quad \text{in } \mathcal{M}(\Omega),$$

as well as

$$\psi_\lambda(\partial_t u(t)) + \frac{\lambda}{2} \langle \sigma(t); \nu \rangle^2 = - \langle \sigma(t); \nu \rangle \partial_t u(t) \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega.$$

By (3.5.3) and standard arguments of convex analysis, this last formula is then equivalent to the boundary condition $\langle \sigma(t); \nu \rangle + \psi'_\lambda(\partial_t u(t)) = 0$ \mathcal{H}^{n-1} -a.e. on $\partial\Omega$. \square

Remark 3.5.11. According to the boundary condition at fixed $\varepsilon > 0$ and the convergence (3.5.20) of the normal stress, we have, as $\varepsilon \rightarrow 0$, $\partial_t u_\varepsilon = \lambda g_\varepsilon - \lambda \langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle \rightarrow -\lambda \langle \sigma; \nu \rangle = \lambda \psi'_\lambda(\partial_t u)$ weakly in $L^2(0, T; L^2(\partial\Omega))$. It enables one to identify the function w (the limit of the trace of u_ε in (3.5.11)) as $\partial_t w = \lambda \psi'_\lambda(\partial_t u)$, and by Proposition 3.5.8,

$$\partial_t u_\varepsilon \rightarrow \lambda \psi'_\lambda(\partial_t u) \quad \text{strongly in } L^2(0, T; L^2(\partial\Omega)). \quad (3.5.32)$$

3.5.4 Uniqueness

In order to establish the uniqueness of the variational solution to the elasto-plastic problem, we derive a general comparison principle between two solutions which, in the context of hyperbolic equations, is known as a *Kato inequality*. This result is much more than needed in order to establish uniqueness. However, it will be useful later to show regularity results for the variational solution of the elasto-plastic problem (see Section 3.6).

Proposition 3.5.12 (Kato inequality). *Let (u, σ, p) (resp. $(\tilde{u}, \tilde{\sigma}, \tilde{p})$) be a variational solution of the elasto-plastic problem associated with the initial condition $(u_0, v_0, \sigma_0, p_0)$ (resp. $(\tilde{u}_0, \tilde{v}_0, \tilde{\sigma}_0, \tilde{p}_0)$) and the source term f (resp. \tilde{f}). Then for all $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$,*

$$\begin{aligned} & \int_0^T \int_\Omega (\partial_t u - \partial_t \tilde{u})^2 \partial_t \varphi dx dt + \int_0^T \int_\Omega |\sigma - \tilde{\sigma}|^2 \partial_t \varphi dx dt + \int_\Omega (v_0 - \tilde{v}_0)^2 \varphi(0) dx + \int_\Omega |\sigma_0 - \tilde{\sigma}_0|^2 \varphi(0) dx \\ & - 2 \int_0^T \int_\Omega \langle \sigma - \tilde{\sigma}; \nabla \varphi \rangle (\partial_t u - \partial_t \tilde{u}) dx dt + 2 \int_0^T \int_\Omega (f - \tilde{f}) (\partial_t u - \partial_t \tilde{u}) \varphi dx dt \geq 0. \end{aligned} \quad (3.5.33)$$

Proof. Let (u, σ, p) (resp. $(\tilde{u}, \tilde{\sigma}, \tilde{p})$) be a variational solution of the elasto-plastic problem associated with the initial condition $(u_0, v_0, \sigma_0, p_0)$ (resp. $(\tilde{u}_0, \tilde{v}_0, \tilde{\sigma}_0, \tilde{p}_0)$) and the source term f (resp. \tilde{f}), and let $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$. Thanks to the equation of motion and since the function $(\partial_t u - \partial_t \tilde{u})\varphi \in L^2(0, T; L^2(\Omega))$, we infer that

$$\int_0^T \int_\Omega \partial_{tt} u (\partial_t u - \partial_t \tilde{u}) \varphi dx dt - \int_0^T \int_\Omega \operatorname{div} \sigma (\partial_t u - \partial_t \tilde{u}) \varphi dx dt = \int_0^T \int_\Omega f (\partial_t u - \partial_t \tilde{u}) \varphi dx dt,$$

and similarly,

$$\int_0^T \int_\Omega \partial_{tt} \tilde{u} (\partial_t u - \partial_t \tilde{u}) \varphi dx dt - \int_0^T \int_\Omega \operatorname{div} \tilde{\sigma} (\partial_t u - \partial_t \tilde{u}) \varphi dx dt = \int_0^T \int_\Omega \tilde{f} (\partial_t u - \partial_t \tilde{u}) \varphi dx dt,$$

Summing up both previous equalities leads to

$$\int_0^T \int_{\Omega} (\partial_{tt}u - \partial_{tt}\tilde{u})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt - \int_0^T \int_{\Omega} (\operatorname{div} \sigma - \operatorname{div} \tilde{\sigma})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt = \int_0^T \int_{\Omega} (f - \tilde{f})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt. \quad (3.5.34)$$

Using Proposition 3.2.4, we get

$$\begin{aligned} - \int_0^T \int_{\Omega} (\operatorname{div} \sigma - \operatorname{div} \tilde{\sigma})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt &= \int_0^T \int_{\Omega} (\partial_t u - \partial_t \tilde{u}) \langle \sigma - \tilde{\sigma}; \nabla \varphi \rangle \, dx \, dt \\ &\quad - \int_0^T \int_{\partial \Omega} (\langle \sigma; \nu \rangle - \langle \tilde{\sigma}; \nu \rangle) (\partial_t u - \partial_t \tilde{u}) \varphi \, d\mathcal{H}^{n-1} \, dt + \int_0^T \int_{\Omega} \varphi \, d[(\sigma(t) - \tilde{\sigma}(t)) \cdot (D\partial_t u(t) - D\partial_t \tilde{u}(t))] \, dt. \end{aligned} \quad (3.5.35)$$

Now, Definition 3.2.2 implies that

$$\begin{aligned} \int_0^T \int_{\Omega} \varphi \, d[(\sigma(t) - \tilde{\sigma}(t)) \cdot (D\partial_t u(t) - D\partial_t \tilde{u}(t))] \, dt \\ = \int_0^T \int_{\Omega} \varphi \, d[(\sigma(t) - \tilde{\sigma}(t)) \cdot (\partial_t p(t) - \partial_t \tilde{p}(t))] \, dt + \int_0^T \int_{\Omega} \varphi \langle \sigma(t) - \tilde{\sigma}(t); \partial_t \sigma(t) - \partial_t \tilde{\sigma}(t) \rangle \, dx \, dt \end{aligned}$$

On the one hand, by the flow rule we have for a.e. $t \in [0, T]$,

$$[\sigma(t) \cdot \partial_t p(t)] = |\partial_t p(t)| \quad \text{and} \quad [\tilde{\sigma}(t) \cdot \partial_t \tilde{p}(t)] = |\partial_t \tilde{p}(t)|,$$

while, the stress constraint and Proposition 3.2.4 imply that for a.e. $t \in [0, T]$

$$[\tilde{\sigma}(t) \cdot \partial_t p(t)] \leq |\partial_t p(t)| \quad \text{and} \quad [\sigma(t) \cdot \partial_t \tilde{p}(t)] \leq |\partial_t \tilde{p}(t)| \quad \text{in } \mathcal{M}(\Omega).$$

Consequently, since φ is non-negative, we have

$$\int_0^T \int_{\Omega} \varphi \, d[(\sigma(t) - \tilde{\sigma}(t)) \cdot (\partial_t p(t) - \partial_t \tilde{p}(t))] \, dt \geq 0.$$

It implies that

$$\begin{aligned} - \int_0^T \int_{\partial \Omega} (\langle \sigma; \nu \rangle - \langle \tilde{\sigma}; \nu \rangle) (\partial_t u - \partial_t \tilde{u}) \varphi \, d\mathcal{H}^{n-1} \, dt &\leq - \int_0^T \int_{\Omega} (\partial_{tt}u - \partial_{tt}\tilde{u})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt \\ - \int_0^T \int_{\Omega} \langle \sigma - \tilde{\sigma}; \partial_t \sigma - \partial_t \tilde{\sigma} \rangle \varphi \, dx \, dt &- \int_0^T \int_{\Omega} \langle \sigma - \tilde{\sigma}; \nabla \varphi \rangle (\partial_t u - \partial_t \tilde{u}) \, dx \, dt + \int_0^T \int_{\Omega} (f - \tilde{f})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt. \end{aligned} \quad (3.5.36)$$

Using next that both functions $\partial_t u - \partial_t \tilde{u} \in W^{1,\infty}([0, T]; L^2(\Omega))$ and $\sigma - \tilde{\sigma} \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$, we can integrate by parts with respect to the time variable to get

$$\begin{aligned} - \int_0^T \int_{\Omega} (\partial_{tt}u - \partial_{tt}\tilde{u})(\partial_t u - \partial_t \tilde{u}) \varphi \, dx \, dt - \int_0^T \int_{\Omega} \langle \sigma - \tilde{\sigma}; \partial_t \sigma - \partial_t \tilde{\sigma} \rangle \varphi \, dx \, dt \\ = \frac{1}{2} \left(\int_0^T \int_{\Omega} (\partial_t u - \partial_t \tilde{u})^2 \partial_t \varphi \, dx \, dt + \int_0^T \int_{\Omega} |\sigma - \tilde{\sigma}|^2 \partial_t \varphi \, dx \, dt + \int_{\Omega} (v_0 - \tilde{v}_0)^2 \varphi(0) \, dx \right. \\ \left. - \int_{\Omega} (\partial_t u(T) - \partial_t \tilde{u}(T))^2 \varphi(T) \, dx + \int_{\Omega} |\sigma_0 - \tilde{\sigma}_0|^2 \varphi(0) \, dx - \int_{\Omega} |\sigma(T) - \tilde{\sigma}(T)|^2 \varphi(T) \, dx \right), \end{aligned}$$

and inserting inside (3.5.36) leads to

$$\begin{aligned} & - \int_0^T \int_{\partial\Omega} (\langle \sigma; \nu \rangle - \langle \bar{\sigma}; \nu \rangle) (\partial_t u - \partial_t \bar{u}) \varphi \, d\mathcal{H}^{n-1} \, dt \leq \frac{1}{2} \int_0^T \int_{\Omega} (\partial_t u - \partial_t \bar{u})^2 \partial_t \varphi \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Omega} |\sigma - \bar{\sigma}|^2 \partial_t \varphi \, dx \, dt \\ & + \frac{1}{2} \left(\int_{\Omega} (v_0 - \bar{v}_0)^2 \varphi(0) \, dx + \int_{\Omega} |\sigma_0 - \bar{\sigma}_0|^2 \varphi(0) \, dx \right) - \int_0^T \int_{\Omega} \langle \sigma - \bar{\sigma}; \nabla \varphi \rangle (\partial_t u - \partial_t \bar{u}) \, dx \, dt + \int_0^T \int_{\Omega} (f - \bar{f}) (\partial_t u - \partial_t \bar{u}) \varphi \, dx \, dt. \end{aligned}$$

Using the boundary condition, the fact that ψ is a convex function and the fact that φ is non-negative, we obtain

$$- \int_0^T \int_{\partial\Omega} (\langle \sigma; \nu \rangle - \langle \bar{\sigma}; \nu \rangle) (\partial_t u - \partial_t \bar{u}) \varphi \, d\mathcal{H}^{n-1} \, dt = \int_0^T \int_{\partial\Omega} (\psi'(\partial_t u) - \psi'(\partial_t \bar{u})) (\partial_t u - \partial_t \bar{u}) \varphi \, d\mathcal{H}^{n-1} \, dt \geq 0,$$

and the inequality (3.5.33) follows. \square

As a consequence of Proposition 3.5.12, if (u_1, σ_1, p_1) and (u_2, σ_2, p_2) are two variational solutions of the elasto-plastic problem given by Theorem 3.5.4 for the same initial data (u_0, v_0, e_0, p_0) and source term f , taking $\varphi(t, x) := T - t$ as test function in (3.5.33) implies that $\sigma_1 = \sigma_2$ and $\partial_t u_1 = \partial_t u_2$. Consequently, since $u_1(0) = u_2(0) = u_0$, we deduce that $u_1 = u_2$, and using the additive decomposition, that $p_1 = p_2$.

Remark 3.5.13. *The uniqueness of the solution shows that there is no need to extract subsequences in all weak and strong convergences obtained before.*

3.5.5 Homogeneous Dirichlet and Neumann boundary conditions

The boundary condition (3.5.27) studied so far does not account for the important Dirichlet and Neumann cases. It is however possible to recover these particular situations by means of asymptotic analysis as the coefficient $\lambda \rightarrow 0^+$ for the Dirichlet, or $\lambda \rightarrow +\infty$ for the Neumann case. The object of this section is to show such rigorous convergence results.

Theorem 3.5.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class \mathcal{C}^1 . Consider a source term $f \in H^1([0, T]; L^2(\Omega))$ and an initial data $(u_0, v_0, \sigma_0, p_0) \in H^1(\Omega) \times H^2(\Omega) \times H(\text{div}, \Omega) \times L^2(\Omega; \mathbb{R}^n)$ such that*

$$\begin{cases} \nabla u_0 = \sigma_0 + p_0 \text{ in } L^2(\Omega; \mathbb{R}^n), \\ |\sigma_0| \leq 1 \text{ a.e. in } \Omega, \end{cases} \quad (3.5.37)$$

and

$$\langle \sigma_0; \nu \rangle = v_0 = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega. \quad (3.5.38)$$

For any $\lambda > 0$, let $(u_\lambda, \sigma_\lambda, p_\lambda)$ the unique variational solution to the elasto-plastic problem given by Theorem 3.5.4. For $\ell \in \{0, +\infty\}$, there exist a unique triple $(u^{(\ell)}, \sigma^{(\ell)}, p^{(\ell)})$ with the regularity

$$\begin{cases} u^{(\ell)} \in W^{2,\infty}([0, T]; L^2(\Omega)) \cap \mathcal{C}^{0,1}([0, T]; BV(\Omega)), \\ \sigma^{(\ell)} \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ p^{(\ell)} \in \mathcal{C}^{0,1}([0, T]; \mathcal{M}(\Omega; \mathbb{R}^n)), \end{cases} \quad (3.5.39)$$

such that, as $\lambda \rightarrow \ell$,

$$\begin{cases} u_\lambda \rightharpoonup u^{(\ell)} & \text{weakly* in } W^{2,\infty}([0, T]; L^2(\Omega)), \\ \sigma_\lambda \rightharpoonup \sigma^{(\ell)} & \text{weakly* in } W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ p_\lambda(t) \rightharpoonup p^{(\ell)}(t) & \text{weakly* in } \mathcal{M}(\Omega; \mathbb{R}^n) \text{ for all } t \in [0, T], \end{cases} \quad (3.5.40)$$

and which satisfies the following properties :

1. The initial conditions :

$$u^{(\ell)}(0) = u_0, \quad \partial_t u^{(\ell)}(0) = v_0, \quad \sigma^{(\ell)}(0) = \sigma_0, \quad p^{(\ell)}(0) = p_0;$$

2. The additive decomposition : for all $t \in [0, T]$

$$Du^{(\ell)}(t) = \sigma^{(\ell)}(t) + p^{(\ell)}(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^n);$$

3. The equation of motion :

$$\partial_{tt}u^{(\ell)} - \operatorname{div}\sigma^{(\ell)} = f \quad \text{in } L^2(0, T; L^2(\Omega));$$

4. The boundary condition :

$$\langle \sigma^{(\infty)}; \nu \rangle = 0 \quad \text{in } L^2(0, T; L^2(\partial\Omega));$$

5. The stress constraint : for all $t \in [0, T]$,

$$|\sigma^{(\ell)}(t)| \leq 1 \quad \text{a.e. in } \Omega;$$

6. The flow rule : for a.e. $t \in [0, T]$,

$$\begin{cases} |\partial_t p^{(0)}(t)| = [\sigma^{(0)}(t) \cdot \partial_t p^{(0)}(t)] & \text{in } \mathcal{M}(\Omega), \\ |\partial_t u^{(0)}(t)| = -\langle \sigma^{(0)}(t); \nu \rangle \partial_t u^{(0)}(t) & \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \end{cases}$$

or

$$|\partial_t p^{(\infty)}(t)| = [\sigma^{(\infty)}(t) \cdot \partial_t p^{(\infty)}(t)] \quad \text{in } \mathcal{M}(\Omega).$$

Remark 3.5.15. Note that in the Dirichlet case ($\ell = 0$), as classical in variational problems with linear growth, the velocity may concentrate on the boundary so that its inner trace might not vanish as required by the boundary condition. It explains why some plastic strain can accumulate on the boundary and that the flow rule is formulated in $\overline{\Omega}$ and not only in Ω .

Remark 3.5.16. The assumption (3.5.38) seems to be artificial, however it allows us to easily satisfy the boundary condition $\langle \sigma_0; \nu \rangle + \lambda^{-1}v_0 = 0$ on $\partial\Omega$ for the initial data for every $\lambda > 0$. It seems more natural to consider initial data that satisfy (3.5.37) and, for example in the Neumann case, only

$$\langle \sigma_0; \nu \rangle = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega. \quad (3.5.41)$$

To do so, one should be able to construct for every $(u_0, v_0, \sigma_0, p_0) \in H^1(\Omega) \times H^2(\Omega) \times H(\operatorname{div}, \Omega) \times L^2(\Omega; \mathbb{R}^n)$, satisfying (3.5.37) and (3.5.41), a sequence $(u_0^\lambda, v_0^\lambda, \sigma_0^\lambda, p_0^\lambda) \in H^1(\Omega) \times H^2(\Omega) \times H(\operatorname{div}, \Omega) \times L^2(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} \nabla u_0^\lambda = \sigma_0^\lambda + p_0^\lambda & \text{in } L^2(\Omega; \mathbb{R}^n), \\ \langle \sigma_0^\lambda; \nu \rangle + \lambda^{-1}v_0^\lambda = 0 & \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \\ |\sigma_0^\lambda| \leq 1 & \text{a.e. in } \Omega, \end{cases}$$

and, at least,

$$\begin{cases} u_0^\lambda \rightharpoonup u_0 & \text{weakly in } L^2(\Omega), \\ v_0^\lambda \rightharpoonup v_0 & \text{weakly in } H^1(\Omega), \\ \sigma_0^\lambda \rightharpoonup \sigma_0 & \text{weakly in } H(\operatorname{div}, \Omega), \\ p_0^\lambda \rightharpoonup p_0 & \text{weakly* in } \mathcal{M}(\Omega; \mathbb{R}^n). \end{cases}$$

This issue will not be addressed in this work and for that reason, we assume (3.5.38) for simplicity.

Proof. The proof follows closely the lines of that of Theorem 3.5.4. It is divided into three steps.

Step 1 : Weak convergences. Passing to the limit as $\varepsilon \rightarrow 0$ in estimates (3.5.6) and (3.5.7) yields

$$\sup_{t \in [0, T]} \|\partial_{tt}u_\lambda(t)\|_{L^2(\Omega)}^2 + \sup_{t \in [0, T]} \|\partial_t \sigma_\lambda(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \leq C \left(\|\operatorname{div}\sigma_0 + f(0)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \left(\int_0^T \|\partial_t f(t)\|_{L^2(\Omega; \mathbb{R}^n)} dt \right)^2 \right),$$

and

$$\begin{aligned} \sup_{t \in [0, T]} \|\partial_t u_\lambda(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \sup_{t \in [0, T]} \|\sigma_\lambda(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \int_0^T \psi_\lambda(\partial_t u_\lambda(t)) dt + \frac{\lambda}{2} \int_0^T \int_{\partial\Omega} \langle \sigma_\lambda; \nu \rangle^2 d\mathcal{H}^{n-1} dt \\ \leq C \left(\|v_0\|_{L^2(\Omega)}^2 + \|\sigma_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \left(\int_0^T \|f(t)\|_{L^2(\Omega; \mathbb{R}^n)} dt \right)^2 \right), \end{aligned} \quad (3.5.42)$$

where the constants $C > 0$ are independent of λ . Using similar arguments than in Subsection 3.5.1, we can find a subsequence (not relabeled) and, for $\ell \in \{0, +\infty\}$, functions $u^{(\ell)}$, $\sigma^{(\ell)}$ and $p^{(\ell)}$ as in (3.5.39) such that the weak convergences (3.5.40) hold. In particular, we can easily derive the initial conditions, the additive decomposition, the stress constraint, and the equation of motion.

Using the equation of motion for fixed $\lambda > 0$, we infer that the sequence $(\sigma_\lambda)_{\lambda > 0}$ is bounded in $L^2(0, T; H(\operatorname{div}, \Omega))$, so that $\langle \sigma_\lambda; \nu \rangle \rightharpoonup \langle \sigma^{(\ell)}; \nu \rangle$ weakly in $L^2(0, T; H^{-1/2}(\partial\Omega))$. On the other hand, according to estimate (3.5.42), we have $\langle \sigma_\lambda; \nu \rangle \rightarrow 0$ strongly in $L^2(0, T; L^2(\partial\Omega))$ as $\lambda \rightarrow +\infty$. Therefore, the Neumann boundary condition $\langle \sigma^{(\infty)}; \nu \rangle = 0$ in $L^2(0, T; L^2(\partial\Omega))$ follows.

Step 2 : Strong convergences. We now show that, as $\lambda \rightarrow \ell$,

$$\begin{cases} \partial_t u_\lambda \rightarrow \partial_t u^{(\ell)} & \text{strongly in } C^0([0, T]; L^2(\Omega)), \\ \sigma_\lambda \rightarrow \sigma^{(\ell)} & \text{strongly in } C^0([0, T]; L^2(\Omega; \mathbb{R}^n)). \end{cases} \quad (3.5.43)$$

To this aim, subtracting the equations of motion, we get

$$\partial_{tt} u_\lambda - \partial_{tt} u^{(\ell)} - \operatorname{div}(\sigma_\lambda - \sigma^{(\ell)}) = 0 \quad \text{in } L^2(0, T; L^2(\Omega)).$$

For $t \in [0, T]$, multiplying by $\mathbb{1}_{[0, t]} \partial_t u_\lambda$, integrating over $(0, T) \times \Omega$, and using Definition 3.2.2 of duality together with the integration by parts formula given by Proposition 3.2.4 yields

$$\begin{aligned} & \int_0^t \int_\Omega (\partial_{tt} u_\lambda - \partial_{tt} u^{(\ell)}) \partial_t u_\lambda \, dx \, ds + \int_0^t \int_\Omega \langle \sigma_\lambda - \sigma^{(\ell)}; \partial_t \sigma_\lambda \rangle \, dx \, ds \\ & + \int_0^t [(\sigma_\lambda(s) - \sigma^{(\ell)}(s)) \cdot \partial_t p_\lambda(s)](\Omega) \, ds - \int_0^t \int_{\partial\Omega} (\langle \sigma_\lambda; \nu \rangle - \langle \sigma^{(\ell)}; \nu \rangle) \partial_t u_\lambda \, d\mathcal{H}^{n-1} \, ds = 0. \end{aligned}$$

According to the flow rule for fixed $\lambda > 0$, the fact that for all $s \in [0, T]$, $\|\sigma^{(\ell)}(s)\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1$ and Proposition 3.2.3, we get $[(\sigma_\lambda(s) - \sigma^{(\ell)}(s)) \cdot \partial_t p_\lambda(s)](\Omega) \geq 0$ for a.e. $s \in [0, T]$. Thus,

$$\begin{aligned} & \int_0^t \int_\Omega (\partial_{tt} u_\lambda - \partial_{tt} u^{(\ell)}) (\partial_t u_\lambda - \partial_t u^{(\ell)}) \, dx \, ds + \int_0^t \int_\Omega \langle \sigma_\lambda - \sigma^{(\ell)}; \partial_t \sigma_\lambda - \partial_t \sigma^{(\ell)} \rangle \, dx \, ds \\ & \leq - \int_0^t \int_\Omega (\partial_{tt} u_\lambda - \partial_{tt} u^{(\ell)}) \partial_t u^{(\ell)} \, dx \, ds - \int_0^t \int_\Omega \langle \sigma_\lambda - \sigma^{(\ell)}; \partial_t \sigma^{(\ell)} \rangle \, dx \, ds + \int_0^t \int_{\partial\Omega} (\langle \sigma_\lambda; \nu \rangle - \langle \sigma^{(\ell)}; \nu \rangle) \partial_t u_\lambda \, d\mathcal{H}^{n-1} \, ds, \end{aligned}$$

and integrating by parts the left hand side with respect to time yields

$$\begin{aligned} & \frac{1}{2} \|\partial_t u_\lambda(t) - \partial_t u^{(\ell)}(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sigma_\lambda(t) - \sigma^{(\ell)}(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \\ & \leq - \int_0^t \int_\Omega (\partial_{tt} u_\lambda - \partial_{tt} u^{(\ell)}) \partial_t u^{(\ell)} \, dx \, ds - \int_0^t \int_\Omega \langle \sigma_\lambda - \sigma^{(\ell)}; \partial_t \sigma^{(\ell)} \rangle \, dx \, ds + \int_0^t \int_{\partial\Omega} (\langle \sigma_\lambda; \nu \rangle - \langle \sigma^{(\ell)}; \nu \rangle) \partial_t u_\lambda \, d\mathcal{H}^{n-1} \, ds. \end{aligned} \quad (3.5.44)$$

The weak convergences (3.5.40) ensure that the first two integrals in the right hand side of (3.5.44) tend to 0 as $\lambda \rightarrow \ell$. Concerning the boundary integral, if $\ell = +\infty$, since $\langle \sigma^{(\infty)}; \nu \rangle = 0$, using the boundary condition for fixed $\lambda > 0$, we infer that

$$(\langle \sigma_\lambda; \nu \rangle - \langle \sigma^{(\infty)}; \nu \rangle) \partial_t u_\lambda = -\psi'_\lambda(\partial_t u_\lambda) \partial_t u_\lambda \leq 0 \quad \text{a.e. on } (0, T) \times \partial\Omega$$

by convexity of ψ_λ . On the other hand, if $\ell = 0$, from the boundary condition for fixed $\lambda > 0$ and the fact that $\|\langle \sigma^{(0)}; \nu \rangle\|_{L^\infty((0, T) \times \partial\Omega)} \leq 1$, we get

$$\begin{aligned} (\langle \sigma_\lambda; \nu \rangle - \langle \sigma^{(0)}; \nu \rangle) \partial_t u_\lambda &= - \left(|\partial_t u_\lambda| \mathbb{1}_{\|\partial_t u_\lambda\| > \lambda} + \frac{|\partial_t u_\lambda|^2}{\lambda} \mathbb{1}_{\|\partial_t u_\lambda\| \leq \lambda} \right) + \langle \sigma^{(0)}; \nu \rangle \partial_t u_\lambda \\ &\leq - \left(\frac{|\partial_t u_\lambda|^2}{\lambda} + \langle \sigma^{(0)}; \nu \rangle \partial_t u_\lambda \right) \mathbb{1}_{\|\partial_t u_\lambda\| \leq \lambda} \leq \lambda \quad \text{a.e. on } (0, T) \times \partial\Omega. \end{aligned}$$

In both cases, we obtain that the right hand side of (3.5.44) is infinitesimal as $\lambda \rightarrow \ell$, which completes the proof of the strong convergences.

Step 3 : The flow rules. If $\ell = +\infty$, writing the energy balance (3.5.5) between two arbitrary times $0 \leq t_1 < t_2 \leq T$

yields

$$\frac{1}{2}\|\partial_t u_\lambda(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sigma_\lambda(t_2)\|_{L^2(\Omega;\mathbb{R}^n)}^2 + \int_{t_1}^{t_2} |\partial_t p_\lambda(t)|(\Omega) dt \leq \frac{1}{2}\|\partial_t u_\lambda(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sigma_\lambda(t_1)\|_{L^2(\Omega;\mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u_\lambda dx dt.$$

Using the strong convergences (3.5.43) together with the sequential lower semi-continuity of the mapping

$$p \mapsto \int_{t_1}^{t_2} |\partial_t p(t)|(\Omega) dt$$

with respect to the weak convergence (3.5.40) (see *e.g.* [30, Appendix]) leads to

$$\begin{aligned} \frac{1}{2}\|\partial_t u^{(\infty)}(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sigma^{(\infty)}(t_2)\|_{L^2(\Omega;\mathbb{R}^n)}^2 + \int_{t_1}^{t_2} |\partial_t p^{(\infty)}(t)|(\Omega) dt \\ \leq \frac{1}{2}\|\partial_t u^{(\infty)}(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sigma^{(\infty)}(t_1)\|_{L^2(\Omega;\mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u^{(\infty)} dx dt. \end{aligned}$$

Deriving this inequality with respect to time, and using the equation of motion then implies that for a.e. $t \in [0, T]$, $|\partial_t p^{(\infty)}(t)|(\Omega) \leq [\sigma^{(\infty)}(t) \cdot \partial_t p^{(\infty)}(t)](\Omega)$. On the other hand, since we have $\|\sigma^{(\infty)}\|_{L^\infty((0,T)\times\Omega)} \leq 1$, Proposition 3.2.3 ensures that $|\partial_t p^{(\infty)}(t)| \geq [\sigma^{(\infty)}(t) \cdot \partial_t p^{(\infty)}(t)]$ in $\mathcal{M}(\Omega)$, from which we deduce that

$$|\partial_t p^{(\infty)}(t)| = [\sigma^{(\infty)}(t) \cdot \partial_t p^{(\infty)}(t)] \quad \text{in } \mathcal{M}(\Omega) \text{ for a.e. } t \in [0, T].$$

Next, if $\ell = 0$, writing the energy balance (3.5.5) between two arbitrary times $0 \leq t_1 < t_2 \leq T$ yields

$$\begin{aligned} \frac{1}{2}\|\partial_t u_\lambda(t_2)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sigma_\lambda(t_2)\|_{L^2(\Omega;\mathbb{R}^n)}^2 + \int_{t_1}^{t_2} |\partial_t p_\lambda(t)|(\Omega) dt + \int_{t_1}^{t_2} \int_{\partial\Omega} \psi_\lambda(\partial_t u_\lambda) d\mathcal{H}^{n-1} dt \\ \leq \frac{1}{2}\|\partial_t u_\lambda(t_1)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sigma_\lambda(t_1)\|_{L^2(\Omega;\mathbb{R}^n)}^2 + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u_\lambda dx dt. \end{aligned}$$

Using the definition of ψ_λ , we have

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\partial\Omega} \psi_\lambda(\partial_t u_\lambda) d\mathcal{H}^{n-1} dt \geq \int_{t_1}^{t_2} \int_{\partial\Omega} |\partial_t u_\lambda| d\mathcal{H}^{n-1} dt - \int_{t_1}^{t_2} \int_{\|\partial_t u_\lambda(t)\| \leq \lambda \cap \partial\Omega} |\partial_t u_\lambda| d\mathcal{H}^{n-1} dt \\ - \frac{\lambda}{2} \int_{t_1}^{t_2} \mathcal{H}^{n-1}(\{|\partial_t u_\lambda(t)| > \lambda\} \cap \partial\Omega) dt \geq \int_{t_1}^{t_2} \int_{\partial\Omega} |\partial_t u_\lambda| d\mathcal{H}^{n-1} dt - \lambda(t_2 - t_1) \mathcal{H}^{n-1}(\partial\Omega). \end{aligned}$$

Thanks to the strong convergences (3.5.43) and the sequential lower semi-continuity of the mapping

$$(u, p) \mapsto \int_{t_1}^{t_2} |\partial_t p(t)|(\Omega) dt + \int_{t_1}^{t_2} \int_{\partial\Omega} |\partial_t u| d\mathcal{H}^{n-1} dt$$

with respect to the convergences (3.5.40) and (3.5.43) (see *e.g.* [30, Appendix]), we get

$$\begin{aligned} \frac{1}{2}\left(\|\partial_t u^{(0)}(t_2)\|_{L^2(\Omega)}^2 + \|\sigma^{(0)}(t_2)\|_{L^2(\Omega;\mathbb{R}^n)}^2\right) + \int_{t_1}^{t_2} |\partial_t p^{(0)}(t)|(\Omega) dt + \int_{t_1}^{t_2} \int_{\partial\Omega} |\partial_t u^{(0)}| d\mathcal{H}^{n-1} dt \\ \leq \frac{1}{2}\left(\|\partial_t u^{(0)}(t_1)\|_{L^2(\Omega)}^2 + \|\sigma^{(0)}(t_1)\|_{L^2(\Omega;\mathbb{R}^n)}^2\right) + \int_{t_1}^{t_2} \int_{\Omega} f \partial_t u^{(0)} dx dt, \end{aligned}$$

or still, by definition of duality

$$|\partial_t p^{(0)}(t)|(\Omega) + \int_{\partial\Omega} |\partial_t u^{(0)}(t)| d\mathcal{H}^{n-1} \leq [\sigma^{(0)}(t) \cdot \partial_t p^{(0)}(t)](\Omega) - \int_{\partial\Omega} \langle \sigma^{(0)}(t); \nu \rangle \partial_t u^{(0)}(t) d\mathcal{H}^{n-1}.$$

On the other hand, since $\|\langle \sigma^{(0)}; \nu \rangle\|_{L^\infty((0,T)\times\partial\Omega)} \leq \|\sigma^{(0)}\|_{L^\infty((0,T)\times\Omega)} \leq 1$, Proposition 3.2.3 ensures that for a.e. $t \in [0, T]$,

$|\partial_t p^{(0)}(t)| \geq [\sigma^{(0)}(t) \cdot \partial_t p^{(0)}(t)]$ in $\mathcal{M}(\Omega)$ and $|\partial_t u^{(0)}(t)| \geq -\langle \sigma^{(0)}(t); \nu \rangle \partial_t u^{(0)}(t)$ \mathcal{H}^{n-1} -a.e. on $\partial\Omega$, from which we deduce that

$$|\partial_t p^{(0)}(t)| = [\sigma^{(0)}(t) \cdot \partial_t p^{(0)}(t)] \quad \text{in } \mathcal{M}(\Omega)$$

and

$$|\partial_t u^{(0)}(t)| = -\langle \sigma^{(0)}(t); \nu \rangle \partial_t u^{(0)}(t) \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega.$$

Finally, in both cases $\ell \in \{0, +\infty\}$, the uniqueness can be recovered as in Subsection 3.5.4. \square

3.6 Short time regularity of the solution

In this section, we prove that the variational solutions to the elasto-plastic problem are smooth in short time, provided the initial data are smooth and compactly supported in space. This kind of regularity result in the context of dynamical elasto-plasticity seems to be new, and the argument strongly rests on the hyperbolic structure of the model. The general idea is similar to the proof of the fact that the (unique) entropic solution to a scalar conservation law with BV initial data is actually BV (instead of just L^1 in the Kruřkov theory [58]). It consists in proving a comparison principle between two solutions associated with different initial data. In [58], an L^1 -contraction principle states that, at time t , the L^1 -distance between two solutions can be estimated by the L^1 -distance of the initial data. In our context, an L^2 -comparison principle has been established in Proposition 3.5.12. Then, translating in space the data enables one to get an L^2 -estimate on the difference quotient of the solution in terms of the L^2 -norm of the difference quotient of the data. In particular, if the data are H^1 , then the solution is H^1 as well (see [37, Lemma 10] in the full space). Since we are dealing with a boundary value problem, one has to be careful that the translated solutions remain inside the domain Ω . This is the reason why we need to ensure that, in short time, if the data are compactly supported in space, then so is the solution, which is a statement of the finite speed propagation property. In that way, the boundary of the domain can be ignored, and one can argue similarly as in the full space.

Proposition 3.6.1 (Finite speed propagation). *Let (u, σ, p) be the variational solution of the elasto-plastic problem given by Theorem 3.5.4 for the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source term f . Suppose that there exists a compact set $K \subset \Omega$ such that*

$$\text{supp}(v_0, \sigma_0, f(t)) \subset K \quad \text{for all } t \in [0, T].$$

Then, for all $T^ \in (0, T]$ be such that $T^* < \text{dist}(K, \partial\Omega)$, there exists a compact set $K^* \subset \Omega$ such that $\text{supp}(\partial_t u, \sigma) \subset [0, T^*] \times K^*$.*

Remark 3.6.2. *The speed propagation is equal to 1 in this case since the modulus of the eigenvalues of the matrices A_i (see (3.3.9) in the case where $n = 2$ and Section 3.7 in the general case) is bounded by 1.*

Proof. By assumption, we know that for all $x \in \partial\Omega$, we have $\text{dist}(x, K) > T^*$, so that we can find some $r_x > 0$ such that $\text{dist}(x, K) = r_x + T^*$. Using the fact that $\partial\Omega$ is compact, we obtain the existence of $p \in \mathbb{N}$ and $x_1, \dots, x_p \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{i=1}^p B\left(x_i, \frac{r_{x_i}}{4}\right).$$

Observe that if $y \in B\left(x_i, \frac{r_{x_i}}{2}\right)$ for some $1 \leq i \leq p$, then

$$|\text{dist}(y, K) - \text{dist}(x_i, K)| \leq |y - x_i| \leq \frac{r_{x_i}}{2},$$

and consequently,

$$\text{dist}(y, K) \geq T^* + \frac{r_{x_i}}{2} > 0,$$

which implies that $v_0 = 0$, $\sigma_0 = 0$ and $f(t) = 0$ in $\bigcup_{i=1}^p B(x_i, r_i/2)$ for all $t \in [0, T]$.

Define $\eta = \min_{1 \leq i \leq p} r_{x_i}/4 > 0$, and consider the boundary layer

$$L_\eta = \{y \in \Omega : 0 < \text{dist}(y, \partial\Omega) < \eta\} \subset \bigcup_{i=1}^p B\left(x_i, \frac{r_{x_i}}{2}\right) \cap \Omega$$

so that $v_0 = 0$, $\sigma_0 = 0$ and $f(t) = 0$ in L_η for all $t \in [0, T]$. Let us show that $\partial_t u = 0$ and $\sigma = 0$ on $[0, T^*] \times L_\eta$. To this aim, let $x_0 \in L_\eta$, and, the set L_η being open, let $\rho_0 \in (0, \eta/2)$ be such that $B(x_0, \rho_0) \subset L_\eta$. Now define the function

$\varphi \in W^{1,\infty}((0, T^*) \times \mathbb{R}^n)$ as

$$\varphi(t, x) = \begin{cases} T^* - t + \rho_0 - |x - x_0| & \text{if } \begin{cases} t \in [0, T^*], \\ \rho_0 < |x - x_0| < \rho_0 + T^* - t, \end{cases} \\ T^* - t & \text{if } \begin{cases} t \in [0, T^*], \\ |x - x_0| < \rho_0, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\varphi \geq 0$, and

$$\begin{cases} \partial_t \varphi = -\mathbb{1}_{\{(t,x): t \in [0, T^*], |x-x_0| < \rho_0 + T^* - t\}}, \\ \nabla \varphi = -\frac{x-x_0}{|x-x_0|} \mathbb{1}_{\{(t,x): t \in [0, T^*], \rho_0 < |x-x_0| < \rho_0 + T^* - t\}}, \end{cases}$$

which implies that $-|\partial_t u|^2 \partial_t \varphi - |\sigma|^2 \partial_t \varphi + 2 \langle \sigma; \nabla \varphi \rangle \partial_t u \geq 0$ a.e in $(0, T^*) \times \Omega$. Consequently, using the comparison principle, see Proposition 3.5.12, (with (u, σ, p) and the null solution) it follows that

$$-\int_0^{T^*} \int_{B(x_0, \rho_0)} (|\partial_t u|^2 + |\sigma|^2) \partial_t \varphi + 2 \langle \sigma; \nabla \varphi \rangle \partial_t u \, dx \, dt \leq \int_{B(x_0, \rho_0 + T^*)} (|v_0|^2 + |\sigma_0|^2) \varphi(0) \, dx + \int_0^{T^*} \int_{B(x_0, \rho_0 + T^*)} |f| |\partial_t u| \varphi \, dx \, dt,$$

and since the spatial derivative of φ vanish on $B(x_0, \rho_0)$, we get

$$\int_0^{T^*} \int_{B(x_0, \rho_0)} (|\partial_t u|^2 + |\sigma|^2) \, dx \, dt \leq \int_{B(x_0, \rho_0 + T^*)} (|v_0|^2 + |\sigma_0|^2) \varphi(0) \, dx + \int_0^{T^*} \int_{B(x_0, \rho_0 + T^*)} |f| |\partial_t u| \varphi \, dx \, dt.$$

Using the definition of x_0 , we obtain that for every $y \in B(x_0, \rho_0 + T^*)$,

$$|\text{dist}(y, K) - \text{dist}(x_0, K)| \leq \rho_0 + T^*,$$

and thus

$$\text{dist}(y, K) > T^* + \eta - \rho_0 - T^* > \frac{\eta}{2} > 0.$$

Consequently,

$$\int_0^{T^*} \int_{B(x_0, \rho_0)} (|\partial_t u|^2 + |\sigma|^2) \, dx \, dt \leq 0$$

which implies that both $\partial_t u$ and σ vanish in $(0, T^*) \times L_\eta$. The conclusion thus follows by setting $K^* = \Omega \setminus L_\eta$. \square

Remark 3.6.3. Since $u \in W^{2,\infty}([0, T]; L^2(\Omega))$, we have, for all $t \in [0, T]$,

$$u(t) = u_0 + \int_0^t \partial_t u(s) \, ds,$$

where the integral is intended as a Bochner integral in $L^2(\Omega)$. Therefore, if further $\text{supp}(u_0) \subset K$, it follows that $\text{supp}(u) \subset [0, T^*] \times K^*$. As a consequence, the measure Du is also compactly supported in $[0, T^*] \times K^*$, and the additive decomposition entails that $\text{supp}(p) \subset [0, T^*] \times K^*$ as well.

We are now in position to state the regularity result.

Theorem 3.6.4 (Short time regularity). *Let (u, σ, p) be the variational solution to the elasto-plastic problem given by Theorem 3.5.4 for the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source term f . Suppose that there exists a compact set $K \subset \Omega$ such that*

$$\text{supp}(u_0, v_0, \sigma_0, p_0, f(t)) \subset K \quad \text{for all } t \in [0, T],$$

and that

$$\sigma_0 \in H^1(\Omega; \mathbb{R}^n), \quad f \in H^1((0, T) \times \Omega).$$

Then, for all $T^* \in (0, T]$ be such that $T^* < \text{dist}(K, \partial\Omega)$, we have

$$\begin{cases} u \in H^1([0, T^*]; H^1(\Omega)), \\ \sigma \in L^2(0, T^*; H^1(\Omega; \mathbb{R}^n)), \\ p \in H^1([0, T^*]; L^2(\Omega; \mathbb{R}^n)). \end{cases}$$

Proof. Using Proposition 3.6.1 and Remark 3.6.3, we know that for all $T^* < \text{dist}(K, \partial\Omega)$, there exists a compact set, $K^* \subset \Omega$ such that $\text{supp}(u, \sigma, p) \subset [0, T^*] \times K^*$. Since K^* is a compact subset of Ω , there exists $\delta > 0$ such that for all $h \in \mathbb{R}^n$ with $|h| < \delta$, the sets $K^* + h$ are also compactly embedded in Ω . Let Ω' be a bounded smooth open subset of \mathbb{R}^n such that $\bar{\Omega} \subset \Omega'$, and for all $h \in \mathbb{R}^n$ with $|h| < \delta$, $\Omega + h \subset \Omega'$.

Step 1 : Extension on $(0, T^*) \times \Omega'$. We denote by \bar{f} , \bar{u} and $\bar{\sigma}$ the extensions of f , u and σ by zero on $(0, T^*) \times \Omega'$. Clearly, one has $\bar{f} \in H^1([0, T^*]; L^2(\Omega'))$, $\bar{u} \in W^{2,\infty}([0, T^*]; L^2(\Omega'))$, $\bar{\sigma} \in W^{1,\infty}([0, T^*]; L^2(\Omega'; \mathbb{R}^n))$ and

$$\|\bar{\sigma}\|_{L^\infty((0, T^*) \times \Omega')} \leq 1.$$

In addition, for all $t \in [0, T^*]$, since the (inner) trace on $\partial\Omega$ of $u(t)$ vanishes, [3, Theorem 3.87] ensures that the function $\bar{u}(t) \in BV(\Omega')$ and $D\bar{u}(t) = Du(t)$ in $\mathcal{M}(\Omega')$. Hence, we get $\bar{u} \in C^{0,1}([0, T^*]; BV(\Omega'))$. Similarly, since the (inner) normal trace on $\partial\Omega$ of σ vanishes, we deduce that $\text{div}\bar{\sigma} \in L^\infty(0, T^*; L^2(\Omega'))$, which implies that $\bar{\sigma} \in L^\infty(0, T^*; H(\text{div}, \Omega'))$.

For every $t \in [0, T^*]$, we define the measure $\bar{p}(t) \in \mathcal{M}(\Omega'; \mathbb{R}^n)$ by

$$\bar{p}(t) = D\bar{u}(t) - \bar{\sigma}(t).$$

Using the regularity of \bar{u} and $\bar{\sigma}$, we obtain that $\bar{p} \in C^{0,1}([0, T^*]; \mathcal{M}(\Omega'; \mathbb{R}^n))$. For a.e. $t \in [0, T^*]$, we consider the Radon measure $[\bar{\sigma}(t) \cdot \partial_t \bar{p}(t)]$ on Ω' as in Definition 3.2.2. Clearly $[\bar{\sigma}(t) \cdot \partial_t \bar{p}(t)] = [\sigma(t) \cdot \partial_t p(t)]$ in $\mathcal{M}(\Omega)$, while Proposition 3.2.3 ensures that $|\bar{\sigma}(t) \cdot \partial_t \bar{p}(t)|(\Omega' \setminus \Omega) \leq |\partial_t \bar{p}(t)|(\Omega' \setminus \Omega) = 0$. It implies that for a.e. $t \in [0, T^*]$, we have $|\partial_t \bar{p}(t)| = [\bar{\sigma}(t) \cdot \partial_t \bar{p}(t)]$ in $\mathcal{M}(\Omega')$.

Step 2 : Spatial translation. For every $h \in \mathbb{R}^n$ be such that $|h| < \delta$, we define the translation operator τ_h of a generic function $F : (0, T^*) \times \Omega' \rightarrow \mathbb{R}$ by

$$\tau_h F(t, x) = F(t, x + h) \quad \text{for all } (t, x) \in (0, T^*) \times \Omega.$$

Then, $\tau_h \bar{f} \in H^1([0, T^*]; L^2(\Omega))$, $\tau_h \bar{u} \in W^{2,\infty}([0, T^*]; L^2(\Omega))$ and $\tau_h \bar{\sigma} \in W^{1,\infty}([0, T^*]; L^2(\Omega; \mathbb{R}^n))$ with $\|\tau_h \bar{\sigma}\|_{L^\infty((0, T^*) \times \Omega)} \leq 1$. According to [3, Remark 3.18], for all $t \in [0, T^*]$, we have $D(\tau_h \bar{u})(t) = \tau_{-h\#} D\bar{u}(t)$ (the push-forward of the measure $D\bar{u}(t)$ by the mapping $x \mapsto x - h$) which implies that $\tau_h \bar{u} \in C^{0,1}([0, T^*]; BV(\Omega))$. Finally, since for all $t \in [0, T^*]$ the push-forward measure $\tau_{-h\#} \bar{p}(t) \in \mathcal{M}(\Omega; \mathbb{R}^n)$ satisfies

$$\tau_{-h\#} \bar{p}(t) = D\tau_h \bar{u}(t) - \tau_h \bar{\sigma}(t),$$

the regularity of $\tau_h \bar{u}$ and $\tau_h \bar{\sigma}$ ensures that $\tau_{-h\#} \bar{p} \in C^{0,1}([0, T^*]; \mathcal{M}(\Omega; \mathbb{R}^n))$.

Step 3 : The translation of the solution is a solution. We define the translation of the solution $(u_h, \sigma_h, p_h) := (\tau_h \bar{u}, \tau_h \bar{\sigma}, \tau_{-h\#} \bar{p})$ and $f_h := \tau_h \bar{f}$. Let us show that

1. Regularity properties : we have $u_h \in W^{2,\infty}([0, T^*]; L^2(\Omega)) \cap C^{0,1}([0, T^*]; BV(\Omega))$, $\sigma_h \in W^{1,\infty}([0, T^*]; L^2(\Omega; \mathbb{R}^n))$ and $p_h \in C^{0,1}([0, T^*]; \mathcal{M}(\Omega; \mathbb{R}^n))$;
2. Equation of motion : $\partial_{tt} u_h - \text{div} \sigma_h = f_h$ in $L^\infty(0, T^*; L^2(\Omega))$;
3. Additive decomposition : for every $t \in [0, T^*]$,

$$Du_h(t) = \sigma_h(t) + p_h(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^n);$$

4. Stress constraint : for every $t \in [0, T^*]$, $|\sigma_h(t)| \leq 1$ a.e. in Ω ;
5. Flow rule : for a.e. $t \in [0, T^*]$, $|\partial_t p_h(t)| = [\sigma_h(t) \cdot \partial_t p_h(t)]$ in $\mathcal{M}(\Omega)$;
6. Boundary condition : for a.e. $t \in [0, T^*]$,

$$\langle \sigma_h(t); \nu \rangle + \psi'_\lambda(\partial_t u_h(t)) = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega;$$

7. Initial conditions :

$$u_h(0) = \tau_h u_0, \quad \partial_t u_h(0) = \tau_h v_0, \quad \sigma_h(0) = \tau_h \sigma_0, \quad p_h(0) = \tau_{-h\#} p_0.$$

Items 1, 3 and 4 have already been proved in step 2. Due to the definition of τ_h , items 2 and 7 are automatically satisfied. Let us examine the point 6. Since $\text{supp}(u, \sigma) \subset [0, T^*] \times K^*$, then for all $t \in [0, T^*]$, $\text{supp}(u_h(t), \sigma_h(t)) \subset (K^* + h)$

which is a compact subset of Ω as long as $|h| < \delta$. Therefore, for all $t \in [0, T^*]$, we have $\partial_t u_h(t) = 0$ and $\langle \sigma_h(t); \nu \rangle = 0$ on $\partial\Omega$, and thus 6 holds. It remains to show item 5. By Definition 3.2.2 of duality, we know that for a.e. $t \in [0, T^*]$ and for all $\varphi \in C_c^\infty(\Omega)$,

$$\langle [\partial_t p_h(t) \cdot \sigma_h(t)] | \varphi \rangle = - \int_{\Omega} \partial_t u_h(t) \operatorname{div} \sigma_h(t) \varphi \, dx - \int_{\Omega} \sigma_h(t) \cdot \nabla \varphi \partial_t u_h(t) \, dx - \int_{\Omega} \langle \partial_t \sigma_h(t); \sigma_h(t) \rangle \varphi \, dx.$$

Using a change of variables (and since $|h| < \delta$), we get

$$\langle [\partial_t p_h(t) \cdot \sigma_h(t)] | \varphi \rangle = - \int_{\Omega} \partial_t \bar{u}(t) \operatorname{div} \bar{\sigma}(t) \varphi(\cdot - h) \, dx - \int_{\Omega} \langle \bar{\sigma}(t); \nabla \varphi(\cdot - h) \rangle \partial_t \bar{u}(t) \, dx - \int_{\Omega} \langle \partial_t \bar{\sigma}(t); \bar{\sigma}(t) \rangle \varphi(\cdot - h) \, dx,$$

and according to the integration by parts formula (Proposition 3.2.4), we obtain

$$\langle [\partial_t p_h(t) \cdot \sigma_h(t)] | \varphi \rangle = \langle |\partial_t \bar{p}| \varphi(\cdot - h) \rangle = \langle |\partial_t p_h| \varphi \rangle,$$

where we used that $\varphi(\cdot - h) \in C_c^\infty(\Omega')$ and $|\partial_t \bar{p}(t)| = [\bar{\sigma}(t) \cdot \partial_t \bar{p}(t)]$ in $\mathcal{M}(\Omega')$.

Step 4 : Regularity. Now that (u_h, σ_h, p_h) is a solution associated with the initial condition $(\tau_h u_0, \tau_h v_0, \tau_h \sigma_0, \tau_{-h\#} p_0)$ and source term f_h , Proposition 3.5.12 (with the test function $\varphi(t, x) = T^* - t$) then implies that

$$\begin{aligned} & \int_0^{T^*} \int_{\Omega} (\partial_t u_h - \partial_t u)^2 \, dx \, dt + \int_0^T \int_{\Omega} |\sigma_h - \sigma|^2 \, dx \, dt \\ & \leq T^* \left(\int_{\Omega} (\tau_h v_0 - v_0)^2 \, dx + \int_{\Omega} |\tau_h \sigma_0 - \sigma_0|^2 \, dx + 2 \int_0^{T^*} \int_{\Omega} |\tau_h f - f| |\tau_h \partial_t u - \partial_t u| \, dx \, dt \right). \end{aligned}$$

According to Young's inequality, and since $v_0 \in H^1(\Omega)$, $\sigma_0 \in H^1(\Omega; \mathbb{R}^n)$, $f \in H^1((0, T^*) \times \Omega)$, we get

$$\begin{aligned} & \int_0^{T^*} \int_{\Omega} (\partial_t u_h - \partial_t u)^2 \, dx \, dt + \int_0^{T^*} \int_{\Omega} |\sigma_h - \sigma|^2 \, dx \, dt \\ & \leq C(T^*) |h|^2 \left(\|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla \sigma_0\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 + \int_0^{T^*} \|\nabla f(t)\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 \, dt \right), \end{aligned}$$

from which we get $\partial_t u \in L^2(0, T^*; H^1(\Omega))$ and $\sigma \in L^2(0, T^*; H^1(\Omega; \mathbb{R}^n))$. Writing for all $t \in [0, T^*]$,

$$u(t) = u_0 + \int_0^t \partial_t u(s) \, ds$$

as a Bochner integral in $L^2(\Omega)$, we obtain that $u \in L^2(0, T^*; H^1(\Omega))$, or still $u \in H^1([0, T^*]; H^1(\Omega))$. Finally, we have $p = \nabla u - \sigma \in L^2(0, T^*; L^2(\Omega; \mathbb{R}^n))$ and $\partial_t p = \nabla \partial_t u - \partial_t \sigma \in L^2(0, T^*; L^2(\Omega; \mathbb{R}^n))$ which shows the desired regularity $p \in H^1([0, T^*]; L^2(\Omega; \mathbb{R}^n))$. \square

A similar result holds for the variational solutions to the elasto-plastic problem with homogeneous Dirichlet or Neumann boundary conditions given by Theorem 3.5.14.

Theorem 3.6.5. *Let ℓ be in $\{0, +\infty\}$. Let $(u^{(\ell)}, \sigma^{(\ell)}, p^{(\ell)})$ be the variational solution to the elasto-plastic problem given by Theorem 3.5.14 for the initial condition $(u_0, v_0, \sigma_0, p_0)$ (satisfying (3.5.37) and (3.5.38)) and the source term f with homogeneous Dirichlet boundary condition (if $\ell = 0$) or homogeneous Neumann boundary condition (if $\ell = +\infty$). Suppose that there exists a compact set $K \subset \Omega$ such that*

$$\operatorname{supp}(u_0, v_0, \sigma_0, p_0, f(t)) \subset K \quad \text{for all } t \in [0, T],$$

and that

$$\sigma_0 \in H^1(\Omega; \mathbb{R}^n), \quad f \in H^1((0, T) \times \Omega).$$

Then, for all $T^* \in (0, T]$ be such that $T^* < \operatorname{dist}(K, \partial\Omega)$, we have

$$\begin{cases} u^{(\ell)} \in H^1([0, T^*]; H^1(\Omega)), \\ \sigma^{(\ell)} \in L^2(0, T^*; H^1(\Omega; \mathbb{R}^n)), \\ p^{(\ell)} \in H^1([0, T^*]; L^2(\Omega; \mathbb{R}^n)). \end{cases}$$

Proof. For any $\lambda > 0$, let $(u^\lambda, \sigma^\lambda, p^\lambda)$ the unique variational solution to the elasto-plastic problem given by Theorem 3.5.4 associated with the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source term f . The proof of Theorem 3.6.4 yields

$$\begin{aligned} \int_0^{T^*} \int_\Omega (\partial_t u_h^\lambda - \partial_t u^\lambda)^2 dx dt + \int_0^{T^*} \int_\Omega |\sigma_h^\lambda - \sigma^\lambda|^2 dx dt \\ \leq C(T^*)|h|^2 \left(\|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla \sigma_0\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 + \int_0^{T^*} \|\nabla f(t)\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 dt \right), \end{aligned}$$

Thanks to the weak convergences (3.5.40), passing to the lower limit leads to

$$\begin{aligned} \int_0^{T^*} \int_\Omega (\partial_t u_h^{(\ell)} - \partial_t u^{(\ell)})^2 dx dt + \int_0^{T^*} \int_\Omega |\sigma_h^{(\ell)} - \sigma^{(\ell)}|^2 dx dt \\ \leq C(T^*)|h|^2 \left(\|\nabla v_0\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \|\nabla \sigma_0\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 + \int_0^{T^*} \|\nabla f(t)\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 dt \right), \end{aligned}$$

from which we conclude as in the proof of Theorem 3.6.4. \square

3.7 Dissipative formulation

In this last Section, we establish precise links between the variational and the dissipative formulations by making rigorous the formal computations done in Subsection 3.3.3. We show that any variational solution generates a dissipative solution by performing the manipulations on the approximate elasto-visco-plastic model (for which the solution is essentially smooth), and then by passing to the limit as the viscosity parameter $\varepsilon \rightarrow 0$. A partial converse statement is proved, provided the dissipative solutions are smoother in time. We then employ measure theoretic arguments to establish that a solution of the variational problem can be constructed.

In order to define precisely the dissipative formulation of the dynamical elasto-plastic problem, we introduce the convex set

$$\mathcal{C} := \mathbb{R} \times \bar{B} \subset \mathbb{R}^{n+1},$$

and, for $i = 1, \dots, n$, we define the matrices

$$A_i := -2\mathbf{e}_1 \odot \mathbf{e}_{i+1},$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ stands for the canonical basis of \mathbb{R}^n . For all $x \in \partial\Omega$, we denote the boundary matrix by

$$A_\nu(x) := \sum_{i=1}^n A_i \nu_i(x),$$

where $\nu(x)$ is the outer unit normal to Ω at $x \in \partial\Omega$. In addition, for each $\lambda > 0$ and all $x \in \partial\Omega$, we define the matrix $M_\nu(x) \in \mathbb{M}^{(n+1) \times (n+1)}$ by

$$M_\nu(x) := \lambda^{-1} \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda \sum_{i,j=1}^n \nu_i(x) \nu_j(x) \mathbf{e}_{i+1} \odot \mathbf{e}_{j+1}.$$

In this section, we always assume that Ω is of class \mathcal{C}^1 , so the normal $\nu \in \mathcal{C}(\partial\Omega)$, and thus both matrices A_ν and $M_\nu \in \mathcal{C}(\partial\Omega; \mathbb{M}^{(n+1) \times (n+1)})$.

Definition 3.7.1. Let $U_0 \in L^2(\Omega; \mathcal{C})$ be an initial data and $F \in L^2((0, T) \times \Omega; \mathbb{R}^{n+1})$ be a source term. A function $U \in L^2((0, T) \times \Omega; \mathcal{C})$ is a dissipative solution to the elasto-plastic problem if for all constant vector $\kappa \in \mathcal{C}$ and all $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$,

$$\begin{aligned} \int_0^T \int_\Omega \left(|U - \kappa|^2 \partial_t \varphi + \sum_{i=1}^n \langle A_i (U - \kappa); U - \kappa \rangle \partial_i \varphi \right) dx dt \\ + \int_\Omega |U_0 - \kappa|^2 \varphi(0) dx + 2 \int_0^T \int_\Omega \langle F; U - \kappa \rangle \varphi dx dt + \int_0^T \int_{\partial\Omega} \langle M_\nu \kappa^+; \kappa^+ \rangle \varphi d\mathcal{H}^{n-1} dt \geq 0, \end{aligned}$$

where κ_+ denotes the orthogonal projection of κ onto $\text{Ker}(A_\nu + M_\nu) \cap \text{Im} A_\nu$.

Remark 3.7.2. Using elementary algebraic computations, we have for all $\kappa = (k, \tau) \in \mathbb{R}^{n+1}$ and all $\xi \in \mathbb{R}^n$

$$\sum_{i=1}^n \xi_i \langle A_i \kappa; \kappa \rangle = -2 \langle \tau; \xi \rangle k.$$

In addition, according to Lemma 2.2.3, we have

$$\mathbb{R}^{n+1} = \text{Ker} A_\nu \oplus \left(\text{Ker}(A_\nu - M_\nu) \cap \text{Im} A_\nu \right) \oplus \left(\text{Ker}(A_\nu + M_\nu) \cap \text{Im} A_\nu \right).$$

For each $\kappa = (k, \tau) \in \mathbb{R}^{n+1}$, denoting by $\kappa^{(0)}$, κ^- and κ^+ the projection of κ onto $\text{Ker} A_\nu$, $\text{Ker}(A_\nu - M_\nu) \cap \text{Im} A_\nu$ and $\text{Ker}(A_\nu + M_\nu) \cap \text{Im} A_\nu$, respectively, we get $\kappa = \kappa^{(0)} + \kappa^- + \kappa^+$ where

$$\begin{cases} \kappa^{(0)} = (0, \tau - \langle \tau; \nu \rangle \nu), \\ \kappa^- = \left(\frac{k - \lambda \langle \tau; \nu \rangle}{2}, \left(\frac{\langle \tau; \nu \rangle}{2} - \frac{k}{2\lambda} \right) \nu \right), \\ \kappa^+ = \left(\frac{k + \lambda \langle \tau; \nu \rangle}{2}, \left(\frac{\langle \tau; \nu \rangle}{2} + \frac{k}{2\lambda} \right) \nu \right). \end{cases}$$

In particular

$$\langle M_\nu \kappa^\pm; \kappa^\pm \rangle = 2\lambda \left(\frac{k}{2\lambda} \pm \frac{\langle \tau; \nu \rangle}{2} \right)^2.$$

The following result states that variational solutions to the elasto-plastic problem generate dissipative solutions.

Proposition 3.7.3. Let (u, σ, p) be the variational solution to the elasto-plastic problem given by Theorem 3.5.4 for the initial data $(u_0, v_0, \sigma_0, p_0)$ and the source term f , and define $U := (\partial_t u, \sigma)$. Then $U \in L^2((0, T) \times \Omega; \mathcal{C})$ is a dissipative solution to the elasto-plastic problem according to Definition 3.7.1 for the initial data $U_0 = (v_0, \sigma_0)$ and the source term $F = (f, 0)$.

Proof. For fixed $\varepsilon > 0$, let $(u_\varepsilon, \sigma_\varepsilon, p_\varepsilon)$ be the solution of the elasto-visco-plastic problem given by Theorem 3.4.1 for the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source terms f and $g_\varepsilon = \varepsilon \langle \nabla v_0; \nu \rangle$. Let $k \in \mathbb{R}$ and $\tau \in \bar{B}$, taking $(\partial_t u_\varepsilon - k)\varphi \in L^2(0, T; H^1(\Omega))$ as test function in the variational formulation (3.4.35), and using the additive decomposition (3.4.1) together with the boundary condition (3.4.2) yields

$$\begin{aligned} & \int_0^T \int_\Omega \partial_{tt} u_\varepsilon (\partial_t u_\varepsilon - k) \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \partial_t \sigma_\varepsilon; \sigma_\varepsilon - \tau \rangle \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\varepsilon - \tau; \partial_t p_\varepsilon \rangle \varphi \, dx \, dt \\ & \quad + \varepsilon \int_0^T \int_\Omega |\nabla \partial_t u_\varepsilon|^2 \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\varepsilon - \tau; \nabla \varphi \rangle (\partial_t u_\varepsilon - k) \, dx \, dt + \int_0^T \int_\Omega \langle \tau; \nabla((\partial_t u_\varepsilon - k)\varphi) \rangle \, dx \, dt \\ & \quad + \varepsilon \int_0^T \int_\Omega \langle \nabla \partial_t u_\varepsilon; \nabla \varphi \rangle (\partial_t u_\varepsilon - k) \, dx \, dt - \int_0^T \int_{\partial\Omega} \langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle (\partial_t u_\varepsilon - k) \varphi \, d\mathcal{H}^{n-1} \, dt = \int_0^T \int_\Omega f (\partial_t u_\varepsilon - k) \varphi \, dx \, dt. \end{aligned}$$

Using next the flow rule (3.4.3) and the fact that $\tau \in \bar{B}$, we infer that $\langle \sigma_\varepsilon - \tau; \partial_t p_\varepsilon \rangle \geq 0$ a.e. in $(0, T) \times \Omega$, and thus

$$\begin{aligned} & \int_0^T \int_\Omega \partial_{tt} u_\varepsilon (\partial_t u_\varepsilon - k) \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \partial_t \sigma_\varepsilon; \sigma_\varepsilon - \tau \rangle \varphi \, dx \, dt + \int_0^T \int_\Omega \langle \sigma_\varepsilon - \tau; \nabla \varphi \rangle (\partial_t u_\varepsilon - k) \, dx \, dt \\ & \quad + \int_0^T \int_\Omega \langle \tau; \nabla((\partial_t u_\varepsilon - k)\varphi) \rangle \, dx \, dt + \varepsilon \int_0^T \int_\Omega \langle \nabla \partial_t u_\varepsilon; \nabla \varphi \rangle (\partial_t u_\varepsilon - k) \, dx \, dt \\ & \quad - \int_0^T \int_{\partial\Omega} \langle \sigma_\varepsilon + \varepsilon \nabla \partial_t u_\varepsilon; \nu \rangle (\partial_t u_\varepsilon - k) \varphi \, d\mathcal{H}^{n-1} \, dt \leq \int_0^T \int_\Omega f (\partial_t u_\varepsilon - k) \varphi \, dx \, dt. \quad (3.7.1) \end{aligned}$$

Since $\partial_t u_\varepsilon \in W^{1,\infty}([0, T]; L^2(\Omega))$ and $\sigma_\varepsilon \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$, integrating by parts with respect to time yields

$$2 \int_0^T \int_\Omega \partial_{tt} u_\varepsilon (\partial_t u_\varepsilon - k) \varphi \, dx \, dt = - \int_0^T \int_\Omega (\partial_t u_\varepsilon - k)^2 \partial_t \varphi \, dx \, dt + \int_\Omega (\partial_t u_\varepsilon(T) - k)^2 \varphi(T) \, dx - \int_\Omega (v_0 - k)^2 \varphi(0) \, dx, \quad (3.7.2)$$

and similarly

$$2 \int_0^T \int_{\Omega} \langle \sigma_{\varepsilon} - \tau; \partial_t \sigma_{\varepsilon} \rangle \varphi \, dx \, dt = - \int_0^T \int_{\Omega} |\sigma_{\varepsilon} - \tau|^2 \partial_t \varphi \, dx \, dt + \int_{\Omega} |\sigma_{\varepsilon}(T) - \tau|^2 \varphi(T) \, dx - \int_{\Omega} |\sigma_0 - \tau|^2 \varphi(0) \, dx. \quad (3.7.3)$$

In addition, since τ is constant and $\partial_t u_{\varepsilon} \in L^2(0, T; H^1(\Omega))$, using an integration by parts formula with respect to the space variable leads to

$$\int_0^T \int_{\Omega} \langle \tau; \nabla((\partial_t u_{\varepsilon} - k)\varphi) \rangle \, dx \, dt = \int_0^T \int_{\partial\Omega} \langle \tau; \nu \rangle (\partial_t u_{\varepsilon} - k) \varphi \, d\mathcal{H}^{n-1} \, dt. \quad (3.7.4)$$

Gathering (3.7.1)–(3.7.4) yields

$$\begin{aligned} & \int_0^T \int_{\Omega} (\partial_t u_{\varepsilon} - k)^2 \partial_t \varphi \, dx \, dt + \int_0^T \int_{\Omega} |\sigma_{\varepsilon} - \tau|^2 \partial_t \varphi \, dx \, dt + \int_{\Omega} (v_0 - k)^2 \varphi(0) \, dx + \int_{\Omega} |\sigma_0 - \tau|^2 \varphi(0) \, dx \\ & - 2 \int_0^T \int_{\Omega} \langle \sigma_{\varepsilon} - \tau; \nabla \varphi \rangle (\partial_t u_{\varepsilon} - k) \, dx \, dt - 2\varepsilon \int_0^T \int_{\Omega} \langle \nabla \partial_t u_{\varepsilon}; \nabla \varphi \rangle (\partial_t u_{\varepsilon} - k) \, dx \, dt \\ & + 2 \int_0^T \int_{\Omega} f(\partial_t u_{\varepsilon} - k) \varphi \, dx \, dt + 2 \int_0^T \int_{\partial\Omega} (\langle \sigma_{\varepsilon} + \varepsilon \nabla \partial_t u_{\varepsilon}; \nu \rangle - \langle \tau; \nu \rangle) (\partial_t u_{\varepsilon} - k) \varphi \, d\mathcal{H}^{n-1} \, dt \geq 0, \end{aligned}$$

and passing to the limit as $\varepsilon \rightarrow 0$, Proposition 3.5.8 and (3.5.32) imply that

$$\begin{aligned} & \int_0^T \int_{\Omega} (\partial_t u - k)^2 \partial_t \varphi \, dx \, dt + \int_0^T \int_{\Omega} |\sigma - \tau|^2 \partial_t \varphi \, dx \, dt + \int_{\Omega} (v_0 - k)^2 \varphi(0) \, dx + \int_{\Omega} |\sigma_0 - \tau|^2 \varphi(0) \, dx \\ & - 2 \int_0^T \int_{\Omega} \langle \sigma - \tau; \nabla \varphi \rangle (\partial_t u - k) \, dx \, dt + 2 \int_0^T \int_{\Omega} f(\partial_t u - k) \varphi \, dx \, dt + 2 \int_0^T \int_{\partial\Omega} (\langle \sigma; \nu \rangle - \langle \tau; \nu \rangle) (\lambda \psi'_{\lambda}(\partial_t u) - k) \varphi \, d\mathcal{H}^{n-1} \, dt \geq 0. \end{aligned}$$

Thanks to algebraic manipulations, we have for a.e. $(t, x) \in \partial(0, T) \times \Omega$,

$$\begin{aligned} 2(\langle \sigma; \nu \rangle - \langle \tau; \nu \rangle) (\lambda \psi'_{\lambda}(\partial_t u) - k) &= 2\lambda \left(\left(\frac{\lambda \psi'_{\lambda}(\partial_t u) - k}{2\lambda} + \frac{\langle \sigma; \nu \rangle - \langle \tau; \nu \rangle}{2} \right)^2 - \left(\frac{\lambda \psi'_{\lambda}(\partial_t u) - k}{2\lambda} - \frac{\langle \sigma; \nu \rangle - \langle \tau; \nu \rangle}{2} \right)^2 \right) \\ &= 2\lambda \left(\left(\frac{k}{2\lambda} + \frac{\langle \tau; \nu \rangle}{2} \right)^2 - \left(\frac{\lambda \psi'_{\lambda}(\partial_t u) - k}{2\lambda} - \frac{\langle \sigma; \nu \rangle - \langle \tau; \nu \rangle}{2} \right)^2 \right), \end{aligned}$$

where we used the boundary condition $\langle \sigma; \nu \rangle + \psi'_{\lambda}(\partial_t u) = 0$ on $(0, T) \times \partial\Omega$. Finally, from Remark 3.7.2, it follows that

$$\begin{aligned} & \int_0^T \int_{\Omega} |U - \kappa|^2 \partial_t \varphi \, dx \, dt + \sum_{i=1}^n \int_0^T \int_{\Omega} \langle A_i(U - \kappa); U - \kappa \rangle \partial_i \varphi \, dx \, dt \\ & + \int_{\Omega} |U_0 - \kappa|^2 \varphi(0) \, dx + 2 \int_0^T \int_{\Omega} \langle F; U - \kappa \rangle \varphi \, dx \, dt + \int_0^T \int_{\partial\Omega} \langle M_{\nu} \kappa^+; \kappa^+ \rangle \varphi \, d\mathcal{H}^{n-1} \, dt \geq 0, \end{aligned}$$

as required. \square

We finally show that, provided additional regularity assumptions, any dissipative solution to the elasto-plastic problem generates a variational solution.

Proposition 3.7.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^2 boundary. Let $U = (v, \sigma) \in W^{1,\infty}([0, T]; L^2(\Omega; \mathcal{E}))$ be a dissipative solution to the elasto-plastic problem according to Definition 3.7.1 for the initial data $U_0 = (v_0, \sigma_0) \in L^2(\Omega; \mathcal{E})$ and the source term $F = (f, 0)$ with $f \in L^{\infty}(0, T; L^2(\Omega))$. Then, for all $u_0 \in BV(\Omega) \cap L^2(\Omega)$, there exists a triple (u, σ, p) which is a variational solution to the elasto-plastic problem associated with the initial data $(u_0, v_0, \sigma_0, Du_0 - \sigma_0)$ and the source term f .*

Proof. We split the proof into several steps.

Step 1 : Initial conditions for v and σ . According to Lemma 2.2.14 (and Remark 2.5.3), we infer that the initial

condition is satisfied in the essential-limit sense, *i.e.*, for all $\phi \in \mathcal{C}_c^1(\Omega)$,

$$\lim_{t \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_{t-\alpha}^t \int_{\Omega} |U(s, x) - U_0(x)|^2 \phi(x) dx ds = 0.$$

On the other hand, since $U \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^{n+1}))$, then $U(t) \rightarrow U(0)$ strongly in $L^2(\Omega; \mathbb{R}^{n+1})$ as $t \rightarrow 0$. It thus follows that $U(0) = U_0$, or still $v(0) = v_0$ and $\sigma(0) = \sigma_0$. By Definition 3.7.1 of dissipative solutions, and using the regularity assumption $U \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^{n+1}))$, we can integrate by parts with respect to time to get that, for all $\varphi \in \mathcal{C}_c^1((0, T) \times \mathbb{R}^n)$ with $\varphi \geq 0$ and all constant vector $\kappa \in \mathcal{C}$,

$$\begin{aligned} \int_0^T \int_{\Omega} \left(\langle -2\partial_t U; U - \kappa \rangle \varphi + \sum_{i=1}^n \langle A_i(U - \kappa); U - \kappa \rangle \partial_i \varphi \right) dx dt \\ + 2 \int_0^T \int_{\Omega} \langle F; U - \kappa \rangle \varphi dx dt + \int_0^T \int_{\partial\Omega} \langle M_\nu \kappa^+; \kappa^+ \rangle \varphi d\mathcal{H}^{n-1} dt \geq 0, \end{aligned}$$

or still

$$\begin{aligned} \int_0^T \int_{\Omega} \left(-2\langle U; \partial_t U \rangle \varphi + \sum_{i=1}^n \langle A_i U; U \rangle \partial_i \varphi + 2\langle F; U \rangle \varphi \right) dx dt \\ + 2 \left\langle \kappa; \int_0^T \int_{\Omega} \left(\partial_t U \varphi - \sum_{i=1}^n (A_i U) \partial_i \varphi - F \varphi \right) dx dt \right\rangle + \int_0^T \int_{\partial\Omega} (\langle M_\nu \kappa^+; \kappa^+ \rangle + \langle A_\nu \kappa; \kappa \rangle) \varphi d\mathcal{H}^{n-1} dt \geq 0. \end{aligned}$$

According to Lemma 2.2.3, we have $\langle M_\nu \kappa^+; \kappa^+ \rangle + \langle A_\nu \kappa; \kappa \rangle = \langle M_\nu \kappa^-; \kappa^- \rangle$, and thus

$$\begin{aligned} \int_0^T \int_{\Omega} \left(-2\langle U; \partial_t U \rangle \varphi + \sum_{i=1}^n \langle A_i U; U \rangle \partial_i \varphi + 2\langle F; U \rangle \varphi \right) dx dt \\ + 2 \left\langle \kappa; \int_0^T \int_{\Omega} \left(\partial_t U \varphi - \sum_{i=1}^n (A_i U) \partial_i \varphi - F \varphi \right) dx dt \right\rangle + \int_0^T \int_{\partial\Omega} \langle M_\nu \kappa^-; \kappa^- \rangle \varphi d\mathcal{H}^{n-1} dt \geq 0. \quad (3.7.5) \end{aligned}$$

Step 2 : Definition of (u, σ, p) and first properties. For all $t \in [0, T]$, let us define the displacement as

$$u(t) := u_0 + \int_0^t v(s) ds \text{ in } L^2(\Omega).$$

According to the regularity assumption on U , we infer that $u \in W^{2,\infty}([0, T]; L^2(\Omega))$ and that $\sigma \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$. In addition, for all $t \in [0, T]$, we have $|\sigma(t)| \leq 1$ a.e. in Ω . Note that, by construction $u(0) = u_0$, and by step 1, $\partial_t u(0) = v_0$. We also define the plastic strain by

$$p := Du - \sigma \in W^{1,\infty}([0, T]; H^{-1}(\Omega; \mathbb{R}^n)). \quad (3.7.6)$$

We now use Remark 3.7.2 to rewrite (3.7.5) as

$$\begin{aligned} - \int_0^T \int_{\Omega} (\partial_t u \partial_{tt} u \varphi + \langle \sigma; \partial_t \sigma \rangle \varphi + \langle \sigma; \nabla \varphi \rangle \partial_t u - f \partial_t u \varphi) dx + k \int_0^T \int_{\Omega} (\partial_{tt} u \varphi + \langle \sigma; \nabla \varphi \rangle - f \varphi) dx dt \\ + \left\langle \tau; \int_0^T \int_{\Omega} (\partial_t \sigma \varphi + \partial_t u \nabla \varphi) dx dt \right\rangle + \lambda \int_0^T \int_{\partial\Omega} \left(\frac{k}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right)^2 \varphi d\mathcal{H}^{n-1} dt \geq 0, \quad (3.7.7) \end{aligned}$$

for all $\kappa = (k, \tau) \in \mathbb{R} \times \bar{B}$ and all $\varphi \in \mathcal{C}_c^1((0, T) \times \mathbb{R}^n)$ with $\varphi \geq 0$. Choosing $k = 0$ and $\tau = 0$, we deduce that

$$- \int_0^T \int_{\Omega} (\partial_t u \partial_{tt} u \varphi + \langle \sigma; \partial_t \sigma \rangle \varphi + \langle \sigma; \nabla \varphi \rangle \partial_t u - f \partial_t u \varphi) dx dt \geq 0.$$

In particular, one can localize in time to get, for all $\phi \in \mathcal{C}_c^1(\mathbb{R}^n)$ with $\phi \geq 0$ and for a.e. $t \in [0, T]$,

$$-\int_{\Omega} \left(\partial_t u(t) \partial_{tt} u(t) \phi + \langle \sigma(t); \partial_t \sigma(t) \rangle \phi + \langle \sigma(t); \nabla \phi \rangle \partial_t u(t) - f(t) \partial_t u(t) \phi \right) dx \geq 0.$$

As a consequence, for a.e. $t \in [0, T]$, there exists a non-negative measure $\mu(t) \in \mathcal{M}(\mathbb{R}^n)$ compactly supported in $\bar{\Omega}$ such that for all $\phi \in \mathcal{C}_c^1(\mathbb{R}^n)$

$$\langle \mu(t) | \phi \rangle = - \int_{\Omega} \left(\partial_t u(t) \partial_{tt} u(t) \phi + \langle \sigma(t); \partial_t \sigma(t) \rangle \phi + \langle \sigma(t); \nabla \phi \rangle \partial_t u(t) - f(t) \partial_t u(t) \phi \right) dx. \quad (3.7.8)$$

Particular, according to Fubini's Theorem, the function $t \mapsto \langle \mu(t) | \phi \rangle$ is measurable, and by density, this property remains true for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. It shows the weak* measurability of the mapping $\mu : t \mapsto \mu(t) \in \mathcal{M}(\mathbb{R}^n)$. In addition, since $\mu(t)$ has compact support in $\bar{\Omega}$, we can take $\phi \equiv 1$ as test function in (3.7.8) which ensures that

$$\mu(t)(\mathbb{R}^n) = \langle \mu(t) | 1 \rangle = - \int_{\Omega} \left(\partial_t u(t) \partial_{tt} u(t) + \langle \sigma(t); \partial_t \sigma(t) \rangle - f(t) \partial_t u(t) \right) dx.$$

Using next the Cauchy-Schwarz inequality, we get

$$\operatorname{ess\,sup}_{t \in [0, T]} \mu(t)(\mathbb{R}^n) < +\infty \quad (3.7.9)$$

which shows that $\mu \in L_{w*}^{\infty}(0, T; \mathcal{M}(\mathbb{R}^n))$.

Step 3 : Equation of motion. In (3.7.7), choosing $\tau = 0$ and $k \in \mathbb{R}$ arbitrary leads to

$$\int_0^T \int_{\Omega} (\partial_{tt} u \varphi + \langle \sigma; \nabla \varphi \rangle - f \varphi) dx dt = 0$$

for every $\varphi \in \mathcal{C}_c^1((0, T) \times \Omega)$, which implies that

$$\partial_{tt} u - \operatorname{div} \sigma = f \text{ in } L^{\infty}(0, T; L^2(\Omega)).$$

In particular $\sigma \in L^{\infty}(0, T; H(\operatorname{div}, \Omega))$, and since by the stress constraint $\sigma \in L^{\infty}((0, T) \times \Omega; \mathbb{R}^n)$, then $\langle \sigma; \nu \rangle \in L^{\infty}((0, T) \times \partial\Omega)$. As a consequence, for all $\varphi \in \mathcal{C}_c^1((0, T) \times \mathbb{R}^n)$,

$$\int_0^T \int_{\Omega} (\partial_{tt} u \varphi + \langle \sigma; \nabla \varphi \rangle - f \varphi) dx dt = \int_0^T \int_{\partial\Omega} \langle \sigma; \nu \rangle \varphi d\mathcal{H}^{n-1} dt.$$

Reporting into (3.7.7) leads to

$$\begin{aligned} & - \int_0^T \int_{\Omega} \left(\partial_t u \partial_{tt} u \varphi + \langle \sigma; \partial_t \sigma \rangle \varphi + \langle \sigma; \nabla \varphi \rangle \partial_t u - f \partial_t u \varphi \right) dx dt + k \int_0^T \int_{\partial\Omega} \langle \sigma; \nu \rangle \varphi d\mathcal{H}^{n-1} dt \\ & + \left\langle \tau; \int_0^T \int_{\Omega} (\partial_t \sigma \varphi + \partial_t u \nabla \varphi) dx dt \right\rangle + \lambda \int_0^T \int_{\partial\Omega} \left(\frac{k}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right)^2 \varphi d\mathcal{H}^{n-1} dt \geq 0. \end{aligned} \quad (3.7.10)$$

Step 3 : Flow rule and additional regularity. Choosing next $k = 0$ in (3.7.10), we get for all $\tau \in \bar{B}$ and all $\varphi \in \mathcal{C}_c^1((0, T) \times \Omega)$ with $\varphi \geq 0$,

$$-\left\langle \tau; \int_0^T \int_{\Omega} (\partial_t \sigma \varphi + \partial_t u \nabla \varphi) dx dt \right\rangle \leq \int_0^T \int_{\Omega} \varphi d\mu(t) dt.$$

Using the definition (3.7.6) of p , we infer that

$$\left\langle \tau; \int_0^T \langle \partial_t p(t) | \varphi(t) \rangle dt \right\rangle \leq \int_0^T \int_{\Omega} \varphi d\mu(t) dt.$$

Since $\partial_t p \in L^2(0, T; H^{-1}(\Omega))$, we can localize with respect to time to get, for a.e. $t \in [0, T]$ and all $\phi \in C_c^1(\Omega)$ with $\phi \geq 0$,

$$\langle \tau; \langle \partial_t p(t) | \phi \rangle \rangle \leq \int_{\Omega} \phi \, d\mu(t).$$

Passing to the supremum with respect to $\tau \in \bar{B}$ and using (3.7.9) yields

$$|\langle \partial_t p(t) | \phi \rangle| \leq \int_{\Omega} \phi \, d\mu(t) \leq C_* \|\phi\|_{L^\infty(\Omega)}, \quad (3.7.11)$$

where $C_* > 0$ is independent of t , which shows that $\partial_t p(t) \in \mathcal{M}(\Omega; \mathbb{R}^n)$. In particular, since $\mu(t) \geq 0$, we obtain

$$|\partial_t p(t)| \leq \mu(t) \quad \text{in } \mathcal{M}(\Omega). \quad (3.7.12)$$

We already know that $p \in W^{1,\infty}([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$ which ensures the measurability of the function $t \mapsto \langle \partial_t p(t) | \phi \rangle$ for all $\phi \in C_c^1(\Omega)$. Then by density this property remains true for all $\phi \in C_c(\Omega)$. It shows the weak* measurability of the mapping $t \mapsto \partial_t p(t) \in \mathcal{M}(\Omega; \mathbb{R}^n)$, and by (3.7.11), that $\partial_t p \in L_{w^*}^\infty(0, T; \mathcal{M}(\Omega; \mathbb{R}^n))$. Then by definition of the distributional derivative we have for all $\phi \in C_c(\Omega)$, $t \mapsto \langle p(t) | \phi \rangle \in W^{1,\infty}([0, T]; \mathbb{R}^n)$ and $\frac{d}{dt} \langle p(t) | \phi \rangle = \langle \partial_t p(t) | \phi \rangle$ for a.e. $t \in [0, T]$. It thus shows that for all $0 \leq t_1 \leq t_2 \leq T$,

$$|\langle p(t_2) - p(t_1) | \phi \rangle| = |\langle p(t_2) | \phi \rangle - \langle p(t_1) | \phi \rangle| = \left| \int_{t_1}^{t_2} \langle \partial_t p(t) | \phi \rangle \, dt \right| \leq C_* \|\phi\|_{L^\infty(\Omega)} (t_2 - t_1).$$

Dividing the previous inequality by $\|\phi\|_{L^\infty(\Omega)}$ and passing to the supremum with respect to $\phi \in C_c(\Omega)$ shows that $p \in C^{0,1}([0, T]; \mathcal{M}(\Omega; \mathbb{R}^n))$. By construction, we have $Du(t) = \sigma(t) + p(t)$ in $\mathcal{M}(\Omega; \mathbb{R}^n)$ for all $t \in [0, T]$, and therefore, thanks to the already established regularity of u and σ , we infer that $u \in C^{0,1}([0, T]; BV(\Omega))$.

According to the definition (3.7.8) of the measure $\mu(t)$ and the equation of motion, we have for all $\phi \in C_c^1(\Omega)$ and for a.e. $t \in [0, T]$,

$$\langle \mu(t) | \phi \rangle = - \int_{\Omega} (\partial_t u(t) \operatorname{div} \sigma(t) \phi + \langle \sigma(t); \partial_t \sigma(t) \rangle \phi + \langle \sigma(t); \nabla \phi \rangle \partial_t u(t)) \, dx = \langle [\sigma(t) \cdot \partial_t p(t)] | \phi \rangle$$

which is well defined according to Definition 3.2.2 since, for a.e. $t \in [0, T]$, $\sigma(t) \in H(\operatorname{div}, \Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$, and $D\partial_t u(t) = \partial_t \sigma(t) + \partial_t p(t)$ with $\partial_t u(t) \in BV(\Omega) \cap L^2(\Omega)$, $\partial_t \sigma(t) \in L^2(\Omega; \mathbb{R}^n)$, and $\partial_t p(t) \in \mathcal{M}(\Omega; \mathbb{R}^n)$. Therefore, (3.7.12) yields $|\partial_t p(t)| \leq [\sigma(t) \cdot \partial_t p(t)]$ in $\mathcal{M}(\Omega)$. On the other hand, using the stress constraint $\|\sigma(t)\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1$ and Proposition 3.2.3, the other inequality $|\partial_t p(t)| \geq [\sigma(t) \cdot \partial_t p(t)]$ in $\mathcal{M}(\Omega)$ follows, so that, finally

$$|\partial_t p(t)| = [\sigma(t) \cdot \partial_t p(t)] \quad \text{in } \mathcal{M}(\Omega).$$

Step 4 : Boundary condition. Let $\varphi \in C_c^1((0, T) \times \mathbb{R}^n)$ with $\varphi \geq 0$. Using, the integration by parts formula in BV , we get

$$\int_0^T \int_{\Omega} (\partial_t \sigma \varphi + \partial_t u \nabla \varphi) \, dx \, dt = - \int_0^T \int_{\Omega} \varphi \, d\partial_t p(t) \, dt + \int_0^T \int_{\partial\Omega} \varphi \partial_t u \nu \, d\mathcal{H}^{n-1} \, dt,$$

while the integration by parts formula given by Proposition 3.2.4 together with the equation of motion yields

$$\langle \mu(t) | \phi \rangle = \langle [\sigma(t) \cdot \partial_t p(t)] | \phi \rangle - \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \partial_t u(t) \phi \, d\mathcal{H}^{n-1}.$$

Reporting inside (3.7.10), it follows that

$$\begin{aligned} \int_0^T \int_{\Omega} \varphi \, d[(\sigma(t) - \tau) \cdot \partial_t p(t)] \, dt &\geq \int_0^T \int_{\partial\Omega} (\langle \sigma; \nu \rangle - \langle \tau; \nu \rangle) \partial_t u \varphi \, d\mathcal{H}^{n-1} \, dt \\ &\quad - k \int_0^T \int_{\partial\Omega} \langle \sigma; \nu \rangle \varphi \, d\mathcal{H}^{n-1} \, dt - \lambda \int_0^T \int_{\partial\Omega} \left(\frac{k}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right)^2 \varphi \, d\mathcal{H}^{n-1} \, dt, \end{aligned}$$

and localizing in time, we get that for a.e. $t \in [0, T]$, and all $\phi \in W^{1,\infty}(\Omega)$ with $\phi \geq 0$.

$$\int_{\Omega} \phi \, d[(\sigma(t) - \tau) \cdot \partial_t p(t)] \geq \int_{\partial\Omega} (\langle \sigma(t); \nu \rangle - \langle \tau; \nu \rangle) \partial_t u(t) \phi \, d\mathcal{H}^{n-1} - k \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \phi \, d\mathcal{H}^{n-1} - \lambda \int_{\partial\Omega} \left(\frac{k}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right)^2 \phi \, d\mathcal{H}^{n-1}. \quad (3.7.13)$$

We next wish to localize the previous relation in space. To this aim, we define,

$$\phi_{\varepsilon}(x) := \begin{cases} 0 & \text{if } \text{dist}(x, \partial\Omega) \geq \varepsilon, \\ \frac{\varepsilon - \text{dist}(x, \partial\Omega)}{\varepsilon} & \text{if } \text{dist}(x, \partial\Omega) < \varepsilon. \end{cases}$$

Note that $\phi_{\varepsilon} \in W^{1,\infty}(\Omega)$, $\phi_{\varepsilon} \geq 0$ and since Ω is of class \mathcal{C}^2 , then for $\varepsilon > 0$ small enough, we have $\nabla \phi_{\varepsilon}(x - \varepsilon s \nu(x)) = \nu(x)/\varepsilon$ for all $x \in \partial\Omega$ and all $s \in [0, 1]$. Using that $\phi_{\varepsilon} = 1$ on $\partial\Omega$, we get for all $\zeta \in W^{1,\infty}(\Omega)$ with $\zeta \geq 0$,

$$\begin{aligned} \int_{\Omega} \phi_{\varepsilon} \zeta \, d[(\sigma(t) - \tau) \cdot \partial_t p(t)] &= - \int_{\Omega} \langle \sigma(t) - \tau; \partial_t \sigma(t) \rangle \phi_{\varepsilon} \zeta \, dx - \int_{\Omega} \partial_t u(t) \text{div} \sigma(t) \phi_{\varepsilon} \zeta \, dx - \int_{\Omega} \langle \sigma(t) - \tau; \nabla \zeta \rangle \phi_{\varepsilon} \partial_t u(t) \, dx \\ &\quad - \int_{\Omega} \langle \sigma(t) - \tau; \nabla \phi_{\varepsilon} \rangle \zeta \partial_t u(t) \, dx + \int_{\partial\Omega} \langle \sigma(t); \nu \rangle - \langle \tau; \nu \rangle \partial_t u(t) \zeta \, d\mathcal{H}^{n-1}. \end{aligned} \quad (3.7.14)$$

Since $\phi_{\varepsilon} \rightarrow 0$ strongly in $L^1(\Omega)$, the dominated convergence theorem ensures that the three first integrals in the right hand side of (3.7.14) tend to zero as $\varepsilon \rightarrow 0$. Then, according to the coarea formula (see [46, Lemma 3.2.34]), denoting $\Omega_{\varepsilon} := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$, the fourth integral writes as

$$\begin{aligned} \int_{\Omega} \langle \sigma(t) - \tau; \nabla \phi_{\varepsilon} \rangle \zeta \partial_t u(t) \, dx &= \int_{\Omega_{\varepsilon}} \langle \sigma(t) - \tau; \nabla \phi_{\varepsilon} \rangle \zeta \partial_t u(t) \, dx \\ &= \int_0^1 \int_{\partial\Omega} \langle \sigma(x - s\varepsilon \nu(x), t) - \tau; \nu(x) \rangle \partial_t u(x - s\varepsilon \nu(x), t) \zeta(x - s\varepsilon \nu(x)) \, d\mathcal{H}^{n-1}(x) \, ds. \end{aligned}$$

Therefore, according to (3.2.2) and (3.2.4), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \langle \sigma(t) - \tau; \nabla \phi_{\varepsilon} \rangle \zeta \partial_t u(t) \, dx = \int_{\partial\Omega} (\langle \sigma(t); \nu \rangle - \langle \tau; \nu \rangle) \partial_t u(t) \zeta \, d\mathcal{H}^{n-1}.$$

Inserting inside (3.7.14), we get

$$\int_{\Omega} \phi_{\varepsilon} \zeta \, d[(\sigma(t) - \tau) \cdot \partial_t p(t)] \rightarrow 0,$$

and (3.7.13) yields for all $\zeta \in W^{1,\infty}(\Omega)$ with $\zeta \geq 0$,

$$0 \geq \int_{\partial\Omega} (\langle \sigma(t); \nu \rangle - \langle \tau; \nu \rangle) \partial_t u(t) \zeta \, d\mathcal{H}^{n-1} - k \int_{\partial\Omega} \langle \sigma(t); \nu \rangle \zeta \, d\mathcal{H}^{n-1} - \lambda \int_{\partial\Omega} \left(\frac{k}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right)^2 \zeta \, d\mathcal{H}^{n-1}.$$

As a consequence, we get for a.e. $t \in [0, T]$ and all $(k, \tau) \in \mathbb{R} \times \bar{B}$, then

$$-(\langle \sigma(t); \nu \rangle - \langle \tau; \nu \rangle) \partial_t u(t) \geq -k \langle \sigma(t); \nu \rangle - \lambda \left(\frac{k}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right)^2 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega. \quad (3.7.15)$$

Note that the maximum of the right-hand side with respect to $k \in \mathbb{R}$, is attained at some $k^* \in \mathbb{R}$, which satisfies according to the first order condition

$$-\langle \sigma(t); \nu \rangle - \left(\frac{k^*}{2\lambda} - \frac{\langle \tau; \nu \rangle}{2} \right) = 0.$$

Replacing in (3.7.15), and taking $\tau = -z\nu$ for any $z \in [-1, 1]$, leads to

$$-(\langle \sigma(t); \nu \rangle + z) \partial_t u(t) \geq \lambda (\langle \sigma(t); \nu \rangle)^2 + \lambda z \langle \sigma(t); \nu \rangle,$$

or still, thanks to Young's inequality

$$\frac{\lambda}{2}z^2 \geq \frac{\lambda}{2}(\langle \sigma(t); \nu \rangle)^2 + \partial_t u(t)(z + \langle \sigma(t); \nu \rangle).$$

Finally, from Remark 3.5.3, we deduce that $\partial_t u(t) \in \partial \psi_\lambda^*(-\langle \sigma(t); \nu \rangle)$, or equivalently that $\langle \sigma(t); \nu \rangle + \psi'_\lambda(\partial_t u(t)) = 0$. \square

Remark 3.7.5. *In the variational framework, the initial data must satisfy the hypotheses (3.5.4). Here, the constraint is also satisfied by the initial data. Note that since the hyperbolic variables are the velocity and the stress, the additive decomposition of the initial data has been ensured by the construction of the plastic strain. However, we did not suppose that the initial data satisfies the boundary condition.*

Numerical study of constrained Friedrichs' systems

4.1 Introduction

The aim of this chapter is to examine the numerical approximation of Friedrichs' equations under constraints (posed in the whole space or in bounded domains). To do so, we use a popular method for hyperbolic problems : the method of finite volumes (for a detailed presentation of this method, we refer to [43, 64]). Although there is an important number of schemes that have been developed, the analysis of the convergence and its rate of schemes on unstructured meshes for multidimensional problems (*i.e.* the domain is a subset of \mathbb{R}^n with $n > 1$ and the solution belongs to \mathbb{R}^m with $m > 1$) is still in its infancy.

However, the article [98] has established a rate of convergence for the RKDG scheme (see [27]), using P0 finite elements in space and the RK1 scheme in time, on unstructured meshes for generic problems of the following form

$$\begin{cases} \partial_t U + \sum_{j=1}^n \partial_j (A_j U) + BU = f, & \text{in } (0, T) \times \mathbb{R}^n, \\ U(0, x) = U_0(x), & \text{in } \mathbb{R}^n, \end{cases} \quad (4.1.1)$$

where $U : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $A_j : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{M}_{\text{sym}}^{m \times m}$, $B : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{M}^{m \times m}$, $f : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbb{M}^{m \times m}$ (resp. $\mathbb{M}_{\text{sym}}^{m \times m}$) is the space of $m \times m$ (resp. symmetric) matrices with real coefficients.

In the framework of this thesis, this article has two major interests. Firstly, this result directly applies to Friedrichs' systems without constraint in the whole space. Secondly, the proof of the rate of convergence relies on a comparison principle between the exact solution and the solution constructed by the scheme which is similar to the Kato inequality derived in [37] (see (1.4.5)). A drawback of this approach is that it needs the partial differential equation in (4.1.1) to be satisfied in the strong sense : but in the constrained case of [37], the partial differential equation is only satisfied in a weak sense.

In addition to the analysis of the RKDG scheme in the unconstrained case in [98], the study of the convergence of a scheme based on the Rusanov scheme (see [64, Section 12.5]) on Cartesian meshes has been performed in [37]. In fact, to show the existence of a solution (in the sense of Definition 4.2.1) to the problem

$$\begin{cases} \partial_t U + \sum_{j=1}^n A_j \partial_j U = 0 & \text{in } (0, T] \times \mathbb{R}^n, \\ U(t, x) \in \mathcal{C} & \text{if } (t, x) \in [0, T] \times \mathbb{R}^n, \\ U(0, x) = U^0(x) & \text{if } x \in \mathbb{R}^n, \end{cases} \quad (4.1.2)$$

where \mathcal{C} is a fixed non empty closed and convex subset of \mathbb{R}^m , the authors construct a numerical solution with a two step scheme. At each time step, the scheme first let the solution evolve according to the Rusanov scheme without taking care about the constraint. Then, on each mesh they project the solution onto the set of constraints.

Thanks to this splitting strategy and to a compactness argument (which relies on the fact that the mesh is Cartesian), they show that the numerical solution admits a convergent subsequence and they prove that the limit of

this subsequence has to be a solution of (4.1.2) in the sense of Definition 4.2.1. This method has a clear benefit : the constraint is taken into account in a simple and natural way thanks to the projection step.

Our first aim is to combine the theoretical results of [98] and the splitting strategy of [37] to construct a stable scheme on unstructured meshes for constrained Friedrichs' systems in the whole space, and to get an explicit rate of convergence for this scheme.

To study the rate of convergence of such a scheme, we need theoretical results about constrained Friedrichs' problems posed in the whole space (in order to overcome the fact that the partial differential is now satisfied in a weak sense). These results, in addition to their uses for the convergence analysis, allow us to justify the formal derivation of the dissipative inequalities in [37] and are exposed in Section 4.2.

Then, we show in Section 4.3 that the finite volume scheme on unstructured meshes, based on the work [98], associated with a projection step has the same rate of convergence (in the space $L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$) as in the unconstrained case (obtained in [98]).

The second aim of this chapter is to develop a scheme on unstructured meshes for constrained Friedrichs' systems on bounded domains. We use the scheme (very similar to the scheme of [98] for the unbounded case), presented in the note [28], that deals with first order (unconstrained) hyperbolic systems in bounded domains and again the splitting strategy of [37] to design two schemes for constrained problems on bounded domains. In Section 4.4, we show that these schemes are stable (under a Courant-Friedrichs-Lewy condition) and we illustrate the solution constructed by these schemes on the example of the simplified model of the dynamical perfect plasticity problem (described in Chapter 3) and show that the interaction between the constraint and the boundary condition that has been underlined theoretically by the nonlinear boundary condition can also be observed numerically.

4.2 Viscous approximation of Friedrichs' systems under convex constraints

This section is an improved version of the article : Jean-François BABADJIAN, Clément MIFSUD et Nicolas SEGUIN, « Relaxation approximation of Friedrichs' systems under convex constraints ». In : *Netw. Heterog. Media* 11.2 (2016), p. 223–237.

The aim of this section is to prove the convergence of a relaxation approximation of weak solutions to Friedrichs' systems under convex constraints. The well-posedness has been established in [37] by means of a numerical scheme. We present here another way to construct such weak solutions thanks to a model that relaxes the constraints. We consider the following Cauchy problem : find $U : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\begin{cases} \partial_t U + \sum_{j=1}^n A_j \partial_j U = 0 & \text{in } (0, T] \times \mathbb{R}^n, \\ U(t, x) \in \mathcal{C} & \text{if } (t, x) \in [0, T] \times \mathbb{R}^n, \\ U(0, x) = U^0(x) & \text{if } x \in \mathbb{R}^n, \end{cases} \quad (4.2.1)$$

where \mathcal{C} is a fixed (*i.e.* independent of the time and space variables) non empty closed and convex subset of \mathbb{R}^m containing 0 in its interior, the matrices A_j are $m \times m$ symmetric matrices independent of time and space, and $T > 0$. The main difficulty is due to the constraints which introduce nonlinear effects to the linear Friedrichs' system [47]. This type of hyperbolic problems has been introduced in [37] where a notion of weak solutions to problem (4.2.1) has been defined.

Definition 4.2.1. Let $U^0 \in L^2(\mathbb{R}^n, \mathcal{C})$, and $T > 0$. A function U is a weak constrained solution of (4.2.1) if $U \in L^2([0, T] \times \mathbb{R}^n, \mathcal{C})$ satisfies

$$\int_0^T \int_{\mathbb{R}^n} \left(|U - \kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle U - \kappa; A_j (U - \kappa) \rangle \partial_j \phi \right) dx dt + \int_{\mathbb{R}^n} |U^0(x) - \kappa|^2 \phi(0, x) dx \geq 0, \quad (4.2.2)$$

for all $\kappa \in \mathcal{C}$ and $\phi \in C_c^\infty([0, T] \times \mathbb{R}^n)$ with $\phi(t, x) \geq 0$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

We recall here the main result of [37].

Theorem 4.2.2. Assume that $U^0 \in L^2(\mathbb{R}^n, \mathcal{C})$. There exists a unique weak constrained solution $U \in L^2([0, T] \times \mathbb{R}^n, \mathcal{C})$ to (4.2.1) in the sense of Definition 4.2.1. In addition, this solution belongs to $C([0, T]; L^2(\mathbb{R}^n, \mathcal{C}))$, and if further $U^0 \in H^1(\mathbb{R}^n, \mathcal{C})$, then $U \in L^\infty([0, T], H^1(\mathbb{R}^n, \mathcal{C}))$.

As already mentioned, the well-posedness of problem (4.2.1) has been established in [37] thanks to a numerical method. We relax here the constraints $U(t, x) \in \mathcal{C}$ for a.e. $(t, x) \in (0, T) \times \mathbb{R}^n$ as

$$\partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon = \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}, \quad (4.2.3)$$

where $P_{\mathcal{C}}$ denotes the projection onto the closed convex set \mathcal{C} , and $\epsilon > 0$ is a small parameter. Formally, if we multiply equation (4.2.3) by ϵ , and let ϵ tend to 0, we get that the formal limit of U_ϵ , denoted by U , satisfies $P_{\mathcal{C}}(U) = U$, which ensures that $U \in \mathcal{C}$. In addition, Definition 4.2.1 has been motivated in [37] by a formal derivation from the relaxation system (4.2.3). To see it, it suffices to take the scalar product of equation (4.2.3) with $U_\epsilon - \kappa$, where $\kappa \in \mathcal{C}$ is arbitrary. We then use the first order characterization of the projection which ensures that the right hand side is non-positive, to get the inequality of Definition 4.2.1. The purpose of this work is to rigorously justify these formal steps.

The relaxation model presented here is very similar to viscous approximation of constrained models found in mechanics, and especially in plasticity. We start from the system of dynamical linear elasticity in three space dimensions which can be written as

$$\begin{cases} \partial_t F + \nabla_x v = 0, \\ \partial_t v + \operatorname{div} \sigma = 0, \end{cases} \quad (4.2.4)$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^3$. In the previous system, $F(t, x)$ is a 3×3 matrix which stands for the displacement gradient, $v(t, x) \in \mathbb{R}^3$ is the velocity, *i.e.* the displacement time derivative, and $\sigma = \mu(F + F^T) + \lambda(\operatorname{tr} F)I_3$ is the symmetric Cauchy stress tensor (here λ and μ are the Lamé coefficients, and I_3 is the identity matrix in \mathbb{R}^3). This system can be rewritten (thanks to a change of variables - see [82]) in the Friedrichs' framework as

$$\partial_t U + A_1 \partial_1 U + A_2 \partial_2 U + A_3 \partial_3 U = 0,$$

where U is a vector in \mathbb{R}^9 (containing the three components of the velocity v , and the six components of the stress σ) and A_1, A_2 and A_3 are 9×9 symmetric matrices.

We now introduce the convex constraint coming from plasticity, see [91]. Indeed, the theory of perfect plasticity is characterized by the fact that the stress tensor σ is constrained to stay inside a fixed closed convex set \mathcal{C} of symmetric 3×3 matrices. The total strain is then additively decomposed as the sum of (i) the elastic strain, denoted by e , which is still related to the stress by the linear relation $\sigma = \lambda \operatorname{tr} e + 2\mu e$; (ii) and the plastic strain, denoted by p , whose rate is oriented in a normal direction to \mathcal{C} at σ . Summarizing, one has

$$\frac{F + F^T}{2} = e + p, \quad \sigma = \lambda \operatorname{tr} e + 2\mu e \in \mathcal{C}, \quad \partial_t p \in \partial I_{\mathcal{C}}(\sigma), \quad (4.2.5)$$

where $\partial I_{\mathcal{C}}(\sigma)$ denotes the sub-differential of $I_{\mathcal{C}}$, the indicator function of \mathcal{C} , at the point σ . Using Fenchel-Moreau regularization of $I_{\mathcal{C}}$ (see [84]), the last condition in (4.2.5) can be relaxed as

$$\partial_t p = \frac{1}{\epsilon} (P_{\mathcal{C}}(\sigma) - \sigma),$$

where $\epsilon > 0$ is a viscosity parameter. We can now reformulate, at least formally, the dynamical problem of viscoplasticity (see [86, 91]) as

$$\begin{cases} \partial_t U + A_1 \partial_1 U + A_2 \partial_2 U + A_3 \partial_3 U = \frac{P_{\mathcal{C}}(U) - U}{\epsilon} & \text{on } (0, T) \times \mathbb{R}^3, \\ U(t, x) \in \mathcal{C} & \text{if } (t, x) \in [0, T] \times \mathbb{R}^3, \\ U(0, x) = U^0(x) & \text{if } x \in \mathbb{R}^3, \end{cases} \quad (4.2.6)$$

where, again, $U \in \mathbb{R}^9$, and $\tilde{\mathcal{C}} = \{U \in \mathbb{R}^9 : \sigma \in \mathcal{C}\}$, and A_1, A_2 and A_3 are the same matrices than in the elasto-dynamic case. As ϵ tends to zero, one expects the solution to (4.2.6) to converge to that of the model of perfect plasticity (see [92] in the quasi-static case).

Notation. In the sequel, we denote by $\langle ; \rangle$ the canonical scalar product of \mathbb{R}^m (and $\|\cdot\|$ the associated norm) and by $\langle | \rangle_{\mathcal{H}}$ the scalar product of an Hilbert space \mathcal{H} (when $\mathcal{H} \neq \mathbb{R}^m$).

This section is organized as follows. In the first subsection, we establish the existence and uniqueness of the relaxation model thanks to a parabolic approximation. In the second subsection, we prove that the relaxed solution U_ϵ to (4.2.3) satisfies the inequalities of Definition 4.2.1. Finally, to get the existence of a solution as a limit when ϵ tends

to zero of relaxed solutions U_ϵ , we prove the strong convergence of the sequence $(U_\epsilon)_{\epsilon>0}$ in the space $L^2((0, T) \times \omega)$, where ω is a open bounded subset of \mathbb{R}^n , to a weak solution to the constrained Friedrichs' systems.

4.2.1 Parabolic approximation

In order to construct a solution to the relaxation problem (4.2.3), we use a parabolic type regularization. To this aim, we consider a classical sequence of mollifiers in \mathbb{R}^n , denoted by $(\rho_\eta)_{\eta>0}$.

Theorem 4.2.3. *Let $U^0 \in H^1(\mathbb{R}^n, \mathcal{C})$. For every $\epsilon > 0$ and $\eta > 0$, the system*

$$\begin{cases} \partial_t U_{\epsilon, \eta} - \eta \Delta U_{\epsilon, \eta} + \sum_{j=1}^n A_j \partial_j U_{\epsilon, \eta} = \frac{P_{\mathcal{C}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta}}{\epsilon}, & \text{on } (0, T] \times \mathbb{R}^n, \\ U_{\epsilon, \eta}(0, x) = U^0(x) * \rho_\eta, & \text{if } x \in \mathbb{R}^n, \end{cases} \quad (4.2.7)$$

admits a unique solution $U_{\epsilon, \eta}$ with the following properties :

$$U_{\epsilon, \eta} \in L^2(0, T; H^2(\mathbb{R}^n, \mathbb{R}^m)), \quad \partial_t U_{\epsilon, \eta} \in L^2(0, T; H^1(\mathbb{R}^n, \mathbb{R}^m)),$$

and

$$\partial_{tt} U_{\epsilon, \eta} \in L^2(0, T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m)).$$

Furthermore, we have the following estimates

$$\sup_{0 \leq t \leq T} \|U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n, \mathbb{R}^m)}^2 \leq \|U^0\|_{L^2(\mathbb{R}^n, \mathbb{R}^m)}^2, \quad (4.2.8)$$

$$\sup_{0 \leq t \leq T} \|U_{\epsilon, \eta}(t)\|_{H^1(\mathbb{R}^n, \mathbb{R}^m)} \leq \|U^0\|_{H^1(\mathbb{R}^n, \mathbb{R}^m)}, \quad (4.2.9)$$

$$\sup_{0 \leq t \leq T} \|\partial_t U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n, \mathbb{R}^m)} \leq C \|U^0\|_{H^1(\mathbb{R}^n, \mathbb{R}^m)}, \quad (4.2.10)$$

for some constant $C > 0$ independent of η and ϵ .

Before giving the proof of this theorem, let us state two technical lemmas.

Lemma 4.2.4. *Let $U \in H^1((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$. Then the function $P_{\mathcal{C}} \circ U$ also belongs to $H^1((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$, and*

$$\|P_{\mathcal{C}} \circ U\|_{H^1((0, T) \times \mathbb{R}^n, \mathbb{R}^m)} \leq \|U\|_{H^1((0, T) \times \mathbb{R}^n, \mathbb{R}^m)}, \quad (4.2.11)$$

and we also have for all $1 \leq k \leq n$ for almost every $t \in (0, T)$ and almost every $x \in \mathbb{R}^n$

$$\begin{cases} |\partial_t (P_{\mathcal{C}} \circ U)|(t, x) \leq |\partial_t U|(t, x) \\ |\partial_k (P_{\mathcal{C}} \circ U)|(t, x) \leq |\partial_k U|(t, x) \end{cases} \quad (4.2.12)$$

Remark 4.2.5. *In particular, inequalities (4.2.12) imply that for every set $B \subset (0, T) \times \mathbb{R}^n$ the following inequalities hold*

$$\langle \partial_t (P_{\mathcal{C}} \circ U) - \partial_t U | \partial_t U \rangle_{L^2(B; \mathbb{R}^m)} \leq 0, \quad (4.2.13)$$

and, for $1 \leq k \leq n$,

$$\langle \partial_k (P_{\mathcal{C}} \circ U) - \partial_k U | \partial_k U \rangle_{L^2(B; \mathbb{R}^m)} \leq 0, \quad (4.2.14)$$

Proof. Let A be a compact subset of $(0, T)$. Using [22, Proposition 9.3], we know that the function $P_{\mathcal{C}} \circ U$ belongs to $H^1((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ if and only if for all $\delta \in \mathbb{R}$ such that $|\delta|$ is strictly less than the distance of A to the boundary of $(0, T)$ and all $h \in \mathbb{R}^n$, we have

$$\|P_{\mathcal{C}} \circ U(\cdot + \delta, \cdot + h) - P_{\mathcal{C}} \circ U\|_{L^2(A \times \mathbb{R}^n, \mathbb{R}^m)} \leq C \sqrt{\delta^2 + |h|^2}.$$

In addition, we know that $\|\nabla_{t,x} (P_{\mathcal{C}} \circ U)\|_{L^2(A \times \mathbb{R}^n, \mathbb{R}^m)} \leq C$. We now observe that for $(t, x) \in A \times \mathbb{R}^n$ and $|\delta|$ strictly less

than the distance of A to the boundary of $(0, T)$, we have

$$\begin{aligned} \left(\int_{A \times \mathbb{R}^n} |P_{\mathcal{E}} \circ U(t + \delta, x + h) - P_{\mathcal{E}} \circ U(t, x)|^2 dt dx \right)^{1/2} &\leq \left(\int_{A \times \mathbb{R}^n} |U(t + \delta, x + h) - U(t, x)|^2 dt dx \right)^{1/2} \\ &\leq \sqrt{\delta^2 + |h|^2} \|\nabla_{t,x} U\|_{L^2(A \times \mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

Consequently, we obtain that $P_{\mathcal{E}} \circ U$ belongs to $H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ and the inequality (4.2.11). To derive the inequalities (4.2.12), it suffices to notice that we have (by similar arguments) for any ω compact subset of \mathbb{R}^n , for $(t, x) \in A \times \omega$, $|\delta|$ strictly less than the distance of A to the boundary of $(0, T)$ and $|h| < \text{dist}(A, \omega)$

$$\begin{aligned} \left(\int_{A \times \omega} |P_{\mathcal{E}} \circ U(t + \delta, x + h) - P_{\mathcal{E}} \circ U(t, x)|^2 dt dx \right)^{1/2} &\leq \left(\int_{A \times \omega} |U(t + \delta, x + h) - U(t, x)|^2 dt dx \right)^{1/2} \\ &\leq \sqrt{\delta^2 + |h|^2} \|\nabla_{t,x} U\|_{L^2(A \times \omega; \mathbb{R}^m)}. \end{aligned}$$

It implies that

$$\|\nabla_{t,x} (P_{\mathcal{E}} \circ U)\|_{L^2(A \times \omega; \mathbb{R}^m)} \leq \|\nabla_{t,x} U\|_{L^2(A \times \omega; \mathbb{R}^m)}.$$

If we localize the previous inequality, we get the inequalities (4.2.12). \square

We also are going to use the following result (whose proof can be found in [41, Section 5.9.2]).

Lemma 4.2.6. *Let $U \in L^2(0, T, H^1(\mathbb{R}^n; \mathbb{R}^m))$ with $\partial_t U \in L^2(0, T; H^{-1}(\mathbb{R}^n; \mathbb{R}^m))$. Then, the function*

$$t \mapsto \|U(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2,$$

is absolutely continuous, and for a.e. $t \in [0, T]$,

$$\frac{d}{dt} \left(\frac{1}{2} \|U(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) = \langle U(t), \partial_t U(t) \rangle_{H^1(\mathbb{R}^n; \mathbb{R}^m), H^{-1}(\mathbb{R}^n; \mathbb{R}^m)}.$$

We now are in position to prove Theorem 4.2.3.

Proof. The proof essentially follows that of Theorem 1 in [41, Part II, Section 7.3.2]. Let $X = L^\infty(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))$ and $V \in X$. We consider the problem

$$\begin{cases} \partial_t U - \eta \Delta U = \frac{P_{\mathcal{E}}(V) - V}{\epsilon} - \sum_{j=1}^n A_j \partial_j V, & \text{on } (0, T) \times \mathbb{R}^n, \\ U(0, x) = U^0(x) * \rho_\eta, & \text{if } x \in \mathbb{R}^n. \end{cases} \quad (4.2.15)$$

Since $0 \in \mathcal{E}$, we have the following inequality

$$\forall k \in \mathbb{R}^m, \quad |(P_{\mathcal{E}}(k) - k)| \leq |k|$$

which shows that $\frac{P_{\mathcal{E}}(V) - V}{\epsilon} - \sum_{j=1}^n A_j \partial_j V \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$. Using the theory of parabolic equations (see [49,

Theorem 1.5] and also [32, Chapter XVI]), we get that equation (4.2.15) admits a unique solution U with $U \in L^2(0, T; H^2(\mathbb{R}^n; \mathbb{R}^m))$ and $\partial_t U \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$. Let $\tilde{V} \in X$ and \tilde{U} be the solution to (4.2.15) associated with \tilde{V} . The function $\hat{U} = U - \tilde{U}$ is a solution to

$$\begin{cases} \partial_t \hat{U} - \eta \Delta \hat{U} = \frac{P_{\mathcal{E}}(V) - P_{\mathcal{E}}(\tilde{V}) - (V - \tilde{V})}{\epsilon} - \sum_{j=1}^n A_j \partial_j (V - \tilde{V}), & \text{on } (0, T) \times \mathbb{R}^n, \\ \hat{U}(0, x) = 0, & \text{if } x \in \mathbb{R}^n. \end{cases} \quad (4.2.16)$$

Thanks to the theory of parabolic equations, we have the following estimate, we obtain the following estimate

$$\begin{aligned} \operatorname{ess-sup}_{0 \leq t \leq T} \|\hat{U}(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} &\leq C(\eta) \left\| \frac{P_{\mathcal{E}}(V) - P_{\mathcal{E}}(\tilde{V}) - (V - \tilde{V})}{\epsilon} - \sum_{j=1}^n A_j \partial_j (V - \tilde{V}) \right\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))} \\ &\leq C(\eta) \max\left(\frac{1}{\epsilon}, \|A_j\|\right) \|V - \tilde{V}\|_{L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))} \\ &\leq C(\eta) \max\left(\frac{1}{\epsilon}, \|A_j\|\right) \sqrt{T} \operatorname{ess-sup}_{0 \leq t \leq T} \|V(t) - \tilde{V}(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

Therefore, the mapping

$$\psi : \begin{cases} X \rightarrow \{U \in L^2(0, T; H^2(\mathbb{R}^n; \mathbb{R}^m)) \text{ and } \partial_t U \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))\} \subset X \\ \hat{V} \mapsto \hat{U} \end{cases}$$

is Lipschitz-continuous with Lipschitz constant bounded by $C(\eta) \max\left(\frac{1}{\epsilon}, \|A_j\|\right) \sqrt{T}$.

We now divide $[0, T]$ into sub-intervals $[0, T_1], [T_1, 2T_1], [2T_1, 3T_1], \dots, [NT_1, T]$ such that

$$C(\eta) \max\left(\frac{2}{\epsilon}, \|A_j\|\right) \sqrt{\max(T_1, T - NT_1)} < 1.$$

For every $i \in \{0, \dots, N-1\}$, the Banach fixed-point Theorem ensures the existence and uniqueness of a solution to (4.2.7) on the interval $[iT_1, (i+1)T_1]$ (with initial condition $U^0 * \rho_\eta$ if $i = 0$, and $U_{\epsilon, \eta}(iT_1)$ if $i \geq 1$ obtained at the previous step). We then obtain a solution $U_{\epsilon, \eta}$ on the entire interval $[0, T]$ by gluing the solutions on each sub-intervals, so that $U_{\epsilon, \eta} \in L^2(0, T; H^2(\mathbb{R}^n; \mathbb{R}^m))$. According to the initial condition on each sub-intervals, the function $t \mapsto U_{\epsilon, \eta}(t)$ is continuous in $L^2(\mathbb{R}^n; \mathbb{R}^m)$ at every $t = iT_1$, so that $\partial_t U_{\epsilon, \eta} \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$, and, in particular $U_{\epsilon, \eta} \in H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$.

To obtain the announced regularity, we use Lemma 4.2.4. It ensures that $(P_{\mathcal{E}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta})/\epsilon \in H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ and then, using again to the regularity theory of parabolic equations, we obtain

$$U_{\epsilon, \eta} \in L^2(0, T; H^2(\mathbb{R}^n, \mathbb{R}^m)), \quad \partial_t U_{\epsilon, \eta} \in L^2(0, T; H^1(\mathbb{R}^n, \mathbb{R}^m)),$$

and

$$\partial_{tt} U_{\epsilon, \eta} \in L^2(0, T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m)).$$

Now we derive the estimates. Using Lemma 4.2.6 to $U_{\epsilon, \eta}$, we get for a.e. $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) &= \langle U_{\epsilon, \eta}(t) | \partial_t U_{\epsilon, \eta}(t) \rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \\ &= \left\langle U_{\epsilon, \eta}(t) \left| \frac{P_{\mathcal{E}}(U_{\epsilon, \eta})(t) - U_{\epsilon, \eta}(t)}{\epsilon} - \sum_{j=1}^n A_j \partial_j U_{\epsilon, \eta}(t) + \eta \Delta U_{\epsilon, \eta}(t) \right. \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)}, \end{aligned} \quad (4.2.17)$$

where we used the fact that $U_{\epsilon, \eta}$ is a solution to the partial differential equation (4.2.7). Since $0 \in \mathcal{E}$, we have

$$\langle U_{\epsilon, \eta} | P_{\mathcal{E}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta} \rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)}(t) = \langle P_{\mathcal{E}}(U_{\epsilon, \eta}) | P_{\mathcal{E}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta} \rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)}(t) - \|P_{\mathcal{E}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta}\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2(t) \leq 0. \quad (4.2.18)$$

On the other hand, if $v \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$, an integration by parts shows that

$$\langle v | \eta \Delta v \rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} = -\eta \|Dv\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq 0,$$

and

$$\left\langle v \left| \sum_{j=1}^n A_j \partial_j v \right. \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} = 0,$$

since the matrices A_j are symmetric and independent of the space variables. By approximation, these formulas are

true for $v \in H^1(\mathbb{R}^n, \mathbb{R}^m)$ as well, and in particular,

$$\left\langle U_{\epsilon, \eta}(t) \left| \sum_{j=1}^n A_j \partial_j U_{\epsilon, \eta}(t) + \eta \Delta U_{\epsilon, \eta}(t) \right. \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq 0. \quad (4.2.19)$$

Gathering (4.2.17), (4.2.18) and (4.2.19), we obtain

$$\frac{d}{dt} \left(\frac{1}{2} \|U_{\epsilon, \eta}\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) \leq 0,$$

and using Gronwall's Lemma, we derive the first estimate (4.2.8).

$$\sup_{0 \leq t \leq T} \|U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 = \|U^0 * \rho_\eta\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|U^0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2. \quad (4.2.20)$$

We apply the same argument to the spatial weak derivatives $\partial_k U_{\epsilon, \eta}$ of $U_{\epsilon, \eta}$. Deriving the partial differential equation (4.2.7) in the sense of distribution, we infer that

$$\partial_t \partial_k U_{\epsilon, \eta} - \eta \Delta \partial_k U_{\epsilon, \eta} + \sum_{j=1}^n A_j \partial_j \partial_k U_{\epsilon, \eta} = \partial_k \frac{P_{\mathcal{C}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta}}{\epsilon}.$$

The previous equality actually holds in $L^2(0, T, L^2(\mathbb{R}^n; \mathbb{R}^m))$ thanks to the regularity of $U_{\epsilon, \eta}$, and we can apply Lemma 4.2.6 to obtain for a.e. $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\partial_k U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) &= \langle \partial_k U_{\epsilon, \eta}(t), \partial_t \partial_k U_{\epsilon, \eta}(t) \rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \\ &= \left\langle \partial_k U_{\epsilon, \eta}(t) \left| \partial_k \frac{P_{\mathcal{C}}(U_{\epsilon, \eta})(t) - U_{\epsilon, \eta}(t)}{\epsilon} - \sum_{j=1}^n A_j \partial_j \partial_k U_{\epsilon, \eta}(t) + \eta \Delta \partial_k U_{\epsilon, \eta}(t) \right. \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

Arguing as in (4.2.19), we get

$$\frac{d}{dt} \left(\frac{1}{2} \|\partial_k U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) \leq \left\langle \partial_k U_{\epsilon, \eta}(t) \left| \partial_k \frac{P_{\mathcal{C}}(U_{\epsilon, \eta})(t) - U_{\epsilon, \eta}(t)}{\epsilon} \right. \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)},$$

and using the inequality (4.2.14), we have

$$\frac{d}{dt} \left(\frac{1}{2} \|\partial_k U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) \leq 0.$$

It yields

$$\sup_{0 \leq t \leq T} \|\partial_k U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|\partial_k U^0 * \rho_\eta\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|\partial_k U^0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2, \quad (4.2.21)$$

which completes the proof of estimate (4.2.9).

We finally derive the last estimate for $\partial_t U_{\epsilon, \eta}$. Again, we derive the partial differential equation (4.2.7) with respect to t in the distributional sense to get

$$\partial_{tt} U_{\epsilon, \eta} - \eta \Delta \partial_t U_{\epsilon, \eta} + \sum_{j=1}^n A_j \partial_j \partial_t U_{\epsilon, \eta} = \partial_t \frac{P_{\mathcal{C}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta}}{\epsilon}$$

in $L^2(0, T, H^{-1}(\mathbb{R}^n; \mathbb{R}^m))$. Using again Lemma 4.2.6, we get for a.e. $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\partial_t U_{\epsilon, \eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) &= \langle \partial_t U_{\epsilon, \eta}(t) | \partial_{tt} U_{\epsilon, \eta}(t) \rangle_{H^1(\mathbb{R}^n; \mathbb{R}^m), H^{-1}(\mathbb{R}^n; \mathbb{R}^m)} \\ &= \langle \partial_t U_{\epsilon, \eta}(t) | \eta \Delta \partial_t U_{\epsilon, \eta}(t) - \sum_{j=1}^n A_j \partial_j \partial_t U_{\epsilon, \eta}(t) + \partial_t \frac{P_{\mathcal{C}}(U_{\epsilon, \eta})(t) - U_{\epsilon, \eta}(t)}{\epsilon} \rangle_{H^1(\mathbb{R}^n; \mathbb{R}^m), H^{-1}(\mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

As before, we infer that

$$\left\langle \partial_t U_{\epsilon,\eta}(t) \left| \sum_{j=1}^n A_j \partial_j \partial_t U_{\epsilon,\eta}(t) \right. \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} = 0, \quad \left\langle \partial_t U_{\epsilon,\eta}(t) \left| \eta \Delta \partial_t U_{\epsilon,\eta}(t) \right. \right\rangle_{H^1(\mathbb{R}^n; \mathbb{R}^m), H^{-1}(\mathbb{R}^n; \mathbb{R}^m)} \leq 0,$$

which shows, again using 4.2.13, that

$$\frac{d}{dt} \left(\frac{1}{2} \|\partial_t U_{\epsilon,\eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) \leq 0.$$

Consequently, we have

$$\sup_{0 \leq t \leq T} \|\partial_t U_{\epsilon,\eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|\partial_t U_{\epsilon,\eta}(0, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2. \quad (4.2.22)$$

Now, we are going to determine the value of $\partial_t U_{\epsilon,\eta}$ at $t = 0$. To this aim, let us take a test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ with $\phi(T, \cdot) = 0$. On the one hand, according to Fubini's Theorem and Green's formula on $[0, T]$, we have

$$\int_0^T \int_{\mathbb{R}^n} \langle \partial_t U_{\epsilon,\eta}; \partial_t \phi \rangle dx dt = - \int_{\mathbb{R}^n} \langle \partial_t U_{\epsilon,\eta}(0, \cdot); \phi(0, \cdot) \rangle dx - \int_0^T \langle \partial_{tt} U_{\epsilon,\eta} | \phi \rangle_{H^1(\mathbb{R}^n; \mathbb{R}^m), H^{-1}(\mathbb{R}^n; \mathbb{R}^m)} dt,$$

since $\partial_t U_{\epsilon,\eta} \in H^1(0, T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m))$ and ϕ is smooth. On the other hand, according to equation (4.2.7)

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \langle \partial_t U_{\epsilon,\eta}; \partial_t \phi \rangle dx dt &= \int_0^T \int_{\mathbb{R}^n} \left\langle \eta \Delta U_{\epsilon,\eta} - \sum_{j=1}^n A_j \partial_j U_{\epsilon,\eta} + \frac{P_{\mathcal{C}}(U_{\epsilon,\eta}) - U_{\epsilon,\eta}}{\epsilon}; \partial_t \phi \right\rangle dx dt \\ &= - \int_0^T \left\langle \partial_t \left[\eta \Delta U_{\epsilon,\eta} - \sum_{j=1}^n A_j \partial_j U_{\epsilon,\eta} + \frac{P_{\mathcal{C}}(U_{\epsilon,\eta}) - U_{\epsilon,\eta}}{\epsilon} \right] \middle| \phi \right\rangle_{H^1(\mathbb{R}^n; \mathbb{R}^m), H^{-1}(\mathbb{R}^n; \mathbb{R}^m)} \\ &\quad - \int_{\mathbb{R}^n} \left\langle \eta \Delta U^0 * \rho_\eta - \sum_{j=1}^n A_j \partial_j U^0 * \rho_\eta + \frac{P_{\mathcal{C}}(U^0 * \rho_\eta) - U^0 * \rho_\eta}{\epsilon}; \phi(0, \cdot) \right\rangle dx. \end{aligned}$$

Since $\phi(0, \cdot)$ is arbitrary, we obtain

$$\partial_t U_{\epsilon,\eta}(0, \cdot) = \eta \Delta U^0 * \rho_\eta - \sum_{j=1}^n A_j \partial_j U^0 * \rho_\eta + \frac{P_{\mathcal{C}}(U^0 * \rho_\eta) - U^0 * \rho_\eta}{\epsilon}.$$

Notice that since U_0 takes its value in \mathcal{C} , $U^0 * \rho_\eta$ takes also its value in \mathcal{C} , it implies that

$$\partial_t U_{\epsilon,\eta}(0, \cdot) = \eta \Delta U^0 * \rho_\eta - \sum_{j=1}^n A_j \partial_j U^0 * \rho_\eta.$$

From (4.2.22) and the fact that

$$\|\Delta U^0 * \rho_\eta\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \frac{C}{\eta} \|\nabla U^0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)},$$

for some constant $C > 0$ independent of η , we deduce that

$$\sup_{0 \leq t \leq T} \|\partial_t U_{\epsilon,\eta}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq C \|U^0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)},$$

where $C > 0$ is another constant independent of η and ϵ , which completes the proof of the last estimate (4.2.10). \square

4.2.2 Approximation of the convex constraints

Thanks to Theorem 4.2.3, we construct the solution to the relaxation problem

$$\begin{cases} \partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon = \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}, & \text{on } (0, T] \times \mathbb{R}^n, \\ U_\epsilon(0, x) = U^0(x), & \text{if } x \in \mathbb{R}^n. \end{cases} \quad (4.2.23)$$

as the limit of the solution of the parabolic problem (4.2.7) when η tends to zero.

Theorem 4.2.7. *There exists a unique solution $U_\epsilon \in H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ to (4.2.23) satisfying, for all $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$,*

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon; \phi \right\rangle dx dt = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; \phi \right\rangle dx dt, \quad (4.2.24)$$

and $U_\epsilon(0, \cdot) = U_0$ in $H^1(\mathbb{R}^n, \mathcal{C})$. In addition, we have the following estimates

$$\sup_{0 \leq t \leq T} \|U_\epsilon(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|U^0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2, \quad (4.2.25)$$

$$\sup_{0 \leq t \leq T} \|U_\epsilon(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U^0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}, \quad (4.2.26)$$

$$\sup_{0 \leq t \leq T} \|\partial_t U_\epsilon(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq C \|U^0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}, \quad (4.2.27)$$

for some constant $C > 0$ independent of η and ϵ .

Proof. Thanks to Theorem 4.2.3, the sequence $(U_{\epsilon, \eta})_{\eta > 0}$ is bounded in the space $H^1([0, T] \times \mathbb{R}^n; \mathbb{R}^m)$. We can thus extract a subsequence (not relabeled) such that

$$U_{\epsilon, \eta} \rightharpoonup U_\epsilon \text{ weakly in } H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m).$$

In particular, (4.2.25), (4.2.26) and (4.2.27) are consequences of the estimates of Theorem 4.2.3 by the lower semi-continuity of the norm with respect to weak convergence. Since the embedding of $H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ into $L^2_{\text{loc}}([0, T] \times \mathbb{R}^n; \mathbb{R}^m)$ is compact (cf [1]), we deduce that

$$U_{\epsilon, \eta} \rightarrow U_\epsilon \text{ strongly in } L^2((0, T) \times (-R, R)^n; \mathbb{R}^m),$$

for each $R > 0$. Let $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ and R such that the support of ϕ is contained in $[-R, R]^{n+1}$. Since $U_{\epsilon, \eta}$ is a weak solution of (4.2.15), we have

$$\int_0^T \int_{\mathbb{R}^n} \left(\langle \partial_t U_{\epsilon, \eta}; \phi \rangle + \eta \sum_{j=1}^n \langle \partial_j U_{\epsilon, \eta}; \partial_j \phi \rangle + \sum_{j=1}^n \langle A_j \partial_j U_{\epsilon, \eta}; \phi \rangle \right) dx dt = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_{\mathcal{C}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta}}{\epsilon}; \phi \right\rangle dx dt.$$

Using the weak convergence, we infer that

$$\int_0^T \int_{\mathbb{R}^n} \left(\langle \partial_t U_{\epsilon, \eta}; \phi \rangle + \eta \sum_{j=1}^n \langle \partial_j U_{\epsilon, \eta}; \partial_j \phi \rangle + \sum_{j=1}^n \langle A_j \partial_j U_{\epsilon, \eta}; \phi \rangle \right) dx dt \rightarrow \int_0^T \int_{\mathbb{R}^n} \left(\langle \partial_t U_\epsilon; \phi \rangle + \sum_{j=1}^n \langle A_j \partial_j U_\epsilon; \phi \rangle \right) dx dt.$$

On the other hand, the strong convergence yields

$$\left\langle \frac{P_{\mathcal{C}}(U_{\epsilon, \eta}) - U_{\epsilon, \eta}}{\epsilon}; \phi \right\rangle \rightarrow \left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; \phi \right\rangle \text{ strongly in } L^1((0, T) \times \mathbb{R}^n),$$

and consequently, we obtain

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon; \phi \right\rangle dx dt = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; \phi \right\rangle dx dt. \quad (4.2.28)$$

We next focus on the initial condition. We take $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^n; \mathbb{R}^m)$ (in particular $\phi(T) = 0$). An integration by parts shows that

$$-\int_0^T \int_{\mathbb{R}^n} \langle U_{\epsilon, \eta}; \partial_t \phi \rangle dx dt + \int_{\mathbb{R}^n} \langle U^0 * \rho_\eta(x); \phi(0, x) \rangle dx = \int_0^T \int_{\mathbb{R}^n} \langle \partial_t U_{\epsilon, \eta}; \phi \rangle dx dt.$$

Letting η tend to zero, and using (4.2.28) leads to

$$\int_0^T \int_{\mathbb{R}^n} \left(-\langle U_\epsilon; \partial_t \phi \rangle + \sum_{j=1}^n \langle A_j \partial_j U_\epsilon; \phi \rangle \right) dx dt + \int_{\mathbb{R}^n} \langle U^0(x); \phi(0, x) \rangle dx = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; \phi \right\rangle dx dt, \quad (4.2.29)$$

since $U^0 * \rho_\eta \rightarrow U^0$ strongly in $L_{\text{loc}}^2(\mathbb{R}^n; \mathbb{R}^m)$. We now integrate by parts in (4.2.28), using the fact that $U_\epsilon \in H^1(0, T, L^2(\mathbb{R}^n; \mathbb{R}^m))$,

$$\int_0^T \int_{\mathbb{R}^n} \left(\langle U_\epsilon; \partial_t \phi \rangle + \sum_{j=1}^n \langle A_j \partial_j U_\epsilon; \phi \rangle \right) dx dt + \int_{\mathbb{R}^n} \langle U_\epsilon(0, x); \phi(0, x) \rangle dx = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; \phi \right\rangle dx dt. \quad (4.2.30)$$

Using the equations (4.2.29) and (4.2.30), it gives that

$$\int_{\mathbb{R}^n} \langle U_\epsilon(0, x); \phi(0, x) \rangle dx = \int_{\mathbb{R}^n} \langle U^0(x); \phi(0, x) \rangle dx,$$

and consequently the initial condition is satisfied in $L^2(\mathbb{R}^n; \mathbb{R}^m)$.

It remains to show the uniqueness. Let us consider two solutions U_ϵ and \tilde{U}_ϵ associated with the same initial condition U^0 . Using the partial differential equation (4.2.23), we obtain

$$\partial_t (U_\epsilon - \tilde{U}_\epsilon) + \sum_{j=1}^n A_j \partial_j (U_\epsilon - \tilde{U}_\epsilon) = \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon - P_{\mathcal{C}}(\tilde{U}_\epsilon) + \tilde{U}_\epsilon}{\epsilon}.$$

As already observed, we know that for a.e. $t \in [0, T]$,

$$\int_{\mathbb{R}^n} \sum_{j=1}^n \langle A_j \partial_j U_\epsilon(t) - \tilde{U}_\epsilon(t); U_\epsilon(t) - \tilde{U}_\epsilon(t) \rangle dx = 0,$$

and also

$$\frac{1}{2} \frac{d}{dt} \|U_\epsilon(t) - \tilde{U}_\epsilon(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 = \left\langle \partial_t U_\epsilon(t) - \partial_t \tilde{U}_\epsilon(t) \mid U_\epsilon(t) - \tilde{U}_\epsilon(t) \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)}.$$

Consequently, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|U_\epsilon(t) - \tilde{U}_\epsilon(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \\ &= \left\langle \frac{P_{\mathcal{C}}(U_\epsilon)(t) - U_\epsilon(t) - P_{\mathcal{C}}(\tilde{U}_\epsilon)(t) + \tilde{U}_\epsilon(t)}{\epsilon} \mid U_\epsilon(t) - \tilde{U}_\epsilon(t) \right\rangle_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \frac{1}{\epsilon} \|U_\epsilon(t) - \tilde{U}_\epsilon(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2, \end{aligned}$$

since $\text{Id} - P_{\mathcal{C}}$ is 1-Lipschitz. Gronwall's Lemma thus implies that $U_\epsilon = \tilde{U}_\epsilon$ since they satisfy the same initial condition. As a consequence of the uniqueness, we deduce that there is no need to extract a subsequence from $(U_{\epsilon, \eta})_{\eta > 0}$ to get the convergences as $\eta \rightarrow 0$. \square

4.2.3 Convergence of the relaxed formulation

In this section, we first show that the solution U_ϵ to the relaxation problem (4.2.23) satisfies the inequality of Definition 4.2.1, and then we prove that we can pass to the limit in this inequality to get a solution to the initial problem (4.2.1).

Lemma 4.2.8. *Let U_ϵ be the unique solution to (4.2.23). For all $\kappa \in \mathcal{C}$ and for all $\phi \in U^{1, \infty}((-\infty, T) \times \mathbb{R}^n, \mathbb{R}^+)$ with compact*

support in $(-\infty, T) \times \mathbb{R}^n$, one has

$$\int_0^T \int_{\mathbb{R}^n} \left(|U_\epsilon - \kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle U_\epsilon - \kappa; A_j(U_\epsilon - \kappa) \rangle \partial_j \phi \right) dx dt + \int_{\mathbb{R}^n} |U^0(x) - \kappa|^2 \phi(0, x) dx \geq 0. \quad (4.2.31)$$

Proof. Since U_ϵ is a solution to (4.2.23), we know that

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon - \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; U_\epsilon - \kappa \right\rangle \phi dx dt = 0.$$

By the first order characterization of the projection, one has

$$\left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; U_\epsilon - \kappa \right\rangle(t, x) \leq 0$$

for a.e. $(t, x) \in [0, T] \times \mathbb{R}^n$. On the other hand, since $U_\epsilon \in H^1(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$ and $U_\epsilon \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))$ we can integrate by parts to obtain the desired result. \square

Remark 4.2.9. Let us stress that, although the function U_ϵ satisfies the same inequality than the weak constrained solution, it is not a weak constrained solution in the sense of Definition 4.2.1 because it does not a priori belong to \mathcal{C} .

To get a weak constrained solution from the sequence of solutions $(U_\epsilon)_{\epsilon>0}$ to the relaxation problem (4.2.23), we need to pass to the limit as $\epsilon \rightarrow 0$ in the previous inequality. This is the purpose of the following result.

Theorem 4.2.10. For every bounded open set $\omega \subset \mathbb{R}^n$, the sequence $(U_\epsilon)_{\epsilon>0}$ converges strongly in $L^2((0, T) \times \omega; \mathbb{R}^m)$ to some function U which is a weak constrained solution to problem (4.2.1).

Proof. The uniform (with respect to ϵ) bounds of Theorem 4.2.7 and the fact that the embedding of $H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ into $L^2_{\text{loc}}([0, T] \times \mathbb{R}^n; \mathbb{R}^m)$ is compact allow us to say that the sequence $(U_\epsilon)_{\epsilon>0}$ admits a subsequence (not relabeled) which converges strongly in $L^2([0, T]; L^2(\omega; \mathbb{R}^m))$ and weakly in $L^2([0, T]; L^2(\mathbb{R}^n; \mathbb{R}^m))$ to some $U \in L^2([0, T]; L^2(\mathbb{R}^n; \mathbb{R}^m))$. Let us take a test function $\phi \in C_c^\infty((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ in (4.2.24). Multiplying this inequality by ϵ and passing to the limit as $\epsilon \rightarrow 0$ yields $U = P_{\mathcal{C}}(U)$ a.e. in $(0, T) \times \mathbb{R}^n$ which shows that $U \in L^2([0, T]; L^2(\mathbb{R}^n; \mathcal{C}))$. Finally, passing to the limit as $\epsilon \rightarrow 0$ in (4.2.31) shows that U is a solution in the sense of Definition 4.2.1 to the problem (4.2.1). Note finally that, by uniqueness of the solution to (4.2.1) (see [37, Lemma 9]), there is no need to extract a subsequence to get the above convergences as $\epsilon \rightarrow 0$. \square

Remark 4.2.11. When $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$, the unique solution U to (4.2.1) belongs to $H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ and satisfies the following a priori estimates

$$\sup_{0 \leq t \leq T} \|U(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|U^0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2, \quad \sup_{0 \leq t \leq T} \|U(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U^0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)},$$

and

$$\sup_{0 \leq t \leq T} \|\partial_t U(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq C \|U^0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}.$$

The construction of the solution U to (4.2.1) rests on the assumption that the initial data $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$. Let us now explain how to construct a solution U when U_0 only belongs to $L^2(\mathbb{R}^n; \mathcal{C})$. We use here the following result whose proof can be found in [37]

Theorem 4.2.12. Let U^0 and $\tilde{U}^0 \in H^1(\mathbb{R}^n; \mathcal{C})$. We denote by U (resp. \tilde{U}) the solution in $L^2(0, T; L^2(\mathbb{R}^n; \mathcal{C}))$ to problem (4.2.1) in the sense of Definition 4.2.1 associated with U^0 (resp. \tilde{U}^0). Then, U and \tilde{U} belong to $C([0, T]; L^2(\mathbb{R}^n; \mathcal{C}))$, and, in addition, we have the following estimate

$$\forall t \in [0, T], \forall r > 0, \quad \|U(t, \cdot) - \tilde{U}(t, \cdot)\|_{L^2(B(0, r); \mathbb{R}^m)} \leq \|U^0 - \tilde{U}^0\|_{L^2(B(0, r+nLT); \mathbb{R}^m)},$$

where L is the maximum of the spectral radii of the matrices A_i .

By mollification, let us construct a sequence $(U_k^0)_{k \in \mathbb{N}}$ such that $U_k^0 \in H^1(\mathbb{R}^n; \mathcal{C})$ for all $k \in \mathbb{N}$, which converges to U^0 in $L^2(\mathbb{R}^n; \mathcal{C})$. The estimates of Theorem 4.2.12 imply that

$$\sup_{t \in [0, T]} \|U_k(t, \cdot) - U_l(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U_k^0 - U_l^0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}, \quad (4.2.32)$$

where U_k (resp. U_l) is the solution to (4.2.1) associated with the initial condition U_k^0 (resp. U_l^0). It follows that the sequence $(U_k)_{k \in \mathbb{N}}$ is of Cauchy type in $L^\infty(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$, and therefore it converges strongly in $L^\infty(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$ to some function $U \in L^\infty(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$. Thanks to the strong convergence, we find that U satisfies the inequality (4.2.2). In addition, since $U_k = P_{\mathcal{C}}(U_k)$ for all $k \in \mathbb{N}$, we deduce that $U = P_{\mathcal{C}}(U)$ which ensures that $U \in L^\infty(0, T; L^2(\mathbb{R}^n; \mathcal{C}))$. The following result has thus been established.

Theorem 4.2.13. *Let $U^0 \in L^2(\mathbb{R}^n; \mathcal{C})$, then there exists a unique solution $U \in L^\infty(0, T, L^2(\mathbb{R}^n; \mathcal{C}))$ to (4.2.1) in the sense of Definition 4.2.1.*

Remark 4.2.14. *In this section, we construct an approximation of the Friedrichs' constrained solution by sending firstly the viscosity parameter η to 0 and then the parameter ϵ (related to the constraint) to 0. Since we proved that the sequence $(U_{\epsilon, \eta})$ is bounded in $H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ uniformly with respect to the two parameters ϵ and η , one can also send first the parameter ϵ to 0 in order to obtain a function $U_\eta \in H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ that satisfy the following inequality*

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \left(|U_\eta - \kappa|^2 \partial_t \phi + \sum_{j=1}^n \langle U_\eta - \kappa; A_j(U_\eta - \kappa) \rangle \partial_j \phi \right) dx dt + \eta \int_0^T \int_{\mathbb{R}^n} |U_\eta - \kappa|^2 \Delta \phi dx dt \\ + \int_{\mathbb{R}^n} |U^0 * \rho_\eta(x) - \kappa|^2 \phi(0, x) dx \geq \int_0^T \int_{\mathbb{R}^n} |\nabla(U_\eta - \kappa)|^2 \phi dx dt \geq 0, \end{aligned} \quad (4.2.33)$$

for all $\kappa \in \mathcal{C}$ and all $\phi \in U^{2, \infty}((-\infty, T) \times \mathbb{R}^n; \mathbb{R}^+)$ with compact support in $(-\infty, T) \times \mathbb{R}^n$. Then one can send the parameter η to 0 to get an approximation of the Friedrichs' constrained solution.

4.2.4 Conclusion

Lastly, the relaxed problem (4.2.23), that was used in [37] to derive formally a definition of weak solutions of hyperbolic constrained problems, is actually a rigorous way to construct weak solutions of hyperbolic constrained problems. It is worth noting that this relaxation procedure is deeply related to viscoplastic models. In order to fully apply this theory to mechanical problems, one should consider problems that are posed in bounded spatial domain. To do so, a new formulation of weak solutions to Friedrichs' systems posed in bounded domains is proposed in Section 2.5.1, without constraints. It remains now to investigate the interactions between the boundary conditions which are considered in Chapter 2 and the convex constraints from the numerical point of view.

4.3 Numerical study of Friedrichs' systems under constraint on the whole space

The goal of this section is to construct a scheme on unstructured meshes for constrained Friedrichs' problems, posed in the whole space, and to analyze the rate of convergence for that scheme towards a regular dissipative solution (defined in [37] - see also Theorem 1.4.2 and Definition 1.4.1).

On the one hand, the analysis of the classical upwind, explicit in time, finite volume scheme on unstructured meshes for unconstrained linear first order hyperbolic systems have been performed in [98] and [55]. More precisely, they examine the approximation of a solution of the following problem

$$\begin{cases} \partial_t U + \sum_{i=1}^n \partial_i (A_i U) + BU = f, & \text{in } (0, T) \times \mathbb{R}^n, \\ U(0, x) = U_0(x), & \text{in } \mathbb{R}^n, \end{cases}$$

where $U : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $A_i : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{M}_{\text{sym}}^{m \times m}$, $B : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{M}^{m \times m}$, $f : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbb{M}^{m \times m}$ (resp. $\mathbb{M}_{\text{sym}}^{m \times m}$) is the space of $m \times m$ (resp. symmetric) matrices with real coefficients. Their result is established for very wide class of systems : there is no restriction on the dimensions m and n , the matrices may depend on t and x , they also add a zero-order term thanks to the matrix B . Since we are interested in a restricted class of such systems (no zero-order term, constant coefficients matrices, no term source), we decide to present the idea of [98] in this special case. The scheme, analyzed in [98], yields a piecewise constant approximate solution V_h . First, it can be shown that this scheme enjoys a stability property (under a Courant-Friedrichs-Lewy condition) : the $L^2(\mathbb{R}^n)$ -norm of V_h decreases in time. Then, the main purpose of [98] is to derive the following rate of convergence

$$\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \leq Ch^{1/2}, \quad (4.3.1)$$

where h is the maximum of diameters of the meshes and C is constant only depending on the data (the matrices A_i and B , the source term f , the initial condition U_0 and the final time T).

Numerous works have been done to measure the rate of convergence of finite volume schemes, associated to linear first order hyperbolic equations, in various L^p spaces. For scalar equations, a rate of convergence of order $h^{1/2}$ in the space $L^\infty((0, T); L^1(\mathbb{R}^n))$ for $BV(\mathbb{R}^n)$ -initial data (when the mesh is Cartesian) can be found in [61]. This result has been extended for several finite volume schemes on unstructured meshes (see for example [21, 24, 26, 44, 97]). In these papers, the authors obtain a (non-optimal) rate of convergence of order $h^{1/4}$ in the space $L^\infty((0, T); L^1(\mathbb{R}^n))$ for $BV(\mathbb{R}^n)$ -initial data. In the context of transport equations, several authors have derived rates of convergences for finite volume schemes on unstructured meshes : for $H^2(\mathbb{R}^n)$ -initial data, Després derives a rate of order $h^{1/2}$ in $L^\infty((0, T); L^2(\mathbb{R}^d))$ in [36], for $C^2(\mathbb{R}^n)$ -initial data ; Bouche, Ghidaglia and Pascal obtain a rate of order h in $L^\infty((0, T) \times \mathbb{R}^d)$ in [19]; for $BV(\mathbb{R}^n)$ -initial data ; Merlet and Vovelle get a rate of order $h^{1/2}$ in $L^\infty((0, T); L^1(\mathbb{R}^d))$ in [78], for Lipschitz initial data ; Merlet derives a rate of order $h^{1/2-\epsilon}$ in $L^\infty((0, T); L^1(\mathbb{R}^d))$ in [77]. The results of [77, 78] have been rederived by Delarue and Lagoutière in [33] thanks to a probabilistic approach. We choose here to focus our attention on the derivation of a rate of convergence, in the constrained case, associated to the norm $L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ since this norm is well suited to the theoretical framework of this kind of problems (see Chapter 2) and the analysis we perform closely follows the lines of [98, 28].

To derive the estimate (4.3.1), the main idea of [98] is to derive an inequality, closely related to the comparison principle derived in [37, Lemma 8] (see also (1.4.5)), of the following form

$$- \int_0^T \int_{\mathbb{R}^n} |V_h - U|^2(t, x) \partial_t \phi(t, x) dx dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h - U; A_i(V_h - U) \rangle(t, x) \partial_i \phi(t, x) dx dt \leq \langle \mu_h^2, \phi \rangle - \langle \mu_h^1, 2U \phi \rangle,$$

where $\phi \in C_c^1((0, T) \times \mathbb{R}^n; \mathbb{R}^+)$, μ_h^1 and μ_h^2 are linear forms encoding the errors due to the numerical approximation. A precise analysis of these error terms allows one to obtain the rate of convergence (4.3.1).

On the other hand, in [37], the authors study the existence and uniqueness of a dissipative solution associated with the problem

$$\begin{cases} \partial_t U + \sum_{i=1}^n A_i \partial_i U = 0, & \text{in } (0, T) \times \mathbb{R}^n, \\ U(0, x) = U_0(x), & \text{in } \mathbb{R}^n, \\ U \in \mathcal{C}, \end{cases}$$

where $U : (t, x) \in (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ and \mathcal{C} is a fixed closed convex subset of \mathbb{R}^m . To prove the existence, they construct a numerical approximation on a Cartesian mesh. The scheme proposed in [37] is based on a splitting strategy. The first step uses the Rusanov scheme (for a detailed presentation of this scheme see [64, Section 12.5]) and in a second step, they project on every mesh onto the convex set \mathcal{C} . It is important to notice that, due to a compactness argument, they prove that their approximation converges to a dissipative solution without estimating the rate of convergence.

The idea of this section is to combined the scheme described in [98] and the splitting strategy of [37] to construct an approximation of constrained Friedrichs' systems on unstructured meshes and to derive an error estimate in the space $L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$, similar to the one obtained in [98].

We organize this Section as follows : in Subsection 4.3.1, we introduce useful notations and describe precisely the scheme that we use. In Subsection 4.3.2, we show that this scheme is stable under a certain Courant-Friedrichs-Lewy condition, in the following sense : the $L^2(\mathbb{R}^n)$ -norm of the numerical solution decreases over time. Then, we derive a dissipative inequality, similar to the one exposed in [98], which includes error terms due to the numerical approximation. Finally, we analyze in details the error terms and obtain an error estimate with the same rate of convergence as in the unconstrained case of [98].

4.3.1 Description of the scheme

In this section, we present the general framework of this work and the scheme we are interested in. Let \mathcal{T}_h be a triangulation of \mathbb{R}^n such that : let $\mathcal{I} \subset \mathbb{N}$, let $\mathcal{T}_h = (K_i)_{i \in \mathcal{I}}$ a family of open nonempty convex polytopes such that $U_{i \in \mathcal{I}} \bar{K}_i = \mathbb{R}^n$, for all $i \neq j$, $K_i \cap K_j = \emptyset$ and $h = \sup_{i \in \mathcal{I}} (\text{diam} K_i) < +\infty$. The set of edges of a polytope K is denoted \mathcal{E}_K . We introduce the following notations (see also Figure 4.1),

$$\begin{aligned} m_K, m_{\partial K} &: \mathcal{L}^n\text{-measure of } K \text{ and } \mathcal{H}^{n-1}\text{-measure } \partial K, \\ e \in \mathcal{E}_K &: \text{an edge } ((n-1)\text{-dimensional polytope) of } K \text{ with } \mathcal{H}^{n-1}\text{-measure } m_e, \\ \nu_{K_e} = (\nu_{K_e}^1, \nu_{K_e}^2, \dots, \nu_{K_e}^n) &: \text{the unit exterior normal of } K \text{ on the edge } e, \\ K_e &: \text{neighboring cell of } K \text{ with } \bar{K} \cap \bar{K}_e = e. \end{aligned}$$

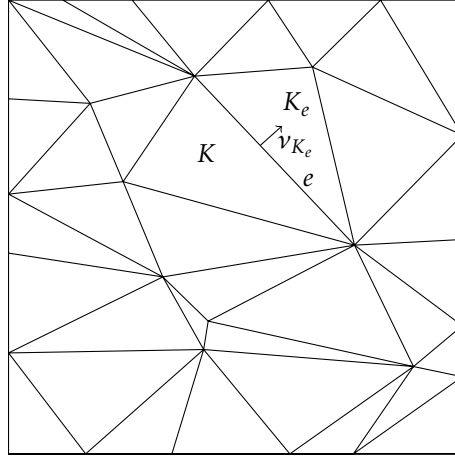


FIGURE 4.1 – An unstructured meshes of the square $[0, 1] \times [0, 1]$. Here the polytopes are triangles.

We also suppose that the triangulation is regular in the sense that there exists a constant $C_1 > 0$ (independent of the triangulation \mathcal{T}_h) such that

$$\forall K \in \mathcal{T}_h, \quad C_1 h^n \leq m_K, \quad \text{and} \quad \forall K \in \mathcal{T}_h, \quad \forall e \in \mathcal{E}_K \quad C_1 h^{n-1} \leq m_e.$$

Remark 4.3.1. The existence of a constant C_2 (independent of the triangulation \mathcal{T}_h), such that the reverse inequality

$$\forall K \in \mathcal{T}_h, \quad m_K \leq C_2 h^n \quad \text{and} \quad \forall e \in \mathcal{E}_K, \quad m_e \leq C_2 h^{n-1}$$

holds, is due to the isodiametric inequality (see [42] for more details).

Remark 4.3.2. If A is a symmetric real matrix of size $n \times n$ and P a unitary matrix such that $A = P^t D P$ where $D = (d_{i,j})_{1 \leq i,j \leq n}$ is a diagonal matrix with

$$d_{1,1} \leq d_{2,2} \leq \dots \leq d_{q,q} \leq 0 < d_{q+1,q+1} \leq \dots \leq d_{n-1,n-1} \leq d_{n,n},$$

for some integer q between 1 and n , then the positive (resp. negative) part, denoted A^+ (resp. A^-), of A is equal to the matrix $A^+ = P^t D^+ P$ (resp. $A^- = P^t D^- P$) where $D^+ = (d_{i,j}^{pos})_{1 \leq i,j \leq n}$ (resp. $D^- = (d_{i,j}^{neg})_{1 \leq i,j \leq n}$) is a diagonal matrix with

$$\begin{aligned} d_{1,1}^{pos} = d_{2,2}^{pos} = \dots = d_{q,q}^{pos} = 0 \quad \text{and} \quad d_{q+1,q+1}^{pos} = d_{q+1,q+1}, \dots, d_{n-1,n-1}^{pos} = d_{n-1,n-1}, d_{n,n}^{pos} = d_{n,n}, \\ d_{1,1}^{neg} = d_{1,1}, d_{2,2}^{neg} = d_{2,2}, \dots, d_{q,q}^{neg} = d_{q,q} \quad \text{and} \quad d_{q+1,q+1}^{neg} = \dots = d_{n-1,n-1}^{neg} = d_{n,n}^{neg} = 0. \end{aligned}$$

One should observe that $A = A^+ + A^-$ and also that these decompositions do not depend on the choice of P .

We use a finite volume scheme for the following initial value problem

$$\begin{cases} \partial_t U + \sum_{i=1}^n A_i \partial_i U = 0, & \text{on } (0, T) \times \mathbb{R}^n, \\ U(0, x) = U_0(x), & \text{on } \mathbb{R}^n, \\ U(t, x) \in \mathcal{C} \subset \mathbb{R}^m, & \text{a.e in } (0, T) \times \mathbb{R}^n \end{cases} \quad (4.3.2)$$

where \mathcal{C} is a closed convex (independent of t and x) such that there exists $r > 0$ such that $B(0, r) \subset \mathcal{C}$ (see also Section 4.2). The equations of (4.3.2) have to be understood in the sense of Definition 4.2.1 (see also [37]). To construct such a scheme for the system (4.3.2), we first let the unknown U evolve and forget the constraint. We integrate the equation of motion in a cell K to get

$$\int_K \partial_t U \, dx + \int_K \sum_{i=1}^n A_i \partial_i U \, dx = 0,$$

Integrating by parts the second integral provides

$$\int_K \partial_t U \, dx + \int_{\partial K} A_\nu U \, d\mathcal{H}^{n-1} = 0,$$

Discretizing the time derivative by the Euler explicit scheme leads us to

$$\frac{1}{t^{p+1} - t^p} \int_K U(t^{p+1}) - U(t^p) \, dx + \int_{\partial K} A_\nu U \, d\mathcal{H}^{n-1} = 0,$$

One needs to discretize the flux $A_\nu U$ at the boundary of K . We observe that we need to discretize the flux exchange at each edge of K . Hence, we rewrite the previous equation as

$$\frac{1}{t^{p+1} - t^p} \int_K U(t^{p+1}) - U(t^p) \, dx + \sum_{e \in \mathcal{E}_K} \int_e A_\nu U \, d\mathcal{H}^{n-1} = 0,$$

Then a simple discretization of the flux at the edge e is the following

$$\int_e A_\nu U \, d\mathcal{H}^{n-1} \approx m_e A_{K_e}^+ v_K^p + m_e A_{K_e}^- v_{K_e}^p,$$

where v_K^p stands for the approximate mean value of U on $[t^p, t^{p+1}) \times K$ and $A_{K_e} = \sum_{i=1}^n A_i v_{K_e}^i$. The previous approximation means that the flux $A_\nu U$ at the edge e between the cell K and K_e is approximated by the sum of the component of U in the cell K (at time t^p) associated to the non-negative eigenvalues of A_ν and the component of U in the cell K_e (at time t^p) associated to the positive eigenvalues of A_ν . This leads us to the following step

$$\frac{m_K}{t^{p+1} - t^p} (v_K^{p,*} - v_K^p) + \sum_{e \in \mathcal{E}_K} m_e A_{K_e}^+ v_K^p + m_e A_{K_e}^- v_{K_e}^p = 0.$$

Here, we decide to denote by $v_K^{p,*}$ the approximate value of U (on $[t^p, t^{p+1}) \times K$) after this first step in which we forget that the original problem was constrained. In order to take account of the constraint, we simply project on each cell K the value $v_K^{p,*}$ onto the set \mathcal{C} . Hence, the second step is

$$v_K^{p+1} = P_{\mathcal{C}}(v_K^{p,*}).$$

It leads us to a piecewise constant approximation of U , denoted by V_h , such that

$$\forall (t, x) \in [t^p, t^{p+1}) \times K, \quad V_h(t, x) = v_K^p,$$

where $0 = t^0 < t^1 < \dots < t^{N+1} = T$ ($t^{p+1} - t^p = \Delta t$) constructed thanks to the following scheme (explicit in time) and if $U_0 \in L^2(\mathbb{R}^n; \mathcal{C})$, it takes the following form

$$\boxed{\begin{cases} v_K^0 &= \frac{1}{m_K} \int_K U_0(x) \, dx. \\ v_K^{p,*} &= v_K^p - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} g_{K_e} m_e, \\ v_K^{p+1} &= P_{\mathcal{C}}(v_K^{p,*}), \end{cases}} \quad (4.3.3)$$

where p is an integer in $[0, N]$, $P_{\mathcal{C}}$ is the projection onto the closed convex \mathcal{C} and

$$g_{K_e} = D_{K_e} v_K^p - C_{K_e} v_{K_e}^p, \quad \text{with} \quad C_{K_e} = -(A_{K_e})^- + \eta \text{Id} \text{ and } D_{K_e} = (A_{K_e})^+ + \eta \text{Id}, \quad (4.3.4)$$

where $A_{K_e} = \sum_{i=1}^n A_i v_{K_e}^i$, $(A_{K_e})^-$ (resp. $(A_{K_e})^+$) is the negative (resp. positive) part of A_{K_e} , η an arbitrary positive constant.

Remark 4.3.3. The term ηId is necessary to compare the space parameter h and the time parameter Δt (see Remark 4.3.6). Without this term, Theorem 4.3.17 would lead to a rate of convergence depending on h and Δt .

In particular, one has $A_{K_e} = D_{K_e} - C_{K_e}$. Using this identity and a discrete version of Green's formula

$$\sum_{e \in \mathcal{E}_K} A_{K_e} m_e = 0, \quad (4.3.5)$$

leads us to a non conservative form of the scheme (4.3.3)

$$v_K^{p,*} = v_K^p - \frac{\Delta t}{m_K} \left(\sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right). \quad (4.3.6)$$

Remark 4.3.4. As in the previous section, we denote by $\langle \cdot; \cdot \rangle$ the canonical scalar product of \mathbb{R}^m and $|\cdot|$ the associated norm. By abuse of notation, we also use the notation $|\cdot|$ for the (matrix) operator norm associated with the canonical norm of \mathbb{R}^m . In addition, if P is a non-negative symmetric matrix, we denote for $y \in \mathbb{R}^m$ by $|y|_P$ the quantity $\langle y; Py \rangle$.

4.3.2 Stability in time of the scheme

In this section, we prove that the scheme enjoys the same stability property as the one in [98]. Namely, the $L^2(\mathbb{R}^n)$ -norm of the solution does not increase in time, under a Courant-Friedrichs-Lewy condition, shortened CFL condition in the following.

Proposition 4.3.5 (Stability of the scheme (4.3.3)). *Suppose that there exists $\epsilon \in]0, 1[$ such that the following CFL condition holds :*

$$\sup_{K, e \in \mathcal{E}_K} \frac{\Delta t m_{\partial K}}{m_K} |C_{K_e}| \leq 1 - \epsilon, \quad (4.3.7)$$

then we have

$$m_K \left(|v_K^{p,*}|^2 - |v_K^p|^2 \right) + \Delta t \sum_{e \in \mathcal{E}_K} \left(\langle C_{K_e} v_K^p; v_K^p \rangle - \langle C_{K_e} v_{K_e}^p; v_{K_e}^p \rangle \right) m_e \leq -\epsilon \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e} (v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e.$$

Remark 4.3.6. The CFL condition implies, in particular, that there exists a constant C (depending only on η and n) such that

$$\Delta t \leq Ch.$$

Indeed, from the CFL condition we deduce that

$$\forall K \in \mathcal{T}_h, \quad \forall e \in \mathcal{E}_K \quad \Delta t |C_{K_e}| \leq \frac{m_K}{m_{\partial K}},$$

and thanks to the definition of C_{K_e} and the fact that $-(A_{K_e})^-$ is non-negative, it yields

$$\forall K \in \mathcal{T}_h, \quad \eta \Delta t \leq \frac{m_K}{m_{\partial K}}.$$

The isoperimetric inequality (see [46, Section 3.2.43]) implies then that there exists a constant C (depending only on n) such that

$$\eta \Delta t \leq C m_K^{1/n}.$$

The isodiametric inequality (see Remark 4.3.1) allows us to conclude.

Proof. Let p be an integer in $[0, N]$. From the non-conservative form (4.3.6), we have

$$v_K^{p,*} = \sum_{e \in \mathcal{E}_K} \frac{m_e}{m_{\partial K}} v_K^{p,*}(e),$$

where we set

$$v_K^{p,*}(e) = v_K^p - \frac{\Delta t m_{\partial K}}{m_K} C_{K_e} (v_K^p - v_{K_e}^p).$$

Observe that,

$$\begin{aligned}
|v_K^{p,*}(e)|^2 &= |v_K^p|^2 - 2 \frac{\Delta t m_{\partial K}}{m_K} \langle v_K^p; C_{K_e}(v_K^p - v_{K_e}^p) \rangle + \left| \frac{\Delta t m_{\partial K}}{m_K} C_{K_e}(v_K^p - v_{K_e}^p) \right|^2 \\
&= |v_K^p|^2 - \frac{\Delta t m_{\partial K}}{m_K} \left(\langle v_K^p; C_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) \\
&\quad - \frac{\Delta t m_{\partial K}}{m_K} \left\langle v_K^p - v_{K_e}^p; \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} C_{K_e} \right) C_{K_e} (v_K^p - v_{K_e}^p) \right\rangle \\
&\leq |v_K^p|^2 - \frac{\Delta t m_{\partial K}}{m_K} \left(\langle v_K^p; C_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) - \epsilon \frac{\Delta t m_{\partial K}}{m_K} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle
\end{aligned}$$

Here we have used the CFL condition to obtain

$$\forall y \in \mathbb{R}^m, \left\langle \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} C_{K_e} \right) y; y \right\rangle \geq \epsilon |y|^2.$$

and consequently,

$$-\frac{\Delta t m_{\partial K}}{m_K} \left\langle v_K^p - v_{K_e}^p; \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} C_{K_e} \right) C_{K_e} (v_K^p - v_{K_e}^p) \right\rangle \leq -\epsilon \frac{\Delta t m_{\partial K}}{m_K} \left| C_{K_e}^{\frac{1}{2}} (v_K^p - v_{K_e}^p) \right|^2.$$

Finally, using convexity,

$$|v_K^{p,*}|^2 \leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} \left(\langle v_K^p; C_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e - \epsilon \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle m_e. \quad (4.3.8)$$

□

Remark 4.3.7. Since $0 \in \mathcal{C}$, we have $|P_{\mathcal{C}}(x)| \leq |x|$ and this implies that $|v_K^{p+1}| \leq |v_K^{p,*}|$. It follows that

$$m_K \left(\left| v_K^{p+1} \right|^2 - \left| v_K^p \right|^2 \right) + \Delta t \sum_{e \in \mathcal{E}_K} \left(\langle C_{K_e} v_K^p; v_K^p \rangle - \langle C_{K_e} v_{K_e}^p; v_{K_e}^p \rangle \right) m_e \leq -\epsilon \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e} (v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e. \quad (4.3.9)$$

Furthermore, if we use the relation (4.3.5), we obtain

$$m_K \left(\left| v_K^{p+1} \right|^2 - \left| v_K^p \right|^2 \right) + \Delta t \sum_{e \in \mathcal{E}_K} \left(\langle D_{K_e} v_K^p; v_K^p \rangle - \langle C_{K_e} v_{K_e}^p; v_{K_e}^p \rangle \right) m_e \leq -\epsilon \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e} (v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e. \quad (4.3.10)$$

Proposition 4.3.8. Assume that $U_0 \in L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ and Δt satisfies the CFL condition for a given $\epsilon \in]0, 1[$, then the approximate solution V_h satisfies

$$\forall t \in [0, T], \quad \|V_h(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}.$$

In addition, we have, for any $q \in [0, N]$

$$\sum_{p=0}^q \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \langle C_{K_e} (v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e \Delta t \leq \frac{1}{\epsilon} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2. \quad (4.3.11)$$

Proof. Let $t \in [0, T]$ and q an integer such that $t \in [t^q, t^{q+1})$ (or $q = N + 1$ if $t = T$). Let p an integer in $[0, N]$. First, we remark

$$\sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \left(\langle D_{K_e} v_K^p; v_K^p \rangle - \langle C_{K_e} v_{K_e}^p; v_{K_e}^p \rangle \right) m_e = 0,$$

thanks to the definition of D_{K_e} and C_{K_e} in (4.3.4) and since each term on an edge e appears twice in the previous sum (depending on which side of e we are) and these two terms cancel out. Then we sum over $K \in \mathcal{T}_h$ in the relation (4.3.10) to get

$$\sum_{K \in \mathcal{T}_h} m_K \left| v_K^{p+1} \right|^2 + \epsilon \sum_{K \in \mathcal{T}_h} \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e} (v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e \leq \sum_{K \in \mathcal{T}_h} m_K \left| v_K^p \right|^2.$$

We rewrite the previous inequality as

$$\|V_h(t^{p+1}, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \epsilon \sum_{K \in \mathcal{T}_h} \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e}(v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e \leq \|V_h(t^p, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

Summing the last inequality from $p = 0$ to $q - 1$ we get,

$$\|V_h(t^q, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \epsilon \sum_{p=0}^{q-1} \sum_{K \in \mathcal{T}_h} \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e}(v_K^p - v_{K_e}^p); (v_K^p - v_{K_e}^p) \rangle m_e \leq \|V_h(t^0, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

Observing that $\|V_h(t^q, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} = \|V_h(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}$, the result follows. \square

Proposition 4.3.9. *Assume that $(U_0, \widetilde{U}_0) \in L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ and Δt satisfies the CFL condition for a given $\epsilon \in]0, 1[$, then the approximate solution V_h (resp. \widetilde{V}_h) associated with U_0 (resp. \widetilde{U}_0) satisfies*

$$\forall t \in [0, T], \quad \|V_h(t, \cdot) - \widetilde{V}_h(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U_0 - \widetilde{U}_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}.$$

Proof. We denote by V_h the approximate solution associated with U_0 using the scheme (4.3.3) and by v_K^p the value of V_h on $[t^p, t^{p+1}) \times K$. Similarly, we denote by \widetilde{V}_h the approximate solution associated with \widetilde{U}_0 using the scheme (4.3.3) and by \widetilde{v}_K^p the value of \widetilde{V}_h on $[t^p, t^{p+1}) \times K$. We denote by $d_K^p = v_K^p - \widetilde{v}_K^p$ and $d_K^{p,*} = v_K^{p,*} - \widetilde{v}_K^{p,*}$, using the linearity of the first step of the scheme we have

$$d_K^{p,*} = d_K^p - \frac{\Delta t}{m_K} \left(\sum_{e \in \mathcal{E}_K} C_{K_e} (d_K^p - d_{K_e}^p) m_e \right).$$

As in the proof of Proposition 4.3.5, we obtain

$$m_K \left(|d_K^{p,*}|^2 - |d_K^p|^2 \right) + \Delta t \sum_{e \in \mathcal{E}_K} \left(\langle D_{K_e} d_K^p; d_K^p \rangle - \langle C_{K_e} d_{K_e}^p; d_{K_e}^p \rangle \right) m_e \leq -\epsilon \Delta t \sum_{e \in \mathcal{E}_K} \langle C_{K_e} (d_K^p - d_{K_e}^p); (d_K^p - d_{K_e}^p) \rangle m_e. \quad (4.3.12)$$

Using the fact that the projection is 1-Lipschitz, we get

$$\left| d_K^{p+1} \right| = \left| P_{\mathcal{C}}(v_K^{p,*}) - P_{\mathcal{C}}(\widetilde{v}_K^{p,*}) \right| \leq \left| v_K^{p,*} - \widetilde{v}_K^{p,*} \right| = \left| d_K^{p,*} \right|.$$

Summing the relation (4.3.12) over $K \in \mathcal{T}_h$, we get (as in the proof of Proposition 4.3.8)

$$\forall t \in [0, T], \quad \|V_h(t, \cdot) - \widetilde{V}_h(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U_0 - \widetilde{U}_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}.$$

\square

4.3.3 Dissipative inequality with error terms

In this section, we derive a Kato inequality (see Theorem 1.4.2 in the introduction of this thesis or [37, Lemma 8]) with error terms (due to the numerical approximation), when $U_0 \in H^1(\mathbb{R}^n)$ takes its values in \mathcal{C} and has a compact support, between the numerical approximation V_h constructed from U_0 and U the dissipative solution (in the sense of [37] - see also Definition 4.2.2) associated with U_0 .

Remark 4.3.10. *Observe that V_h and U have a compact support (in $[0, T] \times \mathbb{R}^n$). It is due to the finite speed propagation property of the explicit scheme for V_h and to an argument similar to the one we have presented in Proposition 3.6.1 for U . Since V_h has a compact support in $[0, T] \times \mathbb{R}^n$ (see Remark 4.3.10), all the sums considered here have a finite number of non-zero terms.*

In the following proposition, we derive an identity that shows that the approximate solution satisfies the partial differential equation up to error terms.

Proposition 4.3.11. *Let $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$ with a compact support. The solution V_h , constructed from U_0 thanks to the scheme (4.3.3), satisfies, for all functions ψ in $H^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$, the following equality*

$$- \int_0^T \int_{\mathbb{R}^n} \langle V_h; \partial_t \psi \rangle dx dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i \psi \rangle dx dt - \int_{\mathbb{R}^n} \langle U_0; \psi(0, \cdot) \rangle dx = \langle \mu_h^1, \psi \rangle, \quad (4.3.13)$$

with

$$\langle \mu_h^1, \psi \rangle = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; \psi_K^p \rangle - \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{N+1}; \psi_K(t^{N+1}) \rangle + \mathcal{R}_h^1(\psi) + \mathcal{R}_h^2(\psi) + \mathcal{R}_h^3(\psi),$$

and

$$\begin{aligned} \mathcal{R}_h^1(\psi) &= \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; \psi_K(t^{p+1}) - \psi_K^p \rangle, & \mathcal{R}_h^2(\psi) &= \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e}(\psi_e^p - \psi_K^p) \rangle, \\ \mathcal{R}_h^3(\psi) &= \sum_{K \in \mathcal{T}_h} m_K \langle v_K^0; \psi_K(t^0) \rangle - \int_{\mathbb{R}^n} \langle U_0; \psi(0, \cdot) \rangle dx, \end{aligned}$$

where

$$\psi_K(t) = \frac{1}{m_K} \int_K \psi(t, x) dx, \quad \psi_K^p = \frac{1}{\Delta t} \int_{t^p}^{t^{p+1}} \psi_K(t) dt, \quad \text{and} \quad \psi_e^p = \frac{1}{\Delta t m_e} \int_{t^p}^{t^{p+1}} \int_e \psi(t, x) d\mathcal{H}^{n-1}(x) dt. \quad (4.3.14)$$

Proof. First observe that

$$- \int_0^T \int_{\mathbb{R}^n} \langle V_h; \partial_t \psi \rangle dx dt = - \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^p; \psi_K(t^{p+1}) - \psi_K(t^p) \rangle.$$

A discrete integration by parts in time implies that

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^n} \langle V_h; \partial_t \psi \rangle dx dt &= \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; \psi_K^p \rangle + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; \psi_K(t^{p+1}) - \psi_K^p \rangle \\ &\quad - \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{N+1}; \psi_K(t^{N+1}) \rangle + \sum_{K \in \mathcal{T}_h} m_K \langle v_K^0; \psi_K(t^0) \rangle. \end{aligned}$$

Now we examine the other terms of the left-hand side of (4.3.13). Since the matrices A_i do not depend on the space variables, we have

$$- \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i \psi \rangle dx dt = - \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \left\langle v_K^p; \int_{t^p}^{t^{p+1}} \int_{\mathbb{R}^n} \sum_{i=1}^n \partial_i (A_i \psi) dx dt \right\rangle.$$

We integrate by parts in space to obtain

$$- \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i \psi \rangle dx dt = - \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \left\langle v_K^p; \sum_{e \in \mathcal{E}_K} m_e A_{K_e} \psi_e^p \right\rangle.$$

Denoting by \mathcal{S}_h the set of all edges e of the mesh, we have

$$- \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \left\langle v_K^p; \sum_{e \in \mathcal{E}_K} m_e A_{K_e} \psi_e^p \right\rangle = - \sum_{p=0}^N \Delta t \sum_{e \in \mathcal{S}_h} m_e \langle v_K^p - v_{K_e}^p, A_{K_e} \psi_e^p \rangle,$$

since the unit exterior normal on the edge e is equal to its opposite depending on which side of e we are. It yields

$$- \sum_{p=0}^N \Delta t \sum_{e \in \mathcal{S}_h} m_e \langle v_K^p - v_{K_e}^p, A_{K_e} \psi_e^p \rangle = - \sum_{p=0}^N \Delta t \sum_{e \in \mathcal{S}_h} m_e \langle v_K^p - v_{K_e}^p, (-C_{K_e} + C_{K_{e'}}) \psi_e^p \rangle,$$

where $K_{e'}$ is the neighboring cell of K_e such that $\overline{K_{e'}} \cap K_e = e$. Thanks to the definition of the matrices C_{K_e} (see (4.3.4)),

we get

$$-\sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i \psi \rangle dx dt = \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} \psi_e^p \rangle$$

Consequently,

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^n} \langle V_h; \partial_t \psi \rangle dx dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i \psi \rangle dx dt - \int_{\mathbb{R}^n} \langle U_0; \psi(0, \cdot) \rangle dx = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; \psi_K^p \rangle \\ & + \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} \psi_K^p \rangle + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; \psi_K(t^{p+1}) - \psi_K^p \rangle + \sum_{K \in \mathcal{T}_h} m_K \langle v_K^0; \psi_K(t^0) \rangle \\ & - \int_{\mathbb{R}^n} \langle U_0; \psi(0, \cdot) \rangle dx + \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (\psi_e^p - \psi_K^p) \rangle - \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{N+1}; \psi_K(t^{N+1}) \rangle \end{aligned}$$

Now using the scheme, we have

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^n} \langle V_h; \partial_t \psi \rangle dx dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i \psi \rangle dx dt - \int_{\mathbb{R}^n} \langle U_0; \psi(0, \cdot) \rangle dx = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; \psi_K^p \rangle \\ & - \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{N+1}; \psi_K(t^{N+1}) \rangle + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; \psi_K(t^{p+1}) - \psi_K^p \rangle \\ & + \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (\psi_e^p - \psi_K^p) \rangle + \sum_{K \in \mathcal{T}_h} m_K \langle v_K^0; \psi_K(t^0) \rangle - \int_{\mathbb{R}^n} \langle U_0; \psi(0, \cdot) \rangle dx \\ & = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; \psi_K^p \rangle - \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{N+1}; \psi_K(t^{N+1}) \rangle + \mathcal{R}_h^1(\psi) + \mathcal{R}_h^2(\psi) + \mathcal{R}_h^3(\psi). \end{aligned}$$

□

Thanks to the previous proposition, we can derive an inequality that contains all the quadratic terms depending on the solution U and the terms that depends on U and V_h simultaneously that are necessary to obtain the Kato inequality in Proposition 4.3.14.

Proposition 4.3.12. *Let $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$ with a compact support. For $\phi \in C_b^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^+)$ (the space of C^1 functions which are bounded and have bounded derivatives) and U a dissipative solution associated with U_0 , which is in $H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ and has a compact support (see Remark 4.2.11 and Remark 4.3.10), we have*

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} |U|^2(t, x) \partial_t \phi(t, x) dx dt + \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle U; A_i U \rangle(t, x) \partial_i \phi(t, x) dx dt \\ & - \int_0^T \int_{\mathbb{R}^n} 2 \langle V_h; U \rangle \partial_t \phi dx dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} 2 \langle V_h; A_i U \rangle \partial_i \phi dx dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) dx \geq \langle \mu_h^1, 2U\phi \rangle \quad (4.3.15) \end{aligned}$$

Proof. Since $U \in H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ and has a compact support and $\phi \in C_b^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^+) \subset H_{loc}^1(\mathbb{R} \times \mathbb{R}^n)$, the function $U\phi$ belongs to $H^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ and has a compact support. Consequently, one can take $\psi = 2U\phi$ as a test function in the equality (4.3.13) to obtain

$$\begin{aligned}
& -2 \int_0^T \int_{\mathbb{R}^n} \langle V_h; U \rangle \partial_t \phi \, dx \, dt - 2 \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i U \rangle \partial_i \phi \, dx \, dt - 2 \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx \\
& \quad - 2 \int_0^T \int_{\mathbb{R}^n} \langle V_h; \partial_t U \rangle \phi \, dx \, dt - 2 \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i \partial_i U \rangle \phi \, dx \, dt = \langle \mu_h^1, 2U \phi \rangle
\end{aligned}$$

Using the fact that $U \in H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$, it yields

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n} |U|^2 \partial_t \phi \, dx \, dt + \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle U; A_i U \rangle \partial_i \phi \, dx \, dt + \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx \\
& \quad = -2 \int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t U + \sum_{i=1}^n A_i \partial_i U; U \right\rangle \phi + \int_{\mathbb{R}^n} |U(T, \cdot)|^2 \phi(T, \cdot) \, dx.
\end{aligned}$$

Adding the last two equalities, we get

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n} |U|^2(t, x) \partial_t \phi(t, x) \, dx \, dt + \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle U; A_i U \rangle(t, x) \partial_i \phi(t, x) \, dx \, dt \\
& \quad - 2 \int_0^T \int_{\mathbb{R}^n} \langle V_h; U \rangle \partial_t \phi \, dx \, dt - 2 \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i U \rangle \partial_i \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx \\
& \quad = \langle \mu_h^1, 2U \phi \rangle + \int_{\mathbb{R}^n} |U(T, x)|^2 \phi(T, x) \, dx - 2 \int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t U + \sum_{i=1}^n A_i \partial_i U; U - V_h \right\rangle \phi.
\end{aligned} \tag{4.3.16}$$

We now use the approximation of U , denoted U_ϵ (constructed in Theorem 4.2.7), such that

$$U_\epsilon \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m)) \text{ and } \partial_t U_\epsilon \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m)),$$

and

$$\begin{cases} \partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon = \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}, & \text{in } L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m), \\ U_\epsilon(0, x) = U^0(x), & \text{in } L^2(\mathbb{R}^n; \mathbb{R}^m). \end{cases} \tag{4.3.17}$$

Furthermore, we know that the sequences $(U_\epsilon)_{\epsilon>0}$, $(\partial_1 U_\epsilon)_{\epsilon>0}$, ..., $(\partial_n U_\epsilon)_{\epsilon>0}$ and $(\partial_t U_\epsilon)_{\epsilon>0}$ are bounded, uniformly with respect to ϵ , in $L^\infty(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$. Using the characterization of the projection $P_{\mathcal{C}}$ we obtain (since $V_h(t, x) \in \mathcal{C}$ for every (t, x))

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t U_\epsilon + \sum_{j=1}^n A_j \partial_j U_\epsilon; U_\epsilon - V_h \right\rangle \phi = \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_{\mathcal{C}}(U_\epsilon) - U_\epsilon}{\epsilon}; U_\epsilon - V_h \right\rangle \phi \leq 0.$$

Since U_ϵ weakly converges to U in $H^1(\mathcal{K}; \mathbb{R}^m)$ with $\text{supp}(U) \subset \mathcal{K}$ and \mathcal{K} compact subset of $(0, T) \times \mathbb{R}^n$ and U_ϵ strongly converges to U in $L^2(\mathcal{K}; \mathbb{R}^m)$, we get

$$\int_0^T \int_{\mathbb{R}^p} \left\langle \partial_t U + \sum_{j=1}^p A_j \partial_j U; U - V_h \right\rangle \phi \leq 0.$$

Using this inequality in (4.3.16), we get the claim. \square

The following proposition contains the quadratic terms depending on the approximate solution V_h that are needed to obtain the Kato inequality in Proposition 4.3.14.

Proposition 4.3.13. *Let $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$ with a compact support. The solution V_h , constructed from U_0 thanks to the*

scheme (4.3.3), satisfies for all $\phi \in C_b^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^+)$

$$-\int_0^T \int_{\mathbb{R}^n} |V_h|^2 \partial_t \phi \, dx \, dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i V_h \rangle \partial_i \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx \leq \langle \mu_h^2, \phi \rangle, \quad (4.3.18)$$

and for all $\phi \in C_b^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^+)$,

$$\begin{aligned} \langle \mu_h^2, \phi \rangle &= \sum_{i=1}^3 \mathcal{E}_h^i(\phi) + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(|v_K^{p+1} - v_K^{p,*}|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle \right) \phi_K^p \\ &\quad - \sum_{K \in \mathcal{T}_h} m_K |v_K^{N+1}|^2 \phi_K(t^{N+1}) - \epsilon \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \Delta t m_e \langle C_{K_e}(v_K^p - v_{K_e}^p); v_K^p - v_{K_e}^p \rangle \phi_K^p, \end{aligned}$$

with

$$\phi_K(t) = \frac{1}{m_K} \int_K \phi(t, x) \, dx, \quad \phi_K^p = \frac{1}{\Delta t} \int_{t^p}^{t^{p+1}} \phi_K(t) \, dt, \quad \text{and} \quad \phi_e^p = \frac{1}{\Delta t m_e} \int_{t^p}^{t^{p+1}} \int_e \phi(t, x) \, d\mathcal{H}^{n-1}(x) \, dt. \quad (4.3.19)$$

and the terms $\mathcal{E}_h^1(\phi)$, $\mathcal{E}_h^2(\phi)$ and $\mathcal{E}_h^3(\phi)$ are defined as follows

$$\mathcal{E}_h^1(\phi) = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(|v_K^{p+1}|^2 - |v_K^p|^2 \right) (\phi_K(t^{p+1}) - \phi_K^p), \quad \mathcal{E}_h^2(\phi) = \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle C_{K_e}(v_K^p - v_{K_e}^p); v_K^p + v_{K_e}^p \rangle (\phi_e^p - \phi_K^p),$$

and

$$\mathcal{E}_h^3(\phi) = \int_{\mathbb{R}^n} \left(|V_h(0^+, x)|^2 - |U_0(x)|^2 \right) \phi(0, x) \, dx.$$

Proof. Let us find the expression of μ_h^2 .

$$-\int_0^T \int_{\mathbb{R}^n} |V_h|^2 \partial_t \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx = - \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K |v_K^p|^2 (\phi_K(t^{p+1}) - \phi_K(t^p)) - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx.$$

Using a discrete integration by parts in time and the definition of $\mathcal{E}_h^1(\phi)$, we get

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^n} |V_h|^2 \partial_t \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx &= \mathcal{E}_h^1(\phi) + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(|v_K^{p+1}|^2 - |v_K^p|^2 \right) \phi_K^p \\ &\quad - \sum_{K \in \mathcal{T}_h} m_K |v_K^{N+1}|^2 \phi_K(t^{N+1}) + \sum_{K \in \mathcal{T}_h} m_K |v_K^0|^2 \phi_K(t^0) - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx. \end{aligned}$$

Then, the definition of $\mathcal{E}_h^3(\phi)$ implies that

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^n} |V_h|^2 \partial_t \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx \\ = \mathcal{E}_h^1(\phi) + \mathcal{E}_h^3(\phi) + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(|v_K^{p+1}|^2 - |v_K^p|^2 \right) \phi_K^p - \sum_{K \in \mathcal{T}_h} m_K |v_K^{N+1}|^2 \phi_K(t^{N+1}). \end{aligned}$$

We now examine the space derivatives. Integrating by parts in space, we have

$$-\sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i V_h \rangle \partial_i \phi \, dx \, dt = - \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \left\langle v_K^p; \left(\sum_{e \in \mathcal{E}_K} m_e A_{K_e} \phi_e^p \right) v_K^p \right\rangle.$$

Again, using the set of all edges, denoted \mathcal{S}_h , we obtain

$$\sum_{K \in \mathcal{T}_h} \left\langle v_K^p; \left(\sum_{e \in \mathcal{E}_K} m_e A_{K_e} \phi_e^p \right) v_K^p \right\rangle = \sum_{e \in \mathcal{S}_h} m_e \langle v_K^p - v_{K_e}^p, A_{K_e} (v_K^p + v_{K_e}^p) \rangle \phi_e^p.$$

Denoting K_{e_e} the neighboring cell of K_e such that $\overline{K_{e_e}} \cap K_e = e$, it implies that

$$\sum_{e \in \mathcal{S}_h} m_e \langle v_K^p - v_{K_e}^p, A_{K_e} (v_K^p + v_{K_e}^p) \rangle \phi_e^p = \sum_{e \in \mathcal{S}_h} m_e \langle v_K^p - v_{K_e}^p, (-C_{K_e} + C_{K_{e_e}}) (v_K^p + v_{K_e}^p) \rangle \phi_e^p$$

Reindexing the sum (in space), we know that

$$\sum_{K \in \mathcal{T}_h} \left\langle v_K^p; \left(\sum_{e \in \mathcal{E}_K} m_e A_{K_e} \phi_e^p \right) v_K^p \right\rangle = - \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p + v_{K_e}^p) \rangle \phi_e^p.$$

Consequently, using the definition of $\mathcal{E}_h^2(\phi)$, it yields

$$- \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i V_h \rangle \partial_i \phi \, dx \, dt = \mathcal{E}_h^2(\phi) + \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p + v_{K_e}^p) \rangle \phi_K^p.$$

Then,

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^n} |V_h|^2 \partial_t \phi \, dx \, dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h; A_i V_h \rangle \partial_i \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx = \sum_{i=1}^3 \mathcal{E}_h^i(\phi) \\ & + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(|v_K^{p+1}|^2 - |v_K^p|^2 \right) \phi_K^p + \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p + v_{K_e}^p) \rangle \phi_K^p - \sum_{K \in \mathcal{T}_h} m_K |v_K^{N+1}|^2 \phi_K(t^{N+1}). \end{aligned}$$

We observe that

$$\begin{aligned} |v_K^{p+1}|^2 - |v_K^p|^2 &= |v_K^{p+1} - v_K^{p,*}|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle + |v_K^{p,*}|^2 - |v_K^p|^2 \\ &= |v_K^{p+1} - v_K^{p,*}|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle + |v_K^{p,*} - v_K^p|^2 + 2 \langle v_K^{p,*} - v_K^p; v_K^p \rangle. \end{aligned}$$

Using the scheme (4.3.6), we obtain

$$|v_K^{p+1}|^2 - |v_K^p|^2 = |v_K^{p+1} - v_K^{p,*}|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle - 2 \left\langle v_K^p; \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right\rangle + \left| \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right|^2.$$

Consequently, we have

$$\begin{aligned} |v_K^{p+1}|^2 - |v_K^p|^2 + \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p + v_{K_e}^p) \rangle &= |v_K^{p+1} - v_K^{p,*}|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle \\ &+ \left| \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle \end{aligned}$$

and, then using convexity, we get

$$\begin{aligned} \left|v_K^{p+1}\right|^2 - \left|v_K^p\right|^2 + \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e}(v_K^p + v_{K_e}^p) \rangle &\leq \left|v_K^{p+1} - v_K^{p,*}\right|^2 \\ + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle + \sum_{e \in \mathcal{E}_K} \frac{m_e}{m_{\partial K}} \left| \frac{\Delta t m_{\partial K}}{m_K} C_{K_e}(v_K^p - v_{K_e}^p) \right|^2 &- \sum_{e \in \mathcal{E}_K} \frac{m_e}{m_{\partial K}} \frac{\Delta t m_{\partial K}}{m_K} \langle v_K^p - v_{K_e}^p; C_{K_e}(v_K^p - v_{K_e}^p) \rangle. \end{aligned}$$

Following the proof of Proposition 4.3.5, we get

$$\begin{aligned} \left|v_K^{p+1}\right|^2 - \left|v_K^p\right|^2 + \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e}(v_K^p + v_{K_e}^p) \rangle \\ \leq \left|v_K^{p+1} - v_K^{p,*}\right|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle - \epsilon \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} m_e \langle C_{K_e}(v_K^p - v_{K_e}^p); v_K^p - v_{K_e}^p \rangle. \end{aligned}$$

Finally, since the function ϕ is non-negative, it leads us to the following inequality

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^n} |V_h|^2 \partial_t \phi - \sum_{i=1}^n \langle V_h; A_i V_h \rangle \partial_i \phi \, dx \, dt - \int_{\mathbb{R}^n} |U_0|^2 \phi(0, \cdot) \, dx = \sum_{i=1}^3 \mathcal{E}_h^i(\phi) - \sum_{K \in \mathcal{T}_h} m_K |v_K^{N+1}|^2 \phi_K(t^{N+1}) \\ + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(\left|v_K^{p+1} - v_K^{p,*}\right|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle \right) \phi_K^p - \epsilon \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle C_{K_e}(v_K^p - v_{K_e}^p); v_K^p - v_{K_e}^p \rangle \phi_K^p. \end{aligned}$$

□

Proposition 4.3.14. *Let $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$ with a compact support. The solution V_h , constructed from U_0 thanks to the scheme (4.3.3), and the dissipative solution U associated with U_0 obey to the following called Kato inequality*

$$- \int_0^T \int_{\mathbb{R}^n} |V_h - U|^2(t, x) \partial_t \phi(t, x) \, dx \, dt - \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \langle V_h - U; A_i(V_h - U) \rangle(t, x) \partial_i \phi(t, x) \, dx \, dt \leq \langle \mu_h^2, \phi \rangle - \langle \mu_h^1, 2U\phi \rangle,$$

for all $\phi \in \mathcal{C}_b^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^+)$.

Proof. Subtract to the inequality (4.3.18) the inequality (4.3.15). □

4.3.4 Convergence analysis

From the dissipative inequality derived in the previous section, we first obtain a rate of convergence, in $L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$, in the case where U_0 (resp. V_h and $U \in H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$) has a compact support in \mathbb{R}^n (resp. in $[0, T] \times \mathbb{R}^n$). Then, we explain how one can get the similar rate of convergence in the general case when $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$ and $U \in H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ and there is no restriction on the supports of these functions.

Theorem 4.3.15. *Let $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$ with a compact support. The solution V_h , constructed from U_0 thanks to the scheme (4.3.3), and the dissipative solution U associated with U_0 , which is in $H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ and has a compact support (see Remark 4.2.11 and Remark 4.3.10), obey to the following inequality*

$$\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \leq C\sqrt{h},$$

for some constant C depending on ϵ , η , the matrices A_i , the dimensions n , the final time T and the initial data U_0 .

Remark 4.3.16. *In the following proof, we decide to emphasize the dependence of the previous constant on the initial data. To do so, we denote by C a constant that may depend only on ϵ , η , α (see the following definition of ϕ), the matrices A_i , the dimensions n , the final time T . Although the constant may change from one line to another, we keep the same letter C by abuse of notation.*

Proof. Since U and V_h are compactly supported, one can take in the previous propositions the function $\phi(t, x) =$

$\exp(-\alpha t)(T-t)\mathbb{1}_{[0,T]}(t)$, where $\alpha > 0$. For this particular function ϕ we have

$$\begin{aligned} \exp(-\alpha T)\|U - V_h\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)}^2 &\leq \int_0^T \int_{\mathbb{R}^n} (\alpha(T-t)+1)\exp(-\alpha t)|V_h - U|^2(t,x) dx dt \\ &= - \int_0^T \int_{\mathbb{R}^n} |V_h - U|^2(t,x)\partial_t\phi(t,x) - \sum_{i=1}^n \langle V_h - U; A_i(V_h - U) \rangle(t,x)\partial_i\phi(t,x) dx dt \leq \langle \mu_h^2, \phi \rangle - \langle \mu_h^1, 2U\phi \rangle. \end{aligned} \quad (4.3.20)$$

We are going to derive an upper-bound for the term $\langle \mu_h^2, \phi \rangle - \langle \mu_h^1, 2U\phi \rangle$. We have, by definition of μ_h^1 and μ_h^2 (see Propositions 4.3.11 and 4.3.13),

$$\begin{aligned} \langle \mu_h^2, \phi \rangle - \langle \mu_h^1, 2\phi \rangle &\leq \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \left(m_K \left(|v_K^{p+1}|^2 - |v_K^p|^2 \right) (\phi_K(t^{p+1}) - \phi_K^p) - 2m_K \langle v_K^{p+1} - v_K^p; (U\phi)_K(t^{p+1}) - (U\phi)_K^p \rangle \right) \\ &\quad + \int_{\mathbb{R}^n} \left(|V_h(0^+, x)|^2 - |U_0(x)|^2 \right) \phi(0, x) dx - 2 \sum_{K \in \mathcal{T}_h} m_K \langle v_K^0; (U\phi)_K(0) \rangle + 2 \int_{\mathbb{R}^n} \langle U_0; (U\phi)(0, \cdot) \rangle dx \\ &\quad + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left(|v_K^{p+1} - v_K^{p,*}|^2 + 2 \langle v_K^{p+1} - v_K^{p,*}; v_K^{p,*} \rangle \right) \phi_K^p - 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; (U\phi)_K^p \rangle \\ &\quad - 2 \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} \left((U\phi)_e^p - (U\phi)_K^p \right) \rangle - \epsilon Q_h. \end{aligned} \quad (4.3.21)$$

In the previous inequality, we use the fact that the function ϕ is independent of the space variable (and also that $\phi(T) = 0$) and we denote (see Remark 4.3.4)

$$Q_h = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \Delta t m_e \langle C_{K_e} (v_K^p - v_{K_e}^p); v_K^p - v_{K_e}^p \rangle \phi_K^p = \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \Delta t m_e |v_K^p - v_{K_e}^p|_{C_{K_e}}^2 \phi_K^p.$$

For simplicity of presentation, let us set \mathcal{L}_1 to be equal to the right-hand side terms of the first line of inequality (4.3.21), \mathcal{L}_2 the terms of the second line, \mathcal{L}_3 the terms of the third line. Let us now examine each line of the right-hand side of the inequality (4.3.21). Concerning the first line, we have, since the function ϕ is independent of the space variable,

$$\begin{aligned} \mathcal{L}_1 &= \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; v_K^{p+1} + v_K^p - 2U_K(t^{p+1}) \rangle (\phi_K(t^{p+1}) - \phi_K^p) \\ &\quad - 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; U_K(t^{p+1}) \phi_K^p - (U\phi)_K^p \rangle - 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; (U\phi)_K(t^{p+1}) - U_K(t^{p+1}) \phi_K(t^{p+1}) \rangle. \end{aligned}$$

Now observing that the last term actually vanishes. Thus we get

$$\mathcal{L}_1 = \underbrace{\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; v_K^{p+1} + v_K^p - 2U_K(t^{p+1}) \rangle (\phi_K(t^{p+1}) - \phi_K^p)}_{:=T_1} - \underbrace{2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; U_K(t^{p+1}) \phi_K^p - (U\phi)_K^p \rangle}_{:=T_2}.$$

Let us start by the analysis of the term T_2 . First, we have

$$\left| U_K(t^{p+1}) \phi_K^p - (U\phi)_K^p \right|^2 \leq \left| \frac{1}{\Delta t m_K} \int_{t^p}^{t^{p+1}} \int_K (u(t^{p+1}, x) - u(t, x)) \phi(t) dx dt \right|^2,$$

and then, since the function ϕ is bounded (by T),

$$|U_K(t^{p+1})\phi_K^p - (U\phi)_K^p|^2 \leq \frac{T^2}{\Delta t m_K} \int_{t^p}^{t^{p+1}} \int_K |u(t^{p+1}, x) - u(t, x)|^2 dx dt \leq C \frac{\Delta t}{m_K} \|\partial_t U\|_{L^2((t^p, t^{p+1}) \times K; \mathbb{R}^m)}^2.$$

Consequently, using Cauchy-Schwarz inequality, we get

$$|T_2| \leq 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K |v_K^{p+1} - v_K^p| |U_K(t^{p+1})\phi_K^p - (U\phi)_K^p| \leq C \sqrt{\Delta t} \|\partial_t U\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K |v_K^{p+1} - v_K^p|^2 \right)^{1/2}.$$

Noticing that

$$|v_K^{p+1} - v_K^p|^2 \leq |v_K^{p,*} - v_K^p|^2 \leq \left| \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} m_e C_{K_e} (v_K^p - v_{K_e}^p) \right|^2 \leq \left| \sum_{e \in \mathcal{E}_K} \frac{m_e}{m_{\partial K}} \frac{\Delta t m_{\partial K}}{m_K} C_{K_e} (v_K^p - v_{K_e}^p) \right|^2,$$

and using convexity, we get that (see Remark 4.3.4)

$$\begin{aligned} |v_K^{p+1} - v_K^p|^2 &\leq \sum_{e \in \mathcal{E}_K} \frac{m_e}{m_{\partial K}} \left(\frac{\Delta t m_{\partial K}}{m_K} \right)^2 |C_{K_e} (v_K^p - v_{K_e}^p)|^2 \\ &\leq \sum_{e \in \mathcal{E}_K} \frac{m_e \Delta t}{m_K} \left(\frac{\Delta t m_{\partial K}}{m_K} \right) |C_{K_e}| |(v_K^p - v_{K_e}^p)|_{C_{K_e}}^2 \leq \sup_{K, e \in \mathcal{E}_K} \left(\left(\frac{\Delta t m_{\partial K}}{m_K} \right) |C_{K_e}| \right) \sum_{e \in \mathcal{E}_K} \frac{m_e \Delta t}{m_K} |(v_K^p - v_{K_e}^p)|_{C_{K_e}}^2. \end{aligned} \quad (4.3.22)$$

Using the CFL condition (4.3.7) and the definition of Q_h , it yields

$$|T_2| \leq C \sqrt{\Delta t} \|\partial_t U\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} Q_h^{1/2}.$$

Using Proposition 4.3.8 (and Remark 4.2.11), we obtain

$$|T_2| \leq C \sqrt{\Delta t} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} Q_h^{1/2}.$$

Let us now examine the term T_1 . Since the function ϕ is independent of the space variables and $|\partial_t \phi(t, x)| \leq \alpha T + 1$, we have

$$|\phi_K(t^{p+1}) - \phi_K^p| \leq \frac{1}{\Delta t} \int_{t^p}^{t^{p+1}} |\phi(t^{p+1}) - \phi(s)| ds \leq C \Delta t.$$

Consequently, we get the following bound for T_1

$$|T_1| \leq C \sqrt{\Delta t} \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K |v_K^{p+1} - v_K^p|^2 \right)^{1/2} \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t |v_K^{p+1} + v_K^p - 2U_K(t^{p+1})|^2 \right)^{1/2}.$$

Now, observe that

$$|v_K^{p+1} + v_K^p - 2U_K(t^{p+1})| \leq |v_K^{p+1} - U_K(t^{p+1})| + |v_K^p - U_K(t^{p+1})|.$$

Then, we have

$$\begin{aligned} \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t |v_K^p - U_K(t^{p+1})|^2 &\leq 2 \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t |v_K^p - U_K^p|^2 + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t |U_K^p - U_K(t^{p+1})|^2 \right) \\ &\leq C \left(\|V_h - U\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)}^2 + (\Delta t)^2 \|\partial_t U\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)}^2 \right). \end{aligned} \quad (4.3.23)$$

Reindexing the (double) sum depending on the solution U on the left-hand side of (4.3.23) to get

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t |v_K^{p+1} - U_K(t^{p+1})|^2 = \sum_{p=1}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t |v_K^p - U_K(t^p)|^2 + \Delta t \|V_h(T, \cdot) - U(T, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2.$$

Thanks to Proposition 4.3.8 (and Remark 4.2.11), we know that

$$\Delta t \|V_h(T, \cdot) - U(T, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq 2\Delta t \left(\|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \sup_{t \in [0, T]} \|U(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) \leq 4\Delta t \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2.$$

As for the inequality (4.3.23), we finally get

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \Delta t \left| v_K^{p+1} - U_K(t^{p+1}) \right|^2 \leq C \left(\|V_h - U\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)}^2 + \Delta t \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right).$$

It implies that

$$|T_1| \leq C \sqrt{\Delta t} \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left| v_K^{p+1} - v_K^p \right|^2 \right)^{1/2} \left(\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} + \sqrt{\Delta t} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \right),$$

and, thanks to (4.3.22) and to (4.3.11), we obtain

$$\begin{aligned} |T_1| &\leq C \sqrt{\Delta t} \left(\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} + \sqrt{\Delta t} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \right) \left(\sum_{p=0}^q \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \Delta t \left| v_K^p - v_{K_e}^p \right|_{C_{K_e}}^2 \right)^{1/2} \\ &\leq C \sqrt{\Delta t} \left(\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} + \sqrt{\Delta t} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right). \end{aligned}$$

Finally, we get that the terms of the first line satisfy the following bound

$$|\mathcal{L}_1| \leq C \left(\sqrt{\Delta t} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} Q_h^{1/2} + \sqrt{\Delta t} \left(\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} + \sqrt{\Delta t} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) \right). \quad (4.3.24)$$

Now let us examine the second line of the right-hand side of (4.3.21). Observe that

$$\mathcal{L}_2 = \int_{\mathbb{R}^n} \left(|V_h(0^+, x) - U_0(x)|^2 \right) \phi(0, x) dx,$$

from which, we easily deduce that

$$|\mathcal{L}_2| \leq Ch^2 \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}^2. \quad (4.3.25)$$

We then analyze the terms of the third line of (4.3.21). We can rewrite the terms of this line as

$$\mathcal{L}_3 = - \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \left| v_K^{p+1} - v_K^{p,*} \right|^2 \phi_K^p + 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; v_K^{p+1} - U_K^p \rangle \phi_K^p - 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; (U\phi)_K^p - U_K^p \phi_K^p \rangle.$$

By definition of the projection step and using the properties of the projection operator, we have

$$\langle v_K^{p+1} - v_K^{p,*}; v_K^{p+1} - U_K^p \rangle = \langle P_{\mathcal{C}}(v_K^{p,*}) - v_K^{p,*}; P_{\mathcal{C}}(v_K^{p,*}) - U_K^p \rangle \leq 0,$$

because the solution takes its value in \mathcal{C} (and so does U_K^p). Since the function ϕ is non-negative, we get

$$\mathcal{L}_3 \leq -2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; (U\phi)_K^p - U_K^p \phi_K^p \rangle. \quad (4.3.26)$$

By definition of the scheme, we obtain

$$\begin{aligned} & \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^{p,*}; (U\phi)_K^p - U_K^p \phi_K^p \rangle \\ &= \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; (U\phi)_K^p - U_K^p \phi_K^p \rangle + \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \Delta t \left\langle \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e; (U\phi)_K^p - U_K^p \phi_K^p \right\rangle. \end{aligned} \quad (4.3.27)$$

Since the function ϕ is bounded (by T), we get

$$|(U\phi)_K^p - U_K^p \phi_K^p|^2 \leq \frac{1}{\Delta t m_K} T^2 \int_{t^p}^{t^{p+1}} \int_K |u(t, x) - U_K^p|^2 dx dt \leq C \Delta t \frac{h^2}{m_K} \|U\|_{H^1((t^p, t^{p+1}) \times K; \mathbb{R}^m)}^2. \quad (4.3.28)$$

Consequently, by the Cauchy-Schwarz inequality,

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; (U\phi)_K^p - U_K^p \phi_K^p \rangle \leq C \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K |v_K^{p+1} - v_K^p|^2 \right)^{1/2} \times \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \Delta t h^2 \|U\|_{H^1((t^p, t^{p+1}) \times K; \mathbb{R}^m)}^2 \right)^{1/2}.$$

The inequality (4.3.22) implies that

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} m_K \langle v_K^{p+1} - v_K^p; (U\phi)_K^p - U_K^p \phi_K^p \rangle \leq C \sqrt{\Delta t h} \|U\|_{H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \left(\sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} m_e |v_K^p - v_{K_e}^p|_{C_{K_e}}^2 \right)^{1/2}.$$

We also have, thanks to the Cauchy-Schwarz inequality and the estimate (4.3.28),

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \Delta t \left\langle \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e; (U\phi)_K^p - U_K^p \phi_K^p \right\rangle \leq C \sqrt{\Delta t h} \|U\|_{H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \frac{(\Delta t)^2}{m_K} \left| \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right|^2 \right)^{1/2}.$$

Now using the inequality (4.3.11), we have

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \frac{(\Delta t)^2}{m_K} \left| \sum_{e \in \mathcal{E}_K} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right|^2 \leq \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} \Delta t m_e |v_K^p - v_{K_e}^p|_{C_{K_e}}^2 \leq \frac{1}{\epsilon} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2.$$

Gathering the inequality (4.3.26) and the decomposition (4.3.27), we obtain

$$\mathcal{L}_3 \leq C \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \sqrt{\Delta t h} \|U\|_{H^1((0, T) \times \mathbb{R}^n; \mathbb{R}^m)},$$

the *a priori* estimates (following from Theorem 4.2.7) provides

$$\mathcal{L}_3 \leq C \sqrt{\Delta t h} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}^2. \quad (4.3.29)$$

Finally, if we look at the first term of the fourth line of (4.3.21) and call it T_3 (the term Q_h will be treated at the end of this proof), we need to bound the following term

$$\begin{aligned} T_3 &= -2 \sum_{p=0}^N \Delta t \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \langle v_K^p - v_{K_e}^p; C_{K_e} ((U\phi)_e^p - (U\phi)_K^p) \rangle \\ &= 2 \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \int_{t^p}^{t^{p+1}} \langle v_K^p - v_{K_e}^p; C_{K_e} (U_e^p(t) - U_K^p(t)) \rangle \phi(t) dt. \end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$|T_3| \leq 2 \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \int_{t^p}^{t^{p+1}} |v_K^p - v_{K_e}^p|_{C_{K_e}} \phi(t) dt \right)^{1/2} \left(\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \int_{t^p}^{t^{p+1}} \|U_e^p(t) - U_K^p(t)\|_{C_{K_e}} \phi(t) dt \right)^{1/2}.$$

Since the matrices A_i have constant coefficients, the supremum of the spectral radii of matrices C_{K_e} is finite. Consequently, we deduce that there exists a constant C such that

$$\sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \int_{t^p}^{t^{p+1}} \|U_e^p(t) - U_K^p(t)\|_{C_{K_e}} \phi(t) dt \leq C \sum_{p=0}^N \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} m_e \int_{t^p}^{t^{p+1}} \int_e |U(t, y) - U_K^p(t)|^2 \phi(t) dt d\mathcal{H}^{n-1}.$$

Using [55, Lemma 4.3], we obtain

$$\sum_{e \in \mathcal{E}_K} m_e \int_e |U(t, y) - U_K^p(t)|^2 d\mathcal{H}^{n-1}(x) \leq Ch \|U(t)\|_{H^1(K; \mathbb{R}^m)}^2.$$

By definition of the term Q_h , it yields

$$|T_3| \leq C\sqrt{h} \|U\|_{L^2((0, T); H^1(\mathbb{R}^n; \mathbb{R}^m))} Q_h^{1/2}.$$

Remark 4.2.11 gives

$$|T_3| \leq C\sqrt{h} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} Q_h^{1/2}. \quad (4.3.30)$$

Using the previous inequalities and the inequalities (4.3.20) and (4.3.21) and gathering the estimates (4.3.24), (4.3.25), (4.3.29) and (4.3.30), we get

$$\begin{aligned} \exp(-\alpha T) \|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)}^2 + \epsilon Q_h \\ \leq C\sqrt{\Delta t} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} Q_h^{1/2} + C\sqrt{\Delta t} \|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} + C\Delta t \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \\ + Ch^2 \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} + C\sqrt{\Delta t} h \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}^2 + C\sqrt{h} Q_h^{1/2} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

Thanks to Remark 4.3.6, we obtain

$$\begin{aligned} \exp(-\alpha T) \|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)}^2 + \epsilon Q_h \\ \leq C\sqrt{h} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} Q_h^{1/2} + C\sqrt{h} \|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} + Ch \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \\ + Ch^2 \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} + Ch^{3/2} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}^2 + C\sqrt{h} Q_h^{1/2} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

It implies that

$$\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \leq C\sqrt{h} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} \quad \text{and} \quad Q_h^{1/2} \leq C\sqrt{h} \|U_0\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)},$$

where the constant $C = C(\epsilon, \eta, \alpha, A_i, n, T)$ depends on ϵ, η, α , the matrices A_i , the dimensions n , the final time T . \square

Let us now derive a similar theorem in the case of non-compactly supported solution.

Theorem 4.3.17. *Let $U \in H^1((0, T) \times \mathbb{R}^n; \mathcal{C})$ be a dissipative solution associated with the initial condition $U_0 \in H^1(\mathbb{R}^n; \mathcal{C})$. Let V_h be the solution constructed from U_0 thanks to the scheme (4.3.3). Then we have,*

$$\|U - V_h\|_{L^2((0, T) \times \mathbb{R}^n; \mathbb{R}^m)} \leq C\sqrt{h},$$

for some constant C depending on ϵ, η the matrices A_i , the dimensions n , the final time T , the initial data U_0 .

Proof. Let $(U_{0,p})_{p \in \mathbb{N}} \in (C_c^\infty(\mathbb{R}^n; \mathcal{C}))^{\mathbb{N}}$ be a sequence of functions that converges to U_0 in $H^1(\mathbb{R}^n; \mathbb{R}^m)$. We denote by $V_{h,p}$ the approximated solution associated with $U_{0,p}$ and by U_p the dissipative solution associated with $U_{0,p}$. Then, we

have

$$\begin{aligned} \|U - V_h\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} &\leq \|U - U_p\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} + \|U_p - V_{h,p}\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} + \|V_{h,p} - V_h\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} \\ &\leq 2\|U_0 - U_{0,p}\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} + \|U_p - V_{h,p}\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} \\ &\leq 2\|U_0 - U_{0,p}\|_{L^2((0,T)\times\mathbb{R}^n;\mathbb{R}^m)} + C\|U_p\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)}\sqrt{h}, \end{aligned}$$

using Proposition 4.3.9 and [37, Lemma 9]. In addition, we know that the constant C depends only on ϵ , η , α , the matrices A_i , the dimensions n , the final time T . Sending p to $+\infty$, in the previous inequality, we get the result. \square

4.4 The simplified model of the dynamical perfect plasticity : a numerical approach

In this section we present two schemes, closely related, adapted for Friedrichs' systems under constraint when the spatial domain is bounded. These problems have been described in a general way in Section 2.5 and we also have studied in details a specific example, the simplified model of dynamical perfect plasticity, in Chapter 3. Let us briefly recall the general framework of Friedrichs' systems under constraint in bounded domains : we look at a partial differential system (that has to be understood in a weak sense)

$$\partial_t U + \sum_{i=1}^n A_i \partial_i U = f, \quad \text{on } (0, T) \times \Omega,$$

where $U : (t, x) \in (0, T) \times \Omega \rightarrow \mathbb{R}^m$, $A_i : (t, x) \in (0, T) \times \Omega \rightarrow \mathbb{M}_{\text{sym}}^{m \times m}$, $f : (t, x) \in (0, T) \times \Omega \rightarrow \mathbb{R}^m$ and $\mathbb{M}_{\text{sym}}^{m \times m}$ is the space of $m \times m$ symmetric matrices with real coefficients. The variable U is constrained to set in a fixed convex \mathcal{C} *i.e.*

$$\forall (t, x) \in (0, T) \times \Omega, \quad U(t, x) \in \mathcal{C}.$$

In addition to a classical initial condition at $t = 0$, we add the boundary condition

$$“(A_\nu - M_\nu)U = 0”, \quad \text{on } (0, T) \times \partial\Omega, \quad (4.4.1)$$

where the matrices A_ν and M_ν obey to the requirements of Section 2.2.1. From the theoretical point of view, we have seen on the example of the simplified model of dynamical perfect plasticity that the linear boundary condition, $(A_\nu - M_\nu)U = 0$, of the unconstrained case has to be replaced by a nonlinear boundary condition due to the constraint (that is why we put the boundary condition in quotation marks in (4.4.1)).

However, we are going to construct schemes (based on the same splitting strategy as in the previous section) that *a priori* do not take account of this non-linearity at the boundary. More precisely, at each time step, we first let the system evolve following an upwind, explicit in time, finite volume scheme on unstructured meshes, based on [28], designed for problems without constraint. In particular, during this first step, the boundary cells are treated as if the boundary condition was the linear one, $(A_\nu - M_\nu)U = 0$. In a second step, we project onto the set of constraints on every cell. We describe in details these schemes in Subsection 4.4.1.

In Subsection 4.4.2, we establish that these schemes are stable in time under a Courant-Friedrichs-Lewy condition. Finally, in Subsection 4.4.3, we show on the simplified model of the dynamical perfect plasticity how these schemes behave and we underline the interaction of the constraint and the boundary condition from the numerical point of view (for the theoretical point of view, see Chapter 3).

4.4.1 Description of schemes

In this subsection, we present the general framework of this work and the schemes we are interested in. Let \mathcal{T}_h be a triangulation of $\Omega \subset \mathbb{R}^n$ (a n -dimensional polytope) such that : let $\mathcal{I} \subset \mathbb{N}$, let $\mathcal{T}_h = (K_i)_{i \in \mathcal{I}}$ a family of open nonempty convex polytope such that $\cup_{i \in \mathcal{I}} \overline{K_i} = \overline{\Omega}$, for all $i \neq j$, $K_i \cap K_j = \emptyset$ and $h = \sup_{i \in \mathcal{I}} (\text{diam} K_i) < +\infty$. We introduce the notations of Section 4.3.1 for m_K , $m_{\partial K}$, e , m_e , K_e and ν_{K_e} . We want to investigate the numerical approximation (using finite

volume schemes) of the following constrained Friedrichs system

$$\begin{cases} \partial_t U + \sum_{i=1}^n A_i \partial_i U = f, & \text{on } (0, T) \times \Omega, \\ U(0, x) = U_0(x), & \text{on } \Omega, \\ (A_\nu - M_\nu)U = 0, & \text{on } (0, T) \times \partial\Omega, \\ U(t, x) \in \mathcal{C} \subset \mathbb{R}^m, & \text{a.e in } (0, T) \times \Omega \end{cases} \quad (4.4.2)$$

where \mathcal{C} is a closed convex (independent of t and x) such that there exists $r > 0$ such that $B(0, r) \subset \mathcal{C}$. The equations of (4.4.2) have to be understood in a weak sense (see Section 2.5 : the solutions of such problems satisfy a family of inequalities and are called dissipative). We construct a piece-wise constant approximation of U , denoted by V_h , such that

$$\forall [t^p, t^{p+1}) \times K, \quad V_h(t, x) = v_K^p,$$

where $0 = t^0 < t^1 < \dots < t^{N+1} = T$ ($t^{p+1} - t^p = \Delta t$). The scheme is explicit in time and takes the following form

$$\begin{cases} v_K^0 &= \frac{1}{m_K} \int_K U_0(x) dx. \\ v_K^{p,*} &= v_K^p - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_K} g_{K_e} m_e + \Delta t f_K^p, \\ v_K^{p+1} &= P_{\mathcal{C}}(v_K^{p,*}), \end{cases} \quad (4.4.3)$$

where p is an integer in $[0, N]$, $P_{\mathcal{C}}$ is the projection onto the closed convex \mathcal{C} ,

$$f_K^p = \frac{1}{m_K \Delta t} \int_{t^p}^{t^{p+1}} \int_K f(t, x) dx dt,$$

and

$$g_{K_e} = D_{K_e} v_K^p - C_{K_e} v_{K_e}^p, \quad \text{with } C_{K_e} = -(A_{K_e})^- \text{ and } D_{K_e} = (A_{K_e})^+, \quad \text{when } e \cap \partial\Omega = \emptyset \quad (4.4.4)$$

$$g_{K_e} = B_{K_e} v_{K_e}^p, \quad \text{otherwise,} \quad (4.4.5)$$

where $A_{K_e} = \sum_{i=1}^n A_i \nu_{K_e}^i$, $(A_{K_e})^-$ (resp. $(A_{K_e})^+$) is the negative (resp. positive) part of A_{K_e} (see Remark 4.3.2) and B_{K_e} is either equal to

$$B_{K_e} = (A_{K_e} + M_{K_e})/2, \quad (4.4.6)$$

where $M_{K_e} = M_{\nu_{K_e}} = M(\nu_{K_e})$ a matrix satisfying the conditions of Section 2.2.1, this scheme is based on the note [28] of Coudière, Vila and Villedieu and in the following we denoted it as the CVV-scheme ; or B_{K_e} is defined as

$$B_{K_e} = M_{K_e}, \quad (4.4.7)$$

in the following we denoted it as the dissipative scheme.

In what follows, we denote by $\mathcal{E}_{K_{\text{int}}}$ the set of interior edges e of K , i.e. $e \cap \partial\Omega = \emptyset$ and by $\mathcal{E}_{K_{\text{b}}}$ the set of boundary edges e of K , i.e. $e \cap \partial\Omega = e$.

Remark 4.4.1. *The edges of K are, as K , open sets.*

We use the following discrete Green formula :

$$\sum_{e \in \mathcal{E}_K} A_{K_e} m_e = 0 \quad \Leftrightarrow \quad \sum_{e \in \mathcal{E}_{K_{\text{b}}}} A_{K_e} m_e + \sum_{e \in \mathcal{E}_{K_{\text{int}}}} D_{K_e} m_e = \sum_{e \in \mathcal{E}_{K_{\text{int}}}} C_{K_e} m_e. \quad (4.4.8)$$

Using this formula, it leads us to a non conservative form of the scheme 4.4.3

$$v_K^{p,*} = v_K^p - \frac{\Delta t}{m_K} \left(\sum_{e \in \mathcal{E}_{K_{\text{int}}}} C_{K_e} (v_K^p - v_{K_e}^p) m_e \right) - \frac{\Delta t}{m_K} \left(\sum_{e \in \mathcal{E}_{K_b}} (B_{K_e} - A_{K_e}) v_K^p m_e \right). \quad (4.4.9)$$

Remark 4.4.2. Inside the domain Ω , the scheme (4.4.3) is the same as (4.3.3) (when $\eta = 0$, see also Remark 4.3.3).

Remark 4.4.3. As in the previous section, we denote by $\langle \cdot; \cdot \rangle$ the canonical scalar product of \mathbb{R}^m and $|\cdot|$ the associated norm. By abuse of notation, we also use the notation $|\cdot|$ for the (matrix) operator norm associated with the canonical norm of \mathbb{R}^m .

4.4.2 Stability in time of schemes

In this section, we prove that the scheme enjoys a stability property under a Courant-Friedrichs-Lewy condition. Namely, the $L^2(\mathbb{R}^n)$ -norm of the solution do not increase in time. For simplicity, we decide to derive this stability property in the case where the source term is null.

Proposition 4.4.4. Suppose that there exists $\epsilon \in]0, 1[$ such that the following CFL condition holds :

$$\max \left(\sup_{K, e \in \mathcal{E}_K} \frac{\Delta t m_{\partial K}}{m_K} |C_{K_e}|, \sup_{K, e \in \mathcal{E}_{K_b}} \frac{\Delta t m_{\partial K}}{m_K} |(M_{K_e} - A_{K_e})/2| \right) \leq 1 - \epsilon, \quad (4.4.10)$$

the CVV-scheme (with (4.4.6)) is stable, i.e. the approximate solution V_h satisfies

$$\forall t \in [0, T], \quad \|V_h(t, \cdot)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)} \leq \|U_0\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}.$$

Suppose that there exists $\epsilon \in]0, 1[$ such that the following CFL condition holds :

$$\max \left(\sup_{K, e \in \mathcal{E}_K} \frac{\Delta t m_{\partial K}}{m_K} |C_{K_e}|, \sup_{K, e \in \mathcal{E}_{K_b}} \frac{\Delta t m_{\partial K}}{m_K} |M_{K_e} - A_{K_e}| \right) \leq 1 - \epsilon, \quad (4.4.11)$$

the dissipative scheme (with (4.4.7)) is stable.

Remark 4.4.5. From the two CFL conditions (4.4.10) and (4.4.11), it appears that one should use, from the practical point of view, the CVV scheme since its CFL condition is twice smaller than that of the dissipative scheme.

Proof. Since the proof for the CVV-scheme and the dissipative scheme are quite similar, we start with a general matrix B_{K_e} for $e \in \mathcal{E}_{K_b}$ and at the end of the proof, we explain how to derive the specific estimate (4.4.10) and (4.4.11).

From the non-conservative form (4.4.9), we have

$$v_K^{p,*} = \sum_{e \in \mathcal{E}_K} \frac{m_e}{m_{\partial K}} v_K^{p,*}(e),$$

where we set

$$v_K^{p,*}(e) = v_K^p - \frac{\Delta t m_{\partial K}}{m_K} C_{K_e} (v_K^p - v_{K_e}^p),$$

if $e \in \mathcal{E}_{K_{\text{int}}}$ and

$$v_K^{p,*}(e) = v_K^p - \frac{\Delta t m_{\partial K}}{m_K} (B_{K_e} - A_{K_e}) v_K^p,$$

if $e \in \mathcal{E}_{K_b}$. Observe that, as in the proof of Proposition 4.3.5, we have for all $e \in \mathcal{E}_{K_{\text{int}}}$,

$$|v_K^{p,*}(e)|^2 \leq |v_K^p|^2 - \frac{\Delta t m_{\partial K}}{m_K} \left(\langle v_K^p; C_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) - \epsilon \frac{\Delta t m_{\partial K}}{m_K} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle.$$

Now, if $e \in \mathcal{E}_{K_b}$, the same reasoning implies that

$$|v_K^{p,*}(e)|^2 = |v_K^p|^2 - \frac{\Delta t m_{\partial K}}{m_K} \langle v_K^p; B_{K_e} v_K^p \rangle + \frac{\Delta t m_{\partial K}}{m_K} \langle v_K^p; A_{K_e} v_K^p \rangle - \frac{\Delta t m_{\partial K}}{m_K} \left\langle \left(B_{K_e} - A_{K_e} - \frac{\Delta t m_{\partial K}}{m_K} (B_{K_e} - A_{K_e})^2 \right) v_K^p; v_K^p \right\rangle.$$

Using convexity,

$$\begin{aligned} |v_K^{p,*}|^2 &\leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \left(\langle v_K^p; C_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e - \epsilon \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle m_e \\ &\quad + \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_b}} \langle v_K^p; A_{K_e} v_K^p \rangle m_e - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_b}} \langle v_K^p; B_{K_e} v_K^p \rangle m_e - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_b}} \left\langle \left(B_{K_e} - A_{K_e} - \frac{\Delta t m_{\partial K}}{m_K} (B_{K_e} - A_{K_e})^2 \right) v_K^p; v_K^p \right\rangle m_e. \end{aligned}$$

Furthermore, if we use the relation (4.4.8), we obtain

$$\begin{aligned} |v_K^{p,*}|^2 &\leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \left(\langle v_K^p; D_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e - \epsilon \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle m_e \\ &\quad - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_b}} \langle v_K^p; B_{K_e} v_K^p \rangle m_e - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_b}} \left\langle \left(B_{K_e} - A_{K_e} - \frac{\Delta t m_{\partial K}}{m_K} (B_{K_e} - A_{K_e})^2 \right) v_K^p; v_K^p \right\rangle m_e. \quad (4.4.12) \end{aligned}$$

Using the general inequality (4.4.12), we explain how to derive the stability property for the dissipative scheme first and then for the CVV scheme.

The dissipative scheme In this case, we have

$$\forall e \in \mathcal{E}_{K_b}, \quad B_{K_e} = M_{K_e} \geq 0.$$

Consequently, the inequality (4.4.12) implies that

$$\begin{aligned} |v_K^{p,*}|^2 &\leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \left(\langle v_K^p; D_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e - \epsilon \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle m_e \\ &\quad - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_b}} \left\langle \left(M_{K_e} - A_{K_e} - \frac{\Delta t m_{\partial K}}{m_K} (M_{K_e} - A_{K_e})^2 \right) v_K^p; v_K^p \right\rangle m_e. \end{aligned}$$

Note that the CFL condition (4.4.11) implies that for all $y \in \mathbb{R}^m$, we have

$$\left\langle \text{Id} - \frac{\Delta t m_{\partial K}}{m_K} (M_{K_e} - A_{K_e}) y; y \right\rangle \geq \epsilon |y|^2.$$

Now observe that, using the notations of Chapter 2 (and using in particular, Lemma 2.2.3), we have

$$\begin{aligned} \left\langle (M_{K_e} - A_{K_e}) \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} (M_{K_e} - A_{K_e}) \right) y; y \right\rangle &= \left\langle (M_{K_e} - A_{K_e}) \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} (M_{K_e} - A_{K_e}) \right) y_+; y_+ \right\rangle \\ &= 2 \left\langle M_{K_e} \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} (M_{K_e} - A_{K_e}) \right) y_+; y_+ \right\rangle = 2 \left\langle \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} (M_{K_e} - A_{K_e}) \right) (M_{K_e}^{1/2} y_+); (M_{K_e}^{1/2} y_+) \right\rangle \geq 2\epsilon \langle M_{K_e} y_+; y_+ \rangle, \end{aligned}$$

Consequently, we obtain in particular that

$$|v_K^{p,*}|^2 \leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K_{\text{int}}}} \left(\langle v_K^p; D_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e. \quad (4.4.13)$$

The CVV-scheme In this case, we have

$$\forall e \in \mathcal{E}_{K_b}, \quad B_{K_e} = \frac{A_{K_e} + M_{K_e}}{2}.$$

Remark that, again using the notations of Chapter 2 (and using in particular, Lemma 2.2.3), for all $e \in \mathcal{E}_{K_b}$

$$\langle v_K^p; B_{K_e} v_K^p \rangle = \left\langle v_K^p; \frac{A_{K_e} + M_{K_e}}{2} v_K^p \right\rangle = \langle (v_K^p)_-; M_{K_e} (v_K^p)_- \rangle \geq 0.$$

Consequently, from the inequality (4.4.12), we obtain

$$\begin{aligned} |v_K^{p,*}|^2 \leq & |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K \text{int}}} \left(\langle v_K^p; D_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e - \epsilon \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K \text{int}}} \langle v_K^p - v_{K_e}^p; C_{K_e} (v_K^p - v_{K_e}^p) \rangle m_e \\ & - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K \text{b}}} \left\langle \left(\frac{M_{K_e} - A_{K_e}}{2} - \frac{\Delta t m_{\partial K}}{m_K} \left(\frac{M_{K_e} - A_{K_e}}{2} \right)^2 \right) v_K^p; v_K^p \right\rangle m_e. \end{aligned}$$

Note that the CFL condition (4.4.11) implies that for all $y \in \mathbb{R}^m$, we have

$$\left\langle \text{Id} - \frac{\Delta t m_{\partial K}}{m_K} \left(\frac{M_{K_e} - A_{K_e}}{2} \right) y; y \right\rangle \geq \epsilon |y|^2.$$

Now, as for the CVV-scheme, we easily obtain

$$\left\langle \left(\frac{M_{K_e} - A_{K_e}}{2} \right) \left(\text{Id} - \frac{\Delta t m_{\partial K}}{m_K} \left(\frac{M_{K_e} - A_{K_e}}{2} \right) \right) y; y \right\rangle \geq \epsilon \langle M_{K_e} y_+; y_+ \rangle,$$

Consequently, we obtain in particular that

$$|v_K^{p,*}|^2 \leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K \text{int}}} \left(\langle v_K^p; D_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e. \quad (4.4.14)$$

In both cases, we derive the same inequality (namely (4.4.13) and (4.4.14)). The projection step provides (see Remark 4.3.7)

$$|v_K^{p+1}|^2 \leq |v_K^p|^2 - \frac{\Delta t}{m_K} \sum_{e \in \mathcal{E}_{K \text{int}}} \left(\langle v_K^p; D_{K_e} v_K^p \rangle - \langle v_{K_e}^p; C_{K_e} v_{K_e}^p \rangle \right) m_e.$$

From this inequality, we can easily conclude that these schemes are stable as in the proof of Proposition 4.3.8. \square

Remark 4.4.6. *The study of the convergence of these two schemes presents technical difficulties and consequently we do not present any convergence analysis for constrained problems in bounded domains (see [28] for the unconstrained case).*

4.4.3 Numerical tests on the simplified model of dynamical perfect plasticity

In this subsection, we present our numerical results, based on the CVV-scheme exposed in Subsection 4.4.1, on the simplified model of dynamical perfect plasticity, described in details in Chapter 3.

Let us briefly recall the equations of this model and the two points of views that one can use to describe its (theoretical) solution. First, the equations, derived from the physics of solids (see Chapter 1), of this simplified model of dynamical perfect plasticity are

$$\begin{cases} \partial_t v - \text{div} \sigma = f, & \nabla v = \partial_t \sigma + \partial_t p, \\ |\sigma| \leq 1, & \text{if } |\sigma| < 1, \partial_t p = 0, \quad \text{if } |\sigma| = 1, \frac{\partial_t p}{|\partial_t p|} = \sigma. \end{cases} \quad (4.4.15)$$

where $v : \Omega \times [0, T] \rightarrow \mathbb{R}$ is the velocity of the material, $\sigma : \Omega \times [0, T] \rightarrow \mathbb{R}^2$ the Cauchy stress tensor and $p : \Omega \times [0, T] \rightarrow \mathbb{R}^2$ the plastic deformation tensor and Ω is a open bounded subset of \mathbb{R}^2 . The tensor σ is constrained to stay in the unit closed Euclidean ball of \mathbb{R}^2 , denoted \bar{B} . To these equations, we add initial and boundary conditions. The boundary condition, that comes from the hyperbolic point of view, is the following nonlinear one

$$\langle \sigma; \nu \rangle + \lambda^{-1} T_\lambda(v) = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (4.4.16)$$

where $T_\lambda(z) = \min(-\lambda, \max(z, \lambda))$. It shows a threshold on the velocity (due to the constraint) in the boundary condition. The initial condition $(v, \sigma)(t = 0) = (v_0, \sigma_0)$ has to satisfy two hypotheses

$$\langle \sigma_0; \nu \rangle + \lambda^{-1} v_0 = 0 \quad \mathcal{H}^1 \text{ on } \partial\Omega, \quad (4.4.17)$$

$$|\sigma_0| \leq 1 \text{ a.e. in } \Omega. \quad (4.4.18)$$

The first condition asserts that the initial condition has to satisfy the boundary condition that we will use in the

unconstrained case and the second one that the initial condition satisfy the constraint. To construct a numerical approximation of the solutions of this problem, we use the hyperbolic formalism, described in Section 2.5 and in Section 3.7, of this problem. We use the new variable $U = (v, \sigma)$ and rewrite the problem (4.4.15) and (4.4.16) using that variable U in the sense of Definition 2.5.1 : the function U satisfies for all $\varphi \in W_{c,+}^{1,\infty}(\mathbb{R} \times \mathbb{R}^2)$, and for all $\kappa \in \mathcal{C} = \mathbb{R} \times \bar{B}$ the following inequality

$$\begin{aligned} & \int_0^T \int_{\Omega} |U(t,x) - \kappa|^2 \partial_t \varphi(t,x) dx dt + \int_0^T \int_{\Omega} \sum_{i=1}^2 \langle U(t,x) - \kappa; A_i(U(t,x) - \kappa) \rangle \partial_i \varphi(t,x) dx dt \\ & + \int_{\Omega} |U_0(x) - \kappa|^2 \varphi(0,x) dx + \int_0^T \int_{\Omega} 2 \langle F(t,x); U(t,x) - \kappa \rangle \varphi(t,x) dx dt + \int_0^T \int_{\partial\Omega} \langle \kappa_+; M_v \kappa_+ \rangle \varphi(t,x) d\mathcal{H}^{n-1}(x) dt \geq 0, \end{aligned} \quad (4.4.19)$$

where $F = (f, 0, 0)$, $U_0 = (v_0, \sigma_0)$ and

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad (4.4.20)$$

and κ_+ stands for the projection onto $(\ker(A_v + M_v)) \cap \text{Im} A_v$. In fact, this definition is due to the hyperbolic approach of the simplified model of perfect plasticity (see Section 1.3.2 and 3.7). We have seen that the equation of motion, the flow rule, the nonlinear boundary condition, the initial condition can be recovered from this definition. Using this hyperbolic point of view, the simplified model of dynamical perfect plasticity can be approached thanks to the two schemes described in Subsection 4.4. One important point to notice first is that these schemes do not include a special treatment at the boundary to model the nonlinear boundary condition (4.4.16). Indeed, we only take into account the constraint thanks to a projection step on every mesh and the first step of these schemes use the linear boundary condition

$$\langle \sigma; v \rangle + \lambda^{-1} v = 0. \quad (4.4.21)$$

Our goals now are to test numerically the interactions between the boundary condition and the constraint for this particular hyperbolic system under constraint and to see if the nonlinear boundary condition is obtained with these schemes. The major point that allows us to bring to light these facts is the velocity threshold overrun in the boundary condition.

To observe this overrun, we present here three different test cases. The spatial domain is the same in the three tests, the differences between them are in the choice of the initial data and the source term. In the first test, the initial data is the following one : the initial elastic tensor σ_0 satisfies the constraint (*i.e.* Equation (4.4.18)) but the constraint is saturated on a part on the boundary (*i.e.* $|\sigma_0| = 1$ on a part of $\partial\Omega$) and the initial velocity satisfies the (nonlinear) boundary condition (4.4.17) (consequently $|v_0| \leq \lambda$ on the boundary) but inside the domain, the initial velocity is important (*i.e.* $|v_0| \gg \lambda$ inside Ω). The initial goal of this test was to observe waves with large velocity coming from the interior of the domain arriving at the boundary in order to see if the scheme takes account of the threshold at the boundary. We will see that the constrained dynamic quickly modifies the magnitude of the velocity inside the domain and consequently, the amplitudes of the waves coming from the initial area where the amplitude of the velocity was larger than the threshold are smaller than the threshold when then touch the boundary of the domain. However, this test case is still interesting for at least two reasons. First it allows us to illustrate the difference between the constrained dynamic (*i.e.* the perfect plasticity model) and the unconstrained dynamic (the linear elasticity equations, see Section 1.3.1). Secondly, since we do not use any term source here, it is a good example to show the stability of the scheme.

The other tests are based on the following formal motivation : we want to observe large velocities near the boundary. But if we look at the equation of motion

$$\partial_t v - \text{div} \sigma = f,$$

we see that if f is positive (for example) near the boundary (for each time) then the velocity is going to increase over time near the boundary. Hence, we present a test case when the source term f is equal to a positive constant near the boundary and to zero elsewhere and another test where f is equal to a positive constant near a certain part of the boundary and to a negative constant near another part of the boundary and is equal to zero elsewhere. These two tests allow us to obtain large velocity near the boundary (*i.e.* $|v| \gg \lambda$ near $\partial\Omega$) and to bring to light that the nonlinear boundary is taken into account by our schemes. In the three following subsections, we present in details these tests and comment the numerical solutions obtained.

Remark 4.4.7. *The schemes (4.3.3) and (4.4.3) have been implemented in a program in C++. We use the meshing tools of*

FreeFem++ (see [52]) and then our program recovers the necessary information (for example, the neighboring cells of each cell of the mesh) about the mesh to construct the approximate solution. At each time step, our program writes a VTK file that allows us to visualize the approximate solution thanks to the application Paraview (see [2]). Our program can handle 2D unstructured meshes (formed with triangles) and problems with $U \in \mathbb{R}^2$ or \mathbb{R}^3 .

Remark 4.4.8. As we have already observed the CFL condition of the CVV scheme is twice times smaller than the CFL condition of the dissipative, that is why all the tests that we present here have been made with the CVV scheme. There is no strong difference between the two schemes.

Remark 4.4.9. In the following, we decide to focus our attention to the case $\lambda = 1$. It means that the hyperbolic boundary condition that we use are the classical transparent boundary condition for the elastic case. In the constrained case, the boundary condition is

$$\langle \sigma; \nu \rangle + T_1(v) = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (4.4.22)$$

where $T_1(z) = \min(-1, \max(z, 1))$.

Remark 4.4.10. In what follows, we use initial data that are compactly supported and smooth. To do so, we define for $c = (c_x, c_y) \in \mathbb{R}^2$ and $\delta = (\delta_x, \delta_y) \in \mathbb{R}^2$ the function (see also Figure 4.3)

$$\rho_{c,\delta} = \exp\left(-\frac{1}{1 - \frac{1}{\delta_x^2}(x - c_x)^2}\right) \exp\left(-\frac{1}{1 - \frac{1}{\delta_y^2}(y - c_y)^2}\right) \mathbb{1}_{[c_x - \delta_x, c_x + \delta_x]}(x) \mathbb{1}_{[c_y - \delta_y, c_y + \delta_y]}(y) \in C_c^\infty(\mathbb{R}^2; \mathbb{R}). \quad (4.4.23)$$

In the three following case, we use the following data

- Spatial domain : $\Omega = [0, 1] \times [0, 1]$. We use the program FreeFem++ (see [52]) to build the mesh for this domain. To be precise, we use the command square with 200 points on the horizontal axes and 200 points on the vertical axes. The mesh contains 80000 triangles.
- Final time : $T = 1$. We use 800 time-steps and consequently the CFL condition (4.4.10) is approximately equal to 0.71.

4.4.3.1 Testcase 1 : High speed dynamic with or without constraint

The purpose of this test is to show the strong differences between the dynamic in the unconstrained case and the constrained case. In the first one, the theoretical model is the linear elasticity equations in 2D (the same anti-plane shear hypotheses have been made here as in Section 3.3.2). In the second one, we are trying to approach the solutions of the simplified model of the dynamical perfect plasticity. From the numerical point of view, the unique difference between these cases is that in the first case we do not use a projection step in (4.4.3) (and consequently, $v_K^{p+1} = v_K^{p,*}$) and in the second case we do. To illustrate these strong dynamical differences, we use the following data (see also Figure 4.4).

- Initial data : The initial velocity v_0 is the sum of two velocities, denoted v_0^1 (see Figure 4.2) and v_0^2 (see Figure 4.3). The first one v_0^1 is null outside the open ball B_1 of radius 0.3 and center $(1, 0.5)$, v_0^1 is equal to -1 on the open ball B_2 of radius 0.25 and center $(1, 0.5)$. In the strip between these two balls, we join these two constants using a C^1 connection. To be precise, we denote, for $(x, y) \in [0, 1]^2$, by $r(x, y)$ the Euclidean distance between the point (x, y) and the point $(1, 0.5)$, and we have

$$v_0^1(x, y) = -(16000r(x, y)^3 - 13200r(x, y)^2 + 3600r(x, y) - 324), \quad \text{on } B_1 \setminus B_2.$$

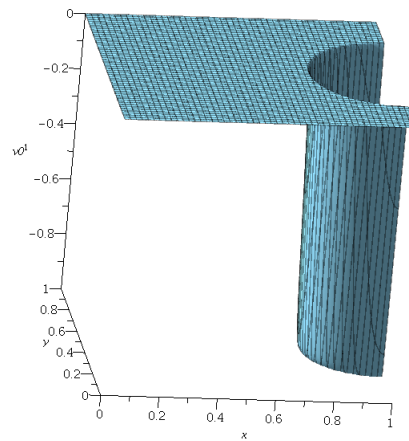
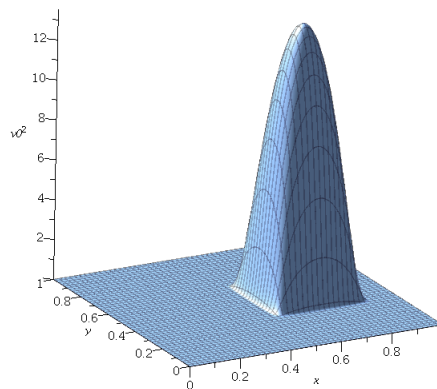
It is important to notice that $-1 \leq v_0^1 \leq 0$. The second velocity v_0^2 has a compact support in $[0, 1]^2$, namely $[0.5, 0.9] \times [0.3, 0.7]$, and is equal to

$$\forall (x, y) \in [0, 1]^2, \quad v_0^2(x, y) = 100\rho_{(0.7, 0.5), (0.2, 0.2)}(x, y).$$

In order to satisfy the (linear) boundary condition at $x = 1$ the first component of σ is equal to $-v_0^1$. The second component of σ is null on Ω . Consequently, we have $v_0 + \langle \sigma; \nu \rangle = 0$ on $\partial\Omega$. Remark also that the initial data belong to the convex set of constraints.

- The term source is null on $[0, T] \times \Omega$.

In Figure 4.5, we compare the velocity of the material in the linear elasticity case (on the right column) and the perfect plasticity case (on the left column) at two different times : at $t = 0.1$ (top row), the waves coming from the initial high-speed velocity v_0^2 have not reached the boundary yet. At $t = 0.3$ (bottom row), the waves has now reached the boundary.

FIGURE 4.2 – Graph of the function v_0^1 on $[0, 1] \times [0, 1]$.FIGURE 4.3 – Graph of the function $v_0^2 = 100\rho_{(0.7,0.5),(0.2,0.2)}$ on $[0, 1] \times [0, 1]$.

First, one can observe, in Figure 4.5, that the waves (coming from v_0^2) are travelling at the same speed in the unconstrained or the constrained case. But, in the elastic case, we see oscillations : at $t = 0$ the maximum of the velocity is around 13.5 and at $t = 0.1$, the maximum is now around 7 (but the velocity is around 5 near the wave-front). On the contrary, the plastic waves (coming from v_0^2) the initial structure of the velocity is preserved. We see that the maximum of the velocity decreases by a less important manner than in the elasticity case (it is around 10) and at the same time, the velocity is slower near the wave-front (around 1).

Now, when the waves touch the boundary, we still observe oscillations in the elastic case. In particular the velocity (see Figure 4.5d), on a important part of the boundary $x = 1$ (on the right of these pictures), is larger than 1 (the plastic threshold for the velocity in the boundary condition). In the perfect plasticity case (see Figure 4.5c), we still observe the structure of the initial velocity v_0^2 (the maximum is now around 5) and the velocity near the wave-front is around 1 (the threshold at the boundary - in fact there is no overrun of the threshold at the boundary).

In Figure 4.6, we observe the dynamical differences between the elastic (second column) and the plastic (first column) case by looking at the magnitude of the variable σ at $t = 0.1$ and $t = 0.3$. One should be careful when looking at these pictures, the scales are different for the elastic and the plastic models. Despite the fact that the scale are different (due to the constraint $|\sigma| \leq 1$ in the plastic model), the structure of the elastic and plastic solutions are quite similar and the fact that these solutions are close to the threshold $|\sigma| = 1$ at the boundary $x = 1$ is a consequence of the initial data.

At time $t = 0.3$, the differences on the variable σ between the elastic and the plastic situations appear clearly. In the elastic case, we observe waves generated by the initial velocity v_0^2 and in addition, the magnitude of the function

σ is larger than 1 on a large part of the boundary. On the contrary, in the plastic case, the areas where the constraint is saturated are located near the domain of the initial velocity v_0^2 and consequently, we can not see any interactions between the constraint and the boundary $x = 1$.

We also want to understand what is the role of the projection step in the scheme (4.4.3). To do so, we have represented in Figure 4.7, the areas (in red) where $|\sigma| > 1$ after the first step of the scheme. We observe that these zones are restricted to the domain of the initial velocity v_0^2 and also that the domain where the projection step is needed shrinks over time. In fact, the projection step is useless from the time $t \approx 0.62$.

Lastly, we show on this particular test without test source that the scheme (4.4.3) is stable in the sense of Proposition 4.4.4 in the constrained and the unconstrained cases. It is worth noticing that the introduction of the constraint induces a strong decrease of the L^2 -norm in short time.

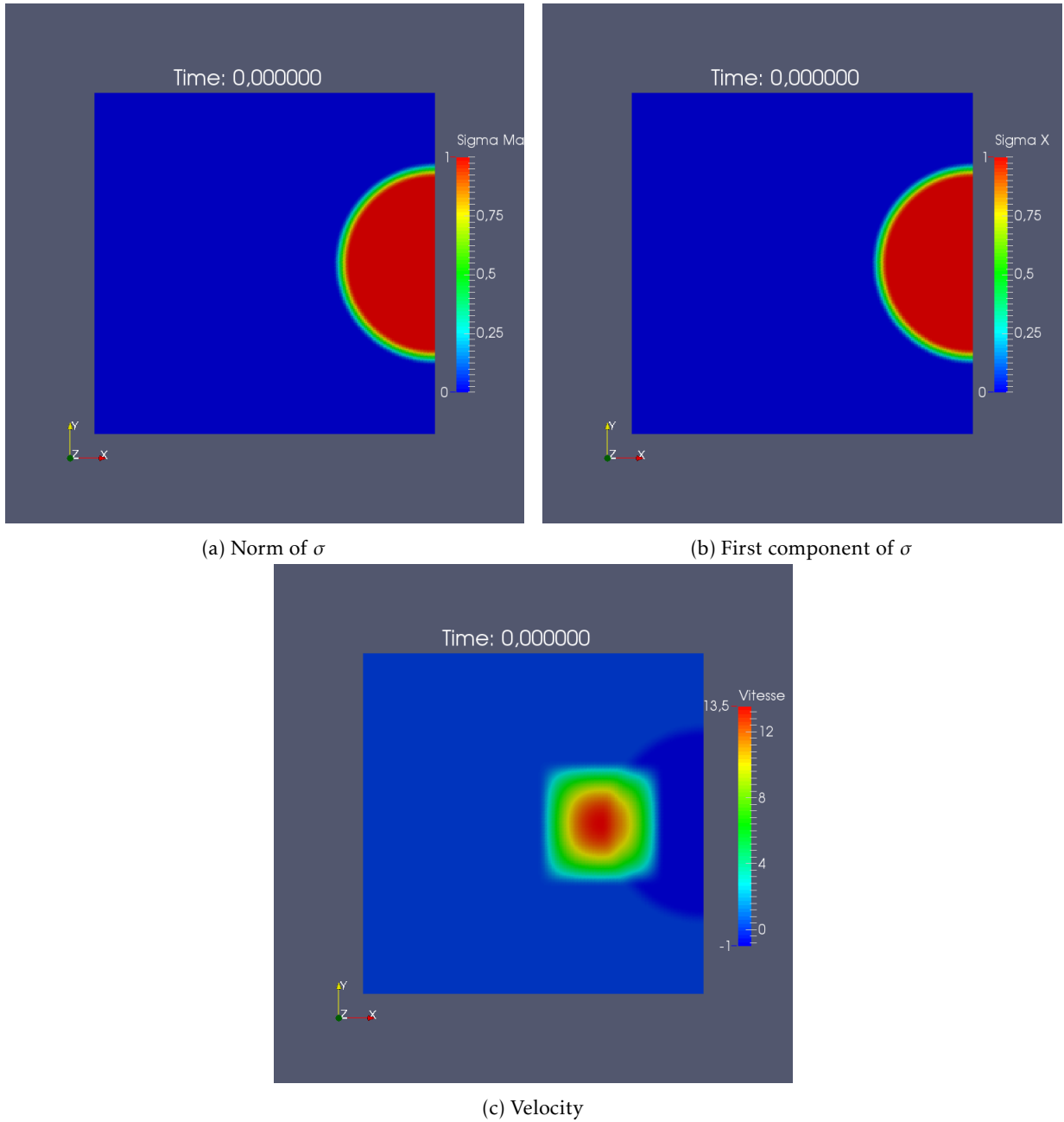


FIGURE 4.4 – Initial configuration for the test case of Subsection 4.4.3.1

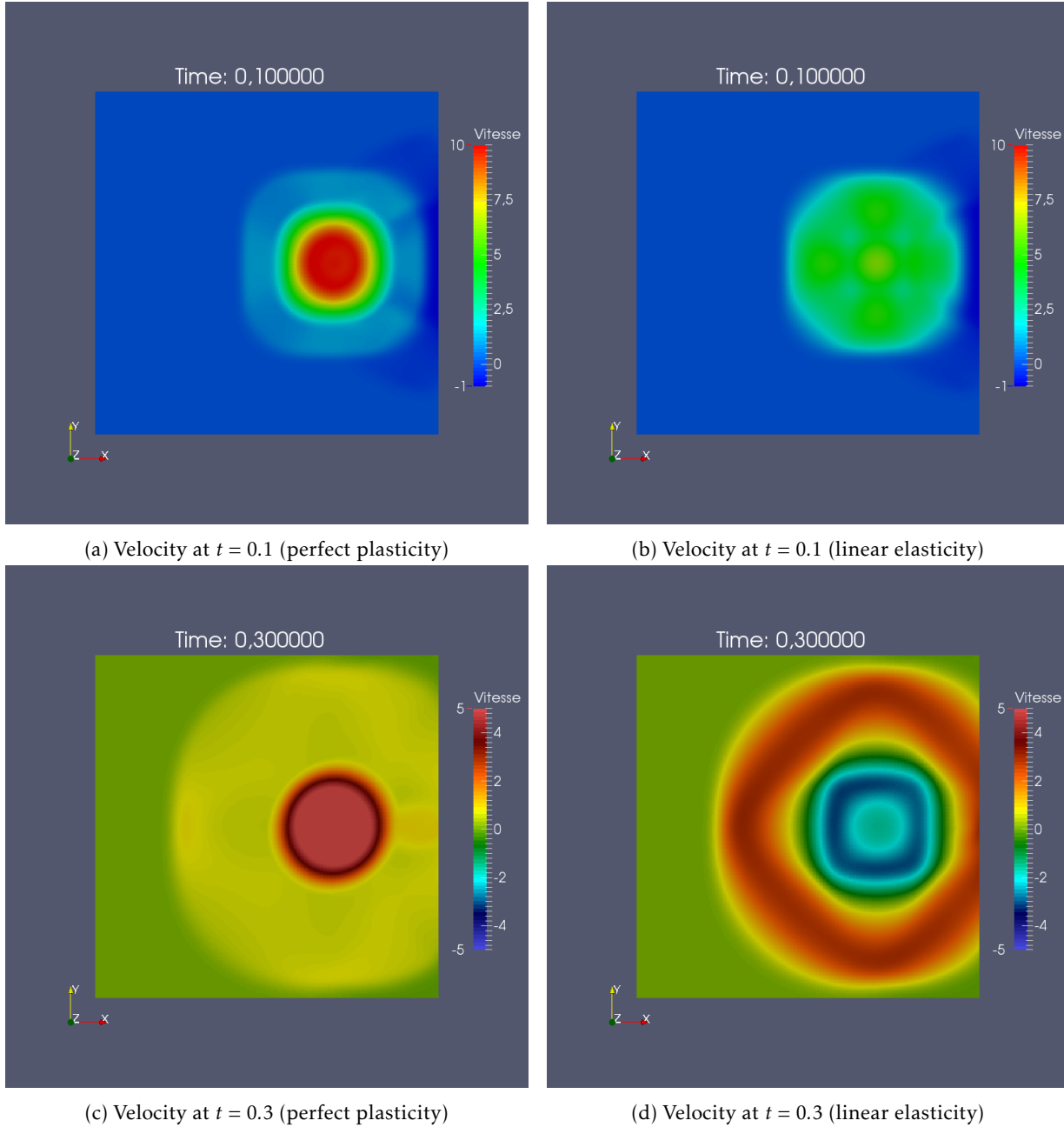


FIGURE 4.5 – Velocity for the test case of Subsection 4.4.3.1 at different times (first row : $t = 0.1$, second row : $t = 0.2$, third row : $t = 0.3$). Left Column : With constraint (perfect plasticity model). Right column : Without constraint (linear elasticity).

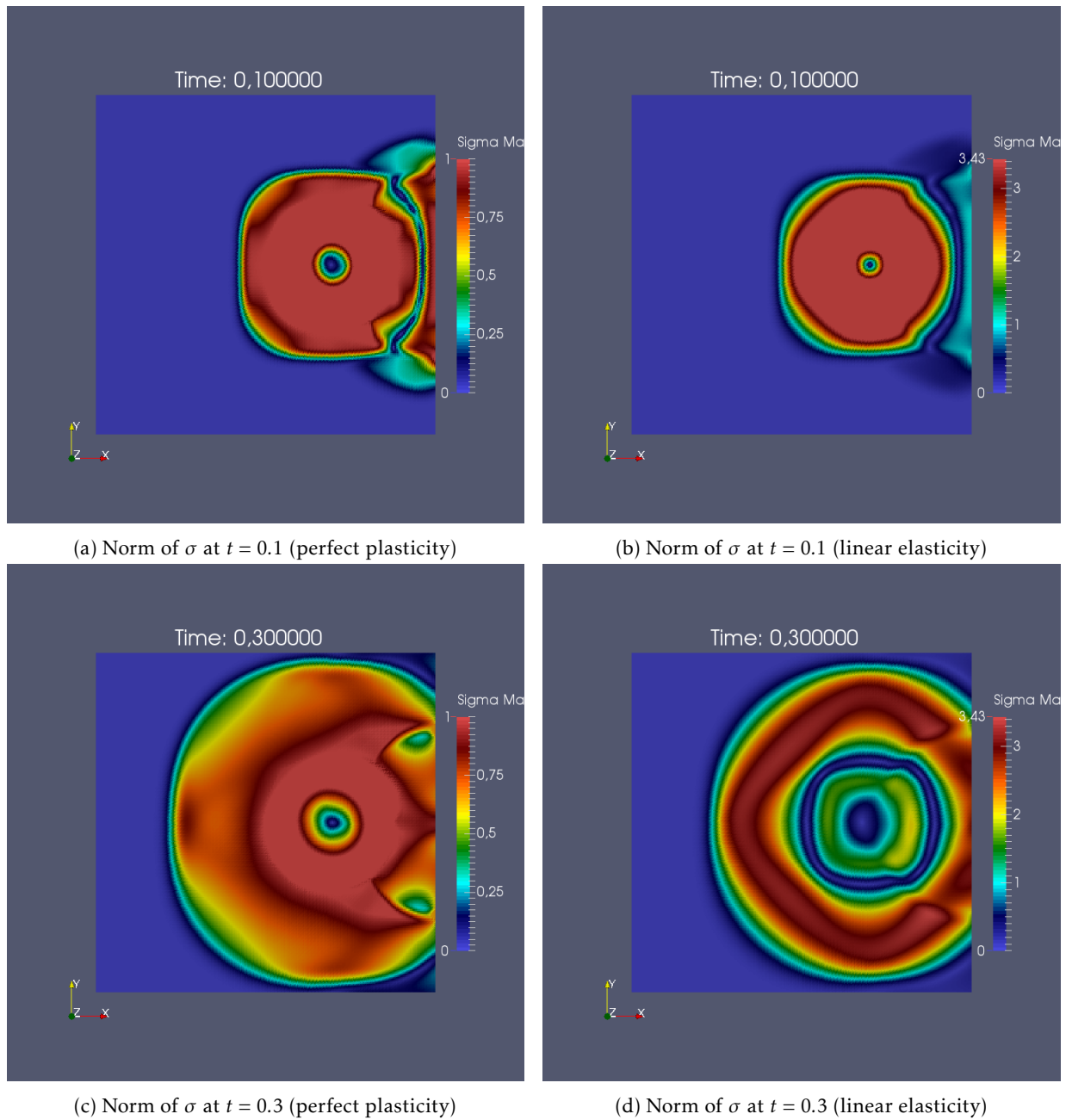


FIGURE 4.6 – Norm of σ for the test case of Subsection 4.4.3.1 at different times (first row : $t = 0.1$, second row : $t = 0.3$). Left Column : With constraint (perfect plasticity model). Right column : Without constraint (linear elasticity).

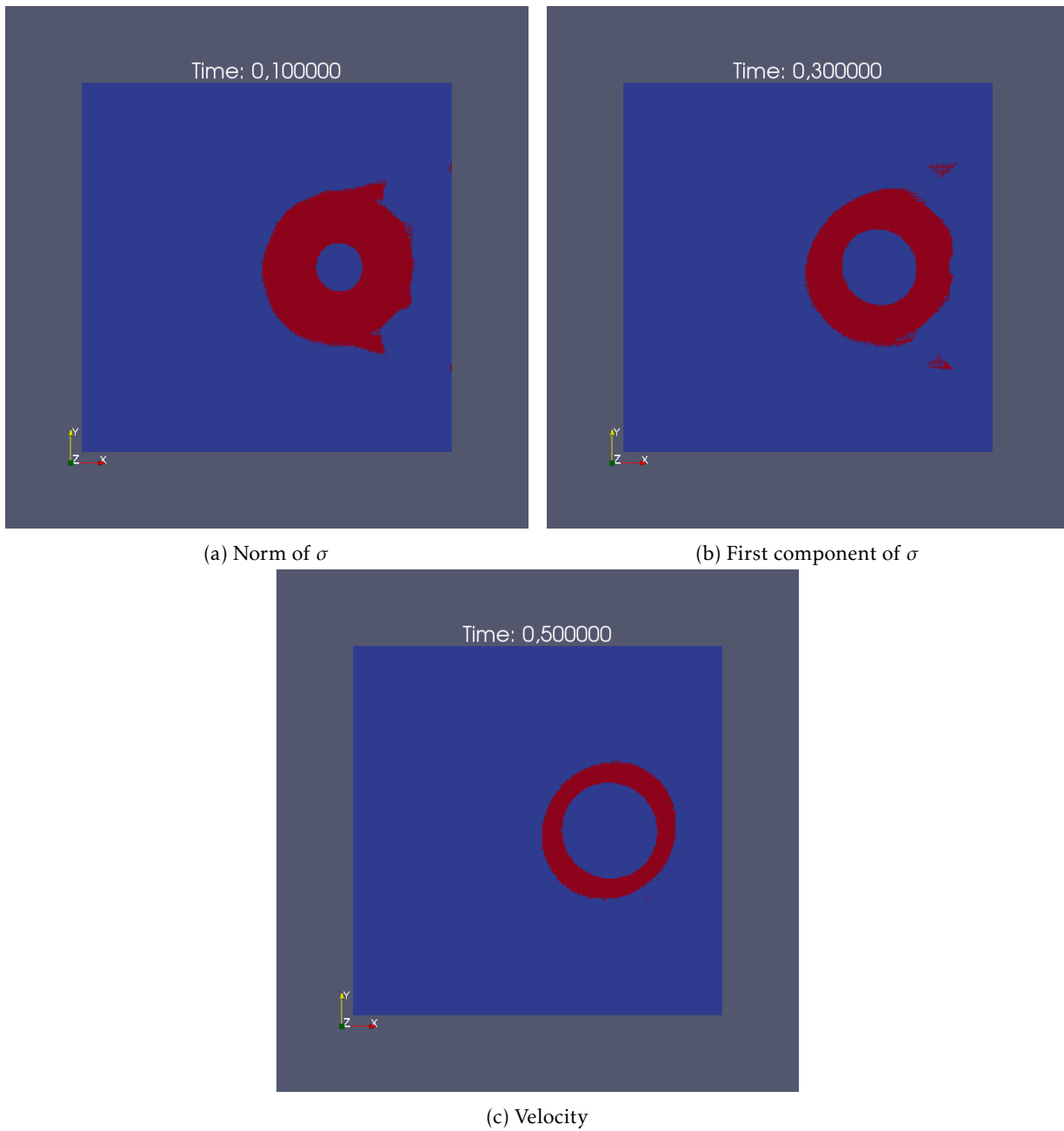


FIGURE 4.7 – Areas where σ do or do not belong to the set of constraints before the projection step of the scheme (4.4.3). In blue, σ belongs to the set of constraints (*i.e.* $|\sigma| \leq 1$). In red, $|\sigma| > 1$.

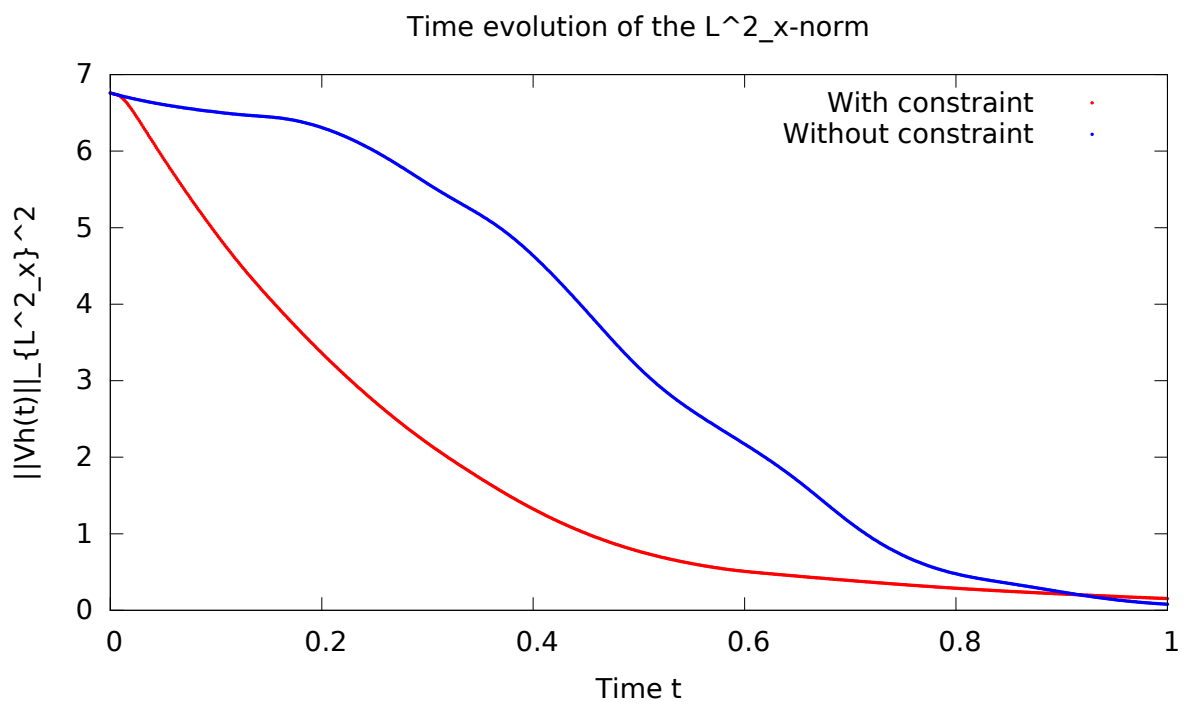


FIGURE 4.8 – Time evolution of the $L^2(\Omega)$ -norm of the numerical solution V_h with constraint (perfect plasticity model) and without constraint.

4.4.3.2 Testcase 2 : Interactions between the boundary condition and the constraint

In this second test case, the initial value of σ is the same than in the previous test case (see Figure 4.4a and Figure 4.4b). The initial velocity is equal to the opposite of the first component of σ . With this test, we focus our attention on the plastic model and, in particular, on the interactions between the constraint and the boundary condition (more precisely, we look at the boundary condition on the right of the domain, *i.e.* $x = 1$). We recall here the main hypotheses on the data we use for this test.

- Initial data : In this test, we use data that touch the boundary $x = 1$. The initial velocity v_0 is null outside the open ball B_1 of radius 0.3 and center $(1, 0.5)$, v_0 is equal to -1 on the open ball B_2 of radius 0.25 and center $(1, 0.5)$. In the strip between these two balls, we join these two constants using a C^1 connection. To be precise, we denote, for $(x, y) \in [0, 1]^2$, by $r(x, y)$ the Euclidean distance between the point (x, y) and the point $(1, 0.5)$, and we have

$$v_0(x, y) = -(16000r(x, y)^3 - 13200r(x, y)^2 + 3600r(x, y) - 324), \quad \text{on } B_1 \setminus B_2.$$

It is important to notice that $-1 \leq v_0 \leq 0$. In order to satisfy the (linear) boundary condition at $x = 1$ the first component of σ is equal to $-v_0$. The second component of σ is null on Ω . Consequently, we have $v_0 + \langle \sigma; \nu \rangle = 0$ on $\partial\Omega$. Remark also that the initial data belong to the convex set of constraints.

- The term source f is equal to 50 for all $t \in [0, T]$, for all $y \in [0, 1]$ and $x > 0.8$ and to 0 elsewhere.

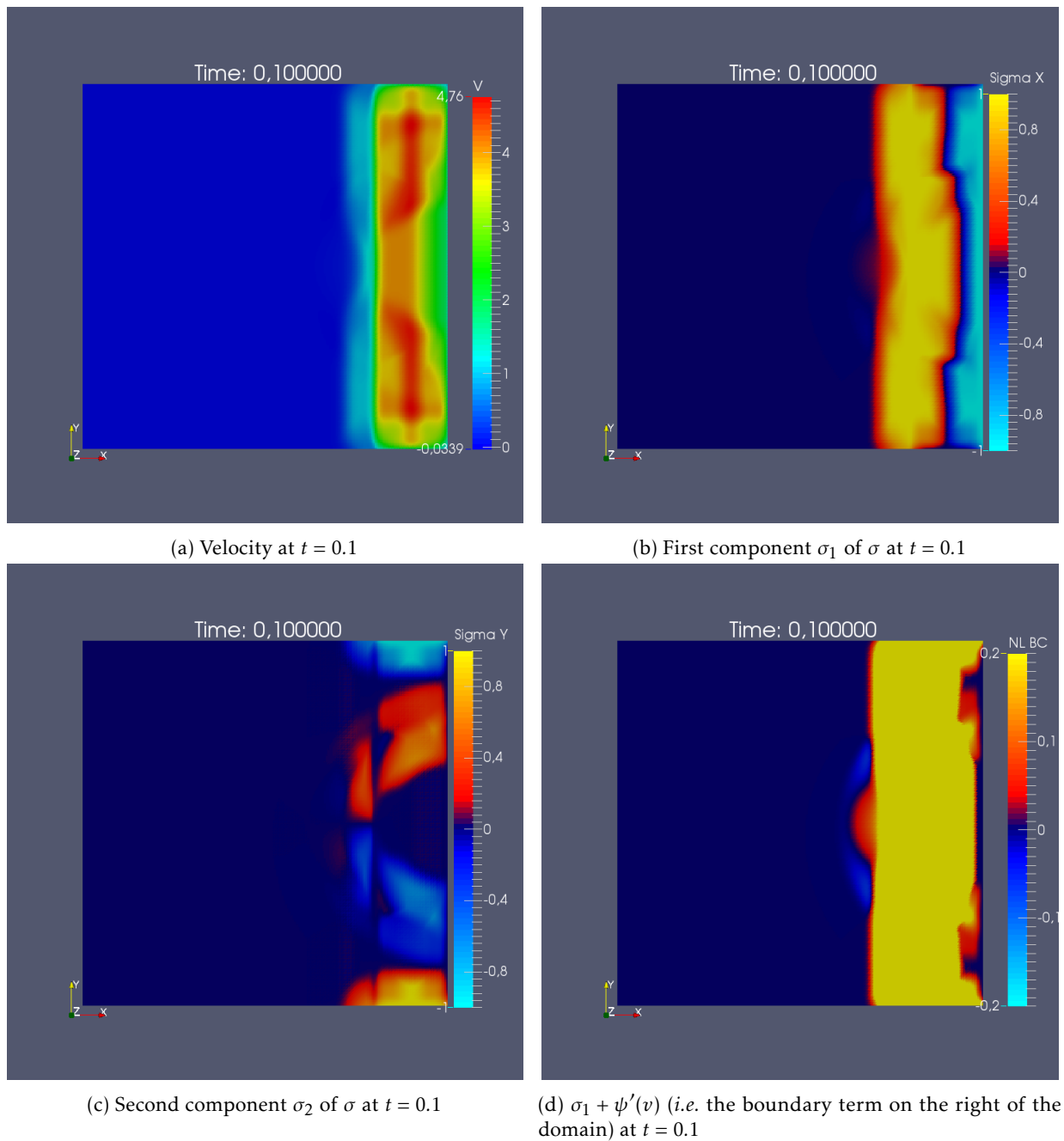
We decide to highlight the interaction between the constraint and the boundary at two different times $t = 0.1$ in Figure 4.9 and $t = 0.5$ in Figure 4.10. In these two figures, we display the velocity (top left of the figure), the first component, denoted σ_1 in the following, of σ (top right), the second component (bottom left), denoted σ_2 , and the term $\sigma_1 + T_1(v)$ (which is involved in the boundary condition at $x = 1$: $\sigma_1 + T_1(v) = 0$).

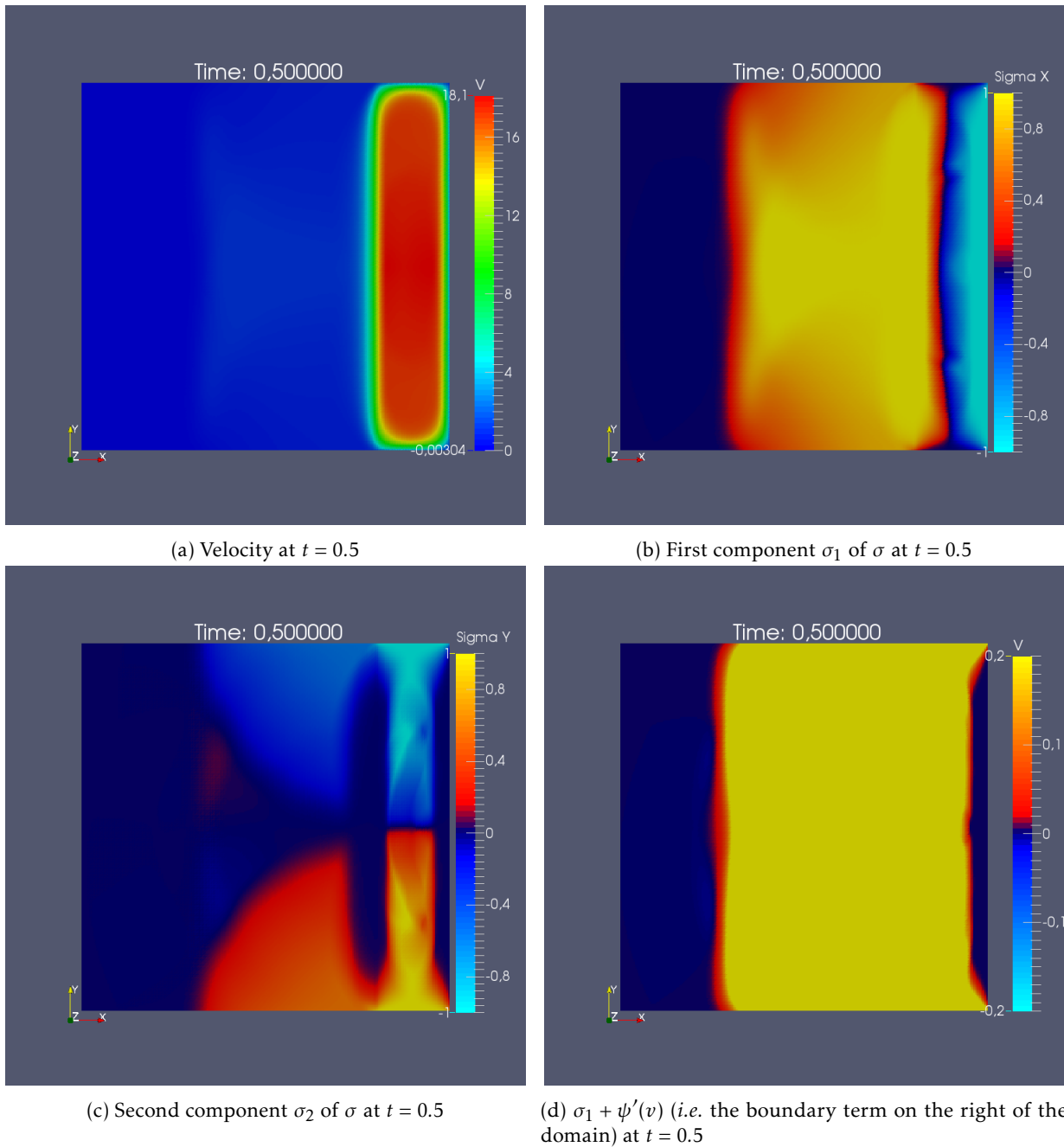
At these two different times, we observe that the introduction of a positive term source in the strip $[0.8, 1] \times [0, 1]$ allows us to get a large velocity (*i.e.* $v \gg 1$) near the boundary $x = 1$ (see Figure 4.9a and 4.10a). The theoretical boundary condition implies that in this situation we should see that $\sigma_1 = -1$ at the boundary $x = 1$ (and consequently, $\sigma_2 = 0$ due to the constraint). Numerically, the scheme produces a solution that matches the mathematical model (see Figure 4.9b, 4.9c, 4.10b and 4.10c). Consequently, the nonlinear boundary condition is satisfied by the numerical approximation (see Figure 4.9d and 4.10d) despite the fact that we have not implemented in particular treatment at the boundary to get this nonlinear boundary condition. This fact may be seen as a first validation of our schemes.

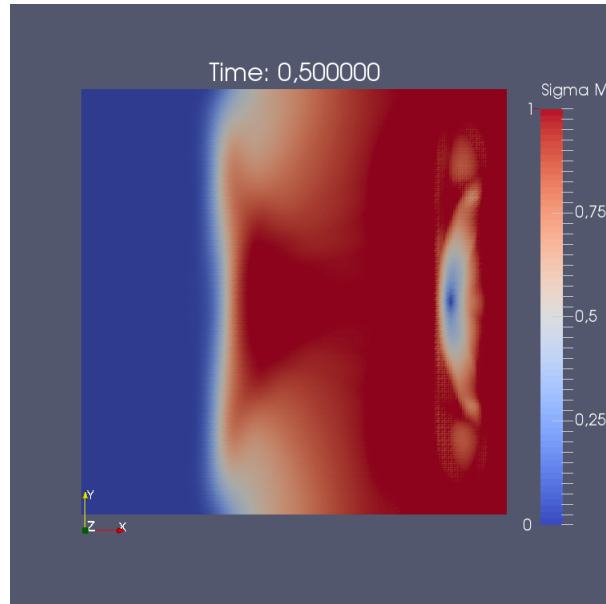
In Figure 4.11, we show the area where the plastic variable $\partial_t p$ is not zero at time $t = 0.5$. We recall that

$$\partial_t p = 0, \quad \text{if } |\sigma| < 1, \quad \partial_t p = \frac{\sigma}{|\sigma|}, \quad \text{if } |\sigma| = 1.$$

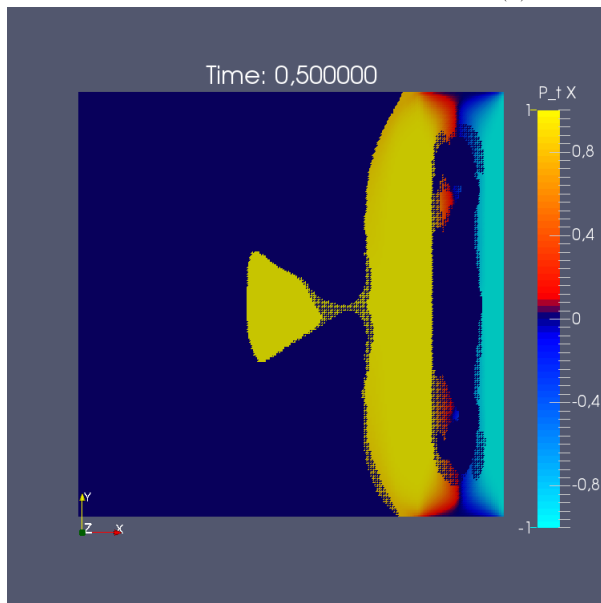
Again, we see that the fact that $\partial_t p$ has to be equal to $(-1, 0)$ at the boundary $x = 1$ is preserved by the scheme.

FIGURE 4.9 – Test case of Subsection 4.4.3.2 at time $t = 0.1$.

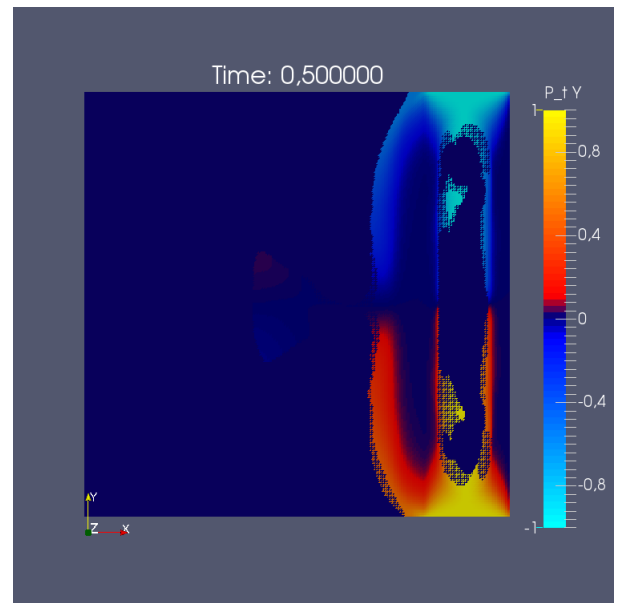
FIGURE 4.10 – Test case of Subsection 4.4.3.2 at time $t = 0.5$.



(a) Norm of σ at $t = 0.5$



(b) First component $\partial_t p_1$ of $\partial_t p$ at $t = 0.5$



(c) Second component $\partial_t p_2$ of $\partial_t p$ at $t = 0.5$

FIGURE 4.11 – Test case of Subsection 4.4.3.2 at time $t = 0.5$.

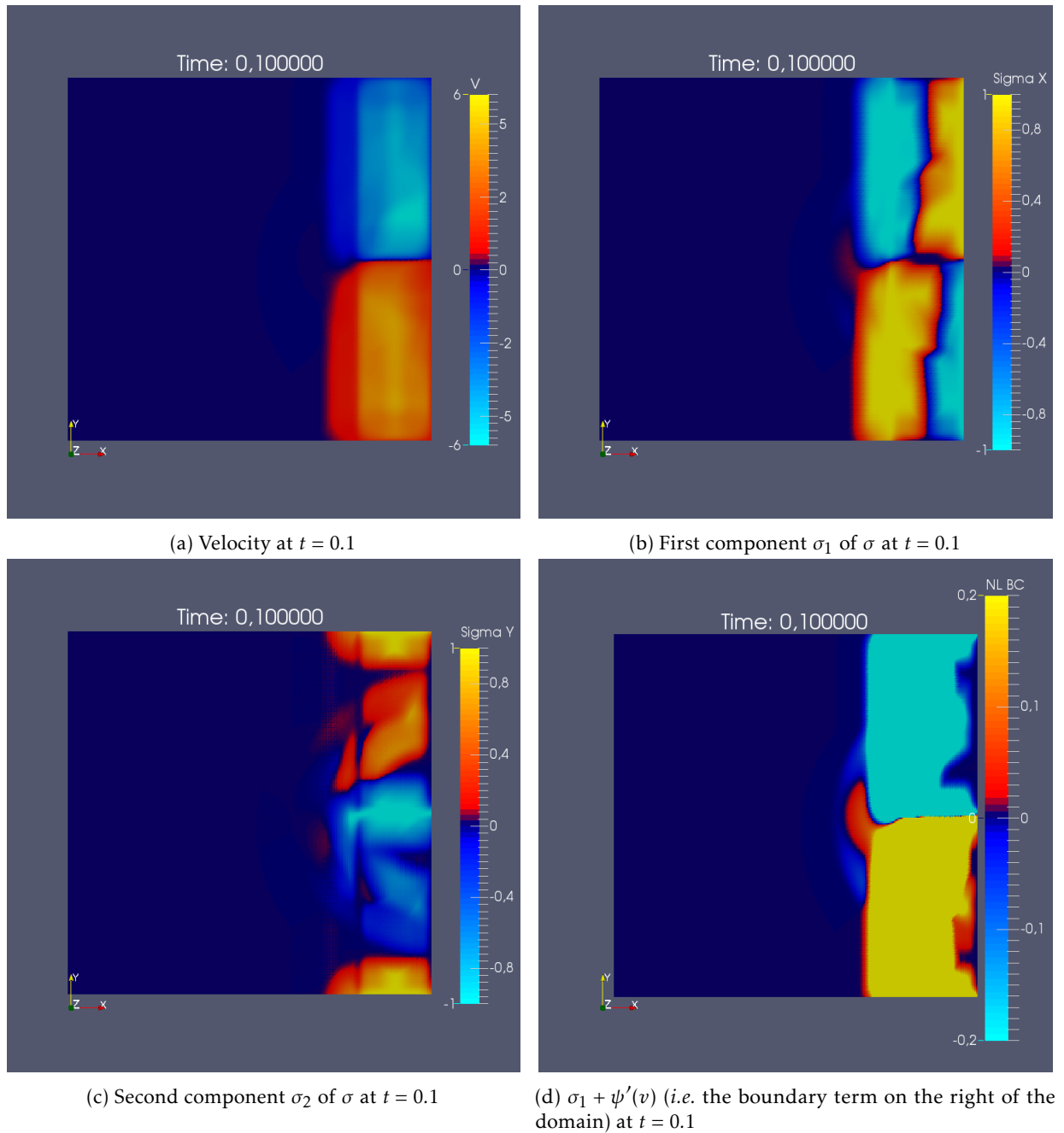
4.4.3.3 Testcase 3 : Interactions between the boundary condition and the constraint

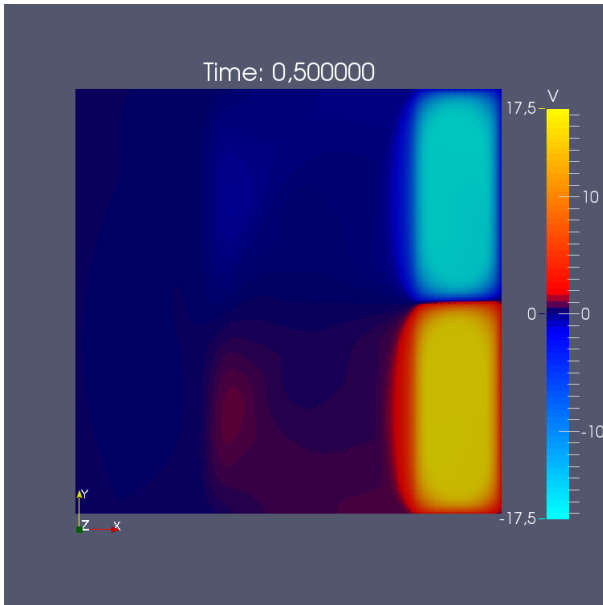
In this last test case, the only change on the data concerns the term source which is now equal to 50 on $[0.8, 1] \times [0, 0.5]$ and to -50 on $[0.8, 1] \times [0.5, 1]$. More precisely, the data we use for this test are the following one.

- The term source f is equal to 50 for all $t \in [0, T]$, for all $y \in [0, 0.5]$ and $x > 0.8$, to -50 for all $t \in [0, T]$, for all $y \in [0.5, 1]$ and $x > 0.8$ and to 0 elsewhere.

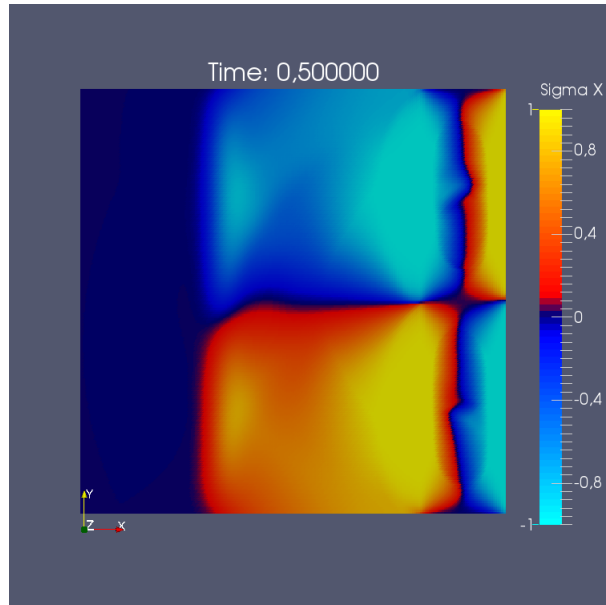
We decide to display the same data as in the previous test : the velocity, the first and second component of σ , the term $\sigma_1 + T_1(v)$ at time $t = 0.1$ in Figure 4.9 and at time $t = 0.5$ in Figure 4.10 ; the norm of σ and the two components of $\partial_t p$ at time $t = 0.5$ in Figure 4.14.

Due to the term source, we see two different zones near the boundary $x = 1$: in the first one the velocity is much higher than 1 and in the second one considerably lower than -1 . Consequently, in the first one, we see that the variable σ follows the nonlinear boundary condition and consequently we have $\sigma_1 = -1$ and $\sigma_2 = 0$. Similarly, in the second area, we have $\sigma_1 = 1$ and $\sigma_2 = 0$ due to the interaction between the constraint and the boundary condition. This interaction can also be highlighted thanks to the variable $\partial_t p$.

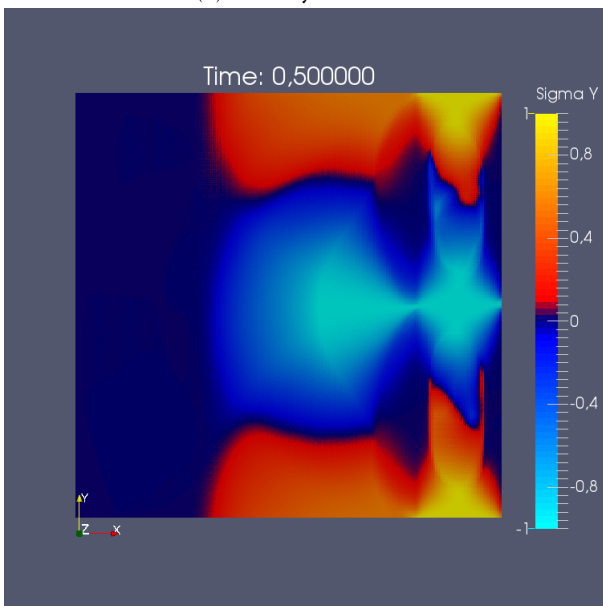
FIGURE 4.12 – Test case of Subsection 4.4.3.3 at time $t = 0.1$.



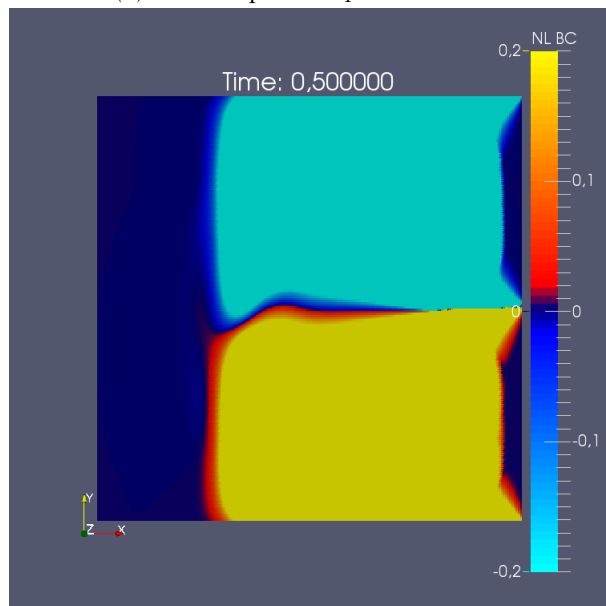
(a) Velocity at $t = 0.5$



(b) First component σ_1 of σ at $t = 0.5$

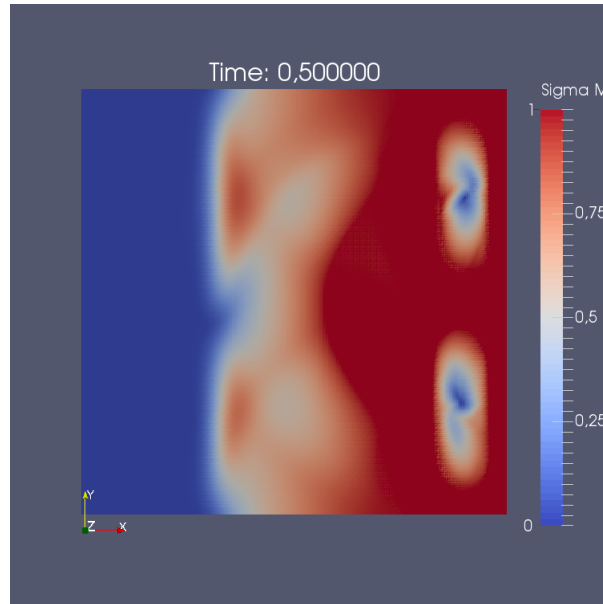


(c) Second component σ_2 of σ at $t = 0.5$

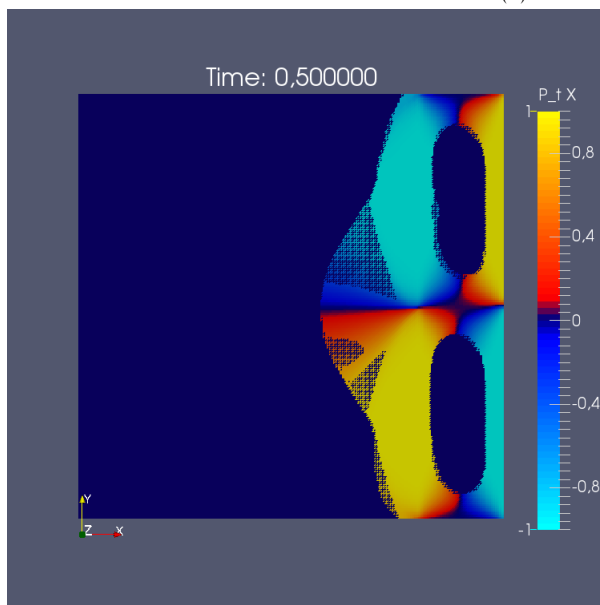


(d) $\sigma_1 + \psi'(v)$ (*i.e.* the boundary term on the right of the domain) at $t = 0.1$

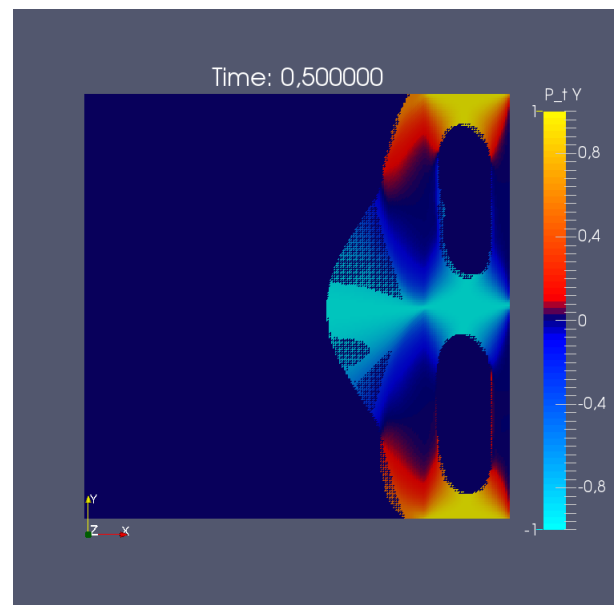
FIGURE 4.13 – Test case of Subsection 4.4.3.3 at time $t = 0.5$.



(a) Norm of σ at $t = 0.5$



(b) First component $\partial_t p_1$ of $\partial_t p$ at $t = 0.5$



(c) Second component $\partial_t p_2$ of $\partial_t p$ at $t = 0.5$

FIGURE 4.14 – Test case of Subsection 4.4.3.3 at time $t = 0.5$.

Conclusion

Cette thèse a permis d'étendre les formulations faibles développées dans [37] pour les systèmes de FRIEDRICHS avec contraintes au cas de problèmes posés dans des domaines avec bord avec ou sans contraintes. Muni de ce nouveau cadre mathématique, nous avons pu explorer en détail l'exemple d'un modèle simplifié de la plasticité parfaite. Nous avons alors comparé les deux approches possibles pour étudier ce problème : la première, provenant du calcul des variations, nous a permis, grâce à des résultats fins d'analyse fonctionnelle de démontrer que le problème que nous regardions était bien posé. La seconde approche, provenant des travaux de cette thèse, a motivé notre choix de condition de bord initial et nous a fourni un exemple d'interaction entre le bord et les contraintes. Ce lien a été mis en évidence de manière théorique et observé numériquement. Ce type d'analogie et l'étude qui en découle mériterait d'être envisager pour d'autres systèmes hyperboliques (contraints ou non).

De plus, ce modèle simplifié de la dynamique de la plasticité parfaite a permis d'effectuer un parallèle précis entre deux notions différentes de solutions : les solutions "variationnelles" et les solutions "hyperboliques sous contrainte". Nous avons réussi à démontrer que ces deux visions du même problème étaient en fait équivalentes. Cette équivalence nous a permis d'utiliser des méthodes numériques, classiques pour les problèmes hyperboliques, pour ce problème provenant du calcul des variations. L'approximation numérique s'est révélée en accord avec les résultats théoriques. Nous avons notamment pu mettre en évidence la condition de bord nonlinéaire qui symbolise l'interdépendance des contraintes et du bord.

Le modèle simplifié de la dynamique de plasticité parfaite a été étudié en détail dans cette thèse sous différents angles : hyperbolique sous contrainte, variationnel et numérique. Cependant ces analyses méritent d'être généralisées dans le futur. Il est notamment intéressant de se demander si les résultats du Chapitre 3 peuvent être étendus à des modèles plus généraux et notamment au modèle 3D de la plasticité parfaite (décrit dans la Section 1.1.1). Il serait aussi pertinent d'examiner les conséquences des résultats présents dans cette thèse pour les modèles quasi-statiques (et non plus uniquement dynamique). Enfin, une analyse asymptotique qui nous permettrait d'obtenir des résultats sur le modèle simplifié avec condition de bord mixte serait aussi bénéfique pour la compréhension du lien entre la vision hyperbolique et la vision variationnelle.

Enfin, d'un point de vue numérique, nous avons vu que le traitement de la contrainte de manière simple (une projection sur chaque élément du maillage à chaque pas de temps) a permis d'obtenir une solution numérique conforme à la théorie. Il serait important de se demander si un traitement plus spécifique au bord ne permettrait pas d'obtenir de meilleurs résultats mais aussi de s'attaquer au problème de l'étude de la convergence de tels schémas et à terme d'obtenir un taux de convergence pour les schémas adaptés aux systèmes hyperboliques en domaine borné avec condition de bord.

Mathematical symbols and notations

We gather here some notations and symbols, which are, for the most part, explained when they are used for the first time in a chapter.

We try to use French mathematical conventions in the French part of this thesis and English mathematical conventions in the chapters written in English.

In the following, n, m, q are positive integers.

Sets, vectors and matrices

If a and $b \in \mathbb{R}^n$, we write $\langle a; b \rangle$ for the Euclidean scalar product, and we denote by $|a| = \sqrt{\langle a; a \rangle}$ the associated norm.

We write $\mathbb{M}^{q \times n}$ for the set of real $q \times n$ matrices, and $\mathbb{M}_{\text{sym}}^{n \times n}$ for that of all real symmetric $n \times n$ matrices. Given a matrix $A \in \mathbb{M}^{n \times n}$, we let $|A| := \text{tr}(AA^T)^{1/2}$ (A^T is the transpose of A , and $\text{tr}A$ is its trace) which defines the usual Euclidean norm over $\mathbb{M}^{n \times n}$. In addition, if P is a non-negative symmetric matrix, we denote for $y \in \mathbb{R}^m$ by $|y|_P$ the quantity $\langle y; Py \rangle$.

We recall that for any two vectors a and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{n \times n}$ stands for the tensor product, *i.e.*, $(a \otimes b)_{ij} = a_i b_j$ for all $1 \leq i, j \leq n$, and $a \odot b := (a \otimes b + b \otimes a)/2 \in \mathbb{M}_{\text{sym}}^{n \times n}$ denotes the symmetric tensor product.

If $x_0 \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x_0, r)$ the Euclidean open ball (in \mathbb{R}^n) of radius r and center x_0 and by $\bar{B}(x_0, r)$ the Euclidean closed ball of radius r and center x_0 . To shorten notation, we let $\bar{B} := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed unit ball in \mathbb{R}^n . and $P_{\bar{B}}$ be the orthonormal projection onto \bar{B} .

Functional spaces and measures

Let \mathcal{U} be an open subset of \mathbb{R}^n , \mathcal{B} a Banach space, $1 \leq p \leq +\infty$ and $\alpha \in]0, 1]$.

We write $\mathcal{M}(\mathcal{U}; \mathbb{R}^m)$ (or simply $\mathcal{M}(\mathcal{U})$ if $m = 1$) for the space of bounded Radon measures in \mathcal{U} with values in \mathbb{R}^m , endowed with the norm $|\mu|(\mathcal{U})$, where $|\mu| \in \mathcal{M}(\mathcal{U})$ is the total variation of the measure μ .

The Lebesgue measure in \mathbb{R}^n is denoted by \mathcal{L}^n , and the $(n-1)$ -dimensional Hausdorff measure by \mathcal{H}^{n-1} .

We denote by $L^p(\mathcal{U}; \mathcal{B})$ the L^p Lebesgue space of measurable functions with values in \mathcal{B} and its associated norm is $\|\cdot\|_{L^p(\mathcal{U}; \mathcal{B})}$. The space of L^p functions from \mathcal{U} to \mathbb{R} is simply denoted by $L^p(\mathcal{U})$ and its norm $\|\cdot\|_{L^p(\mathcal{U})}$. Similarly, we write $W^{q,p}(\mathcal{U}; \mathcal{B})$ (respectively $H^q(\mathcal{U}; \mathcal{B})$) the $W^{q,p}$ (respectively H^q) Sobolev space of functions with values in \mathcal{B} and $\|\cdot\|_{W^{q,p}(\mathcal{U}; \mathcal{B})}$ its associated norm (respectively $\|\cdot\|_{H^q(\mathcal{U}; \mathcal{B})}$). Again, if the functions are real-valued, we simply write $W^{q,p}(\mathcal{U})$ (respectively $H^q(\mathcal{U})$).

The space $\mathcal{C}^p(\mathcal{U}; \mathcal{B})$ (respectively $\mathcal{C}^p(\mathcal{U})$) is the space of all functions φ with values in \mathcal{B} (respectively in \mathbb{R}) which are \mathcal{C}^p with a compact support in \mathcal{U} . We use similar notations for $\mathcal{C}^{p,\alpha}$ Hölder spaces.

The space $\mathcal{C}_c^p(\mathcal{U}; \mathcal{B})$ (respectively $\mathcal{C}_c^p(\mathcal{U})$) is made of all functions φ with values in \mathcal{B} (respectively in \mathbb{R}) which are \mathcal{C}^p with a compact support in \mathcal{U} . We denote by $\mathcal{D}'(\mathcal{U})$ (respectively $\mathcal{D}'(\mathcal{U}; \mathbb{R}^n)$) the space of distributions (respectively vector valued distributions).

We denote by $H(\text{div}, \mathcal{U})$ the Hilbert space of all $\sigma \in L^2(\mathcal{U}; \mathbb{R}^n)$ such that $\text{div} \sigma \in L^2(\mathcal{U})$.

The space $BV(\mathcal{U})$ of functions of bounded variation in \mathcal{U} is made of all functions $u \in L^1(\mathcal{U})$ such that the distributional gradient $Du \in \mathcal{M}(\mathcal{U}; \mathbb{R}^n)$.

We also use tests functions with non-negative values and denote by $W_{c,+}^{1,\infty}(U)$, the space of $W^{1,\infty}(U)$ functions with non-negative values and compact supports and by $\mathcal{C}_b^1(U; \mathbb{R}^+)$ the space of \mathcal{C}^1 non-negative functions which are bounded and have bounded derivatives.

We denote by $\langle \cdot | \cdot \rangle_{\mathcal{B}, \mathcal{B}'}$ the duality pairing between \mathcal{B} and its topological dual \mathcal{B}' (or simply $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ when \mathcal{H} is a Hilbert space). To shorten notation, we write $\langle \cdot | \cdot \rangle$ for the duality pairing between $\mathcal{C}_c^p(\mathcal{U}; \mathbb{R}^m)$ and $\mathcal{M}(\mathcal{U}; \mathbb{R}^m)$.

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