Université de Nice Sophia Antipolis
École Doctorale de Sciences Fondamentales et Appliquées

THÈSE

pour obtenir le titre de

Docteur en Sciences

de l'Université de Nice Sophia Antipolis

Discipline : Mathématiques

présentée et soutenue par

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Théories homotopiques des algèbres unitaires et des opérades

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soutenue le 14 septembre 2016

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Remerciements

Je tiens tout d'abord à remercier mon directeur de thèse, Bruno Vallette qui m'a introduit au monde très riche des opérades avec une énergie communicative. Merci de m'avoir soutenu, formé, de m'avoir orienté dans des directions fructueuses et d'avoir relu maintes fois tous mes écrits mathématiques.

Je souhaiterais également adresser mes remerciements aux membres de mon jury de thèse : merci à Clemens Berger d'avoir accepté de répondre à mes nombreuses questions sur les opérades et les catégories ; merci à Damien Calaque et Kathryn Hess d'avoir accepté de rapporter cette thèse cet été, période peu propice à ce genre d'exercice ; merci à Grégory Ginot de m'avoir présenté Bruno et orienté vers le monde de la topologie algébrique ; merci enfin à Ieke Moerdijk qui n'a pas hésité à me présenter ses idées sur les ensembles dendroidaux et grâce à qui je vais continuer les Mathématiques à Utrecht.

Mes anciens professeurs sont pour beaucoup dans la voie mathématique que j'ai suivie. Je pense notamment à M. Bozec en terminale, à Saab Abou-Jaoudé en prépa et à Gabriele Vezzosi beaucoup plus récemment en master. De nombreux mathématiciens m'ont également beaucoup apporté : je pense particulièrement à Joan Millés et au duo formé par Damien Lejay et Mathieu Anel.

Merci à tous mes amis parisiens pour tous les bons moments passés lors de mes retours dans la capitale : Corentin, François, Jean-Sébastien, Romain, Vincent et Vu Lan.

Ces trois années à Nice ont été riches en rencontres. J'ai été accueilli très chaleureusement dans le bureau 801 par Camilo et Jean-Baptiste en même temps qu'Olivier puis rejoint par Björn. Merci à eux pour toutes ces longues discussions. Les autres chercheurs, doctorants ou chercheurs plus avancés ont instauré une excellente ambiance au sein du Laboratoire Dieudonné : merci à Amine, Arthur, Bienvenu, Carole, Charles, Christophe, Claire, Eduard, Eléonore, David, David, Guillaume, Julie, Julien, Kateryna, Luis, Marcella, Mélisande, Nathalie, Reine, Rinel, Sorin. Merci également aux secrétaires : Manuelle, Isabelle, Angélique et Christine ont toujours parfaitement administré mes ordres de mission qui étaient parfois demandé un peu tardivement.

Merci enfin à toute ma famille et à Xiaojiao pour leur soutien constant.

Chapitre 1

Introduction

1.1 Des espaces aux complexes de chaînes

1.1.1 Une courte introduction à la topologie algébrique

Parmi les nombreuses disciplines mathématiques, la topologie est l'étude de la forme des espaces, de l'agencement des points des espaces : la forme d'une sphère, celle d'un bouquet de cercles, celle d'un tore, celle de la bouteille de Klein, ... Ainsi, la topologie ne se préoccupe pas des exemples d'espaces topologiques exotiques comme les ensembles de Cantor, ni de la différence entre connexité et connexité par arcs, ni encore des subtilités des espaces vectoriels topologiques si chères aux analystes. Elle n'est pas à proprement parler l'étude des espaces topologiques mais plutôt l'étude des formes. Le topologue restreint souvent son étude à une classe particulière d'espaces topologiques, les CW complexes, qui sont fabriqués en recollant successivement des boules D^n de dimension $n \in \mathbb{N}$:

$$D^n := \left\{ x = (x_0, \cdots, x_{n-1}) \in \mathbb{R}^n | \sum x_i^2 \leq 1 \right\} ,$$

le long de leur frontière, la sphère S^{n-1} de dimension $n-1$:

$$S^{n-1} := \left\{ x = (x_0, \cdots, x_{n-1}) \in \mathbb{R}^n | \sum x_i^2 = 1 \right\} .$$

Plus précisément, un espace topologique X est un CW complexe s'il existe une filtration de sous espaces $X_0 \subset X_1 \subset \cdots \subset X$ de X, vérifiant les propriétés suivantes :

- ▷ l'espace X_0 est un ensemble de points, c'est à dire un espace topologique discret.
- ▷ l'espace X_{n+1} est obtenu à partir de X_n par recollement de cellules de dimension $n+1$. Autrement dit il existe un ensemble A_n de n-sphères S^n de X_n tel que le carré suivant est un poussé-en-avant :

$$\begin{tikzcd} \coprod_{a\in A_n} S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod_{a\in A_n} D^{n+1} \arrow[r] & X_{n+1} . \end{tikzcd}$$

- ▷ l'espace X est l'union de tous les X_n :

$$X = \mathrm{colim}\ (X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots) .$$

La topologie algébrique est l'étude de la forme des espaces par des moyens algébriques. Cela consiste à associer des espaces vectoriels, des groupes, ou d'autres objets algébriques à tout espace topologique X afin d'en étudier la forme. Le passage du monde souple et malléable des espaces au monde rigide de l'algèbre se traduit par une identification dans le monde de l'algèbre de deux structures topologiques reliés par une déformation continue, c'est-à-dire deux structures homotopes.

Définition (Homotopie entre fonctions). Soit $f : X \to Y$ et $g : X \to Y$ deux fonctions continues entre espaces topologiques. On dit que f et g sont homotopes s'il existe une fonction $H : X \times [0,1] \to$

Y telle que $H(-,0) = f$ et $H(-,1) = g$. Une équivalence forte d'homotopie est un morphisme $f : X \to Y$ qui possède un inverse homotopique, c'est-à-dire qu'il existe un morphisme $g : Y \to X$ tel que fg est homotope à l'identité de Y et gf est homotope à l'identité de X.

Tous nos invariants algébriques vont donc envoyer une équivalence forte d'homotopie sur un isomorphisme de structures algébriques. A ce titre, le théorème de Whitehead nous montre que les groupes d'homotopie sont les invariants algébriques des CW complexes les plus fins que l'on puisse obtenir.

Définition (Groupes d'homotopie, équivalence faible d'homotopie)**.** Soit X un espace topologique. On note $\pi_0(X)$ l'ensemble des composantes connexes par arcs de X, c'est-à-dire le quotient de l'ensemble des fonctions du point vers X par la relation d'homotopie. Soit x un point de X. Pour tout entier $n \geq 1$, on définit le $n^{\text{ième}}$ groupe d'homotopie de X relativement à x, noté $\pi_n(X, x)$, comme le quotient par la relation d'homotopie de l'ensemble des applications de la sphère $S^n \subset \mathbb{R}^{n+1}$ de dimension n vers X qui envoient le point $(1, 0, \ldots, 0)$ vers x. Une application entre espaces topologiques $f : X \to Y$ qui induit des isomorphismes entre les groupes d'homotopie de X et ceux de Y est appelée une équivalence faible d'homotopie.

Théorème (Whitehead)**.** *Une fonction entre CW complexes est une équivalence faible d'homotopie si et seulement elle est une équivalence forte d'homotopie.*

La catégorie des CW complexes est une sous-catégorie pleine des espaces topologiques. Cependant, lorsque l'on manipule les CW complexes, il peut arriver que l'on sorte de cette sous-catégorie : par exemple, certaines colimites de CW complexes ne sont pas des CW complexes. De plus, l'espace topologique des fonctions entre deux CW complexes est rarement un CW complexe. Le théorème suivant, dit «théoreme d'approximation CW», nous montre que l'on peut alors toujours se ramener à un CW complexe.

Théorème. *Pour tout espace topologique X, il existe un CW complexe X' et une équivalence faible d'homotopie de X' vers X.*

La topologie algébrique étudie les CW complexes à équivalences faibles d'homotopie près. Cependant, ces groupes d'homotopie sont très difficiles à calculer ; ainsi, ceux des sphères sont loin d'être connus. Il convient donc de se pencher sur des invariants moins puissants mais plus facilement manipulables et calculables. Dans la suite, nous allons décrire de tels invariants : les groupes d'homologie. Avant cela, nous allons d'abord faire un détour par un modèle combinatoire des espaces : les ensembles simpliciaux.

1.1.2 Les ensembles simpliciaux

Les ensembles simpliciaux sont des modèles combinatoires des espaces. Un ensemble simplicial est la donnée d'une suite d'ensembles $X_0, X_1, X_2, \ldots$ et d'applications entre ces ensembles que nous allons décrire. On interprète l'ensemble X_0 comme un ensemble de points, X_1 comme une ensemble de segments, X_2 comme un ensemble de triangles, X_3 comme un ensemble de tétraèdres, etc. Un segment possède deux extrémités ; nous avons ainsi deux applications de X_1 vers X_0. De même, un triangle a trois faces, ce qui nous donne trois applications de X_2 vers X_1. Par ailleurs, un point peut être considéré comme un segment contracté ; cela se traduit par une application de X_0 vers X_1. Nous avons de la même façon de nombreuses applications entre X_n et X_{n+1} qui obéissent à la même heuristique pour des entiers n plus grands. Voici la définition précise.

Définition. La catégorie Δ est la catégorie dont les objets sont les entiers $n \in \mathbb{N}$ et dont les morphismes $\hom_\Delta(n, m)$ sont les fonctions croissantes de $\{0, \ldots, n\}$ vers $\{0, \ldots, m\}$. Les ensembles simpliciaux sont les foncteurs contravariants de la catégorie Δ vers la catégorie des ensembles.

Plus simplement, un ensemble simplicial X est une suite d'ensembles $(X_n)_{n\in\mathbb{N}}$ munis d'applications «faces» $(d_i : X_n \to X_{n-1})_{i=0}^n$ et d'applications «dégénérescences» $(s_i : X_n \to X_{n+1})_{i=0}^n$

vérifiant les relations suivantes

$$\begin{cases} d_i d_j = d_{j-1} d_i \text{ si } i < j \ , \\ d_i s_j = s_{j-1} di \text{ si } i < j \ , \\ d_i s_j = Id \text{ si } i = j \text{ ou si } i = j+1 \ , \\ d_i s_j = s_j d_{i-1} \text{ si } i > j+1 \ , \\ s_i s_j = s_{j+1} s_i \text{ si } i \geq j \ . \end{cases}$$

Nous avons introduit les ensembles simpliciaux comme des structures combinatoires d'espaces qui seraient façonnés à partir de points, de triangles, de tétraèdres, etc. Quels sont donc les ensembles simpliciaux qui représentent ces briques élémentaires (les «tétraèdres» de dimension n) ? Ce sont les ensembles simpliciaux représentables suivants :

$$\Delta[n] := \hom_{\Delta}(-, n) \ .$$

Tout ensemble simplicial est la colimite d'un diagramme ne contenant que les ensembles simpliciaux du type $\Delta[n]$.

On peut parler d'une manière plus générale d'objet simplicial dans une catégorie de telle sorte que les ensembles simpliciaux sont les objets simpliciaux de la catégorie des ensembles. On peut alors parler d'espaces topologiques simpliciaux, d'espaces vectoriels simpliciaux, d'anneaux simpliciaux, etc.

Définition (Objet simplicial)**.** Soit C une catégorie. Un objet simplicial de C est un foncteur contravariant de la catégorie Δ vers C.

Les ensembles simpliciaux sont liés aux espaces topologiques de la manière suivante. Considérons l'espace $|\Delta[n]|$ défini pour tout entier $n \in \mathbb{N}$ par :

$$|\Delta[n]| := \left\{ x = (x_0, \ldots, x_n) \subset \mathbb{R}^{n+1} | \ \sum_{i=0}^{n} x_i = 1 \right\} \ .$$

La suite $(|\Delta[n]|)_{n\in\mathbb{N}}$ a une structure naturelle d'espace cosimplicial, c'est à-dire qu'il existe un foncteur covariant de la catégorie Δ vers la catégorie des espaces topologiques dont l'image de l'entier n est $|\Delta[n]|$. A un espace topologique X on peut alors associer l'ensemble simplicial suivant appelé ensemble simplicial singulier de X :

$$S(X)_n = \{ f : |\Delta[n]| \to X \text{ continue} \} \ .$$

D'autre part, à tout ensemble simplicial X, on peut associer le CW complexe $|X|$, appelé réalisation géométrique de X, défini de la manière suivante

$$|X| := \coprod_{n\in\mathbb{N}} X_n \times |\Delta[n]| / \sim$$

où, pour tout morphisme $\phi : n \to m$ de la catégorie Δ, pour tout élément $x \in X_m$ et tout élément $y \in |\Delta[n]|$ la relation $\sim$ identifie $(\phi^*(x), y)$ et $(x, |\phi|(y))$. Ces deux constructions sont fonctorielles et $| - |$ est adjoint à gauche de S.

$$\mathsf{sSet} \underset{S}{\overset{|-|}{\rightleftarrows}} \mathsf{Top}$$

On peut se demander si les constructions algébriques que l'on a faites sur les espaces topologiques peuvent aussi être calculées sur les ensembles simpliciaux. C'est le cas si l'on se restreint à une classe particulière d'ensembles simpliciaux, les complexes de Kan, qui contient en particulier les ensembles simpliciaux singuliers $S(X)$ des espaces topologiques. On peut ainsi définir de manière combinatoire les groupes d'homotopie des complexes de Kan, de sorte que les groupes d'homotopie d'un espace topologique X sont exactement ceux de $S(X)$ et que ceux d'un complexe de Kan Y sont exactement ceux de l'espace $|Y|$. On définit les groupes d'homotopie d'un ensemble simplicial qui n'est pas forcément un complexe de Kan de la façon suivante.

Définition (Groupes d'homotopie et équivalences faibles)**.** Les groupes d'homotopie d'un ensemble simplicial X sont les groupes d'homotopie de l'espace $|X|$. Une équivalence faible d'ensembles simpliciaux est un morphisme $f : X \to Y$ qui induit des isomorphismes entre groupes d'homotopie, c'est-à-dire tel que la fonction continue $|f| : |X| \to |Y|$ est une équivalence faible d'homotopie.

Il existe de plus une notion d'homotopie entre deux morphismes d'ensembles simpliciaux $f : X \to Y$ et $g : X \to Y$ dès que Y est un complexe de Kan. Nous savons que

- ▷ Tout espace topologique X est faiblement équivalent à la réalisation géométrique $|Y|$ d'un ensemble simplicial Y.
- ▷ Tout ensemble simplicial Y est faiblement équivalent à l'ensemble simplicial singulier $S(X)$ d'un CW complexe X.
- ▷ Si Y est un complexe de Kan, toute fonction continue $f : |X| \to |Y|$ est homotope à la réalisation géométrique $|g|$ d'un morphisme d'ensembles simpliciaux $g : X \to Y$.
- ▷ Si X est un CW complexe, tout morphisme $f : S(X) \to S(Y)$ est homotope à l'image par le foncteur S d'une fonction continue $g : X \to Y$.

Il semble donc qu'étudier les espaces à équivalences faibles d'homotopie près revienne à étudier les ensembles simpliciaux à équivalences faibles près. Nous donnerons une véritable substance à cette assertion lorsque nous aborderons les catégories de modèles.

1.1.3 Les complexes de chaînes

Soit $\mathbb{K}$ un anneau commutatif unitaire.

Nous avons vu plus haut que les groupes d'homotopie sont les meilleurs invariants algébriques des espaces que l'on connaisse. Malheureusement, ils sont très difficiles à calculer. D'autres invariants, plus simples en pratique, existent : ce sont les groupes d'homologie. Ceux-ci sont en fait issus de la linéarisation des espaces. Soit X un CW complexe et soit Y un ensemble simplicial qui représente X, c'est-à-dire que $|Y|$ est faiblement équivalent à X ; par exemple, on peut choisir de prendre $Y = S(X)$. On considère alors, $\mathbb{K} \cdot Y$ la $\mathbb{K}$-linéarisation de Y, c'est-à-dire le $\mathbb{K}$-module simplicial tel que pour tout entier n, $(\mathbb{K} \cdot Y)_n$ est le $\mathbb{K}$-module librement engendré par l'ensemble Y_n. Les groupes d'homologie de X sont les groupes d'homotopie du $\mathbb{K}$-module simplicial $\mathbb{K} \cdot Y$. La structure linéaire de $\mathbb{K} \cdot Y$ rend le calcul de ces groupes bien plus simple que ceux de Y. Cela tient au fait que les $\mathbb{K}$-modules simpliciaux sont liés aux complexes de chaînes.

Définition (Complexes de chaînes)**.** Un complexe de chaînes $\mathcal{V} = ((\mathcal{V}_n)_{n\in\mathbb{Z}}, d)$ est la donnée d'une suite de $\mathbb{K}$-modules $(\mathcal{V}_n)_{n\in\mathbb{Z}}$ ainsi que d'applications $d : \mathcal{V}_n \to \mathcal{V}_{n-1}$ telles que la composition

$$\mathcal{V}_{n+1} \xrightarrow{d} \mathcal{V}_n \xrightarrow{d} \mathcal{V}_{n-1}$$

est nulle. Pour tout complexe de chaînes $\mathcal{V}$, le $n^{\text{ième}}$ groupe d'homologie de $\mathcal{V}$ est le quotient du noyau de l'application $d : \mathcal{V}_n \to \mathcal{V}_{n-1}$ par l'image de $d : \mathcal{V}_{n+1} \to \mathcal{V}_n$:

$$H_n(\mathcal{V}) := ker(d : \mathcal{V}_n \to \mathcal{V}_{n-1})/Im(d : \mathcal{V}_{n+1} \to \mathcal{V}_n) .$$

Un morphisme de complexes de chaînes $f : \mathcal{V} \to \mathcal{W}$ est un quasi-isomorphisme s'il induit des isomorphismes entre les groupes d'homologie de $\mathcal{V}$ et ceux de $\mathcal{W}$.

Théorème (Correspondance de Dold-Kan)**.** *Il existe une équivalence de catégories entre la catégorie des complexes de chaînes en degrés positifs et la catégorie des $\mathbb{K}$-modules simpliciaux :*

$$\textit{Complexes de chaînes}_{\geq 0} \underset{N}{\overset{\Gamma}{\rightleftarrows}} \mathbb{K}\textit{-modules simpliciaux. .}$$

De plus, les groupes d'homotopie d'un $\mathbb{K}$-module simplicial sont les groupes d'homologie du complexe de chaînes correspondant.

Pour étudier les espaces topologiques, nous leur associons donc des complexes de chaînes dont nous calculons ensuite les groupes d'homologie. Ce sont des invariants moins fins que les groupes d'homotopie des espaces mais plus faciles à calculer.

Exemple. Les groupes d'homotopie des sphères sont loin d'être connus. Par contre, leurs groupes d'homologie sont très simples :

$$H_k(S^n) = \begin{cases} \mathbb{K} \text{ si } k \in \{0, n\} \\ 0 \text{ sinon.} \end{cases}$$

La topologie algébrique est, comme on l'a vu, l'étude algébrique des espaces topologiques à équivalences faibles d'homotopie près. Une version simplifiée de cette discipline est l'étude des espaces «linéarisés» à équivalences faibles d'homotopie près, c'est-à-dire l'étude des complexes de chaînes à quasi-isomorphismes près.

Nous ne nous restreignons pas aux complexes de chaînes en degrés positifs qui sont des modèles algébriques des espaces, mais manipulons les complexes de chaînes en tous degrés pour plusieurs raisons :

- ▷ Cela nous permet d'une part de travailler avec des duaux de complexes de chaînes. Ainsi, si $\mathcal{V}$ est un complexe de chaînes, on peut définir son complexe de chaînes dual $\mathcal{V}^*$ par

$$\mathcal{V}_n^* := \hom_{\mathbb{K}}(\mathcal{V}_{-n}, \mathbb{K}) \ .$$

- ▷ D'autre part, l'endofoncteur de suspension qui envoie le complexe de chaînes $\mathcal{V}$ sur le complexe $s\mathcal{V}$ défini par

$$\begin{cases} s\mathcal{V}_n := \mathcal{V}_{n-1} \ , \\ d_{s\mathcal{V}}x := (-1)^{|x|} d_{\mathcal{V}}x \ , \end{cases}$$

est un automorphisme de la catégorie des complexes de chaînes en tous degrés.

1.2 Un peu d'algèbre homotopique

Dans la section précédente, nous avons motivé l'étude des espaces topologiques à équivalences faibles d'homotopie près, des ensembles simpliciaux à équivalences près et des complexes de chaînes à quasi-isomorphismes près. Dans toutes ces situations, nous avons une catégorie (espaces topologiques, ensembles simpliciaux, complexes de chaînes) et une notion d'équivalence (équivalence faible d'homotopie, équivalence faible, quasi-isomorphisme). Dans cette section, en évoquant des travaux de Grothendieck, Quillen, Dwyer et Kan, nous décrivons un formalisme qui permet d'étudier de telles situations.

1.2.1 Catégorie homotopique

Soit C une catégorie. Soit W une sous catégorie de C qui contient tous les objets et tous les isomorphismes et que nous appelons sous-catégorie des équivalences. Nous avons vu plusieurs exemples de telles situations :

- ▷ La catégorie des espaces topologiques et la sous-catégorie des équivalences faibles.
- ▷ La catégorie des espaces topologiques et la sous-catégorie des équivalences fortes.
- ▷ La catégorie des ensembles simpliciaux et la sous-catégorie des équivalences faibles.
- ▷ La catégorie des complexes de chaînes et la sous-catégorie des quasi-isomorphismes.

Définition. La catégorie homotopique de C relativement aux équivalences W est la donnée d'une catégorie $Ho(\mathsf{C})$ est d'un foncteur $\pi : \mathsf{C} \to Ho(\mathsf{C})$ qui envoie toute flèche de W sur un isomorphisme et qui est universel pour cette propriété : pour tout foncteur F de C vers une catégorie D qui envoie les équivalences sur des isomorphismes, il existe un unique foncteur F/W de $Ho(\mathsf{C})$ vers D, tel que le diagramme suivant commute.

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathsf{D} \\ & Ho(\mathsf{C}) \arrow[ur, "F/\mathsf{W}"'] & \end{tikzcd}$$

Supposons que C soit une petite catégorie, c'est-à-dire une catégorie dont les objets forment un ensemble. La catégorie homotopique de C a la forme suivante. D'une part, ses objets sont les objets de C. D'autre part, l'ensemble des morphismes dans $Ho(\mathsf{C})$ de X vers Y est le quotient de l'ensemble des chaînes

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \cdots \longrightarrow X_n = Y \ ,$$

où les flèches pointant vers la gauche appartiennent à la sous-catégorie W, par les relations suivantes :

1. La chaîne

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \cdots \xrightarrow{f} X_i \xleftarrow{Id_{X_i}} X_i \xrightarrow{g} \cdots \longrightarrow X_n = Y$$

   est équivalente à la chaîne

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \cdots \longleftarrow X_{i-1} \xrightarrow{gf} X_{i+1} \longleftarrow \cdots \longrightarrow X_n = Y \ .$$

2. Deux chaînes faisant partie d'un hamac comme suit, où les flèches verticales appartiennent à la sous catégorie W, sont équivalentes.

$$\begin{tikzcd}
X = X_{0,1} \arrow[r] \arrow[d, "="'] & X_{1,1} \arrow[d] & X_{2,1} \arrow[l] \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{n,1} = Y \arrow[d, "="] \\
X = X_{0,2} \arrow[r] & X_{1,2} & X_{2,2} \arrow[l] \arrow[r] & \cdots \arrow[r] & X_{n,2} = Y \ .
\end{tikzcd}$$

Les morphismes de la catégorie homotopique sont donc obtenus comme quotient d'une structure plus large. En réalité, ce sont les composantes connexes par arcs d'un espace de morphismes. Ainsi, on peut considérer la catégorie simpliciale dont les objets sont ceux de C et dont l'ensemble simplicial de morphismes de X vers Y a pour n-simplexes les hamacs de largeur n comme suit, où les flèches verticales et les flèches pointant vers la gauche appartiennent à la sous catégorie W.

$$\begin{tikzcd}
X = X_{0,0} \arrow[r] \arrow[d, "="'] & X_{1,0} \arrow[d] & X_{2,0} \arrow[l] \arrow[r] \arrow[d] & \ldots \arrow[r] & X_{k,0} = Y \arrow[d, "="] \\
X = X_{0,1} \arrow[r] \arrow[d, "="'] & X_{1,1} \arrow[d] & X_{2,1} \arrow[l] \arrow[r] \arrow[d] & \ldots \arrow[r] & X_{k,1} = Y \arrow[d, "="] \\
\vdots \arrow[d, "="'] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
X = X_{0,n} \arrow[r] & X_{1,n} & X_{2,n} \arrow[l] \arrow[r] & \ldots \arrow[r] & X_{k,n} = Y
\end{tikzcd}$$

C'est la localisation de Dwyer–Kan de la catégorie C relativement aux équivalences W introduite dans l'article [DK80a]. Alors la catégorie homotopique $Ho(\mathsf{C})$ est la catégorie dont les morphismes sont obtenus comme composantes connexes par arcs des espaces de morphismes de la localisation de Dwyer–Kan.

La donnée d'équivalences W au sein d'une catégorie C donne donc lieu à une catégorie simpliciale, autrement dit à une infini-catégorie. Deux problèmes se posent cependant. D'une part, la construction que l'on a faite ne permet d'obtenir des ensembles simpliciaux petits (c'est-à-dire dont les simplexes sont des ensembles et non des classes) que si la catégorie est petite, ce qui n'est pas le cas des catégories qui nous intéressent. D'autre part, en faisant abstraction des problèmes de taille d'ensembles, il est très difficile d'étudier la «forme» de ces hamacs : en effet, ce sont des ensembles simpliciaux très gros, qui ne sont pas en général des complexes de Kan. Les structures de modèles vont permettre de remédier à ces deux problèmes.

D'autre part, si on se penche sur la catégorie des espaces topologiques et sur les équivalences fortes d'homotopie, la catégorie homotopique a une présentation beaucoup plus simple :
- ▷ Ses objets sont les espaces topologiques.
- ▷ Ses morphismes sont les classes d'homotopie de morphismes.

Les structures de modèles vont permettre d'obtenir une localisation de manière presque aussi simple.

1.2.2 Les structures de modèles

De nombreux problèmes de topologie se traduisent par des problèmes de relèvement. Par exemple si $p : E \to B$ est une fibration de Hurewicz, on peut se demander si un diagramme de la forme suivante

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd},$$

possède un relèvement, c'est-à-dire s'il existe un morphisme $g : X \to E$ tel que $pg = f$. En théorie de l'obstruction, on rencontre la situation analogue où p n'est qu'une fibration de Serre mais où X est un CW complexe. Il arrive également de rencontrer le problème de relèvement dual suivant

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \\ B \arrow[ur, dashed, "?"'] & \end{tikzcd},$$

où $i : A \to B$ est une cofibration de Hurewicz. Sachant que pour tout espace topologique X, le morphisme $\emptyset \to X$ est une cofibration de Hurewicz et le morphisme $X \to *$ est une fibration de Hurewicz, ces problèmes sont les facettes d'un même problème de relèvement de la forme suivante

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "?"'] & Y \end{tikzcd},$$

où i est une cofibration et où p est une fibration. Ce problème a une solution dès que l'une des deux fonctions i ou p est une équivalence d'homotopie.

Par ailleurs, lorsque la fonction $p : X \to Y$ n'est pas une fibration, on peut la «remplacer» par une fibration, au sens où il existe une fonction $j : X \to X'$ qui est à la fois une cofibration et une équivalence d'homotopie et telle que f se factorise par j suivie d'une fibration. De la même façon, toute application peut être remplacée par une cofibration au sens dual où elle peut être factorisée par une cofibration suivie d'une fonction qui est à la fois une fibration et une équivalence d'homotopie. Ceux sont là tous les ingrédients des structures de modèles.

Définition (Catégorie de modèles)**.** Soit C une catégorie. Une structure de modèles sur C est la donnée de trois classes de morphismes, les cofibrations, les fibrations et les équivalences faibles, contenant tous les isomorphismes et qui sont stables par composition. On appelle respectivement cofibrations acycliques et fibrations acycliques les morphismes qui sont à la fois des équivalences faibles et des cofibrations et les morphismes qui sont à la fois des équivalences faibles et des fibrations. Les trois classes de morphismes doivent vérifier les axiomes suivants.

1. Un rétract d'un morphisme d'une de ces classes est encore dans cette classe. Autrement dit, dans le diagramme suivant, si f est une cofibration (resp. fibration, resp. équivalence faible),

alors g est une cofibration (resp. fibration, resp. équivalence faible).

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] \arrow[rr, bend left, "Id"] & X' \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ Y \arrow[r] \arrow[rr, bend right, "Id"'] & Y' \arrow[r] & Y \end{tikzcd}$$

2. Soit $f : X \to Y$ et $g : Y \to Z$ deux flèches de la catégorie C. Si deux des trois flèches f, g et gf sont des équivalences faibles, alors la troisième est également une équivalence faible.
3. Considérons le carré commutatif suivant.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

   Supposons de plus que i soit une cofibration et que p soit une fibration. Dès lors, ce carré possède un relèvement, c'est-à-dire qu'il existe une flèche de C $h : B \to X$ telle que $hi = f$ et $ph = g$, si une des deux flèches verticales, i ou p est également une équivalence faible.
4. Toute flèche f de C se factorise sous la forme d'une cofibration suivie d'une fibration. De plus, une telle factorisation peut être choisie de sorte que la cofibration (resp. la fibration) soit également une équivalence faible.

Une catégorie de modèles est une catégorie qui possède toutes les limites et toutes les colimites et qui est munie d'une structure de modèles. Par ailleurs, on appelle objet cofibrant (resp. objet fibrant) tout objet X tel que le morphisme $\emptyset \to X$ de l'objet initial vers X est une cofibration (resp. le morphisme $X \to *$ de X vers l'objet final est une fibration).

Voici quelques exemples fondamentaux de structures de modèles :

- ▷ La catégorie des espaces topologiques peut être munie de deux structures de modèles : la structure de Quillen et la structure de Hurewicz-Strom. Au sein de la première, les équivalences faibles sont les équivalences faibles d'homotopie et les fibrations sont les fibrations de Serre. Les objets cofibrants sont les CW complexes. Au sein de la seconde, les équivalences faibles sont les équivalences fortes d'homotopie et les fibrations sont les fibrations de Hurewicz.
- ▷ On connaît également deux structures de modèles sur la catégorie des ensembles simpliciaux : la structure de Kan-Quillen et la structure de Joyal–Tierney. Au sein de la première les équivalences faibles sont les équivalences faibles d'homotopie et les cofibrations sont les injections. Les objets fibrants sont les complexes de Kan. Au sein de la seconde les cofibrations sont les injections et les objets fibrants sont les quasi-catégories étudiées en détail par Lurie dans le livre [Lur09].
- ▷ La catégorie des complexes de chaînes possède une structure de modèles, dite structure projective, dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations sont les surjections. Les objets cofibrants sont alors les complexes de chaînes $\mathcal{V}$ tels que le $\mathbb{K}$-module $\mathcal{V}_n$ est projectif pour tout entier n. Il existe sur cette catégorie une autre structure de modèles, dite structure injective, duale de la structure projective. Elle possède les mêmes équivalences faibles et ses cofibrations sont les injections. Les objets fibrants sont alors les complexes de chaînes $\mathcal{V}$ tels que le $\mathbb{K}$-module $\mathcal{V}_n$ est injectif pour tout entier n.
- ▷ La catégorie des complexes de chaînes en degrés positifs possède également un structure de modèles projective (resp. injective) dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations (resp. les cofibrations) sont les surjections (resp. injections) en degrés strictement positifs.

Considérons une catégorie de modèles C. Pour tout objet fibrant X de C, un objet en chemin de X est la donnée de la factorisation du morphisme canonique $X \to X \times X$ par une cofibration acyclique suivie d'une fibration.

$$\begin{tikzcd} X \arrow[r, hook, "\sim"] & PX \arrow[r, two heads] & X \times X \end{tikzcd}$$

Dans ce cadre, on dit que deux morphismes f et g d'un objet cofibrant A vers un objet fibrant X sont homotopes s'il existe un morphisme $h : A \to PX$ tel que le diagramme suivant commute.

$$\begin{tikzcd} A \arrow[r, "h"'] \arrow[rr, bend left, "f \times g"] & PX \arrow[r] & X \times X \end{tikzcd}$$

C'est une relation d'équivalence. On peut alors construire la catégorie homotopique de C de la façon suivante. D'une part, les objets de $Ho(\mathsf{C})$ sont ceux de C. D'autre part, pour tout objet X de C, on choisit un objet fibrant et cofibrant LRX muni d'une équivalence faible le reliant à X. Alors, on peut utiliser la définition suivante

$$\hom_{Ho(\mathsf{C})}(X, Y) := \hom_{\mathsf{C}}(LRX, LRY)/\sim \quad ,$$

où $\sim$ est la relation d'homotopie définie plus haut.

Par ailleurs, de la même façon qu'il existe un objet en chemin de X, il existe un objet triangle, un objet tétraèdre, etc. Bref, il existe un objet simplicial $(X_n)_{n\in\mathbb{N}}$ de C vérifiant certaines relations décrites en détail dans [DK80b] et qui sont les généralisations aux niveaux de dimensions supérieures de la définition d'un objet en chemin. On peut alors définir l'espace des morphismes d'un objet cofibrant A vers un objet fibrant X comme l'ensemble simplicial $(\hom_{\mathsf{C}}(A, X_n))_{n\in\mathbb{N}}$.

Souvent, les catégories de modèles que l'on considère sont enrichies par la catégorie des ensembles simpliciaux, de sorte que pour tout objet cofibrant X et tout objet fibrant Y, l'ensemble simplicial des morphismes de X vers Y est faiblement équivalent à l'ensemble simplicial que l'on aurait obtenu par la procédure précédente. Par exemple, la catégorie des ensembles simpliciaux est enrichie par elle même de la façon suivante

$$\mathrm{Map}(X, Y)_n := \hom_{\mathsf{sSet}}(X \times \Delta[n], Y) \ ,$$

de telle sorte que si Y est un complexe de Kan, $\mathrm{Map}(X, Y)$ est équivalent à l'espace des morphismes attendu par la structure de modèles de Kan–Quillen. Par ailleurs, Dwyer et Kan ont montré que les hamacs de morphismes décrits dans la sous-section précédente sont de bons modèles des espaces de morphismes.

Théorème ([DK80b])**.** *Soit C une catégorie de modèles. Pour tout objets X et Y, l'espace des morphismes de X vers Y défini par la structure de modèles est équivalent à l'ensemble simplicial des hamacs de morphismes décrit au paragraphe précédent.*

La topologie algébrique consistant à relier deux mondes (le monde des espaces et celui de l'algèbre) donc à relier des catégories, elle s'exprime au moyen d'adjonctions et d'équivalences de catégories. Il existe des notions d'adjonctions et d'équivalences pour les catégories de modèles.

Définition (Adjonction et équivalence de Quillen)**.** Une adjonction de Quillen est une adjonction entre deux catégories munies de structures de modèles telle que le foncteur adjoint à gauche préserve les cofibrations et les cofibrations acycliques ; ou de manière équivalente, le foncteur adjoint à droite préserve les fibrations et les fibrations acycliques. Une équivalence de Quillen est une adjonction de Quillen $L \dashv R$ entre deux catégories de modèles C et D telle que pour tout objet cofibrant X de C et tout objet fibrant Y de D, un morphisme $LX \to Y$ est une équivalence si et seulement si son adjoint $X \to RY$ est une équivalence.

Citons quelques exemples fondamentaux d'adjonctions de Quillen et d'équivalences de Quillen.

Exemple. ▷ L'adjonction décrite plus haut entre la catégorie des ensembles simpliciaux et la catégorie des espaces topologiques induit une équivalence de Quillen lorsque les ensembles simpliciaux sont munis de la structure de modèles de Kan–Quillen et lorsque les espaces sont munis de la structure de Quillen dont les équivalences faibles sont les équivalences faibles d'homotopie. Ces deux catégories de modèles décrivent donc la même «infini-catégorie» que l'on appelle «infini-catégorie des types d'homotopie».

▷ L'adjonction entre les ensembles simpliciaux et les complexes de chaînes en degrés positifs induite par la linéarisation et la correspondance de Dold–Kan est une adjonction de Quillen dès que la catégorie des ensembles simpliciaux est munie de la structure de Kan–Quillen et celle des complexes de chaînes de la structure projective ou de la structure injective.

Remarque. Les notions de fibrations et de cofibrations ne sont pas *a priori* des notions fondamentales de la théorie de l'homotopie mais plutôt des artefacts facilitant les manipulations et les calculs. Il est étonnant qu'elles soient devenues les piliers des structures de modèles qui ont un rôle important en théorie de l'homotopie. Cela vient sans doute du fait que l'on rencontre des catégories de modèles dans de nombreuses situations et qu'elles sont un support efficace pour de nombreux calculs ; par exemple le calcul des limites ou des colimites homotopiques. Cette situation déplaît à de nombreux mathématiciens et beaucoup aimeraient «faire sans». Citons par exemple Grothendieck dans une lettre à Thomason datée du 2 avril 1991 et que nous rapporte Maltsiniotis : «Les constructions homotopiques essentielles sont indépendantes de toutes structures supplémentaires, tel un ensemble C de cofibrations ou un ensemble F de fibrations ou les deux à la fois. De telles structures supplémentaires sont utiles, dans la mesure où elles permettent d'expliciter les constructions essentielles, et d'en établir l'existence. Mais elles ne sont pas plus essentielles pour le sens intrinsèque des opérations (qu'elles auraient tendance plutôt à obscurcir, jusqu'à présent) que le choix d'une base plus ou moins arbitraire d'un module, en algèbre linéaire.»

1.2.3 L'hypothèse homotopique

Nous avons décrit plus haut deux notions d'équivalences d'espaces. D'une part, les équivalences faibles d'homotopie et, d'autre part, les équivalences fortes d'homotopies. Etant donné que ces deux notions coïncident lorsque l'on se restreint aux CW complexes et que tout espace topologique est lié par une équivalence faible à un CW complexe, on peut considérer que la première notion concerne l'étude des CW complexes à homotopie près et la seconde notion l'étude de tous les espaces topologiques à homotopie près. Dans tout ce qui précède, nous avons clairement privilégié la première notion d'équivalence. Pourquoi ?

L'homotopie est l'étude des relations entre des objets, des relations entre ces relations, des relations entre ces relations de relations, etc. De plus, ces relations doivent pouvoir être composées et posséder des inverses, les notions de composition et d'inverse étant comprises dans un sens suffisamment large. La donnée combinatoire d'objets, de relations entre ces objets, de relations entre ces relations, etc, est appelée un type d'homotopie. D'une manière générale, la complexité de toute cette combinatoire rend impossible de donner une définition algébrique précise de ce qu'est un type d'homotopie. Il s'agit donc de trouver des modèles de ces types d'homotopie dans le bestiaire mathématique. L'hypothèse homotopique que l'on doit à Grothendieck affirme que les CW complexes à homotopie près sont de bons modèles des types d'homotopie.

Remarque. Certains auteurs considèrent que les complexes de Kan sont une bonne définition des types d'homotopie. L'hypothèse homotopique devient alors un théorème : la catégorie de modèles de Quillen des espaces topologiques est Quillen-équivalente à la catégorie de modèles de Kan–Quillen des ensembles simpliciaux.

1.3 Théorie homotopique des algèbres sur une opérade

1.3.1 Structures algébriques sur les complexes de chaînes

Le produit scalaire des $\mathbb{K}$-modules peut être étendu aux complexes de chaînes de la manière suivante :

$$\begin{cases} (\mathcal{V} \otimes \mathcal{W})_n := \bigoplus_{i+j=n} \mathcal{V}_i \otimes \mathcal{W}_j \ , \\ d(x \otimes y) := dx \otimes y + (-1)^{|x|} x \otimes dy \ . \end{cases}$$

Ce produit scalaire est symétrique au moyen de la transformation suivante connue sous le nom de «règle de Koszul» :

$$\begin{aligned} \mathcal{V} \otimes \mathcal{W} &\simeq \mathcal{W} \otimes \mathcal{V} \\ x \otimes y &\mapsto (-1)^{|x||y|} y \otimes x \ , \end{aligned}$$

où $|x|$ et $|y|$ sont respectivement le degré de x et celui de y. Dès lors, nous pouvons considérer des complexes de chaînes munis de structures d'algèbres associatives, d'algèbres commutatives, d'algèbres de Lie, etc. Ainsi, une structure d'algèbre associative sur un complexe de chaînes $\mathcal{A}$ est la donnée d'un morphisme $\gamma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ tel que le diagramme suivant commute

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\gamma \otimes Id"] \arrow[d, "Id \otimes \gamma"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\gamma"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\gamma"'] & \mathcal{A} \ . \end{tikzcd}$$

Nous pouvons de la même façon considérer des structures «coalgébriques» sur les complexes de chaînes. Par exemple, une structure de cogèbre coassociative sur un complexe de chaînes $\mathcal{C}$ est la donnée d'un morphisme $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ tel que le diagramme suivant commute

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{C} \otimes \mathcal{C} \arrow[d, "Id \otimes \Delta"] \\ \mathcal{C} \otimes \mathcal{C} \arrow[r, "\Delta \otimes Id"'] & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \ . \end{tikzcd}$$

Plusieurs exemples de telles structures algébriques ont un rôle important en Mathématiques.

- ▷ Le complexe de De Rham des formes différentielles d'une variété possède une structure d'algèbre commutative.
- ▷ Tout problème de déformation est codé par une algèbre de Lie différentielle graduée.
- ▷ Une théorie de champs en Physique se traduit par un complexe de chaînes muni d'une structure d'algèbre de Batalin-Vilkoviski ; c'est le formalisme BV.
- ▷ Pour tout ensemble simplicial X, l'application diagonale de X dans le produit $X \times X$ qui à tout élément x associe le couple (x, x) induit une structure d'E_∞-cogèbre sur la construction de Dold-Kan de X.

1.3.2 Opérades et coopérades

Une représentation de l'algèbre $\mathbb{K}[X]/\left(X^2 - 1\right)$ est la donnée d'un $\mathbb{K}$-module $\mathcal{V}$ et d'un morphisme d'algèbres associatives unitaires de $\mathbb{K}[X]/\left(X^2 - 1\right)$ vers $\mathrm{End}(\mathcal{V}) := [\mathcal{V}, \mathcal{V}]$; c'est exactement la donnée de $\mathcal{V}$ et d'une involution de $\mathcal{V}$. En ce sens, l'algèbre unitaire $\mathbb{K}[X]/\left(X^2 - 1\right)$ code les involutions. De la même façon, l'algèbre $\mathbb{K}[X]/\left(X^2\right)$ code les endomorphismes de carré nul. On peut donc voir les algèbres associatives unitaires comme une manière de coder des types d'endomorphismes, c'est-à-dire des types d'opérations à une entrée et une sortie. Les opérades sont des objets mathématiques qui permettent de coder des types d'opérations à plusieurs entrées et une sortie.

Définition (Opérade)**.** Une opérade $\mathscr{P}$ est une suite $\mathcal{P}(0)$, $\mathcal{P}(1)$, $\mathcal{P}(2)$, $\ldots$, $\mathcal{P}(n)$, $\ldots$ d'espaces vectoriels (ou plus généralement de complexes de chaînes), représentant des opérations à n entrées et une sortie, munie pour tout $n \in \mathbb{N}$ d'une action à droite du groupe de permutations $\mathbb{S}_n$ sur $\mathcal{P}(n)$ représentant les permutations des entrées, et munie d'un produit de composition

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$$

pour tout entier $n \in \mathbb{N}$ et pour tout $i \in \{1, \ldots, n\}$, représentant la composition des opérations et vérifiant des relations décrites dans le livre [LV12, §5.3.7]. Il existe également un élément $1 \in \mathcal{P}(1)$ qui est une unité pour ce produit.

A partir de tout complexe de chaînes $\mathcal{V}$, on peut construire une opérade $\mathrm{End}_{\mathcal{V}}$ qui généralise l'algèbre $\mathrm{End}(\mathcal{V})$:

$$\mathrm{End}_{\mathcal{V}}(n) := [\mathcal{V}^{\otimes n}, \mathcal{V}] \ .$$

Définition (Algèbre sur une opérade)**.** Pour toute opérade $\mathscr{P}$, une structure de $\mathscr{P}$-algèbre sur un complexe de chaînes $\mathcal{V}$ est la donnée d'un morphisme d'opérades de $\mathscr{P}$ vers $\mathrm{End}_{\mathcal{V}}$.

Exemple. ▷ Soit $\mathscr{A}ss$ l'opérade définie par

$$\mathscr{A}ss(n) = \begin{cases} \mathbb{K}[\mathbb{S}_n] \text{ si } n > 0 \ , \\ 0 \text{ si } n = 0 \ . \end{cases}$$

Alors, les algèbres de l'opérade $\mathscr{A}ss$ sont exactement les algèbres associatives non unitaires.

▷ Soit $\mathscr{C}om$ l'opérade définie par

$$\mathscr{C}om(n) = \begin{cases} \mathbb{K} \text{ si } n > 0 \ , \\ 0 \text{ si } n = 0 \ . \end{cases}$$

Les algèbres de l'opérade $\mathscr{C}om$ sont exactement les algèbres associatives commutatives non unitaires.

Les algèbres d'une opérade $\mathscr{P}$ forment une catégorie : par exemple, la catégorie des algèbres associatives, la catégorie des algèbres commutatives, la catégorie des algèbres de Lie, etc. De plus un morphisme d'opérades $f : \mathscr{P} \to \mathscr{Q}$ induit un foncteur f^* de la catégorie des $\mathscr{Q}$-algèbres vers la catégorie des $\mathscr{P}$-algèbres qui envoie une $\mathscr{Q}$-algèbre, c'est-à-dire un complexe de chaînes $\mathcal{V}$ muni d'un morphisme d'opérades $g : \mathscr{P} \to \mathrm{End}_{\mathcal{V}}$ vers le morphisme

$$\mathscr{P} \xrightarrow{f} \mathscr{Q} \xrightarrow{g} \mathrm{End}_{\mathcal{V}} \ .$$

Ce foncteur a un adjoint à gauche $f_!$. Un morphisme d'opérades $f : \mathscr{P} \to \mathscr{Q}$ induit donc une adjonction entre les $\mathscr{Q}$-algèbres et les $\mathscr{P}$-algèbres.

$$\mathscr{P} - \mathsf{alg} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathscr{Q} - \mathsf{alg}$$

Exemple. Il existe une opérade $\mathscr{L}ie$ dont les algèbres sont les algèbres de Lie ; on en trouvera une présentation par générateurs et relations dans le livre [LV12, Chapter 13]. L'injection canonique de l'opérade $\mathscr{L}ie$ dans l'opérade $\mathscr{A}ss$ induit une adjonction entre les algèbres de Lie et les algèbres associatives qui n'est autre que celle donnée par le foncteur d'oubli des algèbres associatives vers les algèbres de Lie et le foncteur adjoint «algèbre enveloppante».

Les opérades sont des objets mathématiques qui généralisent les algèbres associatives unitaires (une opérade concentrée en arité 1 est une algèbre associative unitaire). De la même façon, les coopérades généralisent les cogèbres coassociatives counitaires.

Définition (Coopérade)**.** Une coopérade $\mathscr{C}$ est une suite $\mathcal{C}(0)$, $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\ldots$ de complexes de chaînes munie pour tout $n \in \mathbb{N}$ d'une action à droite du groupe de permutations $\mathbb{S}_n$ sur $\mathcal{C}(n)$ et munie d'un coproduit de décomposition

$$\Delta : \mathcal{C}(n) \to \sum_{i_1+\cdots+i_k=n} \mathcal{C}(k) \otimes_{\mathbb{S}_k} \left(\mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k) \otimes_{\mathbb{S}_{i_1} \times \cdots \mathbb{S}_{i_k}} \mathbb{K}[\mathbb{S}_n]\right)$$

pour tout entier n, vérifiant des relations décrites dans le livre [LV12, §5.8]. Il existe également une application $\epsilon : \mathcal{C}(1) \to \mathbb{K}$ se comportant comme une counité pour ce coproduit.

Nous avons décrit la notion d'algèbre sur une opérade. Il existe de même une notion de cogèbre sur une coopérade.

Définition (Cogèbre sur une coopérade)**.** Pour toute coopérade $\mathscr{C}$, une structure de $\mathscr{C}$-cogèbre conilpotente sur un complexe de chaînes $\mathcal{V}$ est la donnée d'un morphisme

$$\mathcal{V} \to \bigoplus_k \mathcal{C}(k) \otimes_{\mathbb{S}_k} \mathcal{V}^{\otimes k} ,$$

vérifiant des relations décrites dans le livre [LV12, §5.8].

Remarque. Il existe également une notion de cogèbre sur une opérade qui inclut les E_∞-cogèbres dont nous avons parlé plus haut. Les cogèbres conilpotentes sur une coopérade vont en fait nous servir à décrire, cela peut paraître étonnant à première vue, des algèbres sur une opérade.

1.3.3 Opérades colorées

Une opérade est donnée par une structure algébrique, non pas sur un complexe de chaînes, mais sur plusieurs complexes. Existe-t-il des objets qui codent des structures algébriques de ce type, de la même façon que les opérades codent des types d'algèbres ?

Les opérades colorées codent ce type de structures. Une opérade colorée $\mathscr{P}$ est la donnée d'un ensemble de couleurs C ainsi que de complexes de chaînes $\mathcal{P}(c_1, \ldots, c_n; c)$ pour tout $(n+1)$-uplet d'élément de C, représentant des opérations à n entrées colorées par $c_1, \ldots, c_n$ et dont la sortie est colorée par c. Ces opérations peuvent être composées de la même manière que dans une opérade, tout en respectant les couleurs. Les opérades colorées s'inscrivent ainsi dans le diagramme de généralisations suivant.

$$\begin{tikzcd}
\text{Algèbres associatives unitaires} \arrow[r, hook, "\text{plusieurs entrées}"] \arrow[d, hook, "\text{plusieurs couleurs}"'] & \text{opérades} \arrow[d, hook] \\
\text{dg catégories} \arrow[r, hook] & \text{opérades colorées.}
\end{tikzcd}$$

Exemple. Il existe une opérade colorée, dont l'ensemble des couleurs est $\mathbb{N}$ et dont les algèbres sont les opérades. Voir [DV15].

1.3.4 Structures de modèles

Lorsque nous travaillons sur un corps de caractéristique zéro, la théorie de l'homotopie des algèbres sur une opérade $\mathscr{P}$ est décrite par une structure de modèles projective, c'est-à-dire une structure dont les équivalences faibles et les fibrations sont les morphismes qui sont respectivement des équivalences faibles et des fibrations pour la structure de modèles projective de la catégorie des complexes de chaînes.

Théorème (Hinich)**.** *Soit $\mathscr{P}$ une opérade. Lorsque l'anneau de base $\mathbb{K}$ est un corps de caractéristique zéro, la catégorie des $\mathscr{P}$-algèbres admet une structure de modèles dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations sont les surjections.*

De plus, pour tout morphisme d'opérades $f : \mathscr{P} \to \mathscr{Q}$, l'adjonction induite

$$\mathscr{P} - \mathsf{alg} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathscr{Q} - \mathsf{alg} ,$$

est une adjonction de Quillen. C'est même une équivalence de Quillen si et seulement si le morphisme $f : \mathscr{P} \to \mathscr{Q}$ est un quasi-isomorphisme entre les complexes de chaînes sous-jacents. Les quasi-isomorphismes sont donc une bonne notion d'équivalences d'opérades.

Théorème (Hinich)**.** *Lorsque $\mathbb{K}$ est un corps de caractéristique zéro, la catégorie des opérades possède une structure de modèles dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations sont les surjections.*

Dans le cas où $\mathbb{K}$ est un corps de caractéristique non nulle, il existe des structures de modèles sur certaines sous-catégories de la catégorie des opérades.

Théorème ([BM03])**.** *Lorsque $\mathbb{K}$ est un corps de caractéristique non nulle, la catégorie des opérades $\mathscr{P} = (\mathcal{P}, \gamma, u)$ telles que $\mathcal{P}(0) = \mathbb{K}$, possède une structure de modèles dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations sont les surjections.*

Toujours en caractéristique non nulle, il est possible de munir la catégorie des algèbres sur une opérade cofibrante d'une structure de modèles projective.

Théorème. *[BM03, 4.2] Lorsque $\mathbb{K}$ est un corps de caractéristique non nulle, la catégorie des algèbres sur une opérade cofibrante (pour la structure de modèles précédente) possède une structure de modèles dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations sont les surjections. De plus une équivalence faible entre opérades cofibrantes $f : \mathscr{P} \to \mathscr{Q}$ induit une équivalence de Quillen entre la catégorie de modèles des $\mathscr{P}$-algèbres et la catégorie de modèles des $\mathscr{Q}$-algèbres.*

Dans chacune de ces structures de modèles, les équivalences faibles sont les quasi-isomorphismes et les surjections sont les fibrations. En particulier, tout objet est fibrant. Dès lors, pour décrire l'homotopie des opérades et de leurs algèbres, il s'agit avant tout de construire des remplacements cofibrants. D'une part, on l'a vu, pour décrire la théorie de l'homotopie des algèbres sur une opérade $\mathscr{P}$ en caractéristique non nulle, il convient de construire un remplacement cofibrant de $\mathscr{P}$; de fait, la catégorie des algèbres sur ce remplacement cofibrant possède une structure de modèles projective. D'autre part, en caractéristique zéro, même si toute catégorie d'algèbres sur une opérade peut être munie d'une structure de modèles, il est tout de même utile de travailler avec des opérades cofibrantes. Par exemple, dans ce cas, on a un «théorème de transfert homotopique».

Théorème ([BM03])**.** *Soit $\mathscr{P}$ une opérade cofibrante et soit $p : \mathcal{A} \to \mathcal{V}$ une fibration acyclique de complexes de chaînes. Supposons que $\mathcal{A}$ soit muni d'une structure de $\mathscr{P}$-algèbre. Alors, il existe une structure de $\mathscr{P}$-algèbre sur $\mathcal{A}$ homotope à la première (au sens d'une homotopie entre morphismes d'opérades de $\mathscr{P}$ vers $\mathrm{End}_{\mathcal{A}}$) et une structure de $\mathscr{P}$-algèbre sur $\mathcal{V}$ faisant de p un morphisme de $\mathscr{P}$-algèbres.*

1.3.5 Adjonction bar-cobar : des algèbres associatives aux cogèbres coassociatives

Dans toute la suite, $\mathbb{K}$ est un corps de caractéristique zéro.

Intéressons nous au cas précis des algèbres associatives non unitaires. Nous savons que cette catégorie peut être munie d'une structure de modèles dont les équivalences faibles sont les quasi-isomorphismes et dont les fibrations sont les surjections. Dès lors, un des principaux problèmes que l'on rencontre en manipulant ces algèbres est de calculer des remplacements cofibrants. L'adjonction bar-cobar reliant ces algèbres aux cogèbres coassociatives conilpotentes offre un cadre très solide pour calculer de telles résolutions.

$$\text{Cogèbres coassociatives conilpotentes} \underset{B}{\overset{\Omega}{\rightleftarrows}} \text{Algèbres associatives}$$

Cette adjonction est liée à la notion de morphisme tordant. Un morphisme tordant entre une cogèbre coassociative conilpotente $\mathscr{C}$ et une algèbre associative $\mathscr{A}$ est la donnée d'une application α de degré -1 de $\mathscr{C}$ vers $\mathscr{A}$, vérifiant une équation de courbure, dite équation de Maurer-Cartan

$$\partial\alpha + \gamma(\alpha \otimes \alpha)\Delta = 0 \ ,$$

où γ est le produit de $\mathscr{A}$, Δ est le coproduit de $\mathscr{C}$, $\partial\alpha$ est le commutateur de α avec les différentielles en présence. Ainsi, la donnée d'un morphisme tordant α entre $\mathscr{C}$ et $\mathscr{A}$ est équivalente à la donnée d'un morphisme d'algèbres de $\Omega\mathscr{C}$ vers $\mathscr{A}$ et équivalente à la donnée d'un morphisme de cogèbres conilpotentes de $\mathscr{C}$ vers $B\mathscr{A}$.

Pour toute algèbre $\mathscr{A}$, le morphisme canonique $\Omega B\mathscr{A} \to \mathscr{A}$ est une résolution cofibrante. De plus, cette résolution est fonctorielle. Sachant que toute algèbre est fibrante, nous sommes tentés, plutôt que d'étudier les morphismes entre deux algèbres $\mathscr{A}$ et $\mathscr{A}'$, de nous pencher sur l'espace des morphismes de $\Omega B\mathscr{A}$ vers $\mathscr{A}'$ qui ne sont autres que les morphismes de cogèbres de $B\mathscr{A}$ vers $B\mathscr{A}'$; nous sommes donc tentés de nous placer dans le monde des cogèbres. Le théorème suivant nous permet de le faire sans perdre d'information homotopique.

Théorème ([LH03]). *Il existe une structure de modèles sur la catégorie des cogèbres coassociatives conilpotentes dont les cofibrations et les équivalences faibles sont respectivement les morphismes f tels que $\Omega(f)$ est une cofibration et les morphismes g tels que $\Omega(g)$ est une équivalence faible, c'est-à-dire un quasi-isomorphisme.*

Intéressons nous aux objets fibrants de cette structure de modèles sur la catégorie des cogèbres coassociatives conilpotentes. Ce sont les cogèbres colibres, c'est-à-dire les cogèbres $\mathscr{C}$ de la forme

$$\mathscr{C} := \overline{\mathbb{T}}\mathcal{V} := \mathcal{V} \oplus \mathcal{V} \otimes \mathcal{V} \oplus \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V} \oplus \cdots \oplus \mathcal{V}^{\otimes n} \oplus \cdots$$

La donnée de la différentielle de la cogèbre $\mathscr{C}$ correspond exactement à une structure de A_∞-algèbre, c'est-à-dire une structure d'algèbre sur l'opérade A_∞ qui est un remplacement cofibrant de l'opérade $\mathscr{A}ss$, sur la désuspension $s^{-1}\mathcal{V}$ de $\mathcal{V}$. Bref, les objets fibrants de la catégorie des cogèbres coassociatives conilpotentes correspondent exactement aux A_∞-algèbres.

Par ailleurs, ces constructions bar et cobar se généralisent au niveau opéradique. En d'autres termes, il existe une adjonction mettant en relation la catégorie des opérades non unitaires et la catégorie des coopérades conilpotentes, qui étend l'adjonction $\Omega \dashv B$ précédente et que l'on note de la même façon.

$$\text{Coopérades conilpotentes} \underset{B}{\overset{\Omega}{\rightleftarrows}} \text{Opérades non unitaires}$$

De la même manière, pour toute opérade non unitaire $\mathscr{P}$, la construction bar-cobar $\Omega B\mathscr{P}$ est un remplacement cofibrant de $\mathscr{P}$.

1.3.6 Adjonction bar-cobar générale

La situation décrite au paragraphe précédent fait partie d'un cadre plus large. Considérons une opérade $\mathscr{P}$ et un remplacement cofibrant de $\mathscr{P}$ de la forme $\Omega\mathscr{C}$ où $\mathscr{C}$ est une coopérade conilpotente. Par exemple, nous pouvons prendre $\mathscr{P} = \mathscr{A}ss$ et $\mathscr{C} = \mathscr{A}ss^{¡}$ qui est une coopérade décrite en détail dans [LV12, Chapitre 9] telle que l'opérade $\Omega\mathscr{A}ss^{¡}$ est l'opérade A_∞ qui un remplacement cofibrant de l'opérade $\mathscr{A}ss$.

Le morphisme d'opérades $\Omega\mathscr{C} \to \mathscr{P}$ correspond à un morphisme morphisme tordant $\alpha : \mathscr{C} \to \mathscr{P}$. Ce dernier induit une adjonction entre la catégorie des $\mathscr{P}$-algèbres et la catégorie des $\mathscr{C}$-cogèbres (conilpotentes) que l'on note $\Omega_\alpha \dashv B_\alpha$.

$$\mathscr{C} - \text{cogèbres} \underset{B_\alpha}{\overset{\Omega_\alpha}{\rightleftarrows}} \mathscr{P} - \text{algèbres}$$

Dans le cas où $\mathscr{P} = \mathscr{A}ss$ et $\mathscr{C} = \mathscr{A}ss^{¡}$, on retrouve à un changement d'indice près l'adjonction entre les algèbres associatives et les cogèbres coassociatives conilpotentes du paragraphe précédent (voir le chapitre 11 du livre [LV12]). On peut alors transférer la structure de modèles projective des $\mathscr{P}$-algèbres vers les $\mathscr{C}$-cogèbres.

Théorème ([Val14]). *Si $\alpha : \mathscr{C} \to \mathscr{P}$ est un morphisme tordant tel que le morphisme d'opérades induit $\Omega\mathscr{C} \to \mathscr{P}$ est un remplacement cofibrant de l'opérade $\mathscr{P}$, alors il existe une structure de modèles sur la catégorie des $\mathscr{C}$-cogèbres telle que les cofibrations et les équivalences faibles sont respectivement les morphismes f tels que $\Omega_\alpha(f)$ est une cofibration et les morphismes g tels que $\Omega_\alpha(g)$ est une équivalence faible. De plus, l'adjonction $\Omega_\alpha \dashv B_\alpha$ est une équivalence de Quillen.*

Les objets fibrants de la catégorie des $\mathscr{C}$-cogèbres correspondent alors exactement aux $\Omega\mathscr{C}$-algèbres. On retrouve ainsi la situation du paragraphe précédent où les cogèbres coassociatives conilpotentes fibrantes correspondaient aux A_∞-algèbres. On peut alors voir la catégorie de modèles des $\mathscr{C}$-cogèbres comme un nouveau contexte homotopique plus large pour étudier les $\Omega\mathscr{C}$-algèbres. Qu'a-t-on gagné en immergeant la catégorie des $\Omega\mathscr{C}$-algèbres dans celle des $\mathscr{C}$-cogèbres ? D'une part, tous les objets que l'on manipule sont désormais cofibrants. D'autre part, les morphismes entre cogèbres fibrantes peuvent être construits par des méthodes d'obstruction.

1.3.7 Problème de l'unité

Nous avons vu que nous pouvions décrire la théorie de l'homotopie des algèbres sur une opérade grâce à celle des cogèbres sur une coopérade. Malheureusement, cette assertion souffre une exception. Le formalisme que l'on a décrit ne s'applique qu'aux opérades non-unitaires, ou, de manière équivalente, aux opérades unitaires augmentées.

Définition (Opérade augmentée)**.** Une opérade augmentée $(\mathscr{P}, \epsilon)$ est la donnée d'une opérade unitaire $\mathscr{P}$ et d'un morphisme d'opérades $\epsilon : \mathscr{P} \to \mathbb{K}$. Alors $\mathscr{P}$ est la somme directe d'une opérade non unitaire (qui est le noyau de ϵ) et de l'unité de $\mathscr{P}$.

De nombreuses opérades sont augmentées : $\mathscr{A}ss$, $\mathscr{C}om$ et $\mathscr{L}ie$ le sont par exemple. Cependant, les opérades qui codent des types d'algèbres avec unités ne sont pas augmentées : par exemple, l'opérade $u\mathscr{A}ss$ dont les algèbres sont les algèbres associatives unitaires ou encore $u\mathscr{C}om$ dont les algèbres sont les algèbres commutatives unitaires. On aimerait étendre le formalisme décrit plus haut à ces types d'algèbres.

Polishchuk et Positselski, dans leur livre [PP05], proposent une généralisation aux algèbres associatives unitaires de la construction bar des algèbres associatives (non unitaires ou augmentées) : pour toute algèbre associative unitaire $\mathscr{A}$, on choisit un $\mathbb{K}$-module gradué supplémentaire de l'unité $\overline{\mathscr{A}}$ qui n'est pas une algèbre associative mais qui possède un produit (obtenu en projetant celui de $\mathscr{A}$) et une différentielle (obtenue en projetant celle de $\mathscr{A}$) ; on peut alors faire le calcul de la construction bar classique sur $\overline{\mathscr{A}}$; on obtient une cogèbre conilpotente courbée, cette courbure apparaissant du fait que $\overline{\mathscr{A}}$ n'est pas une algèbre associative.

Définition (Cogèbre conilptente courbée)**.** Une cogèbre conilpotente courbée $\mathscr{C}$ est la donnée d'un espace vectoriel gradué $\mathcal{C} = (\mathcal{C}_n)_{n\in\mathbb{N}}$ muni d'une structure de cogèbre coassociative conilpotente $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, d'applications $d : \mathcal{C}_n \to \mathcal{C}_{n-1}$ de degré -1 et d'une application $\theta : \mathcal{C} \to \mathbb{K}$ de degré -2 appelée la courbure. Les applications d forment une codérivation de la cogèbre ; en d'autres termes

$$\Delta d = (d \otimes Id + Id \otimes d)\Delta \ .$$

De plus, $d^2 = (\theta \otimes Id)\Delta$.

La construction bar B_{PP} de Polishchuk et Positselski n'est malheureusement pas fonctorielle. Cependant, il existe une construction cobar fonctorielle Ω_u des cogèbres conilpotentes courbées vers les algèbres associatives unitaires telle que, pour toute algèbre associative unitaire $\mathscr{A}$, la construction $\Omega_u B_{PP}\mathscr{A}$ (qui dépend d'un choix de $\mathbb{K}$-module gradué supplémentaire de l'unité) est un remplacement cofibrant de $\mathscr{A}$.

Hirsh et Millès ont généralisé ces constructions au niveau des opérades dans l'article [HM12]. Plus concrètement, il ont produit une construction bar B_{HM} des opérades munies d'un choix de scindage de l'unité à valeurs dans les coopérades conilpotentes courbées et un foncteur cobar Ω_u dans le sens inverse. De la même façon qu'au niveau des algèbres, pour tout opérade $\mathscr{P}$, la construction $\Omega_u B_{HM}\mathscr{P}$ est un remplacement cofibrant de $\mathscr{P}$. En utilisant une méthode appelée «dualité de Koszul», Hirsh et Millès ont calculé un remplacement cofibrant de l'opérade $u\mathscr{A}ss$. D'autre part, ils ont montré que tout morphisme d'opérades de la forme $\Omega_u\mathscr{C} \to \mathscr{P}$ satisfaisant une condition dite de semi-augmentation induit une adjonction entre la catégorie des $\mathscr{P}$-algèbres et la catégorie des cogèbres sur la coopérade courbée $\mathscr{C}$, prolongeant ainsi certains résultats du chapitre 11 du livre [LV12].

$$\mathscr{C} - \text{cogèbres} \rightleftarrows \mathscr{P} - \text{algèbres}$$

Remarque. Nous avons dit plus haut que la construction bar de Positselski (et par conséquent celle de Hirsh-Millès) n'était pas fonctorielle. Cela mérite d'être un peu nuancé. En effet, elle est fonctorielle de la catégorie des algèbres (resp. des opérades) semi-augmentées (c'est-à-dire munies d'un scindage de l'unité) vers les cogèbres courbées (resp. les coopérades courbées). Dans cette thèse, nous introduisons une nouvelle construction bar, fonctorielle cette fois-ci depuis la catégorie des algèbres sans restriction.

1.4 Résumé de la thèse

Nous présentons enfin les trois chapitres qui forment le corps de cette thèse ainsi que les appendices. Ils ont été conçus comme des articles indépendants et peuvent donc être lus dans le désordre, à cela près que le chapitre 3 est la suite et reprend des constructions du chapitre 2.

1.4.1 Chapitre 2

Ce chapitre s'applique à décrire la théorie homotopique des types d'algèbres qui comportent une unité : par exemple les algèbres associatives unitaires ou encore les algèbres commutatives unitaires. Pour cela, on introduit des constructions nouvelles, après [PP05] et [HM12], qui permettent d'obtenir le bon cadre catégoriel.

Considérons l'opérade $u\mathscr{A}ss$ dont les algèbres sont les algèbres associatives unitaires. Dans le cadre du formalisme développé par Hirsh et Millès, il existe une coopérade conilpotente courbée notée $u\mathscr{A}ss^{\text{i}}$ et appelée duale de Koszul de l'opérade $u\mathscr{A}ss$, ainsi qu'un morphisme d'opérades de $\Omega_u u\mathscr{A}ss^{\text{i}}$ vers $u\mathscr{A}ss$ qui est un remplacement cofibrant de $u\mathscr{A}ss$. Ce morphisme induit une adjonction entre les $u\mathscr{A}ss$-algèbres, c'est-à-dire les algèbres associatives unitaires, et les $u\mathscr{A}ss^{\text{i}}$-cogèbres. En remarquant que, à un changement d'indices près, les $u\mathscr{A}ss^{\text{i}}$-cogèbres sont exactement les cogèbres conilpotentes courbées, on obtient une adjonction entre les algèbres associatives unitaires et les cogèbres conilpotentes courbées $\Omega_u \dashv B_c$. Le foncteur Ω_u est exactement le foncteur introduit par Polishchuk et Positselski et le foncteur B_c est une modification de leur construction bar.

$$\text{Cogèbres conilpotentes courbées} \underset{B_c}{\overset{\Omega_u}{\rightleftarrows}} \text{algèbres associatives unitaires}$$

De la même manière que les adjonctions bar-cobar précédentes, celle-ci est liée à une notion de morphisme tordant. Un morphisme tordant entre une cogèbre conilpotente courbée $\mathscr{C}$ et une algèbre $\mathscr{A}$ est une application α de degré -1 entre $\mathscr{C}$ et $\mathscr{A}$ satisfaisant l'équation suivante, dite équation de Maurer-Cartan

$$\partial\alpha + \gamma(\alpha \otimes \alpha)\Delta = \theta(-)1_{\mathscr{P}}$$

où γ est le produit de $\mathscr{A}$, Δ est le coproduit de $\mathscr{C}$, $\partial\alpha$ est le commutateur de α avec les différentielles en présence et θ est la courbure de $\mathscr{C}$. Ainsi, la donnée d'un morphisme tordant α entre $\mathscr{C}$ et $\mathscr{A}$ est équivalente à la donnée d'un morphisme d'algèbres de $\Omega_u\mathscr{C}$ vers $\mathscr{A}$ et équivalente à la donnée d'un morphisme de cogèbres conilpotentes courbées de $\mathscr{C}$ vers $B_c\mathscr{A}$.

Cette adjonction s'étend au niveau opéradique, entre les opérades et les coopérades conilpotentes courbées. De manière similaire au cas des opérades non unitaires développé dans le livre [LV12], tout morphisme tordant $\alpha : \mathscr{C} \to \mathscr{P}$ entre une coopérade conilpotente courbée $\mathscr{C}$ et une opérade $\mathscr{P}$ induit une nouvelle adjonction entre les $\mathscr{C}$-cogèbres et les $\mathscr{P}$-algèbres que l'on note $\Omega_\alpha \dashv B_\alpha$.

Théorème. *Soit $\alpha : \mathscr{C} \to \mathscr{P}$ un morphisme tordant entre une coopérade conilpotente courbée et une opérade ; soit alors $\Omega_\alpha \dashv B_\alpha$ l'adjonction induite entre la catégorie des $\mathscr{P}$-algèbres et celle des $\mathscr{C}$-cogèbres. Il existe une structure de modèles sur la catégorie des $\mathscr{C}$-cogèbres dont les cofibrations et les équivalences faibles sont respectivement les morphismes f tels que $\Omega_\alpha f$ est une cofibration et les morphismes g tels que $\Omega_\alpha g$ est une équivalence faible. L'adjonction $\Omega_\alpha \dashv B_\alpha$ est alors une adjonction de Quillen.*

Ce théorème généralise des résultats de Lefevre-Hasegawa et Vallette cités plus haut. Pour le démontrer, nous n'utilisons pas la méthode initiée par Hinich à laquelle ils ont fait appel. En effet, la présence de courbure rend cette méthode plus difficile à mettre en oeuvre. Nous nous servons d'un résultat récent sur les catégories de modèles démontré dans l'article [BHK$^+$15] et qui repose sur la présentabilité des catégories en présence. On sait déjà depuis [DCH16] que la catégorie des algèbres sur un opérade est présentable. Par contre, nous avons un besoin crucial du résultat dual que nous démontrons.

Théorème. *La catégorie des cogèbres conilpotentes sur une coopérade conilpotente courbée est présentable.*

Pour tout morphisme d'opérades $f : \mathscr{P} \to \mathscr{Q}$ et tout morphisme tordant $\alpha : \mathscr{C} \to \mathscr{P}$, l'application $f\alpha : \mathscr{C} \to \mathscr{Q}$ est encore un morphisme tordant. Si f est un quasi-isomorphisme, la structure de modèles induite sur la catégorie $\mathscr{C}$ par l'adjonction $\Omega_\alpha \dashv B_\alpha$ coïncide avec la structure induite par l'adjonction $\Omega_{f\alpha} \dashv B_{f\alpha}$. On peut alors s'intéresser à la structure induite par le morphisme tordant universel $\iota : \mathscr{C} \to \Omega_u\mathscr{C}$. L'adjonction $\Omega_\iota \dashv B_\iota$ est alors une équivalence de Quillen. De plus, les objets fibrants sont exactement les images par le foncteur bar B_ι des algèbres sur l'opérade $\Omega_u\mathscr{C}$.

De la même façon qu'à la section 1.3.6, en immergeant la catégorie des $\Omega_u\mathscr{C}$-algèbres dans celle des $\mathscr{C}$-cogèbres, on a gagné, d'une part, le fait que les objets que l'on manipule sont désormais cofibrants, d'autre part le fait que les morphismes entre $\mathscr{C}$-cogèbres fibrantes peuvent être construits par des méthodes d'obstruction. De plus, pour montrer que deux $\Omega_u\mathscr{C}$-algèbres sont équivalentes, il suffit désormais de construire un morphisme entre les $\mathscr{C}$-cogèbres correspondantes.

Dans ce chapitre, nous montrons également que la catégorie des algèbres sur une opérade, ainsi que la catégorie des cogèbres sur une coopérade sont enrichies sur la catégorie des cogèbres cocommutatives. Nous savons par ailleurs grâce à Hinich ([Hin01]) que toute cogèbre cocommutative qui possède la propriété de conilpotence représente un problème de déformation. Nous montrons que l'enrichissement en cogèbres cocommutatives que nous avons obtenu est une manière de décrire la déformation des morphismes entre algèbres sur une opérade. De plus, si nous travaillons dans le monde non symétrique (c'est-à-dire avec des opérades et des coopérades qui ne sont pas munies d'actions des groupes symétriques), l'enrichissement peut se faire sur les cogèbres coassociatives. Ces dernières décrivent d'une manière condensée, à la fois la théorie de déformation des morphismes et le type d'homotopie des espaces de morphismes.

Nous terminons ce chapitre par deux exemples d'applications de ce formalisme. D'une part, nous revenons sur l'adjonction $\Omega_u \dashv B_c$ entre les algèbres associatives unitaires et les cogèbres conilpotentes courbées. Nous montrons que la structure de modèles obtenue par transfert sur la catégorie des cogèbres conilpotentes courbées fait de cette adjonction une équivalence de Quillen. On peut donc étudier les algèbres associatives unitaires sous la forme de cogèbres conilpotentes courbées tout comme on pouvait étudier les algèbres associatives non unitaires sous la forme de cogèbres conilpotentes différentielles graduées ([LH03]). D'autre part, en utilisant la dualité de Koszul de Hirsh-Millès, nous introduisons une adjonction bar-cobar entre les algèbres commutatives unitaires et les cogèbres de Lie conilpotentes courbées. Nous montrons également que la structure de modèles obtenue par transfert sur la catégorie des cogèbres de Lie conilpotentes courbées fait de cette adjonction une équivalence de Quillen.

1.4.2 Chapitre 3

Ce chapitre s'attache à décrire la théorie de l'homotopie des opérades (unitaires non nécessairement augmentées). Pour cela, on utilise l'adjonction $\Omega_u \dashv B_c$ entre les opérades et les coopérades conilpotentes courbées introduite au chapitre précédent. On suppose que le corps de base $\mathbb{K}$ est de caractéristique zéro. Dès lors, pour toute opérade $\mathscr{P}$, la construction $\Omega_u B_c \mathscr{P}$ est un remplacement cofibrant de $\mathscr{P}$. Comme précédemment, on peut alors se demander s'il est possible de décrire la théorie de l'homotopie des opérades sous la forme de coopérades courbées.

Théorème. *Il existe une structure de modèles sur la catégorie des coopérades conilpotentes courbées induite à gauche par l'adjonction $\Omega_u \dashv B_c$, c'est-à-dire telle que les cofibrations et les équivalences faibles sont respectivement les morphismes f tels que $\Omega_u f$ est une cofibration et les morphismes g tels que $\Omega_u g$ est une équivalence faible. De plus, l'adjonction $\Omega_u \dashv B_c$ est une équivalence de Quillen.*

Ce théorème est héritier de résultats dus à Hinich, Lefevre-Hasegawa et Vallette, notamment les théorèmes des sous-sections 1.3.6 et 1.3.5. Contrairement au chapitre précédent, nous reprenons leur stratégie de démonstration. Néanmoins, dans sa mise en oeuvre, de nouvelles difficultés apparaissent du fait des actions des groupes symétriques et de la combinatoire des arbres.

Les coopérades conilpotentes courbées qui sont fibrantes correspondent à une bonne notion d'opérade relâchée à homotopie près que nous appelons opérade à homotopie près («homotopy operad» en anglais). Une opérade est constituée d'une collection de complexes chaînes munis d'actions des groupes symétriques, d'un produit de composition et d'une unité pour ce produit. La notion d'opérade à homotopie près correspond au relâchement à homotopie près du produit opéradique et de l'unité. Remarquons que Van Der Laan avait déjà introduit dans [VdL02] une notion d'opérade non unitaire à homotopie près. Les opérades à homotopie près ont des propriétés homotopiques similaires à celles des algèbres sur une opérade cofibrante :

- ▷ Il existe un théorème de transfert homotopique pour les opérades à homotopie près.
- ▷ On peut calculer des objets en chemin d'opérades dans le monde des opérades à homotopie près, ce qui permet de décrire de manière plus simple les homotopies entre les morphismes d'opérades.

Parmi, ces opérades à homotopie près, il en existe dont l'unité a encore un comportement strict. Ce sont les opérades à homotopie près strictement unitaires. Nous montrons qu'elles vérifient certaines propriétés de stabilité. Notamment, un transfert homotopique d'une opérade à homotopie près strictement unitaire est encore une opérade à homotopie près strictement unitaire.

Théorème. *Soit $p : \mathcal{P} \to \mathcal{Q}$ une fibration acyclique de $\mathbb{S}$-modules. Supposons que $\mathcal{P}$ soit muni d'une structure d'opérade à homotopie près strictement unitaire. Alors, il existe une structure d'opérade à homotopie près strictement unitaire sur $\mathcal{P}$, isotope à la première, et une structure d'opérade à homotopie près strictement unitaire sur $\mathcal{Q}$, faisant de p un morphisme d'opérades à homotopie près strictement unitaires.*

Remarque. Les opérades sont elles-mêmes des algèbres sur une opérade colorée. On pourrait alors penser que ce chapitre est une conséquence directe du chapitre précédent, quitte à adapter aux opérades colorées le formalisme développé. Ce n'est pas le cas. En effet, les coopérades ne sont pas des cogèbres sur une coopérade colorée. Cela est dû aux actions des groupes symétriques.

1.4.3 Chapitre 4

Nous avons envisagé plus haut les ensembles simpliciaux comme des modèles combinatoires des espaces. Ils peuvent également être interprétés comme des modèles de catégories dont la composition est relâchée à homotopie près. Soit X un ensemble simplicial. On peut ainsi voir X_0 comme un ensemble d'objets, X_1 comme un ensemble de flèches entre ces objets. Les deux applications faces sont alors interprétées comme donnant la source et le but d'une flèche. Les éléments de X_2 définissent alors une composition à homotopie près et les ensembles X_n pour les entiers n plus grands définissent des cohérences supérieures. De la même façon que les complexes de Kan sont les objets fibrants de la structure de modèles de Kan–Quillen sur les ensembles simpliciaux et sont ainsi les bons représentants des types d'homotopie, les objets fibrants de la structure de Joyal–Tierney sont les bons représentants des infini-catégories : on les appelle les quasi-catégories. Elles sont étudiées en détail dans le livre de Lurie [Lur09].

On a vu plus haut que les opérades algébriques colorées sont aux dg catégories ce que les opérades algébriques sont aux algèbres associatives unitaires. De la même manière, les opérades colorées ensemblistes (c'est-à-dire façonnées à partir d'ensembles au lieu des complexes de chaînes) généralisent à la fois les opérades ensemblistes et les catégories. Suivant le point de vue précédent, Moerdijk et Weiss ont introduit une catégorie généralisant les ensembles simpliciaux qui nous donne des modèles «d'infini-opérades» : il s'agit de la catégorie des ensembles dendroidaux. Plus concrètement, les ensembles dendroidaux sont des foncteurs contravariants d'une catégorie Ω dont les objets sont des arbres et dont les morphismes sont construits à partir de faces et de dégénérescences à la manière de la catégorie Δ, vers la catégorie des ensembles. Parmi les ensembles dendroidaux, les infini-opérades sont ceux qui satisfont une propriété de relèvement similaire à celle définissant les quasi-catégories.

$$\begin{tikzcd}
\text{Quasi-catégories} \arrow[r, hook] \arrow[d, hook] & \text{ensembles simpliciaux} \arrow[d, hook] \\
\text{infini-opérades} \arrow[r, hook] & \text{ensembles dendroidaux.}
\end{tikzcd}$$

Le but de ce chapitre est de comparer ce modèle d'infini-opérade avec les notions d'opérades algébriques relâchées vues aux chapitre précédent. La première étape est de décrire une version à plusieurs couleurs (ou plusieurs objets) de la notion d'opérade à homotopie près strictement unitaire. En d'autre termes, nous remplissons le cube de généralisations qui suit.

$$\begin{tikzcd}
& \text{algèbres} \arrow[rr, "\text{opérations multi-linéaires}"] \arrow[dl, "\text{plusieurs couleurs}"'] \arrow[dd] & & \text{opérades} \arrow[dl] \arrow[dd] \\
\text{dg catégories} \arrow[rr] \arrow[dd, "\text{à homotopie près}"'] & & \text{opérades colorées} \arrow[dd] & \\
& \mathcal{A}_\infty\text{-algèbres} \arrow[rr] \arrow[dl] & & \text{opérades à homotopie près} \arrow[dl] \\
\mathcal{A}_\infty\text{-catégories} \arrow[rr] & & \textbf{opérades colorées à homotopie près strictement unitaires} &
\end{tikzcd}$$

Nous montrons que le foncteur d'inclusion des opérades colorées dans la catégorie des opérades colorées à homotopie près strictement unitaires possède un adjoint à gauche W_H qui généralise la construction de Boardman–Vogt des opérades introduite dans [BM06].

$$\text{Opérades colorées à homotopie près strictement unitaires} \underset{i}{\overset{W_H}{\rightleftarrows}} \text{opérades colorées}$$

Pour comparer les opérades colorées à homotopie près strictement unitaires avec les ensembles dendroidaux, nous construisons un foncteur N^Ω appelé nerf dendroidal allant des premières vers les seconds.

Théorème. *Le nerf dendroidal d'une opérade à homotopie près strictement unitaire est une infini-opérade.*

Ce nerf dendroidal généralise selon deux directions différentes des foncteurs présents dans la littérature mathématique : d'une part, le nerf homotopiquement cohérent des opérades colorées de Moerdijk–Weiss hcN ; d'autre part, le nerf simplicial des A_∞-catégories de Faonte–Lurie. Nous montrons quelques propriétés homotopiques du nerf dendroidal, dont le théorème suivant est une conséquence directe.

Théorème. *Le nerf homotopiquement cohérent* hcN *est un foncteur de Quillen à droite.*

Enfin, nous comparons ce nerf dendroidal avec une autre construction appelée «big nerve», notée $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ et inspirée du travail de Lurie ([Lur12, §1.3.1]). Nous montrons que le nerf homotopiquement cohérent hcN est équivalent à ce nerf $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$.

Théorème. *Il existe une transformation naturelle α^* du nerf $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ vers le nerf homotopiquement cohérent* hcN *telle que, pour toute opérade colorée $\mathscr{P}$, le morphisme $\alpha^*(\mathscr{P}) : \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P}) \to \mathrm{hcN}(\mathscr{P})$ est une équivalence faible d'ensembles dendroidaux.*

1.4.4 Appendices

Appendice A

Dans ce premier appendice, nous démontrons un résultat qui est une conséquence directe d'un théorème de [CLM]. Ce résultat concerne les cogèbres cocommutatives, c'est-à-dire les complexes de chaînes $\mathcal{C}$ munis d'un morphisme $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ coassociatif, possédant une counité ϵ, et tel que le diagramme suivant commute

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[rr, bend right, "\Delta"'] & \mathcal{C} \otimes \mathcal{C} \arrow[r, "\tau"] & \mathcal{C} \otimes \mathcal{C} \end{tikzcd} ,$$

où τ est le morphisme de symétrie

$$\tau : x \otimes y \in \mathcal{C} \otimes \mathcal{C} \mapsto (-1)^{|x||y|} y \otimes x \ .$$

Théorème. *Si $\mathbb{K}$ est un corps de caractéristique zéro algébriquement clos, toute cogèbre cocommutative est la somme directe de cogèbres cocommutatives conilpotentes.*

Comme nous savons depuis [Hin01] que les cogèbres cocommutatives conilpotentes codent les problèmes de déformation, nous pouvons dès lors envisager les cogèbres cocommutatives sur un corps de caractéristique zéro algébriquement clos comme des collections de problèmes de déformation. Nous avons utilisé ce résultat au cours du chapitre 2 ; en effet, nous avons montré que la catégorie des algèbres sur un opérade était enrichie sur les cogèbres cocommutatives et que ces dernières décrivaient la déformation des morphismes d'algèbres sur cette opérade.

Appendice B

Soit D une infini-opérade, c'est-à-dire un ensemble dendroidal vérifiant une certaine propriété de relèvement et soit $c_1, \ldots, c_n, c$ des couleurs de D. On souhaite décrire l'espace des opérations de D à n entrées colorées par $c_1, \ldots, c_n$ et dont la sortie est colorée par c. En effet, connaître cet espace est essentiel pour montrer qu'un morphisme d'ensembles dendroidaux entre infini-opérades est une équivalence faible. Deux constructions sont apparues dans la littérature. Nous montrons dans cet appendice que ces constructions sont équivalentes.

Appendice C

L'adjonction $\Omega_u \dashv B_c$ que l'on a introduite, établit un pont entre les algèbres associatives unitaires et les cogèbres conilpotentes courbées. Elle peut s'étendre à un cadre à plusieurs couleurs. Ainsi, il existe une adjonction reliant les dg catégories aux cocatégories conilpotentes courbées qui généralise $\Omega_u \dashv B_c$ et que l'on note de la même façon.

$$\text{Cocatégories conilpotentes courbées} \underset{B_c}{\overset{\Omega_u}{\rightleftarrows}} \text{dg catégories}$$

Par ailleurs, nous savons grâce au travail de Tabuada ([Tab05]) que la catégorie des dg catégories possède une structure de modèles qui étend celle des algèbres associatives unitaires. On peut alors se demander si la méthode que l'on a utilisée au cours des chapitres 2 et 3 s'applique au cadre des dg catégories. Autrement dit, on peut se demander s'il existe une structure de modèles sur la catégorie des cocatégories conilpotentes courbées transférée depuis celle des dg catégories le long de l'adjonction $\Omega_u \dashv B_c$. Le théorème suivant donne une réponse négative.

Théorème. *Il n'existe pas de structure de modèles sur la catégorie des cocatégories conilpotentes courbées telle que le foncteur Ω_u préserve les cofibrations et les équivalences faibles.*

Conventions et notations

- ▷ Soit $\mathbb{K}$ un corps. Dans de nombreux cas, nous supposerons que $\mathbb{K}$ est de caractéristique zéro.
- ▷ La catégorie des $\mathbb{K}$-modules gradués est notée gMod, tandis que celle des complexes de chaînes est notée dgMod. Ces deux catégories sont munies de leur structure monoïdale symétrique fermée usuelle dont le hom interne est noté $[\ ,\ ]$. La catégorie des complexes de chaînes est également munie de la structure de modèles projective dont les équivalences faibles (resp. les fibrations) sont les quasi-isomorphismes (resp. les surjections). Le degré d'un élément homogène x d'un $\mathbb{K}$-module gradué ou d'un complexe de chaînes est noté $|x|$.
- ▷ Pour tout entier $n \geq 0$, soit D^n le complexe de chaînes engendré par un élément de degré n et son bord de degré $n-1$. Soit S^n le complexe de chaînes engendré par un cycle de degré n.
- ▷ Soit $\mathsf{gMod}^{\mathsf{deg}}$ la catégorie dont les objets sont les $\mathbb{K}$-modules gradués et dont les morphismes sont les morphismes gradués.
- ▷ On note $\mathsf{dgMod}^{\geq 0}$ la sous-catégorie de dgMod des complexes de chaînes nuls en degrés strictement négatifs.
- ▷ On désignera par $[n]$ l'ensemble ordonné $0 < 1 < \cdots < n$ et par $\underline{n}$ l'ensemble $\{1, \ldots, n\}$.
- ▷ On note Set la catégorie des ensembles, Δ la catégorie dont les objets sont les ensembles ordonnés $[n]$, pour tout entier $n \geq 0$, et dont les morphismes sont les fonctions croissantes, et sSet la catégorie des ensembles simpliciaux. Cette dernière est munie de la structure de modèles de Kan–Quillen ; voir [GJ99, I.11.3].
- ▷ Tout diagramme de la forme suivante dénote une adjonction où R est adjoint à droite.

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

- ▷ Soit $\mathcal{V}$ un $\mathbb{K}$-module gradué muni d'une filtration $(F_n\mathcal{V})_{n\in\mathbb{N}}$. Le complexe gradué associé à cette filtration est noté $G\mathcal{V}$. Autrement dit,

$$G\mathcal{V} = \bigoplus_n G_n\mathcal{V}\ ,$$

  où $G_n\mathcal{V} = F_n\mathcal{V}/F_{n-1}\mathcal{V}$.
- ▷ Soit $\mathcal{V}$ un complexe de chaînes. La *suspension* de $\mathcal{V}$ est le complexe de chaînes $s\mathcal{V}$ défini par $(s\mathcal{V})_n := \mathcal{V}_{n-1}$ et $d_{s\mathcal{V}}(sv) := -sd_{\mathcal{V}}(v)$.

Chapitre 2

Homotopy theory of unital algebras

Introduction

Among the various types of algebras, some of them include units like the ubiquitous unital associative algebras and unital commutative algebras or the unital Batalin–Vilkovisky algebras which arose in Mathematical Physics. When working with a chain complex carrying such an algebraic structure, like the de Rham algebra of differential manifolds for instance, one would like to understand the homotopical properties that this algebraic data satisfies with the underlying differential map. The purpose of the present chapter is develop a framework which allows one to prove the homotopical properties carried by types of algebras with units ; that is, their property up to quasi-isomorphisms.

In order to work with types of algebras in a general way, one needs a precise notion which encodes types of algebras. This is achieved by the concept of an operad. Operads are generalizations of associative algebras which encode some types of algebras (associative, commutative, Lie, Batalin-Vilkovisky, . . .) in a way that a representation of an operad $\mathscr{P}$ is a chain complex together with a structure of algebra of the type encoded by $\mathscr{P}$.

Besides, one of the most common and powerful tool to study homotopical algebra, that is to study categories with a notion of equivalences, are the model category structures introduced by Daniel Quillen which make the manipulation of weak equivalences easier by means of other maps called respectively cofibrations and fibrations. Hinich proved in the article [Hin97], that the category of algebras over an operad carries a model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are surjections. In a purely theoretical perspective, this model structure describes all the homotopical data of this category. However, the cofibrant objects are not easy to handle ; they are the retracts of free algebras whose generators carries a particular filtration.

Hinich ([Hin01]) embedded the category of differential graded Lie algebras into the category of dg cocommutative coalgebras. From the model structure of the category of dg Lie algebras he obtained a model structure on the category of dg cocommutative coalgebras which is Quillen equivalent to the first one. In this new model category, any object is cofibrant. Moreover, this context allows one to build structures and morphisms using obstruction methods. So this new context of dg cocommutative coalgebras is more suitable to study the homotopy theory of dg Lie algebras than the category of dg Lie algebras itself. In a similar perspective, Lefevre-Hasegawa embedded the category of nonunital dg associative algebras into the category dg coassociative coalgebras shown to be Quillen equivalent to the first one ; see [LH03]. Vallette generalized these results to all types of algebras encoded by any operad satisfying a technical condition : it is an augmented operad. Augmented operads are related to the dual notion of conilpotent cooperads by an adjunction called the operadic bar-cobar adjunction $\Omega \dashv B$. Vallette embedded the category of algebras over an augmented operad $\mathscr{P}$ into category of coalgebras over a cooperad $\mathscr{P}^{¡}$ called the Koszul dual of $\mathscr{P}$. He transferred the model structure on the category of $\mathscr{P}$-algebras to the category of $\mathscr{P}^{¡}$-coalgebras and got again a Quillen equivalence between these two model categories.

However the operads describing types algebras with units do not satisfy the technical condition to be augmented. To extend the result of Vallette to categories of algebras over any operad, one first needs to modify the operadic bar-cobar adjunction. Inspired by the work of Hirsh and Millès in [HM12], we introduce an adjunction à la bar-cobar relating dg operads to curved conilpotent cooperads.

$$\text{Curved conilpotent cooperads} \underset{B_c}{\overset{\Omega_u}{\rightleftarrows}} \text{dg Operads}$$

Moreover, any morphism of dg operads f from a cobar construction $\Omega_u \mathscr{C}$ of a curved conilpotent cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ comes equipped with an adjunction $\Omega_f \dashv B_f$ relating $\mathscr{P}$-algebras to $\mathscr{C}$-coalgebras.

$$\mathscr{C} - \text{coalgebras} \underset{B_f}{\overset{\Omega_f}{\rightleftarrows}} \mathscr{P} - \text{algebras}$$

The model structure of $\mathscr{P}$-algebras can be transferred to the category of $\mathscr{C}$-coalgebras along this adjunction.

Theorem. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be a twisting morphism of operad and let $\Omega_\alpha \dashv B_\alpha$ be the bar-cobar adjunction between $\mathscr{P}$-algebras and $\mathscr{C}$-coalgebras induced by α. There exists a model structure on the category of $\mathscr{C}$-coalgebras whose cofibrations (resp. weak equivalences) are morphisms whose image under Ω_α is a cofibration (resp. weak equivalence). With this model category structure, the adjunction $\Omega_\alpha \dashv B_\alpha$ is a Quillen adjunction.*

To prove this theorem, we use new techniques coming from category theory. Specifically, we utilize a Theorem of [BHK+15] involving presentable categories.

We study in details the particular case where the morphism of operads f from $\Omega_u \mathscr{C}$ to $\mathscr{P}$ is a quasi-isomorphism ; for instance if f is the identity ι of $\Omega_u \mathscr{C}$. In this case, the Quillen adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. We show that the fibrant $\mathscr{C}$-coalgebras are the images of the $\Omega\mathscr{C}$-algebras under the functor B_ι. So, switching from the category of $\Omega\mathscr{C}$-algebras to the category of $\mathscr{C}$-coalgebras by the functor B_ι amounts to introduce new morphisms between $\Omega\mathscr{C}$-algebras. These new morphisms can be built using obstruction methods. Moreover, any $\Omega\mathscr{C}$-algebra becomes cofibrant in this new context.

This chapter also deals with enrichments of the category of $\mathscr{P}$-algebras for any differential graded operad $\mathscr{P}$, and of the category of $\mathscr{C}$-coalgebras for any curved cooperad $\mathscr{C}$. These two categories are enriched in simplicial sets in a way that recovers the mapping spaces. Moreover, they are tensored, cotensored and enriched in cocommutative coalgebras. These cocommutative coalgebras encode the deformations of morphisms of algebras over an operad. In the context of nonsymmetric operads and nonsymmetric cooperads, this enrichment can be extended to all coassociative coalgebras. These coassociative coalgebras encode in single objects both the mapping spaces and the deformation of morphisms.

Finally, we apply the framework developed here to concrete operads like the operad $u\mathscr{A}s$ of unital associative algebras and the operad $u\mathscr{C}om$ of unital commutative algebras. For these two operads, the process of curved Koszul duality developed in [HM12] relates the curved cooperads $u\mathscr{A}s^¡$ and $u\mathscr{C}om^¡$ to respectively the operads $u\mathscr{A}s$ and $u\mathscr{C}om$.

Layout

The chapter is organized as follows. In the first part, we recall several notions about category theory, and homological algebra. In the second part, we recall the notions of operads, cooperads, algebras over an operad and coalgebras over a cooperad. We also prove some results, as the presentability of the category of coalgebras over a curved cooperad, that we will need in the sequel. The third part deals with enrichments of the categories of algebras over an operad and coalgebras over a curved cooperad ; specifically, we study enrichments over simplicial sets, cocommutative coalgebras and coassociative coalgebras. In the fourth part, we introduce an adjunction à la bar-cobar between operads and curved cooperads related to a notion of twisting morphism. We use it to define the adjunction between $\mathscr{P}$-algebras and $\mathscr{C}$-coalgebras for a twisting morphism from a curved cooperad $\mathscr{C}$ to an operad $\mathscr{P}$. In the fifth section, we recall the projective model structure on the category of algebras over an operad. We describe models for the mapping spaces and we show that the enrichment over cocommutative coalgebras encodes deformation of morphisms. The sixth part transfers the projective model structure on $\mathscr{P}$-algebras along the previous adjunction to obtain a model structure on $\mathscr{C}$-coalgebras and a Quillen adjunction. The seventh part deals with these model structures in the case where the operad $\mathscr{P}$ is the cobar construction $\Omega_u\mathscr{C}$ of $\mathscr{C}$. In particular, the adjunction induced is a Quillen equivalence. Finally, in the eighth part, we applied the formalism developed in the previous sections to study the examples of unital associative algebras and unital commutative algebras.

2.1 Preliminaries

In this first section, we recall some categorical concepts like the presentability and the notions of enrichment, tensoring and cotensoring. Moreover, we describe several notions of coalgebras like coassociative coalgebras and cocommutative coalgebras that have been extensively studied respectively in [GG99] and in [Hin01]. More specifically the category of coassociative coalgebras admits a model structure related by a Quillen adjunction to the category of simplicial sets ; the category of conilpotent cocommutative coalgebras admits a model structure Quillen equivalent to the projective model structure on Lie algebras. Finally, we describe the Sullivan polynomial algebras.

2.1.1 Presentable categories

Definition 1 (Presentable category)**.** Let C be a cocomplete category. An object X of C is called *compact* if for any filtered diagram $F : I \to \mathsf{C}$ the map $\operatorname{colim} \hom_{\mathsf{C}}(X, F) \to \hom_{\mathsf{C}}(X, \operatorname{colim} F)$ is an isomorphism. The category C is said to be *presentable* if there exists a set of compact objects such that any object of C is the colimit of a filtered diagram involving only these compact objects.

The following proposition is a classical result of category theory.

Proposition 1. *[AR94] A functor $L : \mathsf{C} \to \mathsf{D}$ between presentable categories is a left adjoint if and only if it preserves colimits.*

2.1.2 Tensoring, cotensoring and enrichment

In this section, we recall the definition of tensored-cotensored-enriched category over a monoidal category. See [Bor94] for the original reference.

Definition 2 (Action, coaction)**.** Let $(\mathsf{E}, \otimes, \mathcal{I})$ be a monoidal category and let C be a category.

▷ An enrichment of C over E is a bifunctor $[-,-] : \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{E}$ together with functorial morphisms

$$\begin{cases} \gamma_{X,Y,Z} : [Y,Z] \otimes [X,Y] \to [X,Z] \ , \\ \upsilon_X : \mathcal{I} \to [X,X] \end{cases}$$

for any objects X, Y and Z of C and which are composition and unit in terms of the following commutative diagrams.

$$\begin{tikzcd} {[Y,Z] \otimes [X,Y] \otimes [V,X]} \arrow[r, "\gamma_{X,Y,Z} \otimes Id"] \arrow[d, "Id \otimes \gamma_{V,X,Y} \otimes Id"'] & {[X,Z] \otimes [V,X]} \arrow[d, "\gamma_{V,X,Z}"] \\ {[Y,Z] \otimes [V,Y]} \arrow[r, "\gamma_{V,Y,Z}"'] & {[V,Z]} \end{tikzcd}$$

$$\begin{tikzcd} {[X,Y] \otimes [X,X]} \arrow[dr] & {[X,Y]} \arrow[l, "Id \otimes \upsilon_X"'] \arrow[r, "\upsilon_Y \otimes Id"] \arrow[d, "Id"] & {[Y,Y] \otimes [X,Y]} \arrow[dl] \\ & {[X,Y]} & \end{tikzcd}$$

▷ A right action of E on C is a functor

$$- \otimes - : \mathsf{C} \times \mathsf{E} \to \mathsf{C}$$

together with functorial isomorphisms

$$\begin{cases} X \otimes (\mathcal{A} \otimes \mathcal{B}) \simeq (X \otimes \mathcal{A}) \otimes \mathcal{B} \ , \\ X \otimes \mathcal{I} \simeq X \ , \end{cases}$$

for any $X \in \mathsf{C}$, any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$; these functors are compatible with the monoidal structure of E in terms of the following commutative diagrams.

$$\begin{tikzcd} \big((X \otimes \mathcal{A}) \otimes \mathcal{B}\big) \otimes \mathcal{C} \arrow[r] \arrow[d] & \big(X \otimes (\mathcal{A} \otimes \mathcal{B})\big) \otimes \mathcal{C} \arrow[r] & X \otimes \big((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}\big) \arrow[d] \\ (X \otimes \mathcal{A}) \otimes (\mathcal{B} \otimes \mathcal{C}) \arrow[rr] & & X \otimes \big(\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})\big) \end{tikzcd}$$

$$\begin{tikzcd} (X \otimes \mathcal{I}) \otimes \mathcal{A} \arrow[rr] \arrow[dr] & & X \otimes (\mathcal{I} \otimes \mathcal{A}) \arrow[dl] \\ & X \otimes \mathcal{A} & \end{tikzcd}$$

▷ A left coaction of E on C is a functor :

$$\langle -, - \rangle : \mathsf{E}^{op} \times \mathsf{C} \to \mathsf{C}$$

together with functorial isomorphisms

$$\begin{cases} \langle \mathcal{A} \otimes \mathcal{B}, X \rangle \simeq \langle \mathcal{A} \langle \mathcal{B}, X \rangle \rangle \ , \\ \langle \mathcal{I}, X \rangle \simeq X \ . \end{cases}$$

which satisfy the commutative duals of the diagrams above.

Definition 3 (Category tensored-cotensored-enriched over a monoidal category)**.** Let E be a monoidal category and let C be a category. We say that C is *tensored-cotensored-enriched* over E if there exists three functors :

$$\begin{cases} \{-,-\} : \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{E} \\ - \triangleleft - : \mathsf{C} \times \mathsf{E} \to \mathsf{C} \\ \langle -, - \rangle : \mathsf{E}^{op} \times \mathsf{C} \to \mathsf{C} \end{cases}$$

together with functorial isomorphisms

$$\hom_{\mathsf{C}}(X \triangleleft \mathcal{A}, Y) \simeq \hom_{\mathsf{E}}(\mathcal{A}, \{X, Y\}) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A}, Y \rangle) \ ,$$

for any $X, Y \in \mathsf{C}$, any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$ and where $\mathcal{I}$ is the monoidal unit of E, such that $- \triangleleft -$ defines a right action of E on C.

The axioms and terminology of these notions are justified by the following proposition.

Proposition 2. *If the category C is tensored-cotensored-enriched over E, then, it is enriched in the usual sense and the functor $\langle -, - \rangle$ is a left coaction in the sense of Definition 2.*

Proof. Suppose that the category C is tensored-cotensored-enriched over E. On the one hand, let us define the composition relative to the enrichment $\{-,-\}$. For any object X, Y of C, the identity morphism of $\{X, Y\}$ defines a morphism $X \triangleleft \{X, Y\} \to Y$. So for any objects X, Y, Z, we have a map

$$X \triangleleft (\{X, Y\} \otimes \{Y, Z\}) \simeq (X \triangleleft \{X, Y\}) \triangleleft \{Y, Z\} \to Y \triangleleft \{Y, Z\} \to Z$$

and hence a map $\{X, Y\} \otimes \{Y, Z\} \to \{X, Z\}$. So is defined the composition. The coherence diagrams of Definition 2 ensure us that the composition is associative and gives us a unit. On the other hand, let us show that the functor $\langle -,-\rangle$ is a left coaction. For any $X,Y \in \mathsf{C}$ and any $\mathcal{A},\mathcal{B} \in \mathsf{E}$, we have functorial isomorphisms :

$$\begin{aligned}\hom_{\mathsf{C}}(X,\langle \mathcal{A}\otimes\mathcal{B},Y\rangle) &\simeq \hom_{\mathsf{C}}(X \triangleleft (\mathcal{A}\otimes\mathcal{B}),Y) \simeq \hom_{\mathsf{C}}((X\triangleleft\mathcal{A})\triangleleft\mathcal{B},Y)\\ &\simeq \hom_{\mathsf{C}}(X\triangleleft\mathcal{A},\langle\mathcal{B},Y\rangle) \simeq \hom_{\mathsf{C}}(X,\langle\mathcal{A}\langle\mathcal{B},Y\rangle\rangle)\ .\end{aligned}$$

By the Yoneda lemma, this gives us a functorial isomorphism $\langle\mathcal{A}\otimes\mathcal{B},Y\rangle \simeq \langle\mathcal{A}\langle\mathcal{B},Y\rangle\rangle$. This functorial isomorphism satisfy the coherence conditions of Definition 2 because the functorial isomorphism $X\triangleleft(\mathcal{A}\otimes\mathcal{B}) \simeq (X\triangleleft\mathcal{A})\triangleleft\mathcal{B}$ satisfy the coherence conditions of the same definition.

□

Proposition 3. *Let E be a presentable monoidal category and let C be a presentable category.*

- *Suppose that there exists a right action $-\triangleleft-$ of E on C and that for any $\mathcal{A}\in\mathsf{E}$ and for any $X\in\mathsf{C}$, the functors $X\triangleleft - : \mathsf{E}\to\mathsf{C}$ and $-\triangleleft\mathcal{A} : \mathsf{C}\to\mathsf{C}$ preserve colimits. Then, C is tensored-cotensored-enriched over E.*
- *Suppose that there exists a left coaction $\langle -,-\rangle$ of E on C and that there exists a functor*

$$-\triangleleft- : \mathsf{C}\times\mathsf{E}\to\mathsf{C}$$

together with a functorial isomorphism

$$\hom_{\mathsf{C}}(X\triangleleft\mathcal{A},Y) \simeq \hom_{\mathsf{C}}(X,\langle\mathcal{A},Y\rangle)\ .$$

Suppose moreover that the functor $\langle -,Y\rangle : \mathsf{E}^{op}\to\mathsf{C}$ sends colimits in E to limits. Then, C is tensored-cotensored-enriched over E.

Proof. The first point is a direct consequence of Proposition 1. Let us prove the second point. Since, E is left coacts on C, by the same arguments as in the proof of Proposition 2, we can show that the bifunctor $-\triangleleft-$ is a right action of E on E. Moreover, since the functors $\langle -,Y\rangle$ preserve limits, then any functor of the form $X\triangleleft -$ preserves colimits. The result is then a direct consequence of the first point. □

Definition 4 (Homotopical enrichment)**.** Let M be a model category and let E be a model category with a monoidal structure. We say that M is homotopically enriched over E if it enriched over E and if for any cofibration $f : X\to X'$ in M and any fibration $g : Y\to Y'$ in M, the morphism in E :

$$\{X',Y\}\to\{X',Y\}\times_{\{X,Y'\}}\{X,Y\}$$

is a fibration. Moreover, we require this morphism to be a weak equivalence whenever f or g is a weak equivalence.

2.1.3 Coalgebras

Definition 5 (Coalgebras)**.** A *coassociative coalgebra* $\mathscr{C} = (\mathcal{C},\Delta,\epsilon)$ is a chain complex $\mathcal{C}$ equipped with a coassociative coproduct $\Delta\mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$ and a counit $\epsilon : \mathcal{C}\to\mathbb{K}$ such that $Id_{\mathcal{C}} = (Id_{\mathcal{C}}\otimes\epsilon)\Delta = (\epsilon\otimes Id_{\mathcal{C}})\Delta$. The kernel of the map ϵ is denoted $\overline{\mathcal{C}}$. The coalgebra $\mathscr{C}$ is said cocommutative if $\Delta = \tau\Delta$ where

$$\tau(x\otimes y) = (-1)^{|x||y|}y\otimes x\ .$$

A *conilpotent coalgebra* $\mathscr{C} = (\mathcal{C},\Delta,\epsilon,1)$ is the data of a coassociative coalgebra $(\mathcal{C},\Delta,\epsilon)$ together with a graded atom, that is a nonzero element $1\in\mathcal{C}$ such that $\Delta 1 = 1\otimes 1$. In this context, let us define the map $\overline{\Delta} : \overline{\mathcal{C}}\to\overline{\mathcal{C}}\otimes\overline{\mathcal{C}}$ as follows :

$$\overline{\Delta}x := \Delta x - 1\otimes x - x\otimes 1 \in \overline{\mathcal{C}}\otimes\overline{\mathcal{C}}\ .$$

We also require, that for any $x\in\overline{\mathcal{C}}$, there exists an integer n such that

$$\overline{\Delta}^n x := (Id_{\mathcal{C}}^{\otimes n-1}\otimes\overline{\Delta})\cdots(Id_{\mathcal{C}}\otimes\overline{\Delta})\overline{\Delta}(x) = 0\ .$$

A conilpotent cocommutative coalgebra $\mathscr{C}$ is said to be a *Hinich coalgebra* if 1 is a dg atom, that is $d1 = 0$. We denote by uCog be the category of coassociative coalgebras and by uCocom the category of cocommutative coalgebras. Let $\mathsf{uNilCocom}$ (resp. $\mathsf{Hinich-cog}$)be the full subcategory of uCocom made up of conilpotent cocommutative coalgebras (resp. Hinich coalgebras).

Any conilpotent coalgebra $\mathscr{C}$ has a canonical filtration called the coradical filtration :

$$F_n^{rad}\mathcal{C} := \mathbb{K}\cdot 1 \oplus \{x \in \overline{\mathcal{C}} |\ \overline{\Delta}^{n+1} x = 0\}\ ,$$

which is not necessarily stable under the codifferential d.

Proposition 4. *Let f be a morphism of coalgebras between two conilpotent coalgebras $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ and $\mathscr{D} = (\mathcal{D}, \Delta', \epsilon', 1')$. Then, $f(1) = 1'$.*

Proof. Let $x \in \overline{\mathcal{D}}$ such that $f(1) = 1' + x$. Since $\Delta f(1) = (f \otimes f)\Delta(1)$, then $\overline{\Delta}x = x \otimes x$. Since there exits an integer n such that $\overline{\Delta}^n(x) = x \otimes \cdots \otimes x = 0$, then $x = 0$. ☐

Proposition 5. *The categories uCog, uCocom and $\mathsf{uNilCocom}$ and $\mathsf{Hinich-cog}$ are presentable. The forgetful functor from uCog to the category of chain complexes has a right adjoint called the cofree counital coalgebra functor. The same statement holds for the category uCocom. The functor $\mathscr{C} \mapsto \overline{\mathcal{C}}$ from the category $\mathsf{Hinich-cog}$ to the category of chain complexes has a right adjoint. The tensor product of the category of chain complexes induces closed symmetric monoidal structures on the categories uCog and uCocom.*

Proof. The results are proven in [AJ13, 2.1, 2.2,2.5] for the category uCog. The methods used apply mutatis mutandis for the other categories. ☐

Theorem 1 ([GG99])**.** *The full sub category $\mathsf{uCog}^{\geq 0}$ of uCog made up of by nonnegatively graded coalgebras admits a model structure whose cofibrations are the monomorphisms and whose weak equivalences are the quasi-isomorphisms.*

The category $\mathsf{Hinich-cog}$ is related to the category of Lie-algebras by an adjunction described in [Qui69] :

$$\mathsf{Hinich-cog} \underset{\mathcal{C}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathsf{Lie-alg}\ .$$

Theorem 2. *[Hin01] There exists a model structure on the category $\mathsf{Hinich-cog}$ whose cofibrations are monomorphisms and whose weak equivalences are morphisms whose image under the functor $\mathcal{L}$ is a quasi-isomorphism. The class of weak equivalences is contained in the class of quasi-isomorphisms. Moreover, the adjunction $\mathcal{L} \dashv \mathcal{C}$ is a Qullen equivalence when the category of Lie algebras is equipped with its projective model structure whose fibrations (resp. weak equivalences) are surjections (resp. quasi-isomorphisms) (see [Hin97]).*

Definition 6 (Deformation problems)**.** Let $\mathsf{Artin-alg}$ be the category of nonpositively graded local finite dimensional dg commutative algebra. A deformation problem is a functor from the category $\mathsf{Artin-alg}$ to the category of simplicial sets.

Lurie showed in [Lur11] that a suitable infinity-category of deformation problems (called formal moduli problems), is equivalent to the infinity-category of Lie algebras if the characteristic of the base field $\mathbb{K}$ is zero. Therefore, it is equivalent to the infinity-category of Hinich coalgebras. In that perspective, any Hinich coalgebra $\mathscr{C}$ induces a deformation problem as follows. :

$$R \in \mathsf{Artin-alg} \mapsto \mathrm{Map}_{\mathsf{Hinich-cog}}(R^*, \mathscr{C})\ .$$

Remark 1. We use the definition of Hinich of a deformation problem given in [Hin01]. We do not describe here the homotopy theory of such deformation problems nor a precise link with the work of Lurie who uses the framework of quasi categories (see [Lur11]). In the sequel, we will only use the fact that for any morphism of deformation problem $f : X \to Y$, if $f(R)$ is a weak equivalence of simplicial sets for any algebra $R \in \mathsf{Artin-alg}$, then f is an equivalence of deformation problems.

2.1.4 Coalgebras and simplicial sets

In this subsection, we describe a Quillen adjunction between the category of simplicial sets and the category of coassociative coalgebras. This adjunction is part of the Dold–Kan correspondence. From a simplicial set X, one can produce a chain complex $DK(X)$ called the normalized Moore complex. In degree n, $DK(X)_n$ is the sub-vector space of $\mathbb{K} \cdot X_n$ which is the intersection of the kernels of the faces $d_0, \ldots, d_{n-1}$. The differential is $(-1)^n d_n$. Moreover, the Alexander-Whitney map makes the functor DK comonoidal. Then, the diagonal map $X \to X \times X$ gives to $DK(X)$ a structure of coalgebras. Thus, we have a functor DK^c from simplicial sets to the category uCog of coassociative coalgebras. This functor DK^c admits a right adjoint given by

$$N(\mathcal{C})_n := \hom_{\mathsf{uCog}}(DK^c(\Delta[n]), \mathcal{C})$$

Actually, we have the following sequence of adjunctions,

$$\mathsf{sSet} \underset{N}{\overset{DK^c}{\rightleftarrows}} \mathsf{uCog}^{\geq 0} \underset{tr}{\overset{in}{\rightleftarrows}} \mathsf{uCog}$$

where in is the embedding of $\mathsf{uCog}^{\geq 0}$ into uCog and where tr is the truncation.

Proposition 6. *The above adjunction between* $\mathsf{uCog}^{\geq 0}$ *and* sSet *is a Quillen adjunction.*

Proof. The functor DK^c carries monomorphisms to monomorphisms and homotopy weak equivalences to quasi-isomorphisms ; see [GJ99, III.2]. □

2.1.5 The Sullivan algebra of polynomial forms on standard simplicies

Definition 7 (Sullivan polynomial algebras)**.** [Sul77] For any integer $n \in \mathbb{N}$, the n^{th} *algebra of polynomial forms* is the following differential graded unital commutative algebra :

$$\Omega_n := \mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(\Sigma t_i = 1)$$

where the degree of t_i is zero and where $d_{\Omega_n}(t_i) = dt_i$. In particular, $\sum dt_i = 0$.

Any map of finite ordinals $\phi : [n] \to [m]$ defines a morphism of differential graded unital commutative algebra :

$$\begin{aligned} \Omega(\phi) : \Omega_m &\to \Omega_n \\ t_i &\mapsto \sum_{\phi(j)=i} t_j \ . \end{aligned}$$

Therefore, the collection $\{\Omega_n\}_{n \in \mathbb{N}}$ defines a simplicial differential graded commutative algebra. Moreover, one can extend this construction to a contravariant functor $\Omega_\bullet$ from simplicial sets to differential graded unital commutative algebras such that $\Omega_{\Delta[n]} = \Omega_n$. This functor is part of an adjunction.

$$\mathsf{sSet} \overset{\Omega_\bullet}{\rightleftarrows} u\mathscr{C}om - \mathsf{alg}^{op}$$

Proposition 7. *[BG76, 8] When the characteristic of the field* $\mathbb{K}$ *is zero, then the category* $u\mathscr{C}om-\mathsf{alg}$ *of differential graded unital commutative algebras admits a projective model structure where fibrations (resp. weak equivalences) are degreewise surjections (resp. quasi-isomorphisms). In that context, the adjunction between simplicial sets and* $u\mathscr{C}om - \mathsf{alg}$ *is a Quillen adjunction.*

2.2 Operads, cooperads, algebras and coalgebras

The purpose of this section is to recall the definitions of operads, cooperads, algebras over an operad and coalgebras over a cooperad that we will use in the sequel ; we refer the reader to the book [LV12]. Moreover, we prove that the category of coalgebras over a curved cooperad is presentable.

2.2.1 Operads and cooperads

We recall here the definitions of operads and cooperads. We refer to the book [LV12] and the article [HM12].

Definition 8 (Symmetric modules)**.** Let $\mathbb{S}$ be the groupoid whose objects are integers $n \in \mathbb{N}$ and whose morphisms are :

$$\begin{cases} \hom_{\mathbb{S}}(n,m) = \emptyset \text{ if } n \neq m \\ \hom_{\mathbb{S}}(n,n) = \mathbb{S}_n \text{ otherwise .} \end{cases}$$

A graded $\mathbb{S}$-module (resp. dg $\mathbb{S}$-module) is a presheaf on $\mathbb{S}$ valued in the category of graded $\mathbb{K}$-modules (resp. chain complexes). The name $\mathbb{S}$-module will refer both to graded $\mathbb{S}$-modules and dg $\mathbb{S}$-modules. We say that a $\mathbb{S}$-module $\mathcal{V}$ is reduced if $\mathcal{V}(0) = \{0\}$.

The category of $\mathbb{S}$-modules has a monoidal structure which is as follows : for any $\mathbb{S}$-modules $\mathcal{V}$ and $\mathcal{W}$, and for any $n \geq 1$:

$$(\mathcal{V} \circ \mathcal{W})(n) := \bigoplus_{\substack{k \geq 1 \\ X_1 \sqcup \cdots \sqcup X_k = \{1,\ldots,n\}}} \mathcal{V}(k) \otimes_{\mathbb{S}_k} \big(\mathcal{W}(\#X_1) \otimes \cdots \otimes \mathcal{W}(\#X_k)\big)$$

where $\#X_i$ is the cardinal of the set X_i. For $n = 0$,

$$(\mathcal{V} \circ \mathcal{W})(0) := \mathcal{V}(0) \oplus \Big(\bigoplus_{k \geq 1} \mathcal{V}(k) \otimes_{\mathbb{S}_k} \big(\mathcal{W}(0) \otimes \cdots \otimes \mathcal{W}(0)\big)\Big) .$$

The monoidal unit is given by the $\mathbb{S}$-module $\mathcal{I}$ which is $\mathbb{K}$ in arity 1 and $\{0\}$ in other arities.

Notations.

- ▷ For any dg $\mathbb{S}$-module $\mathcal{V}$, we will denote by $\mathcal{V}^{grad}$ the underlying graded $\mathbb{S}$-module.
- ▷ Let $f : \mathcal{V} \to \mathcal{V}'$ and $g : \mathcal{W} \to \mathcal{W}'$ and $h : \mathcal{W} \to \mathcal{W}'$ be three morphisms of $\mathbb{S}$-modules. Then, we denote by $f \circ (g; h)$ the map from $\mathcal{V} \circ \mathcal{W}$ to $\mathcal{V}' \circ \mathcal{W}'$ defined as follows :

$$f \circ (g; h) := \sum_{i+j=n-1} f \otimes_{\mathbb{S}_n} (g^{\otimes i} \otimes h \otimes g^{\otimes j}) .$$

  In the case where g is the identity, we use the notation $f \circ' h$.

$$f \circ' h := f \circ (Id; h) .$$

- ▷ For any two graded $\mathbb{S}$-modules (resp. dg $\mathbb{S}$-modules) $\mathcal{V}$ and $\mathcal{W}$, we denote by $[\mathcal{V}, \mathcal{W}]$ the graded $\mathbb{K}$-modules (resp. chain complex) :

$$[\mathcal{V}, \mathcal{W}]_n := \prod_{\substack{k \geq 0 \\ l \in \mathbb{N}}} \hom_{\mathbb{K}[\mathbb{S}_n]}(\mathcal{V}(k)_l, \mathcal{W}(k)_{l+n}) .$$

  In that context morphisms of chain complex from X to $[\mathcal{V}, \mathcal{W}]$ are in one-to-one correspondence with morphism of $\mathbb{S}$-modules from $X \otimes \mathcal{V}$ to $\mathcal{W}$.

Proposition 8. *[LV12, 6] If the characteristic of the field $\mathbb{K}$ is zero, then the operadic Kunneth holds*

$$H(\mathcal{V} \circ \mathcal{W}) \simeq H(\mathcal{V}) \circ H(\mathcal{W}) ,$$

for any dg $\mathbb{S}$-modules $\mathcal{V}$ and $\mathcal{W}$,where H denotes the homology.

Definition 9. (Operads) A graded operad $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ (resp. dg operad) is a monoid in the category of graded $\mathbb{S}$-modules (resp. dg $\mathbb{S}$-modules). We denote by $\mathsf{dg} - \mathsf{Operad}$ the category of dg operads.

Example 1. For any graded $\mathbb{K}$-module (resp. chain complex) $\mathcal{V}$, $\mathrm{End}_{\mathcal{V}}$ is the graded operad (resp. dg operad) defined by :

$$\mathrm{End}_{\mathcal{V}}(n) := \hom(\mathcal{V}^{\otimes n}, \mathcal{V}) .$$

The composition in the operad $\mathrm{End}_{\mathcal{V}}$ is given by the composition of morphisms of graded $\mathbb{K}$-modules (resp. chain complexes).

A degree k derivation d on a graded operad $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ is the data of degree k maps $d : \mathcal{P}(n) \to \mathcal{P}(n)$ which commute with the action of $\mathbb{S}_n$ and such that

$$d\,\gamma = \gamma\,(d \circ Id + Id \circ' d)\ .$$

Proposition 9. *[LV12, 5] The forgetful functor from operads to $\mathbb{S}$-modules has a left adjoint called the free operad functor and denoted $\mathbb{T}$. For any $\mathbb{S}$-module $\mathcal{V}$, $\mathbb{T}\mathcal{V}$ is the $\mathbb{S}$-module made up of trees whose vertices are filled with elements of $\mathcal{V}$ with coherent arity. The composition is given by the grafting of trees.*

Remark 2. We will define $\mathbb{T}$ in the next chapter.

There is a one-to-one correspondence between the degree k derivation on the graded free operad $\mathbb{T}\mathcal{V}$ and the degree k maps from $\mathcal{V}$ to $\mathbb{T}\mathcal{V}$. Indeed, from such a map u one can produce the derivation D_u such that for any tree T labeled by elements of $\mathcal{V}$:

$$D_u(T) := \sum_v Id \otimes \cdots \otimes u(v) \otimes \cdots \otimes Id\ .$$

where the sum is taken over the vertices of the tree T.

Definition 10 (Cooperads)**.** A cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ is a comonoid in the category of $\mathbb{S}$-modules. We denote by $\overline{\mathscr{C}}$ the kernel of the morphism $\epsilon : \mathscr{C} \to \mathcal{I}$. A cooperad $\mathscr{C}$ is said to be coaugmented if it is equipped with a morphism of cooperads $\mathcal{I} \to \mathscr{C}$. In this case, we denote by 1 the image of the unit of $\mathbb{K}$ into $\mathcal{C}(1)$. A coaugmented cooperad $\mathscr{C}$ is said to be conilpotent if the process of successive decomposition stabilizes in finite time for any element. A precise definition is given in [LV12, 5.8.6].

The forgetful functor from conilpotent cooperads to $\mathbb{S}$-modules which sends $\mathscr{C}$ to $\overline{\mathscr{C}}$ has a right adjoint sending $\mathcal{V}$ to the tree module $\mathbb{T}(\mathcal{V})$ with the decomposition given by the degrafting of trees. We denote it by $\mathbb{T}^c(\mathcal{V})$. We also denote by $\delta : \mathscr{C} \to \mathbb{T}^c(\overline{\mathcal{C}})$ the counit of the adjunction. Any conilpotent cooperad is equipped with a filtration called the coradical filtration

$$F_n^{rad}\mathcal{C}(m) := \{p \in \mathcal{C}(m) | \delta(p) \in \mathbb{T}^{\leq n}(\overline{\mathcal{C}})(m)\}$$

where the symbol $\mathbb{T}^{\leq n}$ denotes the trees with at most n vertices. In particular, $F_0^{rad}\mathcal{C} = \mathcal{I}$.

Notations. Let $\mathcal{C}$ be coaugmented cooperad and m be an integer. We denote by Δ_m and the map

$$\Delta_m :\ \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \longrightarrow \mathbb{T}(\overline{\mathcal{C}}) \twoheadrightarrow \mathbb{T}^m(\overline{\mathcal{C}})\ .$$

A degree k coderivation on a cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ is a degree k map d of $\mathbb{S}$-modules from $\mathcal{C}$ to $\mathcal{C}$ such that

$$\Delta\ d = (d \circ Id + Id \circ' d)\Delta\ .$$

If the cooperad is coaugmented, we also require that $d(1) = 0$. Let $\mathbb{T}^c(\mathcal{V})$ be a cofree conilpotent cooperad. There is a one-to-one correspondence between degree k coderivation on $\mathbb{T}^c(\mathcal{V})$ and degree k maps from $\overline{\mathbb{T}}(\mathcal{V})$to $\mathcal{V}$. Indeed, such a map u is uniquely extended by the following coderivation D_u defined on any tree T labeled by elements of $\mathcal{V}$ as follows :

$$D_u(T) := \sum_{T' \subset T} Id \otimes \cdots \otimes u(T') \otimes \cdots \otimes Id\ ,$$

where the sum is taken on the subtrees T' of T.

Definition 11 (Curved cooperads)**.** A curved cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ is a coaugmented graded cooperad equipped with a degree -2 map $\theta : \mathcal{C}(1) \to \mathbb{K}$ and a degree -1 coderivation d such that

$$\begin{aligned} d^2 &= (\theta \otimes Id - Id \otimes \theta)\Delta_2\ , \\ \theta d &= 0\ . \end{aligned}$$

A morphism curved conilpotent cooperads is a morphism of conilpotent cooperads $\phi : \mathscr{C} \to \mathscr{D}$ which commutes with the coderivations and such that $\theta_{\mathscr{C}} = \theta_{\mathscr{D}}\phi$. We denote by cCoop the category of curved conilpotent cooperads.

The coradical filtration of a conilpotent cooperad has the following property with respect to the decomposition map.

Lemma 1. *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ be a conilpotent cooperad. Then*

$$\Delta(F_n^{rad}\mathcal{C}) \subset \sum_{p_0+\cdots+p_k \leq n} (F_{p_0}^{rad}\mathcal{C})(k) \otimes_{\mathbb{S}_k} (F_{p_1}^{rad}\mathcal{C} \otimes \cdots \otimes F_{p_k}^{rad}\mathcal{C}) \ .$$

Proof. It suffices to prove the result for cofree cooperads. Indeed, any conilpotent cooperad $\mathscr{C}$ is equipped with a map $\delta : \mathcal{C} \to \mathbb{T}^c(\overline{\mathcal{C}})$ such that $F_n^{rad}\mathcal{C} = \delta^{-1}(F_n^{rad}\mathbb{T}^c(\overline{\mathcal{C}}))$. □

Lemma 2. *Let $\mathscr{C} = \mathbb{T}^c(\mathcal{V})$ be a cofree conilpotent graded cooperad equipped with a degree -2 map $\theta : \mathbb{T}(\mathcal{V})(1) \twoheadrightarrow \mathcal{V}(1) \to \mathbb{K}$. Let $\phi : \overline{\mathbb{T}}\mathcal{V} \to \mathcal{V}$ be a degree -1 map and let D_ϕ be the corresponding coderivation on $\mathscr{C}$. Then, the triple $(\mathbb{T}\mathcal{V}, D_\phi, \theta)$ is a curved cooperad if and only if ϕ satisfies the following equation :*

$$\phi D_\phi = (\theta \otimes \pi_\mathcal{V} - \pi_\mathcal{V} \otimes \theta)\Delta_2$$

where $\pi_\mathcal{V}$ is the projection $\mathbb{T}(\mathcal{V}) \to \mathcal{V}$.

Proof. If $(\mathbb{T}\mathcal{V}, D_\phi, \theta)$ is a curved cooperad, then $\phi D_\phi = \pi D_\phi^2 = (\theta \otimes \pi_\mathcal{V} - \pi_\mathcal{V} \otimes \theta)\Delta_2$. Conversely, suppose that $\phi D_\phi = (\theta \otimes \pi_\mathcal{V} - \pi_\mathcal{V} \otimes \theta)\Delta_2$. For any tree T labeled by elements of $\mathcal{V}$, one can prove that

$$D_\phi^2(T) = \sum_{T' \subset T} Id \otimes (\phi D_\phi(T')) \otimes Id \ .$$

Actually, it is the sum over every arity one vertex v of
- ▷ $\pm\theta(v)(T - v)$ if v is the bottom vertex or a top vertex.
- ▷ $\pm(\theta(v)(T - v) - \theta(v)(T - v)) = 0$ otherwise.

Hence $(\mathbb{T}\mathcal{V}, D_\phi, \theta)$ is a curved cooperad. □

There exists notions of $\mathbb{N}$-modules, nonsymmetric operads, nonsymmetric cooperads and their morphisms defined for instance in [LV12, 5.9]. We will speak about the nonsymmetric context to refer to these ones. Notice that the operadic Kunneth formula holds in the nonsymmetric context without the assumption that the characteristic of the field $\mathbb{K}$ is zero.

2.2.2 Algebras over an operad

Definition 12 (Algebras over an operad)**.** Let $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ be an operad. A *$\mathscr{P}$-module* $\mathscr{A} = (\mathcal{A}, \gamma_\mathscr{A})$ is a left module in the category of $\mathbb{S}$-module, that is a $\mathbb{S}$-module $\mathcal{A}$ equipped with a map $\gamma_\mathscr{A} : \mathcal{P} \circ \mathcal{A} \to \mathcal{A}$ such that the following diagrams commute.

$$\begin{tikzcd} \mathcal{P} \circ \mathcal{P} \circ \mathcal{A} \arrow[r, "Id \circ \gamma_\mathscr{A}"] \arrow[d, "\gamma_\mathscr{A} \circ Id"'] & \mathcal{P} \circ \mathcal{A} \arrow[d, "\gamma_\mathscr{A}"] \\ \mathscr{P} \circ \mathcal{A} \arrow[r, "\gamma_\mathscr{A}"'] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I} \circ \mathcal{A} \arrow[r, "1 \circ Id"] \arrow[rr, bend right, "Id"'] & \mathscr{P} \circ \mathcal{A} \arrow[r, "\gamma_\mathcal{A}"] & \mathcal{A} \end{tikzcd}$$

A morphism of $\mathscr{P}$-modules from $\mathscr{A}$ to $\mathscr{B} = (\mathcal{B}, \gamma_\mathscr{B})$ is a morphism of $\mathbb{S}$-modules $f : \mathcal{A} \to \mathcal{B}$ such that $\gamma_\mathscr{B}(Id \circ f) = f\gamma_\mathscr{A}$. A *$\mathscr{P}$-algebra* is a $\mathscr{P}$-module $\mathcal{A}$ concentrated in arity 0. We denote $\mathscr{P} - \mathsf{alg}$ the category of $\mathscr{P}$-algebras.

The forgetful functor from the category of $\mathscr{P}$-modules to the category of $\mathbb{S}$-modules has a left adjoint given by

$$\mathcal{V} \mapsto \mathcal{P} \circ \mathcal{V}$$

The images of this left adjoint functor are called the free $\mathscr{P}$-modules.

Definition 13 (Ideal)**.** An *ideal* of a $\mathscr{P}$-algebra $\mathscr{A}$ is a sub-chain complex $\mathcal{B} \subset \mathcal{A}$ such that for any $p \in \mathcal{P}(n)$ and $(x_i)_{i=1}^n \in \mathcal{A}$ $(n \geq 1)$:

$$\gamma_\mathscr{A}\big(p \otimes_{\mathbb{S}_n} (x_1 \otimes \cdots \otimes x_n)\big) \in \mathcal{B}$$

whenever one of the x_i is in $\mathcal{B}$ (for $n \geq 1$). Then the quotient $\mathcal{A}/\mathcal{B}$ has an induced structure of $\mathscr{P}$-algebra.

Definition 14 (Derivation)**.** Let $\mathscr{P}$ be a graded operad and let $\mathscr{A}$ be an $\mathscr{P}$-module. Suppose that the graded operad $\mathscr{P}$ is equipped with a degree k derivation $d_{\mathscr{P}}$. Then, a *derivation* of $\mathscr{A}$ is a degree k map $d_{\mathscr{A}}$ from $\mathcal{A}$ to $\mathcal{A}$ such that

$$d_{\mathscr{A}}\ \gamma_{\mathscr{A}} = \gamma_{\mathscr{A}}\ \left(d_{\mathscr{P}} \circ Id_{\mathscr{A}} + Id \circ' d_{\mathscr{A}}\right)\ .$$

Let $\mathscr{P}$ be a graded operad equipped with a degree k derivation $d_{\mathscr{P}}$. There is a one-to-one correspondence between the derivations on a free $\mathscr{P}$-module $\mathscr{A} = \mathscr{P} \circ \mathcal{V}$ and the degree k maps $\mathcal{V} \to \mathcal{P} \circ \mathcal{V}$. Indeed, any such map $u : \mathcal{V} \to \mathcal{P} \circ \mathcal{V}$ is uniquely extended by the derivation $D_u = d_{\mathscr{P}} \circ Id + D'$ on $\mathscr{A}$ where :

$$D' :=\ \mathcal{P} \circ \mathcal{V} \xrightarrow{Id \circ u} \mathcal{P} \circ \mathcal{P} \circ \mathcal{V} \xrightarrow{\gamma \circ Id} \mathcal{P} \circ \mathcal{V}\ \ .$$

2.2.3 Coalgebras over a cooperad

Definition 15 (Comodules and coalgebras over a cooperad)**.** Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a cooperad. A *$\mathscr{C}$-comodule* $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ is a left $\mathscr{C}$-comodule in the category of $\mathbb{S}$-modules, that is a $\mathbb{S}$-module $\mathcal{D}$ together with a morphism $\Delta_{\mathscr{D}} : \mathcal{D} \to \mathcal{C} \circ \mathcal{D}$ such that the following diagrams commute.

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\Delta_{\mathscr{D}}"] \arrow[d, "\Delta_{\mathscr{D}}"'] & \mathcal{C} \circ \mathcal{D} \arrow[d, "Id \circ \Delta_{\mathscr{D}}"] \\ \mathcal{C} \circ \mathcal{D} \arrow[r, "\Delta_{\mathscr{C}} \circ Id"'] & \mathcal{C} \circ \mathcal{C} \circ \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "\Delta_{\mathscr{D}}"] \arrow[rr, bend right, "Id"'] & \mathcal{C} \circ \mathcal{D} \arrow[r, "\epsilon \circ Id"] & \mathcal{D} \end{tikzcd}$$

A *$\mathscr{C}$-coalgebra* is a $\mathscr{C}$-comodule concentrated in arity 0.

REMARK 3. Our notion of $\mathscr{C}$-coalgebra recovers actually a notion called sometimes in the literature conilpotent $\mathscr{C}$-coalgebra ; see [LV12, 5.4.8].

Let $\mathscr{C}$ be a coaugmented cooperad. Then, the forgetful functor from the category of $\mathscr{C}$-comodules to the category of $\mathbb{S}$-modules has a right adjoint which sends $\mathcal{V}$ to $\mathscr{C} \circ \mathcal{V}$. The images of the right adjoint are called the cofree $\mathscr{C}$-comodules.

Definition 16 (Coderivation)**.** Let $\mathscr{C}$ be a graded cooperad and let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a $\mathscr{C}$-comodule. Suppose that $\mathscr{C}$ is equipped with a degree k coderivation $d_{\mathscr{C}}$. A *coderivation* on $\mathscr{D}$ is a degree k map $d_{\mathscr{D}}$ from $\mathcal{D}$ to $\mathcal{D}$ such that

$$\Delta_{\mathscr{D}} d_{\mathscr{D}} = (d_{\mathscr{C}} \circ Id + Id \circ' d_{\mathscr{D}})\Delta_{\mathscr{D}}\ .$$

Let $\mathscr{C}$ be a cooperad equipped with a degree k coderivation and let $\mathcal{V}$ be a graded $\mathbb{K}$-module. Then, there is a one-to-one correspondence between the coderivations on the $\mathscr{C}$-coalgebra $\mathscr{C} \circ \mathcal{V}$ and the degree k maps $\mathscr{C} \circ \mathcal{V} \to \mathcal{V}$. Indeed, any such map u induces the coderivation

$$D_u := (d_{\mathscr{C}} \circ Id_{\mathcal{V}}) + (Id \circ (\pi; u))(\Delta_{\mathscr{C}} \circ Id_{\mathcal{V}})\ ,$$

where $\pi = \epsilon \circ Id : \mathscr{C} \circ \mathcal{V} \to \mathcal{V}$.

Definition 17 (Coalgebra over a curved cooperad)**.** Let $\mathscr{C}$ be a curved cooperad. A *$\mathscr{C}$-coalgebra* is a graded $\mathscr{C}^{grad}$-coalgebra $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ together with a coderivation $d_{\mathcal{D}}$ such that

$$d_{\mathcal{D}}^2 = (\theta_{\mathscr{C}} \circ Id)\Delta_{\mathscr{D}}$$

Proposition 10. *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a conilpotent curved cooperad and let $\mathcal{V}$ be a graded $\mathbb{K}$-module. There is a one-to-one correspondence between the degree -1 maps $\phi : \mathcal{C} \circ \mathcal{V} \to \mathcal{V}$ such that*

$$\phi D_\phi := \phi\big(Id \circ (\pi; \phi)\big)(\Delta_{\mathscr{C}} \circ Id_{\mathcal{V}}) + \phi(d_{\mathcal{C}} \circ Id_{\mathcal{V}}) = \theta \circ Id_{\mathcal{V}}$$

and the structures of $\mathscr{C}$-coalgebra (where $\mathscr{C}$ is considered as a curved cooperad) on the graded cofree coalgebra $\mathscr{C}^{grad} \circ \mathcal{V}$.

Proof. A structure of $\mathscr{C}$-coalgebra on $\mathscr{C}^{grad} \circ \mathcal{V}$ amounts to the data of a degree -1 coderivation D_ϕ such that $D_\phi^2 = (\theta \circ Id_{\mathscr{C}} \circ Id_{\mathcal{V}})(\Delta_{\mathscr{C}} \circ Id_{\mathcal{V}})$. If $D_\phi^2 = (\theta \circ Id_{\mathscr{C}} \circ Id_{\mathcal{V}})(\Delta_{\mathscr{C}} \circ Id_{\mathcal{V}})$, then $\phi D_\phi = \theta \circ Id_{\mathcal{V}}$. Conversely, suppose that $\phi D_\phi = \theta \circ Id$. We have :

$$
\begin{aligned}
D_\phi^2 &= (d_{\mathscr{C}}^2 \circ Id_{\mathcal{C}}) + \big(Id \circ (\pi;\phi)\big)(\Delta \circ Id)(d_{\mathscr{C}} \circ Id_{\mathcal{C}}) + \big(d_{\mathscr{C}} \circ (\pi;\phi)\big)(\Delta \circ Id) \\
&\quad + \big(Id \circ (\pi;\phi)\big)(\Delta \circ Id)\big(Id \circ (\pi;\phi)\big)(\Delta \circ Id) \ .
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
&\big(Id \circ (\pi;\phi)\big)(\Delta \circ Id)\big(Id \circ (\pi;\phi)\big)(\Delta \circ Id) + \big(Id \circ (\pi;\phi)\big)(\Delta \circ Id)(d_{\mathscr{C}} \circ Id_{\mathcal{C}}) + \big(d_{\mathscr{C}} \circ (\pi;\phi)\big)(\Delta \circ Id) \\
&= \big(Id \circ (\pi;\phi)\big)(Id \circ' D_\phi)(\Delta \circ Id) = \big(Id \circ (\pi;\phi D_\phi)\big)(\Delta \circ Id) = \Big(\big(Id \circ (\epsilon;\theta)\big)\Delta\Big) \circ Id \ .
\end{aligned}
$$

On the other hand,

$$
(d_{\mathscr{C}}^2 \circ Id) = \big((\theta \circ Id)\Delta\big) \circ Id + \Big(\big(\sum Id \otimes_{\mathbb{S}} (\epsilon^{\otimes i} \otimes \theta \otimes \epsilon^{\otimes j})\big)\Delta\Big) \circ Id \ .
$$

Hence $D_\phi^2 = \big((\theta \circ Id)\Delta\big) \circ Id_{\mathcal{V}}$. $\square$

Definition 18. Any $\mathscr{C}$-coalgebra $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ over a conilpotent cooperad $\mathscr{C}$ admits a filtration called the *coradical filtration* and defined as follows.

$$
F_n^{rad}\mathcal{D} := \{x \in \mathcal{D} | \Delta_{\mathscr{D}}(x) \in (F_n^{rad}\mathcal{C}) \circ \mathcal{D}\} \ .
$$

Proposition 11. *Let $\mathscr{C}$ be a conilpotent cooperad and let $\mathscr{D}$ be a $\mathscr{C}$-coalgebra. For any integer n :*

$$
\Delta_{\mathscr{D}}(F_n^{rad}\mathcal{D}) \subset \sum_{i_0+i_1+\cdots+i_k=n} (F_{i_0}^{rad}\mathcal{C})(k) \otimes_{\mathbb{S}_k} (F_{i_1}^{rad}\mathcal{D} \otimes \cdots \otimes F_{i_k}^{rad}\mathcal{D}) \ .
$$

Lemma 3. *Let $\mathcal{V}$ and $\mathcal{W}$ be two graded $\mathbb{K}$-modules equipped with filtrations $(F_n\mathcal{V})_{n\in\mathbb{N}}$ and $(F_n\mathcal{W})_{n\in\mathbb{N}}$, and let $\phi : \mathcal{V} \to \mathcal{W}$ be an injection such that $F_n\mathcal{V} = \phi^{-1}(F_n\mathcal{W})$ for any integer n. Then, there exists a map $\psi : \mathcal{W} \to \mathcal{V}$ such that $\psi\phi = Id$ and $\psi(F_n\mathcal{W}) = F_n\mathcal{V}$ for any $n \in \mathbb{N}$.*

Proof. For an integer $n \geq -1$, suppose that we have built a sub graded $\mathbb{K}$-module $\mathcal{U}_n$ of $F_n\mathcal{W}$ such that $F_m\mathcal{W} = \phi(F_m\mathcal{V}) \oplus (\mathcal{U}_n \cap F_m\mathcal{W})$ for any $m \leq n$. Let $\mathcal{U}'_n$ be a sub-graded $\mathbb{K}$-module of $F_{n+1}\mathcal{W}$ algebraic complement to $\phi(F_{n+1}\mathcal{V}) \oplus \mathcal{U}_n$. Then, let $\mathcal{U}_{n+1} := \mathcal{U}_n \oplus \mathcal{U}'_n$. Finally, let $\mathcal{U} := \operatorname{colim} \mathcal{U}_n$. We define ψ by

$$
\psi = \begin{cases} \phi^{-1} \text{ on } \phi(\mathcal{V}) \ , \\ 0 \text{ on } \mathcal{U} \ . \end{cases}
$$

$\square$

Proof of Proposition 11 : The map $\Delta_{\mathscr{D}} : \mathcal{D} \to \mathcal{C} \circ \mathcal{D}$ is actually a morphism of $\mathscr{C}$-coalgebras such that $\Delta_{\mathscr{D}}^{-1}(F_n^{rad}\mathcal{C} \circ \mathcal{D}) = F_n^{rad}\mathcal{D}$. By Lemma 3, there exists a map of graded $\mathbb{K}$-modules $\nabla : \mathcal{C} \circ \mathcal{D} \to \mathcal{D}$ such that $\nabla\Delta_{\mathscr{D}} = Id_{\mathcal{D}}$ and $\nabla(F_n\mathcal{C} \circ \mathcal{D}) = F_n\mathcal{D}$. Then, the following diagram is commutative.

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{C} \circ \mathcal{D} \arrow[d, "\Delta \circ Id"] & \\
\mathcal{C} \circ \mathcal{D} \arrow[r, "Id \circ \Delta"'] \arrow[rr, bend right, "Id"'] & \mathcal{C} \circ \mathcal{C} \circ \mathcal{D} \arrow[r, "Id \circ \nabla"'] & \mathcal{C} \circ \mathcal{D}
\end{tikzcd}
$$

By Lemma 1, we know that :

$$
(\Delta \circ Id)\Delta(F_n^{rad}\mathcal{D}) \subset \sum_{i_0+\cdots+i_k=n} F_{i_0}^{rad}\mathcal{C}(k) \otimes_{\mathbb{S}_k} (F_{i_1}^{rad}\mathcal{C} \circ \mathcal{D} \otimes \cdots \otimes F_{i_k}^{rad}\mathcal{C} \circ \mathcal{D}) \ ,
$$

Moreover, we know that :

$$(Id \circ \nabla)\big(F^{rad}_{i_0}\mathcal{C}(k) \otimes_{\mathbb{S}_k} (F^{rad}_{i_1}\mathcal{C} \circ \mathcal{D} \otimes \cdots \otimes F^{rad}_{i_k}\mathcal{C} \circ \mathcal{D})) \subset F^{rad}_{i_0}\mathcal{C}(k) \otimes_{\mathbb{S}_k} (F^{rad}_{i_1}\mathcal{D} \otimes \cdots \otimes F^{rad}_{i_k}\mathcal{D}) \ .$$

So, we have :

$$\Delta(F^{rad}_n\mathcal{D}) = (Id \circ \nabla)(\Delta \circ Id)\Delta(F^{rad}_n\mathcal{D}) \subset \sum_k \sum_{i_0+\cdots+i_k=n} F^{rad}_{i_0}\mathcal{C}(k) \otimes_{\mathbb{S}_k} (F^{rad}_{i_1}\mathcal{D} \otimes \cdots \otimes F^{rad}_{i_k}\mathcal{D}) \ .$$

□

2.2.4 Presentability

This subsection deals with the presentability of the category of algebras over an operad and the presentability of the category of coalgebras over a conilpotent curved cooperad.

Theorem 3. *[DCH16, 5.2] Let $\mathscr{P}$ be a dg-operad. Then the category $\mathscr{P} - \mathsf{alg}$ of $\mathscr{P}$-algebras is presentable.*

The essence of the last theorem is that any $\mathscr{P}$-algebra is the colimit of a filtered diagram of finitely presented $\mathscr{P}$-algebras.

Theorem 4. *Let $\mathscr{C}$ be a conilpotent curved cooperad. The category $\mathscr{C} - \mathsf{cog}$ of $\mathscr{C}$-coalgebras is presentable.*

The essence of this theorem is that any $\mathscr{C}$-coalgebra is the colimit of a filtered diagram of finite dimensional $\mathscr{C}$-coalgebras. Since the category of $\mathscr{C}$-coalgebras does not seem to be comonadic over a known presentable category, we cannot use the same kind of arguments as in the proof of [DCH16, 5.2].

Lemma 4. *The category $\mathscr{C} - \mathsf{cog}$ is cocomplete.*

Proof. The colimit of a diagram of $\mathscr{C}$-coalgebras is its colimit in the category of graded $\mathbb{K}$-modules, together with the obvious decomposition map and coderivation map. □

Lemma 5. *For any $\mathscr{C}$-coalgebra $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and finite dimensional sub-graded $\mathbb{K}$-module $\mathcal{V} \in \mathcal{C}$, there exists a finite dimensional sub-$\mathscr{C}$-coalgebra $\mathscr{E}$ of $\mathscr{D}$ which contains $\mathcal{V}$.*

Proof. Let us prove the result by induction on the coradical filtration of $\mathscr{D}$. Suppose first that $\mathcal{V} \subset F_0\mathcal{D}$. Then, $\mathcal{V} + d\mathcal{V}$ is a sub $\mathscr{C}$-coalgebra of $\mathscr{D}$. Then, suppose that, for any finite dimensional sub-graded $\mathbb{K}$-module $\mathcal{W} \in F^{rad}_n\mathcal{D}$, there exists a finite dimensional sub $\mathscr{C}$-coalgebra $\mathscr{E}$ of $F^{rad}_n\mathscr{D}$ which contains $\mathcal{W}$. Consider now a finite dimensional sub-graded $\mathbb{K}$-module $\mathcal{V} \subset F_{n+1}\mathcal{D}$. By Proposition 11, for any element $x \in F^{rad}_{n+1}\mathcal{D}$, $\Delta_{\mathscr{D}}(x) - 1 \otimes x \in \mathcal{C} \circ F^{rad}_n\mathcal{D}$. Since we are working with conilpotent $\mathscr{C}$-coalgebras, there exists a finite dimensional sub graded $\mathbb{K}$-module $\mathcal{V}(x)$ of $F^{rad}_n\mathcal{D}$ such that $\Delta_{\mathscr{D}}(x) - 1 \otimes x \in \mathcal{C} \circ \mathcal{V}(x)$. Let $(e_i)_{i=1}^k$ be a linearly free family of elements of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{V} \cap F^{rad}_n\mathcal{D} \oplus \bigoplus_{i=1}^k \mathbb{K}.e_i$. By the induction hypothesis, let $\mathcal{E}$ be a finite dimensional sub $\mathscr{C}$-coalgebra of $\mathscr{D}$ which contains

$$\mathcal{V} \cap F^{rad}_n\mathcal{D} \oplus \sum \mathcal{V}(e_i) + \mathcal{V}(d_{\mathcal{D}}e_i) \ .$$

Then, the sum

$$\mathcal{E} + \sum_i (\mathbb{K}.e_i \oplus \mathbb{K}.d_{\mathcal{D}}e_i)$$

is a finite dimensional sub $\mathscr{C}$-coalgebra of $\mathscr{D}$ which contains $\mathcal{V}$. □

Finally, we show that a finite dimensional $\mathscr{C}$-coalgebras is a compact object.

Proposition 12. *A finite dimensional $\mathscr{C}$-coalgebra is a compact object.*

We will need the following technical lemma.

Lemma 6. *Let $D : I \to \mathscr{C} - \mathsf{cog}$ be a filtered diagram. Let $x \in D(i)$ for an object i of I. If the image of x in* $\mathrm{colim}D$ *is zero, then, there exists an object i' of I and a map $\phi : i \to i'$ such that $D(\phi)(x) = 0$.*

Proof. The colimit of the diagram D is the cokernel of the map

$$g : \bigoplus_{f:j\to j'} D(j) \to \bigoplus_{i\in Ob(I)} D(i)$$

such that for any morphism $f : j \to j'$of I, the morphism g sends $x \in D(j)$ to $x - D(f)(x)$. Let $x \in D(i)$ whose image in $\mathrm{colim}D$ is zero. Then, there exists an element $y = \sum y_f$ of $\bigoplus_{f:j\to j'} D(j)$ such that $g(y) = x$. Let i' be a cocone in I of the finite diagram made up of the morphisms f such that $y_f \neq 0$. Then, the image in $D(i')$ of $\sum y_f$ is the same as the image in $D(i')$ of $\sum D(f)(y_f)$. Hence, the image of x in $D(i')$ is zero. □

Proof of Proposition 12. Let $D : I \to \mathscr{C} - \mathsf{cog}$ be a filtered diagram and let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a finite dimensional $\mathscr{C}$-coalgebra. We have to show that the canonical map

$$\mathrm{colim}\big(\hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, D)\big) \to \hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, \mathrm{colim}D)$$

is bijective.

- ▷ Let us first show that it is surjective. Let $f : \mathscr{D} \to \mathrm{colim}D$ be a map of $\mathscr{C}$-coalgebra and let $\mathcal{D}'$ be the image of f inside $\mathrm{colim}D$ which is also a sub-$\mathscr{C}$-coalgebra of $\mathrm{colim}D$. Let $\{e_a\}_{a=1}^n$ be a basis of the graded $\mathbb{K}$-module $\mathcal{D}'$. Since the diagram D is filtered, there exists an object i of I and elements $x_a \in D(i)$ whose image in $\mathrm{colim}D$ is e_a. Let $\mathscr{E}$ be the smallest sub $\mathscr{C}$-coalgebra of $D(i)$ which contains all the x_a and let $\mathscr{E}'$ be the image of $\mathscr{E}$ in $\mathrm{colim}D$. Notice that $\mathscr{E}'$ contains $\mathcal{D}'$ and that the map $\mathscr{E} \to \mathscr{E}'$ is surjective. By Lemma 6 and since $\mathscr{E}$ is finite dimensional, there exists an object i' and a map $\phi : i \to i'$ such that, the map $\mathscr{E}'' := D(\phi)(\mathscr{E}) \to \mathscr{E}'$ is an isomorphism of $\mathscr{C}$-coalgebras. So, let $\mathscr{D}''$ be the sub $\mathscr{C}$-coalgebra of $\mathscr{E}''$ which is the image of $\mathcal{D}'$ through the inverse isomorphism $\mathscr{E}' \to \mathscr{E}''$. Hence, the map $\mathscr{D} \to \mathscr{D}' \to \mathrm{colim}D$ factors through the map $\mathscr{D} \to \mathscr{D}' \simeq \mathscr{E}'' \to D(i')$ and so the canonical map $\mathrm{colim}\big(\hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, D)\big) \to \hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, \mathrm{colim}D)$ is surjective.
- ▷ Let us show that it is injective. Let $f \in \hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, D(i))$ and $g \in \hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, D(j))$ be two maps whose images in $\hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, \mathrm{colim}D)$ are the same ; it is denoted h. Since the category I is filtered, there exists an object k together with maps $\phi : i \to k$ and $\psi : j \to k$. Then $D(\phi)f(\mathscr{D}) + D(\psi)g(\mathscr{D})$ is a finite dimensional sub $\mathscr{C}$-coalgebra of $D(k)$ whose image in $\mathrm{colim}D$ is $h(\mathscr{D})$. As in the previous point (by Lemma 6), the exists a map $\zeta : k \to k'$ in I such that the map $u : D(\zeta)\big(D(\phi)f(\mathscr{D}) + D(\psi)g(\mathcal{C})\big) \to h(\mathscr{D})$ is an isomorphism. Since the dimension (as a graded $\mathbb{K}$-module) of $D(\zeta)D(\phi)f(\mathscr{D})$ and the dimension of $D(\zeta)D(\psi)g(\mathscr{D})$ are both greater than the dimension of $h(\mathscr{D})$, then we must have :

  $$D(\zeta)\big(D(\phi)f(\mathscr{D}) + D(\psi)g(\mathscr{D})\big) = D(\zeta)D(\phi)f(\mathscr{D}) = D(\zeta)D(\psi)g(\mathscr{D}) \ .$$

  In this context, we have

  $$D(\zeta)D(\phi)f = u^{-1}h = D(\zeta)D(\psi)g \ .$$

  Hence, f and g represent the same element of $\mathrm{colim}\big(\hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}, D)\big)$.

□

Proof of Theorem 4. The isomorphisms classes of finite dimensional $\mathscr{C}$-coalgebras form a set. By Proposition 12, any finite dimensional $\mathscr{C}$-coalgebra is a compact object of the category $\mathscr{C} - \mathsf{cog}$. Moreover, any $\mathscr{C}$-coalgebra is the colimit of the diagram of its finite dimensional sub $\mathscr{C}$-coalgebras (with inclusions between them) ; this is a filtered diagram (and even a directed set). Hence, the category $\mathscr{C} - \mathsf{cog}$ is presentable. □

2.3 Enrichment

This section deals with several enrichment of the category of algebras of an operad and of the category of coalgebras of a curved conilpotent cooperad. Specifically, we prove that both the category of algebras over an operad and the category of coalgebras over a curved conilpotent cooperad are tensored, cotensored and enriched over cocommutative coalgebras and enriched over simplicial sets. In the nonsymmetric context, algebras over an operad and coalgebras over a curved conilpotent cooperad are tensored, cotensored and enriched over coassociative coalgebras ; this leads to another enrichment in simplicial sets.

2.3.1 Enrichment over coassociative coalgebras and cocommutative coalgebras

We will show in this subsection that the category of algebras over an operad and the category of coalgebras over a curved conilpotent cooperad are tensored-cotensored-enriched (see Definition 3) over the category uCocom of counital cocommutative coalgebras. Moreover, in the nonsymmetric context, they are tensored-cotensored-enriched over the category uCog of coassociative coalgebras. We will use these enrichments in the sequel to describe respectively deformation of morphisms and mapping spaces

Enrichment of $\mathscr{P}$-algebras over coalgebras

Let $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ be a dg operad. For any counital cocommutative coalgebra $\mathscr{C} = (\mathcal{C}, \Delta_{\mathscr{C}}, \epsilon)$ and any $\mathscr{P}$-algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, then the chain complex $[\mathcal{C}, \mathcal{A}]$ has a canonical structure of $\mathscr{P}$-algebra as follows.

▷ For any $p \in \mathcal{P}(n)$ $(n \geq 1)$, and for any $f_1, \ldots, f_n \in [\mathcal{C}, \mathcal{A}]$ and any $x = \mathcal{C}$, then

$$\gamma_{[\mathcal{C},\mathcal{A}]}\big(p \otimes_{\mathbb{S}_n} (f_1 \otimes \cdots \otimes f_n)\big)(x) = \gamma_{\mathscr{A}}(p \otimes -)(f_1 \otimes \cdots \otimes f_n)\Delta_{\mathscr{C}}^{n-1}(x)$$

▷ For any $p \in \mathcal{P}(0)$:

$$\gamma_{[\mathcal{C},\mathcal{A}]}(p) = \gamma_{\mathscr{A}}(p)\epsilon_{\mathcal{C}}$$

The chain complex $[\mathcal{C}, \mathcal{A}]$ together with its structure of $\mathscr{P}$-algebra is denoted $[\mathscr{C}, \mathscr{A}]$.

Lemma 7. *The assignment $\mathscr{C}, \mathscr{A} \mapsto [\mathscr{C}, \mathscr{A}]$ defines a left coaction (see Definition 2) of the category* uCocom *of counital cocommutative coalgebras on the category $\mathscr{P} - \mathsf{alg}$ of $\mathscr{P}$-algebras .*

Proof. The construction is functorial covariantly with respect to $\mathscr{P}$-algebras and contravariantly with respect to counital cocommutative coalgebras. Moreover, for any counital cocommutative coalgebras $\mathscr{C}$ and $\mathscr{D}$, and any $\mathscr{P}$-algebra $\mathscr{A}$ there is an isomorphism of chain complexes

$$\rho_{\mathscr{C},\mathscr{D},\mathscr{A}} : [\mathcal{C} \otimes \mathcal{D}, \mathcal{A}] \to [\mathcal{C}, [\mathcal{D}, \mathcal{A}]]$$

such that $\rho_{\mathscr{C},\mathscr{D},\mathscr{A}}(f)(x)(y) = f(x \otimes y)$. This is a morphism of $\mathscr{P}$-algebra which is functorial in $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{A}$, and it satisfies the coherence conditions of Definition 2. □

One can define a left adjoint to the functor $[\mathscr{C}, -]$ as follows. Let $\mathscr{A} \triangleleft \mathscr{C}$ be the following the quotient of the free $\mathscr{P}$-algebra $\mathcal{P} \circ (\mathcal{A} \otimes \mathcal{C})$ by the ideal I generated by the relations

$$\begin{cases} \gamma_{\mathscr{A}}\big(p \otimes_{\mathbb{S}_n} (y_1 \otimes \cdots \otimes y_n)\big) \otimes x \sim \sum (-1)^{\sum_{i<j} |x_{(i)}||y_j|} p \otimes_{\mathbb{S}_n} \big((y_1 \otimes x_{(1)}) \otimes \cdots \otimes (y_n \otimes x_{(n)})\big) \\ \gamma_{\mathscr{A}}(p) \otimes x \sim \epsilon(x)p \text{ for any } p \in \mathcal{P}(0) \ , \end{cases}$$

with $\Delta^{n-1}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n)}$.

Theorem 5. *The category of $\mathscr{P}$-algebras is tensored-cotensored-enriched over the category* uCocom *of counital cocommutative coalgebras. The right action is given by the functor $-\triangleleft-$, the left coaction is given by the functor $[-,-]$. We denote the enrichment by $\{-,-\}$.*

Proof. Since the functor $[-,-]$ defines a coaction of the category of counital cocommutative coalgebras on the category of $\mathscr{P}$-algebras, since the functor $[-, \mathcal{A}]$ sends colimits to limits and since the functor $[\mathscr{C}, -]$ is left adjoint to the functor $- \triangleleft \mathscr{C}$, then we can conclude by Proposition 3. □

Let us describe $\{\mathscr{A}, \mathscr{A}'\}$ for two $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{A}'$. This is the biggest sub-coalgebra of the cofree cocommutative coalgebra $F([\mathcal{A}, \mathcal{A}'])$ such that the following diagram commutes

$$\begin{tikzcd}
\{\mathscr{A}, \mathscr{A}'\} \arrow[rrr] \arrow[d, "(\epsilon, Id, \Delta, \ldots)"'] & & & {[\mathcal{A}, \mathcal{A}']} \arrow[d] \\
\prod_{n\geq 0}\{\mathscr{A}, \mathscr{A}'\}^{\otimes n}/\mathbb{S}_n \arrow[r] & \prod_{n\geq 0}[\mathcal{A}, \mathcal{A}']^{\otimes n}/\mathbb{S}_n \arrow[r] & {[\mathcal{P}\circ\mathcal{A}, \mathcal{P}\circ\mathcal{A}']} \arrow[r] & {[\mathcal{P}\circ\mathcal{A}, \mathcal{A}']} .
\end{tikzcd}$$

where the map $\prod_{n\geq 0}[\mathcal{A}, \mathcal{A}']^{\otimes n}/\mathbb{S}_n \to [\mathcal{P}\circ\mathcal{A}, \mathcal{P}\circ\mathcal{A}']$ sends $f_1\otimes\cdots\otimes f_n$ to $Id_{\mathcal{P}(n)}\otimes_{\mathbb{S}_n}(f_1\otimes\cdots\otimes f_n)$.

Enrichment of $\mathscr{C}$-coalgebras over coalgebras

Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a curved conilpotent cooperad.

For any $\mathscr{C}$-coalgebra $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and any counital cocommutative coalgebra $\mathscr{E} = (\mathcal{E}, \Delta_{\mathscr{E}}, \epsilon)$, the tensor product $\mathcal{D}\otimes\mathcal{E}$ has a structure of $\mathscr{C}$-coalgebra given by

$$\mathcal{D}\otimes\mathcal{E} \xrightarrow{\bigoplus_n \Delta_n\otimes\Delta^{n-1}} \bigoplus_n(\mathcal{C}(n)\otimes_{\mathbb{S}_n}\mathcal{D}^{\otimes n})\otimes\mathcal{E}^{\otimes n} \longrightarrow \bigoplus_n \mathcal{C}(n)\otimes_{\mathbb{S}_n}(\mathcal{D}\otimes\mathcal{E})^{\otimes n} .$$

Theorem 6. *The category $\mathscr{C}-\mathsf{cog}$ of $\mathscr{C}$-coalgebras is tensored-cotensored-enriched over the category of cocommutative counital coalgebras. The right action is given by the construction $-\otimes-$. We denote the left coaction by $\langle -,-\rangle$ and the enrichment by $\{-,-\}$.*

Proof. The assignment $\mathscr{D},\mathscr{E}\mapsto\mathscr{D}\otimes\mathscr{E}$ defines a right action of the category of counital cocommutative coalgebras on the category of $\mathscr{C}$-coalgebras. Moreover, the functor $\mathscr{D}\otimes -$ and the functor $-\otimes\mathscr{E}$ preserve colimits. We conclude by Proposition 3. □

If $\mathscr{D}$ and $\mathscr{D}'$ are two $\mathscr{C}$-coalgebras, then the cocommutative counital hom coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ is the terminal sub-coalgebra of the cofree counital cocommutative coalgebra $F([\mathcal{D}, \mathcal{D}'])$ over the chain complex $[\mathcal{D}, \mathcal{D}']$ such that the following diagram commutes.

$$\begin{tikzcd}
\{\mathscr{D}, \mathscr{D}'\} \arrow[rrr] \arrow[d, "(\epsilon, Id, \Delta, \ldots)"'] & & & {[\mathcal{D}, \mathcal{D}']} \arrow[d] \\
\prod_{n\geq 0}\{\mathscr{D}, \mathscr{D}'\}^{\otimes n}/\mathbb{S}_n \arrow[r] & \prod_{n\geq 0}[\mathcal{D}, \mathcal{D}']^{\otimes n}/\mathbb{S}_n \arrow[r] & {[\mathcal{C}\circ\mathcal{D}, \mathcal{C}\circ\mathcal{D}']} \arrow[r] & {[\mathcal{D}, \mathcal{C}\circ\mathcal{D}']}
\end{tikzcd}$$

Morphisms are atoms

Definition 19. Let $\mathscr{E} = (\mathcal{E}, \Delta, \epsilon)$ be a cocommutative coalgebra. A *graded atom* of $\mathscr{E}$ is a nonzero element $a\in\mathcal{E}_0$ such that $\Delta a = a\otimes a$. A *dg atom* is a graded atom a such that $d_{\mathcal{E}}a = 0$.

Proposition 13. *For any two $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{A}'$, the dg atoms of the cocommutative coalgebra $\{\mathscr{A}, \mathscr{A}'\}$ are the morphisms of $\mathscr{P}$-algebras from $\mathscr{A}$ to $\mathscr{A}'$. Similarly, for any two $\mathscr{C}$-coalgebras $\mathscr{D}$ and $\mathscr{D}'$, the dg atoms of the cocommutative coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ are the morphisms of $\mathscr{C}$-coalgebras from $\mathscr{D}$ to $\mathscr{D}'$.*

Proof. We have

$$\hom_{\mathsf{uCocom}}(\mathbb{K}, \{\mathscr{A}, \mathscr{A}'\}) \simeq \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}\lhd\mathbb{K}, \mathscr{A}') \simeq \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \mathscr{A}') .$$

□

Nonsymmetric context

In the nonsymmetric context, we can get rid of the cocommutativity condition.

Proposition 14.

▷ *If $\mathscr{P}$ is a nonsymmetric operad, then the category of $\mathscr{P}$-algebras is tensored-cotensored-enriched over the category* uCog *of counital coassociative coalgebras.*
▷ *If $\mathscr{C}$ is a nonsymmetric conilpotent curved cooperad, then the category of $\mathscr{C}$-coalgebras is tensored-cotensored-enriched over the category* uCog *of counital coassociative coalgebras.*

We denote by $\{-,-\}^{ns}$ these two enrichments over counital coassociative coalgebras.

Proof. The proof is similar to the proofs of Theorem 5 and of Theorem 6. ☐

The inclusion functor $\mathsf{uCocom} \hookrightarrow \mathsf{uCog}$ is a left adjoint (since it preserves colimits). Let R be its right adjoint. It sends any counital coassociative coalgebra to its biggest cocommutative subcoalgebra.

Proposition 15. *For any $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{A}'$, the cocommutative coalgebra, $\{\mathscr{A},\mathscr{A}'\}$ is the biggest cocommutative subcoalgebra $R(\{\mathscr{A},\mathscr{A}'\}^{ns})$ of $\{\mathscr{A},\mathscr{A}'\}^{ns}$. Similarly, for any $\mathscr{C}$-coalgebras $\mathscr{D}$ and $\mathscr{D}'$, the cocommutative coalgebra, $\{\mathscr{D},\mathscr{D}'\}$ is the biggest cocommutative subcoalgebra $R(\{\mathscr{D},\mathscr{D}'\}^{ns})$ of $\{\mathscr{D},\mathscr{D}'\}^{ns}$.*

Proof. For any cocommutative coalgebra $\mathscr{E}$, we have :

$$\hom_{\mathsf{uCocom}}(\mathscr{E},\{\mathscr{A},\mathscr{A}'\}) \simeq \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}\triangleleft\mathscr{E},\mathscr{A}') \simeq \hom_{\mathsf{uCog}}(\mathscr{E},\{\mathscr{A},\mathscr{A}'\}^{ns}) \simeq \hom_{\mathsf{uCocom}}(\mathscr{E},R(\{\mathscr{A},\mathscr{A}'\}^{ns}))\ .$$

Since these isomorphisms are functorial, $R(\{\mathscr{A},\mathscr{A}'\}^{ns})$ is isomorphic to $\{\mathscr{A},\mathscr{A}'\}$. ☐

2.3.2 Simplicial enrichment

In this section, we recall the fact that the Sullivan polynomials forms algebras allows one to enrich the category of algebras over an operad. See for instance [Hin01].

General case

Let A be a unital commutative $\mathbb{K}$-algebra. The functor $A\otimes - : \mathsf{dgMod} \to \mathsf{dgMod}_A$ is strong symmetric monoidal and comonoidal. Hence it induces several functors :
▷ from operads to operads enriched in A-modules,
▷ from cooperads to cooperads enriched in A-modules,
▷ from $\mathscr{P}$-algebras (in the category of $\mathbb{K}$-modules) to $A\otimes\mathscr{P}$-algebras (in the category of A-modules),
▷ from $\mathscr{C}$-coalgebras (in the category of $\mathbb{K}$-modules) to $A\otimes\mathscr{C}$-coalgebras (in the category of A-modules).

Applying this to the case of the Sullivan algebras of polynomial forms on standard simplicies leads us to the following proposition.

Proposition 16. *Let $\mathscr{P}$ be a dg operad and let $\mathscr{C}$ be a curved conilpotent cooperad. The category of $\mathscr{P}$-algebras and the category of $\mathscr{C}$-coalgebras are enriched in simplicial sets as follows :*

$$\mathrm{HOM}(\mathscr{A},\mathscr{A}')_n := \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{A}') \simeq \hom_{\Omega_n\otimes\mathscr{P}-\mathsf{alg}}(\Omega_n\otimes\mathscr{A},\Omega_n\otimes\mathscr{A}')\ ,$$

$$\mathrm{HOM}(\mathscr{D},\mathscr{D}')_n := \hom_{\Omega_n\otimes\mathscr{C}-\mathsf{cog}}(\Omega_n\otimes\mathscr{D},\Omega_n\otimes\mathscr{D}')\ .$$

Proof. The only point that needs to be cleared up is the simplicial structure on $\mathrm{HOM}(\mathscr{D},\mathscr{D}')$. Let $\phi : [m] \to [n]$ be a map between finite ordinals . We want to define $\phi^* : \mathrm{HOM}(\mathscr{D},\mathscr{D}')_n \to \mathrm{HOM}(\mathscr{D},\mathscr{D}')_m$. An element of $\mathrm{HOM}(\mathscr{D},\mathscr{D}')_n$ is a morphism of graded $\mathbb{K}$-modules f from $\mathcal{D}$ to $\Omega_n\otimes\mathcal{D}'$ such that $fd_{\mathcal{D}} = (d_{\Omega_n}\otimes Id_{\mathcal{D}'} + Id_{\Omega_n}\otimes d_{\mathcal{D}'})$ and such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & \Omega_n\otimes\mathcal{D}' \arrow[d, "Id\otimes\Delta"] \\
\mathcal{C}\circ\mathcal{D} \arrow[r, "Id\circ f"'] & \mathcal{C}\circ(\Omega_n\otimes\mathcal{D}) \arrow[r] & \Omega_n\otimes(\mathcal{C}\circ\mathcal{D}')\ ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "f"] \arrow[d, "\theta_{\mathcal{D}}"'] & \Omega_n\otimes\mathcal{D}' \arrow[d, "Id_A\otimes\theta_{\mathcal{D}'}"] \\
\mathbb{K} \arrow[r] & \Omega_n\ ,
\end{tikzcd}$$

where the map $\mathcal{C} \circ (\Omega_n \otimes \mathcal{D}') \to \Omega_n \otimes (\mathcal{C} \circ \mathcal{D}')$ in the first diagram is the following map

$$x \otimes_{\mathbb{S}_k} \big((a_1 \otimes x_1) \otimes \cdots \otimes (a_k \otimes x_k)\big) \mapsto (-1)^{|x|(\sum |a_i|)}(-1)^{\sum_{i>j}|a_i||x_j|}(a_1 \cdots a_k) \otimes \big(x \otimes_{\mathbb{S}_k} (x_1 \otimes \cdots \otimes x_k)\big) .$$

Then, $\phi^*(f) = (\Omega[\phi] \otimes Id)f$ where $\Omega[\phi] : \Omega_n \to \Omega_m$ is the structural map induced by ϕ. $\square$

Proposition 17. *For any simplicial set X which is the colimit of a finite diagram of simplicies $\Delta[n]$ and for any $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{A}'$, we have :*

$$\hom_{\mathsf{sSet}}(X, \mathrm{HOM}(\mathscr{A}, \mathscr{A}')) \simeq \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Omega_X \otimes \mathscr{A}') .$$

Proof. It suffices to notice that the functor from commutative algebras to $R \otimes \mathscr{P}$-algebras $R \mapsto R \otimes \mathscr{A}'$ preserves finite limits. $\square$

Remark 4. The enrichment of the category of $\mathscr{P}$-algebras and of the category of $\mathscr{C}$-coalgebras over simplicial sets that we described above is a part of a more general enrichment over functors from the category of unital commutative algebras to simplicial sets :

$$\begin{aligned}
&R \mapsto (\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes R \otimes \mathscr{B}))_{n\in\mathbb{N}} \\
&R \mapsto (\hom_{\Omega_n \otimes R \otimes \mathscr{C}-\mathsf{cog}}(\Omega_n \otimes R \otimes \mathscr{D}, \Omega_n \otimes R \otimes \mathscr{D}') \ .)_{n\in\mathbb{N}} \ .
\end{aligned}$$

Nonsymmetric context

In the nonsymmetric context, we can use some associative algebras instead of the commutative Sullivan algebras to define a simplicial enrichment. Let Λ_n be the linear dual of the Dold-Kan coalgebra over the standard simplex :

$$\Lambda_n := DK^c(\Delta[n])^* .$$

This defines a simplicial unital associative algebra.

Besides, let $\mathscr{P}$ be a nonsymmetric dg operad. For any $\mathscr{P}$-algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, and for any associative algebra A, $A \otimes \mathcal{A}$ has a canonical structure of $\mathscr{P}$-algebra.

Definition 20 (Nonsymmetric simplicial enrichment of algebras over an operad)**.** For any two $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{B}$, let $\mathrm{HOM}^{ns}(\mathscr{A}, \mathscr{B})$ be the following simplicial set :

$$\mathrm{HOM}^{ns}(\mathscr{A}, \mathscr{B})_n := \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Lambda_n \otimes \mathscr{B}) .$$

This defines a simplicial enrichment of the category of $\mathscr{P}$-algebras over simplicial sets.

Let $\mathscr{C}$ be a nonsymmetric curved conilpotent cooperad. For any associative algebra A and for any two $\mathscr{C}$-coalgebras $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and $\mathscr{E} = (\mathcal{E}, \Delta_{\mathcal{E}})$, we denote by $\hom_{A,\mathscr{C}}(\mathscr{D}, \mathscr{E})$ the set of morphisms f of from $\mathcal{D}$ to $A \otimes \mathcal{E}$ which commute with the coderivations such that the following diagrams commute.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & A \otimes \mathcal{E} \arrow[d, "Id_A \otimes \Delta_{\mathscr{E}}"] \\
\mathcal{C} \circ_{ns} \mathcal{D} \arrow[r] & \mathcal{C} \circ_{ns} (A \otimes \mathcal{E}) \arrow[r] & A \otimes (\mathcal{C} \circ_{ns} \mathcal{E})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "f"] \arrow[d, "\theta_{\mathcal{D}}"'] & A \otimes \mathcal{E} \arrow[d, "Id_A \otimes \theta_{\mathcal{E}}"] \\
\mathbb{K} \arrow[r] & A
\end{tikzcd}$$

Definition 21 (Nonsymmetric simplicial enrichment of coalgebras over a curved cooperad)**.** For any two $\mathscr{C}$-coalgebras $\mathscr{D}$ and $\mathscr{D}'$, let $\mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{D}')_n$ be the following simplicial set :

$$\mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{D}')_n := \hom_{\Lambda_n, \mathscr{C}}(\mathscr{D}, \mathscr{D}') .$$

This defines a simplicial enrichment of the category of $\mathscr{C}$-coalgebras over simplicial sets.

These simplicial enrichments are related to the enrichments over coassociative coalgebras that we described above.

Proposition 18. *For any two $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{B}$ and for any two $\mathscr{C}$-coalgebras $\mathscr{D}$ and $\mathscr{D}'$, we have functorial isomorphisms*

$$\mathrm{HOM}^{ns}(\mathscr{A},\mathscr{B}) \simeq N(\{\mathscr{A},\mathscr{B}\}^{ns}) \ ,$$

$$\mathrm{HOM}^{ns}(\mathscr{D},\mathscr{D}') \simeq N(\{\mathscr{D},\mathscr{D}'\}^{ns}) \ .$$

Proof. The proof for $\mathscr{P}$-algebras is straightforward. Let us prove the result for the $\mathscr{C}$-coalgebras. A morphism of graded $\mathbb{K}$-modules f from $\mathcal{D}$ to $\Lambda_n \otimes \mathcal{D}'$ is equivalent to a morphism from $\mathcal{D} \otimes DK^c(\Delta[n])$ to $\mathcal{D}'$. In that context, f belongs to $\hom_{\Lambda_n,\mathscr{C}}(\mathscr{D},\mathscr{D}')$ if and only if the corresponding morphism from $\mathcal{D} \otimes DK^c(\Delta[n])$ to $\mathcal{D}'$ is a morphism of $\mathscr{C}$-coalgebras. So

$$\begin{aligned}\mathrm{HOM}^{ns}(\mathscr{A},\mathscr{B})_n &:= \hom_{\Lambda_n,\mathscr{C}}(\mathscr{D},\mathscr{D}') \simeq \hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D} \otimes DK^c(\Delta[n]),\mathscr{D}') \\ &\simeq \hom_{\mathsf{uCog}}(DK^c(\Delta[n]),\{\mathscr{D},\mathscr{D}'\}^{ns}) \simeq \hom_{\mathsf{sSet}}(\Delta[n], N(\{\mathscr{D},\mathscr{D}'\}^{ns})) \ .\end{aligned}$$

□

2.4 Bar-cobar adjunctions

The usual bar-cobar adjunction relates nonunital algebras to non-counital conilpotent coalgebras, see [LV12, Chapter 2]. It can be extended to nonunital operads and conilpotent cooperads, see [GJ94]. Besides, as a direct consequence of the work of Hirsh and Millès, [HM12], there exists an adjunction a la bar-cobar relating unital algebras with curved conilpotent coalgebras. We extend it to operads and curved conilpotent cooperads.

The bar-cobar adjunction $\Omega_u \dashv B_c$ is a tool to compute resolutions of operads. But it has other aspects : any morphism of operad from the cobar construction $\Omega_u\mathscr{C}$ of a curved conilpotent cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ gives rise to a new adjunction à la bar cobar between $\mathscr{C}$-coalgebras and $\mathscr{P}$-algebras.

2.4.1 Operadic bar-cobar construction

The usual operadic bar-cobar adjunction (see [LV12, Chapter 6]) relates augmented operads to differential graded conilpotent cooperads. The bar construction $B\mathscr{P}$ of an operad $\mathscr{P}$ does use the augmentation of $\mathscr{P}$ as it is the graded cofree cooperad on the suspension of $\overline{\mathscr{P}}$. If $\mathscr{P}$ is not augmented, one can try to add an element to $\mathscr{P}$ whose boundary is the unit of $\mathscr{P}$ and try the same computation. This is the new bar-construction ; its output is no more a differential graded cooperad but a curved cooperad.

The new curved bar functor B_c has also a left adjoint Ω_u whose formula looks like the usual operadic cobar functor. Again, as in [LV12, Chapter 6], this adjunction is related to a notion of twisting morphism.

Definition 22 (Operadic bar construction)**.** The *operadic bar construction* of a dg operad $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1)$ is a curved conilpotent cooperad $B_c\mathscr{P} = (\mathbb{T}^c(s\mathcal{P} \oplus v), D, \theta)$ where $s\mathcal{P}$ is the suspension of the $\mathbb{S}$-module $\mathcal{P}$ and v is an arity 1, degree 2 element. It is equipped with the coderivation D which extends the following map from $\overline{\mathbb{T}}(s\mathcal{P} \oplus \mathbb{K}\cdot v)$ to $s\mathcal{P} \oplus \mathbb{K}\cdot v$:

$$\begin{aligned}\mathbb{T}(s\mathcal{P} \oplus v) \to \mathbb{T}^{\leq 2}(s\mathcal{P} \oplus v) &\to s\mathcal{P} \oplus v \\ sx &\mapsto -sd_{\mathcal{P}}x \\ sx \otimes sy &\mapsto (-1)^{|x|} s\gamma_{\mathscr{P}}(x \otimes y) \\ v &\mapsto s1 \ .\end{aligned}$$

It has the following curvature map :

$$\begin{aligned}\theta : \overline{\mathbb{T}}(s\mathcal{P} \oplus v) \to s\mathscr{P} \oplus \mathbb{K}\cdot v \to \mathbb{K}\cdot v &\to \mathbb{K} \\ v &\mapsto 1 \ .\end{aligned}$$

Proposition 19. *The map θ is actually a curvature for the coderivation, that is $D^2 = (\theta \otimes Id - Id \otimes \theta)\Delta_2$.*

Proof. Let π be the projection from $B_c\mathscr{P}$ to $s\mathcal{P}$. By Proposition 2, it suffices to prove that $\pi D^2 = (\theta \otimes \pi - \pi \otimes \theta)\Delta_2$. This is a straightforward calculation. $\square$

Definition 23 (Operadic cobar construction)**.** The *operadic cobar construction* of a curved conilpotent cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, \theta)$ is the dg operad $\Omega_u\mathscr{C} = (\mathbb{T}s^{-1}\mathcal{C}, D)$ where D is the following degree -1 derivation

$$s^{-1}x \mapsto \theta(x)1 - s^{-1}dx - \sum(-1)^{|x_{(1)}|}s^{-1}x_{(1)} \otimes s^{-1}x_{(2)} \ .$$

where $\Delta_2(x) = \sum x_1 \otimes x_2$.

Proposition 20. *The derivation D squares to zero.*

Proof. It suffices to prove the result for any element of the form $s^{-1}x$, which is a straightforward calculation. $\square$

Definition 24 (Operadic twisting morphism)**.** Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a curved conilpotent cooperad and let $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}}, d)$ be a dg operad. An *operadic twisting morphism* from $\mathscr{C}$ to $\mathscr{P}$ is a degree -1 map of $\mathbb{S}$-modules (or $\mathbb{N}$-modules in the nonsymmetric case) :

$$\alpha : \overline{\mathcal{C}} \to \mathcal{P}$$

such that

$$\partial(\alpha) + \gamma(\alpha \otimes \alpha)\Delta_2 = \Theta$$

where $\Theta(x) = \theta(x)1_{\mathscr{P}}$ for any $x \in \mathscr{C}$. We denote by $Tw(\mathscr{C}, \mathscr{P})$ the set of operadic twisting morphisms from $\mathscr{C}$ to $\mathscr{P}$.

Proposition 21. *We have the following functorial isomorphisms :*

$$\hom_{\mathsf{dg-Operad}}(\Omega_u\mathscr{C}, \mathscr{P}) \simeq Tw(\mathscr{C}, \mathscr{P}) \simeq \hom_{\mathsf{cCoop}}(\mathscr{C}, B_c\mathscr{P}) \ .$$

Proof. Proving the existence of the functorial isomorphism $\hom_{\mathsf{dg-Operad}}(\Omega_u\mathscr{C}, \mathscr{P}) \simeq Tw(\mathscr{C}, \mathscr{P})$ is similar to the proof of [HM12, 3.4.1]. Let us show that we have a functorial isomorphism $Tw(\mathscr{C}, \mathscr{P}) \simeq \hom_{\mathsf{cCoop}}(\mathscr{C}, B_c\mathscr{P})$. Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. We obtain a degree zero map from $\overline{\mathcal{C}}$ to $s\mathcal{P} \oplus \mathbb{K} \cdot v$ as follows :

$$\begin{aligned} \overline{\mathcal{C}} &\to s\mathcal{P} \oplus \mathbb{K} \cdot v \\ c &\mapsto s\alpha(x) + \theta_{\mathscr{C}}(x) \ . \end{aligned}$$

This induces a morphism of graded cooperads $f_\alpha : \mathscr{C} \to B_c\mathscr{P} = \mathbb{T}^c(s\mathcal{P} \oplus \mathbb{K} \cdot v)$ such that $\theta_{\mathscr{C}} = \theta_{B_c\mathscr{P}} f_\alpha$. Since $\partial(\alpha) + \gamma_{\mathscr{P}}(\alpha \otimes \alpha)\Delta_2 = \Theta$, then f_α commutes with the coderivations and so is a morphism of curved cooperads. Conversely, from any morphism of curved cooperads f from $\mathscr{C}$ to $B_c\mathscr{P}$, on obtains a twisting morphism as follows :

$$\mathscr{C} \xrightarrow{f} B_c\mathscr{P} \twoheadrightarrow s\mathscr{P} \longrightarrow \mathscr{P} \ .$$

The two constructions that we described are inverse to another. $\square$

Hence, the functors Ω_u and B_c realize an adjunction between the category of dg operads and the category of curved conilpotent cooperads.

$$\mathsf{cCoop} \underset{B_c}{\overset{\Omega_u}{\rightleftarrows}} \mathsf{dg-Operad}$$

2.4.2 Twisted products

Let $\alpha : \overline{\mathscr{C}} \to \mathscr{P}$ be an operadic twisting morphism.

Definition 25 (Twisted $\mathscr{P}$-module)**.** For any $\mathscr{C}$-comodule $\mathscr{D}$, let $\mathscr{P} \circ_\alpha \mathscr{D}$ be the free $\mathscr{P}^{grad}$-module $\mathcal{P} \circ \mathcal{D}$ equipped with the unique derivation which extends the map

$$\begin{aligned} \mathcal{D} &\to \mathcal{P} \circ \mathcal{D} \\ x &\mapsto d_{\mathcal{D}}(x) - (\alpha \circ Id)\Delta(x) \ . \end{aligned}$$

Definition 26 (Twisted $\mathscr{C}$-comodule)**.** For any $\mathscr{P}$-module $\mathscr{A}$, let $\mathscr{C} \circ_\alpha \mathscr{A}$ be the cofree $\mathscr{C}^{grad}$-comodule $\mathcal{C} \circ \mathcal{A}$ equipped with a unique coderivation which extends the map

$$\begin{aligned} \mathcal{C} \circ \mathcal{A} &\to \mathcal{A} \\ x &\mapsto \big(d_{\mathcal{A}}(\epsilon_{\mathscr{C}} \circ Id) + \gamma_{\mathscr{A}}(\alpha \circ Id)\big)(x) \ . \end{aligned}$$

Proposition 22. *The derivation of $\mathscr{P} \circ_\alpha \mathscr{D}$ squares to zero. Hence, $\mathscr{P} \circ_\alpha \mathscr{D}$ is a dg $\mathscr{P}$-module. Similarly, the coderivation of $\mathscr{C} \circ_\alpha \mathscr{A}$ squares to $(\theta \circ Id)\Delta$. Hence, $\mathscr{C} \circ_\alpha \mathscr{A}$ is a $\mathscr{C}$-comodule.*

Proof. To prove the first point, it suffices to show that $\pi D^2 = 0$, which is a straightforward calculation. To prove the second point, it suffices to show that $\pi D^2 = (\theta \circ Id)$, which is a straightforward calculation. □

Definition 27 (Twisting morphism relative to an operadic twisting morphism)**.** For any $\mathscr{C}$-comodule $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and any $\mathscr{P}$-module $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ an α-twisting morphism from $\mathscr{D}$ to $\mathscr{A}$ is a degree 0 map $\phi : \mathcal{D} \to \mathcal{A}$ such that

$$\partial(\phi) + \gamma_{\mathcal{A}}(\alpha \circ \phi)\Delta_{\mathcal{C}} = 0 \ .$$

We denote by $Tw_\alpha(\mathscr{D}, \mathscr{A})$ the set of α-twisting morphism from $\mathscr{D}$ to $\mathscr{A}$.

Proposition 23. *There are functorial isomorphisms*

$$\hom_{\mathscr{P}-\mathsf{mod}}(\mathscr{P} \circ_\alpha \mathscr{D}, \mathscr{A}) \simeq Tw_\alpha(\mathscr{D}, \mathscr{A}) \simeq \hom_{\mathscr{C}-\mathsf{comod}}(\mathscr{D}, \mathscr{C} \circ_\alpha \mathscr{A})$$

for any $\mathscr{C}$-comodule $\mathscr{D}$ and any $\mathscr{P}$-module $\mathscr{A}$.

Proof. The proof is similar to [LV12, 11.3.2]. □

2.4.3 Bar-cobar adjunction for algebras over an operad and coalgebras over a cooperad

Following [LV12, Chapter 11], we call the previous functors respectively the bar construction for $\mathscr{P}$-algebras and the cobar construction for $\mathscr{C}$-coalgebras.

Definition 28 (Bar construction and cobar construction relative to an operadic twisting morphism)**.** Let $\alpha : \overline{\mathscr{C}} \to \mathscr{P}$ be an operadic twisting morphism. The α-bar construction is the functor from $\mathscr{P}$-algebras to $\mathscr{C}$-coalgebras defined by :

$$B_\alpha \mathscr{A} := \mathscr{C} \circ_\alpha \mathscr{A} \ .$$

The α-cobar construction is the functor from $\mathscr{C}$-coalgebras to $\mathscr{P}$-algebras defined by :

$$\Omega_\alpha \mathscr{D} := \mathscr{P} \circ_\alpha \mathscr{D} \ .$$

We already know, by Proposition 23 that Ω_α is left adjoint to B_α. Moreover, this adjunction is enriched over cocommutative coalgebras and simplicial sets.

Proposition 24. *The functors Ω_α and B_α induces functorial isomorphisms of counital cocommutative coalgebras and of simplicial sets :*

$$\{\Omega_\alpha \mathscr{D}, \mathscr{A}\} \simeq \{\mathcal{C}, B_\alpha \mathscr{A}\} \ , \quad \mathrm{HOM}(\Omega_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{HOM}(\mathscr{D}, B_\alpha \mathscr{A}) \ ,$$

for any $\mathscr{C}$-coalgebra $\mathscr{D}$ and any $\mathscr{P}$-algebra $\mathscr{A}$;

Lemma 8. *We have a functorial isomorphism :*

$$Tw_\alpha(\mathscr{D}\otimes\mathscr{E},\mathscr{A}) \simeq Tw_\alpha(\mathscr{D},[\mathscr{E},\mathscr{A}]) \ .$$

for any $\mathscr{C}$-coalgebra $\mathscr{D}$, any $\mathscr{P}$-algebra $\mathscr{A}$ and any counital cocommutative coalgebra $\mathscr{E}$.

Proof. The set of morphisms of graded $\mathbb{K}$-modules from $\mathcal{D}\otimes\mathcal{E}$ to $\mathcal{A}$ is in bijection with the set of morphisms of graded $\mathbb{K}$-modules from $\mathcal{C}$ to $[\mathcal{D},\mathcal{A}]$. This bijection and its inverse preserve α-twisting morphisms. □

Proof of Proposition 24. On the one hand, for any cocommutative coalgebra $\mathscr{E}$, we have :

$$\begin{aligned}
\hom_{\mathsf{uCocom}}(\mathscr{E},\{\Omega_\alpha\mathscr{D},\mathscr{A}\}) &\simeq \hom_{\mathscr{P}-\mathsf{alg}}(\Omega_\alpha\mathscr{D},[\mathscr{E},\mathscr{A}])\\
&\simeq Tw_\alpha(\mathscr{D},[\mathscr{E},\mathscr{A}]) \simeq Tw_\alpha(\mathscr{D}\otimes\mathscr{E},\mathscr{A})\\
&\simeq \hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D}\otimes\mathscr{E},B_\alpha\mathscr{A})\\
&\simeq \hom_{\mathsf{uCocom}}(\mathscr{E},\{\mathscr{D},B_\alpha\mathscr{A}\}) \ .
\end{aligned}$$

On the other hand, there is a functorial isomorphism $\mathrm{HOM}(\Omega_\alpha\mathscr{D},\mathscr{A}) \simeq \mathrm{HOM}(\mathscr{D},B_\alpha\mathscr{A})$ because the bar-cobar adjunction still works when we work with Ω_n as base ring instead of $\mathbb{K}$.

□

2.5 Homotopy theory of algebras over an operad

In this section, we recall the result of Hinich stating that for any dg operad $\mathscr{P}$, the category of $\mathscr{P}$-algebras admits a projective model structure whose weak equivalences are quasi-isomorphisms (see [Hin97] and [BM03]). Moreover, we show that the simplicial enrichment of the category of $\mathscr{P}$-algebras that we described above gives models for the mapping spaces. Finally, we show that the enrichment over cocommutative coalgebras introduced in Section 2.3 encodes deformation of morphisms of $\mathscr{P}$-algebras.

2.5.1 Model structure on algebras over an operad

We recall here results about model structures on the categories of algebras over an operad.

Definition 29 (Right induced model structures)**.** Consider the following adjunction.

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

Suppose that C admits a cofibrantly generated model structure. We say that D admits a model structure *right induced* by the adjunction $L \dashv R$ if it admits a model structure whose weak equivalences (resp. fibrations) are the morphisms f such that $R(f)$ is a weak equivalence (resp. a fibration) and whose generating cofibrations (resp. generating acyclic cofibrations) are the images through L of the generating cofibrations (resp. generating acyclic cofibrations) of C.

Definition 30 (Admissible operad)**.** An operad $\mathscr{P}$ is said to be *admissible* if the category of $\mathscr{P}$-algebras admits a projective model structure, that is a model structure right induced by the adjunction

$$\mathsf{dgMod} \overset{\mathscr{P}\circ -}{\rightleftarrows} \mathscr{P}-\mathsf{alg}$$

whose right adjoint is the forgetful functor.

Theorem 7. *[Hin97] Any nonsymmetric operad is admissible. When the characteristic of the field $\mathbb{K}$ is zero, then any operad is admissible.*

2.5.2 Mapping spaces

The simplicial enrichments of the category of $\mathscr{P}$-algebras described above give us models for the mapping spaces.

Proposition 25. *Suppose that the characteristic of the field $\mathbb{K}$ is zero. Let $\mathscr{P}$ be a dg operad. The assignment $\mathscr{A}, \mathscr{A}' \mapsto \mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ (resp. $\mathscr{A}, \mathscr{A}' \mapsto \mathrm{HOM}(\mathscr{A}, \mathscr{A}')^{ns}$ in the nonsymmetric context) defines an homotopy enrichment of the category of $\mathscr{P}$-algebras over the category of simplicial sets. Moreover, for any cofibrant $\mathscr{P}$-algebra $\mathscr{A}$ and any fibrant $\mathscr{P}$-algebra $\mathscr{A}'$, the simplicial set $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ (resp. $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')^{ns}$ in the nonsymmetric context) is a model of the mapping space $Map(\mathscr{A}, \mathscr{A}')$.*

Remark 5. The characteristic zero assumption is not necessary in the nonsymmetric context.

Proof. Let $f : \mathscr{A}, \to \mathscr{A}'$ and $g : \mathscr{B} \to \mathscr{B}'$ be respectively a cofibration and a fibration of $\mathscr{P}$-algebras. Let $h : X \to Y$ be a monomorphism of simplicial sets. We suppose that X and Y are colimits of finite diagrams made up of simplicies $\Delta[n]$. Consider a square as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{HOM}(\mathscr{A}', \mathscr{B}) \arrow[d] \\ Y \arrow[r] & \mathrm{HOM}(\mathscr{A}', \mathscr{B}') \times_{\mathrm{HOM}(\mathscr{A}, \mathscr{B}')} \mathrm{HOM}(\mathscr{A}, \mathscr{B}) \end{tikzcd}$$

By Proposition 17, it induces the following square

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \Omega_Y \otimes \mathscr{B} \arrow[d] \\ \mathcal{A}' \arrow[r] & \Omega_Y \otimes \mathscr{B}' \times_{\Omega_X \otimes \mathscr{B}'} \Omega_X \otimes \mathscr{B} \end{tikzcd} ,$$

which has a lifting whenever f, g or h is a weak equivalence; indeed, by Propostion 7, the map $\Omega_Y \to \Omega_X$ is a fibration and it is an acyclic fibration whenever h is an acyclic cofibration. Besides, to prove that $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ is a model of the mapping space $Map(\mathscr{A}, \mathscr{A}')$, it suffices to notice that $\{\Omega_n \otimes \mathscr{A}'\}_{n \in \mathbb{N}}$ is a Reedy fibrant resolution of the constant simplicial $\mathscr{P}$-algebra $\mathscr{A}'$. The result in the nonsymmetric context can be proved in a similar way. $\square$

2.5.3 Deformation theory of morphism of algebras over an operad

We know that the category of $\mathscr{P}$-algebras is enriched over the category uCocom of cocommutative coalgebras. In this subsection, we show that for any $\mathscr{P}$-algebras $\mathscr{A}$ and $\mathscr{B}$, the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$ encodes the deformation theory of morphisms from $\mathscr{A}$ to $\mathscr{B}$. We suppose in this subsection that the field $\mathbb{K}$ is algebraically closed.

Any morphism of $\mathscr{P}$-algebras $f : \mathscr{A} \to \mathscr{B}$ defines a defomation problem $\mathrm{Def}(f)$.

$$\begin{aligned} &\mathsf{Artin-alg} \to \mathsf{sSet} \\ &\qquad R \mapsto \mathrm{Map}(\mathscr{A}, \mathscr{B} \otimes R) \times^h_{\mathrm{Map}(\mathscr{A}, \mathscr{B})} \{f\} . \end{aligned}$$

The following theorem is a direct consequence of a result by Chuang, Lazarev and Mannan ([CLM, Theorem 2.9]). It is proved in Appendix.

Theorem 8. *Suppose that the base field $\mathbb{K}$ is algebraically closed and that its characteristic is zero. Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a cocommutative coalgebra and let A be the set of graded atoms of $\mathscr{C}$. There exists a unique decomposition $\mathscr{C} \simeq \bigoplus_a \mathscr{C}_a$ where $\mathscr{C}_a = (\mathcal{C}_a, \Delta, \epsilon_a)$ is a sub-coalgebra of $\mathscr{C}$ which contains the atom a and which belongs to the category* uNilCocom*. Moreover, a morphism f of cocommutative coalgebras from $\mathscr{C} = \bigoplus_{a \in A} \mathscr{C}_a$ to $\mathscr{D} = \bigoplus_{b \in B} \mathscr{D}_b$ consists in an element $\phi(a) \in B$ and a morphism $f_a : \mathscr{C}_a \to \mathscr{D}_{\phi(a)}$ for any $a \in A$*

We know from Proposition 13 that a morphism f of $\mathscr{P}$-algebras from $\mathscr{A}$ to $\mathscr{B}$ is a dg atom of the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$. Applying Theorem 8 to the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$, we obtain the conilpotent cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$. This is in particular an Hinich coalgebra which encodes a deformation problem $R \mapsto \mathrm{Map}(R^*, \{\mathscr{A}, \mathscr{B}\}_f)$. We show in the next proposition that this deformation problem is $\mathrm{Def}(f)$.

Theorem 9. *Suppose that $\mathscr{A}$ is a cofibrant $\mathscr{P}$-algebra. Then, the deformation problem induced by the conilpotent cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$ is $Def(f)$.*

Lemma 9. *If $\mathscr{A}$ is a cofibrant $\mathscr{P}$-algebra, the simplicial Hinich coalgebra*

$$\{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f$$

is a Reedy fibrant replacement of the constant simplicial Hinich coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$.

Proof. Let $g : X \to Y$ be a monomorphism of simplicial sets which are finite colimits of standard simplicies $\Delta[n]$. Let $h : \mathscr{C}_1 \to \mathscr{C}_2$ be a monomorphism of Hinich coalgebras. Consider the following square :

$$\begin{tikzcd}
\mathscr{C}_1 \arrow[r] \arrow[d] & \{\mathscr{A}, \Omega_Y \otimes \mathscr{B}\}_f \arrow[d, "{\{\mathscr{A}, \Omega[g] \otimes \mathscr{B}\}}"] \\
\mathscr{C}_2 \arrow[r] & \{\mathscr{A}, \Omega_X \otimes \mathscr{B}\}_f .
\end{tikzcd}$$

Any morphism of cocommutative coalgebra from a conilpotent cocommutative coalgebra $\mathscr{C}$ to $\{\mathscr{A}, \mathscr{B}\}$ such that the atom of $\mathscr{C}$ targets the atom f of $\{\mathscr{A}, \mathscr{B}\}$ is a morphism from $\mathscr{C}$ to $\{\mathscr{A}, \mathscr{B}\}_f$. So, lifting the previous square amounts to lift the following square of $\mathscr{P}$-algebra.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & {[\mathscr{C}_2, \Omega_Y \otimes \mathscr{B}]} \arrow[d] \\
\mathscr{A} \arrow[r] & {[\mathscr{C}_1, \Omega_Y \otimes \mathscr{B}] \times_{[\mathscr{C}_1, \Omega_X \otimes \mathscr{B}]} [\mathscr{C}_2, \Omega_Y \otimes \mathscr{B}]}
\end{tikzcd}$$

This is possible whenever, g or h is a weak equivalence, since any weak equivalence of Hinich coalgebras is in particular a quasi-isomorphism. So, in particular, for any face map $\{\mathscr{A}, \Omega_{n+1} \otimes \mathscr{B}\} \to \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}$ is an acyclic fibration of Hinich coalgebra and for any integer $n \in \mathbb{N}$, the morphism $\{\mathscr{A}, \Omega_n \otimes \mathscr{B}\} \to \{\mathscr{A}, \Omega_{\partial\Delta[n]} \otimes \mathscr{B}\}$ is a fibration. □

Proof of Theorem 9. By Lemma 9, the deformation problem induced by the Hinich coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$ is equivalent to the following deformation problem :

$$R \in \mathsf{Artin-alg} \mapsto (\hom_{\mathsf{Hinich-cog}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f)_{n \in \mathbb{N}} .$$

We have :

$$\begin{aligned}
\hom_{\mathsf{Hinich-cog}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f) &\simeq \hom_{\mathsf{uCocom}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}) \times_{\hom_{\mathsf{uCocom}}(\mathbb{K}, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\})} \{f\} \\
&\simeq \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A} \lhd R^*, \Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{B})} \{f\} \\
&\simeq \hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, R \otimes \Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{B})} \{f\} .
\end{aligned}$$

Since the simplicial sets $(\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, R \otimes \Omega_n \otimes \mathscr{B})_{n \in \mathbb{N}}$ and $(\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{B}))_{n \in \mathbb{N}}$ are Kan complexes and models of respectively $\mathrm{Map}(\mathscr{A}, R \otimes \mathscr{B})$ and $\mathrm{Map}(\mathscr{A}, \mathscr{B})$ and since the map between them is a fibration, then the simplicial set

$$(\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, R \otimes \Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}-\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{B})} \{f\})_{n \in \mathbb{N}}$$

is a model of the homotopy pullback $\mathrm{Map}(\mathscr{A}, R \otimes \mathscr{B}) \times^h_{\mathrm{Map}(\mathscr{A}, \mathscr{B})} \{f\}$. □

2.6 Model structures on coalgebras over a cooperad

In this section, we show, that for any operadic twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$, the projective model structure on the category of $\mathscr{P}$-algebras can be transferred through the cobar construction functor Ω_α to the category of $\mathscr{C}$-coalgebras. This result is in the vein of similar results by Hinich [Hin01], Lefevre-Hasegawa [LH03], Vallette [Val14] and Positselski [Pos11]. However, we use a new method for the proof that uses the presentability of the category of algebras over an operad and of the category of coalgebras over a curved conilpotent cooperad ; specifically, we use a theorem proved by Bayeh, Hess, Karpova, Kedziorek, Riehl and Shipley in [BHK+15] and [HKRS15].

2.6.1 Model structure induced by a twisting morphism

Definition 31 (Left induced model structures)**.** Consider the following adjunction.

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

Suppose that D admits a model structure. We say that C admits a model structure left induced by the adjunction $L \dashv R$ if it admits a model structure whose weak equivalences (resp. cofibrations) are the morphisms f such that $L(f)$ is a weak equivalence (resp. a cofibration).

Here is the main theorem of the present chapter.

Theorem 10. *Let $\mathscr{P}$ be a dg operad, let $\mathscr{C}$ be a curved conilpotent cooperad and let α be an operadic twisting morphism between them. Suppose that the field $\mathbb{K}$ is of characteristic zero. We know that the category of $\mathscr{P}$-algebras admits a projective model structure. Then, the category of $\mathscr{C}$-coalgebras admits a model structure left induced by the adjunction $\Omega_\alpha \dashv B_\alpha$. We call it the α-model structure. In the nonsymmetric context, we can drop the assumption that the characteristic of the field $\mathbb{K}$ is zero.*

To prove this theorem, we will use the following result.

Theorem 11. *[BHK+15][HKRS15] Consider an adjunction*

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{M}$$

between presentable categories. Suppose that M is endowed with a cofibrantly generated model structure. Then, there exists a left induced model structure on C if the morphisms which have the right lifting property with respect to left induced cofibrations are left induced weak equivalences. In particular, this is true if the category C has a cofibrant replacement functor, and if any cofibrant object has a cylinder.

From now on, a *weak equivalence* (resp. *cofibration*) of $\mathscr{C}$-coalgebras is a morphism whose image under Ω_α is a weak equivalence (resp. cofibration). An *acyclic cofibration* is a morphism which is both a cofibration and a weak equivalence. A *fibration* is a morphism which has the right lifting property with respect to all acyclic cofibration and an *acyclic fibration* is a morphism which is both a fibration and a weak equivalence. Here is the proof.

Proof of Theorem 10. Proposition 26 ensures us that the cofibrations of the category of $\mathscr{C}$-coalgebras are the monomorphisms. Hence, any object is cofibrant. Then, Propostion 28 provides us with a cylinder for any object if the characteristic of $\mathbb{K}$ is zero. In the nonsymmetric context, Proposition 29 provides us with a cylinder. We conclude by Theorem 11. □

2.6.2 Cofibrations

Proposition 26. *The class of cofibrations of $\mathscr{C}$-coalgebras is the class of monomorphisms.*

Lemma 10. *Let $f : \mathscr{D} \to \mathscr{E}$ be a monomorphism of $\mathscr{C}$-coalgebras such that $\Delta(\mathcal{E}) \subset \mathcal{C} \circ \mathcal{D}$. Then, f is a cofibration.*

Proof. We can decompose the graded $\mathbb{K}$-module $\mathcal{E}$ as $\mathcal{E} = \mathcal{D} \oplus \mathcal{F}$. The coderivation $d_{\mathcal{E}}$ corresponds then to the following matrix.

$$\begin{pmatrix} d_{\mathcal{D}} & \phi \\ 0 & d_{\mathcal{F}} \end{pmatrix}$$

Consider the following diagram of $\mathscr{P}$-algebras :

$$\begin{tikzcd} \mathcal{P} \circ (s^{-1}\mathcal{F}) \arrow[r] \arrow[d, hook, "\mathcal{P}(\phi)"] & \Omega_\alpha \mathcal{D} \\ \mathcal{P} \circ (s^{-1}\mathcal{F} \oplus \mathcal{F}) & \end{tikzcd}$$

where the horizontal map sends $s^{-1}x$ to $\phi(x) + (\alpha \circ Id)\Delta(x)$. Then, the vertical map is a cofibration and f is the pushout of this vertical map along the horizontal map. Hence, f is a cofibration. $\square$

Proof of Proposition 26. Let $f : \mathscr{D} \to \mathscr{E}$ be a cofibration. Then $\Omega_\alpha(f)$ it is a monomorphism. Since the following square is commutative, then f is a monomorphism.

$$\begin{tikzcd} \Omega_\alpha \mathscr{D} \arrow[r] & \Omega_\alpha \mathscr{E} \\ \mathscr{D} \arrow[r] \arrow[u] & \mathscr{E} \arrow[u] \end{tikzcd}$$

Conversely, if f is a monomorphism, then, it can be decomposed into the transfinite composition of the maps $f_n = \mathscr{D} + F_{n-1}\mathscr{E} \to \mathscr{D} + F_n\mathscr{E}$. Since the maps f_n satisfy the conditions of Lemma 10, they are cofibrations. So f is a cofibration. $\square$

2.6.3 Filtered quasi-isomorphism

Definition 32 (Filtered quasi-isomorphism)**.** Let $\mathscr{D}$ and $\mathscr{E}$ be two $\mathscr{C}$-coalgebras. A morphism of $\mathscr{C}$-coalgebras f from $\mathscr{D}$ to $\mathscr{E}$ is said to be a *filtered quasi-isomorphism* if the induced morphisms between the graded complexes relative to the coradical filtrations are quasi-isomorphisms, that is, if for any integer n, the morphism from $G_n^{rad}\mathscr{D}$ to $G_n^{rad}\mathscr{E}$ is a a quasi-isomorphism.

Proposition 27. *If the characteristic of $\mathbb{K}$ is zero, a filtered quasi-isomorphism is a weak equivalence of $\mathscr{C}$-coalgebras. The characteristic zero assumption is not necessary in the nonsymmetric context.*

We will use the following classical result.

Theorem 12. *[ML95, XI.3.4] Let $f : A \to B$ be a map of filtered chain complexes. Suppose that the filtrations are bounded below and exhaustive. If for any integer n, the map $G_nA \to G_nB$ is a quasi-isomorphism, then f is a quasi-isomorphism.*

Proof of Proposition 27. Consider the following filtration on $\Omega_\alpha\mathscr{D}$ (resp. $\Omega_\alpha\mathscr{E}$)

$$F_n\Omega_\alpha\mathscr{D} = \mathcal{P}(0) \oplus \sum_{\substack{k\geq 1 \\ p_1+\cdots+p_k=n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} \left(F_{p_1}^{rad}\mathcal{D} \otimes \cdots \otimes F_{p_k}^{rad}\mathcal{D}\right) .$$

It is clear that $\Omega_\alpha(f)$ sends $F_n\Omega_\alpha\mathscr{D}$ to $F_n\Omega_\alpha\mathscr{E}$ for any integer n. Then,

$$G_n\Omega_\alpha\mathscr{D} = \sum_{\substack{k\geq 1 \\ p_1+\cdots+p_k=n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} \left(G_{p_1}^{rad}\mathcal{D} \otimes \cdots \otimes G_{p_k}^{rad}\mathcal{D}\right)$$

Then, by the operadic Kunneth formula, $G_n(\Omega_\alpha(f)) : G_n\Omega_\alpha\mathscr{D} \to G_n\Omega_\alpha\mathscr{E}$ is a quasi-isomorphism for any $n \in \mathbb{N}$. Hence, by Theorem 22, $\Omega_u(f)$ is a quasi-isomorphism. $\square$

Remark 6. The coradical filtration is not the only filtration whose notion of filtered quasi-isomorphism gives us weak equivalences. An exhaustive filtration $(F_n\mathcal{D})_{n\in\mathbb{N}}$ is called *admissible* if :

$$\begin{cases} \Delta(F_n\mathcal{D}) \subset \sum_{p_0+p_1+\cdots+p_k=n} F_0^{rad}\mathcal{C} \otimes_{\mathbb{S}_k} \left(F_{p_1}\mathcal{D} \otimes \cdots \otimes F_{p_k}\mathcal{D}\right) , \\ d(F_n\mathcal{D}) \subset F_n\mathcal{D} , \\ d^2(F_n\mathcal{D}) \subset F_{n-1}\mathcal{D} . \end{cases}$$

2.6.4 Cylinder object

Proposition 28. *Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a $\mathscr{C}$-coalgebra. Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ be a cylinder of $\Omega_\alpha(\mathscr{D})$ such that the structural map $p : \mathscr{A} \to \Omega_\alpha(\mathscr{D})$ is an acyclic fibration. The following diagram*

$$\begin{tikzcd}
B_\alpha\Omega_\alpha(\mathscr{D}\oplus\mathscr{D}) \arrow[r] & B_\alpha(\mathscr{A}) \arrow[r, "B_\alpha p"] & B_\alpha\Omega_\alpha(\mathscr{D}) \\
\mathscr{D}\oplus\mathscr{D} \arrow[r] \arrow[u] & \mathscr{E} \arrow[r] \arrow[u] & \mathscr{D} \arrow[u]
\end{tikzcd}$$

where $\mathscr{E} := B_\alpha(\mathscr{A}) \times_{B_\alpha(\Omega_\alpha\mathscr{D})} \mathscr{D}$, provides us with a cylinder $\mathscr{E} = (\mathcal{E}, \Delta_{\mathscr{E}})$ for the $\mathscr{C}$-coalgebra $\mathscr{D}$.

Lemma 11. *The pullback $\mathscr{E}$ is the biggest sub-graded $\mathscr{C}^{grad}$-coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of the morphism $\mathscr{D} \to B_\alpha\Omega_\alpha\mathscr{D}$.*

Proof. Let $\mathscr{F} = (\mathcal{F}, \Delta_{\mathscr{F}})$ be the biggest sub-graded $\mathscr{C}^{grad}$-coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of the morphism $\mathscr{D} \to B_\alpha\Omega_\alpha\mathscr{D}$. Proving that $\mathscr{F}$ is the underlying $\mathscr{C}^{grad}$-coalgebra of $\mathscr{E}$ amounts to prove that $\mathcal{F}$ is stable under the coderivation D of $B_\alpha\mathscr{A}$. We will prove it by induction on the coradical filtration of $\mathcal{F}$. First, by the maximality property of $\mathscr{F}$, $F_0^{rad}\mathcal{F}$ is stable under D. Then suppose that $F_n^{rad}\mathcal{F}$ is stable under D for an integer $n \geq 0$. Let x be an element of $F_{n+1}^{rad}\mathcal{F}$. On the one hand $B_\alpha(p)D(x) = D(B_\alpha(p)(x))$. Since $B_\alpha(p)(x)$ is in the image of $\mathscr{D}$ and since this image is stable under the coderivation of $B_\alpha\Omega_\alpha\mathscr{D}$, then $B_\alpha(p)D(x)$ is the image of $\mathscr{D}$. On the other hand, we have

$$\Delta(D(x)) = (d_{\mathcal{C}} \circ Id + Id \circ' D)\Delta(x)$$

So, since $\Delta(x) - 1_{\mathcal{C}} \otimes x \in \mathcal{C} \circ (F_n^{rad}\mathcal{F})$, and since $F_n^{rad}\mathcal{F}$ is stable under D by the inductivity assumption, then

$$\Delta(D(x)) - 1_{\mathscr{C}} \otimes D(x) \in \mathscr{C} \circ (F_n^{rad}\mathcal{F}) \ .$$

By these two points, $\mathcal{F} + \mathbb{K} \cdot D(x)$ is a sub- graded $\mathscr{C}^{grad}$-coalgebra of $B_\alpha(\mathscr{A})$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of $\mathscr{D}$. By the maximality property of $\mathscr{F}$, then $D(x) \in \mathcal{F}$. So, $F_{n+1}^{rad}\mathcal{F}$ is stable under D. Hence, by induction $\mathcal{F}$ is stable under D. ☐

To prove Proposition 28, we will show that the pullback map $\mathscr{E} \to \mathscr{D}$ is a filtered quasi-isomorphism. Since $\Omega_\alpha\mathscr{D}$ is a cofibrant $\mathscr{P}$-algebra, there exists a right inverse $q : \Omega_\alpha\mathscr{D} \to \mathscr{A}$ to the acyclic fibration $p : \mathscr{A} \to \Omega_\alpha\mathscr{D}$. Then, let us decompose $\mathcal{A}$ as $\mathcal{A} = \Omega_\alpha\mathscr{D} \oplus K$. The chain complex K is acyclic. So, let $h : K \to K$ be a degree 1 map such that $\partial(h) = Id_K$. It can be extended to a map

$$\begin{aligned} B_\alpha\mathscr{A} \twoheadrightarrow \mathcal{A} \twoheadrightarrow K &\to \mathcal{A} \\ x &\mapsto h(x) \ . \end{aligned}$$

The zero map is a coderivation on the graded cooperad $\mathscr{C}^{grad}$. Then, let D_h be the degree 1 coderivation of $(B_\alpha\mathscr{A})^{grad}$ relative to the zero coderivation on $\mathscr{C}^{grad}$ and whose projection on $\mathcal{A}$ is h. In other words, $D_h = Id_{\mathscr{C}} \circ' h$.

Lemma 12. *The sub-$\mathscr{C}$-coalgebra $\mathscr{E}$ of $B_\alpha\mathscr{A}$ is stable under D_h.*

Proof. By Lemma 11, it suffices to prove that the biggest sub-graded $\mathscr{C}^{grad}$-coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ lies inside $\mathscr{D}$, is stable under D_h. To that purpose, we use the same arguments as in the proof of Lemma 11 and the fact that $B_\alpha(p)D_h = 0$. ☐

Proof of Proposition 28. Since the map $\mathscr{D} \oplus \mathscr{D} \to \mathscr{E}$ is a monomorphism and so a cofibration, it suffices to show that the map $\mathscr{E} \to \mathscr{D}$ is a weak equivalence. We will show that it is a filtered quasi-isomorphism. Let $n \in \mathbb{N}$; let us show that the map $G_n\mathcal{E} \to G_n\mathcal{D}$ is a quasi-isomorphism. To that purpose, consider the following filtration on $B_\alpha\mathscr{A}$:

$$F'_k B_\alpha\mathscr{A} := \sum_{i \leq k} \mathcal{C} \otimes_{\mathbb{S}} (K^{\otimes i} \otimes (\Omega_\alpha\mathscr{D})^{\otimes j})$$

This filtration is stable under the coderivations d and D_h and it induces a filtration on $G_n^{rad}\mathcal{E}$. It is clear that the morphism $G'_0 G_n^{rad}\mathcal{E} \to G_n^{rad}\mathcal{D}$ is an isomorphism. Moreover, for any integer $k \geq 1$, $\partial(D_h) = k.Id$ on $G'_k G_n^{rad}\mathcal{E}$. Since the characteristic of $\mathbb{K}$ is zero, $G'_k G_n^{rad}\mathcal{E}$ is acyclic. By Theorem 22, the map $G_n\mathcal{E} \to G_n\mathcal{D}$ is a quasi-isomorphism. $\square$

Proposition 29. *In the nonsymmetric context, $\mathscr{D} \otimes DK^c(\Delta[1])$ provides us with a cylinder for the $\mathscr{C}$-coalgebra $\mathscr{D}$.*

Proof. Since $G_n^{rad}(\mathcal{D} \otimes DK^c(\Delta[1])) = G_n^{rad}(\mathcal{D}) \otimes DK^c(\Delta[1])$ and since the map $DK^c(\Delta[1]) \to \mathbb{K}$ is a quasi-isomorphism, then $\mathcal{D} \otimes DK^c(\Delta[1]) \to \mathcal{D}$ is a filtered quasi-isomorphism. $\square$

2.6.5 Simplicial enrichment in the nonsymmetric context

Proposition 30. *In the nonsymmetric context, the assignment $\mathscr{D}, \mathscr{D}' \mapsto \mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{D}')$ defines an homotopy enrichment of the category of $\mathscr{C}$-coalgebras together its α-model structure over the category of simplicial sets. Moreover, if $\mathscr{D}'$ is fibrant $\mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{D}')$ provides a model for the mapping space* $\mathrm{Map}(\mathscr{D}, \mathscr{D}')$.

Proof. Let $f : \mathscr{D} \to \mathscr{D}'$ be a cofibration of $\mathscr{C}$-coalgebras, let $g : \mathscr{E} \to \mathscr{E}'$ be a fibration of $\mathscr{C}$-coalgebras and let $h : X \to Y$ be a cofibration (i.e. a monomorphism) of simplicial sets. Consider the following square.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{HOM}^{ns}(\mathscr{D}', \mathscr{E}) \arrow[d] \\ Y \arrow[r] & \mathrm{HOM}^{ns}(\mathscr{D}', \mathscr{E}') \times_{\mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{E}')} \mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{E}) \end{tikzcd}$$

It induces a square

$$\begin{tikzcd} \mathscr{D}' \otimes DK^c(X) \coprod_{\mathscr{D} \otimes DK^c(X)} \mathscr{D} \otimes DK^c(Y) \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ \mathscr{D}' \otimes DK^c(Y) \arrow[r] & \mathscr{E}' . \end{tikzcd}$$

The left vertical map is a monomorphism and so a cofibration.

- ▷ If the morphism $g : \mathscr{E} \to \mathscr{E}'$ is an acyclic fibration, then the square has a lifting.
- ▷ Suppose that the morphism $h : X \to Y$ is an acyclic cofibration. Then the morphism $\mathscr{D} \otimes DK^c(X) \to \mathscr{D} \otimes DK^c(Y)$ is a filtered quasi-isomorphism and a cofibration, so it is an acyclic cofibration. Hence, its pushout $\mathscr{D}' \otimes DK^c(X) \to \mathscr{D}' \otimes DK^c(X) \coprod_{\mathscr{D} \otimes DK^c(X)} \mathscr{D} \otimes DK^c(Y)$ is also an acyclic cofibration. Moreover, the map $\mathscr{D}' \otimes DK^c(X) \to \mathscr{D}' \otimes DK^c(Y)$ is a filtered quasi-isomorphism and so a weak equivalence. So, by the 2-out-of-3 rule, the morphism $\mathscr{D}' \otimes DK^c(X) \coprod_{\mathscr{D} \otimes DK^c(X)} \mathscr{D} \otimes DK^c(Y) \to \mathscr{D}' \otimes DK^c(Y)$ is a weak equivalence. Since, it is a cofibration, it is an acyclic cofibration and the square has a lifting.
- ▷ Suppose that the morphism $f : \mathscr{D} \to \mathscr{D}'$ is an acyclic cofibration. Then, the morphism $\mathscr{D} \otimes DK^c(X) \to \mathscr{D}' \otimes DK^c(X)$ is an acyclic cofibration. This is a consequence of the fact that, $\Omega_\alpha(\mathscr{D} \otimes DK^c(X)) = (\Omega_\alpha \mathscr{D}) \lhd DK^c(X)$, and that for any fibration of $\mathscr{P}$-algebras $\mathscr{A} \to \mathscr{A}'$, the morphism $[DK^c(X), \mathscr{A}] \to [DK^c(X), \mathscr{A}']$ is also a fibration. Then, the same arguments as in the previous point show us that the morphism $\mathscr{D}' \otimes DK^c(X) \coprod_{\mathscr{D} \otimes DK^c(X)} \mathscr{D} \otimes DK^c(Y) \to \mathscr{D}' \otimes DK^c(Y)$ is an acyclic cofibration and so the square has a lifting.

In particular, we have proved that any coface map $\mathscr{D} \otimes DK^c(\Delta[n]) \to \mathscr{D} \otimes DK^c(\Delta[n+1])$ is an acyclic cofibration. Moreover, for any integer n, the morphism $\mathscr{D} \otimes DK^c(\partial\Delta[n]) \to \mathscr{D} \otimes DK^c(\Delta[n])$ is a cofibration. So, the cosimplicial $\mathscr{C}$-coalgebra $(\mathscr{D} \otimes DK^c(\Delta[n]))_{n \in \mathbb{N}}$ is a Reedy cofibrant replacement of the constant cosimplicial $\mathscr{C}$-coalgebra $\mathscr{D}$. So, if $\mathscr{D}'$ is fibrant the simplicial set $(\hom_{\mathscr{C}-\mathsf{cog}}(\mathscr{D} \otimes DK^c(\Delta[n]), \mathscr{D}'))_{n \in \mathbb{N}}$ which is isomorphic to $\mathrm{HOM}^{ns}(\mathscr{D}, \mathscr{D}')$ is a model for the mapping space $\mathrm{Map}(\mathscr{D}, \mathscr{D}')$. $\square$

2.6.6 Changing of operads and cooperads

In this subsection, we explore how the left induced model structure on coalgebras over a curved conilpotent cooperad is modified when we change the underlying operadic twisting morphism. This is inspired by [DCH16] where a similar study is done in the context augmented dg operads and dg conilpotent cooperads.

Recall first that a morphism of dg operads $f : \mathscr{P} \to \mathscr{Q}$ induces an adjunction between their categories of algebras

$$\mathscr{P} - \mathsf{alg} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathscr{Q} - \mathsf{alg}$$

whose right adjoint f^* sends a $\mathscr{Q}$-algebra $\mathscr{A}$ to the same underlying chain complex. This adjunction is a Quillen adjunction with respect to the projective model structures ; see [BM03]. Similar things happen for coalgebras over curved conilpotent cooperads.

Proposition 31. *Let $f : \mathscr{C} \to \mathscr{D}$ be a morphism of curved conilpotent cooperads. It induces an adjunction between their categories of coalgebras*

$$\mathscr{C} - \mathsf{cog} \underset{f^!}{\overset{f_*}{\rightleftarrows}} \mathscr{D} - \mathsf{cog}$$

whose left adjoint f_* sends a $\mathscr{C}$-coalgebra $\mathscr{E}$ to the same underlying graded $\mathbb{K}$-module.

Proof. Let $\mathscr{E} = (\mathcal{E}, \Delta, d)$ be a $\mathscr{C}$-coalgebra. It has a structure of $\mathscr{D}$-coalgebra defined by the composite map

$$\mathcal{E} \xrightarrow{\Delta} \mathscr{C} \circ \mathcal{E} \xrightarrow{f \circ Id} \mathscr{D} \circ \mathcal{E} \ .$$

This defines the functor f_*. Since it preserves colimits and since the category of $\mathscr{C}$-coalgebras and the category of $\mathscr{D}$-coalgebras are presentable, then f_* has a right adjoint by Proposition 1. $\square$

Besides, let us fix a dg operad $\mathscr{P}$. The canonical operadic twisting morphism $\pi : B_c\mathscr{P} \to \mathscr{P}$ is universal in the sense that any operadic twisting morphism α from a curved conilpotent cooperad $\mathscr{C}$ to $\mathscr{P}$ is equivalent to a morphism of curved cooperad f from $\mathscr{C}$ to $B_c\mathscr{P}$; then, $\alpha = \pi f$. In that context, the cobar functor Ω_α can be decomposed as $\Omega_\alpha = \Omega_\pi\ f_*$, and the α-model structure on the category of $\mathscr{C}$-coalgebra is the model structure left induced by the π-model structure on the category of $B_c\mathscr{P}$-coalgebras.

On the other hand, let fix a curved conilpotent cooperad $\mathscr{C}$. The canonical operadic twisting morphism $\iota : \mathscr{C} \to \Omega_u\mathscr{C}$ is universal in the sense that any operadic twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$ is equivalent to the data of a morphism of operads f from $\Omega_u\mathscr{C}$ to $\mathscr{P}$; then $\alpha = f\iota$. A direct consequence of the following proposition is that the model structure on $\mathscr{C}$-coalgebras induced by the universal operadic twisting morphism $\iota : \mathscr{C} \to \Omega_u\mathscr{C}$ is universal in the sense that any α-model structure is a left Bousfield localization of this ι-model structure.

Proposition 32. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism and let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of dg operads. The $(f\alpha)$-model structure on the category of $\mathscr{C}$-coalgebras is the left Bousfield localization of the α-model structure with respect to $(f\alpha)$-weak equivalences. Moreover, if the Quillen adjunction $f_! \dashv f^*$ is a Quillen equivalence, the $(f\alpha)$-model structure coincides with the α-model structure.*

Proof. The cofibrations of the α-model structure and the cofibrations of the $(f\alpha)$-model structure are both the monomorphisms. Moreover, the functor $f_!$ is a left Quillen adjoint functor. So, for any α-weak equivalence g, since $\Omega_\alpha(g)$ is a weak equivalence between cofibrant objects, then $\Omega_{(f\alpha)}(g) = f_!\Omega_\alpha(g)$ is a weak equivalence. So the α-weak equivalences are in particular $(f\alpha)$-weak equivalences. So is proven that the $(f\alpha)$-model structure is a left Bousfield localization of the α-model structure. Suppose now that the adjunction $f_! \dashv f^*$ is a Quillen equivalence. Then for any $\mathscr{C}$-coalgebra $\mathscr{E}$, the morphism

$$\Omega_\alpha\mathscr{E} \to f^* f_! \Omega_\alpha\mathscr{E} = f^*\Omega_{\zeta\alpha}\mathscr{E}$$

is a quasi-isomorphism. Since the functor f^* is the identity on the underlying chain complexes, the following commutative square ensures us that a morphism $g : \mathscr{E} \to \mathscr{E}'$ of $\mathscr{C}$-coalgebras is a α-weak equivalence if and only if it is an $f\alpha$-weak equivalence.

$$\begin{tikzcd}
f^*\Omega_{(f\alpha)}\mathscr{E} \arrow[r, "f^*\Omega_{(f\alpha)}(g)"] & f * \Omega_{(f\alpha)}\mathscr{E}' \\
\Omega_\alpha \mathscr{E} \arrow[u] \arrow[r, "\Omega_\alpha(g)"'] & \Omega_\alpha \mathscr{E}' \arrow[u] .
\end{tikzcd}$$

□

2.7 The universal model structure

In the previous section, we studied model structures on categories of coalgebras over a curved conilpotent cooperad which are induced by an operadic twisting morphism α. In this section, we will investigate the particular case where the operadic twisting morphism is the universal twisting morphism $\iota : \mathscr{C} \to \Omega_u \mathscr{C}$ for any curved conilpotent cooperad $\mathscr{C}$. This model structure is universal in the sense that, for any operadic twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$, the α-model structure on the category of $\mathscr{C}$-coalgebras is obtained from the ι-model structure by Bousfield localization. We will show that the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence, that the fibrant $\mathscr{C}$-coalgebras in the ι-model structure are the images of the $\Omega_u\mathscr{C}$-algebras under the functor B_ι, and we will describe the cofibrations, the weak equivalences and the fibrations between them. Moreover, we will prove that enrichment of $\mathscr{C}$-coalgebras over simplicial sets computes the mapping spaces expected by the model structure.

We suppose here that the characteristic of the field $\mathbb{K}$ is zero. This assumption is not necessary in the nonsymmetric context.

2.7.1 Quillen equivalence

Theorem 13. *The adjunction $\Omega_\iota \dashv B_\iota$ relating $\mathscr{C}$-coalgebras to $\Omega_u\mathscr{C}$-algebras is a Quillen equivalence.*

Proof. Let us show that for any $\Omega_u\mathscr{C}$-algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, the map $\Omega_\iota B_\iota \mathscr{A} = \Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \to \mathscr{A}$ is a quasi isomorphism. The coradical filtration of $\mathscr{C}$ induces a filtration on $\Omega_u\mathscr{C}$:

$$\begin{aligned}
&F_0\Omega_u\mathscr{C} := \mathbb{K}.1 \\
&F_n\Omega_u\mathscr{C} := \mathbb{K}.1 \oplus \sum_{\substack{i_1+\ldots+i_k=n\\ k\geq 1}} s^{-1}F^{rad}_{i_1}\overline{\mathcal{C}} \otimes \cdots \otimes s^{-1}F^{rad}_{i_k}\overline{\mathcal{C}} \text{ ,for } n \geq 1 \ .
\end{aligned}$$

It induces a filtration on $\Omega_u\mathscr{C} \circ_\iota \mathscr{C}$ and on $\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A}$:

$$\begin{aligned}
&F_n(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) := F_n\Omega_u\mathscr{C}(0) \oplus \sum_{\substack{i_0+\cdots+i_k=n\\ k\geq 1}} F_{i_0}(\Omega_u\mathscr{C})(k) \otimes_{\mathbb{S}_k} \left(F^{rad}_{i_1}\mathcal{C} \otimes \cdots \otimes F^{rad}_{i_k}\mathcal{C}\right) , \\
&F_n(\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A}) := F_n(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) \circ \mathcal{A} \ .
\end{aligned}$$

Then $G(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) = \Omega_u(G\mathscr{C}) \circ_{G\iota} G\mathscr{C}$. By [LV12, Lemma 6.5.14], the map $\Omega_u(G\mathscr{C}) \circ_{G\iota} G\mathscr{C} \to \mathcal{I}$ is a quasi-isomorphism. So, the map

$$G(\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathcal{A}) \to G\mathcal{A}$$

is a quasi-isomorphism (here $G\mathcal{A}$ is the graded complex corresponding to the constant filtration $F_n\mathcal{A} = \mathcal{A}$). Hence, by Theorem 22, the map $\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. Besides, for any $\mathscr{C}$-coalgebra $\mathscr{D}$, the morphism $\mathscr{D} \to B_\iota\Omega_\iota\mathscr{D}$ is a weak equivalence by the 2-out-of-3 and by the fact that the map $\Omega_\iota B_\iota \Omega_\iota \mathscr{D} \to \Omega_\iota\mathscr{D}$ is a quasi-isomorphism. Hence, the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. □

2.7.2 Fibrant objects

The purpose of this subsection is to describe the fibrant objects of the ι-model structure.

Definition 33 (Quasi-cofree $\mathscr{C}$-coalgebras)**.** A $\mathscr{C}$-coalgebra is said to be *quasi-cofree* if its underlying $\mathscr{C}^{grad}$-coalgebra is cofree, that is of the form $\mathscr{C}^{grad} \circ \mathcal{V}$. A morphism of quasi-cofree $\mathscr{C}$-coalgebras $F : \mathscr{C} \circ \mathcal{V} \to \mathscr{C} \circ \mathcal{W}$ is said to be *strict* if there exists a map $f : \mathcal{V} \to \mathcal{W}$ such that $F = Id \circ f$.

Proposition 33. *The quasi-cofree $\mathscr{C}$-coalgebras and the strict morphisms are the images of the functor B_ι.*

Proof. The images of the functor B_ι are in particular quasi-cofree $\mathscr{C}$-coalgebras and strict morphisms. Conversely, let $\mathscr{D} := \mathscr{C} \circ \mathcal{A}$ be a quasi-cofree $\mathscr{C}$-coalgebra. Its coderivation extends the degree -1 map $d_{\mathcal{A}} \oplus \gamma : \mathcal{A} \oplus \overline{\mathcal{C}} \circ \mathcal{A} \to \mathcal{A}$. The map γ gives us a degree -1 map from $\overline{\mathcal{C}}$ to the operad $\mathrm{End}_{\mathcal{A}}$. Since, the coderivation which extends $d_{\mathcal{A}} \oplus \gamma$ squares to $(\theta \circ Id)\Delta$, then α is a twisting morphism and so induces a morphism of operads from $\Omega_u\mathscr{C}$ to $\mathrm{End}_{\mathcal{A}}$, which is an $\Omega_u\mathscr{C}$-algebra structure on $\mathcal{A}$. Then, $\mathscr{D} = B_\iota\mathcal{A}$. Besides, let $F = Id \circ f$ be a strict morphism from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Since F commutes with the coderivations, then f is a morphism of $\Omega_u\mathscr{C}$-algebras. □

Theorem 14. *The fibrant $\mathscr{C}$-coalgebras in the ι-model structure are the objects isomorphic to the quasi-cofree $\mathscr{C}$-coalgebras, that is the objects isomorphic to the image under the functor B_ι of a $\Omega_u\mathscr{C}$-algebra*

Proof. Let $\mathscr{D}$ be a fibrant object. Since the morphism $\mathscr{D} \to B_\iota\Omega_\iota\mathscr{D}$ is an acyclic cofibration, the following square has a lifting .

$$\begin{tikzcd} \mathscr{D} \arrow[r, "Id"] \arrow[d] & \mathscr{D} \arrow[d] \\ B_\alpha\Omega_\alpha\mathscr{D} \arrow[r] & * . \end{tikzcd}$$

Hence, $\mathscr{D}$ is a retract of a quasi-cofree $\mathscr{C}$-coalgebra. Hence, by Lemma 14, it is isomorphic to a quasi-cofree $\mathscr{C}$-coalgebra. Conversely, a quasi-cofree $\mathscr{C}$-coalgebra is fibrant since it is the image under B_ι of a $\Omega_u\mathscr{C}$-algebra which is fibrant. □

Lemma 13. *Let $f : \mathcal{V} \to \mathcal{W}$ be a morphism of graded $\mathbb{K}$-modules. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are endowed with exhaustive filtrations $(F_n\mathcal{V})_{n\in\mathbb{N}}$ and $(F_n\mathcal{W})_{n\in\mathbb{N}}$ such that f maps $F_n\mathcal{V}$ into $F_n\mathcal{W}$ for any integer n. If the morphism $Gf : G\mathcal{V} \to G\mathcal{W}$ is an isomorphism, then f is an isomorphism.*

Proof. Suppose that Gf is an isomorphism. Then, the map $F_0(f) : F_0\mathcal{V} \to F_0\mathcal{W}$ which is the restriction of f on $F_0\mathcal{V}$ is an isomorphism. Suppose now that $F_n(f)$ is an isomorphism for an integer n. Let us show that $F_{n+1}(f)$ is surjective and injective. Let $y \in F_{n+1}\mathcal{W}$; it represents the element $[y] \in G_{n+1}\mathcal{W}$. Since $G_{n+1}f$ is surjective, there exists an element $[x] \in G_{n+1}\mathcal{V}$ such that $G_{n+1}(f)([x]) = [y]$. Then, $f(x) - y \in F_n\mathcal{W}$. So let $x' \in F_n\mathcal{V}$ such that $f(x) - y = f(x')$. Hence, $y = f(x - x')$ is in the image of $F_{n+1}(f)$ and so $F_{n+1}(f)$ is surjective. Let $x'' \in F_{n+1}\mathcal{V}$ such that $f(x'') = 0$. Then, since $G_{n+1}(f)$ is injective, then $[x''] = 0 \in G_{n+1}\mathcal{V}$. So $x'' \in F_n\mathcal{V}$ and since $F_n(f)$ is injective, $x'' = 0$. So $F_{n+1}(f)$ is injective. Moreover, since the two filtrations are exhaustive, then f is both injective and surjective and so it is an isomorphism. □

Lemma 14. *A retract of a graded cofree $\mathscr{C}^{grad}$-coalgebra is isomorphic to a graded cofree $\mathscr{C}^{grad}$-coalgebra.*

Proof. Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a graded $\mathscr{C}^{grad}$-coalgebra which is a retract of $\mathscr{C} \circ \mathcal{V}$. On the one hand, the following diagram is a retract, that is the compositions of the horizontal maps give the identity on the bottom and on the top

$$\begin{tikzcd} G_n^{rad}\mathcal{D} \arrow[r] \arrow[d] & G_n^{rad}(\mathcal{C} \circ \mathcal{V}) \arrow[r] \arrow[d] & G_n^{rad}\mathcal{D} \arrow[d] \\ (G_n^{rad}\mathcal{C}) \circ F_0^{rad}\mathcal{D} \arrow[r] & (G_n^{rad}\mathcal{C}) \circ F_0^{rad}(\mathcal{C} \circ \mathcal{V}) \arrow[r] & (G_n^{rad}\mathcal{C}) \circ F_0^{rad}\mathcal{D} . \end{tikzcd}$$

Since the middle vertical map is an isomorphism, then all the vertical maps are isomorphisms. On the other hand, the map $\epsilon \circ Id : \mathcal{C} \circ \mathcal{V} \to \mathcal{V} = F_0^{rad}\mathcal{C} \circ \mathcal{V}$ gives us a map $\mathcal{D} \to F_0\mathcal{D}$ and hence a morphism of graded $\mathscr{C}$-coalgebras $f : \mathscr{D} \to \mathscr{C} \circ F_0\mathcal{D}$. Let us show that f is an isomorphism. It is clear that the map $F_0\mathcal{D} \to F_0(\mathcal{C} \circ F_0\mathcal{D})$ is an isomorphism. For any integer $n \geq 1$, the following diagram is commutative

$$\begin{tikzcd} G_n(\mathcal{D}) \arrow[r, "f"] \arrow[d, "\Delta"'] & G_n(\mathcal{C} \circ F_0\mathcal{D}) \arrow[d, "\Delta"] \\ (G_n\mathcal{C}) \circ F_0\mathcal{D} \arrow[r, "Id \circ f"'] & (G_n\mathcal{C}) \circ F_0(\mathcal{C} \circ F_0\mathcal{D}) = (G_n\mathcal{C}) \circ F_0\mathcal{D}\ . \end{tikzcd}$$

Since, the vertical maps are isomorphisms and since the bottom horizontal map is an isomorphism, then the top horizontal map is also an isomorphism. Hence, the map $Gf : G\mathcal{D} \to G(\mathcal{C} \circ F_0\mathcal{D})$ is an isomorphism. By Lemma 13, f is an isomorphism. ☐

2.7.3 Cofibration, fibrations and weak equivalences between fibrant objects

We show here that cofibrations, weak equivalences and fibrations between quasi-cofree $\mathscr{C}$-coalgebras are easily characterized.

Proposition 34. *Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ and $\mathscr{B} = (\mathcal{B}, \gamma_{\mathscr{B}})$ be two $\Omega_u\mathscr{C}$-algebras and let $F : B_\iota\mathscr{A} \to B_\iota\mathscr{B}$ be a morphism between their bar constructions. We denote by $f : B_\iota\mathscr{A} \to \mathcal{B}$ its projection $f = \pi_{\mathcal{B}}F$ on $\mathcal{B}$.*

- ▷ *The morphism F is a cofibration if and only if the restriction $f_{|\mathcal{A}}$ is a monomorphism.*
- ▷ *The morphism F is a weak equivalence if and only $f_{|\mathcal{A}}$ is a quasi-isomorphism.*
- ▷ *The morphism F is a fibration if and only if $f_{|\mathcal{A}}$ is an epimorphism.*

Lemma 15. *The morphism of chain complexes $\mathcal{A} \to \Omega_\iota B_\iota\mathscr{A}$ which is the restriction to $\mathcal{A}$ of the canonical morphism $B_\iota\mathscr{A} \to B_\iota\Omega_\iota B_\iota\mathscr{A}$ is a quasi-isomorphism.*

Proof. It is a right inverse of the morphism canonical morphism of $\Omega_u\mathscr{C}$-algebras $\Omega_\iota B_\iota\mathscr{A} \to \mathscr{A}$. ☐

Proof of Proposition 34. Note first that $f_{|\mathcal{A}} = F_{|\mathcal{A}}$.

- ▷ Suppose that F is a cofibration, i.e. a monomorphism. Then, its restriction $F_{|\mathcal{A}}$ is also a monomorphism. Conversely, suppose that the map f is a monomorphism. Then, using the same arguments as in Lemma 13, we can prove by induction that for any integer n, the map $F : F_n^{rad}B_\iota\mathscr{A} \to F_n^{rad}B_\iota\mathscr{B}$ is a monomorphism.
- ▷ By Lemma 15, the maps $\mathcal{A} \to \Omega_\iota B_\iota\mathscr{A}$ and $\mathcal{B} \to \Omega_\iota B_\iota\mathscr{B}$ are quasi-isomorphisms. Consider the following diagram.

$$\begin{tikzcd} \Omega_\iota B_\iota\mathscr{A} \arrow[r, "\Omega_\iota F"] & \Omega_\iota B_\iota\mathscr{B} \\ \mathcal{A} \arrow[r, "f"'] \arrow[u] & \mathcal{B} \arrow[u] \end{tikzcd}$$

  It ensures us that f is a quasi-isomorphism if and only if $\Omega_\iota F$ is a quasi-isomorphism, that is if and only if F is a weak equivalence.
- ▷ Suppose that F is a fibration. Since any chain complex can be considered as a $\mathscr{C}$-coalgebra, then any square of $\mathscr{C}$-coalgebras has follows has a lifting

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_\iota\mathscr{A} \arrow[d] \\ D^n \arrow[r] & B_\iota\mathscr{B}\ . \end{tikzcd}$$

  This ensures us that the map f is an epimorphism. Conversely, suppose that f is an epimorphism. By Lemma 16, there exists an isomorphism $G : \mathscr{C} \circ \mathcal{V} \to B_\iota\mathscr{A}$ such that FG is in the image of the functor B_ι. If we denote by g the map from $\mathcal{V}$ to $\mathcal{A}$ which underlies G, then g is an isomorphism. So fg is surjective and so $FG = B_\iota(fg)$ is a fibration. Since G is an isomorphism, then F is a fibration.

□

Lemma 16. *Let $F : B_\iota\mathscr{A} \to B_\iota\mathscr{B}$ be a morphism of $\mathscr{C}$-coalgebras such that the underlying morphism $f : \mathcal{A} \to \mathcal{B}$ is surjective. Then, there exists an $\Omega_u\mathscr{C}$-algebra $\mathscr{A}'$ and and isomorphism of quasi-cofree $\mathscr{C}$-coalgebra $G : B_\iota\mathscr{A}' \to B_\iota\mathscr{A}$ such that FG is a strict morphism, that is the image under the functor B_ι of a morphism of $\Omega_u\mathscr{C}$-algebras $\mathscr{A}' \to \mathscr{B}$.*

Proof. We build an isomorphism of graded $\mathscr{C}^{grad}$-coalgebras $G : \mathscr{C} \circ \mathcal{A} \to \mathscr{C} \circ \mathcal{A}$ such that FG is a strict morphism, that is of the from $Id_{\mathscr{C}} \circ h$. To that purpose we define inductively maps $g_n : F_n^{rad}\mathscr{C} \circ \mathcal{A} \to \mathcal{A}$ such that g_{n_1} is the restriction of g_n on $F_{n-1}^{rad}\mathscr{C} \circ \mathcal{A}$ and such that we have the following equality between maps from $F_n^{rad}\mathscr{C} \circ \mathcal{A}$ to $\mathcal{A}$:

$$fg_n + f(Id \circ g_{n-1})(\overline{\Delta} \circ Id) = f\pi_{\mathcal{A}} \ , \tag{2.1}$$

where $\pi_{\mathcal{A}} = \epsilon \circ Id$ is the projection of $\mathscr{C} \circ \mathcal{A}$ on $\mathcal{A}$. First let us choose $g_0 = Id_{\mathcal{A}}$. Then, suppose that we have built g_n satisfying Equation 2.1. The map $f : \mathcal{A} \to \mathcal{B}$ and the injection of $F_n^{rad}\mathscr{C} \circ \mathcal{A}$ into $F_{n+1}^{rad}\mathscr{C} \circ \mathcal{A}$ give us the following square

$$\begin{tikzcd}
\hom_{\mathsf{gMod}}(F_{n+1}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \arrow[r] \arrow[d] & \hom_{\mathsf{gMod}}(F_{n+1}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{B}) \arrow[d] \\
\hom_{\mathsf{gMod}}(F_{n}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \arrow[r] & \hom_{\mathsf{gMod}}(F_{n}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{B}) \ .
\end{tikzcd}$$

The following map is surjective :

$$\hom_{\mathsf{gMod}}(F_{n+1}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \to \hom_{\mathsf{gMod}}(F_{n}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \times_{\hom_{\mathsf{gMod}}(F_{n}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{B})} \hom_{\mathsf{gMod}}(F_{n+1}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{B}) \ .$$

So there exists an element of $\hom_{\mathsf{gMod}}(F_{n+1}^{rad}\mathscr{C} \circ \mathcal{A}, \mathcal{A})$ whose image under this map is the pair $(g_n, f\pi_{\mathcal{A}} - f_{n+1}(Id \circ g_n)(\overline{\Delta} \circ Id))$. We can choose this element to be g_{n+1}. Thus, let g be the map from $\mathscr{C} \circ \mathcal{A}$ to $\mathcal{A}$ whose restriction to $F_n^{rad}\mathscr{C} \circ \mathcal{A}$ is g_n for any n. Let G be the map of graded $\mathscr{C}^{grad}$-coalgebras which extend g. By Lemma 17, the map G is an isomorphism. Let us transfer the coderivation of $B_\iota\mathscr{A}$ to $\mathscr{C} \circ \mathcal{A}$ along the isomorphism G. This gives us a new $\Omega_u\mathscr{C}$-algebra structure on the chain complex $\mathcal{A}$ that we denote $\mathscr{A}'$. Finally, the morphism FG is the image under the functor B_ι of the morphism of $\Omega_u\mathscr{C}$-algebras $fg_0 : \mathcal{A} \to \mathcal{B}$. □

Lemma 17. *Let $F : \mathcal{C} = \mathscr{C} \circ \mathcal{V} \to \mathcal{D} = \mathscr{C} \circ \mathcal{W}$ be a morphism of quasi-cofree $\mathscr{C}$-coalgebras. Then, F is an isomorphism if and only if its underlying map $f : \mathcal{V} \to \mathcal{W}$ is an isomorphism.*

Proof. Suppose first that F is an isomorphism with inverse G. Let us denote $g : \mathcal{W} \to \mathcal{V}$ the map underlying G. Then the map g is inverse to f and so f is an isomorphism. Conversely, suppose that f is an isomorphism. A straightforward induction shows that F is both injective and surjective. □

2.7.4 Mapping spaces and deformation theory

Proposition 35. *For any cofibrant $\mathscr{C}$-coalgebra $\mathscr{D}$ and any fibrant $\mathscr{C}$-coalgebra $\mathscr{E}$, the simplicial set $\mathrm{HOM}(\mathscr{D}, \mathcal{E})$ is a Kan complex and is a model for the mapping space $\mathrm{Map}(\mathcal{C}, \mathcal{D})$.*

Proof. Any fibrant $\mathscr{C}$-coalgebra $\mathscr{E}$ is isomorphic to the image under B_ι of a $\Omega_u\mathscr{C}$-algebra $\mathscr{A}$. So, we have :

$$\mathrm{HOM}(\mathscr{D}, \mathscr{E}) \simeq \mathrm{HOM}(\mathscr{D}, B_\iota\mathscr{A}) \simeq \mathrm{HOM}(\Omega_\iota\mathscr{D}, \mathscr{A}) \simeq \mathrm{Map}(\Omega_\iota\mathscr{D}, \mathscr{A}) \simeq \mathrm{Map}(\mathscr{D}, B_\iota\mathscr{A}) \ .$$

□

Corollary 1. *Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. Let us endow the category of $\mathscr{C}$-coalgebras with the α-model structure. For any cofibrant $\mathscr{C}$-coalgebra $\mathscr{D}$ and any fibrant $\mathscr{C}$-coalgebra $\mathscr{E}$, the simplicial set* $\mathrm{HOM}(\mathscr{D}, \mathscr{E})$ *is a Kan complex and is a model for the mapping space* $\mathrm{Map}(\mathscr{D}, \mathscr{E})$.

Proof. It suffices to notice that fibrations and acyclic fibrations in the α-model structure are in particular fibrations and acyclic fibrations in the ι-model structure. Then, we can conclude by Proposition 35. □

Let $f : \mathscr{D} \to B_\iota \mathscr{A}$ be a morphism of $\mathscr{C}$-coalgebras. We know from Proposition 13 that it is a dg atom of the cocommutative coalgebra $\{\mathscr{D}, B_\iota \mathscr{A}\}$. Consider the Hinich coalgebra $\{\mathscr{D}, B_\iota \mathscr{A}\}_f$ that appears from the decomposition described in Theorem 8.

Proposition 36. *The deformation problem induced by $\{\mathscr{D}, B_\iota \mathscr{A}\}_f$ is equivalent to the deformation problem*

$$R \in \mathsf{Artin} - \mathsf{alg} \mapsto (\hom_{R\otimes\Omega_n\otimes\mathscr{C}-\mathsf{cog}}(R \otimes \Omega_n \otimes \mathscr{C}, R \otimes \Omega_n \otimes B_\iota \mathscr{A}))_{n\in\mathbb{N}} \ .$$

Proof. This is a direct consequence of Proposition 24 and Theorem 9. □

2.7.5 Algebras of the operad $\Omega_u \mathscr{C}$

We have shown above that the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. Moreover, in Proposition 33, we have shown that fibrant $\mathscr{C}$-coalgebras are $\Omega_u \mathscr{C}$-algebras. So switching from the model category of $\Omega_u \mathscr{C}$-algebras to the model category of $\mathscr{C}$-coalgebras amounts to add new morphisms between any two $\Omega_u \mathscr{C}$-algebras. The weak equivalences and the fibrations of $\Omega_u \mathscr{C}$-algebras remain respectively weak equivalences and fibrations under this embedding but, in the category of $\mathscr{C}$-coalgebras, any monomorphism is a cofibration. In particular, any object is cofibrant. Besides, the following proposition provides a tool to decide whether two $\Omega_u \mathscr{C}$-algebras are equivalent.

Proposition 37. *Let $\mathscr{A}$ and $\mathscr{B}$ be two $\Omega_u \mathscr{C}$-algebras. There exists a chain of weak equivalences of $\Omega_u \mathscr{C}$-algebras between $\mathscr{A}$ and $\mathscr{B}$*

$$\mathscr{A} = \mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}_1 \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mathscr{A}_{n-1} \xleftarrow{\sim} \mathscr{A}_n = \mathscr{B}$$

if and only if there exists an weak equivalence of $\mathscr{C}$-coalgebras between $B_\iota \mathscr{A}$ and $B_\iota \mathscr{B}$.

Proof. Suppose that there exists a chain of weak equivalences from $\mathscr{A}$ to $\mathscr{B}$. Then, there exists a chain of weak equivalences between $B_\iota \mathscr{A}$ and $B_\iota \mathscr{B}$. Moreover, the object of this chain are fibrant and cofibrant. So any morphism of this chain has an homotopical inverse. So there exists a weak equivalence from $B_\iota \mathscr{A}$ to $B_\iota \mathscr{B}$. Conversely, consider a weak equivalence F from $B_\iota \mathscr{A}$ to $B_\iota \mathscr{B}$. Then, the following chain of weak equivalences of $\Omega_u \mathscr{C}$-algebras links $\mathscr{A}$ to $\mathscr{B}$.

$$\mathscr{A} \xleftarrow{\sim} \Omega_\iota B_\iota \mathscr{A} \xrightarrow{\Omega_\iota(f)} \Omega_\iota B_\iota \mathscr{B} \xrightarrow{\sim} \mathscr{B} \ .$$

□

2.7.6 Koszul morphism

In this subsection, we study the operadic twisting morphisms $\alpha : \mathscr{C} \to \mathscr{P}$ such that the α-model structure on the category of $\mathscr{C}$-coalgebras coincides with the universal ι-model structure that we described above. Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. We denote by $\phi : \Omega_u(\mathscr{C}) \to \mathscr{P}$ the morphism of operads induced by α.

Theorem 15. *The following assertions are equivalent.*

1. *The adjunction*

$$\Omega_u(\mathscr{C}) - \mathsf{alg} \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathscr{P} - \mathsf{alg}$$

is a Quillen equivalence.

2. *The morphism of operads $\Omega_u(\mathscr{C}) \to \mathscr{P}$ is a quasi-isomorphism.*
3. *The α-model structure coincides with the ι-model structure and $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence.*
4. *For any $\mathscr{P}$-algebra $\mathscr{A}$, the map $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism, and for any $\mathscr{C}$-coalgebra $\mathscr{D}$, the morphism $\mathscr{D} \to \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}$ is a ι-equivalence (it is the case if, for instance, it is a filtered quasi-isomorphism).*
5. *The morphisms of $\mathbb{S}$-modules $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}$ and $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P}$ are quasi-isomorphisms.*

Lemma 18. *Let $f : \mathcal{V} \to \mathcal{V}'$ be a morphism of $\mathbb{S}$-modules. Suppose that, for any chain complex $\mathcal{W}$ (that is a $\mathbb{S}$-modules concentrated in arity zero), the morphism $\mathcal{V} \circ \mathcal{W} \to \mathcal{V}' \circ \mathcal{W}$ is a quasi-isomorphism. Then, f is a quasi-isomorphism.*

Proof. By the operadic Kunneth formula, for any graded $\mathbb{K}$-module $\mathcal{W}$, the map $H(\mathcal{V}) \circ \mathcal{W} \to H(\mathcal{V}') \circ \mathcal{W}$ is an isomorphism. So, for any integer n, the map $f_n : H(\mathcal{V})(n) \otimes_{\mathbb{S}_n} \mathbb{K}^n \to H(\mathcal{V}')(n) \otimes_{\mathbb{S}_n} \mathbb{K}^n$ is an isomorphism. Let $(e_i)_{i=1}^n$ be a basis of $\mathbb{K}^n$. The map

$$p \in H(\mathcal{V})(n) \mapsto p \otimes (e_1 \otimes \cdots \otimes e_n) \mapsto f_n(p) \otimes (e_1 \otimes \cdots \otimes e_n) \mapsto f_n(p) \in H(\mathcal{V}')(n)$$

is an isomorphism. So, the morphism $H(\mathcal{V}) \to H(\mathcal{V}')$ is an isomorphism. □

Proof of Theorem 15.

- ▷ Let us first prove the equivalence between (1) and (2). Suppose (2). Let $\mathscr{A}$ be a cofibrant $\Omega_u\mathscr{C}$-algebra and let $\mathscr{B}$ be a fibrant $\mathscr{P}$-algebra. Consider a map $f : \phi_!(\mathscr{A}) \to \mathscr{B}$ and its adjoint map $g : \mathscr{A} \to \phi^*(\mathscr{B})$. The following diagram of $\Omega_u\mathscr{C}$-algebras is commutative.

$$\begin{tikzcd}
\Omega_\iota B_\iota \mathscr{A} \arrow[r] \arrow[d] & \phi^*\phi_!\Omega_\iota B_\iota \mathscr{A} \arrow[d] & \\
\mathscr{A} \arrow[r] \arrow[rr, bend right, "g"'] & \phi^*\phi_!\mathscr{A} \arrow[r, "\phi^*(f)"] & \phi^*\mathscr{B}
\end{tikzcd}$$

  The left vertical map is a quasi-isomorphism. Since a left Quillen functor preserves weak equivalences between cofibrant objects and since ϕ^* preserves quasi-isomorphisms, then the right vertical map is a quasi-isomorphism. Besides $\phi_!\Omega_\iota B_\iota \mathscr{A}$ is actually $\Omega_\alpha B_\iota \mathscr{A}$. Since the morphism ϕ is a quasi-isomorphism, we can prove that the map $\Omega_\iota B_\iota \mathscr{A} \to \phi^*\Omega_\alpha B_\iota \mathscr{A}$ is a filtered quasi-isomorphism for a well chosen filtration, and so is a quasi-isomorphism. So, by the 2-out-of-3 rule, the map $\mathscr{A} \to \phi^*\phi_!\mathscr{A}$ is a quasi-isomorphism. Hence, f is a quasi-isomorphism if and only if $\phi^*(f)$ is a quasi-isomorphism, if and only if g is a quasi-isomorphism. So the assertion (1) is true. Conversely, suppose (1). Then, for any chain complex (considered as a $\mathscr{C}$-coalgebra) $\mathcal{V}$, the map $\Omega_\iota \mathcal{V} \to \Omega_\alpha \mathcal{V}$ is a quasi-isomorphism. So, by Lemma 18, (2) is true.
- ▷ Suppose (1) and let us show (3). By Proposition 32, the α-model structure coincides with the ι-model structure. Moreover, since the adjunctions $\phi_! \dashv \phi^*$ and $\Omega_\iota \dashv B_\iota$ are both Quillen equivalence, then the adjunction $\phi_!\Omega_\iota \dashv B_\iota \phi^*$ which is $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence.
- ▷ Suppose (3) and let us show (4). Since $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence, then $\Omega_\alpha B_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism for any $\mathscr{P}$-algebra $\mathscr{A}$ and $\mathscr{D} \to B_\alpha \Omega_\alpha \mathscr{D}$ is an α-weak equivalence for any $\mathscr{C}$-coalgebra $\mathscr{D}$. Since the α-model structure coincides with the ι-model structure, then $\mathscr{D} \to B_\alpha \Omega_\alpha \mathscr{D}$ is a ι-weak equivalence. So (4) is true.
- ▷ Suppose (4) and let us show (5). For any $\mathscr{P}$-algebra $\mathscr{A}$, the morphism $\Omega_\alpha B_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. Applying this to free $\mathscr{P}$-algebras and using Lemma 18, we conclude that the map $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P}$ is a quasi-isomorphism. Moreover, for any $\mathscr{P}$-algebra $\mathscr{A}$, the following diagram commutes

$$\begin{tikzcd}
\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \arrow[r] \arrow[dr] & \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{A} \arrow[d] \\
& \mathscr{A} \; .
\end{tikzcd}$$

Since the composite map and the vertical map are quasi-isomorphisms (because $\Omega_\iota \dashv B_\iota$ and $\Omega_\alpha \dashv B_\alpha$ are Quillen equivalences), then, by the 2-out-of-3 rule, the horizontal map is a quasi-isomorphism. Applying this to free $\mathscr{P}$-algebras and using Lemma 18, we conclude that the map $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}$ is a quasi-isomorphism. Besides, for any $\mathscr{C}$-coalgebra $\mathscr{D}$ the following diagram commutes.

$$\begin{tikzcd}
\mathscr{D} \arrow[r] \arrow[dr] & \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \circ_\iota \mathscr{D} \arrow[d] \\
& \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}
\end{tikzcd}$$

By the 2-out-of-3 rule, the vertical map is a ι-weak equivalence. So the map $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \circ_\iota \mathscr{D} \to \Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}$ is a quasi-isomorphism. Applying this for $\mathscr{C}$-coalgebras which are just chain complexes and using Lemma 18, we obtain that the map $\Omega_u(\mathscr{C}) \circ_\iota \Omega_u(\mathscr{C}) \to \Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P}$ is a quasi-isomorphism.

▷ Suppose (5) and let us show (2). The following square of $\mathbb{S}$-modules is commutative.

$$\begin{tikzcd}
\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \arrow[r] \arrow[d] & \Omega_u(\mathscr{C}) \arrow[d] \\
\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \arrow[r] & \mathscr{P}
\end{tikzcd}$$

Since the left vertical map and the horizontal maps are quasi-isomorphisms, then the right vertical map is also a quasi-isomorphism.

□

Definition 34 (Koszul morphisms)**.** An operadic twisting morphism $\alpha : \mathscr{C} \to \mathscr{P}$ satisfying the properties of Theorem 15 is called a *Koszul morphism*.

In the next section, we will see that the Koszul duality is a method to produce Koszul morphisms from a presentation of an operad.

2.8 Examples

The purpose of this section is to apply the general framework described in the previous sections to the case of common nonaugmented operads like the operads $u\mathscr{A}s$ and $u\mathscr{C}om$ whose algebras are respectively the unital associative algebras and the unital commutative algebras. So, for any of these operads $\mathscr{P}$, one looks after a curved conilpotent cooperad $\mathscr{C}$ together with an operadic twisting morphism α from $\mathscr{C}$ to $\mathscr{P}$ such that the induced morphism of operads from $\Omega_u\mathscr{C}$ to $\mathscr{P}$ is a quasi-isomorphism ; that is, α is a Koszul morphism. One can use the universal twisting morphism $B_c\mathscr{P} \to \mathscr{P}$. However, the bar construction is always very big. Instead, one usually tries to produce a sub-cooperad of $B_c\mathscr{P}$ whose cobar construction will be a resolution of $\mathscr{P}$. The Koszul duality theory is a way to produce such a sub-cooperad when the operad $\mathscr{P}$ has a quadratic presentation or a quadratic-linear presentation. This construction has been extended to quadratic-linear-constant presentations by Hirsh and Millès in [HM12], generalizing to operads the curved Koszul duality of algebras developed by Polishchuk and Positselski in [PP05].

2.8.1 Koszul duality

Koszul duality is a way to build a cooperad $\mathscr{P}^{\text{¡}}$ together with a canonical operadic twisting morphism from $\mathscr{P}^{\text{¡}}$ to $\mathscr{P}$, when $\mathscr{P}$ has a "nice enough" presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$. Here, we present a particular case of the construction of Hirsh and Millès in [HM12], but in a slightly different way due to the fact that our operadic bar construction is different.

Let $\mathscr{P}$ be a dg operad equipped with a presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$, where $\mathcal{V}$ is a dg $\mathbb{S}$-module and where $(\mathcal{R})$ is the operadic ideal generated by a sub-dg-$\mathbb{S}$-module $\mathcal{R}$ of $\mathcal{I} \oplus \mathbb{T}^2(\mathcal{V}) \subset \mathbb{T}^{\leq 2}(\mathcal{V})$ such that $\mathcal{R} \cap \mathcal{I} = \{0\}$ (otherwise, $\mathscr{P}$ would be zero).

Definition 35 (Cooperad Koszul dual of an operad)**.** The Koszul dual cooperad $\mathscr{P}^{¡}$ of $\mathscr{P}$, associated to the presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$, is the final graded sub-cooperad of $\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$ (where v is an element of arity 1 and degree 2 such that $dv = 0$) such that the composition

$$\mathscr{P}^{¡} \to \mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v) \to (s\mathcal{V} \otimes s\mathcal{V} \oplus \mathbb{K} \cdot v)/s^2\mathcal{R}$$

is zero. It is equipped with a degree -2 map :

$$\begin{aligned} \theta : \mathscr{P}^{¡} \to s\mathcal{V} \oplus \mathbb{K}.v \to \mathbb{K}.v &\to \mathbb{K} \\ v &\mapsto 1 \ . \end{aligned}$$

Proposition 38. *The cooperad $\mathscr{P}^{¡}$ is stable under the coderivation of $\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$ induced by the differential of $\mathcal{V}$. When equipped with this coderivation and the map $\theta : v \mapsto 1$, the cooperad $\mathscr{P}^{¡}$ is a curved conilpotent cooperad. Moreover, there is a monomorphism of curved cooperads $\mathscr{P}^{¡} \to B_c\mathscr{P}$.*

Proof. By the similar arguments as in the proof of Lemma 11, $\mathscr{P}^{¡}$ is stable under the coderivation of $\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$. Consider the map $s\mathcal{V} \oplus \mathbb{K}.v \to s\mathscr{P} \oplus \mathbb{K}.v$. It induces a monomorphism of graded cooperads $j : \mathscr{P}^{¡} \to \mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K}.v) \to B_c\mathscr{P}$. Let us show that j commutes with the coderivations. Since $B_c\mathscr{P}$ is a quasi-cofree cooperad, by Proposition [LV12, 10.5.5] it suffices to show that

$$\pi_1 d_{B_c\mathscr{P}} j = \pi_1 j \ d_{\mathscr{P}^{¡}}$$

where π_1 is the projection of $B_c\mathscr{P}$ on $F_1 B_c\mathscr{P}$. Since j is a monomorphism and because of the shape of the coderivation of $B_c\mathscr{P}$, we have :

$$\pi_1 d_{B_c\mathscr{P}} j = \pi_1 d_{B_c\mathscr{P}} j \pi_{\leq 2} \ ,$$

$$\pi_1 j \ d_{\mathscr{P}^{¡}} = \pi_1 j \ d_{\mathscr{P}^{¡}} \pi_1 \ .$$

Since $F_2^{rad}\mathscr{P}^{¡} = \mathbb{K}.1 \oplus s\mathcal{V} \oplus s^2\mathcal{R}$, we can check by a direct computation that

$$\pi_1 d_{B_c\mathscr{P}} j \pi_{\leq 2} = \pi_1 d_{B_c\mathscr{P}} j \pi_1 \ .$$

It is straightforward to check that

$$\pi_1 d_{B_c\mathscr{P}} j \pi_1 = \pi_1 j \ d_{\mathscr{P}^{¡}} \pi_1 \ .$$

Hence, j commutes with the coderivations. Since $\theta_{\mathscr{P}^{¡}} = \theta_{B_c\mathscr{P}} j$, then $\mathscr{P}^{¡}$ is a sub curved cooperad of $B_c\mathscr{P}$. □

Definition 36 (Koszul operad)**.** The operad $\mathscr{P}$ (or more accurately the presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/\mathcal{R}$) is said to be Koszul if the twisting morphism $\kappa : \mathscr{P}^{¡} \to \mathscr{P}$ is Koszul, that is if the map $\Omega_c\mathscr{P}^{¡} \to \mathscr{P}$ is a quasi-isomorphism.

The following theorem is a powerful tool to show that an operad is Koszul;

Theorem 16. *[HM12, Theorem 4.3.1] Let $q\mathscr{P}$ be the dg operad $q\mathscr{P} := \mathbb{T}(\mathcal{V})/q\mathcal{R}$ where $q\mathcal{R}$ is the projection of $\mathcal{R} \subset \mathbb{K} \cdot v \oplus \mathbb{T}^2\mathcal{V}$ onto $\mathbb{T}^2\mathcal{V}$. Notice that $q\mathscr{P}^{¡}$ is a dg cooperad. Suppose that the canonical morphism*

$$q\mathscr{P} \circ_\kappa q\mathscr{P}^{¡} \to \mathcal{I}$$

is a quasi-isomorphism. Then $\mathscr{P}$ is Koszul.

2.8.2 Coalgebras over a Koszul dual

In this subsection, we describe the category of $\mathscr{P}^{¡}$-coalgebras, where $\mathscr{P}^{¡}$ is the Koszul dual of the "quadratic homogeneous operad" $\mathscr{P}$ as defined above. We will need the following definition.

Definition 37 (Pre-coradical filtration)**.** Let $\mathscr{W}$ be a graded $\mathbb{S}$-module and $\mathcal{C}$ be a graded $\mathbb{K}$-module equipped with a map $\Delta^{(1)} : \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$. Let $(F_n^{prad}\mathcal{C})_{n\in\mathbb{N}}$ be the following (non-necessarily exhaustive) filtration on $\mathcal{C}$ called the pre-coradical filtration.

$$\begin{cases} F_0^{prad}(\mathcal{C}) := ker(\Delta^{(1)}) \ , \\ F_n^{prad}(\mathcal{C}) := (\Delta^{(1)})^{-1}\big(\mathcal{W}(0) \oplus \sum_{\substack{i_1+\cdots+i_k=n-1 \\ k\geq 1}} \mathcal{W}(k) \otimes_{\mathbb{S}_k} (F_{i_1}^{prad}\mathcal{C} \otimes \cdots \otimes F_{i_k}^{prad}\mathcal{C})\big) \ . \end{cases}$$

Theorem 17. *Suppose that the characteristic of the field $\mathbb{K}$ is zero (this assumption is not necessary in the nonsymmetric context). The $\mathscr{P}^{¡}$-coalgebras are the graded-$\mathbb{K}$-modules $\mathcal{C}$ equipped with a map $\Delta^{(1)} : \to (s\mathcal{V} \oplus \mathbb{K} \cdot v) \circ \mathcal{C}$ and a degree -1 map $d_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ such that the pre-coradical filtration $(F_n^{prad}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive, such that*

$$\begin{cases} d_{\mathcal{C}}^2 = (\theta \circ Id)\Delta^{(1)} \\ \Delta^{(1)} d_{\mathcal{C}} = (d \circ Id + id \circ' d_{\mathcal{C}})\Delta^{(1)} \end{cases}$$

and such that the composite map :

$$\mathcal{C} \xrightarrow{\Delta^{(1)}+(Id\circ'\Delta^{(1)})\Delta^{(1)}} \overline{\mathbb{T}}^{\leq 2}(s\mathcal{V} \oplus \mathbb{K} \cdot v) \circ \mathcal{C} \twoheadrightarrow (\mathbb{T}^2(s\mathcal{V}) \oplus \mathbb{K} \cdot v) \circ \mathcal{C} \twoheadrightarrow \left((\mathbb{T}^2(s\mathcal{V}) \oplus \mathbb{K} \cdot v)/s^2 R\right) \circ \mathcal{C}$$

is zero.

Lemma 19. *Let $\mathscr{C} = \mathbb{T}^c(\mathcal{W})$ be graded cofree conilpotent cooperad. Coalgebras over $\mathscr{C}$ are graded-$\mathbb{K}$-modules $\mathcal{C}$ equipped with a map $\Delta^{(1)} : \mathcal{C} \to \mathcal{W}\circ\mathcal{C}$ such that the pre-coradical filtration $(F_n^{prad}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive.*

Proof. Let $\mathcal{C}$ be a graded-$\mathbb{K}$-module with a map $\Delta^{(1)} : \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ such that the pre-coradical filtration $(F_n^{prad}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive. Then, let us define $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathbb{T}(\mathcal{W}) \circ \mathcal{C}$ by induction as follows.

$$\begin{cases} \Delta_{\mathcal{C}}(x) := 1 \otimes x \text{ if } x \in F_0^{prad}\mathcal{C} \ , \\ \Delta_{\mathcal{C}}(x) := 1 \otimes x + (Id \circ \Delta_{\mathcal{C}})\Delta^{(1)} \text{ if } x \in F_n^{prad}\mathcal{C} \ . \end{cases}$$

This defines a structure of $\mathscr{C}$-coalgebra on $\mathcal{C}$. Moreover, if the map $\Delta^{(1)}$ arises from a structure of $\mathscr{C}$-coalgebra on $\mathcal{C}$, then the construction we just described recovers this $\mathscr{C}$-coalgebra structure. ☐

Proof of Theorem 17. Let $\mathcal{C}$ be a graded-$\mathbb{K}$-modules $\mathcal{C}$ equipped with a map $\Delta^{(1)} : \mathcal{C} \to (s\mathcal{V}\oplus\mathbb{K}\cdot v)\circ\mathcal{C}$ and a degree -1 map $d_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ satisfying the conditions of Theorem 17. By Lemma 19, $\mathcal{C}$ is a graded $\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$-coalgebra. Besides, for any $x \in \mathcal{C}$, let $\mathcal{C}(x)$ be a finite dimensional sub-$\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$ of $\mathcal{C}$ which contains x. By Lemma 20, the map $\Delta_{\mathcal{C}(x)} : \mathcal{C}(x) \to \mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v) \circ \mathcal{C}(x)$ factorizes through a map $\mathcal{C}(x) \to \mathscr{P}^{¡}(x) \circ \mathcal{C}(x)$. Hence, $\mathcal{C}$ has a structure of graded $(\mathscr{P}^{¡})^{grad}$-coalgebra. Moreover, the condition of Theorem 17 ensures us that $\mathcal{C}$ is a $\mathscr{P}^{¡}$-coalgebra ☐

Lemma 20. *Let $\mathcal{C}(x)$ be the graded $\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$-coalgebra defined in the proof of Theorem 17. Then, $\mathcal{C}(x)$ is a $\mathscr{P}^{¡}$-coalgebra.*

Proof. Remember that $\mathcal{C}(x)$ is a finite dimensional sub-graded-$\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$-coalgebra of $\mathcal{C}$. Let $(e_i)_{i=1}^m$ be a basis of $\mathcal{C}(x)$. Then, for any $i \in \{1, \dots, m\}$, , let $p_{i,0} \in \overline{\mathbb{T}}(s\mathcal{V} \oplus \mathbb{K} \cdot v)(0)$, and for any integer $k \geq 1$ for any nondecreasing function s from $\{1, \dots, k\}$ to $\{1, \dots, m\}$, let $p_{i,k,s} \in \overline{\mathbb{T}}(s\mathcal{V} \oplus \mathbb{K} \cdot v)(k)$ such that

$$\Delta(e_i) = 1 \otimes e_i + p_{i,0} + \sum_{k=0}^{\infty} \sum_{s} p_{i,k,s} \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)})$$

For any non decreasing function s from $\{1, \dots, k\}$ to $\{1, \dots, m\}$ and for any $\sigma \in \mathbb{S}_k$, let $\epsilon(s, \sigma)$ be the element of $\mathbb{Z}/2\mathbb{Z}$ such the structural action of σ on $\mathcal{C}^{\otimes k}$ sends $e_{s(1)}\otimes\cdots\otimes e_{s(k)}$ to $(-1)^{\epsilon(s,\sigma)} e_{s\sigma^{-1}(1)} \otimes \cdots \otimes e_{s\sigma^{-1}(k)}$. Besides, let $Inv(s)$ be the subgroup of $\mathbb{S}_k$ of permutation σ such that $s = s\sigma^{-1}$. Then, we can choose $p_{i,k,s}$ such that $p_{i,k,s}^{\sigma} = (-1)^{\epsilon(s,\sigma)} p_{i,k,s}$ for any $\sigma \in Inv(s)$. Indeed, if it is not the case, we can replace $p_{i,k,s}$ by

$$\frac{1}{\#Inv(s)} \sum_{\sigma\in Inv(s)} (-1)^{\epsilon(s,\sigma)} p_{i,k,s}^{\sigma} \ .$$

Let $\mathscr{D}$ be the sub-graded $\mathbb{S}$-module of $\mathbb{T}(s\mathcal{V} \oplus \mathbb{K} \cdot v)$ generated by 1 and the elements $p_{i,k,s}$. Since $(\Delta \circ Id)\Delta(e_i) = (Id \circ \Delta)\Delta(e_i)$ for any i, then there exists an element of $q_{i,k,s} \in (\mathscr{D} \circ \mathscr{D})(k)$ such that

$$\Delta(p_{i,k,s}) \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}) = q_{i,k,s} \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}) \ .$$

It implies that

$$\Delta(p_{i,k,s}) = \frac{1}{\#Inv(s)} \sum_{\sigma \in Inv(s)} (-1)^{\epsilon(s,\sigma)} q^{\sigma}_{i,k,s} \ .$$

So, $\Delta(p_{i,k,s}) \in \mathscr{D} \circ \mathscr{D}$. Hence $\mathscr{D}$ is a sub-graded cooperad of $\mathbb{T}^c(s\mathcal{V} \oplus \mathbb{K} \cdot v)$. Moreover, for any i, $(\pi \circ Id)\Delta(e_i) = 0$ where π is the projection of $\mathbb{T}(s\mathcal{V} \oplus \mathbb{K} \cdot v)$ onto $\left(\mathbb{K} \cdot v \oplus \mathbb{T}^2(s\mathcal{V})\right) / s^2 R$. So, $\pi(p_{i,k,s}) = 0$ for any 3-tuple (i,k,s) and so $\pi_{|\mathscr{D}} = 0$. Hence $\mathscr{D} \subset \mathscr{P}^{¡}$. □

2.8.3 Unital associative algebras up to homotopy

A presentation of the operad $u\mathscr{A}s$

Let $u\mathscr{A}s$ be the nonsymmetric operad defined by the presentation $u\mathscr{A}s := \mathbb{T}(\mathbb{K} \cdot \mu \oplus \mathbb{K} \cdot v)/(R)$ where μ is an arity two element and v is an arity zero element. The nonsymmetric module $R \subset \mathcal{I} \oplus \mathbb{T}^2(\mathbb{K} \cdot \mu \oplus \mathbb{K} \cdot v)$ is made up of the following relations

$$\begin{cases} \mu \otimes_{ns} (v \otimes 1) - 1 \ , \\ \mu \otimes_{ns} (1 \otimes v) - 1 \ , \\ \mu \otimes_{ns} (\mu \otimes 1) - \mu \otimes_{ns} (1 \otimes \mu) \ . \end{cases}$$

Coalgebras over $u\mathscr{A}s^{¡}$

The Koszul dual curved cooperad $u\mathscr{A}s^{¡}$ of the operad $u\mathscr{A}s$ is described in [HM12].

Proposition 39. *The endofunctor of the category of chain complexes $\mathcal{V} \mapsto s\mathcal{V}$ induces an equivalence between $u\mathscr{A}s^{¡}$-coalgebras and non-counital curved conilpotent coassociative coalgebras.*

Proof. This is a consequence of Theorem 17. □

Bar-Cobar adjunction

On the one hand, there exists an adjunction relating $u\mathscr{A}s$-algebras to $u\mathscr{A}s^{¡}$-coalgebras, induced by the operadic twisting morphism $\alpha : u\mathscr{A}s^{¡} \to u\mathscr{A}s$. On the other hand the category of $u\mathscr{A}s^{¡}$-coalgebras is equivalent to the category of curved conilpotent coalgebras. Thus, we obtain a bar-cobar adjunction between unital associative algebras and curved conilpotent coalgebras which is the restriction to arity one of the operadic bar-cobar adjunction described in Section 2.4.1 (with the exception that we can consider noncounital coalgebras instead of coaugmented counital coalgebras). For this reason, we will denote this adjunction with the same symbols as in the operadic context, that is $\Omega_u \dashv B_c$. So we have :

$$\Omega_u \mathscr{C} := \overline{\mathbb{T}}(s^{-1}\mathscr{C})$$

$$B_c \mathscr{A} := \overline{\mathbb{T}}(s\mathscr{A} \oplus \mathbb{K} \cdot v)$$

for any curved conilpotent coalgebra $\mathscr{C}$ and for any unital algebra $\mathscr{A}$. The derivation of $\Omega_u(\mathscr{C})$ and the coderivation of $B_c(\mathscr{A})$ are defined as in Section 2.4.1.

The Koszul property

Proposition 40. *[HM12, 6.1.8] The operad $u\mathscr{A}s$ is Koszul.*

Remark 7. The model structure on curved conilpotent coalgebras that we get by transfer along the adjunction $\Omega_u \dashv B_c$ is the model structure that Positselski described in [Pos11].

2.8.4 The operad $uCom$

The operad $u\mathscr{C}om$

The operad $u\mathscr{C}om$ is the quotient of the free operad generated by an element of arity two μ, invariant under the action of $\mathbb{S}_2$ and an element of arity zero element υ, by the operadic ideal generated by the relations

$$\begin{cases} \mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - \mu \otimes_{\mathbb{S}_2} (1 \otimes \mu) \ , \\ \mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - \left(\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1)\right)^{(2,3)} \ , \\ \mu \otimes_{\mathbb{S}_2} (\upsilon \otimes 1) - 1 \ . \end{cases}$$

Coalgebras over $u\mathscr{C}om^{¡}$

We will show that the category of $u\mathscr{C}om^{¡}$-coalgebras is equivalent to the category of curved conilpotent Lie coalgebra.

Definition 38 (Curved Lie coalgebra)**.** A *curved Lie coalgebra* $\mathscr{C} = (\mathcal{C}, \delta, d, \theta)$ is a graded $\mathbb{K}$-module $\mathcal{C}$ equipped with a skewsymmetric map $\delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that

$$(\delta \otimes Id)\delta = (Id \otimes \delta)\delta + (Id \otimes \tau)(\delta \otimes Id)\delta \ ,$$

where the exchange map is $\tau(x \otimes y) = (-1)^{|x||y|}$. It is also equipped with a degree -1 map $d : \mathcal{C} \to \mathcal{C}$ which is a coderivation, that is

$$\delta d = (d \otimes Id + Id \otimes d)\delta \ ,$$

and with a degree -2 map $\theta : \mathcal{C} \to \mathbb{K}$ which is a curvature, that is

$$d^2 = (\theta \otimes Id)\delta \ .$$

A curved Lie coalgebra $\mathcal{C}$ is said to be *conilpotent* if for any $x \in \mathcal{C}$, there exists an integer n such that any composite element of the form

$$(Id \otimes \cdots \otimes \delta \otimes \cdots \otimes Id) \cdots \delta(x)$$

is zero whenever δ appears n times.

Proposition 41. *The endofunctor of the category of chain complexes $\mathcal{V} \mapsto s\mathcal{V}$ induces an equivalence between $u\mathscr{C}om^{¡}$-coalgebras and curved conilpotent Lie coalgebras.*

Proof. This is a consequence of Theorem 17. □

The bar-cobar adjunction

If we compose the bar-cobar adjunction between $u\mathscr{C}om$-algebras and $u\mathscr{C}om^{¡}$-coalgebras with the equivalence between $u\mathscr{C}om^{¡}$-coalgebras and curved conilpotent Lie coalgebras, then we obtain an adjunction $\Omega_C \dashv B_L$ between unital commutative algebras and curved conilpotent Lie coalgebras which is as follows.

Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}}, 1)$ be a unital commutative algebra. Let $B_L(\mathscr{A})$ be the quasi-cofree curved conilpotent Lie coalgebras

$$B_L(\mathcal{A}) := \mathrm{Lie}^c \circ (s\mathcal{A} \oplus \mathbb{K}v)$$

where Lie^c denotes the Lie cooperad which is the linear dual of the Lie operad. The coderivation extends the map

$$\begin{aligned} \mathrm{Lie}^c(s\mathcal{A} \oplus \mathbb{K}v) \twoheadrightarrow s\mathcal{A} \wedge s\mathcal{A} \oplus s\mathcal{A} \oplus \mathbb{K}v &\to s\mathcal{A} \\ sx \wedge sy &\mapsto (-1)^{|x|} s\gamma_{\mathcal{A}}(x \otimes y) \\ v &\mapsto s1 \\ sx &\mapsto -sdx \ . \end{aligned}$$

The curvature is the map

$$\begin{aligned} \mathrm{Lie}^c(s\mathcal{A} \oplus \mathbb{K} \cdot v) \twoheadrightarrow \mathbb{K} \cdot v &\to \mathbb{K} \\ v &\mapsto 1 \ . \end{aligned}$$

Let $\mathscr{C} = (\mathcal{C}, \delta, d_{\mathcal{C}}, \theta)$ be a curved Lie coalgebra. Let $\Omega_C(\mathscr{C})$ be the free unital commutative algebra

$$\Omega_C\mathcal{C} := S(s^{-1}\mathcal{C}) := \bigoplus_{n \in \mathbb{N}} (s^{-1}\mathcal{C})^{\otimes n}/\mathbb{S}_n \ ,$$

whose coderivation extends the map

$$\begin{aligned} &s^{-1}\mathcal{C} \to S(s^{-1}\mathcal{C}) \\ &s^{-1}x \mapsto \theta(x)1 - s^{-1}d_{\mathcal{C}}x - \sum (-1)^{|x_1|} s^{-1}x_1 \otimes_{\mathbb{S}_2} s^{-1}x_2 \ , \end{aligned}$$

where $\delta(x) = \sum x_1 \wedge x_2$. A twisting morphism from a curved conilpotent Lie coalgebra $\mathscr{C}$ to a unital commutative algebra $\mathscr{A}$ is a degree -1 map $\alpha : \mathcal{C} \to \mathcal{A}$ such that

$$\partial\alpha + \gamma_{\mathscr{A}}(\alpha \otimes \alpha)\delta_{\mathscr{C}} = \theta(-)1_{\mathscr{A}} \ .$$

The Koszul property

Theorem 18. *The operad $u\mathscr{C}om$ is Koszul.*

Proof. We use Theorem 16. We have :

$$qu\mathscr{C}om \circ_\kappa qu\mathscr{C}om^{¡} \simeq \mathbb{K} \cdot v \oplus \bigoplus_{S \subset \{1,\dots,p\}} \mathscr{C}om \circ_\kappa \mathscr{C}om^{¡}(n) \ .$$

We already know that the the canonical morphism $\mathscr{C}om \circ_\kappa \mathscr{C}om^{¡} \to \mathcal{I}$ is a quasi-isomorphism. So for any $n > 1$, the morphism $qu\mathscr{C}om \circ_\kappa qu\mathscr{C}om^{¡}(n) \to \mathcal{I}(n)$ is a quasi-isomorphism. Moreover, in arity zero, $qu\mathscr{C}om \circ_\kappa qu\mathscr{C}om^{¡}(0) \to 0$ is a quasi-isomorphism since the $d(1 \otimes sv) = v$. $\square$

Chapitre 3

Algebraic operads up to homotopy

Introduction

In representation theory, algebras encode some types of endomorphisms on vector spaces, that is linear operations satisfying some relations. More generally, the notion of operads is a tool which governs multilinear operations. More specifically, an operad encodes a type of algebras like associative, commutative, Lie or Batalin–Vilkovisky algebras, in a way that a representation of this operad amounts to the data of a vector space together with a structure of algebra of that type. For instance, the representations of the operad called the Lie operad, see [LV12, 13.2], are vector spaces together with a Lie-algebra structure. The correspondence between operads and their types of algebras is functorial. Indeed, any morphism of operads $f : \mathscr{P} \to \mathscr{Q}$ induces an adjunction between the category of $\mathscr{Q}$-algebras and the category of $\mathscr{P}$-algebras.

This chapter deals with the homotopy theory of differential graded operads over a field of characteristic zero. For any dg operad $\mathscr{P}$, the category of $\mathscr{P}$-algebras admits a projective model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are surjections. Moreover, for any morphism of operads f from $\mathscr{P}$ to $\mathscr{Q}$, the resulting adjunction between the category of $\mathscr{P}$-algebras and the category of $\mathscr{Q}$-algebras is a Quillen equivalence if and only if f is a quasi-isomorphism on the underlying chain complexes of $\mathscr{P}$ and $\mathscr{Q}$. So, quasi-isomorphisms provide a suitable notion of equivalence of dg operads. We know that the category of dg operads carries a model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are surjections, see [Hin97], [Spi01] and [BM03].

Several issues appear when describing the homotopy theory of dg operads with this model structure. For instance, one can ask whether a two dg operads are weakly equivalent ; for example, whether a dg operad is formal, that is weakly equivalent to its homology. Moreover, how to describe in a concrete manner homotopies between morphism ? This last issue is related to the computation of cofibrant resolutions of dg operads. A general tool to produce such cofibrant resolutions is provided by the operadic bar-cobar adjunction introduced first by Getzler and Jones [GJ94] which relates augmented dg operads to dg cooperads. This is generalized in the previous chapter to any kind of dg operads. The absence of augmentation of a dg operad is encoded into a curvature at the level of cooperads. Thus, we have an adjunction relating the category of dg operads to the category curved conilpotent cooperads.

$$\text{Curved conilpotent cooperads} \underset{B_c}{\overset{\Omega_u}{\rightleftarrows}} \text{dg Operads}$$

The importance of this adjunction with respect to the computation of cofibrant operads lies in the fact that for any operad $\mathscr{P}$, the counit map $\Omega_u B_c \mathscr{P} \to \mathscr{P}$ is a cofibrant resolution of $\mathscr{P}$. So, to describe homotopy between morphisms of dg operads from $\mathscr{P}$ to $\mathscr{Q}$, it is convenient to take place in the larger framework of morphisms from $\Omega_u B_c \mathscr{P}$ to $\mathscr{Q}$, which are equivalent to morphisms of curved conilpotent cooperads from $B_c \mathscr{P}$ to $B_c \mathscr{Q}$; so it is convenient to encode the homotopy theory of dg operads not in the category of dg operads itself but in the category of curved conilpotent cooperads. This leads us to the following result.

Theorem. *There exists a model structure on the category of curved conilpotent cooperads whose cofibrations and weak equivalences are created by the cobar construction functor Ω_u. Moreover, the adjunction $\Omega_u \dashv B_c$ is a Quillen equivalence.*

This theorem generalizes results of Lefevre-Hasegawa ([LH03]) and Positselski ([Pos11]) respectively about the homotopy theory of nonunital associative algebras and about the homotopy theory of unital associative algebras. The proof relies on the same kind of method initiated by Hinich. New difficulties appear with the combinatorics of trees and the interplay of symmetric groups.

Why switching from dg operads to curved conilpotent cooperads ? First, all the objects of the model category of curved conilpotent cooperads are cofibrant. Then, the sub-category of fibrant curved conilpotent cooperads is equivalent to a category whose objects and morphisms are an homotopy loosening of respectively the notion of dg operads and the notion of morphisms of dg operads, that we call homotopy operads and ∞-morphisms. These new structures can be built on objects using obstruction methods. Moreover, it is a convenient framework to study formality of dg operads. Indeed, a dg operad $\mathscr{P}$ is formal if and only if there exists an ∞-morphism from $\mathscr{P}$ to its homology and whose first level map is a quasi-isomorphism. Finally, there exists a transfer theorem for homotopy operads as follows.

Theorem. *Let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of dg-$\mathbb{S}$-modules which is both a surjection and a quasi-isomorphism. Suppose that $\mathscr{P}$ has a structure of homotopy operad. Then, there exists a structure of homotopy operad on $\mathscr{Q}$ and an extension of f into an ∞-morphism of homotopy operads.*

One could think that dg operads are themselves algebras over a colored operad and apply the results of the previous chapter to get the present theorems. Actually, the actions of the symmetric groups underlying curved conilpotent cooperads seem to prevent them to be coalgebras over a colored cooperad.

Layout

The chapter is organized as follows. The first section recalls the notions of operads, cooperads and the operadic bar-cobar adjunction. The second one recalls the Hinich model structure on dg operads and describes their cofibrations and a simplicial enrichment computing mapping spaces. The core of the chapter is the third part which establishes the model structure on curved conilpotent cooperads. The fourth section studies in details the fibrant objects of this model category which are a notion of operads up to homotopy that we call homotopy operads.

Preliminaries

We suppose the characteristic of the base field to be zero. The following theorem will be of major use.

Theorem 19 (Maschke)**.** *When the characteristic of the field $\mathbb{K}$ is zero, any module over the ring $\mathbb{K}[\mathbb{S}_n]$ is projective and injective.*

3.1 Complement on operads and cooperads

In this first section, we recall the notions of operads and cooperads that are not already stated in the previous chapter. First, we recall the construction of the tree module and we introduce the alternated tree module. Then, we show that the category of operads is presentable and we prove that it is also the case of rthe category of of curved conilpotent cooperads. Finally, we recall the refined bar construction of Hirsh-Millès adjunction introduced in [HM12].

3.1.1 The tree module

Definition 39 (Tree module)**.** Let $\mathcal{V}$ be an $\mathbb{S}$-module (graded or differential graded) and let T be a nonplanar tree with p vertices and q leaves. Let $\bigotimes_T \mathcal{V}$ be the following graded $\mathbb{K}$-module (resp. chain complex)

$$\bigotimes_T \mathcal{V} := \big(\bigotimes_{v_1,\ldots,v_p} \mathcal{V}(\#v_1) \otimes \cdots \otimes \mathcal{V}(\#v_q)\big)_{\mathbb{S}_p} ,$$

where the multi-tensor product is taken over the bijections from the set $\{1, \cdots, p\}$ to the set of vertices of T. Moreover, for any vertex v_i, $\#v_i$ denotes the number of inputs of v_i. Besides, let $T(\mathcal{V})$ be the $\mathbb{S}$-module such that $T(\mathcal{V})(k) = 0$ for $q \neq k$ and

$$T(\mathcal{V})(q) := \bigoplus_{l_1,\ldots,l_q} \bigotimes_T \mathcal{V} ,$$

where the sum is taken over the bijections from the set $\{1, \cdots, q\}$ to the set of leaves of T. Finally, let $\mathbb{T}\mathcal{V}$ be the following $\mathbb{S}$-module

$$\mathbb{T}\mathcal{V} := \bigoplus_{[T]} T(\mathcal{V}) ,$$

where the sum is taken over the isomorphism class of trees.

Notations.

- ▷ For any $\mathbb{S}$-module $\mathcal{V}$, we denote by $\pi_{\mathcal{V}}$ the canonical projection of $\mathbb{T}\mathcal{V}$ onto $\mathcal{V}$.
- ▷ We denote by $\mathbb{T}^{\leq n}\mathcal{V}$ (resp. $\mathbb{T}^n\mathcal{V}$) the sub-$\mathbb{S}$-module of $\mathbb{T}\mathcal{V}$ made up of trees with n or less than n vertices (resp. made up of n vertices).
- ▷ Let T be a tree. We denote by $\{T = T_1 \sqcup \ldots \sqcup T_k\}$ the partition of T by k subtrees.
- ▷ Let $f : \mathbb{T}\mathcal{V} \to \mathcal{W}$ be a morphism of $\mathbb{S}$-modules. We denote by $f(T)$ the restriction of f on $T(\mathcal{V}) \subset \mathbb{T}\mathcal{V}$. Moreover, if the tree T decomposes into a partition of subtrees $T = T_1 \sqcup \cdots \sqcup T_k$, then we denote by
$$f(T_1) \otimes \cdots \otimes f(T_k)$$
the map from $T(\mathcal{V})$ to $T/T_1, \ldots, T_k(\mathcal{W})$ which consists in applying $f(T_i)$ on any subtree T_i of T.

Proposition 42. *[LV12, §5.5] For any $\mathbb{S}$-module $\mathcal{V}$, the tree module $\mathbb{T}\mathcal{V}$ has a structure of operad given by the grafting of trees. Moreover, the functor $\mathbb{T}$ from the category of $\mathbb{S}$-modules to the category of operads which is left adjoint to the forgetful functor.*

There is a one-to-one correspondence between the degree k derivation on the graded free operad $\mathbb{T}\mathcal{V}$ and the degree k maps from $\mathcal{V}$ to $\mathbb{T}\mathcal{V}$. Indeed, from such a map u one can produce the derivation D_u such that

$$D_u(p_1 \otimes \cdots \otimes p_n) := \sum (-1)^{k(|p_1|+\cdots+|p_{i-1}|)} p_1 \otimes \cdots \otimes u(p_i) \otimes \cdots \otimes p_n .$$

Definition 40 (Alternated tree module)**.** Let $\mathcal{V}$ and $\mathcal{W}$ be two $\mathbb{S}$-modules and let T be a tree. The alternated tree module $T(\mathcal{V}, \mathcal{W})$ is the sub $\mathbb{S}$-module of $T(\mathcal{V} \oplus \mathcal{W})$ made up of labelings of the tree T such that, if a vertex is labelled by an element of $\mathcal{V}$ (resp. $\mathcal{W}$), then its neighbors are labelled by $\mathcal{W}$ (resp. $\mathcal{V}$). Moreover, $\mathbb{T}(\mathcal{V}, \mathcal{W})$ is the sum over the isomorphism classes of trees T of $T(\mathcal{V}, \mathcal{W})$.

Proposition 43. *Let $f : \mathcal{V} \to \mathcal{V}'$ and $g : \mathcal{W} \to \mathcal{W}'$ be two quasi-isomorphisms of dg-$\mathbb{S}$-modules and let T be a tree. Then, the map $T(f) : T(\mathcal{V}) \to T(\mathcal{V}')$ and the map $T(f, g) : T(\mathcal{V}, \mathcal{W}) \to T(\mathcal{V}', \mathcal{W}')$ are quasi-isomorphisms.*

Proof. It follows from the definition of the tree module and the Kunneth formula. □

3.1.2 Presentability

In this section, we prove that both the category of differential graded operads and the category of curved conilpotent cooperads are presentable.

Proposition 44. *The category* $\mathsf{dg-Operad}$ *of differential graded operads is presentable.*

Proof. The tree module endofunctor $\mathbb{T}$ of the category of $\mathbb{S}$-modules is a monad and the category of operads is monadic over this monad. Moreover the functor $\mathbb{T}$ preserves filtered colimits and so is accessible. We conclude by [AR94, Theorem 2.78]. ☐

We will now prove that the category of curved conilpotent cooperads is presentable. The essence of this result is that any cooperad is the colimit of the filtered diagram of its finite dimensional sub-cooperads.

Proposition 45. *The category of curved conilpotent cooperads is presentable.*

Lemma 21. *The category of curved conilpotent cooperads is cocomplete. The forgetful functor from the category of curved conilpotent cooperads to the category of graded conilpotent cooperads preserves and reflects colimits*

Proof. Straightforward. ☐

Definition 41. We say that a $\mathbb{S}$-module $\mathcal{V}$ is a finite dimensional if the $\mathbb{K}$-module $\bigoplus_{n\in\mathbb{N}} \mathcal{V}(n)$ is finite dimensional. We say that $\mathcal{V}$ is aritywise finite dimensional if $\mathcal{V}(n)$ is of finite dimension for any integer $n \in \mathbb{N}$.

Proposition 46. *[AC03, 2.2.5] For any graded cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ and any element $x \in \mathcal{C}(k)$, there exists a finite dimensional sub-cooperad of $\mathscr{C}$ which contains x.*

Corollary 2. *For any curved cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, \theta)$ and any element $x \in \mathcal{C}(k)$, there exists a finite dimensional sub-cooperad of $\mathscr{C}$ which contains x.*

Proof. Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}}, \epsilon)$ be a finite dimensional sub-graded cooperad of $\mathscr{C}$ which contains x. Then $\mathcal{D} + d\mathcal{D}$ is a finite dimensional sub-curved cooperad of $\mathscr{C}$ which contains x. ☐

Lemma 22. *Any finite dimensional curved conilpotent cooperad is a compact object in the category* cCoop.

Proof. The arguments of the proof of Proposition 12 apply mutatis mutandis. ☐

Proof of Proposition 45. By Corollary 2 and Lemma 21, any curved conilpotent cooperad is the colimit of the filtered colimit of its finite dimensional sub-curved conilpotent cooperads which are compact objects by Lemma 22. Moreover, the subcategory of cCoop of finite dimensional objects is equivalent to a small category. ☐

3.1.3 Product of two coaugmented cooperads

We use a result by Aubry and Chataur ([AC03]) relating cooperads to operads in order to describe the product of coaugmented cooperads.

Definition 42 (Profinite operads)**.** Let $\mathsf{prof-gOperad}$ be the pro-category of the category of finite dimensional graded operads. Its objects are complete graded operads, that is graded operads $\mathscr{P}$ such that

$$\lim \mathscr{P}/I \simeq \mathscr{P}$$

where the limit is taken over the finite codimensional ideals of the operad $\mathscr{P}$. The morphisms are the morphisms of operads $f : \mathscr{P} \to \mathscr{Q}$ such that for any finite codimensional ideal I of $\mathscr{Q}$, $f^{-1}(I)$ contains a finite codimensional ideal of $\mathscr{P}$. The inclusion functor from the category of operads to the category of profinite graded operads has a left adjoint $\mathscr{P} \to \hat{\mathscr{P}}$ called the profinite completion. The profinite completion $\hat{\mathscr{P}}$ is the following limit

$$\lim \mathscr{P}/I$$

over the finite codimensional ideals of $\mathscr{P}$.

Proposition 47. *[AC03, 2.2.8] The linear dual of a graded cooperad has a structure of operad. This induces an antiequivalence between the category of graded cooperads* gCoop *and the category of profinite operads* $\mathsf{prof-gOperad}$*.*

Proposition 48. *Let* $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ *and* $\mathscr{D} = (\mathcal{D}, \Delta', \epsilon', 1')$ *be two coaugmented cooperads (graded or differential graded). Then, the product* $\mathscr{C} \times \mathscr{D}$ *in the category of cooperads is the* $\mathbb{S}$*-module* $\mathbb{T}(\overline{\mathcal{C}}, \overline{\mathcal{D}})$ *equipped with the decomposition* $\mathbb{T}(\overline{\mathcal{C}}, \overline{\mathcal{D}}) \to \mathbb{T}(\overline{\mathcal{C}}, \overline{\mathcal{D}}) \circ \mathbb{T}(\overline{\mathcal{C}}, \overline{\mathcal{D}})$ *given by the degrafting of trees and the decomposition inside the cooperads* $\mathscr{C}$ *and* $\mathscr{D}$*.*

Remark 8. Actually, if $\mathscr{C}$ and $\mathscr{D}$ are conilpotent, $\mathbb{T}(\overline{\mathcal{C}}, \overline{\mathcal{D}})$ with its structure of cooperad is a sub-cooperad of $\mathbb{T}^c(\overline{\mathcal{C}} \oplus \overline{\mathcal{D}})$. The inclusion is the following map.

$$\mathbb{T}(\overline{\mathcal{C}}, \overline{\mathcal{D}}) \xrightarrow{\delta_{\mathscr{C}}, \delta_{\mathscr{D}}} \mathbb{T}(\overline{\mathbb{T}}\, \overline{\mathcal{C}}, \overline{\mathbb{T}}\, \overline{\mathcal{D}}) \to \mathbb{T}(\overline{\mathcal{C}} \oplus \overline{\mathcal{D}}) \ .$$

Lemma 23. *Let* $F : I \to \mathsf{gCoop}$ *and* $G : J \to \mathsf{gCoop}$ *be two filtered diagrams of graded cooperads whose images are finite dimensional and whose colimits are respectively* $\mathscr{C}$ *and* $\mathscr{D}$*. Then* $\mathscr{C} \times \mathscr{D}$ *is the colimit of the diagram*

$$F \times G : I \times J \to \mathsf{gCoop}$$
$$F \times G(i,j) := F(i) \times G(j)$$

Proof. Let $\mathscr{E}$ be a cooperad and let $u \times v : \mathscr{E} \to \mathscr{C} \times \mathscr{D}$ be a morphism of cooperads. By Proposition 46, $\mathscr{E}$ is the colimit of the filtered diagram $H : K \to \mathsf{gCoop}$ of its finite dimensional sub-cooperads. For any such sub-cooperad $\mathscr{E}_k$ the morphism u and v factorize respectively through cooperads $F(i)$ and $G(j)$. So $u \times v$ factorizes through $F(i) \times G(j)$. Thus, we obtain a morphism from $\mathscr{E}_k$ to the colimit of the diagram $F \times G$. We even obtain a morphism from $\mathscr{E}$ to $\mathrm{colim} F \times G$. Conversely, any morphism from $\mathscr{E}$ to $\mathrm{colim} F \times G$ induces a morphism from $\mathscr{E}$ to $\mathscr{C} \times \mathscr{D}$. ☐

Proof of Proposition 48. Suppose first that $\mathscr{C}$ and $\mathscr{D}$ are finite dimensional. Let $\mathscr{E}$ be a graded cooperad. Since $\mathscr{C}$ is finite dimensional, then

$$\hom_{\mathsf{prof-gOperad}}(\mathscr{C}^*, \mathscr{E}^*) \simeq \hom_{\mathsf{gOperad}}(\mathscr{C}^*, \mathscr{E}^*) \ .$$

The same statement holds for $\mathscr{D}^*$. So, we have

$$\begin{aligned}
\hom_{\mathsf{prof-gOperad}}(\mathscr{C}^*, \mathscr{E}^*) \times \hom_{\mathsf{prof-gOperad}}(\mathscr{D}^*, \mathscr{E}^*) &\simeq \hom_{\mathsf{gOperad}}(\mathscr{C}^*, \mathscr{E}^*) \times \hom_{\mathsf{gOperad}}(\mathscr{C}^*, \mathscr{E}^*) \\
&\simeq \hom_{\mathsf{gOperad}}(\mathbb{T}(\overline{\mathscr{C}}^*, \overline{\mathscr{D}}^*), \mathscr{E}^*) \\
&\simeq \hom_{\mathsf{prof-gOperad}}(\hat{\mathbb{T}}(\overline{\mathscr{C}}^*, \overline{\mathscr{D}}^*), \mathscr{E}^*) \ .
\end{aligned}$$

Since, the profinite graded operad $\hat{\mathbb{T}}(\overline{\mathscr{C}}^*, \overline{\mathscr{D}}^*)$ is the linear dual of the cooperad $\mathbb{T}(\overline{\mathscr{C}}, \overline{\mathscr{D}})$, then this last cooperad is the product $\mathscr{C} \times \mathscr{D}$. The general case is a consequence of Lemma 23. ☐

3.1.4 Truncated bar construction

In this section, we recall the operadic bar construction of Hirsh-Millès that we call the truncated bar construction ; see [HM12] for the original reference. Let $\mathscr{P} := (\mathcal{P}, \gamma, 1)$ be a dg operad. Suppose that $\mathscr{P}$ is equipped with semi-augmentation, that is a morphism of graded $\mathbb{S}$-modules $\epsilon : \mathcal{P} \to \mathcal{I}$ such that $\epsilon(1) = Id$. We denote by $\overline{\mathcal{P}}$ the kernel of ϵ and we denote by $\pi_{\mathcal{P}}$ the projection of $\mathcal{P}$ onto $\overline{\mathcal{P}}$ along the unit 1. The truncated bar construction of $\mathscr{P}$ relatively to the semi-augmentation ϵ is the cofree conilpotent cooperad $B_r\mathscr{P} := \mathbb{T}^c s\overline{\mathcal{P}}$. It is equipped with the coderivation which extends the map

$$\begin{aligned}
\overline{\mathbb{T}}(s\overline{\mathcal{P}}) \hookrightarrow \overline{\mathbb{T}}^{\leq 2}(s\overline{\mathcal{P}}) &\to s\overline{\mathcal{P}} \\
sx \otimes sy &\mapsto (-1)^{|x|} s\pi\gamma(x \otimes y) \\
sx &\mapsto -s\pi dx
\end{aligned}$$

The curvature θ is the following map

$$\begin{aligned}\mathbb{T}(s\overline{\mathcal{P}}) \hookrightarrow \overline{\mathbb{T}}^{\leq 2}(s\overline{\mathcal{P}}) \to \mathbb{K} \\ sx \otimes sy \mapsto (-1)^{|x|+1}\epsilon\gamma(x \otimes y) \\ sx \mapsto \epsilon(dx)\end{aligned}$$

A truncated twisting morphism from a curved conilpotent cooperad $\mathscr{C}$ to a semi-augmented operad $\mathscr{P}$ is a twisting morphism $\alpha : \overline{\mathcal{C}} \to \mathcal{P}$ such that $\epsilon\alpha = 0$. We denote by $trTw(\mathscr{C}, \mathscr{P})$ the set of twisting morphism from a curved conilpotent cooperad to a semi-augmented dg operad $\mathscr{P}$.

Proposition 49 ([HM12])**.** *For any semi-augmented operad $\mathscr{P}$ and any curved conilpotent cooperad $\mathscr{C}$, we have functorial isomorphisms :*

$$\hom_{\mathsf{sa-dg-Operad}}(\Omega_u\mathscr{C}, \mathscr{P}) \simeq trTw(\mathscr{C}, \mathscr{P}) \simeq \hom_{\mathsf{cCoop}}(\mathscr{C}, B_r\mathscr{P}) \ ,$$

where $\mathsf{sa} - \mathsf{dg} - \mathsf{Operad}$ *is the category of semi-augmented dg operads.*

For any semi-augmented dg operad, the universal truncated twisting morphism

$$B_r\mathscr{P} = \mathbb{T}s\overline{\mathcal{P}} \twoheadrightarrow s\overline{\mathcal{P}} \xrightarrow{s^{-1}} \overline{\mathcal{P}} \hookrightarrow \mathcal{P}$$

is in particular a twisting morphism. So it induces a morphism of curved conilpotent cooperad from $B_r\mathscr{P}$ to $B_c\mathscr{P}$.

3.2 Model structure on operads

This section deals with the homotopy theory of dg operads. We recall the result proved by Hinich in [Hin97] that there exists a model structure on the category of dg operads whose weak equivalences are quasi-isomorphisms and whose fibrations are surjections. Then, we describe the cofibrations in a convenient way to be able to use it in the sequel.

Notations. We denote by S^k_n (resp. D^k_n) the dg-$\mathbb{S}$-module which is the chain complex S^k (resp. D^k) is arity n and zero in other arities.

3.2.1 Model structure on $\mathbb{S}$-modules

For any integer $n \in \mathbb{N}$, there exists a cofibrantly generated model structure on the category of chain complexes of $\mathbb{K}[\mathbb{S}_n]$-modules whose fibrations (resp. weak equivalences) are epimorphisms (resp. quasi-isomorphisms) (see [Hovey, 2.3]). Moreover, the cofibrations are exactly the monomorphisms whose cokernel is degreewise a projective $\mathbb{K}[\mathbb{S}_n]$-module. Notice that any monomorphism of $\mathbb{K}[\mathbb{S}_n]$-modules with projective cokernel is split. Since the characteristic of $\mathbb{K}$ is zero, by Maschke's Theorem, any monomorphism of chain complex of $\mathbb{K}[\mathbb{S}_n]$-modules is a cofibration.

Subsequently, there exists a cofibrantly generated model structure on the category of dg $\mathbb{S}$-modules whose cofibrations (resp. fibrations, resp. weak equivalences) are exactly monomorphims (resp. epimorphisms, resp. quasi-isomorphisms). Moreover, a set of generating cofibration is made up of the maps $0 \to S^0_n$ and $S^k_n \to D^{k+1}_n$; a set of generating acyclic cofibrations is made up of the maps $0 \to D^k_n$.

3.2.2 Model structure on operads

Consider the adjunction

$$\mathrm{dg}\ \mathbb{S} - \mathrm{mod} \underset{U}{\overset{\mathbb{T}}{\rightleftarrows}} \mathsf{dg} - \mathsf{Operad}\ ,$$

where U is the forgetful functor. Transferring this model structure along this adjunction gives the following Theorem.

Theorem 20 ([Hin97]). *The category of dg operads admits a cofibrantly generated model structure where the weak equivalence (resp. fibrations) are the componentwise quasi-isomorphisms (resp. epimorphisms). The generating cofibrations are the map $\mathcal{I} \to \mathbb{T}(S_n^0)$ and the maps $\mathbb{T}(S_n^k) \to \mathbb{T}(D_n^{k+1})$. The generating acyclic cofibrations are the maps $\mathcal{I} \to \mathbb{T}(D_n^k)$.*

3.2.3 Cofibrations of operads

We prove the following proposition in the vein of [MV09, Appendix 1].

Proposition 50. *Cofibrations of operads are exactly retracts of morphisms $\mathscr{P} \to \mathscr{P} \vee \mathbb{T}S$ where S is a $\mathbb{S}$-module endowed with an exhaustive filtration*

$$\{0\} = S_0 \subset S_1 \subset ... \subset \mathrm{colim}_{\beta<\alpha} S_\alpha \ ,$$

indexed by an ordinal α and such that

$$d(S_{i+1}) \subset S_{i+1} \oplus \mathscr{P} \vee \mathbb{T}(S_i) \ .$$

Proof. Given the generating cofibrations of the category of operads given in Theorem 20 and by (Hovey), any cofibration of operads is a retract of a morphism $\mathscr{P} \to \mathscr{P} \vee \mathbb{T}S$ as in Proposition 50 with the additional conditions that the cokernel of the inclusions $S_i \to S_{i+1}$ are free $\mathbb{S}$-modules (that is the cokernel is a free $\mathbb{K}[\mathbb{S}_n]$-module in arity n) and that

$$d(S_{i+1}) \subset \mathscr{P} \vee \mathbb{T}(S_i) \ .$$

Conversely, consider a morphism of operads $f : \mathscr{P} \to \mathscr{P} \vee \mathbb{T}S$ such that $d(S) \subset S \oplus \mathscr{P}$. It fill the following pushout diagram

$$\begin{tikzcd}
\mathbb{T}(s^{-1}S) \arrow[r] \arrow[d] & \mathscr{P} \arrow[d] \\
\mathbb{T}(S \oplus s^{-1}S) \arrow[r] & \mathscr{P} \vee \mathbb{T}S
\end{tikzcd}$$

where $S \oplus s^{-1}S$ is endowed with the differential $x + s^{-1}y \mapsto dx + s^{-1}x - s^{-1}dy$. Since the map $s^{-1}S \to S \oplus s^{-1}S$ is a cofibration of dg $\mathbb{S}$-modules (since it is a monomorphism), then $\mathbb{T}(s^{-1}S) \to \mathbb{T}(S \oplus s^{-1}S)$ is a cofibration of operads and so f is a cofibration. Any morphism $\mathscr{P} \to \mathscr{P} \vee \mathbb{T}S$ as in Proposition 50 is a transfinite composite of morphisms as f and so is a cofibration. $\square$

3.2.4 Enrichment in simplicial sets

Let $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}})$ be an operad and let $\mathscr{A} := (\mathcal{A}, \gamma_{\mathscr{A}}, 1_{\mathscr{A}})$ be a unital commutative algebra. Let $\mathcal{P} \otimes \mathcal{A}$ be the $\mathbb{S}$-module defined by

$$(\mathcal{P} \otimes \mathcal{A})(m) := \mathcal{P}(m) \otimes \mathcal{A} \ .$$

It has an obvious structure of operad. This construction is functorial.

Besides for any integer $n \in \mathbb{N}$, let Ω_n be the unital commutative algebra

$$\Omega_n := \mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(\Sigma t_i = 1; \Sigma dt_i = 0) \ .$$

The construction $n \mapsto \Omega_n$ defines a simplicial unital commutative algebra. This provides an enrichment of the category of dg operads over simplicial sets as follows :

$$HOM(\mathscr{P}, \mathscr{Q})_n := \hom_{\mathsf{dg-Operad}}(\mathscr{P}, \mathscr{Q} \otimes \Omega_n) = \hom_{\mathsf{dg-Operad}/\Omega_n}(\mathscr{P} \otimes \Omega_n, \mathscr{Q} \otimes \Omega_n) \ .$$

Proposition 51. *For any dg operads $\mathscr{P}$ and $\mathscr{Q}$ with $\mathscr{P}$ cofibrant, the simplicial set $HOM(\mathscr{P}, \mathscr{Q})$ is a model for the mapping space* $\mathrm{Map}(\mathscr{P}, \mathscr{Q})$.

Proof. It suffices to notice that the simplicial operad $(\mathscr{Q} \otimes \Omega_n)_{n\in\mathbb{N}}$ is a Reedy fibrant replacement of the constant simplicial operad $\mathscr{Q}$. $\square$

3.3 Model structure on curved conilpotent cooperads

In this section, we show that the model structure on the category of dg operads can be transferred through the cobar construction functor to the category of curved conilpotent cooperads. This result is in the vein of earlier results by Hinich [Hin01], Lefevre-Hasegawa [LH03], Vallette [Val14] and Positselski [Pos11]. Our proof relies on the same kind of method ; however new difficulties appear with the combinatorics of trees and actions of symmetric groups.

3.3.1 Statement of the result

Here is the main result of this chapter. The remaining of this section will be made up of its proof.

Theorem 21. *There exists a model structure on the category of curved conilpotent cooperads whose cofibrations (resp. weak equivalences) are the morphisms whose image under the cobar construction functor Ω_u is a cofibration (resp. a weak equivalence). Moreover, the adjunction $\Omega_u \dashv B_c$ is a Quillen equivalence.*

From now on, we call *cofibration* (resp. *weak equivalence*) of curved conilpotent cooperads the morphisms whose image under the cobar functor Ω_u is a cofibration (resp. weak equivalence). Moreover, we call *acyclic cofibrations* the morphisms which are both cofibrations and weak equivalences and we call *fibrations* the morphisms which have the right lifting property with respect to acyclic cofibrations. Finally, we call *acyclic fibrations* the morphisms which are both fibrations and weak equivalences.

3.3.2 Cofibrations

We describe the cofibrations of curved conilpotent cooperads.

Proposition 52. *The cofibrations of curved conilpotent cooperads are the monomorphisms.*

Proof. Let $f : \mathscr{C} \to \mathscr{D}$ be a monomorphism of curved conilpotent cooperads. It is the transfinite composite of the morphisms

$$f_n : \mathscr{D}_{[n]} \to \mathscr{D}_{[n+1]} \ ,$$

where $\mathscr{D}_{[k]} = \mathscr{C} + F_n^{rad}\mathscr{D} \subset \mathscr{D}$. Then, $\Omega_u(f_n)$ is a morphism of operads of the form $\mathscr{P} \to \mathscr{P} \vee \mathbb{T}(S)$ as in Proposition 50. So, $\Omega_u(f_n)$ is a cofibration. Since cofibration of operads are stable under transfinite composition, then the transfinite composition of the morphism $\Omega_u(f_n)$ is a cofibration ; this morphism is $\Omega_u(f)$. So f is a cofibration. Conversely, if $f : \mathscr{C} \to \mathscr{D}$ is a cofibration, then $\Omega_u(f)$ is a cofibration and in particular a monomorphism. So the composite map $s^{-1}\overline{\mathcal{C}} \hookrightarrow \Omega_u\mathscr{C} \to \Omega_u\mathscr{D}$ is a monomorphism. Hence, the map $\overline{\mathcal{C}} \to \overline{\mathcal{D}}$ is a monomorphism and so f is a monomorphism. □

3.3.3 Weak equivalences and filtered quasi-isomorphisms

Weak equivalences of curved conilpotent cooperads are morphisms whose image under the functor cobar Ω_u is a quasi-isomorphism. Giving their explicit description is not an easy task. A sufficient condition for a morphism to be a weak equivalence is to be a filtered quasi-isomorphism.

Definition 43 (Admissible filtrations and filtered quasi-isomorphisms)**.** Let $\mathscr{C} := (\mathcal{C}, \Delta, \epsilon, 1, \theta)$ be a curved conilpotent cooperad. An admissible filtration $(F_n\mathcal{C})_{n\in\mathbb{N}}$ of $\mathscr{C}$ is an exhaustive filtration of the $\mathbb{S}$-module $\mathcal{C}$ satisfying the following conditions.

$$\begin{cases} d(F_n\mathcal{C}) \subset F_n\mathcal{C} \ , \\ \Delta(F_n\mathcal{C})(m) \subset \sum_{\substack{p_0+\cdots+p_k \leq n \\ X_1\sqcup\cdots\sqcup X_k=\{1,\ldots,m\}}} (F_{p_0}\mathcal{C})(k) \otimes_{\mathbb{S}_k} (F_{p_1}\mathcal{C}(\#X_1) \otimes \cdots \otimes F_{p_k}\mathcal{C}(\#X_k)) \ , \\ F_0\mathcal{C} := \mathcal{I}. \end{cases}$$

Let $\mathscr{C}$ and $\mathscr{D}$ be two curved conilpotent cooperads both equipped with an admissible filtration. A filtered quasi-isomorphism from $\mathscr{C}$ to $\mathscr{D}$ relative to these two filtrations is a morphism $f : \mathscr{C} \to \mathscr{D}$, such that the induced morphism

$$Gf : G\mathscr{C} \to G\mathscr{D}$$

is a quasi-isomorphism.

EXAMPLE 2. We know from the previous chapter that the coradical fitration of a curved conilpotent cooperad is admissible.

Proposition 53. *A filtered quasi-isomorphism is a weak equivalence.*

We will use the following Theorem to prove this proposition.

Theorem 22. *[ML95, XI.3.4] Let $f : \mathcal{V} \to \mathcal{W}$ be a map of filtered chain complexes. Suppose that the filtrations are bounded below and exhaustive. If for any integer n, the map $G_n\mathcal{V} \to G_n\mathcal{W}$ is a quasi-isomorphism, then f is a quasi-isomorphism.*

Proof of Proposition 53. Let $f : \mathscr{C} \to \mathscr{D}$ be a filtered quasi-isomorphism. Consider the following filtration on $\Omega_u\mathscr{C}$:

$$F_n\Omega_u\mathscr{C} := \mathbb{K} \cdot 1 \oplus \bigoplus_k \sum_{i_1+\cdot+i_k=n} s^{-1}F_{i_1}\overline{\mathcal{C}} \otimes \cdots \otimes s^{-1}F_{i_k}\overline{\mathcal{C}}$$

for n varying from 0 to ∞. Let us endow $\Omega_u\mathscr{D}$ with a filtration built in the same fashion. Then, for any integer n, let us endow $G_n\Omega_u\mathscr{C}$ with the following filtration

$$F'_kG_n\Omega_u\mathscr{C} := \bigoplus_{-p\leq k} \sum_{i_1+\cdots+i_p=n} G_{i_1}\overline{\mathcal{C}} \otimes \cdots \otimes s^{-1}G_{i_p}\overline{\mathcal{C}}$$

for k varying from $-p$ to 0. Again, we endow $G_n\Omega_u\mathscr{D}$ with a filtration built in the same fashion. Then, the map

$$G'G_nf : G'G_n\Omega_u\mathscr{D} \to G'G_n\Omega_u\mathscr{C}$$

is a quasi-isomorphism. We conclude by Theorem 22. □

Proposition 54. *Let $\mathscr{P}$ be an operad together with a semi-augmentation $\epsilon : \mathcal{P} \to \mathcal{I}$. Then the canonical morphism $B_r\mathscr{P} \to B_c\mathscr{P}$ is a filtered quasi-isomorphism with respect to the coradical filtrations.*

Proof. It suffices to notice that the morphism of chain complex

$$\begin{aligned} s\overline{\mathcal{P}} &\to s\mathcal{P} \oplus \mathbb{K} \cdot v \\ sx &\mapsto sx + \theta(sx)v \end{aligned}$$

is a quasi-isomorphism. □

3.3.4 Bar-cobar and cobar-bar resolutions

Let $\mathcal{V}$ be a dg $\mathbb{S}$-module. The tree module $\mathbb{T}(s^{-1}\overline{\mathbb{T}}\mathcal{V})$ has both a structure of operad and cooperad. Let D be the derivation which makes of $\mathbb{T}(s^{-1}\overline{\mathbb{T}}\mathcal{V})$ the cobar construction of the dg conilpotent cooperad $\mathbb{T}\mathcal{V}$, that is :

$$s^{-1}A \to -s^{-1}dA - \sum(-1)^{|A_1|}s^{-1}A_1 \otimes s^{-1}A_2$$

for any $A \in \overline{\mathbb{T}}\mathcal{V}$, where $\Delta_2A = \sum A_1 \otimes A_2$. Besides, let h be the degree 1 coderivation of the cooperad $\mathbb{T}(s^{-1}\overline{\mathbb{T}}\mathcal{V})$ which extends the following map

$$\begin{aligned} \mathbb{T}(s^{-1}\overline{\mathbb{T}}\mathcal{V}) \twoheadrightarrow \mathbb{T}^2(s^{-1}\overline{\mathbb{T}}\mathcal{V}) &\to \overline{\mathbb{T}}\mathcal{V} \\ s^{-1}A_1 \otimes s^{-1}A_2 &\mapsto s^{-1}(A_1 \otimes A_2) \end{aligned}$$

Lemma 24. *Let T be a tree with k vertices ordered from 1 to k and let T_1, ..., T_k be non trivial trees. Consider the sub $\mathbb{S}$-module of $\mathbb{T}(s^{-1}\overline{\mathbb{T}}\mathcal{V})$ made up of the tree T whose i^{th} vertex is labelled by $T_i(\mathcal{V})$. On this submodule, we have :*

$$Dh + hD = qId \ ,$$

where q is the sum of the numbers of inner edges of T and of the trees T_i.

Proof. Let a be an inner edge of the tree T. It links two vertices which are labelled respectively by the tree modules $s^{-1}T_i(\mathcal{V})$ and the tree module $s^{-1}T_j(\mathcal{V})$. The derivation h consists in grafting the tree T_i with the tree T_j for any inner edge a. We can write :

$$h(T) := \sum_{a \in inner(T)} \mathrm{graft}(a) \ ,$$

where $inner(T)$ is the set of inner edges of T. Moreover, let x be a vertex of the tree T. It is labelled by the tree module $sT_i(\mathcal{V})$. The derivation D consists in cutting the tree module $s^{-1}T_i(\mathcal{V})$ into two trees $s^{-1}T - i, 1(\mathcal{V}) \otimes s^{-1}T_{i,2}(\mathcal{V})$ along any inner edge of T_i, plus apply the differential d of $\mathcal{V}$; and that is done for any vertex x of the tree T. So, we can write

$$D(T) = \sum_{a \in inner(T_1,\ldots,T_k)} \mathrm{cut}(a) \sum_{v \in vert(T_1,\ldots,T_k)} \mathrm{diff}(x) \ ,$$

where $inner(T_1, \ldots, T_k)$ is the set of inner edges of the trees T_1, ..., T_k and $vert(T_1, \ldots, T_k)$ is the set of vertices of the trees T_1, ..., T_k. So, we have :

$$hD = \sum_{\substack{a \in inner(T_1,\ldots,T_k) \\ b \in inner(T)}} \mathrm{graft}(b)\mathrm{cut}(a) + \sum_{a \in inner(T_1,\ldots,T_k)} \mathrm{graft}(a)\mathrm{cut}(a) + \sum_{\substack{v \in vert(T_1,\ldots,T_k) \\ a \in inner(T)}} \mathrm{graft}(a)\mathrm{diff}(x) \ .$$

On the other hand,

$$Dh = \sum_{a \in inner(T)} \mathrm{cut}(a)\mathrm{graft}(a) + \sum_{\substack{a \in inner(T_1,\ldots,T_k) \\ b \in inner(T)}} \mathrm{cut}(a)\mathrm{graft}(b) + \sum_{\substack{v \in vert(T_1,\ldots,T_k) \\ a \in inner(T)}} \mathrm{diff}(x)\mathrm{graft}(a) \ .$$

For any inner edge a of the trees T_1, ..., T_k and for any inner edge b of the tree T, $\mathrm{cut}(a)\mathrm{graft}(b) + \mathrm{graft}(b)\mathrm{cut}(a)$. Moreover, for any inner edge a of the tree T and for any vertex x of the trees T_1, ..., T_k, $\mathrm{diff}(x)\mathrm{graft}(a) + \mathrm{diff}(x)\mathrm{graft}(a) = 0$. Finally, for any inner edge a of the trees T_1, ..., T_k and T, $\mathrm{cut}(a)\mathrm{graft}(a) + \mathrm{graft}(a)\mathrm{cut}(a) = Id$. ☐

Similarly, the $\mathbb{S}$-module $\mathbb{T}(s\overline{\mathbb{T}}\mathcal{V})$ has a structure of operad and a structure of cooperad. Let D be the coderivation which makes of $\mathbb{T}(s(\mathbb{T}\mathcal{V}) \oplus \mathbb{K} \cdot v)$ the truncated bar construction of the dg operad $\mathbb{T}\mathcal{V}$, that is : derivation

$$\begin{gathered} sA_1 \otimes sA_2 \mapsto (-1)^{|A_1|} s(A_1 \otimes A_2) \\ sA \mapsto -sdA \end{gathered}$$

for any $A \in \overline{\mathbb{T}}\mathcal{V}$. Moreover, let h be the degree 1 derivation which extends the following map.

$$\begin{aligned} s\overline{\mathbb{T}}\mathcal{V} &\to \mathbb{T}(s\overline{\mathbb{T}}\mathcal{V}) \\ sA &\mapsto \sum (-1)^{|A_1|} sA_1 \otimes sA_2 \end{aligned}$$

where $\Delta_2 A = \sum A_1 \otimes A_2$ for any $A \in \overline{\mathbb{T}}\mathcal{V}$.

Lemma 25. *Let T be a tree with k vertices ordered from 1 to k and let T_1, ..., T_k be non trivial trees. Consider the sub $\mathbb{S}$-module of $\mathbb{T}(s\overline{\mathbb{T}}\mathcal{V})$ made up of the tree T whose i^{th} vertex is labelled by $T_i(\mathcal{V})$. On this submodule, we have :*

$$Dh + hD = qId \ ,$$

where q is the sum of the numbers of inner edges of T and of all the trees T_i.

Proof. The proof relies on the same techniques as Lemma 24. □

Proposition 55. *Let $\mathscr{P}$ be a dg operad. Then the canonical morphism $p : \Omega_u B_c \mathscr{P} \to \mathscr{P}$ is a weak equivalence.*

Proof. Consider the following filtration on $\Omega_u B_c \mathscr{P}$:

$$F_n \Omega_u B_c \mathscr{P} := \mathcal{I} \oplus \bigoplus_{k \geq 1} \sum_{i_1 + \cdots + i_k = n} s^{-1} F^{rad}_{i_1} \overline{B}_c \mathscr{P} \otimes \cdots \otimes s^{-1} F^{rad}_{i_1} \overline{B}_c \mathscr{P} \ , \ n \geq 1 \ .$$

Consider also the constant filtration $(F_n \mathscr{P})_{n \geq 1}$ on $\mathscr{P}$. We show that the morphisms $G_n p : G_n \Omega_u B_c \mathscr{P} \to G_n \mathscr{P}$ are quasi-isomorphisms. On the one hand, $G_1 p : \mathbb{K} \cdot 1 \oplus \mathbb{K} \cdot s^{-1} v \oplus s^{-1} s \mathscr{P} \to \mathscr{P}$ is a quasi-isomorphism. On the other hand, for any $n > 1$, $G_n \mathscr{P} = 0$ and $G_n \Omega_u B_c \mathscr{P}$ is contractible by Lemma 24. We conclude by Theorem 22. □

A straightforward consequence of the above Proposition 55 is that for any curved conilpotent cooperad $\mathscr{C}$, the map $\mathscr{C} \to B_r \Omega_u \mathscr{C}$ is a weak equivalence. Indeed, since the morphism $\Omega_u B_r \Omega_u \mathscr{C} \to \Omega_u B_r \Omega_u \mathscr{C}$ is a quasi-isomorphism, then $\Omega_u B_r \Omega_u \mathscr{C} \to \Omega_u \mathscr{C}$ is a quasi-isomorphism and so its right inverse $\Omega_u \mathscr{C} \to \Omega_u B_r \Omega_u \mathscr{C}$ is also a quasi-isomorphism. The following proposition is a more precise statement.

Proposition 56. *Let $\mathscr{C}$ be a curved conilpotent cooperad. Let us endow $\mathscr{C}$ with its coradical filtration and let us endow $B_r \Omega_u \mathscr{C}$ with the following filtration :*

$$F_n B_r \Omega_u \mathscr{C} := \mathcal{I} \oplus \sum_{\substack{i_1 + \cdot + i_k = n \\ k \geq 1}} s F_{i_1} \overline{\Omega}_u \mathscr{C} \otimes \cdots \otimes s F_{i_k} \overline{\Omega}_u \mathscr{C} \ , \ n \geq 0 \ ,$$

where

$$F_i \overline{\Omega}_u \mathscr{C} := \sum_{\substack{j_1 + \cdot + j_k = i \\ k \geq 1}} s^{-1} F^{rad}_{j_1} \overline{\mathcal{C}} \otimes \cdots \otimes s^{-1} F^{rad}_{j_k} \overline{\mathcal{C}}, \ i \geq 1 \ .$$

These two filtration are admissible and the canonical morphism $\mathscr{C} \to B_c \Omega_u \mathscr{C}$ is a filtered quasi-isomorphism.

Proof. Let $n \geq 1$. Let us show that the morphism $G_n \mathscr{C} \to G_n B_r \Omega_u \mathscr{C}$ is a quasi-isomorphism. Consider the filtration $(F'_k G_n B_r \Omega_u \mathscr{C})_{k=-n}^{-1}$ on $G_n B_c \Omega_u \mathscr{C}$ where $F'_k G_n B_c \Omega_u \mathscr{C}$ is made up of the trees whose vertices are labeled by trees whose total number of vertices is at least $-k$. Consider also the filtration $(F'_k G^{rad}_n \mathscr{C})_{k=-n}^{-1}$ of $G^{rad}_n \mathscr{C}$ such that $F'_k G^{rad}_n \mathscr{C} = 0$ for $k < -1$ and $F'_{-1} G^{rad}_n \mathscr{C} = G^{rad}_n \mathscr{C}$. The map $G'_{-1} G^{rad}_n \mathscr{C} \to G'_{-1} G_n B_r \Omega_u \mathscr{C}$ is a quasi-isomorphism; that is the identity of $G^{rad}_n \mathscr{C}$. Moreover, $G'_k G_n B_r \Omega_u \mathscr{C}$ for $k \neq -1$ is contractible by Lemma 25. □

3.3.5 Key lemma

Lemma 26 (Key Lemma)**.** *Let $\mathscr{C}$ be a curved conilpotent cooperad and let $p : \mathscr{P} \to \Omega_u \mathscr{C}$ be a fibration of operads (that is a surjection). Consider the following square :*

$$\begin{tikzcd} B_r \mathscr{P} \arrow[r, "B_r p"] & B_r \Omega_u \mathscr{C} \\ \mathscr{D} \arrow[u] \arrow[r] & \mathscr{C} \arrow[u] \end{tikzcd}$$

where $\mathscr{D}$ is the pullback $B_r \mathscr{P} \times_{B_r \Omega_u \mathscr{C}} \mathscr{C}$. Then, the morphism $\mathscr{D} \to B_r \mathscr{P}$ is a weak equivalence.

Lemma 27. *The curved conilpotent cooperad $\mathscr{D}$ is the biggest sub-graded-cooperad of $B_r \mathscr{P}$ whose image under $B_r(p)$ lies inside $\mathscr{C}$.*

Proof. Let $\mathscr{E}$ be the biggest sub-graded-cooperad of $B_r \mathscr{P}$ whose image under $B_r(p)$ lies inside $\mathscr{C}$. It suffices to prove that $\mathscr{E}$ is stable under the coderivation of $B_c \mathscr{P}$. □

Proof of Lemma 26. By Maschke's Theorem, there exists a map of graded $\mathbb{S}$-modules $\tilde{i} : \Omega_u \mathscr{C} \to \mathcal{P}$ such that $p\tilde{i} = Id_{\Omega_u \mathscr{C}}$. The restriction of $\tilde{i}$ to $s^{-1}\mathcal{C}$ extends to a morphism of graded operads $i : \Omega_u \mathscr{C} \to \mathscr{P}$. We again have $pi = Id$. Subsequenty, let K be the kernel of p. We have the following isomorphism of graded cooperads :

$$\mathscr{D} \simeq \mathscr{C} \times \mathbb{T}^c(sK) = \mathbb{T}^c(\mathscr{C}, \mathbb{T}(sK))$$

Let us endow $B_r \mathscr{P}$ with the following filtration

$$F_n B_r \mathscr{P} := \mathcal{I} \oplus \sum_{\substack{i_1 + \cdots + i_k = n \\ k \geq 1}} sF_{i_1}\overline{\mathcal{P}} \otimes \cdots \otimes sF_{i_k}\overline{\mathcal{P}},\ n \geq 0\ ,$$

where

$$F_i \overline{\mathcal{P}} := K \oplus \sum_{\substack{j_1 + \cdots + j_k = i \\ k \geq 1}} s^{-1} F^{grad}_{j_1} \overline{\mathcal{C}} \otimes \cdots \otimes s^{-1} F^{grad}_{j_k} \overline{\mathcal{C}}\ ,\ i \geq 1$$

This induces a filtration on $\mathscr{D}$. These two filtrations are admissible. Let us show that the morphism $i : \mathscr{D} \to B_r \mathscr{P}$ is a filtered quasi-isomorphism. The dg $\mathbb{S}$-modules $G_n \mathscr{D}$ and $G_n B_r \mathscr{P}$ are made up of trees whose vertices are labelled alternatively by sK and $\mathscr{C}$ (resp. $B_r \Omega_u \mathscr{C}$). If we denote by $F'_k G_n \mathscr{C}$ the sub dg $\mathbb{S}$-module made up of trees such that at least $-k$ vertices are labelled by sK, we obtained a bounded below filtration on $G_n \mathscr{C}$; moreover, we define the filtration $F' G_n B_r \Omega_u \mathscr{C}$ in the same fashion. The map

$$G' G_n \mathscr{C} \to G' G_n B_r \Omega_u \mathscr{C}$$

is a quasi-isomorphism by Proposition 56. We conclude by Theorem 22 and Proposition 53. □

Remark 9. The curved conilpotent cooperad $\mathscr{D}$ of the key lemma is also the pullback $B_c \mathscr{P} \times_{B_c \Omega_u \mathscr{C}} \mathscr{C}$.

3.3.6 Proof of Theorem 21

We gather the results proven above to prove Theorem 21. We use the same steps as the proof of Theorem 3.1 in [Hin01].

Proof of Theorem 21.

- ▷ The category of curved conilpotent is presentable. So, it is complete and cocomplete.
- ▷ Let f and g be two composable morphisms of curved conilpotent cooperads. It is clear that f, and g and fg are all weak equivalences if two of them are weak equivalences since it is the case for $\Omega_u f$, $\Omega_u g$ and $\Omega_u fg$.
- ▷ Cofibrations and weak equivalences are stable under retracts because it is the case for cofibrations and weak equivalences of operads. Since they are the morphisms which satisfy the right lifting property with respect to acyclic cofibrations, the fibrations are also stable under retracts.
- ▷ Let $f : \mathscr{C} \to \mathscr{D}$ be a morphism of curved conilpotent cooperads. Let us factorize the morphism of operads $\Omega_u(f)$ by a cofibration followed by an acyclic fibration $\Omega_u \mathscr{C} \to \mathscr{P} \to \Omega_u \mathscr{D}$ (resp. an acyclic cofibration followed by a fibration). By Lemma 26, the following diagram provides us with a factorization of f by a cofibration followed by an acyclic fibration (resp. an acyclic cofibration followed by a fibration).

$$\begin{tikzcd}
B_c \Omega_u \mathscr{C} \arrow[r] & B_c \mathscr{P} \arrow[r] & B_c \Omega_u \mathscr{D} \\
\mathscr{C} \arrow[r] \arrow[u] & B_c \mathscr{P} \times_{B_c \Omega_u \mathscr{D}} \mathscr{D} \arrow[r] \arrow[u] & \mathscr{D} \arrow[u]
\end{tikzcd}$$

- ▷ Consider the following square of curved conilpotent cooperads,

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d, "f"'] & \mathscr{E} \arrow[d, "g"] \\
\mathscr{D} \arrow[r] & \mathscr{F}
\end{tikzcd}$$

where f is a cofibration and g is an acyclic fibration. By Lemma 26, g can be factorized as follows

$$\mathscr{E} \xrightarrow{g_1} B_c\mathscr{P} \times_{B_c\Omega_u\mathscr{F}} \mathscr{F} \xrightarrow{g_2} \mathscr{F}$$

where g_1 is an acyclic cofibration and where g_2 is the pullback of a map $B_c\mathscr{P} \to B_c\Omega_u\mathscr{F}$ which is the image under the functor B_c of an acyclic fibration of operads $\mathscr{P} \to \Omega_u\mathscr{F}$. Since, f has the left lifting property with respect to this map $\mathscr{P} \to \Omega_u\mathscr{F}$, then it has the left lifting property with respect to g_2 Moreover, the following square has a lifting.

$$\begin{tikzcd} \mathscr{E} \arrow[r, "="] \arrow[d, "g_1"'] & \mathscr{E} \arrow[d, "g"] \\ B_c\mathscr{P} \times_{B_c\Omega_u\mathscr{F}} \mathscr{F} \arrow[r, "g_2"'] & \mathscr{F} \end{tikzcd}$$

The composition of these two liftings gives us a lifting of the first square.

- ▷ At this point, we have proved the existence of the model structure on the category of curved conilpotent cooperads. Obviously, the adjunction $\Omega_u \dashv B_c$ is a Quillen adjunction. It is a Quillen equivalence by Proposition 55.

□

3.3.7 Fibrations

Proposition 57. *The fibrations are the retract of pullback of maps of the form $B_c(f) : B_c\mathscr{P} \to B_c\mathscr{Q}$ where $f : \mathscr{P} \to \mathscr{Q}$ is a surjection of operads.*

Lemma 28. *A fibration of curved conilpotent cooperads is surjective.*

Proof. Let $g : \mathscr{C} \to \mathscr{D}$ be a fibration of curved conilpotent cooperads. Let $\mathscr{E}$be the curved conilpotent cooperad $\mathscr{E} := \mathcal{I} \oplus \overline{\mathscr{D}} \oplus s^{-1}\overline{\mathscr{D}}$. The decomposition Δ is defined as follows.

$$\begin{cases} \Delta x := \Delta_{\mathscr{D}} x, \text{ if } x \in \mathscr{D} \subset \mathscr{E} \ , \\ \Delta s^{-1}x := (s^{-1} \circ Id + Id \circ' s^{-1})\Delta x \ . \end{cases}$$

Moreover, the coderivation sends x to $s^{-1}x$ and $s^{-1}x$ to 0. Consider the following square.

$$\begin{tikzcd} \mathcal{I} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "g"] \\ \mathscr{E} \arrow[r] & \mathscr{D} \end{tikzcd}$$

Since the morphism $\mathcal{I} \to \mathscr{E}$ is a filtered quasi-isomorphism and a monomorphism, then it is an acyclic fibration. So, the square has a lifting. Subsequently the morphism $\mathscr{C} \to \mathscr{D}$ is surjective. □

Proof of Proposition 57. It is clear a retract of a pullback of a map $B_c(f)$ where f is a surjection is a fibration. Conversely, let $g : \mathscr{C} \to \mathscr{D}$ be a fibration of curved conilpotent cooperads. Consider the following diagram

$$\begin{tikzcd} B_c\Omega_u\mathscr{C} \arrow[rr] & & B_c\Omega_u\mathscr{D} \\ \mathscr{C} \arrow[u] \arrow[r] & \mathscr{E} \arrow[ul] \arrow[r] & \mathscr{D} \arrow[u] \end{tikzcd}$$

where $\mathscr{E}$ is the pullback $B_c\Omega_u\mathscr{C} \times_{B_c\Omega_u\mathscr{D}} \mathscr{D}$. By Lemma 28, g is a surjection and so $\Omega_u(g)$ is also a surjection. So, $B_c\Omega_u(g)$ is a fibration. By the key lemma (Theorem 26), the morphism $\mathscr{E} \to B_c\Omega_u\mathscr{C}$ is a weak equivalence. Since the map $\mathscr{C} \to B_c\Omega_u\mathscr{C}$ is an acyclic fibration, then the map $\mathscr{C} \to \mathscr{E}$ is also a weak equivalence and a monomorphism; that is an acyclic cofibration. Hence, the following diagram has a lifting.

$$\begin{tikzcd} \mathscr{C} \arrow[r, "="] \arrow[d] & \mathscr{C} \arrow[d, "g"] \\ \mathscr{E} \arrow[r] & \mathscr{D} \end{tikzcd}$$

So g is a retract of the morphism $\mathscr{E} \to \mathscr{D}$. □

3.4 Curved conilpotent cooperads as models for homotopy operads

In Section 3.3, we have transferred the model structure of the category of dg operads to the category of curved conilpotent cooperads along the cobar construction functor in order to obtain a Quillen equivalence. So curved conilpotent cooperads encode as well the homotopy theory of dg operads. In this section, we make this statement more concrete ; indeed, we show that the cofibrant-fibrant objects of the category of curved conilpotent cooperads correspond to a notion of operads up to homotopy.

3.4.1 Homotopy operads

Definition 44 (Homotopy unital operad)**.** A *homotopy operad* $\mathscr{P}$ is a dg-$\mathbb{S}$-module $\mathcal{P}$ with a distinguished element $1_{\mathscr{P}} \in \mathcal{P}(1)_0$ together with the data of a curved conilpotent cooperad on $\mathbb{T}^c(s\mathcal{P} \oplus \mathbb{K} \cdot v)$ whose derivation restricts to $sd_{\mathcal{P}}$ on $s\mathcal{P}$ and such that $dv = s1_{\mathscr{P}}$ and whose curvature θ is the following map.

$$\begin{aligned} \mathbb{T}^c(s\mathcal{P} \oplus \mathbb{K} \cdot v) \twoheadrightarrow s\mathcal{P} \oplus \mathbb{K} \cdot v \twoheadrightarrow \mathbb{K} \cdot v &\to \mathbb{K} \\ v &\mapsto 1 \end{aligned}$$

The curved conilpotent cooperad $\mathbb{T}^c(s\mathcal{P} \oplus \mathbb{K} \cdot v)$ is called the *bar construction* of the homotopy operad $\mathscr{P}$ and is denoted $B_c\mathscr{P}$. An ∞-morphism of homotopy operads from $\mathscr{P}$ to $\mathscr{Q}$ is a morphism of curved conilpotent cooperads from $B_c\mathscr{P}$ to $B_c\mathscr{Q}$.

Notations. For any homotopy operad $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}})$, we denote by $B_c^{\leq n}\mathscr{P}$ the sub curved conilpotent cooperad of $B_c\mathscr{P}$ whose underlying $\mathbb{S}$-module is $\mathbb{T}^{\leq n}(s\mathcal{P} \oplus \mathbb{K} \cdot v)$.

Example 3. The functor bar B_c from the category of operads to the category of curved conilpotent cooperads factorises through an inclusion functor from the category of operads to the category of homotopy operads.

Proposition 58. *Let $(\mathscr{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}})$ and $(\mathscr{Q}, \gamma_{\mathscr{Q}}, 1_{\mathscr{Q}})$ be two homotopy operads. There is a one-to-one correspondence between the ∞-morphisms of curved cooperads from $\mathscr{P}$ to $\mathscr{Q}$ and the degree 0 maps $f : \overline{\mathbb{T}}(s\mathcal{P} \oplus \mathbb{K} \cdot v) \to s\mathcal{Q} \oplus \mathbb{K} \cdot v$ such that on any tree T :*

$$\sum_{T = T_1 \sqcup \cdots \sqcup T_k} \gamma_{\mathscr{Q}}(T/T_1, \ldots, T_k)(f(T_1) \otimes \cdots \otimes f(T_k)) = \sum_{T' \subset T} f(T/T')(Id \otimes \cdots \otimes \gamma_{\mathscr{P}}(T') \otimes \cdots \otimes Id) \ ,$$

and such that $\theta_{\mathscr{Q}} f = \theta_{\mathscr{P}}$.

Proof. The proof relies on the same techniques as the proof of [LV12, 10.5.5]. □

Definition 45 (Infinity-quasi-isomorphisms)**.** Let $\mathscr{P}$ and $\mathscr{Q}$ be two homotopy operads. Letet $f : \overline{\mathbb{T}}(s\mathcal{P} \oplus \mathbb{K} \cdot v) \to s\mathcal{Q}$ be an ∞-morphism from $\mathscr{P}$ to $\mathscr{Q}$. We say that f is an ∞-*isomorphism* (resp. ∞-*monomorphism*, ∞-*epimorphism*, ∞-*quasi-isomorphism*, ∞-*isotopy*) if $f_{|s\mathscr{P}}$ is an isomorphism (resp. monomorphism, epimorphism, quasi-isomorphism, the identity of the $\mathbb{S}$-module $s\mathcal{P}$). An ∞-morphism $f : \overline{\mathbb{T}}(s\mathcal{P} \oplus \mathbb{K} \cdot v) \to s\mathcal{Q}$ is said to be strict if $f(T)$ is zero on trees with two vertices or more.

Example 4. Morphisms of operads are examples of strict morphisms.

Proposition 59. *An ∞-morphism is a monomorphism (resp. isomorphism) if and only if it is an ∞-monomorphism (resp. ∞-isomorphism)*

Proof. The fact that ∞-morphism is a monomorphism, if and only if it is an ∞-monomorphism follows from a straightforward induction. A similar induction shows that an ∞-monomorphism is an isomorphism if and only if it an ∞-isomorphism. So ∞-isomorphisms are isomorphisms. □

Proposition 60. *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, \theta)$ be a curved conilpotent cooperad whose underlying graded cooperad is cofree cogenerated by a graded $\mathbb{S}$-module $\mathcal{V}$; that is $\mathscr{C} \simeq \mathbb{T}^c\mathcal{V}$ in the category of graded cooperads. Suppose that there exists $v \in \mathcal{V}(1)_2$ such that $\theta(v) = 1$. Then, $\mathscr{C}$ is isomorphic to the bar construction of a homotopy operad.*

Proof. Let $s\mathcal{P} \subset \mathcal{V}$ be the kernel of the restriction of the curvature θ to $\mathcal{V}$. We have an isomorphism of dg $\mathbb{S}$-modules $f_1 : \mathcal{V} \simeq s\mathcal{P} \oplus \mathbb{K} \cdot v$. Consider, the following morphism

$$\begin{aligned} f : \mathbb{T}\mathcal{V} &\to s\mathcal{P} \oplus \mathbb{K}v \\ x \in \mathcal{V} &\mapsto f_1(x) \\ x_1 \otimes \cdots \otimes x_n &\mapsto \theta(x_1 \otimes \cdots \otimes x_n)v \ . \end{aligned}$$

It induces an isomorphism of graded conilpotent coalgebras between $\mathscr{C}$ and $\mathbb{T}(s\mathcal{P} \oplus v)$. Let us endow $\mathbb{T}(s\mathcal{P} \oplus v)$ with the structure of curved cooperad obtained by transfer of the coderivation of $\mathscr{C}$ and the curvature of $\mathscr{C}$ along this isomorphism. Then, $\mathbb{T}(s\mathcal{P} \oplus v)$ becomes the bar construction of a homotopy operad. □

Proposition 61. *Let f be an ∞-epimorphism (resp. ∞-monomorphism) from $\mathscr{P}$ to $\mathscr{Q}$. There exists an ∞-isotopy g such that fg (resp. gf) is a strict morphism.*

Proof. The proof relies on the same arguments as [LH03, 1.3.3.3]. □

3.4.2 Obstruction theory of homotopy operads and ∞-morphisms

Proposition 62. *Let $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}})$ and $\mathscr{Q} = (\mathcal{Q}, \gamma_{\mathscr{Q}}, 1_{\mathscr{Q}})$ be two homotopy operads. Let l be a map from $B_c^{\leq n-1}\mathscr{P}$ to $s\mathcal{Q} \oplus \mathbb{K} \cdot v$ which can be extended to a morphism of curved conilpotent cooperads from $B_c^{\leq n-1}\mathscr{P}$ to $B_c^{\leq n-1}\mathscr{Q}$. Let m be the degree -1 map from $\mathbb{T}^n(s\mathcal{P} \oplus \mathbb{K} \cdot v)$ to $s\mathcal{Q}\mathbb{K} \cdot v$ defined on any tree T with n vertices by*

$$m := \sum_{\substack{T' \subset T \\ \#T' \geq 2}} l(Id \otimes \cdots \otimes \gamma_{\mathscr{P}}(T') \otimes \cdots \otimes Id) - \sum_{\substack{T = T_1 \sqcup \cdots \sqcup T_k \\ k \geq 2}} \gamma_{\mathscr{Q}}(l(T_1) \otimes \cdots \otimes l(T_k)) \ .$$

Then, m is a cycle of the chain complex $[\mathbb{T}^n(s\mathcal{P} \oplus \mathbb{K} \cdot v), s\mathcal{Q}\mathbb{K} \cdot v]$ whose differential is induced by the differential of $s\mathcal{P} \oplus \mathbb{K} \cdot v$ and the differential of $s\mathcal{Q}\mathbb{K} \cdot v$.

Proof. Let L be the morphism of cooperads from $B_c^{\leq n-1}\mathscr{P}$ to $B_c^{\leq n-1}$ induced by the map l; that is

$$L(T) := \sum_{\substack{T = T_1 \sqcup \cdots \sqcup T_k \\ k \geq 2}} l(T_1) \otimes \cdots \otimes l(T_k) \ .$$

Moreover, let M be the map from $\mathbb{T}^n(s\mathcal{P} \oplus \mathbb{K} \cdot v)$ to $B_c\mathscr{Q}$ defined as follows on any tree T with n vertices :

$$M := \sum_{T' \subset T; \#T' \geq 2} L(Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id) + \sum_{T = T_1 \sqcup \cdots \sqcup T_k; k \geq 2} (l(T_1) \otimes \cdots \otimes l(T_k))d - D(l(T_1) \otimes \cdots \otimes l(T_k)) \ ,$$

where D denotes the coderivation of either $B_c\mathscr{P}'$ and $B_c\mathscr{Q}$ and where d is the differential of $\mathbb{T}(s\mathcal{Q} \oplus \mathbb{K} \cdot v)$ induced by the differential of $s\mathcal{Q} \oplus \mathbb{K} \cdot v$. Then

$$\begin{aligned} DM + Md &= \sum_{\substack{T' \subset T \\ \#vert(T') \geq 2}} DL(Id \otimes \cdots \otimes \gamma_{\mathscr{P}}(T') \otimes \cdots \otimes Id) - \sum_{\substack{T = T_1 \sqcup \cdots \sqcup T_k \\ k \geq 2}} D^2(l(T_1) \otimes \cdots \otimes l(T_k)) \\ &+ \sum_{\substack{T' \subset T \\ \#vert(T') \geq 2}} L(Id \otimes \cdots \otimes \gamma_{\mathscr{P}}(T') \otimes \cdots \otimes Id)d \\ &= LD^2 - \sum_{\substack{T = T_1 \sqcup \cdots \sqcup T_k \\ k \geq 2}} D^2(l(T_1) \otimes \cdots \otimes l(T_k)) \\ &= (\theta \otimes L - L \otimes \theta)\Delta_2 - (\theta \otimes L - L \otimes \theta)\Delta_2 = 0 \ . \end{aligned}$$

Moreover, let $\pi_{>1}M$ be the projection of M on $\mathbb{T}^{>1}(s\mathscr{Q} \oplus \mathbb{K} \cdot v)$. We have :

$$\pi_{>1}M := \sum_{\substack{T=T_1\sqcup\cdots\sqcup T_k\\ k\geq 2}} \sum_i (l(T_1) \otimes \cdots \otimes l(T_i)D \otimes \cdots \otimes l(T_k)) - \sum_{\substack{T=T_1\sqcup\cdots\sqcup T_k\\ k\geq 2}} \pi_{>1}D(l(T_1) \otimes \cdots \otimes l(T_k))$$

Since

$$\begin{aligned} &\sum_{\substack{T=T_1\sqcup\cdots\sqcup T_k\\ k\geq 2}} \pi_{>1}D(l(T_1) \otimes \cdots \otimes l(T_k)) \\ &= \sum_{\substack{T=T_1\sqcup\cdots\sqcup T_k\\ k\geq 2}} \sum_{T'\subsetneq T/T_1,\ldots,T_k} (Id \otimes \cdots \otimes \gamma_{\mathscr{Q}}(T') \otimes \cdots \otimes Id)(l(T_1) \otimes \cdots \otimes l(T_k)) \\ &= \sum_{T'\subset T} \sum_{\substack{T=T_1\sqcup\cdots\sqcup T'\sqcup\ldots\sqcup T_k\\ k\geq 2}} (l(T_1) \otimes \cdots \otimes \gamma_{\mathscr{Q}}L(T') \otimes \cdots \otimes l(T_k)) \ , \end{aligned}$$

then $\pi_{>1}M = 0$. So $M = m$ and so $\partial m = DM + Md = 0$. □

Proposition 63. *Let $\mathcal{P}$ be a graded $\mathbb{S}$-module together with a degree -1 map $\gamma : \mathbb{T}^{\leq n-1}(s\mathcal{P}\oplus\mathbb{K}\cdot v) \to s\mathcal{P}$ such that on any tree T with $n-1$ or less vertices :*

$$\sum_{T'\subset T} \gamma(T/T')(Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id) = (\theta \otimes \pi - \pi \otimes \theta)\Delta_2 \ ,$$

where θ and π are defined in the obvious way. In particular, γ extends a differential d on $s\mathcal{P}$. Let κ be the degree -2 map from $\mathbb{T}^n(s\mathcal{P} \oplus \mathbb{K} \cdot v)$ to $s\mathcal{P}$ defined on any tree T with n vertices by

$$\kappa := \sum_{\substack{T'\subsetneq T\\ \#vert(T')\geq 2}} \gamma(T/T')(Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id) - (\theta \otimes \pi - \pi \otimes \theta)\Delta_2 \ .$$

Then κ is a cycle of the chain complex $[\mathbb{T}^n(s\mathcal{P} \oplus \mathbb{K} \cdot v), s\mathcal{P}]$.

Proof. If $n = 2$, then $\kappa = -(\theta\otimes\pi-\pi\otimes\theta)\Delta_2$ is a cycle. If $n \geq 3$, then let $\kappa = \sum_{\substack{T'\subsetneq T\\ \#vert(T')\geq 2}} \gamma(T/T')(Id\otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id)$. We use the same technique as in the proof of Proposition62 to prove that it is a cycle. □

The following proposition is a consequence of Proposition 62 and will allows us to show that bar constructions of homotopy operads are fibrant curved conilpotent cooperads.

Proposition 64. *Consider the following square of homotopy operads with ∞-morphisms.*

$$\begin{tikzcd} \mathscr{P} \arrow[r, "u"] \arrow[d, "f"'] & \mathscr{Q} \arrow[d, "g"] \\ \mathscr{P}' \arrow[r, "v"'] & \mathscr{Q}' \end{tikzcd}$$

where f is both an ∞-quasi-isomorphism and an ∞-monomorphism and g is an ∞-epimorphism. Then, this square has a lifting.

Proof. By Proposition 61, we can suppose that f and g are strict morphisms. We will build by induction maps

$$l_n : \mathbb{T}^n(s\mathcal{P}' \oplus \mathbb{K} \cdot v) \to s\mathcal{Q} \oplus \mathbb{K} \cdot v \ , n \geq 1 \ ,$$

such that

$$\begin{cases} \partial(l_n) = m_n \ , \\ g_1 l_n = v_n \ , \\ l_n(f_1 \otimes \cdots \otimes f_1) = u_n \ . \end{cases}$$

where

$$m_n := \sum_{\substack{T' \subset T \\ \#vert(T') \geq 2}} l_{<n}(Id \otimes \cdots \otimes \gamma_{T'} \otimes \cdots \otimes Id) - \sum_{\substack{T = T_1 \sqcup \cdots \sqcup T_k \\ k \geq 2}} \gamma(l_{T_1} \otimes \cdots \otimes l_{T_k})$$

Suppose that we have constructed $l_1, \ldots, l_{n-1}$. Since by Proposition 62, m_n is a cycle in the chain complex $[\mathbb{T}^n(s\mathcal{P}' \oplus v), s\mathcal{Q} \oplus \mathbb{K} \cdot v]$ (whose differential is induced by the differential of $s\mathcal{P}' \oplus \mathbb{K} \cdot v$ and the differential of $s\mathcal{Q} \oplus \mathbb{K} \cdot v$), constructing l_n amounts to lift the following square.

$$\begin{tikzcd} S^{-1} \arrow[r, "m_n"] \arrow[d] & {[\mathbb{T}^n(s\mathcal{P}' \oplus v), s\mathcal{Q} \oplus v]} \arrow[d] \\ D^0 \arrow[r, "(u_n, v_n)"'] & {[\mathbb{T}^n(s\mathcal{P} \oplus v), s\mathcal{Q} \oplus v] \times_{[\mathbb{T}^n(s\mathcal{P} \oplus v), s\mathcal{Q}' \oplus v]} [\mathbb{T}^n(s\mathcal{P}' \oplus v), s\mathcal{Q}' \oplus v]} \end{tikzcd}$$

Since g_1 is a fibration and since f_1 is an acyclic cofibration of dg $\mathbb{S}$-modules, then the right vertical map is an acyclic fibration of chain complexes. So the square has a lifting. Thus, we obtain l_n. Then, let $L : B_c\mathscr{P}' \to B_c\mathscr{Q}$ the morphism of graded cooperads induced by the maps $(l_k)_{k=1}^{\infty}$. It is a morphism of curved conilpotent cooperads since $\theta_{B_c\mathscr{P}'} = \theta_{B_c\mathscr{Q}'} B_c v = \theta_{B_c\mathscr{P}'}(B_c g)L = \theta_{B_c\mathscr{Q}} L$. ☐

3.4.3 Fibrant curved conilpotent cooperads

Proposition 65. *The fibrant curved conilpotent cooperads are the curved conilpotent cooperads isomorphic to the bar construction $B_c\mathscr{P}$ of a homotopy operad $\mathscr{P}$.*

The proof of the theorem consists in showing that a cofree curved conilpotent cooperad $\mathscr{C} = \mathbb{T}(s\mathscr{P} \oplus \mathbb{K} \cdot v)$ is a retract of $B_\beta\Omega_\beta\mathcal{C}$ which is fibrant. For that purpose, we will build by induction a left inverse p to the map $i : \mathcal{C} \to B_\beta\Omega_\beta\mathcal{C}$.

Lemma 29. *A retract of a cofree graded cooperad $\mathbb{T}^c\mathcal{V}$ is cofree.*

Proof. Let $\mathscr{C}$ be a retract of the cofree curved conilpotent cooperad $\mathbb{T}^c(\mathcal{V} \oplus \mathbb{K} \cdot v)$. On the one hand, the map $\mathscr{C} \to \mathbb{T}(\mathcal{V} \oplus \mathbb{K} \cdot v) \twoheadrightarrow \mathcal{V} \oplus \mathbb{K} \cdot v \to F_0^{rad}\mathscr{C}$ gives a map of graded cooperads $\mathscr{C} \to \mathbb{T}^c(F_0^{rad}\mathscr{C})$.blabla ☐

Proof of Proposition 65. Let $\mathscr{C}$ be a fibrant curved conilpotent cooperad. Since the map $\mathscr{C} \to B_c\Omega_u\mathscr{C}$ is an acyclic cofibration, it has a right inverse p and so, $\mathscr{C}$ is a retract of $B_c\Omega_u\mathscr{C}$. So, by Lemma 29, $\mathscr{C}$ is cofree : $\mathscr{C} := \mathbb{T}(\mathcal{V})$. Moreover, $p(v)$ is an element of $\mathcal{V}$ such that $\theta(p(v)) = 1$. So, by Proposition 60, $\mathscr{C}$ is isomorphic to the bar construction of a homotopy operad. Conversely let $\mathscr{P}$ be a homotopy operad. The canonical morphism $B_c\mathscr{P} \to B_c\Omega_u B_c\mathscr{P}$ is an ∞-monomorphism. So, by Proposition 64, it has a left inverse; so $B_c\mathscr{P}$ is a retract of $B_c\Omega_u B_c\mathscr{P}$. Since $B_c\Omega_u B_c\mathscr{P}$ is fibrant, then $B_c\mathscr{P}$ is fibrant. ☐

Proposition 66. *An ∞-morphism of homotopy operads is a cofibration (resp. a fibration, a weak equivalence) if and only if it is a ∞-monomorphism (resp. ∞-epimorphism, ∞-quasi-isomorphism).*

Proof. We have already proven (Proposition 59) that an ∞-morphism is a monomorphism (that is a cofibration) if an only if it is an ∞-monomorphism. Let $f : \mathscr{P} \to \mathscr{Q}$ be a ∞-morphism of homotopy operads. Consider the following square of $\mathbb{S}$-modules.

$$\begin{tikzcd} \Omega_u\mathscr{C} \arrow[r, "\Omega_u f"] & \Omega_u\mathscr{D} \\ \mathscr{P} \arrow[u] \arrow[r, "f_0"'] & \mathscr{Q} \arrow[u] \end{tikzcd}$$

where f_0 is the restriction of f to $\mathscr{P}$. The two vertical maps are quasi-isomorphisms. So the lower horizontal map is a quasi-isomorphism if and only if the upper horizontal map is a quasi-isomorphism; that is f is a weak equivalence if and only if f_0 is a quasi-isomorphism. Finally, suppose that f is an ∞-epimorphism. Since it is surjective, then $\Omega_u(f)$ is surjective and so $B_c\Omega_u(f)$ is a fibration. Let us show that f is a retract of $B_c\Omega_u f$. We already know (Proposition 65) that $B_c\mathscr{Q}$ is a retract of $B_c\Omega_u B_c\mathscr{Q}$. Consider the following diagram.

$$\begin{tikzcd}
B_c\mathscr{P} \arrow[r] \arrow[d, "f"'] & B_c\Omega_u B_c\mathscr{P} \arrow[r, dashed] \arrow[d, "B_c\Omega_u f"] & B_c\mathscr{P} \arrow[d, "f"] \\
B_c\mathscr{Q} \arrow[r] & B_c\Omega_u B_c\mathscr{Q} \arrow[r] & B_c\mathscr{Q}
\end{tikzcd}$$

Finding a morphism $B_c\Omega_u B_c\mathscr{P} \to B_c\mathscr{P}$ making the diagram commute amounts to lift the following square, which is possible by Proposition 64.

$$\begin{tikzcd}
B_c\mathscr{P} \arrow[r, "="] \arrow[d] & B_c\mathscr{P} \arrow[d, "f"] \\
B_c\Omega_u B_c\mathscr{P} \arrow[r] & B_c\mathscr{Q}
\end{tikzcd}$$

Conversely, suppose that f is a fibration. It is an ∞-epimorphism because the following diagram of curved conilpotent cooperads,

$$\begin{tikzcd}
\mathcal{I} \arrow[r] \arrow[d] & B_c\mathscr{P} \arrow[d, "f"] \\
\mathcal{I} \oplus (s\mathcal{Q} \oplus \mathbb{K}\cdot v) \oplus s^{-1}(s\mathcal{Q} \oplus \mathbb{K}\cdot v) \arrow[r] & B_c\mathscr{Q}\ ,
\end{tikzcd}$$

where $\mathcal{I} \oplus (s\mathcal{Q} \oplus \mathbb{K}\cdot v) \oplus s^{-1}(s\mathcal{Q} \oplus \mathbb{K}\cdot v)$ is a dg $\mathbb{S}$-module considered as a curved conilpotent cooperad with trivial decomposition Δ. □

Proposition 67. *Let $\mathscr{P}$ and $\mathscr{Q}$ be two dg operads. They are linked by a zig-zag of quasi-isomorphisms of dg operads if and only if they are linked by an ∞-quasi-isomorphism.*

Proof. Suppose that $\mathscr{P}$ and $\mathscr{Q}$ are linked by an ∞-quasi-isomorphism f. Then, they are linked by a zig-zag of quasi-isomorphisms of operads as follows.

$$\mathscr{P} \longleftarrow \Omega_u B_c\mathscr{P} \xrightarrow{\Omega_u f} \Omega_u B_c\mathscr{Q} \longrightarrow \mathscr{Q}$$

Conversely, suppose that $\mathscr{P}$ and $\mathscr{Q}$ are linked by a zig-zag of quasi-isomorphisms of operads. Any quasi-isomorphism of operads has an homotopy inverse which is an ∞-quasi-isomorphism. So there exists a ∞-quasi-isomorphism from $\mathscr{P}$ to $\mathscr{Q}$. □

3.4.4 Homotopy transfer theorem

Consider an acyclic fibration of dg $\mathbb{S}$-modules $p : \mathcal{P} \to \mathcal{Q}$.

Theorem 23. *Suppose that $\mathcal{P}$ has a structure of homotopy operad denoted by $\mathscr{P}$. Then, there exists an ∞-isotopy $f : \mathscr{P} \to \mathscr{P}'$ of homotopy operads and a structure of homotopy operad on $\mathcal{Q}$ such that the map $p : \mathcal{P}' \to \mathcal{Q}$ is a morphism of homotopy operads.*

Proof. We build by induction this ∞-isotopy and this structure of homotopy operad on $\mathcal{Q}$; that is we build by induction maps

$$\begin{cases} \gamma_n : \mathbb{T}^n(s\mathcal{Q} \oplus \mathbb{K}\cdot v) \xrightarrow{|-1|} s\mathcal{Q}\ , \\ f_n : \mathbb{T}^n(s\mathcal{P} \oplus \mathbb{K}\cdot v) \to s\mathcal{P}\ , \end{cases}$$

for $n \geq 2$ such that on any tree T with n vertices :

$$\begin{cases} \partial\gamma_n = -\sum_{\substack{T' \subsetneq T \\ \#vert(T') \geq 2}} \gamma_{<n}(Id \otimes \cdots \otimes \gamma_{<n}(T') \otimes \cdots \otimes Id)\ , \\ \gamma_n p^{\otimes n} + p\partial(f_n) = \sum_{\substack{T' \subset T \\ \#vert(T') \geq 2}} pf_{<n}(Id \otimes \cdots \otimes \gamma_{\mathscr{P}}(T') \otimes \cdots \otimes Id) \\ \quad - \sum_{\substack{T = T_1 \sqcup \ldots \sqcup T_k \\ 1<k<n}} \gamma_{<n}(pf_{<n}(T_1) \otimes \cdots \otimes pf_{<n}(T_k))\ . \end{cases}$$

Suppose that we have built γ_2, f_2, $\ldots$, γ_{n-1}, f_{n-1}. Consider the chain complex

$$[\mathbb{T}^n(s\mathcal{Q}\oplus v), s\mathcal{Q}] \oplus [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \oplus s^{-1}[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}]$$

where $[\mathbb{T}^n(s\mathcal{Q}\oplus v), s\mathcal{Q}]$ is endowed with the differential induced by the differential of $s\mathcal{Q}\oplus v$ and the differential of $s\mathcal{Q}$. Moreover, the differential on the other summands is the adding of s^{-1} to any element of $[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}]$. The following morphisms of chain complexes

$$[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \oplus s^{-1}[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \xrightarrow{Id\oplus\partial} [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \xrightarrow{p} [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{Q}]$$
$$[\mathbb{T}^n(s\mathcal{Q}\oplus v), s\mathcal{Q}] \xrightarrow{p} [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{Q}]$$

are respectively a fibration and a weak equivalence. Then, the morphism

$$[\mathbb{T}^n(s\mathcal{Q}\oplus v), s\mathcal{Q}] \oplus [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \oplus s^{-1}[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \to [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{Q}]$$

is an acyclic fibration. Moreover, by Proposition 63, the element

$$\kappa_n := -\sum_{\substack{T'\subsetneq T\\ \#vert(T')\geq 2}} \gamma_{<n}(Id\otimes\cdots\otimes\gamma_{<n}(T')\otimes\cdots\otimes Id)$$

is a cycle of the chain complex $[\mathbb{T}^n(s\mathcal{Q}\oplus v), s\mathcal{Q}]$. This gives us the following square of chain complexes.

$$\begin{tikzcd}
S^{-2} \arrow[r, "{(\kappa_n,0,0)}"] \arrow[d] & {[\mathbb{T}^n(s\mathcal{Q}\oplus v), s\mathcal{Q}] \oplus [\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}] \oplus s^{-1}[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{P}]} \arrow[d] \\
D^{-1} \arrow[r, "\chi_n"'] & {[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{Q}]}
\end{tikzcd}$$

where χ_n is the following element of $[\mathbb{T}^n(s\mathcal{P}\oplus v), s\mathcal{Q}]$:

$$\chi_n = \sum_{\substack{T'\subset T\\ \#vert(T')\geq 2}} pf_{<n}(Id\otimes\cdots\otimes\gamma_{\mathscr{P}}(T')\otimes\cdots\otimes Id) - \sum_{\substack{T=T_1\sqcup\ldots\sqcup T_k\\ 1<k<n}} \gamma_{<n}(pf_{<n}(T_1)\otimes\cdots\otimes pf_{<n}(T_k))\ .$$

This square has a lifting which gives us γ_n and f_n. ☐

This homotopy transfer theorem may for instance be applied to the homology of a homotopy operad. Indeed, a dg $\mathbb{S}$-module is linked to its homology by an acyclic fibration.

Proposition 68. *Let $\mathcal{V}$ be a dg $\mathbb{S}$-module and let $H(\mathcal{V})$ be its homology. There exists an acyclic fibration of dg $\mathbb{S}$-modules from $\mathcal{V}$ to $H(\mathcal{V})$.*

Proof. Let $Z(\mathcal{V})$ be the $\mathbb{S}$-module of cycles of $\mathcal{V}$. Consider the following diagram of $\mathbb{S}$-modules.

$$\mathcal{V} \longleftarrow Z(\mathcal{V}) \longrightarrow H(\mathcal{V})$$

Since any graded $\mathbb{K}[\mathbb{S}_n]$ module is projective, the surjective morphism $Z(\mathcal{V})\to H(\mathcal{V})$ has a right inverse. Thus, we obtain an inclusion $H(\mathcal{V})\to\mathcal{V}$ which is a quasi-isomorphism. It has a right inverse which is an acyclic fibration. ☐

REMARK 10. Let $\mathscr{P} = (\mathcal{P},\gamma,1)$ be a dg operad and let p be an acyclic fibration of $\mathbb{S}$-modules from $\mathcal{P}$ to its homology $H(\mathcal{P})$. The homotopy transfer theorem applied to p defines to operadic Massey products of $\mathscr{P}$. We refer to [Liv15] for a computation of operadic Massey products of the Swiss cheese operad.

3.4.5 Path object

For any integer $n \in \mathbb{N}$ let $\Phi[n]$ be the linear dual of the normalized Moore complex of the simplicial set Δ_n. For instance $\Phi[1]$ is as follows :

$$\begin{cases} \Phi[1]_0 := \mathbb{K} \cdot (0) \oplus \mathbb{K} \cdot (1) \ , \\ \Phi[1]_{-1} := \mathbb{K} \cdot (01) \ , \\ \Phi[1]_n = 0 \ , \ n \notin \{-1, 0\} \ , \\ d(0) = -(01) \ , \\ d(1) = (01) \ . \end{cases}$$

Proposition 69. *Let $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}})$ be an homotopy operad. The, dg $\mathbb{S}$-module $\mathcal{P} \otimes \Phi[1]$ has a structure of homotopy operad that we denote $\mathscr{P} \otimes \Phi[1]$ and which is a path object of the homotopy operad $\mathscr{P}$.*

Proof. For convenience, we will denote $\mathbb{T}^m(s\mathcal{P} \otimes \Phi[1] \oplus \mathbb{K} \cdot v)$ by $B_c^m \mathcal{P}_1$ and $\mathbb{T}^m(s\mathcal{P} \otimes \oplus \mathbb{K} \cdot v)$ by $B_c^m \mathcal{P}$. We build by induction maps

$$\gamma_m : B_c^m \mathcal{P}_1 \xrightarrow{|-1|} s\mathcal{P} \otimes \Phi[1]$$

such that on any tree T with m or less than m vertices

$$\partial \gamma(T) + \sum_{\substack{T' \subsetneq T \\ \#vert(T') \geq 2}} \gamma(T/T')(Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes) = (\theta \otimes \pi - \pi \otimes \theta)\Delta_2$$

where π is the projection of $B_c \mathcal{P}_1$ onto $s\mathcal{P} \otimes \Phi[1]$ and θ is the map $B_c \mathcal{P}_1 \twoheadrightarrow \mathbb{K} \cdot v \to \mathbb{K}$. Moreover, we require the following equality between maps from $B_c^m \mathcal{P}_1$ to $s\mathcal{P}$ (resp. $B_c^m \mathcal{P}$ to $s\mathcal{P} \otimes \Phi[1]$) :

$$\gamma_{\mathscr{P}}(Id_{s\mathcal{P}} \otimes \delta_i \oplus Id_v)^{\otimes m} = (Id_{s\mathcal{P}} \otimes \delta_i)\gamma_m$$

$$\gamma_m(Id_{s\mathcal{P}} \otimes \sigma \oplus Id_v)^{\otimes m} = (Id_{s\mathcal{P}} \otimes \sigma)\gamma_{\mathscr{P}}$$

for the two face maps $\delta_i : \Phi[1] \to \mathbb{K}$ and for any degeneracy map $\sigma : \mathbb{K} \to \Phi[1]$. Suppose that we have built $\gamma_2, \ldots, \gamma_{m-1}$. Using the same techniques as in the proof of Proposition 64, building γ_m amounts to find a lifting to the following square :

$$\begin{tikzcd} S^{-1} \arrow[r] \arrow[d] & {[B_c^m \mathcal{P}_1, s\mathcal{P} \otimes \Phi[1]]} \arrow[d] \\ D^{-1} \arrow[r] & {[B_c^m \mathcal{P}, s\mathcal{P} \otimes \Phi[1]] \times_{[B_c^m \mathcal{P}, s\mathcal{P} \oplus s\mathcal{P}]} [B_c^m \mathcal{P}_1, s\mathcal{P} \oplus s\mathcal{P}]} \ . \end{tikzcd}$$

Since the map $B_c^m \mathcal{P} \to B_c^m \mathcal{P}_1$ induced by the degeneracy map $\sigma : \mathbb{K} \to \Phi[1]$ is an acyclic cofibration and since the morphism $s\mathcal{P} \otimes \Phi[1] \to s\mathcal{P} \oplus s\mathcal{P}$ induced by the two face maps $\Phi[1] \to \mathbb{K}$ is a fibration, then the right vertical map of the diagram is an acyclic fibration. So the square has a lifting. $\square$

3.4.6 Strict unital homotopy operads

Definition 46 (Strict unital homotopy operads)**.** A strict unital homotopy operad is a homotopy operad $(\mathscr{P}, \gamma_{\mathscr{P}}, 1)$ such that :

$$\begin{cases} \gamma_{\mathscr{P}}(v) = dv = s1_{\mathscr{P}} \ , \\ \gamma_{\mathscr{P}}(s1_{\mathscr{P}} \otimes sx) = sx \\ \gamma_{\mathscr{P}}(sx \otimes s1_{\mathscr{P}}) = (-1)^{|x|} sx \\ \gamma_{\mathscr{P}}(sx \otimes v) = \gamma_{\mathscr{P}}(v \otimes sx) = 0 \\ \gamma_{\mathscr{P}}(sx \otimes \cdots \otimes s1_{\mathscr{P}} \otimes \cdots \otimes sy) = 0, \text{ on } \mathbb{T}^{\geq 3}(s\mathscr{P} \oplus \mathbb{K} \cdot v) \ , \\ \gamma_{\mathscr{P}}(sx \otimes \cdots \otimes v \otimes \cdots \otimes sy) = 0, \text{ on } \mathbb{T}^{\geq 3}(s\mathscr{P} \oplus \mathbb{K} \cdot v) \ . \end{cases}$$

Let $\mathscr{P}$ and $\mathscr{Q}$ be two strict unital homotopy operads. A strict unital ∞-morphism from $\mathscr{P}$ to $\mathscr{Q}$ is an ∞-morphism $f : \overline{\mathbb{T}}(s\mathscr{P} \oplus \mathbb{K} \cdot v) \to s\mathscr{Q}$ such that

$$\begin{cases} f(v) = 0 \ , \\ f(sx \otimes \cdots \otimes v \otimes \cdots \otimes sy) = 0, \ \text{on } \mathbb{T}^{\geq 2}(s\mathscr{P} \oplus \mathbb{K} \cdot v) \ , \\ f(sx \otimes \cdots \otimes s1_{\mathscr{P}} \otimes \cdots \otimes sy) = 0, \ \text{on } \mathbb{T}^{\geq 2}(s\mathscr{P} \oplus \mathbb{K} \cdot v) \ . \end{cases}$$

In particular $f^+(v) = v$ and $f(s1_{\mathscr{P}}) = s1_{\mathscr{Q}}$.

Definition 47 (Truncated bar construction of a strict unital homotopy operad)**.** A semi-augmentation of a strict unital homotopy operad $(\mathscr{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}})$ is a morphism of graded $\mathbb{S}$-modules $\epsilon : \mathscr{P} \to \mathcal{I}$ such that $\epsilon 1_{\mathscr{P}} = 1$. We denote by $\overline{\mathscr{P}}$ the kernel of ϵ and by π the projection of $\mathscr{P}$ on $\overline{\mathscr{P}}$ parallel to $1_{\mathscr{P}}$. Let $(\mathscr{P}, \gamma_{\mathscr{P}})$ be a strict unital operad equipped with a semi augmentation ϵ. The truncated bar construction of $\mathscr{P}$ is the conilpotent cooperad $B_r\mathscr{P} := \mathbb{T}(s\overline{\mathscr{P}})$ equipped with the coderivation which extends the map

$$\overline{\gamma}_{\mathscr{P}} \overline{\mathbb{T}} s\overline{\mathscr{P}} \to s\overline{\mathscr{P}}$$

defined by $\overline{\gamma}_{\mathscr{P}} := \pi\gamma_{\mathscr{P}}$. It is also equipped with the degree -2 map $\theta := \epsilon(s^{-1})\gamma_{\mathscr{P}}$.

Proposition 70. *Let $(\mathscr{P}, \gamma_{\mathscr{P}}, 1, \epsilon)$ be a semi-augmented strict unital operad. The truncated bar construction $B_r\mathscr{P}$ is a curved conilpotent cooperad with curvature θ. Moreover, the map*

$$B_r\mathscr{P} \twoheadrightarrow \overline{\mathscr{P}} \hookrightarrow \mathscr{P} \oplus \mathbb{K} \cdot v$$

induces a morphism of curved conilpotent cooperad from $B_r\mathscr{P}$ to $B_c\mathscr{P}$. This morphism is universal, in the sense that for any strict unital operad $\mathscr{Q}$, and for any morphism of curved conilpotent cooperad $f : B_r\mathscr{P} \to B_c\mathscr{Q}$, there exists a unique strict unital ∞-morphism which extends f.

$$\begin{tikzcd} B_r\mathscr{P} \arrow[r, "f"] \arrow[d] & B_c\mathscr{Q} \\ B_c\mathscr{P} \arrow[ur, dashed] & \end{tikzcd}$$

Proof. It follows from straightforward calculations. ☐

Proposition 71. *Any ∞-morphism between strict unital homotopy morphism is homotopic to a strict unital ∞-morphism.*

Proof. Let $\mathscr{P}$ and $\mathscr{Q}$ be strict unital homotopy operads and let $f : B_c\mathscr{P} \to B_c\mathscr{Q}$ be a morphism of curved conilpotent cooperads. First choose a semi-augmentation of $\mathscr{P}$. Then, denote by g the composite morphism $B_r\mathscr{P} \hookrightarrow B_c\mathscr{P} \xrightarrow{f} B_c\mathscr{Q}$. Let $h : B_c\mathscr{P} \to B_c\mathscr{Q}$ be the unique morphism of strict unital homotopy operads which extends the morphism g. Consider the following square

$$\begin{tikzcd} B_r\mathscr{P} \arrow[r] \arrow[d] & \mathrm{path}(B_c\mathscr{Q}) \arrow[d] \\ B_c\mathscr{P} \arrow[r, "{f,h}"'] & B_c\mathscr{Q} \times B_c\mathscr{Q}, \end{tikzcd}$$

where the horizontal upper arrow is the composite morphism $B_r\mathscr{P} \xrightarrow{g} B_c\mathscr{Q} \to \mathrm{path}(B_c\mathscr{Q})$. Since the inclusion $B_r\mathscr{P} \to B_c\mathscr{Q}$ is an acyclic cofibration and since the map $\mathrm{path}(B_c\mathscr{Q}) \to B_c\mathscr{Q} \times B_c\mathscr{Q}$ is a fibration (by definition of a path object), then this square has a lifting. ☐

Proposition 72. *Let $p : \mathcal{P} \to \mathcal{Q}$ be an acyclic fibration of dg $\mathbb{S}$-modules together with a structure of strict unital homotopy on $\mathcal{P}$. Then, there exists a structure of strict unital homotopy operad on $\mathcal{Q}$ and a strict unital ∞-isotopy $f : \mathcal{P} \to \mathcal{P}'$ such that pf is a strict unital ∞-morphism.*

Proof. We can impose the strict unital conditions at every steps of the proof of Theorem 23. ☐

Chapitre 4

From homotopy operads to infinity-operads

Introduction

In Algebra, the structure relations hold strictly. This is for instance the case for sets, groups, rings, vector spaces, etc. So all of these examples are well encoded by categories. In this context, two objects are considered to represent equivalent notions if they are related by an isomorphism. However, in some areas of Mathematics, this equivalence relation is too strong and one would like to consider only weakly equivalent objects. For example, two categories are considered to be essentially the same if they are equivalent, two chain complexes give rise to the same homology groups when they are quasi-isomorphic, and two topological spaces have the same homotopy type if they are related by weak homotopy equivalences. In these examples, one can consider the Dwyer–Kan localization with respect to the class of weak equivalences. This provides us with a higher category structure made up of 2-morphisms, which are morphisms between morphisms, and, in general, n-morphisms, which are morphisms between $(n-1)$-morphisms, for integers n. These n-morphisms are invertible for $n \geq 2$ and they encode coherent higher homotopies. In this way, one can perform higher algebra in these $(\infty, 1)$-categories, which can actually be either categories enriched in simplicial sets, like the Dwyer–Kan localization, or Joyal's quasi-categories, called ∞-categories by Jacob Lurie [Lur09]. Recall that a quasi-category is a simplicial set satisfying a lifting condition, which defines the composition of morphisms up to homotopy.

Categories enriched in chain complexes, called dg categories, are a kind of linear version of ∞-categories. In his book [Lur12], Jacob Lurie presents several ways to interpret dg categories as ∞-categories. Relaxing the associativity relation of the composition rule in a dg category leads to the definition of an $\mathcal{A}_\infty$-category, notion which now plays a key role in symplectic geometry [FOOO09]. In [Fao13], Giovanni Faonte extends one of Lurie's constructions to the case of $\mathcal{A}_\infty$-categories and so builds a simplicial nerve of $\mathcal{A}_\infty$-categories. Let $[n]$ be the poset $0 < 1 < \cdots < n$ canonically enriched into a dg category ; the simplicial nerve of an $\mathcal{A}_\infty$-category C is defined by the following simplicial set :

$$\mathrm{N}_{\mathcal{A}_\infty}(\mathsf{C})_n := \mathrm{Hom}_{\mathcal{A}_\infty\text{-}\mathsf{cat}}([n], \mathsf{C}) \ ,$$

for any integer n. Faonte's main result asserts that this actually forms a quasi-category.

Representations of associative algebras are made up of linear operators. To encode multi-linear operators, which are operators with many inputs but one output, one can use representations of operads. Furthermore, to encode multi-linear operators on a many components object, one can use reprensentations of colored operads. A colored operad $\mathscr{P}$ is the data of a set of colors and sets of multi-linear operations, with inputs and output labeled by the colors. One can only compose operations with matching colors. Moreover, the action of the symmetric groups allows one to permute the inputs. Note that any category may be viewed as a colored operad, where the objects are the colors and where the maps are operations of arity one.

On the one hand, the notion of quasi-category has an analogue in the world of operads which is called an ∞-operad ; this notion has been developed by Ieke Moerijk and Ittay Weiss in [MW07]. First one needs an operadic generalization of the notion of a simplicial set : it is defined as a contravariant functor from trees to sets and called a dendroidal set. In the same way as a quasi-category is a simplicial set satisfying a lifting property, an ∞-operad is a dendroidal set satisfying a similar lifting property. On the other hand, the differential graded notion of an $\mathcal{A}_\infty$-algebra has also been extended by Pepijn Van der Laan [VdL03] to a notion called homotopy operad. A homotopy operad is the data of operations, with zero or many inputs but one output, structured in a family of differential graded $\mathbb{K}$-modules. The group actions and composition maps are well defined but the latter ones are "associative" only up to a family of higher homotopies. Actually, a homotopy operad whose operations only have one input is an $\mathcal{A}_\infty$-algebra.

The present chapter has the following two main goals. The first one is to define a suitable notion of homotopy colored operads, with a homotopy control of the unit, and which completes the following commutative diagram.

$$\begin{tikzcd}
 & \text{algebras} \arrow[rr, "\text{multi-linear operations}"] \arrow[dl, "\text{many colors}"'] \arrow[dd] & & \text{operads} \arrow[dl] \arrow[dd] \\
\text{dg categories} \arrow[rr] \arrow[dd, "\text{up to homotopy}"'] & & \text{colored operads} \arrow[dd] & \\
 & \mathcal{A}_\infty\text{-algebras} \arrow[rr] \arrow[dl] & & \text{homotopy operads} \arrow[dl] \\
\mathcal{A}_\infty\text{-categories} \arrow[rr] & & \textbf{strict unital homotopy colored operads} &
\end{tikzcd}$$

The second goal is to extend the nerve of Faonte–Lurie to the operadic level, study this extension and compare it with other constructions which appear in the literature. More precisely, we build a nerve functor N^Ω from the category of strict unital homotopy colored operads to dendroidal sets, such that for any $\mathcal{A}_\infty$-category A, the simplicial set induced by the dendroidal set $\mathrm{N}^\Omega(A)$ is equal to $\mathrm{N}_{\mathcal{A}_\infty}(A)$.

$$\begin{tikzcd}
\mathcal{A}_\infty\text{-categories} \arrow[r, "\mathrm{N}_{\mathcal{A}_\infty}"] \arrow[d, hook] & \text{simplicial sets} \\
\text{strict unital homotopy colored operads} \arrow[r, "\mathrm{N}^\Omega"] & \text{dendroidal sets} \arrow[u]
\end{tikzcd}$$

This *dendroidal nerve* is defined as follows. For any tree T, let us denote by $\mathbb{K}\Omega(T)$ the canonical algebraic colored operad induced by T and consider it as a strict unital homotopy colored operad. The dendroidal nerve $\mathrm{N}^\Omega(\mathscr{P})$ of a strict unital homotopy colored operad $\mathscr{P}$ is defined by

$$\mathrm{N}^\Omega(\mathscr{P})_T := \mathrm{Hom}_{\mathsf{suOp}_\infty}(\mathbb{K}\Omega(T), \mathscr{P}) \ ,$$

where suOp_∞ is the category of strict unital homotopy colored operads, with their morphisms, sometimes called ∞-morphisms. The following theorem gives the first comparison statement between the two worlds of homotopy operads and ∞-operads.

Theorem. *The dendroidal nerve of a strict unital homotopy colored operad is an ∞-operad.*

Moreover, we extend the Boardman–Vogt construction of dg colored operads to the category of strict unital homotopy colored operads. This later one is shown to provide us with a rectification functor, that is it is left adjoint to the inclusion functor from the category of dg colored operads to the category of strict unital homotopy colored operads. As a direct corollary, the dendroidal nerve

N^{Ω} generalises the homotopy coherent nerve hcN of colored operads on chain complexes built by Ieke Moerdijk and Ittay Weiss in [MW07] : there is a canonical isomorphism

$$\mathrm{hcN}(\mathscr{P}) \simeq \mathrm{N}^{\Omega}(\mathscr{P}) \ ,$$

which is natural in dg colored operads $\mathscr{P}$.

We prove that the dendroidal nerve functor satisfies some nice homotopy properties. For instance, if we endow the category of dendroidal sets with the Cisinski–Moerdijk model structure introduced in [CM11], we can characterize the morphisms of strict unital homotopy colored operads whose image under the nerve N^{Ω} are weak equivalences (resp. fibrations). As a consequence, if we consider the category of colored operads on chain complexes over a field of characteristic zero with the model structure introduced in [Cav14], then we can show that the homotopy coherent nerve hcN is a right Quillen functor.

Theorem. *There is a Quillen adjunction*

$$\mathsf{dSet} \underset{\mathrm{hcN}}{\overset{W_!^{dg}}{\rightleftarrows}} \mathsf{dg\text{-}Op} \ .$$

Finally, from a category enriched in chain complexes, one can truncate the mapping spaces and apply to them the Dold–Kan construction to obtain a simplicial category. Then, one can apply the simplicial nerve of simplicial categories and get a quasi-category. This procedure, called the big nerve, is introduced by Lurie in [Lur12, Section 1.3.1]. He shows that this nerve is equivalent to the simplicial nerve $\mathrm{N}_{\mathcal{A}_\infty}$. The big nerve and the latter result can be extended to the operadic level.

Theorem. *There exists a transformation of functors α^* from the big nerve of dg operads to the homotopy coherent nerve such that, for any dg colored operad $\mathscr{P}$, the morphism $\alpha^*(\mathscr{P}) : \mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}(\mathscr{P}) \to \mathrm{hcN}(\mathscr{P})$ is a weak equivalence of dendroidal sets.*

On the one hand, in topology, algebraic structures like products and compositions are often only defined up to homotopy. With no surprise, the same phenomenon appears on the operadic level. For instance, the space of configurations of discs inside the unit disc is canonically endowed with an operad structure, known and widely used as the little discs operad. However, the homotopy equivalent spaces made up of configurations of points on the plane, which are as ubiquitous as the latter ones, can only be endowed with an operad structure *up to homotopy*, see [VdL02]. For this reason, one needs a notion of topological or simplicial operad up to homotopy. A first model is provided by the above-mentioned notion of an ∞-operad. Other models exist as the Lurie's model extensively studied in [Lur12] or as the dendroidal Segal spaces. Moreover these notions of operads up to homotopy allows one to describes algebras in the context of ∞-categories. For instance, the notion of algebras of an ∞-operad in the ∞-category of spaces has been studied by Gijs Heuts in [Heu11]. On the other hand, the notion of strict unital homotopy operad lies in the algebraic framework of chain complexes. It appears in many situation. For instance a deformation retract of an algebraic operad inherits a structure of a homotopy operad. This chapter fits in the perspective of algebraic topology by showing that the usual functor sending a simplicial operad to an operad in chain complexes is part of a larger picture.

Layout In the first section of this chapter, we recall the definitions of colored operads, dendroidal sets, and ∞-operads; we provide the reader with more details on the relations between trees and operads. In the second section, we define colored generalizations of the notions of cooperads, homotopy operads and strict unital homotopy operads. Moreover, we extend the Boardman–Vogt construction to strict unital homotopy colored operads. The third section is the main part of this chapter. There, we introduce the dendroidal nerve which goes from the category of strict unital colored homotopy operads to the category of dendroidal sets. We prove that its image actually lands in the category of ∞-operads and we show that its restriction to dg operads is equal to the homotopy coherent nerve of Moerdijk–Weiss. Then we investigate its homotopical behavior and show that the homotopy coherent nerve is a right Quillen functor. In the fourth section, we introduce the big nerve of dg colored operads which extends the big nerve of dg categories of Lurie. Finally, we show that it is pointwise equivalent to the homotopy coherent nerve. In the appendix, we prove a equivalence between two notions of operations spaces of ∞-operads.

4.1 Recollections on colored operads and dendroidal sets

In this section, we recall the notions of colored operads, dendroidal sets and ∞-operads. These concepts are intimately related to the properties of trees. First, we make more precise the appendix of [BM07] defining colored operads as monoids. This clear presentation will allow us to introduce the relevant new operadic notions in the next section. Then, we give a short survey on dendroidal sets and ∞-operads together with their homotopical properties after the original references [MW07], [CM11] and [CM13].

4.1.1 Colored operads as monoids

We give a monoidal definition of colored operads over a symmetric monoidal category $(\mathsf{E}, \otimes, \mathbb{1}_{\mathsf{E}})$ with colimits preserved by the monoidal product. This notion can be found in the appendix of the paper [BM07] by Berger–Moerdijk. We start working over a fixed *set of colors* C.

Definition 48 (The groupoid Bij_C)**.** Let Bij_C be the category whose objects are pairs $(\chi : X \to C; c)$, where c is a color in C and χ is a function from a finite set X to the set C. A morphism from $(c; \chi : X \to C)$ to $(\chi' : Y \to C; c)$ consists of a a bijection $\beta : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "\beta"] \arrow[dr, "\chi"'] & & Y \arrow[dl, "\chi'"] \\ & C\, . & \end{tikzcd}$$

There are no morphisms between objects $(\chi; c)$ and $(\chi'; c')$ when $c \neq c'$.

Definition 49 ($(C, \mathbb{S})$-module)**.** A $(C, \mathbb{S})$*-module* is an E-presheaf on Bij_C, i.e. a functor from the category Bij_C^{op} to E. The category of $(C, \mathbb{S})$-modules is the category of E-presheaves on Bij_C. We denote it by $(C, \mathbb{S})$-Mod.

NOTATION Let $\mathcal{V}$ be a $(C, \mathbb{S})$-module, let c, c_1, $\ldots$, c_m be elements of C and let ϕ the function from $\underline{n}$ to C which sends i to c_i. We will sometimes write $\mathcal{V}(c_1, \ldots, c_m; c)$ to denote $\mathcal{V}(\phi; c)$.

One can interpret a $(C, \mathbb{S})$-module as a collection of operations with one output and zero or many inputs labeled by colors. For example, let $\mathcal{V}$ be a $(C, \mathbb{S})$-module, let $\chi : X \to C$ be a function and let c be an element of C. Then $\mathcal{V}(\chi; c)$ represents the operations whose inputs are the elements $x \in X$ colored by $\chi(x)$ and whose output is colored by c. To compose operations of two $(C, \mathbb{S})$-modules with respect to the colors, we introduce the following composite product.

Definition 50 (Composite product)**.** Let $\mathcal{V}$ and $\mathcal{W}$ be two $(C, \mathbb{S})$-modules. Their *composite product* is the $(C, \mathbb{S})$-module $\mathcal{V} \circ \mathcal{W}$ which sends every object $(\chi : X \to C; c)$ of the category Bij_C to the following colimit : $\mathcal{V} \circ \mathcal{W}(\chi : X \to C; c) :=$

$$\coprod_{k \geq 1} \left(\coprod_{\psi:\, \underline{k} \to C} \ \coprod_{\alpha:\, X \to \underline{k}} \mathcal{V}(\psi; c) \otimes \mathcal{W}(\chi_{|\alpha^{-1}(1)}; \psi(1)) \otimes \cdots \otimes \mathcal{W}(\chi_{|\alpha^{-1}(k)}; \psi(k)) \right)_{\mathbb{S}_k} ,$$

where the second coproduct is taken over all functions $\psi : \underline{k} \to C$ and where the third coproduct is taken over all functions $\alpha : X \to \underline{k}$. This colimit is a coset under the following actions of the symmetric groups $\mathbb{S}_k$, for integers $k \geq 1$.

- ▷ A permutation σ in $\mathbb{S}_k$ induces an isomorphism $(\psi\sigma^{-1}; c) \to (\psi; c)$ in the groupoid Bij_C and so an isomorphism $\mathcal{V}(\sigma) : \mathcal{V}(\psi; c) \to \mathcal{V}(\psi\sigma^{-1}; c)$ in E.

▷ This permutation also induces an isomorphism σ^* from $\mathcal{W}(\phi_{|\alpha^{-1}(1)}; \psi(1)) \otimes \cdots \otimes \mathcal{W}(\phi_{|\alpha^{-1}(k)}; \psi(k))$ to $\mathcal{W}(\phi_{|\alpha^{-1}(\sigma^{-1}(1))}; \psi(\sigma^{-1}(1))) \otimes \cdots \otimes \mathcal{W}(\phi_{|\alpha^{-1}(\sigma^{-1}(k))}; \psi(\sigma^{-1}(k)))$ through the symmetric monoidal structure of E.

The global action of $\mathbb{S}_k$ is given by the isomorphisms $V(\sigma) \otimes \sigma^*$, for every permutation $\sigma \in \mathbb{S}_k$.

Let I_C be the $(C, \mathbb{S})$-module defined by $I_C(c; c) := I_C(* \mapsto c; c) = \mathbb{1}_{\mathsf{E}}$, for any $c \in C$, and by $I_C(\chi : X \to C; c) = \emptyset$ (the initial object in the category E) if the cardinal of X is not 1 or if the image of χ is different from the color $\{c\}$ of the output.

Proposition 73. *The category of $(C, \mathbb{S})$-modules together with the composite product $\circ$ and the unit object I_C forms a monoidal category.*

Proof. The proof is similar to the classical case of non-colored operads, see [LV12, Section 5.1]. □

Definition 51 (C-colored operad)**.** A *C-colored operad* is a monoid $(\mathcal{P}, \gamma, \eta)$ in the monoidal category of $(C, \mathbb{S})$-modules : the composition map $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ is associative and the map $\eta : I_C \to \mathcal{P}$ is a unit.

4.1.2 Morphisms of colored operads

We provide here a detailed definition of the suitable notion of morphism between two operads colored over possibly different sets of colors. From now on, we consider the set of colors to be part of the data and we work over varying sets of colors.

Definition 52 (Colored $\mathbb{S}$-module)**.** A *colored $\mathbb{S}$-module* $(C, \mathcal{V})$ is made up of a set C of colors and a $(C, \mathbb{S})$-module $\mathcal{V}$.

We will define morphisms of $(C, \mathbb{S})$-modules in an analogous way as morphisms of presheaves on topological spaces are defined. Let us recall that an E-presheaf on a topological space X is a E-presheaf over the category Open_X of open subsets of X with inclusions. Let X and Y be topological spaces and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves respectively on X and Y. Any continuous function $f : X \to Y$ induces a functor F from Open_Y to Open_X. The presheaf $f^{-1}\mathcal{G}$ on X is defined by $f^{-1}\mathcal{G}(U) := \lim_{f(U) \subset V} \mathcal{G}(V)$ and the presheaf $f_*\mathcal{F}$ on Y by $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)) = \mathcal{F}(F(U))$. Furthermore, a morphism of presheaves on topological spaces from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ is the data of a continuous function f from X to Y, and hence a functor F from Open_Y to Open_X, together with a morphism of presheaves over Open_Y from $\mathcal{G}$ to $f_*\mathcal{F}$, or equivalently, a morphism of presheaves over Open_X from $f^{-1}\mathcal{G}$ to $\mathcal{F}$.

Definition 53 (Pullback and pushforward of colored $\mathbb{S}$-module)**.** Let $\phi : C \to D$ be a function between two sets of colors and let $\mathcal{V}$ and $\mathcal{W}$ be a $(C, \mathbb{S})$-module and a $(D, \mathbb{S})$-module respectively. We define $\phi^*\mathcal{W}$ to be the following $(C, \mathbb{S})$-module

$$\phi^*\mathcal{W}(\chi; c) := \mathcal{W}(\phi\chi; \phi(c))$$

and we define $\phi_!\mathcal{V}$ to be the following $(D, \mathbb{S})$-module

$$\phi_!\mathcal{V}(\rho; d) := \coprod_{\substack{\rho = \phi\chi, \\ \phi(c) = d}} \mathcal{V}(\chi; c) ,$$

for any $\rho : X \to D$ and $d \in D$. The coproduct is taken over the colors c in C such that $\phi(c) = d$ and the functions $\chi : X \to C$ such that $\rho = \phi\chi$.

These two constructions are functorial.

Lemma 30. *For any function $\phi : C \to D$, the functor ϕ^* is right adjoint to the functor $\phi_!$; equivalently there exist natural bijections*

$$\mathrm{Hom}_{(D,\mathbb{S})\text{-}\mathsf{Mod}}(\phi_!\mathcal{V}, \mathcal{W}) \cong \mathrm{Hom}_{(C,\mathbb{S})\text{-}\mathsf{Mod}}(\mathcal{V}, \phi^*\mathcal{W}) .$$

Proof. The proof is straightforward and left to the reader. □

These functors behave well with respect to the composition of functions : for any functions $\phi : B \to C$ and $\psi : C \to D$, $(\psi\phi)^* = \phi^*\psi^*$ and $(\psi\phi)_! = \psi_!\phi_!$.

Definition 54 (Morphism of colored $\mathbb{S}$-modules)**.** A *morphism of colored* $\mathbb{S}$*-modules* from $(C, \mathcal{V})$ to $(D, \mathcal{W})$ amounts to the data of a function ϕ from C to D and a morphism f^* of $(C, \mathbb{S})$-modules from $\mathcal{V}$ to $\phi^*\mathcal{W}$, or equivalently, a morphism $f_!$ of $(D, \mathbb{S})$-modules from $\phi_!\mathcal{V}$ to $\mathcal{W}$. Such a morphism, defined either by the couple (ϕ, f^*) or by the couple $(\phi, f_!)$, will be denoted simply by f.

A morphism $f : (C, \mathcal{V}) \to (D, \mathcal{W})$ of colored $\mathbb{S}$-modules is therefore the data of a function $\phi : C \to D$ and morphisms $f^*(\chi; c) : \mathcal{V}(\chi; c) \to \mathcal{W}(\phi\chi; \phi(c))$ for any object $(\chi : X \to C; c)$ of Bij_C such that, for any bijection $\beta : Y \to X$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{V}(c;\chi) \arrow[r, "f^*(\chi;c)"] \arrow[d, "\mathcal{V}(\beta)"'] & \mathcal{W}(\phi\chi;\phi(c)) \arrow[d, "\mathcal{W}(\beta)"] \\
\mathcal{V}(\chi\beta;c) \arrow[r, "f^*(\chi\beta;c)"'] & \mathcal{W}(\phi\chi\beta;\phi(c))
\end{tikzcd} .$$

Proposition 74. *Colored* $\mathbb{S}$*-modules and their morphisms form a category, denoted by* $\mathbb{S}\text{-}\mathsf{Mod}$.

Proof. One defines the composite of two morphisms $f = (\phi, f^*)$ and $g = (\psi, g^*)$ by

$$gf := (\psi\phi, \phi^*(g^*)f^*) .$$

□

Remark 11. For any set C, there is an inclusion of categories $(C, \mathbb{S})\text{-}\mathsf{Mod} \hookrightarrow \mathbb{S}\text{-}\mathsf{Mod}$ which sends a $(C, \mathbb{S})$-module $\mathcal{V}$ to the $\mathbb{S}$-module $(C, \mathcal{V})$. A morphism $f = (\phi, f^*)$ is in the image of this inclusion if and only if ϕ is the identity of the set C.

Lemma 31. *For any function* $\phi : C \to D$*, the functor* ϕ^* *is lax monoidal, i.e. there are morphisms* $\phi^*_{\mathcal{V},\mathcal{W}} : \phi^*\mathcal{V} \circ \phi^*\mathcal{W} \to \phi^*(\mathcal{V} \circ \mathcal{W})$*, natural in* $\mathcal{V}$ *and* $\mathcal{W}$ *and* $\phi^*_I : I_C \to \phi^* I_D$*, satisfying associativity and unitality conditions, see [ML98] for more details.*

Proof. For any $(D, \mathbb{S})$-modules $\mathcal{V}$ and $\mathcal{W}$, the morphism $\phi^*_{\mathcal{V},\mathcal{W}}$ is built from the following equality :

$$\begin{gathered}
\phi^*(\mathcal{V})(\psi; c) \otimes \phi^*(\mathcal{W})(\chi_{|\alpha^{-1}(1)}; \psi(1)) \otimes \cdots \otimes \phi^*(\mathcal{W})(\chi_{|\alpha^{-1}(k)}; \psi(k)) = \\
\mathcal{V}(\phi\psi; \phi(c)) \otimes \mathcal{W}(\phi\chi_{|\alpha^{-1}(1)}; \phi(\psi(1))) \otimes \cdots \otimes \mathcal{W}(\phi\chi_{|\alpha^{-1}(k)}; \phi(\psi(k))) .
\end{gathered}$$

□

Remark 12. The monoidal functor ϕ^* is strong if and only if the map ϕ is bijective.

Proposition 75. *Let* $\phi : C \to D$ *be a function, and let* $(\mathcal{P}, \gamma, \eta)$ *be a* D*-colored operad. The* $(C, \mathbb{S})$*-module* $\phi^*\mathcal{P}$ *has a canonical structure of* C*-operad* $(\phi^*\mathcal{P}, \gamma^\phi, \eta^\phi)$ *induced by the structure of* D*-operad on* $\mathcal{P}$.

Proof. This is a corollary of the previous lemma since the image of a monoid under a lax monoidal functor is again a monoid. □

Definition 55 (The category of colored operads)**.**

- ▷ A *colored operad* $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ is the data of a set of colors C and a C-colored operad $(\mathcal{P}, \gamma, \eta)$.
- ▷ A *morphism of colored operads* from $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ to $\mathscr{Q} = (D, \mathcal{Q}, \nu, \theta)$ is a morphism of colored $\mathbb{S}$-modules $f = (\phi, f^*)$ such that f^* is a morphism of C-operads from $(\mathcal{P}, \gamma, \eta)$ to $(\phi^*\mathcal{Q}, \nu^\phi, \theta^\phi)$, i.e. a morphism of monoids in the monoidal category of $(C, \mathbb{S})$-modules.

Proposition 76. *Colored operads together with their morphisms form a category denoted* Op.

Proof. We first prove that the composite of two morphisms of colored operads is a morphism of colored operads. Let $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$, $\mathscr{Q} = (D, \mathcal{Q}, \nu, \theta)$ and $\mathscr{R} = (E, \mathcal{R}, \mu, \upsilon)$ be three colored operads, and let $f = (\phi, f^*) : \mathscr{P} \to \mathscr{Q}$ and $g = (\psi, g^*) : \mathscr{Q} \to \mathscr{R}$ be two morphisms of colored operads. Their composite is equal to $gf := (\psi\phi, \phi^*(g^*)f^*)$. Since the functor ϕ^* is a lax monoidal functor, it preserves morphisms of monoids. So the composite $\phi^*(g^*)f^*$ is a morphism of $(C, \mathbb{S})$-operads. The unit morphisms for the composite of morphisms of colored operads are the unit morphisms of the category of colored $\mathbb{S}$-modules. $\square$

Definition 56 (Restriction of colored operads to categories)**.** Let C be a set of colors and let $j : \mathsf{Bij}_C(1) \hookrightarrow \mathsf{Bij}_C$ be the embedding of the full sub-category of Bij_C made up of objects $(\chi : X \to C; c)$ where X has one elements. It induces a restriction functor j^* from the category of C-colored operads to the category of categories enriched over E and whose set of objects is C. It extends canonically to a functor $j^* : \mathsf{Op} \to \mathsf{Cat}_{\mathsf{E}}$ from colored operads enriched over E to small categories enriched over E.

Convention Since we will only work with colored operads throughout the text, we will often use the simpler terminology of $\mathbb{S}$-modules and operads, i.e. dropping the understood adjective "colored".

4.1.3 The category of trees

The theory of operads is intrinsically related to the combinatorics of trees. So we begin this section with a precise definition of the notion of tree used here. The formalism of trees will be used in the next section to introduce the concept of dendroidal set [MW07], which is an operadic generalization of the concept of simplicial set. Finally, we will recall the definition of an ∞-operad, which is to dendroidal sets what ∞-categories are to simplicial sets, that is a weak Kan object.

Definition 57 (Graph)**.** A *graph* is a quadruple $G = (V, F, \upsilon, \rho)$ where V is a finite set of elements called the *vertices*, F is a finite set of elements called the *flags* or *half-edges*, υ is a function from the set F of flags to the set V of vertices and ρ is an involution of the set F. The orbits of this involution are called *edges*. An edge is *inner* if it contains two flags and *outer* otherwise.

Example 5. For instance, two vertices linked by an edge is just a set of vertices with two objects $\{v_1, v_2\}$, a set of flags with two objects $\{f_1, f_2\}$, a function υ such that $\upsilon(f_i) = v_i$ and an involution ρ such that $\rho(f_1) = f_2$.

Definition 58 (Tree)**.** A *(rooted) tree* $T = (V, F, \upsilon, \rho, r)$ is a connected graph (V, F, υ, ρ) with no cycles and a distinguished outer edge r called the *root*. The remaining outer edges are called *leaves*. In this context, each vertex has one output edge and possibly many input edges (there can be no input edge). The number of inputs of a vertex is called its *arity*.

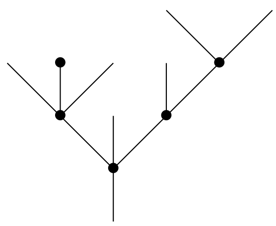


Figure 4.1 – Example of a tree.

Figure 4.2 – The *trivial* tree with no vertex but one edge.

Definition 59 (Sub-tree)**.** Let $T = (V, F, \upsilon, \rho, r)$ be a tree. A *non trivial sub-tree* T' of T is a non-empty subset V' of V which is connected, i.e. for any two vertices v_1 and v_2 of V', there exists a path between them in the tree T which only visits vertices in V'. This subset determines a new tree also denoted T' which is the 5-tuple $(V', F', \upsilon', \rho', r')$ where F' is the set of flags f of F such that $\upsilon(f)$ is in V', the function υ' is the restriction of υ to the set F' and for any flag f in F', then $\rho'(f) = \rho(f)$ if $\rho(f) \in F'$ and $\rho(f) = f$ otherwise. The root r' is the edge of T' which is the flag closest to the root r of T.

Definition 60 (Partition of a tree)**.** Let $T = (V, F, u, \rho, r)$ be a tree. A *partition* of T with no trivial component is the data of non-trivial sub-trees $T_1, \ldots, T_k$ with no common vertices such that their union contains every vertex of T. Moreover, we denote by $T/T_1 \cdots T_n$ the tree obtained from T by contracting into one vertex each sub-tree T_i.

$T = T_1 \sqcup T_2 \sqcup T_3$ $\qquad\qquad$ $T/T_1T_2T_3$

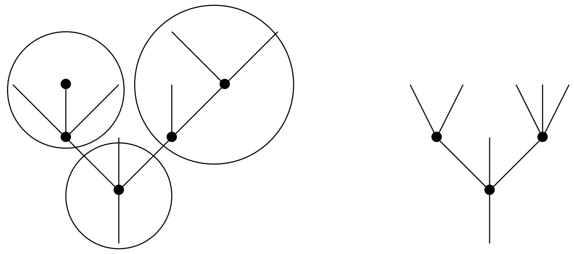


FIGURE 4.3 – An example of a partitioned tree and its associated contraction.

Following Moerdijk–Weiss [MW07], we can consider the following set theoretical colored operad $\Omega(T)$ generated by any tree T.

Definition 61 (The operad $\Omega(T)$)**.** For any tree T, $\Omega(T)$ is the operad colored by the set of edges of T, freely generated by the set of vertices of T. In details, $\Omega(T)(c; \chi : X \to C) := \{*\}$ if there is a (possibly trivial) sub-tree of T with output c and inputs $\chi(x_1), \ldots, \chi(x_n)$. Otherwise $\Omega(T)(c; \chi : X \to C) := \emptyset$. The composite map is given by the grafting of sub-trees inside T.

Definition 62 (The category of trees)**.** The category of trees, written Tree is made up of trees, as defined above ; the morphisms from a tree T to a tree T' are the morphisms of operads from $\Omega(T)$ to $\Omega(T')$, i.e.

$$\mathrm{Hom}_{\mathsf{Tree}}(T, T') := \mathrm{Hom}_{\mathsf{Op}}(\Omega(T), \Omega(T')) \ .$$

Here are three families of simple morphisms of trees. An *outer coface* δ_v adds a new external vertex v, i.e. a vertex attached to at most one other vertex. An *inner coface* δ_e introduces an inner edge e. A *codegeneracy* σ erase an arity 1 vertex. The inner and outer cofaces, the codegeneracies and the isomorphisms generate all the morphisms of trees. More details can be found in the paper [MW07].

4.1.4 Dendroidal sets

We introduce here the concept of dendroidal set due to Moerdijk–Weiss [MW07], which is an operadic generalization of the concept of simplicial set.

Definition 63 (Dendroidal set)**.** A *dendroidal set* is presheaf on the category Tree, i.e. it is a functor from the category Tree^{op} to the category Set of sets. We denote the category of dendroidal sets by dSet. By analogy with simplices of simplicial sets, we call *dendrices* of a dendroidal set D the elements of the image sets D_T, for any tree T.

Let D be a dendroidal set. Any outer coface $\delta_v : T - \{v\} \to T$, any inner coface $\delta_e : T/e \to T$ and any codegeneracy $\sigma : T \to T'$ give respectively an outer face $d_v : D_T \to D_{T-\{v\}}$, an inner face $d_e : D_T \to D_{T/e}$ and a degeneracy $s : D_{T'} \to D_T$.

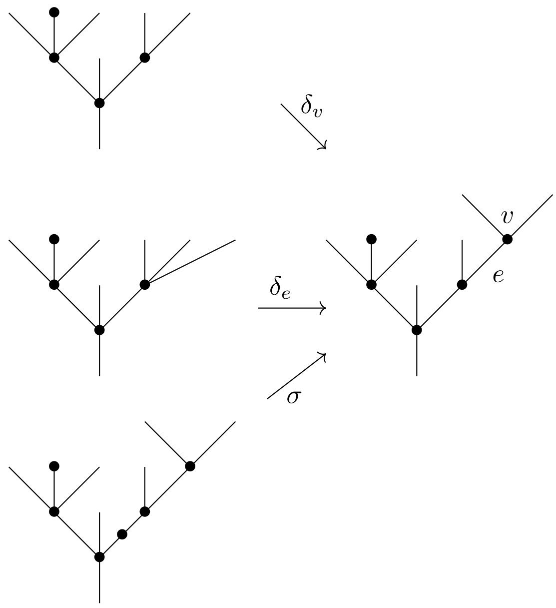


FIGURE 4.4 – Examples of an outer coface, an inner coface and a codegeneracy

We can consider the poset $[n] := 0 < 1 < \cdots < n$ as the linear tree (ladder) with n vertices and $n+1$ edges labeled by the set $\{0,\ldots,n\}$ from bottom to top. Then we can consider the category Δ made up of sets $\{0,\ldots,n\}$ and non-decreasing functions as a full sub-category of the category Tree. Let us denote $i : \Delta \to \mathsf{Tree}$ this embedding. It induces a restriction functor $i^* : \mathsf{dSet} \to \mathsf{sSet}$ which sends a dendroidal set $D : \mathsf{Tree}^{op} \to \mathsf{Set}$ to the following simplicial set :

$$i^*D : \Delta^{op} \overset{i}{\hookrightarrow} \mathsf{Tree}^{op} \xrightarrow{D} \mathsf{Set} \ .$$

Moreover, the restriction functor i^* has a left adjoint $i_!$ sending a simplicial set S to the dendroidal $i_!S$ such that $(i_!S)_{[n]} = S_n$ for any linear tree $[n]$ and such that $(i_!S)_T = \emptyset$ for any tree T which is not in the image of $i : \Delta \to \mathsf{Tree}$.

Dendroidal sets are thus a generalization of simplicial sets. Moreover, the simplicial sets $\Delta[n]$, $\partial\Delta[n]$ and $\Lambda^k[n]$ have dendroidal analogues which we define below.

Definition 64 (The dendroidal sets $\Omega[T]$, $\partial\Omega[T]$, $\Lambda^e[T]$ and $\partial^{ext}\Omega[T]$)**.** For any tree T, let $\Omega[T]$ be the dendroidal set given by the Yoneda embedding of trees into dendroidal sets, that is

$$\Omega[T] := \mathrm{Hom}_{\mathsf{Tree}}(-,T) \ .$$

It has several canonical sub-objects that we will use.

▷ Let $\partial\Omega[T]$ be the sub-dendroidal set of $\Omega[T]$ generated by all the faces (inner and outer).

$$\partial\Omega[T]_{T'} := \{f \in \mathrm{Hom}_{\mathsf{Tree}}(T',T) \mid \exists g,\ \exists\delta,\ f = \delta g\} \ ,$$

where g is a morphism of trees and δ is a coface targeting T.

▷ For any inner edge e of T, let $\Lambda^e[T]$ be the sub-dendroidal set of $\Omega[T]$ generated by all the faces (inner and outer) except the one corresponding to e, that is

$$\Lambda^e[T]_{T'} := \{f \in \mathrm{Hom}_{\mathsf{Tree}}(T',T) \mid \exists g,\ \exists\delta \neq \delta_e,\ f = \delta g\} \ ,$$

where g is a morphism of trees and δ is a coface targeting T which is different from the inner coface related to the edge e.

▷ Finally let $\partial^{ext}\Omega[T]$ be the the sub-dendroidal set of $\Omega[T]$ generated by all the outer faces, that is
$$\partial^{ext}\Omega[T]_{T'} := \{f \in \mathrm{Hom}_{\mathsf{Tree}}(T', T) \mid \exists g,\ \exists \delta,\ f = \delta g\}\ ,$$
where g is a morphism of trees and δ is an outer coface targeting T.

The category of set-theoretical colored operads is canonically embedded in the category of dendroidal sets through the nerve functor $\mathrm{N}_d : \mathsf{Op} \to \mathsf{dSet}$ defined as follows :
$$\mathrm{N}_d(\mathscr{P})_T := \mathrm{Hom}_{\mathsf{Op}}(\Omega(T), \mathscr{P})\ .$$
This functor has a left adjoint τ_d such that $\tau_d(\Omega[T]) = \Omega(T)$.

Convention For any dendroidal set D, the elements of the set $D_|$, where $|$ is the trivial tree, will be called the colors of the dendroidal set D. Moreover, if we denote the corolla with m inputs by C_m, then the elements of D_{C_m} will be called the arity m operations of D. For such a corolla and any leaf l of it, there is a face map $D_T \to D_|$, which gives the color of the input corresponding to l. The face map $D_T \to D_|$ corresponding to the root gives the color of the output. Indeed, for any set-theoretical colored operad $\mathscr{P}$, the set $\mathrm{N}_d(\mathscr{P})_|$ is the set of colors of $\mathscr{P}$ and $\mathrm{N}_d(\mathscr{P})_{C_m}$ are the operations of $\mathscr{P}$ of arity m.

4.1.5 Infinity-operads

We give here the definition of an ∞-operad, which is to dendroidal sets what ∞-categories are to simplicial sets. Recall that an *∞-category* (or *quasi-category* or *weak Kan complex*) is a simplicial set X such that for all $n \geq 2$ and $1 \leq k \leq n-1$ and every morphism $\Lambda^k[n] \to X$, there is a morphism $\Delta[n] \to X$ which lifts the horn inclusion $\Lambda^k[n] \to \Delta[n]$.

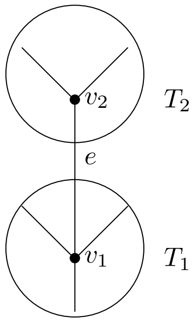
$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Quasi-categories are models of $(\infty, 1)$-categories where the objects are the 0-vertices, the 1-morphisms are the 1-vertices and where the higher vertices encode higher homotopical data. Indeed, the above lifting property for $n = 2$ and $k = 1$ means that any two 1-morphisms such that the target of the first one is the source of the second one can be composed up to homotopy. More details can be found in [Lur09]. The following notion of ∞-operad is an operadic generalization of this notion of ∞-category.

Definition 65 (∞-operad)**.** An *∞-operad*, or *infinity-operad* in plain words, is a dendroidal set D such that for every tree T and any inner edge e of T, every morphism from $\Lambda^e[T]$ to D can be lifted to a morphism from $\Omega[T]$ to D :

$$\begin{tikzcd} \Lambda^e[T] \arrow[r] \arrow[d, hook] & D \\ \Omega[T]\ . \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

The "horn" condition of the definition for two-vertex trees means that two operations with compatible colors can be composed up to homotopy. Indeed, let T be a tree with two vertices : it is made up of two sub-trees T_1 and T_2 joined by an edge e. Let v_1 (resp. v_2) be the unique vertex of the tree T_1 (resp. T_2).



Let T' be the corolla obtained from T by contracting the edge e. The morphisms $\delta_{v_2} : T_1 \to T$, $\delta_{v_1} : T_2 \to T$ and $\delta_e : T' \to T$ are the three coface maps targeting the tree T. Let $x \in D_{T_1}$ and $y \in D_{T_2}$ such that $d_e(x) = d_e(y) \in D_|$. Through the two outer face maps, they determine a morphism $\Lambda^e[T] \to D$, which induces a morphism $\Omega[T] \to D$, i.e. a element z of D_T. The inner face $d_e(z)$ can be thought of as the composition of x with y along the edge e. Moreover, the fact that the morphism from $\Omega[T]$ to D extending x and y is not necessarily unique means that their composite is not necessarily strictly unique.

4.1.6 The homotopy theory of dendroidal sets

In their paper [CM11], Denis-Charles Cisinski and Ieke Moerdijk endow the category of dendroidal sets with a model structure in order to provide a homotopical interpretation for the notion of ∞-operad.

Definition 66 (Categorical fibration)**.** A *categorical fibration* (or *isofibration*) is a functor $f : \mathsf{C} \to \mathsf{C}'$ such that, given any isomorphism $\phi : c_0' \to c_1'$ in C', and any object c_1 in C such that $f(c_1) = c_1'$, there exists an isomorphism $\psi : c_0 \to c_1$ in C, such that $f(\psi) = \phi$.

Theorem 24. *[CM11, Theorem 2.4, Proposition 2.6] The category of dendroidal sets is endowed with a model structure such that :*

- ▷ *the fibrant objects are ∞-operads.*
- ▷ *the cofibrations are the normal monomorphisms, i.e. the monomorphisms $A \to B$ such that for every tree T the action of the automorphism group $Aut(T)$ on $B_T - A_T$ is free.*
- ▷ *the fibrations between fibrant objects, i.e. ∞-operads, are the morphisms f such that $i^*\tau_d(f)$ is a categorical fibration and such that f satisfies the right lifting property with respect to the inner horn inclusions, i.e. the morphisms $\Lambda^e[T] \hookrightarrow \Omega[T]$ where T is a tree and e is an inner edge of T.*

Furthermore, this model structure is left proper and combinatorial.

Remark 13. The restriction of this model structure to the category of simplicial sets, which is the slice category $\mathsf{dSet}/\Delta[0]$, is exactly the Joyal model structure. See [Lur09, 2.2.5]. Furthermore, in [Joy, Proposition E.1.10] André Joyal shows that a model structure on a category is determined by its class of cofibrations and its class of fibrant objects.

We will need to have a precise description of the weak equivalences between ∞-operads.

Definition 67 (Essentially surjective morphisms)**.** A morphism of dendroidal sets $A \to B$ is *essentially surjective* if the induced functor $i^*\tau_d A \to i^*\tau_d B$ is essentially surjective, i.e. any object of $i^*\tau_d B$ is isomorphic to the image of an object of $i^*\tau_d A$.

For any integers $m \geq 0$ and $n \geq 0$, let $C_{m,n}$ be the tree made up of a corolla having m inputs and the linear tree with n vertices $[n]$ under it ; see Figure 4.1.6. There is a canonical morphism of trees from $[n]$ to $C_{m,n}$. The data of such morphisms $[n] \to C_{m,n}$ gives us a cosimplicial object in the category of trees for any $m \geq 0$.

Let us number the leaves of C_m from 1 to m. Let A be a dendroidal set and let c, c_1, ..., c_m be colors of A, i.e. the elements of the set $A_|$. Let $A^L(c_1, \ldots, c_m; c)$ be the sub-simplicial set of $\mathrm{Hom}_{\mathsf{dSet}}(\Omega[C_{m,-}], A)$ whose n simplices are the morphisms of $\mathrm{Hom}_{\mathsf{dSet}}(\Omega[C_{m,n}], A)$ such that their restriction to the leave i is the color c_i and their restriction to $[n]$ is the degeneracy of the color c.

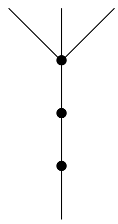


FIGURE 4.5 – Example : the tree $C_{3,2}$.

Definition 68 (Fully faithful morphisms)**.** A morphism of ∞-operads $f : P \to Q$ is *fully faithful* if for any integer $m \geq 0$ and for any colors c, c_1, ..., c_m of P, the morphism

$$P^L(c_1, \ldots, c_m; c) \to Q^L(f(c_1), \ldots, f(c_m); f(c))$$

is a weak homotopy equivalence of simplicial set, i.e. a weak equivalence in the Quillen model structure on simplicial sets.

Proposition 77. *[Heu11, Proposition 4.17] A morphism of ∞-operads $f : P \to Q$ is a weak equivalence if and only it is essentially surjective and fully faithful.*

REMARK 14. In the paper [CM13], Cisinski–Moerdijk give an other definition of fully faithful morphisms : a morphism of ∞-operads $f : P \to Q$ is “fully faithful“ if for any integer $m \geq 0$ and for any colors c, c_1, ..., c_m of P, the morphism

$$P(c_1, \ldots, c_m; c) \to Q(f(c_1), \ldots, f(c_m); f(c))$$

is a weak homotopy equivalence of simplicial set, where $P(c_1, \ldots, c_m; c)$ is a simplicial set defined by the following collection of pullbacks

$$\begin{tikzcd}
P(c_1, \ldots, c_m; c)_n \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathsf{dSet}}(\Omega[C_m], P^{(\Delta[n])}) \arrow[d] \\
\Delta[0] \arrow[r, "(c, c_1, \ldots, c_m)"'] & \mathrm{Hom}_{\mathsf{dSet}}(\coprod_{i=0}^{m} \Delta[0], P^{(\Delta[n])}) .
\end{tikzcd}$$

where $(P^{(\Delta[n])})_{n \in \mathbb{N}}$ is a Reedy fibrant replacement of P which is defined in [CM13, 3.1]. In the appendix we give a direct proof of the fact that $P(c_1, \ldots, c_m; c)$ is homotopy equivalent to $P^L(c_1, \ldots, c_m; c)$. This implies that the two definitions of the fully-faithful morphisms of ∞-operads are equivalent.

4.1.7 The Dold Kan correspondance

We recall here the Dold–Kan correspondance between simplicial $\mathbb{K}$-modules and nonnegatively graded chain complex of $\mathbb{K}$-modules. See [GJ99, 3.2] and [ML95, Section 8.8] for more details on this subject.

Let X be a simplicial $\mathbb{K}$-module and let V be chain complex of $\mathbb{K}$-modules.

▷ Let $C(X)$ be the Moore complex of X, that is the nonnegatively graded chain complex such that $C(X)_n := X_n$ for any integer $n \geq 1$, and whose differential $d : C(X)_n \to C(X)_{n-1}$ is given by the formula $d := \sum_{i=0}^{n} (-1)^i d_i$ where d_i is the i^{th} face.

▷ Let $N(X)$ be the normalized Moore complex of X, that is the sub-chain complex of $C(X)$ defined by $N(X) := \cap_{i=0}^{n-1} ker(d_i)$.

▷ Let $D(X)$ be the sub-chain complex of $C(X)$ generated by the images of the degeneracies $D(X) := \sum_{i=0}^{n-1} im(s_i)$. Besides, let $\pi(X)$ the functorial projection of $C(X)$ on

$C/D(X) := C(X)/D(X)$. Then, the composite map $N(X) \hookrightarrow C(X) \twoheadrightarrow C/D(X)$ is a functorial isomorphism.

▷ Let $\Gamma(V)$ be the simplicial $\mathbb{K}$-module such that for any integer n, $\Gamma(V)_n := \bigoplus_{[n]\twoheadrightarrow[p]} V_p$, where the sum is a taken over the maps of ordered sets $[n] \twoheadrightarrow [p]$ which are surjections. The faces and degeneracies are defined as follows. Let $\eta : [n] \twoheadrightarrow [p]$ be a surjection and $v \in V_p \subseteq \Gamma(V)_n$ an element in the η-summand of $\Gamma(V)_n$.
  ▷ For any codegeneracy $\sigma_i : [n+1] \twoheadrightarrow [n]$, the corresponding degeneracy $s_i(v)$ of v is the element $v \in V_p \subseteq \Gamma(V)_{n+1}$ in the $\eta\sigma_i$-summand of $\Gamma(V)_{n+1}$.
  ▷ Let $\delta_i : [n-1] \hookrightarrow [n]$ be a coface. If $\eta\delta_i$ is still a surjection, then the corresponding face $d_i(v)$ of v is the element $v \in V_p \subseteq \Gamma(V)_{n-1}$ in the $\eta\delta_i$-summand of $\Gamma(V)_{n-1}$. Otherwise, $\eta\delta_i$ can be uniquely factorized as the composition $[n-1] \twoheadrightarrow [p-1] \hookrightarrow [p]$ of an ordered surjection η' followed by a coface δ_j. If $j = p$, then $d_i(v) := (-1)^p d(v) \in V_{p-1}$ in the η'-summand of $\Gamma(V)_{n-1}$. Otherwise, $d_i(v) := 0$.

The functor Γ from the category $\mathsf{dg\text{-}Mod}^{\geq 0}$ of nonnegatively graded chain complexes to the category of simplicial $\mathbb{K}$-modules is both right adjoint and left adjoint to the functor C/D. Furthermore, these two functors give us an equivalence of categories. This is the Dold–Kan correspondence in the context of $\mathbb{K}$-modules.

The functor C/D from the category of simplicial $\mathbb{K}$-modules to the category of nonnegatively graded chain complexes is lax symmetric monoidal through the Eilenberg–Zilber map. Furthermore, it is lax comonoidal through the Alexander–Whitney map ; see [ML95, Section 8.8] for a description of these maps. Hence, its adjoint Γ is lax monoidal. However, Γ is not symmetric monoidal.

4.2 Strict unital homotopy colored operads

In this section we introduce the new notion of homotopy colored operads with strict unit, which is the operadic generalization of the notion of $\mathcal{A}_\infty$-category [FOOO09]. For that purpose, we introduce the notions of colored cooperads, conilpotent cofree colored cooperads and coderivations, which are generalizations from the non-colored case in the framework developed in the previous section. The propositions are often proved in the same way as in the non-colored case, but we recall the key properties that will be used later on.

4.2.1 Colored cooperads

We first consider the category of colored cooperads.

Definition 69 (Colored cooperads)**.**
▷ For any set C, a *C-cooperad* is a comonoid $(\mathcal{C}, \Delta, \varepsilon)$ in the category of $(C, \mathbb{S})$-modules.
▷ More generally, a *colored cooperad* is a quadruple $(C, \mathcal{C}, \Delta, \varepsilon)$, where C is a set and where $(\mathcal{C}, \Delta, \varepsilon)$ is a C-cooperad.

Lemma 32. *For any function $\phi : C \to D$, the functor $\phi_! : (C, \mathbb{S})\text{-}\mathsf{Mod} \to (D, \mathbb{S})\text{-}\mathsf{Mod}$ is a lax comonoidal functor, i.e. there is a natural morphism $\phi_!(\mathcal{V} \circ \mathcal{W}) \to \phi_!\mathcal{V} \circ \phi_!\mathcal{W}$ and a morphism $\phi_! I_C \to I_D$ satisfying coherence conditions.*

Proof. For any $(C, \mathbb{S})$-modules $\mathcal{V}$ and $\mathcal{W}$ and any object $(c'; \chi' : X \to D)$ of Bij_D, we have :

$$\phi_!(\mathcal{V} \circ \mathcal{W})(\chi'; c') = \coprod_{k\geq 1} \Big(\coprod_{c,\chi,\alpha,\upsilon} \mathcal{V}(\upsilon; c) \otimes \mathcal{W}(\chi_{|\alpha^{-1}(1)}; \upsilon(1)) \otimes \cdots \otimes \mathcal{W}(\chi_{|\alpha^{-1}(k)}; \upsilon(k)) \Big)_{\mathbb{S}_k} ,$$

where the second coproduct is taken over the colors c in C such that $\phi(c) = c'$, the functions $\chi : X \to C$ such that $\phi\chi = \chi'$, the functions $\alpha \to \underline{k}$ and the functions $\upsilon : \underline{k} \to C$. Furthermore we have,

$$(\phi_!\mathcal{V} \circ \phi_!\mathcal{W})(\chi'; c') = \coprod_{k\geq 1} \Big(\coprod_{\substack{c,\chi,\alpha,\upsilon \\ (c_1,\ldots,c_k)}} \mathcal{V}(\upsilon; c) \otimes \mathcal{W}(\chi_{|\alpha^{-1}(1)}; c_1) \otimes \cdots \otimes \mathcal{W}(\chi_{|\alpha^{-1}(k)}; c_k) \Big)_{\mathbb{S}_k} ,$$

where the second coproduct is taken over the colors c in C such that $\phi(c) = c'$, the functions $\chi : X \to C$ such that $\phi\chi = \chi'$, the functions $\alpha \to \underline{k}$, the functions $\upsilon : \underline{k} \to C$, and the k-tuples of colors $(c_1, \ldots, c_k)$ such that $\phi(c_i) = \phi(\upsilon(i))$. The map

$$\begin{aligned} \{\upsilon : \underline{k} \to C\} &\to \{(\upsilon : \underline{k} \to C, (c_1, \ldots, c_k)) \mid \phi(c_i) = \phi(\upsilon(i)),\ \forall i\} \\ \upsilon &\mapsto \big(\upsilon, (\upsilon(1), \ldots, \upsilon(k))\big) \end{aligned}$$

induces a monomorphism $\phi_!(\mathcal{V} \circ \mathcal{W})(\chi'; c') \hookrightarrow (\phi_!\mathcal{V} \circ \phi_!\mathcal{W})(\chi'; c')$ which satisfies the required properties. $\square$

Proposition 78. *Let $\phi : C \to D$ be a function, and let $(\mathcal{C}, \Delta, \varepsilon)$ be a C-cooperad. The $(D, \mathbb{S})$-module $\phi_!\mathcal{C}$ has a canonical structure of D-cooperad $(\phi_!\mathcal{C}, \Delta_\phi, \varepsilon_\phi)$ induced by the structure of C-cooperad of $\mathcal{C}$.*

Proof. This is a corollary of the previous lemma since comonoids induce comonoids through lax comonoidal functors. $\square$

Definition 70 (Morphisms of colored cooperads)**.** A *morphism of colored cooperads* from $\mathscr{C} = (C, \mathcal{C}, \Delta, \varepsilon)$ to $\mathscr{D} = (D, \mathcal{D}, \Delta', \varepsilon')$ is a morphism of $\mathbb{S}$-modules $f = (\phi, f_!)$ such that $f_!$ is a morphism of D-cooperads from $(\phi_!\mathcal{C}, \Delta_\phi, \varepsilon_\phi)$ to $(\mathcal{D}, \Delta', \varepsilon')$.

Proposition 79. *Colored cooperads with their morphisms form a category denoted by* Coop.

Proof. This proof is similar to the proof of Proposition 76 for colored operads. $\square$

4.2.2 Coaugmented colored cooperads

Throughout this section, E is an abelian monoidal category.

Definition 71 (Coaugmented colored cooperad)**.** A *coaugmented colored cooperad* $\mathscr{C} = (C, \mathcal{C}, \Delta, \varepsilon, u)$ is the data of a colored cooperad $(C, \mathcal{C}, \Delta, \varepsilon)$ together with a morphism of C-cooperads $u : I_C \to \mathcal{C}$.

Since E is an abelian category, any coaugmented colored cooperad $\mathscr{C}$ has the form $\mathcal{C} = I_C \oplus \overline{\mathcal{C}}$ where $\overline{\mathcal{C}}$ is the kernel of the counit map $\varepsilon : \mathcal{C} \to I_C$. Furthermore, the restriction $\Delta_{|\overline{\mathcal{C}}}$ of the coproduct Δ to $\overline{\mathcal{C}}$ is equal to

$$\Delta_{|\overline{\mathcal{C}}} = I_C \circ \mathrm{Id} + \mathrm{Id} \circ I_C + \overline{\Delta}\ ,$$

where the map $\overline{\Delta}$ is made up of the image of Δ living in the summand of $\overline{\mathcal{C}} \circ (I_C \oplus \overline{\mathcal{C}})$ where $\overline{\mathcal{C}}$ appears more than once on the right-hand side of the composite product $\circ$. Moreover the restriction of the coproduct Δ to I_C is the canonical morphism $I_C \to I_C \circ I_C$.

Definition 72 (Morphisms of coaugmented colored cooperads)**.** A *morphism* of coaugmented colored cooperads from $(C, \mathcal{C}, \Delta, \varepsilon, u)$ to $(D, \mathcal{D}, \Delta', \varepsilon', u')$ is a morphism of colored cooperads $f = (\phi, f_!)$, whose restriction to I_C is equal to the identity :

$$I_C(c; c) = \mathbb{1}_{\mathsf{E}} \xrightarrow{\mathrm{Id}} \mathbb{1}_{\mathsf{E}} = I_D(\phi(c); \phi(c))\ .$$

This defines the category of coaugmented colored cooperads.

4.2.3 The tree module and the free colored operad

The tree module is the underlying construction of the free colored operad and the cofree colored cooperad on an arbitrary colored $\mathbb{S}$-module.

Definition 73 (Colored trees)**.** Let C be a set. A *C-colored tree* $t = (T, \kappa)$ is the data of a tree $T = (V, F, u, \rho, r)$ and a coloring function κ from the set of edges of T to the set of colors C. A *morphism* of C-colored trees from t to t' is a morphism of trees such that the induced function on edges commutes with the coloring functions.

Notations. Note that trees are denoted by capital letters, whereas colored trees are denoted by small letters. Let $t = (T, \kappa)$ be a C-colored tree and $\mathcal{V}$ be a $(C, \mathbb{S})$-module. For any vertex v of t, we denote by $\mathrm{in}(v)$ and $\mathrm{out}(v)$ respectively the set of inputs and the one-point-set of the output of v. Then we will denote the object $\mathcal{V}(\kappa(\mathrm{out}(v)); \kappa_{|\mathrm{in}(v)})$ simply by $\mathcal{V}(v)$.

For any $(C, \mathbb{S})$-module $\mathcal{V}$ and any C-colored tree $t = ((V, F, u, \rho, r), \kappa)$, we denote by $t(\mathcal{V})$ the following colimit in the category E

$$t(\mathcal{V}) := \Big(\coprod_{\phi : \underline{n} \to V} \mathcal{V}(\phi(1)) \otimes \cdots \otimes \mathcal{V}(\phi(n)) \Big)_{\mathbb{S}_n} , \tag{4.1}$$

which is made up of all the possible ways of labeling the vertices of the tree t with elements of $\mathcal{V}$. This colimit is taken over the set of bijections from the set $\underline{n}$ to the set V of vertices of t, n being the number of vertices of t, modulo the action of $\mathbb{S}_n$.

For any object $(\chi : X \to C; c)$ in the category Bij_C, we consider the category $\mathsf{Tree}_C(\chi; c)$ where the objects are pairs (t, α), with t a C-colored tree whose root is colored by c and α a bijection from X to the leaves of t such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "\chi"'] & & \mathrm{leaves}(t) \arrow[dl, "\kappa"] \\ & C . & \end{tikzcd}$$

The morphisms in $\mathsf{Tree}_C(\chi; c)$ from (t, α) to (t', α') are the isomorphisms of C-colored trees $\beta : t \to t'$ such that $\alpha' = \beta\alpha$ on the leaves. For any $(C, \mathbb{S})$-module, Formula (4.1) induces a functor from the category $\mathsf{Tree}_C(\chi; c)$ to the category E, i.e. a diagram in E.

Definition 74 (Tree module)**.** For any $(C, \mathbb{S})$-module $\mathcal{V}$, the *tree module* $\mathbb{T}\mathcal{V}$ is defined, for any object $(\chi; c)$ of the category of Bij_C, by the following colimit :

$$\mathbb{T}\mathcal{V}(\chi; c) := \operatorname*{colim}_{(t,\alpha) \in \mathsf{Tree}_C(\chi; c)} t(\mathcal{V}) .$$

This construction is functorial in $(\chi; c)$ and thus defines a $(C, \mathbb{S})$-module.

For any C-colored tree t, we can consider the object $t(\mathcal{V})$ of E as a $(C, \mathbb{S})$-module by taking the above colimit only over the pairs (t', α) such that t' is isomorphic, as a C-colored tree, to t, that is

$$t(\mathcal{V})(\chi; c) := \operatorname*{colim}_{\substack{(t',\alpha) \in \mathsf{Tree}_C(\chi; c) \\ t' \simeq t}} t'(\mathcal{V}) .$$

Note that the coproduct $\coprod_{[t]} t(\mathcal{V})$ over the isomorphism classes $[t]$ of C-colored trees is isomorphic to the tree module of $\mathcal{V}$,

$$\mathbb{T}\mathcal{V} \cong \coprod_{[t]} t(\mathcal{V}) .$$

Finally, the tree module $\mathbb{T}\mathcal{V}$ is functorial in $\mathcal{V}$ and thus defines an endofunctor $\mathbb{T}$ of the category $(C, \mathbb{S})$-Mod. It canonically extends to an endofunctor $\mathbb{T}$ of the whole category of colored $\mathbb{S}$-modules. In the sequel, we will also work with the *augmented tree module* $\overline{\mathbb{T}}\mathcal{V}$ made up of non trivial trees.

$$\overline{\mathbb{T}}\mathcal{V} \cong \coprod_{[t] \neq |} t(\mathcal{V}) .$$

Remark 15. Let $t = t_1 \sqcup \ldots \sqcup t_k$ be a partition of the C-colored tree t into sub-trees. Let $\phi : C \to D$ be a function and let $f(t_i)$ be a morphism of colored $\mathbb{S}$-modules over ϕ from $t_i(\mathcal{V})$ to $(D, \mathcal{W})$, for any i. Then, it induces a morphism of colored $\mathbb{S}$-modules over ϕ from $t(\mathcal{V})$ to $(D, \mathcal{W})$, that we denote by $f(t_1) \otimes \cdots \otimes f(t_k)$.

Proposition 80. *The colored $\mathbb{S}$-module $\mathbb{T}\mathcal{V}$ has a canonical structure of a colored operad given by the grafting of trees. We denote this colored operad by $\mathbb{T}^o\mathcal{V}$. It gives rise to a functor $\mathbb{T}^o : \mathbb{S}\text{-}\mathsf{Mod} \to \mathsf{Op}$ from the category of $\mathbb{S}$-modules to the category Op of colored operads, which is left adjoint to the forgetful functor $\mathsf{Op} \to \mathbb{S}\text{-}\mathsf{Mod}$.*

Proof. The proof is similar to the non-colored case, see [LV12, Section 5.8]. The extension of morphisms from $\mathcal{V}$ to $\mathbb{T}^o\mathcal{V}$ is given by the construction mentioned in the above remark. □

In plain words, the colored operad $\mathbb{T}^o\mathcal{V}$ is the *free colored operad* on the colored $\mathbb{S}$-module $\mathcal{V}$.

Notations. If f is a morphism of $\mathbb{S}$-modules from $(C, \mathbb{T}\mathcal{V})$ to $(D, \mathcal{W})$, then we denote by $f(t)$ the restriction of f to $t(\mathcal{V})$, for any C-colored tree t.

Definition 75. Suppose that E is a concrete category, that is made up of sets with additional structures. Let $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ be a colored operad. An ideal J of $\mathscr{P}$ is a sub-$(C, \mathbb{S})$-module of $\mathcal{P}$ which is stable under the composition with any element of $\mathcal{P}$. In other word, for any elements $x_0, x_1, \ldots, x_n$ of $\mathcal{P}$ which are composable, the composite $\gamma\big(x_0 \otimes (x_1 \otimes \cdots \otimes x_n)\big)$ belongs to J whenever of the x_i is in J. If R is a sub-graded-$(C, \mathbb{S})$-module of $\mathcal{P}$, we denote by (R) the smallest ideal of $\mathscr{P}$ which contains R.

Proposition 81. *With the notation of Definition 75, the $\mathbb{S}$-module $\mathcal{P}/(R)$ has a canonical structure of C-colored operad induced by the structure on $\mathcal{P}$. We denote this operad $\mathscr{P}/(R) = (C, \mathcal{P}/(R), \gamma, \eta)$.*

Proof. It suffices to notice that the composite morphism

$$\mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P} \twoheadrightarrow \mathcal{P}/(R)$$

factors uniquely through the map $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}/(R) \circ \mathcal{P}/(R)$. □

4.2.4 Conilpotent colored cooperads

We suppose again that E is an abelian category.

Definition 76 (Conilpotent colored cooperads)**.** A *conilpotent* colored cooperad $\mathscr{C} = (C, \mathcal{C}, \Delta, \varepsilon, u)$ is a coaugmented colored cooperad such that the images of any element under the right-hand side iterations of the decomposition map $\bar{\Delta} + \mathrm{Id} \circ I_C$ stabilize at some point.(We refer the reader to [LV12, Section 5.8] for more details in the non-colored case.) The full subcategory of the category of coaugmented colored cooperads made up of the conilpotent colored cooperads is denoted by $\mathsf{ConilCoop}$.

Proposition 82. *For any $(C, \mathbb{S})$-module $\mathcal{V}$, the tree module $\mathbb{T}\mathcal{V}$ has a canonical structure of conilpotent C-cooperad given by the degrafting of trees. We denote this colored cooperad by $\mathbb{T}^c\mathcal{V}$. This defines a functor $\mathbb{T}^c : \mathbb{S}\text{-}\mathsf{Mod} \to \mathsf{ConilCoop}$ from the category of colored $\mathbb{S}$-modules to the category of conilpotent colored cooperads, which is right adjoint to the forgetful functor $\mathscr{C} \mapsto \overline{\mathcal{C}}$.*

Proof. The proof works in the same way as in the non-colored case, see [LV12, Section 5.8]. For any conilpotent cooperad $\mathscr{C} = (C, \mathcal{C}, \Delta, \varepsilon, u)$, there is a canonical morphism of conilpotent C-cooperads $\delta : \mathscr{C} \to \mathbb{T}^c\mathcal{C}$ [LV12, Proof of Theorem 5.8.9]. The natural isomorphism

$$\mathrm{Hom}_{\mathbb{S}\text{-}\mathsf{Mod}}((C, \overline{\mathcal{C}}), (D, \mathcal{V})) \cong \mathrm{Hom}_{\mathsf{ConilCoop}}(\mathscr{C}, (D, \mathbb{T}^c\mathcal{V})) \ ,$$

for any conilpotent colored cooperad $\mathscr{C}$ and any $\mathbb{S}$-module $(D, \mathcal{V})$ is given as follows. Any morphism of $\mathbb{S}$-modules $f : (C, \overline{\mathcal{C}}) \to (D, \mathcal{V})$ extends to a morphism of conilpotent colored cooperads Rf from $\mathscr{C}$ to $\mathbb{T}^c\mathcal{V}$ using the following formula :

$$Rf = (\mathbb{T}^c f)\,\delta \ .$$

□

In plain words, the colored cooperad $\mathbb{T}^c\mathcal{V}$ is the *cofree conilpotent colored cooperad* on the colored $\mathbb{S}$-module $\mathcal{V}$. In the case where $\mathscr{C}$ is a cofree conilpotent colored cooperad $\mathscr{C} = (C, \mathbb{T}^c\mathcal{W})$, the adjoint morphism $Rf : \mathbb{T}^c\mathcal{W} \to \mathbb{T}^c\mathcal{V}$ is given by the more simple formula

$$Rf(t) = \sum_{t = t_1 \sqcup t_2 \sqcup \ldots \sqcup t_k} f(t_1) \otimes f(t_2) \otimes \cdots \otimes f(t_k) \ , \tag{4.2}$$

where the sum is taken over the partitions with no trivial component of the C-colored tree t.

4.2.5 Derivations and coderivations

In this paragraph, the monoidal category E is, most of the time, the category gMod of graded $\mathbb{K}$-modules with degree zero linear maps. It is a subcategory of the category $\mathsf{gMod}^{\mathsf{deg}}$ of graded $\mathbb{K}$-modules with graded morphisms. We will just allow ourselves to use morphisms of degree different from zero to build codifferentials.

Let $\mathcal{V}$ and $\mathcal{W}$ be two $(C,\mathbb{S})$-modules. By definition, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{V}\circ\mathcal{W}$ are gMod-presheaves over Bij_C. Let $f:\mathcal{V}\to\mathcal{V}$ and $g:\mathcal{W}\to\mathcal{W}$ be two endomorphisms of $\mathsf{gMod}^{\mathsf{deg}}$-presheaves over Bij_C. For any homogeneous elements $x\in\mathcal{V}(\phi;c)$ and $x_i\in\mathcal{W}(\psi_i;\phi(i))$, for $1\leq i\leq k$, we consider the following map :

$$x\otimes x_1\otimes\cdots\otimes x_k\mapsto f(x)\otimes x_1\otimes\cdots\otimes x_k+\sum_{i=1}^{n}(-1)^{|g|(|x|+|x_1|+\cdots+|x_{i-1}|)}x\otimes x_1\otimes\cdots\otimes g(x_i)\otimes\cdots\otimes x_k$$

The collection of these maps can be lifted to a morphism

$$f\circ \mathrm{Id}_{\mathcal{W}}+\mathrm{Id}_{\mathcal{V}}\circ' g:\mathcal{V}\circ\mathcal{W}\to\mathcal{V}\circ\mathcal{W}$$

of $\mathsf{gMod}^{\mathsf{deg}}$-presheaves over Bij_C, which is a linearization of the morphism $f\circ g$.

Definition 77 (Derivations, differentials, coderivations, and codifferentials)**.**

- ▷ A *derivation* of a colored operad $\mathscr{P}=(C,\mathcal{P},\gamma,\eta)$ is a morphism $d:\mathcal{P}\to\mathcal{P}$ of $\mathsf{gMod}^{\mathsf{deg}}$-presheaves over Bij_C such that

  $$\gamma\ (d\circ\mathrm{Id}_{\mathcal{P}}+\mathrm{Id}_{\mathcal{P}}\circ' d)=d\ \gamma\ .$$

  A *differential* is a degree -1 square-zero derivation.
- ▷ A *coderivation* of a colored cooperad $\mathscr{C}=(C,\mathcal{C},\Delta,\varepsilon)$ is a morphism $d:\mathcal{C}\to\mathcal{C}$ of $\mathsf{gMod}^{\mathsf{deg}}$-presheaves over Bij_C such that

  $$(d\circ\mathrm{Id}_{\mathcal{C}}+\mathrm{Id}_{\mathcal{C}}\circ' d)\ \Delta=\Delta\ d\ .$$

  A *codifferential* is a degree -1 square-zero coderivation.

A dg colored operad is an operad equipped with a differential. Morphisms of dg colored operads are morphisms of graded colored operads commuting with the differentials. We denote this category by $\mathsf{dg\text{-}Op}$. The same phenomenon holds for dg colored cooperads, i.e. colored cooperads equipped with a codifferential. For any coaugmented colored cooperad $\mathscr{C}=(C,\mathcal{C},\Delta,\varepsilon,u)$, we require moreover that coderivations satisfy $\varepsilon\ d=0$ and $d\ u=0$.

A derivation of a free operad is determined by its value on the generators.

Proposition 83. *Let $u:\mathcal{V}\to\mathbb{T}\mathcal{V}$ be a graded morphism of graded $(C,\mathbb{S})$-modules. There is a unique derivation of the graded colored operad $\mathbb{T}\mathcal{V}$ whch extends this morphism.*

Proof. The proof relies on the same arguments as the proof of [LV12, 6.3.6]. The derivation d_u which extends u is the following.

$$d_u(x_1\otimes\cdots\otimes x_n):=\sum_{i=1}^{n}(-1)^{|x_1|+\cdots+|x_{i-1}|}x_1\otimes\cdots\otimes dx_i\otimes\cdots\otimes x_n\ .$$

□

Proposition 84. *Let $\mathscr{P}=(C,\mathcal{P},\gamma,\eta,d)$ be a colored operad equipped with a derivation d. Let R be a sub-$(C,\mathbb{S})$-module of $\mathcal{P}$ such that $d(R)$ is contained in the smallest ideal (R) which contains R. Then, d induces a derivation on the C-colored operad $\mathscr{P}/(R)$.*

Proof. Since d is a derivation of $\mathscr{P}$, the sub$(C,\mathbb{S})$-module $d^{-1}((R))\subset\mathcal{P}$ is an ideal of $\mathscr{P}$. Since, it contains R, then it contains (R). So the graded map $\mathcal{P}\xrightarrow{d}\mathcal{P}\to\mathcal{P}/(R)$ factors uniquely through the projection $\mathcal{P}\to\mathcal{P}/(R)$. Thus we get a map $d_{\mathcal{P}/(R)}$ which a derivation of the colored operad $\mathscr{P}/(R)$. □

Coderivations on cofree colored cooperads are completely characterized by their projections onto their generators.

Proposition 85. *Let $\gamma : \overline{\mathbb{T}}\mathcal{V} \to \mathcal{V}$ be a graded morphism. There is a unique coderivation d_γ on the cofree colored cooperad $\mathbb{T}^c\mathcal{V}$ which extends γ ; it is given by the following formula :*

$$d_\gamma(t) = \sum_{s \subset t} \mathrm{Id} \otimes \cdots \otimes \gamma(s) \otimes \cdots \otimes \mathrm{Id} \ ,$$

where the sum is taken over the non-trivial sub-trees s of a colored tree t. In this context, the coderivation d_γ squares to zero if and only if $\gamma \circ d_\gamma = 0$.

Proof. The proof is similar to the non-colored case, see [LV12, Chapter 6]. □

Notations. For a coderivation d_γ on a cofree colored cooperad $\mathbb{T}^c\mathcal{V}$, we denote by γ its projection onto $\mathcal{V}$.

As we have already seen through Equation (4.2), a morphism of conilpotent cofree colored cooperads from $(C, \mathbb{T}^c\mathcal{V})$ to $(D, \mathbb{T}^c\mathcal{W})$ is equivalent to the data of a morphism of $\mathbb{S}$-modules f from $(C, \overline{\mathbb{T}}\mathcal{V})$ to $(D, \mathcal{W})$. If the cofree colored cooperads are equipped with codifferentials, then the following proposition gives the condition on f under which Rf is a morphism of dg cooperads.

Proposition 86. *Let $(C, \mathcal{V})$ and $(D, \mathcal{W})$ be $\mathbb{S}$-modules, let d_γ and d_ν be two codifferentials on $\mathbb{T}^c\mathcal{V}$ and $\mathbb{T}^c\mathcal{W}$ respectively, and let $f : (C, \overline{\mathbb{T}}\mathcal{V}) \to (D, \mathcal{W})$ be a morphism of $\mathbb{S}$-modules. Then Rf is a morphism of coaugmented dg cooperads, i.e. it commutes with the codifferentials, if and only if*

$$\nu Rf = f d_\gamma \ . \tag{4.3}$$

Proof. The proof is similar to the proof of Proposition 10.5.3 of [LV12]. On the one hand, if $d_\nu Rf = Rf d_\gamma$, then $\nu\ Rf = f\ d_\gamma$ as ν (resp. f) is the projection onto W of d_ν (resp. Rf). On the other hand, since Rf is given by Formula (4.2) and since d_γ is given by Proposition 85, we have

$$(Rf d_\gamma)(t) = \sum_{s \subset t} \sum_{t = t_1 \sqcup \ldots \sqcup s \sqcup \ldots \sqcup t_k} f(t_1) \otimes \cdots \otimes (f d_\gamma)(s) \otimes \cdots \otimes f(t_k) \ ,$$

for any C-colored tree t. We also have :

$$(d_\nu Rf)(t) = \sum_{s \subset t} \sum_{t = t_1 \sqcup \ldots \sqcup s \sqcup \ldots \sqcup t_k} f(t_1) \otimes \cdots \otimes (\nu Rf)(s) \otimes \cdots \otimes f(t_k) \ .$$

So, if $\nu Rf = f d_\gamma$, then $d_\nu Rf = Rf d_\gamma$. □

Convention In the case where the category E is the category dgMod of chain complexes, a codifferential on the cofree cooperad $\mathbb{T}^c\mathcal{V}$ has the from $d_1 + d_{\geq 2}$, where d_1 is the internal codifferential induced by the differential of $\mathcal{V}$ through the formula

$$d_1(x_1 \otimes \cdots \otimes x_m) = \sum_{i=1}^{n} (-1)^{|x_1| + \cdots + |x_{i-1}|} (x_1 \otimes \cdots \otimes d_{\mathcal{V}}(x_i) \otimes \cdots \otimes x_n) \ ,$$

and where $d_{\geq 2}$ is an additional codifferential, which is nonzero only on $\mathbb{T}^{\geq 2}\mathcal{V}$, the summand made up of trees with at least two vertices. We refer the reader to [LV12, Chapter 6] for more details.

4.2.6 The categories of homotopy colored operads

The concept of homotopy operads in the differential graded context was introduced by Pepijn Van der Laan in the non-colored case in the paper [VdL03]. In order to compare this notion to ∞-operads, we need to extend it by including colors and adding a homotopy coherent unit. In this section, the category E is the category dgMod of chain complexes.

Definition 78 (Strict unital homotopy colored operads)**.**

▷ A *nonunital homotopy colored operad* $\mathscr{P} = (C, \mathcal{P}, \gamma)$ is the data of a colored $\mathbb{S}$-module $(C, \mathcal{P})$ and a codifferential d_γ on the cofree colored cooperad $\mathbb{T}^c(s\mathcal{P})$ on the suspension $s\mathcal{P}$ of $\mathcal{P}$.
A *strict unital homotopy colored operad* $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ is the data of a nonunital homotopy operad $(C, \mathcal{P}, \gamma)$ together with a morphism $\eta : I_C \to \mathcal{P}$ of $\mathbb{S}$-modules called the *unit*. For each color c in C, we denote by id_c the image of the unit $1_{\mathbb{K}}$ of the ground field $\mathbb{K}$ under the map $\eta(c;c)$ from $\mathbb{K} = I_C(c;c)$ to $\mathcal{P}(c;c)$. Furthermore we require that the unit satisfies the following homotopy coherences :

$$\begin{cases} \gamma(s\mathrm{id}) = 0 & \\ \gamma(t)(s\mathrm{id}_c \otimes sp) = sp, & \text{for colored trees } t \text{ with 2 vertices;} \\ \gamma(t)(sp \otimes s\mathrm{id}_c) = (-1)^{|p|} sp, & \text{for colored trees } t \text{ with 2 vertices;} \\ \gamma(t)(sp_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sp_{n-1}) = 0, & \text{for colored trees } t \text{ with at least 3 vertices.} \end{cases}$$

In the second (resp. the third) equation $s\mathrm{id}_c$ (resp. sp) labels the vertex attached to the root of t.

▷ A *morphism of nonunital homotopy colored operads* $\mathcal{P} = (C, \mathcal{P}, \gamma) \rightsquigarrow \mathcal{Q} = (D, \mathcal{Q}, \nu)$ is a morphism of coaugmented dg cooperads $Rf : (\mathbb{T}^c(s\mathcal{P}), d_\gamma) \to (\mathbb{T}^c(s\mathcal{Q}), d_\nu)$.
A *morphism of strict unital homotopy colored operads* from $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ to $\mathscr{Q} = (D, \mathcal{Q}, \nu, \theta)$ is a morphism Rf of nonunital homotopy colored operads such that its composite with the projection onto the generators $f : \overline{\mathbb{T}}(s\mathcal{P}) \to s\mathcal{Q}$ satisfies :

$$\begin{cases} f(s\mathrm{id}_c) = s\mathrm{id}_{\phi(c)} \\ f(sp_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sp_{n-1}) = 0 \ . \end{cases}$$

The category of strict unital homotopy colored operads is denoted by suOp_∞.

Interpretation Let us unfold this definition a little bit. A nonunital homotopy colored operad $\mathscr{P} = (C, \mathcal{P}, \gamma)$ can actually be viewed as a colored $\mathbb{S}$-module endowed with a partial composition "associative up to higher homotopies". To be precise, it is necessary to give an orientation to the trees. This orientation will allow us to deal with the signs inherent to the underlying symmetric monoidal structure of the category dgMod and which come from the suspension. For example, let $t = (\{v_0, v_1, v_2\}, F, u, \rho, r, \kappa)$ be a C-colored tree with three vertices represented in the following picture.

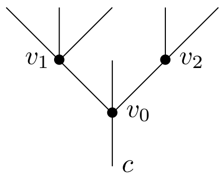


Figure 4.6 – The tree t.

The set of leaves is denoted by l and the color of the root is $c \in C$. The sub-tree containing v_0 and v_1 (respectively v_0 and v_2) is denoted by t_1 (resp. t_2) and its leaves by l_1 (resp. l_2). Let x be an element of the $\mathbb{K}$-module $t(\mathcal{P})$. It is equal to a sum of elements $a_0 \otimes a_1 \otimes a_2$ where $a_i \in \mathcal{P}(v_i)$. The choice of such a representation of x is related to the way we travel along the tree : in, this case, the path is (v_0, v_1, v_2). This defines an orientation of the colored tree t. This orientation induces a morphism $\mathcal{P}(v_0) \otimes \mathcal{P}(v_1) \otimes \mathcal{P}(v_2) \to s\mathcal{P}(v_0) \otimes s\mathcal{P}(v_1) \otimes s\mathcal{P}(v_2)$ through the mapping

$$a_0 \otimes a_1 \otimes a_2 \mapsto (-1)^{|a_1|}(sa_0) \otimes (sa_1) \otimes (sa_2) \ .$$

Then, applying γ and the desuspension map $s\mathcal{P} \to \mathcal{P}$ gives a morphism

$$\gamma_3 : \mathcal{P}(v_0) \otimes \mathcal{P}(v_1) \otimes \mathcal{P}(v_2) \to \mathcal{P}(l)$$

of degree 1. Furthermore, the orientation that we have chosen for t induces orientations on t_1, t_2, t/t_1 and t/t_2. Applying the same procedure produces degree 0 morphisms respectively from

$\mathcal{P}(v_0)\otimes\mathcal{P}(v_1)$ to $\mathcal{P}(l_1)$, from $\mathcal{P}(v_0)\otimes\mathcal{P}(v_2)$ to $\mathcal{P}(l_2)$, from $\mathcal{P}(l_1)\otimes\mathcal{P}(v_2)$ to $\mathcal{P}(l)$ and from $\mathcal{P}(l_2)\otimes\mathcal{P}(v_1)$ to $\mathcal{P}(l)$. We denote all of them by γ_2, since they amount to composing 2 vertices labeled by $\mathcal{P}$. Let τ be the canonical isomorphism $\mathcal{P}(v_0)\otimes\mathcal{P}(v_1)\otimes\mathcal{P}(v_2)\simeq\mathcal{P}(v_0)\otimes\mathcal{P}(v_2)\otimes\mathcal{P}(v_1)$. The fact that d_γ squares to zero implies :

$$\gamma_2(\gamma_2\otimes\mathrm{Id})-\gamma_2(\gamma_2\otimes\mathrm{Id})\tau=\partial(\gamma_3)$$

where $\partial(\gamma_3)=d_{\mathcal{P}}\ \gamma_3+\gamma_3\ d_{\mathcal{P}^{\otimes 3}}$. We interpret γ_2 as a partial composition ; so the above equation shows that the parallel composition is not strictly associative but "associative up to homotopy" ; and this homotopy is precisely γ_3. In the same way, γ_3 applied to trees with 3 vertices one above another provides us with a homotopy for the sequential composite of γ_2. The other maps $\gamma(t)$, for bigger trees t, are higher homotopies. Indeed, given an orientation on a colored tree t, we have the following equation :

$$\partial(\gamma(t))=\sum_{s\subset t}\pm\gamma(t/s)(\mathrm{Id}\otimes\cdots\otimes\gamma(s)\otimes\cdots\otimes\mathrm{Id})\ ,$$

where the sum is taken over all the sub-trees s of t, with at least 2 vertices. In strict unital homotopy colored operads, the composition is relaxed up to homotopy but the unit remains strict.

Interpretation A morphism of nonunital homotopy colored operads from $\mathscr{P}=(C,\mathcal{P},\gamma)$ to $\mathscr{Q}=(D,\mathcal{Q},\nu)$ is a morphism of colored $\mathbb{S}$-modules $f:\overline{\mathbb{T}}^c(s\mathcal{P})\to s\mathcal{Q}$ such that $\nu Rf=fd_\gamma$, according to Proposition 86. Again, a choice of orientation of a colored tree t with n vertices $v_0,\ldots,v_{n-1}$ gives a morphism of graded $\mathbb{K}$-modules $f_n:\mathcal{P}(v_0)\otimes\cdots\otimes\mathcal{P}(v_{n-1})\to\mathcal{Q}(l)$ and morphisms $\gamma_n:\mathcal{P}(v_0)\otimes\cdots\otimes\mathcal{P}(v_{n-1})\to\mathcal{P}(l)$ and $\nu_n:\mathcal{Q}(v_0)\otimes\cdots\otimes\mathcal{Q}(v_{n-1})\to\mathcal{Q}(l)$. In the case where the tree t has two vertices, the fact that f is a morphism of dg cooperads implies :

$$\nu_2(f_1\otimes f_1)-f_1\gamma_2=\partial(f_2)\ ,$$

where $\partial(f_2)=d_{\mathcal{Q}}\ f_2+f_2\ d_{\mathcal{P}^{\otimes 2}}$. Since γ_2 and ν_2 are interpreted as composite maps, f_1 commutes with these compositions up to homotopy ; and this homotopy is precisely f_2. The other maps $f(t)$ for bigger trees t are the data of a higher homotopical control. A morphism of strict unital homotopy colored operads commutes with the composite maps up to higher homotopies but strictly with the units.

Proposition 87.

- ▷ *Nonunital homotopy colored operad concentrated in arity one is the same notion as nonunital $\mathcal{A}_\infty$-category. Strict unital homotopy colored operad concentrated in arity one is the same notion as $\mathcal{A}_\infty$-category.*
- ▷ *The forgetful functor from strict unital colored operads to nonunital colored operads has a left adjoint which is an embedding of category of nonunital homotopy colored operads into the category of strict unital homotopy colored operads.*
- ▷ *The category* dg-Op *of differential graded colored operads embeds canonically into the category* suOp_∞ *of strict unital homotopy colored operads.*

Proof. The proof of the first point is straightforward with the various definitions from [FOOO09]. For the second point, any nonunital colored operad $(C,\mathcal{P},\gamma)$ is sent to $(C,\mathcal{P}\oplus I_C,\gamma)$. For the third point, any dg colored operad $\mathscr{P}=(C,\mathcal{P},\gamma,\eta)$ can be seen as the strict unital homotopy colored operad $(C,\mathcal{P},\widetilde{\gamma},\widetilde{\eta})$ as follows. The structure map $\widetilde{\gamma}$ is defined by $\widetilde{\gamma}(t):=0$ for colored trees t with more than 3 vertices, by

$$\widetilde{\gamma}(t)(sa_0\otimes sa_1):=(-1)^{|a_0|}s\gamma(a_0\otimes a_1)$$

for colored trees t with 2 vertices, and by $\widetilde{\gamma}(t):=d_{s\mathcal{P}}=-d_{\mathcal{P}}$ for colored trees t with 1 vertex. □

In this context, a morphism of dg colored operads $\mathscr{P}\to\mathscr{Q}$ is a morphism of strict unital homotopy colored operads Rf such that the corresponding morphism of colored $\mathbb{S}$-modules $f:\overline{\mathbb{T}}(s\mathcal{P})\to s\mathcal{Q}$ vanishes on the trees with two vertices or more.

$$\begin{tikzcd}
\mathsf{dg\text{-}cat} \arrow[r, hook] \arrow[d, hook] & \mathsf{dg\text{-}Op} \arrow[d, hook] \\
\mathcal{A}_\infty\text{-}\mathsf{cat} \arrow[r, hook] & \mathsf{suOp}_\infty
\end{tikzcd}$$

4.2.7 The Boardman–Vogt construction of strict unital homotopy operads

We recall here the Boardman–Vogt construction W_H for dg colored operads and we extend this construction to strict unital homotopy colored operads, that we still denote by W_H. We show that this new functor from the category of strict unital homotopy colored operads to the category of dg colored operads is left adjoint to the inclusion functor from the category of dg colored operads to the category of strict unital homotopy colored operads. For more details about the Boardman–Vogt construction W_H for dg colored operads, we refer the reader to the papers [BM07, BM06, Wei07].

Definition 79 (An interval in the category of chain complexes)**.** Let H be the chain complex made up of two generators h_0 and h_1 in degree 0 and one generator h in degree 1 such that the differential $d(h)$ is equal to $h_1 - h_0$. It is equipped with a symmetric product $\vee : H \otimes H \to H$ such that h_0 is a unit, h_1 is idempotent, h is nilpotent, and such that $h \vee h_1 = 0$; it is also equipped with a map $\epsilon : H \to \mathbb{K}$ such that $\epsilon(h_i) = 1_{\mathbb{K}}$ and $\epsilon(h) = 0$.

Definition 80 (The Boardman–Vogt construction)**.** For any dg colored operad $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$, the *Boardman–Vogt construction* $W_H(\mathscr{P})$ of $\mathscr{P}$ is the dg C-colored operad made up of C-colored trees whose vertices are labeled by elements of $\mathcal{P}$ and whose inner edges are labeled by elements of the interval H. This is subject to the following two identifications.

- ▷ If a vertex v with one input is labeled by an identity, then the tree is identified with the same tree with the vertex v removed and the two adjacent edges glued together. If the resulting edge is inner, then it is labeled by the element of H given by the product of the two elements labeling the former adjacent edges. And if the resulting edge is outer, then the tree is multiplied by the image under ϵ of the former inner adjacent edge.
- ▷ If an inner edge e is labeled by h_0, then the tree is identified with its contraction along this edge. The resulting vertex is labeled by the composition in the operad $\mathscr{P}$ of the labelings of the two former adjacent vertices.

The operadic composition is given by the grafting of trees where the new inner edge is labeled by h_1.

Since the labeling of the inner edges by h_0 leads to a contraction of the edge in the construction of the operad $W_H(\mathscr{P})$, then we can consider that the underlying graded operad of $W_H(\mathscr{P})$ is made up of C-colored trees whose vertices are labeled by elements of $\mathcal{P}$ and whose inner edges are labeled either by h or h_1 ; this is subject to some relations. Since an inner edge labeled by h_1 is the result of an operadic composition, then the underlying graded operad of $W_H(\mathscr{P})$ is generated by the C-colored trees whose vertices are labeled by elements of $\mathcal{P}$ and whose inner edges are labeled by h. This graded $\mathbb{S}$-module of generators is isomorphic to the graded $\mathbb{S}$-module $s^{-1}\mathbb{T}(s\mathcal{P})$. So the underlying graded operad of $W_H(\mathscr{P})$ is isomorphic to a quotient of the free graded colored operad $\mathbb{T}\left(s^{-1}\mathbb{T}(s\mathcal{P})\right)$. Inspired by this presentation, we define the extended Boardman–Vogt construction which applies to strict unital homotopy colored operads. In Proposition 89, we show that the extended Boardman–Vogt construction is actually an extension of the Boardman–Vogt construction in the sense that the two constructions coincide on dg colored operads.

Definition 81 (Extended Boardman–Vogt construction W_H)**.** For any strict unital homotopy colored operad $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$, the *extended Boardman–Vogt construction* $W_H\mathscr{P} = (V_H(\mathcal{P}), d_W)$ is the data of a graded C-colored operad $V_H(\mathcal{P})$ together with a degree -1 derivation d_W. On the one hand, $V_H(\mathcal{P})$ is the quotient of the free graded C-colored operad $\mathbb{T}\left(s^{-1}\mathbb{T}(s\mathcal{P})\right)$ by the operadic ideal generated by the relations

$$\begin{cases} s^{-1}s\mathrm{id}_c = 1_c \ , \\ s^{-1}(sx_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sx_n) = 0, \text{ for } n \geq 1 \ , \end{cases}$$

where 1_c is the unit of the C-colored operad $\mathbb{T}\left(s^{-1}\mathbb{T}(s\mathcal{P})\right)$ for the color c. On the other hand, for any tree T whose vertices are labeled by elements $sx_1, \ldots, sx_n$ of $s\mathcal{P}$, the value of d_W on $s^{-1}(sx_1 \otimes \cdots \otimes sx_n)$ is the following

$$d_W\left(s^{-1}(sx_1 \otimes \cdots \otimes sx_n)\right) = -s^{-1}d_\gamma(sx_1 \otimes \cdots \otimes sx_n) - \sum_{a \in \mathrm{inner}(T)} \mathrm{cut}_a\left(s^{-1}(s_1 \otimes \cdots \otimes sx_n)\right) \ ,$$

where the sum is taken over the inner edges of T and where cut_a is the operation which cuts the tree T into two parts along the edge a as follows :

$$\mathrm{cut}_a\left(s^{-1}(sx_1 \otimes \cdots \otimes sx_n)\right) = (-1)^{|x_1|+\cdots+|x_p|+p}s^{-1}(sx_1 \otimes \cdots \otimes sx_p) \otimes s^{-1}(sx_{p+1} \otimes \cdots \otimes sx_n) \ ,$$

with sx_p and sx_{p+1} the two labelings of the vertices which border the edge a.

Remark 16. Since the formula that we give for d_W sends an element of the form $s^{-1}(sx_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sx_n)$ to the operadic ideal generated by the elements of the same form $s^{-1}(sy_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sy_k)$, then Proposition 84 ensures us that the derivation d_W is well defined.

Proposition 88. *The derivation d_W squares to zero.*

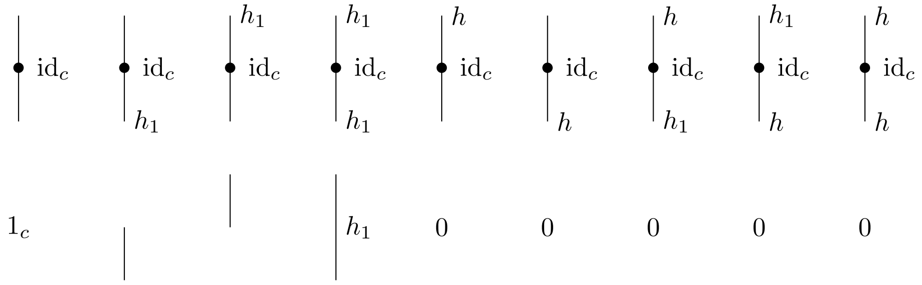
Proof. On the desuspension of any colored tree T whose vertices are labeled by elements of $s\mathcal{P}$, we have :

$$\begin{aligned} d_W^2 &= Id_{s^{-1}} \otimes d_\gamma^2 \\ &+ \sum_{\substack{T' \subsetneq T \\ a > T'}} -\mathrm{cut}_a(Id_{s^{-1}} \otimes Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id) \\ &+ \sum_{\substack{T' \subsetneq T \\ a < T'}} -\mathrm{cut}_a(Id_{s^{-1}} \otimes Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id) \\ &+ \sum_{\substack{a \in inner(T) \\ T' > a}} -(Id_{s^{-1}} \otimes Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id)\mathrm{cut}_a \\ &+ \sum_{\substack{a \in inner(T) \\ T' < a}} -(Id_{s^{-1}} \otimes Id \otimes \cdots \otimes \gamma(T') \otimes \cdots \otimes Id)\mathrm{cut}_a \ , \end{aligned}$$

where the sums are taken over the subtrees T' of T and the inner edges a of T such that a is not an inner edge of T' but is over T' ($a > T'$) or under T' ($a < T'$). The sum of the second line (resp. third line) with the fourth line (resp. fifth line) is zero. Since $d_\gamma^2 = 0$, then $d_W^2 = 0$. ☐

Proposition 89. *The Boardman–Vogt construction coincides with the extended Boardman–Vogt construction on dg colored operads.*

Proof. Let $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ be a dg colored operad. We know that the Boardman–Vogt construction is generated by colored trees whose vertices are labeled by elements of $\mathcal{P}$ and whose inner edges are labeled by $h \in H$. The composition is the grafting of trees where the new inner edges are labeled by $h_1 \in H$. These trees are subjects to some relations involving the units id_c of the operad $\mathscr{P}$ which are described in the following picture. The second row represents the value in the Boardman–Vogt construction of the elements in the first row.



These relations are summed up by the following two points.

- ▷ The colored tree with one vertex labeled by id_c is the unit for the color c.
- ▷ Any generating tree (that is a tree whose inner edges are labeled by h) with at least two vertices and with one of them labeled by a unit id_c is zero.

Since the $\mathbb{S}$-module of generating trees of isomorphic to $s^{-1}\mathbb{T}(s\mathcal{P})$, then the underlying graded colored operad of the Boardman–Vogt construction of $\mathscr{P}$ is the quotient of $\mathbb{T}\left(s^{-1}\mathbb{T}(s\mathcal{P})\right)$ by the relations

$$\begin{cases} s^{-1}s\mathrm{id}_c = 1_c \ , \\ s^{-1}(sx_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sx_n) = 0, \ \text{for } n \geq 1 \ . \end{cases}$$

So, the underlying graded colored operad of the Boardman–Vogt construction of $\mathscr{P}$ is canonically isomorphic to $V_H(\mathcal{P})$. Moreover, on any colored tree whose vertices are labeled elements of $\mathcal{P}$and whose inner edges are labeled by h, the derivation of the Boardman–Vogt construction consists in the following three operations.

- ▷ For any vertex, apply the differential d of $\mathcal{P}$ to the labeling of this vertex.
- ▷ For any edge labeled by h, replace h by h_0, that is remove h and contract the edge. The labeling of the two adjacent vertices are merged using the composition in the operad $\mathscr{P}$.
- ▷ For any edge labeled by h, replace h by h_1.

The two first points correspond to the coderivation d_γ of the cofree conilpotent colored cooperad $\mathbb{T}(s\mathcal{P})$ and the third point correspond to the operation cut_a. □

Theorem 25. *The extended Boardman-Vogt construction is a functor left adjoint to the inclusion functor from the category of dg colored operads to the category of strict unital homotopy colored operads.*

$$\mathsf{suOp}_\infty \underset{i}{\overset{W_H}{\rightleftarrows}} \mathsf{dg\text{-}Op}$$

Proof. Let $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$ be a strict unital homotopy colored operad and let $\mathscr{Q} = (C', \mathcal{Q}, \gamma', \eta')$ be a colored dg operad (seen as a strict unital homotopy colored operad). A morphism of strict unital homotopy colored operads from $\mathscr{P}$ to $\mathscr{Q}$ is the data of a morphism of graded colored $\mathbb{S}$-modules :

$$f : \overline{\mathbb{T}}(s\mathcal{P}) \to s\mathcal{Q}$$

such that $\gamma' R f = f d_\gamma$ and such that $f(s\mathrm{id}_c) = s\mathrm{id}_{\phi(c)}$, where ϕ is the underlying function on colors, and $f(sx_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sx_n) = 0$. Since $\mathcal{Q}$ has a structure of graded colored operad, it is also the data of a morphism of graded colored operads g from $\mathbb{T}(s^{-1}\overline{\mathbb{T}}(s\mathcal{P}))$ to $\mathcal{Q}$ such that $g(s^{-1}s\mathrm{id}_c) = \mathrm{id}_{\phi(c)}$, $g(s^{-1}(sx_1 \otimes \cdots \otimes s\mathrm{id}_c \otimes \cdots \otimes sx_n)) = 0$ and $gd_W = d_{\mathcal{Q}}g$. So it is equivalent to the data of a morphism of dg colored operads from $W_H\mathscr{P}$ to $\mathscr{Q}$ □

Remark 17. The counit of this adjunction recovers the Boardman–Vogt resolution $W_H\mathscr{P} \to \mathscr{P}$ of any dg colored operad $\mathscr{P}$. The unit of the adjunction provides a rectification $\mathscr{P} \to W_H\mathscr{P}$ for any strict unital homotopy colored operad $\mathscr{P}$.

Remark 18. This adjunction may be understood as a strict unital operadic bar-cobar adjunction $\Omega \dashv B$; see [LV12, Chapter 6]. Indeed, as proved by Berger an Moerdijk in [BM06], the Boardman–Vogt construction coincides with the functor cobar-bar ΩB on augmented dg operads.

4.3 The dendroidal nerve of strict unital homotopy colored operads

In this section, we introduce the dendroidal nerve of strict unital homotopy colored operads ; we then show that its image actually produces an infinity-operad. So it provides us with a new functor which relates these two notions. To compare this dendroidal nerve with the existing constructions, we prove that it extends both Faonte–Lurie's simplicial nerve of strict unital $\mathcal{A}_\infty$-categories and Moerdijk–Weiss' homotopy coherent nerve of dg operads. Then, we characterize the morphisms of strict unital homotopy colored operads whose images under the dendroidal nerve are respectively weak equivalences and fibrations for the Cisinski–Moerdijk model structure. Finally, we endow the category of dg colored operads with a model structure introduced in [Cav14] and show that the homotopy coherent nerve is a right Quillen functor.

From now on, the word "colored" will often be understood. For instance, we call "strict unital homotopy colored operads" simply by *"su homotopy operads"*.

4.3.1 Trees as operads

Let Lin be the functor which associates to any set the free $\mathbb{K}$-module on it

$$Lin \ : \ X \in \mathsf{Set} \ \mapsto \ \bigoplus_{x\in X} \mathbb{K}.x \in \mathsf{Mod} \subset \mathsf{gMod} \subset \mathsf{dgMod} \ .$$

This $\mathbb{K}$-module can be considered as a differential graded $\mathbb{K}$-module concentrated in degree zero with trivial differential. Therefore, we view the functor Lin as mapping into the category dgMod. This functor is a strong symmetric monoidal functor. So it can be extended to a functor from the category of colored operads on sets to the category of differential graded operads.

We denote by $\mathbb{K}\Omega(T) := Lin(\Omega(T))$ the image under this functor of the set-theoretical operad $\Omega(T)$ (Definition 61). The colors of $\mathbb{K}\Omega(T)$ are the edges of T and $\mathbb{K}\Omega(T)(a; \chi : X \to \mathrm{edges}(T)) = \mathbb{K}$ when χ is injective and when there is a sub-tree (possibly trivial) of T with root a and leaves $\chi(X)$. Since $\mathbb{K}\Omega(T)$ is a dg operad, it can be consider as a su homotopy operad. Furthermore, the map $T \in \mathsf{Tree} \mapsto \Omega(T) \in \mathsf{Op}$ defines a codendroidal object in the category of set-theoretical operads. Therefore, the map $T \mapsto \mathbb{K}\Omega(T)$ is a codendroidal object in the category $\mathsf{dg\text{-}Op}$ of dg operads and so in the category suOp_∞ of su homotopy operads.

4.3.2 The dendroidal nerve

The usual nerve of a category is a functor which associates, to any small category C, the simplicial set

$$\mathrm{N}(\mathsf{C})_n := \mathrm{Hom}_{\mathsf{Cat}}([n], \mathsf{C})$$

where Cat is the category of small categories and where $[n]$ is the poset $0 < \cdots < n$ viewed as a category. More generally, the nerve of an object X in a category A associated to a functor $F : \Delta \to \mathsf{A}$ is the simplicial set $\mathrm{N}(X)_n := \mathrm{Hom}_{\mathsf{A}}(F([n]), X)$. In the same way, a dendroidal nerve of an object X associated to a functor $F : \mathsf{Tree} \to \mathsf{A}$ is the dendroidal set

$$\mathrm{N}^\Omega(X)_T := \mathrm{Hom}_{\mathsf{A}}(F(T), X) \ .$$

We apply this construction to the functor $T \in \mathsf{Tree} \mapsto \mathbb{K}\Omega(T) \in \mathsf{suOp}_\infty$.

Definition 82 (Dendroidal nerve)**.** The *dendroidal nerve* of su homotopy operads N^Ω is the functor from the category suOp_∞ to the category dSet of dendroidal sets defined by the following formula :

$$\mathrm{N}^\Omega(\mathscr{P})_T := \mathrm{Hom}_{\mathsf{suOp}_\infty}(\mathbb{K}\Omega(T), \mathscr{P}) \ ,$$

for any tree T and any su homotopy operad $\mathscr{P}$.

Let us describe the dendrices of the dendroidal nerve of a su homotopy operad $\mathscr{P} = (C, \mathcal{P}, \gamma, \eta)$. We denote by d_ν the structural codifferential on $\mathbb{T}^c(s\mathbb{K}\Omega(T))$, which comes from the operad structure on $\mathbb{K}\Omega(T)$.

Lemma 33. *A T-dendrex $\mathbb{K}\Omega(T) \rightsquigarrow \mathscr{P}$ is equivalent to the following data :*

- ▷ *an underlying function ϕ from the set of edges of T to the set of colors C,*
- ▷ *maps of graded $\mathbb{S}$-modules $f(t) : t(s\mathbb{K}\Omega(T)) \to s\mathcal{P}$ over the function ϕ for any tree $t = T'/T_1 \cdots T_k$, which is the contraction of a sub-tree T' of T along a partition $T' = T_1 \sqcup \ldots \sqcup T_k$ and which is canonically colored by the set of edges of T,*

satisfying the following equations, for the same class of trees t,

$$\sum_{t = t_1 \sqcup \ldots \sqcup t_l} \gamma(t/t_1 \cdots t_l)\Big(f(t_1) \otimes \cdots \otimes f(t_l)\Big) = f d_\nu(t) \ , \tag{4.4}$$

where the sum runs over the partitions with no trivial component of the colored tree t.

Remark 19. As the trees $t = T'/T_1 \cdots T_k$ are canonically colored by the set of edges of T, the set of such colored trees is canonically bijective with the set of partitioned sub-trees of T under the inverse of the mapping $T' = T_1 \sqcup \ldots \sqcup T_k \mapsto T'/T_1 \cdots T_k$.

Proof. On the one hand, a T-dendrex is a morphism of su homotopy operads $\mathbb{K}\Omega(T) \rightsquigarrow \mathscr{P}$, which can be described as a morphism of $\mathbb{S}$-modules $f : \overline{\mathbb{T}}(s\mathbb{K}\Omega(T)) \to s\mathcal{P}$ satisfying Relation (4.3). In particular, it gives us an underlying function ϕ from the set of edges of T to the set of colors C of $\mathscr{P}$ and morphisms $f(t)$ for any colored tree $t = T'/T_1 \cdots T_k$ satisfying Relation (4.4). On the other hand, let us consider a function ϕ and morphisms $\{f(t)\}_{t=T'/T_1 \cdots T_k}$ satisfying Relation (4.4). Let us fix :

- ▷ for any colored tree t with one vertex v and two edges having the same color $e \in \mathrm{edges}(T)$,

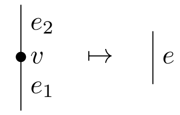
$$\begin{aligned} f(t) : t\big(s\mathbb{K}\Omega(T)\big) &\to s\mathcal{P} \\ sv &\mapsto s\mathrm{id}_{\phi(e)} \ ; \end{aligned}$$

- ▷ for any other tree t, colored by the edges of T, and which is different from a contraction of a sub-tree of T, $f(t) := 0$.

The data of the function ϕ and the maps $\{f(t)\}_t$ amounts to a morphism $f : \overline{\mathbb{T}}(s\mathbb{K}\Omega(T)) \to s\mathcal{P}$. Furthermore, since the morphisms $\{f(t)\}_{t=T'/T_1 \cdots T_k}$ satisfy Relation (4.4), then the morphism f satisfy Relation (4.3) and so is a T-dendrex of $\mathscr{P}$. ☐

The definition of morphisms of su homotopy operads induces the following description of the images of dendrices under the face and degeneracy maps of the dendroidal nerve $\mathrm{N}^\Omega(\mathscr{P})$. Let $x = (\phi; \{f(t)\}_{t=T'/T_1 \cdots T_k})$ be a T-dendrex of the dendroidal nerve $\mathrm{N}^\Omega(\mathscr{P})$. For any outer vertex v of T, the outer face $\delta_v(x)$ of x is given by the restriction of ϕ to the set $\mathrm{edges}(T) \setminus \mathrm{in}(v)$ and by the restriction of the family $\{f(t)\}_{t=T'/T_1 \cdots T_k}$ to the contractions of the sub-trees T' of $T - \{v\}$. For any inner edge e of T, the corresponding inner face $\delta_e(x)$ of x is given by the restriction of ϕ to the set $\mathrm{edges}(T) \setminus \{e\}$ and by the restriction of the family $\{f(t)\}_{t=T'/T_1 \cdots T_k}$ to the partitioned sub-trees T' of T such that the edge e is inside one of the trees T_i. Finally, let e be an edge of T and let T_σ be the tree obtained from T replacing e by two edges e_1 and e_2 separated by a vertex v. There is a codegeneracy $s : T_\sigma \to T$ sending T_σ to T.

$$T_\sigma \quad \to \quad T$$



The corresponding degeneracy $\sigma(x)$ of x is a T_σ-dendrex $(\phi_\sigma; \{f_\sigma(t)\}_t)$ of $\mathrm{N}^\Omega(\mathscr{P})$ described as follows. On the one hand, we have $\phi_\sigma(e_1) = \phi_\sigma(e_2) = \phi(e)$ and $\phi_\sigma(a) = \phi(a)$ for the other edges a of T_σ which can be considered as edges of T. On the other hand, let $t = T'/T_1 \cdots T_k$ be a contracted sub-tree of T_σ.

▷ If $k \geq 2$ and if one of the T_i is the one-vertex tree made up of the vertex v and the edges e_1 and e_2, then $f_\sigma(t) = 0$.
▷ If t is the one-vertex tree made up of the vertex v and the edges e_1 and e_2, then $f_\sigma(t)$ is equal to :

$$f_\sigma(t)(e_1; e_2) : t\big(s\mathbb{K}\Omega(T_\sigma)\big)(e_1; e_2) \simeq s\mathbb{K} \to s\mathcal{P}(\phi(e); \phi(e))$$
$$sv \mapsto s\mathrm{id}_{\phi(e)} \ .$$

▷ Otherwise, $f_\sigma(t) = f(t)$ since $t\big(s\mathbb{K}\Omega(T_\sigma)\big) \simeq t\big(s\mathbb{K}\Omega(T)\big)$.

4.3.3 The dendroidal nerve is an infinity-operad

Lemma 34. *([MT10, Corollary 3.2.7]) A morphism of dendroidal sets $\Lambda^e[T] \to D$ is the data of dendrices $x_v \in D_{T\backslash v}$ for any external vertex v and $x_a \in D_{T/a}$ for any inner edge a different from e, which agree on common faces.*

This lemma applied to the case of the dendroidal nerve of a su homotopy operads gives the following description.

Corollary 3. *For any tree T and any inner edge e, a morphism of dendroidal sets $\Lambda^e[T] \to \mathrm{N}^\Omega(\mathscr{P})$ is equivalent to the data of :*
▷ *a function ϕ from the set of edges of T to the set C and*
▷ *morphisms of $\mathbb{S}$-modules $f(t) : t\big(s\mathbb{K}\Omega(T)\big) \to s\mathcal{P}$ over ϕ, for every contracted colored sub-tree $t = T'/T_1 \cdots T_k$ along a partition $T' = T_1 \sqcup \ldots \sqcup T_k$, except for the full tree T with no contraction and the tree T/e where only the two-vertices sub-tree spanned by the edge e is contracted,*

satisfying Equation (4.4) for each of these trees.

Proof. The result is a direct corollary of Lemma 34 and the description of faces given in the previous section. ☐

Theorem 26. *The dendroidal nerve of a strict unital homotopy colored operad is an ∞-operad.*

Proof. Consider a morphism of dendroidal sets f from $\Lambda^e[T]$ to $\mathrm{N}^\Omega(\mathscr{P})$ given by a function ϕ from the edges of T to C and morphisms of $\mathbb{S}$-modules $f(t) : t(s\mathbb{K}\Omega(T)) \to s\mathcal{P}$ over ϕ for the trees t described in Corollary 3. Recall that a morphism from $\Omega[T] = \mathrm{Hom}_{\mathsf{Tree}}(-, T)$ to $\mathrm{N}^\Omega(\mathcal{P})$ amounts to the data of a T-dendrex. So to extend f to a morphism from $\Omega[T]$ to $\mathrm{N}^\Omega(\mathcal{P})$, we have to build $f(T)$ and $f(T/e)$ so that Equation (4.4) is fulfilled for these two trees. Let us recall that d_ν denotes the structural codifferential on $\mathbb{T}^c(s\mathbb{K}\Omega(T))$. We fix $f(T) := 0$. Then, because of Equation (4.4) for the tree T, the map $f(T/e)$ must satisfy the following formula :

$$f(T/e)\Big(\mathrm{Id} \otimes \cdots \otimes \nu(e) \otimes \cdots \otimes \mathrm{Id}\Big) = -\sum_{a \neq e} f(T/a)\Big(\mathrm{Id} \otimes \cdots \otimes \nu(a) \otimes \cdots \otimes \mathrm{Id}\Big)$$
$$+ \sum_{T = T_1 \sqcup \ldots \sqcup T_k} \gamma(T/T_1 \cdots T_k)\Big(f(T_1) \otimes \cdots \otimes f(T_k)\Big) \ ,$$

where the first sum runs over the inner edges of the tree T different from e and where the second runs over all the partitions of the tree T with no trivial component. Since $\mathrm{Id} \otimes \cdots \otimes \nu(e) \otimes \cdots \otimes \mathrm{Id}$ is an isomorphism of $\mathbb{S}$-modules, we have built $f(T/e)$. We know that Equation (4.4) is satisfied for every tree $t = T'/T_1 \cdots T_k$ which is the contraction of a sub-tree T' of T along a partition $T' = T_1 \sqcup \ldots \sqcup T_k$ except for $t = T/e$. As in the proof of Proposition 86, we have :

$$(Rfd_\nu)(T) = \sum_{\substack{S \subset T \\ T = T_1 \sqcup \ldots \sqcup S \sqcup \ldots \sqcup T_k}} f(T_1) \otimes \cdots \otimes (fd_\nu)(S) \otimes \cdots \otimes f(T_k) \ ,$$

and

$$(d_\gamma Rf)(T) = \sum_{\substack{S \subset T \\ T = T_1 \sqcup \ldots \sqcup S \sqcup \ldots \sqcup T_k}} f(T_1) \otimes \cdots \otimes (\gamma Rf)(S) \otimes \cdots \otimes f(T_k) \ .$$

Therefore, we have
$$(d_\gamma Rf)(T) = (Rfd_\nu)(T)\ ,$$
and so
$$(fd_\nu d_\nu)(T) = 0 = (\gamma d_\gamma Rf)(T) = (\gamma Rfd_\nu)(T)\ .$$
The above equation rewrites
$$\sum_a (fd_\nu)(T/a)\Big(\mathrm{Id}\otimes\cdots\otimes\nu(a)\otimes\cdots\otimes\mathrm{Id}\Big) = \sum_a (\gamma Rf)(T/a)\Big(\mathrm{Id}\otimes\cdots\otimes\nu(a)\otimes\cdots\otimes\mathrm{Id}\Big)\ ,$$
where the two sums run over the inner edges of the tree T. We already know that $(fd_\nu)(T/a) = \gamma Rf(T/a)$ for all the inner edges a different from e and that $\mathrm{Id}\otimes\cdots\otimes\nu(e)\otimes\cdots\otimes\mathrm{Id}$ is an isomorphism. Therefore, we get
$$fd_\nu(T/e) = \gamma Rf(T/e)\ .$$
So $fd_\nu = \gamma Rf$ and the morphism f induces a T-dendrex of $\mathrm{N}^\Omega(\mathscr{P})$, which extends the initial morphism $\Lambda^e[T]\to\mathrm{N}^\Omega(\mathscr{P})$. □

So, the image of $\mathrm{N}^\Omega(-)$ lies in the category of ∞-operads. Therefore, we can consider it as a functor from the category of su homotopy operads to the category of ∞-operads :
$$\mathrm{N}^\Omega : \mathsf{suOp}_\infty \to \infty\text{-}\mathsf{Op}\ .$$

Recall from Proposition 87, that strict unital A_∞-categories are the su homotopy operads concentrated in arity one. G. Faonte already defined in [Fao13] a simplicial nerve $\mathrm{N}_{\mathcal{A}_\infty}$ for strict unital A_∞-categories, generalizing a first construction of J. Lurie [Lur12]. The present dendroidal nerve is actually a generalization of Faonte's simplicial nerve.

Proposition 90. *The simplicial part of the restriction to strict unital A_∞-categories of the dendroidal nerve is equal to Faonte's simplicial nerve :*
$$\pi\left(\mathrm{N}^\Omega{}_{|\mathcal{A}_\infty\text{-}\mathsf{cat}}\right) = \mathrm{N}_{\mathcal{A}_\infty}\ ,$$
where π be the restriction of dendroidal sets onto simplicial sets.

Finally, we show that the dendroidal nerve forgets the information contained in nonnegative degrees.

Definition 83. Let tr be the truncation endofunctor of the category $\mathsf{dg\text{-}Mod}$ which sends a chain complex of $\mathbb{K}$-module V to the bounded below chain complex of $\mathbb{K}$-module $tr(V)$ defined as follows :
- ▷ for any negative integer $n<0$, $tr(V)_n := \{0\}$
- ▷ for any negative integer $n<0$, $tr(V)_n := V_n$
- ▷ at degree 0, $tr(V)_0 := \ker(d : V_0 \to V_{-1})$ is the kernel of the differential.

This functor extends to an endofunctor also denoted by tr of the category of su homotopy operads sending $\mathscr{P} = (C,\mathcal{P},\nu,\eta)$ to $(C, tr(\mathcal{P}), tr\nu, \eta)$.

Proposition 91. *The dendroidal nerve is equal to its pre-composition with the truncation functor.*
$$\mathrm{N}^\Omega tr = \mathrm{N}^\Omega$$

Proof. The proof is a straightforward consequence of the description of the nerve in Section 4.3.2. □

4.3.4 The dendroidal nerve for operads is the homotopy coherent nerve

Notations. For any tree T, we denote the operad $W_H(K\Omega(T))$ by $W_H(T)$.

Definition 84 (The homotopy coherent nerve)**.** The *homotopy coherent nerve* of a dg operad $\mathscr{P}$ is defined by the following dendroidal set
$$\mathrm{hcN}(\mathscr{P})_T := \mathrm{Hom}_{\mathsf{dg\text{-}Op}}(W_H(T),\mathscr{P})\ .$$

For dg operads, this construction is equal to the dendroidal nerve.

Proposition 92. *There is a canonical isomorphism*

$$\mathrm{hcN}(\mathscr{P}) \simeq \mathrm{N}^{\Omega}(\mathscr{P}) \ ,$$

which is natural in dg colored operads $\mathscr{P}$.

Proof. This is a direct consequence of the fact that the functor W_H is left adjoint to the inclusion functor from the category of dg colored operads to the category of strict unital homotopy colored operads. □

4.3.5 Homotopical properties of the dendroidal nerve

In this section, we explore the homotopical properties of the dendroidal nerve N^{Ω}. More precisely, we give a description of morphisms of su homotopy colored whose image under N^{Ω} are weak equivalences (resp. fibrations) in the Cisinski–Moerdijk model structure. Then we relate these results to the model structure on nonunital $\mathcal{A}_\infty$-algebras existing in [LH03, Theorem 1.3.3.1]. More precisely, we show that the simplicial nerve of $\mathcal{A}_\infty$-categories $\mathrm{N}_{\mathcal{A}_\infty}$ sends weak equivalences (resp. fibrations) of $\mathcal{A}_\infty$-algebras to weak equivalences (resp. fibrations) of the Joyal model structure on simplicial sets.

Lemma 35. *The zero-homology-group functor $H_0 : \mathsf{dg\text{-}Mod} \to \mathsf{Mod}$ induces a functor also denoted H_0 from the category suOp_∞ to the category Op of colored operads enriched in $\mathbb{K}$-modules. Forgetting the many-inputs-elements, we get a functor j^*H_0 from su homotopy operads to categories enriched in $\mathbb{K}$-modules.*

Proof. Straightforward. □

Theorem 27. *Let $f : \mathscr{P} = (C, \mathcal{P}, \nu) \to \mathscr{Q} = (C, \mathcal{Q}, \omega)$ be a morphism of su homotopy operads. The following assertions are equivalent.*

1. *The morphism of ∞-operads $\mathrm{N}^{\Omega}(f)$ is a weak equivalence.*
2. *The functor $j^*H_0(f)$ is an equivalence of categories and for any colors $c_1, \ldots, c_m$ and c, the first level morphism of chain complex of $\mathbb{K}$-modules $f : s\mathcal{P}(c_1, \ldots, c_m; c) \to s\mathcal{Q}(\phi(c_1), \ldots, \phi(c_m); \phi(c))$ induces isomorphisms of homology groups of positive degrees.*

The proof of Theorem 27 requires the following lemmata.

Lemma 36. *Let $\mathscr{P}$ be a su homotopy operad. For any colors $c, c_1, \ldots, c_m$, the simplicial set $\mathrm{N}^{\Omega}(\mathscr{P})^L(c_1, \ldots, c_m; c)$ is a simplicial $\mathbb{K}$-module and its normalized chain complex (see [GJ99, 3.2]) is isomorphic the chain complex $tr(\mathcal{P}(c_1, \ldots, c_m; c))$.*

Proof of Lemma 36. Let us unfold what is the simplicial set $X := \mathrm{N}^{\Omega}(\mathscr{P})^L(c_1, \ldots, c_m; c)$. An n-simplex of X is the data of a morphism of su homotopy operads over $(c_1, \ldots, c_m, c)$ from $\mathbb{K}\Omega(C_{m,n})$ to $\mathscr{P}$ and whose restriction to $\Delta[n]$ is the degeneracy of the color c. It is then the data of maps of graded $\mathbb{S}$-modules $t(s\mathbb{K}\Omega(C_{m,n})) \to s\mathcal{P}$ over $(c_1, \ldots, c_m, c)$ for any colored tree t obtained from $C_{m,n}$ by contracting a subtree. As the restriction of the morphism of su homotopy operads to $[n]$ is given by c, we can restrict our attention to such colored trees t which contain the corolla C_m. Let us number the edges of $[n] \subset C_{m,n}$ from bottom to top and by 0 to n. Then, there is a bijection between the set of such contracted trees t and the set of sequences of integers $0 \leq i_0 < \cdots < i_k \leq n$ given by the numbers of the edges of $[n] \subset C_{m,n}$ which still appear in t. A map $t(s\mathbb{K}\Omega(C_{m,n})) \to s\mathcal{P}$ corresponds to an element $p_{i_0 < \cdots < i_k} \in \mathcal{P}(c_1, \ldots, c_m; c)$ whose degree is the number of vertices of t minus 1, that is k. Relation (4.4) applied to these elements give the following equation :

$$d_{\mathcal{P}}(p_{i_0 < \cdots < i_k}) = \sum_{\alpha=0}^{k} (-1)^{\alpha} p_{i_0 < \cdots < \widehat{i_\alpha} < \cdots < i_k}$$

where $i_0 < \cdots < \widehat{i_\alpha} < \cdots < i_k$ is obtained from the sequence $i_0 < \cdots < i_k$ by retrieving the integer i_α. A face d_i corresponding to a coface δ_i sends the collection $\{p_{i_0 < \cdots < i_k}\}_{0 \leq i_0 < \cdots < i_k \leq n}$ to the collection $\{p_{\delta_i(j_0)<\cdots<\delta_i(j_k)}\}_{0\leq j_0<\cdots<j_k\leq n-1}$. A degeneracy s_i corresponding to a codegeneracy σ_i sends the collection $\{p_{i_0<\cdots<i_k}\}_{0\leq i_0<\cdots<i_k\leq n}$ to the collection $(p'_{j_0<\cdots<j_k})_{0\leq j_0<\cdots<j_k\leq n+1}$ where $p'_{j_0<\cdots<j_k} = p_{\sigma_i(j_0)<\cdots<\sigma_i(j_k)}$ if we have indeed $\sigma(j_0)_i < \cdots < \sigma_i(j_k)$ and $p'_{(j_0<\cdots<j_k)} = 0$ otherwise. Then, it is clear that $X = \mathrm{N}^\Omega(\mathscr{P})^L(c_1,\ldots,c_m;c)$ is a simplicial $\mathbb{K}$-module. Its normalized chain complex is the chain complex $N(X)$ concentrated in non-negative degrees such that $N(X)_n = \bigcap_{i=0}^{n-1} ker(d_i)$ and the differential $d : N(X)_n \to N(X)_{n-1}$ is $(-1)^n d_n$ where d_n is the n^{th} face map. This shows finally that $tr(\mathcal{P}(c_1,\ldots,c_m;c))$ is isomorphic to $N(X)$. □

Lemma 37. *The functor $\tau_d\mathrm{N}^\Omega$ from su homotopy operads to set–theoretical colored operads is isomorphic to the functor H_0.*

Proof of Lemma 37. We know from Theorem 26 that for any su homotopy operad $\mathscr{P}$, $\mathrm{N}^\Omega(\mathscr{P})$ is an ∞-operad. Thus Section 3.5 of [Wei07] gives us a concrete description of the operad $\tau_d\mathrm{N}^\Omega(\mathscr{P})$: it has the same colors as $\mathscr{P}$ and for any such colors c_1, ..., c_m and c, the set $\tau_d\mathrm{N}^\Omega(\mathscr{P})(c_1,\ldots,c_m;c)$ is the π_0 of the simplicial set $\mathrm{N}^\Omega(\mathscr{P})(c_1,\ldots,c_m;c)$. A straightforward computation shows that the operads $\tau_d\mathrm{N}^\Omega(\mathscr{P})$ and $H_0(\mathscr{P})$ are canonically isomorphic. □

Proof of Theorem 27. The theorem is a straightforward consequence of Lemma 36 and Lemma 37. □

Theorem 28. *Let $f : \mathscr{P} = (C,\mathcal{P},\nu) \to \mathscr{Q} = (C,\mathcal{Q},\omega)$ be a morphism of su homotopy operads. The following assertions are equivalent.*

1. *The morphism of ∞-operads $\mathrm{N}^\Omega(f)$ is a fibration.*
2. *The functor $j^*H_0(f)$ is a categorical fibration (also called isofibraton) and for any colors c_1, ..., c_m and c, the first level morphism of chain complex of $\mathbb{K}$-modules $f : s\mathcal{P}(c_1,\ldots,c_m;c) \to s\mathcal{Q}(\phi(c_1),\ldots,\phi(c_m);\phi(c))$ is a degreewise epimorphism for degrees $n \geq 2$.*

Proof. As usual, we denote by ϕ the underlying function of f.

(2) ⇒ (1) By Lemma 37 we know that $i^*\tau_d\mathrm{N}^\Omega(f)$ is a categorical fibration. Now, suppose that we have the following commutative diagram

$$\begin{tikzcd} \Lambda^e[T] \arrow[r] \arrow[d] & \mathrm{N}^\Omega(\mathscr{P}) \arrow[d, "\mathrm{N}^\Omega(f)"] \\ \Omega[T] \arrow[r] & \mathrm{N}^\Omega(\mathscr{Q}) \end{tikzcd}$$

where T is a tree with a root r and leaves l_1, ..., l_k. This diagram corresponds to morphisms of graded $\mathbb{S}$-modules :

- ▷ on the one hand, $h(t) : t(s\mathbb{K}\Omega(T)) \to s\mathcal{Q}$ for any tree t colored by the edges of T and obtained by contracting a sub-tree of T.
- ▷ on the other hand, $g(t) : t(s\mathbb{K}\Omega(T)) \to s\mathcal{P}$ for the same colored trees t except for $t = T$ and $t = T/e$.

The underlying function from the set of edges of T to C (resp. C') of the graded morphisms $g(t)$ (resp. $h(t)$) is denoted ϕ_g (resp. ϕ_h). We have $\phi\phi_g = \phi_h$. Let $\tilde{h}(T) : T(s\mathbb{K}\Omega(T)) \to s\mathcal{Q}$ be the following composition of graded morphisms :

$$\tilde{h}(T) := h(T) - \sum_{\substack{T=T_1\sqcup\ldots\sqcup T_k \\ k\geq 2}} f(T/T_1\cdots T_k)\Big(g(T_1)\otimes\cdots\otimes g(T_k)\Big) .$$

Let x be a generator of $T(s\mathbb{K}\Omega(T))(l_1,\ldots,l_k;r)$. Its degree is the number of vertices of T and so is equal or higher than 2. The surjectivity condition for f ensures us that $\tilde{h}(T)(x)$ has an antecedent through the first level map $f : s\mathcal{P}(\phi_g(l_1),\ldots,\phi_g(l_k);\phi_g(r)) \to s\mathcal{Q}(\phi_h(l_1),\ldots,\phi_h(l_k);\phi_h(r))$. This antecedent corresponds to a map of graded $\mathbb{S}$-modules $g(T) : T(s\mathbb{K}\Omega(T)) \to s\mathcal{P}$ over ϕ_g. Furthermore, we have $\tilde{h}(T) = fg(T)$ and therefore, we have $h(T) = fRg(T)$.

As in the proof of Theorem 26, let us define $g(T/e)$ by

$$g(T/e)\Big(\mathrm{Id}\otimes\cdots\otimes\nu(e)\otimes\cdots\otimes\mathrm{Id}\Big) = -\sum_{a\neq e} g(T/a)\Big(\mathrm{Id}\otimes\cdots\otimes\nu(a)\otimes\cdots\otimes\mathrm{Id}\Big)$$
$$+\sum_{T=T_1\sqcup\ldots\sqcup T_k}\gamma(T/T_1\cdots T_k)\Big(g(T_1)\otimes\cdots\otimes g(T_k)\Big)\ ,$$

where the first sum runs over the inner edges of the tree T different from e and where the second sum runs over all the partitions of the tree T with no trivial component. The same method as in the proof of Theorem 26 shows that the data of the maps $g(t)$ defines a T-dendrex p of $\mathrm{N}^\Omega(\mathscr{P})$. Let us prove that $\mathrm{N}^\Omega(f)(p) = q$ where q is the T-dendrex of $\mathrm{N}^\Omega(\mathscr{Q})$ corresponding to h. This amounts to prove that $h(t) = fRg(t)$ for any colored tree obtained by contracting a subtree of T. We already know that this is true for every such t except $t = T/e$. In the spirit of Proposition 86, we prove that $RfRg(T) = Rh(T)$ for any such tree except $t = T/e$. We consider the following commutative diagram :

$$\begin{tikzcd}
T(s\mathbb{K}\Omega(T)) \arrow[r, "Rg(T)"'] \arrow[d, "d_\gamma"'] \arrow[rr, bend left, "Rh(T)"] & \mathbb{T}(s\mathcal{P}) \arrow[r, "Rf"] \arrow[d, "d_\nu"] & \mathbb{T}(s\mathcal{Q}) \arrow[d, "\omega"] \\
\mathbb{T}(s\mathbb{K}\Omega(T)) \arrow[r, "Rg"'] & \mathbb{T}(s\mathcal{P}) \arrow[r, "f"'] & \mathcal{Q}
\end{tikzcd}$$

where d_γ is the structural coderivation of the cooperad $\mathbb{T}^c(s\mathbb{K}\Omega(T))$. As $\omega Rh = hd_\gamma$, we have $hd_\gamma(T) = \omega Rh(T) = \omega RfRg(T) = fd_\nu Rg(T) = fRgd_\gamma(T)$. Then, applying the same procedure as in the proof of Theorem 26, we get that $h(T/e) = fRg(T/e)$. This proves that $h = fRg$ and so that the square above has a lifting. So, by the characterization of the fibrations between fibrant dendroidal sets given in Theorem 24, $\mathrm{N}^\Omega(f)$ is a fibration.

(1) $\Rightarrow$ (2) By Lemma 37, we know that $j^*H_0(f)$ is a categorical fibration. Furthermore, let $n \geq 1$ be an integer and let $q \in \mathcal{Q}(\phi(c_1),\ldots,\phi(c_m);\phi(c))_n$ be an element of degree n of $\mathscr{Q}$. We know that the tree $C_{m,n}$ is made up of a corolla with m leaves above a linear tree of length n. Let e be the lowest inner edge of $C_{m,n}$. For any tree t colored by the edges of $C_{m,n}$ and which is a contraction of a subtree of $C_{m,n}$, let sq_t be the element of $s\mathcal{Q}$ as follows.

- ▷ If t is the whole tree $C_{m,n}$, then $sq_t := sq \in s\mathcal{Q}(\phi(c_1),\ldots,\phi(c_m);\phi(c))_{n+1}$.
- ▷ If t is the contracted tree $C_{m,n}/e$, then $sq_t = -sdq \in s\mathcal{Q}(\phi(c_1),\ldots,\phi(c_m);\phi(c))_n$.
- ▷ If t has only one vertex and does not contain the corolla C_m, then $sq_t = s\mathtt{id}_{\phi(c)} \in s\mathcal{Q}(\phi(c);\phi(c))_1$.
- ▷ Otherwise, $sq_t = 0 \in s\mathcal{Q}(\phi(c_1),\ldots,\phi(c_m);\phi(c))$ if t contains the corolla and $sq_t = 0 \in s\mathcal{Q}(\phi(c);\phi(c))$ if not.

According to Section 4.3.2, these elements sq_t define a $C_{m,n}$-dendrex of the dendroidal set $\mathrm{N}^\Omega(\mathscr{Q})$, i.e. a morphism $\Omega[C_{m,n}] \to \mathrm{N}^\Omega(\mathscr{Q})$. For any tree t colored by the edges of $C_{m,n}$ and which is a contraction of a subtree of $C_{m,n}$, except for $t = C_{m,n}$ and $t = C_{m,n}/e$, let sp_t be the element of $s\mathcal{P}$ as follows.

- ▷ If t has only one vertex and does not contain the corolla C_m, then $sp_t = s\mathtt{id}_c \in s\mathcal{P}(c;c)_1$.
- ▷ Otherwise, $sp_t = 0 \in s\mathcal{P}(c_1,\ldots,c_m;c)$ if t contains the corolla and $sp_t = 0 \in s\mathcal{P}(c;c)$ if not.

According to Corollary 3, these elements p_t define a morphism of dendroidal sets from $\Lambda^e[C_{m,n}]$ to $\mathrm{N}^\Omega(\mathscr{P})$. The two morphisms of dendroidal sets that we have built fit in the following commutative square of dendroidal sets.

$$\begin{array}{ccc}
\Lambda^e[T] & \longrightarrow & \mathrm{N}^\Omega(\mathscr{P}) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{N}^\Omega(f)} \\
\Omega[T] & \longrightarrow & \mathrm{N}^\Omega(\mathscr{Q})
\end{array}$$

As $\mathrm{N}^{\Omega}(f)$ is a fibration, the square has a lifting. This provides an element $sp \in s\mathcal{P}(c_1, \ldots, c_m; c)$ such that $f(sp) = sq$.

$\square$

Recall from [LH03, Theorem 1.3.3.1] that, if $\mathbb{K}$ is a field, the category of nonunital $\mathcal{A}_\infty$-algebras admits a model structure without limits where

- ▷ the weak equivalences are the morphisms $f : \mathscr{A} = (\mathcal{A}, \gamma) \to \mathscr{A}' = (\mathcal{A}', \gamma')$ such that the first level map $f_1 : s\mathcal{A} \to s\mathcal{A}'$ is a quasi-isomorphism.
- ▷ the fibrations are the morphisms $f : \mathscr{A} = (\mathcal{A}, \gamma) \to \mathscr{A}' = (\mathcal{A}', \gamma')$ such that the first level map $f_1 : s\mathcal{A} \to s\mathcal{A}'$ is a degreewise epimorphism.
- ▷ the cofibrations are the morphisms $f : \mathscr{A} = (\mathcal{A}, \gamma) \to \mathscr{A}' = (\mathcal{A}', \gamma')$ such that the first level map $f_1 : s\mathcal{A} \to s\mathcal{A}'$ is a degreewise monomorphism.

We know from Section 4.2.6 that the category of nonunital $\mathcal{A}_\infty$-algebras is embedded in the category of strict unital $\mathcal{A}_\infty$-categories. Then, Theorem 27 and Theorem 28 have the following consequence.

Corollary 4. *The simplicial nerve $\mathrm{N}_{\mathcal{A}_\infty}$ sends weak equivalences (resp. fibrations) of nonunital $\mathcal{A}_\infty$-algebras to weak equivalences (resp. fibrations) of simplicial sets for the Joyal model structure.*

Proof. This result is a straightforward consequence of Theorem 27 and Theorem 28. $\square$

To extend such a result to the operadic level, one would need a homotopy theory of homotopy operads. This will be the subject of another paper.

4.3.6 The homotopy coherent nerve is a right Quillen functor

There is a pair of adjoint functors

$$\mathsf{dSet} \underset{\mathrm{hcN}}{\overset{W_!^{dg}}{\rightleftarrows}} \mathsf{dg\text{-}Op} \ , \tag{4.5}$$

where the functor $W_!^{dg}$, left adjoint to the homotopy coherent nerve, is constructed as follows. For any tree T, we set $W_!^{dg}(\Omega[T]) := W_H(T)$ and then, since a dendroidal set is a colimit of a diagram made up of trees, the image of a dendroidal set is the corresponding colimit of the diagram made up of the images of the trees. By definition, this functor preserves colimits. In this section, we will show that this adjunction is a Quillen adjunction with respect to the model category structure on dg operads introduced in [Cav14], when the characteristic of the field $\mathbb{K}$ is 0.

Proposition 93 ([Cav14] Theorem 4.22 and Proposition 5.3)**.** *Assume that $\mathbb{K}$ is a characteristic 0 field. The category* $\mathsf{dg\text{-}Op}$ *admits a right proper cofibrantly generated model structure where*

- ▷ *the weak equivalences are the morphisms $f : \mathscr{P} \to \mathscr{Q}$ such that $j^*H_0(f)$ is an essentially surjective functor and such that the morphism of chain complexes $f : \mathcal{P}(c_1, \ldots, c_m; c) \to \mathcal{Q}(\phi(c_1), \ldots, \phi(c_m); \phi(c))$ is a quasi-isomorphism, for any integer $m \geq 0$ and for any colors $c_1, \ldots, c_m$ and c, where ϕ is the function underlying f.*
- ▷ *the fibrations are the morphisms $f : \mathscr{P} \to \mathscr{Q}$ such that the functor $j^*H_0(f)$ is an isofibration and the morphism of chain complexes $f : \mathcal{P}(c_1, \ldots, c_m; c) \to \mathcal{Q}(\phi(c_1), \ldots, \phi(c_m); \phi(c))$ is a degreewise epimorphism, for any colors $c_1, \ldots, c_m$ and c.*

Remark 20. The model structure given here may seem different from the definition given by Caviglia. It is not the case. In fact :

- ▷ The two notions of weak equivalences coincide by [Cav14, Proposition 5.3].
- ▷ Let $f : \mathscr{P} \to \mathscr{Q}$ of be a morphism of dg operads such that the morphism of chain complexes $f : \mathcal{P}(c_1, \ldots, c_m; c) \to \mathcal{Q}(\phi(c_1), \ldots, \phi(c_m); \phi(c))$ is a degreewise epimorphism, for any colors $c_1, \ldots, c_m$ and c. Then, f is a fibration in the sense of [Cav14] if and only the functor $j^*f : j^*\mathscr{P} \to j^*\mathscr{Q}$ is a fibration for the canonical model structure on dg categories introduced in [BM13, Definition 1.6]. Besides, this canonical model structure coincides with the model structure introduced by Tabuada in [Tab05] ; then f is a fibration if and only if j^*f is a fibration in the sense of Tabuada, so if and only if it is a fibration in the sense of Proposition 93.

Theorem 29. *When the characteristic of the field $\mathbb{K}$ is 0, the adjunction (4.5) is a Quillen adjunction.*

Proof. This is a straightforward consequence of Theorem 27 and Theorem 28. $\square$

REMARK 21. If $\mathbb{K}$ is not a characteristic 0 field, a model structure as in Proposition 93 exists on the category of reduced dg operads, i.e. dg operads with no elements of arity 0. Moreover, we have a similar Quillen adjunction between reduced dg operads and reduced dendroidal sets.

4.4 The big nerve of dg categories and dg colored operads

In [Lur12, 1.3.1], Lurie introduces another functor from dg categories to quasi-categories called the big nerve $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ and he proves that it is point-wise equivalent to the homotopy coherent nerve hcN. In this section, we extend Lurie's big nerve functor to dg colored operads and show that it is point-wise equivalent to the homotopy coherent nerve of dg operads hcN. To do so, we have to reformulate Lurie's arguments since the Alexander–Whitney map is not symmetric, that is his formula cannot be applied mutatis mutandis on the operadic level.

4.4.1 The Boardman–Vogt construction for simplicial operads

We recall here the Boardman–Vogt construction for the simplicial operad $\Omega(T)$ for any tree T, see [BM06] for more details.

NOTATION For any integer $n \geq 0$, the set $\Delta[1]_n$ has $n+2$ elements that we denote $e_{0,n} = (0\,0\cdots 0)$, $e_{1,n} = (0\,0\cdots 0\,1)$, ..., $e_{n+1,n} = (1\,1\cdots 1)$.

For any tree T, let $W_{\Delta[1]}(T)$ be the simplicial operad whose colors are the edges of T and such that

- ▷ $W_{\Delta[1]}(T)(e_1, \ldots, e_m; e) := \prod_{in(T')} \Delta[1]$ if there is a subtree T' of T whose leaves are e_1, ..., e_m and whose root is e, where $in(T')$ is the set of inner edges of T'.
- ▷ $W_{\Delta[1]}(T)(e_1, \ldots, e_m; e) := \emptyset$, the initial simplicial set, otherwise.

The operadic composition is given by the grafting of trees where the new inner edge is labeled by the degeneracies of $1 \in \Delta[1]_0$.

As in Definition 79, the simplicial set $\Delta[1]$ has a structure of interval given by the following morphisms.

- ▷ the morphism $\Delta[1] \to \Delta[0]$
- ▷ the morphism $\max : \Delta[1] \times \Delta[1] \to \Delta[1]$ which sends the couple $(e_{i,n}, e_{j,n})$ to $\max(e_{i,n}, e_{j,n}) := e_{max(i,j),n}$.

This interval structure induces a cosimplicial structure on the mapping $T \mapsto W_{\Delta[1]}(T)$.

4.4.2 The big nerve of dg operads

Originally, Lurie defined the big nerve as follows. From any dg category $\mathscr{C}$, one can truncate the mapping spaces and then apply the Dold–Kan functor Γ defined in Section 4.1.7. Since this last one is monoidal, one gets a simplicial category. One can then can apply the nerve of simplicial categories to obtain a quasi-category. In other words, the big nerve $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{C})$ of a dg category $\mathscr{C}$ is the following simplicial set

$$\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{C})_n := \mathrm{Hom}_{\mathsf{sSet\text{-}cat}}\big(W_{\Delta[1]}([n]), \Gamma(tr(\mathscr{C}))\big) \ ,$$

where $\mathsf{sSet\text{-}cat}$ is category of simplicial categories. However, notice that the functor Γ from non-negatively graded chain complexes to simplicial $\mathbb{K}$-modules is not symmetric monoidal. Therefore, the big nerve cannot be directly extended to dg operads using Lurie's formula. We first have to reformulate its definition.

Let $\mathbb{K}$-sSet-cat be the category of categories enriched in simplicial $\mathbb{K}$-modules, and let dg-$\mathsf{cat}^{\geq 0}$ be the category of categories enriched over the monoidal category dg-$\mathsf{Mod}^{\geq 0}$ of nonnegatively graded chain complexes. The functors Γ, C/D and N are lax monoidal and so extend respectively to a functor from the category dg-$\mathsf{cat}^{\geq 0}$ (resp. $\mathbb{K}$-sSet-cat) to the category $\mathbb{K}$-sSet-cat (resp. dg-$\mathsf{cat}^{\geq 0}$).

Lemma 38. *The functor $C/D : \mathbb{K}\text{-}\mathsf{sSet}\text{-}\mathsf{cat} \to \mathsf{dg}\text{-}\mathsf{cat}^{\geq 0}$ is left adjoint to the functor $\Gamma : \mathsf{dg}\text{-}\mathsf{cat}^{\geq 0} \to \mathbb{K}\text{-}\mathsf{sSet}\text{-}\mathsf{cat}$*

Proof. On the one hand, the endofunctor $N\Gamma$ of the category dg-$\mathsf{cat}^{\geq 0}$ is exactly the identity functor. On the other hand, the functor $N : \mathbb{K}\text{-}\mathsf{sSet}\text{-}\mathsf{cat} \to \mathsf{dg}\text{-}\mathsf{cat}^{\geq 0}$ is fully faithful. So for any category $\mathscr{C}$ enriched in simplicial $\mathbb{K}$-modules and for any category $\mathscr{D}$ enriched over dg-$\mathsf{Mod}^{\geq 0}$, we have :

$$\begin{aligned}\mathrm{Hom}_{\mathbb{K}\text{-}\mathsf{sSet}\text{-}\mathsf{cat}}\big(\mathscr{C},\Gamma\mathscr{D}\big) &\simeq \mathrm{Hom}_{\mathsf{dg}\text{-}\mathsf{cat}^{\geq 0}}\big(N\mathscr{C},N\Gamma\mathscr{D}\big)\\ &\simeq \mathrm{Hom}_{\mathsf{dg}\text{-}\mathsf{cat}^{\geq 0}}\big(N\mathscr{C},\mathscr{D}\big)\ .\end{aligned}$$

Since the functor $C/D : \mathbb{K}\text{-}\mathsf{sSet}\text{-}\mathsf{cat} \to \mathsf{dg}\text{-}\mathsf{cat}^{\geq 0}$ is isomorphic the functor N, it is also left adjoint to Γ. □

Subsequently, the big nerve $\mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}(\mathscr{C})$ can be rewritten as

$$\mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}(\mathscr{C})_n \simeq \mathrm{Hom}_{\mathsf{dg}\text{-}\mathsf{cat}}\big(C/D(\mathbb{K}W_{\Delta[1]}([n])),\mathscr{C}\big)$$

where $\mathbb{K}W_{\Delta[1]}([n])$ is the category enriched in simplicial $\mathbb{K}$-modules freely obtained from the simplicial category $W_{\Delta[1]}([n])$. Since the functor C/D from simplicial $\mathbb{K}$-modules to nonnegatively graded chain complexes is symmetric monoidal, then this last formula can be extended to the operadic level.

Definition 85 (The big nerve of dg operads)**.** Let $\mathscr{P}$ be a dg operad. The *big nerve* $\mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}(\mathscr{P})$ of $\mathscr{P}$ is the following dendroidal set.

$$\mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}(\mathscr{P})_T := \mathrm{Hom}_{\mathsf{dg}\text{-}\mathsf{Op}}\big(C/D(\mathbb{K}W_{\Delta[1]}(T)),\mathscr{P}\big)\ .$$

The big nerve is a functor from the category of dg colored operads to the category of dendroidal sets.

As the homotopy coherent nerve, the big nerve admits a left adjoint.

Proposition 94. *Let $W_!^{\Delta}$ be the colimit preserving functor from the category of dendroidal sets to the category of simplicial operads such that $W_!^{\Delta}(\Omega[T]) := W_{\Delta[1]}(T)$ for any tree T. Then the functor $C/D(\mathbb{K}W_!^{\Delta})$ is left adjoint to the big nerve.*

Proof. It follows from the fact that the functor C/D preserves colimits. □

Proposition 95. *The big nerve of a dg colored operad is an ∞-operad.*

Proof. Let us consider a tree T together with and inner edge e and a morphism f of dendroidal sets from $\Lambda^e[T]$ to the big nerve $\mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}(\mathscr{P})$ of a dg colored operad $\mathscr{P}$. We denote by $l_1, \ldots, l_m$ the leaves of the tree T and by r its root. The images of l_i (resp. r) under the morphism f are denoted c_i (resp. c). The morphism f and the adjunction $C/D\,\mathbb{K}W_!^{\Delta} \dashv \mathrm{N}^{\mathrm{big}}_{\mathrm{dg}}$ give a map of chain complexes from $C/D(\mathbb{K}W_!^{\Delta}(\Lambda^e[T]))(l_1,\ldots,l_m;r)$ to $\mathcal{P}(c_1,\ldots,c_m;c)$. Through the adjunction $C/D \vdash \Gamma$, it corresponds to a map f' of simplicial sets from $W_!^{\Delta}(\Lambda^e[T])(l_1,\ldots,l_m;r)$ to $\Gamma\mathcal{P}(c_1,\ldots,c_m;c)$. Since the map $W_!^{\Delta}(\Lambda^e[T])(l_1,\ldots,l_m;r) \hookrightarrow W_!^{\Delta}(\Omega[T])(l_1,\ldots,l_m;r)$ is anodyne (see [MW09, Section 7]) and since $\Gamma\mathcal{P}(c_1,\ldots,c_m)$ is a Kan complex, there is a lifting f'' of the map f' :

$$\begin{tikzcd} W_!^{\Delta}(\Lambda^e[T])(l_1,\ldots,l_m;r) \arrow[r, "f'"] \arrow[d, hook] & \Gamma\mathcal{P}(c_1,\ldots,c_m;c) \\ W_!^{\Delta}(\Omega[T])(l_1,\ldots,l_m;r)\ . \arrow[ur, "f''"'] & \end{tikzcd}$$

As the map $i \in \{1, \ldots, m\} \mapsto l_i$ is an injection, for any permutation $\sigma \in \mathbb{S}_m$ the structural isomorphisms $W_!^\Delta(\Omega[T])(l_1, \ldots, l_m; r) \simeq W_!^\Delta(\Omega[T])(l_{\sigma(1)}, \ldots, l_{\sigma(m)}; r)$ and $\Gamma\mathcal{P}(c_1, \ldots, c_m; c) \simeq \Gamma\mathcal{P}(c_{\sigma(1)}, \ldots, c_{\sigma(m)}; c)$ give us a morphism

$$W_!^\Delta(\Omega[T])(l_{\sigma(1)}, \ldots, l_{\sigma(m)}; r) \to \Gamma\mathcal{P}(c_{\sigma(1)}, \ldots, c_{\sigma(m)}; c) \ .$$

Moreover, for any other inputs $e_1, \ldots, e_l$ and output e_0, the simplicial set $W_!^\Delta(\Omega[T])(e_1, \ldots, e_l; e_0)$ is exactly $W_!^\Delta(\Lambda^e[T])(e_1, \ldots, e_l; e_0)$. So we have maps from $W_!^\Delta(\Omega[T])(e_1, \ldots, e_l; e_0)$ to $\Gamma\mathcal{P}$. All these maps induce a morphism of dg colored operads from $C/D(\mathbb{K}W_!^\Delta(\Omega[T]))$ to $\mathscr{P}$ and so a morphism of dendroidal sets from $\Omega[T]$ to $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})$ which extends the morphism f. □

4.4.3 From the big nerve to the homotopy coherent nerve

In this section, we construct a morphism of functors α^* from $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ to hcN.

The mappings $h \mapsto e_{1,1}$, $h_0 \mapsto e_{0,0}$ and $h_1 \mapsto e_{1,0}$ induce an isomorphism from the chain complex H defined in Section 4.2.7 to the chain complex $C/D(\mathbb{K}\Delta[1]) := C(\mathbb{K}\Delta[1])/D(\mathbb{K}\Delta[1])$. The functor C/D is symmetric monoidal through the Eilenberg–Zilber map ; see [Fao13, 3.1]. So, we get morphisms α_k from $H^{\otimes k}$ to $C/D(\mathbb{K}\Delta[1]^k)$ for any integer $k \geq 1$. Let us describe them. On the one hand, we have :

$$\alpha_k(h \otimes \cdots \otimes h) = \sum_{\sigma \in \mathbb{S}_k} \mathrm{sign}(\sigma) e_{\sigma(k),k} \otimes \cdots \otimes e_{\sigma(1),k} \ .$$

On the other hand, let $A \in H^{\otimes k}$ and $B \in H^{\otimes l}$ be homogeneous elements whose cumulate degree is n and such that $\alpha_{k+l}(A \otimes B) = \sum_{i \in I} A_i \otimes B_i$ where $A_i \in C/D(\mathbb{K}\Delta[1]^k)$ and $B_i \in C/D(\mathbb{K}\Delta[1]^l)$. Then we have

$$\alpha_{k+l+1}(A \otimes h_0 \otimes B) = \sum_{i \in I} A_i \otimes e_{0,n} \otimes B_i,$$
$$\alpha_{k+l+1}(A \otimes h_1 \otimes B) = \sum_{i \in I} A_i \otimes e_{n+1,n} \otimes B_i \ .$$

Note also that the morphism $\alpha_{k,l}$ from $C/D(\mathbb{K}\Delta[1]^k) \otimes C/D(\mathbb{K}\Delta[1]^l)$ to $C/D(\mathbb{K}\Delta[1]^{k+l})$ given by the Eilenberg–Zilber map satisfy the equation $\alpha_{k,l}(\alpha_k \otimes \alpha_l) = \alpha_{k+l}$. This follows from the fact that the Eilenberg–Zilber map is the structural map making C/D into a symmetric monoidal functor.

Proposition 96. *These maps α_k induce a morphism of dg operads $\alpha_T : W_H(T) \to C/D\mathbb{K}W_{\Delta[1]}(T)$, for any tree T. Moreover, these morphisms are functorial in T. Subsequently, they induce*
- ▷ *a morphism of functors α from $W_!^{dg}$ to $C/D\mathbb{K}W_!^\Delta$,*
- ▷ *and a morphism of functors $\alpha^* : \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}} \to \mathrm{hcN}$.*

Proof. The maps α_k induce morphisms of dg $\mathbb{S}$-modules $\alpha_T : W_H(T) \to C/D\mathbb{K}W_{\Delta[1]}(T)$ for any tree T. We have to show that the morphisms α_T are morphisms of dg operads and that they are functorial with respect to the trees T.
- ▷ The former property follows from the fact that for any integers $k, l \geq 0$, the following square is commutative.

$$\begin{tikzcd}
H^{\otimes k} \otimes H^{\otimes l} \arrow[r, "\alpha_k \otimes \alpha_l"] \arrow[dd] & C/D(\mathbb{K}\Delta[1]^k) \otimes C/D(\mathbb{K}\Delta[1]^l) \arrow[d, "\alpha_{k,l}"] \\
 & C/D(\mathbb{K}\Delta[1]^{k+l}) \arrow[d] \\
H^{\otimes k+1+l} \arrow[r, "\alpha_{k+1+l}"'] & C/D(\mathbb{K}\Delta[1]^{k+1+l})
\end{tikzcd} ,$$

where the left vertical map sends $A \otimes B \in H^{\otimes k} \otimes H^{\otimes l}$ to $A \otimes h_1 \otimes B$ and where the bottom-right vertical map is the functorial image under $C/D\,\mathbb{K}(-)$ of the morphism from $\Delta[1]^k \times \Delta[1]^l$ to $\Delta[1]^{k+1+l}$ which sends (A, B) to $(A, e_{n+1,n}, B)$.

▷ Let us show that the morphisms α_T are functorial with respect to the trees T. It is straightforward to show that for any coface $\delta : T \to T'$, we have $C/D\,\mathbb{K}W_{\Delta[1]}(\delta)\alpha_T = \alpha_{T'}W_H(\delta)$. To prove that the same equation holds for a codegeneracy, it suffices to note that the map

$$h \otimes A \in H^{\otimes k} \xmapsto{\alpha_k} \sum_i \pm e_{i,n} \otimes A_i \longmapsto \sum_i \pm A_i \in C/D(\mathbb{K}\Delta[1]^{k-1})$$

is 0.

□

The goal of the end of this Section is to prove the following theorem.

Theorem 30. *For any dg colored operad $\mathscr{P}$, the morphism of dendroidal sets $\alpha^*(\mathscr{P}) : \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P}) \to \mathrm{hcN}(\mathscr{P})$ is a weak equivalence.*

We already know that the colors of $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})$ and the colors $\mathrm{hcN}(\mathscr{P})$ are both the colors of $\mathscr{P}$ and that $\alpha^*(\mathscr{P})$ is the identity on these. Hence, the morphism $\alpha^*(\mathscr{P})$ is essentially surjective. So, as both $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})$ and $\mathrm{hcN}(\mathscr{P})$ are ∞-operads, we only have to prove that $\alpha^*(\mathscr{P})$ is fully faithful to prove the theorem ; that is, we have to show that for any integer m and for any colors $c_1, \ldots, c_m$ and c of $\mathscr{P}$, the map :

$$\alpha^*(\mathscr{P})(c_1, \ldots, c_m; c) : \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})^L(c_1, \ldots, c_m; c) \to \mathrm{hcN}(\mathscr{P})^L(c_1, \ldots, c_m; c)$$

is a weak equivalence of simplicial sets for the Kan–Quillen model structure.

4.4.4 The cosimplicial simplicial set Q

We introduce here a cosimplicial simplicial set denoted Q which will allow us to deal with the operations space $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})^L(c_1, \ldots, c_m; c)$ of the big nerve. Lurie introduced in [Lur09, 2.2.2] a very similar cosimplicial simplicial set $Q^\bullet$ in a slightly different context and with different conventions. The purpose of this subsection is to recall some results of [Lur09, Section 2.2.2] about $Q^\bullet$ which extend directly to Q.

For any integer $n \geq 0$, let $D_{m,n}$ be the following colimit of dendroidal sets.

$$D_{m,n} := C_{m,n} \coprod_{\Delta[n]} \Delta[0]$$

The colors of of $D_{m,n}$ are the leaves $l_1, \ldots, l_m$ and the root r of the tree $C_{m,n}$. Since $\Delta[-]$ is a cosimplicial simplicial set, $D_{m,-}$ is a cosimplicial dendroidal set.

Definition 86 (The cosimplicial simplicial set Q)**.** Let Q be the cosimplicial simplicial set defined by

$$Q[n] := W_!^{\Delta}(D_{m,n})(l_1, \ldots, l_m; r)$$

for any integer $n \geq 0$.

Proposition 97. *There is an isomorphism of simplicial sets :*

$$\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})^L(c_1, \ldots, c_m; c) \simeq \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(C/D\mathbb{K}Q[-], \mathcal{P}(c_1, \ldots, c_m; c)\big) \ .$$

Proof. A n-vertex of $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})^L(c_1, \ldots, c_m; c)$ is a morphism of dendroidal sets from $D_{m,n}$ to $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})$ which sends the colors l_i to c_i and r to c. So it is a morphism of dg operads from $C/D\mathbb{K}W_!^{\Delta}(D_{m,n})$ to $\mathscr{P}$ which sends the colors l_i to c_i and r to c and so it is a morphism of chain complexes from $C/D\mathbb{K}(Q[n])$ to $\mathcal{P}(c_1, \ldots, c_m; c)$. □

The simplicial set $Q[n]$ admits the following description.

$$Q[n] \simeq \Delta[1]^n / \sim$$

where $(A, e_{k+1,k}, B) \sim (A', e_{k+1,k}, B)$. We now describe the cosimplicial structure of $Q[-]$. Let $(q_1, \ldots, q_n) \in Q[n]_l$ be a l-vertex of $Q[n]$ represented by a l-vertex of $\Delta[1]^n$ and let $\delta_i : [n] \to [n+1]$ (resp. $\sigma_i : [n] \to [n-1]$) be a coface (resp. a codegeneracy).

▷ If $i \geq 1$:

$$Q(\delta_i)(q_1, \ldots, q_n) = (q_1, \ldots, q_{i-1}, e_{0,l}, q_i, \ldots, q_n) \ ,$$
$$Q(\sigma_i)(q_1, \ldots, q_n) = (q_1, \ldots, \max(q_i, q_{i+1}), \ldots, q_n) \ .$$

▷ If $i = 0$:

$$Q(\delta_i)(q_1, \ldots, q_n) = (e_{l+1,l}, q_1, \ldots, q_n) \ ,$$
$$Q(\sigma_i)(q_1, \ldots, q_n) = (q_2, \ldots, q_n) \ .$$

Definition 87 (A morphism from Q to Δ, [Lur09] Remark 2.2.2.6)**.** Let β be the morphism of cosimplicial simplicial sets $\beta : Q[-] \to \Delta[-]$ which sends the element $e_{i_1,k} \otimes \cdots \otimes e_{i_n,k} \in Q[n]_k$ to

$$[\max\{j | i_j = k+1\} \leq \max\{j | i_j \geq k\} \leq \cdots \leq \max_j\{j | i_j \geq 1\}] \in \Delta[n]_k$$

with the convention $\max(\emptyset) = 0$. We denote by b the morphism of cosimplicial chain complexes $b := C/D\mathbb{K}\beta : C/D\mathbb{K}Q[-] \to C/D\mathbb{K}\Delta[-]$.

Lemma 39. *The morphism of functors induced by the morphism b*

$$b^* : \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(C/D\mathbb{K}\Delta[-], -\big) \to \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(C/D\mathbb{K}Q[-], -\big)$$

is a point-wise equivalence, that is the morphism of simplicial sets $b^*(V)$ is a weak equivalence of simplicial sets for the Kan–Quillen model structure, for any chain complex V.

Proof. It is a consequence of [Lur09, Proposition 2.2.2.7] and [Lur09, Proposition 2.2.2.9]. □

4.4.5 The big nerve is equivalent to the homotopy coherent nerve

We now prove Theorem 30. For that purpose, we use the method of the proof of [Lur12, Proposition 1.3.1.17].

Proposition 98. *There is an isomorphism of simplicial sets :*

$$\mathrm{hcN}(\mathscr{P})^L(c_1, \ldots, c_m; c) \simeq \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(C/D\mathbb{K}\Delta[-], \mathcal{P}(c_1, \ldots, c_m; c)\big) \ .$$

Proof. We know that

$$\mathrm{hcN}(\mathscr{P})^L(c_1, \ldots, c_m; c) \simeq \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(W_!^{dg}(D_{m,-})(l_1, \ldots, l_m; r), \mathcal{P}(c_1, \ldots, c_m; c)\big) \ .$$

The chain complex $W_!^{dg}(D_{m,n})(l_1, \ldots, l_m; r)$ can be described as $H^{\otimes n}/\sim$, where $A \otimes h_1 \otimes B \sim 0$, if the degree of A is different from 0 and $A \otimes h_1 \otimes B \sim h_1 \otimes \cdots \otimes h_1 \otimes B$ otherwise. Let A be an homogeneous element of $H^{\otimes n}$. It has the form $A = h_1^{\otimes i_0} \otimes A'$ where h occupies the places (counted from 1 to n) $i_1 < \cdots < i_k$ and A' does not contain h_1. Then, the mapping which sends A to the element $[i_0 \leq i_1 \leq \cdots \leq i_k]$ gives us an isomorphism of cosimplicial chain complexes :

$$W_!^{dg}(D_{m,-})(l_1, \ldots, l_m; r) \simeq C/D\mathbb{K}\Delta[-] \ .$$

□

Definition 88 (From Δ to Q)**.** We have introduced a morphism of functors α from $W_!^{dg}$ to $C/D\mathbb{K}W_!^{\Delta}$. If we apply it to the dendroidal sets $(D_{m,n})_{n\in\mathbb{N}}$, we get

▷ a morphism of cosimplicial chain complexes a from $C/D\mathbb{K}\Delta[-]$ to $C/D\mathbb{K}Q[-]$,

▷ and a morphism of functors

$$a^* : \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(C/D\mathbb{K}Q[-], -\big) \to \mathrm{Hom}_{\mathsf{dg\text{-}Mod}}\big(C/D\mathbb{K}\Delta[-], -\big) \ .$$

Lemma 40. *The endomorphism ba of the cosimplicial chain complexes $C/D(\mathbb{K}\Delta[-])$ is the identity.*

Proof. It follows from the description of the morphisms a and b given above, and a straightforward computation. $\square$

Proof of Theorem 30. By Proposition 97 and Proposition 98, the map of simplicial sets $a^*(\mathcal{P}(c_1, \dots, c_m; c))$ from $\mathrm{Hom}_{\mathsf{dg\text{-}Mod}}(C/D\mathbb{K}Q[-], \mathcal{P}(c_1, \dots, c_m; c))$ to $\mathrm{Hom}_{\mathsf{dg\text{-}Mod}}(C/D\mathbb{K}\Delta[-], \mathcal{P}(c_1, \dots, c_m; c))$ can be considered as a map from $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})^L(c_1, \dots, c_m; c)$ to $\mathrm{hcN}(\mathscr{P})^L(c_1, \dots, c_m; c)$. By construction, it is exactly

$$\alpha^*(\mathscr{P})(c_1, \dots, c_m; c) : \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})^L(c_1, \dots, c_m; c) \to \mathrm{hcN}(\mathscr{P})^L(c_1, \dots, c_m; c) .$$

Since its composition with the map $b^*(\mathcal{P}(c_1, \dots, c_m; c))$ is the identity and since the map $b^*(\mathcal{P}(c_1, \dots, c_m; c))$ is a weak equivalence, then $a^*(\mathcal{P}(c_1, \dots, c_m; c))$ is a weak equivalence. So, for any dg colored operad $\mathscr{P}$, the morphism of ∞-operads $\alpha^*(\mathscr{P})$ from $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P})$ to $\mathrm{hcN}(\mathscr{P})$ is fully faithful. Since it is essentially surjective, then it is an equivalence. $\square$

Corollary 5. *When the characteristic of the field $\mathbb{K}$ is zero, the adjunction $C/D\mathbb{K}W_!^{\Delta} \dashv \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ is a Quillen adjunction, with respect to the model structure on the category* $\mathsf{dg\text{-}Op}$ *of dg operads given in Proposition 93.*

Proof. It suffices to prove that the functor $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ preserves weak equivalences and fibrations. Since it is equivalent to the homotopy coherent nerve which preserves weak equivalences, then it preserves weak equivalences. Let $f : \mathscr{P} \to \mathscr{Q}$ be a fibration of dg operads and let ϕ its underlying function between colors. On the one hand, since the functors from dg operads to set-theoretical operads $\tau_d \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}$ and H_0 are isomorphic, then the functor $i^* \tau_d \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(f)$ is an isofibration. On the other hand, let T be a tree with $m \geq 0$ leaves $l_1, \dots, l_m$, a root denoted by r and an inner edge e. Consider the following commutative square of dendroidal sets

$$\begin{tikzcd} \Lambda^e[T] \arrow[r] \arrow[d] & \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{P}) \arrow[d, "\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(f)"] \\ \Omega[T] \arrow[r] & \mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(\mathscr{Q}) . \end{tikzcd}$$

It induces a commutative square of chain complexes.

$$\begin{tikzcd} C/D\mathbb{K}W_!^{\Delta}\Lambda^e[T](l_1, \dots, l_m; r) \arrow[r] \arrow[d] & \mathcal{P}(c_1, \dots, c_m; c) \arrow[d, "f(c_1, \dots, c_m; c)"] \\ C/D\mathbb{K}W_!^{\Delta}\Omega[T](l_1, \dots, l_m; r) \arrow[r] & \mathcal{Q}(\phi(c_1), \dots, \phi(c_m); \phi(c)) \end{tikzcd}$$

Since the morphism $f(c_1, \dots, c_m; c)$ is a fibration of chain complexes and since the left vertical map is a trivial cofibration, then this square has a lifting. Since the function $i \in \{1, \dots, m\} \mapsto l_i$ is injective, we get, for any permutation $\sigma \in \mathbb{S}_m$, a lifting of the same square but where $(l_1, \dots, l_m)$ (resp. $(c_1, \dots, c_m)$) is replaced by $(l_{\sigma(1)}, \dots, l_{\sigma(m)})$ (resp. $(c_{\sigma(1)}, \dots, c_{\sigma(m)})$). This gives us a lifting of the first square. By the characterization of the fibrations between fibrant dendroidal sets given in Theorem 24, then $\mathrm{N}_{\mathrm{dg}}^{\mathrm{big}}(f)$ is a fibration. $\square$

Appendices

Appendix A

The purpose of this appendix is to describe the category of cocommutative coalgebras over an algebraically closed field of characteristic zero in the vein of the article [CLM].

Definition 89 (Pseudo-compact algebras)**.** The category of pseudo compact algebras is the category antiequivalent to the category of cocommutative coalgebras and made up of inverse limits of diagrams of finite dimensional commutative algebras. The morphisms are the morphisms of algebras which are continuous with respect to the induced topology. A pseudo compact algebra $\mathscr{A}$ is called local if it its underlying graded algebra is local.

The antiequivalence between pseudo compact algebras and cocommutative coalgebras is realized as follows : the linear dual of a cocommutative coalgebra is a pseudo compact algebra ; conversely, the topological linear dual of a pseudo compact algebra is a cocommutative coalgebra. Moreover, for any cocommutative coalgebra $\mathscr{C}$, the sub-coalgebras of $\mathscr{C}$ are in correspondence with the ideals of $\mathscr{C}^*$.

Definition 90 (Orthogonal ideals and sub-coalgebras)**.** Let $\mathscr{D} = (\mathcal{D}, \Delta, \epsilon)$ be a sub-coalgebra of $\mathscr{C}$. The orthogonal of $\mathscr{D}$ is the sub-chain complex $\mathscr{D}^\perp := \{f \in \mathscr{C}^* | \ \forall x \in \mathcal{D},\ f(x) = 0\} \subset \mathscr{C}^*$ which is an ideal of $\mathscr{C}^*$. Let I be an ideal of the commutative algebra $\mathscr{C}^*$. The orthogonal of I is the sub-chain complex $I^\perp := \{x \in \mathscr{C} | \ \forall f \in I,\ f(x) = 0\} \ \subset \mathscr{C}$ which is a sub-coalgebra of $\mathscr{C}$.

One of the consequences of this relation between ideals of $\mathscr{C}$ and sub-coalgebras of $\mathscr{C}$ is the following proposition.

Proposition 99. *[GG99] Let $\mathscr{C}$ be a cocommutative coalgebra and let x be an element of $\mathscr{C}$. There exists a finite dimensional sub-coalgebra of $\mathscr{C}$ which contains x.*

Chuang, Lazarev and Mannan showed that any pseudo compact algebra can be decomposed into a product of local pseudo compact algebras.

Theorem 31. *[CLM, 2.9] Any pseudo compact algebra $\mathscr{A}$ is isomorphic to the product of local pseudo compact algebras $\mathscr{A} \simeq \prod_{i\in I} \mathscr{A}_i$. Moreover, a morphism of product of local pseudo compact algebras $f : \prod_{i\in I} \mathscr{A}_i \to \prod_{j\in J} \mathscr{B}_j$ is the data of an element $\phi(j) \in I$ and a morphism $f_j : \mathscr{A}_{\phi(j)} \to \mathscr{B}_j$ for any $j \in J$.*

We show that local pseudo compact algebras are linear duals of conilpotent cocommutative coalgebras.

Definition 91. A nonzero graded cocommutative coalgebra is said to be irreducible if any two nonzero sub-coalgebras have nonzero intersection.

Proposition 100. *A graded cocommutative coalgebra is irreducible if and only if dual algebra is local.*

Proof. Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a graded cocommutative coalgebra. We first suppose that it is irreducible. Let M_1 and M_2 be two maximal ideals of the commutative algebra $\mathscr{C}^*$. Since, $\mathscr{C}$ is irreducible, then the sub-coalgebras $M_1^\perp$ and $M_2^\perp$ have nonzero intersection. So $M_1 + M_2 \subset (M_1^\perp \cap M_2^\perp)^\perp$ is a proper ideal. Since M_1 and M_2 are maximal ideals, then $M_1 = M_1 + M_2 = M_2$. So $\mathscr{C}^*$ is local. Conversely, suppose that $\mathscr{C}^*$ is local. We denote by M its maximal ideal. By Lemma 41, M is the kernel of an augmentation $\mathscr{C}^* \to \mathbb{K}$. By the antiequivalence between pseudo compact algebra and cocommutative coalgebras, we obtain a morphism of coalgebras $\mathbb{K} \to \mathscr{C}$, that is an atom a of $\mathscr{C}$. For any nonzero sub-coalgebra $\mathscr{D}$ of $\mathscr{C}$, the orthogonal $\mathscr{D}^\perp$ is contained in M. Thus $\mathbb{K} \cdot a = M^\perp \subset (\mathscr{D}^\perp)^\perp = \mathscr{D}$. So any nonzero sub-coalgebra of $\mathscr{C}$ contains a. Subsequently, $\mathscr{C}$ is irreducible. ☐

Lemma 41. *Let $\mathscr{A}$ be a graded local pseudo compact algebra. Then, the maximal ideal M of $\mathscr{A}$ is the kernel of an augmentation $\mathscr{A} \to \mathbb{K}$.*

Proof. Since $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}}, 1)$ is the inverse limit of finite dimensional algebras and since M is maximal, then M is the kernel of a surjection $\mathscr{A} \to \mathscr{B}$ where $\mathscr{B} = (\mathcal{B}, \gamma_{\mathcal{B}}, 1)$ is a finite dimensional commutative algebras. Since M is maximal, then any nonzero element of $\mathcal{B}$ is invertible. Since, the elements in nonzero degrees are nilpotent, then $\mathcal{B}$ is concentrated in degree zero. So $\mathcal{B}$ is a finite dimensional field extension of $\mathbb{K}$. Finally, $\mathcal{B} \simeq \mathbb{K}$ because $\mathbb{K}$ is an algebraically closed field of characteristic zero. ☐

Proposition 101. *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be an irreducible cocommutative coalgebra. Then, the non-counital cocommutative coalgebra $(\overline{\mathcal{C}}, \overline{\Delta})$ is conilpotent.*

Proof. Let x an element of $\overline{\mathcal{C}}$ and be $\mathscr{D} = (\mathcal{D}, \Delta, \epsilon)$ be a finite dimensional sub-coalgebra of $\mathscr{C}$ which contains x. The commutative algebra $\mathscr{D}^*$ is local; its maximal ideal is $M := \overline{\mathcal{D}}^*$. Then, $\mathscr{D}_0^*$ is also local with maximal ideal M_0. By Nakayama's lemma, M_0 is nilpotent. So, M is nilpotent and so $\overline{\mathcal{D}}$ is a conilpotent noncounital cocommutative coalgebra. ☐

Corollary 6. *The antiequivalence between the category of pseudo compact algebras and the category* uCocom *of cocommutative coalgebras restricts to an antiequivalence between the category of local pseudo compact algebras and the category* $\mathsf{uNilCocom}$ *of conilpotent cocommutative coalgebras.*

Proof. It is a direct consequence of Proposition 100 and Proposition 101. ☐

Theorem 32. *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a dg cocommutative coalgebra over an algebraically closed field of characteristic zero and let A be its set of graded atoms. There exists a unique decomposition $\mathscr{C} \simeq \bigoplus_{a\in A} \mathscr{C}_a$ where $\mathscr{C}_a$ is a sub-coalgebra of $\mathscr{C}$ which contains a and which belongs to the category* $\mathsf{uNilCocom}$. *Moreover, a morphism of dg cocommutative coalgebras $f : \bigoplus_{a\in A} \mathscr{C}_a \to \bigoplus_{b\in B} \mathscr{D}_b$ is the the data of an element $\phi(a) \in B$ and a morphism $f_a : \mathscr{C}_a \to \mathscr{D}_{\phi(a)}$ for any $a \in A$.*

Proof. The only point that needs to be cleared up is that the decomposition $\mathscr{C} = \bigoplus_{i\in I} \mathscr{C}_i$ is indexed by the set graded atoms of $\mathscr{C}$. A graded atom of $\mathscr{C}$ is a morphism of graded cocommutative coalgebras from $\mathbb{K}$ to $\mathscr{C}$, that is a morphism of graded pseudo compact algebras from $\prod_{i\in I} \mathscr{C}_i^*$ to $\mathbb{K}$. So it is the choice of an element of I. ☐

Appendix B

In the literature, there are two ways to define the fully faithful morphisms of ∞-operads $f : P \to Q$:

- ▷ for any integer m and any colors $c_1, \dots, c_m, c$ of P, the map $P(c_1, \dots, c_m; c) \to Q(f(c_1), \dots, f(c_m); f(c))$ is a weak homotopy equivalence of simplicial sets.
- ▷ for any integer m and any colors $c_1, \dots, c_m, c$ of P, the map $P^L(c_1, \dots, c_m; c) \to Q^L(f(c_1), \dots, f(c_m); f(c))$ is a weak homotopy equivalence of simplicial sets.

The goal of this appendix is to prove that these two definitions are equivalent that is to prove the following theorem.

Theorem 33. *For any ∞-operad P, for any integer m and any colors c, $c_1, \dots, c_m$ of P, the simplicial set $P(c_1, \dots, c_m; c)$ is homotopy equivalent to the simplicial set $P^L(c_1, \dots, c_m; c)$. In other words, there is a chain of weak homotopy equivalences between $P(c_1, \dots, c_m; c)$ and $P^L(c_1, \dots, c_m; c)$. Furthermore, this construction is functorial with respect to P.*

Let $P^{(\Delta[q])}$ be the functorial Reedy fibrant resolution of P defined in [CM13, 3.2]. The two following squares are pullbacks :

$$\begin{tikzcd}
P(c_1,\ldots,c_m;c)_q \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathsf{dSet}}(\Omega[C_m], P^{(\Delta[q])}) \arrow[d] \\
\Delta[0] \arrow[r, "{(c,c_1,\ldots,c_m)}"'] & \mathrm{Hom}_{\mathsf{dSet}}(\Delta[0] \sqcup \coprod_{i=1}^m \Delta[0], P^{(\Delta[q])})
\end{tikzcd} ,$$

$$\begin{tikzcd}
P^L(c_1,\ldots,c_m;c)_p \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathsf{dSet}}(\Omega[C_{m,p}], P) \arrow[d] \\
\Delta[0] \arrow[r, "{(c,c_1,\ldots,c_m)}"'] & \mathrm{Hom}_{\mathsf{dSet}}(\Delta[p] \sqcup \coprod_{i=1}^m \Delta[0], P)
\end{tikzcd} .$$

The left vertical maps of these two diagrams are induced by cofaces maps targeting the trees C_m and $C_{m,n}$. Let $\{M^P_{p,q}\}_{p,q\in\mathbb{N}}$ be the pullback of the following diagrams :

$$\begin{tikzcd}
M^P_{p,q} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathsf{dSet}}(\Omega[C_{m,p}], P^{(\Delta[q])}) \arrow[d] \\
\Delta[0] \arrow[r, "{(c,c_1,\ldots,c_m)}"'] & \mathrm{Hom}_{\mathsf{dSet}}(\Delta[p] \sqcup \coprod_{i=1}^m \Delta[0], P^{(\Delta[q])})
\end{tikzcd}$$

The collection $\{M^P_{p,q}\}_{p,q}$ has a canonical structure of a bisimplicial set. The simplicial set $M^P_{0,-}$ (resp. $M^P_{-,0}$) is equal to $P(c_1,\ldots,c_m;c)$ (resp. $P^L(c_1,\ldots,c_m;c)$). From now on, we will denote $P^{(\Delta[q])}$ simply by P_q.

Lemma 42. *For any face map $d_i : P_q \to P_{q-1}$, the induced morphism of simplicial sets $M^P_{-,q} \to M^P_{-,q-1}$ is a trivial fibration.*

Proof of Lemma 42. Let k be an integer and suppose that we have the following diagram :

$$\begin{tikzcd}
\partial\Delta[k] \arrow[r] \arrow[d] & M^P_{-,q} \arrow[d, "d_i"] \\
\Delta[k] \arrow[r] & M^P_{-,q-1}
\end{tikzcd} .$$

It corresponds to elements $b_0, \ldots, b_k$ in $M^P_{k-1,q}$ and a in $M^P_{k,p-1}$ having coherent face relations. The color c of P induces a morphism $c : \Delta[k] \to P_q$. Then, c, b_0, ...and b_k together give a map $\partial\Omega[C_{m,k}] \to P_q$ which fits into the following diagram :

$$\begin{tikzcd}
\partial\Omega[C_{m,k}] \arrow[r, "{(b_0,\ldots,b_k,c)}"] \arrow[d] & P_q \arrow[d, "d_i"] \\
\Omega[C_{m,k}] \arrow[r] & P_{q-1}
\end{tikzcd} .$$

As the face $d_i : P_q \to P_{q-1}$ is a trivial fibration, then the former square has a lifting ; therefore, the first diagram has a lifting. As this is true for any integer k and for any such diagram, this proves the lemma. ☐

Lemma 43. *let δ_p be the unique external coface $C_{m,p-1} \to C_{m,p}$ which omits the lower vertex of $C_{m,p}$. The face induced $M^P_{p,-} \to M^P_{p-1,-}$ is a trivial fibration.*

Proof of Lemma 43. Let $k \geq 1$ be an integer. On the one hand, let $\lim(d_*, P)_{k-1}$ be the projective limit of the diagram of dendroidal sets made up of :

- ▷ for any integer $0 \leq i \leq n$ a copy of P_{k-1} denoted by (d_i, P_{k-1}).
- ▷ if $k \geq 2$, for any pair of integers (i, j) such that $0 \leq i < j \leq n$, a copy $(d_i d_j, P_{k-2})$ of P_{k-2} and maps

$$(d_i, P_k) \xrightarrow{d_{j-1}} (d_i d_j, P_k) \xleftarrow{d_i} (d_j, P_k) \quad .$$

Then, the canonical map $P_k \to \lim(d_*, P)_{k-1}$ is a fibration ; see [DK80b, 4.3] for more details. Suppose that we have the following diagram for an integer $k \geq 0$:

$$\begin{tikzcd} \partial\Delta[k] \arrow[r] \arrow[d] & M^P_{p,-} \arrow[d, "d_p"] \\ \Delta[k] \arrow[r] & M^P_{p-1,-} \end{tikzcd}$$

If $k = 0$ this diagram corresponds to the data of an element of $x \in M^P_{p-1,0}$. The degeneracy $s_{p-1}x$ of x gives a lifting of the diagram. Suppose now that $k \geq 1$. The diagram induces an element $b \in (\lim(d_*, P)_{k-1})_{C_{m,p}}$ and an element $a \in (P_k)_{C_{m,p-1}}$. They fit in the following square :

$$\begin{tikzcd} \partial^{ext}\Omega[C_{m,p}] \arrow[r, "(a,c)"] \arrow[d] & P_k \arrow[d] \\ \Omega[C_{m,p}] \arrow[r, "b"'] & \lim(d_*, P)_{k-1} \end{tikzcd} \quad .$$

The right vertical map is a fibration and the left verical map is a trivial cofibration ; see [MW09, Lemma 5.1]. Then the square has a lifting which induces a lifting of the first square. □

Proof of Theorem 33. By the two former lemmata, it is straightforward to prove that all the face maps $M^P_{p,-} \to M^P_{p-1,-}$ and $M^P_{-,q} \to M^P_{-,q-1}$ and degeneracies $M^P_{p,-} \to M^P_{p+1,-}$ and $M^P_{-,q} \to M^P_{-,q+1}$ are weak homotopy equivalences of simplicial sets. Then by [Hir03, Corollary 15.11.12], we have a chain of weak homotopy equivalences :

$$P(c_1, \ldots, c_m; c) \longrightarrow diag(M^P) \longleftarrow P^L(x_1, \ldots, x_m; x) \ ,$$

where $diag(M^P)$ is the diagonal of the bisimplicial set M^P. Furthermore, as the Reedy fibrant resolution $P^{(\Delta[-])}$ of P is functorial in P, a morphism $f : P \to Q$ of ∞-operads induces the following diagram :

$$\begin{tikzcd} P(c_1, \ldots, c_m; c) \arrow[r] \arrow[d] & diag(M^P) \arrow[d] & P^L(c_1, \ldots, c_m; c) \arrow[l] \arrow[d] \\ Q(f(c_1), \ldots, f(c_m); f(c)) \arrow[r] & diag(M^Q) & Q^L(f(c_1), \ldots, f(c_m); f(c)) \arrow[l] \end{tikzcd} \ ,$$

where the horizontal maps are weak homotopy equivalences and the vertical ones are canonically induced by f. □

Corollary 7. *The two definitions of the fully-faithful morphisms of ∞-operads are equivalent.*

Proof. Consider the former diagram. As the horizontal maps are weak homotopy equivalences, then the left vertical map is a weak equivalence if and only if the central vertical map is a weak equivalence if and only if the right vertical map is a weak equivalence. □

Appendix C

Consider the adjunction $\Omega_u \dashv B_c$ described in Section 2.8.3 and relating curved conilpotent coalgebras to unital algebras. We have shown that the projective model structure of the category of unital algebras may be transferred to the category of curved conilpotent coalgebras along this adjunction. In other words, there exists a model structure on the category of curved conilpotent cooperads whose cofibrations (resp. weak equivalences) are the morphisms whose image under Ω_u is a cofibration (resp. weak equivalence). This have been extended to the operadic level in the chapter 3. In this appendix, we show that this method cannot be extended to the multi-colors framework, that is to dg categories and curved conilpotent cocategories. As an immediate consequence, it cannot be extended to colored operads.

Definition 92. A curved conilpotent cocategory $\mathscr{C} = (C, \mathcal{C}, \Delta, d, \theta)$ is the data of a graded conilpotent cocategory (that is a graded conilpotent colored cooperad concentrated in arity one) $(C, \mathcal{C}, \Delta, \epsilon)$ together with a degree -1 coderivation d and a degree -2 map of $(C, \mathbb{S})$-modules

$$\theta : \mathcal{C} \to I_C \ ,$$

where I_C is the $(C, \mathbb{S})$-module defined in section 4.1.1. Moreover, we require that

$$d^2 = (\theta \otimes Id)\Delta \ .$$

Curved conilpotent cocategories are related to dg categories (that is dg colored operads concentrated in arity one) by an adjunction à la bar cobar that we denote $\Omega_u \dashv B_c$ since it extends the adjunction between unital algebras and curved conilpotent coalgebras that we described in Section 2.8.3 and that was already denoted $\Omega_u \dashv B_c$. On the one hand, let $\mathscr{A} := (C, \mathcal{A}, \gamma, (1_c)_{c \in C})$ be dg category. Its bar construction is the curved conilpotent cocategory $B_c\mathscr{A} := \mathbb{T}^c(s\mathcal{A} \oplus s^2 I_C)$. It is equipped with the coderivation which extends the following map.

$$\begin{aligned} \mathbb{T}(s\mathcal{A} \oplus s^2 I_C) \twoheadrightarrow \overline{\mathbb{T}}^{\leq 2}(s\mathcal{A} \oplus s^2 I_C) &\to s\mathcal{A} \oplus s^2 I_C \\ sx \otimes sy &\mapsto (-1)^{|x|} s\gamma_{\mathscr{A}}(x \otimes y) \\ sx \otimes s^2 1_c &\mapsto 0 \\ s^2 1_c &\mapsto s1_c \\ sx &\mapsto -sdx \ . \end{aligned}$$

Its curvature is the degree -2 map.

$$\begin{aligned} \mathbb{T}(s\mathcal{A} \oplus s^2 I_C) \twoheadrightarrow s^2 I_C &\to I_C \\ s^2 1_c &\mapsto 1_c \ . \end{aligned}$$

On the other hand, let $\mathscr{C} := (C, \mathcal{C}, \Delta, d, \theta)$ be a curved conilpotent cooperad. Its cobar construction is made up of the graded category

$$\Omega_u \mathscr{C} := \mathbb{T}(s^{-1}\overline{\mathcal{C}}) \ ,$$

together with the following derivation,

$$s^{-1}x \mapsto \theta(x)1 - s^{-1}dx - \sum (-1)^{|x_1|} s^{-1}x_1 \otimes s^{-1}x_2 \ ,$$

where $\overline{\Delta}_2 x = \sum x_1 \otimes x_2$.

Proposition 102. *The bar construction and the cobar construction are both functors. Moreover, the functor Ω_u is left adjoint to the functor B_c.*

Proof. The proof relies on the same arguments as the proof of Proposition 23. □

Tabuada proved in [Tab05] that the category of dg categories may be equipped with a model structure that is the restriction to dg categories of the more recent model structure of colored operads of Caviglia that we described in the chapter 4 (Proposition 93).

Theorem 34. *There does not exists a model structure on the category of curved conilpotent cocategories such that the functor Ω_u preserves cofibrations and weak equivalences (and so is a left Quillen functor).*

Proof. Let I be the curved conilpotent cocategory with one object 0 and such that $I(0;0) := \mathbb{K}$. Then, $\Omega_u(I)$ is the dg category I_0. Let J be the dg category with two objects 0 and 1 and such that

$$J(i,j) = \mathbb{K} \ , \forall i,j \in \{0,1\} \ ,$$

with obvious units and composition. It is clear that the functor $J \to I_0$ given by the identity of $\mathbb{K}$ is an acyclic fibration of dg categories. If such a model structure exists on the category of curved conilpotent cocategories, then the morphism

$$B_cJ \times_{B_cI_0} I \to I$$

is an acyclic fibration and the morphism $\Omega_u(B_cJ \times_{B_cI_0} I) \to \Omega_u I = I_0$ is a weak equivalence of dg categories. By Lemma 44, $\Omega_u(B_cJ \times_{B_cI_0} I) = I_{\{0,1\}}$. Since the morphism $I_{\{0,1\}} \to I_0$ is a not a weak equivalence, then such a model structure does not exist. □

Lemma 44. *The pullback $B_cJ \times_{B_cI_0} I$ of the proof of Theorem 34 is the cocategory with two objects* 0 *and* 1 *and such that*

$$B_cJ \times_{B_cI_0} I(i,j) = \begin{cases} \mathbb{K} \text{ if } i = j \ , \\ 0 \text{ otherwise.} \end{cases}$$

Proof. It is clear that $B_cJ \times_{B_cI_0} I$ is the biggest sub cocategory of B_cJ whose image in B_cI_0 is in the image of I. Then, $F_0^{rad} B_cJ \times_{B_cI_0} I$ lies inside $F_0^{rad} B_cJ$ and its image in B_cI_0 is zero. So, a straightforward checking shows that $F_0^{rad} B_cJ \times_{B_cI_0} I$ is zero and hence $B_cJ \times_{B_cI_0} I$ is as described in the lemma. □

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Théories homotopiques des algèbres unitaires et des opérades

Résumé Dans cette thèse, nous nous intéressons aux propriétés homotopiques des algèbres sur une opérade, des opérades elles-mêmes et des opérades colorées, dans le monde des complexes de chaînes. Nous introduisons une nouvelle adjonction bar-cobar entre les opérades unitaires et les coopérades conilpotentes courbées. Ceci nous permet de munir ces dernières d'une structure de modèles induite par la structure projective des opérades le long de cette adjonction, qui devient alors une équivalence de Quillen. Ce résultat permet de passer, sans perte d'information homotopique, dans le monde des coopérades qui est plus puissant : on peut y décrire, par exemple, les objets fibrants-cofibrants en termes d'opérades à homotopie près. Nous appliquons ensuite la même stratégie aux algèbres sur une opérade. Pour cela, on munit la catégorie des cogèbres sur la coopérade duale de Koszul d'une structure de modèles induite par celle de la catégorie des algèbres d'origine le long de leur adjonction bar-cobar, qui devient une équivalence de Quillen. Cela nous permet de décrire explicitement pour la première fois des propriétés homotopique des algèbres sur une opérade non nécessairement augmentée. Dans une dernière partie, nous introduisons la notion d'opérade colorée à homotopie près que nous arrivons à comparer aux infinies-opérades de Moerdijk–Weiss au moyen d'un foncteur : le nerf dendroidal. Nous montrons qu'il étend des constructions dues à Lurie et à Faonte et nous étudions ses propriétés homotopiques. En particulier, sa restriction aux opérades colorées est un foncteur de Quillen à droite. Tout ceci permet de relier explicitement deux mondes des opérades supérieures.

Mots clés : Opérades, algèbre homotopique, algèbre homologique, dualité de Koszul, constructions bar et cobar, ensembles dendroidaux.

Homotopy theories of unital algebras and operads

Abstract This thesis deals with the homotopical properties of algebras over an operad, of operads themselves and of colored operads, in the framework of chain complexes. We introduce a new bar-cobar adjunction between unital operads and curved conilpotent cooperads. This allows us to endow the latter with a model structure induced by the projective model structure on operads along this adjunction, which then becomes a Quillen-equivalence. This result allows us to study the homotopy theory of operads in the world of cooperads which is more powerful : for instance, fibrant-cofibrant objects can be described in terms of operads up to homotopy. We then apply the same strategy to algebras over an operad. More specifically, we endow the category of coalgebras over the Koszul dual cooperad with a model structure induced by that of the category of algebras along their bar-cobar adjunction, which becomes a Quillen-equivalence. This allows us to describe explicitly for the first time some homotopy properties of algebras over a not necessarily augmented operad. In the last part, we introduce the notion of homotopy colored operad that we compare to Moerdijk–Weiss' infinity-operads by means of a functor : the dendroidal nerve. We show that it extends existing constructions due to Lurie and Faonte and we study its homotopical properties. In particular, we show that its restriction to colored operads is a right Quillen functor. All this allows us to connect explicitly two different worlds of higher operads.

Keywords : Operads, homotopical algebra, homological algebra, Koszul duality, bar and cobar constructions, dendroidal sets.