Méthodes de théorie des modèles pour l'étude de groupes topologiques

Tomás Ibarlucía

Institut Camille Jordan
UMR 5208 du CNRS

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Tomás Ibarlucía

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Tomás Ibarlucía

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devant le jury composé de :

M. Itaï Ben Yaacov, Université Lyon 1, *Directeur de thèse*
M. David Evans, Imperial College London, *Rapporteur*
M. Damien Gaboriau, ENS de Lyon, *Examinateur*
M. Lionel Nguyen Van Thé, Université d'Aix-Marseille, *Examinateur*
M. Christian Rosendal, University of Illinois at Chicago, *Rapporteur*

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Introduction en français

Avant-propos

Cette thèse en logique appliquée et théorie descriptive des ensembles contribue à la théorie des groupes topologiques et leurs systèmes dynamiques, et à ses liens avec la théorie des modèles. Les travaux inclus dans ce manuscrit sont le résultat de deux projets de recherche différents, correspondant aux deux parties en lesquelles la thèse est divisée, comme expliqué ci-dessous. Les sujets abordés sont ainsi divers, mais peuvent s'inscrire dans un cadre général commun : l'étude de groupes d'automorphismes de structures homogènes.

Le sujet de la première partie de la thèse, réalisée sous la direction d'Itaï Ben Yaacov, est l'étude des groupes polonais Roelcke précompacts à travers les strutures $\aleph_0$-catégoriques qui leur sont associées, et réciproquement. Elle est constituée des articles suivants :

– Chapter 1 : [**Iba14**] *The dynamical hierarchy for Roelcke precompact Polish groups*, à apparaître dans Israel Journal of Mathematics.
– Chapter 2 : [**BIT15**] *Eberlein oligomorphic groups* (travail en commun avec **Itaï Ben Yaacov** et **Todor Tsankov**), soumis pour publication.
– Chapter 3 : [**Iba16**] *Automorphism groups of randomized structures*, soumis pour publication.

La deuxième partie correspond à un travail en commun avec **Julien Melleray**, portant sur les homéomorphismes minimaux de l'espace de Cantor et leurs groupes pleins. Les résultats de ce travail en commun ont été recueillis dans les articles suivants :

– Chapter 4 : [**IM14**] *Full groups of minimal homeomorphisms and Baire category methods*, Ergodic Theory Dynam. Systems, vol. 36 (2016), no. 2, pp. 550-573.
– Chapter 5 : [**IM15**] *Dynamical simplices and minimal homeomorphisms*, soumis pour publication.

Dans les sections qui suivent, nous donnons une introduction aux sujets de la thèse ainsi qu'une présentation de ses résultats principaux.

Groupes et structures

Le point de rencontre entre logique et dynamique considéré dans cette thèse réside dans l'étude de groupes d'automorphismes de structures. De nombreux exemples intéressants de groupes topologiques se présentent naturellement sous la forme $G = \mathrm{Aut}(M)$, où M est une structure *classique* (discrète) et la topologie de G est celle de la convergence ponctuelle sur M. En fait, quitte à laisser tomber le mot *naturellement*, la situation précédente comprend une très large classe de groupes : tout *groupe de permutations fermé* (c-à-d, tout sous-groupe fermé du groupe S_∞ des bijections d'un ensemble dénombrable X) est le groupe d'automorphismes d'une structure dans le sens de la logique classique, à savoir, la structure obtenue en additionnant des prédicats pour les orbites de G dans toutes les puissances finies de X. Nous considérons X dénombrable car nous nous intéressons uniquement à des groupes topologiques à base dénombrable.

Comme l'a remarqué Julien Melleray [**Mel10**], la situation s'étend à la classe plus large de tous les *groupes polonais*, en passant de la logique classique à la logique continue, telle que développée dans [**BU10, BBHU08**]. On rappelle qu'un espace topologique est polonais s'il est séparable et admet une métrique compatible complète. Dans la logique continue, les structures sont des espaces métriques complets, bornés, et les prédicats sont des fonctions à valeurs réelles, uniformément continues, bornées. La métrique remplace la relation d'identité : en particulier, tous les automorphismes sont des isométries. Le groupe d'automorphismes G d'une structure séparable M, muni de la topologie de la convergence ponctuelle, est ainsi un sous-groupe polonais du groupe $\mathrm{Iso}(M)$ des isométries de M. Réciproquement, tout groupe polonais peut être obtenu de cette façon. En effet, par un résultat classique de Birkhoff–Kakutani, tout groupe polonais G admet une métrique d_L compatible et invariante à gauche. Il suffit alors de considérer $M = \widehat{G_L}$, la complétion de G par rapport à d_L, puis d'ajouter des prédicats pour les fonctions distance à chaque orbite dans toutes les puissances finies de M. On a donc que G agit sur M par isométries et, en fait, $G = \mathrm{Aut}(M)$. La construction est décrite plus explicitement dans le Chapitre 1, §1.3.

Dans les exemples présentés naturellement, ce n'est pas surprenant que les propriétés modèle-théoriques de la structure M donnent des informations importantes sur le groupe G, tout comme les propriétés particulières de toute autre action intéressante de G. En revanche, les structures *ad hoc* décrites ci-dessus pourraient difficilement dire quelque chose de nouveau sur un groupe G donné (plutôt, elles pourraient servir comme des exemples modèle-théoriques étranges). Néanmoins, elles permettent le transfert de techniques du côté logique vers le côté dynamique (comme montré déjà dans [**Mel10**]), et suggèrent même que des concepts et résultats théoriques généraux devraient aussi passer avec. Il se trouve que cela est particulièrement le cas dans une certaine famille de groupes polonais, que nous introduirons ensuite. La première partie de cette thèse porte sur l'élaboration d'un dictionnaire précis entre les deux côtés de l'équation $G = \mathrm{Aut}(M)$ pour cette classe particulière.

Un groupe topologique G est *Roelcke précompact* si pour tout ensemble ouvert $U \subset G$ il y a un ensemble fini $F \subset G$ tel que $UFU = G$. Afin d'expliquer le nom, nous rappelons que tout groupe topologique est muni de quatre structures uniformes naturelles. Ce sont l'*uniformité gauche* (selon laquelle deux éléments $x, y \in G$ sont proches si le produit $x^{-1}y$ appartient à un entourage petit de l'identité), l'*uniformité droite* (x, y sont proches si xy^{-1} est proche de l'identité), l'*uniformité supérieure* (le suprémum des deux dernières) et l'*uniformité inférieure* (leur infimum, appelée aussi *uniformité de Roelcke* par Uspenskij [**Usp02**]). Les uniformités gauche, droite et inférieure sont toujours compatibles avec la topologie du groupe.

Les complétions de G par rapport aux trois uniformités compatibles sont des objets très intéressants, et une question basique est celle de savoir quand il leur arrive d'être compacts ; c'est-à-dire, quand G est *précompact* par rapport à ces uniformités. Si G est polonais, alors on peut voir que la complétion à gauche (qui est juste l'espace $\widehat{G_L}$) est compacte si et seulement si G est compact (si et seulement si la complétion à droite est compacte), donc cette condition ne donne rien de nouveau. En revanche, le fait que la complétion par rapport à l'uniformité inférieure soit compacte (ce qui revient à la condition $UFU = G$ indiquée plus haut, expliquant le nom) s'avère définir une nouvelle et très riche classe de groupes polonais. Hors le cas compact, ce sont toujours des groupes «infini-dimensionnels» non-abéliens, en quelque sorte orthogonaux à la classe des groupes localement compacts.

Des exemples fondamentaux de groupes polonais Roelcke précompacts, qui apparaîtront à un moment ou un autre lors de cette thèse, incluent :

- le groupe symétrique infini, S_∞ ;
- le groupe de bijections monotones des rationnels, $\mathrm{Aut}(\mathbb{Q},<)$;
- le groupe d'automorphismes du graphe aléatoire, $\mathrm{Aut}(RG)$;
- le groupe d'homéomorphismes de l'espace de Cantor, $\mathrm{Homeo}(2^\omega)$;
- le groupe unitaire, $U(\ell^2)$;
- le groupe de transformations invariantes d'un espace de Lebesgue, $\mathrm{Aut}(\Omega)$;
- le groupe de transformations quasi-invariantes d'un espace de Lebesgue, $\mathrm{Aut}^*(\Omega)$;
- le groupes d'isométries de l'espace d'Urysohn borné, $\mathrm{Iso}(\mathbb{U}_1)$;
- le groupe d'homéomorphismes croissants de l'intervalle, $H_+[0,1]$.

Les quatre premiers exemples de cette liste appartiennent à une classe de groupes qui ont été étudiés en relation avec la logique depuis un moment. Ce sont les groupes de permutations *oligomorphes*. Le terme, qui est censé signifier «peu de formes», a été introduit par Cameron [**Cam90**] : un groupe de permutations fermé $G \curvearrowright X$ est oligomorphe s'il n'y a qu'un nombre fini de configurations possibles de n-uplets de X, à G près. C'est-à-dire, si les espaces d'orbites X^n/G sont finis pour tout n. (Dans le cas de $\mathrm{Homeo}(2^\omega)$, l'ensemble dénombrable X sous-jacent est juste l'algèbre des sous-ensembles ouvert-fermés de 2^ω.) Or, si X est regardé comme une structure logique M appropriée de sorte que $G = \mathrm{Aut}(M)$, alors un théorème classique de Ryll-Nardzewski montre que le fait que $G \curvearrowright X$ soit oligomorphe équivaut à que M soit $\aleph_0$-*catégorique* (aussi dit *dénombrablement catégorique*) : toute structure dénombrable ayant les mêmes propriétés du premier ordre que M, est isomorphe à M. Par conséquent, les groupes oligomorphes sont précisément les groupes d'automorphismes de structures classiques dénombrablement catégoriques, à une seule sorte. Dans les exemples précédents, on reconnaît : l'unique ensemble dénombrable, l'unique ordre linéaire dense dénombrable sans extrémités, l'unique graphe dénombrable homogène universel, et l'unique algèbre de Boole dénombrable sans atomes.

Todor Tsankov [**Tsa12**] a observé que tout groupe oligomorphe est Roelcke précompact. De plus, il a montré que les sous-groupes Roelcke précompacts fermés de S_∞ sont précisément les limites inverses de groupes oligomorphes. C'est pourquoi nous appellerons ces groupes *pro-oligomorphes*. Un groupe polonais G est pro-oligomorphe si et seulement s'il peut être présenté comme $G = \mathrm{Aut}(M)$ pour une structure classique dénombrablement catégorique M à plusieurs sortes.

Plus tard, Itaï Ben Yaacov et Todor Tsankov [**BT14**], et indépendamment Christian Rosendal [**Ros13**], ont réalisé qu'un groupe polonais arbitraire est Roelcke précompact si et seulement s'il admet une action fidèle *approximativement oligomorphe*. Autrement dit, s'il peut être vu comme le groupe d'automorphismes d'une structure $\aleph_0$-catégorique dans le sens de la logique continue (aussi dit *séparablement catégorique*). Des exemples basiques de telles structures (à part les cas discrets) sont l'espace d'Hilbert séparable de dimension infinie, l'algèbre de mesure d'un espace de Lebesgue sans atomes, les treillis de Banach L_p sans atomes, et l'espace d'Urysohn borné de diamètre 1, chacun unique dans son genre.

La catégoricité séparable est un phénomène très intéressant et assez ubiquitaire (il se manifeste tout au long du *spectre de la stabilité*), qui implique néanmoins des propriétés très fortes pour la structure (saturation, homogénéité, définissabilité des $\emptyset$-types partiels, par exemple). Or, une caractéristique commune importante des structures $\aleph_0$-catégoriques est qu'elles sont déterminées, à bi-interprétabilité près, par leur groupes d'automorphismes, comme il a été montré dans [**AZ86, BK13**]. Donc, on pourrait s'attendre à ce que les propriétés modèle-théoriques d'une structure séparablement catégorique (du moins, celles qui sont préservées par bi-interprétations) soient codées par des propriétés naturelles de son groupe d'automorphismes. Ce principe directeur a motivé les résultats du travail de Ben Yaacov et Tsankov [**BT14**], dans lesquels ils ont étudié l'une de ces propriétés fondamentales : la *stabilité*. L'idée a été développée et poussée plus loin dans les articles qui forment les premiers chapitres de cette thèse.

Avant que l'on puisse aller dans plus de détails, nous examinerons d'autres idées récentes reliant l'étude des groupes topologiques à encore un autre domaine complètement différent.

Pour finir cette section, nous notons que les structures $\aleph_0$-catégoriques sont un cas spécial de structures homogènes, mais que de nombreuses structures intéressantes à beaucoup de symétries ne sont pas $\aleph_0$-catégoriques. Une autre connexion importante entre logique et dynamique, qui est pertinente pour la deuxième partie de cette thèse, réside dans l'étude de structures homogènes et leurs groupes d'automorphismes par le biais de la théorie de Fraïssé. Nous reviendrons sur ce lien dans une section ultérieure.

Systèmes dynamiques et leurs représentations

Un sujet classique en dynamique topologique est l'étude des propriétés de périodicité des fonctions continues sur des groupes. Au-dehors du domaine des groupes compacts, les orbites des fonctions continues $f\colon G \to \mathbb{R}$ peuvent être assez compliquées, et les notions classiques qui rentrent en jeu sont celles de *presque périodicité* (lorsque l'orbite Gf est précompacte au sens de la norme dans $\mathrm{C}(G)$) et *faible presque périodicité* (lorsque Gf est faiblement précompact). À leur tour, ces notions sont associées à des systèmes dynamiques intéressants qui viennent attachés au groupe.

Si G est un groupe topologique, un *G-ambit* est donné par une action continue de G sur un espace compact séparé X, accompagné d'un point distingué $x_0 \in X$ tel que l'orbite Gx_0 est dense dans X. (C'est la même chose qu'une *compactification* $G \to X$, définie par le fait d'envoyer 1 sur x_0.) L'algèbre de toutes les fonctions bornées $f : G \to \mathbb{R}$ qui sont uniformément continues par rapport à l'uniformité droite est dénotée par $\mathrm{RUC}(G)$. Puis, il existe une correspondance bijective entre les G-ambits, à isomorphisme près, et les sous-algèbres fermées et invariantes à gauche de $\mathrm{RUC}(G)$, qui de plus préserve l'ordre : des inclusions de sous-algèbres correspondent à des facteurs entre des ambits. Des sous-algèbres intéressantes, comme celles qui correspondent aux fonctions presque périodiques et faiblement presque périodiques, induisent des ambits avec des propriétés algébriques intéressantes. Le *compactifié de Bohr*, bG (qui correspond aux fonctions presque périodiques), est un groupe topologique compact, tandis que la *compactification WAP*, $W(G)$, est un semi-groupe semi-topologique. L'algèbre $\mathrm{RUC}(G)$ elle-même induit le *plus grand ambit*, βG, qui est un semi-groupe semi-topologique à droite. Ces trois sont des ambits universels pour leur propriétés correspondantes, par exemple, tout G-ambit qui est un semi-groupe semi-topologique est un facteur de $W(G)$.

Récemment, Eli Glasner et Michael Megrelishvili ont proposé une approche complètement nouvelle de ce sujet, l'enrichissant et l'élargissant (un résumé en est donné dans [**GM14b**]). Si $G \curvearrowright X$ est un système dynamique compact et que V est un espace de Banach, ils définissent une *représentation de X dans V* comme étant une application faible*-continue

$$\iota\colon X \to V^*$$

accompagnée d'un homomorphisme $\pi\colon G \to \mathrm{Iso}(V)$ tel que, par rapport à l'action duale induite $G \curvearrowright V^*$, l'application ι est G-équivariante. Puis, si $\mathcal{K}$ est une classe d'espaces de Banach, le système X est *$\mathcal{K}$-représentable* s'il admet suffisamment de représentations dans des espaces de Banach de la classe $\mathcal{K}$. (On peut s'attendre à avoir une représentation injective, soit juste des représentations qui séparent les points, selon le contexte.)

En considérant la représentation injective $X \to \mathrm{C}(X)^*$ évidente, on voit que tout système dynamique est Banach-représentable. Le fait intéressant est que la représentabilité dans des bonnes classes d'espaces de Banach correspond à des bonnes propriétés dynamiques. Ainsi, par exemple, par un résultat de Megrelishvili [**Meg03**], les G-ambits WAP (c-à-d, les facteurs de $W(G)$) sont précisément les ambits *réflexif*-représentables. De façon similaire, les ambits presque périodiques sont ceux qui sont *euclidien*-représentables.

Ensuite, dans les travaux [**GM06, GM12**], les auteurs étudient les propriétés dynamiques des systèmes qui sont représentables dans certaines généralisations naturelles des espaces réflexifs : les espaces *Asplund* et *Rosenthal*. Un espace de Banach est Asplund si le dual de tout sous-espace séparable est séparable ; par exemple, les espaces $c_0(\Gamma)$. Il est Rosenthal s'il ne contient pas une copie de ℓ^1 ; par exemple, tout espace Asplund est Rosenthal. L'inclusion est en fait stricte, mais il n'est pas facile d'en exhiber un exemple. Il s'avère que les notions d'Asplund et

Rosenthal-représentabilité sont robustes, admettant plusieurs présentations équivalentes. Par exemple, pour des systèmes *métrisables* compacts $G \curvearrowright X$, la table suivante montre comment la propriété d'être $\mathcal{K}$-représentable est liée à la complexité des orbites de fonctions $f \in \mathrm{C}(X)$. (Ci-dessous, O_f dénote la clôture ponctuelle de Gf dans $\mathbb{R}^X$, et $\mathcal{B}_1(X) \subset \mathbb{R}^X$ est l'espace des fonctions de première classe sur X.)

Nom	**Classe $\mathcal{K}$**	**Complexité des orbites**
WAP	Reflexif	$O_f \subset \mathrm{C}(X)$
HNS	Asplund	O_f est métrisable
Tame	Rosenthal	$O_f \subset \mathcal{B}_1(X)$

Dans la direction opposée, on peut considérer la sous-classe des systèmes WAP qui ne sont pas seulement reflexif-représentables mais *Hilbert*-représentables. La sous-algèbre de $\mathrm{RUC}(G)$ correspondante est étroitement liée à l'*algèbre de Fourier–Stieltjes* du groupe, qui consiste des coefficients matriciels des représentations unitaires de G.

Les systèmes Hilbert-représentables ont été étudiés dans [**Meg08, GW12**]. Il s'avère que ceux-là forment une classe beaucoup plus subtile que les précédentes, dont on manque encore d'une caractérisation dynamique satisfaisante. De plus, la question basique suivante est toujours ouverte, même pour $G = \mathbb{Z}$. Soient $H(G)$, G^{Asp} et G^{Ros} les *plus larges* G-ambits Hilbert, Asplund et Rosenthal-représentables, respectivement. Alors, il est connu que les facteurs de G^{Asp} sont précisément les G-ambits Asplund-représentables, et un énoncé analogue est vrai pour G^{Ros} par rapport aux systèmes Rosenthal (et pour $W(G)$ par rapport aux systèmes reflexifs, comme dit auparavant). En revanche, on ne sait pas si tous les facteurs de $H(G)$ sont Hilbert-représentables.

Un autre problème intéressant est celui de comprendre sous quelles conditions les fonctions WAP peuvent être approximées par des coefficients matriciels de représentations unitaires de G. De façon équivalente, sous quelles conditions $H(G)$ est égal à $W(G)$. Les groupes avec cette propriété sont dits *Eberlein*.

La classification des ambits en termes de leurs représentations de Banach que nous venons de discuter induit la hiérarchie suivante de systèmes dynamiques universels associés à un groupe donné :

$$\beta G \to G^{\mathrm{Ros}} \to G^{\mathrm{Asp}} \to W(G) \to H(G) \to bG.$$

Il est connu, par exemple, que ces ambits sont tous distincts pour le cas de $G = \mathbb{Z}$.

Nous étudierons ces ambits et leur algèbres de fonctions correspondantes pour le cas des groupes polonais Roelcke précompacts. Dans ce cas, il est important de considérer en plus la *compactification de Roelcke*, $R(G)$, qui correspond à l'algèbre des fonctions qui sont uniformément continues par rapport à l'uniformité inférieure. Par l'hypothèse de précompacité, ceci est juste la complétion du groupe par rapport à l'uniformité inférieure. En particulier, $R(G)$ est métrisable. L'ambit $R(G)$ est toujours plus large que G^{Asp}, mais pas toujours plus large que G^{Ros}.

Un dictionnaire pour des logiciens

L'observation fondamentale suivante a été faite dans [**BT14**]. Si G est le groupe d'automorphismes d'une structure $\aleph_0$-catégorique (métrique), alors une fonction $f \in \mathrm{C}(G)$ est WAP si et seulement si elle peut s'écrire de la forme

$$(*) \qquad f(g) = \varphi(a, gb)$$

pour une formule continue *stable* $\varphi(x,y)$ et des paramètres a,b de M (qui peuvent être des uplets infinis). La même idée montre que si l'on enlève la condition de stabilité et l'on considère donc des formules $\varphi(x,y)$ arbitraires, alors on récupère, par l'expression (*), la famille de toutes les fonctions basse-uniformément continues (c'est-à-dire, uniformément continues pour l'uniformité inférieure). En particulier, G est un *groupe WAP* (c-à-d, $W(G) = R(G)$) précisément si M est une structure stable.

Ceci était notre point de départ pour le travail du Chapitre 1, dont le but était de compléter la table de la section précédente avec une colonne pour la théorie des modèles. Il s'avère que la représentabilité dans des espaces Rosenthal correspond à une généralisation importante de la stabilité. Disons ici qu'une fonction continue $f\colon G \to \mathbb{R}$ est *basse-apprivoisée* si elle se factorise par G^{Ros} et que de plus f est basse-uniformément continue.

Proposition (Ch. 1, §3). *Une fonction continue $f\colon G \to \mathbb{R}$ est basse-apprivoisée si et seulement si elle peut s'écrire de la forme $f(g) = \varphi(a, gb)$ pour une formule NIP $\varphi(x,y)$.*

La correspondance entre fonctions WAP et formules stables est une conséquence d'un critère classique de Grothendieck pour la compacité faible dans les espaces $\mathrm{C}(X)$ (voir [**Ben13a**]) ; à son tour, la correspondance entre fonctions basse-apprivoisées et formules NIP repose sur un théorème de dichotomie célèbre de Bourgain, Fremlin et Talagrand [**BFT78**] pour les clôtures ponctuelles des sous-ensembles de $\mathrm{C}(X)$.

Il s'en suit que la notion de fonction Asplund-représentable devrait correspondre à une propriété modèle-théorique intermédiaire entre stabilité et NIP, soit collapser à l'une de ces deux.

Théorème (Ch. 1, §2). *Si G est le groupe d'automorphismes d'une structure $\aleph_0$-catégorique, alors $W(G) = G^{\mathrm{Asp}}$.*

D'un point de vue modèle-théorique, la preuve est assez simple. L'ingrédient principal est l'astuce standard de passer d'une suite indexée par $\mathbb{N}$ à une suite indexée par $\mathbb{Q}$ préservant une propriété demandée : dans ce cas, témoigner de l'instabilité. Cependant, ceci est nouveau du point de vue de la topologie, et nous ne sommes pas au courant d'une preuve alternative d'inspiration topologique.

En fait, notre argument montre une identité plus forte[2], à savoir :

$$W(G) = G^{\mathrm{SUC}},$$

[2]On a toujours les facteurs $R(G) \to G^{\mathrm{SUC}} \to G^{\mathrm{Asp}} \to W(G)$. Or, on remarque que la métrisabilité de $R(G)$ implique déjà que $G^{\mathrm{Asp}} = G^{\mathrm{SUC}}$, comme on peut le déduire du résultat principal de [**GMU08**].

où G^{SUC} est le plus grand facteur de la compactification de Roelcke qui est un semi-groupe compact semi-topologique à droite. De plus, cette identité peut être interprétée comme une reformulation de l'équivalence entre stabilité et définissabilité de types.

Corollaire. *Soit G un groupe polonais Roelcke précompact. Si G n'est pas un groupe WAP, alors $R(G)$ n'admet pas de structure de semi-groupe semi-topologique à droite étendant la structure de groupe de G.*

Une autre conséquence de l'identification $W(G) = G^{\mathrm{Asp}}$ est la suivante. Comme il a été remarqué auparavant, tout système compact $G \curvearrowright X$ est représentable dans l'espace de Banach $\mathrm{C}(X)$. Cependant, l'espace $\mathrm{C}(X)$ est généralement, sous plusieurs aspects, aussi compliqué qu'un espace de Banach peut l'être. Une exception à ceci se présente lorsque X est dénombrable (ou, plus généralement, *épars*) : dans ce cas, $\mathrm{C}(X)$ est en fait un espace Asplund.

Corollaire. *Si G est un groupe polonais Roelcke précompact, alors tout G-ambit dénombrable (ou épars) est WAP.*

Les résultat énoncés jusqu'ici fournissent aussi une bonne source d'exemples de systèmes dynamiques. Toute structure NIP, instable, séparablement catégorique, par exemple, donne lieu à un système Rosenthal-représentable non Asplund-représentable. De plus, on peut profiter des outils modèle-théoriques comme l'élimination de quantificateurs pour donner des descriptions précises des algèbres de fonctions presque périodiques (qui correspondent à des imaginaires dans la clôture algébrique du vide), faiblement presque périodiques (en décrivant les formules stables) et basse-apprivoisées (en décrivant les formules NIP) dans de nombreux exemples concrets, commençant par ceux que nous avons listé au début de l'introduction. Pour les fonctions WAP ceci a été fait dans [**BT14**]. Nous nous occupons des autres algèbres dans le Ch. 1, §4. Par exemple, on a le résultat suivant.

Corollaire. *Toute fonction basse-apprivoisée sur* $\mathrm{Iso}(\mathbb{U}_1)$ *est constante.*

Glasner et Megrelishvili ont posé la question de savoir si tout groupe polonais peut être plongé dans le groupe d'isométries d'un espace de Banach Rosenthal. Le dernier corollaire suggère que $\mathrm{Iso}(\mathbb{U}_1)$ pourrait ne pas avoir de représentations Rosenthal non-triviales du tout. Néanmoins, puisqu'il peut y avoir des fonctions apprivoisées qui ne sont pas basse-uniformément continues, notre résultat ne suffit pas pour assurer cette conclusion.[3]

Lors de l'analyse de quelques groupes d'automorphismes de structures classiques, tels que $\mathrm{Aut}(RG)$ ou $\mathrm{Homeo}(2^{\omega})$, il y a une difficulté technique qui survient, qui a de l'intérêt en soi. Normalement, on peut donner des bonnes descriptions des formules stables et NIP *classiques* (c-à-d, $\{0,1\}$-valuées). Cependant, pour décrire les fonctions WAP et basse-apprivoisées on a besoin de rendre compte de toutes les formules *continues* de ces types (de plus, on considère stabilité et NIP sur des domaines restreints). Ainsi, on voudrait pouvoir assurer que les formules continues stables/NIP sont des limites uniformes des combinaisons de formules stables/NIP classiques, ce qui n'est pas clair *a priori*.

[3] Une réponse complète pourrait être obtenue par une étude appropriée des prédicats *extérieurement* définissables dans $\mathbb{U}_1$, mais cela n'a pas encore été accompli.

La difficulté a été résolue pour les formules stables dans [**BT14**], avec une preuve topologique utilisant le théorème de point fixe de Ryll-Nardzewski pour des ensembles convexes faiblement compacts dans des espaces de Banach. Leur argument ne marche pas pour le cas NIP, et en fait nous ne savons pas si cette propriété d'approximation est vraie en général pour NIP. Néanmoins, nous avons identifié une propriété, que nous appelons *avoir des extensions définissables de types sur des ensembles finis*, qui donne une condition suffisante. Cette condition est vérifiée pour les formules stables, récupérant ainsi le résultat de [**BT14**], mais elle est vérifiée aussi dans plusieurs structures instables, ce qui permet de comprendre leurs formules NIP continues en termes des formules NIP classiques. Nous discutons cette condition dans le Ch. 1, §4.1.

Avant de passer aux contenus du Chapitre 2, nous rappelons d'autres notions et résultats importants de [**BT14**]. Supposons comme précédemment que l'on a $G = \mathrm{Aut}(M)$ pour une structure séparablement catégorique M. Par l'homogénéité de M, la complétion à gauche $\widehat{G_L}$ peut être identifiée avec le semi-groupe topologique d'endomorphismes élémentaires de M. Puis, la compactification de Roelcke $R(G)$ peut être présentée comme l'espace de types $\mathrm{tp}(x,y)$ où $x,y \in \widehat{G_L} \subset M^M$. Si l'on considère la restriction des types $\mathrm{tp}(x,y)$ aux formules stables, alors on obtient précisément la compactification WAP de G. De plus, la loi de semi-groupe de $W(G)$ peut être décrite en termes de la relation d'indépendance stable de M.

Le travail du Chapitre 2, en commun avec Itaï Ben Yaacov et Todor Tsankov, s'occupe du cas des ambits et fonctions Hilbert-représentables, pour la famille des groupes pro-oligomorphes. C'est-à-dire, nous nous limitons aux structures dénombrablement catégoriques *classiques*.

La raison principale de cette restriction est l'existence d'un théorème de classification pour les représentations unitaires des groupes pro-oligomorphes, prouvé dans [**Tsa12**]. Avec cette classification sous la main, nous pouvons montrer que l'algèbre des fonctions Hilbert-représentables sur ces groupes est engendrées par les fonctions de la forme $f(g) = \varphi(a, gb)$ où $\varphi(x,y)$ est une relation d'équivalence définissable dans la structure classique M associée.

À son tour, ceci nous permet de donner une bonne description de la *compacitifaction hilbertienne* $H(G)$. Cet ambit a toujours la structure d'un semi-groupe semi-topologique.

Théorème (Ch. 2, §3.2). *Soit G le groupe d'automorphismes d'une structure classique $\aleph_0$-catégorique M. Alors $H(G)$ est le semi-groupe semi-topologique des endomorphismes élémentaires* partiels *de M^{eq} avec domaine algébriquement clos.*

Ainsi, par exemple, la compactification hilbertienne de $\mathrm{Aut}(\mathbb{Q},<)$ est le semi-groupe des bijections monotones partielles de $\mathbb{Q}$, et la compactification hilbertienne de $\mathrm{Aut}(RG)$ est le semi-groupe des automorphismes de graphe partiels du graphe aléatoire. En fait, ceux-là sont aussi leur compactifications WAP : ce sont des groupes Eberlein.

Une conséquence du théorème précédent est que $H(G)$ est un *semi-groupe inversif*, c'est-à-dire, que l'on a

$$pp^*p = p \tag{**}$$

pour tout $p \in H(G)$. Ici, $^*\colon H(G) \to H(G)$ désigne l'involution naturelle qui étend l'opération d'inverse sur G. Il se trouve que cette propriété fournit une caractérisation des groupes pro-oligomorphes Eberlein.

En fait, la condition (**) a une interprétation modèle-théorique intéressante dans $W(G)$. Si $p = [x,y] \in W(G)$ est le type stable d'une paire $x,y \in \widehat{G_L}$, alors on a $pp^*p = p$ si et seulement si $x \mathop{\smile\!\!\!\!\!\mid}_{x\cap y} y$ (où l'intersection est calculée dans M^{eq}). Ceci explique comment la condition d'être *mono-basé* apparaît dans notre résultat principal du Chapitre 2.

Théorème (Ch. 2, §3.3-§4). *Soit G le groupe d'automorphismes d'une structure classique dénombrablement catégorique M. Alors, sont équivalents :*

(1) $W(G)$ est un semi-groupe inversif.
(2) M est mono-basée pour la relation d'indépendance stable, c-à-d, $A \mathop{\smile\!\!\!\!\!\mid}_{A\cap B} B$ pour tous ensembles algébriquement clos $A, B \subset M^{\mathrm{eq}}$.
(3) G est Eberlein, c-à-d, $H(G) = W(G)$.

De plus, un ambit de $*$-semi-groupe semi-topologique est Hilbert-représentable si et seulement s'il est un semi-groupe inversif.

Par conséquent, de la même façon que bG est l'ambit de *groupe topologique* universel et que $W(G)$ est l'ambit de *semi-groupe semi-topologique* universel, pour les groupes pro-oligomorphes nous pouvons caractériser $H(G)$ comme l'ambit de *$*$-semi-groupe inversif semi-topologique* universel.[4]

Ensuite, par un résultat classique de théorie des modèles, notre théorème implique aussi la chose suivante.

Corollaire. *On a $H(G) = R(G)$ si et seulement si M est $\aleph_0$-stable.*

En considérant l'exemple célèbre de Hrushovski d'un pseudo-plan $\aleph_0$-catégorique stable, on peut alors montrer un exemple d'un groupe WAP qui n'est pas Eberlein. Cela répond à une question de Glasner et Megrelishvili.

Ensuite nous passons à l'analyse de la représentabilité des facteurs arbitraires de $H(G)$. Pour cela nous sommes amenés à prouver un résultat intermédiaire, intéressant en soi. Notons que les compactifications $H(G)$, $W(G)$ et $R(G)$ du groupe d'automorphismes d'une structure $\aleph_0$-catégorique classique sont toujours zero-dimensionnelles, ce qui suit de leurs descriptions modèle-théoriques (pour le cas de $W(G)$, on rappelle que les formules stables classiques déterminent toutes les formules stables continues).

Théorème (Ch. 2, §4). *Si G est un groupe pro-oligomorphe, alors tous les facteurs de $H(G)$ sont zero-dimensionnels.*

En revanche, il y a des exemples de facteurs non-zero-dimensionnels de la compactification de Roelcke. Nous ne savons pas si le résultat précédent est valable également pour les facteurs de la compactification WAP.

Finalement, nous pouvons répondre, pour les groupes pro-oligomorphes, à la question ouverte mentionnée dans la section précédente.

[4]On note, cependant, que cette caractérisation algébrique est fausse sous des hypothèses plus faibles ; par exemple, $H(G)$ n'est pas un semi-groupe inversif dans le cas du groupe discret $\mathbb{Z}$, ni dans le cas du groupe polonais Roelcke précompact $U(\ell^2)$.

Théorème (Ch. 2, §4). *Si G est un groupe pro-oligomorphe, alors tous les facteurs de $H(G)$ sont Hilbert-représentables.*

Le Chapitre 2 finit avec une remarque et une question sur la complexité des ambits dénombrables de groupes pro-oligomorphes. Si M est classique, $\aleph_0$-catégorique et $\aleph_0$-stable, alors les espaces de types $S_x(M)$ en un nombre fini de variables sont dénombrables de rang fini. Nous remarquons que ceci peut se généraliser à des groupes pro-oligomorphes arbitraires de la façon suivante.

Proposition (Ch. 2, §4). *Tout ambit dénombrable Hilbert-représentable d'un groupe pro-oligomorphe a rang de Cantor–Bendixson fini.*

Puisque dans le cas $\aleph_0$-stable l'hypothèse de Hilbert-représentabilité n'est pas nécessaire (elle suit des autres), on peut demander si le résultat précédent est toujours valable pour des ambits dénombrables arbitraires. Même si, comme on le remarque avec un exemple dans Ch. 2, §3.3, un ambit dénombrable d'un groupe pro-oligomorphe n'est pas forcément Hilbert-représentable.

Avec les résultats des Chapitres 1 et 2, nous accomplissons notre but déclaré au début de cette section, qui était d'étendre le dictionnaire de la section précédente avec une colonne pour la théorie des modèles. Par simplicité, nous le présentons ici sous la forme suivante : nous relions les propriétés de la structure $\aleph_0$-catégorique M avec les classes $\mathcal{K}$ d'espaces de Banach pour lesquelles la compactification de Roelcke $R(G)$ (du groupe d'automorphismes de M) est $\mathcal{K}$-représentable. La deuxième ligne (pour Hilbert) est valable dans le cadre classique.[5]

Classe $\mathcal{K}$	**Structure M**
Euclidien	compact
Hilbert	$\aleph_0$-stable
Reflexif	stable
Asplund	stable
Rosenthal	NIP

Structures randomisées

Dans plusieurs situations mathématiques, lorsque l'on considère un ensemble ou bien une structure M d'un certain genre, il est intéressant de réfléchir aux éléments *aléatoires* de M, et d'étudier leurs propriétés *attendues*. Dans [**Kei99**], Jerome Keisler a proposé une approche modèle-théorique de ce thème : si l'on commence avec une structure logique, alors les éléments aléatoires devraient former une nouvelle structure, dans laquelle les prédicats originaux $\varphi(x)$ seraient remplacés par des nouvelles formules $\mathbb{E}[\![\varphi(x)]\!]$ interprétées comme leurs espérances. En fait, il y a différentes façons possibles de randomiser une structure, or, comme Keisler l'a montré, elles partagent toutes la même théorie de premier ordre. Ainsi, étant

[5] Dans un travail en cours, non inclus dans ce manuscrit, nous avons trouvé des indices qui suggèrent que ceci n'est pas valable dans le cadre métrique.

donnée une théorie complète T, on obtient de façon canonique une *théorie randomisée* T^R.

Initialement, cela a été fait dans le cadre de la logique du premier ordre classique, ce qui rendait la construction plutôt encombrante. Après le développement de la logique continue, la théorie des randomisations a été adaptée à ce formalisme par Ben Yaacov et Keisler. En effet, la logique continue offre un cadre parfait pour l'étude des randomisations et, réciproquement, les structures randomisées sont une source intéressante de structures métriques.

Avec le formalisme de la logique continue, les randomisations préservent beaucoup de propriétés modèle-théoriques, dont la $\aleph_0$-catégoricité. Ainsi, dans le cas particulier où M est une structure $\aleph_0$-catégorique, il n'y a essentiellement qu'une seule randomisation séparable de M, disons M^R. Cela veut dire que, étant donné un groupe polonais Roelcke précompact G (disons, le groupe d'automorphismes de M), il y a un groupe polonais G^R qui lui est canoniquement associé (le groupe d'automorphismes de M^R), et qui est lui aussi Roelcke précompact. Donc, quel est ce groupe ? Le travail du Chapitre 3 était motivé par cette question.

Soit Ω un espace de probabilité standard. Étant donné un groupe polonais G quelconque, nous considérons le produit semi-direct de $L^0(\Omega, G)$, le groupe des éléments aléatoires de G, et $\mathrm{Aut}(\Omega)$, le groupe des transformations préservant la mesure de Ω, qui agit naturellement sur le premier. Nous l'appelons le *produit en couronne mesurable* de G et $\mathrm{Aut}(\Omega)$, et on le dénote par

$$G \wr \Omega := L^0(\Omega, G) \rtimes \mathrm{Aut}(\Omega).$$

Le groupe $G \wr \Omega$ est naturellement un groupe polonais.

Nous montrons ce qui suit, où M^R désigne la *randomisation borélienne* d'une structure séparable M. Ceci est un exemple canonique de randomisation ; essentiellement, M^R est donnée par l'ensemble $L^0(\Omega, M)$ avec la structure appropriée.

Theorem (Ch. 3, §2.2). *On a* $\mathrm{Aut}(M^R) \simeq \mathrm{Aut}(M) \wr \Omega$ *en tant que groupes topologiques.*

Au-delà de la motivation modèle-théorique, les produits en couronne mesurables sont une source intéressante de nouveaux groupes polonais. Dans le Ch. 3, §2-3, nous étudions les propriétés des $G \wr \Omega$-actions induites par des actions de G. Nous montrons le résultat suivant, dont la première partie donne une preuve alternative de la préservation de l'$\aleph_0$-catégoricité par des randomisations.

Théorème (Ch. 3, §2-3). *Si un groupe polonais G agit fidèlement et de façon approximativement oligomorphe sur un espace métrique polonais M, alors $G \wr \Omega$ agit fidèlement et de façon approximativement oligomorphe sur $L^0(\Omega, M)$. En particulier, si G est Roelcke précompact, alors $G \wr \Omega$ l'est aussi.*

Dans ce cas, la compactification de Roelcke $R(G \wr \Omega)$ peut être identifiée avec l'espace

$$\mathcal{M}(\Omega, R(G)) := \{\lambda \in \mathfrak{R}(\Omega_0 \times R(G) \times \Omega_1) : \lambda|_{\Omega_0} = \mu_0,\ \lambda|_{\Omega_1} = \mu_0\}.$$

(Ici, $\mathfrak{R}(X)$ est l'espace compact des mesures de probabilité boréliennes sur X ; chaque Ω_i est un espace de probabilité standard de mesure μ_i, et $\lambda|_{\Omega_i}$ est la mesure image de λ sur Ω_i.)

Ensuite, nous étudions quelques propriétés de préservation des espaces $\mathcal{M}(\Omega, X)$ introduits ci-dessus. Si X est un G-ambit, alors $\mathcal{M}(\Omega, X)$ est naturellement un $G \wr \Omega$-ambit. De plus, cette construction se comporte bien par rapport aux semi-groupes semi-topologiques.

Proposition (Ch. 3, §3). *Si S est un semi-groupe semi-topologique compact métrisable, alors $\mathcal{M}(\Omega, S)$ admet une loi de semi-groupe semi-topologique naturelle. Si S est un G-ambit, alors la loi de $\mathcal{M}(\Omega, S)$ étend l'action naturelle de $G \wr \Omega$ sur $\mathcal{M}(\Omega, S)$.*

Si S est représentable par des contractions sur un espace d'Hilbert, alors $\mathcal{M}(\Omega, S)$ l'est aussi.

Supposons que G est un groupe Roelcke précompact polonais. Il suit du théorème et de la proposition précédents que si G est un groupe WAP (c-à-d, si $W(G) = R(G)$), alors $G \wr \Omega$ l'est aussi. De même, si G satisfait $H(G) = R(G)$, alors on a aussi $H(G \wr \Omega) = R(G \wr \Omega)$.[6]

On sait, par le dictionnaire discuté dans la section précédente, que la préservation des groupes WAP a une contrepartie modèle-théorique. À savoir : la stabilité est préservée par des randomisations. Cela avait été prouvé en toute généralité dans [**Ben13b**]. Un autre résultat important du même genre, la préservation des théories (et des formules) NIP par des randomisations, avait été établie dans [**Ben09**]. Puisque la preuve du dernier est particulièrement compliquée, il nous a paru intéressant de chercher aussi une preuve alternative de ce fait, même limitée au cas $\aleph_0$-catégorique.

Nous avons fini par prouver un résultat de préservation général pour des représentations de Banach de certains flots randomisés. Pour cela, on définit qu'une classe $\mathcal{K}$ d'espaces de Banach est *R-fermée* si l'espace de Bochner $L^2(\Omega, V)$ est dans $\mathcal{K}$ dès que V est dans $\mathcal{K}$ (plus une condition technique sur les fonctions $\mathcal{K}$-représentables, voir Ch. 3, Definition 3.15). Utilisant des résultats classiques de la théorie des espaces de Banach (et des résultats récents de Glasner et Megrelishvili, pour la condition technique), on voit que les classes des espaces d'Hilbert, reflexifs, Asplund et Rosenthal sont toutes R-fermées.

Soit $\mathcal{S}(\Omega, X)$ l'espace compact $\{\lambda \in \mathfrak{R}(\Omega \times X) : \lambda|_\Omega = \mu\}$. Nous montrons que ceci correspond à des espaces de types à paramètres : par exemple, $S_x^{T^R}(M^R) \simeq \mathcal{S}(\Omega, S_x^T(M))$. Puis, le résultat de préservation est le suivant.

Théorème (Ch. 3, §3.3). *Soit $\mathcal{K}$ une classe R-fermée d'espaces de Banach. Soit $G \curvearrowright X$ une action continue d'un groupe polonais sur un espace compact métrisable, et supposons que $G \curvearrowright X$ est $\mathcal{K}$-représentable. Alors, le système $G \wr \Omega \curvearrowright \mathcal{S}(\Omega, X)$ est aussi $\mathcal{K}$-représentable.*

Combiné avec les résultats du Chapitre 1, cela donne des nouvelles preuves, dans le cas $\aleph_0$-catégorique, des faits suivants.

Corollaire. *Si $\varphi(x, y)$ est une formule NIP pour T, alors $\mathbb{E}[\![\varphi(x, y)]\!]$ est une formule NIP pour T^R. Ceci reste vrai si l'on remplace NIP par stable.*

[6] Il est naturel de conjecturer que $W(G \wr \Omega) \simeq \mathcal{M}(\Omega, W(G))$ et que $H(G \wr \Omega) \simeq \mathcal{M}(\Omega, H(G))$, mais nous n'avons pas tranché cette question.

D'autre part, notre dictionnaire pour l'Hilbert-représentabilité ne marche, pour l'instant, que pour la logique classique. Ainsi, par exemple, on peut déduire de nos résultats que si M est classique, $\aleph_0$-catégorique, $\aleph_0$-stable, alors $R(\mathrm{Aut}(M^R))$ est Hilbert-représentable ; mais nous ne savons pas ce que cela dit, en termes modèle-théoriques, sur M^R. Puisque les résultats du Chapitre 2 étaient basés sur la notion de théorie mono-basée, nous devrions chercher une généralisation métrique de cette propriété. Dans la littérature, il existe une telle généralisation proposée. À savoir, la notion des théories *fortement finiment basées (SFB)*, introduite dans [**BBH14**]. Une théorie stable, $\aleph_0$-catégorique T est SFB si et seulement si la théorie T_P des belles paires de modèles de T est $\aleph_0$-catégorique. Cela coïncide avec les théories mono-basées dans le cadre classique, mais ça s'applique aussi à quelques théories métriques importantes, comme celle des espaces d'Hilbert infini-dimensionnels et celle de l'algèbre de mesure de l'intervalle.

Ainsi, on pourrait s'attendre à ce que la randomisation d'une théorie SFB soit à nouveau SFB. À notre surprise, il s'avère que ceci n'est pas le cas. Plutôt, le contraire est vrai.

Théorème (Ch. 3, §4). *La théorie $(T^R)_P$ de belles paires de modèles d'une théorie randomisée stable T^R n'est jamais $\aleph_0$-catégorique, à moins que T ne soit la théorie d'une structure compacte.*

En fait, nous montrons que si $N \prec M$ est une paire élémentaire de modèles séparables d'une théorie stable T, alors on peut construire un modèle séparable de $(T^R)_P$ comme suit. Soit M^R la randomisation donnée par l'ensemble de toutes les variables aléatoires $\Omega^2 \to M$, et soit $S \subset M^R$ le sous-ensemble des variables aléatoires prenant des valeurs dans N qui sont mesurables par rapport à la première coordonnée de Ω^2 (c-à-d, celles qui se factorisent de la forme $\Omega^2 \to \Omega \to N \to M$). Alors, avec la structure appropriée, la paire (M^R, S) est un modèle de $(T^R)_P$. Cela donne des modèles séparables non-isomorphes, par exemple pour les cas $N = M$ versus $N \subsetneq M$.

De plus, on a le résultat suivant. Soit S_1 l'ensemble correspondant au cas $N = M$ comme ci-dessus (et supposons que M est $\aleph_0$-catégorique). Si l'on prend $h \in \mathrm{End}(M)^{\Omega^2}$, on peut définir $S_h = \{hr : r \in S_1\}$. Alors, la paire $\mathbb{P}_h := (M^R, S_h)$ est aussi un modèle de $(T^R)_P$.

Théorème (Ch. 3, §4). *Supposons que T est stable, $\aleph_0$-catégorique. Alors, les modèles séparables de $(T^R)_P$ sont précisément les paires isomorphes à $\mathbb{P}_h$ pour quelque $h \in \mathrm{End}(M)^{\Omega^2}$.*

En plus des résultats précédents, nous prouvons que la randomisation d'une théorie métrique $\aleph_0$-stable est à nouveau $\aleph_0$-stable (cela avait été prouvé pour T classique dans [**BK09**]). Finalement, le Chapitre 3 termine avec une description des groupes d'automorphismes de paires de randomisations de la forme (M^R, S) induites par des modèles M, N comme ci-dessus.

Aspects descriptifs des groupes pleins

Nous passons maintenant aux sujets de la deuxième partie de la thèse. Les résultats mentionnés dans cette section et les suivantes ont été obtenus avec Julien Melleray.

Nous oublierons les groupes d'automorphismes de structures logiques pour un court moment —mais seulement avec le propos de convaincre le lecteur d'y revenir.

Les groupes pleins sont un type particulier de groupes d'automorphismes qui apparaissent en théorie ergodique et dans sa plus jeune sœur, la dynamique sur l'espace de Cantor. Dans le dernier contexte, si g est un homéomorphisme minimal de l'espace de Cantor $X := 2^\omega$, on peut considérer la «structure» enrichie sur l'espace X résultant de nommer les orbites de g sur X (N.B. ce n'est pas une structure au sens de la logique), puis considérer le groupe d'automorphismes correspondant. Ceci est appelé le *groupe plein* de g, dénoté par $[g]$, et consiste donc de tous les homéomorphismes $h \in \mathrm{Homeo}(X)$ tels que $h(O_g(x)) = O_g(x)$ pour tout $x \in X$, où $O_g(x)$ est la g-orbite de x.

Les groupes pleins de transformations préservant la mesure d'un espace de probabilité standard Ω sont définis de façon similaire (et aussi les groupes pleins d'actions de groupes dénombrables, ce qui est plus intéressant dans ce cas).

Un fait crucial sur les groupes pleins est qu'ils sont des invariants algébriques complets pour l'équivalence orbitale. C'est-à-dire, si g et h sont des homéomorphismes minimaux de X tels que $[g] \simeq [h]$ en tant que groupes abstraits, alors il y a un homéomorphisme de X tel que des éléments dans la même g-orbite sont envoyés sur des éléments dans la même h-orbite, et des éléments dans des g-orbites distinctes sont envoyés sur des éléments dans des h-orbites distinctes. Cela a été prouvé par Giordano, Putnam et Skau [**GPS99**]. Le théorème analogue dans le cadre de la théorie ergodique avait été établi plusieurs décennies plus tôt par Dye, et c'était à l'origine d'une étude intensive des groupes pleins.

Or, une caractéristique précieuse des groupes pleins dans le cadre de la *théorie ergodique* est qu'ils ont un bon comportement du point de vue topologique. D'abord, ils admettent une topologie de groupe polonaise. (C'est ne pourtant pas la topologie induite par le groupe polonais ambiant $\mathrm{Aut}(\Omega)$; plutôt, les groupes pleins deviennent des groupes polonais sous la topologie uniforme de $\mathrm{Aut}(\Omega)$.) Cette topologie joue un rôle remarquable dans plusieurs résultats de la littérature, par exemple lors des caractérisations de propriétés invariantes par l'équivalence orbitale. Deuxièmement, les groupes pleins sont toujours des sous-ensembles boréliens de $\mathrm{Aut}(\Omega)$ de complexité relativement basse : ils sont $\mathbf{\Pi}^0_3$ dans la hiérarchie borélienne.

En revanche, aucune topologie polonaise n'a jamais été proposée pour les groupes pleins d'homéomorphismes minimaux de l'espace de Cantor. Le travail du Chapitre 4 s'occupe de cette situation, et son premier résultat est le suivant.

Théorème (Ch. 4, §2). *Soit $\Gamma \subset \mathrm{Homeo}(X)$ un groupe dénombrable agissant de façon minimale sur l'espace de Cantor. Alors, il n'y a pas de topologie de groupe Hausdorff, Baire, à base dénombrable sur le groupe plein* $[\Gamma]$.

En particulier, le groupe plein $[g]$ d'un homéomorphisme minimal n'admet pas de topologie de groupe polonaise.

Notons que, contrairement à la situation dans le cadre mesurable, ici on peut fixer un point $x \in X$ et regarder les permutations de l'orbite $O_g(x)$ induites par des éléments de $[g]$; puisque $O_g(x)$ est dénombrable, cela donne un homomorphisme naturel de $[g]$ dans le groupe de permutations d'un ensemble dénombrable. La preuve du théorème précédent montre qu'une topologie Hausdorff, Baire sur $[g]$ rendrait cet homomorphisme continu. En fait, la topologie devrait raffiner la topologie de la convergence ponctuelle sur X, ce qui rendrait impossible pour $[g]$ d'avoir une base dénombrable d'ouverts.

Ensuite, nous avons étudié la question de la complexité de $[g]$ dedans $\mathrm{Homeo}(X)$. Ici encore, la réponse s'avère être la pire possible.

Théorème (Ch. 4, §3). *Soit g un homéomorphisme minimal de l'espace de Cantor. Alors, $[g]$ est un sous-ensemble coanalytique non-borélien de* $\mathrm{Homeo}(X)$.

Ces résultats suggèrent que les groupes pleins dans le cadre de la dynamique sur l'espace de Cantor risquent de ne pas être aussi utiles, en tant qu'invariant, que leurs contreparties ergodiques ont démontré l'être. Ainsi, on pourrait essayer d'étudier à leur place un objet relié qui se comporte mieux. Un candidat naturel est la clôture du groupe plein dedans $\mathrm{Homeo}(X)$.[7] Étant donné un homéomorphisme minimal g, dénotons par $G_g = \overline{[g]}$ sa clôture.

Le groupe G_g présente deux caractéristiques importantes, en plus de sa topologie de groupe polonaise évidente. La première est qu'il est aussi un invariant algébrique complet pour l'équivalence orbitale, comme on le remarque dans le Ch. 4, §4 ; cela suit d'un autre résultat profond de Giordano, Putnam et Skau.

La deuxième est que G_g admet une bonne description. En effet, il suit d'un théorème de Glasner et Weiss que G_g est précisément l'ensemble des $h \in \mathrm{Homeo}(X)$ qui préservent les mesures de probabilité g-invariantes sur X.

On rappelle que $\mathrm{Homeo}(X)$ peut être identifié avec le groupe d'automorphismes de l'algèbre de Boole des sous-ensembles ouvert-fermés de X. Puisque G_g est un sous-groupe fermé du groupe d'homéomorphismes de l'espace de Cantor, il doit être le groupe d'automorphismes d'une *expansion* de l'algèbre des ensembles ouvert-fermés de X. Comme nous le verrons dans la section suivante, la description de G_g mentionnée plus haut nous permettra de donner une bonne présentation de cette structure.

Une question intéressante que l'on peut se poser sur G_g est celle de savoir s'il admet des éléments *génériques*. Un élément $h \in G_g$ est générique si sa classe de conjugaison $\{khk^{-1} : k \in G_g\}$ est comaigre dans G_g. L'existence d'un élément générique a des fortes conséquences pour la clôture d'un groupe plein. Par exemple, elle implique que G_g doit être simple, et que tout homomorphisme dans un groupe

[7] Un autre candidat naturel serait le *normalisateur* du groupe plein, $N[g]$, qui est l'ensemble de tous les $h \in \mathrm{Homeo}(X)$ tels que $h(O_g(x)) = O_g(h(x))$ pour tout $x \in X$ (c-à-d, $N[g]$ est le groupe d'automorphismes de la relation d'équivalence induite par les orbites de g). On peut voir que $N[g]$ est aussi un invariant complet pour l'équivalence orbitale. Cependant, comme l'a observé Julien Melleray lors d'une communication privée, utilisant que $[g]$ est non-borélien dans $\mathrm{Homeo}(X)$, on peut voir que $N[g]$ n'est pas borélien non plus et qu'il n'admet pas de topologie de groupe polonaise.

topologique séparable est automatiquement continu. Cette question peut être approchée au moyen des méthodes expliquées ci-dessous.

Théorie de Fraïssé pour les bonnes mesures

Un point de convergence majeur de la dynamique, la logique et la combinatoire fut inauguré par Kecrhis, Pestov et Todorcevic dans leur célèbre papier [**KPT05**]. Dans ce travail, ils étaient concernés par l'étude des flots minimaux universels de groupes polonais, ce dont on ne s'occupe pas dans cette thèse ; pourtant, leur outil principal, l'étude des groupes polonais au moyen de la théorie de Fraïssé, a été adapté depuis, de nombreuses façons, à l'étude d'autres problèmes avec ou sans rapport avec le leur. Ici, nous serons concernés par le travail de Kechris et Rosendal [**KR07**] (qui a été aussi inspiré par des travaux précédents de Hodges *et al.* [**HHLS93**] et de Truss [**Tru92**]). Ils ont étudié le problème de déterminer sous quelles conditions un groupe polonais admet une classe de conjugaison dense ou comaigre.

Dorénavant, toutes les structures que nous considérons le sont au sens de la logique classique. Une structure $\mathbb{K}$ est *ultrahomogène* si tout isomorphisme entre des sous-structures finiment engendrées de $\mathbb{K}$ s'étend à un automorphisme de $\mathbb{K}$. Fraïssé a montré qu'une structure dénombrable ultrahomogène $\mathcal{K}$ est déterminée, à isomorphisme près, par son *âge*, qui est la classe des structures *finiment engendrées* qui peuvent être plongées dans $\mathbb{K}$. Ainsi, l'âge code toute l'information modèle-théorique de $\mathbb{K}$. Les idées de [**KPT05**] ou [**KR07**] sont basées sur le principe que certaines propriétés combinatoires de l'âge de $\mathbb{K}$ codent des propriétés dynamiques du groupe d'automorphismes $\mathrm{Aut}(\mathbb{K})$.

Soit $\mathbb{K}$ une structure ultrahomogène dénombrable et soit $\mathcal{A}$ son âge. Alors, on peut considérer la classe $\mathcal{A}_p$ des paires (A, f) où $A \in \mathcal{A}$ et f est un automorphisme partiel de A. Kechris et Rosendal ont prouvé que $\mathrm{Aut}(\mathbb{K})$ a une classe de conjugaison dense si et seulement si $\mathcal{A}_p$ satisfait la *propriété de plongement joint (JEP)*, c-à-d, si n'importe quels deux automorphismes partiels de structures dans $\mathcal{A}$ peuvent être fusionnés dans un automorphisme partiel d'une super-structure commune dans $\mathcal{A}$. De plus, $\mathrm{Aut}(\mathbb{K})$ a une classe de conjugaison comaigre si et seulement si $\mathcal{A}_p$ satisfait JEP ainsi qu'une autre condition plus sophistiquée, appelée la *propriété d'amalgamation faible (WAP)*[8]. Par exemple, ils ont réussi à montrer que JEP et WAP sont vérifiées pour la classe des algèbres de Boole finies, qui est l'âge de l'algèbre des ensembles ouvert-fermés de l'espace de Cantor. Par conséquent, ils ont établi que l'espace de Cantor a un homéomorphisme générique.

De plus, ils ont étudié une autre propriété combinatoire forte et importante dont quelques classes de structures finiment engendrées bénéficient, appelée la *propriété de Hrushovski*, qui est liée à la possibilité d'étendre des automorphismes partiels à des (vrais) automorphismes de structures plus larges.

Nous expliquons le rapport de tout cela avec notre sujet. Comme auparavant, soit X l'espace de Cantor, et soit g un homéomorphisme minimal de X. On dénote l'ensemble de mesures de probabilité g-invariantes sur X par K_g.

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[8]N.B. sans aucun rapport avec la faible presque périodicité !

Il suit d'un résultat de Glasner et Weiss que la minimalité de g implique la propriété de K_g suivante : *si $A, B \subset X$ sont des ensemble ouvert-fermés tels que $\mu(A) < \mu(B)$ pour toute $\mu \in K_g$, alors il existe un ensemble ouvert-fermé $C \subset B$ tel que $\mu(C) = \mu(A)$ pour toute $\mu \in K_g$*. Lorsqu'un ensemble K de mesures de probabilité sans atomes à support plein sur X bénéficie de cette propriété, on dit que K es *bon* (généralisant la définition de [**Aki05**]). Un argument de va-et-vient standard montre la chose suivante.

Lemme. *Supposons que K est un bon simplexe de mesures de probabilité à support plein sur X. Soit $G_K = \{h \in \mathrm{Homeo}(X) \colon \forall \mu \in K\ h_* \mu = \mu\}$. Si A, B sont des ensembles ouvert-fermés avec $\mu(A) = \mu(B)$ pour toute $\mu \in K$, alors il existe $h \in G_K$ tel que $h(A) = B$.*

Ceci dit qu'une certaine structure est ultrahomogène. En effet, soit K comme dans l'énoncé précédent, puis soit $\mathbb{K}$ l'algèbre de Boole des ensembles ouvert-fermés de X augmentée avec des prédicats $P_{\mu,r}$ pour chaque $\mu \in K$ et $r \in [0, 1]$, interprétés de manière à ce qu'un ensemble A vérifie $P_{\mu,r}$ si et seulement si $\mu(A) = r$. (En fait, par la séparabilité de l'espace de mesures de probabilité, on peut supposer que le langage est dénombrable.) Alors on a $G_K = \mathrm{Aut}(\mathbb{K})$, et le lemme implique facilement que $\mathbb{K}$ est ultrahomogène.

En particulier, si l'on dénote par $\mathbb{K}_g$ la structure construite à partir de $K = K_g$, on obtient que la clôture du groupe plein $[g]$ est $G_g = \mathrm{Aut}(\mathbb{K}_g)$. Ainsi, on peut appliquer la théorie de [**KR07**] à l'étude de G_g. Dans cette généralité, nous pouvons montrer ce qui suit (grâce à une observation de Kostya Medynets ; notre résultat original était plus faible).

Théorème (Ch. 4, §4). *Soit g un homéomorphisme minimal de l'espace de Cantor, et soit $\mathbb{K}_g$ comme ci-dessus. Alors, l'âge de $\mathbb{K}_g$ a la propriété de Hrushovski.*

Cela a d'abord la conséquence suivante.

Corollaire. *Le groupe G_g est moyennable.*

De plus, combiné avec un résultat de Bezuglyi et Medynets [**BM08**], le théorème implique aussi ce qui suit.

Corollaire. *Le groupe G_g est topologiquement simple.*

Ensuite, nous avons étudié les conditions JEP et WAP dans un cas particulier, à savoir, lorsque g est uniquement ergodique. Fixons donc un homéomorphisme minimal g, et supposons qu'il n'y a qu'une seule mesure g-invariante, μ, qui est ainsi une *bonne mesure* au sens de [**Aki05**]. Soit V_μ l'ensemble (dénombrable) des valeurs $\mu(A) \in [0, 1]$ que μ prend sur des ensembles ouvert-fermés $A \subset X$. Alors, l'âge de $\mathbb{K}_g$ ne dépend que de V_μ, et donc cet ensemble code toute l'information importante de $\mathbb{K}_g$ et G_g. En étudiant la propriété de plongement joint en termes de V_μ, nous obtenons la caractérisation technique suivante.

Proposition (Ch. 4, §5). *Il existe une classe de conjugaison dense dans G_g si et seulement si V_μ satisfait la condition suivante : à chaque fois que l'on a $a_i, b_j \in V_\mu$ et $n_i, m_j \in \mathbb{N}$ tels que $\sum_{i=1}^p n_i a_i = 1 = \sum_{j=1}^q m_j b_j$, on peut trouver $c_{i,j} \in V_\mu$ tels que : $m_j b_j = \sum_{i=1}^p \mathrm{ppcm}(n_i, m_j) c_{i,j}$ pour tout j, et $n_i a_i = \sum_{j=1}^q \mathrm{ppcm}(n_i, m_j) c_{i,j}$ pour tout i.*

En particulier, cela est vrai lorsque $V_\mu + \mathbb{Z}$ est un $\mathbb{Q}$-sous-espace vectoriel de $\mathbb{R}$, et lorsque $V_\mu + \mathbb{Z}$ est un sous-anneau de $\mathbb{R}$.

Malheureusement, nous n'avons pas réussi à donner une caractérisation du même genre pour l'existence d'une classe de conjugaison comaigre dans G_g. Contrairement à ce qui se passe dans de nombreux exemples de [**KR07**], où la propriété de Hrushovski peut être utilisée pour prouver une forme forte de la propriété d'amalgamation faible, cela semble ne pas suffire dans notre cas. Néanmoins, cette méthode marche tout à fait pour une famille particulière de bonnes mesures, ce qui nous permet de donner une nouvelle preuve du résultat d'Ethan Akin suivant [**Aki05**].

Proposition (Ch. 4, §5). *Supposons que $V_\mu + \mathbb{Z}$ est un $\mathbb{Q}$-sous-espace vetoriel de $\mathbb{R}$. Alors, il existe un homéomorphisme générique dans G_g.*

Lors des résultats précédents, on a supposé que l'homéomorphisme minimal uniquement ergodique g était donné. Or, si V est n'importe quel sous-ensemble infini dénombrable de $[0,1]$ contenant 0 et 1 et clos par des additions modulo 1, alors on peut montrer qu'il existe une bonne mesure μ sur X telle que $V = V_\mu$. De plus, Akin a prouvé qu'alors il existe un homéomorphisme g tel que $K_g = \{\mu\}$. Par conséquent, par exemple, on peut utiliser la caractérisation donnée plus haut pour montrer qu'il y a un homéomorphisme minimal g pour lequel G_g n'admet pas de classe de conjugaison dense.

Simplexes dynamiques

Comme nous venons de le mentionner, étant donnée une bonne mesure μ sur l'espace de Cantor, Akin [**Aki05**] a prouvé l'existence d'un homéomorphisme minimal dont μ est la seule mesure de probabilité invariante. Ainsi, les bonnes mesures sont précisément les mesures invariantes des homéomorphismes minimaux uniquement ergodiques de l'espace de Cantor. Existe-il une caractérisation similaire pour les ensembles de mesures de probabilité invariantes des homéomorphismes minimaux arbitraires de l'espace de Cantor ? Nous nous occupons de ce problème dans le travail du Chapitre 5.

Une réponse à cette question avait été donnée dans un travail non publié de Heidi Dahl [**Dah08**]. Soit K un ensemble de mesures de probabilité de l'espace de Cantor X. Dahl a montré qu'il y a un homéomorphisme minimal g de X tel que $K = K_g$ si, et seulement si, K est un simplexe de Choquet de mesures sans atomes à support plein vérifiant les propriétés suivantes : les points extrémaux de K sont mutuellement singuliers, K est bon, et les fonctions de la forme $\mu \mapsto \int f \, d\mu$ pour $f \in \mathrm{C}(X, \mathbb{Z})$ sont denses dans les fonctions affines continues sur K. Sa preuve est basée sur les complexes et puissantes méthodes de Giordano, Herman, Putnam et Skau, qui dépendent des outils de la K-théorie, la théorie des groupes à dimension et des diagrammes de Bratelli–Vershik.

Dans le Chapitre 5 nous adoptons une approche plus élémentaire, puis nous donnons une caractérisation des simplexes K_g pour g minimal qui reste plus proche dans l'esprit de la condition d'Akin pour le cas uniquement ergodique. Plus précisément, nous proposons la définition suivante. Soit K un ensemble compact, convexe, non vide, de mesures de probabilité sans atomes à support plein sur X. Nous disons que K est un *simplexe dynamique* si, en plus, K est bon (tel que défini dans la section précédente) et *approximativement divisible*, c'est-à-dire, pour tous $\epsilon > 0$, $n \in \mathbb{N}^*$ et tout ensemble ouvert-fermé $A \subset X$ il existe un ensemble ouvert-fermé $B \subset A$ tel que, pour tout $\mu \in K$, $\mu(A) - \epsilon \leq n\mu(B) \leq \mu(A)$.

Ensuite, commençant avec un simplexe dynamique K, nous prouvons l'existence d'un homéomorphisme minimal g tel que $K = K_g$ en construisant, par découpage et empilement, une suite de partitions ouvert-fermées de X visant à constituer des partitions de Kakutani–Rokhlin de g.

Théorème (Ch. 5). *Un ensemble K de mesures de probabilité sur X est un simplexe dynamique si et seulement si $K = K_g$ pour un homéomorphisme minimal g de X.*

De plus, l'homéomorphisme g que nous obtenons est *saturé*, ce qui veut dire que le *groupe plein topologique* $[[g]]$ est dense dans G_g. Nous rappelons que $[[g]]$ consiste des $h \in [g]$ pour lesquels il y a une partition ouvert-fermée $A_1, \dots, A_k$ de X et des entiers $n_1, \dots, n_k$ tels que $h|_{A_i} = g^{n_i}|_{A_i}$ pour chaque $i = 1, \dots, k$. Ainsi, notre construction donne une preuve élémentaire de la chose suivante.

Corollaire. *Si g est un homéomorphisme minimal de X, alors il existe un homéomorphisme minimal saturé g' avec $G_g = G_{g'}$.*

Cela était connu antérieurement par des moyens plus sophistiqués. Nous avions utilisé ce résultat lors de la preuve de la propriété de Hrushovski énoncée dans la section précédente.

On peut se demander si l'hypothèse de divisibilité approximée est vraiment nécessaire, ou bien, de façon équivalente, s'il y a des bons simplexes qui ne sont pas des simplexes dynamiques. Nous ne connaissons pas la réponse en général. Cependant, la divisibilité approximée est en effet redondante lorsque l'on considère des simplexes fini-dimensionnels.

Corollaire. *Soit K un bon simplexe de mesures de probabilité sans atomes à support plein sur X. Si K n'a qu'un nombre fini de points extrémaux, alors il existe un homéomorphisme minimal g de X tel que $K = K_g$.*

Ce corollaire avait déjà été obtenu par Dahl, sous l'hypothèse supplémentaire que les points extrémaux de K sont mutuellement singuliers.

Introduction

Foreword

This thesis in applied logic and descriptive set theory contributes to the theory of topological groups and their dynamical systems, and to its connections with model theory. The works included in this manuscript are the result of two different research projects, corresponding to the two parts in which the thesis is divided, as explained below. The topics covered are thus diverse, but fit together within a common, general frame: the study of automorphism groups of homogeneous structures.

The subject of the first part of the dissertation, realized under the supervision of Itaï Ben Yaacov, is the study of Roelcke precompact Polish groups via the $\aleph_0$-categorical structures associated to them, and conversely. It consists of the following research papers:

- Chapter 1: [**Iba14**] *The dynamical hierarchy for Roelcke precompact Polish groups*, to appear in Israel Journal of Mathematics.
- Chapter 2: [**BIT15**] *Eberlein oligomorphic groups* (joint work with **Itaï Ben Yaacov** and **Todor Tsankov**), submitted for publication.
- Chapter 3: [**Iba16**] *Automorphism groups of randomized structures*, submitted for publication.

The second part corresponds to a collaboration with **Julien Melleray**, about minimal homeomorphisms of the Cantor space and their full groups. The results of this collaboration have been collected in the following works:

- Chapter 4: [**IM14**] *Full groups of minimal homeomorphisms and Baire category methods*, Ergodic Theory Dynam. Systems, vol. 36 (2016), no. 2, pp. 550-573.
- Chapter 5: [**IM15**] *Dynamical simplices and minimal homeomorphisms*, submitted for publication.

In the following sections, we give an introduction to the subjects of the thesis along with a presentation of its main results.

Groups and structures

The meeting point of logic and dynamics considered in this thesis lies in the study of automorphism groups of structures. Many interesting examples of topological groups are naturally presented as $G = \mathrm{Aut}(M)$ where M is a *classical* (discrete) structure and the topology on G is that of pointwise convergence over M. In fact, if one is willing to drop the word *naturally*, the latter situation encompasses a very large class of groups: every *closed permutation group* (i.e., every closed subgroup of the group S_∞ of bijections of a countable set X) is the automorphism group of a structure in the sense of classical logic, namely, the structure obtained by adding predicates for the orbits of G in all finite powers of X. We ask X to be countable since we are only interested in second-countable topological groups.

As pointed out by Julien Melleray [**Mel10**], the situation extends to the wider class of all *Polish groups*, by passing from classical to *continuous* logic, as developped in [**BU10, BBHU08**]. We recall that a topological space is Polish if it is separable and admits a complete, compatible metric. In continuous logic, structures are complete, bounded, metric spaces, and the predicates are uniformly continuous, bounded, real-valued functions. The metric takes the place of the equality relation: in particular, all automorphisms are isometries. The automorphism group G of a separable structure M, endowed with the topology of pointwise convergence, is thus a Polish subgroup of the group $\mathrm{Iso}(M)$ of isometries of M. Conversely, any Polish group can be obtained in this fashion. Indeed, by a classical result of Birkhoff–Kakutani, any Polish group G admits a left-invariant compatible metric d_L. Then it suffices to consider $M = \widehat{G}_L$, the completion of G with respect to d_L, and add predicates for the distance functions to each orbit in every finite power of M. Hence G acts on M by isometries and, moreover, $G = \mathrm{Aut}(M)$. The construction is described more explicitly in Chapter 1, §1.3.

In the naturally presented examples, it is not surprising that the model-theoretic features of a structure M give important information about its automorphism group G, as would do the particular features of any interesting action of G. On the other hand, the *ad hoc* structures M described above could hardly tell anything new about a given group G (rather, they may serve as peculiar model-theoretic examples). However, they allow for a transfer of techniques from the logical side to the dynamical side (as shown already in [**Mel10**]), and even suggest that some general theory should also go through. As it turns out, this is especially the case within a certain family of Polish groups, which we shall now introduce. The first part of this thesis deals with the elaboration of a precise dictionary between both sides of the equation $G = \mathrm{Aut}(M)$ within this particular class.

A topological group G is *Roelcke precompact* if for every open set $U \subset G$ there is a finite set $F \subset G$ such that $UFU = G$. In order to explain the name, we recall that every topological group comes endowed with four natural uniform structures. These are the *left uniformity* (where two elements $x, y \in G$ are close if the product $x^{-1}y$ belongs to a small neighborhood of the identity), the *right uniformity* (where x, y are close if xy^{-1} is close to the identity), the *upper uniformity* (the supremum of the latter two) and the *lower uniformity* (their infimum, also called the *Roelcke uniformity* by Uspenskij [**Usp02**]). The left, right and lower uniformities are always compatible with the topology of the group.

The completions of G with respect to the three compatible uniformities are very interesting objects, and a basic thing to ask is when they happen to be compact; that is, when G is *precompact* with respect to these uniformities. If G is Polish, then one can see that the left completion (which is just the space $\widehat{G}_L$) is compact if and only if G is compact (if and only if the right completion is compact), so this condition does not bring out anything new. On the other hand, the fact that the completion with respect to the lower uniformity be compact (which boils down to the condition $UFU = G$ stated above, explaining the name) turns out to define a new, very rich class of Polish groups. Outside the compact case, these are always "infinite-dimensional" non-abelian groups, in a sense orthogonal to the class of locally compact groups.

Fundamental examples of Roelcke precompact Polish groups, which will appear at one time or another along the thesis, include:

- the infinite symmetric group, S_∞;
- the group of monotone bijections of the rationals, $\mathrm{Aut}(\mathbb{Q},<)$;
- the automorphism group of the random graph, $\mathrm{Aut}(RG)$;
- the homeomorphism group of the Cantor space, $\mathrm{Homeo}(2^\omega)$;
- the unitary group, $U(\ell^2)$;
- the group of measure-preserving transformations of a Lebesgue space, $\mathrm{Aut}(\Omega)$;
- the group of quasi-invariant transformations of a Lebesgue space, $\mathrm{Aut}^*(\Omega)$;
- the isometry group of the bounded Urysohn space, $\mathrm{Iso}(\mathbb{U}_1)$;
- the group of increasing homeomorphisms of the interval, $H_+[0,1]$.

The first four examples in this list belong to a class of groups that has been studied in connection to logic already for a while. These are *oligomorphic* permutation groups. The term, intended to mean "few shapes", was introduced by Cameron [**Cam90**]: a closed permutation group $G \curvearrowright X$ is oligomorphic if, modulo G, there are only finitely many possible configurations of n-tuples of X. That is, if the orbit spaces X^n/G are finite for all n. (In the case of $\mathrm{Homeo}(2^\omega)$, the underlying countable set X is just the algebra of clopen subsets of 2^ω.) Now, if X is seen as an appropriate logic structure M in such a way that $G = \mathrm{Aut}(M)$, then a classical theorem of Ryll-Nardzewski shows that $G \curvearrowright X$ being oligomorphic is equivalent to saying that M is $\aleph_0$-*categorical* (also said *countably categorical*): any countable structure sharing the same first-order properties of M, is isomorphic to M. Thus, oligomorphic groups are precisely the automorphism groups of classical, one-sorted, countably categorical structures. In the former examples, one recognizes: the unique countable set, the unique countable dense linear order without endpoints, the unique countable homogeneous universal graph, and the unique countable atomless Boolean algebra.

Todor Tsankov [**Tsa12**] observed that every oligomorphic group is Roelcke precompact. Moreover, he showed that Roelcke precompact closed subgroups of S_∞ are precisely the inverse limits of oligomorphic groups. This is why we shall call these groups *pro-oligomorphic*. A Polish group G is pro-oligomorphic if and only if it can be presented as $G = \mathrm{Aut}(M)$ for a classical, *multi-sorted*, countably categorical structure M.

Later, Itaï Ben Yaacov and Todor Tsankov [**BT14**], and independently Christian Rosendal [**Ros13**], realized that an arbitrary Polish group is Roelcke precompact if and only if it admits an *approximately oligomorphic* faithful action. In other words, if it can be seen as the automorphism group of an $\aleph_0$-categorical structure in the sense of continuous logic (also called *separably categorical*). Basic examples of such structures (other than the discrete ones) are the separable infinite-dimensional Hilbert space, the measure algebra of an atomless Lebesgue space, the atomless L_p Banach lattices and the bounded Urysohn space of diameter 1, each one unique in its kind.

Separable categoricity is a very interesting and quite ubiquitous phenomenon (it occurs all along the *stability spectrum*), which nevertheless implies very strong properties for the structure (saturation, homogeneity, definability of partial $\emptyset$-types, for example). Now, an important common feature of $\aleph_0$-categorical structures is that they are determined, up to bi-interpretability, by their automorphism groups, as shown in [**AZ86, BK13**]. Hence one may expect that the model-theoretic properties of a separably categorical structure (at least those that are preserved by bi-interpretations) be coded by natural properties of its automorphism group. This guiding principle motivated the results of Ben Yaacov and Tsankov's work [**BT14**], in which they studied one of these fundamental properties: *stability*. The idea was further developed in the articles that form the first chapters of this thesis.

Before we can get into more details, we will review other recent ideas connecting the study of topological groups to yet a completely different domain.

To end this section, let us remark that $\aleph_0$-categorical structures are a special case of homogeneous structures, but many interesting highly symmetric structures are not $\aleph_0$-categorical. Another important connection between logic and dynamics, which is relevant for the second part of this thesis, lies in the study of homogeneous structures and their automorphism groups via Fraïssé theory. We will review this link in a subsequent section.

Dynamical systems and their representations

An old theme in topological dynamics is the study of periodicity properties of continuous functions on groups. Outside the realm of compact groups, the orbits of continuous functions $f: G \to \mathbb{R}$ can be quite complicated, and the classical notions that come into play are those of *almost periodicity* (when the orbit Gf is norm precompact in $\mathrm{C}(G)$) and *weak almost periodicity (WAP)* (when Gf is weakly precompact). In turn, these notions are associated with interesting dynamical systems that come attached to the group.

If G is a topological group, a *G-ambit* is given by a continuous action of G on a compact Hausdorff space X, together with a distinguished point $x_0 \in X$ such that the orbit Gx_0 is dense in X. (This is the same as a *compactification* $G \to X$, defined by sending 1 to x_0.) The algebra of all bounded functions $f: G \to \mathbb{R}$ that are uniformly continuous with respect to the right uniformity of G is denoted by $\mathrm{RUC}(G)$. Then, there is a one-to-one correspondence between G-ambits and closed, left-invariant subalgebras of $\mathrm{RUC}(G)$, which is moreover order-preserving: inclusions of subalgebras correspond to factor maps between ambits. Interesting subalgebras, such as those corresponding to almost periodic and weakly almost periodic functions, induce ambits with interesting algebraic structures. The *Bohr compactification*, bG (corresponding to almost periodic functions), is a compact topological group, whereas the *WAP compactification*, $W(G)$, is a semitopological semigroup. The algebra $\mathrm{RUC}(G)$ itself induces the *greatest ambit*, βG, which is a right topological semigroup. These three are universal ambits for their corresponding properties, e.g., every semitopological semigroup G-ambit is a factor of $W(G)$.

Recently, Eli Glasner and Michael Megrelishvili have proposed a completely new approach to this subject, enriching and enlarging it (a survey of which is given in [**GM14b**]). If $G \curvearrowright X$ is a compact dynamical system and V is a Banach space, they define a *representation of X on V* to be a weak*-continuous map

$$\iota \colon X \to V^*$$

together with a continuous homomorphism $\pi\colon G \to \mathrm{Iso}(V)$ such that, with respect to the induced dual action $G \curvearrowright V^*$, the map ι is G-equivariant. Then, if $\mathcal{K}$ is a class of Banach spaces, the system X is *$\mathcal{K}$-representable* if it admits sufficiently many representations on Banach spaces of the class $\mathcal{K}$. (One may expect to have an injective representation, or just representations separating points, depending on the context.)

By considering the obvious injective representation $X \to \mathrm{C}(X)^*$, one sees that any dynamical system is Banach-representable. The interesting fact is that representability on classes of well-behaved Banach spaces corresponds to good dynamical properties. Thus, for instance, by a result of Megrelishvili [**Meg03**], WAP G-ambits (i.e., the factors of $W(G)$) are precisely the *reflexive*-representable ambits. Similarly, almost periodic ambits are the *Euclidean*-representable ones.

Then, in the works [**GM06, GM12**], the authors studied the dynamical properties of systems that are representable on some natural generalizations of reflexive spaces: *Asplund* and *Rosenthal* spaces. A Banach space is Asplund if the dual of every separable subspace is separable, e.g., $c_0(\Gamma)$ spaces. It is Rosenthal if it does not contain an isomorphic copy of ℓ^1, e.g., every Asplund space is Rosenthal; the inclusion is actually strict, but it is not easy to exhibit an example. As it turns out, Asplund and Rosenthal-representability are robust notions, admitting several equivalent presentations. For instance, for *metrizable* compact systems $G \curvearrowright X$, the following table shows how the property of being $\mathcal{K}$-representable is related to the complexity of the orbits of functions $f \in \mathrm{C}(X)$. (Here, O_f denotes the pointwise closure of Gf inside $\mathbb{R}^X$, and $\mathcal{B}_1(X) \subset \mathbb{R}^X$ is the space of Baire 1 functions on X.)

Name	**Class $\mathcal{K}$**	**Complexity of orbits**
WAP	Reflexive	$O_f \subset \mathrm{C}(X)$
HNS	Asplund	O_f is metrizable
Tame	Rosenthal	$O_f \subset \mathcal{B}_1(X)$

In the opposite direction, one can consider the subclass of WAP systems that are not only reflexive but also *Hilbert*-representable. The corresponding subalgebra of $\mathrm{RUC}(G)$ is closely related to the *Fourier–Stieltjes algebra* of the group, which consists of the matrix coefficients of unitary representations of G.

Hilbert-representable systems have been investigated in [**Meg08, GW12**]. It turns out to be a much subtler class than the previous ones, and a satisfactory dynamical characterization of it is still missing. Moreover, the following basic question is open, even for $G = \mathbb{Z}$. Let us denote by $H(G)$, G^{Asp} and G^{Ros} the *largest* Hilbert, Asplund and Rosenthal-representable G-ambits, respectively. Then, it is known that the factors of G^{Asp} are precisely the Asplund-representable G-ambits, and an analogous statement holds for G^{Ros} with respect to Rosenthal systems (and for $W(G)$ with respect to reflexive systems, as said before). In contrast, it is unknown whether all factors of $H(G)$ are Hilbert-representable.

Another interesting problem is that of understanding when WAP functions can be approximated by matrix coefficients of unitary representations of G. Equivalently, when $H(G)$ equals $W(G)$. Groups with this property are called *Eberlein*.

The classification of ambits in terms of their Banach representations we have just discussed gives the following hierarchy of universal dynamical systems associated to a given group:

$$\beta G \to G^{\mathrm{Ros}} \to G^{\mathrm{Asp}} \to W(G) \to H(G) \to bG.$$

It is known, for instance, that these ambits are all distinct for the case of $G = \mathbb{Z}$.

We will study these ambits and their corresponding function algebras for the case of Polish Roelcke precompact groups. In this case, it is also important to consider the *Roelcke compactification*, $R(G)$, which corresponds to the algebra of functions that are uniformly continuous with respect to the lower uniformity. By the precompactness assumption, this is just the completion with respect to the lower uniformity. In particular, $R(G)$ is metrizable. The ambit $R(G)$ is always larger than G^{Asp}, but not always larger than G^{Ros}.

A dictionary for logicians

The following key observation was made in [**BT14**]. If G is the automorphism group of an $\aleph_0$-categorical (metric) structure M, then a function $f \in \mathrm{C}(G)$ is WAP if and only if it can be written in the form

$$(*) \qquad f(g) = \varphi(a, gb)$$

for a *stable* continuous formula $\varphi(x,y)$ and parameters a, b from M (which might be infinite tuples). The same idea shows that if one drops the condition of stability and considers arbitrary formulas φ, one recovers, via the expression (*), the family of all lower uniformly continuous functions on G (that is, right-and-left uniformly continuous). In particular, G is a *WAP group* (i.e., $W(G) = R(G)$) precisely if M is a stable structure.

This was the starting point of our work of Chapter 1, whose purpose was to complete the table of the previous section with a column for model theory. As it is, representability on Rosenthal spaces corresponds to an important generalization of stability. Let us say here that a continuous function $f\colon G \to \mathbb{R}$ is *lower tame* if it factors through G^{Ros}, and moreover f is lower uniformly continuous.

Proposition (Ch. 1, §3). *A continuous function $f\colon G \to \mathbb{R}$ is lower tame if and only if it can be written in the form $f(g) = \varphi(a, gb)$ for an NIP formula $\varphi(x,y)$.*

The correspondence between WAP functions and stable formulas is a consequence of a classical criterion of Grothendieck for weak-compactness in $\mathrm{C}(X)$ spaces (see [**Ben13a**]); in turn, the correspondence between lower tame functions and NIP formulas reposes on a famous dichotomy theorem of Bourgain, Fremlin and Talagrand [**BFT78**] for pointwise closures of subsets of $\mathrm{C}(X)$.

It follows that the notion of Asplund-representable functions should correspond to a model-theoretic property intermediate between stability and NIP, or collapse to one of these two.

Theorem (Ch. 1, §2). *If G is the automorphism group of an $\aleph_0$-categorical structure, then $W(G) = G^{\mathrm{Asp}}$.*

From a model-theoretic point of view, the proof is quite simple. The main ingredient is the standard trick of passing from a sequence indexed by $\mathbb{N}$ to a sequence indexed by $\mathbb{Q}$ preserving some required property: in this case, witnessing unstability. However, this is novel from the standpoint of topology, and we are not aware of an alternative, topologically inspired proof.

Actually, our argument shows a stronger[9] identity, namely:

$$W(G) = G^{\mathrm{SUC}},$$

where G^{SUC} is the largest compact right topological semigroup factor of the Roelcke compactification. This identity can moreover be interpreted as a rephrasing of the equivalence between stability and definability of types.

Corollary. *Let G be a Polish Roelcke precompact group. If G is not a WAP group, then $R(G)$ does not admit the structure of a right topological semigroup extending the group structure of G.*

Another consequence of the identification $W(G) = G^{\mathrm{Asp}}$ is the following. As pointed out earlier, every compact system $G \curvearrowright X$ is representable on the Banach space $\mathrm{C}(X)$. However, the space $\mathrm{C}(X)$ is usually, in most respects, as complicated as a Banach space can be. An exception to this is when X is countable (or, more generally, *scattered*): in this case, $\mathrm{C}(X)$ is actually an Asplund space.

Corollary. *If G is a Polish Roelcke precompact group, then every countable (or scattered) G-ambit is WAP.*

The results stated so far also provide a nice source of examples of dynamical systems. Any NIP, unstable, separably categorical structure, for instance, gives rise to a Rosenthal-representable, non Asplund-representable system. One can also take advantage of model-theoretic tools such as quantifier elimination to give precise descriptions of the algebras of almost periodic functions (which correspond to imaginaries in the algebraic closure of the empty set), weakly almost periodic functions (by describing the stable formulas) and lower tame functions (by describing the NIP formulas) in various concrete examples, starting with those listed at the beginning of the introduction. For WAP functions this was done in [**BT14**]. We take care of the other algebras in Ch. 1, §4. For instance, one has the following.

Corollary. *Every lower tame function on* $\mathrm{Iso}(\mathbb{U}_1)$ *is constant.*

Glasner and Megrelishvili have asked whether every Polish group can be embedded in the isometry group of a Rosenthal Banach space. The latter corollary

[9]One always has factor maps $R(G) \to G^{\mathrm{SUC}} \to G^{\mathrm{Asp}} \to W(G)$. However, we should remark that the metrizability of $R(G)$ already implies $G^{\mathrm{Asp}} = G^{\mathrm{SUC}}$, as can be deduced from the main result of [**GMU08**].

suggests that $\mathrm{Iso}(\mathbb{U}_1)$ might not admit non-trivial Rosenthal representations at all. However, since there may be tame functions which are not left uniformly continuous, our result is not enough to ensure that conclusion.[10]

In the analysis of some automorphism groups of classical structures, such as $\mathrm{Aut}(RG)$ or $\mathrm{Homeo}(2^\omega)$, there is a technical difficulty that arises, which has some intrinsic interest. One can usually give good descriptions of *classical* (i.e., $\{0,1\}$-valued) stable and NIP formulas. However, to describe WAP and lower tame functions one needs to give an account of all *continuous* such formulas (in fact, moreover, one only considers stability and NIP in a restricted domain). Thus, one would like to ensure that continuous stable/NIP formulas are uniform limits of combinations of classical stable/NIP formulas, which is not obvious *a priori*.

The difficulty was solved for stable formulas in [**BT14**], with a topological proof that uses Ryll-Nardzewski's fixed-point theorem for weakly compact convex sets in Banach spaces. Their argument does not carry on to the NIP case, and in fact we do not know whether this approximation property is true in general for NIP. Nevertheless, we have identified a property, which we call *having definable extensions of types over finite sets*, which gives a sufficient condition. This condition holds for stable formulas, thus recovering the result of [**BT14**], and also holds in many unstable structures, which permits to understand their NIP continuous formulas in terms of the classical ones. We discuss this condition in Ch. 1, §4.1.

Before we go on to the contents of Chapter 2, let us recall some other important notions and results from [**BT14**]. Suppose, as before, that we have $G = \mathrm{Aut}(M)$ for a separably categorical structure M. By the homogeneity of M, the left completion $\widehat{G}_L$ can be identified with the topological semigroup of elementary endomorphisms of M. Then, the Roelcke compactification $R(G)$ can be presented as the space of types $\mathrm{tp}(x,y)$ where $x,y \in \widehat{G}_L \subset M^M$. If one considers the restriction of the types $\mathrm{tp}(x,y)$ to stable formulas, then one obtains precisely the WAP compactification of G. Moreover, the semigroup law on $W(G)$ can be described in terms of the stable independence relation of M.

The work of Chapter 2, a collaboration with Itaï Ben Yaacov and Todor Tsankov, addresses the case of Hilbert-representable ambits and functions, for the family of pro-oligomorphic groups. That is, we restrict our attention to countably categorical *classical* structures.

The main reason for this restriction is the existence of a classification theorem for the unitary representations of pro-oligomorphic groups, proven in [**Tsa12**]. With the classification at hand, we can show that the algebra of Hilbert-representable functions on such a group is generated by the functions of the form $f(g) = \varphi(a, gb)$ where $\varphi(x,y)$ is a definable equivalence relation on the associated classical structure M.

In turn, this allows us to give a nice description of the *Hilbert-compactification* $H(G)$. This ambit always has the structure of a semitopological semigroup.

[10] A complete answer might follow from an appropriate study of *externally* definable predicates in $\mathbb{U}_1$, but this has not been achieved yet.

Theorem (Ch. 2, §3.2). *Let G be the automorphism group of an $\aleph_0$-categorical classical structure M. Then $H(G)$ is the semitopological semigroup of* partial *elementary endomorphisms of M^{eq} with algebraically closed domain.*

Thus, for instance, the Hilbert compactification of $\mathrm{Aut}(\mathbb{Q},<)$ is the semigroup of monotone partial bijections of $\mathbb{Q}$, and the Hilbert compactification of $\mathrm{Aut}(RG)$ is the semigroup of partial graph automorphisms of the random graph. In fact, these are also their corresponding WAP compactifications: they are Eberlein groups.

A consequence of the previous theorem is that $H(G)$ is an *inverse semigroup*, that is, we have

$$pp^*p = p \tag{**}$$

for all $p \in H(G)$. Here, $^*\colon H(G) \to H(G)$ denotes the natural involution extending the inverse operation on G. As it turns out, this property provides a characterization of Eberlein pro-oligomorphic groups.

In fact, condition (**) has an interesting model-theoretic interpretation in $W(G)$. If $p = [x,y] \in W(G)$ is the stable type of a pair $x,y \in \widehat{G}_L$, then we have $pp^*p = p$ if and only if $x \mathop{\smile\!\!\!\!\!\!\;|}_{x\cap y} y$ (where the intersection is taken in M^{eq}). This explains how the condition of *one-basedness* appears in our main result of Chapter 2.

Theorem (Ch. 2, §3.3-§4). *Let G be the automorphism group of a countably categorical classical structure M. Then the following are equivalent:*

(1) $W(G)$ is an inverse semigroup.
(2) M is one-based for stable independence, i.e., $A \mathop{\smile\!\!\!\!\!\!\;|}_{A\cap B} B$ for any algebraically closed sets $A,B \subset M^{\mathrm{eq}}$.
(3) G is Eberlein, i.e., $H(G) = W(G)$.

Moreover, a semitopological $*$-semigroup G-ambit is Hilbert-representable if and only if it is an inverse semigroup.

Thus, in the same way that bG is the universal *topological group* G-ambit and $W(G)$ is the universal *semitopological semigroup* G-ambit, for pro-oligomorphic groups we can characterize $H(G)$ as the universal *semitopological inverse $*$-semigroup* G-ambit.[11]

Now, by a classical result from model theory, our theorem also implies the following.

Corollary. *We have $H(G) = R(G)$ if and only if M is $\aleph_0$-stable.*

By resorting to the celebrated example of Hrushovski of an $\aleph_0$-categorical, stable pseudoplane, we can then show an example of a WAP group which is not Eberlein. This answered a question of Glasner and Megrelishvili.

Then we go on to analyse the representability of arbitrary factors of $H(G)$. For this we are led to prove an intermediate result, of independent interest. Remark that the compactifications $H(G)$, $W(G)$ and $R(G)$ of the automorphism group of a countably categorical classical structure are always zero-dimensional, as follows

[11]Note, however, that this algebraic characterization fails completely under weaker assumptions, e.g., $H(G)$ is not an inverse semigroup in the case of the discrete group $\mathbb{Z}$, nor in the case of the Roelcke precompact Polish group $U(\ell^2)$.

from the their model-theoretic descriptions (for the case of $W(G)$, one should recall that classical stable formulas determine all continuous stable formulas).

Theorem (Ch. 2, §4). *If G is a pro-oligomorphic group, then all factors of $H(G)$ are zero-dimensional.*

In contrast, there are examples of non-zero-dimensional factors of the Roelcke compactification. We do not know whether the previous result is valid, instead, for the factors of the WAP compactification.

Finally, we can address, for pro-oligomorphic groups, the open question mentioned in the previous section.

Theorem (Ch. 2, §4). *If G is a pro-oligomorphic group, then all factors of $H(G)$ are Hilbert-representable.*

Chapter 2 ends with a remark and a question about the complexity of countable ambits of pro-oligomorphic groups. If M is classical countably categorical and $\aleph_0$-stable, then the type spaces $S_x(M)$ in finite variables are countable of finite rank. We observe that this can be generalized to arbitrary pro-oligomorphic groups in the following way.

Proposition (Ch. 2, §4). *Every countable Hilbert-representable ambit of a pro-oligomorphic group has finite Cantor–Bendixson rank.*

Since in the $\aleph_0$-stable case the hypothesis of Hilbert-representability is unnecessary (it follows from the others), one may ask whether the previous result actually holds for arbitrary countable ambits. Although, as we point out with an example in Ch. 2, §3.3, a countable ambit of a pro-oligomorphic group need not be Hilbert-representable.

With the results of Chapters 1 and 2, we achieve the purpose declared at the beginning of this section, which was to expand the dictionary of the previous section with a column for model theory. For simplicity, let us state it in this manner: we relate the properties of the $\aleph_0$-categorical structure M with the classes $\mathcal{K}$ of Banach spaces for which the Roelcke compactification $R(G)$ (of the automorphism group of M) is $\mathcal{K}$-representable. The second line (for Hilbert) holds in the classical setting.[12]

Class $\mathcal{K}$	**Structure M**
Euclidean	compact
Hilbert	$\aleph_0$-stable
Reflexive	stable
Asplund	stable
Rosenthal	NIP

[12] In an ongoing work, not included in this manuscript, we have found some evidence that suggests that this fails in the metric setting.

Randomized structures

In many mathematical situations, when considering a set or structure M of some kind, it is interesting or convenient to think about *random* elements of M, and to study their *expected* properties. In [**Kei99**], Jerome Keisler proposed a model-theoretic approach to this theme: if we start with a logic structure, then the random elements should form a new structure, where the original predicates $\varphi(x)$ are replaced by new formulas $\mathbb{E}[\![\varphi(x)]\!]$ that account for their expected values. Actually, there might be several different ways of randomizing a structure, but, as Keisler showed, they all share the same first-order theory. Hence, given a complete theory T, we obtain a canonical *randomized theory* T^R.

This was done, originally, in a classical first-order setting, which made the construction rather cumbersome. After continuous logic was developed, the theory of randomizations was adapted to this setting by Ben Yaacov and Keisler. Indeed, continuous logic offers a perfect framework for studying randomizations, and, conversely, randomized structures are an interesting source of metric structures.

With the formalism of continuous logic, randomizations preserve a lot of model-theoretic properties, including $\aleph_0$-categoricity. Thus, in the special case that M is an $\aleph_0$-categorical structure, there is an essentially unique separable randomization of M, say M^R. This means that, given a Roelcke precompact Polish group G (say, the automorphism group of M), there is a canonically associated Polish group G^R (the automorphism group of M^R), which is Roelcke precompact too. So, what is this group? The work of Chapter 3 was motivated by this question.

Let Ω be a standard probability space. Given any Polish group G, we consider the semidirect product of $L^0(\Omega,G)$, the group of random elements of G, and $\mathrm{Aut}(\Omega)$, the group of measure-preserving transformations of Ω, which acts naturally on the former. We call this the *measurable wreath product* of G and $\mathrm{Aut}(\Omega)$, and we denote it by

$$G\wr\Omega := L^0(\Omega,G)\rtimes\mathrm{Aut}(\Omega).$$

The group $G\wr\Omega$ is naturally a Polish group.

We show the following, where M^R denotes the *Borel randomization* of a separable structure M. This is a canonical example of randomization; essentially, M^R is given by the set $L^0(\Omega,M)$ with the appropriate structure.

Theorem (Ch. 3, §2.2). *We have* $\mathrm{Aut}(M^R)\simeq\mathrm{Aut}(M)\wr\Omega$ *as topological groups.*

Beyond the model-theoretic motivation, measurable wreath products can be seen as an interesting source of new Polish groups. In Ch. 3, §2-3, we investigate the properties of $G\wr\Omega$-actions induced by actions of G. We show the following, the first part of which gives an alternative proof of the preservation of $\aleph_0$-categoricity by randomizations.

Theorem (Ch. 3, §2-3). *If a Polish group G acts faithfully and approximately oligomorphically on a Polish metric space M, then $G\wr\Omega$ acts faithfully and approximately oligomorphically on $L^0(\Omega,M)$. In particular, if G is Roelcke precompact, then so is $G\wr\Omega$.*

In that case, the Roelcke compactification $R(G\wr\Omega)$ can be identified with the space

$$\mathcal{M}(\Omega,R(G)) := \{\lambda\in\mathfrak{K}(\Omega_0\times R(G)\times\Omega_1) : \lambda|_{\Omega_0}=\mu_0,\ \lambda|_{\Omega_1}=\mu_0\}.$$

(Here, $\mathfrak{R}(X)$ is the compact space of Borel probability measures on X; each Ω_i is a standard probability space with measure μ_i, and $\lambda|_{\Omega_i}$ is the pushforward of λ to Ω_i.)

Then, we study some preservation properties of the spaces $\mathcal{M}(\Omega, X)$ introduced above. If X is a G-ambit, then $\mathcal{M}(\Omega, X)$ is naturally a $G \wr \Omega$-ambit. Moreover, this construction behaves well with respect to semitopological semigroups.

Proposition (Ch. 3, §3). *If S is a metrizable compact semitopological semigroup, then $\mathcal{M}(\Omega, S)$ admits a natural semitopological semigroup law. If S is a G-ambit, then the law of $\mathcal{M}(\Omega, S)$ extends the natural action of $G \wr \Omega$ on $\mathcal{M}(\Omega, S)$.*

If S is representable by contractions on a Hilbert space, then so is $\mathcal{M}(\Omega, S)$.

Suppose G is a Roelcke precompact Polish group. It follows from the previous theorem and proposition that if G is a WAP group (i.e., if $W(G) = R(G)$), then so is $G \wr \Omega$. Similarly, if G is satisfies $H(G) = R(G)$, then also $H(G \wr \Omega) = R(G \wr \Omega)$.[13]

We know, by the dictionary discussed in the previous section, that the preservation of WAP groups has a model-theoretic counterpart. Namely: stability is preserved under randomizations. This had been proved in all generality in [**Ben13b**]. Another important result of the same kind, the preservation of NIP theories (and formulas) under randomizations, had been established in [**Ben09**]. Since the proof of the latter is particularly involved, we found it interesting to look for an alternative proof of this fact too, even if limited to the $\aleph_0$-categorical case.

We ended up proving a general preservation result for Banach representations of certain randomized flows. For this, we define a class of Banach spaces $\mathcal{K}$ to be *R-closed* if, whenever $V \in \mathcal{K}$, the Bochner space $L^2(\Omega, V)$ is also in $\mathcal{K}$ (plus a technical condition about $\mathcal{K}$-representable functions, see Ch. 3, Definition 3.15). Using some classical results of Banach space theory (together with some recent results of Glasner and Megrelishvili, for the technical condition), one sees that the classes of Hilbert, reflexive, Asplund and Rosenthal spaces are all R-closed.

Let $\mathcal{S}(\Omega, X)$ denote the compact space $\{\lambda \in \mathfrak{R}(\Omega \times X) : \lambda|_\Omega = \mu\}$. We show that this corresponds to randomized type spaces with parameters: for instance, $S_x^{T^R}(M^R) \simeq \mathcal{S}(\Omega, S_x^T(M))$. Then, the preservation result is the following.

Theorem (Ch. 3, §3.3). *Let $\mathcal{K}$ be an R-closed class of Banach spaces. Let $G \curvearrowright X$ be a continuous action of a Polish group on a compact metrizable space, and suppose that $G \curvearrowright X$ is $\mathcal{K}$-representable. Then, the system $G \wr \Omega \curvearrowright \mathcal{S}(\Omega, X)$ is $\mathcal{K}$-representable too.*

Combined with the results of Chapter 1, this gives us new proofs, in the $\aleph_0$-categorical setting, of the following facts.

Corollary. *If $\varphi(x,y)$ is an NIP formula for T, then $\mathbb{E}[\![\varphi(x,y)]\!]$ is an NIP formula for T^R. Similarly for stable formulas.*

On the other hand, our dictionary for Hilbert-representability only works, for the moment, for classical logic. Thus, for instance, we can deduce from our results that if M is classical, $\aleph_0$-categorical and $\aleph_0$-stable, then $R(\mathrm{Aut}(M^R))$ is Hilbert-representable; but we do not know what this says, model-theoretically, about M^R.

[13] It is natural to conjecture that $W(G \wr \Omega) \simeq \mathcal{M}(\Omega, W(G))$ and $H(G \wr \Omega) \simeq \mathcal{M}(\Omega, H(G))$, though we have not settled this question.

Since the results of Chapter 2 were based on the notion of one-basedness, we should look for a metric generalization of this property. In the literature, there is one such proposed generalization. Namely, the notion of *strongly finitely based (SFB)* theories, introduced in [**BBH14**]. An $\aleph_0$-categorical, stable theory T is SFB if and only if the theory T_P of beautiful pairs of models of T is $\aleph_0$-categorical. This coincides with one-basedness in the classical setting, but it applies in addition to some important well-behaved metric theories, such as the theories of an infinite-dimensional Hilbert space and of the measure algebra of the interval.

Hence, one may expect that the randomization of an SFB theory be again SFB. Surprisingly for us, this turns out to be false. Rather, the opposite holds.

Theorem (Ch. 3, §4). *The theory $(T^R)_P$ of beautiful pairs of models of a randomized stable theory T^R is never $\aleph_0$-categorical, unless T is the theory of a compact structure.*

In fact, we show that if $N \prec M$ is any elementary pair of separable models of a stable theory T, then we can construct a separable model of $(T^R)_P$ as follows. We let M^R be given by the set of all random variables $\Omega^2 \to M$, and we let $S \subset M^R$ be the subset of random variables taking values in N that are measurable with respect to the first coordinate of Ω^2 (i.e., those that factor as $\Omega^2 \to \Omega \to N \to M$). Then, with the appropriate structure, the pair (M^R, S) is a model of $(T^R)_P$. This yields non-isomorphic separable models, for instance for the cases $N = M$ versus $N \subsetneq M$.

Moreover, we have the following. Let S_1 be the set S corresponding to the case $N = M$ (and assume M is $\aleph_0$-categorical). If we let $h \in \mathrm{End}(M)^{\Omega^2}$, we can define $S_h = \{hr : r \in S_1\}$. Then, the pair $\mathbb{P}_h := (M^R, S_h)$ is also a model of $(T^R)_P$.

Theorem (Ch. 3, §4). *Suppose T is stable, $\aleph_0$-categorical. Then, the separable models of $(T^R)_P$ are exactly the pairs isomorphic to $\mathbb{P}_h$ for some $h \in \mathrm{End}(M)^{\Omega^2}$.*

In addition to the previous results, we prove that the randomization of an arbitrary metric $\aleph_0$-stable theory T is again $\aleph_0$-stable (this had been proved for classical T in [**BK09**]). Finally, Chapter 3 ends with a description of the automorphism groups of pairs of randomizations of the form (M^R, S) induced by models M, N as above.

Descriptive aspects of full groups

We now turn to the subjects of the second part of the thesis. The results referred to in this and the following sections have been obtained in collaboration with Julien Melleray.

We will forget about automorphism groups of logic structures for a short moment —only with the purpose of convincing the reader of the importance of coming back to them.

Full groups are a special kind of automorphism groups that appear in ergodic theory and in its younger topological sibling, Cantor dynamics. In the latter context, if g is a minimal homeomorphism of the Cantor space $X := 2^\omega$, one may consider the enriched "structure" on the compact space X resulting of naming the orbits of g on X (N.B. this is not a structure in the sense of logic), then consider the corresponding automorphism group. This is called the *full group* of g, denoted by $[g]$, and consists thus of all homeomorphisms $h \in \mathrm{Homeo}(X)$ such that $h(O_g(x)) = O_g(x)$ for every $x \in X$, where $O_g(x)$ is the g-orbit of x.

Full groups of measure-preserving tranformations of a standard probability space Ω are defined similarly (and, more interestingly in this case, also full groups of actions of countable groups).

A crucial fact about full groups is that they are complete algebraic invariants for orbit equivalence. That is, if g, h are minimal homeomorphisms of X such that $[g] \simeq [h]$ as abstract groups, then there is a homeomorphism of X such that elements in the same g-orbit are mapped into elements in the same h-orbit, and elements in different g-orbits are mapped into elements in different h-orbits. This was proved by Giordano, Putnam and Skau [**GPS99**]. The analogous theorem in the framework of ergodic theory had been established several decades earlier by Dye, and was at the origin of an intensive study of full groups.

Now, a valuable feature of full groups in the *ergodic-theoretic* setting is that they are topologically well-behaved. First, they admit a Polish group topology. (This is not the topology induced by the ambient Polish group $\mathrm{Aut}(\Omega)$, though; rather, full groups become Polish under the uniform topology of $\mathrm{Aut}(\Omega)$.) This topology plays a prominent role in various results in the literature, for instance in characterizations of properties that are invariant under orbit equivalence. Second, full groups are always Borel subsets of $\mathrm{Aut}(\Omega)$ of relatively low complexity: they are $\mathbf{\Pi}^0_3$ in the Borel hierarchy.

On the other hand, no Polish topology has ever been proposed for full groups of minimal homeomorphisms of the Cantor space. The work of Chapter 4 addresses this situation, and its first result is the following.

Theorem (Ch. 4, §2). *Let $\Gamma \subset \mathrm{Homeo}(X)$ be a countable group acting minimally on the Cantor space. Then, there is no second countable, Hausdorff, Baire group topology on the full group* $[\Gamma]$.

In particular, the full group $[g]$ of a minimal homeomorphism does not admit any Polish group topology.

We may remark that, unlike the situation in the measurable context, here one may fix a point $x \in X$ and look at the permutations of the orbit $O_g(x)$ induced by elements of $[g]$; since $O_g(x)$ is countable, this gives a natural homomorphism from $[g]$ into the permutation group of a countable set. The proof of the previous theorem shows that a Hausdorff, Baire topology on $[g]$ would make this homomorphism continuous. In fact, the topology would have to refine the topology of pointwise convergence on X, which would make it impossible for $[g]$ to be second countable.

Then, we investigated the question of the complexity of $[g]$ inside $\mathrm{Homeo}(X)$. Here again, the answer turns out to be worst possible.

Theorem (Ch. 4, §3). *Let g be a minimal homeomorphism of the Cantor space. Then, $[g]$ is a coanalytic non-Borel subset of* $\mathrm{Homeo}(X)$.

These results suggest that full groups in the Cantor minimal setting might fail to be as useful an invariant as their ergodic counterparts have proven to be. Hence, one may attempt to study a gentler but related object instead. A natural candidate is the closure of the full group inside Homeo(X).[14] Given a minimal homeomorphism g, let $G_g = \overline{[g]}$ denote this closure.

The group G_g offers two important features, in addition to its obvious Polish group structure. The first one is that it is also a complete algebraic invariant for orbit equivalence, as we point out in Ch. 4, §4; this follows from another deep theorem of Giordano, Putnam and Skau.

The second is that G_g admits a nice description. Indeed, it follows from a theorem of Glasner and Weiss that G_g is precisely the set of $h \in \mathrm{Homeo}(X)$ that preserve every g-invariant probability measure on X.

We recall that Homeo(X) can be identified with the automorphism group of the Boolean algebra of clopen subsets of X. Since G_g is a closed subgroup of the homeomorphism group of the Cantor space, it must be the automorphism group of a *expansion* of the algebra of clopen sets of X. As we will see in the next section, the description of G_g mentioned above will allow us to give a nice presentation of this structure.

An interesting question one may ask about G_g is whether it admits a *generic* element. An element $h \in G_g$ is generic if its conjugacy class $\{khk^{-1} : k \in G_g\}$ is comeager in G_g. The existence of a generic element has strong consequences for the closure of a full group. For instance, it implies that G_g must be simple, and that any homomorphism into a separable topological group is automatically continuous. This question can be approached by means of the methods explained below.

Fraïssé theory for good measures

A major point of convergence of dynamics, logic and combinatorics was brought to light by Kechris, Pestov and Todorcevic in their celebrated work [**KPT05**]. There, they were concerned with the study of universal minimal flows of Polish groups, which we are not in this thesis; however, their main tool, the study of Polish groups by means of Fraïssé theory, has since then been adapted in a number of different ways to the study of other related or unrelated problems. Here, we will be concerned with the work of Kechris and Rosendal [**KR07**] (which was also inspired by previous work of Hodges et al. [**HHLS93**] and Truss [**Tru92**]). They addressed the problem of determining when a Polish group admits a dense or comeager conjugacy class.

From now on, all structures we consider are in the sense of classical logic. A structure $\mathbb{K}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathbb{K}$ extends to an automorphism of $\mathbb{K}$. Fraïssé showed that a countable ultrahomogeneous structure $\mathbb{K}$ is determined, up to isomorphism, by its *age*, which is the class of all *finitely generated* structures that can be embedded in $\mathbb{K}$. Hence, the age codes all the model-theoretic information of $\mathbb{K}$. The ideas of

[14] Another natural candidate is the *normalizer* of the full group, $N[g]$, which is the set of all $h \in \mathrm{Homeo}(X)$ such that $h(O_g(x)) = O_g(h(x))$ for every $x \in X$ (that is, $N[g]$ is the automorphism group of the equivalence relation induced by the orbits of g). It can be seen that $N[g]$ is also a complete invariant for orbit equivalence. However, as observed by Julien Melleray in a private communication, using that $[g]$ is non-Borel inside Homeo(X), one can see that $N[g]$ is also non-Borel and does not admit a Polish group topology.

[**KPT05**] or [**KR07**] are based on the principle that certain combinatorial properties of the age of $\mathbb{K}$ code dynamical properties of the automorphism group $\mathrm{Aut}(\mathbb{K})$.

Let $\mathbb{K}$ be a countable ultrahomogeneous structure and let $\mathcal{A}$ denote its age. Then, we may consider the class $\mathcal{A}_p$ of pairs (A, f) where $A \in \mathcal{A}$ and f is a partial automorphism of A. Kechris and Rosendal proved that $\mathrm{Aut}(\mathbb{K})$ has a dense conjugacy class if and only if $\mathcal{A}_p$ satisfies a the *joint embedding property (JEP)*, i.e., if any two partial automorphisms of structures in $\mathcal{A}$ can be merged into the partial automorphism of a common superstructure in $\mathcal{A}$. Moreover, $\mathrm{Aut}(\mathbb{K})$ has a comeager conjugacy class if and only if $\mathcal{A}_p$ satisfies JEP together with a more sophisticated condition called the *weak amalgamation property (WAP)*[15]. For instance, they managed to show that JEP and WAP hold true for the class of finite Boolean algebras, which is the age of the algebra of clopen sets of the Cantor space. As a result, they established that the Cantor space has a generic homeomorphism.

In addition, they studied another important, strong combinatorial property that some classes of finitely generated structures enjoy, called the *Hrushovski property*, which is related to the possibility of extending partial automorphisms to (full) automorphisms of larger structures.

Let us explain how this relates to our problems. As before, let X denote the Cantor space, and let g be a minimal homeomorphism of X. We denote the set of all g-invariant probability measures on X by K_g.

Now, it follows from a result of Glasner and Weiss that the minimality of g implies the following property of K_g: *if $A, B \subset X$ are clopen sets such that $\mu(A) < \mu(B)$ for all $\mu \in K_g$, then there is a clopen subset $C \subset B$ such that $\mu(C) = \mu(A)$ for all $\mu \in K_g$.* When a set K of atomless probability measures with full support on X enjoys this property, we say that K is *good* (generalizing the definition of [**Aki05**]). A standard back-and-forth argument shows the following.

Lemma. *Assume K is a good simplex of probability measures on X with full support. Let $G_K = \{h \in \mathrm{Homeo}(X) \colon \forall \mu \in K\ h_*\mu = \mu\}$. If A, B are clopen sets with $\mu(A) = \mu(B)$ for all $\mu \in K$, then there is $h \in G_K$ such that $h(A) = B$.*

This says that a certain structure is ultrahomogeneous. Indeed, let K be as in the statement above, and let $\mathbb{K}$ be the Boolean algebra of clopen sets of X expanded with predicates $P_{\mu,r}$ for each $\mu \in K$ and $r \in [0,1]$, interpreted so that a clopen set A verifies $P_{\mu,r}$ if and only if $\mu(A) = r$. (Actually, by separability of the space of probability measures, the language can be assumed to be countable.) Then we have $G_K = \mathrm{Aut}(\mathbb{K})$, and the lemma implies readily that $\mathbb{K}$ is ultrahomogeneous.

In particular, if we let $\mathbb{K}_g$ denote the structure constructed from $K = K_g$, we obtain that the closure of the full group $[g]$ is $G_g = \mathrm{Aut}(\mathbb{K}_g)$. Thus we can apply the theory of [**KR07**] to the study of G_g. In this generality, we were able to show the following (thanks to an observation of Kostya Medynets; our original result was weaker).

Theorem (Ch. 4, §4). *Let g be a minimal homeomorphism of the Cantor space, and let $\mathbb{K}_g$ be as above. Then, the age of $\mathbb{K}_g$ has the Hrushovski property.*

This already has the following consequence.

[15]N.B. completely unrelated to *weak almost periodicity*!

Corollary. *The group G_g is amenable.*

Moreover, combined with a result of Bezuglyi and Medynets [**BM08**], the theorem also yields the following.

Corollary. *The group G_g is topologically simple.*

Then we studied the conditions JEP and WAP in a particular case, namely, when g is uniquely ergodic. So let us fix a minimal homeomorphism g, and suppose there is a unique g-invariant measure, μ, which is then a *good measure* in the sense of [**Aki05**]. Let V_μ be the (countable) set of values $\mu(A) \in [0,1]$ that μ takes on clopen subsets $A \subset X$. Then, the age of $\mathbb{K}_g$ depends only on V_μ, so this set codes all the relevant information of $\mathbb{K}_g$ and G_g. By studying the joint embedding property in terms of V_μ, we obtain the following technical characterization.

Proposition (Ch. 4, §5). *There is a dense conjugacy class in G_g if and only if V_μ satisfies the following condition: whenever $a_i, b_j \in V_\mu$ and $n_i, m_j \in \mathbb{N}$ are such that $\sum_{i=1}^{p} n_i a_i = 1 = \sum_{j=1}^{q} m_j b_j$, one can find $c_{i,j} \in V_\mu$ such that: $m_j b_j = \sum_{i=1}^{p} lcm(n_i, m_j) c_{i,j}$ for all j, and $n_i a_i = \sum_{j=1}^{q} lcm(n_i, m_j) c_{i,j}$ for all i.*

This holds true in particular when $V_\mu + \mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$, and when $V_\mu + \mathbb{Z}$ is a subring of $\mathbb{R}$.

Unfortunately, we have not succeeded in giving a characterization of the same kind for the existence of a comeager conjugacy class in G_g. Unlike what happens in several examples of [**KR07**], where the Hrushovski property can be used to prove a strong form of the weak amalgamation property, this does not seem to suffice in our case. Nevertheless, this method does work for a particular family of good measures, which allows us to give a new proof of the following result of Ethan Akin [**Aki05**].

Proposition (Ch. 4, §5). *Suppose $V_\mu + \mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$. Then, there is a generic homeomorphism in G_g.*

In the previous results, we have supposed that a uniquely ergodic minimal homeomorphism g was given. Now, if V is any countable infinite subset of $[0,1]$ containing 0 and 1 and closed under addition modulo 1, then one can show that there is a good measure μ on X such that $V = V_\mu$. Moreover, Akin proved that there exists then a minimal homeomorphism g such that $K_g = \{\mu\}$. Thus, for instance, one can use the characterization given above to show that there are minimal homeomorphisms g for which G_g does not admit a dense conjugacy class.

Dynamical simplices

As we just mentioned, given a good measure μ on the Cantor space, Akin [**Aki05**] proved the existence of a minimal homeomorphism for which μ is the only invariant probability measure. Hence, good measures are exactly the invariant measures of uniquely ergodic minimal homeomorphisms of the Cantor space. Is there a similar characterization of the sets of invariant probability measures of arbitrary minimal homeomorphisms of the Cantor space? This problem is addressed in the work of Chapter 5.

An answer to this question had been given in an unpublished work of Heidi Dahl [**Dah08**]. Let K be a set of probability measures on the Cantor space X. Dahl showed that there is a minimal homeomorphism g of X such that $K = K_g$ if, and only if, K is a Choquet simplex of atomless probability measures with full support satisfying the following properties: the extreme points of K are mutually singular, K is good, and the functions of the form $\mu \mapsto \int f \, d\mu$ for $f \in \mathrm{C}(X, \mathbb{Z})$ are dense within the continuous affine functions on K. Her proof is based on the complex and powerful methods of Giordano, Herman, Putnam and Skau, which rely on tools from K-theory, the theory of dimension groups, and of Bratelli–Vershik maps.

In Chapter 5 we take a more elementary approach, and give a characterization of the simplices K_g for minimal g that stays closer in spirit to Akin's condition for the uniquely ergodic case. More precisely, we propose the following definition. Let K be a non-empty compact convex set of atomless probability measures of full support on X. We say that K is a *dynamical simplex* if, in addition, K is good (as defined in the previous section) and *approximately divisible*, that is, for every $\epsilon > 0$, $n \in \mathbb{Z}_{>0}$ and every clopen set $A \subset X$ there is a clopen subset $B \subset A$ such that, for all $\mu \in K$, $\mu(A) - \epsilon \leq n\mu(B) \leq \mu(A)$.

Then, starting with a dynamical simplex K, we prove the existence of a minimal homeomorphism g with $K = K_g$ by constructing, via cutting and stacking, a series of clopen partitions of X that are aimed to be Kakutani–Rokhlin partitions for g.

Theorem (Ch. 5). *A set K of probability measures on X is a dynamical simplex if and only if $K = K_g$ for a minimal homeomorphism g of X.*

Moreover, the homeomorphism g that we obtain is *saturated*, which means that the *topological full group* $[[g]]$ is dense in G_g. We recall that $[[g]]$ consists of those $h \in [g]$ for which there is a clopen partition $A_1, \dots, A_k$ of X and integers $n_1, \dots, n_k$ such that $h|_{A_i} = g^{n_i}|_{A_i}$ for each $i = 1, \dots, k$. Hence, our construction gives an elementary proof of the following.

Corollary. *If g is a minimal homeomorphism of X, then there is a saturated minimal homeomorphism g' with $G_g = G_{g'}$.*

This was previously known by more sophisticated means. We had used this result in the proof of the Hrushovski property stated in the previous section.

One may ask whether the hypothesis of approximate divisibility is actually needed or, equivalently, whether there are good simplices which are not dynamical simplices. We do not know the answer in general. However, approximate divisibility is indeed redundant in the case of finite-dimensional simplices.

Corollary. *Let K be a good simplex of atomless probability measures of full support on X. If K has finitely many extreme points, then $K = K_g$ for some minimal homeomorphism g of X.*

This corollary had already been obtained by Dahl, under the additional assumption that the extreme points of K are mutually singular.

Part 1

Automorphism groups of separably categorical structures

CHAPTER 1

The dynamical hierarchy for Roelcke precompact Polish groups

Abstract. [Article to appear in Israel Journal of Mathematics.] We study several distinguished function algebras on a Polish group G, under the assumption that G is Roelcke precompact. We do this by means of the model-theoretic translation initiated by Ben Yaacov and Tsankov: we investigate the dynamics of $\aleph_0$-categorical metric structures under the action of their automorphism group. We show that, in this context, every strongly uniformly continuous function (in particular, every Asplund function) is weakly almost periodic. We also point out the correspondence between tame functions and NIP formulas, deducing that the isometry group of the Urysohn sphere is Tame $\cap$ UC-trivial.

Contents

Introduction

In a series of recent papers, Glasner and Megrelishvili [**GM06, GM08, Meg08, GM12, GM13**] have studied different classes of functions on topological dynamical systems, arising from compactifications with particular properties. Thus, for example, a real-valued continuous function on a G-space X might be *almost periodic*, *Hilbert-representable*, *weakly almost periodic*, *Asplund-representable* or *tame*, and this classes form a hierarchy

$$\mathrm{AP}(X) \subset \mathrm{Hilb}(X) \subset \mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{Tame}(X) \subset \mathrm{RUC}(X)$$

of subalgebras of the class of *right uniformly continuous* functions. These algebras can be defined in different ways. The latter coincides with the class of functions that can be in some sense represented through a Banach space, and from this point of view the previous subalgebras can be identified, respectively, with the cases when the Banach space is asked to be Euclidean, Hilbert, reflexive, Asplund or

Rosenthal. When $X = G$ and the action is given by group multiplication, functions might also be *left uniformly continuous*, and if they are simultaneously in $\mathrm{RUC}(G)$ they form part of the algebra $\mathrm{UC}(G)$ of *Roelcke uniformly continuous* functions.

We study these algebras for the case of Roelcke precompact Polish groups, by means of the model-theoretic translation developed by Ben Yaacov and Tsankov [**BT14**]. As established in their work, Roelcke precompact Polish groups are exactly those arising as automorphism groups of $\aleph_0$-categorical metric structures. Moreover, one might turn continuous functions on the group into definable predicates on the structure. Under this correlation, the authors showed, weakly almost periodic functions translate into *stable* formulas: a most studied concept of topological dynamics leads to one of the crucial notions of model theory. This provides a unified understanding of several previously studied examples: the permutation group $S(\mathbb{N})$, the unitary group $\mathcal{U}(\ell^2)$, the group of measure preserving transformations of the unit interval $\mathrm{Aut}(\mu)$, the group $\mathrm{Aut}(RG)$ of automorphisms of the random graph or the isometry group $\mathrm{Iso}(\mathbb{U}_1)$ of the Urysohn sphere, among many other "big" groups, are automorphism groups of $\aleph_0$-categorical structures, thus Roelcke precompact. In the first three cases the structures are stable, thus $\mathrm{WAP}(G) = \mathrm{UC}(G)$: their WAP and Roelcke compactifications coincide. Using model-theoretic insight, the authors were able to prove for example that, whenever the latter is the case, the group G is totally minimal.

The so-called *dynamical hierarchy* presented above has been partially described for some of the habitual examples. For the groups $S(\mathbb{N})$, $\mathcal{U}(\ell^2)$ or $\mathrm{Aut}(\mu)$ we have in fact $\mathrm{Hilb}(G) = \mathrm{UC}(G)$; see [**GM14b**, §6.3–6.4]. From [**BT14**, §6] we know, for instance, that the inclusion $\mathrm{WAP}(G) \subset \mathrm{UC}(G)$ is strict for the group $\mathrm{Aut}(\mathbb{Q},<)$ of monotone bijections of the rationals. More drastically, Megrelishvili [**Meg01a**] had shown that the group $H_+[0,1]$ of orientation preserving homeomorphisms of the unit interval, also Roelcke precompact, has a trivial WAP-compactification: $\mathrm{WAP}(G)$ is the algebra of constants; in [**GM08**, §10] this conclusion was extended to the algebra $\mathrm{Asp}(G)$ (and indeed to the algebra $\mathrm{SUC}(G)$ of *strongly uniformly continuous functions*, containing $\mathrm{Asp}(G)$). The same was established for the group $\mathrm{Iso}(\mathbb{U}_1)$. If one drops the requirement of Roelcke precompactness, all inclusions in the hierarchy are known to be strict in appropriate examples.

We show that in fact $\mathrm{WAP}(G) = \mathrm{Asp}(G) = \mathrm{SUC}(G)$ for every Roelcke precompact Polish group G. In addition, we observe that Roelcke uniformly continuous tame functions correspond to NIP formulas on the model-theoretic side. Thus, for instance, $\mathrm{Asp}(G) \subsetneq \mathrm{Tame}(G) \cap \mathrm{UC}(G) = \mathrm{UC}(G)$ for $G = \mathrm{Aut}(\mathbb{Q},<)$, while $\mathrm{WAP}(G) = \mathrm{Tame}(G) \cap \mathrm{UC}(G) \subsetneq \mathrm{UC}(G)$ for $G = \mathrm{Aut}(RG)$ or $G = \mathrm{Homeo}(2^\omega)$. We also deduce that the $\mathrm{Tame} \cap \mathrm{UC}$-compactification of $\mathrm{Iso}(\mathbb{U}_1)$ is trivial.

Our approach is model-theoretic, and we shall assume some familiarity with continuous logic as presented in [**BU10**] or [**BBHU08**]; nevertheless, we give an adapted introduction to $\aleph_0$-categorical metric structures that we hope can be helpful to an interested reader with no background in logic. We will mainly study the dynamics of $\aleph_0$-categorical structures, then derive the corresponding conclusions for their automorphism groups.

The algebra Hilb(G) will not be addressed in this paper. Unlike the properties of stability and dependence, which can be studied locally (that is, formula-by-formula), the model-theoretic interpretation of the algebra Hilb(G) presents a different phenomenon, and will be considered in a future work.

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1. The setting and basic facts

1.1. *G*-spaces and compactifications. Most of the material on topology in this and subsequent sections comes from the works of Glasner and Megrelishvili referred to in the introduction.

A *G-space* X is given by a continuous left action of a topological group G on a topological space X. Then G acts as well on the space $\mathrm{C}(X)$ of continuous, bounded, real-valued functions on X, by $gf(x) = f(g^{-1}x)$. If X is not compact, however, the action on $\mathrm{C}(X)$ need not be continuous for the topology of the uniform norm on $\mathrm{C}(X)$. The functions $f \in \mathrm{C}(X)$ for which the orbit map $g \in G \mapsto gf \in Gf \subset \mathrm{C}(X)$ is norm-continuous are called *right uniformly continuous* (RUC). That is, $f \in \mathrm{RUC}(X)$ if for every $\epsilon > 0$ there is a neighborhood U of the identity of G such that

$$|f(g^{-1}x) - f(x)| < \epsilon$$

for all $x \in X$ and $g \in U$. When $X = G$ is considered as a G-space with the regular left action, we also have the family $\mathrm{LUC}(G)$ of *left uniformly continuous functions*, where the condition is that $|f(xg) - f(x)|$ be small for all $x \in G$ and g close to the identity. The intersection $\mathrm{UC}(G) = \mathrm{RUC}(G) \cap \mathrm{LUC}(G)$ forms the algebra of *Roelcke uniformly continuous functions* on G. The family $\mathrm{RUC}(X)$ is a uniformly closed G-invariant subalgebra of $\mathrm{C}(X)$, and the same is true for $\mathrm{LUC}(G)$ and $\mathrm{UC}(G)$ in the case $X = G$.

If X is compact, then $\mathrm{RUC}(X) = \mathrm{C}(X)$; in the case $X = G$, $\mathrm{UC}(G) = \mathrm{C}(G)$. Moreover, recall that a compact Hausdorff space X admits a unique compatible uniformity (see, for example, [**Bou71**, II, §4, №1]), and that any continuous function from X to another uniform space is automatically uniformly continuous.

Note 1.1. Our spaces, when not compact, will be metric, and G will act on X by uniformly continuous transformations (in practice, by isometries). In this case, we will usually restrict our attention to those functions $f \in \mathrm{RUC}(X)$ that are also uniformly continuous with respect to the metric on X; we denote this family of functions by $\mathrm{RUC}_u(X)$. It is a uniformly closed G-invariant subalgebra. The same subscript u might be added to the other function algebras in the dynamical hierarchy, in order to keep this restriction in mind.

Our groups will be Polish. When we take $X = G$, we assume that a left-invariant, compatible, bounded metric d_L on G has been fixed; its existence is ensured by Birkhoff–Kakutani theorem; see for example [**Ber74**, p. 28]. The subscript u will then refer to this metric; notice that $\mathrm{RUC}_u(G) = \mathrm{UC}(G)$. The algebra $\mathrm{SUC}(G)$, containing $\mathrm{Asp}(G)$ (both to be defined later), is always a subalgebra of $\mathrm{UC}(G)$ (see Section 2); in particular, $\mathrm{SUC}_u(G) = \mathrm{SUC}(G)$ and $\mathrm{Asp}_u(G) = \mathrm{Asp}(G)$. As pointed out to us by M. Megrelishvili, this is not the case for the algebra $\mathrm{Tame}(G)$ (see the discussion after Theorem 4.15), so we will mind the distinction between $\mathrm{Tame}(G)$ and $\mathrm{Tame}_u(G) = \mathrm{Tame}(G) \cap \mathrm{UC}(G)$.

From the equality $\mathrm{RUC}_u(G) = \mathrm{UC}(G)$ we see that $\mathrm{RUC}_u(G)$ does not depend on the particular choice of d_L. Thus, so far, we could omit the metric d_L and consider simply the natural uniformities on G (see for instance [**Bou71**, III, §3, №1] for an explanation of these). However, our approach will require to consider metric spaces, and in fact complete ones. This is why we will consider the space (G, d_L), and mainly its completion $\widehat{G}_L = \widehat{(G, d_L)}$, which is naturally a G-space. We remark that the restriction map $\mathrm{RUC}_u(\widehat{G}_L) \to \mathrm{UC}(G)$ is a norm-preserving G-isomorphism.

A *compactification* of a G-space X is a continuous G-map $\nu\colon X \to Y$ into a compact Hausdorff G-space Y, whose range is dense in Y. In our context it will be important to consider compactifications that are uniformly continuous: in this case we shall say, to make the distinction, that ν is a *u-compactification* of X. A function $f \in \mathrm{C}(X)$ *comes from a compactification* $\nu\colon X \to Y$ if there is $\tilde{f} \in \mathrm{C}(Y)$ such that $f = \tilde{f}\nu$; note that the extension $\tilde{f}$ is unique.

If f comes from a compactification of X, then certainly $f \in \mathrm{RUC}(X)$. The converse is true. In fact, there is a canonical one-to-one correspondence between compactifications of X and uniformly closed G-invariant subalgebras of $\mathrm{RUC}(X)$ (a subalgebra is always assumed to contain the constants). The subalgebra $\mathcal{A}_\nu$ corresponding to a compactification $\nu\colon X \to Y$ is given by the family of all functions $f \in \mathrm{C}(X)$ that come from ν. Conversely, the compactification $X^{\mathcal{A}}$ corresponding to one such algebra $\mathcal{A} \subset \mathrm{RUC}(X)$ is the space $X^{\mathcal{A}}$ of maximal ideals of $\mathcal{A}$ together with the map $\nu_{\mathcal{A}}\colon X \to X^{\mathcal{A}}$, $\nu_{\mathcal{A}}(x) = \{f \in \mathcal{A} : f(x) = 0\}$. We recall that the topology on $X^{\mathcal{A}}$ is generated by the basic open sets $U_{f,\delta} = \{p \in X^{\mathcal{A}} : |\tilde{f}(p)| < \delta\}$ for $f \in \mathcal{A}$ and $\delta > 0$; here, $\tilde{f}(p)$ is the unique constant $r \in \mathbb{R}$ such that $f - r \in p$.

In this way we always have the equality $\mathcal{A} = \mathcal{A}_{\nu_{\mathcal{A}}}$ and a unique G-homeomorphism $j_\nu\colon X^{\mathcal{A}_\nu} \to Y$ with $\nu = j_\nu \nu_{\mathcal{A}_\nu}$. In particular, if $f \in \mathcal{A}$, then f comes from $\nu_{\mathcal{A}}$ (and the extension $\tilde{f} \in \mathrm{C}(X^{\mathcal{A}})$ is defined as above). Finally, the correspondence is functorial, in the sense that inclusions $\mathcal{A} \subset \mathcal{B}$ of subalgebras correspond bijectively to continuous G-maps $j\colon X^{\mathcal{B}} \to X^{\mathcal{A}}$ such that $\nu_{\mathcal{A}} = j\nu_{\mathcal{B}}$. When we say that a given compactification is minimal or maximal within a certain family, we refer to the order induced by these morphisms; in the previous situation, for example, $\nu_{\mathcal{B}}$ is larger than $\nu_{\mathcal{A}}$.

More details on this correspondence can be found in [**dV93**, IV, §5], particularly Theorem 5.18 (though the construction given there is quite different, not based on maximal ideal spaces; for the basics on maximal ideal spaces see [**Con90**, VII, §8]).

We point out here that the correspondence restricts well to our metric setting, namely, it induces a one-to-one correspondence between u-compactifications of X and uniformly closed G-invariant subalgebras of $\mathrm{RUC}_u(X)$. Of course, if ν is a u-compactification of X then any function coming from ν is uniformly continuous, so $A_\nu \subset \mathrm{RUC}_u(X)$. Conversely, we have the following.

Fact 1.2. *If $\mathcal{A}$ is a uniformly closed G-invariant subalgebra of $\mathrm{RUC}_u(X)$ then $\nu_{\mathcal{A}} \colon X \to X^{\mathcal{A}}$ is uniformly continuous.*

Proof. Suppose to the contrary that there is an entourage ϵ of the uniformity of $X^{\mathcal{A}}$ such that for every n there are $x_n, y_n \in X$ with distance $d(x_n, y_n) < 1/n$ but such that $(\nu_{\mathcal{A}}(x_n), \nu_{\mathcal{A}}(y_n)) \notin \epsilon$. We can assume the entourage is of the form $\epsilon = \bigcup_{i<k} U_i \times U_i$ for some cover of $X^{\mathcal{A}}$ by basic open sets

$$U_i = \{p \in X^{\mathcal{A}} : |\tilde{f}_i(p)| < \delta_i\}$$

given by functions $f_i \in \mathcal{A}$ and positive reals δ_i.

Passing to a subnet we can assume that $\nu_{\mathcal{A}}(x_n)$ converges to $p \in X^{\mathcal{A}}$, say $p \in U_i$ for some $i < k$. Since $\mathcal{A}$ is contained in $\mathrm{RUC}_u(X)$ (not merely in $\mathrm{RUC}(X)$) for n big enough we have $|f_i(x_n) - f_i(y_n)| < \frac{1}{2}(\delta_i - |\tilde{f}_i(p)|)$, and also $|f_i(x_n) - \tilde{f}_i(p)| < \frac{1}{2}(\delta_i - |\tilde{f}_i(p)|)$. Thus for the same n we have $|f_i(x_n)| < \delta_i$ and $|f_i(y_n)| < \delta_i$. This implies $(\nu_{\mathcal{A}}(x_n), \nu_{\mathcal{A}}(y_n)) \in \epsilon$, a contradiction. □

Remark 1.3. Let G be a Polish group. Every u-compactification of G factorizes through the left completion $\widehat{G_L}$, and we have a canonical one-to-one correspondence between u-compactifications of G and of $\widehat{G_L}$.

The maximal u-compactification of a Polish group G, that is, the compactification G^{UC} associated to the algebra $\mathrm{UC}(G)$, is called the *Roelcke compactification* of G. If we fix any $g \in G$, the function $d_g(h) = d_L(g, h)$ is in $\mathrm{UC}(G)$. This implies that the compactification $G \to G^{\mathrm{UC}}$ is always a topological embedding.

On the other hand, for any $f \in \mathrm{RUC}(X)$ there is a minimal compactification of X from which f comes, namely the one corresponding to the closed unital algebra generated by the orbit Gf in $\mathrm{C}(X)$. It is called the *cyclic G-space* of f, and denoted by X_f.

An important part of the project developed in [**GM06, GM12, GM14b**] has been to classify the dynamical systems (and particularly their compactifications) by the possibility of representing them as an isometric action on a "good" Banach space. Although we will not make use of it in the present paper, the precise meaning of a *representation* of a G-space X on a Banach space V is given by a pair

$$h \colon G \to \mathrm{Iso}(V),\ \alpha \colon X \to V^*,$$

where h is a continuous homomorphism and α is a weak*-continuous bounded G-map with respect to the dual action $G \times V^* \to V^*$, $(g\phi)(v) = \phi(h(g)^{-1}(v))$. The topology on $\mathrm{Iso}(V)$ is that of pointwise convergence. The representation is *faithful* if α is a topological embedding.

For a family $\mathcal{K}$ of Banach spaces, a G-space X is *$\mathcal{K}$-representable* if it admits a faithful representation on a member $V \in \mathcal{K}$, and it is *$\mathcal{K}$-approximable* if it can be topologically G-embedded into a product of $\mathcal{K}$-representable G-spaces.

1.2. Roelcke precompact Polish groups. Following Uspenskij [**Usp02**, §4], the infimum of the left and right uniformities on a Polish group G is called the *Roelcke uniformity* of the group. Accordingly, G is *Roelcke precompact* if its completion with respect to this uniformity is compact —and thus coincides with the Roelcke compactification of G as defined above. This translates to the condition that for every non-empty neighborhood U of the identity there is a finite set $F \subset G$ such that $UFU = G$.

Let G be a Polish group acting by isometries on a complete metric space X. Given a point $x \in X$, we denote by $[x] = \overline{Gx}$ the closed orbit of x under the action. Then, we define the metric quotient $X /\!\!/ G$ as the space $\{[x] : x \in X\}$ of closed orbits endowed with the induced metric $d([x],[y]) = \inf_{g\in G} d(gx,y)$.

In the rest of the paper, given a countable (possibly finite) set α, we will identify it with an ordinal $\alpha \leq \omega$ and consider the power X^α as a metric G-space with the distance $d(x,y) = \sup_{i<\alpha} 2^{-i} d(x_i,y_i)$ and the diagonal action $gx = (gx_i)_{i<\alpha}$. Of course, the precise choice of the distance is arbitrary and we will only use that it is compatible with the product uniformity and that the diagonal action is by isometries.

The action of G on X is *approximately oligomorphic* if the quotients $X^\alpha /\!\!/ G$ are compact for every $\alpha < \omega$ (equivalently, for $\alpha = \omega$). Then, Theorem 2.4 in [**BT14**] showed the following.

Theorem 1.4. *A Polish group G is Roelcke precompact if and only if the action of G on its left completion $\widehat{G}_L$ is approximately oligomorphic or, equivalently, if G can be embedded in the group of isometries of a complete metric space X in such a way that the induced action of G on X is approximately oligomorphic.*

Recall that the group of isometries of a complete metric space is considered as a Polish group with the topology of pointwise convergence.

Roelcke precompact Polish groups provide a rich family of examples of topological groups with interesting dynamical properties. By means of the previous characterization, Ben Yaacov and Tsankov initiated the study of these groups from the viewpoint of continuous logic.

1.3. $\aleph_0$-categorical metric structures as G-spaces. Thus we turn to logic. We present the basic concepts and facts of the model theory of metric structures. About the general theory we shall be terse, and we refer the reader to the thorough treatments of [**BU10**] and [**BBHU08**]; in fact, we will mostly avoid the syntactical aspect of logic. Instead, we will give precise topological reformulations for the case of $\aleph_0$-categorical structures. At the same time, we explain the relation to the dynamical notions introduced before.

A metric first-order structure is a complete metric space (M,d) of bounded diameter together with a family of distinguished *basic predicates* $f_i \colon M^{n_i} \to \mathbb{R}$ ($n_i < \omega$), $i \in I$, which are uniformly continuous and bounded. (The structure may also have distinguished elements and basic functions from finite powers of M into M, as is the case of the Boolean algebra $\mathcal{B}$ considered in the examples; but these can be coded with appropriate basic predicates.) An automorphism of the structure is an isometry $g \in \mathrm{Iso}(M)$ such that each basic predicate f_i is invariant for the diagonal action of g on M^{n_i}, that is, $f_i(gx) = f_i(x)$ for all $x \in M^{n_i}$. For a separable structure M, the space $\mathrm{Aut}(M)$ of all automorphisms of M is a Polish group under the topology of pointwise convergence.

If M is separable and isomorphic to any other separable structure with the same *first-order properties*, then M is $\aleph_0$*-categorical*. A classical result in model-theory (see [**BBHU08**], Theorem 12.10) implies that this is equivalent to say that M is separable and the action of $\mathrm{Aut}(M)$ on M is approximately oligomorphic. In particular, by Theorem 1.4, $\mathrm{Aut}(M)$ is Roelcke precompact.

The structure M is *classical* if d is the Dirac distance and the basic predicates are $\{0,1\}$-valued. In this case, M is $\aleph_0$-categorical if and only if it is countable and the action of $\mathrm{Aut}(M)$ on M is *oligomorphic*, i.e., the quotients $M^n /\!\!/ \mathrm{Aut}(M)$ are finite for every $n < \omega$.

A *definable predicate* is a function $f\colon M^\alpha \to \mathbb{R}$, with α a countable set, constructed from the basic predicates and the distance by continuous combinations, rearranging of the variables, approximate quantification (i.e., suprema and infima) and uniform limits. Every definable predicate is $\mathrm{Aut}(M)$-invariant, uniformly continuous and bounded. If M is $\aleph_0$-categorical, then $f\colon M^\alpha \to \mathbb{R}$ is a definable predicate if and only if it is continuous and $\mathrm{Aut}(M)$-invariant; see for example [**BK13**], Proposition 2.2.

Definition 1.5. In this paper, we shall use the term *formula* to denote a definable predicate in two countable sets of variables, i.e., a function $f\colon M^\alpha \times M^\beta \to \mathbb{R}$, for countable sets α, β, which is a definable predicate once we rewrite the domain as a countable power of M. We will denote it by $f(x,y)$ to specify the two variables of the formula. Given a formula $f(x,y)$ and an a parameter $a \in M^\alpha$, we denote by $f_a \in \mathrm{C}(M^\beta)$ the continuous function defined by $f_a(b) = f(a,b)$. When we make no reference to α or β, we will assume that $\alpha = \omega$ and $\beta = 1$.

Whenever we talk of a metric structure M as a G-space, we understand that the group is $G = \mathrm{Aut}(M)$ and that it acts on M in the obvious way. This G-space comes with a distinguished function algebra: the family of functions of the form f_a for a formula $f(x,y)$ and a parameter $a \in M^\omega$. We will denote it by $\mathrm{Def}(M)$, and it is in fact a uniformly closed G-invariant subalgebra of $\mathrm{C}(M)$. More generally, if $a \in A^\omega$ for a subset $A \subset M$ (and the variable y is of any length β), we will say that f_a is an *A-definable predicate* in the variable y. A $\emptyset$-definable predicate is just a definable predicate. The family of A-definable predicates in y is clearly a subalgebra of $\mathrm{C}(M^\beta)$, which is uniformly closed as the following shows.

Fact 1.6. *A uniform limit of A-definable predicates is an A-definable predicate.*

Proof. Say we have formulas $f^n(x,y)$ and parameters $a_n \in A^\omega$ such that $f^n_{a_n}$ converges uniformly; without loss of generality we can assume that the tuples are the same, say $a = a_n$. Passing to a subsequence we can assume that $f^n_{a_n}$ converges fast enough, then define $f(x,y)$ as the *forced limit* of the formulas $f^n(x,y)$ (see [**BU10**, §3.2], and compare with Lemma 3.11 therein). Then the limit of the predicates f^n_a is f_a. □

The starting point for our analysis is the following observation, based on the ideas from [**BT14**, §5].

Proposition 1.7. *For a metric structure M we have* $\mathrm{Def}(M) \subset \mathrm{RUC}_u(M)$. *If M is $\aleph_0$-categorical, then moreover* $\mathrm{Def}(M) = \mathrm{RUC}_u(M)$.

Proof. For the first part consider a formula $f(x,y)$ together with a parameter $a \in M^\omega$. Take a neighborhood U of the identity such that $d(a,ga) < \Delta_f(\epsilon)$ for $g \in U$, where Δ_f is a modulus of uniform continuity for $f(x,y)$. Thus $\|gf_a - f_a\| = \|f_{ga} - f_a\| < \epsilon$ whenever $g \in U$. This shows that every $f_a \in \mathrm{Def}(M)$ is in $\mathrm{RUC}_u(M)$.

Now let $h \in \mathrm{RUC}_u(M)$, and set $a \in M^\omega$ to enumerate a dense subset of M. We define $f\colon Ga \times M \to \mathbb{R}$ by

$$f(ga,b) = gh(b) = h(g^{-1}b).$$

This is well defined because a is dense in M; note also that f is G-invariant and uniformly continuous. Indeed, we have

$$|f(ga,b) - f(g'a,b')| \leq |gh(b) - gh(b')| + |gh(b') - g'h(b')|.$$

The first term on the right side is small if b and b' are close: simply observe that $d(g^{-1}b, g^{-1}b') = d(b,b')$, so we use the uniform continuity of h. For the second, given $\epsilon > 0$ there is a neighborhood U of the identity of G such that $\|gh - g'h\| < \epsilon$ whenever $g^{-1}g' \in U$, because h is RUC; since a is dense, there is $\delta > 0$ such that $d(ga, g'a) < \delta$ implies $g^{-1}g' \in U$; thus if $d(ga,g'a) < \delta$ we have $|gh(b') - g'h(b')| < \epsilon$.

This means that f can be extended continuously to $[a] \times M$ (we recall the notation $[a] = \overline{Ga}$). The extension remains G-invariant, so we may regard f as defined on $([a]\times M) /\!\!/ G$, which is a closed subset of the metric space $(M^\omega \times M) /\!\!/ G$. Then we can apply Tietze extension theorem to get a continuous extension to $(M^\omega \times M) /\!\!/ G$. Composing with the projection we get a G-invariant continuous function

$$f\colon M^\omega \times M \to \mathbb{R}.$$

Finally, if M is $\aleph_0$-categorical then the G-invariant continuous function f is in fact a formula $f(x,y)$. Hence we have $h = f_a$, as desired. □

In light of this result, if M is $\aleph_0$-categorical, we can attempt to study the subalgebras of $\mathrm{RUC}_u(M)$ with model-theoretic tools; this is our aim.

Our conclusions will translate easily from structures to groups, the latter being the main subject of interest from the topological viewpoint. Indeed, if G is a Polish group, there is a canonical construction (first described by J. Melleray in [**Mel10**, §3]) that renders the left completion $M = \widehat{G}_L$ a metric first-order structure with automorphism group $\mathrm{Aut}(M) = G$. It suffices to take for I the set of all closed orbits in all finite powers of $\widehat{G}_L$, that is $I = \bigsqcup_{n<\omega} M^n /\!\!/ G$, then define the basic predicates $P_i\colon M^{n_i} \to \mathbb{R}$ (if $i \in M^{n_i} /\!\!/ G$) as the distance functions to the corresponding orbits: $P_i(y) = \inf_{x\in i} d(x,y)$. By Theorem 1.4, if G is Roelcke precompact then G acts approximately oligomorphically on its left completion and hence M is an $\aleph_0$-categorical structure. In addition we have the natural norm-preserving G-isomorphism $\mathrm{RUC}_u(\widehat{G}_L) \simeq \mathrm{UC}(G)$. By this means, our conclusions about the dynamics of $\aleph_0$-categorical structures will carry immediately to Roelcke precompact Polish groups.

Nevertheless, for the analysis of the examples done in Section 4 we shall use the approach initiated in [**BT14**, §5–6] for the study of WAP(G). That is, we will describe the functions on G in terms of the formulas of the "natural" structure M for which $G = \mathrm{Aut}(M)$ (see particularly Lemma 5.1 of the referred paper). To this end we have the following version of Proposition 1.7.

Proposition 1.8. *Let M be a metric structure, $G = \mathrm{Aut}(M)$. If $f(x,y)$ is an arbitrary formula and a,b are tuples from M of the appropriate length, then the function $g \mapsto f(a,gb)$ is in* UC(G). *If M is $\aleph_0$-categorical and $h \in \mathrm{UC}(G)$, then there are a formula $f(x,y)$ in ω-variables x,y and a parameter $a \in M^\omega$ such that $h(g) = f(a,ga)$ for every $g \in G$.*

Proof. The proof of Proposition 1.7 can be adapted readily. Alternatively, we remark that if $a \in M^\omega$ enumerates a dense subset of M then $[a]$ can be identified with $\widehat{G}_L$ (see [**BT14**], Lemma 2.3). Thus the basic predicates $P_i\colon (\widehat{G}_L)^{n_i} \to \mathbb{R}$ defined above are simply the restrictions to $[a]^{n_i}$ of the functions $f_i\colon (M^\omega)^{n_i} \to \mathbb{R}$, $f_i(y) = \inf_{x\in i} d(x,y)$, which are definable predicates if M is $\aleph_0$-categorical; similarly for the general definable predicates on $\widehat{G}_L$. The second claim in the statement then follows from this together with the identifications $\mathrm{UC}(G) \simeq \mathrm{RUC}_u(\widehat{G}_L) = \mathrm{Def}(\widehat{G}_L)$. □

1.4. Types, extensions, indiscernibles. Before we go on, we recall some additional terminology from model theory that we use in our expositions and proofs. Most of it could be avoided if we decided to give a prevailingly topological presentation of our results, but we have chosen to emphasize the interplay between the two domains.

Let M be a metric structure, $A \subset M$ a subset and let y be a variable of length β. A *(complete) type over A (in M) in the variable y* can be defined as a maximal ideal of the uniformly closed algebra of A-definable predicates of M in the variable y. The type over A of an element $b \in M^\beta$ is defined by

$$\mathrm{tp}(b/A) = \{f_a : a \in A^\omega, f(x,y) \text{ a formula with } f(a,b) = 0\}.$$

For $A = \emptyset$ we denote $\mathrm{tp}(b/\emptyset) = \mathrm{tp}(b)$. A more model-theoretic presentation of types in continuous logic is given in [**BBHU08**, §8] or in [**BU10**, §3]; there, a type is a set of *conditions* which an element may eventually satisfy. A type p given as an ideal is identified with the set of conditions of the form $h(y) = 0$ for $h \in p$.

The space of types over A (that is, the maximal ideal space of the algebra of A-definable predicates, with its natural topology) is denoted by $S_y^M(A)$, or by $S(A)$ when $\beta = 1$ and the structure is clear from the context. If A is G-invariant, then the algebra of A-definable predicates is G-invariant and there is a natural action of G on $S_y^M(A)$. Thus, for example, the type space $S(M)$ (together with the natural map $\mathrm{tp}\colon M \to S(M)$) is just the compactification $M^{\mathrm{Def}(M)}$. In particular, if G is Roelcke precompact, then by Proposition 1.7, Remark 1.3 and the discussion about the structure $\widehat{G}_L$ above, we have that $S(\widehat{G}_L) = G^{\mathrm{UC}}$ is just the Roelcke compactification of G.

Remark 1.9. Let $f = f(x,y)$ be an arbitrary formula and let $a \in M^\alpha$ be a parameter. The cyclic G-space of f_a (as defined after Remark 1.3) also has a name in the model-theoretic literature, at least for some authors: it coincides with the space of f*-types* over the orbit Ga as defined in [**TZ12**, p. 132]. Their definition is in the classical setting, but we can adapt it to the metric case by defining a (complete) f-type over $A \subset M^\alpha$ to be a maximal consistent set of conditions of the form $f(a',y) = r$ for $a' \in A$ and $r \in \mathbb{R}$. In other words, an f-type is a maximal ideal of the closed unital algebra generated by $\{f_{a'} : a' \in A\}$. The space of f-types over A is denoted by $S_f(A)$, and the identification $S_f(Ga) = X_{f_a}$ follows.

N.B. This does not coincide in general with the space $S_f(A)$ as defined in [**BU10**], Definition 6.6, or in [**Pil96**, p. 14]. To make the comparison simpler, say $A = B^\alpha$ for some $B \subset M$. The two definitions agree when $B = M$. In the case $B \subset M$, the latter authors define $S_f(A)$ (or $S_f(B)$ in their notation) as the maximal ideal space of the algebra of B-definable predicates in M that come from the compactification $S_f(M)$. This is larger than the one defined above, and it fits better for the study of local stability.

We shall understand $S_f(A)$ in the former sense (except in Lemma 4.2).

A tuple $b \in M^\beta$ *realizes* a type $p \in S_y^M(A)$ if we have $\mathrm{tp}(b/A) = p$. A set q of M-definable predicates in the variable y is *approximately finitely realized in* $B \subset M$ if for every $\epsilon > 0$ and every finite set of predicates $f_i \in q$, $i < k$, there is $b \in B^\beta$ such that $|f_i(b)| < \epsilon$ for each $i < k$. Remark that any $p \in S_y^M(M)$ is approximately finitely realized in M: if for example $f_a \in p$ is bounded away from zero in M, then $1/f_a$ is an A-definable predicate, hence $1 = 1/f_a \cdot f_a \in p$ and p is not a proper ideal. Conversely, by Zorn's Lemma, any set of A-predicates in y approximately finitely realized in M can be extended to a type $p \in S_y^M(A)$.

The following terminology is not standard, so we single it out.

Definition 1.10. We will say that a structure M is $\emptyset$*-saturated* if every type $p \in S_y^M(\emptyset)$ in any countable variable y is realized in M.

Suppose M is $\aleph_0$-categorical. Then the projection $M^\beta \to M^\beta /\!\!/ G$ is a compactification, and the functions that come from it are precisely the continuous G-invariant ones, i.e., the $\emptyset$-definable predicates. Hence the projection to $M^\beta /\!\!/ G$ can be identified with the compactification $\mathrm{tp}\colon M^\beta \to S_y^M(\emptyset)$. A first consequence of this identification is the following homogeneity property: if $\mathrm{tp}(a) = \mathrm{tp}(b)$ for $a, b \in M^\beta$ and we have $\epsilon > 0$, then there is $g \in G$ with $d(a, gb) < \epsilon$. A further consequence is the following.

Fact 1.11. *Every* $\aleph_0$*-categorical structure is* $\emptyset$*-saturated.*

A stronger saturation property is true for $\aleph_0$-categorical structures (they are *approximately* $\aleph_0$*-saturated*, see Definition 1.3 in [**BU07**]), but we will not use it.

Remark 1.12. The left completion $M = \widehat{G}_L$, when seen as a metric structure as defined before, is $\emptyset$-saturated if and only if it is $\aleph_0$-categorical. Indeed, if it is not $\aleph_0$-categorical then the quotient $M^n /\!\!/ G$ is not compact for some $n < \omega$, which means that there are $\epsilon > 0$ and a sequence of orbits $(i_k)_{k<\omega} \subset M^n /\!\!/ G$ any two of which are at distance at least ϵ. We may moreover assume that $(i_k)_{k<\omega}$ is maximal such, since M^n is separable. If, as before, $P_i\colon M^n \to \mathbb{R}$ denotes the distance to the orbit $i \in M^n /\!\!/ G$, then the conditions $\{P_{i_k}(y) \geq \epsilon\}_{k<\omega}$ induce a type over $\emptyset$ not realized in M.

Now suppose that we have a metric structure M given by the basic predicates $f_i\colon M^{n_i} \to \mathbb{R}$, $i \in I$. An *elementary extension* of M is a structure N with basic predicates $\tilde{f}_i\colon N^{n_i} \to \mathbb{R}$, $i \in I$, such that: (i) M is a metric subspace of N, (ii) each $\tilde{f}_i$ extends f_i, and (iii) every type $p \in S_y^N(M)$ is approximately finitely realized in $M \subset N$. One can deduce that the M-definable predicates of M are exactly the restrictions to M of the M-definable predicates of N (essentially, because (iii) ensures that approximate quantification over M and over N coincide), and the restriction is one-to-one. Hence, the spaces $S_y^M(M)$ and $S_y^N(M)$ can be identified. A metric ultrapower construction as in [**BBHU08**, §5] can be used to prove the following.

Fact 1.13. *Every metric structure M admits an elementary extension N such that every type in $S_y(M)$ in any countable variable y is realized in N. (In particular, every structure has a $\emptyset$-saturated elementary extension.)*

Thus, for most purposes, we can refer to types over M or to elements in elementary extensions of M interchangeably. For example, if $p \in S(M)$, $f_a \in \mathrm{Def}(M)$, and b is an element in an elementary extension N of M realizing p, we may prefer to write $f(a,b)$ instead of $\tilde{f}_a(p)$. We recall that the formula $f(x,y)$ of M extends uniquely to a formula of N, and we identify them.

An *indiscernible sequence* in a structure M is a sequence $(a_i)_{i<\omega} \subset M^\beta$ such that, for any $i_1 < \cdots < i_k < \omega$, we have $\mathrm{tp}(a_{i_1} \ldots a_{i_k}) = \mathrm{tp}(a_1 \ldots a_k)$. In a finitary version, if Δ is a finte set of definable predicates and δ is a positive real, then a sequence $(a_i)_{i<\omega}$ is Δ-δ-*indiscernible* if $|\phi(a_{i_1},\ldots,a_{i_k}) - \phi(a_{j_1},\ldots,a_{j_k})| \leq \delta$ for every $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ and every definable predicate $\phi(y_1,\ldots,y_k) \in \Delta$.

Finally, we shall say that a subset $A \subset M^\alpha$ is *type-definable* if it is of the form $\{a \in M^\alpha : f_j(a) = 0 \text{ for all } j \in J\}$ for a family of definable predicates $f_j(x)$, $j \in J$. In particular, every type-definable set is G-invariant and closed. If M is $\aleph_0$-categorical, then any G-invariant closed set is type-definable, even by a single predicate, namely the (continuous G-invariant) distance function $P_A(x) = d(x,A)$. In general, A is called *definable* precisely when the distance function P_A is a definable predicate. In the latter case, if $f(x,y)$ is any formula, then $F(y) = \sup_{x\in A} f(x,y)$ is a definable predicate too, and similarly for the infimum (see [**BBHU08**], Theorem 9.17). If A is definable and N is an elementary extension of M, P_A will denote the definable predicate that coincides with $d(x,A)$ on M^α; thus an element $a \in N^\alpha$ satisfying $P_A(a) = 0$ need not be in A.

1.5. Almost periodic functions. We end this section with some comments about the smallest function algebra presented in the introduction. A continuous bounded function h on a metric G-space X is *almost periodic (*AP*)* if the orbit Gh is a precompact subset of $\mathrm{C}(X)$ (with respect to the topology of the norm). As is easy to check, the family $\mathrm{AP}(X)$ of almost periodic functions on X is a uniformly closed G-invariant subalgebra of $\mathrm{RUC}(X)$. Moreover, if h comes from a compactification $\nu\colon X \to Y$, it is clear that h is AP if and only if its extension to Y is AP. By the Arzelà–Ascoli theorem, we have that $h \in \mathrm{AP}(Y)$ if and only if for every $\epsilon > 0$ and $y \in Y$ there is an open neighborhood O of y such that

$$|h(gy) - h(gy')| < \epsilon$$

for every $y' \in O$ and $g \in G$. From the point of view of Banach space representations, almost periodic functions are precisely those coming from Euclidean-approximable compactifications of X; see [**GM14b**, §5.2] and [**Meg08**], Proposition 3.7.2.

The definition given in the following proposition will be useful for the description of AP functions in the examples of Section 4. (The terminology is not standard.)

Proposition 1.14. *Let M be a $\emptyset$-saturated structure. Let $f(x,y)$ be a formula and $A \subset M^\alpha$, $B \subset M^\beta$ be definable sets. The following are equivalent, and in any of these cases we will say that $f(x,y)$ is* algebraic *on $A \times B$.*

(1) the set $\{f_a|_B : a \in A\}$ is precompact in $\mathrm{C}(B)$;
(2) for every indiscernible sequence $(a_i)_{i<\omega} \subset A$, the predicates $f(a_i,y)$ are all equivalent in B, i.e., we have $f(a_i,b) = f(a_j,b)$ for all i,j and $b \in B$.

Proof. (1) $\Rightarrow$ (2). By precompactness, the sequence $(f_{a_i})_{i<\omega}$ has a Cauchy subsequence, so in particular there are i and j such that $\sup_{y\in B}|f(a_i,y) - f(a_j,y)| \le \epsilon$. By indiscernibility, this is true for all i,j, and the claim follows.

(2) $\Rightarrow$ (1). Let $\epsilon > 0$. If the set of conditions in the variables $(x_i)_{i<\omega}$ given by

$$|\phi(x_{i_1},\ldots,x_{i_k}) - \phi(x_{j_1},\ldots,x_{j_k})| = 0,\ P_A(x_i) = 0,\ \sup_{y\in B}|f(x_i,y) - f(x_j,y)| \ge \epsilon$$

(where ϕ varies over the definable predicates of M, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$), was approximately finitely realized in M, then by $\emptyset$-saturation we could get an indiscernible sequence in M contradicting (2). Therefore, there are a finite set Δ of definable predicates and $\delta > 0$ such that any Δ-δ-indiscernible sequence $(a_i)_{i<\omega} \subset A$ satisfies $\sup_{y\in B}|f(a_i,y) - f(a_j,y)| < \epsilon$ for all i,j.

For every $n < \omega$ let Δ_n, δ_n correspond to $\epsilon = 1/n$ as before. Starting with an arbitrary sequence $(a_i)_{i<\omega} \subset A$, by Ramsey's theorem we can extract a Δ_1-δ_1-indiscernible subsequence, say $(a_i^1)_{i<\omega}$. Inductively, let $(a_i^{n+1})_{i<\omega}$ be a Δ_{n+1}-δ_{n+1}-indiscernible subsequence of $(a_i^n)_{i<\omega}$. If we take $a_j^\omega = a_j^j$ then $(a_j^\omega)_{j<\omega}$ is a subsequence of $(a_i)_{i<\omega}$ and $(f_{a_j^\omega})_{j<\omega}$ is a Cauchy sequence in $\mathrm{C}(B)$. □

Remark 1.15. As the reader can check, the previous proposition holds true if A and B are merely type-definable. In particular, one may consider the case where $A = \{a' \in M^\alpha : \mathrm{tp}(a') = \mathrm{tp}(a)\}$ for some $a \in M^\alpha$, and $B = M^\beta$. If the above equivalent conditions hold in this case (for a saturated model M), it is standard terminology to say that (the *canonical parameter* of) f_a is *algebraic over the empty set*, in symbols $f_a \in \mathrm{acl}(\emptyset)$. Alternatively, in the terminology of Pillay [**Pil96**, p. 9], f_a is *almost $\emptyset$-definable*.

Set $\beta = 1$. Suppose M is $\aleph_0$-categorical, so in particular $A = [a] = \overline{Ga}$. Now, since $f(x,y)$ is uniformly continuous, the families Gf_a and $\{f_{a'} : a' \in [a]\}$ have the same closure in $\mathrm{C}(M)$. We can conclude by Proposition 1.7 that $h \in \mathrm{AP}_u(M)$ if and only if $h = f_a$ for some predicate $f_a \in \mathrm{acl}(\emptyset)$.

The compactification $b\colon G \to bG = G^{\mathrm{AP}}$ associated to the algebra $\mathrm{AP}(G)$ is the *Bohr compactification* of G. The space bG has the structure of a (compact) group making b a homomorphism (see [**dV93**, (D.12)3 and IV(6.15)3]). In fact, the compactification b is the *universal group compactification* of G: if $\nu\colon G \to K$ is a compactification and also a homomorphism into a compact group K, it is easy to see that $\mathcal{A}_\nu \subset \mathrm{AP}(G)$, whence ν factors through b. I. Ben Yaacov has observed the following fact.

Theorem 1.16. *The Bohr compactification $b\colon G \to bG$ of a Roelcke precompact Polish group is always surjective.*

See [**Ben15**], Corollary 5.3. As mentioned there in the introduction, the model-theoretic counterpart of this result is the fact that $\aleph_0$-categoricity is preserved after naming the algebraic closure of the empty set (see Proposition 1.15).

One could call a metric G-space X *almost periodic* if $\mathrm{AP}(X) = \mathrm{RUC}_u(X)$. This is a very strong condition. Indeed, for an action of a topological group G by isometries on a complete bounded metric space (X,d), the function $P_a(y) = d(a,y)$ (which is in $\mathrm{RUC}_u(X)$) is AP if and only if the closed orbit $[a]$ is compact. If the space of closed orbits $X /\!\!/ G$ is compact, we can deduce that X is almost periodic if and only if X is compact (the reverse implication following from Arzelà–Ascoli theorem). This is the case for $\aleph_0$-categorical structures. Also, if G is any Polish group with $\mathrm{AP}(G) = \mathrm{UC}(G)$ then $b\colon G \to bG$ is a topological embedding into a compact Hausdorff group, which implies that G is already compact (see [**dV93**, D.12.4] together with [**BK96**, p. 3–4]).

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2. WAP = Asp = SUC

Let $f\colon M^\alpha \times M^\beta \to \mathbb{R}$ be any formula on a metric structure M, and let $A \subset M^\alpha$, $B \subset M^\beta$ be any subsets. We recall that $f(x,y)$ has the *order property*, let us say, *on* $A \times B$ if there are $\epsilon > 0$ and sequences $(a_i)_{i<\omega} \subset A$, $(b_j)_{j<\omega} \subset B$ such that $|f(a_i,b_j) - f(a_j,b_i)| \geq \epsilon$ for all $i < j < \omega$. If $f(x,y)$ lacks the order property on $A \times B$ we say that it is *stable on* $A \times B$. We invoke the following crucial result, essentially due to Grothendieck, as pointed out by Ben Yaacov in [**Ben13a**] (see Fact 2 and the discussion before Theorem 3 therein).

Fact 2.1. *The formula $f(x,y)$ is stable on $A \times B$ if and only if $\{f_a|_B : a \in A\}$ is weakly precompact in* $\mathrm{C}(B)$.

(In the rest of this section we will only need the case $B = M$ ($\beta = 1$), so we shall only specify A when referring to stability or the order property.)

On the other hand, a function $h \in \mathrm{C}(X)$ on a G-space X is *weakly almost periodic (*WAP*)* if the orbit $Gh \subset \mathrm{C}(X)$ is weakly precompact (that is, precompact with respect to the weak topology on $\mathrm{C}(X)$). It is not difficult to check that the family $\mathrm{WAP}(X)$ of weakly almost periodic functions on X is a uniformly closed G-invariant subalgebra of $\mathrm{C}(X)$ (for instance, resorting to Grothendieck's double limit criterion: Fact 2 in [**Ben13a**]), but it is a bit involved to prove that $\mathrm{WAP}(X)$ is in fact a subalgebra of $\mathrm{RUC}(X)$; see Fact 2.7 in [**Meg03**] and the references thereof. If one knows that a function $h \in \mathrm{C}(X)$ comes from a compactification $\nu\colon X \to Y$, it is an immediate consequence of Grothendieck's double limit criterion (in the form stated in [**Ben13a**]) that $h \in \mathrm{WAP}(X)$ if and only if $\tilde{h} \in \mathrm{WAP}(Y)$.

From Fact 2.1 above we have, for M-definable predicates, that f_a is WAP if and only if $f(x,y)$ is stable on $A = Ga$ (equivalently, on its closure $[a]$). By Proposition 1.7 one concludes the following (compare with Lemma 5.1 in [**BT14**]).

Lemma 2.2. *If M is $\aleph_0$-categorical, then a continuous function is in* $\mathrm{WAP}_u(M)$ *if and only if it is of the form f_a for a formula $f(x,y)$ stable on $[a]$.*

The algebra WAP(X) can also be characterized as the class of functions coming from a reflexive-representable compactification of X. This was first proven in [**Meg03**], Theorem 4.6; for an alternative exposition see Theorem 2.9 in [**Meg08**]. (We also point out the paper of Iovino [**Iov99**] for an earlier treatment of the connection between stability and reflexive Banach spaces.)

A natural generalization of weak almost periodicity is thus to replace *reflexive* by *Asplund* in the latter characterization. Recall that a Banach space is Asplund if the dual of every separable subspace is separable, and that every reflexive space has this property. In this way one gets the family of *Asplund functions*, Asp(X). This is a uniformly closed G-invariant subalgebra of RUC(X). See [**Meg03**, §7].

For a *compact* G-space Y, a function $h \in \mathrm{C}(Y)$ is shown to be Asplund if and only if the orbit $Gh \subset \mathrm{C}(Y)$ is a *fragmented family*; see Theorem 9.12 in [**GM06**]. This means that for any nonempty $B \subset Y$ and any $\epsilon > 0$ there exists an open set $O \subset Y$ such that $B \cap O$ is nonempty and

$$|h(gy) - h(gy')| < \epsilon$$

for every $g \in G$ and $y, y' \in B \cap O$. If X is an arbitrary G-space, then $h \in \mathrm{C}(X)$ belongs to Asp(X) if and only if it comes from an Asplund function on some compactification of X. If a function h comes from two compactifications Y and Z with Y larger than Z, it is an exercise (using the characterization by fragmentability) to check that the extension of h to Y is Asplund if and only if so is its extension to Z (see the proof of Lemma 6.4 in [**GM06**]). That is, any extension of h to some compactification can be used to check whether h is Asplund; for example, a predicate $f_a \in \mathrm{Def}(M)$ is Asplund if and only if its extension to $S(M)$ or to $S_f(Ga)$ satisfies the fragmentability condition.

It will be interesting to bring in a further weaker notion, introduced in [**GM08**]. A function $h \in \mathrm{C}(Y)$ on a compact G-space is *strongly uniformly continuous (*SUC*)* if for every $y \in Y$ and $\epsilon > 0$ there exists a neighborhood U of the identity of G such that

$$|h(gy) - h(guy)| < \epsilon$$

for all $g \in G$ and $u \in U$. In this case it is immediate that, if $j: Y \to Z$ is a compactification between compact G-spaces, then $h \in \mathrm{C}(Z)$ is SUC if and only if $hj \in \mathrm{C}(Y)$ is SUC. A function $h \in \mathrm{RUC}(X)$ on an arbitrary G-space X is called SUC if its extension to some (any) compactification (from which h comes) is SUC. One can see readily that: (i) the family of all strongly uniformly continuous functions on a G-space X forms a uniformly closed G-invariant subalgebra SUC(X) of RUC(X); (ii) every Asplund function is SUC: in the fragmentability condition we take $B = Gy \subset Y$, then use the continuity of the action of G on Y.

It follows from our remarks so far that, in general,

$$\mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{SUC}(X) \subset \mathrm{RUC}(X).$$

It is also clear that $\mathrm{SUC}(G) \subset \mathrm{UC}(G)$ for the regular left action of G on itself: we apply the property defining SUC to the compactification $Y = G^{\mathrm{RUC}}$ (for instance) and the identity element $y = 1 \in G \subset Y$.

An important motivation for the algebra of SUC functions comes from the viewpoint of semigroup compactifications of G. We have already mentioned the universal property of G^{AP}. Similarly, G^{WAP} is the *universal semitopological semigroup compactification* of G (see [**Usp02**, §5]). For their part, G^{Asp} and G^{RUC} are *right topological semigroup compactifications* of G. In their work [**GM08**], the authors showed that the compactification G^{SUC} is also a right topological semigroup compactification of G, and that $\mathrm{SUC}(G)$ is the largest subalgebra of $\mathrm{UC}(G)$ with this property (see Theorem 4.8 therein). In particular, the Roelcke compactification G^{UC} has the structure of a right topological semigroup if and only if $\mathrm{SUC}(G) = \mathrm{UC}(G)$.

We aim to prove the equality WAP = SUC (restricted to RUC_u) for $\aleph_0$-categorical structures and for their automorphism groups.

Switching to logic language, let us say that a formula $f(x,y)$ is SUC *on a subset* $A \subset M^\alpha$ if for any b in any elementary extension of M and every $\epsilon > 0$ there are $\delta > 0$ and a finite tuple c from M such that for every $a \in A$ and every automorphism $u \in G$ satisfying $d(uc,c) < \delta$ we have

$$|f(a,b) - f(ua,b)| < \epsilon.$$

We readily get the following.

Lemma 2.3. *For a metric structure M, a function $f_a \in \mathrm{Def}(M)$ is* SUC *if and only if the formula $f(x,y)$ is* SUC *on $[a]$.*

The most basic example of a non-stable formula in an $\aleph_0$-categorical structure is the order relation on the countable dense linear order without endpoints. It is worth looking into this case.

Example 2.4. The order relation $x < y$ on the (classical) structure $(\mathbb{Q},<)$ is not SUC on $\mathbb{Q}$. Indeed, let $r \in \mathbb{R} \setminus \mathbb{Q}$, $c_1,\dots,c_n \in \mathbb{Q}$. Suppose $c_i < c_{i+1}$ for each i, and say $c_{i_0} < r < c_{i_0+1}$. Take $a \in \mathbb{Q}$, $c_{i_0} < a < r$. There is a monotone bijection u fixing every c_i and such that $r < ua < c_{i_0+1}$. The claim follows.

We will generalize the analysis of this simple example to any non-stable formula in any $\aleph_0$-categorical structure. To this end we shall use the following standard lemma.

Fact 2.5. *Let M be $\emptyset$-saturated, $A \subset M^\omega$ a type-definable subset. If a formula $f(x,y)$ has the order property on A, then there are an elementary extension N of M, distinct real numbers $r,s \in \mathbb{R}$ and elements $(a_i)_{i\in\mathbb{Q}} \subset A$, $(b_j)_{j\in\mathbb{R}} \subset N$ such that $f(a_i,b_j) = r$ for $i < j$ and $f(a_i,b_j) = s$ for $j \leq i$.*

Proof. Suppose there are $\epsilon > 0$ and sequences $(a'_k)_{k<\omega} \subset A$, $(b'_l)_{l<\omega} \subset M$ such that $|f(a'_k,b'_l) - f(a'_l,b'_k)| \geq \epsilon$ for all $k < l < \omega$. Since f is bounded, passing to subsequences carefully we can assume that $\lim_k \lim_l f(a'_k,b'_l) = r$ and $\lim_k \lim_l f(a'_l,b'_k) = s$, necessarily with $|r-s| \geq \epsilon > 0$. Now we consider the conditions in the countable variables $(x_i)_{i\in\mathbb{Q}}$, $(y_j)_{j\in\mathbb{Q}}$ asserting, for each pair of rational numbers $i < j$,

$$x_i \in A,\ f(x_i,y_j) = r \text{ and } f(x_j,y_i) = s.$$

The elements a'_k, b'_l can be used to show that these conditions are approximately finitely realized in M. By saturation, there are $(a_i)_{i\in\mathbb{Q}} \subset A$, $(b_j)_{j\in\mathbb{Q}} \subset M$ satisfying the conditions.

Finally, the conditions $f(a_i, y_j) = r$ for $i < j$, $i \in \mathbb{Q}$, $j \in \mathbb{R} \setminus \mathbb{Q}$, together with $f(a_i, y_j) = s$ for $j \le i$, $i \in \mathbb{Q}$, $j \in \mathbb{R} \setminus \mathbb{Q}$, are approximately finitely realized in $\{b_j\}_{j\in\mathbb{Q}}$. Hence they are realized by elements $(b_j)_{j\in\mathbb{R}\setminus\mathbb{Q}}$ in some elementary extension of M. □

Proposition 2.6. *Let M be $\aleph_0$-categorical. If $f(x,y)$ has the order property on a definable set A, then $f(x,y)$ is not* SUC *on A.*

Proof. We apply the previous fact to find elements $(a_i)_{i\in\mathbb{Q}} \subset A$, $(b_j)_{j\in\mathbb{R}}$ in some elementary extension of M and real numbers $r \neq s$ such that $f(a_i, b_j) = r$ if $i < j$, $f(a_i, b_j) = s$ if $j \le i$. Suppose $f(x,y)$ has the SUC property for $\epsilon = |r-s|/2$; since G is second countable and $\mathbb{R}$ is uncountable, there is an open neighborhood U of the identity that witnesses the property for an infinite number of elements b_j, say for every b_j with j in an infinite set $J \subset \mathbb{R}$. By passing to a subset we may assume that J is discrete, and thus for each $j \in J$ we may take a rational $i(j) < j$ such that $j' < i(j)$ for every $j' < j$, $j' \in J$. We may assume that U is the family of automorphisms moving a finite tuple c at a distance less than δ; say n is the length of the tuple c. Now let $\eta = \Delta_f(|r-s|/2)$, where Δ_f is a modulus of uniform continuity for $f(x,y)$.

Since M is $\aleph_0$-categorical the quotient $M^\omega /\!\!/ G$ is compact, so there must be a pair $j < j'$ in J and an automorphism u such that

$$d(u(ca_{i(j)}), ca_{i(j')}) < \min(\delta, \eta/2^n).$$

In particular $d(uc, c) < \delta$, so $u \in U$. In addition, since $d(ua_{i(j)}, a_{i(j')}) < \eta$, $f(a_{i(j')}, b_j) = s$ and $f(a_{i(j)}, b_j) = r$, we have

$$|f(a_{i(j)}, b_j) - f(ua_{i(j)}, b_j)| \ge |r-s|/2,$$

contradicting the fact that U witnesses the SUC property for b_j and $\epsilon = |r-s|/2$. □

Remark 2.7. We can offer a maybe more conceptual argument to a model-theorist. Suppose M is $\aleph_0$-categorical, take $f_a \in \mathrm{SUC}_u(M) \subset \mathrm{Def}(M)$ (we recall Proposition 1.7) and let p be a type in $S_f(Ga)$. Consider $d_p f\colon Ga \to \mathbb{R}$ given by $d_p f(ga) = g\tilde{f}_a(p)$, which is well-defined and uniformly continuous. Now, the SUC condition for the extension $\tilde{f}_a\colon S_f(Ga) \to \mathbb{R}$ gives, for every $\epsilon > 0$, a neighborhood U of the identity of G such that

$$|d_p f(u^{-1}ga) - d_p f(ga)| < \epsilon$$

for every $g \in G$ and $u \in U$. That is to say, $d_p f \in \mathrm{RUC}_u(Ga)$. A mild adaptation of Proposition 1.7 allows us to deduce that $d_p f$ is an M-definable predicate on Ga. In other words, *every f-type over Ga is definable in M*, which (bearing in mind that M is saturated and that $[a]$ is definable) is well-known to be equivalent to the stability of $f(x,y)$ on Ga. For more on definability of types in continuous logic see [**BU10**, §7] (particularly Proposition 7.7 for the equivalences of stability), and the topical discussion of [**Ben13a**]. Yet an argument based on some variation of Fact 2.5 is needed to prove that definability of types implies stability.

The proposition and previous lemmas yield the desired conclusion.

Corollary 2.8. *Let M be an $\aleph_0$-categorical structure. Then* $\mathrm{WAP}_u(M) = \mathrm{Asp}_u(M) = \mathrm{SUC}_u(M)$.

Theorem 2.9. *Let G be a Roelcke precompact Polish group. Then* $\mathrm{WAP}(G) = \mathrm{Asp}(G) = \mathrm{SUC}(G)$.

Proof. From Remark 1.3 we can deduce that the isomorphism $\mathrm{RUC}_u(\widehat{G_L}) \simeq \mathrm{UC}(G)$ preserves WAP and SUC functions. Thus if $f \in \mathrm{SUC}(G)$ then its continuous extension $\tilde{f}$ to $\widehat{G_L}$ is SUC, so by the previous corollary $\tilde{f} \in \mathrm{WAP}(\widehat{G_L})$; hence $f \in \mathrm{WAP}(G)$. □

For the case of Asplund functions we can give a slight generalization, which applies for example to any M-definable predicate in an approximately $\aleph_0$-saturated separable structure.

If c is an n-tuple (of tuples) and I a is an n-tuple of intervals of $\mathbb{R}$, let us write $f(c,d) \in I$ instead of $(f(c_k,d))_{k<n} \in \prod_{k<n} I_k$. Let us call a formula $f(x,y)$ *Asplund on a subset $A \subset M^\alpha$* of a metric structure M if it *lacks* the following property: (SP) There exist $\epsilon > 0$ and a set B in some elementary extension of M such that, if $f(c,d) \in I$ for some $d \in B$, some tuple c from A and some tuple I of open intervals of $\mathbb{R}$, then there are $b, b' \in B$, $a \in A$ with $f(c,b), f(c,b') \in I$ and $|f(a,b) - f(a,b')| \geq \epsilon$. This makes a function $f_a \in \mathrm{Def}(M)$ Asplund in the topological sense if and only if $f(x,y)$ is Asplund on the orbit Ga, or on its closure $[a]$.

Proposition 2.10. *Let M be a separable $\emptyset$-saturated structure. Let $f(x,y)$ be a formula and $a \in M^\alpha$ a parameter, and suppose that the closed orbit $[a]$ is type-definable. If $f_a \in \mathrm{Asp}(M)$, then $f_a \in \mathrm{WAP}(M)$.*

Proof. Suppose $f(x,y)$ has the order property on $[a]$. Let $(a_i)_{i\in\mathbb{Q}} \subset [a]$, $(b_j)_{j\in\mathbb{R}}$ and $r, s \in \mathbb{R}$ be as given by Fact 2.5. Since M is separable, it is enough to check the condition SP for a countable family C of pairs (c,I). There is at most a countable number of reals l such that, for some $(c,I) \in C$, we have $f(c,b_j) \in I$ if and only if $j = l$. So by throwing them away we may assume that, whenever $f(c,b_l) \in I$, $(c,I) \in C$, there is $j \neq l$ with $f(c,b_j) \in I$; if we then choose $i \in \mathbb{Q}$ lying between l and j, we have $|f(a_i,b_l) - f(a_i,b_j)| = |r - s|$. Hence $f(x,y)$ has SP for $\epsilon = |r - s|$ and $B = \{b_j\}$. □

The previous proposition can be used to get information about certain continuous functions on some (non Roelcke precompact) Polish groups, but not via the structure $M = \widehat{G_L}$, which in general is not $\emptyset$-saturated as mentioned in Remark 1.12. Instead, it may be applied to automorphism groups of saturated structures and functions of the form $g \mapsto f(a, gb)$.

Example 2.11. Let us consider the linearly ordered set $M = (\mathbb{Z}, <)$ (which, as a G-space, can be identified with its automorphism group, $G = \mathbb{Z}$). The basic predicate is given by $P_<(x,y) = 0$ if $x < y$, $P_<(x,y) = 1$ otherwise. The indicator function of the non-positive integers, $f = \mathbb{1}_{\mathbb{Z}_{\leq 0}} \in \mathrm{C}(\mathbb{Z})$, is an M-definable predicate, $f(y) = P_<(0,y)$. It is clearly not in $\mathrm{WAP}(\mathbb{Z})$. However, it comes from the two-point compactification $X = \mathbb{Z} \cup \{-\infty, +\infty\}$, and it is easy to check that its extension to X

satisfies the fragmentability condition, whence in fact $f \in \mathrm{Asp}(\mathbb{Z})$ (more generally, see [**GM06**], Corollary 10.2). Of course, M is not $\emptyset$-saturated.

On the other hand, we can consider the linearly ordered set $N = \bigsqcup_{i\in(\mathbb{Q},<)}(\mathbb{Z},<)_i$ (where each $(\mathbb{Z},<)_i$ is a copy of $(\mathbb{Z},<)$), which is a $\emptyset$-saturated elementary extension of M (say $M = (\mathbb{Z},<)_0$). The automorphism group of N is $G = \mathbb{Z}^{\mathbb{Q}} \rtimes \mathrm{Aut}(\mathbb{Q},<)$. As an M-definable predicate on N, f is the indicator function of the set of elements of N that are not greater than $0 \in M \subset N$. As before, $f \notin \mathrm{WAP}(N)$, but then by Proposition 2.10 we have $f \notin \mathrm{Asp}(N)$ either (note that the orbit of 0 is N). As per Proposition 1.8, the function $h\colon g \mapsto f(g(0))$ is in $\mathrm{UC}(G)$. Since the continuous G-map $g \in G \mapsto g(0) \in N$ is surjective, any compactification of N induces a compactification of G. It follows that $h \in \mathrm{UC}(G) \setminus \mathrm{Asp}(G)$. (However, here one can also adapt the argument of Example 2.4 to show that in fact $f \notin \mathrm{SUC}(N)$ and hence $h \notin \mathrm{SUC}(G)$.)

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3. Tame ∩ UC = NIP = Null ∩ UC

Tame functions have been studied by Glasner and Megrelishvili in [**GM12**], after the introduction of tame dynamical systems by Köhler [**Köh95**] (who called them *regular systems*) and later by Glasner in [**Gla06**]. If the translation of Ben Yaacov and Tsankov for Roelcke precompact Polish groups identifies WAP functions with stable formulas, we remark in this section that tame functions correspond to NIP (or *dependent*) formulas. The study of this model-theoretic notion, a generalization of local stability introduced by Shelah [**She71**], is an active and important domain of research, mainly in the classical first-order setting —though, as the third item of the following proposition points out, the notion has a very natural metric presentation.

Proposition 3.1. *Let M be a $\emptyset$-saturated structure. Let $f(x,y)$ be a formula and $A \subset M^\alpha$, $B \subset M^\beta$ be definable sets. The following are equivalent; in any of these cases, we will say that $f(x,y)$ is* NIP *on $A\times B$.*

1. *There do not exist real numbers $r \neq s$, a sequence $(a_i)_{i<\omega} \subset A$ and a family $(b_I)_{I\subset\omega}$ in some elementary extension, with $P_B(b_I) = 0$ for all $I \subset \omega$, such that for all $i < \omega$, $I \subset \omega$,*
$$f(a_i,b_I) = r \text{ if } i \in I \text{ and } f(a_i,b_I) = s \text{ if } i \notin I.$$
2. *For every indiscernible sequence $(a_i)_{i<\omega} \subset A$ and every $b \in B$ (equivalently, for every b in any elementary extension satisfying $P_B(b) = 0$), the sequence $(f(a_i,b))_{i<\omega}$ converges in $\mathbb{R}$.*
3. *Every sequence $(a_i)_{i<\omega} \subset A$ admits a subsequence $(a_{i_j})_{j<\omega}$ such that $(f(a_{i_j},b))_{j<\omega}$ converges in $\mathbb{R}$ for any b in any elementary extension satisfying $P_B(b) = 0$.*

Proof. (1) ⇒ (2). Let $(a_i)_{i<\omega} \subset A$ be indiscernible, b arbitrary with $P_B(b) = 0$. If $(f(a_i,b))_{i<\omega}$ does not converge, there exist reals $r \neq s$ such that (replacing $(a_i)_{i<\omega}$ by a subsequence) $f(a_{2i},b) \to r$, $f(a_{2i+1},b) \to s$. By $\emptyset$-saturation we may assume that $f(a_{2i},b) = r$ and $f(a_{2i+1},b) = s$ for all i. Given $I \subset \omega$, take a strictly increasing function $\tau\colon \omega \to \omega$ such that $\tau(i)$ is even if and only if i is in I. By indiscernibility, the set of conditions

$$\{f(a_i,y) = t : t \in \{r,s\},\ f(a_{\tau(i)},b) = t\},\ P_B(y) = 0,$$

is approximately finitely realized in M; take b_I to be a realization in some model. Thus, for all i and I, $f(a_i,b_I) = r$ if $i \in I$ and $f(a_i,b_I) = s$ if $i \notin I$, contradicting (1).

(2) ⇒ (3). We claim that for every $\epsilon > 0$ there are some $\delta > 0$ and a finite set of formulas Δ such that, for any b with $P_B(b) = 0$ and every Δ-δ-indiscernible sequence $(a_i)_{i<\omega} \subset A$, there exists $N < \omega$ with $|f(a_i,b) - f(a_j,b)| < \epsilon$ for all $i,j \geq N$. Otherwise, there are $\epsilon > 0$ and, for any Δ, δ as before, a Δ-δ-indiscernible sequence $(a_i)_{i<\omega} \subset A$ and a tuple b with $P_B(b) = 0$ such that $|f(a_{2i},b) - f(a_{2i+1},b)| \geq \epsilon$ for all $i < \omega$. By $\emptyset$-saturation, we can assume that $(a_i)_{i<\omega}$ is indiscernible and $b \in B$. Then, by (2), the sequence $(f(a_i,b))_{i<\omega}$ should converge, but cannot. The claim follows.

Now suppose that Δ_n, δ_n correspond to $\epsilon = 1/n$ as per the previous claim. Given any sequence $(a_i^n)_{i<\omega}$ we can extract, using Ramsey's theorem, a Δ_{n+1}-δ_{n+1}-indiscernible subsequence $(a_i^{n+1})_{i<\omega}$. As in the proof of Proposition 1.14, starting with any $(a_i)_{i<\omega} = (a_i^0)_{i<\omega}$, proceeding inductively and taking the diagonal, we get a subsequence $(a_{i_j})_{j<\omega}$ such that $(f(a_{i_j},b))_{j<\omega}$ converges for any b satisfying $P_B(b) = 0$.

(3) ⇒ (1). Assume we have $(a_i)_{i<\omega}$, $(b_I)_{I\subset\omega}$ and r,s contradicting (1). If $(a_{i_j})_{j<\omega}$ is as given by (3) and $J \subset \omega$ is infinite and coinfinite in $\{i_j : j < \omega\}$, then $(f(a_{i_j},b_J))_{j<\omega}$ converges to both r and s, a contradiction. □

A subset of a topological space is said *sequentially precompact* if every sequence of elements of the subset has a convergent subsequence; we can restate the third item of the previous proposition in the following manner.

Corollary 3.2. *Let M be $\emptyset$-saturated and $A \subset M^\alpha$, $B \subset M^\beta$ be definable sets. A formula $f(x,y)$ is* NIP *on $A \times B$ if and only if $\{\tilde{f}_a|_{B^*} : a \in A\}$ is sequentially precompact in $\mathbb{R}^{B^*}$, where $\tilde{f}_a|_{B^*}$ is the extension of f_a to $B^* = \{p \in S_y(M) : P_B \in p\}$. If $A' \subset A$ is a dense subset, it is enough to check that $\{\tilde{f}_a|_{B^*} : a \in A'\}$ is sequentially precompact in $\mathbb{R}^{B^*}$.*

The proposition and corollary hold true, with the same proof and the obvious adaptations regarding P_B, if A and B are merely type-definable. In the literature, a formula in a given theory is said simply NIP if the previous conditions are satisfied on $M^\alpha \times M^\beta$ for some saturated model M of the theory.

We turn to the topological side. Tame dynamical systems were originally introduced in terms of the enveloping semigroup of a dynamical system, and admit several equivalent presentations. The common theme are certain dichotomy theorems that have their root in the fundamental result of Rosenthal [**Ros74**]: a Banach space either contains an isomorphic copy of ℓ_1 or has the property that every bounded sequence has a weak-Cauchy subsequence.

A Banach space is thus called *Rosenthal* if it contains no isomorphic copy of ℓ_1. Then, a continuous function $f \in \mathrm{C}(X)$ on an arbitrary G-space is *tame* if it comes from a Rosenthal-representable compactification of X. See [**GM12**], Definition 5.5 and Theorem 6.7. See also Lemma 5.4 therein and the reference after Definition 5.5 to the effect that the family Tame(X) of all tame functions on X forms a uniformly closed G-invariant subalgebra of $\mathrm{RUC}(X)$. For metric X we shall mainly consider the restriction $\mathrm{Tame}_u(X) = \mathrm{Tame}(X) \cap \mathrm{RUC}_u(X)$, as per Note 1.1.

From Proposition 5.6 and Fact 4.3 from [**GM12**] we have the following characterization of tame functions on compact systems.

Fact 3.3. *A function $f \in C(Y)$ on a compact G-space Y is tame if and only if every sequence of functions in the orbit Gf admits a weak-Cauchy subsequence or, equivalently, if Gf is sequentially precompact in $\mathbb{R}^Y$.*

Remark 3.4. A direct consequence of this characterization is the following property: if $j\colon Y \to Z$ is a compactification between compact G-spaces and $h \in \mathrm{C}(Z)$, then $h \in \mathrm{Tame}(Z)$ if and only if $hj \in \mathrm{Tame}(Y)$, which says that a function on an arbitrary G-space X is tame if and only if all (or any) of its extensions to compactifications of X are tame. (We had already pointed out the same property for AP, WAP, Asplund and SUC functions.) In fact, observe that the property holds true if $j\colon Y \to Z$ is just a continuous G-map with dense image between arbitrary G-spaces, since in this case j induces a compactification $j_h\colon Y_{hj} \to Z_h$ between the corresponding (compact) cyclic G-spaces.

The link with NIP formulas is then immediate.

Proposition 3.5. *Let M be an $\aleph_0$-categorical structure. Then $h \in \mathrm{Tame}_u(M)$ if and only if $h = f_a$ for a formula $f(x,y)$ that is* NIP *on $[a] \times M$. More generally, if $f(x,y)$ is a formula, $a \in M^\alpha$, and $B \subset M^\beta$ is definable, we have $f_a|_B \in \mathrm{Tame}(B)$ if and only if $f(x,y)$ is* NIP *on $[a] \times B$.*

Proof. The first claim follows from the second by Proposition 1.7. Fixed $f(x,y)$, a and B, the function $f_a \in \mathrm{RUC}_u(B)$ is tame if and only if its extension to $\overline{B}$ is tame, where $\overline{B}$ is the closure in $S_y(M)$ of the image of B under the compactification $M^\beta \to S_y(M)$. Then the second claim follows from Fact 3.3 and Corollary 3.2, taking $A' = Ga$. For this, one can see that Corollary 3.2 holds true with $\overline{B}$ instead of B^* or, alternatively, that $\overline{B} = B^*$ using that M is $\aleph_0$-categorical. We show the latter. Clearly, $\overline{B} \subset B^*$. Let $p \in B^*$ and take b a realization of p in a separable elementary extension M' of M. Let $\phi(z,y)$ be a formula, $c \in M^{|z|}$ and $\epsilon > 0$. By $\aleph_0$-categoricity there is an isomorphism $\sigma\colon M' \to M$. Then $\mathrm{tp}(c) = \mathrm{tp}(\sigma c)$, so, by homogeneity, there is also an automorphism $g \in \mathrm{Aut}(M)$ with $d(c, g\sigma c) < \Delta_\phi(\epsilon)$. Hence $g\sigma b \in B$ and $|\phi(c,b) - \phi(c, g\sigma b)| < \epsilon$. We deduce that $p \in \overline{B}$. □

During the writing of this paper we came to know that, independently from us, A. Chernikov and P. Simon also noticed the connection between tameness in topology and NIP in logic, in the somehow parallel context of definable dynamics [**CS15**]. More on this connection has been elaborated by P. Simon in [**Sim14**].

In fact, it is surprising that the link was not made before, since the parallelism of these ideas in logic and topology is quite remarkable. As we have already said, NIP formulas were introduced by Shelah [**She71**] in 1971, in the classical first-order context. He defined them by the lack of an *independence property (IP)*, whence the name NIP. This independence property is the condition negated in the first item of Proposition 3.1. In the classical first-order setting it can be read like this: a formula $\varphi(x,y)$ has IP if for some sequence of elements $(a_i)_{i<\omega}$ and every pair of non-empty finite disjoint subsets $I, J \subset \omega$, there is b in some model that satisfies the formula

$$\bigwedge_{i\in I} \varphi(a_i, y) \wedge \bigwedge_{j\in J} \neg\varphi(a_j, y).$$

In other words, $\varphi(x,y)$ has IP if for some $(a_i)_{i<\omega}$ the sequence $(\{b : \varphi(a_i, b)\})_{i<\omega}$ of the sets defined by $\varphi(a_i, y)$ on some big enough model of the theory is an *independent sequence* in the sense of mere sets: all Boolean intersections are non-empty.

In the introductory section 1.5 of the survey [**GM14b**] on Banach representations of dynamical systems, Glasner and Megrelishvili write: «*In addition to those characterizations already mentioned, tameness can also be characterized by the lack of an "independence property", where combinatorial Ramsey type arguments take a leading role [...]*». The characterization they allude to is Proposition 6.6 from Kerr and Li [**KL07**], and the independence property involved there can indeed be seen as a topological generalization of Shelah's IP (see also Fact 3.6 below). But the notion of independence is already present in the seminal work of Rosenthal from 1974 [**Ros74**], where a crucial first step towards his dichotomy theorem implies showing that a sequence of subsets of a set S with no convergent subsequence (in the product topology of 2^S) admits a Boolean independent subsequence. Moreover, as pointed out in [**Sim14**], the (not) independence property of Shelah, in its continuous form, appears unequivocally in the work of Bourgain, Fremlin and Talagrand [**BFT78**]; see 2F.(vi).

On the other hand, this is not the first time that the concept of NIP is linked with a notion of another area. In 1992 Laskowski [**Las92**] noted that a formula $\varphi(x,y)$ has the independence property if and only if the family of definable sets of the form $\varphi(a,y)$ is a *Vapnik–Chervonenkis class*, a concept coming from probability theory, and also from the 70's [**VC71**]. He then profited of the examples provided by model theory to exhibit new Vapnik–Chervonenkis classes. In Section 4 we shall do the same thing with respect to tame dynamical systems, complementing the analysis of the examples done by Ben Yaacov and Tsankov [**BT14**, §6].

We end this section by pointing out that $\mathrm{Tame}_u(G)$ coincides, for Roelcke precompact Polish groups, with the restriction to $\mathrm{UC}(G)$ of the algebra $\mathrm{Null}(G)$ of *null functions* on G. Null functions arise from the study of topological sequence entropy of dynamical systems, initiated in [**Goo74**]. A compact G-space Y is null if its topological sequence entropy along any sequence is zero; we refer to [**KL07**, §5] for the pertinent definitions. We shall say that a function f on an arbitrary G-space X is null if it comes from a null compactification of X, and by Corollary 5.5 in [**KL07**] this is equivalent to checking that the cyclic G-space of f is null. For compact X this definition coincides with Definition 5.7 of the same reference (the G-spaces considered there are always compact), as follows from the statements 5.8 and 5.4.(2-4) thereof. The resulting algebra $\mathrm{Null}(X)$ is always a uniformly closed G-invariant subalgebra of $\mathrm{Tame}(X)$ (closedness is proven as for $\mathrm{Tame}(X)$; for the inclusion $\mathrm{Null}(X) \subset \mathrm{Tame}(X)$ compare §5 and §6 in [**KL07**]).

The following fact is a rephrasing of the characterizations of Kerr and Li.

Fact 3.6. *A function* $f \in \mathrm{RUC}(X)$ *is null if and only if there are no real numbers* $r < s$ *such that for every* n *one can find* $(g_i)_{i<n} \subset G$ *and* $(x_I)_{I \subset n} \subset X$ *such that*

$$f(g_i x_I) < r \text{ if } i \in I \text{ and } f(g_i x_I) > s \text{ if } i \notin I.$$

Proof. If f is non-null then its extension to any compactification is non-null, and the existence of elements r, s and, for every n, $(g_i)_{i<n}$ and $(x_I)_{I\subset n}$ as in the statement follows readily from Proposition 5.8 (and Definitions 5.1 and 2.1) in [**KL07**]; we obtain the elements x_I in the compactification, but we can approximate them by elements $\tilde{x}_I$ in X, since we only need that $f(g_i\tilde{x}_I)$ be close to $f(g_i x_I)$ for the finitely many indices $i < n$.

Conversely, if we have $r < s$ with the property negated in the statement, take u, v with $r < u < v < s$ and consider the sets $A_0 = \{p \in X_f : \tilde{f}(p) \leq u\}$, $A_1 = \{p \in X_f : \tilde{f}(p) \geq v\}$; here, $\tilde{f}$ is the extension of f to the cyclic G-space X_f. Then A_0 and A_1 are closed sets with arbitrarily large finite *independence sets*. Hence by Proposition 5.4.(1) in the same paper there is an *IN-pair* $(x, y) \in A_0 \times A_1$, and by 5.8 we deduce that f is non-null. □

When $X = M$ is an $\aleph_0$-categorical structure, it is immediate by $\emptyset$-saturation that a formula $f(x, y)$ is NIP on $[a] \times M$ if and only if f_a is null. Thus $\mathrm{Null}_u(M) = \mathrm{Tame}_u(M)$, and by considering $M = \widehat{G_L}$ (and recalling Remark 1.3) one gets $\mathrm{Null}_u(G) = \mathrm{Tame}_u(G)$ for every Roelcke precompact Polish G.

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4. The hierarchy in some examples

Several interesting Polish groups are naturally presented as automorphism groups of well-known first-order structures. Moreover, most of these structures admit *quantifier elimination*, which enables to describe their definable predicates in a simple way. As a result, the subalgebras of $\mathrm{UC}(G)$ that correspond to nice families of formulas can be understood pretty well in these examples.

Let G be the automorphism group of an $\aleph_0$-categorical structure M. We recall from Proposition 1.8 that the functions h in $UC(G)$ are exactly those of the form $h(g) = f(a, gb)$ for a formula $f(x, y)$ and tuples a, b from M. Then h factors through the orbit map $g \in G \mapsto gb \in [b]$. Bearing in mind Remark 3.4, it follows that

(1) $h \in \mathrm{AP}(G)$ if and only if $f(x, y)$ is algebraic on $[a] \times [b]$ (Proposition 1.14);
(2) $h \in \mathrm{WAP}(G)$ if and only if $f(x, y)$ is stable on $[a] \times [b]$ (Fact 2.1);
(3) $h \in \mathrm{Tame}_u(G)$ if and only if $f(x, y)$ is NIP on $[a] \times [b]$ (Proposition 3.5).

However, a technical difficulty is that $f(x, y)$ may be a formula in infinite variables, whereas it is usually easier to work with predicates involving only finite tuples. This is especially the case in the study of classical structures, for which, moreover, the results in the literature are stated, naturally, for $\{0, 1\}$-valued formulas in finitely many variables. In the following subsection we elaborate a way to deal with this difficulty. The reader willing to go directly to the examples may skip the details and retain merely the conclusion of Theorem 4.6.

4.1. Approximation by formulas in finite variables. In this subsection x and y will denote variables of length ω, and M will be a $\emptyset$-saturated structure. Any formula $f(x, y)$ is, by construction, a uniform limit of formulas defined on finite sub-variables of x,y. Moreover, if $f(x,y)$ is, for instance, stable on $M^\omega \times M^\omega$, then one can uniformly approximate f by stable formulas depending only on finite sub-variables of x,y. It suffices to take $n < \omega$ large enough so that, by uniform continuity, $|f(a,b) - f(a',b')| < \epsilon$ whenever $a_{<n} = a'_{<n}$ and $b_{<n} = b'_{<n}$; then define for example $f_n(x,y) = f(x',y')$, where $x'_{nk+i} = x_i$ and $y'_{nk+i} = y_i$ for all $i < n$, $k < \omega$.

However, if $f(x,y)$ is only known to be stable on $A \times B$ for some subsets $A, B \subset M^\omega$, then the previous simple construction does not ensure the stability of f_n. Besides, it may not be possible to find a formula stable on $M^\omega \times M^\omega$ that agrees with f on $A \times B$.

In [**BT14**], Proposition 4.7, a topological argument is given that permits to approximate WAP functions by stable formulas in finitely many variables. In what follows we give an alternative model-theoretic argument for this fact that can also be applied, in several cases, to NIP formulas.

In what follows, given a set $A \subset M$, the term $\mathrm{acl}(A)$ will denote the *algebraic closure* of A, including *imaginary elements* of M. The reader may wish to consult [**BBHU08**, §10–11] for an account of algebraic closure and imaginary sorts in continuous logic. Alternatively, and with no loss for the examples considered later, the reader may assume that $\mathrm{acl}(A) = A$.

Definition 4.1. Let M be a metric structure, $f(x,y)$ a formula.

(1) We will say that $f(x,y)$ is *in finite variables* if there is $n < \omega$ such that $f(a,b) = f(a',b')$ whenever $a_{<n} = a'_{<n}$ and $b_{<n} = b'_{<n}$.
(2) Let $a \subset M^\omega$ be a tuple and $B \subset M^\omega$ a definable set. We will say that $f(x,y)$ has *definable extensions of types over finite sets on a, B* if for every large enough $n < \omega$ there are an $\mathrm{acl}(a_{<n})$-definable predicate $df(y)$ and a realization a' of $\mathrm{tp}(a/a_{<n})$ (in some elementary extension of M) such that
$$f(a',b) = df(b)$$
for every $b \in B$.
(3) We will say that M has *definable extensions of types over finite sets* if the previous condition is true on a, M^ω for every formula $f(x,y)$ and any $a \in M^\omega$.

Lemma 4.2. *Suppose $f(x,y)$ is stable on $A \times B$ for definable sets A, B. If $a \in A$, then $f(x,y)$ has definable extensions of types over finite sets on a, B.*

Proof. Let $n < \omega$. Since A and B are definable sets we can consider them as sorts in their own right (say, of an expanded structure M'), and consider $f(x,y)$ as a formula defined only on $A \times B$ (so x and y become 1-variables of the corresponding sorts). Then f is a stable formula in the usual sense of [**BU10**], Definition 7.1, and we may apply the results thereof. More precisely, we can consider the f-type of a over $C = \mathrm{acl}(a_{<n})$, call it $p \in S_f(C)$. Here, p is an f-type (in the variable x) in the sense of [**BU10**], Definition 6.6. By Proposition 7.15 of the same paper, p admits a definable extension $q \in S_f(M')$. Moreover, the type q is consistent with $\mathrm{tp}(a/C)$, by the argument explained in [**BU10**, §8.1]; note that, by adding dummy variables, each predicate $h(x) \in \mathrm{tp}(a/C)$ can be seen as a formula $h(x,y)$ (in the structure expanded with constants for the elements of C), which is trivially stable. Then it is enough to take for a' any realization of $q \cup \mathrm{tp}(a/C)$. □

Lemma 4.3. *Let $A, B \subset M^{\omega}$ be definable sets. Given a formula $f(x,y)$, define $\tilde{f}(y,x) = f(x,y)$. Then $f(x,y)$ is algebraic, stable or* NIP *on $A \times B$ if and only if so is $\tilde{f}(y,x)$ on $B \times A$.*

Proof. This is clear for the stable case. For the NIP case, the proof is as in [**Sim15**], Lemma 2.5. If $f(x,y)$ is algebraic on $A \times B$ this means that $K = \{f_a|_B : a \in A\}$ is precompact in $\mathrm{C}(B)$, so given $\epsilon > 0$ there are $a_i \in A$, $i < n$, such that the functions $f_{a_i}|_B$ form an ϵ-net for K. Let $I_j \subset \mathbb{R}$, $j < m$, be a partition of the image of f on sets of diameter less than ϵ. For each function $\tau \colon n \to m$ let $b_\tau \in B$ be such that $f(a_i, b_\tau) \in I_{\tau(i)}$ for every $i < n$, if such an element exists. Then the functions $\tilde{f}_{b_\tau}|_A$ form a 3ϵ-net for $\tilde{K} = \{\tilde{f}_b|_A : b \in B\} \subset \mathrm{C}(A)$. This shows that $\tilde{K}$ is also precompact, hence that $\tilde{f}$ is algebraic on $B \times A$. □

In the following theorem we ask M to be $\aleph_0$-categorical to ensure that the closed orbits we consider are definable sets. The addition of imaginary sorts does not affect the $\aleph_0$-categoricity of M.

Proposition 4.4. *Let M be $\aleph_0$-categorical, $f(x,y)$ a formula, $a, b \in M^{\omega}$. Suppose either*

(1) $f(x,y)$ is algebraic on $[a] \times [b]$,
(2) $f(x,y)$ is stable on $[a] \times [b]$, or
(3) M has definable extensions of types over finite sets, and $f(x,y)$ is NIP *on $[a] \times [b]$.*

Then for every $\epsilon > 0$ there is a formula $f_0(x,y)$ in finite variables such that

$$\sup_{x \in [a], y \in [b]} |f(x,y) - f_0(x,y)| \leq \epsilon$$

and $f_0(x,y)$ is algebraic, stable or NIP, *respectively, on $[a] \times [b]$.*

Proof. Let n be large enough, so that in particular $|f(u,v) - f(u',v)| \leq \epsilon/2$ for any tuples u, u', v with $u_{<n} = u'_{<n}$. Using Lemma 4.2 for cases (1) and (2) (remark that a formula algebraic on $A \times B$ is stable on $A \times B$) we have that in any case there are a formula $df(z,y)$, a parameter $c \in \mathrm{acl}(a_{<n})$ and a realization a' of $\mathrm{tp}(a/a_{<n})$ in some elementary extension of M, such that $f(a',b') = df(c,b')$ for every $b' \in [b]$.

Let C be the set of realizations of $\mathrm{tp}(c/a_{<n})$. Since $c \in \mathrm{acl}(a_{<n})$, this set is compact, $a_{<n}$-definable and contained in the appropriate imaginary sort of M (see [**BBHU08**], Exercise 10.8 and Proposition 10.6). Here, $a_{<n}$-definable means that C is a definable set in the structure M augmented with constants for the elements of $a_{<n}$ (that is, $d(x,C)$ is an $a_{<n}$-definable predicate), and hence we can quantify over C in this augmented structure: in particular, $\sup_{z \in C} df(z,y)$ is an $a_{<n}$-definable predicate. This says that there is a formula $f' \colon M^n \times M^{\omega} \to \mathbb{R}$ such that, for every $b' \in [b]$,

$$f'(a_{<n}, b') = \sup_{z \in C} df(z, b').$$

For any a'' with $a''_{<n} = a_{<n}$ we have $\sup_{y \in [b]} |f(a'',y) - df(c,y)| \leq \epsilon/2$, and the same is true if we replace c by any $c' \in C$. We obtain $\sup_{y \in [b]} |f(a,y) - f'(a_{<n},y)| \leq \epsilon/2$, and thus

$$\sup_{x \in [a], y \in [b]} |f(x,y) - f'(x_{<n},y)| \leq \epsilon/2.$$

Now we consider each of the cases of the statement separately.

(1) Let $(b_j)_{j<\omega}$ be an indiscernible sequence in $[b]$. By the hypothesis and Lemma 4.3, the value of $f(a,b_j)$ is constant in j, and the same holds for a' instead of a. Thus $df(c,b_j)$ is constant in j, and we can deduce that $df(z,y)$ is algebraic on $[c]\times[b]$. Since $C\subset[c]$, it follows that $f'(a_{<n},b_j)$ is constant too. We can conclude that $f'(x_{<n},y)$ is algebraic on $[a_{<n}]\times[b]$.

(2) Since $f(x,y)$ is stable on (the definable sets) $[a]\times[b]$ and M is $\emptyset$-saturated, no sequences a'_i,b'_j, in any elementary extension, with $\mathrm{tp}(a'_i)=\mathrm{tp}(a)$, $\mathrm{tp}(b'_j)=\mathrm{tp}(b)$, can witness the order property for $f(x,y)$. Hence, the function $f_{a'}\in\mathrm{C}([b])$ is WAP. Since $f_{a'}=df_c$ on $[b]$, it follows that $df(z,y)$ is stable on $[c]\times[b]$. Since C is compact, it is not difficult to deduce that $f'(x_{<n},y)$ is stable on $[a_{<n}]\times[b]$. For example, we know that $\max_{l<k}df_{c_l}$ is in $\mathrm{WAP}([b])$ for every $(c_l)_{l<k}\subset C$, and $f'_{a_{<n}}|_{[b]}$ is a uniform limit of functions of this form.

(3) Here, if $(b_j)_{j<\omega}\subset[b]$ is an indiscernible sequence and g is an automorphism of M, the sequence $(df(gc,b_j))_{i<\omega}$ must converge in $\mathbb{R}$. Indeed, $df(gc,b_j)=f(a',g^{-1}b_j)$, so the claim follows from the fact that $f(x,y)$ is NIP on $[a]\times[b]$ and $(g^{-1}b_j)_{j<\omega}$ is also indiscernible. By uniform continuity and a density argument, the same is true if we replace gc with any $c'\in[c]$. We deduce that $df(z,y)$ is NIP on $[c]\times[b]$. As in the previous item, this implies that $f'(x_{<n},y)$ is NIP on $[a_{<n}]\times[b]$.

This is half what we intended. To complete the proof it suffices to apply the same construction to the formula $\tilde{f}'(y,x)=f'(x_{<n},y)$. We obtain a formula $f''(y_{<m},x)$; we define $f_0(x,y)=f''(y_{<m},x)$, then $f_0(x,y)$ is in finite variables and satisfies the other conditions of the statement. □

Question 4.5. Is the previous result true in the NIP case without the assumption on M?

We remark that a $\{0,1\}$-valued formula is necessarily in finite variables. Also, any formula with finite range can be written as a linear combination of $\{0,1\}$-valued formulas. If M is *classical* $\aleph_0$-categorical, then, conversely, any formula in finite variables has finite range, since it factors through the finite space $M^k /\!\!/ G$ for some $k<\omega$. For $G=\mathrm{Aut}(M)$ it follows that $\mathrm{UC}(G)$ is the closed algebra generated by the functions of the form $g\mapsto f(a,gb)$ where a,b are parameters and $f(x,y)$ is a *classical* (i.e., $\{0,1\}$-valued) formula.

Let us define $c\mathrm{Tame}_u(G)$ (respectively, $c\mathrm{AP}(G)$, $c\mathrm{WAP}(G)$) as the closed subalgebra of $\mathrm{UC}(G)$ generated by the functions of the form $g\mapsto f(a,gb)$ for $\{0,1\}$-valued NIP (resp., algebraic, stable) formulas $f(x,y)$. That is, these are the algebras generated by classical formulas of the appropriate corresponding kind. Here, assuming M is classical $\aleph_0$-categorical, it is indifferent to ask $f(x,y)$ to be NIP only on $[a]\times[b]$ or in its whole domain, since one can easily modify f so that it be NIP (resp., algebraic, stable) everywhere, without changing the function $g\mapsto f(a,gb)$. Indeed, one can assume that $a,b\in M^k$ for some $k<\omega$, then set f to be 0 outside $[a]\times[b]$ (since M^k is discrete and $[a]\times[b]$ is definable, the modified f is still definable).

From the previous proposition and discussion we obtain the following conclusion, which extends Theorem 5.4 in [**BT14**].

Theorem 4.6. *Let M be a classical $\aleph_0$-categorical structure, G its automorphism group. Then $c\mathrm{AP}(G) = \mathrm{AP}(G)$ and $c\mathrm{WAP}(G) = \mathrm{WAP}(G)$. If M has definable extensions of types over finite sets, then also $c\mathrm{Tame}_u(G) = \mathrm{Tame}_u(G)$.*

As we will see shortly, the assumption that M has definable extension of types over finite sets is satisfied in many interesting cases. The following is a useful sufficient condition.

Lemma 4.7. *Suppose M is classical, $\aleph_0$-categorical, and that for every $a \in M^\omega$ and $n < \omega$ there is a type $p \in S_x(M)$ such that p extends $\mathrm{tp}(a/a_{<n})$ and p is $a_{<n}$-invariant (i.e., p is fixed under all automorphisms of M fixing the tuple $a_{<n}$). Then M has definable extensions of types over finite sets.*

Proof. Let $a \in M^\omega$, $n < \omega$; take p as in the hypothesis of the lemma, a' a realization of p. Given a formula $f(x,y)$, the function df defined by $df(b) = f(a',b)$ is $a_{<n}$-invariant. Since M is classical $\aleph_0$-categorical, the structure M expanded with constants for the elements of $a_{<n}$ is $\aleph_0$-categorical too (see [**TZ12**], Corollary 4.3.7). It follows that $df(y)$ is an $a_{<n}$-definable predicate, hence the conditions of Definition 4.1 are satisfied. □

4.2. The examples. We describe the dynamical hierarchy of function algebras for the automorphism groups of some well-known (unstable) $\aleph_0$-categorical structures. We start with the oligomorphic groups $\mathrm{Aut}(\mathbb{Q},<)$, $\mathrm{Aut}(RG)$ and $\mathrm{Homeo}(2^\omega)$.

The unique countable dense linear order without endpoints, $(\mathbb{Q},<)$, admits quantifier elimination (see [**TZ12**, §3.3.2]). This implies, for $G = \mathrm{Aut}(\mathbb{Q},<)$, that $\mathrm{UC}(G)$ is the closed unital algebra generated by the functions of the form $g \mapsto (a = gb)$ and $g \mapsto (a < gb)$ for elements $a,b \in \mathbb{Q}$ —where we think of the classical predicates $x = y$ and $x < y$ as $\{0,1\}$-valued functions. The formula $x < y$ is NIP (and $x = y$ is of course stable), whence we deduce that every UC function is tame. On the other hand, $x < y$ is unstable, so $g \mapsto (a < gb)$ is not WAP (in fact, as follows from [**BT14**], Example 6.2, $\mathrm{WAP}(G)$ is precisely the unital algebra generated by the functions of the form $g \mapsto (a = gb)$).

Now suppose $f(x,y)$ is a formula algebraic on $[a] \times [b]$. For slight convenience we may assume, by Theorem 4.6, that f is classical and the tuples involved are finite. For tuples c,d, let us write $c < d$ to mean that every element of the tuple c is less than every element of d. Let $b' \in [b]$. We can choose a sequence of tuples $(a^i)_{i<\omega}$ in $\mathbb{Q}$ (or in an elementary extension if we did not assume the tuples are finite) such that $a \simeq a^i$ as linear orders, $a = a_0$, $b < a^1$, $b' < a^1$, and $a^i < a^j$ if $i < j$. By quantifier elimination, the type of a tuple depends only on its isomorphism type as a linear order; hence $(a^i)_{i<\omega}$ is an indiscernible sequence. By the hypothesis on f we have that $(f(a^i,b))_{i<\omega}$ is constant, and the same with b' instead of b. But, again by quantifier elimination, $f(a^1,b) = f(a^1,b')$. It follows that $f(a,b) = f(a,b')$. We have thus shown that $g \mapsto f(a,gb)$ is constant, and can deduce that G is AP-trivial. (In fact, as is well-known, G is extremely amenable ([**Pes98**]), which is a much stronger property: if $f \in \mathrm{AP}(G)$ then the compact G-space $\overline{Gf}$ must have a fixed point; since the action of G on $\overline{Gf}$ is by isometries we conclude that $\overline{Gf}$ is a singleton, i.e., that f is constant.)

Putting these conclusions together we get the following (where $\mathbb{R}$ stands for the algebra of constant functions on G).

Corollary 4.8. *For $G = \mathrm{Aut}(\mathbb{Q},<)$ we have $\mathbb{R} = \mathrm{AP}(G) \subsetneq \mathrm{WAP}(G) \subsetneq \mathrm{Tame}_u(G) = \mathrm{UC}(G)$.*

The situation is different for the random graph RG, the unique countable, homogeneous, universal graph. It has quantifier elimination, which in this case implies that $\mathrm{UC}(\mathrm{Aut}(RG))$ is the closed unital algebra generated by the functions of the form $g \mapsto (a = gb)$ and $g \mapsto (a\,R\,gb)$ (where R denotes the adjacency relation of the graph). Also, stable formulas on $[a]\times[b]$ are again exactly those expressible in the reduct of RG to the identity relation ([**BT14**], Example 6.1). But in this case no other formula is NIP on $[a]\times[b]$.

Lemma 4.9. *On the random graph, every classical* NIP *formula is stable.*

Proof. Theorem 4.7 in [**She90**, Ch. II] shows that if there is an unstable NIP formula then there is a formula with the *strict order property*. The theory of the random graph, being *simple*, does not admit a formula with the strict order property; see [**TZ12**], Corollary 7.3.14 and Exercise 8.2.4. □

It follows for $G = \mathrm{Aut}(RG)$ that $c\,\mathrm{Tame}_u(G) = c\,\mathrm{WAP}(G)$. Now we argue that RG has definable extensions of types over finite sets, whence $\mathrm{Tame}_u(G) = \mathrm{WAP}(G)$ by Theorem 4.6. For any $a \in RG^\omega$ and $n < \omega$, the *free amalgam* of a and RG over $a_{<n}$ is a graph containing RG and a copy $a' \simeq a$ such that $a_{<n} = a'_{<n}$ and, for every $i \geq n$, a_i is not R-related to any element of RG outside $a_{<n}$. The homogeneity and universality of RG ensure that such a copy a' is realized as a tuple in some elementary extension of RG. Since RG has quantifier elimination it is clear that a' realizes $\mathrm{tp}(a/a_{<n})$ and that the type $\mathrm{tp}(a'/RG)$ is $a_{<n}$-invariant, thus Lemma 4.7 applies. We can conclude that $\mathrm{Tame}_u(G)$ is the closed unital algebra generated by the functions of the form $g \mapsto (a = gb)$, $a, b \in RG$. An example of a non-tame function in $\mathrm{UC}(G)$ is of course $g \mapsto (a\,R\,gb)$.

If $f(x,y)$ is a formula algebraic on $[a]\times[b]$ and $b' \in [b]$, we can take a sequence $(a^i)_{i<\omega}$ of disjoint copies of a such that $a = a^0$ and no element of a^i, $i \geq 1$, is R-related to an element of b, b' nor a^j for $j \neq i$. It follows by quantifier elimination that $(a^i)_{i<\omega}$ is indiscernible and that $f(a^1,b) = f(a^1,b')$. Since, by hypothesis, $(f(a^i,b))_{i<\omega}$ and $(f(a^i,b'))_{i<\omega}$ must be constant, we obtain $f(a,b) = f(a,b')$. Thus $g \mapsto f(a,gb)$ is constant.

Corollary 4.10. *For $G = \mathrm{Aut}(RG)$ we have $\mathbb{R} = \mathrm{AP}(G) \subsetneq \mathrm{WAP}(G) = \mathrm{Tame}_u(G) \subsetneq \mathrm{UC}(G)$.*

The group $G = \mathrm{Homeo}(2^\omega)$ of homeomorphisms of the Cantor space, carrying the compact-open topology, can be identified naturally with the automorphism group of the Boolean algebra $\mathcal{B}$ of clopen subsets of 2^ω, with the topology of pointwise convergence. Up to isomorphism, $\mathcal{B}$ is the unique countable atomless Boolean algebra. We consider it as a structure in the language of Boolean algebras, that is, we have basic functions $\wedge, \vee \colon \mathcal{B}^2 \to \mathcal{B}$ and $\neg\colon \mathcal{B} \to \mathcal{B}$ for meet, joint and complementation in the algebra, and constants 0 and 1 for the minimum and maximum of $\mathcal{B}$. In this language $\mathcal{B}$ admits quantifier elimination (see [**Poi85**], Théorème 6.21). This means that two tuples c,d of the same length have the same type over $\emptyset$ if and only if $c \simeq d$ (i.e., the map $c_i \mapsto d_i$ extends to an isomorphism of the generated Boolean algebras). It also implies that UC(G) is the algebra generated by the functions of the form $g \mapsto (0 = t(a,gb))$, where $t(x,y)$ is a Boolean term in finite variables, i.e., a function $\mathcal{B}^n \times \mathcal{B}^m \to \mathcal{B}$ constructed with the basic Boolean operations $\wedge, \vee, \neg$.

Here it is easy to see that $0 = x \wedge y$ is not NIP on $[a]\times[b]$ (for $a,b \notin \{0,1\}$), so the function $g \mapsto (0 = a \wedge gb)$ is not tame. With this in mind, and following the idea of [**BT14**], Example 6.3, one sees the following.

Lemma 4.11. *On the countable atomless Boolean algebra, every classical* NIP *formula is stable.*

Proof. Let $f(x,y)$ be a classical formula. If f is not stable, then it is not stable on $[a]\times[b]$ for some tuples a, b from $\mathcal{B}$. We may modify $f(x,y)$, a and b (without changing the function $g \mapsto f(a,gb)$) so that the elements of the tuple a form a finite partition of 1, and the same for b.

Say $a \in \mathcal{B}^n$, $b \in \mathcal{B}^m$. In [**BT14**], Example 6.3, it is shown that $f(x,y)$ is unstable on $[a]\times[b]$ if and only if there is $b' \in [b]$ such that $f(a,b) \neq f(a,b')$ but the xy-tuples ab and ab' satisfy the same formulas of the form $t(x) = s(y)$ for Boolean terms t,s. Moreover, it is shown that in this case one can choose b' so that, for some indices $i_0, i_1 < n$, $j_0, j_1 < m$, we have (possibly changing b by a conjugate):

(1) $a_{i_0} \wedge b_{j_0} = 0$, $a_{i_0} \wedge b_{j_1} \neq 0$, $a_{i_1} \wedge b_{j_0} \neq 0$ and $a_{i_1} \wedge b_{j_1} \neq 0$;
(2) $a_i \wedge b'_j \neq 0$ for $i \in \{i_0, i_1\}$ and $j \in \{j_0, j_1\}$;
(3) $a_i \wedge b_j = 0$ if and only if $a_i \wedge b'_j = 0$, for every pair $(i,j) \neq (i_0, j_0)$;
(4) $f(a,b) \neq f(a,b')$.

Now we fix an arbitrary $l < \omega$ and choose a partition $c_0, \dots, c_l$ of $a_{i_1} \wedge b_{j_1}$. For each $k < l$ we let $a^k_{i_0} = a_{i_0} \vee c_k$ and $a^k_{i_1} = a_{i_1} \wedge \neg c_k$. We also let $a^k_i = a_i$ for every $i \notin \{i_0, i_1\}$, thus defining a tuple $a^k \in [a]$. Similarly, for every $K \subset l$ we let $b^K_{j_0} = b_{j_0} \vee (\bigvee_{k\in K} c_k)$ and $b^K_{j_1} = b_{j_1} \wedge \neg(\bigvee_{k\in K} c_k)$. We let $b^K_j = b_j$ for $j \notin \{j_0, j_1\}$, and this defines a tuple $b^K \in [b]$. By quantifier elimination, the type of the tuple $a^k b^K$ is determined by the set of pairs i,j such that $a^k_i \wedge b^K_j = 0$. It follows that

$$f(a^k, b^K) = f(a,b') \text{ if } k \in K, \text{ and } f(a^k, b^K) = f(a,b) \text{ if } k \notin K.$$

Hence $f(x,y)$ is not NIP on $[a]\times[b]$. □

The lemma shows that $c\mathrm{Tame}_u(G) = c\mathrm{WAP}(G)$ for $G = \mathrm{Aut}(\mathcal{B})$, and that this algebra is generated by the functions of the form $g \mapsto (a = gb)$. Indeed, the proof shows that an NIP formula on $[a]\times[b]$ is a combination of formulas of the kind $t(x) = s(y)$ for Boolean terms t,s; then simply note that $s(gb) = gs(b)$, so the function on G associated to a formula of latter kind is $g \mapsto (c = gd)$ where $c = t(a)$ and $d = s(b)$. Next we show that $\mathcal{B}$ has definable extensions of types over finite sets, in order to conclude, by Theorem 4.6, that these functions actually generate $\mathrm{Tame}_u(G)$.

Let $a \in \mathcal{B}^\omega$, $n < \omega$; with no loss of generality we may assume that $a_{<n}$ is a partition of 1. We consider the free amalgam of a and $\mathcal{B}$ over $a_{<n}$, which is a Boolean

algebra generated by $\mathcal{B}$ together with a copy $a' \simeq a$ such that $a'_{<n} = a_{<n}$ and, for every $i < n$, every $d \in \mathcal{B}$ and every c in the Boolean algebra generated by a', we have $c \wedge a_i \wedge d \neq 0$ unless $c \wedge a_i = 0$ or $a_i \wedge d = 0$. Such a copy a' is realized as a tuple in some elementary extension of $\mathcal{B}$, and the type $\mathrm{tp}(a'/\mathcal{B})$ is clearly $a_{<n}$-invariant. By Lemma 4.7, $\mathcal{B}$ has definable extensions of types over finite sets.

Finally, as with $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Aut}(RG)$, we show that every AP function on $\mathrm{Homeo}(2^\omega)$ is constant. If $f(x,y)$ is algebraic on $[a]\times[b]$ and $b' \in [b]$, we can find copies a^i of a such that $a = a^0$ and each a^i, $i \geq 1$, forms a free amalgam with B_i over $\emptyset$, where B_i is the algebra generated by b, b' and all a^j, $j < i$. That is, $c \wedge d \neq 0$ for every non-zero $d \in B_i$ and every non-zero c in the algebra generated by a^i. Then $(a^i)_{i<\omega}$ is indiscernible and $f(a^1,b) = f(a^1,b')$. By hypothesis $(f(a^i,b))_{i<\omega}$ and $(f(a^i,b'))_{i<\omega}$ are constant, and therefore $f(a,b) = f(a,b')$.

Corollary 4.12. *For* $G = \mathrm{Homeo}(2^\omega)$ *we have* $\mathbb{R} = \mathrm{AP}(G) \subsetneq \mathrm{WAP}(G) = \mathrm{Tame}_u(G) \subsetneq \mathrm{UC}(G)$.

The previous examples come from classical structures; we consider now a purely metric one: the Urysohn sphere $\mathbb{U}_1$. This is, up to isometry, the unique separable, complete and homogeneous metric space of diameter 1 that is universal for countable metric spaces of diameter at most 1: any such metric space can be embedded in $\mathbb{U}_1$. As a metric structure with no basic predicates (other than the distance), $\mathbb{U}_1$ is $\aleph_0$-categorical and has quantifier elimination, which in this case means that the type of a tuple b depends only on the isomorphism class of b as a metric space; see [**Usv08**, §5]. It also says that $\mathrm{UC}(\mathrm{Iso}(\mathbb{U}_1))$ is generated by the functions of the form $g \mapsto d(a, gb)$ for $a, b \in \mathbb{U}_1$.

We show that $\mathrm{Iso}(\mathbb{U}_1)$ is Tame_u-trivial.

Theorem 4.13. *Every function in* $\mathrm{Tame}_u(\mathrm{Iso}(\mathbb{U}_1))$ *is constant.*

Proof. Suppose $f(x,y)$ is not constant on $[a]\times[b]$, so we have $f(a,b) \neq f(a',b')$ where $a \simeq a'$ and $b \simeq b'$ as metric spaces. We will need to assume that the elements of a are separated enough from the elements of b, (the same for a', b'), and that the metric space ab is similar to $a'b'$; so we precise and justify this. Let $0 < \epsilon < 1$. Note first that, by the universality and homogeneity of the Urysohn sphere, for any tuples x, y in $\mathbb{U}_1$ (or in an elementary extension thereof) we can find $\tilde{y}$ in an elementary extension such that $y \simeq \tilde{y}$, $d(y_n, \tilde{y}_n) = \epsilon$ and $d(x_n, \tilde{y}_m) = (d(x_n, y_m) + \epsilon) \wedge 1$ for all coordinates n, m (where $r \wedge s$ denotes $\min(r,s)$). Note secondly that finitely many iterations of the process of replacing y by $\tilde{y}$ eventually end with $d(x_n, y_m) = 1$ for all n, m.

If we chose ϵ small enough and do one iteration of the previous process for $xy = ab$, by continuity of f we can assume (replacing ab by $a\tilde{b}$) that

$$d(a_n, a_m) \leq d(a_n, b_k) + d(b_k, a_m) - \epsilon,\ d(b_n, b_m) \leq d(b_n, a_k) + d(a_k, b_m) - \epsilon$$

for all n, m, k —and still $f(a,b) \neq f(a',b')$. So we have separated the elements of a from those of b, and we do the same for a', b'.

Next we iterate the process described above for $\epsilon/2$, starting with $xy = ab$, thus producing a finite sequence of copies of b, the last copy $\tilde{b}$ verifying $d(a_n, \tilde{b}_m) = 1$ for all n, m. We do the same starting with a', b', finishing with a copy $\tilde{b}'$ with the analogous property. Since the theory of $\mathbb{U}_1$ has quantifier elimination, we have $f(a,\tilde{b}) = f(a',\tilde{b}')$. So f differs in two consecutive steps of the process, and by replacing our tuples ab, $a'b'$ by these consecutive tuples we may assume also that $a = a'$ and $|d(a_n, b_m) - d(a_n, b'_m)| \leq \epsilon/2$ for all n, m.

With the previous assumptions in mind, we now construct a metric space containing a sequence $(a^i)_{i<\omega}$ of different copies of a and, for each $I \subset \omega$, a copy b^I of $b \simeq b'$ such that $a^i b^I \simeq ab$ if $i \in I$ and $a^i b^I \simeq ab'$ if $i \notin I$. For $i \neq j$, $I \neq J$ and each n, m we define $d(a^i_n, a^j_m) = (d(a_n, a_m) + \epsilon/2) \wedge 1$, $d(b^I_n, b^J_m) = (d(b_n, b_m) + \epsilon/2) \wedge 1$. The triangle inequalities are satisfied; for example, for a^i_n, a^j_m, b^I_k, $i \neq j$, $i \in I$, we have

$$d(a^i_n, a^j_m) = (d(a_n, a_m) + \epsilon/2) \wedge 1 \leq d(a_n, b_k) + d(b_k, a_m) - \epsilon/2 \leq d(a^i_n, b^I_k) + d(b^I_k, a^j_m),$$

and also

$$d(a^i_n, b^I_k) \leq (d(a^i_n, a^i_m) + \epsilon/2) \wedge 1 + d(a^i_m, b^I_k) - \epsilon/2 \leq d(a^i_n, a^j_m) + d(a^j_m, b^I_k).$$

The other inequalities are proved similarly.

By the universality of the Urysohn sphere we can assume that the tuples a^i lie in $\mathbb{U}_1$, the tuples b^I in some elementary extension. By quantifier elimination, $f(a^i, b^I) = f(a, b)$ if $i \in I$ and $f(a^i, b^I) = f(a, b')$ if $i \notin I$. This shows that $f(x, y)$ is not NIP on $[a] \times [b]$. It follows that every tame function of the form $g \mapsto f(a, gb)$ is constant, which proves the theorem. □

In Question 7.10 from [**GM13**] it was asked whether the algebra Tame(G) separates points and closed subsets of G for every Polish group G. As we have seen, this can fail drastically for the algebra Tame$_u(G)$. Unfortunately, we do not know how big the gap between Tame(G) and Tame$_u(G)$ may be.

Question 4.14. Are there tame non-constant functions on Iso($\mathbb{U}_1$)? Is there a way to *regularize* a (non-constant) function $f \in \mathrm{RUC}(G)$ to get (a non-constant) $\tilde{f} \in \mathrm{UC}(G)$, in such a manner that tameness is preserved?

Finally, we consider the group $G = H_+[0,1]$ of increasing homeomorphisms of $[0,1]$ with the compact-open topology —which coincides on G with those of pointwise or uniform convergence. In spite of not being naturally presented as an automorphism group of some $\aleph_0$-categorical metric structure, this group is Roelcke precompact. See [**Usp02**], Example 4.4, for a description of its Roelcke compactification.

The following result was explained to us by M. Megrelishvili.

Theorem 4.15. $\mathrm{UC}(H_+[0,1]) \subset \mathrm{Tame}(H_+[0,1])$.

See [**GM14a**], Theorem 8.1. As remarked there, the inclusion is strict: the function $f : G \to \mathbb{R}$ given by $f(g) = g(1/2)$, for example, is tame (it comes from the Helly space, which is a Rosenthal compactification of G) but not left uniformly continuous: $\sup_g |f(gh) - f(g)| = 1$ for any $h \in G$ with $h(1/2) \neq 1/2$. In fact, f is even null: it is clear that, for reals $r < s$, there are no increasing functions $g_0, g_1 \in [0,1]^{[0,1]}$ and elements $x_{\{0\}}, x_{\{1\}} \in [0,1]$ such that $g_i(x_I) < r$ if $i \in I$ and $g_i(x_I) > s$ if $i \notin I$. One deduces that $\mathrm{UC}(G) \subsetneq \mathrm{Null}(G)$.

Additionally, as we have already recalled, the celebrated result of [**Meg01a**] says that $H_+[0,1]$ is WAP-trivial. On the other hand, one of the main results of [**GM08**] (Theorem 8.3) is the stronger fact that $H_+[0,1]$ is SUC-trivial. In turn, this allows the authors to deduce that $\mathrm{Iso}(\mathbb{U}_1)$ is also SUC-trivial ([**GM08**, §10]). By our Theorem 2.9 we can recover these facts directly from the WAP-triviality of these groups, and extend the conclusion to another interesting Roelcke precompact Polish group that is also known to be WAP-trivial.

Corollary 4.16. *The groups $H_+[0,1]$ and $\mathrm{Iso}(\mathbb{U}_1)$ are SUC-trivial. The same is true for the homeomorphism group of the Lelek fan.*

Proof. The WAP-triviality of $\mathrm{Iso}(\mathbb{U}_1)$ was first observed in [**Pes07**], Corollary 1.4, using the analogous result for $H_+[0,1]$; an alternative proof is given in [**BT14**], Example 6.4, and of course also follows from Theorem 4.13 above. For the homeomorphism group of the Lelek fan, WAP-triviality was proven in [**BT14**]: see the discussion after Corollary 4.10 and the references therein. □

The previous facts about the group $H_+[0,1]$ lead to an interesting model-theoretic example, addressed in the following corollary.

If $f(x,y)$ is a formula in the variables x,y (of arbitrary length), let us say that $f(x,y)$ is *separated* if it is equivalent to a continuous combination of definable predicates $f_i(z_i)$ where, for each i, $z_i = x$ or $z_i = y$. Equivalently, $f(x,y)$ is separated if it factors through the product of type spaces $S_x(\emptyset)\times S_y(\emptyset)$ (by the Stone–Weierstrass theorem, the continuous functions on a product $X\times Y$ of compact Hausdorff spaces is the closed algebra generated by the continuous functions that depend only on X or on Y). Of course, separated formulas are stable. Let us say that a structure is *purely unstable* if every stable formula $f(x,y)$ is separated. No infinite classical structure can be purely unstable, since the identity relation $x=y$ is always stable and never separated.

Corollary 4.17. *The $\aleph_0$-categorical structure $M=\widehat{G}_L$ associated to $G=H_+[0,1]$ is purely unstable and* NIP.

Proof. Of course, WAP-triviality implies that M is purely unstable: if $f(x,y)$ is stable and a,b are parameters, then the function $g\mapsto f(a,gb)$ belongs to $\mathrm{WAP}(G)$ and so is constant. It follows that the value of f on a,b only depends on $[a],[b]$, that is, on $\mathrm{tp}(a),\mathrm{tp}(b)$ since M is $\aleph_0$-categorical; hence $f(x,y)$ is separated.

On the other hand, Theorem 4.15 and Proposition 3.5 imply that every formula $f(x,y)$ with $|y|=1$ is NIP. A well-known argument (see for example Proposition 2.11 in [**Sim15**], which adapts easily to the metric setting), shows that then every formula is NIP. □

We finish with a remark relating sections 2 and 3 of this paper. Since reflexive-representable functions correspond to stable formulas and Rosenthal-representable functions correspond to NIP formulas, it is not surprising that, as we have seen, the natural intermediate subalgebra of Asplund-representable functions collapses to one of the other two: on the model-theoretic side, there is no known natural notion between stable and NIP. However, one might be slightly surprised to find that WAP = Asp rather than Asp = Tame_u (although, in fact, this was already known for $G = H_+[0,1]$). Indeed, Asplund and Rosenthal Banach spaces were once difficult to distinguish, with the first examples coming in the mid-seventies from independent works of James and of Lindenstrauss and Stegall. It is thus worthy to remark that, via our results and the Banach space construction of Glasner and Megrelishvili [**GM12**] (Theorem 6.3), every NIP unstable $\aleph_0$-categorical structure yields an example of a Rosenthal non-Asplund Banach space.

CHAPTER 2

Eberlein oligomorphic groups

Abstract. [Joint work with Itaï Ben Yaacov and Todor Tsankov, submitted for publication.] We study the Fourier–Stieltjes algebra of Roelcke precompact, non-archimedean, Polish groups and give a model-theoretic description of the Hilbert compactification of these groups. We characterize the family of such groups whose Fourier–Stieltjes algebra is dense in the algebra of weakly almost periodic functions: those are exactly the automorphism groups of $\aleph_0$-stable, $\aleph_0$-categorical structures. This analysis is then extended to all semitopological semigroup compactifications S of such a group: S is Hilbert-representable if and only if it is an inverse semigroup. We also show that every factor of the Hilbert compactification is Hilbert-representable.

Contents

Introduction

It has long been recognized in model theory that the action of the automorphism group of an $\aleph_0$-categorical structure on the structure (and its powers) captures all model-theoretic information about it. Moreover, by a classical result of Ahlbrandt and Ziegler [**AZ86**], the automorphism group remembers the structure up to bi-interpretability. As most interesting model-theoretic properties are preserved by interpretations, it is reasonable to expect that those would correspond to natural properties of the automorphism group.

It turns out that many model-theoretic properties of the structure are reflected in the behavior of a certain universal dynamical system associated to the group that we proceed to describe. First, recall that automorphism groups of $\aleph_0$-categorical structures are Roelcke precompact in the following sense.

Definition 0.1. A topological group G is called *Roelcke precompact* if for every neighborhood U of the identity, there exists a finite set F such that $UFU = G$.

To each Roelcke precompact, Polish group G is naturally associated its *Roelcke compactification* $R(G)$, the completion of G with respect to its Roelcke (or lower) uniformity; see Section 2.3 for more details. The natural action $G \curvearrowright R(G)$ renders it a topological dynamical system. From the model-theoretic point of view, if we represent G as the automorphism group of an $\aleph_0$-categorical structure M, $R(G)$ can be considered as a suitable closed subspace of the type space $S_\omega(M)$ in infinitely many variables over the model. Thus, there is a natural correspondence between formulas with parameters from the model, on the one hand, and continuous functions on $R(G)$, on the other. This allows building a dictionary between the model-theoretic and the dynamical setting. For example, *stable* formulas correspond to *weakly almost periodic* functions and *NIP* formulas correspond to *tame* functions.

Particularly relevant to us is the theory of Banach representations of dynamical systems as developed by Glasner and Megrelishvili in a series of papers (see [**GM14b**] and the references therein). If $G \curvearrowright X$ is a topological dynamical system and V is a Banach space, a *representation of X on V* is a pair of continuous maps $\iota\colon X \to B$, $\pi\colon G \to \mathrm{Iso}(V)$, where B is the unit ball of V^* equipped with the weak* topology, $\mathrm{Iso}(V)$ is the group of linear isometries of V, equipped with the strong operator topology, π is a homomorphism, and

$$\langle v, \iota(gx)\rangle = \langle \pi(g)^{-1}v, \iota(x)\rangle,$$

for all $x \in X$, $v \in V$, $g \in G$. A representation is *faithful* if ι is an embedding. If $\mathcal{K}$ is a class of Banach spaces, we say that $G \curvearrowright X$ is *$\mathcal{K}$-representable* if it admits a faithful representation on a Banach space in the class $\mathcal{K}$.

All dynamical systems are representable on some Banach space; however, if one restricts to some (well-chosen) class of Banach spaces $\mathcal{K}$, the $\mathcal{K}$-representable systems usually form an interesting family. Somewhat unexpectedly, in the $\aleph_0$-categorical setting, there are some precise connections between model-theoretic properties of the structure and the classes of Banach spaces $R(G)$ can be represented on: for example, M is stable iff $R(G)$ can be represented on a reflexive Banach space [**BT14**, §5][**GM14b**, §5.1] and M is NIP iff $R(G)$ can be represented on a Banach space that does not contain a copy of ℓ^1 [**Iba14**, §4][**GM14b**, §8.1]. One of the main motivating questions for this paper was what the appropriate model-theoretic condition is for $R(G)$ to be representable on a Hilbert space.

For some classes $\mathcal{K}$ of Banach spaces, there are dynamical systems that are universal for the $\mathcal{K}$-representable ones. For example, $W(G)$, the *WAP compactification* of G is universal for reflexively representable systems and $H(G)$, the *Hilbert compactification*, is universal for Hilbert-representable systems. Both $W(G)$ and $H(G)$ carry the structure of a compact *semitopological semigroup* and $H(G)$ is a factor of $W(G)$.

The main focus of the paper are the automorphism groups of $\aleph_0$-categorical *classical*, *discrete* (multi-sorted) structures, or, equivalently, Roelcke precompact, Polish, *non-archimedean* groups. (A group is *non-archimedean* if it admits an open basis at the identity consisting of open subgroups.) We make this assumption tacitly throughout the paper: when we say "$\aleph_0$-categorical structure", we will always mean a classical one, as opposed to metric. A non-archimedean, Polish, Roelcke precompact group will be called *pro-oligomorphic*; it is *oligomorphic* if the structure can be chosen one-sorted.

For every non-archimedean group G, the compactification $G \to H(G)$ is a topological embedding. Our first result is a concrete description of $H(G)$ for pro-oligomorphic groups, in model-theoretic terms. More precisely, we have the following.

Theorem 0.2. *Let M be an $\aleph_0$-categorical structure and let $G = \mathrm{Aut}(M)$. Then $H(G)$ is isomorphic to the semigroup of partial elementary embeddings $M^{\mathrm{eq}} \to M^{\mathrm{eq}}$ with algebraically closed domains.*

Using this description, we give two characterizations of pro-oligomorphic groups for which $W(G) = H(G)$: one model-theoretic, and one in terms of the semigroup $W(G)$. This is the main result of the paper.

Theorem 0.3. *Let M be an $\aleph_0$-categorical structure and let $G = \mathrm{Aut}(M)$. The following are equivalent:*

(1) The idempotents of $W(G)$ commute;
(2) M is one-based for stable independence;
(3) $W(G) = H(G)$.

Using Theorem 0.3 and a classical, deep result in model theory, we can now give a satisfactory answer of our initial question.

Corollary 0.4. *Let M be an $\aleph_0$-categorical structure and let $G = \mathrm{Aut}(M)$. Then the following are equivalent:*

(1) M is $\aleph_0$-stable;
(2) $R(G)$ is Hilbert-representable.

Corollary 0.4 and a well-known example of an $\aleph_0$-categorical, stable, non-$\aleph_0$-stable structure, due to Hrushovski, give us the following corollary (cf. Example 3.14) which answers a question of Glasner and Megrelishvili [**GM14b**, Question 6.10].

Corollary 0.5. *There exists an oligomorphic group G that satisfies $R(G) = W(G) \neq H(G)$.*

While all factors of $W(G)$ are known to be reflexively representable (or *reflexively approximable*, for a general topological group G), it is an open question whether all factors of $H(G)$ are Hilbert-representable [**GM14b**, Question 5.12.3]. We can give a positive answer to this question in the case of pro-oligomorphic groups (cf. Theorem 4.8).

Theorem 0.6. *Let G be a pro-oligomorphic group. Then all factors of $H(G)$ are Hilbert-representable.*

The correspondence between model-theoretic properties of $\aleph_0$-categorical structures and dynamical properties of their automorphism groups is not restricted to the non-archimedean case. The correct model-theoretic setting for dealing with general Roelcke precompact, Polish groups is that of continuous logic and in both [**BT14**] and [**Iba14**], the results are proved in full generality. However, the two most important tools used in this paper are currently only available in the non-archimedean setting: namely, the classification of the unitary representations on the dynamical side and the notion of one-basedness on the model-theoretic side. For the moment, we do not even have a plausible conjecture of what the model-theoretic characterization of Hilbert-representable functions on a Roelcke precompact Polish group should be in general. Theorem 0.3 clearly fails in the continuous setting (for example, for the unitary group). While we do not have a counterexample to Corollary 0.4 for general separably categorical structures, we strongly suspect that it also fails.

As one of the goals of this paper is to provide a dictionary between model theory and abstract topological dynamics, we have tried to make the exposition fairly self-contained (apart from a couple of difficult model-theoretic results) and accessible to people working in both areas.

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1. Preliminaries

1.1. Compactifications of topological groups. Let G be a topological group. The algebra of complex-valued, continuous, bounded functions on G will be denoted by $\mathrm{C}(G)$. This algebra carries always the uniform norm, $\|f\| = \sup_{g\in G}|f(g)|$. The group G admits a left and a right action on $\mathrm{C}(G)$, given, respectively, by $(gf)(h) = f(g^{-1}h)$ and $(fg)(h) = f(hg^{-1})$ for every $f \in \mathrm{C}(G)$ and $g,h \in G$. These actions are isometric but in general not continuous.

When considering subalgebras of $\mathrm{C}(G)$, we will always assume that these are unital and closed under complex conjugation. If we say that a subalgebra is *closed*, we mean closed with respect to the uniform norm. *Left-invariant*, *right-invariant* and *bi-invariant* refer to the actions of G defined above.

A *compactification* of G is a compact Hausdorff space X with a continuous left action of G, together with a continuous G-map $\alpha\colon G \to X$ with dense image (where G carries the natural left action on itself). Given a compactification $\alpha\colon G \to X$, we denote by $\mathcal{A}(\alpha) := \mathrm{C}(X) \circ \alpha$ the subalgebra of $\mathrm{C}(G)$ consisting of those functions that factor through α. We may also denote it by $\mathcal{A}(X)$, if no confusion arises. The algebra $\mathcal{A}(\alpha)$ is always left-invariant, and the compactification will be called *bi-invariant* if $\mathcal{A}(\alpha)$ is moreover right-invariant.

Given compactifications $\alpha_X\colon G \to X$ and $\alpha_Y\colon G \to Y$, we say that α_Y is a *G-factor* of α_X (or shortly, that Y is a factor of X) if there is a continuous surjective G-map $X \to Y$ commuting with α_X and α_Y. In this case, this G-map is unique.

A function $f \in \mathrm{C}(G)$ is *right uniformly continuous* if the orbit map $g \in G \mapsto gf \in \mathrm{C}(G)$ is norm-continuous. Let $\mathrm{RUC}(G)$ denote the family of all right uniformly continuous, bounded functions on G, which is a bi-invariant closed subalgebra of $\mathrm{C}(G)$. Any function $f \in \mathrm{C}(G)$ that factors through a compactification of G is in $\mathrm{RUC}(G)$. In fact, there is a one-to-one correspondence between compactifications of G (up to isomorphism) and left-invariant closed subalgebras of $\mathrm{RUC}(G)$. It takes a compactification α to the algebra $\mathcal{A}(\alpha)$. Conversely, it takes a left-invariant closed subalgebra $A \subset \mathrm{RUC}(G)$ to the maximal ideal space G^A (with its usual compact Gelfand topology) together with the canonical G-map $G \to G^A$. Under this correspondence, A is a subalgebra of B if and only if G^B is a factor of G^A.

1.2. The Fourier–Stieltjes algebra and the WAP algebra of a topological group. We start by recalling the definition of positive definite functions and the GNS construction. See, for example, [**BdlHV08**], Theorem C.4.10.

Definition 1.1. A function $f\colon G \to \mathbb{C}$ is *positive definite* if the following equivalent conditions hold.

(1) For every $g_1,\dots,g_n \in G$, the matrix $(f(g_j^{-1}g_i))_{ij}$ is positive semi-definite, i.e., for any $c_1,\dots,c_n \in \mathbb{C}$ we have
$$\sum_{ij} c_i\overline{c_j} f(g_j^{-1}g_i) \geq 0.$$

(2) The exists a unitary representation $\pi\colon G \to U(\mathcal{H})$ and a vector $v \in \mathcal{H}$ such that, for all $g \in G$,
$$f(g) = \langle v, \pi(g)v\rangle.$$

In particular, every positive definite function is bounded: $|f(g)| \leq f(1)$ for all $g \in G$. Besides, we can assume that v is a *cyclic vector* for the representation $\pi\colon G \to U(\mathcal{H})$ (that is, $\mathcal{H}$ is the closed subspace generated by $\pi(G)v$); then π is continuous if and only if $f \in \mathrm{C}(G)$.

The family of continuous positive definite functions on G is denoted by $P(G)$. The (not necessarily closed) subalgebra of $\mathrm{C}(G)$ generated by $P(G)$ is the *Fourier–Stieltjes algebra of G*, and is denoted by $B(G)$.

Definition 1.2. A *(unitary) matrix coefficient* of a topological group G is a function $f \in \mathrm{C}(G)$ of the form
$$f(g) = \langle v, \pi(g)w\rangle$$
for a continuous unitary representation $\pi\colon G \to U(\mathcal{H})$ and vectors $v, w \in \mathcal{H}$. We will use the notation $f = m_{v,w}$, or $f = m^{\pi}_{v,w}$ if we wish to specify π.

Fact 1.3. *$B(G)$ is the family of matrix coefficients of G.*

That matrix coefficients form an algebra follows from considering orthogonal sums, tensor products, and duals of representations.

In fact, every matrix coefficient is a linear combination of four positive definite functions,

$$4m_{v,w} = m_{v+w,v+w} - m_{v-w,v-w} + im_{v+iw,v+iw} - im_{v-iw,v-iw},$$

so $B(G)$ coincides with the linear span of $P(G)$.

Next we recall weakly almost periodic functions, Grothendieck's double limit criterion and the reflexive representation theorem of Megrelishvili (see [**Meg03**], Theorem 5.1).

Definition 1.4. A function $f \in \mathrm{C}(G)$ is *weakly almost periodic* if the following equivalent conditions hold:

(1) The orbit Gf is precompact for the weak topology on $\mathrm{C}(G)$.
(2) For all sequences $g_i, h_j \in G$, we have

$$\lim_i \lim_j f(g_i h_j) = \lim_j \lim_i f(g_i h_j)$$

whenever both limits exist.
(3) There exists a continuous, isometric representation $\pi\colon G \to \mathrm{Iso}(V)$ on a reflexive Banach space V and vectors $v \in V$, $w \in V^*$ such that, for all $g \in G$,

$$f(g) = \langle v, \pi(g)w\rangle.$$

It follows easily that the family $\mathrm{WAP}(G)$ of weakly almost periodic functions on G is a closed bi-invariant subalgebra of $\mathrm{RUC}(G)$ containing $B(G)$. On the other hand, $B(G)$ is almost never closed in $\mathrm{C}(G)$ (see the beginning of Section 3). Following [**GM14b**, §6], we will denote the closure $\overline{B(G)}$ by $\mathrm{Hilb}(G)$. It consists precisely of the continuous functions on G that factor through Hilbert-representable compactifications of G (see the next subsection). The algebra $B(G)$ is bi-invariant, hence so is $\mathrm{Hilb}(G)$.

Thus we have

$$\mathrm{Hilb}(G) \subset \mathrm{WAP}(G),$$

or equivalently: the *Hilbert compactification* $G \to H(G)$ associated to the closed left-invariant algebra $\mathrm{Hilb}(G)$ is a G-factor of the *WAP compactification* $G \to W(G)$ associated to $\mathrm{WAP}(G)$. We will review the main properties of these compactifications in Section 2.

Finally, we recall that a function $f\colon G \to \mathbb{C}$ is *Roelcke uniformly continuous* if the map $(g, g') \in G \times G \mapsto gfg' \in \mathrm{C}(G)$ is norm-continuous. The family of all Roelcke uniformly continuous functions on G is a closed bi-invariant subalgebra of $\mathrm{RUC}(G)$, denoted by $\mathrm{UC}(G)$. We always have $\mathrm{WAP}(G) \subset \mathrm{UC}(G)$.

Definition 1.5. Let G be a topological group.

(i) G is *Eberlein* if $\mathrm{Hilb}(G) = \mathrm{WAP}(G)$.
(ii) G is a *WAP group* if $\mathrm{WAP}(G) = \mathrm{UC}(G)$.
(iii) G is *strongly Eberlein* if $\mathrm{Hilb}(G) = \mathrm{UC}(G)$.

In his fundamental work [**Ebe49**], Eberlein introduced weakly almost periodic functions (in the context of locally compact abelian groups) and proved the inclusion $B(G) \subset \mathrm{WAP}(G)$. In fact, all his examples of WAP functions lied in the closure of $B(G)$. Rudin writes in [**Rud59**] that Eberlein asked him whether the closure of

$B(G)$ may in fact coincide with WAP(G). Of course, by the Peter–Weyl theorem, this is the case for compact groups (indeed, $\mathrm{Hilb}(G) = \mathrm{C}(G)$). However, Rudin showed in the referred paper that this is not true in general. As an example, he exhibited a concrete function $f \in \mathrm{WAP}(\mathbb{Z}) \setminus \mathrm{Hilb}(\mathbb{Z})$. Later, Chou proved that, more generally, the inclusion $\mathrm{Hilb}(G) \subset \mathrm{WAP}(G)$ is strict for any non-compact locally compact nilpotent group [**Cho82**]. On the other hand, he remarked that equality does hold for some non-compact locally compact groups, and introduced the name *Eberlein* for this class.

The definitions of *WAP groups* and *strongly Eberlein groups* were introduced by Glasner and Megrelishvili in [**GM14b**].

Examples of non-compact Eberlein groups include $SL_n(\mathbb{R})$ (and any semisimple Lie group with finite center; see [**Vee79**]), the unitary group $U(\ell^2)$ [**Meg08**], the group $\mathrm{Aut}(\mu)$ of measure preserving transformations of the unit interval [**Gla12**] and the symmetry group of a countable set, S_∞ [**GM14b**]. The latter three are in fact strongly Eberlein. We will give some new examples in Section 3.3.

1.3. Representations on Hilbert spaces. Let X be a compactification of a Polish group G. We say that X is *Hilbert-representable* if there exist a Hilbert space $\mathcal{H}$, an embedding $\iota\colon X \to \mathcal{H}$ (where $\mathcal{H}$ carries the weak topology) and a unitary representation $\pi\colon G \to U(\mathcal{H})$ such that $\iota(gx) = \pi(g)\iota(x)$ for all $x \in X$ and $g \in G$. This definition coincides, for the class of Hilbert spaces, with the notion of $\mathcal{K}$-representability given in the introduction.

Given a function $f \in \mathrm{RUC}(G)$, let X_f be the compactification of G associated to the invariant closed subalgebra of $\mathrm{RUC}(G)$ generated by f. In [**GW12**, §2], it is observed that X_f is Hilbert-representable whenever f is positive definite (in the case $G = \mathbb{Z}$); more generally, the following holds.

Lemma 1.6. *If $f \in B(G)$, then X_f is Hilbert-representable.*

Proof. Write $f = m^{\pi_0}_{v_0,w_0}$ for some continuous unitary representation $\pi_0\colon G \to U(\mathcal{H}_0)$. Let $\mathcal{H}_1$ be the closed linear span of $\pi(G)w_0$ and let $v = \Pi_{\mathcal{H}_1} v_0$ be the orthogonal projection of v_0 to $\mathcal{H}_1$. Next let $\mathcal{H}$ be the closed linear span of $\pi(G)v$ and let $w = \Pi_{\mathcal{H}} w_0$ be the orthogonal projection of w_0 to $\mathcal{H}$. Consider the restriction $\pi = \pi_0|_{\mathcal{H}}$. Then $f = m^{\pi}_{v,w}$.

Let Z be the weak closure of $\pi(G)w$ in $\mathcal{H}$, which is naturally a (Hilbert-representable) compactification of G via the map $g \mapsto gw$. Consider for each $h \in G$ the function $F_h \in \mathrm{C}(Z)$, $F_h(z) = \langle \pi(h)v, z\rangle$, and remark that $F_h(gw) = hf(g)$. Since $\mathcal{H}$ is generated by $\pi(G)v$, the functions F_h separate points of Z. Hence, by the Stone–Weierstrass theorem, $\mathcal{A}(Z)$ is the closed algebra generated by Gf. In other words, $Z = X_f$ up to isomorphism. □

In contrast, if instead of a matrix coefficient we take any $f \in \mathrm{Hilb}(G)$, it is unknown whether X_f is necessarily Hilbert-representable; see Question 1.9 below.

Proposition 1.7. *Let $\alpha\colon G \to X$ be a metrizable compactification of G. Then the following are equivalent:*

(1) α is Hilbert-representable.
(2) $\mathcal{A}(\alpha) = \overline{\mathcal{A}(\alpha) \cap B(G)}$.

Proof. Suppose (ι,π) is a representation of (X,G) on a Hilbert space $\mathcal{H}$. The functions $F_v\colon w \mapsto \langle v,w\rangle$ separate points of $\mathcal{H}$, hence the algebra generated by $\{F_v\iota\alpha\}_{v\in\mathcal{H}}$ is dense in $\mathcal{A}(\alpha)$ and contained in $B(G)$.

Conversely, suppose that (2) holds. The metrizability assumption on X says that $\mathcal{A}(\alpha)$ is separable. Thus, let $B \subset \mathcal{A}(\alpha)\cap B(G)$ be a countable dense subset. By the previous lemma, for each $f \in B$ there is a representation (ι_f,π_f) of (X_f,G) on a Hilbert space $\mathcal{H}_f$. We consider

$$\mathcal{H} = \bigoplus_{f\in B}\mathcal{H}_f$$

and let $\pi = \bigoplus_{f\in B}\pi_f\colon G \to U(\mathcal{H})$ be the orthogonal sum of the representations π_f. For each $f \in B$ let $w_f = \iota\alpha(1) \in \mathcal{H}_f$. Since B is countable, by rescaling we may assume that $w = (w_f)_{f\in B}$ is summable, i.e., $w \in \mathcal{H}$. Now we define $\alpha'\colon G \to \mathcal{H}$ by $\alpha'(g) = \pi(g)w$, and let Z be the weak closure of $\alpha'(G)$ in $\mathcal{H}$. Then the restriction $\alpha'\colon G \to Z$ is a Hilbert-representable compactification of G, which we claim is isomorphic to α. Indeed, Z is homeomorphic to a subspace of the product $\prod_{f\in B}X_f$, so the projection maps $Z \to X_f$ separate points of Z; hence $\mathcal{A}(\alpha')$ is the closed algebra generated by the algebras $\mathcal{A}(X_f)$, $f \in B$. Since B is dense in $\mathcal{A}(\alpha)$, we deduce that $\mathcal{A}(\alpha') = \mathcal{A}(\alpha)$, which proves our claim. $\square$

Remark 1.8. A basic consequence of the first implication of the above proposition (which does not use the metrizability assumption) is that all Hilbert-representable compactifications of G are factors of $H(G)$.

Question 1.9 ([**GM14b**], Question 5.12.3; [**Meg07**], Question 7.6). Are Hilbert-representable dynamical systems closed under factors; equivalently (for ambits), are all factors of $H(G)$ Hilbert-representable?

This question has also been investigated in [**GW12**]. In Section 4, we will see that the answer is positive for pro-oligomorphic groups.

We should note that *reflexively representable* dynamical systems are preserved under factors. In fact, the reflexively representable (or rather, when $W(G)$ is not metrizable, *reflexively approximable*) compactifications of G are exactly the factors of $W(G)$. See [**Meg08**] and the references therein.

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2. Semitopological semigroup compactifications

2.1. Definitions. A *semitopological semigroup* is a semigroup that carries a topological structure such that the product operation is separately continuous. That is to say, multiplying by an arbitrary fixed element to the left is continuous, and similarly to the right. We shall be interested in semitopological semigroups arising in the following manner.

Definition 2.1. A compactification $\alpha\colon G \to S$ is a *semitopological semigroup compactification* if S admits a semitopological semigroup law that makes of α a homomorphism.

Remark 2.2. Suppose $\alpha\colon G \to S$ is a semitopological semigroup compactification.

(1) Then S is in fact a monoid: $\alpha(1)$ is an identity.
(2) By Lawson's joint continuity theorem, $\alpha(G)$ is a topological group ([**Law74**], Corollary 6.3).
(3) The compactification is bi-invariant.

Both the Hilbert and WAP compactifications introduced before, $H(G)$ and $W(G)$, are semitopological semigroup compactifications. Moreover, $W(G)$ is universal among these, in the following sense.

Fact 2.3. *Let S be a bi-invariant compactification of G. The following are equivalent.*

(1) S is a semitopological semigroup compactification.
(2) S is a factor of $W(G)$.

Proof. See, for instance, [**BJM78**, Ch. III, §8], Corollary 8.5. □

Given a reflexive Banach space V, the semigroup $\Theta(V)$ of linear contractions of V,

$$\Theta(V) = \{T \in L(V) : \|T\| \leq 1\},$$

endowed with the weak operator topology, is compact and semitopological. It turns out that every compact semitopological semigroup can be seen as a closed subsemigroup of $\Theta(V)$ for some reflexive Banach space V [**Sht94, Meg01b**]. Thus, in this sense, every compact semitopological semigroup is *reflexively representable*. A stricter notion of representability of semigroups is the following.

Definition 2.4. A semitopological semigroup S is *Hilbert-representable* if it can be embedded in the compact semitopological semigroup $\Theta(\mathcal{H})$ of linear contractions of a Hilbert space $\mathcal{H}$.

It is not difficult to see the following.

Fact 2.5. *Let G be a topological group.*

(1) $H(G)$ is Hilbert-representable as a semitopological semigroup.
(2) G is Eberlein if and only if $W(G)$ is Hilbert-representable as a semitopological semigroup.

In fact, the two notions of representability on Hilbert spaces discussed so far essentially coincide. See Lemma 4.5 in [**Meg08**].

Fact 2.6. *Let $\alpha\colon G \to S$ be a metrizable semitopological semigroup compactification of G. Then S is a Hilbert-representable semitopological semigroup if and only if α is a Hilbert-representable compactification.*

In the non-metrizable case, Definition 2.4 is the correct property to consider, while Hilbert-representability of dynamical systems has to be relaxed. However, the semigroups that we study in this paper are metrizable.

Definition 2.7. Let $\alpha\colon G \to S$ be a semitopological semigroup compactification. We will say that α is *$*$-closed* or, equivalently, that α is a *semitopological $*$-semigroup compactification*, if the inverse operation on the group $\alpha(G)$ extends to a continuous involution $^*\colon S \to S$.

Remark 2.8. $\alpha\colon G \to S$ is $*$-closed if and only if, whenever $f \in \mathcal{A}(\alpha)$, the function $g \mapsto f(g^{-1})$ is also in $\mathcal{A}(\alpha)$.

It follows readily that both $H(G)$ and $W(G)$ are $*$-closed.

Proposition 2.9. *Every Hilbert-representable semitopological semigroup compactification is $*$-closed.*

Proof. Let $\alpha\colon G \to S$ be a compactification with an embedding $\beta\colon S \to \Theta(\mathcal{H})$. It suffices to see that the image of β is closed under the adjoint operation ${}^*\colon \Theta(\mathcal{H}) \to \Theta(\mathcal{H})$; indeed, then we can define s^* as the preimage of $\beta(s)^*$, and this gives a continuous map ${}^*\colon S \to S$ that extends the inverse operation on $\alpha(G)$. Now, if $s \in S$ is the limit of a net $\alpha(g_i) \in \alpha(G)$, then $\beta\alpha(g_i^{-1})$ converges to $\beta(s)^*$; by compactness we may assume that $\alpha(g_i^{-1})$ converges to some $s' \in S$, so $\beta(s') = \beta(s)^*$. Hence $\beta(s)^* \in \beta(S)$. □

2.2. Inverse semigroups. In this short subsection, we review some general notions of the theory of semigroups, and some particular facts that hold for compact semitopological $*$-semigroups with a dense subgroup.

An element e in a semigroup S is an *idempotent* if $e^2 = e$. If S has an involution *, then $e \in S$ is *self-adjoint* if $e^* = e$.

Definition 2.10. Let S be a semigroup.

(i) An element $p \in S$ is *regular* if there exists $q \in S$ such that $pqp = p$.
(ii) S is *regular* if every element is regular.
(iii) An element $q \in S$ is an *inverse* for $p \in S$ if $pqp = p$ and $qpq = q$.
(iv) S is an *inverse semigroup* if every element has a unique inverse.

The canonical example of an inverse semigroup is the *symmetric inverse semigroup* of all partial bijections of a set, with composition where it is defined.

A proof of the following general characterization can be found in [**How95**], Theorem 5.1.1.

Fact 2.11. *The following are equivalent for a semigroup S.*

(1) S is an inverse semigroup.
(2) S is regular and the idempotents commute.

When a compact semitopological structure is available, and the semigroup contains a dense subgroup, much more is true. We formulate these additional properties in the case that we are interested in.

Fact 2.12. *Let $G \to S$ be a semitopological $*$-semigroup compactification.*

(1) For every $p, q \in S$ we have $Sq = Spq$ if and only if $q = p^*pq$.
(2) Every idempotent is self-adjoint.
(3) Let $e, f \in S$ be idempotents. The following are equivalent:
  (a) e and f commute.
  (b) ef is also an idempotent.
(4) Let $p \in S$. The following are equivalent:
  (a) p is regular.
  (b) $pp^*p = p$.

   (c) p has a unique inverse.

(5) *In particular, the following are equivalent:*
   (a) S in an inverse semigroup.
   (b) S is regular.

Proof.

(1) See Lawson [**Law84**], Proposition 4.1.
(2) This follows easily from (1).
(3) One implication is immediate and the other is clear using that idempotents are self-adjoint.
(4) Suppose p is regular, so $pqp = q$ for some q. Hence $Sq = Spq$, so by (1) we have $q = p^*pq$, and then $p = pqp = pp^*pqp = pp^*p$.
If now we suppose that $p = pp^*p$, then $p^* = p^*pp^*$, so p and p^* are inverses. If q is another inverse, then as before we have $q = p^*pq$, whence $qp = p^*pqp = p^*p$. Dually, $pq = pp^*$. Then $q = qpq = qpp^* = p^*pp^* = p^*$.
(5) Clear. □

2.3. The WAP compactification of pro-oligomorphic groups. In this subsection we will recall the model-theoretic description of the WAP compactification given in [**BT14**] for Roelcke precompact Polish groups. Since the results of the present paper are concerned with pro-oligomorphic groups, our presentation here will be restricted to these, i.e., to automorphism groups of *classical* $\aleph_0$-categorical structures (as opposed to *metric*). Still, it will be convenient to consider formulas as real-valued functions, taking values in $\{0,1\}$.

We refer to [**TZ12**] for the necessary background in model theory and for the basics of $\aleph_0$-categorical structures. Let us recall the definition of the family of groups we will study.

Definition 2.13. A group G is *oligomorphic* if it can be presented as a closed permutation group $G \leq S(X)$ of a countable set X such that the orbit spaces X^n/G are finite for every $n < \omega$. *Equivalently:* if G is the automorphism group of an $\aleph_0$-categorical one-sorted structure.

A Polish group G obtained as an inverse limit of oligomorphic groups will be called *pro-oligomorphic*. *Equivalently:* G can be presented as the automorphism group of an $\aleph_0$-categorical multi-sorted structure. These are exactly the Roelcke precompact, non-archimedean, Polish groups: see [**Tsa12**], Theorem 2.4.

Throughout this paper, whenever G is a pro-oligomorphic group and we write $G = \mathrm{Aut}(M)$, we understand that M is an $\aleph_0$-categorical structure and G is its automorphism group. By the homogeneity of $\aleph_0$-categorical structures, we have the following.

Fact 2.14. *Let $G = \mathrm{Aut}(M)$ be a pro-oligomorphic group and $\widehat{G_L}$ be the completion of G with respect to its left uniformity, which is a topological semigroup. Then $\widehat{G_L}$ can be identified with the topological semigroup $E(M)$ of elementary embeddings of M into itself with the topology of pointwise convergence.*

Proof. Let $\xi \in M^\omega$ be an enumeration of M and define the distance d_L on $E(M)$ by $d_L(x,y) = \sup_{i<\omega} 2^{-i} d(x(\xi_i), y(\xi_i))$, where d is the discrete distance on M. It induces the topology of pointwise convergence on $E(M)$. The restriction of d_L to G is a compatible left-invariant metric, thus inducing the left uniformity of G. By homogeneity, G is dense on $E(M)$ with respect to d_L. Since moreover $E(M)$ is complete with respect to d_L, it is the left completion of G. □

Recall that if (X,d) is a metric space and G acts on X by isometries,

$$X /\!\!/ G = \{\overline{Gx} : x \in X\}$$

is a metric space with distance

$$d(\overline{Gx}, \overline{Gy}) = \inf\{d(g_1 x, g_2 y) : g_1, g_2 \in G\}.$$

The completion with respect to the *Roelcke uniformity* of a Polish group G (the infimum of the left and right uniformities) can be described as the space of orbit closures $R(G) = (\widehat{G}_L \times \widehat{G}_L) /\!\!/ G$, where G acts diagonally on $\widehat{G}_L \times \widehat{G}_L$ by left translation. The group G is Roelcke precompact precisely when $R(G)$ is compact. That is, when the completion $R(G)$ coincides with the compactification of G associated to the algebra $\mathrm{UC}(G)$. Note that the completion $R(G)$ is metrizable; thus, for Roelcke precompact Polish groups, this is a metrizable compactification, and so are all its factors.

For the rest of this section, we fix a pro-oligomorphic group $G = \mathrm{Aut}(M)$. By Fact 2.14, in this case we can write $R(G) = (E(M) \times E(M)) /\!\!/ G$. Given elements $x, y \in E(M)$, we denote the class of (x,y) in $R(G)$ by $[x,y]_R$. Now, each $x \in E(M)$ can be coded by a tuple $\tilde{x} \in M^\omega$: $\tilde{x}_i = x(\xi_i)$ (if $\xi \in M^\omega$ is a fixed enumeration of M). Then, the element $[x,y]_R \in R(G)$ can be seen as the type $\mathrm{tp}(\tilde{x}, \tilde{y}/\emptyset)$. In other words, $[x,y]_R$ is determined by the values

$$\varphi(x(a), y(b)),$$

where $\varphi(u,v)$ varies over the formulas of M and a, b vary over tuples of M of the appropriate length. In this fashion, a sequence $[x_n, y_n]_R$ in $R(G)$ converges to $[x,y]_R$ if the truth value $\varphi(x_n(a), y_n(b))$ converges to $\varphi(x(a), y(b))$ for all φ, a, and b.

Since $\mathrm{WAP}(G)$ is a subalgebra of $\mathrm{UC}(G)$, the WAP compactification $W(G)$ is a factor of $R(G)$. We will denote the image of $[x,y]_R$ in $W(G)$ simply by $[x,y]$. That is to say, $[x,y]$ is determined by the values $\varphi(x(a), y(b))$, as before, only that $\varphi(u,v)$ ranges over the stable formulas of M. In particular, G is a WAP group if and only if the structure M is stable.

The canonical G-map $G \to W(G)$ is given by

$$g \mapsto [1_G, g],$$

and the G-action by

$$g[x,y] = [xg^{-1}, y].$$

The involution $^*\colon W(G) \to W(G)$ extending the inverse on the image of G is given by

$$[x,y]^* = [y,x].$$

Moreover, the semitopological semigroup law of $W(G)$ can be described in terms of the *stable independence relation* of M. In order to explain this, we first recall the definition of *imaginaries* and some notions from stability theory.

Let M be a structure. An *imaginary element* of M is the a class of a definable equivalence relation on some finite power of M. In other words, if a formula $\varphi(u,v)$ defines an equivalence relation on M^n, then each class $[a]_\varphi \in M^n/\varphi$ is an imaginary of M.

A standard model-theoretic construction allows to consider all the imaginaries of M as actual elements in a larger (multi-sorted) structure, denoted M^{eq}. See [**TZ12**, §8.4] for the details. This enlargement of M is in many senses innocuous; in particular, the natural restriction map $\mathrm{Aut}(M^{\mathrm{eq}}) \to \mathrm{Aut}(M)$ is an isomorphism between their automorphism groups. Thus, for many purposes, it is convenient to work directly with the structure M^{eq}.

Moreover, imaginary elements of $\aleph_0$-categorical structures are in correspondence with the open subgroups of its automorphism group. Indeed, a subgroup $V \leq G$ is open if and only if it is the stabilizer of an imaginary element of M. That is to say, if and only if there is a definable equivalence relation $\varphi(u,v)$ and a tuple c such that

$$V = \{g \in G : \varphi(c, gc)\} = \{g \in G : [c]_\varphi = g[c]_\varphi\}.$$

See, for example, [**Tsa12**, §5].

A special kind of imaginaries is given as follows. If $\varphi(u,v)$ is any formula, we can define a formula $E_\varphi(u,u')$ by

$$E_\varphi(u,u') := \forall v(\varphi(u,v) \leftrightarrow \varphi(u',v)).$$

Then E_φ defines an equivalence relation on $M^{|u|}$. An imaginary $[c]_{E_\varphi} \in M^{|u|}/E_\varphi$ should be seen as representing the formula $\varphi(c,v)$; $[c]_{E_\varphi}$ (also denoted simply by $[c]_\varphi$) is called the *canonical parameter* of $\varphi(c,v)$,

Given a type $t \in S_u(M)$ and a formula $\varphi(u,v)$, the *φ-definition* of t is the function $d_t\varphi\colon M^{|v|} \to \{0,1\}$ given by

$$d_t\varphi(b) := \varphi(u,b)^t,$$

where the right term denotes the value of $\varphi(u,b)$ in the type t. Then, the formula $\varphi(u,v)$ is *stable* if and only if $d_t\varphi$ is an M-definable predicate for every $t \in S_u(M)$. In this case we can write $d_t\varphi(v)$ in the form $\psi(c,v)$, and then consider the canonical parameter of this formula; we denote this canonical parameter by $\mathrm{Cb}_\varphi(t)$. (The choice of the formula ψ can be done uniformly in t, that is, c depends on t but $\psi(w,v)$ does not.) The tuple

$$\mathrm{Cb}(t) = (\mathrm{Cb}_\varphi(t))_{\varphi \text{ stable}}$$

is the *canonical base* of t.

Finally, an element $d \in M^{\mathrm{eq}}$ is in the *algebraic closure* of a set $A \subset M^{\mathrm{eq}}$ if, for some finite tuple $a \subset A$, d has only finitely many conjugates by automorphisms fixing a. We denote the algebraic closure of A by $\mathrm{acl}(A)$ (which is always a subset of M^{eq}). The set A is *algebraically closed* if $A = \mathrm{acl}(A)$.

Fact 2.15. *Let $a \in (M^{\mathrm{eq}})^{|u|}$ be a tuple and $B \subset M^{\mathrm{eq}}$ be any subset. There is an extension of the type* $\mathrm{tp}(a/\mathrm{acl}(B))$ *to a type $t \in S_u(M)$ such that* $\mathrm{Cb}(t) \subset \mathrm{acl}(B)$. *Moreover,* $\mathrm{Cb}(t)$ *does not depend on the particular extension; in other words, if $s \in S_u(M)$ is another such extension, then $d_t\varphi = d_s\varphi$ for every stable formula $\varphi(u,v)$.*

Definition 2.16.

(i) If a, B and t are as in the previous fact, we define $\mathrm{Cb}_\varphi(a/B) := \mathrm{Cb}_\varphi(t)$, $\mathrm{Cb}(a/B) := \mathrm{Cb}(t)$.

(ii) Given any sets $A, B, C \subset M^{\mathrm{eq}}$, we say that A *is stably independent from* C *over* B, denoted

$$A \mathop{\smile\!\!\!\!\!\!\;|}_{B} C,$$

if for any tuple $a \in A^{|u|}$ we have $\mathrm{Cb}(a/B) = \mathrm{Cb}(a/BC)$.

(iii) If a, c are tuples from M^{eq} and B is any subset, we write $a \equiv^s_B c$ to mean that a and c have the same *stable type* over B, that is, $\varphi(a,b) = \varphi(c,b)$ for any stable formula $\varphi(u,v)$ and parameter $b \in B^{|v|}$. When B is empty we shall write simply $a \equiv c$, since $a \equiv^s_\emptyset c$ is indeed equivalent to $\mathrm{tp}(a/\emptyset) = \mathrm{tp}(c/\emptyset)$.

Note that the natural identification of $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M^{\mathrm{eq}})$ extends to an identification of $E(M)$ and $E(M^{\mathrm{eq}})$.

Convention 2.17. We may consider the elements of $E(M)$ as sets (notably, to apply the relations $\mathop{\smile\!\!\!\!\!\!\;|}$ and $\equiv^s$ to them), and this shall be done in the following way: an element $x \in E(M)$ is interpreted as the set $x(M^{\mathrm{eq}})$, that is, the imaginary completion of the image of x. For instance, $x \cap y$ will denote $x(M^{\mathrm{eq}}) \cap y(M^{\mathrm{eq}})$.

If appearing as arguments of the relation $\equiv^s$, the elements of $E(M)$ will be considered as infinite tuples indexed by M (or by ω via a fixed enumeration ξ, as before).

In these contexts, the juxtaposition xy will denote the juxtaposed tuple (or merely the union of sets).

The pair $(\mathop{\smile\!\!\!\!\!\!\;|}, \equiv^s)$ satisfies the following usual properties.

Fact 2.18. *Let x, y, z, w be any tuples from M^{eq}.*

(1) (Invariance) If $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$ and $xyz \equiv x'y'z'$, then $x' \mathop{\smile\!\!\!\!\!\!\;|}_{y'} z'$. If $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$ and $x \equiv^s_{yz} x'$, then $x' \mathop{\smile\!\!\!\!\!\!\;|}_y z$.

(2) (Symmetry) $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$ if and only if $z \mathop{\smile\!\!\!\!\!\!\;|}_y x$.

(3) (Transitivity) $x \mathop{\smile\!\!\!\!\!\!\;|}_y zw$ if and only if $x \mathop{\smile\!\!\!\!\!\!\;|}_{yz} w$ and $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$.

(4) (Existence) There exist x', y', z' such that $x'y' \equiv xy$, $y'z' \equiv yz$ and $x' \mathop{\smile\!\!\!\!\!\!\;|}_{y'} z'$.

(5) (Stationarity) Suppose y is algebraically closed. If $x \equiv^s_y z$, $x \mathop{\smile\!\!\!\!\!\!\;|}_y w$ and $z \mathop{\smile\!\!\!\!\!\!\;|}_y w$, then $x \equiv^s_{yw} z$.

(6) (Non-triviality) If $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$, then $\mathrm{acl}(x) \cap \mathrm{acl}(z) \subset \mathrm{acl}(y)$.

Proof. We refer the reader to [**Pil96**, Ch. 1, §2]. □

Fact 2.19. *The semigroup law in $W(G)$ is given by*

$$[x,y][y,z] = [x,z] \ \text{ if } \ x \mathop{\smile\!\!\!\!\!\!\;|}_{y} z.$$

The properties of the independence relation stated above ensure that, for any $p, q \in W(G)$, we can always find $x, y, z \in E(M)$ such that $p = [x,y]$, $q = [y,z]$ and $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$.

The latter allows for a model-theoretic description of the idempotents of $W(G)$. This was given in [**BT14**, §5]. Let us end this section by recalling this description and giving a complete proof. Moreover, we complement it with a characterization of the regular elements of $W(G)$, which will be used in our main result.

For the definition and properties of the φ-rank see [**Pil96**, Ch. 1, §3].

Lemma 2.20. *Let $p = [x,y] \in W(G)$, $C = x \cap y$.*

(1) The following are equivalent.
  (a) p is an idempotent (i.e., $pp = p$).
  (b) $x \equiv^s_C y$ and $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C y$.

(2) The following are equivalent.
  (a) p is regular (i.e., $pp^*p = p$).
  (b) $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C y$.

Proof. (1) Suppose p is an idempotent. By replacing x,y by an equivalent pair if necessary, we can find $z \in E(M)$ such that $x \mathop{\smile\!\!\!\!\!\!\;\mid}_y z$ and $xy \equiv yz$. Then $[x,y] = [x,y][y,z] = [x,z]$, so $y \equiv^s_x z$. It is easy to deduce that $C = x \cap z = y \cap z$, and further that $x \equiv^s_C y$.

Next we argue that $x \mathop{\smile\!\!\!\!\!\!\;\mid}_z y$. This is equivalent to show that for every stable formula φ the φ-rank of x over yz equals the φ-rank of x over z: $R_\varphi(x/yz) = R_\varphi(x/z)$. Indeed, since $x \mathop{\smile\!\!\!\!\!\!\;\mid}_y z$ and $xy \equiv xz$, we have

$$R_\varphi(x/yz) = R_\varphi(x/y) = R_\varphi(x/z).$$

Thus $x \mathop{\smile\!\!\!\!\!\!\;\mid}_y z$ and $x \mathop{\smile\!\!\!\!\!\!\;\mid}_z y$, so we have $\mathrm{Cb}(x/yz) \subset y \cap z$. This implies that $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C y$.

Conversely, if $x \equiv^s_C y$ and $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C y$, take z with $x \mathop{\smile\!\!\!\!\!\!\;\mid}_y z$ and $xy \equiv yz$. It follows that $C = x \cap z$, $x \equiv^s_C z$ and $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C z$. This implies $y \equiv^s_x z$, i.e., $[x,y] = [x,z]$, which means that p is an idempotent.

(2) Suppose p is regular. Replacing x,y by an equivalent pair if necessary, we can find $z,w \in E(M)$ with $x \mathop{\smile\!\!\!\!\!\!\;\mid}_y z$, $xy \equiv zy$, $x \mathop{\smile\!\!\!\!\!\!\;\mid}_z w$ and $xy \equiv zw$. Then the condition $p = pp^*p$ becomes

$$[x,y] = [x,y][y,z][z,w] = [x,w],$$

so $y \equiv^s_x w$. It is easy to check that $C = x \cap z = x \cap w$. Moreover, since $e = pp^* = [x,z]$ is an idempotent, we have that $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C z$. From $x \mathop{\smile\!\!\!\!\!\!\;\mid}_z w$ we get then $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C w$. Since $y \equiv^s_x w$, we have $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C y$.

Suppose conversely that $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C y$, and find z,w as above, so that in particular $pp^*p = [x,w]$. From the hypothesis we get $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C z$, and then $x \mathop{\smile\!\!\!\!\!\!\;\mid}_C w$. The condition $xy \equiv zw$ implies $y \equiv^s_C w$, whence $y \equiv^s_x w$. That is, $pp^*p = [x,w] = [x,y] = p$. □

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3. The Fourier–Stieltjes algebra of pro-oligomorphic groups

3.1. Examples of functions in $\mathrm{Hilb}(G)\setminus B(G)$. As mentioned before, the Fourier–Stieltjes algebra $B(G)$ is, as a general rule, strictly contained in its closure $\mathrm{Hilb}(G)$. For example, if G is compact, then $B(G)$ is not closed in $\mathrm{C}(G)$ unless G is finite (see for instance [**HR70**], Theorem 37.4). Let us begin this section with a model-theoretic argument showing that the same holds for pro-oligomorphic groups.

For locally compact groups, the algebra $C_0(G)$ of functions vanishing at infinity is always contained in $\mathrm{Hilb}(G)$. Indeed, the functions in $C_0(G)$ factor through the one-point compactification of G, and the latter is a Hilbert-representable semi-topological semigroup: it can be embedded into the contractions of $L^2(G,\mu)$ by sending the point at infinity to the zero operator, and otherwise extending the regular representation of G.

Similarly, for closed subgroups of S_∞, we have a simple way of producing functions in $\mathrm{Hilb}(G)$. Recall that if a group G acts continuously on a discrete set X, then we have a natural unitary representation $\pi\colon G \to U(\ell^2(X))$ defined (on the canonical basis of $\ell^2(X)$) by $\pi(g)e_x = e_{gx}$.

Lemma 3.1. *Let M be a structure, $G = \mathrm{Aut}(M)$. Let $F\colon M^n \to \mathbb{C}$ be a function vanishing at infinity and let $a \in M^n$. Then the function $f \in \mathrm{C}(G)$ given by $f(g) = F(ga)$ belongs to* $\mathrm{Hilb}(G)$.

Proof. We can assume that F is zero everywhere except on a finite set $B \subset M^n$, since the general case can be uniformly approximated by instances of this form. Take the natural representation $\pi\colon G \to U(\ell^2(M^n))$ and the vectors $v = \sum_{b\in B} F(b)e_b$, $w = e_a$. Then we have $f(g) = \langle v, \pi(g)w\rangle$, which shows that $f \in B(G)$. □

Next, we see that, in contrast, functions in $B(G)$ must satisfy a non-trivial decay condition. Given an action by isometries $G \curvearrowright X$ and a sequence $(x_i)_{i<\omega} \subset X$, let us say that (x_i) is *indiscernible* if for all indices $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ we have the equality

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}] = [x_{j_1}, x_{j_2}, \ldots, x_{j_k}]$$

in $X^k /\!\!/ G$.

Suppose G is pro-oligomorphic, so that $E(M) = \widehat{G_L}$. Note then that every function $f \in B(G)$, being left uniformly continuous, extends to a function on $E(M)$.

Proposition 3.2. *Let G be a pro-oligomorphic group, say $G = \mathrm{Aut}(M)$. Let $F\colon M^n \to \mathbb{C}$ be a function vanishing at infinity, $a \in M^n$, and let $f\colon E(M) \to \mathbb{C}$ be given by $f(x) = F(x(a))$.*

Suppose $(x_i)_{i<\omega} \subset E(M)$ is an indiscernible sequence such that $(x_i(a))_{i<\omega}$ is non-constant. If $f|_G \in B(G)$, then

$$\left|\frac{1}{n}\sum_{i<n} f(x_i)\right| = O(\frac{1}{\sqrt{n}}),$$

the implicit constant depending only on f.

Proof. Suppose we have a continuous unitary representation $\pi\colon G \to U(\mathcal{H})$ such that $f(g) = \langle v, \pi(g)w\rangle$ for all $g \in G$. Being a homomorphism, π is left uniformly continuous, so it extends to a representation $\pi\colon E(M) \to E(\mathcal{H})$. (Here, $E(\mathcal{H})$ is the semigroup of isometric linear endomorphisms of $\mathcal{H}$, which is also the left completion of $U(\mathcal{H})$.) We have $f(x) = \langle v, \pi(x)w\rangle$ for all $x \in E(M)$.

Since $(x_i) \subset E(M)$ is indiscernible, so is $(w_i) \subset \mathcal{H}$ for $w_i = \pi(x_i)w$. Now, as the reader can check, an indiscernible sequence (w_i) in a Hilbert space is always of the form $w_i = w' + w'_i$, where $w'_i \perp w'$, $\|w'_i\| = \|w'_j\|$ and $w'_i \perp w'_j$ for $i \neq j$. In particular,

w' is the weak limit of (w_i). Since F vanishes at infinity and $(x_i a)$ is indiscernible and non-constant, we have that $f(x_i) \to 0$. That is, $\langle v, w' \rangle = 0$. We deduce that

$$\left| \frac{1}{n} \sum_{i<n} f(x_i) \right| = \left| \langle v, \frac{1}{n} \sum_{i<n} w'_n \rangle \right| \leq \frac{\|v\| \cdot \| \sum_{i<n} w'_i \|}{n} = \frac{\|v\| \sqrt{\sum_{i<n} \|w'_i\|^2}}{n} = \frac{\|v\| \cdot \|w'_0\|}{\sqrt{n}} \leq \frac{\|v\| \cdot \|w\|}{\sqrt{n}}.$$

□

Corollary 3.3. *Let G be pro-oligomorphic and infinite. Then $B(G)$ is not closed in the uniform norm.*

Proof. Choose any non-constant indiscernible sequence $(x_i) \subset E(M)$ (which always exists if M is $\aleph_0$-categorical) and an element $a \in M$ such that $(x_i(a))$ is non-constant. Then take $F\colon M \to \mathbb{C}$ vanishing at infinity and such that $F(x_i(a)) = 1/i^{1/3}$. Then, by Lemma 3.1 and Proposition 3.2, we obtain that the function defined by $f(g) = F(ga)$ is in $\mathrm{Hilb}(G)$ and not in $B(G)$. □

3.2. A model-theoretic description of the Hilbert compactification. As explained in Section 2.3, the WAP compactification of a pro-oligomorphic group G is the space of types of pairs of embeddings $x, y \in E(M)$ *restricted to stable formulas*. Dually, this can be stated by saying that $\mathrm{WAP}(G)$ is the closed algebra generated by the functions of the form

$$g \mapsto \varphi(a, gb),$$

where $\varphi(u, v)$ is a stable formula and a, b are tuples from M. See [**BT14**, §5] or [**Iba14**, §4]. Hence it is natural to ask which formulas $\varphi(u, v)$ give rise, in the preceding way, to functions in the subalgebra $\mathrm{Hilb}(G)$.

We start with the following basic observation.

Lemma 3.4. *Let M be a structure, $G = \mathrm{Aut}(M)$. Let $\varphi(u, v)$ be a formula defining an equivalence relation on M^n and let $a, b \in M^n$. Then the function*

$$f(g) = \varphi(a, gb)$$

(which takes the value 1 if the elements are related and 0 otherwise) is in $B(G)$.

Proof. It suffices to consider the natural representation $\pi\colon G \to U(\ell^2(M^n/\varphi))$, then observe that $f(g) = \langle e_{[a]_\varphi}, \pi(g) e_{[b]_\varphi} \rangle$. □

The reader can also check that f belongs to $B(G)$ under the weaker assumption that $\varphi(x, b)$ defines a *weakly normal set*, that is to say, that the canonical parameter of $\varphi(x, b)$ is in the algebraic closure of any tuple a that satisfies the formula.

We want to give a converse to the previous lemma, for $\aleph_0$-categorical structures. For this we invoke the classification theorem of unitary representations of pro-oligomorphic groups proved in [**Tsa12**].

Fact 3.5 (Classification Theorem). *Let G be a pro-oligomorphic group.*

(1) Every continuous unitary representation of G is a direct sum of irreducible representations.
(2) Every irreducible continuous unitary representation is a subrepresentation of the quasi-regular representation $\pi_V\colon G \to U(\ell^2(G/V))$ for some open subgroup $V \leq G$.

Proposition 3.6. *Let G be pro-oligomorphic, $G = \mathrm{Aut}(M)$. Then* $\mathrm{Hilb}(G)$ *is the closed linear span of the functions of the form*

$$g \mapsto \varphi(a, gb)$$

where $\varphi(u,v)$ is a definable equivalence relation on some power M^n and a,b are tuples in M^n.

Proof. It suffices to show that every $f \in B(G)$ can be uniformly approximated by linear combinations of functions of this form. By the classification theorem, every continuous unitary representation is a subrepresentation of one of the form $\pi\colon G \to U\left(\bigoplus_k \ell^2(G/V_k)\right)$, where each V_k is an open subgroup of G. Now, every matrix coefficient of π can be uniformly approximated by a linear combination of *basic* matrix coefficients, that is, given by

$$g \mapsto \langle e_{g_0 V_k}, \pi(g) e_{g_1 V_k} \rangle$$

for vectors $e_{g_0 V_k}, e_{g_1 V_k}$ from the canonical basis of $\ell^2(G/V_k)$. Fix an open subgroup $V = V_k$; it is the stabilizer of some imaginary element $[c]_\varphi \in M^{\mathrm{eq}}$. If we take $a = g_0 c$ and $b = g_1 c$, we have that $g_0 V = g g_1 V$ if and only if $[a]_\varphi = g[b]_\varphi$. In other words,

$$\langle e_{g_0 V}, \pi(g) e_{g_1 V} \rangle = \varphi(a, gb).$$

The proposition follows. □

Dually, this characterization of $\mathrm{Hilb}(G)$ will provide a nice model-theoretic description of the Hilbert compactification $H(G)$.

We fix a pro-oligomorphic group $G = \mathrm{Aut}(M)$. We recalled earlier that the left completion of G can be identified with the topological semigroup $E(M)$ of elementary embeddings $M \to M$, which is the pointwise closure of G inside M^M. A natural generalization is to consider *partial* elementary maps of M.

In fact, the correct framework will be M^{eq}. Let $K = M^{\mathrm{eq}} \cup \{\infty\}$ be the one-point compactification of M^{eq}, and let

$$\Xi = \{p \in K^K : p(\infty) = \infty \text{ and } p \text{ is injective on } p^{-1}(M^{\mathrm{eq}})\}.$$

Then Ξ, equipped with composition and the product topology, is a compact semitopological inverse semigroup (in fact, isomorphic to the semigroup of partial bijections of M^{eq}). Let $P(M) = \overline{G} \subset \Xi$ be the closure of G in the product space K^K (where we set $g(\infty) = \infty$ for every $g \in G$). Then, if we think of an element $p \in K^K$ as a partial map $M^{\mathrm{eq}} \to M^{\mathrm{eq}}$ (undefined on a whenever $p(a) = \infty$), we get the following.

Proposition 3.7. *The elements of $P(M)$ are precisely the partial elementary maps of M with algebraically closed domain. Besides, $P(M)$ is closed under composition, and with this operation it becomes a semitopological $*$-semigroup compactification of G, which is moreover an inverse semigroup.*

Proof. It is clear that any $p \in P(G)$ is a partial elementary map of M, and also that its domain must be algebraically closed. Conversely, let $p\colon A \to M^{\mathrm{eq}}$ be an elementary map with A algebraically closed. Fix a finite tuple a from A, a finite tuple b disjoint from A and a finite subset $C \subset M^{\mathrm{eq}}$ (intended as the complement of a neighborhood of ∞ in K); denote $a' = p(a)$. Choose a tuple b' such that $ab \equiv a'b'$, then take b'' satisfying $b''a' \equiv b'a'$ and $b'' \perp_{a'} C$. Since b is disjoint from $\mathrm{acl}(a) \subset A$ we have that b'' is disjoint from $\mathrm{acl}(a')$, whence b'' is disjoint from C. Now, by homogeneity there is $g \in G$ such that $ga = a'$ and $gb = b''$. This shows that p can be approximated by elements of G in the topology of K^K.

Finally, $P(M)$, being a closed subsemigroup of Ξ closed under inverses, is also a compact inverse semitopological semigroup.

□

We remark that we have defined $P(M)$ directly as a family of partial maps on M^{eq}, and not on M. Unlike the case of $E(M)$, which can be identified with $E(M^{\mathrm{eq}})$, the previous construction applied to M would yield a smaller object (a factor of $P(M)$), which may loose information. However, it is easy to check the following.

Remark 3.8. The structure M has *weak elimination of imaginaries* (see for instance [**TZ12**, §8.4]) if and only if $P(M)$ coincides with its factor consisting of partial elementary maps of M with (relatively) algebraically closed domain.

We also observe that $P(M)$ can be alternatively defined as the closure of the image of G inside $\Theta(\ell^2(M^{\mathrm{eq}}))$, induced by the natural unitary representation $G \to U(\ell^2(M^{\mathrm{eq}}))$. Indeed, by identifying $\infty \in K$ with the zero of the Hilbert space, we have natural topological embeddings

$$G \subset \Xi \subset \Theta(\ell^2(M^{\mathrm{eq}})).$$

In particular, $P(M)$ is a factor of the Hilbert compactification.

Theorem 3.9. *Let $G = \mathrm{Aut}(M)$ be a pro-oligomorphic group. Then $P(M)$ coincides with the Hilbert compactification $H(G)$.*

Proof. This follows from the previous observation and the fact, implied by the classification theorem, that every separable continuous unitary representation of G is a subrepresentation of $G \to U(\ell^2(M^{\mathrm{eq}}))$. Nevertheless, let us give an explicit isomorphism $H(G) \to P(M)$ based on the model-theoretic description of $W(G)$. Given endomorphisms $x, y \in E(M^{\mathrm{eq}})$, let $[x,y]_H$ denote the image of $[x,y] \in W(G)$ under the canonical G-map $W(G) \to H(G)$. Proposition 3.6 says that $[x,y]_H$ is the type determined by the values $\varphi(x(a), y(b))$ for definable equivalence relations $\varphi(u,v)$ and parameters a, b from M. Equivalently, it is the type determined by the values $x(a) = y(b)$ for parameters $a, b \in M^{\mathrm{eq}}$. We consider the map

$$[x,y]_H \in H(G) \mapsto x^{-1} \circ y \in \Xi,$$

where, on the right, x, y are seen as elements of Ξ. By our description of $H(G)$, this is well-defined and injective, and it is clearly a continuous G-map. Since $H(G)$ is compact, its image is $P(M)$. □

3.3. Characterization of Eberlein pro-oligomorphic groups. A corollary of the previous results is that if a pro-oligomorphic group G is Eberlein (that is, if we have $W(G) = H(G)$), then $W(G)$ must be an inverse semigroup. As it turns out, this is a sufficient condition. Moreover, this property is related to the following model-theoretic notion.

Definition 3.10. Let M be a structure. We will say that M is *one-based for stable independence* if for any algebraically closed sets $A, B \subset M^{\mathrm{eq}}$ we have

$$A \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} B.$$

Equivalently: if for any tuple a and set B we have $\mathrm{Cb}(a/B) \subset \mathrm{acl}(a)$.

Theorem 3.11. *Let $G = \mathrm{Aut}(M)$ be a pro-oligomorphic group. The following are equivalent.*

(1) $W(G)$ is an inverse semigroup.
(2) The idempotents of $W(G)$ commute.
(3) M is one-based for stable independence.
(4) G is Eberlein.

Proof. (1) $\Rightarrow$ (2) is just a consequence of the general characterization referred in Fact 2.11.

(2) $\Rightarrow$ (1): Let $p \in W(G)$, say $p = [x,y]$ for $x,y \in E(M)$. We identify x,y with their images under the embedding $E(M) \to W(G)$. Then we can write $p = [x,y] = [x,1][1,y] = x^*y$. Now, for any $z \in E(M)$ we have $z^*z = 1$, so the element zz^* is an idempotent. If idempotents commute, we obtain $pp^*p = x^*yy^*xx^*y = x^*xx^*yy^*y = x^*y = p$.

(1) $\Rightarrow$ (3): By hypothesis, every element is regular, so by Lemma 2.20 we have $x \mathop{\smile\!\!\!\!\!\!|}_{x\cap y} y$ for any $x,y \in E(M)$. Now take algebraically closed sets $A,B \subset M^{\mathrm{eq}}$. By replacing AB by an equivalent copy if necessary, we can find $x \in E(M)$ such that $A \subset x$ and $x \mathop{\smile\!\!\!\!\!\!|}_A B$. Again, by replacing xAB by an equivalent copy, we can find $y \in E(M)$ such that $B \subset y$ and $x \mathop{\smile\!\!\!\!\!\!|}_B y$. In particular, $x \cap y = x \cap B = A \cap B$. Since $x \mathop{\smile\!\!\!\!\!\!|}_{x\cap y} y$ and $x \cap y = A \cap B \subset y$, we have $x \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} y$. Hence $A \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} B$.

(3) $\Rightarrow$ (4): We want to show that the canonical G-map $W(G) \to H(G)$ is injective. Given $p,q \in W(G)$, we can always choose $x,y,z \in E(M)$ such that $p = [x,y]$ and $q = [x,z]$. If the images of p and q in $H(G)$ coincide, then $x \cap y = x \cap z =: C$, and moreover $y \equiv^s_C z$. Since M is one-based for stable independence, $y \mathop{\smile\!\!\!\!\!\!|}_C x$ and $z \mathop{\smile\!\!\!\!\!\!|}_C x$. By stationarity, we get $y \equiv^s_x z$, that is to say, $p = q$.

(4) $\Rightarrow$ (1): Clear from the identification $H(G) = P(M)$. □

Corollary 3.12. *The following are equivalent.*

(1) G is strongly Eberlein.
(2) M is $\aleph_0$-stable (i.e., the space of types $S_u(M)$, in any finite variable u, is countable).
(3) The intersection $\bigcap_{x\in E(M)} E(M)\cdot x$ is non-empty.

Moreover, if the previous conditions hold, the action of G on $S_u(M)$ is oligomorphic.

Proof. (1) $\Leftrightarrow$ (2): As mentioned before, G is a WAP group if and only if M is stable. By the previous theorem, G is strongly Eberlein if and only if M is stable and one-based. A classical result of Zilber (see Theorem 5.12 in [**Pil96**, Ch. 2], and also [**BBH14**], Proposition 3.12) states that an $\aleph_0$-categorical stable structure is one-based if and only if it is $\aleph_0$-stable.

(2) $\Leftrightarrow$ (3): This is just a topological reformulation. Indeed, it is easy to see that

$$y \in \bigcap_{x \in E(M)} E(M) \cdot x$$

if and only if every type over the image $M' = y(M)$ (possibly in countably many variables) is realized in M. In turn, there exists a submodel M' of M with this property if and only if M is $\aleph_0$-stable.

For the moreover part, it suffices to show that $S_u(M)/G$ is finite. We sketch the (standard) argument. To every indiscernible sequence $(a_i)_{i<\omega} \subset M^{|u|}$ we assign its limit type $p \in S_u(M)$. This is a surjective G-map. Since M is one-based, the type of an indiscernible sequence $(a_i)_{i<\omega}$ is determined by $\mathrm{tp}(a_0 a_1)$. By $\aleph_0$-categoricity, there are only finitely many types $\mathrm{tp}(a_0 a_1)$. □

Example 3.13. As mentioned before, the group S_∞ of permutations of a countable set X is (strongly) Eberlein; its Roelcke compactification is the semigroup of partial bijections of X. We can give some new examples. Consider the following oligomorphic groups:

(1) the automorphism group of a dense linear order, $\mathrm{Aut}(\mathbb{Q},<)$;
(2) the homeomorphism group of the Cantor space (or, equivalently, the automorphism group of its algebra of clopen sets), $\mathrm{Homeo}(2^\omega)$;
(3) the automorphism group of the random graph.

It follows from the results in [**BT14**, §6] (see also [**Iba14**, §4.2]) that for each of these groups (as well as for S_∞) the algebra $\mathrm{WAP}(G)$ is generated by the functions of the form

$$g \mapsto a = gb$$

for elements a, b in the respective structures. Since these are obviously in $\mathrm{Hilb}(G)$, we deduce that these groups are Eberlein. In fact, fixed some $G = \mathrm{Aut}(M)$ from the above list, we deduce that the WAP compactification consists of the partial elementary maps $M \to M$ with relatively algebraically closed domain. (In particular, as is well-known, these structures have weak elimination of imaginaries.)

Example 3.14. A famous conjecture of Zilber claimed that an $\aleph_0$-categorical stable structure should be $\aleph_0$-stable (equivalently, one-based, or still: not encoding a *pseudoplane*). This was refuted by Hrushovski, who constructed an $\aleph_0$-categorical stable pseudoplane. The details of the construction can be found in [**Wag94**]. It follows from Theorem 3.11 that the automorphism group of this pseudoplane is an oligomorphic WAP group that is not Eberlein. This answers Question 6.10 in [**GM14b**].

Example 3.15. The previous example can be used to produce a countable compact dynamical system of finite Cantor–Bendixson rank that is faithfully representable on a reflexive Banach space, but not on a Hilbert space, in the sense of representability defined in the introduction (see [**Meg08**] for more background). Indeed, let M be Hrushovski's stable pseudoplane, $G = \mathrm{Aut}(M)$, and choose some formula $\varphi(u,v)$ and parameters a, b such that $f \colon g \mapsto \varphi(a, gb)$ is not in $\mathrm{Hilb}(G)$. Now, the space $S_\varphi(M)$ of φ-types in the variable v, with parameters from M, induces a compactification X of G via the map $g \mapsto \mathrm{tp}_\varphi(gb/M)$. Since f belongs to the associated algebra, the dynamical system $G \curvearrowright X$ is not Hilbert-representable; but it is reflexively representable, since φ is stable. Finally, as is well-known, the space of local types $S_\varphi(M)$ of a stable formula over a countable structure is a countable compact zero-dimensional space of finite Cantor–Bendixson rank (see, for instance, [**Pil96**], Remark 2.3 and Lemma 3.1).

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4. Hilbert-representable factors

In this section we extend our analysis to the factors of $H(G)$ and $W(G)$. We start by showing that all factors of $H(G)$ are zero-dimensional.

We recall that if $\pi\colon G \to U(\mathcal{H})$ is a continuous unitary representation, then π extends naturally to a homomorphism $\pi\colon H(G) \to \Theta(\mathcal{H})$.

Lemma 4.1. *Let $\pi\colon G \to U(\mathcal{H})$ be a continuous unitary representation of a Roelcke precompact Polish group. Let $\eta \in \mathcal{H}$ be a vector such that $\pi(V)\eta = \eta$ for some open subgroup $V \leq G$ (i.e., $\pi(v)\eta = \eta$ for all $v \in V$). Then $\pi(H(G))\eta$ is countable.*

Proof. Since G is Polish, we can assume $\mathcal{H}$ is separable. As G is Roelcke precompact and V is open, the space of double cosets $V\backslash G/V$ is finite. Since η is fixed by V, the function $g \mapsto \langle \eta, \pi(g)\eta\rangle$ factors through $V\backslash G/V$, hence the set

$$\{\langle \pi(g_1)\eta, \pi(g_2)\eta\rangle : g_1, g_2 \in G\}$$

is finite. By continuity, $\{\langle \pi(p_1)\eta, \pi(p_2)\eta\rangle : p_1, p_2 \in H(G)\}$ is equal to it, and therefore also finite. Now the separability of $\mathcal{H}$ implies that $\pi(H(G))\eta$ is countable. $\square$

Proposition 4.2. *Let G be a Roelcke precompact Polish group and let $f \in \mathrm{C}(H(G))$ be a function such that $Vf = f$ for some open subgroup $V \leq G$. Then $f(H(G))$ is countable.*

Proof. Let $f = \lim_n f_n$, where $f_n(g) = \langle \xi_n, \pi(g)\eta_n\rangle$ for some representation π and vectors ξ_n, η_n. First, we may assume that each ξ_n is fixed by $\pi(V)$. Indeed, let n be such that $\|f - f_n\| \leq \epsilon$ and let ξ_n' be the element of minimal norm of $\overline{\mathrm{co}}(\pi(V)\xi_n)$. Note that ξ_n' is fixed by $\pi(V)$ and for every $g \in G$ and $v \in V$,

$$|\langle \xi_n, \pi(g)\eta_n\rangle - \langle \pi(v)\xi_n, \pi(g)\eta_n\rangle| = |f_n(g) - vf_n(g)| \leq 2\epsilon,$$

implying that

$$|\langle \xi_n, \pi(g)\eta_n\rangle - \langle \xi_n', \pi(g)\eta_n\rangle| \leq 2\epsilon$$

and thus we can replace ξ_n by ξ_n' without losing much.

Next, by replacing π with a sum of infinitely many copies of itself and rescaling if necessary, we may assume that $\xi_n = \xi$ for all n. Finally, apply Lemma 4.1 to obtain that $\pi(H(G))\xi$ is countable and let E be the equivalence relation on $H(G)$ given by $p\,E\,q \iff \pi(p^*)\xi = \pi(q^*)\xi$ (so that E has countably many classes). Now all f_n and f factor through E, so, in particular, the image of f is countable. $\square$

Lemma 4.3. *Suppose G is pro-oligomorphic and let $A \subset \mathrm{WAP}(G)$ be a closed subalgebra. Let $A_0 \subset A$ be the subalgebra of functions f such that $Vf = f$ for some open subgroup $V \leq G$, and let $A_1 \subset A$ be the subalgebra of functions with finite range.*

(1) If A is invariant, then A_0 is dense in A.

(2) If A is bi-invariant, then A_1 is dense in A.

Proof. This follows almost verbatim from the proofs of Proposition 4.7 and Theorem 4.8 in [**BT14**]. □

Theorem 4.4. *If G is a pro-oligomorphic group, then every factor of $H(G)$ is zero-dimensional.*

Proof. Let S be a factor of $H(G)$. Since $\mathcal{A}(S)$ is invariant, using Lemma 4.3.(1) and Proposition 4.2, we see that continuous functions on S with countable range separate points in S. This implies the conclusion of the theorem. □

Question 4.5. Is the same true for all factors of $W(G)$?

The automorphism group of the dense, countable circular order acts minimally on the circle and this dynamical system is a quotient of the Roelcke compactification of the group. So certainly some hypothesis is necessary to obtain zero-dimensionality.

The previous theorem, restated as follows, is useful to show that Hilbert-representability is preserved under factors.

Corollary 4.6. *Let $A \subset \mathrm{Hilb}(G)$ be an invariant closed subalgebra, and let $A_1 \subset A$ be the subalgebra of functions with finite range. Then A_1 is dense in A.*

Proposition 4.7. *Let G be a pro-oligomorphic group. If $f \in \mathrm{Hilb}(G)$ has finite range, then $f \in B(G)$.*

Proof. Suppose first that f is $\{0,1\}$-valued. By Proposition 3.6, we know that f can be approximated in norm by a linear combination of $\{0,1\}$-valued matrix coefficients $m_0,\dots,m_{n-1} \in B(G)$, say

$$\Big\|f - \sum_{i<n} \lambda_i m_i\Big\| < 1/2.$$

Hence there is a Boolean function $b\colon \{0,1\}^n \to \{0,1\}$ such that $b\Big((m_i(g))_{i<n}\Big) = f(g)$ for every $g \in G$. This implies that f can be written as a Boolean combination of the matrix coefficients m_i. Now it is enough to note that, first, the negation of a $\{0,1\}$-valued function $m \in B(G)$ is again in $B(G)$, since we can write it as the difference $\neg m = 1 - m$, and, second, the conjunction of two $\{0,1\}$-valued functions $m_0, m_1 \in B(G)$ is again in $B(G)$, since it is simply the product $m_0 \wedge m_1 = m_0 m_1$. We conclude that f is a matrix coefficient.

Finally, it is clear that every $f \in \mathrm{Hilb}(G)$ of finite range is a linear combination of $\{0,1\}$-valued functions in $\mathrm{Hilb}(G)$. Hence $f \in B(G)$. □

We can finally give an answer to Question 1.9 for pro-oligomorphic groups.

Theorem 4.8. *Let G be a pro-oligomorphic group. Every factor of $H(G)$ is Hilbert-representable.*

Proof. Let A be a subalgebra of $\mathrm{Hilb}(G)$ and let $A_1 \subset A$ be the subalgebra of functions with finite range. By Corollary 4.6 and Proposition 4.7, we have that A_1 is dense in A and contained in $A \cap B(G)$. Hence $A = \overline{A \cap B(G)}$ and, by Proposition 1.7, the factor of $H(G)$ corresponding to A is Hilbert-representable. □

Corollary 4.9. *Every semitopological semigroup factor of $H(G)$ is $*$-closed and is an inverse semigroup.*

Proof. The first claim follows from the previous theorem and Proposition 2.9. Now, any such G-factor $H(G) \to S$ must preserve the involution. It follows that $pp^*p = p$ for every $p \in S$, hence S is an inverse semigroup. □

It turns out that the converse of the above corollary also holds. The following is a generalization of Theorem 3.11.

Theorem 4.10. *Let S be a semitopological $*$-semigroup compactification of a pro-oligomorphic group G. The following are equivalent:*

(1) S is an inverse semigroup;
(2) the idempotents of S commute;
(3) S is Hilbert-representable.

Proof. The equivalence (1) ⇔ (2) is exactly as in Theorem 3.11. The implication (3) ⇒ (1) is clear, for example by the previous corollary.

(1) ⇒ (3): Let $A \subset \mathrm{WAP}(G)$ be the algebra generated by the union of $\mathcal{A}(S)$ and $\mathrm{Hilb}(G)$. By Theorem 4.8, to prove (3), it is enough to show that $A = \mathrm{Hilb}(G)$.

Let S_H be the compactification of G associated to the subalgebra $A \subset \mathrm{WAP}(G)$. It follows from Fact 2.3 and Remark 2.8 that S_H is a semitopological $*$-semigroup compactification. Since S and $H(G)$ are inverse semigroups, so is S_H. Indeed, if $q \in S_H$ and $f \in \mathcal{A}(S)$ or $f \in \mathrm{Hilb}(G)$, then $f(qq^*q) = f(q)$, for this holds in S and in $H(G)$; hence this holds for any $f \in A$, which shows that $qq^*q = q$.

Let $\phi_0 \colon W(G) \to S_H$ and $\phi_1 \colon S_H \to H(G)$ be the canonical factor maps. We need to show that ϕ_1 is injective, so that $S_H = H(G)$. The proof of Theorem 3.11 shows that the canonical factor map $\phi_1\phi_0 \colon W(G) \to H(G)$ is injective on the set of regular elements of $W(G)$.

Let $p \in W(G)$ be any element, and let P be the closed subsemigroup of $W(G)$ generated by p^*p. Then $P^* = P$, so by [**BT14**], Lemma 3.6, there exists an idempotent $e_p \in P$ such that the set $W(G)e_p$ is contained in every set $W(G)s$ for $s \in P$. Then $W(G)e_p \subset W(G)p^*pe_p \subset W(G)pe_p \subset W(G)e_p$, so $W(G)pe_p = W(G)e_p$. It follows from Fact 2.12.(1) that $e_p = p^*pe_p$. We set $\sigma(p) = pe_p$, and we remark that $\sigma(p)$ is regular.

We claim that $\phi_1(\sigma(p)) = \phi_1(p)$. Since P is generated by p^*p, there is a sequence $n_i < \omega$ such that $(p^*p)^{n_i} \to e_p$. By continuity, $\phi_1(p(p^*p)^{n_i}) \to \phi_1(pe_p)$, but we also have $\phi_1(p(p^*p)^{n_i}) = \phi_1(p)$ since ϕ_1 is a homomorphism and S_H is an inverse semigroup.

Finally, let $q, q' \in S_H$ and suppose $\phi_1(q) = \phi_1(q')$. Let $p, p' \in W(G)$ be any elements with $\phi_0(p) = q$ and $\phi_0(p') = q'$. The associated elements $\sigma(p)$ and $\sigma(p')$ are regular and have the same image in $H(G)$, hence $\sigma(p) = \sigma(p')$. It follows that $q = q'$. □

We point out that, even for the group of integers $G = \mathbb{Z}$, a Hilbert-representable semitopological semigroup compactification of G need not be an inverse semigroup. The semigroup considered in [**BLM01**] serves as a counterexample.

To conclude, we give a bound on the complexity of the countable factors of $H(G)$.

Proposition 4.11. *Let $Z \subset \mathcal{H}$ be a countable, weakly compact subset of a Hilbert space $\mathcal{H}$. Denote $D(Z) = \{\|\xi - \eta\| : \xi, \eta \in Z\}$ and assume that $D(Z)$ is finite. Then $D(Z') \subsetneq D(Z)$ (here Z' is the Cantor–Bendixson derivative of Z). In particular, the Cantor–Bendixson rank of Z is bounded by $|D(Z)|$.*

Proof. Let $\delta = \max D(Z)$; we show that $\delta \notin D(Z')$. Suppose, towards a contradiction, that $\xi, \eta \in Z'$ are such that $\|\xi - \eta\| = \delta$. By translating Z, we can assume that $\xi = 0$. Let $\eta_n \to^w \eta$ with η_n distinct elements of Z. As δ is maximal, $D(Z)$ is finite, $\eta_n \to^w \eta$, and the norm is lower semicontinuous in the weak topology, we must have that eventually $\|\eta_n\| = \|\eta\|$. This, in turn, implies that $\eta_n \to \eta$ in norm, which means that η_n is eventually constant (again, since $D(Z)$ is finite). This contradicts the choice of the sequence η_n. □

Recall that a *G-ambit* (X, x_0) is just a compactification of G where x_0 is the image of 1. If X is countable then x_0 must be isolated.

Corollary 4.12. *Let G be a Roelcke precompact group and let (X, x_0) be a Hilbert-representable G-ambit. Then X is countable if and only if the stabilizer G_{x_0} is open, and in this case we have*

$$\operatorname{rank} X \leq |\{G_{x_0} g G_{x_0} : g \in G\}|.$$

Proof. Follows from Lemma 4.1 (and its proof) and Proposition 4.11. □

Note that the previous result is a generalization of the following model-theoretic fact, which follows from one-basedness: in an $\aleph_0$-categorical $\aleph_0$-stable theory, the Morley rank of any finite tuple a is bounded by the number of distinct types $\operatorname{tp}(b/a)$ with $\operatorname{tp}(b) = \operatorname{tp}(a)$.

Question 4.13. Do countable ambits of pro-oligomorphic groups necessarily have finite Cantor–Bendixson rank?

We remark that every countable compactification of a Roelcke precompact Polish group G is a factor of $W(G)$. Indeed, since countable compact systems are *Asplund-representable* (see, for instance, [**GM06**], Corollary 10.2), this follows from [**Iba14**], Theorem 2.9.

Akin and Glasner [**AG14**] construct countable WAP $\mathbb{Z}$-ambits of arbitrarily high rank. But of course, the group $\mathbb{Z}$ is not pro-oligomorphic.

CHAPTER 3

Automorphism groups of randomized structures

Abstract. We study automorphism groups of randomizations of separable structures, with focus on the $\aleph_0$-categorical case. We give a description of the automorphism group of the Borel randomization in terms of the group of the original structure. In the $\aleph_0$-categorical context, this provides a new source of Roelcke precompact Polish groups, and we describe the associated Roelcke compactifications. This allows us also to recover and generalize preservation results of stable and NIP formulas previously established in the literature, via a Banach-theoretic translation. Finally, we study the separable models of the theory of beautiful pairs of randomizations, and we show that this theory is in general not $\aleph_0$-categorical.

Contents

Introduction

A randomization of a structure M is a metric structure whose elements are random variables taking values on M, and whose predicates account for the expected values of the original predicates of M. The idea goes back to Keisler [**Kei99**], where it was developed in a classical first-order framework. Later, in [**BK09**], Keisler and Ben Yaacov adapted the construction so that randomizations could be regarded as *metric* structures (that is, in the sense of continuous first-order logic), and with this approach they proved several preservation results, supporting the claim that this is the correct frame to develop the idea.

The construction was further adapted by Ben Yaacov in [**Ben13b**], so that randomizations of metric structures could also be considered. In [**Ben09**], another important and difficult preservation result was proved, concerning NIP formulas. Further model-theoretic analysis of randomized structures have been carried out in [**AGK15a, AK15, AGK15b**].

In the present work, we approach the subject from the viewpoint of descriptive set theory. Our main motivation is the study of the symmetries of randomized structures. More precisely, we describe and study the automorphism group of the *Borel randomization* of a separable metric structure. This is the most basic example of a randomization, yet encompassing most of the intuition of the subject. If M is a separable structure with automorphism group G, then the automorphism group of its Borel randomization M^R is a *measurable wreath product*,

$$G \wr \Omega := L^0(\Omega, G) \rtimes \mathrm{Aut}(\Omega),$$

where Ω denotes a standard probability space. For instance, the symmetry group of a randomized countable set, $\mathbb{N}^R$, is the semidirect product of the random permutations of $\mathbb{N}$ and the measure-preserving transformations of Ω.

The group $G \wr \Omega$, induced by a given Polish group G, is an interesting object in itself. It had already been considered by Kechris in [**Kec10**, §19ff.]. Here, we investigate in detail the continuous actions of $G \wr \Omega$ induced by actions of G. First, in Section 2, we study the isometric actions of $G \wr \Omega$ of this kind. We show that every approximately oligomorphic faithful action of G induces an approximately oligomorphic faithful action of $G \wr \Omega$. In particular, if G is *Roelcke precompact*, then so is $G \wr \Omega$. Afterwards, we make the link with the automorphism groups of randomized structures, as explained above.

Later, in Section 3, we investigate some compact $G \wr \Omega$-flows. This corresponds to the study of type spaces in randomized structures. When G is Roelcke precompact, we give an explicit description of the Roelcke compactification of $G \wr \Omega$, denoted by $R(G \wr \Omega)$, in terms of $R(G)$. Furthermore, we show how some eventual properties of $R(G)$ (existence of a compatible semigroup law, representability by contractions on Hilbert spaces) pass on to $R(G \wr \Omega)$. We also prove a general preservation result concerning Banach representations of randomized type spaces.

If M is the separable model of an $\aleph_0$-categorical theory T, then most of the model-theoretic information of T is coded by dynamical properties of $G = \mathrm{Aut}(M)$, as per the recent works [**BT14, Iba14, BIT15**]. On the other hand, the randomized theory T^R is also $\aleph_0$-categorical; hence, in this case, the Borel randomization encompasses all the model-theoretic information of T^R, and this can be recovered from the group $G \wr \Omega$. The preservation results of Section 3 get then a precise model-theoretic meaning, and allow us to give new proofs (in the $\aleph_0$-categorical setting) of the theorems of preservation of stability and NIP from [**BK09, Ben13b**] and [**Ben09**].

Finally, in Section 4, we come back to more model-theoretic concerns, and we study the theory $(T^R)_P$ of beautiful pairs of models of a randomized theory T^R. This is motivated by the results of [**BBH14**] on the problem of generalizing the notion of one-basedness to the metric setting. When T is $\aleph_0$-categorical, we classify the separable models of $(T^R)_P$, and show in particular that $(T^R)_P$ is never $\aleph_0$-categorical (except when T is the theory of a compact structure). We also extend to the metric setting the result of preservation of $\aleph_0$-stability from [**BK09**]. We end with a description of the automorphism groups of some canonical models of $(T^R)_P$.

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1. Preliminaries

1.1. Notation. Throughout the paper we fix an atomless Lebesgue space Ω, say the unit interval $[0,1]$ with Lebesgue measure μ.

If (X,d) is a Polish, bounded, metric space, then

$$X^\Omega := L^0(\Omega, X)$$

will denote the space of random variables $r\colon \Omega \to X$ (where X carries the Borel σ-algebra), up to equality almost everywhere, endowed with the induced L^1-distance,

$$d^\Omega(r,s) := \int d(r(\omega), s(\omega)) d\mu(\omega).$$

If X is just a topological Polish space, then we may choose any compatible, bounded distance d and define X^Ω to be space $L^0(\Omega, X)$ with the topology induced by d^Ω. This is independent of the choice of d: a sequence r_n converges to r in X^Ω if and only if every subsequence of r_n has a further subsequence that converges almost surely to r. In both cases, as a metric or a topological space, X^Ω is Polish. See, for instance, [**LM14**, Annexe C] or [**Kec10**, §19].

An important particular case is the metric space $X = [0,1]$. Throughout the paper, we denote

$$\mathbb{A} := [0,1]^\Omega = L^0(\Omega, [0,1]).$$

Thus, the metric on $\mathbb{A}$ is given by $\mathbb{E}|R-S|$, where $\mathbb{E}\colon \mathbb{A} \to [0,1]$ is the expectation function $\mathbb{E}R = \int R(\omega) d\mu(\omega)$.

In the previous constructions, Ω might be replaced, later, by the unit square Ω^2. We remark that $X^{\Omega_0 \times \Omega_1}$ is naturally isomorphic to $(X^{\Omega_1})^{\Omega_0}$ via the natural map $r \mapsto \tilde{r}$, $\tilde{r}(\omega_0)(\omega_1) = r(\omega_0, \omega_1)$. For a proof, see [**Fre06**, 418R].

Given a compact metrizable space K, we will denote by $\mathfrak{R}(K)$ the compact space of Borel probability measures on K. The topology is the weak* topology as a subset of the dual space of continuous functions on K.

1.2. A measurable wreath product. We denote by $\mathrm{Aut}(\Omega)$ the group of invertible measure-preserving transformations of Ω, up to equality almost everywhere. If X is a Polish space, then $\mathrm{Aut}(\Omega)$ acts on X^Ω by the formula

$$(tr)(\omega) = r(t^{-1}(\omega)),$$

where $t \in \mathrm{Aut}(\Omega)$ and $r \in X^\Omega$. If (X,d) is metric, then this action is by isometries on (X^Ω, d^Ω). In particular, $\mathrm{Aut}(\Omega)$ can be seen as a subgroup of the isometry group of $\mathbb{A}$, which is a Polish group under the topology of pointwise convergence. With the induced topology, $\mathrm{Aut}(\Omega)$ is a Polish group. In addition, the action $\mathrm{Aut}(\Omega) \curvearrowright X^\Omega$ is continuous.

Given a Polish group G, the space G^Ω is also a Polish group with the operation of pointwise multiplication. Hence we have an action of $\mathrm{Aut}(\Omega)$ on the group G^Ω. In this case, given $g \in G^\Omega$ and $t \in \mathrm{Aut}(\Omega)$, we will denote the action of t on g by $t \cdot g$. (When $\mathrm{Aut}(\Omega)$ and G^Ω act simultaneously on a space Z, the term tg will denote their product as homeomorphisms of Z.)

We introduce the following definition.

Definition 1.1. The *measurable wreath product* of G and $\mathrm{Aut}(\Omega)$ is the semidirect product

$$G \wr \Omega := G^\Omega \rtimes \mathrm{Aut}(\Omega),$$

which is a Polish group endowed with the product topology.

If G acts continuously on a Polish space X, then G^Ω acts continuously on X^Ω, by the formula

$$(gr)(\omega) = g(\omega)(r(\omega)),$$

where $g \in G^\Omega$ and $r \in X^\Omega$. Note that gr is indeed a random variable: if $U \subset X$ is open and $\{U_i\}_{i\in\mathbb{N}}$ is a countable base for the topology of X, then

$$(gr)^{-1}(U) = \bigcup_{i\in\mathbb{N}} \Big(r^{-1}(U_i) \cap g^{-1}(\{h \in G : h(U_i) \subset U\}) \Big),$$

which is a measurable set since $\{h \in G : h(U_i) \subset U\} = \{h \in G : h^{-1}(U^c) \subset U_i^c\}$ is a closed subset of G. That the action $G^\Omega \curvearrowright X^\Omega$ is continuous can be deduced from the fact that $g_n r_n$ converges almost surely to gr if g_n and r_n converge almost surely to g and r. If moreover the action $G \curvearrowright (X,d)$ is by isometries, then so is $G^\Omega \curvearrowright (X^\Omega, d^\Omega)$.

Now, given a continuous action $G \curvearrowright X$, we have, simultaneously, continuous actions of $\mathrm{Aut}(\Omega)$ and G^Ω on X^Ω. These induce a continuous action $G \wr \Omega \curvearrowright X^\Omega$. Indeed, inside the group of homeomorphisms of X^Ω, the elements $t \in \mathrm{Aut}(\Omega)$, $g \in G^\Omega$ satisfy the relation $t \cdot g = tgt^{-1}$.

We say that an action by isometries $G \curvearrowright (X,d)$ is *faithful* if the corresponding group homomorphism $G \to \mathrm{Iso}(X)$ is a topological embedding. Here, $\mathrm{Iso}(X)$ is the isometry group of X with the topology of pointwise convergence.

Lemma 1.2. *Let G be a Polish group acting continuously by isometries on a Polish metric space (X,d) with at least two elements. If the action is faithful, then so is the induced action $G \wr \Omega \curvearrowright (X^\Omega, d^\Omega)$.*

Proof. The action $\mathrm{Aut}(\Omega) \curvearrowright (X^\Omega, d^\Omega)$ is faithful since X has at least two elements (we omit the details). Since the action $G \curvearrowright (X,d)$ is faithful, we see that a sequence in G^Ω converges to the identity if and only if every subsequence has a further subsequence g_n such that, almost surely, $g_n(\omega)(x)$ converges to x for every $x \in X$. Let $D \subset X$ be a countable dense subset and let $C \subset X^\Omega$ be the family of constant random variables taking a value from D. It follows that $g_n \to 1$ in G^Ω if and only if $d^\Omega(g_n c, c) \to 0$ for every $c \in C$. In particular, the action $G^\Omega \curvearrowright X^\Omega$ is faithful.

To see that the action $G \wr \Omega \curvearrowright X^\Omega$ is faithful we must check that, whenever $g_n t_n \to 1$ in $\mathrm{Iso}(X^\Omega)$ for $g_n \in G^\Omega$ and $t_n \in \mathrm{Aut}(\Omega)$, we have $g_n \to 1$ and $t_n \to 1$. Now, if $d^\Omega(g_n t_n r, r) \to 0$ for every random variable r, specializing on the constants $c \in C$ (which are fixed under $\mathrm{Aut}(\Omega)$) we see that $g_n \to 1$. Then also $t_n \to 1$. □

1.3. Semigroup completions. Let G be a Polish group, and let d_L be a left-invariant compatible metric on G, which exists by the Birkhoff–Kakutani theorem (see [**Ber74**, p. 28]). The completion of G with respect to d_L will be denoted by $\widehat{G}_L$. Then, the group law on G extends to a jointly continuous semigroup operation on $\widehat{G}_L$. As a topological semigroup, $\widehat{G}_L$ does not depend on the particular choice of d_L (it is the completion of G with respect to its left uniformity). Similarly, the completion of G with respect to a compatible right-invariant metric d_R will be denoted by $\widehat{G}_R$, and is also a topological semigroup. The inverse operation on G extends to a homeomorphic anti-isomorphism ${}^*\colon \widehat{G}_R \to \widehat{G}_L$.

The right completion of the group $\mathrm{Aut}(\Omega)$ is the semigroup $\mathrm{End}(\Omega)$ of measure-preserving transformations of Ω, up to equality almost everywhere (we will revisit this fact later). Then, given any Polish metric space (X,d), the left completion $\mathrm{End}(\Omega)^*$ acts by isometries on (X^Ω, d^Ω) by the formula $(s^* r)(\omega) = r(s(\omega))$, where $s \in \mathrm{End}(\Omega)$, $r \in X^\Omega$.

Lemma 1.3. *Let G be a Polish group. Then, the left completion of $G \wr \Omega$ is the topological semigroup $(\widehat{G}_L)^\Omega \rtimes \mathrm{End}(\Omega)^*$.*

Proof. Fix a compatible left-invariant metric d_L on the left completion $\widehat{G}_L$, and consider the induced metric d_L^Ω on $(\widehat{G}_L)^\Omega$. Since $(\widehat{G}_L, d_L)$ is complete, so is $((\widehat{G}_L)^\Omega, d_L^\Omega)$ (see [**Kec10**], Proposition 19.6). Let d_Ω be a compatible left-invariant metric d_Ω on $\mathrm{End}(\Omega)^*$. Then, the metric $d = d_L^\Omega + d_\Omega$ is a complete, left-invariant metric on $(\widehat{G}_L)^\Omega \rtimes \mathrm{End}(\Omega)^*$, compatible with the topology of $G \wr \Omega$. Since $G \wr \Omega$ is moreover dense in $(\widehat{G}_L)^\Omega \rtimes \mathrm{End}(\Omega)^*$, this must be its left completion. □

1.4. Borel randomizations. Given a structure or a class of models of a certain first-order theory, there are several ways of producing randomizations from them. For the most part of this paper, however, we will only be interested in one particular construction, which can be considered the basic, canonical example of a randomization. In the $\aleph_0$-categorical setting, which is of particular interest to us, this is actually the one and only example one needs to consider, since randomizations preserve separable categoricity.

Unless otherwise stated, all structures and theories we consider are in the sense of first-order *continuous* logic, as per [**BU10, BBHU08**]. (Traditional discrete structures and theories, which form a particular case, are refer to as *classical*.) In particular, structures are complete metric spaces. Furthermore, we assume all our structures to be separable in a countable language.

Let M be a metric structure with at least two elements, in a language L, which we shall assume to be one-sorted for simplicity. We introduce below the *Borel randomization of* M, denoted in this paper by M^R, which is a structure in a two-sorted language L^R. We have borrowed the name from [**AK15**], Definition 2.1, although, there, the term refers to the natural pre-structure whose completion gives M^R; also, they only define it for classical M.

The *main sort* of M^R is the metric space (M^Ω, d^Ω), and the *auxiliary sort* of M^R is the space $\mathbb{A}$ with its natural metric. For each definable predicate $\varphi\colon M^n \to [0,1]$ there is a definable function

$$[\![\varphi(x)]\!]\colon (M^\Omega)^n \to \mathbb{A},$$

given by

$$[\![\varphi(r)]\!](\omega) = \varphi(r(\omega)),$$

where $r \in (M^\Omega)^n \simeq (M^n)^\Omega$. In addition, the auxiliary sort $\mathbb{A}$ is equipped with a predicate for the expectation function

$$\mathbb{E}\colon \mathbb{A} \to [0,1],$$

and with definable functions for the basic arithmetic operations between random variables, which in fact permit to define all continuous pointwise-defined functions $\mathbb{A}^n \to \mathbb{A}$ ([**Ben13b**], Lemma 2.13). Thus, as a reduct, $\mathbb{A}$ is a model of the *theory of* $[0,1]$*-valued random variables*, in the sense of [**Ben13b**, §2].

For a syntactical presentation and an explicit description of the language L^R, see [**Ben13b**, §3] or, in the classical setting, [**BK09**, §2]. Let T denote the first-order theory of M in the language L. Then, the *randomization theory* T^R is the theory of M^R in the language L^R.

Proposition 1.4. *The theory T^R has quantifier elimination. More precisely, if x and y are tuples from the main and the auxiliary sort, respectively, then the T^R-type of xy is determined by the values $\mathbb{E}\tau(x,y)$ where $\tau(x,y) \in \mathbb{A}$ is a term on xy, that is, the result of applying any operations of the auxiliary sort to any random variables from y and any random variables of the form $[\![\varphi(x)]\!]$ for an L-formula φ.*

Proof. See [**Ben13b**, §3.5]. □

One could present the Borel randomization M^R as a structure in the sort M^Ω alone, by considering as predicates the functions $\mathbb{E}[\![\varphi(x)]\!]$. However, $\mathbb{A}$ would then be present as an imaginary sort (and a very important one, which justifies to make it a sort in its own right). Remark, in this respect, that the structure MALG_μ, the measure algebra of Ω (which can be thought of as 2^Ω), is bi-interpretable with $\mathbb{A}$ (see [**Ben13b**, §2]).

1.5. Bochner spaces. Let V be a Banach space, which we will assume to be separable. The *Bochner space* $L^2(\Omega, V)$ is the space of measurable functions $f : \Omega \to V$, modulo equality almost everywhere, for which the norm

$$\|f\|_2 := \Big(\int \|f(\omega)\|_V^2 d\omega\Big)^{1/2}$$

is finite. Equipped with this norm, $L^2(\Omega, V)$ is a Banach space.

If G is a Polish group and $G \curvearrowright V$ is a continuous action by isometries, then the action $G^\Omega \curvearrowright V^\Omega$ (defined topologically, since the norm metric on V is not bounded) restricts to an isometric action on $L^2(\Omega, V)$. Similarly for the action of $\mathrm{Aut}(\Omega)$. Thus, we obtain an isometric continuous action $G \wr \Omega \curvearrowright L^2(\Omega, V)$.

We recall a description of the dual space $L^2(\Omega, V)^*$. This can be identified with the space $L^2_{w^*}(\Omega, V^*)$ of weakly* measurable functions $\psi\colon \Omega \to V^*$, modulo equality almost everywhere, for which there exists $h \in L^2(\Omega)$ with $\|\psi(\omega)\|_{V^*} \leq h(\omega)$ for almost every ω. There is a natural linear map $L^2_{w^*}(\Omega, V^*) \to L^2(\Omega, V)^*$, defined implicitly by the relation

$$\langle f, \psi\rangle = \int \langle f(\omega), \psi(\omega)\rangle d\omega$$

for $f \in L^2(\Omega, V)$ and $\psi \in L^2_{w^*}(\Omega, V^*)$. This is in fact a bijection; see [**CM97**, §1.5]. We endow the space $L^2_{w^*}(\Omega, V^*)$ with the weak* topology as the dual of $L^2(\Omega, V)$.

Every isometric continuous action $G \curvearrowright V$ induces a dual action $G \curvearrowright V^*$, given by $\langle v, g\psi\rangle = \langle g^{-1}v, \psi\rangle$ for $v \in V$ and $\psi \in V^*$. This action is continuous for the weak* topology on V^*. In particular, we have a continuous action $G \wr \Omega \curvearrowright L^2_{w^*}(\Omega, V^*)$, which satisfies the relation $(gt\psi)(\omega) = g(\omega)(\psi(t^{-1}(\omega)))$ for $g \in G^\Omega$, $t \in \mathrm{Aut}(\Omega)$ and $\psi \in L^2_{w^*}(\Omega, V^*)$.

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2. Actions on randomized metric spaces

2.1. Quotients of isometric actions. If $G \curvearrowright (X, d)$ is an action by isometries, we define the *metric quotient* of X by G as the space of orbit closures

$$X /\!\!/ G := \{\overline{Gx} : x \in X\}$$

endowed with the distance $d(\overline{Gx}, \overline{Gy}) = \inf_{g\in G} d(x, gy)$. If (X, d) is a Polish metric space, then so is $(X /\!\!/ G, d)$. The action $G \curvearrowright (X, d)$ is *approximately oligomorphic* if the metric quotient $X^n /\!\!/ G$ of the diagonal action $G \curvearrowright (X^n, d)$ is compact for all $n \in \mathbb{N}$. Equivalently, if the metric quotient $X^{\mathbb{N}} /\!\!/ G$ is compact. Here, the metric d on X^n or $X^{\mathbb{N}}$ is any compatible distance for which the diagonal action of G is isometric.

A Polish group G is *Roelcke precompact* if it admits a faithful approximately oligomorphic action on a Polish metric space (X, d) (this is actually an equivalent property, we will recall the original definition in §3.1). For example, the group $\mathrm{Aut}(\Omega)$ is Roelcke precompact and the action $\mathrm{Aut}(\Omega) \curvearrowright \mathbb{A}$ is approximately oligomorphic. See [**BT14, Ben13b**].

Let $\pi\colon X \to Y$ be a surjective map between topological spaces. A *Borel selector* for π is a measurable map $\sigma\colon Y \to X$ such that $\pi\sigma$ is the identity of Y.

Lemma 2.1.

(1) Let $G \curvearrowright (X,d)$ be an action by isometries on a Polish metric space X. Then, the quotient map $\pi\colon X \to X /\!\!/ G$ admits a Borel selector.

(2) Let $\pi\colon K_0 \to K_1$ be a continuous surjective map between compact metrizable spaces. Then π admits a Borel selector.

Proof. (1). We adapt the proof of Theorem 12.16 from [**Kec95**]. Let $F(X)$ be the *Effros Borel space* of closed subsets of X with the σ-algebra generated by the sets $[U] = \{F \in F(X) : F \cap U \neq \emptyset\}$, where U varies over the open subsets of X. As a set, the quotient $X /\!\!/ G$ is contained in $F(X)$. Moreover, we have $[U] \cap (X /\!\!/ G) = \pi(U)$, which is an open subset of $X /\!\!/ G$ if U is open in X. Indeed, let $u \in U$ and take $\epsilon > 0$ such that $d(u,v) < \epsilon$ implies $v \in U$. Then, if $d(\pi(u),\pi(x)) < \epsilon$, there is $g \in G$ such that $d(u,gx) < \epsilon$, whence $\pi(x) = \pi(gx) \in \pi(U)$. It follows that $A \cap (X /\!\!/ G)$ is a Borel subset of $X /\!\!/ G$ whenever A is Borel in $F(X)$.

Now, by Theorem 12.13 in [**Kec95**], there is a measurable map $d\colon F(X) \to X$ such that $d(F) \in F$ for every non-empty closed set $F \subset X$. The restriction of d to $X /\!\!/ G$ is thus a Borel selector for π.

(2). As before, we know that there is a measurable map $d\colon F(K_0) \to K_0$ with $d(F) \in F$ for $F \neq \emptyset$, thus it suffices to show that the fiber map $\pi^{-1}\colon K_1 \to F(K_0)$ is measurable. Then again, if $F \subset K_0$ is any subset, we have $(\pi^{-1})^{-1}([F]) = \pi(F)$. Now, the σ-algebra of $F(K_0)$ is also generated by the sets $[F]$ where F varies over the closed subsets of K_0. Since K_0 is compact and π is continuous, $\pi(F)$ is closed if F is closed, so the fiber map is measurable. □

Remark that if G has a normal Polish subgroup N, then any isometric action $G \curvearrowright (X,d)$ induces an isometric action $G \curvearrowright (X /\!\!/ N, d)$, by the formula $g\overline{Nx} = \overline{Ngx}$.

Lemma 2.2. *Let G be a Polish group and let $G \curvearrowright (X,d)$ be an action by isometries on a Polish metric space. Suppose G can be written as a product $G = NH$ for Polish subgroups $N,H < G$ with N normal in G. Then we have the following natural isometric isomorphisms:*

(1) $X /\!\!/ G \simeq (X /\!\!/ N) /\!\!/ H$.

(2) $X^\Omega /\!\!/ G^\Omega \simeq (X /\!\!/ G)^\Omega$.

(3) $X^\Omega /\!\!/ (G \wr \Omega) \simeq (X /\!\!/ G)^\Omega /\!\!/ \mathrm{Aut}(\Omega)$.

Proof. (1). We verify that the map $\overline{Gx} \mapsto \overline{H\overline{Nx}}$ is isometric. Indeed,

$$\inf_{h\in H} d(\overline{Nx}, h\overline{Ny}) = \inf_{h\in H} d(\overline{Nx}, \overline{Nhx}) = \inf_{n\in N}\inf_{h\in H} d(x, nhx) = \inf_{g\in G} d(x, gx).$$

(2). The isomorphism is given by $\overline{G^\Omega r} \mapsto r'$, where, for $r \in X^\Omega$, we let $r' \in (X /\!\!/ G)^\Omega$ be the random variable defined almost surely by $r'(\omega) = \overline{Gr(\omega)}$. Lemma 2.1.(1) ensures this map is surjective. We verify that it is isometric. Indeed, we have

$$d^\Omega(r',s') = \mathbb{E}\inf_{g\in G} d(r,gs) = \inf_{g\in G^\Omega} d^\Omega(r,gs) = d^\Omega(\overline{G^\Omega r}, \overline{G^\Omega s}),$$

where the second identity can be seen by approximating $r,s \in X^\Omega$ by random variables of finite range.

(3). Follows from (1) and (2). □

Proposition 2.3. *Suppose G is a Polish group acting approximately oligomorphically on a Polish metric space (X,d). Then, the induced action $G \wr \Omega \curvearrowright (X^\Omega, d^\Omega)$ is also approximately oligomorphic.*

Proof. Since $(X^\Omega)^n \simeq (X^n)^\Omega$, by the previous lemma we have

$$(X^\Omega)^n /\!\!/ (G \wr \Omega) \simeq K^\Omega /\!\!/ \mathrm{Aut}(\Omega),$$

where the quotient $K = X^n /\!\!/ G$ is assumed to be compact. As such, K is a continuous image of the Cantor space, i.e., there exists a continuous surjective map $2^\mathbb{N} \to K$. This induces a natural continuous map $(2^\mathbb{N})^\Omega \to K^\Omega$, which is surjective by Lemma 2.1.(2). Finally, this gives us a continuous surjective map $(2^\Omega)^\mathbb{N} /\!\!/ \mathrm{Aut}(\Omega) \to K^\Omega /\!\!/ \mathrm{Aut}(\Omega)$. Since the action $\mathrm{Aut}(\Omega) \curvearrowright 2^\Omega$ is approximately oligomorphic (2^Ω is a closed subset of $\mathbb{A}$), we deduce that $K^\Omega /\!\!/ \mathrm{Aut}(\Omega)$ is compact, as we wanted. (In other words, K^Ω is an imaginary sort of the $\aleph_0$-categorical structure $\mathbb{A}$.) □

Corollary 2.4. *If G is a Polish Roelcke precompact group, then so is $G \wr \Omega$.*

Proof. Follows from the previous proposition and Lemma 1.2. □

It is true in general that the semidirect product of two Roelcke precompact groups is again Roelcke precompact (see [**Tsa12**], Proposition 2.2), but in our case G^Ω is not expected to be Roelcke precompact. For instance, if $G = 2$ is the finite group with 2 elements, then G^Ω is non-compact and abelian, thus not Roelcke precompact. In fact, the group G^Ω alone may have rather unusual properties; see [**KLM15**].

It will be useful to have an explicit description of the quotients $K^\Omega /\!\!/ \mathrm{Aut}(\Omega)$ for compact K: the orbit closures of compact-valued random variables should be seen as probability distributions.

Lemma 2.5. *Let (K,d) be a compact metric space, and consider the induced action $\mathrm{Aut}(\Omega) \curvearrowright (K^\Omega, d^\Omega)$. Then, the quotient $K^\Omega /\!\!/ \mathrm{Aut}(\Omega)$ is homeomorphic to the space $\mathfrak{R}(K)$ of Borel probability measures on K.*

Proof. Let $\pi\colon K^\Omega \to K^\Omega /\!\!/ \mathrm{Aut}(\Omega)$ be the quotient map. Given $r \in K^\Omega$, the pushforward of the Lebesgue measure by r is the measure $r_*\mu \in \mathfrak{R}(K)$ defined by $\int_K f\,dr_*\mu = \int_\Omega f r\,d\mu$. We consider the map $\theta\colon \pi(r) \mapsto r_*\mu$, which is clearly well-defined and continuous.

Suppose $r_*\mu = s_*\mu$. Then, given any finite algebra B of Borel subsets of K, the preimages $r^{-1}(B)$ and $s^{-1}(B)$ are isomorphic measure algebras. Hence, by the homogeneity of MALG_μ, there is $t \in \mathrm{Aut}(\Omega)$ such that $t^{-1}(r^{-1}(B)) = s^{-1}(B)$. By duality, this yields $\pi(r) = \pi(s)$, so θ is injective. Conversely, given any measure $\nu \in \mathfrak{R}(K)$, the associated measure algebra $\mathrm{MALG}(K,\nu)$ is separable, thus it embeds into the measure algebra of Ω. By duality, this induces a measure preserving transformation $r_\nu\colon \Omega \to (K,\nu)$, that is, $(r_\nu)_*\mu = \nu$. Hence, θ is a continuous bijection. Since $K^\Omega /\!\!/ \mathrm{Aut}(\Omega)$ is compact, it is moreover a homeomorphism. □

Before passing to the next section we record the following expected counterpoint to the previous facts.

Lemma 2.6. *Let (X,d) be a Polish metric space and let $G \leq \mathrm{Iso}(X,d)$ be any Polish subgroup of isometries of X. We may see G as the subgroup of constant elements of G^Ω. Then, the quotient $X^\Omega /\!\!/ (G \times \mathrm{Aut}(\Omega))$ is compact if and only if X is compact.*

Proof. We already know one implication. For the converse, if X is not compact, let $x_i \in X$, $i \in \mathbb{N}$, be such that $d(x_i, x_j) > \epsilon$ for $i \neq j$. For $n \in \mathbb{N}$, let $\{A_i^n\}_{i<2^n}$ be a partition of Ω by sets of measure $1/2^n$. Take $r_n = \sum_{i<n} x_i \chi_{A_i^n}$. We claim that the sequence r_n has no convergent subsequence in $X^\Omega /\!\!/ G \times \mathrm{Aut}(\Omega)$.

Suppose for a contradiction that there are a subsequence $\tilde{r}_n = r_{m(n)}$, $\tilde{A}_i^n = A_i^{m(n)}$, and elements $g_n \in G$, $t_n \in \mathrm{Aut}(\Omega)$ such that $g_n t_n \tilde{r}_n = \sum_{i<m(n)} g_n x_i \chi_{t_n \tilde{A}_i^n}$ converges in X^Ω to a random variable r. Let $\delta = \epsilon/16$. For some N we have $\int d(r, \tilde{r}_n) d\mu < \delta$ whenever $n \geq N$. By Chebyshev's inequality, the set $A \subset \Omega$ where we have simultaneously $d(r, \tilde{r}_N) \leq 4\delta$ and $d(r, \tilde{r}_{N+1}) \leq 4\delta$ has measure $\mu(A) > 1/2$. We can deduce that there are i, j, k, $j \neq k$, such that both $A \cap \tilde{A}_i^N \cap \tilde{A}_j^{N+1}$ and $A \cap \tilde{A}_i^N \cap \tilde{A}_k^{N+1}$ are non-empty; say ω_j is in the former intersection and ω_k in the latter. Then

$$\begin{aligned}\epsilon < d(g_{N+1}x_j, g_{N+1}x_k) &= d(\tilde{r}_{N+1}(\omega_j), \tilde{r}_{N+1}(\omega_k)) \leq 8\delta + d(r(\omega_j), r(\omega_k)) \\ &\leq 16\delta + d(\tilde{r}_N(\omega_j), \tilde{r}_N(\omega_k)) = \epsilon + d(g_N x_i, g_N x_i) = \epsilon,\end{aligned}$$

a contradiction. □

2.2. The automorphism group of the Borel randomization. In this subsection we fix a (separable, metric) logic structure M, with at least two elements. The automorphism group of M, $\mathrm{Aut}(M)$, is a Polish subgroup of the group of isometries of M (that is, with the topology of pointwise convergence). For simplicity of notation we will denote $G = \mathrm{Aut}(M)$.

Let M^R be the Borel randomization of M, and let $G^R = \mathrm{Aut}(M^R)$. The groups G^Ω and $\mathrm{Aut}(\Omega)$ act faithfully on M^Ω, the main sort of M^R. Additionally, $\mathrm{Aut}(\Omega)$ acts on the auxiliary sort $\mathbb{A}$, and we can consider the trivial action of G^Ω on $\mathbb{A}$, i.e., each $g \in G^\Omega$ acts as the identity of $\mathbb{A}$. Combining the actions on each sort, we obtain actions $G^\Omega \curvearrowright M^R$, $\mathrm{Aut}(\Omega) \curvearrowright M^R$, which are clearly by isomorphisms. Using Lemma 1.2, we deduce that $G \wr \Omega$ is a topological subgroup of G^R.

Lemma 2.7. *If $g \in G^R$ is the identity on the auxiliary sort, then $g \in G^\Omega$.*

Proof. If g is the identity on the auxiliary sort, then for every L-formula φ and random variable $r \in M^\Omega$ we have $[\![\varphi(gr)]\!] = g[\![\varphi(r)]\!] = [\![\varphi(r)]\!]$, that is,

$$\varphi((gr)(\omega)) = \varphi(r(\omega)) \text{ for almost every } \omega. \tag{1}$$

Let $D \subset M$ be a countable dense subset and consider $C \subset M^R$ the family of constant random variables taking a value from D. Since the language L is countable, by (1) we have

$$\varphi((gc)(\omega)) = \varphi(c)$$

for every formula φ, every tuple c from C and every ω in a common full-measure set $F \subset \Omega$. Now, for $\omega \in F$ and $c \in D$ we define

$$g(\omega)(c) := (gc)(\omega)$$

(where the c on the right is the corresponding constant function on C), and this induces and elementary map $g(\omega)\colon D \to M$, which extends by continuity to an endomorphism of M.

Next we check that, for every $r \in M^\Omega$,

$$(gr)(\omega) = g(\omega)(r(\omega)) \text{ for almost every } \omega. \tag{2}$$

By (1), for each $r \in M^\Omega$ there is a full-measure subset $F' \subset F$ such that

$$d(gr(\omega), gc(\omega)) = d(r(\omega), c)$$

for every $c \in C$ and $\omega \in F'$. Let $\omega \in F'$, and take a sequence $c_n \in C$ such that $c_n \to r(\omega)$. Then we have $d(gr(\omega), gc_n(\omega)) = d(r(\omega), c_n) \to 0$, and on the other hand $gc_n(\omega) = g(\omega)(c_n) \to g(\omega)(r(\omega))$. That is to say, $gr(\omega) = g(\omega)(r(\omega))$. This proves (2).

Note that $g(\omega)$ is surjective (i.e., $g(\omega) \in G$) for almost every ω. Indeed, since the image of $g(\omega)$ is closed, it is enough to see that it contains D. But for a constant $c \in C$ we have, by (2), $g(\omega)(g^{-1}c(\omega)) = g(g^{-1}c(\omega)) = c(\omega) = c$ for every ω in a full-measure set that depends on c. Since D is countable, we are done.

Finally, we check that the map $\omega \in F \subset \Omega \mapsto g(\omega) \in G$ is measurable, which shows that it belongs to G^Ω. It is enough to see that, for every $c, d \in D$ and $\epsilon > 0$, the set $A = \{\omega \in F : d(g(\omega)(c), d) < \epsilon\}$ is measurable. This is clear, since $A = \{\omega \in F : d(gc(\omega), d) < \epsilon\}$ and $gc \in M^\Omega$ is measurable. □

Let $\operatorname{Aut}(\mathbb{A})$ be the automorphism group of $\mathbb{A}$ as a reduct of M^R. Now, it is easy to check that the map $t \in \operatorname{Aut}(\Omega) \mapsto t^* \in \operatorname{Aut}(\mathbb{A})$ is surjective, where $(t^*R)(\omega) = R(t(\omega))$ for $R \in \mathbb{A}$. (That is, the left actions $\operatorname{Aut}(\Omega) \curvearrowright \mathbb{A}$ and $\operatorname{Aut}(\mathbb{A}) \curvearrowright \mathbb{A}$ are anti-isomorphic; remark also that, as topological groups, $\operatorname{Aut}(\Omega) \simeq \operatorname{Aut}(\mathbb{A})$, since every group is anti-isomorphic to itself.)

Theorem 2.8. *For every separable structure M we have* $\operatorname{Aut}(M^R) = \operatorname{Aut}(M) \wr \Omega$.

Proof. We have already established that $G \wr \Omega$ is a topological subgroup of G^R. Let $\sigma \in G^R$. The restriction of σ to the auxiliary sort induces an automorphism of $\mathbb{A}$, say t^* for $t \in \operatorname{Aut}(\Omega)$. Let $g = \sigma t$. Then g is the identity on the auxiliary sort, so by the previous lemma we have $g \in G^\Omega$. We conclude that G^R is the product of G^Ω and $\operatorname{Aut}(\Omega)$, so the proof is complete. □

As an application we get a new proof of the following preservation result of [**BK09, Ben13b**]. Recall that M is $\aleph_0$*-categorical* if every separable model of its first-order theory is isomorphic to M.

Corollary 2.9. *If M is $\aleph_0$-categorical, then so is M^R.*

Proof. By the continuous version of the theorem of Ryll-Nardzewski, a structure N is $\aleph_0$-categorical if and only if the action $\operatorname{Aut}(N) \curvearrowright N$ is approximately oligomorphic (see [**BT14**, §5]). Thus, the result follows from the previous theorem and Proposition 2.3. □

In addition to the automorphism group of M, one can consider the topological semigroup of endomorphisms (elementary self-embeddings) of M, which we denote by $\operatorname{End}(M)$. For $\aleph_0$-categorical structures we have the following pleasant fact, observed in [**BT14**, §2.2].

Proposition 2.10. *Suppose M is $\aleph_0$-categorical, $G = \mathrm{Aut}(M)$. Then $\widehat{G}_L = \mathrm{End}(M)$, that is,* $\mathrm{End}(M)$ *is the left completion of* $\mathrm{Aut}(M)$.

Proof. See [**BIT15**], Fact 2.14. The proof adapts readily to the case of metric structures, as per [**BT14**], Lemma 2.3. $\square$

For instance, the left completion of $\mathrm{Aut}(\Omega) \simeq \mathrm{Aut}(\mathbb{A})$ is $\mathrm{End}(\mathbb{A})$, and the latter is anti-isomorphic to $\mathrm{End}(\Omega)$ by the map $t \in \mathrm{End}(\Omega) \mapsto t^* \in \mathrm{End}(\mathbb{A})$. This shows that $\mathrm{End}(\Omega)$ is the right completion of $\mathrm{Aut}(\Omega)$.

Corollary 2.11. *Let M be an $\aleph_0$-categorical structure, $G = \mathrm{Aut}(M)$. Then* $\mathrm{End}(M^R) = \widehat{(G^R)}_L = \mathrm{End}(M)^\Omega \rtimes \mathrm{End}(\Omega)^*$.

Proof. Combine Corollary 2.9, Theorem 2.8, Proposition 2.10 and Lemma 1.3. $\square$

Remark 2.12. Suppose M is $\aleph_0$-categorical. Since M^R is $\aleph_0$-categorical too, the elementary submodels of M^R are the images of its elementary self-embeddings. We deduce from the previous corollary that (the main sort of) a submodel of M^R consists of random variables of the form hs^*r for $r \in M^\Omega$ and some fixed pair $h \in \mathrm{End}(M)^\Omega$, $s \in \mathrm{End}(\Omega)$. In other words, a submodel is given by choosing (measurably) for each $\omega \in \Omega$ a submodel $M_\omega \prec M$ (the image of $h(\omega)$), then considering all sections $\Omega \to \bigcup_{\omega\in\Omega} M_\omega$ that are measurable with respect to a fixed factor of Ω (the σ-algebra generated by s).

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3. Randomized compactifications

3.1. The Markov randomization. The results of the previous section support the view that the randomization of an isometric action $G \curvearrowright (X,d)$ should be considered to be the action $G \wr \Omega \curvearrowright (X^\Omega, d^\Omega)$. A natural question is what should be considered the randomization of a G-flow, that is, of a continuous action $G \curvearrowright K$ on a compact space. Seemingly, there is not a canonical answer for this question. In this subsection, we will introduce a construction that provides a satisfactory answer for the *Roelcke compactification* of Roelcke precompact Polish groups.

Let G be a Polish group, $\widehat{G}_L$ its left completion. As observed in [**BT14**, §2.1], the metric quotient $R(G) = (\widehat{G}_L \times \widehat{G}_L) \mathbin{/\!/} G$ can be identified with the *Roelcke completion* of G. Note that $R(G)$ is a Polish space, and G acts continuously on it by the formula

$$g\overline{G(x,y)} = \overline{G(xg^{-1},y)},$$

where $g \in G$ and $x,y \in \widehat{G}_L$. Theorem 2.4 in [**BT14**] shows that G admits a faithful approximately oligomorphic action if and only if $R(G)$ is compact (i.e., G is *Roelcke precompact*). In that case, we call $R(G)$ the *Roelcke compactification* of G.

Given a compact metrizable space K and a probability measure $\lambda \in \mathfrak{R}(\Omega \times K)$, we will denote by $\lambda|_\Omega$ the pushforward of λ by the projection $\Omega \times K \to \Omega$. In what follows, we fix two copies Ω_0 and Ω_1 of Ω; we still denote the Lebesgue measure on each of them by μ.

Definition 3.1. Let K be a compact metrizable space. We define the *Markov randomization* of K as the compact space

$$\mathcal{M}(\Omega,K) := \{\lambda \in \mathfrak{R}(\Omega_0 \times K \times \Omega_1) : \lambda|_{\Omega_0} = \mu,\ \lambda|_{\Omega_1} = \mu\}.$$

For instance, if 1 denotes the one-point space, then $\mathcal{M}(\Omega,1)$ is just the space of self-joinings of the Lebesgue measure, which can be identified with the space of Markov operators of $L^2(\Omega)$ (see, for instance, [**Gla03**, Ch. 6, §2]).

If $G \curvearrowright K$ is a continuous action of a Polish group on K, then we have an induced continuous action $G \wr \Omega \curvearrowright \mathcal{M}(\Omega,K)$. In order to describe it, we observe first that we can identify

$$\mathcal{M}(\Omega,K) \simeq E \,/\!\!/\, \mathrm{Aut}(\Omega),$$

where $E = \{(s,k,r) \in (\Omega_0 \times K \times \Omega_1)^\Omega : s_*\mu = \mu,\ r_*\mu = \mu\}$, and $\mathrm{Aut}(\Omega)$ acts on it by restriction of the action $\mathrm{Aut}(\Omega) \curvearrowright (\Omega_0 \times K \times \Omega_1)^\Omega$. Indeed, the identification follows from a straightforward adaptation of the proof of Lemma 2.5; the measure λ corresponding to the class of a triple (r,k,s) is defined by the relation $\int f\,d\lambda = \int f(r,k,s)d\omega$, for $f \in C(\Omega_0 \times K \times \Omega_1)$. Next, we remark that E is naturally homeomorphic to the product $\mathrm{End}(\Omega) \times K^\Omega \times \mathrm{End}(\Omega)$. The corresponding action of $\mathrm{Aut}(\Omega)$ is given by $t \cdot (s,k,r) = (st^{-1}, kt^{-1}, rt^{-1})$, for $t \in \mathrm{Aut}(\Omega)$, $s,r \in \mathrm{End}(\Omega)$ and $k \in K^\Omega$. Hence, by considering the quotient with respect to this action, we have

$$\mathcal{M}(\Omega,K) \simeq \Big(\mathrm{End}(\Omega) \times K^\Omega \times \mathrm{End}(\Omega)\Big) \,/\!\!/\, \mathrm{Aut}(\Omega).$$

Let us denote by $[s,k,r] \in \mathcal{M}(\Omega,K)$ the image by the previous homeomorphism of the class of the triple (s,k,r). Then, given $t \in \mathrm{Aut}(\Omega)$, $g \in G^\Omega$, we have natural actions

$$t[s,k,r] := [ts,k,r],\ g[s,k,r] := [s,(s^* \cdot g)k,r].$$

Here, $(s^* \cdot g)k(\omega) = g(s(\omega))(k(\omega))$. These actions are compatible with the multiplication law of the semidirect product of G^Ω and $\mathrm{Aut}(\Omega)$, thus they induce an action $G \wr \Omega \curvearrowright \mathcal{M}(\Omega,K)$, which is moreover continuous.

Theorem 3.2. *Let G be a Polish Roelcke precompact group. Then we have a $G \wr \Omega$-equivariant homeomorphism*

$$R(G \wr \Omega) \simeq \mathcal{M}(\Omega, R(G)).$$

Proof. Using Corollary 1.3 and Lemma 2.2,

$$\begin{aligned} R(G \wr \Omega) &\simeq \Big((\widehat{G}_L)^\Omega \times \mathrm{End}(\Omega)^* \times (\widehat{G}_L)^\Omega \times \mathrm{End}(\Omega)^*\Big) \,/\!\!/\, (G \wr \Omega) \\ &\simeq \Big(\mathrm{End}(\Omega)^* \times \Big((\widehat{G}_L \times \widehat{G}_L)^\Omega \,/\!\!/\, G^\Omega\Big) \times \mathrm{End}(\Omega)^*\Big) \,/\!\!/\, \mathrm{Aut}(\Omega) \\ &\simeq \Big(\mathrm{End}(\Omega) \times R(G)^\Omega \times \mathrm{End}(\Omega)\Big) \,/\!\!/\, \mathrm{Aut}(\Omega) \\ &\simeq \mathcal{M}(\Omega, R(G)). \end{aligned}$$

It is easy to verify that the given homeomorphism respects the actions of $G \wr \Omega$. □

We remark next that the Markov randomization behaves well with respect to *semitopological semigroups*. Recall that a topological space with a semigroup law is semitopological if multiplication is separately continuous.

Let S be a compact metrizable semitopological semigroup. Then, the space S^Ω is a semitopological semigroup with pointwise multiplication. Notice that, since S is separable metrizable, then the product in S, being separately continuous, is in fact jointly measurable. Hence the pointwise product of two elements of S^Ω is again in S^Ω.

Now we can define a product on $\mathcal{M}(\Omega, S)$, as follows. Given $\lambda, \nu \in \mathcal{M}(\Omega, S)$, it is always possible to find $t, s, r \in \mathrm{End}(\Omega)$ and $p, q \in S^\Omega$ such that $\lambda = [t, p, s]$, $\nu = [s, q, r]$, and such that the σ-algebra on Ω generated by t, p is relatively independent from the σ-algebra generated by q, r over the σ-algebra generated by s. Then, we set

$$\lambda\nu := [t, pq, r].$$

The relative independence condition ensures the good definition. Alternatively, the measure $\lambda\nu$ can be defined by the formula

$$\int f \, d\lambda\nu = \iiint f(\omega_0, xy, \omega_1) d\lambda^\omega(\omega_0, x) d\nu_\omega(y, \omega_1) d\omega$$

for $f \in \mathrm{C}(\Omega_0 \times S \times \Omega_1)$, where λ^ω, ν_ω are given by the disintegrations of λ, ν over Ω_1 and Ω_0, respectively, i.e.,

$$\lambda = \int \lambda^{\omega_1} \times \delta_{\omega_1} d\omega_1, \ \nu = \int \delta_{\omega_0} \times \nu_{\omega_0} d\omega_0.$$

The product thus defined is associative and separately continuous; we omit the (routine) verification. Hence we have the following.

Proposition 3.3. *If S is a compact, metrizable, semitopological semigroup, then so is $\mathcal{M}(\Omega, S)$, with the product defined above.*

In particular, if the Roelcke compactification is a semitopological semigroup (that is, if it admits a semitopological semigroup law compatible with the group law of G), then $R(G \wr \Omega)$ is a semitopological semigroup too (compatible with $G \wr \Omega$). Suppose that $G = \mathrm{Aut}(M)$ for an $\aleph_0$-categorical structure M. It follows from [**BT14**], Theorem 5.5, that M is *stable* if and only if $R(G)$ is a semitopological semigroup. Hence, from these two facts together we get a new proof (in the $\aleph_0$-categorical case) of the preservation of stability by randomizations: if T is stable, then the randomized theory T^R is stable ([**Ben13b**, §4.2][**BK09**, §5.3]).

Given a continuous function $\varphi \in \mathrm{C}(K)$, we can define an associated function $\mathbb{E}[\![\varphi]\!] \in \mathrm{C}(\mathcal{M}(\Omega, K))$, given by

$$\mathbb{E}[\![\varphi]\!]([s, k, r]) = \int \varphi(k(\omega)) d\omega.$$

Suppose then that $K = W(G)$ is the *WAP compactification* of G, that is, the largest semitopological semigroup compactification of G. Since $\mathcal{M}(\Omega, W(G))$ is a semitopological semigroup, it is a factor of $W(G \wr \Omega)$, the WAP compactification of $G \wr \Omega$. Thus, if φ is a continuous function on the WAP compactification of G, then $\mathbb{E}[\![\varphi]\!]$ factors through the WAP compactification of $G \wr \Omega$.

As per [**BT14**, §5], every function $\varphi \in \mathrm{C}(W(G))$ can be seen as a stable formula $\varphi(x, y)$ on the $\aleph_0$-categorical structure M (defined on a certain domain), and conversely. Under this translation, the function $\mathbb{E}[\![\varphi]\!]$ corresponds to the formula

$\mathbb{E}[\![\varphi(x,y)]\!]$. Hence, by the previous discussion, we recover also (for $\aleph_0$-categorical theories) the strong form of the preservation of stability ([**Ben13b**], Theorem 4.9).

COROLLARY 3.4. *If $\varphi(x,y)$ is stable for T, then $\mathbb{E}[\![\varphi(x,y)]\!]$ is stable for T^R.*

The naïve converse of the previous fact is obvious: if $\varphi(x,y)$ is unstable, then $\mathbb{E}[\![\varphi(x,y)]\!]$ is unstable (with the order property witnessed even by constant random variables). However, one may ask for a more subtle converse: is every *stable* formula $\Phi(x,y)$ in T^R (say, with variables from the main sort) equivalent to a continuous combination of formulas of the form $\mathbb{E}[\![\varphi(x,y)]\!]$ for *stable* formulas $\varphi(x,y)$? We prefer to pose the question in the following terms.

QUESTION 3.5. Do we have $W(G \wr \Omega) \simeq \mathcal{M}(\Omega, W(G))$ for every Roelcke precompact Polish group G?

REMARK 3.6. The *Bohr compactification* of $G \wr \Omega$, that is, the largest topological group compactification of $G \wr \Omega$, is trivial (i.e., a singleton). Indeed, the Bohr compactification of the automorphism group of an $\aleph_0$-categorical structure N can be identified with the automorphism group of $\operatorname{acl}^N(\emptyset)$, the (imaginary) algebraic closure of the empty set in N, as follows from [**Ben15**] (see also [**Iba14**, §1.5]). However, the algebraic closure of $\emptyset$ in M^R is trivial (regardless of M), in the sense that it coincides with the definable closure of $\emptyset$, as follows from [**Ben13b**], Theorem 5.9.

3.2. Hilbert-representability. As mentioned above, if an $\aleph_0$-categorical structure M is stable, then the Roelcke completion of its automorphism group, $R(G)$, is a compact semitopological semigroup, and conversely. In that case, by a general result of Shtern [**Sht94**], $R(G)$ can be embedded (topologically and homomorphically) into the compact semitopological semigroup

$$\Theta(V) = \{T \in L(V) : \|T\| \leq 1\}$$

of linear contractions of a *reflexive* Banach space V (endowed with the weak operator topology). Thus, an interesting stronger property is satisfied if the space V can be chosen to be a Hilbert space.

DEFINITION 3.7. A semitopological semigroup S is *Hilbert-representable* if it can be embedded into $\Theta(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

For the case of $R(G)$, this property is therefore a strengthening of stability, and has been investigated as such in [**BIT15**]. We showed there that, for a *classical* $\aleph_0$-categorical structure M, $R(G)$ is a Hilbert-representable semitopological semigroup if and only if M is stable and one-based (equivalently, $\aleph_0$-stable). It is unclear how to generalize this for metric structures; we will come back to this discussion in Section 4. Here, we show that this property is preserved under randomizations.

Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{H}^{\otimes n}$ the n-fold tensor product $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ of Hilbert spaces. Also, we write

$$\mathcal{H}^{\otimes} := \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n}$$

for the direct sum of all the n-fold tensor self-products of $\mathcal{H}$. We recall that every linear contraction of $\mathcal{H}$ acts naturally as a linear contraction on each $\mathcal{H}^{\otimes n}$ (satisfying the identity $T(u_1 \otimes \cdots \otimes u_n) = Tu_1 \otimes \cdots \otimes Tu_n$), and hence also on the direct sum $\mathcal{H}^{\otimes}$. That is, we have an inclusion of semitopological semigroups, $\Theta(\mathcal{H}) < \Theta(\mathcal{H}^{\otimes})$.

Theorem 3.8. *Let S be a compact metrizable semitopological semigroup. If S is Hilbert-representable, then so is $\mathcal{M}(\Omega, S)$.*

Proof. Let $\beta \colon S \to \Theta(\mathcal{H})$ be an embedding into the semigroup of contractions of a Hilbert space $\mathcal{H}$, which we can also see as an embedding $\beta \colon S \to \Theta(\mathcal{H}^{\otimes})$. We consider the map

$$\beta^R \colon \mathcal{M}(\Omega, S) \to \Theta(L^2(\Omega, \mathcal{H}^{\otimes}))$$

defined implicitly by the inner product

$$\langle f_0, \beta^R(\lambda) f_1 \rangle = \int \langle f_0(\omega_0), \beta(x) f_1(\omega_1) \rangle d\lambda(\omega_0, x, \omega_1),$$

where $\lambda \in \mathcal{M}(\Omega, S)$ and $f_0, f_1 \in L^2(\Omega, \mathcal{H}^{\otimes})$. It is checked easily that $\beta^R(\lambda)$ is a linear contraction of $L^2(\Omega, \mathcal{H}^{\otimes})$ for every λ. Also, if the functions $f_0, f_1 : \Omega \to \mathcal{H}^{\otimes}$ are continuous, then the function $(\omega_0, x, \omega_1) \mapsto \langle f_0(\omega_0), \beta(x) f_1(\omega_1) \rangle$ is continuous; hence, if λ_n converge to λ, then the integrals $\langle f_0, \beta^R(\lambda_n) f_1 \rangle$ converge to $\langle f_0, \beta^R(\lambda) f_1 \rangle$. If f_0, f_1 are not continuous, we can approximate them in norm by continuous functions f_0', f_1'; in particular, the inner product $\langle f_0, \beta^R(\lambda) f_1 \rangle$ is approximated by $\langle f_0', \beta^R(\lambda) f_1' \rangle$, uniformly on λ. We see, thence, that β^R is continuous.

We now check that β^R is a homomorphism. Let $\lambda, \nu \in \mathcal{M}(\Omega, S)$. We note that, for almost every ω, $\beta^R(\nu) f_1(\omega)$ equals the vector-valued integral $\int \beta(y) f_1(\omega_1) d\nu_\omega(y, \omega_1)$. Hence,

$$\begin{aligned}
\langle f_0, \beta^R(\lambda)\beta^R(\nu) f_1 \rangle &= \iint \langle f_0(\omega_0), \beta(x)(\beta^R(\nu) f_1(\omega)) \rangle d\lambda^\omega(\omega_0, x) d\omega \\
&= \iiint \langle f_0(\omega_0), \beta(x)\beta(y) f_1(\omega_1)) \rangle d\nu_\omega(y, \omega_1) d\lambda^\omega(\omega_0, x) d\omega \\
&= \langle f_0, \beta^R(\lambda\nu) f_1 \rangle.
\end{aligned}$$

Since $\mathcal{M}(\Omega, S)$ is compact, we are only left to show that β^R is injective. Since β is an embedding, the continuous functions $x \mapsto \langle u_0, \beta(x) u_1 \rangle$ for $u_0, u_1 \in \mathcal{H}$ separate points of S. Hence, by the Stone–Weierstrass theorem, the unital algebra A generated by them is dense in $\mathrm{C}(S)$. The key observation then is that a function $h \in A$ is always of the form $h(x) = \langle w_0, \beta(x) w_1 \rangle$ for appropriate vectors $w_0, w_1 \in \mathcal{H}^{\otimes}$. Now, if h is one such function and we are given $e_0, e_1 \in \mathrm{C}(\Omega)$, we consider $f_0, f_1 \in L^2(\Omega, \mathcal{H}^{\otimes})$ defined by $f_0(\omega_0) = e_0(\omega_0) w_0$, $f_1(\omega_1) = e_1(\omega_1) w_1$. Then, if $\beta^R(\lambda) = \beta^R(\nu)$, the identity $\langle f_0, \beta^R(\lambda) f_1 \rangle = \langle f_0, \beta^R(\nu) f_1 \rangle$ becomes

$$\int e_0 h e_1 \, d\lambda = \int e_0 h e_1 \, d\nu.$$

Since this holds for arbitrary $e_0, e_1 \in \mathrm{C}(\Omega)$, $h \in A$, it follows that $\lambda = \nu$. □

As a particular case we obtain the following.

COROLLARY 3.9. *If $G = \mathrm{Aut}(M)$ for a classical $\aleph_0$-categorical $\aleph_0$-stable structure M, then $R(G \wr \Omega)$ is a Hilbert-representable semitopological semigroup.*

PROOF. Follows from the previous results together with [**BIT15**], Corollary 3.12. □

For any topological group G, there is always a largest Hilbert-representable semitopological semigroup compactification of G, which we denote by $H(G)$.

QUESTION 3.10. Is it $H(G \wr \Omega) \simeq \mathcal{M}(\Omega, H(G))$ for every Roelcke precompact Polish group G?

3.3. A general preservation result. We end this section with a general preservation result about Banach representations of randomized type spaces (Theorem 3.18 below). Modulo some additional theory, this is indeed a generalization of the preservation results discussed above, concerning stability and Hilbert-representability. Moreover, it allows us to recover, in the $\aleph_0$-categorical case, the main result of [**Ben09**] of preservation of NIP formulas.

In the previous subsections, we have considered a particular way of randomizing G-flows or, more generally, some interesting compact spaces. From a model-theoretic point of view, the main compact spaces associated to a structure M are its type spaces. In particular, the space $S(M)$ of complete types with parameters from M captures a large amount of model-theoretic information about the structure (and, in some cases, even about its theory). With the natural map $M \to S(M)$, this type space is a compactification of M in which M embeds. Thus, the natural randomized object to consider in this context is the compactification $M^R \to S(M^R)$, which we describe below.

Given a compact metrizable space K, we will consider the subspace of $L^2_{w^*}(\Omega, \mathrm{C}(K)^*)$ (with the weak* topology) consisting of those elements p that take values in the Borel probability measures on K. We observe that we have a homeomorphism

$$\{p \in L^2_{w^*}(\Omega, \mathrm{C}(K)^*) : p(\omega) \in \mathfrak{R}(K)\ \mu\text{-a.e.}\} \simeq \{\lambda \in \mathfrak{R}(\Omega \times K) : \lambda|_\Omega = \mu\},$$

where each p corresponds to the measure λ that can be disintegrated as $\lambda_\omega = p(\omega)$ almost everywhere. For convenience, we introduce a notation for this space.

DEFINITION 3.11. For a compact metrizable space K we define

$$\mathcal{S}(\Omega, K) := \{\lambda \in \mathfrak{R}(\Omega \times K) : \lambda|_\Omega = \mu\},$$

which we may identify with $\{p \in L^2_{w^*}(\Omega, \mathrm{C}(K)^*) : p(\omega) \in \mathfrak{R}(K)\ \mu\text{-a.e.}\}$ when convenient.

If $G \curvearrowright K$ is a continuous action, then we have an induced continuous action $G \wr \Omega \curvearrowright \mathcal{S}(\Omega, K)$. Indeed, observe first that G acts continuously by isometries on $\mathrm{C}(K)$, by $(gf)(x) = f(g^{-1}x)$ for $f \in \mathrm{C}(K)$, $g \in G$ and $x \in K$. Hence we have an induced action $G \wr \Omega \curvearrowright L^2_{w^*}(\Omega, \mathrm{C}(K)^*)$ (as per §1.5), which restricts to a continuous action on $\mathcal{S}(\Omega, K)$.

Now, fix any separable metric structure M in a countable language L. Given a set Δ of L-formulas $\varphi(x, y)$, we let $S_\Delta(M)$ be the space of quantifier-free Δ-types in the variable x with parameters from M, which is a compact metrizable space. The value of a type $q \in S_x(M)$ on a formula $\varphi(x, b)$ is denoted by $\varphi(x, b)^q$ (this is a real number in $[0,1]$). If $G = \mathrm{Aut}(M)$, then G acts continuously on $S_\Delta(M)$ by the relation $\varphi(x,b)^{gp} = \varphi(x, g^{-1}b)^p$.

In addition, we let Δ^R be the set of L^R-formulas of the form $\mathbb{E}\tau(x,y,z)$ where x,y are tuples of variables from the main sort, z is a tuple of variables from the auxiliary sort, and $\tau(x,y,z)$ is a term on xyz built upon the formulas of Δ. More precisely, $\tau(x,y,z)$ is constructed by applying any operations of the auxiliary sort to any variables from z and to any term of the form $[\![\varphi(x,y)]\!]$ for $\varphi \in \Delta$. Then, we can consider the space $S_{\Delta^R}(M^R)$ of quantifier-free Δ^R-types in the variable x (thus, from the main sort) with parameters from M^R, and the corresponding action $G \wr \Omega \curvearrowright S_{\Delta^R}(M^R)$.

Remark 3.12. Let $\tau(x,y,z)$ be a term as above. Then, if we substitute y by a tuple b from M, and we substitute z by a tuple of real numbers c, then $\tau(x,b,c)$ can be interpreted naturally as an L-formula with parameters from b, which is moreover obtained by a combination of formulas $\varphi(x,b)$ for $\varphi(x,y) \in \Delta$. In particular, for $q \in S_\Delta(M)$ the value $\tau(x,b,c)^q$ is defined, and this induces a continuous function $\tau(x,b,c)\colon S_\Delta(M) \to [0,1]$.

For the rest of the paper, for simplicity of notation, given $f \in \mathrm{C}(K)$ and $\nu \in \mathcal{R}(K)$, we may denote the expected value $\int f\,d\nu$ by $\mathbb{E}^\nu(f)$.

Lemma 3.13. *For any set of L-formulas Δ, we have a $G \wr \Omega$-equivariant homeomorphism $S_{\Delta^R}(M^R) \simeq \mathcal{S}(\Omega, S_\Delta(M))$.*

Under this identification, a type $p \in S_\Delta(M^R)$ can be seen as a random variable with values in $\mathcal{R}(S_\Delta(M))$, and such that

$$\mathbb{E}\tau(x,r,s)^p = \int \mathbb{E}^{p(\omega)}(\tau(x,r(\omega),s(\omega))d\omega$$

for every $\mathbb{E}\tau(x,y,z) \in \Delta^R$, $r \in (M^\Omega)^{|y|}$ and $s \in \mathbb{A}^{|z|}$.

Proof. Let us denote $K = S_\Delta(M)$. For each measure $\lambda \in \mathcal{S}(\Omega, K)$ we define a type $p_\lambda \in S_{\Delta^R}(M^R)$ by setting the value of p_λ on a formula $\mathbb{E}\tau(x,r,s)$ (as in the statement) to be

$$\mathbb{E}\tau(x,r,s)^{p_\lambda} := \int \tau(x,r(\omega),s(\omega))^q\,d\lambda(\omega,q) = \int \mathbb{E}^{\lambda_\omega}(\tau(x,r(\omega),s(\omega)))d\omega.$$

To see that this defines a type, we may write λ as the class of a pair (t,k) in the quotient $(\mathrm{End}(\Omega) \times K^\Omega) /\!\!/ \mathrm{Aut}(\Omega) \simeq \mathcal{S}(\Omega, K)$. Suppose first that t is actually in $\mathrm{Aut}(\Omega)$ and that k takes values on a finite set of realized types of $S_\Delta(M)$, so that we may write $k(\omega) = \mathrm{tp}_\Delta(k'(\omega))$ for some $k' \in (M^\Omega)^{|x|}$. Then p_λ is a realized type, namely $p_\lambda = \mathrm{tp}_{\Delta^R}(k't^{-1})$. In the general case, (t,k) is a limit of pairs of the previous form, which readily implies that p_λ is approximately finitely satisfiable, i.e., a type.

The map $f\colon \omega \mapsto \tau(x,r(\omega),s(\omega))$ is in $L^2(\Omega, \mathrm{C}(K))$, and we have $\mathbb{E}\tau(x,r,s)^{p_\lambda} = \langle f, \lambda\rangle$. Thus, the map $\theta\colon \lambda \mapsto p_\lambda$ is clearly continuous. By representing measures as we did in the previous paragraph, it is also clear that every realized type in $S_{\Delta^R}(M^R)$ is in the image of θ, hence that θ is surjective.

Checking that θ is $G \wr \Omega$-equivariant is a straightforward verification. Finally, if $\lambda \neq \nu$, then there are a set $A \subset \Omega$ and a function $h \in \mathrm{C}(K)$ such that $\mathbb{E}^\lambda(\chi_A h) \neq$

$\mathbb{E}^{\nu}(\chi_A h)$. Now, h is a continuous combination of functions induced by formulas $\varphi(x,b)$ for $\varphi(x,y) \in \Delta$ and $b \in M^{|y|}$. It follows that $\chi_A(\omega)h(q) = \tau(x,r(\omega),s(\omega))^q$ for some appropriate term τ, where we can choose $s = \chi_A$ and r to be a tuple of constant random variables. Hence, p_λ and p_ν differ in the formula $\mathbb{E}\tau(x,r,s)$, so we conclude that θ is injective. □

Next we recall some notions from the theory of Banach representations of dynamical systems as developed by Glasner and Megrelishvili; see, for instance, the survey paper [**GM14b**]. A *representation* of a (compact, Hausdorff) flow $G \curvearrowright X$ on a Banach space V is given by an isometric continuous action $G \curvearrowright V$ together with a weakly* continuous map

$$\alpha : X \to V^*$$

that is G-equivariant with respect to the dual action $G \curvearrowright V^*$. The representation is *faithful* if α is injective. If the representation is faithful and $\mathcal{K}$ is any class of Banach space containing V, then the flow is said *$\mathcal{K}$-representable*.

We introduce in addition the following definitions.

Definition 3.14. Let X be a G-flow and $\mathcal{K}$ a class of Banach spaces. We say that $f \in \mathrm{C}(X)$ is *$\mathcal{K}$-vector-representable* if there are a representation α of $G \curvearrowright X$ on a Banach space $V \in \mathcal{K}$ and a vector $v \in V$ such that, for all $x \in X$,

$$f(x) = \langle v, \alpha(x) \rangle.$$

We denote the family of all $\mathcal{K}$-vector-representable continuous functions on X by $B_{\mathcal{K}}(X)$.

We remark that if the class $\mathcal{K}$ is closed under Banach subspaces and G is separable, then in the previous definition we can always assume that V is separable. Indeed, it suffices to replace V by the closed subspace generated by Gv.

Definition 3.15. Let $\mathcal{K}$ be a class of Banach spaces closed under isomorphisms and subspaces. We say that $\mathcal{K}$ is *R-closed* if, in addition, the following conditions hold.

(1) If $V \in \mathcal{K}$, then $L^2(\Omega, V) \in \mathcal{K}$.
(2) If X is a $\mathcal{K}$-representable G-flow, then $B_{\mathcal{K}}(X)$ is dense in $\mathrm{C}(X)$.

The main classes of Banach spaces considered in [**GM14b**], and in the related works of the same authors, are R-closed. Following them, we say that a Banach space is *Rosenthal* if it does not contain an isomorphic copy of ℓ^1.

Lemma 3.16. *The classes of Hilbert, reflexive, Asplund and Rosenthal Banach spaces are R-closed.*

Proof. We comment on the two conditions of the definition separately.

(1). This is obvious for Hilbert spaces. Asplund spaces can be characterized by the property that $L^2(\Omega, V)^*$ is naturally identified with $L^2(\Omega, V^*)$ (see, for instance, [**DU77**, IV., §1]), and from this fact the claim follows easily for reflexive and Asplund spaces. For Rosenthal spaces this was proved by Pisier in [**Pis78**]; see also [**CM97**, §2.2].

(2). For the classes of reflexive, Asplund and Rosenthal spaces, it follows from the works of Glasner and Megrelishvili that every $\mathcal{K}$-representable G-flow X satisfies $B_{\mathcal{K}}(X) = \mathrm{C}(X)$. Particularly, for Rosenthal spaces, this is a consequence of [**GM12**], Theorem 6.7. For the class $\mathcal{K}$ of Hilbert spaces, by considering sums and tensor products we see that $B_{\mathcal{K}}(X)$ forms a unital subalgebra of $\mathrm{C}(X)$; if X is $\mathcal{K}$-representable, then $B_{\mathcal{K}}(X)$ separates points of X, hence the Stone–Weierstrass theorem implies that $B_{\mathcal{K}}(X)$ is dense in $\mathrm{C}(X)$. □

Remark 3.17. Let $\mathcal{K}$ be an R-closed class of Banach spaces, and let X be a metrizable G-flow. Suppose that the representations of X on Banach spaces of the class $\mathcal{K}$ separate points of X. Then, X is actually $\mathcal{K}$-representable. Indeed, since $\mathcal{K}$ is closed under forming L^2-spaces and subspaces, it follows that $\mathcal{K}$ is closed under ℓ^2-sums; then, using that X is second countable, we can choose countably many representations separating points and use them to construct a faithful representation on the ℓ^2-sum of the corresponding spaces, as is done in [**Meg08**], Lemma 3.3.

Let X be a metrizable flow of a Polish group G. We construct, for every representation $\alpha\colon X \to V^*$ of $G \curvearrowright X$ on a separable Banach space V, an induced representation
$$\alpha^R\colon \mathcal{S}(\Omega, X) \to L^2(\Omega, V)^*$$
of the action $G \wr \Omega \curvearrowright \mathcal{S}(\Omega, X)$ on the Bochner space $L^2(\Omega, V)$.

The induced action of $G \wr \Omega$ on $L^2(\Omega, V)$ was described in §1.5. As for α^R, we may define it by the relation
$$\langle f, \alpha^R(\lambda)\rangle = \int \langle f(\omega), \alpha(x)\rangle d\lambda(\omega, x)$$
for $f \in L^2(\Omega, V)$ and $\lambda \in \mathcal{S}(\Omega, X)$. By approximating f by continuous functions, it is clear that α^R is continuous, since α is. In addition, it is convenient to define $\alpha^R(\lambda)$ as an element of $L^2_{w^*}(\Omega, V^*)$. Given a measure $m \in \mathfrak{R}(X)$, the weak* expectation of α with respect to m is the functional $\mathbb{E}^m(\alpha) \in V^*$ defined by
$$\langle v, \mathbb{E}^m(\alpha)\rangle = \mathbb{E}^m(\langle v, \alpha\rangle) = \int \langle v, \alpha(x)\rangle dm(x)$$
for every $v \in V$. Then, given $p \in \mathcal{S}(\Omega, X)$, we define $\alpha^R(p)\colon \Omega \to V^*$ by
$$\alpha^R(p)(\omega) = \mathbb{E}^{p(\omega)}(\alpha).$$
Since α is continuous, its image in V^* is bounded, and this ensures that $\alpha^R(p) \in L^2_{w^*}(\Omega, V^*)$. Clearly, the two definitions of α^R coincide. Also, it is straightforward to check that α^R is $G \wr \Omega$-equivariant.

Theorem 3.18. *Let $G \curvearrowright X$ be a continuous action of a Polish group on a compact metrizable space. Let $\mathcal{K}$ be an R-closed class of Banach spaces. If $G \curvearrowright X$ is $\mathcal{K}$-representable, then so is $G \wr \Omega \curvearrowright \mathcal{S}(\Omega, X)$.*

Proof. By Remark 3.17, it suffices to show that the representations on Banach spaces of the class $\mathcal{K}$ separate points of $\mathcal{S}(\Omega, X)$. Suppose that $p, q \in \mathcal{S}(\Omega, X)$ cannot be separated in this way. In particular, since $\mathcal{K}$ is R-closed, $\alpha^R(p) = \alpha^R(q)$ for every representation $\alpha\colon X \to V^*$ on a separable Banach space $V \in \mathcal{K}$. For such α, if $v \in V$, we have then

$$\langle v, \mathbb{E}^{p(\omega)}(\alpha)\rangle = \langle v, \mathbb{E}^{q(\omega)}(\alpha)\rangle$$

for almost every ω.

Since X is $\mathcal{K}$-representable, our hypothesis on $\mathcal{K}$ ensures that we can find a countable dense family $F \subset \mathrm{C}(X)$ consisting of $\mathcal{K}$-vector-representable functions. For each $f \in F$, let α_f be a representation of X on a separable Banach space $V_f \in \mathcal{K}$ with a vector $v_f \in V_f$ such that $f(x) = \langle v_f, \alpha_f(x)\rangle$ for every $x \in X$. Since F is countable, we have that $\langle v_f, \mathbb{E}^{p(\omega)}(\alpha_f)\rangle = \langle v_f, \mathbb{E}^{q(\omega)}(\alpha_f)\rangle$ for all $f \in F$ and every ω in a common full-measure set. That is, $\mathbb{E}^{p(\omega)}(f) = \mathbb{E}^{q(\omega)}(f)$ for every $f \in F$ and almost every ω. Since F is dense in $\mathrm{C}(X)$, it follows that $p(\omega) = q(\omega)$ almost everywhere. That is, $p = q$, and the theorem follows. □

By thinking of X as a type space, the previous result can be thought of as a Banach-theoretic counterpart to the preservation results of model-theoretic properties by randomizations, studied within [**BK09, Ben13b, Ben09**]. In the case of $\aleph_0$-categorical theories, by the translation discussed in [**Iba14**], this correspondence is exact. Indeed, suppose T is separably categorical, and let $\varphi(x,y)$ be any formula. We obtain a new proof of the following (see [**Ben09**, §5]).

Corollary 3.19. *If $\varphi(x,y)$ is NIP for T, then $\mathbb{E}[\![\varphi(x,y)]\!]$ is NIP for T^R.*

Proof. Let us fix a model M of T and $G = \mathrm{Aut}(M)$. It follows from [**Iba14**, §3] that a formula $\varphi(x,y)$ is NIP if and only if, for some set of formulas Δ containing φ, the action $G \curvearrowright S_\Delta(M)$ is Rosenthal-representable. In that case, by our previous results, the action $\mathrm{Aut}(M^R) \curvearrowright S_{\Delta^R}(M^R)$, which is the same as the action $G \wr \Omega \curvearrowright \mathcal{S}(\Omega, S_\Delta(M))$, is Rosenthal-representable too. Since Δ^R contains the formula $\mathbb{E}[\![\varphi(x,y)]\!]$, we deduce that the latter is NIP. □

If instead of considering Rosenthal spaces we consider reflexive spaces, then the same argument yields yet another proof of Corollary 3.4.

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4. Beautiful pairs of randomizations

In this section we study the theory of beautiful pairs of models of a randomized $\aleph_0$-categorical theory. Let us first explain our motivation to do so.

For *classical*, stable, $\aleph_0$-categorical structures, a number of very dissimilar properties turn out be equivalent. For instance, if M is one such structure, $T = \mathrm{Th}(M)$, $G = \mathrm{Aut}(M)$, then each of the following conditions implies the other:

(1) M is $\aleph_0$-stable.
(2) M is one-based.
(3) The theory T_P of beautiful pairs of models of T is $\aleph_0$-categorical.
(4) The Roelcke compactification $R(G)$ is Hilbert-representable.

Within these, the central notion is (2), which has direct, strong model-theoretic consequences for M. That (1) is equivalent to (2) is a classical, intricate theorem. The equivalence of (2) and (3) was proved in [**BBH14**], and the equivalence of (2) and (4) was shown in [**BIT15**].

When we pass from classical to continuous logic, the most basic and well-behaved new structures we get fail to be one-based. Thus the question arises of whether there exists an appropriate generalization of this notion to the metric setting. In [**BBH14**], the authors propose a generalization (for metric, stable theories) that does hold in some important examples, which they call *SFB* (for *strongly finitely based*). They focus on (metric, stable) $\aleph_0$-categorical theories, and there they show the following: T is SFB if and only if the theory T_P of beautiful pairs of models of T is $\aleph_0$-categorical. We may take this as a definition.

We will point out here a weakness of this proposed generalization, by proving a non-preservation result: the property SFB is not preserved by randomizations. In fact, it fails badly for most randomized theories (even though it does hold for the theory of the measure algebra of Ω).

4.1. The theory T_P. We recall the basic definitions and facts about beautiful pairs of models of a stable metric theory, and refer to [**Ben12**, §4] for more details. An *elementary pair* of models of a theory T consists of a model $M \models T$ together with an elementary substructure $N \prec M$. A *beautiful pair* of models of T is an elementary pair (M,N) such that N is approximately $\aleph_0$-saturated (as per [**BU07**], Definition 1.3) and M is approximately $\aleph_0$-saturated over N, that is to say, the structure M augmented with constants for the elements of N is approximately $\aleph_0$-saturated. (We follow the definition of [**Ben12**], which is broader than the one given in [**BBH14**], although both induce the same theory T_P.)

Elementary pairs of models of an L-theory T are considered in the language L_P, which is L expanded with a predicate P for the distance to the smaller model of the pair. We denote by T_P the common theory of all beautiful pairs of models of T in this language, and we write $(M,N) \models T_P$ to say that M together with the interpretation $P(x) = d(x,N)$ forms a model of T_P.

When T is $\aleph_0$-categorical, it can be shown that any saturated model of T_P is again a beautiful pair; this fact is expressed by saying that the class of beautiful pairs of models of T is *almost elementary*. However, we will work with separable models of T_P, which need not be beautiful pairs. To this end, we will use the following general characterization of the models of T_P, which follows from the proof of Theorem 4.4 of [**Ben12**].

Theorem 4.1. *Suppose T is a stable L-theory whose class of beautiful pairs of models is almost elementary. Let $N \prec M$ be models of T. Then, $(M,N) \models T_P$ if and only if the following holds: for every $\epsilon > 0$, every finite z-tuple c from M, every 1-type $p \in S_x(Nc)$ and every finite set of L-formulas $\varphi_i(x,yz)$, $i < n$, there is $a \in M$ such that*

$$|\varphi_i(a,bc) - \varphi_i(x,bc)^p| < \epsilon$$

for every y-tuple b in N and each $i < n$. In other words, types over finite expansions of N are approximately finitely realized in M uniformly on the parameters.

Remark 4.2. If $(M,N) \models T_P$ and $\tilde{N} \prec N$, then also $(M,\tilde{N}) \models T_P$.

4.2. Separable models of $(T^R)_P$. We will consider two copies of the unit interval Ω, say Ω_0 and Ω_1. Then, Ω^2 will denote the product space $\Omega_0 \times \Omega_1$, and Ω will stand for its factor induced by Ω_0; that is, Ω will denote the measure space $\Omega_0 \times \Omega_1$ restricted to the sub-σ-algebra generated by the projection $\Omega_0 \times \Omega_1 \to \Omega_0$.

In this way, if X is a subset of a Polish space Y, then X^Ω becomes a subset of Y^{Ω^2}. The measure on each of the spaces Ω_0, Ω_1, Ω or Ω^2 will still be denoted by μ.

Let us denote $\mathbb{A}_\Omega := [0,1]^\Omega$ and $\mathbb{A}_{\Omega^2} := [0,1]^{\Omega^2}$. Hence, $\mathbb{A}_\Omega$ is a substructure of $\mathbb{A}_{\Omega^2}$. Let ARV denote $\mathrm{Th}(\mathbb{A})$, that is, the theory of $[0,1]$-valued random variables over atomless probability spaces. Finally, we denote by $\mathbb{A}_P$ the pair $(\mathbb{A}_{\Omega^2}, \mathbb{A}_\Omega)$, which is a structure in the language of pairs of models of ARV.

Proposition 4.3. *The theory ARV_P of beautiful pairs of models of ARV is $\aleph_0$-categorical, and we have $\mathbb{A}_P \models ARV_P$.*

Proof. See [**BBH14**], Corollary 3.15. □

From now on, we fix an $\aleph_0$-categorical, stable theory T and a separable model $M \models T$. As before, the theory of beautiful pairs of models of T is T_P and the randomization of T is T^R. The theory of beautiful pairs of models of T^R is $(T^R)_P$. Rather than describing the beautiful pairs of models of T^R, we are interested in the separable models of $(T^R)_P$.

Given any elementary pair $\mathbb{P}$ of models of T^R, we can consider the reduct formed by the pair of their auxiliary sorts. This is an elementary pair of models of ARV, which we denote by $\mathbb{A}_\mathbb{P}$, and which we may call the *auxiliary sort of* $\mathbb{P}$.

Remark 4.4. If $\mathbb{P} \models (T^R)_P$, then, clearly, $\mathbb{A}_\mathbb{P} \models ARV_P$.

Hence, if we have a separable model $\mathbb{P} \models (T^R)_P$, then $\mathbb{A}_\mathbb{P} \simeq \mathbb{A}_P$. It follows that we can identify the large model of the pair $\mathbb{P}$ with the Borel randomization M^R based on Ω^2 (that is, with main sort M^{Ω^2} and auxiliary sort $\mathbb{A}_{\Omega^2}$) and the small model of the pair $\mathbb{P}$ with some substructure $S \prec M^R$ whose auxiliary sort is $\mathbb{A}_\Omega$. Thus, in order to classify the separable models $\mathbb{P} \models (T^R)_P$ up to isomorphism, we are left to understand the different possibilities for the main sort of S.

Notation 4.5. From now on, unless otherwise stated, M^R will denote the randomization of M based on Ω^2, as above. Given a submodel $S \prec M^R$ with main sort S_0 and auxiliary sort $\mathbb{A}$, we will denote by $(M^{\Omega^2}, S_0)_{\mathbb{A}_P}$ the elementary pair (M^R, S) of models of T^R.

It is natural to expect that, if (M,N) is a model of T_P, then $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$ should be a model of $(T^R)_P$. This is correct, but we will prove that this does in no way exhaust the models of $(T^R)_P$ (except in trivial cases). Given $h \in \mathrm{End}(M)^{\Omega^2}$, let

$$\mathbb{P}_h := (M^{\Omega^2}, S_h)_{\mathbb{A}_P},$$

where $S_h := \{hs : s \in M^\Omega\}$. The following is the main result of this section.

Theorem 4.6. *Let $\mathbb{P}$ be a separable elementary pair of models of T^R. Then, $\mathbb{P} \models (T^R)_P$ if and only if $\mathbb{A}_\mathbb{P} \simeq \mathbb{A}_P$. Moreover, in that case, $\mathbb{P} \simeq \mathbb{P}_h$ for some $h \in \mathrm{End}(M)^{\Omega^2}$.*

In particular, for any $N \prec M$, the pair $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$ is a model of $(T^R)_P$. Clearly, this leads to non-isomorphic models of $(T^R)_P$, as long as there exists a pair $N \prec M$ with $N \subsetneq M$. Indeed, for such N, the pairs $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$ and $(M^{\Omega^2}, M^\Omega)_{\mathbb{A}_P}$ are distinct models of $(T^R)_P$. Conversely, if M does not have proper elementary substructures (which happens if and only if M is compact), then $\mathrm{End}(M) = \mathrm{Aut}(M)$, and in that case there is only one model of $(T^R)_P$ up to isomorphism.

Corollary 4.7. *The theory $(T^R)_P$ is not $\aleph_0$-categorical, unless T is the theory of a compact structure.*

As said before, this shows that the property SFB defined in [**BBH14**] is not preserved by randomizations.

Example 4.8. The theory of the randomization of a countable set (with no further structure) is not SFB. On the other hand, it is $\aleph_0$-stable, and the Roelcke compactification of $S_\infty \wr \Omega$ is Hilbert-representable.

We turn to the preliminaries for the proof of Theorem 4.6. Let $S \prec M^R$ be a submodel with auxiliary sort equal to $\mathbb{A}_\Omega$. We have discussed the elementary substructures of M^R in Remark 2.12. We know that S is the image of an endomorphism ht^* of M^R, where $h \in \mathrm{End}(M)^{\Omega^2}$ and $t \in \mathrm{End}(\Omega^2)$.

Then the image of $t^* \in \mathrm{End}(\mathbb{A}_{\Omega^2})$ must be $\mathbb{A}_\Omega$, and hence the main sort of S is

$$\{ht^*r : r \in M^{\Omega^2}\} = \{hs : s \in M^\Omega\} = S_h.$$

It follows that every model of $(T^R)_P$ is isomorphic to $\mathbb{P}_h$ for some $h \in \mathrm{End}(M)^{\Omega^2}$. We want to see that all these are indeed models of $(T^R)_P$.

Since S_h is an elementary substructure of $S_1 = M^\Omega$ (when endowed with the auxiliary sort $\mathbb{A}$), by Remark 4.2 it is enough to prove that $\mathbb{P}_1 \models (T^R)_P$. In fact, we will give a proof of the following (which, incidentally, does not use the $\aleph_0$-categoricity of M).

Proposition 4.9. *Let $N \prec M$ be any elementary substructure. Then, $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P} \models (T^R)_P$.*

We need some preliminary lemmas.

Lemma 4.10. *Let $A \subset \Omega^2$ be a Borel subset. Suppose we are given a compact metrizable space X together with a weakly* measurable family $(p_{\omega_0})_{\omega_0 \in \Omega_0}$ of finite Borel measures on X,*

$$\omega_0 \in \Omega_0 \mapsto p_{\omega_0} \in \mathrm{C}(X)^*,$$

with $p_{\omega_0}(X) = \mu(A_{\omega_0})$; here, $A_{\omega_0} \subset \Omega_1$ is the section of A at ω_0. Then, there is a measurable function $h\colon A \to X$ such that

$$\int_X f\,dp_{\omega_0} = \int_{A_{\omega_0}} f \circ h_{\omega_0}\,d\omega_1 \tag{*}$$

for every $f \in \mathrm{C}(X)$ and almost every $\omega_0 \in \Omega_0$; here, $h_{\omega_0}(\omega_1) = h(\omega_0, \omega_1)$.

Proof. We first note that for each ω_0 we can find a measurable function $h_{\omega_0}\colon A_{\omega_0} \to X$ satisfying (*). Indeed, the measure algebra of (X, p_{ω_0}) is separable, so it can be embedded in the measure algebra of (A_{ω_0}, μ), since the latter is atomless. By duality we get a measure preserving map $h_{\omega_0}\colon A_{\omega_0} \to X$, which is what we wanted.

We have to ensure that we can choose the h_{ω_0} in a measurable way. We consider partial measurable functions from Ω_1 to X: say $S = (X \cup \{*\})^{\Omega_1}$, and for $h_0 \in S$ define $s(h_0) = h_0^{-1}(X)$ to be the support of h. Now, it is not difficult to see that the set

$$E = \{(\omega_0, h_0) \in \Omega_0 \times S : s(h_0) = A_{\omega_0} \text{ and } \mathbb{E}^{p_{\omega_0}}(f) = \mathbb{E}^{\mu}(\chi_{A_{\omega_0}} f \circ h_0) \text{ for all } f \in \mathrm{C}(X)\}$$

is Borel. By the previous paragraph, the projection of E to Ω_0 is Ω_0, so from the Jankov–von Neumann uniformization theorem (see [**Kec95**, §18.A]; note that analytic sets are Lebesgue measurable) we obtain a measurable function $h\colon \Omega_0 \to S$ such that $h(\omega_0) \in E_{\omega_0}$ almost surely. By the natural identification $S^{\Omega_0} \simeq (X \cup \{*\})^{\Omega_0 \times \Omega_1}$, this induces a measurable function $h\colon A \to X$ satisfying the requirements of the lemma. □

Lemma 4.11. *Let M be a separable stable metric structure and $\varphi(x,y)$ be any formula.*

(1) The uniform pseudo-distance

$$d_\varphi(p,q) = \sup_{b \in M^{|y|}} |\varphi(x,b)^p - \varphi(x,b)^q|$$

on the type space $S_x(M)$ is separable. Moreover, every open set for the distance d_φ is Borel measurable for the logic topology of $S_x(M)$.

(2) For every $\epsilon > 0$ there is a natural number $m \in \mathbb{N}$ such that for every $q \in S_x(M)$ there exists a sequence $(a_l)_{l \in \mathbb{N}} \subset M^{|x|}$ with the property that, for any tuple $b \in M^{|y|}$,

$$|\{l \in \mathbb{N} : |\varphi(a_l, b) - \varphi(x,b)^q| \geq \epsilon\}| \leq m.$$

Proof. (1). The first assertion follows from the separability of M and the definability of types in stable theories. See for example [**Ben13a**], Corollary 4. The second follows directly from the separability of M.

(2). By stability, there is a natural number m such that, whenever a_l is an indiscernible sequence with limit type q and b is any tuple, the counting inequality displayed in the statement is satisfied. The following argument by compactness shows that there are a finite set of formulas Δ and a positive number δ such that the same is true whenever a_l is a Δ-δ-*indiscernible* sequence (in the sense defined in [**Iba14**, §1.4]) converging to q. Suppose this does not hold. Let $d_q\varphi$ be the φ-definition of q. Then, for any finite Δ and $\delta > 0$ we can find a large finite Δ-δ-indiscernible sequence a_l and an element b such that, for odd l,

$$|\varphi(a_l, b) - d_q\varphi(b)| \geq \epsilon,$$

whereas, for even l,

$$|\varphi(a_l, b) - d_q\varphi(b)| < \epsilon/2.$$

By compactness, we get an infinite indiscernible sequence a_l and an element b with this property, which yields a contradiction. The bound m can be chosen uniformly in q since types are uniformly definable.

Now, if $q \in S_x(M)$ is any type, we can take any sequence in M converging to q and extract by Ramsey's theorem a Δ-δ-indiscernible subsequence a_l. The lemma follows. □

Proof of Proposition 4.9. We have to check the condition of Theorem 4.1 for the pair $(M^{\Omega^2}, N^{\Omega})_{\mathbb{A}_p}$. For simplicity, we will only check it for basic formulas of the form $\mathbb{E}[\![\varphi_i(x,yz)]\!]$, so in particular the tuples xyz will only contain variables from the *main sort*. It is not difficult to see that this is actually enough. Moreover, it suffices to check the condition for z-tuples consisting of *simple* elements of M^{Ω^2}, that is, random variables of finite range.

So we fix formulas $\mathbb{E}[\![\varphi_i(x,yz)]\!]$, $i<n$, we fix a simple z-tuple t, and a type $p \in S_x^{T^R}(N^{\Omega}t)$. Let $C \subset M^{|z|}$ be the (finite) range of t. We fix $c \in C$, then set $A_c = t^{-1}(c) \subset \Omega^2$. By taking any extension of p to a type over M^{Ω^2}, we may apply Lemma 3.13 and see p as a random variable $p\colon \Omega^2 \to \mathfrak{R}(S_x^T(M)) \to \mathfrak{R}(S_x^T(Nc))$.

We consider the type space $X = S_x^T(Nc)$. For each $\omega_0 \in \Omega_0$ and $f \in \mathrm{C}(X)$, say with f induced by a formula $\varphi(x,bc)$, we set

$$\langle f, p_{\omega_0}\rangle = \int_{(A_c)_{\omega_0}} \mathbb{E}^{p(\omega_0,\omega_1)}(\varphi(x,bc))d\omega_1.$$

This defines a finite measure $p_{\omega_0} \in \mathrm{C}(X)^*$ with $p_{\omega_0}(X) = \mu((A_c)_{\omega_0})$. Applying Lemma 4.10, we get a measurable function $h_c\colon A_c \to X$ that satisfies

$$\int_{(A_c)_{\omega_0}} \mathbb{E}^{p(\omega_0,\omega_1)}(\varphi(x,bc))d\omega_1 = \int_{(A_c)_{\omega_0}} \varphi(x,bc)^{h_c(\omega_0,\omega_1)}d\omega_1$$

for almost every ω_0 and every Nc-formula $\varphi(x,bc)$.

Let $\epsilon > 0$. Using the first item of Lemma 4.11, we can find a countable set $J \subset S_x^T(M)$ and measurable functions $j_c\colon A_c \to J$ for each $c \in C$ such that

$$|\varphi_i(x,bc)^{h_c(\omega)} - \varphi_i(x,bc)^{j_c(\omega)}| \le \epsilon$$

for every $b \in N^{|y|}$, $c \in C$, $i<n$ and $\omega \in \Omega^2$. Next, we apply the second item of the lemma to get sequences $(a_q^l)_{l\in\mathbb{N}} \subset M$ for each $q \in J$ and a natural number $m \in \mathbb{N}$ such that, for each of the formulas $\varphi_i(x,yz)$, $i<n$, and for any $b \in N^{|y|}$, $c \in C$ and $q \in J$, we have

$$|\{l \in \mathbb{N} : |\varphi_i(a_q^l, bc) - \varphi_i(x,bc)^q| \ge \epsilon\}| \le m.$$

In any case, $|\varphi_i(a_q^l, bc) - \varphi_i(x,bc)^q| \le 1$ since we assume formulas are $[0,1]$-valued.

Take $k \in \mathbb{N}$ with $m/k < \epsilon$. For each $c \in C$ and $q \in J$ we can choose a Borel partition $\{A_{c,q}^l\}_{l<k}$ of $A_{c,q} := (h_c)^{-1}(q) \subset A_c$ such that, for almost every $\omega_0 \in \Omega_0$, we have

$$\mu((A_{c,q}^l)_{\omega_0}) = \frac{1}{k}\mu((A_{c,q})_{\omega_0}).$$

Finally, we define $r : \Omega^2 \to M$ by

$$r = \sum_{c\in C, q\in J, l<k} a_q^l \chi_{A_{c,q}^l}.$$

In this way, for any $i < n$ and any tuple of random variables $s \in (N^{|y|})^\Omega$ (which depends only on the variable $\omega_0 \in \Omega_0$), we get

$$
\begin{aligned}
|\mathbb{E}[\![\varphi_i(r,st)]\!] - \mathbb{E}[\![\varphi_i(x,st)]\!]^p| &\leq \sum_{c\in C}\left|\int_{A_c} \varphi_i(r(\omega_0,\omega_1),s(\omega_0)c) - \mathbb{E}^{p(\omega_0,\omega_1)}(\varphi_i(x,s(\omega_0)c))d\mu\right| \\
&\leq \sum_{c\in C}\int_{\Omega_0}\left|\int_{(A_c)_{\omega_0}} \varphi_i(r(\omega_0,\omega_1),s(\omega_0)c) - \mathbb{E}^{p(\omega_0,\omega_1)}(\varphi_i(x,s(\omega_0)c))d\omega_1\right| d\omega_0 \\
&= \sum_{c\in C}\int_{\Omega_0}\left|\int_{(A_c)_{\omega_0}} \varphi_i(r(\omega_0,\omega_1),s(\omega_0)c) - \varphi_i(x,s(\omega_0)c)^{h_c(\omega_0,\omega_1)}d\omega_1\right| d\omega_0 \\
&\leq \epsilon + \sum_{c\in C}\int_{\Omega_0}\int_{(A_c)_{\omega_0}} \left|\varphi_i(r(\omega_0,\omega_1),s(\omega_0)c) - \varphi_i(x,s(\omega_0)c)^{j_c(\omega_0,\omega_1)}\right| d\omega_1 d\omega_0 \\
&= \epsilon + \sum_{c\in C, q\in J, l<k}\int_{\Omega_0}\int_{(A^l_{c,q})_{\omega_0}} \left|\varphi_i(a^l_q,s(\omega_0)c) - \varphi_i(x,s(\omega_0)c)^{q}\right| d\omega_1 d\omega_0 \\
&= \epsilon + \sum_{c\in C, q\in J}\int_{\Omega_0}\frac{1}{k}\mu((A_{c,q})_{\omega_0})\sum_{l<k} \left|\varphi_i(a^l_q,s(\omega_0)c) - \varphi_i(x,s(\omega_0)c)^{q}\right| d\omega_0 \\
&\leq \epsilon + \sum_{c\in C, q\in J}\int_{\Omega_0}\left(\frac{m}{k}+\epsilon\right)\mu((A_{c,q})_{\omega_0}) d\omega_0 \\
&< 3\epsilon.
\end{aligned}
$$

We have thus verified the condition of Theorem 4.1 for (M^{Ω^2}, N^Ω). □

Together with the discussion preceding Proposition 4.9, this completes the proof of Theorem 4.6.

Remark 4.12. Suppose M is a classical structure, and that (M,N) is in fact a beautiful pair of (countable) models of T. In this case, the argument of Proposition 4.9 becomes much simpler, and yields more. Indeed, resuming the argument after the third paragraph, for every type $q \in S^T_x(Nc)$ we can choose a realization $a_c(q) \in M^{|x|}$. Then we take $r = \sum_{c\in C}(a_c \circ h_c)\chi_{A_c}$, and we see readily that r is a realization of p. It follows that $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_p}$ is a beautiful pair of models of T^R.

By using the same ideas we obtain the following, which is the metric generalization of [**BK09**], Theorem 4.1.

Theorem 4.13. *Let T be a metric theory in a countable language. If T is $\aleph_0$-stable, then so is T^R.*

Proof. Since T is $\aleph_0$-stable, there is an elementary pair (M,N) of separable models of T such that N realizes every type over the empty set (possibly in countably many variables) and M realizes every type over N. (In fact, there is even a separable beautiful pair, as can be seen using [**BU07**], Proposition 1.16.) That is, the canonical map $\pi\colon M^{|x|} \to S^T_x(N)$ is surjective. We claim that it admits a Borel selector. By the same argument of Lemma 2.1, it suffices to check that $\pi(U)$ is Borel for every open set $U \subset M^{|x|}$. We will check it first for the *metric topology* of $S^T_x(M)$.

Let $a \in U$ and take $\epsilon > 0$ such that $b \in U$ whenever $d(a,b) < \epsilon$. If $d(\pi(a),q) < \epsilon$, then by saturation there are $a', b \in M^{|x|}$ such that $\pi(a') = \pi(a)$, $\pi(b) = q$ and $d(a',b) < \epsilon$. Hence, $\pi(U)$ is open for the metric topology. Now, since T is $\aleph_0$-stable and N is separable, the metric topology of the type space is Polish, and it follows that U is an F_σ set for the usual compact topology. We deduce that there is a Borel selector for π, say $a\colon S_x^T(N) \to M$.

Now we may proceed as in Remark 4.12 (ignoring the variable z), to show that M^{Ω^2} realizes every type over N^Ω. Indeed, given $p \in S_x^{T^R}(N^\Omega)$, there is $h\colon \Omega^2 \to S_x^T(N)$ such that $\int \mathbb{E}^{p(\omega_0,\omega_1)}(\varphi(x,b))d\omega_1 = \int \varphi(x,b)^{h(\omega_0,\omega_1)}d\omega_1$ for any ω_0, b, φ. If we define $r = a \circ h \in M^{\Omega^2}$, then r is a realization of p.

Similarly, N^Ω realizes every type over the empty set. Then, the pair $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$ witnesses that T^R is $\aleph_0$-stable. $\square$

It had already been observed in [**BBH14**] that an $\aleph_0$-stable, $\aleph_0$-categorical theory need not be SFB. Namely, the theory AL_pL of atomless L^p Banach lattices (for any fixed $p \in [1,\infty)$) is $\aleph_0$-stable and admits only one separable model up to isomorphism, but the corresponding theory AL_pL_P of beautiful pairs admits exactly two non-isomorphic models. To this example we can now add any randomized theory T^R where T is an $\aleph_0$-stable, $\aleph_0$-categorical theory with a non-compact model.

Nevertheless, they point out in [**BBH14**] that the theory AL_pL_P is $\aleph_0$-categorical *up to arbitrarily small perturbations of the predicate P*. For a general theory T, this means that for every $\epsilon > 0$ and any two separable models $(M,N),(M',N') \models T_P$, there exists an isomorphism $\rho\colon M \to M'$ such that

$$|d(x,N) - d(\rho(x),N')| \le \epsilon$$

for every $x \in M$. If T_P has this property, let us say that T is *approximately SFB*. Then, it was conjectured in [**BBH14**] that an $\aleph_0$-stable, $\aleph_0$-categorical theory should be approximately SFB. Our new examples give an interesting family to test this conjecture.

Question 4.14. Let T be an SFB theory, for instance, a classical $\aleph_0$-stable, $\aleph_0$-categorical theory. Is it true that the randomized theory T^R is approximately SFB?

4.3. Automorphisms of pairs. Since this work originated in the study of automorphism groups of randomized structures, let us finish with a description of the automorphism group of a pair $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$, in the fashion of Theorem 2.8.

We begin by describing the automorphism group of the auxiliary sort, $H_P^R :=$ $\mathrm{Aut}(\mathbb{A}_P)$. Via the natural isomorphism $[0,1]^{\Omega^2} \simeq ([0,1]^{\Omega_1})^{\Omega_0}$, we see that the structure $\mathbb{A}_P$ (a model of ARV_P) can be in a sense identified with the randomization $(\mathbb{A}_{\Omega_1})^R$ (a model of ARV^R): they are bi-interpretable. In particular, they have the same automorphism groups:

$$H_P^R = \mathrm{Aut}(\Omega_1) \wr \Omega_0.$$

Now we let M be a separably categorical, stable structure, and we take $G =$ $\mathrm{Aut}(M)$. We fix an elementary substructure $N \prec M$, and we consider the automorphism group of the pair,

$$G_P := \mathrm{Aut}(M,N),$$

which is the subgroup of all $g \in G$ that preserve the predicate $P(x) = d(x, N)$ (equivalently: that fix N setwise). Similarly, we let

$$G_P^R := \mathrm{Aut}((M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P})$$

be the automorphism group of the induced model of $(T^R)_P$, which is the subgroup of $\mathrm{Aut}(M^R)$ ($=G \wr \Omega^2$) fixing N^Ω and $\mathbb{A}_\Omega$ setwise. Furthermore, we consider the subgroup

$$G_P^* := (G_P)^{\Omega^2} \cap G_P^R.$$

Lemma 4.15. *We have $G_P^* = \{g \in G^{\Omega^2} : g|_{N^\Omega} \in \mathrm{Aut}(N)^\Omega\}$. Moreover, if $g \in G_P^R$ is the identity on the auxiliary sort $\mathbb{A}_P$, then $g \in G_P^*$.*

Proof. Let N^R denote the smaller model of the pair $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$, that is, N^Ω together with the auxiliary sort $\mathbb{A}_\Omega$. If $g \in G_P^R$, then the restriction of g to the N^R is an automorphism of N^R, that is, $g|_{N^R} \in \mathrm{Aut}(N) \wr \Omega$. If moreover g is the identity on $\mathbb{A}_P$ (which is the case if $g \in G_P^*$), then $g \in G^{\Omega^2}$ and $g|_{N^R}$ is the identity on $\mathbb{A}_\Omega$, so $g|_{N^R} \in \mathrm{Aut}(N)^\Omega$.

Conversely, if we have $g \in G^{\Omega^2}$ and $g|_{N^\Omega} \in \mathrm{Aut}(N)^\Omega$, then g fixes N^Ω setwise (and $\mathbb{A}_\Omega$ pointwise), so $g \in G_P^R$. Also, $g \in (G_P)^{\Omega^2}$. Indeed, let b be an element in N, which we may see as a constant element of M^{Ω^2}. Since $g \in G_P^R$, we have $d(gb, N^\Omega) = d(b, N^\Omega) = 0$, hence $g(\omega)(b) \in N$ almost surely. By the separability of N, this is true for every $b \in N$ and every ω in a common full measure set. Similarly for g^{-1}. Thus, $g(\omega) \in G_P$ almost surely. □

Corollary 4.16. *$G_P^R \simeq G_P^* \rtimes (\mathrm{Aut}(\Omega_1) \wr \Omega_0)$ as topological groups.*

Proof. As in the proof of Theorem 2.8, the moreover part of the previous lemma shows that G_P^* is the normal complement of H_{RP}, which is what we want. □

The previous description can be used to show that the action $G_R^P \curvearrowright (M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P}$ is not approximately oligomorphic if M is not compact (adapting the method of proof of Proposition 2.3, but with the opposite conclusion; for this, Lemma 2.6 is useful). This gives an alternative proof of Corollary 4.7, modulo showing that $(M^{\Omega^2}, N^\Omega)_{\mathbb{A}_P} \models (T^R)_P$ for some $N \prec M$ (which is easier by assuming $(M, N) \models T_P$). This was actually our original proof of Corollary 4.7.

Part 2

Minimal homeomorphisms of the Cantor space

CHAPTER 4

Full groups of minimal homeomorphisms and Baire category methods

Abstract. [Joint work with Julien Melleray, published in Ergodic Theory and Dynamical Systems, vol. 36(02), pp. 550-573.] We study full groups of minimal actions of countable groups by homeomorphisms on a Cantor space X, showing that these groups do not admit a compatible Polish group topology and, in the case of $\mathbb{Z}$-actions, are coanalytic non-Borel inside Homeo(X). We point out that the full group of a minimal homeomorphism is topologically simple. We also study some properties of the closure of the full group of a minimal homeomorphism inside Homeo(X).

Contents

Introduction

When studying a mathematical structure, one is often led to consider the properties of its automorphism group, and then it is tempting to ask to what extent the group characterizes the structure. A particularly striking example is provided by a theorem of Dye ([**Dye59**], [**Dye63**]) in ergodic theory: assume that two countable groups Γ, Δ act on the unit interval $[0,1]$ by measure-preserving automorphisms, without any non-trivial invariant sets (i.e., the actions are *ergodic*), and consider the groups $[\Gamma]$ (resp. $[\Delta]$) made up of all measurable bijections that map each Γ-orbit (resp. Δ-orbit) onto itself. Then the groups $[\Gamma]$ and $[\Delta]$ are isomorphic if, and only if, there exists a measure-preserving bijection of $[0,1]$ which maps Γ-orbits onto Δ-orbits. One then says that the relations are orbit equivalent; $[\Gamma]$ is called the *full group* of the action. Using this language, Dye's theorem says that the full group of an ergodic action of a countable group on a standard probability space completely remembers the associated equivalence relation up to orbit equivalence.

This result was the motivation for an intensive study of full groups in ergodic theory, for which we point to [**Kec10**] as a general reference. More recently, it came to light, initially via the work of Giordano–Putnam–Skau, that a similar phenomenon takes place in *topological dynamics*. In that context, one still considers actions of countable groups, replacing probability-measure-preserving actions with actions by homeomorphisms of a Cantor space. The two settings are related: for instance, when Γ is a countable group, one could consider the Bernoulli shift action of Γ on $\{0,1\}^\Gamma$ as a measure-preserving action (say, for the $(1/2,1/2)$-Bernoulli measure) or as an action by homeomorphisms. As in the measure-theoretic setting, one can define the full group of an action of a countable group Γ on a Cantor space X: this time, it is made up of all homeomorphisms of X which map Γ-orbits onto themselves. The counterpart of ergodicity here is *minimality*, i.e., the assumption that all the orbits of the action are dense; the analog of Dye's theorem for minimal group actions was proved by Giordano–Putnam–Skau [**GPS99**].

The measure-theoretic and topological settings may appear, at first glance, to be very similar; however, there are deep differences, for instance all ergodic group actions of countable amenable groups are orbit equivalent (Connes–Feldman–Weiss [**CFW81**]) while there exists a continuum of pairwise non-orbit equivalent actions of $\mathbb{Z}$ by minimal homeomorphisms of a Cantor space. Still, it is interesting to investigate properties of full groups in topological dynamics, which has been done by several authors over the last twenty years or so.

In both contexts discussed above, it is natural to consider the full group of an action as a topological group, the topology being induced by the topology of the ambient Polish group (measure-preserving bijections of $[0,1]$ in one case, homeomorphisms of the Cantor space in the other). The usefulness of this approach is however limited by the fact that the full group of an ergodic group action, or a minimal group action, is not closed in the ambient group; in the first case the full group is dense, in the second it seems that the closure is currently only understood for actions of $\mathbb{Z}^d$.

It then comes as a blessing that, in the measure-theoretic context, one can endow the full group with a stronger topology which turns it into a Polish group: the *uniform topology*, induced by the distance given by $d(g,h) = \mu(\{x\colon g(x) \neq h(x)\})$. This paper grew out of the following question: can one do the same thing in the topological context? It is interesting to note that, shortly after the publication of [**GPS99**], Bezyglyi and Kwiatkowski [**BK02**] introduced an analog of the uniform topology in the context of topological dynamics, which is however far from being as nice as the uniform topology of ergodic theory. This provides further motivation for trying to understand whether a nice group topology exists at all.

Theorem. *Let Γ be a countable group acting minimally by homeomorphisms on a Cantor space X. Then any Hausdorff, Baire group topology on $[\Gamma]$ must extend the topology of pointwise convergence for the discrete topology on X. Consequently, there is no second countable, Hausdorff, Baire group topology on $[\Gamma]$.*

This is bad news, but certainly not surprising —if a Polish group topology existed for that group, it would have been considered a long time ago. In the same spirit, one can then wonder about the complexity of full groups inside the ambient automorphism group; in ergodic theory, full groups are always fairly tame, in the sense that they can be written as countable intersections of countable unions of closed sets [**Wei05**]. Yet again, the situation turns out to be more dire in topological dynamics.

Theorem. *The full group of a minimal homeomorphism of a Cantor space X is a coanalytic non-Borel subset of* Homeo(X).

This led us to study the *closure* of a full group inside the homeomorphism group; this is a Polish group, and is also a complete invariant for orbit equivalence if one is willing to restrict one's attention to actions of $\mathbb{Z}$, which we do in the last sections of this article. It follows from a theorem of Glasner–Weiss [**GW95**] that the closure of the full group of a minimal homeomorphism φ coincides with the group of homeomorphisms which preserve all φ-invariant measures. Using work of Bezuglyi–Medynets and Grigorchuk–Medynets, we obtain the following result[1].

Theorem. *The closure of the full group of a minimal homeomorphism of the Cantor space is topologically simple (hence, the full group itself is also topologically simple).*

It is an open problem whether the full group of a minimal homeomorphism is simple.

In the case of uniquely ergodic homeomorphisms, we also provide a criterion for the existence of dense conjugacy classes in the closure of the full group (in terms of the values taken by the unique invariant measure on clopen sets), and use a Fraïssé theoretic approach to recover a result of Akin which describes a class of uniquely ergodic homeomorphisms with the property that the closure of their full group admits a comeager conjugacy class.

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1. Background and terminology

We now go over some background material and discuss in more detail some facts that were mentioned briefly in the introduction.

Recall that a *Cantor space* is a nonempty, zero-dimensional, perfect compact metrizable space; any two Cantor spaces are homeomorphic. Given a Cantor space X, we denote by Clop(X) the Boolean algebra of all clopen subsets of X, and by Homeo(X) the group of homeomorphisms of X. This group can be endowed with the topology whose basic open sets are of the form

$$\{g \in \mathrm{Homeo}(X)\colon \forall i \in \{1,\dots,n\}\ g(U_i) = V_i\},$$

[1] We originally proved this only for uniquely ergodic homeomorphisms; we thank K. Medynets for explaining how to make the argument work in general.

where n is an integer and U_i, V_i are clopen subsets of X. This turns Homeo(X) into a *topological group*, namely the group operations $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous with respect to this topology.

DEFINITION 1.1. A *Polish group* is a topological group whose topology is induced by a complete, separable metric.

Picking a compatible distance d on X, one can check that the topology defined above on Homeo(X) is a Polish group topology, a compatible complete distance being given by

$$d(g,h) = \max_{x\in X} d(g(x),h(x)) + \max_{y\in X} d(g^{-1}(y),h^{-1}(y)).$$

It might be a bit surprising at first that the two topologies we defined coincide; actually, this is a hint of a more general phenomenon: the unique second–countable group topologies on Homeo(X) are the coarse topology and the Polish group topology we defined above (this follows from results of [**And58**], [**Gam91**] and [**RS07**]).

Polish groups form a fairly general class of groups, yet the combination of separability and the use of Baire category methods make them relatively tame. The fact that the Baire category theorem holds in Polish groups is particularly important; we recall that, whenever G is a topological group for which the Baire category theorem holds, H is a separable topological group and $\varphi\colon G \to H$ is a Borel homomorphism, then φ must actually be continuous (see e.g. [**Kec95**, Theorem 9.10]).

DEFINITION 1.2. Let Γ be a countable group acting by homeomorphisms on a Cantor space X. We denote by R_Γ the associated equivalence relation and define its *full group* as the group of all homeomorphisms of X which preserve each Γ-orbit; in symbols,

$$[R_\Gamma] = \{g \in \mathrm{Homeo}(X)\colon \forall x \in X \exists \gamma \in \Gamma\ g(x) = \gamma \cdot x\}.$$

As is the case in ergodic theory, the full group of an action of a countable group by homeomorphisms of a Cantor space X completely remembers the associated equivalence relation, a fact made precise by the following definition and theorem.

DEFINITION 1.3. Let Γ_1, Γ_2 be two countable groups acting by homeomorphisms on a Cantor space X, and let $R_{\Gamma_1}, R_{\Gamma_2}$ be the associated equivalence relations. We say that R_{Γ_1} and R_{Γ_2} are *orbit-equivalent* if there is a homeomorphism g of X such that

$$\forall x,y \in X\ \left(xR_{\Gamma_1}y\right) \Leftrightarrow \left(g(x)R_{\Gamma_2}g(y)\right).$$

THEOREM 1.4 ([**GPS99**], [**Med11**]). *Let Γ_1, Γ_2 be countable groups acting by homeomorphisms on a Cantor space X; assume that all orbits for both actions have cardinality at least 3, and for any nonempty $U \in \mathrm{Clop}(X)$ and $i = 1,2$ there exists $x \in U$ such that $\Gamma_i \cdot x$ intersects U in at least two points.*

Denote by R_{Γ_1} and R_{Γ_2} the associated equivalence relations, and suppose that there exists an isomorphism Φ from $[R_{\Gamma_1}]$ to $[R_{\Gamma_2}]$. Then there must exist $g \in \mathrm{Homeo}(X)$ such that $\Phi(h) = ghg^{-1}$ for all $h \in [R_{\Gamma_1}]$.

Consequently, $[R_{\Gamma_1}]$ and $[R_{\Gamma_1}]$ are isomorphic if, and only if, R_{Γ_1} and R_{Γ_2} are orbit equivalent.

The above result was first proved by Giordano–Putnam–Skau [**GPS99**] for *minimal* actions, which we define now, and then extended by Medynets [**Med11**].

Definition 1.5. Let Γ be a countable group acting by homeomorphisms on a Cantor space. We say that the action is *minimal* if every point has a dense orbit.

Minimal actions are particularly well-studied when $\Gamma = \mathbb{Z}$. In that case the action is simply induced by one homeomorphism φ; accordingly, we will use the notation $[\varphi]$ to denote the full group of the associated equivalence relation. Similarly, when the $\mathbb{Z}$-action associated to a homeomorphism φ is minimal we simply say that φ is minimal. In the case of minimal actions of $\mathbb{Z}$, a particular subgroup plays an important role and is well understood, mainly thanks to work of Matui.

Definition 1.6. Let φ be a homeomorphism of a Cantor space X. Its *topological full group* $[[\varphi]]$ is the set of elements $g \in \mathrm{Homeo}(X)$ for which there is a finite clopen partition $U_1, \dots, U_n$ of X such that on each U_i g coincides with some power of φ.

The topological full group $[[\varphi]]$ is a countable subgroup of $[\varphi]$; note that, if all orbits of φ are infinite, then for any element g in $[\varphi]$ there exists a unique $n_x \in \mathbb{Z}$ such that $g(x) = \varphi^{n_x}(x)$; the group $[[\varphi]]$ is simply made up of all g for which the associated cocycle $x \mapsto n_x$ is continuous. Another equivalent (though apparently weaker) definition is that $[[\varphi]]$ is exactly the set of elements of $[\varphi]$ for which the map $x \mapsto n_x$ has a finite range. Indeed, each set of the form $\{x \in X \colon g(x) = \varphi^n(x)\}$ is closed, and these sets cover X if g belongs to $[\varphi]$, so if there are only finitely many nonempty such sets then they are clopen and g belongs to $[[\varphi]]$.

Though it will not be featured prominently in this paper, the topological full group plays an important part in the study of minimal homeomorphisms; it appears naturally as a subgroup of the C^*-algebra associated to (X, φ) (see for instance [**GPS95**], [**BT98**]), and topological full groups of two minimal homeomorphisms φ_1, φ_2 are isomorphic if and only if the associated systems are *flip-conjugate*, i.e., φ_1 is conjugate to φ_2 or φ_2^{-1} (Boyle–Tomiyama [**BT98**]). Recently, Juschenko–Monod [**JM13**] proved that topological full groups of minimal homeomorphisms of Cantor spaces are amenable, a result which had been conjectured by Grigorchuk–Medynets [**GM**]; in conjunction with work of Matui [**Mat06**], this provided the first example of simple, finitely generated, infinite amenable groups. For further information about these groups, we refer to de Cornulier's thorough survey paper [**dC13**].

The next result elucidates the action of the full group of a minimal homeomorphism on the algebra of clopen sets. Before stating it we set some notation for the sequel.

Notation. Let X be a Cantor space and φ a homeomorphism of X. A Borel probability measure μ on X is *φ-invariant* if $\mu(A) = \mu(\varphi^{-1}(A))$ for any Borel subset $A \subseteq X$ (if this equality holds for clopen sets then it must hold for all Borel sets). Given a homeomorphism φ, we denote by $\mathcal{M}_\varphi$ its set of invariant probability measures, which is a nonempty compact, convex subset of the space of all probability measures on X.

Theorem 1.7 ([**GW95**], Lemma 2.5 and Proposition 2.6). *Let φ be a minimal homeomorphism of a Cantor space X, and A, B be clopen subsets of X. Then the following facts hold:*

- *If $\mu(A) < \mu(B)$ for all $\mu \in \mathcal{M}_\varphi$ then there exists $g \in [[\varphi]]$ such that $g(A) \subset B$.*
- $\Big(\forall \mu \in \mathcal{M}_\varphi\ \mu(A) = \mu(B)\Big) \Leftrightarrow (\exists g \in [\varphi] g(A) = B)$.

Remark 1.8. In both cases, one can add the assumption that $g^2 = 1$ to the right-hand statement. It is useful that in the first statement above, one can find g in the topological full group and not merely in the full group; this is not always possible when it comes to the second statement.

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2. Topologies on full groups

Throughout this section we let Γ denote a countable group acting on a Cantor space X, and R denote the associated equivalence relation, i.e., xRy if and only if $x = \gamma \cdot y$ for some $\gamma \in \Gamma$. We make the following assumption on the action of Γ: given any nonempty open $U \subseteq X$, there exist $x \neq y \in U$ and $\gamma \in \Gamma$ such that $\gamma \cdot x = y$. Equivalently, there exists no nonempty open subset $U \subseteq X$ such that the restriction of R to U is trivial.

Note that this assumption implies that, given any nonempty open U, there exists $g \in [R]$ and a clopen V such that $g^2 = 1$, $g(V)$ and V are nonempty disjoint clopen subsets of U and g coincides with the identity outside of $V \cup g(V)$. We now point out a further consequence.

Lemma 2.1. *The set $\Omega = \{x \in X\colon x \text{ is an accumulation point of } \Gamma \cdot x\}$ is dense G_δ in X.*

Proof. Fix a compatible metric d on X. Recall that x is an accumulation point of $\Gamma \cdot x$ if, and only if, the following condition holds:

$$\forall \varepsilon > 0\ \exists \gamma \in \Gamma\ \ (\gamma \cdot x \neq x \text{ and } d(\gamma \cdot x, x) < \varepsilon)$$

The condition between brackets is open, showing that Ω is indeed G_δ. By the Baire category theorem, to check that Ω is dense we only need to prove that for any $\varepsilon > 0$ the set $\{x\colon \exists \gamma \in \Gamma\ \gamma \cdot x \neq x \text{ and } d(\gamma \cdot x, x) < \varepsilon\}$ is dense, and this follows immediately from our assumption on the action. □

For the next two lemmas and proposition, we let τ denote a group topology on $[R]$ which is Hausdorff and such that $([R], \tau)$ is a Baire space.

Lemma 2.2. *For any nonempty clopen subset U of X, the set $\Delta_U = \{g \in [R] : g_{\restriction U} = id_{\restriction U}\}$ is τ-closed.*

Proof. We claim that $g \in [R]$ coincides with the identity on U if, and only if, $gh = hg$ for any $h \in [R]$ whose support is contained in U. Each set $\{g\colon gh = hg\}$ is closed since τ is a Hausdorff group topology, hence if we prove this claim we can conclude that Δ_U is an intersection of closed subsets of $[R]$ so Δ_U is closed.

Now to the proof of the claim: one inclusion is obvious; to see the converse, assume that there exists $x \in U$ such that $x \neq g(x)$. This gives us a clopen subset

W of U such that W and $g(W)$ are disjoint. By assumption, there exists a clopen subset V of W and an involution $h \in [R]$ with support contained in W (hence in U) and such that V and $h(V)$ are disjoint subsets of W. Then $hg(V) = g(V)$ is disjoint from $gh(V)$, showing that g and h do not commute. □

Lemma 2.3. *For any clopen subset U of X, the set $\Sigma_U = \{g \in [R] : g(U) = U\}$ is τ-closed.*

Proof. We may assume that U is nonempty; also, since τ is a group topology and $(g(U) = U) \Leftrightarrow (g(U) \subseteq U$ and $g^{-1}(U) \subseteq U)$, we only need to show that $\{g \in [R] : g(U) \subseteq U\} = \Sigma'_U$ is closed in $[R]$. To that end, one can use the same strategy as above: this time, we claim that $g \in \Sigma'_U$ if and only if, for any h which coincides with the identity on U, $g^{-1}hg$ coincides with the identity on U. Proving this will show that Σ'_U is an intersection of closed sets (by Lemma 2.2), which gives the result.

Again, one inclusion is obvious; to see the converse, we assume that $g(U)$ is not contained in U. Then there exists a nonempty clopen subset W of U such that $g(W) \cap U = \emptyset$. One can find a nontrivial involution h with support in $g(W)$. This gives us a nonempty clopen $V \subseteq U$ such that $hg(V)$ and $g(V)$ are disjoint, hence $g^{-1}hg$ does not coincide with the identity on U. □

Proposition 2.4. *The set $\{g \in [R] : g(x) = x\}$ is τ-clopen for all $x \in X$.*

Proof. The result of Lemma 2.3 shows that the natural inclusion map from $([R], \tau)$ to $\mathrm{Homeo}(X)$ is Borel. Since τ is assumed to be Baire (this is the first time we are using that assumption) and $\mathrm{Homeo}(X)$ is separable, the inclusion map must be continuous, showing that each set $\{g \in [R] : g(x) = x\}$ is τ-closed. Now, fix $x \in X$ and let H denote the permutation group of the countable set $\Gamma \cdot x$, endowed with its permutation group topology, which is the topology of pointwise convergence on $\Gamma \cdot x$ considered as a discrete set.

Since $\{g \in [R] : g(\gamma_1 \cdot x) = \gamma_2 \cdot x\}$ is closed for all $\gamma_1, \gamma_2 \in \Gamma$, we see that the natural homomorphism from $([R], \tau)$ to H (given by $g \mapsto (\gamma \cdot x \mapsto g(\gamma \cdot x))$ is Borel. Thus this homomorphism must be continuous, and $\{g \in [R] : g(x) = x\}$ is τ-clopen. □

Let us sum up what we just proved.

Theorem 2.5. *Let Γ be a countable group acting by homeomorphisms on a Cantor space X. Assume that the restriction of the associated equivalence relation to a nonempty open subset of X is never trivial.*

Then any Hausdorff, Baire group topology on $[R]$ must extend the topology of pointwise convergence for the discrete topology on X. Consequently, there is no second countable, Hausdorff, Baire group topology on $[R]$.

Proof. The statement in the first sentence corresponds exactly to the result of Proposition 2.4. To see why the second statement holds, let us proceed by contradiction and assume that there exists a second countable, Hausdorff, Baire group topology on $[R]$.

We recall the result of Lemma 2.1: the set Ω made up of all x such that x is an accumulation point of $\Gamma \cdot x$ is dense G_δ in X, thus in particular uncountable. Assuming τ is second countable, the Lindelöff property implies that there exists a sequence $(x_i)_{i<\omega}$ of elements of Ω such that

$$\{g \in [R] : \exists x \in \Omega \ g(x) = x\} = \bigcup_{i<\omega} \{g \in [R] : g(x_i) = x_i\}.$$

However, we claim that, for any countable subset $\{x_i\}_{i<\omega}$ of Ω and any $x \in \Omega \setminus \{x_i\}_{i<\omega}$, there exists $g \in [R]$ such that $g(x) = x$ and $g(x_i) \neq x_i$ for all i ; granting this, we obtain the desired contraditcion.

To conclude the proof, we briefly explain why the claim holds. Using the fact that each x_i is an accumulation point of $\Gamma \cdot x_i$, one can construct inductively a sequence of clopen sets U_j and elements γ_j of Γ with the following properties:

- for all i $x_i \in \bigcup_{j\leq i}(U_j \cup \gamma_j U_j)$;
- for all j, the diameter of $U_j \cup \gamma_j U_j$ is less than 2^{-j}, and $\gamma_j U_j \cap U_j = \emptyset$;
- for all $j \neq k$, $(U_j \cup \gamma_j U_j) \cap (U_k \cup \gamma_k U_k) = \emptyset$;
- for all j $x \notin \gamma_j U_j \cup U_j$.

One can then define a bijection g of X by setting

$$g(y) = \begin{cases} \gamma_j(y) & \text{if } y \in U_j \text{ for some } j, \\ \gamma_j^{-1}(y) & \text{if } y \in \gamma_j(U_j) \text{ for some } j, \\ y & \text{otherwise.} \end{cases}$$

The fact that the diameter of $U_j \cup \gamma_j U_j$ vanishes ensures that g is continuous, so g belongs to $[R]$ and satisfies $g(x) = x$, $g(x_i) \neq x_i$ for all i. □

Remark 2.6. It is not clear whether the Hausdorfness assumption is really needed: if $[R]$ is a simple group, then a non-Hausdorff group topology is necessarily the coarse topology; indeed, the elements that cannot be separated from 1 by an open subset form a normal subgroup of $[R]$. So, if $[R]$ is simple, then the above result says that the unique Baire, second countable group topology on $[R]$ is the coarse topology. However, it is an open question whether $[R]$ is simple, even in the case when R is induced by a minimal action of $\mathbb{Z}$.

The techniques of this section are close to those employed by Rosendal in [**Ros05**], but it seems that his results do not cover the case studied here. It was pointed out by the referee that the subgroups which appear in Lemma 2.2 were introduced by Dye [**Dye63**] in the measurable context, and that he called them *local subgroups*.

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3. Borel complexity of the full group of a minimal homeomorphism

The following question was suggested to us by T. Tsankov: what is the complexity (in the sense of descriptive set theory) of the full group of an equivalence relation induced by a minimal action of a countable group on the Cantor space? We answer that question for $\Gamma = \mathbb{Z}$. Below we use standard results and notations of descriptive set-theory, borrowed from Kechris's book [**Kec95**].

In particular, we recall that if A is a countable set then a *tree* on A is a subset T of the set $A^{<\omega}$ of finite sequences of elements of A, closed under taking initial segments (see [**Kec95**, §2] for information on descriptive-set-theoretic trees and a detailed exposition of related notions). The set $\mathcal{T}$ of trees on A can be endowed with a topology that turns it into a Cantor space, by setting as basic open sets all sets of the form

$$\{T \in \mathcal{T} : \forall s \in S\ s \in T \text{ and } \forall s' \in S'\ s' \notin T\},$$

where S and S' are finite subsets of $A^{<\omega}$.

When $s \in A^{<\omega}$ and $a \in A$, we denote by $s \frown a$ the sequence of length length$(s)+1$ obtained by appending a to s.

A tree T is said to be *well-founded* if it has no infinite branches; in this case one can define inductively the *rank* of an element s of $A^{<\omega}$ by setting

$$\rho_T(s) = \sup\{\rho_T(s \frown a) + 1 : s \frown a \in T\}.$$

In particular, elements not in T and terminal nodes in T all have rank 0; then one defines the rank $\rho(T)$ of T as being equal to the supremum of all $\rho_T(s)+1$ for $s \in A^{<\omega}$; when T is nonempty, this supremum is equal to $\rho_T(\emptyset)+1$.

Having said all this, we can begin to work, which we do by pointing out the obvious: whenever Γ is a countable group acting by homeomomorphisms on a Cantor space X, the full group of the associated equivalence relation R is a co-analytic subset of the Polish group Homeo(X). This is simply due to the fact that each set $\{(g,x) \in \mathrm{Homeo}(X) \times X : g(x) = \gamma \cdot x\}$ is closed, and for all $g \in \mathrm{Homeo}(X)$ one has

$$g \in [R] \Leftrightarrow \forall x \in X \exists \gamma \in \Gamma\ g(x) = \gamma \cdot x.$$

The above line shows that $[R]$ is the co-projection of an F_σ subset of Homeo$(X) \times X$, hence is co-analytic.

One might expect that the descriptive complexity of $[R]$ is not that high in the Borel hierarchy. For instance, in the measure-preserving context, full groups are always Borel of very low complexity: it is shown in [**Wei05**] that the full group of an aperiodic, probability-measure-preserving equivalence relation is a $\mathbf{\Pi}^0_3$-complete subset of the group of measure-preserving automorphisms (that is, in that case the full group is a countable intersection of countable unions of closed sets). Perhaps surprisingly, it turns out that full groups of minimal homeomorphisms are not Borel.

Below, we denote by φ a minimal homeomorphism of a Cantor space X and recall that $[\varphi]$ denotes the full group of the associated equivalence relation. We also denote by $\mathcal{T}$ the space of all trees on Clop(X), endowed with the topology discussed above.

Definition 3.1. To each $g \in \mathrm{Homeo}(X)$ we associate a tree T_g on Clop(X) as follows: for any sequence $(U_0, \dots, U_n)$ of clopen sets, $(U_0, \dots, U_n)$ belongs to T_g iff each U_j is nonempty, $U_{j+1} \subseteq U_j$ for all $j \in \{0, \dots, n-1\}$ and $g(x) \neq \varphi^{\pm j}(x)$ for all $x \in U_j$.

Lemma 3.2. *The map $g \mapsto T_g$ is a Borel mapping from* Homeo(X) *to $\mathcal{T}$. For any $g \in$* Homeo(X)*, g belongs to $[\varphi]$ if, and only if, T_g is well-founded.*

Proof. We need to prove that for any finite sequence of nonempty clopen subsets $(U_0, \dots, U_n)$ the set $\{g \in \mathrm{Homeo}(X) : (U_0, \dots, U_n) \in T_g\}$ is Borel. For this, it is enough to show that for any nonempty clopen subset U of X, the set $\{g : \forall x \in$

$U\ g(x) \neq x\}$ is Borel. The complement of this set is $\{g\colon \exists x \in U\ g(x) = x\}$, which is closed because U is clopen in X and thus compact: if g_n is a sequence of homeomorphisms of X such that for all n there exists $x_n \in U$ such that $g_n(x_n) = x_n$, and g_n converges to g in Homeo(X), then up to some extraction we can assume that x_n converges to $x \in U$; the distance from $g(x_n)$ to $g_n(x_n)$ must converge to 0, so $g_n(x_n)$ converges to $g(x)$, showing that $g(x) = x$. This concludes the proof that $g \mapsto T_g$ is Borel.

Next we fix $g \in$ Homeo(X). We first assume that g does not belong to $[\varphi]$, i.e., there exists $x \in X$ such that $g(x) \neq \varphi^n(x)$ for all $n \in \mathbb{Z}$. Then, using the continuity of g and φ, one can build by induction a decreasing sequence of clopen neighborhoods U_i of x such that for all i and all $y \in U_i$ one has $g(y) \neq \varphi^{\pm i}(y)$, which yields an infinite branch of T_g. Conversely, assume that T_g is not well-founded and let $(U_i)_{i<\omega}$ be an infinite branch of T_g. Then $F = \bigcap_{i<\omega} U_i$ is nonempty, and for all $x \in F$ $g(x)$ is different from $\varphi^n(x)$ for all $n \in \mathbb{Z}$, showing that g does not belong to $[\varphi]$. □

If $[\varphi]$ were Borel, the set $\mathcal{T}_\varphi = \{T_g\colon g \in [\varphi]\}$ would be an analytic subset of $\mathcal{T}$, hence the boundedness principle for coanalytic ranks (see [**Kec95**, Theorem 35.23]) would imply the existence of a countable ordinal α such that the rank of any element of $\mathcal{T}_\varphi$ is less than α. We want to prove that it is not the case, so we need to produce elements of $[\varphi]$ such that the associated tree has arbitrarily large rank. Let us introduce some notation in order to simplify the work ahead.

Definition 3.3. For any $g \in [\varphi]$ we let $\rho(g)$ denote the rank of T_g. For any finite sequence of clopen sets $(U_0,\dots,U_n)$, we let $\rho_g(U_0,\dots,U_n)$ denote the rank of $(U_0,\dots,U_n)$ with regard to the tree T_g. If α and β are ordinals, we write $\alpha \sim \beta$ to express that there are only finitely many ordinals between them; we write $\alpha \gtrsim \beta$ when $\alpha \geq \beta$ or $\alpha \sim \beta$.

An encouraging sign that our introduction of ρ is a good way to capture information about elements of $[\varphi]$ is that the topological full group $[[\varphi]]$ is exactly made up of all $g \in$ Homeo(X) such that $\rho(g) < \omega$.

In order to build elements of $[\varphi]$ such that the associated tree has arbitrarily large rank, the following observation will be crucial.

Lemma 3.4. *Let g be an element of $[\varphi]$, and assume that $\rho(g) \geq \omega$. Then for any $h \in [[\varphi]]$ one has $\rho(g) \sim \rho(hg)$.*

Since $\rho(g) = \rho(g^{-1})$, the above lemma also holds true when multiplying on the right by an element of the topological full group.

Proof. For $k < \omega$, let

$$T_g^k = \{(U_k,\dots,U_n) : (U_0,\dots,U_n) \in T_g \text{ for some } U_0,\dots,U_{k-1}\}.$$

If $h \in [[\varphi]]$, then for some $k < \omega$ and for all $x \in X$ there is j such that $|j| \leq k$ and $h(x) = \varphi^j(x)$. So if $g(x) \neq \varphi^j(x)$ for all $|j| \leq n$ but $hg(x) = \varphi^m(x)$ for some m, then $|m| > n - k$. This implies that $T_g^k \subseteq T_{hg}$. Since $\rho(T_g^k) \geq \rho(T_g) - k$, we get $\rho(T_{hg}) \geq \rho(T_g) - k$, and similarly $\rho(T_g) = \rho(T_{h^{-1}hg}) \geq \rho(T_{hg}) - k$, proving the claim. □

When U is a subset of X and g belongs to $[\varphi]$, we set

$$n(g,U) = \min(\{|k|\colon \exists x \in U\ g(x) = \varphi^k(x)\})$$

Lemma 3.5. *Let α be an infinite ordinal belonging to $\{\rho(g)\colon g \in [\varphi]\}$, and N be an integer. For any nonempty clopen $U \subseteq X$, there exists $h \in [\varphi]$ with support S contained in U (in particular, $h(U) = U$), such that $\rho(h) \gtrsim \alpha$ and $n(h,S) > N$.*

Proof. Pick $g \in [\varphi]$ such that $\rho(g) = \alpha$ is infinite and fix a nonempty clopen $U \subseteq X$ and an integer N. Using compactness of the space of φ-invariant probability measures and the fact that they are all atomless, one can find a nonempty clopen $\tilde{U} \subseteq U$ such that $(2N+2)\mu(\tilde{U}) < \mu(U)$ for any φ-invariant measure μ.

Since φ is minimal, there exist $i_1, \dots, i_n$ such that $X = \bigcup_{j=1}^n \varphi^{i_j}(\tilde{U})$. For all $j \in \{1, \dots, n\}$, denote $U_j = \varphi^{i_j}(\tilde{U})$, and consider the tree T_j defined by

$$(V_0, \dots, V_n) \in T_j \Leftrightarrow (V_0, \dots, V_n) \in T_g \text{ and } V_0 \subseteq U_j.$$

Denote by ρ_j the rank function associated to the well-founded tree T_j, and by $\rho(T_j)$ the rank of T_j. For any finite sequence $(V_0, \dots, V_k)$ of clopen subsets of X, we have

$$\Big(\forall j \in \{1, \dots, n\}\ \rho_j(V_0 \cap U_j, \dots, V_k \cap U_j) = 0\Big) \Rightarrow \rho_g(V_0, \dots, V_k) = 0.$$

From this, we see by transfinite induction that $\rho(g) = \max\{\rho(T_j)\colon j \in \{1, \dots, n\}\}$, so there exists j such that $\rho(T_j) = \alpha$. Fix such a j; any element of $[\varphi]$ coinciding with g on U_j must have rank larger than α.

Applying the Glasner–Weiss result recalled as Theorem 1.7, we can find $f \in [[\varphi]]$ such that $f(U_j) = W \subseteq U$. We also have

$$\mu(g(U_j)) < \mu\left(U \setminus \bigcup_{i=-N}^{N} \varphi^i(W)\right)$$

for any φ-invariant μ, so applying Theorem 1.7 again we can find $k \in [[\varphi]]$ such that $k(g(U_j))$ is contained in U and disjoint from $\bigcup_{i=-N}^{N} \varphi^i(W)$. Now, let h be equal to kgf^{-1} on W, to $fg^{-1}k^{-1}$ on $kgf^{-1}(W)$, and to the identity elsewhere. We set $S = W \cup h(W)$. Using the fact that f, k belong to $[[\varphi]]$ and Lemma 3.4, we see that $\rho(h) \gtrsim \rho(g)$. The construction ensures that $h(W)$ is disjoint from $\bigcup_{i=-N}^{N} \varphi^i(W)$, so $n(h,W) > N$; since h is an involution, $n(h,h(W)) = n(h,W)$ is also strictly larger than N. This ensures that $n(h,S) > N$ and all the desired conditions are satisfied. □

Theorem 3.6. *The full group of a minimal homeomorphism of a Cantor space X is a coanalytic non-Borel subset of* Homeo(X).

Proof. Let φ be a minimal homeomorphism of a Cantor space X. We explain how to produce elements of $[\varphi]$ with arbitrarily large rank. To that end, we fix for the remainder of the proof a compatible distance on X, an element g of $[\varphi]$ and a countable family (V_i) of nonempty disjoint clopen subets of X with the following property: the tree generated by terminal nodes $(U_0, \dots, U_n)$ of T_g such that $U_n = V_i$ for some i has rank at least ω. Note that the value n associated to such a terminal node is determined by i: we must have $g = \varphi^{\pm(n+1)}$ on V_i. We note $n = N_i$, and our hypothesis is that (N_i) is unbounded.

We then pick an infinite sequence $(W_i)_{i<\omega}$ of nonempty clopen subsets of X such that the diameter of each W_i is less than 2^{-i} and $W_i \subseteq V_i$ for all i. Now, let g_i be any sequence of elements of $[\varphi]$ of infinite rank; using Lemma 3.5, we can find elements h_i of $[\varphi]$ with support S_i contained in W_i and such that $\rho(h_i) \gtrsim \rho(g_i)$. We shall also ask that $n(h_i, S_i) > 2N_i + 1$. We then define $h\colon X \to X$ by setting

$$h(x) = \begin{cases} gh_i(x) & \text{if } x \text{ belongs to some } W_i \\ g(x) & \text{otherwise} \end{cases}.$$

Note that, since the sets W_i are pairwise disjoint, h is well-defined. We next show that h is continuous. Let (x_i) be a sequence of elements of X converging to some $x \in X$. If x belongs to W_j for some j then $x_i \in W_j$ for i large enough and continuity of g and h_j ensure that $h(x_i)$ converges to $h(x)$. So we may assume that x does not belong to $\cup W_j$; in that case $h(x) = g(x)$ and since g is continuous we may also assume that x_i belongs to some W_{j_i} for all i. Each W_i is clopen, so we must have $j_i \to +\infty$, hence the diameter of W_{j_i} converges to 0. Since g is uniformly continuous, the diameter of $g(W_{j_i})$ also converges to 0; $h(x_i)$ and $g(x_i)$ both belong to this set, showing that $d(g(x_i), h(x_i))$ converges to 0. Hence $h(x_i)$ converges to $g(x) = h(x)$, proving that h is continuous. The construction also ensures that h is bijective, so h is a homeomorphism of X, and h belongs to $[\varphi]$. The definition of h and the argument of Lemma 3.4 (using the fact that $g = \varphi^{\pm(N_i+1)}$ on W_i) ensure that $\rho(h) \gtrsim \rho(h_i)$ for all i. If we have $\rho(g_i) \geq \alpha_i$ for limit ordinals α_i, we then obtain $\rho(h) \geq \sup \alpha_i$.

Now let α be a countable limit ordinal, and f an element of $[\varphi]$ such that $\rho(f) \geq \alpha$; we now explain how to produce an element of $[\varphi]$ with rank greater than $\alpha + \omega$, which will conclude the proof. The element in question, again denoted by h, is obtained by applying the construction above with $g_i = f$ for all i. Let $(U_0, \dots, U_n)$, $U_0 = U_n = V_i$, be a terminal node of T_g, so in particular $g = \varphi^{\pm(n+1)}$ on U_0, $n = N_i$. Since $n(h_i, S_i) > 2n + 1$, the only way to have $gh_i(x) = \varphi^m(x)$ for $x \in U_0$ is with $|m| > n$, and this says that $(U_0, \dots, U_n)$ belongs to T_{gh_i} (and to T_h) as well. Once we know this, the argument of Lemma 3.4 implies that $\rho_h(U_0, \dots, U_n) \gtrsim \rho(h_i) \geq \alpha$: if $(U'_0, \dots, U'_k) \in T_{h_i}$, $U'_0 \subseteq U_n$, $2n + 1 < k$, then $(U_0, \dots, U_n, U'_{2n+2}, \dots, U'_k) \in T_h$. We conclude that $\rho(h) \geq \alpha + n = \alpha + N_i$. Since this is true for every i, we get $\rho(h) \geq \alpha + \omega$, as expected. □

Remark 3.7. Given the result we just proved, it seems likely that the full group of an equivalence relation induced by a minimal action of a countable group Γ on a Cantor space is never Borel. The above argument may be adapted in large part, but it is not clear to the authors how one can modify Lemma 3.5 in a context where Theorem 1.7 does not hold.

One can nevertheless note that the above result extends to relations induced by actions of $\mathbb{Z}^d$ for all integers d, though this extension of the result is not really meaningful, indeed it is trivial once one knows that full groups associated to minimal $\mathbb{Z}^d$-actions are the same as full groups associated to minimal $\mathbb{Z}$-actions, a powerful result proved in [**GMPS10**].

4. Closures of full groups

We saw above that there does not exist a Hausdorff, Baire group topology on the full group of a minimal homeomorphism φ of a Cantor space X. This precludes the usage of Baire category methods; however, the same cannot be said of the closure of $[\varphi]$, which is of course a Polish group since it is a closed subgroup of Homeo(X) (and the arguments of Section 2 show that the topology induced by that of Homeo(X) is the unique Polish topology on the closure of $[\varphi]$ which is compatible with the group operations). As pointed out in [**GPS99**], the closure of $[\varphi]$ is easy to describe thanks to Theorem 1.7: letting $\mathcal{M}_\varphi$ denote the (compact, convex) set of all φ-invariant probability measures, we have

$$\overline{[\varphi]} = \{g \in \mathrm{Homeo}(X) \colon \forall \mu \in \mathcal{M}_\varphi \ g_*\mu = \mu\}.$$

Notation. Below we denote the closure of the full group of φ in Homeo(X) by G_φ.

This group is relevant when studying topological orbit equivalence of minimal homeomorphisms, because of a theorem of Giordano–Putnam–Skau which implies the following result.

Proposition 4.1. *Let φ_1, φ_2 be two minimal homeomorphisms of a Cantor space X; assume that G_{φ_1} and G_{φ_2} are isomorphic (as abstract groups). Then φ_1 and φ_2 are orbit equivalent.*

Proof. Assume that $\Phi \colon G_{\varphi_1} \to G_{\varphi_2}$ is a group isomorphism. First, the usual reconstruction techniques (see e.g. [**Med11**]) show that there exists a homeomorphism $h \in \mathrm{Homeo}(X)$ such that $\Phi(g) = hgh^{-1}$ for all $g \in G_{\varphi_1}$.

So we have that

$$\forall g \in \mathrm{Homeo}(X) \quad g \in G_{\varphi_1} \Leftrightarrow hgh^{-1} \in G_{\varphi_2}.$$

Since $\mathcal{M}_{\varphi_i}$ is equal to the set of measures which are invariant under translation by elements of G_{φ_i} (for $i = 1, 2$), this means that

$$\forall \mu \quad \mu \in \mathcal{M}_{\varphi_1} \Leftrightarrow h_*\mu \in \mathcal{M}_{\varphi_2}.$$

Then ([**GPS95**, Theorem 2.2(iii)]) implies that φ_1 and φ_2 are orbit equivalent. □

Of course, the converse of the above statement is true: if φ_1 and φ_2 are orbit equivalent, then their full groups are conjugated inside Homeo(X), so the closures of the full groups are also conjugated. However, the statement above is only valid a priori for actions of $\mathbb{Z}$: while it is true that for any minimal actions of countable groups an isomorphism between the closures of the respective full groups must be implemented by an homeomorphism of X, there is no reason why this homeomorphism would be sufficient to prove that the full groups themselves are isomorphic. Indeed, using ideas from ergodic theory, one can see that there are plenty of examples of actions of countable groups Γ_1, Γ_2 on a Cantor space X such that $\overline{[R_{\Gamma_1}]} = \overline{[R_{\Gamma_2}]}$, yet the two associated relations are not orbit equivalent. The example below was explained to us by D. Gaboriau.

Proposition 4.2. *There exists an action of $\mathbb{Z}$ and an action of the free group F_3 on three generators on a Cantor space X, such that the closures of the full groups of the two actions coincide, yet the relations are not orbit equivalent.*

Proof. Let $\mathbb{Z}$ act on the Cantor space $\{0,1\}^\omega$ via the usual odometer map. Consider the free group F_2 on two generators acting by the Bernoulli shift on $\{0,1\}^{F_2}$; using a bijection between ω and F_2, one can see this as an action of F_2 on $\{0,1\}^\omega = X$. Let $F_3 = F_2 * \mathbb{Z}$ act on X, where the action of F_2 is the Bernoulli shift and the action of $\mathbb{Z}$ is via the odometer map. Then the actions of $\mathbb{Z}$ and F_3 on $\{0,1\}^\omega$ both preserve the $(1/2,1/2)$-Bernoulli measure μ on 2^ω; since the odometer is uniquely ergodic, we see that for both actions the closure of the full group is equal to the set of all homeomorphisms which preserve μ. Yet, there cannot even exist a μ-preserving bijection h of X such that, for μ-almost all $x, x' \in X$, one has

$$(xR_{\mathbb{Z}}x') \Leftrightarrow (h(x)R_{F_3}h(x')).$$

Indeed, the relation induced by the action of $\mathbb{Z}$ is hyperfinite, while the relation induced by the action of F_3 contains a subrelation which is induced by a free action of F_2, so it cannot be hyperfinite (see for instance [**Kec10**] for information on probability-measure-preserving group actions and the properties we use here without details). Since a homeomorphism realizing an orbit equivalence between $R_{\mathbb{Z}}$ and R_{F_3} would have to preserve μ, we see that while the closures of both full groups coincide, the associated relations cannot be orbit equivalent. □

In view of this, the following question might be interesting.

Question 4.3. Let Γ_1, Γ_2 be two countable *amenable* groups acting minimally on a Cantor space X. Assume that the closures of the corresponding full groups are isomorphic as abstract groups. Must the two actions be orbit equivalent?

If one knew that any minimal action of a countable amenable group is orbit equivalent to a $\mathbb{Z}$-action then the answer to the question above would be positive; the result of [**GMPS10**] mentioned at the end of the previous section implies that the above question has a positive answer when Γ_1, Γ_2 are finitely generated free abelian groups.

We have already pointed out that it is unknown whether the full group of a minimal homeomorphism φ is simple. This reduces to deciding whether the full group coincides with its derived subgroup. Indeed, it is proved in [**BM08**, Theorem 3.4] that any normal subgroup of $[\varphi]$ contains its derived subgroup; the same is true for G_φ, as can be seen by following the proof of [**BM08**].

Unfortunately, it seems to be hard in general to decide which elements of $[\varphi]$ are products of commutators (though one might conjecture that every element has this property; partial results in this direction can be found in [**BM08**]). The use of Baire category methods might make things simpler in the case of G_φ, especially in view of the following folklore result.

Proposition 4.4. *Let G be a Polish group; assume that G has a comeager conjugacy class. Then every element of G is a commutator.*

Proof. Assume that Ω is a comeager conjugacy class in G, and let $g \in G$. The intersection $g\Omega \cap \Omega$ is nonempty; picking an element g_0 in this intersection, we see that there exists $k \in G$ such that $gkg_0k^{-1} = g_0$, in other words $g = g_0kg_0^{-1}k^{-1}$. □

It is thus interesting to understand when G_φ has a comeager conjugacy class, even more so because of the following observation.

Proposition 4.5. *Let φ be a minimal homeomorphism of a Cantor space X, and assume that G_φ has a comeager conjugacy class. Then G_φ has the* automatic continuity property, *i.e., any homomorphism from G_φ to a separable topological group is continuous.*

Proof. The argument in [**RS07**, Theorem 12] adapts straightforwardly. □

In the next section, we will discuss in more detail the problem of existence of comeager conjugacy classes in G_φ in the particular case when φ is uniquely ergodic, recovering in particular a result of Akin that provides many examples of this phenomenon.

In the first version of this article, we proved a weaker version of the result below, which worked only in the case when $n = 1$ and φ is a uniquely ergodic homeomorphism (see the next section); we are grateful to K. Medynets for pointing out to us the following stronger result.

Theorem 4.6 (Grigorchuk–Medynets [**GM**]). *Let φ be a minimal homeomorphism of a Cantor space X. Then $\{(g_1,\dots,g_n)\colon (g_1,\dots,g_n)$ generates a finite group$\}$ is dense in $[\varphi]^n$ for all n.*

Proof. Since this statement is not explicitly written down in [**GM**] (though it is very close to Theorem 4.7 there), we describe the argument for the reader's convenience. We simply prove that the set of elements of finite order is dense in $[\varphi]$; the proof of the general case is an easy consequence of this argument.

We may assume, replacing φ by a minimal homeomorphism which is orbit equivalent to it (which does not affect the full group) that the topological full group $[[\varphi]]$ is dense in $[\varphi]$. This fact is pointed out in [**BK02**, Theorem 1.6], and follows from a combination of [**GW95**, Theorem 2.2] and [**GPS99**, Lemma 3.3]. Under this assumption, we only need to prove that the set of elements of finite order is dense in $[[\varphi]]$.

We fix $\gamma \in [[\varphi]]$. We let $D_i = \{x\colon \gamma(x) = \varphi^i(x)\}$, and $E_k = \{x\colon \gamma^{-1}(x) = \varphi^k(x)\}$. The sets D_i form a clopen partition of X, as do the sets E_k ; we let $j_\gamma\colon X \to \mathbb{Z}$ (resp. k_γ) be the continuous function defined by $j_\gamma(x) = j$ iff $x \in D_j$ (resp. $k_\gamma(x) = k$ iff $x \in E_k$). We also pick K such that $D_j = \emptyset = E_j$ for all $|j| > K$, fix a compatible distance d on X for the remainder on the proof, and let $\delta > 0$ be such that $d(D_i, D_j) > \delta$ for all nonempty $D_i \neq D_j$, and $d(E_i, E_j) > \delta$ for all nonempty $E_i \neq E_j$.

Recall that a *Kakutani–Rokhlin partition* associated to φ is a clopen partition of X of the form $\{\varphi^i(B_n)\colon 0 \le n \le N, 0 \le i \le h_n - 1\}$. The *base* of the partition is $B = \bigcup_{n=0}^N B_n$, while its *top* is $T = \bigcup_{n=0}^N \varphi^{h_n-1}B_n$. Note that $\varphi(T) = B$. For all i, we set

$$Y_i = \bigcup_{n=0}^{N} \varphi^i(B_n) \quad \text{and} \quad Z_i = \bigcup_{n=0}^{N} \varphi^{-i}(B_n)$$

Kakutani–Rokhlin partitions exist because φ is minimal; actually, one can use minimality to ensure that the following conditions are satisfied (see for instance [**GM**] for a discussion of these partitions and references):

(1) The functions j_γ and k_γ are constant on each atom of the partition.
(2) $\min\{h_n\colon 0 \le n \le N\} \ge 2K+2$.
(3) The diameter of each Y_i and each Z_i is less than δ for all $i \in \{0,\dots K\}$ (this can be ensured because of the uniform continuity of φ and φ^{-1}, and the fact that one can build Kakutani–Rokhlin partitions whose base has arbitrarily small diameter).

Fix such a partition. Note that the third condition ensures that j_γ and k_γ are constant on each Y_i, Z_i ($|i| \le K$), and the second condition guarantees that the sets $(Y_i)_{0\le i\le K}$, $(Z_i)_{1\le i\le K}$ are pairwise disjoint. We now define $P \in [[\varphi]]$ as follows. For all n, and all $i \in \{0,\dots,h_{n-1}\}$, let $j_\gamma(n,i)$ be the value of j_γ on $\varphi^i(B_n) = B_{n,i}$.

- If $0 \le j_\gamma(n,i)+i \le h_{n-1}$, then $P(x) = \gamma(x)$ for all $x \in B_{n,i}$.
- If $j_\gamma(n,i)+i < 0$, then necessarily $i < K$, so $B_{n,i} = Y_i$, and since j_γ is constant on Y_i one has $\gamma(Y_i) \subseteq Z_l$ for some $1 \le l \le K$. The inclusion must be an equality since k_γ is constant on Z_l. Then set $P(x) = \varphi^{-i+h_n-l}(x)$ for all $x \in B_{n,i}$.
- If $j_\gamma(n,i)+i \ge h_n$, then one must similarly have $B_{n,i} = Z_j$ for some $1 \le j \le K$, $\gamma(Z_j) = Y_l$ for some $0 \le l < K$, and one can set $P(x) = \varphi^{-i+l}(x)$ for all $x \in B_{n,i}$.

It is straightforward to check that P has finite order; also, the fact that the diameter of each Y_i, Z_i for $|i| \le K$ is small ensures that for all x one has both $d(P(x),\gamma(x)) \le \delta$ and $d(P^{-1}(x),\gamma(x)) \le \delta$. Thus γ belongs to the closure of the set of elements of finite order, which concludes the proof. □

Actually, as pointed out by K. Medynets, this argument shows that $[\varphi]$ contains a dense locally finite subgroup (the group of all elements which preserve a positive semi-orbit; see the remarks in [**GM**, §5]). We will not need this fact so do not give any details.

Even though we do not know whether G_φ or $[\varphi]$ are simple in general, we can use Theorem 4.6 to prove that these groups do not have any non-trivial closed normal subgroups.

Theorem 4.7. *Let φ be a minimal homeomorphism of a Cantor space X. Then G_φ and $[\varphi]$ are topologically simple.*

Proof. We show that the derived subgroup of $[\varphi]$ is dense, which implies the simplicity of both groups by the result of Bezuglyi–Medynets recalled in the paragraph before Proposition 4.4. Say that $g \in [\varphi]$ is a *p-cycle* on a clopen set U if U is the support of g, $g^p = 1$ and there exists a clopen A such that $U = A \sqcup g(A) \dots \sqcup g^{p-1}A$ (this is the same as saying that the g-orbit of every element of U has cardinality p, and every element outside U is fixed by g). Theorem 4.6 implies that products of cycles are dense in $[\varphi]$, so it is enough for our purposes to show that p-cycles are products of commutators for any integer p.

Let g be a p-cycle on a clopen U, with $U = \sqcup_{i=0}^{p-1} g^i(A)$. Given a permutation σ belonging to the permutation group S_p on p elements, we denote by g_σ the element of $[\varphi]$ defined by setting $g_\sigma(x) = x$ for all x outside U and

$$\forall i \in \{0,\dots,p-1\}\ \forall x \in g^i(A) \quad g_\sigma(x) = g^{\sigma(i)-i}(x).$$

The map $\sigma \mapsto g_\sigma$ is a homomorphism from S_p to $[\varphi]$. Since the commutator subgroup of S_p is the alternating subgroup A_p, we thus see that whenever σ belongs to A_p g_σ is a product of commutators. In particular, g has this property if p is odd. If p is even, let τ be the transposition of S_p which exchanges 0 and 1. Then [**BM08**, Corollary 4.8] tells us that g_τ is a product of 10 commutators in $[\varphi]$; since $gg_\tau = g_\sigma$ for some $\sigma \in A_p$, g is also a product of commutators. □

It was pointed out by the referee that, since one can assume that $[[\varphi]]$ is dense in $[\varphi]$, the density of the derived subgroup of $[\varphi]$ directly follows from the fact that that $[[\varphi]]$ is contained in the derived subgroup of φ, a fact which is mentioned without proof on [**BM08**, p. 419].

Let us mention another reason why we think it might be interesting to further study the properties of closures of full groups.

Proposition 4.8. *Let φ be a minimal homeomorphism of a Cantor space X. Then G_φ is an amenable Polish group.*

Proof. By Theorem 4.6 there exists an increasing sequence of compact subgroups of G_φ whose union is dense in G_φ (see [**KR07**, Proposition 6.4]), which must then be amenable.

The result would also follow immediately from the stronger fact that G_φ actually contains a dense locally finite subgroup.

□

This fact is particularly interesting in view of a question of Angel–Kechris–Lyons [**AKL12**, Question 15.1] asking whether, whenever an amenable Polish group has a metrizable universal minimal flow, the universal minimal flow is uniquely ergodic. A positive answer to the following problem would then show that the answer to Angel–Kechris–Lyons' question is negative.

Question 4.9. Let φ be a minimal homeomorphism of a Cantor space X. Is the universal minimal flow of G_φ metrizable?

Remark 4.10. Proving that there exists *one* minimal homeomorphism φ which is not uniquely ergodic, yet has a metrizable universal minimal flow would be enough to answer negatively the question of Angel–Kechris–Lyons mentioned above. In the opposite direction, proving that the universal minimal flows of these groups are not metrizable as soon as the homeomorphism is not uniquely ergodic, and are metrizable otherwise, would point towards a positive answer to their question.

At the moment, this seems out of reach: for instance, when φ is equal to the usual binary odometer, G_φ is just the set of all homeomorphisms of the Cantor space $\{0,1\}^\omega$ which preserve the usual $(1/2,1/2)$-Bernoulli measure on $\{0,1\}^\omega$. Identifying the universal minimal flow of this group is already a very complicated problem, studied in [**KST12**] where a candidate (which is metrizable) is proposed. Thus it seems that the current state of the art does not, for the moment, allow us to hope for an easy answer to our question.

5. Uniquely ergodic homeomorphisms and Fraïssé theory

From now on, we focus on the case when φ is uniquely ergodic, i.e., there is a unique φ-invariant probability measure.

Definition 5.1. A Borel probability measure μ on a Cantor space X is said to be a *good measure* if μ is atomless, has full support, and satisfies the following property: whenever A, B are clopen subsets of X such that $\mu(A) \leq \mu(B)$, there exists a clopen subset C of B such that $\mu(C) = \mu(A)$.

Note that in the definition above the fact that A, B, C are clopen is essential. Good measures are relevant in our context because of the following fact.

Theorem 5.2 ([**Aki05**]; Glasner–Weiss [**GW95**]). *Let μ be a probability measure on a Cantor space X. There exists a minimal homeomorphism φ of X such that $\{\mu\} = \mathcal{M}_\varphi$ if, and only if, μ is a good measure.*

The fact that the goodness of μ is a necessary condition in the result above is due to Glasner–Weiss (it follows directly from the result we recalled as Theorem 1.7); the fact that is is sufficient is due to Akin.

It seems natural to ask the following question, which we only mention in passing.

Question 5.3. Can one give a similar characterization of compact, convex subsets K of the set of probability measures on a Cantor space X for which there exists a minimal homeomorphism φ of X such that K is the set of all φ-invariant measures?

The following invariant of good measures is very useful.

Definition 5.4 (Akin [**Aki05**]). Let μ be a good measure on a Cantor space X. Its *clopen value set* is the set

$$V(\mu) = \{r \in [0,1] \colon r = \mu(A) \text{ for some clopen } A \subseteq X\}.$$

A good measure μ on a Cantor space X is completely characterized by its clopen value set, in the sense that for any two good measures μ, ν on X with the same clopen value set there must exist a homeomorphism g of X such that $g_*\mu = \nu$ (see [**Aki05**, Theorem 2.9]; we discuss a different proof below). If μ is a good measure, then $V(\mu)$ is the intersection of a countable subgroup of $(\mathbb{R}, +)$ and $[0,1]$, contains 1, and is dense in the interval; conversely it is not hard to see that any such set is the clopen value set of some good measure μ. The density condition corresponds to the fact that μ is atomless, and is equivalent (since $1 \in V$) to saying that V is not contained in $\frac{1}{p}\mathbb{Z}$ for any integer p.

Definition 5.5. Given a good measure μ on a Cantor space X, we follow [**Aki05**] and denote by H_μ the set of all homeomorphisms of X which preserve μ. For a countable $V \subset [0,1]$, we denote by $\langle V \rangle$ the intersection of the subroup of $(\mathbb{R}, +)$ generated by $V \cup \{1\}$ with $[0,1]$; we say that V is *group-like* when V is not contained in $\frac{1}{p}\mathbb{Z}$ for any integer p and $V = \langle V \rangle$. In that case, we denote by μ_V the good measure whose clopen value set is equal to V.

Of course, μ_V above is only defined up to isomorphism; since we focus on isomorphism-invariant properties we allow ourselves this small abuse of terminology.

We would like to understand when there exists a comeager conjugacy class in H_μ. Akin [**Aki05**, Theorem 4.17] proved that this holds true whenever $V(\mu)+\mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$, or equivalently whenever any clopen subset can be partitioned into m clopen subsets of equal measure for any integer m. One can check that this also holds true, for instance, when μ is a Bernoulli measure (this is explicitly pointed out in [**KR07**]), and it was our hope that this property would be satisfied by all good measures. Unfortunately, such is not the case, as we will see shortly; since we approach this problem via techniques developed by Kechris–Rosendal [**KR07**], we quickly recall the framework for their results.

A *signature* L is a set $\{\{(f_i,n_i)\}_{i\in I},\{(R_j,m_j)\}_{j\in J},\{c_k\}_{k\in K}\}$ where each f_i is a *function symbol* of arity n_i, each R_j is a *relation symbol* of arity m_j, and each c_k is a *constant symbol*.

Given a signature L, an *L-structure* $\mathcal{M}$ consists of a set M along with a family $\{\{(f_i^{\mathcal{M}})\}_{i\in I},\{R_j^{\mathcal{M}}\}_{j\in J},\{c_k^{\mathcal{M}}\}_{k\in K}\}$ where each $f_i^{\mathcal{M}}$ is a function from M^{k_i} to M, each $R_j^{\mathcal{M}}$ is a subset of M^{m_j}, and each $c_k^{\mathcal{M}}$ is an element of M. In our context, one might for instance consider the signature containing constant symbols 0 and 1, binary functional symbols $\wedge$ and $\vee$, and consider the class of structures in that signature which are Boolean algebras with minimal element (the empty set) corresponding to the constant 0, and maximal element (the whole set) corresponding to the constant 1. It might also simplify matters to add a unary function symbol standing for complementation. Here, we are not concerned merely with Boolean algebras, but with probability algebras. One way to fit those into our framework is to first fix a set $V\subseteq[0,1]$ (the set of values allowed for the probability measure), and add a unary predicate μ_v for each $v\in V$. Then, one can naturally consider the class of probability algebras with measure taking values in V as a class of structures in this signature L_V.

There are natural notions of embedding/isomorphism of L-structures. Assume that we have fixed a countable signature L (that is, each set I, J, K above is at most countable), and that $\mathcal{K}$ is a class of finite L-structures. Then one says that $\mathcal{K}$ is a *Fraïssé class* if it satisfies the four following conditions:

(1) $\mathcal{K}$ countains only countably many structures up to isomorphism, and contains structures of arbitrarily large finite cardinality.
(2) $\mathcal{K}$ is *hereditary*, i.e., if $A\in\mathcal{K}$ and B embeds in $\mathcal{K}$, then $B\in\mathcal{K}$.
(3) $\mathcal{K}$ satisfies the *joint embedding property (JEP)*, that is, any two elements of $\mathcal{K}$ embed in a common element of $\mathcal{K}$.
(4) $\mathcal{K}$ satisfies the *amalgamation property (AP)*, that is, given $A,B,C\in\mathcal{K}$ and embeddings $i\colon A\to B$, $j\colon A\to C$, there exists $D\in\mathcal{K}$ and embeddings $\beta\colon B\to D$ and $\gamma\colon C\to D$ such that $\beta\circ i=\gamma\circ j$.

The point is that, given a Fraïssé class $\mathcal{K}$, there exists a unique (up to isomorphism) L-structure $\mathbb{K}$ whose *age* is $\mathcal{K}$ and which is *homogeneous*. Here, the age of a structure is the class of finite L-structures which embed in it, and a structure $\mathbb{K}$ is homogeneous if any isomorphism between finite substructures of $\mathbb{K}$ extends to an automorphism of $\mathbb{K}$. Conversely, if $\mathbb{K}$ is a countable homogeneous L-structure whose finitely generated substructures are finite, then its age is a Fraïssé class.

For instance, the class of finite Boolean algebras is a Fraïssé class and its limit is the unique countable atomless Boolean algebra, whose Stone space is the Cantor space —so the automorphism group of the limit is just the homeomorphism group of the Cantor space in another guise. Note that the automorphism group of any countable structure $\mathbb{K}$ may be endowed with its permutation group topology, for which a basis of neighborhoods of the neutral element is given by pointwise stabilizers of finite substructures.

Let us fix a good measure μ on a Cantor space X, set $V = V(\mu)$, and consider the probability algebra $(\mathrm{Clop}(X), \mu)$ made up of all clopen subsets of X endowed with the measure μ, in the signature L_V discussed above. Then it follows from Theorems 5.2 and 1.7 that this is a homogeneous structure: any measure-preserving isomorphism between two finite clopen subalgebras of X is induced by a measure-preserving homeomorphism of X, i.e., an automorphism of the Boolean algebra $\mathrm{Clop}(X)$ which preserves the measure μ. Also, an easy induction on the cardinality of finite subalgebras of $(\mathrm{Clop}(X), \mu)$ shows that its age consists of the finite probability algebras whose measure takes values in V. Hence this is a Fraïssé class; note that this implies that two good measures μ_1, μ_2 such that $V(\mu_1) = V(\mu_2)$ must be isomorphic, by the uniqueness of the Fraïssé limit (this was first proved by Akin [**Aki05**]).

Now we can return to the question of existence of dense/comeager conjugacy classes in H_μ, when μ is a good measure. Assume again that $\mathcal{K}$ is a Fraïssé class in some countable signature L, let $\mathbb{K}$ be its Fraïssé limit and let $\mathcal{K}_1$ denote the class of structures of the form (A, φ), where A belongs to $\mathcal{K}$ and φ is a *partial* automorphism of A, i.e., an isomorphism from a substructure of A onto another substructure of A. An embedding between two such structures (A, φ) and (B, ψ) is an embedding α of A into B such that $\psi \circ \alpha$ extends $\alpha \circ \varphi$. Then, the existence of a dense conjugacy class in $\mathrm{Aut}(\mathbb{K})$ is equivalent to saying that the class $\mathcal{K}_1$ satisfies the joint embedding property (see [**KR07**, Theorem 2.1]).

The existence of a comeager conjugacy class is a little harder to state. Retaining the notations above, say that a class of structures $\mathcal{K}$ satisfies the *weak amalgamation property (WAP)* if for any $A \in \mathcal{K}$ there exists $B \in \mathcal{K}$ and an embedding $i\colon A \to B$ such that for any $C, D \in \mathcal{K}$ and any embeddings $r\colon B \to C$, $s\colon B \to D$, there exists $E \in \mathcal{K}$ and embeddings $\gamma\colon C \to E$ and $\delta\colon D \to E$ such that $\gamma \circ r \circ i = \delta \circ s \circ i$. Then [**KR07**, Theorem 3.4] states that there exists a comeager conjugacy class in $\mathrm{Aut}(\mathbb{K})$ if and only if $\mathcal{K}_1$ satisfies both (JEP) and (WAP).

We now know what combinatorial properties to study when looking at the automorphism groups of good measures; fix a good measure μ and consider the corresponding Fraïssé class $\mathcal{K}^\mu$, which is made up of all finite probability algebras whose measure takes its values inside $V(\mu)$. Theorem 4.6 provides a good starting point: indeed, it shows that any element of $\mathcal{K}_1^\mu$ can be embedded in an element of the form (A, φ), where φ is a *global* automorphism of A. We denote this class by $\mathcal{K}^\mu_{\mathrm{aut}}$.

It follows from Theorem 4.6 that $\mathcal{K}^\mu_{\mathrm{aut}}$ is cofinal in $\mathcal{K}^\mu_1$. Hence, in order to understand when $\mathcal{K}^\mu_1$ satisfies (JEP), we only need to consider automorphisms of finite algebras; explicitly, we now see that H_μ has a dense conjugacy class if and only if the following condition is satisfied: whenever $\mathcal{A},\mathcal{B}$ are finite subalgebras of $\mathrm{Clop}(X)$, and a,b are automorphisms of $(\mathcal{A},\mu)$, $(\mathcal{B},\mu)$ respectively, there exists a finite subalgebra $\mathcal{C}$ of $\mathrm{Clop}(X)$ and an automorphism c of $(\mathcal{C},\mu)$ such that there exist μ-preserving embeddings $\alpha\colon \mathcal{A}\to\mathcal{C}$ and $\beta\colon \mathcal{B}\to\mathcal{C}$ satisfying $c(\alpha(A)) = \alpha(a(A))$ for all $A\in\mathcal{A}$, and $c(\beta(B)) = \beta(b(B))$ for all $B\in\mathcal{B}$.

Unfortunately, this property does not always hold. Indeed, assume that μ satisfies (JEP), and that there exists $A\in\mathrm{Clop}(X)$ such that $\mu(A)=\frac{1}{n}$ for some integer n. Then, there exists an element $a\in H_\mu$ such that X is the disjoint union of $A,\dots,a^{n-1}(A)$. Let now r be any element of $V(\mu)$, B a clopen subset of X such that $\mu(B)=r$, and consider:

- the algebra $\mathcal{A}$ generated by $A,\dots,a^{n-1}(A)$, with the automorphism a;
- the algebra $\mathcal{B}$ made up of B and its complement, with the identity automorphism b.

Assume one can jointly embed $(\mathcal{A},a)$ and $(\mathcal{B},b)$ in $(\mathcal{C},c)$; identify $\mathcal{A},\mathcal{B}$ with the subalgebras of $\mathcal{C}$ associated with these embeddings. Then $B = B\cap \sqcup_{i=0}^{n-1} c^i(A) = \sqcup_{i=0}^{n-1} c^i(B\cap A)$, so B is cut into n clopen subsets of equal measure. This means that $\frac{r}{n}$ must belong to $V(\mu)$. Hence, the joint embedding property fails for instance when $V=\langle \frac{1}{2},\frac{1}{\pi}\rangle$.

Analysing the above example, one can extract a combinatorial condition on V that is equivalent to the existence of a dense conjugacy class in H_{μ_V}.

Proposition 5.6. *Let V be a group-like subset of $[0,1]$. Then there is a dense conjugacy class in H_{μ_V} if, and only if, V satisfies the following condition: whenever $a_i,b_j\in V$ and $n_i,m_j\in\mathbb{N}$ are such that $\sum_{i=1}^p n_ia_i = 1 = \sum_{j=1}^q m_jb_j$, there exist $c_{i,j}\in V$ such that*

$$\forall j\ m_jb_j = \sum_{i=1}^p lcm(n_i,m_j)c_{i,j} \quad \text{and} \quad \forall i\ n_ia_i = \sum_{j=1}^q lcm(n_i,m_j)c_{i,j}.$$

This holds true in particular when $V+\mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$, and when $V+\mathbb{Z}$ is a subring of $\mathbb{R}$.

As we already mentioned above, Akin [**Aki05**] actually proved that H_μ has a comeager conjugacy class when $V(\mu)+\mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$, a fact that we will recover below.

Proof of Proposition 5.6. To simplify the notation below we sometimes do not mention the measure; in particular, all automorphisms are to be understood as preserving μ.

Assume that the joint embedding property for partial automorphisms holds, and consider $(a_i,n_i)_{1\le i\le p}$, $(b_j,m_j)_{1\le j\le q}$ as above. Then one can consider a finite algebra $\mathcal{A}$ with clopen atoms $A_{i,k}$ for $k\in\{0,\dots,n_i-1\}$ such that each $A_{i,k}$ has measure a_i, and an automorphism a of $\mathcal{A}$ such that $a(A_{i,k}) = A_{i,k+1}$ for all i,k (where addition is to be understood modulo n_i); similarly one can consider a finite algebra $\mathcal{B}$ with clopen atoms $B_{j,k}$ ($k \in \{0,\dots,m_j-1\}$) and the corresponding automorphism b of $\mathcal{B}$. For all i we let $A_i = \cup A_{i,k}$ and $B_j = \cup B_{j,k}$.

Then we pick $(\mathcal{C},c)$ such that $(\mathcal{A},a)$ and $(\mathcal{B},b)$ can be embedded in $(\mathcal{C},c)$, where c is an automorphism of the finite algebra $\mathcal{C}$, and we identify them with the corresponding subalgebras of $\mathcal{C}$. If for some i,j $A_i \cap B_j$ is nonempty, then it is a c-invariant clopen set. Any atom of $\mathcal{C}$ contained in some $A_{i,k} \cap B_{j,l}$ must have an orbit whose cardinality is a multiple of $\mathrm{lcm}(n_i,m_j)$, so $c_{i,j} = \frac{1}{\mathrm{lcm}(n_i,m_j)}\mu(A_i \cap B_j)$ belongs to V. Then we have for all i:

$$n_i a_i = \mu(A_i) = \sum_{j=1}^{q} \mu(A_i \cap B_j) = \sum_{j=1}^{q} c_{i,j}\mathrm{lcm}(n_i,m_j).$$

The same reasoning holds for $m_j b_j$.

This proves one implication; to prove the converse, let us first note that, given a clopen U and two cycles a,b on U of orders n,m respectively and such that $\frac{1}{\mathrm{lcm}(n,m)}\mu(U)$ belongs to V, there exists a cycle on U of order $N = \mathrm{lcm}(n,m)$ in which both a and b embed. Such a cycle is obtained by cutting U in N disjoint pieces C_i ($0 \le i \le N-1$) of equal measure, and setting $c(C_i) = C_{i+1}$ (modulo N). Then, let $N = nr = ms$; letting $A_0,\dots,A_{n-1}$ denote the atoms contained in U of the algebra on which a is defined, one obtains the desired embedding by identifying each A_i with $\sqcup_{k=0}^{r-1} C_{nk+i}$, and each B_j with $\sqcup_{k=0}^{s-1} C_{mk+j}$.

Now, let α,β in H_{μ_V} be such that $X = \sqcup_{i=1}^{p} A_i$, where each A_i is clopen and α is a product of cycles α_i of order n_i on A_i, and $X = \sqcup_{j=1}^{q} B_j$, where each B_j is clopen and β is a product of cycles β_j of order m_j on B_j. It is enough to prove that α,β embed in a common element of H_{μ_V}. Let $n_i a_i = \mu(A_i)$ and $m_j b_j = \mu(B_j)$, and apply our assumption on V to get $c_{i,j}$ as in the lemma's statement. Let I denote the set of all (i,j) such that $c_{i,j} \neq 0$; we may find a finite subalgebra of $\mathrm{Clop}(X)$ whose atoms $C_{i,j}^k$ ($(i,j) \in I$, $1 \le k \le \mathrm{lcm}(n_i,m_j)$) are of measure $c_{i,j}$. For each $(i,j) \in I$, set

$$D_{i,j} = \bigsqcup_{k=1}^{\mathrm{lcm}(n_i,m_j)} C_{i,j}^k.$$

We saw that there exists a cycle $\delta_{i,j}$ on $D_{i,j}$, of order $\mathrm{lcm}(n_i,m_j)$, in which a cycle $\alpha_{i,j}$ on $D_{i,j}$ of order n_i and a cycle $\beta_{i,j}$ on $D_{i,j}$ of order m_j both embed. Let δ be the product of all $\delta_{i,j}$; α embeds in δ as the product of all $\alpha_{i,j}$, and β embeds in δ as the product of all $\beta_{i,j}$.

To see that the property under discussion holds true when $V+\mathbb{Z}$ is a subring of $\mathbb{R}$, simply note that in that case $a_i b_j$ belongs to V; thus $c_{i,j} = a_i b_j \frac{n_i m_j}{\mathrm{lcm}(n_i,m_j)} = a_i b_j \mathrm{gcd}(n_i,m_j)$ works.

When $V+\mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$, which is equivalent to saying that $\frac{a}{n} \in V$ for any positive integer n and any $a \in V$, we skip the proof since we will show a stronger property below. $\square$

The above criterion is probably of minimal practical interest, since it appears to be fairly hard to check (certainly, it does not help much when tackling the case when $V+\mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$).

Definition 5.7. Following Akin [**Aki05**], we say that a group-like subset $V \subseteq [0,1]$ is $\mathbb{Q}$-*like* if $V+\mathbb{Z}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$; this is equivalent to saying that V is group-like and $\frac{1}{n}V \subseteq V$ for any positive integer n.

Proposition 5.8. *If V is $\mathbb{Q}$-like, then $\mathcal{K}_{aut}^{\mu_V}$ satisfies the amalgamation property. (The converse is also true.) Hence H_{μ_V} has a comeager conjugacy class in that case.*

Proof. Suppose that (A,φ) embeds in (B,ψ) and in (C,θ). We construct the *Boolean* amalgam $(B\otimes_A C, \psi\otimes\theta)$ of (B,ψ) and (C,θ) over (A,φ) in the standard way (see for example [**KST12**]), and only need to define the measures. We give an argument in the fashion of the one contained in Theorem 2.1 of [**KST12**].

Fix an atom $a\in A$, and list the atoms of B and C contained in a by $\{b_i^k\}_{i<n_k}^{k<n}$ and $\{c_j^l\}_{j<m_l}^{l<m}$ respectively, where b_i^k and $b_{i'}^{k'}$ are in the same ψ-orbit iff $k=k'$, and analogously for the c_j^l. We want to define the values $x_{ij}^{kl}=\mu(b_i^k\otimes c_j^l)$. Then we would translate these values in the obvious manner to the products of the atoms of B and C contained in the φ-translates of a; finally, we would proceed analogously for the other orbits of (A,φ).

Other than being in V, the values x_{ij}^{kl} have to satisfy:

$$0\le x_{ij}^{kl},$$
$$x_{ij}^{kl}=x_{i'j'}^{kl},$$
$$\sum_{ki} x_{ij}^{kl}=\mu(c_j^l),\ \sum_{lj} x_{ij}^{kl}=\mu(b_i^k).$$

Denoting $x^{kl}=x_{ij}^{kl}$, we can reformulate the conditions as:

$$\sum_k n_k m_l x^{kl}=m_l\mu(c_0^l),\ \sum_l n_k m_l x^{kl}=n_k\mu(b_0^k).$$

Considered as a system in the variables $y^{kl}=n_k m_l x^{kl}$, we can find a solution in $\mathbb{R}$, namely $y^{kl}=n_k m_l\frac{\mu(b_0^k)\mu(c_0^l)}{\mu(a)}$. Since V is group-like and dense, there must also be solutions y^{kl} in V. Since it is also $\mathbb{Q}$-like, we can take $x^{kl}=\frac{y^{kl}}{n_k m_l}$ and we are done. □

The amalgamation property for $\mathcal{K}_{\mathrm{aut}}^{\mu}$ is stronger than the existence of a comeager conjugacy class in H_μ; for instance, if $V(\mu)$ is the set of dyadic numbers, then it follows from [**KR07**, discussion after the statement of Theorem 6.5] that H_μ has a comeager conjugacy class, but it is easy to see that $\mathcal{K}_{\mathrm{aut}}^{\mu}$ does not have the amalgamation property in that case. It does, however, admit a cofinal class which satisfies the amalgamation property, which is sufficient to obtain (WAP) (that class is made up of finite subalgebras all of whose atoms have the same measure). *A priori*, the cofinal amalgamation property for $\mathcal{K}_{\mathrm{aut}}^{\mu}$ is itself stronger than (WAP); yet we do not know of an example of measure for which $\mathcal{K}_1^\mu$ has (WAP) but $\mathcal{K}_{\mathrm{aut}}^\mu$ does not have the cofinal amalgamation property.

CHAPTER 5

Dynamical simplices and minimal homeomorphisms

Abstract. [Joint work with Julien Melleray, submitted for publication] We give a characterization of sets K of probability measures on a Cantor space X with the property that there exists a minimal homeomorphism g of X such that the set of g-invariant probability measures on X coincides with K. This extends theorems of Akin (corresponding to the case when K is a singleton) and Dahl (when K is finite-dimensional). Our argument is elementary and different from both Akin's and Dahl's.

Contents

Introduction

The study of minimal homeomorphisms (those for which all orbits are dense) on a Cantor space is a suprisingly rich and active domain of research. In a foundational series of papers (see [**HPS92**], [**GPS95**] and [**GPS99**]), Giordano, Herman, Putnam and Skau have pursued the analysis of minimal actions of $\mathbb{Z}$ (and later $\mathbb{Z}^d$), and developed a deep theory. In particular, it is proved in [**GPS95**] that the partition of a Cantor space X induced by the orbits of a minimal homeomorphism g is completely determined, up to a homeomorphism of X, by the collection of all g-invariant measures.

Gaining a better understanding of sets of invariant measures then becomes a natural concern, and that is our object of study here: given a Cantor space X, and a simplex K of probability measures, when does there exist a minimal homeomorphism g of X such that K is exactly the simplex of all g-invariant measures? Downarowicz [**Dow91**] proved that any abstract Choquet simplex can be realized in this way; here we are not given K as an abstract simplex, but already as a simplex of measures, so the problem has a different flavour.

A theorem of Glasner–Weiss [**GW95**] imposes a necessary condition: if g is a minimal homeomorphism, K is the simplex of all g-invariant measures, and A,B are clopen subsets of X such that $\mu(A) < \mu(B)$ for all $\mu \in K$, then there exists a clopen subset $C \subseteq B$ such that $\mu(C) = \mu(A)$ for all $\mu \in K$. This is already a strong, nontrivial assumption when K is a singleton; in that case the Glasner–Weiss condition is essentially sufficient, as was proved by Akin.

Theorem (Akin [**Aki05**]). *Assume that μ is a probability measure on a Cantor space X which is atomless, has full support, and is* good, *that is, for any clopen sets A, B, if $\mu(A) < \mu(B)$ then there exists a clopen $C \subseteq B$ such that $\mu(C) = \mu(A)$.*

Then there exists a minimal homeomorphism g of X such that the unique g-invariant measure is μ.

Following Akin, we say that a simplex is *good* if it satisfies the necessary condition established by Glasner and Weiss. Akin's theorem suggests that, modulo some simple additional necessary conditions, any good simplex of measures could be the simplex of all invariant measures for some minimal homeomorphism. This idea is further reinforced by an unpublished result of Dahl, which generalizes Akin's theorem.

Theorem (Dahl [**Dah08**]). *Let K be a Choquet simplex made up of atomless probability measures with full support on a Cantor space X. Assume that K is good and has finitely many extreme points, which are mutually singular. Then there exists a minimal homeomorphism whose set of invariant probability measures coincides with K.*

Dahl actually obtains a more general result. To formulate it, we recall her notation: given a simplex K of probability measures on a Cantor space X, let $\mathrm{Aff}(K)$ denote the set of all continuous affine functions on K, and $G(K) \subseteq \mathrm{Aff}(K)$ be the set of all functions $\mu \mapsto \int_X f\,d\mu$, where f belongs to $\mathrm{C}(X, \mathbb{Z})$.

Theorem (Dahl [**Dah08**]). *Let K be a Choquet simplex made up of atomless probability measures with full support on X. Assume K is good and the extreme points of K are mutually singular. If $G(K)$ is dense in $\mathrm{Aff}(K)$, then there exists a minimal homeomorphism whose set of invariant measures is exactly K.*

As pointed out in the third section of [**Dah08**], it follows from Theorem 4.4 in [**Eff81**] that $G(K)$ being dense in $\mathrm{Aff}(K)$ is necessary for $G(K)$ to be a so-called simple dimension group, which is in turn necessary for the existence of g as above. The other conditions are also necessary, so Dahl could have formulated her theorem as an equivalence.

Using Lyapunov's theorem, Dahl proves that any finite-dimensional Choquet simplex of probability measures on X with mutually singular extreme points is such that $G(K)$ is uniformly dense in $\mathrm{Aff}(K)$, thus deducing the theorem we stated previously from the one we just quoted. Dahl's proof of her second theorem above uses some high-powered machinery following the Giordano–Herman–Putnam–Skau approach to topological dynamics via dimension groups, K-theory, Bratteli diagrams and Bratteli–Vershik maps. By contrast, Akin's proof is elementary and rather explicit, though somewhat long.

Here, we take an approach which is different from both Akin's and Dahl's: we build a minimal homeomorphism preserving a prescribed set of probability measures by constructing inductively a sequence of partitions which will turn out to be Kakutani–Rokhlin partitions for that homeomorphism (we recall the definition of Kakutani–Rokhlin partitions and other basic notions of topological dynamics in the next section). While pursuing this approach, we unearthed a necessary condition for K to be the simplex of invariant measures for some minimal homeomorphism. This led us to the following definition.

Definition 0.1. We say that a nonempty set K of probability measures on a Cantor space X is a *dynamical simplex* if it satisfies the following conditions:

- K is compact and convex.
- All elements of K are atomless and have full support.
- K is good.
- K is *approximately divisible*, i.e., for any clopen A, any integer n and any $\varepsilon > 0$, there exists a clopen $B \subseteq A$ such that $n\mu(B) \in [\mu(A) - \varepsilon, \mu(A)]$ for all $\mu \in K$.

Note that, assuming that K is good, the assumption that $B \subseteq A$ in the last item above is redundant; we nevertheless include it because this is how approximate divisibility is used in our arguments.

We borrow the terminology "dynamical simplex" from Dahl, but our definition is different. Using Lyapunov's theorem as in [**Dah08**], it is easy to see that the condition of approximate divisibility is redundant when K is finite dimensional. The main result of this paper is the following.

Theorem 0.2. *Given a simplex K of probability measures on a Cantor space X, there exists a minimal homeomorphism g whose set of invariant measures is K if, and only if, K is a dynamical simplex.*

Our construction produces a minimal homeomorphism g whose set of invariant measures is K and such that the topological full group $[[g]]$ is dense in the closure of the full group $[g]$ (in the terminology of [**BK02**], g is *saturated*). When one starts off by assuming that K is the simplex of T-invariant measures for some minimal homeomorphism T, the existence of such a homeomorphism follows from a combination of theorems of Giordano–Putnam–Skau and Glasner–Weiss, see [**BK02**, Theorem 1.6]. Here we provide an elementary proof of that fact, which seems interesting on its own.

For finite dimensional simplices, our theorem generalizes Dahl's result, showing that the assumption that extreme points are mutually singular is actually a consequence of her other hypotheses. It would be interesting to gain a better understanding of the relationship between her conditions and ours (see Remark 3.4 at the end of the paper).

We would like to point out that the ideas of our construction are different from both Akin's and Dahl's, and relatively elementary; in particular our argument completely bypasses the use of dimension groups, Bratteli diagrams, etc. It is our hope that such ideas could be used to give elementary dynamical proofs of some other theorems of topological dynamics.

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1. Background and notations

Throughout, X is a Cantor space; we fix some compatible distance on X, and whenever we mention the diameter of a set it will be with respect to this distance.

We denote by $\mathrm{Prob}(X)$ the compact space of all probability measures on X, endowed with its usual topology (which comes from seeing $\mathrm{Prob}(X)$ as a subset of the dual of $\mathrm{C}(X)$, endowed with the weak-$*$ topology).

The group $\mathrm{Homeo}(X)$ of all homeomorphisms of X is a Polish group when endowed with the topology of uniform convergence on X. Since X is the Cantor space, one can also describe this topology using Stone duality: a homeomorphism of X corresponds to an automorphism of the Boolean algebra of clopen sets of X, $\mathrm{Clop}(X)$; identifying $\mathrm{Homeo}(X)$ with automorphisms of this algebra yields that a basis of neighbourhoods of identity is given by sets of the form $\{g \in \mathrm{Homeo}(X) \colon \forall A \in \mathcal{A}\ g(A) = A\}$, where $\mathcal{A}$ runs over all clopen partitions of X (note that by compactness all clopen partitions are finite). It is readily checked that the two topologies we just described coincide on $\mathrm{Homeo}(X)$.

Definition 1.1. Given $g \in \mathrm{Homeo}(X)$, its *topological full group* $[[g]]$ is the group of all homeomorphisms h of X such that there exists a clopen partition $A_1, \dots, A_n$ and integers n_i with the property that for all $x \in A_i$ one has $g(x) = h^{n_i}(x)$.

The *full group* $[g]$ is the group of all homeomorphisms h such that for all x there exists n satisfying $h(x) = g^n(x)$.

By definition, the topological full group is countable (there are only countably many clopen sets) and contained in the full group. The reason these groups are relevant to our concerns is the following, which follows easily from Proposition 2.6 of [**GW95**].

Theorem 1.2 (Glasner–Weiss). *Let g be a minimal homeomorphism. The closure of $[g]$ in $\mathrm{Homeo}(X)$ consists of all homeomorphisms which preserve all g-invariant probability measures on X.*

Proof. Let H denote the group of all homeomorphisms which preserve each g-invariant measure. By definition, H is closed and $[g] \subseteq H$, so that $\overline{[g]} \subseteq H$. Towards proving the converse inclusion, pick $h \in H$ and an open neighborhood $O = \{k \in H \colon \forall A \in \mathcal{A}\ k(A) = h(A)\}$ of h, where $\mathcal{A}$ is a clopen partition of X. For any $A \in \mathcal{A}$ there exists, by Proposition 2.6 of [**GW95**], some $k_A \in [g]$ such that $k_A(A) = h(A)$. Then, the map k defined by setting $k(x) = k_A(x)$ whenever $x \in A$ is a homeomorphism (because $\mathcal{A}$ is a clopen partition, and h as well as each k_A are homeomorphisms) and belongs to $O \cap [g]$. □

It is not always true that $[[g]]$ is dense in $[g]$; when that happens we say that g is *saturated* (this follows terminology introduced in [**BK02**]).

We next recall the definition of a *Kakutani–Rokhlin* partition associated to a minimal homeomorphism.

Definition 1.3. A *Kakutani–Rokhlin partition $\mathcal{T}$* associated to a minimal homeomorphism g is a clopen partition of X of the form $A_0 \sqcup \dots A_k$, where each A_i is further subdivided in $B_{i,0}, \dots, B_{i,j_i}$ (possibly $j_i = 0$) and for all i and all $r \in \{0, \dots, j_i - 1\}$, $g(B_{i,r}) = B_{i,r+1}$.

The union of all $B_{i,0}$ is called the *base* of the partition, and the union of all B_{i,j_i} is its *top*. Each A_i is called a *column* of the partition, and the definition ensures that g must map the top of the partition onto its base.

To obtain such a partition, one can first choose a clopen base B; then subdivide it into $B_1,\dots,B_N$, with B_i made up of all $x \in B$ such that $i = \min\{j > 0\colon g^j(x) \in B\}$; and set $B_{i,j} = g^j(B_i)$ for all $j \in \{0,\dots,i-1\}$.

Below we represent a Kakutani–Rokhlin partition; the arrows correspond to the action of the homeomorphism on the partition, which is prescribed on all atoms except those contained in the top (all we know there is that the top is mapped onto the base). The base is colored in blue and the top in red; note that on the picture the base and top do not intersect. They are allowed to, but will not intersect as soon as we take a small enough base.

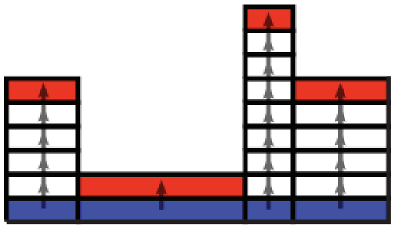

FIGURE 1. A Kakutani–Rokhlin partition

It is a standard, important fact in topological dynamics that, given a minimal homeomorphism g, one can produce a sequence of Kakutani–Rokhlin partitions for g whose atoms generate the algebra of clopen sets, and whose top and base have vanishing diameter. Such a sequence naturally defines a basis of neighborhoods of g in $\mathrm{Homeo}(X)$.

Below, to obtain a minimal homeomorphism with prescribed set of invariant measures, we will define a sequence of partitions which will turn out to be Kakutani–Rokhlin partitions for that homeomorphism.

We recall the definition of a dynamical simplex given in the introduction.

DEFINITION 1.4. We say that a nonempty set K of probability measures on X is a *dynamical simplex* if it satisfies the following conditions:

- K is compact and convex.
- All elements of K are atomless and have full support.
- K is *good*, i.e., for any two clopen sets A, B such that $\mu(A) < \mu(B)$ for all $\mu \in K$, there exists a clopen subset $C \subseteq B$ such that $\mu(C) = \mu(A)$ for all $\mu \in K$.
- K is *approximately divisible*, i.e., for any clopen set A, any integer n and any $\varepsilon > 0$, there exists a clopen $B \subseteq A$ such that $n\mu(B) \in [\mu(A) - \varepsilon, \mu(A)]$ for all $\mu \in K$.

Given a set K of probability measures, and two clopen sets A, B, we use the notation $A \sim_K B$ to denote the fact that $\mu(A) = \mu(B)$ for all $\mu \in K$. Note that, modulo goodness, approximate divisibility may be stated equivalently by saying that there exist $B_1,\dots,B_n$ such that $B_1,\dots,B_n$ are disjoint, contained in A, $B_i \sim_K B_j$ for all i, j, and $A \setminus \bigcup B_i$ has measure less than ε for all $\mu \in K$.

We should point out again that we borrow the term "dynamical simplex" from Dahl [**Dah08**], and that our definition is, at least formally, different from Dahl's:

the definition given in [**Dah08**] includes the assumption that K is a Choquet simplex and extreme points of K are mutually singular, and does not mention approximate divisibility. We quickly discuss the relations between our conditions and Dahl's in Remark 3.4 at the end of the paper.

We note that, when K has finitely many extreme points, the assumption of approximate divisibility is redundant, as follows from the proposition below.

Proposition 1.5. *Assume that K is a compact subset of* $\mathrm{Prob}(X)$, *and all the measures in K are atomless and have full support. Then the following properties hold.*

(1) *For any nonempty clopen set A,* $\inf\{\mu(A)\colon \mu\in K\}>0$.
(2) *For any $\varepsilon>0$, there exists $\delta>0$ such that, for any clopen set A of diameter less than δ, one has $\mu(A)\le\varepsilon$ for all $\mu\in K$.*
(3) *If K has finitely many extreme points, then K is approximately divisible.*

Proof. The first two items are well-known when K is the simplex of all invariant measures for a minimal homeomorphism, and the proofs are simple and similar to that case. We give them for the reader's convenience.

For the first item, assume that there exists a sequence (μ_n) of elements of K and a nonempty clopen set A such that $\mu_n(A)$ converges to 0. Then by compactness of K we find some $\mu\in K$ such that $\mu(A)=0$, contradicting the fact that μ has full support (it is perhaps worth recalling that a sequence μ_n of elements of $\mathrm{Prob}(X)$ converges to $\mu\in\mathrm{Prob}(X)$ exactly if $\mu_n(A)$ converges to $\mu(A)$ for all clopen set A).

The second item requires a bit more work; we follow the argument of [**BM08**, Proposition 2.3] and assume for a contradiction that there exists a sequence of clopen subsets (A_n) of vanishing diameter and $\varepsilon>0$ such that $\mu_n(A_n)\ge\varepsilon$ for all n. Up to passing to a subsequence, we may assume that (A_n) converges to a singleton $\{x\}$ for the Hausdorff distance on compact subsets of X, and that μ_n converges to $\mu\in K$. Let O be any clopen neighborhood of x; we will have $A_n\subseteq O$ for all large enough n, so that $\mu_n(O)\ge\varepsilon$ for all large n. As above, this implies that $\mu(O)\ge\varepsilon$ for all n, so that (taking the intersection over all clopen neighborhoods of x) $\mu(\{x\})\ge\varepsilon$, contradicting the fact that μ is atomless.

To see why the third point holds, we argue in a way similar to [**Dah08**]. Fix $\varepsilon>0$ and let $\mu_1,\dots,\mu_n$ denote the extreme points of K. Lyapunov's theorem on vector measures tells us that

$$\{(\mu_1(B),\dots,\mu_n(B))\colon B \text{ a Borel subset of } A\}$$

is convex. In particular, there exists a Borel subset B of A such that $\mu_i(B)=\frac{1}{n}\mu_i(A)$ for $i\in\{1,\dots,n\}$. Using the regularity of $\mu_1,\dots,\mu_n$ we obtain a clopen subset $C\subset A$ such that $\mu_i(C)\in[\frac{\mu_i(A)-\varepsilon}{n},\frac{\mu_i(A)}{n}]$ for all i, which is what we wanted. □

Remark 1.6. If we had assumed that the extreme points of K were mutually singular, we would not have needed Lyapunov's theorem to conclude that K is approximately divisible when K has finitely many extreme points; but we do not need to make this assumption. We also do not include the assumption that K is a Choquet simplex in our definition of a dynamical simplex, as we do not need it in the arguments. Both these assumptions are clearly necessary for K to be the simplex of all invariant measures of an homeomorphism, hence follow from the others, given the main result of the paper.

We do not know if, in general, approximate divisibility is a consequence of the other assumptions (to which one could add the fact that K is a Choquet simplex with mutually singular extreme points, if necessary) as is the case when K is finite-dimensional. The proof above does not seem to adapt: Lyapunov's theorem does extend to more general situations, but this extension (known as Knowles' theorem, see [**DU77**, IX.1.4]) requires the existence of a finite control measure ν, which will exist only when K has finitely many extreme points. More precisely, one would like to apply the Extension Theorem [**DU77**, I.5.2] to the vector-valued measure $F : \mathrm{Clop}(X) \to \mathrm{C}(K)$, $F(A)(\mu) = \mu(A)$, but then, assuming K has countably many mutually singular extreme points, it is not difficult to see that the second item of the theorem fails. Nevertheless, one can certainly prove that approximate divisibility is redundant in some infinite-dimensional situations, for instance when the extreme boundary of K has only one non-isolated point (this was remarked during a conversation with I. Farah).

Proposition 1.7. *Assume that g is a minimal homeomorphism of X, and that K is the simplex of all g-invariant probability measures. Then K is a dynamical simplex.*

Proof. Clearly g-invariant measures always form a compact convex subset of $\mathrm{Prob}(X)$, and when g is minimal any g-invariant measure must be atomless and have full support. The fact that the simplex K is good follows from the theorem of Glasner and Weiss recalled in the introduction, so we only need to explain why K is approximately divisible.

Start from a nonempty clopen set A, and consider the map g_A defined by $g_A(x) = g^n(x)$, where $n = \min\{i > 0 \colon g^i(x) \in A\}$. Then g_A is a homeomorphism of A, and is minimal. The restriction of any $\mu \in K$ defines a g_A-invariant measure on A, which we still denote by μ. Pick $N \geq n$ such that $n/N < \varepsilon$. Since g_A is aperiodic, we can find a clopen set U such that $U, g_A U, \ldots, g_A^N U$ are disjoint. In particular, $\mu(U)$ is less than $\mu(A)/N$ for all $\mu \in K$.

Now, consider a Kakutani–Rokhlin partition of A associated to g_A, with base U. Let $C_0, \ldots, C_M$ denote the columns of this partition; we have $C_i = C_{i,0} \sqcup C_{i,1} \ldots \sqcup C_{i,n_i}$, with $n_i \geq N \geq n$. Denote $n_i + 1 = k_i n + p$, $p \in \{0, \ldots, n-1\}$. For $i \in \{0, \ldots, M\}$ and $j \in \{0, \ldots, n-1\}$, let $B_{i,j}$ denote the union of the levels of C_i of height $< k_i n$ and equal to j modulo n, and let B_j be the union of all $B_{i,j}$. Then we have $B_j \sim_K B_l$ for all j, l, and the complement of their union is of mesure less than $n\mu(U) \leq \varepsilon\mu(A)$ for all $\mu \in K$.

The following picture is supposed to illustrate the procedure we just described: below, $n = 3$; the domains in blue, red and green are $\sim_K$ equivalent, and the measure of the remainder is less than 3 times the measure of the base, for all $\mu \in K$. □

The following proposition states an homogeneity property of the algebra $\mathrm{Clop}(X)$ in relation to good simplices.

Proposition 1.8. *Assume K is a good simplex of probability measures on a Cantor space X with full support. Let $G = \{g \in \mathrm{Homeo}(X) \colon \forall \mu \in K\ g_*\mu = \mu\}$. If U, V are clopen sets with $U \sim_K V$, then there is $g \in G$ such that $gU = V$.*

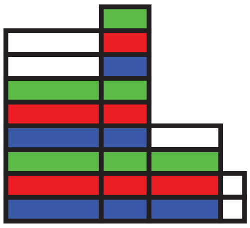

FIGURE 2. A partition in three $\sim_K$ pieces plus a rest of small measure

PROOF. We construct a K-preserving automorphism g of the algebra $\mathrm{Clop}(X)$ by a standard back-and-forth argument. Let $\{A_n\}$, $\{B_n\}$ be two enumerations of $\mathrm{Clop}(X)$, with $A_0 = U$ and $B_0 = V$. Let $\mathcal{A}_0$ be the partition of X into A_0 and its complement. We set $gA_0 = B_0$, $g(X \setminus A_0) = X \setminus B_0$. Now assume inductively that we have defined g on the atoms of a finite clopen partition $\mathcal{A}_n$ such that: (i) the sets $\{A_i\}_{i<n}$ are contained in the algebra generated by $\mathcal{A}_n$, (ii) the image $g\mathcal{A}_n = \{gC : C \in \mathcal{A}_n\}$ is a partition of X, (iii) $\mu(gC) = \mu(C)$ for all $C \in \mathcal{A}_n$, $\mu \in K$, and (iv) the sets $\{B_i\}_{i<n}$ are contained in the algebra generated by $g\mathcal{A}_n$.

Let $C_1, \ldots, C_m$ be the atoms of the partition $\mathcal{A}_n$. We take $C_i^0 = C_i \cap A_n$, $C_i^1 = C_i \setminus A_n$. Since K is good, we can find $D_i^0 \subseteq gC_i$ such that $\mu(D_i^0) = \mu(C_i^0)$ for all $\mu \in K$; we set $D_i^1 = gC_i \setminus D_i^0$. Now we take $D_i^{j,0} = D_i^j \cap B_n$, $D_i^{j,1} = D_i^j \setminus B_n$. Again, since K is good, we can find a clopen partition $\mathcal{A}_{n+1} = \{C_i^{j,k}\}_{i<m}^{j,k<2}$ such that $C_i^{j,k} \subseteq C_i^j$ and $\mu(C_i^{j,k}) = \mu(D_i^{j,k})$ for each i, j, k and all $\mu \in K$. Then we extend the definition of g to $\mathcal{A}_{n+1}$ by setting $gC_i^{j,k} = D_i^{j,k}$. The construction ensures that properties (i)-(iv) are preserved.

At the end we get a K-preserving automorphism of $\mathrm{Clop}(X)$ sending U to V. By Stone duality, this induces an homeomorphism g as required. □

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2. Construction of a saturated element

In this section and the next, we fix a dynamical simplex of measures K on a Cantor space X, and we let G denote the group of all homeomorphisms g of X such that $g_*\mu = \mu$ for all $\mu \in K$.

DEFINITION 2.1. We say that $g \in G$ is *K-saturated* if for any clopen sets U, V such that $U \sim_K V$ there exists $h \in [[g]]$ with $h(U) = V$.

PROPOSITION 2.2. *Assume that g is K-saturated. Then $[[g]]$ is dense in G and g is minimal.*

PROOF. It is easy to see that $[[g]]$ being dense in G is equivalent to g being K-saturated once one has Theorem 1.2 in hand. To see that a K-saturated element is minimal, pick any nonempty clopen set U. Given any $x \in X$, a sufficiently small clopen neighborhood V of x will be such that $\sup_K \mu(V) < \inf_K \mu(U)$, thus there exists $g \in G$ such that $gU \supseteq V \ni x$. Hence $X = \bigcup_{g \in G} gU$.

Now, for all $h \in G$ there exists $k \in [[g]]$ such that $kU = gU$, so that $X = \bigcup_{k\in[[g]]} kU$. Since for all $k \in [[g]]$ we have $kU \subseteq \bigcup_{i\in\mathbb{Z}} g^i U$, we obtain that $X = \bigcup_{i\in\mathbb{Z}} g^i U$. This means that X is the unique nonempty open g-invariant set, which is the same as saying that g is minimal. □

Next, we introduce partitions which resemble Kakutani–Rokhlin partitions. Essentially, we are trying to build a homeomorphism from a sequence of partitions, rather than the other way around.

Definition 2.3. A *KR-partition* $\mathcal{T}$ is a clopen partition of X of the form $A_0 \sqcup \dots A_k$, where each A_i is further subdivided into $B_{i,0},\dots,B_{i,j_i}$ (possibly $j_i = 0$) and for all i and all $r,s \in \{0,\dots,j_i\}$ $B_{i,r} \sim_K B_{i,s}$.

The union of all $B_{i,0}$ is called the *base* of the partition, and the union of all B_{i,j_i} is its *top*. Each A_i is called a *column* of the partition.

To each KR-partition, one can associate the algebra $\mathcal{A}_\mathcal{T}$ whose atoms are all $B_{i,r}$ with $r < j_i$, and the top of the partition; and the partial automorphism of $\mathrm{Clop}(X)$ with domain $\mathcal{A}$ which maps each $B_{i,r}$ to $B_{i,r+1}$ for all i and $r < j_i$, and maps the top of the partition to its base. Note that the ordering of atoms within each column matters.

We say that a KR-partition $\mathcal{S}$ refines a KR-partition $\mathcal{T}$ if the base (respectively, top) of $\mathcal{S}$ is contained in the base (respectively, top) of $\mathcal{T}$, and $\mathcal{S}$-towers are obtained by cutting and stacking towers of $\mathcal{T}$ on top of each other (see Figure 2). More precisely: for each column $C = (D_j)_{0\le j\le J}$ of $\mathcal{S}$ there exist columns $A_{i_k} = (B_{i_k,j})_{0\le j\le j_{i_k}}$ of $\mathcal{T}$ $(0 \le k \le K)$ and clopen subsets $S_j^k \subseteq B_{i_k,j}$ such that $J = \sum_{0\le k\le K}(j_{i_k}+1)$ and such that, for each k, we have $D_{j+\sum_{l<k}(j_{i_l}+1)} = S_j^k$ for every $0 \le j \le j_{i_k}$.

Note that if $\mathcal{S}$ refines $\mathcal{T}$ then the algebra and partial automorphism associated to $\mathcal{S}$ refine those that are associated to $\mathcal{T}$.

Proposition 2.4. *Given a KR-partition $\mathcal{T}$, and $\varepsilon > 0$, there exists a KR-partition $\mathcal{S}$ which refines $\mathcal{T}$ and which is such that the base and top of $\mathcal{S}$ both have diameter less than ε.*

Proof. We let again $A_0,\dots,A_k$ denote the columns of $\mathcal{T}$, $B_{i,0}$, B_{i,j_i} denote respectively the base and top of the i-th column, and B be the base of $\mathcal{T}$.

We begin by describing how to deal with a very favorable particular case, where there exists an integer $n \ge 2$ such that $\mu(B_{0,0}) = \mu(B_{1,0}) = \frac{1}{n}\mu(B)$ and both $B_{0,0}$ and B_{1,j_1} have diameter less than ε. Let $R = B \setminus (B_{0,0} \cup B_{1,0})$. Then we have $\mu(R) = (n-2)\mu(B_{0,0})$ for all $\mu \in K$, so, by goodness, as long as $n > 2$, we can find a copy C of $B_{0,0}$ inside R. Now, the bases of $A_2,\dots,A_k$ induce a partition of C, $C = \bigsqcup_{j=2}^k (C \cap B_{j,0})$. Furthermore, by goodness (or homogeneity), we can find an equivalent cutting of $B_{0,0}$ into pieces $P_j \sim_K (C \cap B_{j,0})$. We pass on this cutting of $B_{0,0}$ throughout the column A_0, and similarly we pass on the cutting of C throughout the columns $A_2,\dots,A_k$. Finally, we stack each new column starting with $C \cap B_{j,0}$ on the top of the new column based on P_j; this gives us a refinement $\mathcal{T}'$ of $\mathcal{T}$.

As long as $n' = n-1 > 2$, we repeat this process, only that now we find copies $C_j \subseteq R' = R \setminus C$ of each P_j. As before, we cut them with the bases of $A_2,\dots,A_k$, then imitate this cutting on the corresponding P_j and pass it on throughout the columns; then we stack the new columns based on subsets of R' on top of the corresponding new columns based on subsets of $B_{0,0}$. Once this process has been repeated $n-2$ times, we apply it, lastly, on A_1. What we obtain is a refinement $\mathcal{S}$ of $\mathcal{T}$ whose base is $B_{0,0}$ and whose top is B_{1,j_1}.

The picture below illustrates the procedure we just described. The column containing the new base is colored in blue, and the one containing the top is red; we keep track of what happens to them in the picture (in a very simple case for readability).

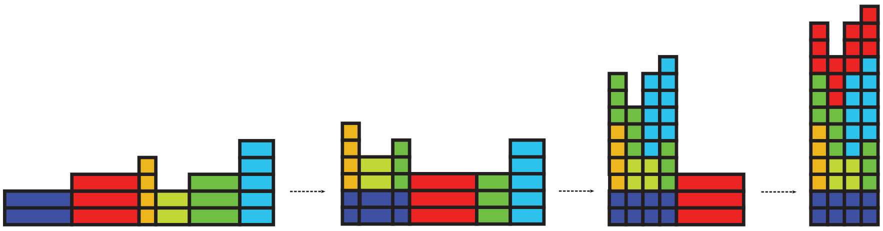

FIGURE 3. Cutting and stacking in the favorable case

That is the last picture that we will include in this article, as the next arguments are a bit harder to illustrate; nevertheless, we invite the reader to draw her own pictures, since we feel that the ideas become more transparent in this way.

To deal with the general case, we use the fact that K is good and approximately divisible to reduce to this favorable case, modulo a small error (which would not appear if K were exactly divisible). First, by cutting $B_{0,0}$ (and throughout A_0) if necessary, we can ensure that $B_{0,0}$ has small diameter. Moreover, by picking a subset of B_{0,j_0} of small diameter and cutting again, we can ensure that both the top and base of the first column of our partition have small diameter. Cutting yet again, we make sure that the union $B_{0,j_0} \cup B_{1,j_1}$ has small diameter. Cutting and using goodness once more, we ensure $\mu(B_{0,j_0}) = \mu(B_{1,j_1})$ for all $\mu \in K$.

Next, pick some integer n such that $\frac{1}{n} < \mu(B_{0,0})$ for all $\mu \in K$. Using the fact that K is good and approximately divisible, we find clopen sets $C_0 \sim_K C_1 \sim_K \dots \sim_K C_{n-1}$ contained in B, pairwise disjoint, such that $C_0 \subseteq B_{0,0}$, $C_1 \subseteq B_{1,0}$ and $E := B \setminus \bigcup_{i=0}^{n-1} C_i \subseteq B_{0,0}$. We cut $B_{0,0}$ into two pieces, one of which is the error E, inducing a further KR-partition, one column of which has base E. We set apart this column A_E, that is, we consider $Y = X \setminus A_E$. Then, our current KR-partition of X induces a KR-partition of Y; cutting one last time we obtain columns based on C_0 and on C_1, which we set to be, respectively, the first and the second column of that partition. We have thus obtained a KR-partition (of Y) which satisfies the assumptions of the favorable case described above. Applying the stacking procedure given for that case, we obtain a new KR-partition of Y whose base is contained in $B_{0,0}$ and whose top is contained in B_{1,j_1}. Finally, considering this partition together with the column A_E, we get a KR-partition of X whose base (contained in $B_{0,0}$) and whose top (contained in $B_{0,j_0} \cup B_{1,j_1}$) have both small diameter. $\square$

We say that a partition is *compatible* with a clopen set U if U is a union of atoms of the partition.

Proposition 2.5. *Given a KR-partition $\mathcal{T}$, and two clopen subsets $U \sim_K V$, one can find a KR-partition $\mathcal{S}$ refining $\mathcal{T}$, compatible with U and V, and such that in each column there are as many atoms contained in U as atoms contained in V.*

Note that then, if g is any element of G which extends the partial automorphism associated to $\mathcal{S}$, there exists an $h \in [[g]]$ such that $h(U) = V$ (because one can map U to V while only permuting atoms within each column of $\mathcal{S}$).

Proof. First, note that by goodness there is a KR-partition $\mathcal{S}$ refining $\mathcal{T}$ and compatible with U, V: consider the algebra generated by $\mathcal{T}$ and U, V, then pull back the associated partion of atoms of $\mathcal{T}$ to the base of $\mathcal{T}$ (via an automorphism $g_{i,s} \in G$ mapping $B_{i,s}$ to $B_{i,0}$), and push it back up (using $g_{i,s}^{-1}$). We obtain a new KR-partition $\mathcal{S}$, refining $\mathcal{T}$, compatible with U and V (each column has been subdivided into smaller columns, and no stacking has taken place).

Now, for all such KR-partitions, we can associate to any column C the numbers

$$u_C = \sharp\{\text{atoms of } C \text{ contained in } U\},\ v_C = \sharp\{\text{atoms of } C \text{ contained in } V\},$$

and $n_C = u_C - v_C$. Our aim is to find a KR-partition with $u_C = v_C$ for all columns. We distinguish the columns of the following types: $\mathcal{C}^+$, the set of columns with $n_C > 0$, and $\mathcal{C}^-$, those where $n_C < 0$. We let $n_{\mathcal{S}}$ denote the maximum value of $|n_C|$ among all columns C, and finally let n denote the smallest possible $n_{\mathcal{S}}$ among all $\mathcal{S}$ refining $\mathcal{T}$ and compatible with U, V.

We suppose for a contradiction that $n \neq 0$, and we pick $\mathcal{S}$ with $n_{\mathcal{S}} = n$. Without loss of generality, we assume that $n = n_C$ for some $C \in \mathcal{C}^+$, and we consider the set $\mathcal{D}$ of all columns C such that $n_C = n$. Let B be the union of all the bases of columns in $\mathcal{D}$, and B' the union of the bases of elements of $\mathcal{C}^-$. Then we observe that either $\mu(B) = \mu(B')$ for all $\mu \in K$, or $\mu(B) < \mu(B')$ for all $\mu \in K$ (otherwise $\mu(U)$ and $\mu(V)$ would not be equal). Thus one can build a new KR-partition by cutting and stacking on top of each column of $\mathcal{D}$ some element of $\mathcal{C}^-$ (just map arbitrarily the union of the tops of elements of $\mathcal{D}$ into the union of the bases of $\mathcal{C}^-$, then refine accordingly). This has the effect of producing a new KR-partition such that every column in $\mathcal{C}^+$ satisfies $n_C < n$. Doing the same (if necessary) with $\mathcal{C}^-$, we obtain a contradiction to the minimality of n. □

The previous two propositions provide us with the tools to construct the K-saturated homeomorphism we were looking for.

Proposition 2.6. *There exists a K-saturated element in G.*

Proof. Fix an enumeration (U_n, V_n) of all pairs of clopen sets (U, V) such that $U \sim_K V$. Using Propositions 2.4 and 2.5, we build a sequence of KR-partitions $\mathcal{S}_n$ with the following properties:

(1) For all n, $\mathcal{S}_{n+1}$ refines $\mathcal{S}_n$.
(2) For all n, $\mathcal{S}_n$ is compatible with U_n, V_n, and in each column of $\mathcal{S}_n$ there are as many atoms contained in U_n as atoms contained in V_n.
(3) The diameters of the base and top of $\mathcal{S}_n$ converge to 0.

Then, there exists a unique $g \in \mathrm{Homeo}(X)$ which extends the partial automorphisms associated to $\mathcal{S}_n$. The construction ensures that $g \in G$, and that g is K-saturated. □

Remark 2.7. The set of all K-saturated homeomorphisms, as well as the set of all minimal homeomorphisms in G, are G_δ subsets of G. The argument above proves that the closure of the set of K-saturated elements contains all minimal elements of G; thus, in G, a generic minimal homeomorphism if K-saturated. It would be interesting to determine precisely the closure of the set of all minimal homeomorphisms. It is tempting to believe that it corresponds to the set of all $g \in G$ such that, for any clopen A different from $\emptyset$ and X, one has $g(A) \neq A$ (see [**BDK06**, Theorem 5.9] for the analogous result in Homeo(X)).

So far, we have managed to build a K-saturated, hence minimal, element which preserves all measures in a given dynamical simplex K. It is *a priori* possible that this element preserves measures not belonging to K; saturation prevents this from happening. That was the original motivation for trying to build a K-saturated element of G rather than merely a minimal homeomorphism belonging to G. To deal with this issue, we have one remaining task: proving that the set of G-invariant probability measures, which is by definition larger than K, coincides with K.

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3. Saturated elements cannot preserve unwanted measures

We will denote the simplex of G-invariant probability measures by K_G. Given $g \in G$, the simplex of g-invariant probability measures will be denoted K_g.

Proposition 3.1. *Let g be a K-saturated element. Then $K_g = K_G$.*

Proof. Clearly, $K_G \subseteq K_g$. For any $\mu \in K_g$ and any $h \in [[g]]$ we have $h_*\mu = \mu$. Since $\{h\colon h_*\mu = \mu\}$ is closed in Homeo(X), and the closure of $[[g]]$ is G since g is K-saturated, we obtain as desired that $h_*\mu = \mu$ for all $h \in G$. □

The last remaining piece of our puzzle is thus the following proposition.

Proposition 3.2. *We have $K_G \subset K$.*

Proof. We proceed by contradiction, and assume that $\nu \notin K$ is such that $g_*\nu = \nu$ for all $g \in G$. By homogeneity, if U, V are clopen sets with $U \sim_K V$, then $\nu(U) = \nu(V)$.

Note first that, if $\mu(A) \leq \frac{1}{n}$ for all $\mu \in K$, then there exist disjoint $A_1, \ldots, A_n$ such that $A_i \sim_K A$ for all $i \in \{1, \ldots, n\}$; since by assumption $\nu(A_1) = \ldots = \nu(A_n) = \nu(A)$, we also have $\nu(A) \leq \frac{1}{n}$. This observation will be used twice below.

Using the Hahn–Banach theorem, we know that there exists a continuous function $f : X \to \mathbb{R}$ such that

$$\forall \mu \in K \ \int_X f\,d\mu < \int_X f\,d\nu.$$

Replacing f by $f + \max\{|f(x)|\colon x \in X\}$, and using the fact that all our measures are probability measures, we may assume that $f(x) \geq 0$ for all x. Using uniform continuity of f, we may further assume that f only takes finitely many values, which are all non-negative rational numbers; multiplying by a large enough integer, we finally reduce to the case when f takes finitely many integer values. Hence we have finitely many clopen sets $(A_i)_{1\le i\le N}$ and positive integers n_i such that

$$\forall \mu \in K \ \sum_{i=1}^{N} n_i\mu(A_i) < \sum_{i=1}^{N} n_i\nu(A_i).$$

Pick integers $p,q>1$ such that

$$\forall \mu \in K \ \sum_{i=1}^{N} n_i\mu(A_i) < \frac{p}{q} < \sum_{i=1}^{N} n_i\nu(A_i).$$

Using the fact that K is approximately divisible, we may find for all i some clopen sets $B_{i,1} \sim_K B_{i,2} \sim_K \ldots \sim_K B_{i,p}$ contained in A_i such that $\mu(A_i \setminus \bigcup B_{i,j})$ is arbitrarily small for all $\mu \in K$, hence also $\nu(A_i \setminus \bigcup B_{i,j})$ is arbitrarily small.

Thus, $\frac{\nu(A_i)}{p} - \nu(B_{i,1})$ can be made arbitrarily small, so we can ensure that

$$\forall \mu \in K \ \sum_{i=1}^{N} n_i\mu(B_{i,1}) < \frac{1}{q} < \sum_{i=1}^{N} n_i\nu(B_{i,1}).$$

We can then build a set B which is a disjoint union of n_1 copies of $B_{1,1}$ (that is, n_1 clopen sets which are $\sim_K$-equivalent to $B_{1,1}$), n_2 copies of $B_{2,1}$, etc; we have

$$\forall \mu \in K \ \mu(B) < \frac{1}{q} \quad \text{and} \quad \nu(B) > \frac{1}{q}.$$

This contradicts the observation made at the beginning of the proof. □

Corollary 3.3. *Assume that g is a K-saturated element of G. Then $K = K_g$.*

Proof. We have $K \subset K_g$ by definition of G, and the converse inclusion follows from the previous propositions. □

We have finally proved Theorem 0.2.

Remark 3.4. Using the idea of the proof of the previous proposition, one can check that goodness and approximate divisibility imply that any affine function on K of the form $\mu \mapsto \int_X f\,d\mu$, where $f \in \mathrm{C}(X,[0,1])$, can be approximated arbitrarily well by an affine function of the form $\mu \mapsto \int_X \chi_B$, where χ_B is the characteristic function of a clopen set. This implies, in the terminology of Dahl, that $G(K)$ is dense in $\mathrm{Aff}(K)$.

It seems clear that $G(K)$ being dense in $\mathrm{Aff}(K)$ and approximate divisibility are related conditions; probably, whenever K is a Choquet simplex and $G(K)$ is dense in $\mathrm{Aff}(K)$, K must be approximately divisible. This seems to follow from Dahl's arguments in her paper but, not being experts in dimension groups, we cannot be certain.

We conclude by discussing a further line of enquiry suggested by our work here. As mentioned at the end of the introduction, our argument gives an elementary proof of the fact that, given a minimal homeomorphism g, there exists a saturated homeomorphism h preserving the same measures as g. It follows from a theorem of Krieger [**Kri80**], the proof of which is relatively short and elementary, that any two saturated homeomorphisms preserving the same measures are orbit equivalent. It then becomes interesting to try and find a dynamical proof of the fact that, if g is a minimal homeomorphism, S is a saturated minimal homeomorphism, and $K_g = K_S$, then g and S are orbit-equivalent: combining such a proof, the main result of this paper, and Krieger's theorem, one would obtain a new dynamical proof of the theorem of Giordano–Putnam–Skau relating orbit equivalence and invariant measures.

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Model theory methods for topological groups

Abstract : This thesis gathers different works approaching subjects of topological dynamics by means of logic and descriptive set theory, and conversely.

The first part is devoted to the study of Roelcke precompact Polish groups, which are the same as the automorphism groups of $\aleph_0$-categorical structures. They form a rich family of examples of infinite-dimensional topological groups, including several interesting permutation groups, isometry groups and homeomorphism groups of distinguished mathematical objects. Building on previous work of Ben Yaacov and Tsankov, we develop a model-theoretic translation of several dynamical aspects of these groups, related to the complexity of the orbits of continuous functions and to Banach representations of associated flows, as studied by Glasner and Megrelishvili. Then we use this translation to prove some new results.

In Chapter 1, we prove that every strongly uniformly continuous function on a Roelcke precompact Polish group is weakly almost periodic. We also show that lower tame functions correspond to NIP formulas, and we use this to describe lower tame functions in a number of important examples.

In Chapter 2 (with I. Ben Yaacov and T. Tsankov), we provide a model-theoretic description of the Hilbert-compactification of oligomorphic groups, and we show that Eberlein oligomorphic groups are precisely the automorphism groups of $\aleph_0$-stable, $\aleph_0$-categorical discrete structures. We also give an account of their Hilbert-representable ambits.

In Chapter 3, we study automorphism groups of randomized structures. This gives new examples of Roelcke precompact Polish groups, and we study some associated flows. We give new proofs of several preservation results, and show that Hilbert-representability is preserved by randomizations. We also study the separable models of the theory of beautiful pairs of randomizations, and we classify them in the $\aleph_0$-categorical case.

The second part (with J. Melleray) studies full groups of minimal homeomorphisms of the Cantor space, and their invariant measures. Full groups are complete algebraic invariants for orbit equivalence. Their counterparts in ergodic theory enjoy good, important topological properties.

In Chapter 4, we show that, in contrast, full groups of minimal homeomorphisms do not admit a Polish group topology, and are moreover non-Borel subsets of the homeomorphism group of the Cantor space. We then study the closures of full groups by means of Fraïssé theory.

Finally, in Chapter 5 we give a characterization of the sets of invariant measures of minimal homeomorphisms of the Cantor space. We also present new, elementary proofs of some results previously established by complex means.

Méthodes de théorie des modèles pour l'étude de groupes topologiques

Résumé : Cette thèse rassemble des travaux qui abordent des sujets de la dynamique topologique par le biais de la logique et de la théorie descriptive des ensembles, et réciproquement.

La première partie est consacrée à l'étude des groupes polonais Roelcke précompacts. Cette famille comprend plusieurs groupes de permutations, d'isométries et d'homéomorphismes d'objets mathématiques distingués. Basés sur des travaux précédents de Ben Yaacov et Tsankov, nous développons une traduction modèle-théorique de plusieurs aspects dynamiques de ces groupes. Puis nous utilisons cette traduction pour obtenir une compréhension précise, dans ce cas, de la hiérarchie dynamique étudiée par Glasner et Megrelishvili. Ensuite (avec I. Ben Yaacov et T. Tsankov), nous donnons une description modèle-théorique de la compactification hilbertienne des groupes oligomorphes, et nous caractérisons les groupes oligomorphes Eberlein.

Nous étudions également les groupes d'automorphismes des structures randomisées, ainsi que les modèles séparables de la théorie des belles paires de randomisations.

Dans la deuxième partie (avec J. Melleray), nous étudions les groupes pleins d'homéomorphismes minimaux de l'espace de Cantor et leurs mesures invariantes. Nous montrons que les groupes pleins des homéomorphismes minimaux n'admettent pas de topologie polonaise, puis qu'ils sont des sous-ensembles non-boréliens du groupe d'homéomorphismes de l'espace de Cantor. Ensuite, nous étudions les clôtures des groupes pleins au moyen de la théorie de Fraïssé. Finalement, nous donnons une caractérisation des ensembles de mesures invariantes des homéomorphismes minimaux de l'espace de Cantor.

Image en couverture : Benito Quinquela Martín, *Barcos en la Boca.*

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