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Predictive control and estimation of uncertain systems with an input delay

Vincent Léchappé

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Thèse de Doctorat

Vincent LÉCHAPPÉ

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Predictive control and estimation of uncertain systems with an input delay

JURY

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Abbreviations and notations

List of abbreviations

DOF	Degree Of Freedom	LTV	Linear Time Varying
ESO	Extended State Observer	MIMO	Multi Input Multi Output
FSA	Finite Spectrum Assignment	NCS	Network Control System
HL	High Level	OS	Operating System
ISS	Input-to-State Stability	PDE	Partial Differential Equation
LL	Low Level	ROS	Real-time Operating System
LMI	Linear Matrix Inequality	SISO	Single Input Single Output
LTI	Linear Time Invariant	TDE	Time Delay Estimation

List of notations

\mathbb{N}	Set of positive integers
\mathbb{N}^*	Set of strictly positive integers
\mathbb{R}	Set of real numbers
\mathbb{R}^+	Set of positive real numbers
$0_{n \times m}$	Matrix of 0 with n rows and m columns
$P_{(i,j)}$	Component of the i^{th} row and j^{th} column of matrix P
I_n	Identity matrix of order n
$P > 0$	The matrix P is positive definite
$\lambda(P)$	Spectrum of matrix P
$\lambda_{min}(P)$	Smallest eigenvalue of the symmetric matrix P
$\lambda_{max}(P)$	Largest eigenvalue of the symmetric matrix P
$\ x\ $	Euclidean norm of the vector x
$\ P\ $	Spectral norm of the matrix P
$ a $	Absolute value of the scalar a
$x \rightarrow y$	“ x converges to y ” means $\lim_{t \rightarrow +\infty} \ x(t) - y(t)\ = 0$
$[a]$	Integer part of the scalar a
$C^m(U, V)$	Set of functions defined in U with values in V which has m continuous derivatives
$u^{(i)}(t)$	i^{th} derivative of u with respect to time
$\mathcal{K}([0, a], \mathbb{R}^+)$	Set of strictly increasing and continuous functions defined on $[0, a[$ with values in \mathbb{R}^+ which take the value 0 at 0
$\mathcal{KL}([0, a[\times \mathbb{R}^+, \mathbb{R}^+)$	Set of continuous functions that belong to the class \mathcal{K} with respect to the first parameter and that is decreasing with respect to the second parameter and tends to 0 in $+\infty$.

Contexte et organisation de la thèse

Les motivations

Cette thèse s'intéresse à la commande basée sur la prédiction de systèmes incertains avec une entrée retardée. Les causes de l'apparition d'un retard sont généralement de deux types:

- Elles sont parfois intrinsèques au système comme c'est le cas pour les phénomènes de transport de matière. On peut citer les exemples classiques de la douche, du chauffage ou un peu moins connus mais tout aussi important dans l'industrie: la recirculation des gaz d'échappement ou le procédé de fraisage.
- Elles sont dues à la nature du système de commande. Par exemple, un temps de communication apparaît si le système et l'organe de commande sont séparés dans l'espace. Les temps de calcul peuvent aussi être à l'origine de retards non négligeables.

Ces derniers retards sont de plus en plus présents du fait du développement des communications sans fil et donc de la commande à distance. En particulier, de nombreux travaux s'intéressent aux systèmes commandés par réseaux (NCS) en prenant en compte notamment les retards et la perte de paquets introduite par le canal de communication. A titre d'exemple, on peut citer la commande à distance d'engins volants par communication sans fil. L'objectif final est de prendre en compte les phénomènes induits par les retards de communication afin d'améliorer la navigation de tels engins en totale autonomie. Depuis peu, la communauté liée à la commande des systèmes est très active sur le pilotage de drones. Cependant, la plupart des travaux ne prend pas en compte les retards dans la conception des lois de commande. A long terme, l'idée sous-jacente est la conception d'engins équipés de capteurs bon marché et de modules de communication capable d'envoyer des mesures et de recevoir des ordres calculés par un calculateur à distance.

Le schéma de la Figure 1 montre les différentes latences qui peuvent survenir dans la boucle de commande d'un quadricoptère. On voit que plusieurs sources de retard interviennent simultanément dans la boucle. On peut mentionner les retards dus aux temps de communication, ceux dus aux temps de calcul et enfin ceux dus aux temps de mesure. En général, il est difficile d'évaluer ces retards et la plupart du temps, seules des valeurs approximatives sont disponibles. On peut aussi remarquer que les temps de calcul sont variables et que les OS temps réels peuvent seulement garantir des valeurs maximales qui ne seront pas dépassées. En ce qui concerne la communication sans fil, les retards introduits vont dépendre de plusieurs facteurs :

- les standards de télécommunication,
- la configuration des modules émetteurs/récepteurs,
- l'état de la batterie,
- l'environnement (temps, obstacles)

qui influenceront sur le canal de propagation. L'influence directe de ces différents facteurs n'étant pas quantifiable précisément, l'exacte valeur du retard est souvent mal connue. La Figure 2

montre une trame reçue par deux modules Xbee Pro® Dig [2009] dont les configurations sont différentes. La même trame est envoyée sur les Figures 2a and 2b et pourtant les informations sont reçues différemment selon la configuration du récepteur. Dans les deux cas, l'information est reçue avec du retard. De plus, ce retard n'est pas constant car l'amplitude de ces retards peut varier de 19 ms pour la configuration 1 à 5 ms pour la configuration 2. Les raisons mentionnées ci-dessus justifient notre objectif de considérer des retards variants et mal connus.

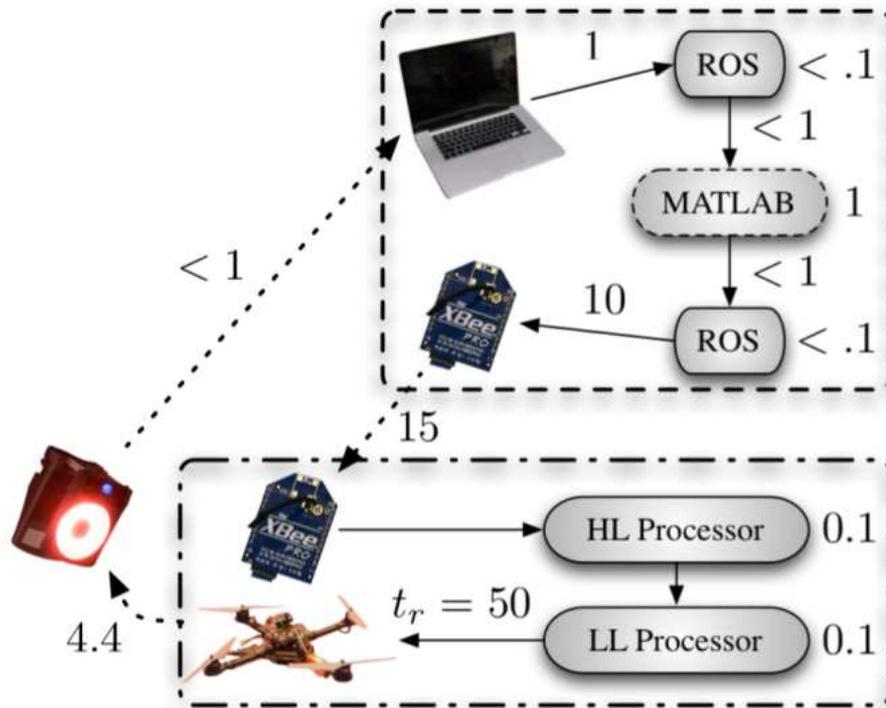


Figure 1 – Latences (en ms) affectant la boucle de commande d'un quadrirotor, extrait de [Mellinger, 2012, pp. 24]. LL: Low Level, HL: High Level

Dans l'exemple de la Figure 1, le retard cumulé est d'environ 80 ms. Cela signifie que 80 ms séparent le moment entre lequel la commande est calculée et l'instant pour lequel elle va être appliquée au drone. Pour des systèmes dont les dynamiques sont très faibles, un tel retard n'affecterait que très peu le comportement du système. Cependant, les systèmes ayant des dynamiques très rapides (comme les engins volants) peuvent être déstabilisés par la présence de très petits retards. Une solution simple pour réduire l'impact de ces retards est donc de diminuer les gains du contrôleur afin d'obtenir une dynamique du système bouclé plus lente que la valeur du retard. En contrepartie, cette méthode conduira inévitablement à une dégradation de l'atténuation des perturbations et du suivi de trajectoire: il est donc nécessaire de trouver des alternatives plus efficaces. La méthode classique utilisée pour compenser les retards dans la commande est de prédire le futur état du système afin de pouvoir compenser parfaitement le retard. Pour calculer une prédiction précise, il est impératif

- 1) d'avoir un modèle précis du système,
- 2) de connaître la valeur du retard.

En reprenant l'exemple des engins volants, le point 1) n'est généralement pas satisfait à cause de nombreux phénomènes tels que les couplages, les effets de sols¹. De plus, les inerties

1. Dans le cas sans retard, les modèles très précis ne sont pas compatibles avec la conception de lois de commande du fait de leur complexité.

sont souvent difficiles à identifier. De la même manière, des perturbations extérieures telles que des rafales de vent ne sont pas prédictibles et vont impacter fortement la qualité de la prédiction. **Par conséquent, l'un des objectifs de la thèse est d'améliorer la qualité de la prédiction malgré les incertitudes de modèle et les perturbations extérieures.** En ce qui concerne le point 2), nous venons de voir un peu plus haut qu'il n'est pas toujours possible de connaître la valeur du retard. En pratique, seulement les bornes maximum et minimum du retard sont généralement connues. **C'est pourquoi, une partie de la thèse sera dédiée au développement de techniques d'estimation de retards. En conclusion, l'objectif de la thèse est d'appliquer des méthodes de commande basée sur la prédiction pour piloter des systèmes incertains en présence d'un retard inconnu et variant dans le temps.**

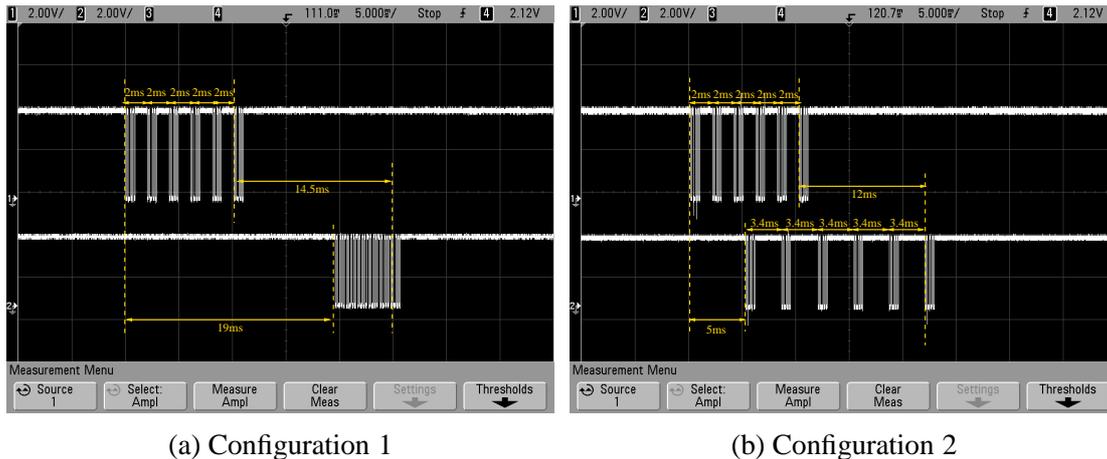


Figure 2 – Deux configurations différentes du récepteur Xbee Pro[®]

Organisation et contribution de la thèse

Pour répondre à ces objectifs, la thèse est divisée en trois parties:

- la première partie présente des solutions d'estimation d'état et de retard,
- la deuxième partie introduit des nouvelles techniques de commande basée sur la prédiction,
- la troisième partie illustre les précédents résultats sur la commande expérimentale d'un moteur à courant continu.

L'organisation détaillée est décrite ci-dessous et est résumée sur le schéma de la Figure 3. Les contributions sont rappelées dans la présentation de chaque chapitre. **Les théorèmes encadrés, présentés dans cette thèse sont des résultats nouveaux constituant les contributions de la thèse.**

La **Partie I** est dédiée à la présentation de nouvelles méthodes d'estimation à la fois du retard et de l'état du système. L'intérêt d'avoir une estimation du retard est double. Tout d'abord, cela permet d'améliorer la qualité de l'observation de l'état puisque les résultats existants considèrent généralement des solutions avec des grands gains pour atténuer l'erreur d'observation due aux incertitudes sur le retard. De plus, une prédiction peut être calculée à partir du retard estimé. La première contribution de cette partie est donc de proposer des solutions d'observation qui permettent d'estimer simultanément le retard et l'état. La seconde contribution réside dans le fait que ces résultats s'appliquent à des retards variants dans le temps. Dans cette partie, on étudiera seulement les performances des estimateurs en boucle ouverte (pas de contrôleur).

Dans le **Chapitre 2**, une méthode d'observation basée sur la construction d'un unique observateur est présentée. Une première version de ce travail a été publiée dans [Léchappé et al. \[2015a\]](#) pour des systèmes linéaires. Une extension aux systèmes non linéaires est disponible dans [Léchappé et al. \[2015b\]](#). L'idée originale se trouve dans la construction d'un observateur à grand gain sur un système étendu qui considère le retard comme un nouvel état. La convergence de l'erreur d'observation est prouvée formellement et des simulations illustrent les performances de cette méthode

Une méthode basée sur la théorie des observateurs interconnectés est proposée dans le **Chapitre 3**. Dans un premier temps, un nouvel observateur de perturbation est présenté afin d'estimer l'entrée retardée. La nouveauté est d'introduire artificiellement un retard pour approximer la dérivée de la dynamique inconnue par la méthode des différences finies. Ensuite, l'idée générale de l'interconnexion des observateurs d'état et de retard est présentée. Il s'agit de coupler un observateur d'état standard (disponible pour des systèmes sans retard) avec un observateur de retard. Les conditions de stabilité sont données dans le cas général puis illustrées sur un exemple particulier. L'efficacité de la méthode est validée en simulation. Une comparaison avec l'observateur du **Chapitre 2** est aussi réalisée en simulation. Le contenu de ce chapitre est résumé dans l'article [Léchappé et al. \[2016\]](#).

La **Partie II** est dédiée à la présentation de techniques de prédiction. L'objectif principal est de proposer des méthodes nouvelles qui offrent des performances satisfaisantes en présence d'incertitudes paramétriques, de perturbations extérieures et de retard inconnu. Des critères explicites de stabilité sont obtenus grâce à une analyse de Lyapunov-Krasovskii. Le sens physique de ces critères est analysé dans le détail.

Le **Chapitre 4** se concentre sur le cas d'un retard connu et est divisé en trois sections. Dans la première section (**Section 4.1**), la prédiction standard, originellement conçue pour un retard constant, est étendue à un retard variant. La stabilité est préservée du moment que le retard ne varie pas trop vite. Ce résultat, déjà connu pour des lois de commande par retour d'état statique, est étendue ici à une classe plus large de lois de commande et au cas d'une connaissance partielle de l'état. Une condition de stabilité est donnée à l'aide d'une fonctionnelle de Lyapunov. La deuxième section (**Section 4.2**) vise à étendre un résultat récent qui permet de calculer une prédiction à partir d'un système dynamique. Le résultat original s'appliquant aux retards constants et aux retours d'état statique est étendu aux retards variants, à une plus grande classe de lois de commande et aux cas où l'état est partiellement mesuré. Enfin, la dernière section (**Section 4.3**) décrit une nouvelle prédiction qui présente de meilleures propriétés de robustesse que la prédiction classique vis à vis des perturbations extérieures et des incertitudes paramétriques. L'idée est de concevoir une prédiction qui inclut indirectement de l'information des dynamiques non modélisées. Ce travail a été présenté dans [Léchappé et al. \[2015c\]](#) pour le cas d'une connaissance totale de l'état et dans [Léchappé et al. \[2015d\]](#) pour le cas d'une connaissance partielle. L'extension de ces travaux à des retards variants dans le temps est aussi traitée dans cette section. Tous les résultats sont illustrés par des exemples académiques et font l'objet d'une étude détaillée en simulation.

Dans le **Chapitre 5**, les résultats des Sections 4.1 et 4.2 sont étendus aux retards inconnus, respectivement dans les **Sections 5.1** et **5.2**. Dans chaque cas, des fonctionnelles de Lyapunov sont utilisées pour étudier la stabilité de la boucle fermée. La construction systématique de ces fonctionnelles est une des contributions de ce chapitre. L'autre contribution est la proposition de lois de commande basée sur la prédiction pour contrôler des systèmes avec des retards inconnus et variants. Ces lois de commande sont couplées, en simulation, à l'observateur de retard du **Chapitre 3**.

Les résultats présentés dans les chapitres précédents sont illustrés expérimentalement dans la **Partie III**.

Ainsi, dans le **Chapitre 6**, un moteur à courant continu (DC) est piloté en présence d'incertitudes paramétriques, d'une perturbation extérieure et d'une entrée retardée. Le retard étant artificiellement introduit, il a été possible de tester de nombreuses configurations. L'observateur de retard du **Chapitre 3** est combiné aux méthodes prédictives de la **Partie II**. Des tests avec des retards sur l'entrée et sur la mesure, inconnus et variés, ont été faits pour montrer que ces méthodes pourront être étendues au cadre plus large de "sortie et entrée retardées". Les résultats de ce chapitre ont été publiés dans [Léchappé et al. \[2015f\]](#) pour un retard constant et connu. L'extension à un retard inconnu est disponible dans [Léchappé et al. \[2015e\]](#).

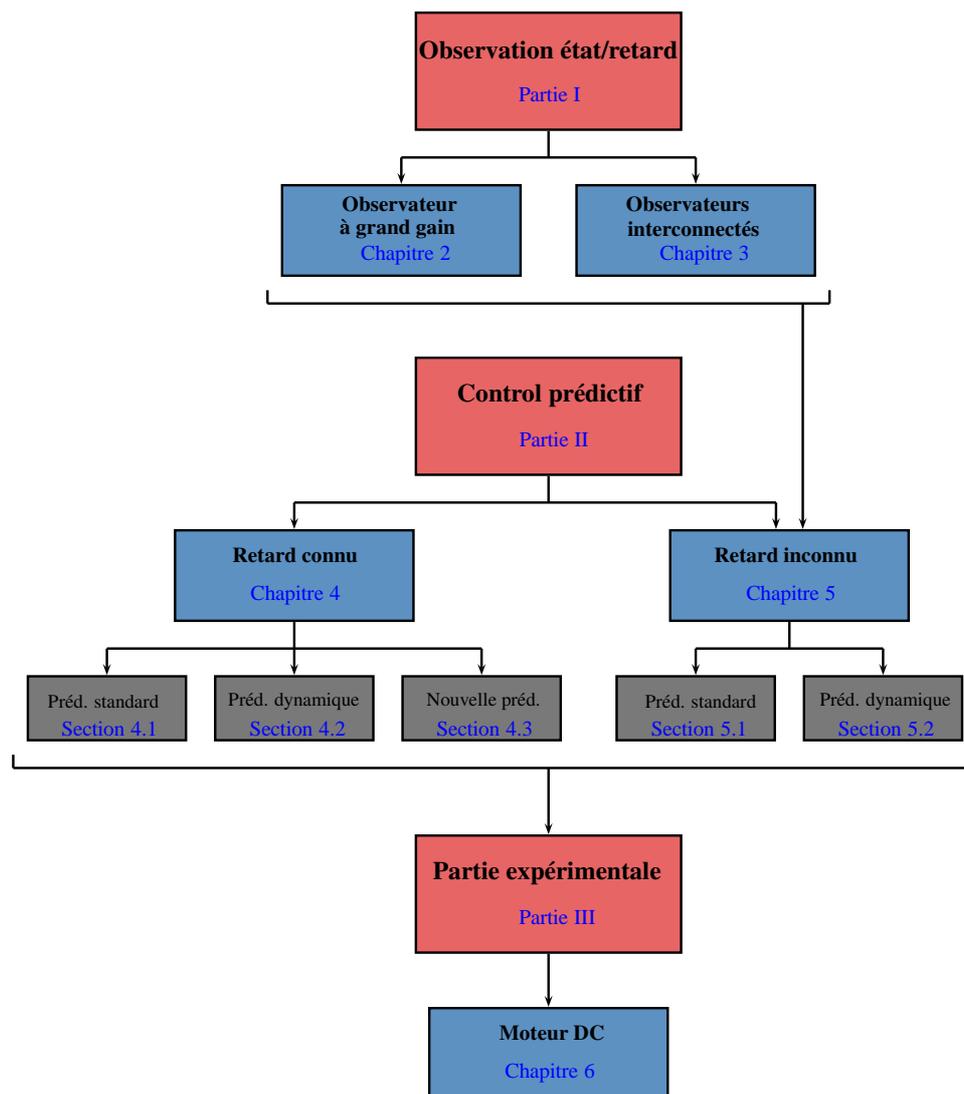


Figure 3 – Organisation de la thèse (préd.:prédiction)

Certains des résultats de la thèse ont été publiés ou sont en cours de révision pour une future publication dans des conférences ou des revues.

Revues internationales

- 1) V. Léchappé, J. De León, E. Moulay, F. Plestan, A. Glumineau, *Delay-state observer for SISO nonlinear systems*, en deuxième lecture à Automatica.
[Léchappé et al. \[2015b\]](#)
- 2) V. Léchappé, S. Rouquet, M. Gonzáles, F. Plestan, J. De León, E. Moulay, A. Glumineau *Delay estimation and predictive control of uncertain systems with input delay: application to a DC motor*, accepté dans IEEE Transactions on Industrial Electronics.
[Léchappé et al. \[2015e\]](#)
- 3) V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, A. Chriette, *New predictive scheme for the control of LTI systems with input delay and unknown disturbances*, Automatica, 52(2), 179-184, 2015.
[Léchappé et al. \[2015c\]](#)

Conférences internationales

- 1) V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, *Interconnected delay and state observer for nonlinear systems with time-varying input delay*, soumis à American Control Conference, Boston, USA, 2016
[Léchappé et al. \[2016\]](#)
- 2) V. Léchappé, J. De León, E. Moulay, F. Plestan, A. Glumineau, *Delay and state observer for SISO LTI systems*, American Control Conference, Chicago, USA, 2015
[Léchappé et al. \[2015a\]](#)
- 3) V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, A. Chriette, *Predictive scheme for observer-based control of LTI systems with unknown disturbances*, European Control Conference, Linz, Autriche, 2015
[Léchappé et al. \[2015d\]](#)
- 4) V. Léchappé, O. Salas, J. De León, F. Plestan, E. Moulay, A. Glumineau, *Predictive control of disturbed systems with input delay: experimental validation on a DC motor*, Time-Delay System workshop, Ann Arbor, USA, 2015
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General introduction

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1.1 Motivations

This thesis is dedicated to the prediction-based control of uncertain input delay systems. The input delay can appear because of the physical nature of the plant: it is usually due to transport phenomenon. Consequently, these systems are very common in chemical, hydraulic and pneumatic industries. The shower, the exhaust gas recirculation, the air heater and the crushing mill are classical examples. The other source of input delay, extrinsic to the system, can arise from communication latencies, computation time or sensor measurements. These delays are more and more common because of the fast development of remote controllers. In particular, lots of works are tackling the problem of network control systems (NCS) that can introduce time-varying and/or packet drops as in [Tipsuwan and Chow \[2003\]](#), [Zhang et al. \[2001\]](#), [Bemporad et al. \[2010\]](#). Some more examples and further details are provided in [Bresch-Pietri \[2012\]](#), [Chiasson and Loiseau \[2007\]](#), [Vyhlídal et al. \[2014\]](#), [Zhong \[2006\]](#). In our case, we are interested in the wireless control of remote devices. The final goal is to improve the control of flying objects. Recently, the control community has been very active on this topic as shown by the numerous works as in [Bouabdallah \[2007\]](#), [Mellinger \[2012\]](#), [Tayebi and McGilvray \[2006\]](#) and references therein. Except a few articles by [Lozano et al. \[2004\]](#), [Ordaz et al. \[2013\]](#),

Ossa-Gomez et al. [2011], most of existing works do not take into account delays in the control design. In the long run, the underlying idea is to design devices equipped with low cost sensors and communication modules able to send measurements and to receive control orders computed by remote super computers.

The scheme given on Figure 1.1 displays the different latencies that can show up in the control loop of a quadrotor. It can be observed that the communication delay comes along with computation and sensor delays. These delays are difficult to measure and most of the time, only an approximation is available. Remark that, computation delays are time-varying and real-time OS can only guarantee a maximum bound on these delays. As for the wireless channel, the delay value depends on various factors such as

- telecommunication standards,
- configuration of the emitter/receiver modules,
- battery life,
- environment (weather conditions, obstacles).

These phenomena are not easy to quantify and the value of the delay is generally not known accurately. Figure 1.2 shows a frame received by two Xbee Pro[®] modules Dig [2009] which have different configurations. The same frame is sent on Figures 1.2a and 1.2b and yet the information is received differently according to the configuration of the receiver. In both cases, it is clear that the information is delayed. In addition, note that this delay is not constant: it varies from 5 ms for configuration 2 to 19 ms for configuration 1. That is why the final objective of the thesis is to deal with unknown and time-varying delays.

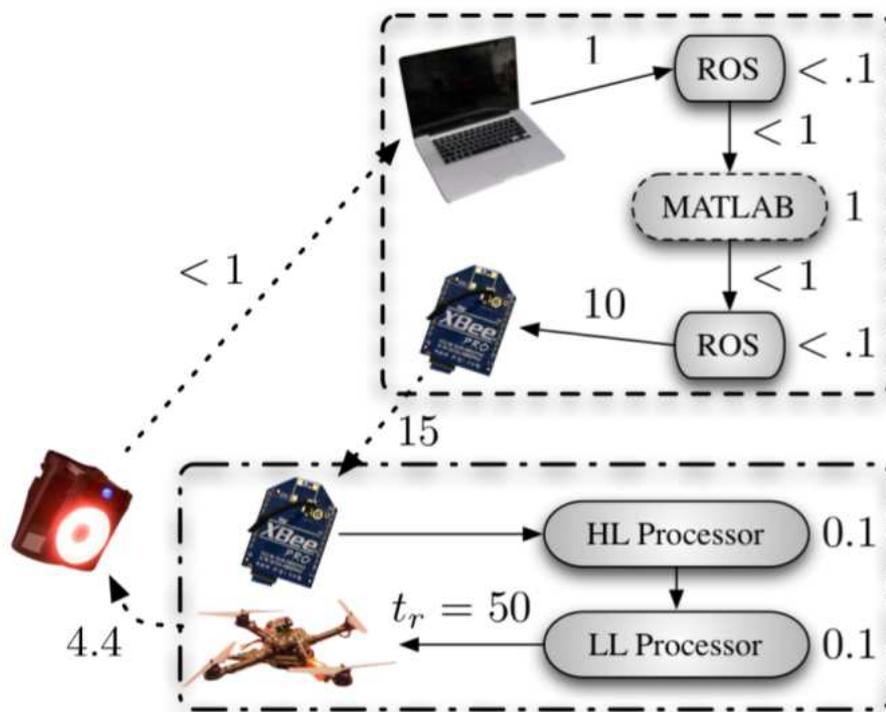


Figure 1.1 – Latencies in the experimental setup (ms) from [Mellinger, 2012, pp. 24] LL: Low Level, HL: High Level.

In the application of Figure 1.1, a total amount of about 80 ms can separate the control value computation from its real application to the drone. For systems with very slow dynamics, such a delay would hardly affect their behaviors. However, unstable systems with fast dynamics like

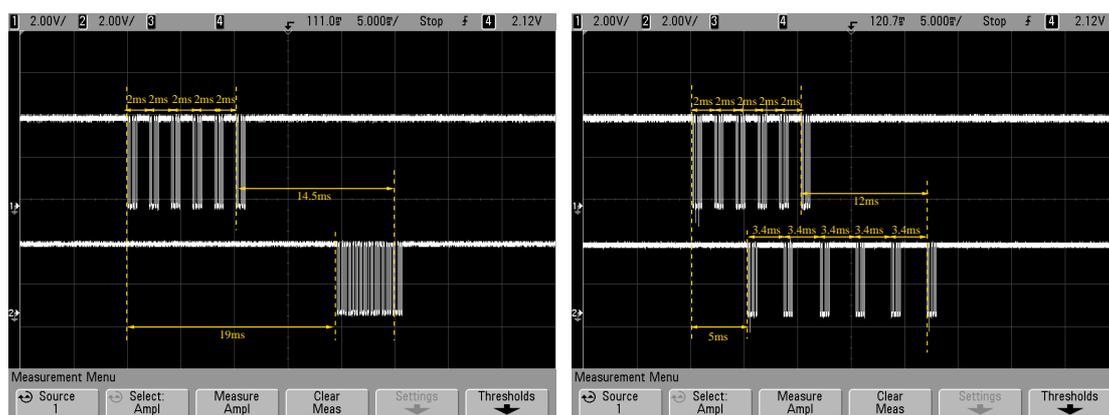
flying objects can be destabilized by the presence of small latencies. The simplest way to reduce the delay effect is to decrease control gains in order to have a closed-loop dynamics slower than the delay value. This will necessarily lead to worst disturbance attenuation and slower trajectory tracking so it is necessary to find alternatives. The well-known method to control systems with input delays is to predict the future state of the plant in order to perfectly compensate the delay. To compute an accurate prediction, it is imperative

- 1) to have a precise model of the plant and
- 2) to know the delay.

Coming back to the flying objects example, the first point is not fulfilled because some phenomena like coupling, ground effect, asymmetric inertia are difficult to model¹. Furthermore, unknown disturbances such as wind gusts are very detrimental to the accuracy of the prediction. **Then, one of the objective of the thesis is to improve the quality of the prediction in spite of model uncertainties and external disturbances.** Note that very recent articles by [Armah and Yi \[2015\]](#), [Sanz et al. \[2015\]](#), [Sanz et al. \[2014\]](#), [Alatorre et al. \[2014\]](#) have developed predictive techniques to improve the control of flying objects which confirms that this is a trendy topic. As for the second point, we have just seen that it is not always possible to know the delay. Usually in practice, only lower and upper bounds are available. **That is why an other part of this thesis is focused on the development of delay estimators.** Unlike most existing works, it has been decided to address this problem from the observation point of view. To sum up, we can say that **the objective of this work is to apply prediction-based control to uncertain systems in presence of an unknown time-varying delay and an external disturbance.**

To deal with this challenging objective, the work has been divided into three parts:

- the first part presents some delay-state estimators for unknown and time-varying delays,
- the second part presents new predictive techniques to overcome some limitation of the standard predictive control,
- the third part illustrates previous results by performing experimental tests on a DC motor.



(a) Configuration 1

(b) Configuration 2

Figure 1.2 – Two different configurations of the Xbee Pro[®] receiver

1. For the delay free case, very accurate models are not compatible with the design of controllers because of their complexity.

1.2 Overview of prediction-based control methods

In this section, the state of the art on predictive control is given to set the theoretical context of the thesis. It is important to note that all the systems depicted on Figure 1.3 can be controlled by the same predictive techniques presented thereafter. There exist two major streams in the control of input delay systems: memoryless and memory controllers.

Memory free controllers In Mazenc et al. [2003], it is shown that a chain of integrators with an arbitrarily large delay in the input can be stabilized² by a bounded feedback. This conclusion is extended to feedforward systems in Mazenc et al. [2004]. Some works have also been dedicated to the control of open-loop stable systems with saturated feedbacks Lin and Haijun [2007], Zhou et al. [2010], Yakoubi and Chitour [2007]. The disadvantage of these results is that they generally lead to slow response time because the control is designed for arbitrarily large delays. Recent papers have also considered saturated control but for a more general class of systems Wang et al. [2013], Liu and Fridman [2014]. In that case, some conditions on the delay size appear to preserve the closed-loop stability since the exact delay compensation is not possible and the systems are possibly unstable.

The works of Richard et al. [2001], Fridman et al. [2002] and Gouaisbaut et al. [2002] study the influence of delay for sliding mode control. It has been shown that the delay can induce oscillations of finite frequency³ around the sliding surface and even instability. Specific sliding mode controllers are presented in Richard et al. [2001] and upper delay bounds preserving the stability are provided. A descriptor approach is used in Fridman et al. [2003a] to compute a delay upper bound that guarantees the closed stability for a class of sliding mode controllers. For more details about sliding modes for delay systems, one can refer to Han et al. [2012].

Some other control techniques have been revisited in presence of delays. In Michiels et al. [2002], a continuous pole placement method is proposed to assign the right most eigenvalue of the closed-loop system. Adaptive output feedbacks have been designed to stabilize a chain of integrators: with a constant delay in the input Choi and Lim [2006], Choi and Lim [2010b], with an unknown but bounded time-varying delay Choi and Lim [2010a]. A H_∞ delay-scheduled controller is proposed in Briat et al. [2009] for known time-varying delays. For more results about infinite horizon LQR, H_∞ control with time-varying delays, the reader can refer to Fridman [2014a] and references therein.

To sum up, some particular systems can remain stable with arbitrarily long delays but it is often at the cost of very sluggish dynamics. For a large class of systems, the stability using memoryless controllers is guaranteed only for sufficiently small delays.

Memory controllers This interest has been explained in Mirkin and Raskin [2003] where it is shown that every stabilizing dead-time controller has an observer/predictor structure. Quoting Mirkin and Raskin [2003], *state prediction is a fundamental concept for delay systems, much like state observation is for systems with incomplete state measurements*. This control technique is especially required for unstable systems with large input delays. As a result, a large part of the thesis will be focused on this technique. Since the first step in the 1950s by Smith, the active 1970s and the introduction of LMIs in 1990s, a vast literature is available on that topic. Here, only a quick insight into the most important results will be provided in order to contextualize our contribution. For more information, one can refer to survey papers Richard [2003], Gu and Niculescu [2003] and to books Hale and Verduyn Lunel [1993], Bekiaris-Liberis and Krstic [2013], Fridman [2014a], Krstic [2009], Malisoff et al. [2015] and Zhong [2006].

2. The global uniform asymptotic stability is proven.

3. The amplitude of such oscillations have been quantified in Richard et al. [2001].

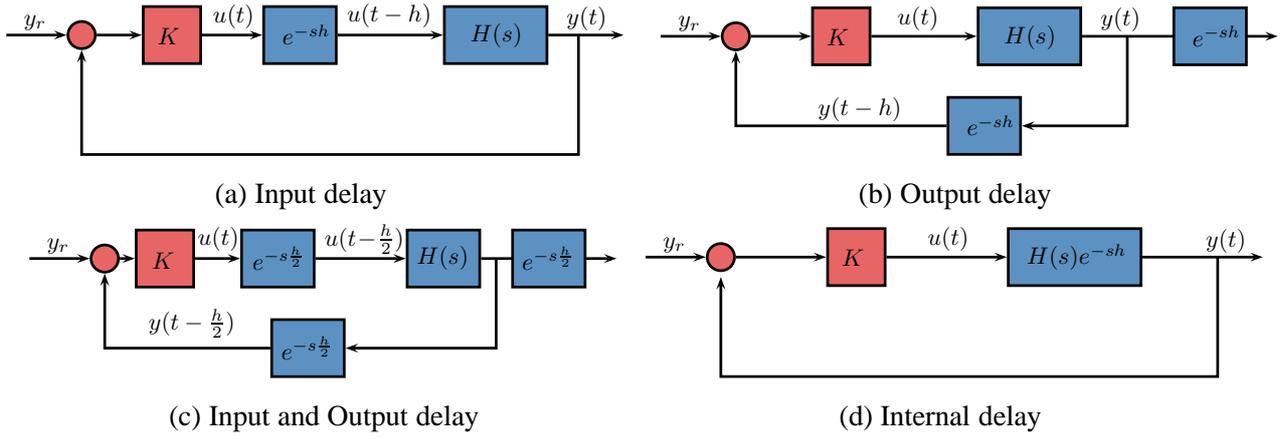


Figure 1.3 – Equivalent systems from the transfer function point of view.

1.2.1 Prediction-based control for constant delays

Smith Predictor

The most well-known method to control dead time systems is probably the Smith predictor. This frequency approach was introduced by Smith at the end of the 1950s in Smith [1957] and Smith [1959]. The idea is based on the design of an inner loop that computes a prediction of the output (see Figure 1.4). The weakness of this controller is that it can only stabilize stable plants. Let us illustrate the cause of instability by studying a first order example.

Example 1.2.1. Consider the system

$$\dot{x} = ax + bu(t - h) \quad (1.1)$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $h > 0$. Its transfer representation is

$$H_0(p) = \frac{X(p)}{U(p)} = \frac{be^{-ph}}{p - a} \quad (1.2)$$

where $X(p)$ and $U(p)$ are respectively the Laplace transform of $x(t)$ and $u(t)$ and p is the Laplace variable. Computing the transfer function of the inner loop of scheme 1.4, one obtains

$$\frac{U(p)}{E(p)} = \frac{(p - a)K}{p - a + Kb(1 - e^{-ph})}, \quad (1.3)$$

then

$$\frac{Y(p)}{E(p)} = \frac{Kbe^{-ph}}{p - a + Kb(1 - e^{-ph})}. \quad (1.4)$$

Finally, the transfer function of the whole system is given by

$$\frac{Y(p)}{Y_r(p)} = \frac{Kb}{p - (a - Kb)}e^{-ph}. \quad (1.5)$$

It can be seen that the delay does not affect the stability so the gain K can be selected as if the system was delay free. With this input/output approach, the instable zero is masked by the pole-zero cancellation in (1.4). However, the characteristic equation is

$$\det \begin{bmatrix} p - a & -be^{-ph} \\ (p - a)K & p - a + Kb(1 - e^{-ph}) \end{bmatrix} = (p - a)(p - (a - Kb)) = 0. \quad (1.6)$$

Thus, if the open-loop system is unstable ($a > 0$), then the closed-loop system is also unstable (independently of the controller). This conclusion can be generalized to any plant H_0 as in [Furukuwa and Shimemura \[1983\]](#). In order to compare with the next techniques, the time domain expression of the feedback u is computed by taking the inverse Laplace transform of (1.3)

$$u(t) = -K \left[x(t) + \int_0^t e^{a(t-s)} bu(s) ds - e^{-ah} \int_{-h}^{t-h} e^{a(t-s)} bu(s) ds \right] \quad (1.7)$$

with $y_r = 0$.

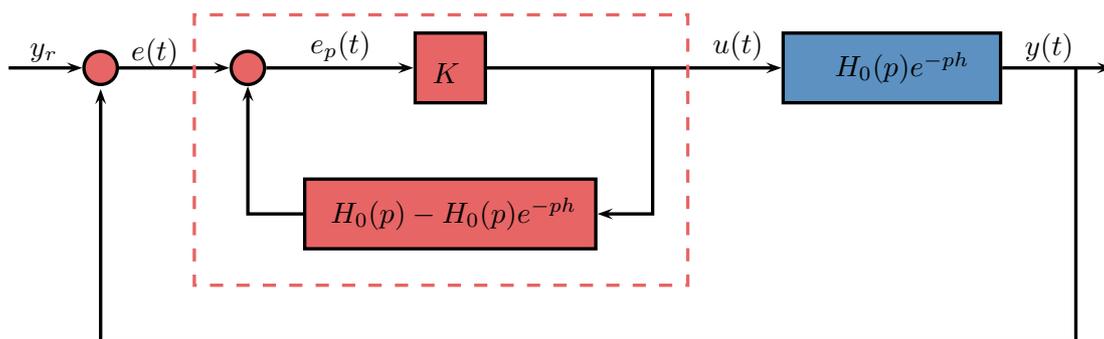


Figure 1.4 – Smith controller is displayed in red. The transfer H_0 represents the delay free plant. Noting that $H_0(p)U(p)$ is the prediction of the state of $H_0(p)e^{-sh}U(p)$ and computing the error $e_p(t) = y_r - y(t+h)$, then the control $u = Ke_p$ is able to perfectly compensate h for stable plants H_0 .

Some works have extended the Smith predictor to unstable systems as in [Kwak et al. \[1999\]](#), [De Paor \[1985\]](#), [Watanabe and Ito \[1981b\]](#), [Normey-Rico and Camacho \[2009\]](#) or to multivariable systems as in [Wang et al. \[2000\]](#). An adaptive Smith Predictor for possibly open-loop unstable plants is presented in [Niculescu and Annaswamy \[2003\]](#). Various articles also deal with the disturbance robustness of the predictor: see [Astrom et al. \[1994\]](#), [Liu et al. \[2005\]](#), [Matausek and Micic \[1999\]](#) and [Watanabe and Ito \[1981b\]](#). The resulting controllers are sometimes called modified (or generalized) Smith predictor (MSP). An analysis of the digital version of these controllers can be found in [Palmor and Halevi \[1990\]](#). For a more extensive review on the Smith Predictor and these modifications the reader can refer to [Palmor \[1999\]](#) and [Normey-Rico and Camacho \[2007\]](#).

Finite Spectrum Assignment (FSA) and model reduction

At the end of the 1970's and the beginning of the 1980's, the result of Smith has been extended to state-space representation and unstable systems by [Artstein \[1982\]](#), [Kwon and Pearson \[1980\]](#), [Olbrot \[1978\]](#). The idea of these new approaches is described thereafter. Consider the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t-h) \quad (1.8)$$

where $x \in \mathbb{R}^n$, $h > 0$ and (A, B) is controllable. To obtain $\dot{x} = (A + BK)x(t)$, it is necessary to have $u(t-h) = Kx(t)$ so

$$u(t) = Kx(t+h). \quad (1.9)$$

This is a predictive feedback since it requires the knowledge of the future state of the system $x(t+h)$. Reminding that the solutions of (1.8) can be computed from any initial solution at

$x(t_0)$, by the formula

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-s)}Bu(s-h)ds \quad (1.10)$$

for all $t \geq 0$, it can be deduced that

$$x(t+h) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds. \quad (1.11)$$

Thus, (1.9) becomes

$$u(t) = K \left[e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds \right]. \quad (1.12)$$

The computation of the above control law requires previous values of the control input u because of the distributed delay term $\int_{t-h}^t e^{A(t-s)}Bu(s)ds$. Therefore, controller (1.12) is called a memory feedback, otherwise, feedbacks that do not have any distributed delay terms are called memoryless feedbacks. The closed-loop dynamics of (1.8) with controller (1.12) reads as

$$\dot{x}(t) = \begin{cases} e^{At}x(0) + \int_0^t e^{A(t-s)}B\phi(s)ds & \text{if } 0 \leq t \leq h \\ (A+BK)x(t) & \text{if } t > h \end{cases} \quad (1.13)$$

where ϕ is the initial condition of u : $u(s-h) = \phi_u(s)$ for all $s \in [0, h]$. As shown by (1.13), delayed system (1.8) is turned into a delay free system with a finite spectrum. Besides, this spectrum can be assigned thanks to the gain K , that is why the method is called Finite Spectrum Assignment (FSA). It has been proposed by [Kwon and Pearson \[1980\]](#) and [Manitius and Olbrot \[1979\]](#) for input delay systems and extended to state delay and input delay systems in [Yanushevsky \[1991\]](#), [Fiagbedzi and Pearson \[1986\]](#), [Jankovic \[2010\]](#) and [Kharitonov \[2014\]](#) and to neutral systems with input delay in [Kharitonov \[2015\]](#). Input and output delays are also considered in [Zhou et al. \[2013\]](#). The recent paper by [Yoon and Lin \[2015\]](#) deals with input, output and state delays. Note that, in the context of optimal control of systems with time-lag, some articles have paved the way of the FSA by introducing memory feedbacks [Kleinman \[1969\]](#), [Koivo and Lee \[1972\]](#), [Lewis \[1979\]](#), [Slater and Wells \[1972\]](#). Similarly, the work of [Mayne \[1968\]](#) has received little attention though it can be considered as a seminal work on FSA technique.

The reduction method proposed by [Artstein \[1982\]](#) is similar to the FSA technique in the sense that it uses a transformation based on the prediction. However, it is more general because any kind of predictive controllers can be designed on the reduced system. Besides, Artstein's work deals with LTV systems with distributed delay in the input. Here, only the point wise delay case for LTI systems is described. Denoting

$$z(t) = x(t) + \int_{t-h}^t e^{A(t-s-h)}Bu(s)ds, \quad (1.14)$$

system (1.8) can be rewritten in the following form

$$\dot{z}(t) = Az(t) + e^{-Ah}Bu(t). \quad (1.15)$$

System (1.15) is now input delay free, the delay only appears as a parameter⁴. Remark that

4. It is was proven in [Banks et al. \[1971\]](#) that (A, B) is controllable if and only if $(A, e^{-Ah}B)$ is controllable.

$z(t) = e^{Ah}x(t+h)$ is a predictive transformation. Similarly, the transformation can be carried out with

$$x_p(t) = x(t+h). \quad (1.16)$$

Therefore, (1.8) becomes

$$\dot{x}_p(t) = Ax_p(t) + Bu(t). \quad (1.17)$$

System (1.17) is delay free which means that all the standard control techniques developed for this class of systems can be applied. In particular, the feedback $u(t) = Kx_p(t)$ that leads to the assignment of the poles of $(A + BK)$. It is clear that the feedback $u(t) = Kx_p(t)$ is the same as the controller (1.12) designed from the FSA technique. The advantage of the reduction method is that much more controllers can be designed not only a state feedback. For instance, sliding mode controllers have been designed by Roh and Oh [1999] and Roh and Oh [2000] based on the reduced system. The reduction method has also been extended to the partial state knowledge case by Di Loreto et al. [2008]. In this later work, an H_∞ controller is designed on the reduced model. Other works by Tadmor [2000] and Zhong [2003] have also combined H_∞ control with predictive control. The reduction method has also been used to control a special class of feedforward systems with a dynamic gain in Koo et al. [2012a].

Comparing feedback (1.7) obtained from the Smith predictor and feedback (1.12) from the FSA technique, a difference can be observed as it was mentioned in [Krstic, 2009, p. 23]. This minor difference comes from the way both controllers are computed. In practice, this difference is not significant as it will be displayed on the next simulation (Figure 1.5a).

Example 1.2.2. *In order to illustrate the features of these predictive techniques, some simulations have been run on the same model as in example 1.2.1. Three controllers have been compared:*

- a memory less controller: $u(t) = Kx(t)$ (no prediction),
- a predictive controller from FSA defined in (1.12),
- a Smith predictor presented in subsection (1.2.1).

The results are displayed on Figure 1.5. On Figure 1.5a, the parameter a is equal to -1 : the open-loop system is stable. On the contrary, on Figure 1.5b, the parameter a is equal to 1. Three conclusions can be drawn from these graphs. First, the Smith predictor and the predictive controller from FSA give very similar results for a stable system (Fig. 1.5a). It seems logical since they both compensate perfectly the delay. Secondly, the Smith predictor is unstable for an open-loop unstable system. Finally, the memory less feedback displays a larger overshoot than the predictive controller in both cases (Figs. 1.5a and 1.5b). Increasing the delay and/or increasing the gain K would have magnify this overshoot and inevitably lead to the destabilization of the system with the memory less feedback. On the contrary, the predictive technique from FSA can stabilize a perfectly known linear system for an arbitrarily long (known) delay.

PDE Backstepping design

This method has been introduced by Krstic [2009]. The idea is to model the delayed input as a transport PDE

$$\begin{cases} \frac{\partial \nu(p,t)}{\partial t} &= \frac{\partial \nu(p,t)}{\partial p}, \\ \nu(h,t) &= u(t). \end{cases} \quad (1.18)$$

The solution of the following first order-hyperbolic PDE is

$$\nu(p,t) = u(t+p-h) \quad (1.19)$$

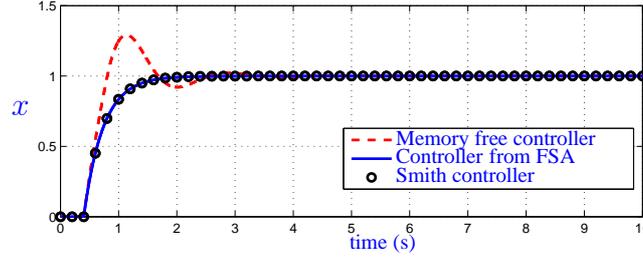
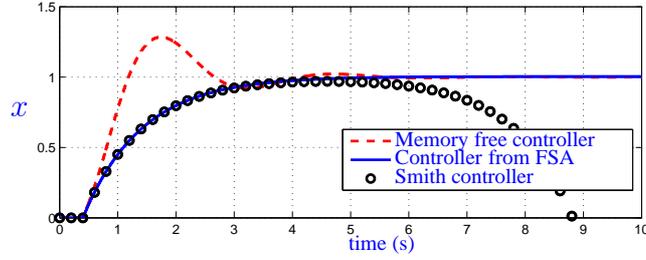
(a) Stable model $a = -1$ (b) Unstable model $a = 1$

Figure 1.5 – Controllers comparison for $\dot{x} = ax + bu(t-h)$ with $b = 1$, $h = 0.5$ s and $x(0) = 0$. In both case, the gain is $K = 2$.

and then

$$\nu(0, t) = u(t - h). \quad (1.20)$$

Therefore, delayed system (1.8) can be represented by the combination of the ODE $\dot{x}(t) = Ax(t) + B\nu(0, t)$ with PDE (1.18) driven by the input u at its boundary as illustrated on Figure 1.6. As a result, the following ODE-PDE system is obtained

$$\begin{cases} \dot{x}(t) &= Ax(t) + B\nu(0, t), \\ \frac{\partial \nu(p, t)}{\partial t} &= \frac{\partial \nu(p, t)}{\partial p}, \\ \nu(h, t) &= u(t). \end{cases} \quad (1.21)$$

Then the transformation

$$w(p, t) = \nu(p, t) - Ke^{Ap}x(t) - \int_0^p Ke^{A(p-y)}B\nu(y, t)dy \quad (1.22)$$

leads to the following systems

$$\begin{cases} \dot{x}(t) &= (A + BK)x(t) + Bw(0, t), \\ \frac{\partial w(p, t)}{\partial t} &= \frac{\partial w(p, t)}{\partial p}, \\ w(h, t) &= 0. \end{cases} \quad (1.23)$$

The value of w is equal to 0 after h seconds so the exponential convergence is guaranteed provided that $A + BK$ is Hurwitz. In addition, reminding that $u(t) = \nu(h, t)$ it is clear that the controller is the same as (1.12). The advantage of the transformed system (1.23) is that the boundary condition is null; then a stability analysis can be performed using the candidate Lyapunov functional

$$V(t) = x(t)^t P x(t) + bh \int_0^h (1 + p)w(p, t)^2 dp. \quad (1.24)$$

In the ideal case, the proof of stability is obvious since it is possible to compute the exact prediction; thus the delayed system is turned into a delay free systems. However, as soon as some uncertainties or external disturbances appear, it is necessary to use a different method since the delay can not be compensated exactly. Lyapunov functionals for memoryless feedbacks have been known for a long time (see [Fridman \[2014b\]](#)) but the PDE backstepping technique is the first result to extend the Lyapunov analysis to predictive feedbacks. This analysis method has been particularly used in the adaptive framework; for delay adaptive control in [Bresch-Pietri and Krstic \[2010\]](#), for delay and plant parameters adaptation in [Bresch-Pietri and Krstic \[2009\]](#) and [Zhu et al. \[2015\]](#) and also for disturbance estimation in [Bresch-Pietri et al. \[2012\]](#) or for inverse optimal redesign in [Krstic \[2008a\]](#).

After these first results on Lyapunov analysis for predictive control, some other Lyapunov functionals have been proposed. A technique to design Lyapunov-Krasovskii functionals for input delay systems stabilized by the reduction approach is proposed by [Mazenc et al. \[2012\]](#). The resulting functionals allow to establish the ISS property of the closed-loop system with respect to additive external disturbances. In [Li et al. \[2014b\]](#), a Lyapunov functional is used to study the robustness of the predictor to delay uncertainties. In [Li et al. \[2014a\]](#), a new functional is constructed for linear systems with multiple input delays. Some more references can be found in [Li et al. \[2014a\]](#).

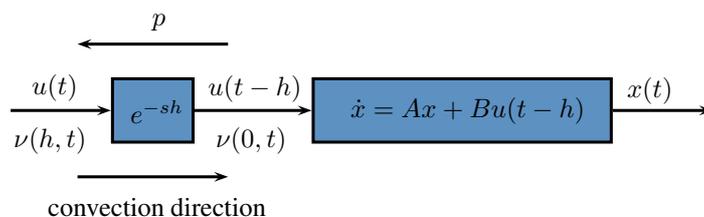


Figure 1.6 – PDE representation of the actuator delay from [Krstic \[2009\]](#).

Predictive control of broader classes of systems

Predictive techniques have recently been extended to broader classes of systems. The finite spectrum has been adapted to nonlinear systems very recently in [Oguchi \[2015\]](#). The reduction approach has been used to prove the stability of an approximated prediction for LTV systems with a constant and arbitrarily large input delay in [Mazenc et al. \[2014\]](#). This work has been extended to a class of nonlinear systems in [Mazenc and Malisoff \[2014\]](#). Other works that do not use the reduction method but aim at compensating input delay has been done for nonlinear systems (that maybe open-loop unstable) by [Krstic \[2008b\]](#) and [Krstic \[2010b\]](#). More details on this approach are available in the book of [Bekiaris-Liberis and Krstic \[2013\]](#). An insight of the method is given thereafter.

Consider forward-complete systems (systems that have bounded solutions time for any bounded input functions)

$$\dot{x}(t) = f(x(t), u(t-h)) \quad (1.25)$$

for which there exists a continuous controller $u(t) = k(x(t))$ that guarantees the global asymptotic stability to the origin of the delay free system. Then, the feedback $u(t) = k(x_p(t))$, where the prediction is computed as follows

$$x_p(t) = x(t) + \int_{t-h}^t f(x_p(s), u(s)) ds, \quad (1.26)$$

will exactly compensate the input delay h ⁵. The forward completeness property is necessary to guarantee that the system does not escape (in finite time) before the control "kicks in" at time $t = h$. In the nonlinear framework, the prediction is not always computable: Karafyllis [2011] presents the stabilization of globally Lipschitz systems thanks to approximate predictors. In Georges et al. [2007] and Besançon et al. [2009], a practical implementation of the prediction based on an implicit Euler discretization and the online computation of the fixed point by a dynamic inversion is proposed⁶. The problem of sampled measurement and zero order-hold controller for nonlinear systems with arbitrarily long input and output delays is dealt with by Karafyllis and Krstic [2012]: Approximate predictors and high-gain observers are designed to globally stabilize a class of globally Lipschitz systems. The case of output feedback for nonlinear systems with input and output delays is worked out in Karafyllis and Krstic [2013b]. In Koo et al. [2012b], an adaptive method is proposed to control a particular class of feedforward and non-feedforward nonlinear systems using past values of the input. Saturated control involving an integral term is designed in Fischer et al. [2013] to stabilize a class of uncertain nonlinear systems. For further details on the control of nonlinear systems with delays, one can refer to Bekiaris-Liberis and Krstic [2013] and Malisoff et al. [2015].

1.2.2 Prediction-based control for time-varying delays

In real-world applications, the delay is often time-varying. A common example is remote control. In the case of NCS, the delay often depends on the network traffic (see Lixian et al. [2013]). In the case of wireless communication, obstacles or distance can induce non constant delays as in Guerrero et al. [2013], Ploplys et al. [2004], Colandairaj et al. [2005], Bresch-Pietri and Petit [2014]. Similar predictive techniques available for constant delays can be extended to time-varying delays. The first work is by Nihtila [1989] where a predictive adaptive controller is designed. In Nihtila [1991], the FSA property is worked out. The difference between predictive control for constant and time-varying delays is illustrated below. Consider the system

$$\dot{x} = Ax + Bu(\eta(t)) \quad (1.27)$$

where $\eta(t) = t - h(t)$. Similarly to the constant delay case, the feedback $u(t) = Kx(t + h(t))$ is applied. The resulting equation is

$$\dot{x} = Ax + BKx(t + \underbrace{h(t) - h(t + h(t))}_{\text{do not cancel anymore}}). \quad (1.28)$$

When the delay is slow-varying, this solution can be efficient but, in order to compensate perfectly the delay, the feedback $u(t) = Kx(\eta^{-1}(t))$ with

$$x(\eta^{-1}(t)) = e^{A(\eta^{-1}(t)-t)}x(t) + \int_t^{\eta^{-1}(t)} e^{A(\eta^{-1}(t)-s)}Bu(\eta(s))ds \quad (1.29)$$

has to be applied. The inverse function η^{-1} represents the prediction horizon that is constantly changing with time. To guarantee its existence, the delay has to be differentiable, bounded $h(t) \in [0, h_{max}]$ and its derivative has to satisfy $|\dot{h}| \leq \delta$ with $\delta < 1$. The practical meaning of this condition is that the delay is causal, it cannot vary faster than the absolute time. Remark

5. The discretization of the integral is an important step that has to be carefully study in particular for nonlinear systems.

6. For more details see the thesis of Benayache [2009].

that in the constant delay case, one has $\eta^{-1}(t) = t + h$. However, in practical applications η^{-1} is difficult to compute since it requires the knowledge of $\eta(t)$ at all instant t . This means that the delay has to be known in advance in order to compute η^{-1} and then the exact prediction (1.29).

As a consequence of this strong constraint, some results have computed approximate predictions and study their stability properties. An estimated horizon is computed in [Witrant et al. \[2007\]](#) thanks to a model of the communication channel. The results are tested on a real experimentation in [Witrant et al. \[2011\]](#) and have been combined with a numerical approximation of the feedback law in [Georges et al. \[2007\]](#) and [Besançon et al. \[2009\]](#). A truncated predictor (without the integral part) has been proposed in [Yoon and Lin \[2013\]](#). However, these controllers do not cope with large delays because they do not have any memory of past instant. The stability of the ideal predictive feedback by a Lyapunov analysis has been solved in [Krstic \[2010a\]](#) by means of a backstepping technique similar to the one of Section 1.2.1. **An approximated predictive feedback will be studied for time-varying delays and some stability conditions will be obtained by a Lyapunov analysis for known and unknown delays in Sections 4.1 and 5.1 respectively.**

Predictive control and Networked Control Systems (NCS) The presence of a communication channel between the system and the controller induces new issues in the control design such as packetisation, packet loss, quantization and of course transmission delays. Recently, this control method has become more and more common and numerous works have been published on that topic creating a new field in the control literature [Zhang et al. \[2001\]](#), [Hespanha et al. \[2007\]](#) [Gupta and C. \[2010\]](#). Indeed, control over a network necessarily involves discrete phenomenon such as sampled measurement and hold control. As a consequence, continuous control methods have been modified and extended to fit the context of NCS. In [Zhang et al. \[2001\]](#), a sampled feedback based on the standard prediction is used to compensate for a known time-varying delay. In [Hetel et al. \[2011\]](#), a delay dependent (not predictive) feedback is proposed based on a delay estimation. However, no delay estimation method is given. In [Kruszewski et al. \[2012\]](#), a switching gain controller is designed for known delays⁷. Some very recent works by [Mazenc and Fridman \[2016\]](#) and [Ahmed-Ali et al. \[2016\]](#) deals with discrete-time measurements and a sampled input with a constant and known pointwise delay in the input. The work of [Mazenc and Fridman \[2016\]](#) uses a dynamic output feedback based on the standard prediction whereas an approximate (asymptotic) predictor is used in [Ahmed-Ali et al. \[2016\]](#).

In the thesis, we will focus on continuous methods so we will consider that sample and hold phenomenon can be neglected.

1.2.3 The limit of predictive control

In previous sections, it has been shown that predictive techniques are available to control systems with input delays. When the delay and the model are perfectly known, exact delay compensation can be achieved for arbitrarily long delay and the controller can be designed as if there were no delay. However, as soon as uncertainties appear, the exact compensation is not possible anymore and the robustness of the controller has to be investigated using tools for time-delay systems. Another important point is the computation of the integral term in the prediction (1.11).

7. The switches are governed by the QoS of the communication channel.

Sensitivity to parameter mismatch and external disturbance

External disturbance Systems are often subject to external disturbances. Since these signals are unknown, they cannot be included directly in the computation of the prediction. As a consequence, it is not possible to compute the exact prediction and to compensate efficiently the perturbation. On Figure 1.7a, it is clear that the prediction is not exact when the disturbance (Figure 1.7b) affects the system between $t = 10$ s and $t = 20$ s. As a consequence, the trajectory tracking is not accurate during this time interval. Some methods have adapted the Smith predictor structure to reject step disturbance as in Astrom et al. [1994], Watanabe and Ito [1981b]. However, these methods apply only to stable plants. For the state space approach, an ISS Lyapunov functional is given in Mazenc et al. [2012] for a system controlled by a standard predictive feedback but no solution is proposed to improve disturbance attenuation. In Polyakov et al. [2013b], the attractive ellipsoid method is used to tune the gains of the predictive controller in order to reduce measurement noise and exogenous disturbance effect. Nevertheless, this method do not allow to reject perfectly a constant disturbance. Adaptive control is used to reject constant perturbation Bresch-Pietri et al. [2012] and sine perturbation Pyrkin et al. [2010] and Pyrkin and Bobstov [2013]. In this framework, controllers are designed as follows: $u(t) = Kx_p(t) - \hat{d}(t)$ with \hat{d} a disturbance estimation. However, the estimated disturbance is not used to compute a more accurate prediction. **In Section 4.3, a new prediction that indirectly includes some disturbance information is presented. It is proved that designing a controller from this new prediction leads to perfect rejection of a constant disturbance. It is also shown that the new predictive scheme⁸ performs a better disturbance attenuation for a large class of time-varying disturbances.**

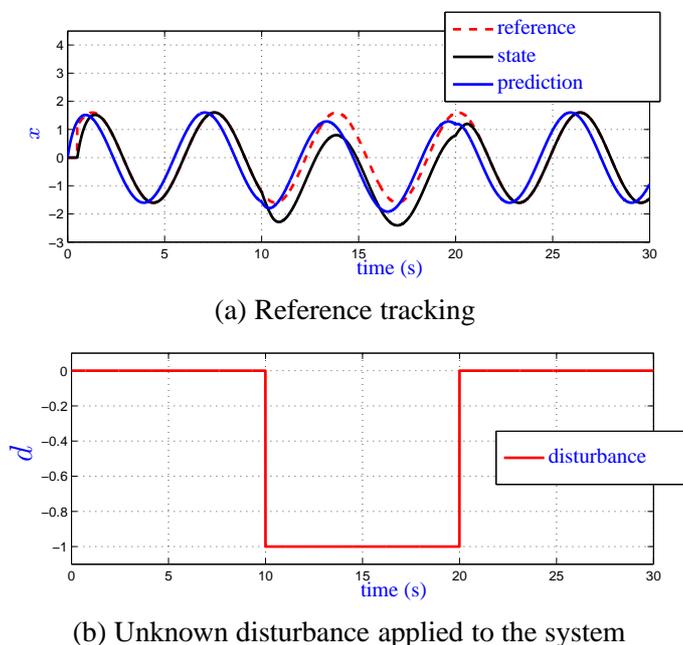


Figure 1.7 – Robustness of the prediction for system $\dot{x} = ax + bu(t - h)$ with $a = 1$, $b = 1$, $h = 0.5$ s and $x(0) = 0$. An unknown disturbance $d = -1$ is applied between $t = 10$ s and $t = 20$ s. The controller is defined by (1.12) with a correction term for the reference tracking.

8. The term "scheme" refers to the prediction and the controller design. The standard predictive scheme is based on the standard prediction (1.11).

Sensitivity to parameter uncertainties The work of Palmor [1980] provides one of the first result to analyze the closed-loop performances in presence of delay mismatch; an explicit criterion is given thanks to the Nyquist criterion. Assuming that only approximations \tilde{A} (respectively \tilde{B}) of A (respectively B) defined in system (1.8), then the prediction reads as

$$\tilde{x}_p(t) = e^{\tilde{A}h}x(t) + \int_{t-h}^t e^{\tilde{A}(t-s)}\tilde{B}u(s)ds. \quad (1.30)$$

It is stated by Furukuwa and Shimemura [1983] that, from the following equation

$$\begin{bmatrix} sI_n - A & -Be^{-sh} \\ -Ke^{\tilde{A}h} & I_m - K(I - e^{-(sI - \tilde{A})h})(sI - \tilde{A})^{-1}\tilde{B} \end{bmatrix} = 0, \quad (1.31)$$

the stability is preserved if the difference between the plant and the model are sufficiently small but no quantitative criterion is given. Another classic method is to use Lyapunov functionals to derive LMIs. The reduction method can be used to rewrite the system and derive LMI conditions for known and constant delay Moon et al. [2001], for unknown and constant input delay Yue [2004] or for unknown and time-varying Chen and Zheng [2006], Yue and Han [2005]. **The properties of the new predictive scheme presented in Section 4.3 will be also analyzed in presence of parameter uncertainties. It will be shown that the new prediction allows better trajectory tracking in presence of parameter uncertainties than the standard one.**

Sensitivity to delay uncertainty In real applications, the delay is usually very difficult to measure with a good accuracy. That is why some delay estimation techniques will be presented in Part I of the manuscript. Furthermore, it is intuitive that control performances can be deteriorated with delay mismatch. That is why a lot of works have tried to quantify the amount of delay uncertainty that can be accepted without destabilizing the system. The first works deal with a frequency domain analysis. The papers by Yamanaka and Shimemura [1987], Michiels and Niculescu [2003] are focused on the effect of delay mismatched for Smith controller. A maximal delay mismatch is derived in Michiels and Niculescu [2003]. For more details about the frequency approach, the reader can refer to the book of Niculescu [2001].

More recently, a Lyapunov functional analysis has been given in Li et al. [2014b]. In Karafyllis and Krstic [2013a], an upper-bound on the delay uncertainty is found by carrying out a particular form of small-gain analysis. In Krstic [2008a], Krstic [2009] and Bekiaris-Liberis and Krstic [2013], a particular Lyapunov functional based on the backstepping technique (Section 1.2.1) is used to show the existence of an admissible upper bound on the delay estimation error. **In Section 5.1, some conditions on the delay error will be derived for constant or time-varying delays by using a new Lyapunov functional.**

Remark 1.2.1. *Note that it is very well-known that predictive control is not robust to parameter uncertainties or delay mismatch. On the one hand, performances are degraded when exact model values are not perfectly known as shown on Figure 1.8 (especially for delay mismatch). On the other hand, for large input delays, this is the only available technique and it can be observed that an unstable system can preserve closed-loop stability with about 10 to 15% error uncertainty.*

Computation of the distributed control law

Predictive controllers are memory controllers because of the integral term in equation (1.12). If the system is open-loop stable, prediction (1.11) can be generated by the following process

$$x_{\hat{p}}(t) = e^{Ah}x(t) + \xi_h(t) \quad (1.32)$$

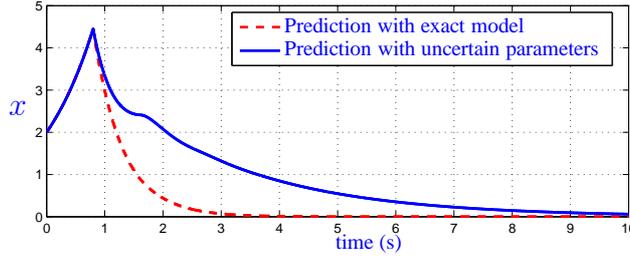
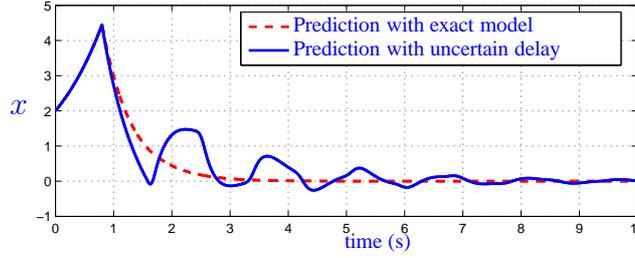
(a) Parameters errors: $\hat{a} = 0.85$ (−15%) and $\hat{b} = 1.15$ (+15%)(b) Delay error: $\hat{h} = 0.9$ s (12.5% error)

Figure 1.8 – Robustness of the prediction for system $\dot{x} = ax + bu(t - h)$ with $a = 1$, $b = 1$, $h = 0.8$ s and $x(0) = 2$. The predictive controllers are computed taking into account some uncertainties on the parameters a and b (Figure 1.8a) and on the delay value h (Figure 1.8b). In both case, the controller gain is the same.

with

$$\begin{aligned}
 \xi_h(t) &= \int_{t-h}^t e^{A(t-s)} Bu(s) ds \\
 &= e^{At} \xi(0) + \int_0^t e^{A(t-s)} bu(s) ds \\
 &\quad - e^{Ah} \left[e^{A(t-h)} \xi(0) + \int_0^{t-h} e^{A(t-h-s)} bu(s) ds \right] \\
 &= \xi(t) - e^{Ah} \xi(t-h)
 \end{aligned} \tag{1.33}$$

where $\xi(t)$ is solution of the equation

$$\dot{\xi}(t) = A\xi(t) + Bu(t). \tag{1.34}$$

In this case, no integral discretization is necessary⁹. However, for open-loop unstable systems, these methods are not applicable anymore and the integral has to be discretized. This step has to be carefully executed because it can make the closed-loop system unstable. This destabilizing effect has been pointed out in Van Assche et al. [1999] and the instability mechanism have been explained in Engelborghs et al. [2001]. In order to explain the destabilizing phenomena and illustrate safe implementation techniques, consider the following scalar example

$$\dot{x} = x(t) + u(t - 1). \tag{1.35}$$

To stabilize this open-loop unstable system with a delay equal to one, the following controller is chosen

$$u(t) = -2x_p(t) = -2e^1 \left[x(t) + \int_0^1 e^{s-1} u(t-s) ds \right]. \tag{1.36}$$

9. See Watanabe and Ito [1981b] and Furukuwa and Shimemura [1983] for more methods to compute the prediction.

Since this system is open-loop unstable, the integral has to be discretized. With the trapezoidal rule, the following approximation is proposed:

$$\int_0^1 e^s u(t-s) ds \approx T_s \left[\frac{1}{2} u(t) + \sum_{i=1}^{N-1} e^{iT_s} u(t-iT_s) + \frac{1}{2} e^1 u(t-1) \right] \quad (1.37)$$

with $T_s = \frac{1}{N}$ and N the number of discretization points. The closed-loop response for (1.35) with feedback (1.36) and discretization (1.37) ($T_s = 100$ ms) is displayed on Figure 1.9a along with the ideal response. It can be seen that the response is correct till about 15 s where high frequency oscillations appear. Indeed, equations (1.35) and (1.36) with discretization (1.37) are a system of retarded type with an infinite number of roots. However, a finite number of roots are in the right half plane. In Engelborghs et al. [2001], stability regions are plotted in function of the delay value and the controller gain. To fix this problem, a low pass filter is introduced in Mondié and Michiels [2003]:

$$\begin{cases} \dot{z}(t) &= -fz(t) - 2f \left(e^1 x(t) + T_s \left[\frac{1}{2} u(t) + \sum_{i=1}^{N-1} e^{iT_s} u(t-iT_s) + \frac{1}{2} e^1 u(t-1) \right] \right) \\ u(t) &= z(t) \end{cases} \quad (1.38)$$

with f the cutoff frequency. It is proved that there always exists N large enough such that the closed-loop approximation (1.35)-(1.38) is asymptotically stable provided that the ideal closed-loop system is asymptotically stable. The results for different values of f are shown on Figure 1.9b for $T_s = 100$ ms. It is clear that the larger f is, the closer is the implementation to the ideal response. However, if the cut off frequency is too large then high frequency oscillations appear again.

Other methods have been proposed. In Popescu and Rasvan [2004], a sampled feedback is designed on the discretized system. It is shown that there always exists a sufficiently small sampling time such that the closed-loop system is asymptotically stable. In Zhong [2004], implementations are based on approximations of (1.33) and (1.34) in the z -domain and in the s -domain. Based on similar approximations, some solutions are provided in Mirkin [2004]. For more details on implementation of distributed control laws, see the book by Zhong [2006].

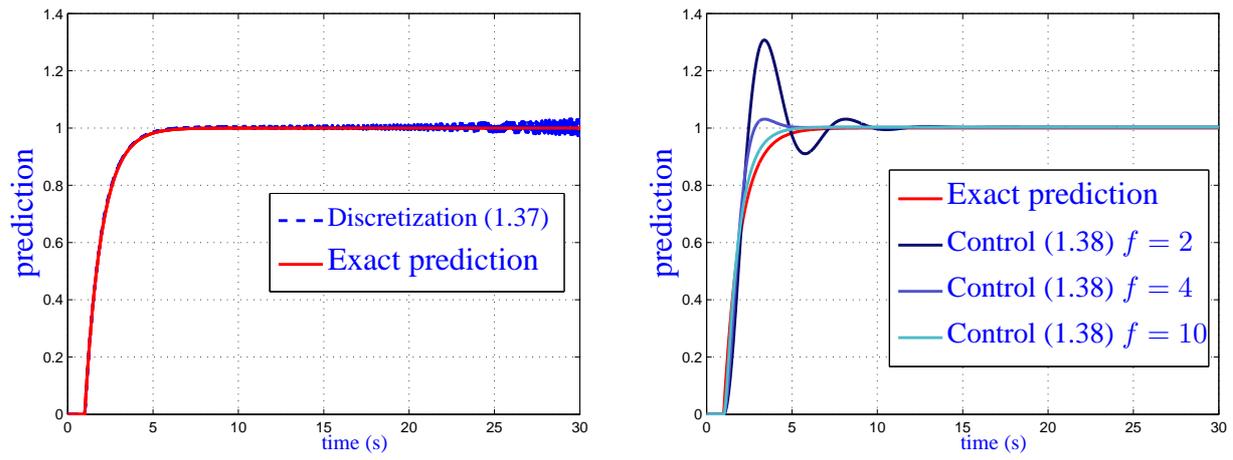
In the sequel of the thesis, all the prediction have been computed with a time domain discretization (with sample-and-hold):

$$\int_0^1 e^s u(t-s) ds \approx T_s \left[\frac{1}{2} u(nT) + \sum_{i=1}^{N-1} e^{iT_s} u_i + \frac{1}{2} e^1 u(nT-1) \right] \quad (1.39)$$

with $u_i = u(nT - iT_s)$ for $t \in [(n-1)T, nT[$ where T is the step size of the solver ($T_s \geq T$). As it is shown on Figure 1.9c for $T_s = 100$ ms and $T = 10$ ms, this approximation is very accurate provided T is sufficiently small.

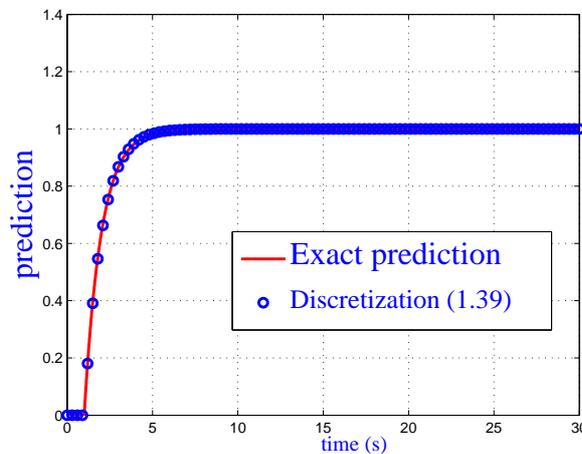
Some works have been done in order to drop the integral part of the prediction in order to limit the implementation complexity: truncated predictor (TP) in Yoon and Lin [2013], Zhou et al. [2012] or pseudo-predictor feedback (PPF) for time-varying delay in Zhou [2014]. A sequential sub predictors method, presented in Najafi et al. [2013], allows to use the process (1.33)-(1.34) even for open-loop unstable systems provided that the constant delay is sufficiently small. **This result is extended to partial measurement knowledge and known and unknown time-varying delay respectively in Section 4.2 and 5.2.**

Even if the integral computation is often pointed out as a weakness of predictive control, the efficiency of such techniques on real systems is illustrated on a DC motor in Chapter 6 of the manuscript.



(a) Integral discretization (1.37) and high frequency destabilizing effect

(b) Discretization and low pass filter from Mondié and Michiels [2003]: (1.37)-(1.38)



(c) Discretization by trapezoidal rule and sample and hold method (1.39)

Figure 1.9 – Comparison of different methods to compute prediction (1.11).

1.3 Observation in presence of delays

In order to compute the prediction, the state and the delays are necessary. However, they are not always available so observers are needed to estimate both the state and the delay. In this section, some background literature is given on that topic and the review is divided into three points

- state observation in presence of delay,
- delay estimation,
- combination of state observation and delay estimation.

1.3.1 State observers for delayed systems

Numerous works have tackled the problem of state observation in presence of delays. These works can be organized according to the type of delays that affect the system: state delays, input delays or output delays. Some important contributions are mentioned in the sequel.

State delay

One of the first works is by [Bhat and Koivo \[1976\]](#) where a state observer is designed for linear systems with a constant and known delay. Other results for linear systems with constant and known delays are also given by [Choi and Chung \[1996\]](#) and [Hou et al. \[2002\]](#). The main limitation of these works is that they consider constant and known delays. The works of [Choi and Chung \[1997\]](#) and [Darouach et al. \[1999\]](#) have considered an unknown but constant delay whereas articles by [Darouach \[2001\]](#) and [Fridman et al. \[2003b\]](#) treat the time-varying but known delay case.

All the previous references study linear systems. References about nonlinear systems are much more scarce. Among the few references, [Ibrir \[2009\]](#) deals with a constant and known delay for nonlinear (triangular) systems and [Wang et al. \[2002\]](#) considers nonlinear disturbances and a time-varying and known delay. Some extra references are given in the survey papers by [Sename \[2001\]](#) and [Richard \[2003\]](#).

State and input delays

More recently some works have been dedicated to state observation for systems with both input and state delays. Most of the papers are restricted to known delays such as [Leyva-Ramos and Pearson \[1995\]](#), [Zheng et al. \[2011\]](#)¹⁰ and [Márquez-Martínez et al. \[2002\]](#) for constant delays or [Farza et al. \[2010\]](#) for time-varying delays. The works of [Seuret et al. \[2009\]](#) and [Ghanes et al. \[2013\]](#) deal with unknown delays. The idea in these last two references is to substitute the real delay $h(t)$ by a known and constant approximation \tilde{h} ¹¹ and then to design a robust observer¹² to attenuate the delay estimation error. The disadvantage of this method is that the observer gains have to be large to compensate for the delay estimation errors.

Input and output delays

The seminal paper by [Watanabe and Ito \[1981a\]](#) works out the case of multiple input and output (constant and known) delays for linear systems. Very recently, a finite dimensional observer has been proposed by [Zhou et al. \[2013\]](#) for the same class of systems. Cascaded observers are proposed by [Germani et al. \[2002\]](#), [Kazantzi and Wright \[2005\]](#) and [Ahmed-Ali et al. \[2012\]](#) to observe systems with constant and known delays in the output. Finally, an interval observation technique has been introduced by [Polyakov et al. \[2013a\]](#) to reconstruct the state of linear systems with an unknown and time-varying delay in the input. The method consists in designing two observers which generate solutions \bar{x} and \underline{x} . Then, it is shown that the state x is always in $[\underline{x}, \bar{x}]$ in the component wise sense. None of the above papers consider an unknown and time-varying delay. Furthermore, they are focused on linear systems. **In Part I, two state observation solutions will be presented for nonlinear systems with time-varying and unknown delays in the input.**

1.3.2 Delay estimation

Signal processing approach

Time delay identification has often been based on a signal processing approach and particularly in the acoustic field as in [Clifford \[1981\]](#) and [Knapp and Carter \[1976\]](#). In this case, the

10. This work considers also a delay in the measurement. An algebraic framework is used to transform nonlinear systems with commensurable delays into a canonical form more appropriate to estimate the state.

11. For instance $\tilde{h} = (h_{min} + h_{max})/2$ in [Seuret et al. \[2009\]](#) and $\tilde{h} = h_{max}$ in [Ghanes et al. \[2013\]](#)

12. A sliding mode observer in [Seuret et al. \[2009\]](#) and a high gain observer in [Ghanes et al. \[2013\]](#).

general idea of TDE techniques is the following: a criterion $J(\hat{h})$ is designed¹³ and the evolution law for \hat{h} has to minimize the criterion $J(\hat{h})$. A standard time-delay estimator is the direct correlator (DC) that seeks the maximum value of the cross correlation between the reference and the delayed signal (see [Knapp and Carter \[1976\]](#)). Other estimators exist such as the average square difference function estimator (ASDF) for discrete time techniques as in [Jacovitti and Scarano \[1993\]](#). These methods are not well adapted in the control context because they are usually offline methods (DC, ASDF) and therefore suffer from a long computation time due to the poor rate of convergence of the optimization algorithm. A survey of TDE techniques with a signal processing focus is given in [Björklund \[2003\]](#).

Control oriented methods

In these approaches, the delay is usually considered as a parameter of the system and its identification is often combined to the identification of other parameters. A vast literature exists on that topic: see the works of [Agarwal and Canudas \[1986\]](#), [Belkoura et al. \[2009\]](#), [Bresch-Pietri and Krstic \[2009\]](#), [Gomez et al. \[2007\]](#), [Na et al. \[2014\]](#), [Orlov et al. \[2003\]](#), [Tuch et al. \[1994\]](#). Some authors use the frequency domain where the delay appears as a parameter in the term $e^{-\tau s}$. In [Agarwal and Canudas \[1986\]](#), the term $e^{-\tau s}$ is approximated by a rational transfer function of the Padé form; then a standard discrete least-square algorithm is used to minimize an objective function. The work of [Tuch et al. \[1994\]](#) is also based on a frequency domain analysis. They proposed a continuous recursive least square algorithm. However, this method does not work if the initial conditions of the system state are not perfectly known. In addition, their solution works only for strictly positive or strictly negative controllers. In [Nihtilä et al. \[1997\]](#), a PDE approximation is used to extract the delay. In [Diop et al. \[2001\]](#), a similar techniques as in [Tuch et al. \[1994\]](#) is applied but the value of $u(t - \tau)$ is required. In [Drakunov et al. \[2006\]](#), observers have been used to identify the delay. However, all the state and its time derivatives are needed; so the method is very sensitive to noise measurement. In [Belkoura et al. \[2008\]](#) and [Belkoura et al. \[2009\]](#), an identification method using algebraic techniques based on annihilation and integration is proposed for transfer functions. Very recently, an adaptive method is proposed in [Na et al. \[2014\]](#) to estimate the system parameters and the delay of SISO systems. The initial conditions of the estimated parameters have to be very close to the exact value to guarantee the convergence of this method. A comparison of three estimation methods is shown on [Figure 1.10](#). One can observe that the technique of [Drakunov et al. \[2006\]](#) has a lot of oscillations whose magnitude depends on the delay size. This chattering phenomenon cannot be reduced by decreasing the step size. It is clear that the methods from [Tuch et al. \[1994\]](#) and [Na et al. \[2014\]](#) are not suited for a time-varying delay. Indeed, both methods converge quickly to the constant delays between 0 and 20 seconds but they cannot converge anymore when the delay varies from 0.4 to 1 s. Note that this problem is intrinsic to the estimator conception and that it does not come from a bad tuning of the gain. A review of delay estimation techniques has been done by [O'Dwyer \[2000\]](#).

Finally, four recent works design controllers that estimate the delay. Unlike previously mentioned papers, next references deal with closed-loop identification method. In [Bresch-Pietri and Krstic \[2009\]](#), a delay estimator is designed from a Lyapunov analysis. In [Bresch-Pietri and Krstic \[2010\]](#), the delay estimator is obtained from the transport PDE representation of the system. However, in both cases, only the state stabilization is proven but not the convergence of the delay estimator. A delay identification law is proposed in [Gaudette and Miller \[2014\]](#) for sampled systems. This law is based on a particular form of systems with two distinct real eigenvalues of multiplicity one. In [Herrera and Ibeas \[2012\]](#), a multi-model scheme is used to

13. The delay is often denoted by the letter h or τ . In this thesis, the notation h will be used.

estimate the delay and then this estimation is plugged into a modified Smith predictor. However, this latter work do not consider external disturbances neither parameter uncertainties.

Note that, in all previously mentioned articles except [Gaudette and Miller \[2014\]](#) and [Herrera and Ibeas \[2012\]](#), the delay is constant. **In Part I, two solutions will be introduced to estimate an unknown and time-varying delay in the input.**

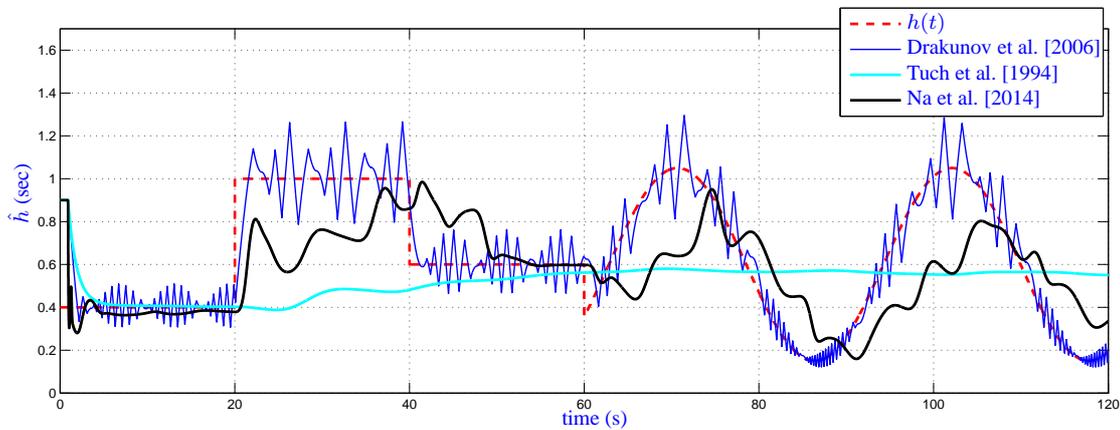


Figure 1.10 – Comparison of delay estimation techniques on the system $\dot{x} = x + u(t - h(t))$

1.3.3 Delay and state observation

As it has been mentioned above, many papers deal with state observation for delay systems. Similarly, various methods exist for delay estimation. However, very few papers deal with both delay identification and state reconstruction. Only two references about both delay and state estimation are discussed thereafter. The first one, by [Bresch-Pietri et al. \[2012\]](#), is restricted to linear systems with a constant delay. In addition, the example of delay estimator proposed in this paper is quite intricate to implement since it involves various integral terms. The second one, by [Gaudette and Miller \[2014\]](#), is also focused on linear systems since the delay estimation technique relies on the existence of two different eigenvalues of A . Besides, the design of the state observer is largely based on a particular sampling/holding technique. Finally, these articles are interested in proving the closed-loop convergence but the delay estimation error convergence is not established. **In Chapters 2 and 3, two methods will be presented to estimate the state and the delay of a class of nonlinear systems with an unknown and time-varying delay in the input.**

1.4 Organization and contribution of the thesis

This thesis is divided into three parts:

- The first part is dedicated to the presentation of two solutions to estimate the state and the delay of a class of nonlinear systems with a time-varying delay in the input.
- The second part presents three predictions and studies their behaviors for known and unknown delays.
- The third part is an experimental part that illustrates previous results.

A detail organization is given thereafter and displayed on Figure 1.11. The contributions are detailed throughout the presentation of each chapter. **Note that, the theorems presented in framed boxes in the thesis are new results that constitute the contributions of this work.**

Part I is dedicated to the presentation of new methods to estimate the delay and the state. The objective of delay observation is twofold. First, one looks for improving the quality of state observation; secondly, one wants to compute an accurate prediction. One of the contribution comes from the simultaneous observation of the state and the delay. The other main contribution is to deal with a time-varying delay. The stability analysis of the observer is studied in open-loop (no feedback).

In **Chapter 2**, a new observation method that allows to estimate both state and delay thanks to a single extended observer is proposed. A first version of this method has been published in [Léchappé et al. \[2015a\]](#) for linear systems. An extension to nonlinear systems is given in [Léchappé et al. \[2015b\]](#). The original idea lies in the construction of high gain observer on an extended system which includes the unknown delay as a part of its augmented state. A formal convergence proof is given and some simulations illustrate the performance of this method.

An interconnected observers scheme is introduced in **Chapter 3**. First, a new disturbance observer, based on Extended State Observer (ESO) method, is designed in order to estimate the retarded input. The novelty consists in introducing an artificial delay to approximate the time derivative of the unknown dynamics by a finite difference scheme. Then, the general framework of interconnected state-delay observers is proposed. It is based on the coupling of a standard state observer (available for delay free systems) with a delay observer. A stability condition is given for the general case and illustrated by a particular example. The efficiency of the method is confirmed by simulations. A comparison with the method of Chapter 2 is also performed by simulations. The content of this chapter is partially sum up in [Léchappé et al. \[2016\]](#).

Part II presents various results on predictive control. The main objective is to offer new techniques to achieve better performances in presence of parameter uncertainties, external disturbances and to deal with unknown delays. Explicit stability conditions are provided for each technique thanks to a Lyapunov functional analysis. The physical meaning of these conditions is thoroughly analyzed.

Chapter 4 deals with known delays and is divided into three sections. The first one (**Section 4.1**) reminds that the standard prediction originally designed for constant delays can be used for time-varying delays under some constraints on the delay rate. The result, already existing for static feedbacks and full state knowledge, is extended to a broader class of controllers and to partial state measurement. A stability condition is provided thanks to a simple Lyapunov functional. The second section (**Section 4.2**) aims at extending a recent result that allows to compute a prediction from a dynamic system. The result originally available for constant delays, static state feedback is extended to time-varying delays, and a broader class of controllers with partial measurement knowledge. The exponential stability is also achieved. Finally, the last section (**Section 4.3**) describes a new predictive scheme that is more robust to external perturbation and parameter uncertainties than the standard predictive scheme. The basic idea is to design a prediction that indirectly includes information about unmodeled dynamics. This work has been presented in [Léchappé et al. \[2015c\]](#) for the state feedback case and it is extended to the output feedback in [Léchappé et al. \[2015d\]](#). The extension of these works to time-varying delays is also treated in this section. All the results are illustrated by academic examples and supported by a thorough simulation analysis.

Chapter 5 extends the results of Sections 4.1 and 4.2 to unknown delays respectively in **Sections 5.1** and **5.2**. In each case, Lyapunov functionals are exhibited to study the stability of the closed-loop system. A contribution of this chapter lies in the systematic Lyapunov analysis that leads to explicit stability conditions. The other contribution is the presentation of predictive techniques to control systems with an unknown and time-varying delay. The predictive control laws are coupled with the delay estimator of Chapter 3 in simulation.

Theoretical results presented in above chapters are illustrated in **Part III**.

In **Chapter 6**, a DC motor is controlled in presence of parameter uncertainties, external disturbances and a time-varying and unknown delay. Since the delay is artificially added, it has been possible to test a lot of configurations seen before. The delay observer of **Chapter 3** is combined with the predictive techniques of **Part II**. Some experiments with input and output unknown and time-varying delays are carried out to show that the result of this thesis can be adapted to this larger framework. The results of this chapter have been published in [Léchappé et al. \[2015f\]](#) for the constant and known delay case. The extension to an unknown delay is available in [Léchappé et al. \[2015e\]](#).

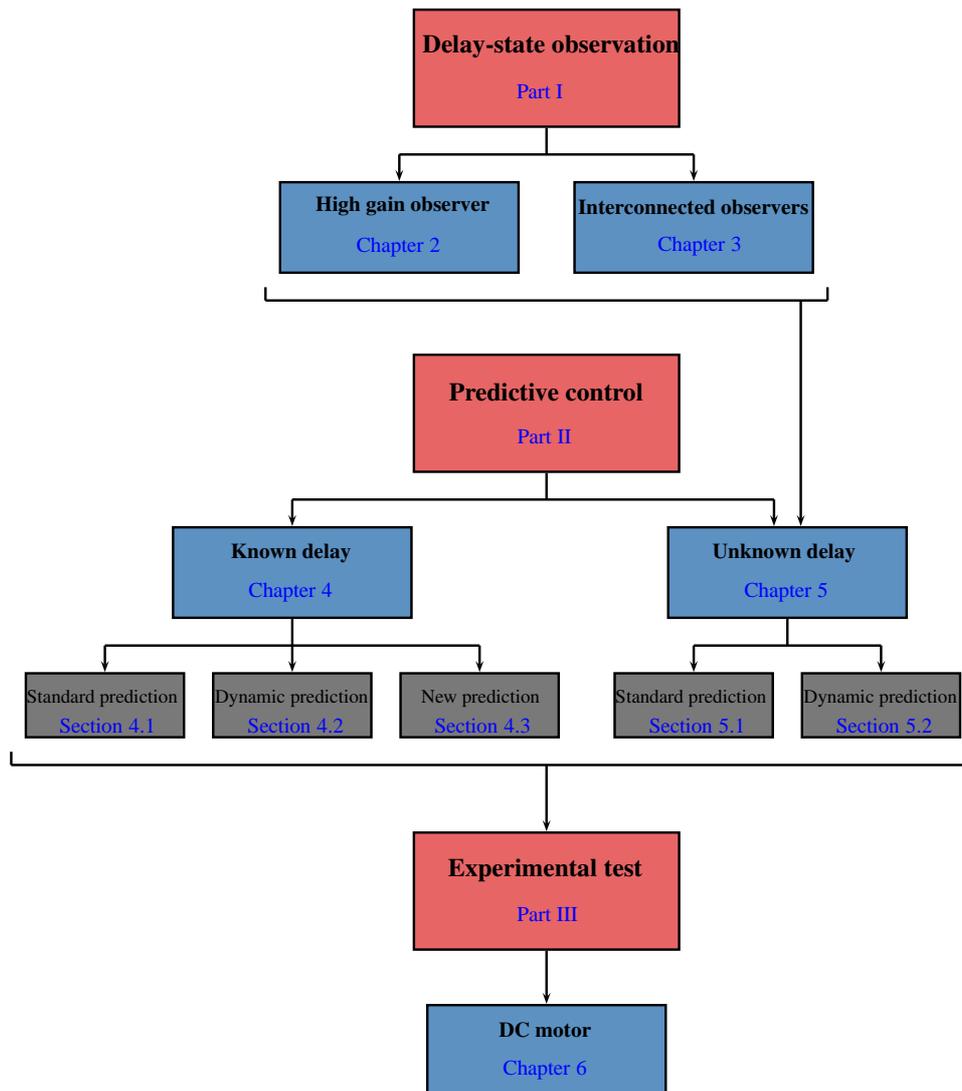


Figure 1.11 – Organization of the thesis

Some of the results presented in this thesis have been published or are under revision process for publication in journals and conferences.

Journal papers

- 1) V. Léchappé, J. De León, E. Moulay, F. Plestan, A. Glumineau, *Delay-state observer for SISO nonlinear systems*, second revision in *Automatica*.
[Léchappé et al. \[2015b\]](#)
- 2) V. Léchappé, S. Rouquet, M. Gonzáles, F. Plestan, J. De León, E. Moulay, A. Glumineau *Delay estimation and predictive control of uncertain systems with input delay: application to a DC motor*, accepted in *IEEE Transactions on Industrial Electronics*.
[Léchappé et al. \[2015e\]](#)
- 3) V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, A. Chriette, *New predictive scheme for the control of LTI systems with input delay and unknown disturbances*, *Automatica*, 52(2), 179-184, 2015.
[Léchappé et al. \[2015c\]](#)

Conference papers

- 1) V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, *Interconnected delay and state observer for nonlinear systems with time-varying input delay*, submitted to American control Conference, Boston, USA, 2016
[Léchappé et al. \[2016\]](#)
- 2) V. Léchappé, J. De León, E. Moulay, F. Plestan, A. Glumineau, *Delay and state observer for SISO LTI systems*, American Control Conference, Chicago, USA, 2015
[Léchappé et al. \[2015a\]](#)
- 3) V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, A. Chriette, *Predictive scheme for observer-based control of LTI systems with unknown disturbances*, European Control Conference, Linz, Austria, 2015
[Léchappé et al. \[2015d\]](#)
- 4) V. Léchappé, O. Salas, J. De León, F. Plestan, E. Moulay, A. Glumineau, *Predictive control of disturbed systems with input delay: experimental validation on a DC motor*, Time-Delay System workshop, Ann Arbor, USA, 2015
[Léchappé et al. \[2015f\]](#)



Delay and State observation of input delay systems

Introduction

Organization

It has been seen in the literature review that many solutions exist for the state observation problem in presence of delay. The delay estimation issue has also been widely studied. However, both problems are rarely combined. And yet, as soon as delay value increases, it is the only way to guarantee an accurate state observation. Besides, in the prediction framework, both state and delay values are necessary to design predictive controllers. That is why this part is dedicated to the presentation of new methods to observe the state and the delay of systems with input delay. Chapter 2 presents the design of a single observer that reconstructs the delay and the state. On the contrary, Chapter 3 introduces a scheme that allows to interconnect standard state observers (used in the delay free case) with a delay observer. The additional contribution, apart from the combined state and delay observation results, is that both methods apply to a class of nonlinear systems with a time-varying delay. Moreover, formal convergence proofs are given and the methods are illustrated and compared in simulation. Note that this part is focused on observation: no closed-loop results will be provided. The organization of this part is reminded on Figure 1.12.

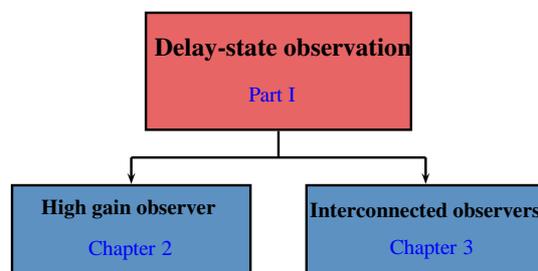


Figure 1.12 – Organization of the part on delay-state estimation

General considerations

System presentation

The systems considered throughout this part are SISO nonlinear systems with a time-varying delay $h(t)$ acting on the control input u . More precisely, the class of systems is defined by

$$(\Sigma) \quad \begin{cases} \dot{x} = \Lambda x + f(x) + g(y)u(t - h(t)) \\ y = Cx = x_1 \\ x(0) = x_0 \end{cases} \quad (1.40)$$

with $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$,

$$\Lambda = \begin{bmatrix} 0_{n \times 1} & I_{n-1} \\ 0 & 0_{1 \times n} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad g(y) = [0, \dots, 0, G(y)]^T \in \mathbb{R}^n, \quad (1.41)$$

$$f(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} \in \mathbb{R}^n \quad \text{and} \quad C = [1, 0, \dots, 0]^T \in \mathbb{R}^n. \quad (1.42)$$

Assumption 1.4.1. *The functions f and G are respectively in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $C^1(\mathbb{R}, \mathbb{R})$ and f is globally Lipschitz, i.e. there exists $c_1 > 0$ such that for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$*

$$\|f(x) - f(z)\| \leq c_1 \|x - z\| \quad (1.43)$$

Assumption 1.4.2. *The function G is bounded, i.e. there exists $\bar{G} > 0$ such that for all $y \in \mathbb{R}$*

$$|G(y)| \leq \bar{G}. \quad (1.44)$$

Assumption 1.4.3. *The delay function h is either piecewise constant or differentiable and satisfies $h(t) \in [h_{min}, h_{max}]$. If it is differentiable, its dynamics, denoted $\dot{h}(t) = \eta(t)$, is unknown and bounded*

$$|\eta(t)| \leq \bar{\eta}. \quad (1.45)$$

Delay observability

The objective of this part is to propose new methods to estimate both the delay and the state of system (Σ) in (1.40). Usually, the delay is considered as a parameter of the system when it is constant [Agarwal and Canudas \[1986\]](#), [Na et al. \[2014\]](#). However, when the delay is time-varying, it has been decided to treat it as a state of an extended system in order to apply observation techniques. Denoting $X = [x^T \ h]^T$, one obtains

$$(\tilde{\Sigma}) \quad \begin{cases} \dot{X} = \tilde{f}(X, u) \\ y = Y(X) = C_{ext}X = x_1 \end{cases} \quad (1.46)$$

with

$$\tilde{f}(X(t), u(t-h(t))) = \begin{pmatrix} \Lambda x(t) + f(x(t)) + g(y(t))u(t-h(t)) \\ \eta(t) \end{pmatrix} \in \mathbb{R}^{n+1} \quad (1.47)$$

and $C_{ext} = [C, 0] \in \mathbb{R}^{n+1}$. Note that in open-loop, (1.46) is a finite dimension system because u is an external signal that does not depend on x . However, (1.46) would become an infinite dimensional system in closed loop when u is a function of x .

Systems (Σ) in (1.40) and $(\tilde{\Sigma})$ in (1.46) are equivalent in the sense that they have the same state trajectories. As a result, the observer design and the stability analysis can be performed on this augmented system. First, the observability analysis of system $(\tilde{\Sigma})$ is going to be worked out. The concept of observability rank condition used in [Theorem 1.1](#) is detailed in [Hermann and Krener \[1977\]](#). The definition is recalled here for clarity: denoting by \mathcal{O} the smallest vector space containing x_1 and closed under the Lie derivatives $L_{\tilde{f}_u}$ (for all $\tau \in \mathcal{O}, L_{\tilde{f}_u}(\tau) \in \mathcal{O}$). This vector space is the classical observation space. Let $\tilde{\mathcal{O}}$ be the co-distribution spanned by $\{d\tau, \tau \in \mathcal{O}\}$, system (1.46) is said to be rank observable at x if $\dim \tilde{\mathcal{O}}(x) = n + 1$. It is said to be rank observable if for all x , $\dim \tilde{\mathcal{O}}(x) = n + 1$. This concept is very convenient to check the observability of nonlinear systems: this is why it will be used in the observability analysis performed thereafter. The following result is achieved.

Theorem 1.1 (Léchappé et al. [2015b]). System $(\tilde{\Sigma})$ is rank-observable if and only if

$$G(y) \neq 0 \quad (1.48)$$

and

$$\dot{u}(t) \neq 0 \quad \forall t \geq -h_{max}. \quad (1.49)$$

Proof. The n first time derivatives of the output y read as

$$\begin{cases} y & = x_1 \\ \dot{y} & = x_2 + \alpha_1(x_1) \\ \ddot{y} & = x_3 + \alpha_2(x_1, x_2) \\ \vdots & = \vdots \\ y^{(n-1)} & = x_n + \alpha_{n-1}(x_1, \dots, x_{n-1}) \\ y^{(n)} & = G(y)u(t-h) + \alpha_n(x_1, \dots, x_n) \end{cases} \quad (1.50)$$

where the α_i are combinations of the f_i 's and their derivatives. Then, the Jacobian matrix of y and its time derivatives has the following form

$$\frac{\partial(y, \dot{y}, \dots, y^{(n)})}{\partial X} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \dots & * & 1 & 0 \\ * & \dots & \dots & * & G(y) \frac{\partial u(t-h)}{\partial h} \end{bmatrix} \quad (1.51)$$

Therefore, system $(\tilde{\Sigma})$ satisfies the rank observability condition (see [Hermann and Krener \[1977\]](#))

$$\text{rank} \left[\frac{\partial(y, \dot{y}, \dots, y^{(n)})}{\partial X} \right] = n + 1 \quad (1.52)$$

if and only if $G(y) \frac{\partial u(t-h)}{\partial h}$ does not cancel. In addition, for all $t \geq 0$ and $h \in [h_{min}, h_{max}]$, one has

$$\frac{\partial u(t-h)}{\partial h} = -\frac{\partial u(t-h)}{\partial t}. \quad (1.53)$$

Then, $\frac{\partial u(t-h)}{\partial h} \neq 0$ is equivalent to $\frac{\partial u(t-h)}{\partial t} \neq 0$. Therefore, (1.52) is equivalent to (1.48) and (1.49) for all $t \geq 0$ and $h \in [h_{min}, h_{max}]$ since the delay is differentiable. This ends the proof. \square

Condition (1.49) is natural because if the input is constant, the delay has no influence on the system: it cannot be observed. Nevertheless, it is restrictive because it means that the input should be strictly monotonic. For a certain class of observers the notion of persistency¹⁴ can be used to relax this condition. Another way to avoid the singularity problem in $\dot{u}(t) = 0$ is to assume that

14. Roughly speaking, an input is said to be persistent for a system if it is rich enough to excite this system in order to estimate some parameters or to reconstruct its state. This concept is presented in [Shimkin and Feuer \[1987\]](#) from a parameter estimation point of view. See Proposition 2 in [Besançon \[2007\]](#) for a characterization of persistent inputs in the observation framework.

Assumption 1.4.4 (Diop et al. [2001]). *There are time intervals I , sufficiently large, where the first derivative of u is continuous and bounded: for all $t \in I$,*

$$|\dot{u}(t)| > \alpha' \tag{1.54}$$

with $\alpha' > 0$.

Indeed, if (1.54) holds, then (1.49) is automatically verified on I . Conditions (1.48) and Assumption 1.4.4 are assumed to be verified in the next chapters.

High gain delay-state observer

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In this chapter, a new solution to estimate the state and the delay of system (Σ) in (1.40) is presented. It is based on the design of a high gain observer on an extended system. The main idea is to substitute the retarded input by its Taylor expansion. That is why, the following assumption is required:

Assumption 2.0.5. *Let $m \in \mathbb{N}^*$, the input u is in $C^{m+1}([-h_{max}, +\infty[, \mathbb{R})$ and its derivatives are known and bounded, i.e. for all $i = 1, \dots, m + 1$, there exist constants $U_i > 0$ such that for all $t \geq -h_{max}$*

$$|u^{(i)}(t)| \leq U_i. \quad (2.1)$$

2.1 Design of an extended system

System $(\tilde{\Sigma})$ in (1.46) is not under an adequate form to estimate h since the delay appears only through the delayed input $u(t - h)$. Therefore, the Taylor's theorem is used to take the delay out of the input. From Assumption 2.0.5, the input is m -times differentiable. Then, there exists a function $\gamma_m : [-h_{max}; +\infty[\rightarrow \mathbb{R}$ such that for all $t_1 > -h_{max}$,

$$u(t_1) = \sum_{i=0}^m \frac{(t_1 - t)^i}{i!} u^{(i)}(t) + \gamma_m(t, t_1 - t) \quad (2.2)$$

where γ_m is called the remainder. In particular taking $t_1 = t - h(t) > -h_{max}$ leads to

$$u(t - h(t)) = \sum_{i=0}^m \frac{(-1)^i h(t)^i}{i!} u^{(i)}(t) + \gamma_m(t, -h(t)) \quad (2.3)$$

for all $t > -h_{max}$. From Assumption 2.0.5, since u is $m + 1$ -differentiable, the remainder γ_m is such that

$$|\gamma_m(t, -h(t))| \leq \frac{h(t)^{m+1}}{(m+1)!} U_{m+1} \leq \frac{h_{max}^{m+1}}{(m+1)!} U_{m+1}. \quad (2.4)$$

Note that the expression $\sum_{i=0}^m \frac{(-1)^i h(t)^i}{i!} u^{(i)}(t)$ is called the Taylor approximation of $u(t - h(t))$ up to order m . By substituting (2.3) in $(\tilde{\Sigma})$, the following system is obtained

$$(\Sigma_{ext}) \quad \begin{cases} \dot{X} = \Lambda_{ext}(\xi(t))X + f_{ext}(X) + g_{ext}(y)u(t) + \Gamma(t, y, h) \\ y = C_{ext}X = x_1 \end{cases} \quad (2.5)$$

with $X = [x^T, h]^T \in \mathbb{R}^{n+1}$,

$$\Lambda_{ext}(\xi(t)) = \begin{bmatrix} 0 & I_{n-1} & 0_{n-1 \times 1} \\ 0 & 0_{1 \times n-1} & \xi(t) \\ 0 & 0_{1 \times n-1} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}, \quad (2.6)$$

$$f_{ext}(X) = [f(x)^T, 0]^T \in \mathbb{R}^{n+1}, \quad (2.7)$$

$$g_{ext}(y) = [g(y)^T, 0]^T \in \mathbb{R}^{n+1}, \quad (2.8)$$

$$\Gamma(t, y, h(t)) = [g(y(t))^T \gamma_m(t, -h(t)), \eta(t)]^T \in \mathbb{R}^{n+1} \quad (2.9)$$

and

$$\xi(t) = G(y(t)) \sum_{i=1}^m \frac{(-1)^i h^{i-1}(t)}{i!} u^{(i)}(t) \in \mathbb{R}. \quad (2.10)$$

Remark 2.1.1. *The use of the Taylor approximation (2.3) requires the computations of the input time derivatives. In open-loop, this is not a problem since the analytical expression of the input is known: the exact time derivatives can be computed analytically. In closed-loop, the standard way to obtain time derivatives is to use numerical differentiators¹. It will be shown that the error introduced by numerical differentiation can be lumped into $\gamma_m(t, -h(t))$. Furthermore, provided that this error is sufficiently small, it will not change the convergence property of the delay-state observer. An alternative to avoid input time differentiation is to insert a chain of integrators preprocessing the input (dynamic feedback).*

Extended system (Σ_{ext}) in (2.5) is equivalent to initial system (Σ) in the sense that they exhibit the same trajectories. However, system (Σ_{ext}) is delay free, with respect to the input, thanks to (2.3). The delayed nature of the plant is still present through $\gamma_m(t, -h(t))$ in Γ . The Taylor's approximation is a convenient way to rewrite the system in order to apply existing results from the observers literature for delay free systems. The error of approximation γ_m is going to be considered as a perturbation in the design of the observer as well as the dynamics of the delay $\eta(t)$. This is a key point of the method: considering the delay dynamics as a perturbation and designing an observer that is able to reconstruct the system state in spite of uncertainty Γ . Indeed, it is often a hard task to find a good approximation of the delay dynamics.

1. See for instance the work by Levant [2003].

2.2 Observer design and convergence analysis

Because of the form of (Σ_{ext}) and because it only has one tuning parameter, the following high gain observer² has been chosen in this work. Let $\lambda(\xi) \in \mathbb{R}^{(n+1) \times (n+1)}$ be the diagonal matrix defined by

$$\lambda(\xi) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & \xi \end{bmatrix} \quad (2.11)$$

and $\bar{A} \in \mathbb{R}^{(n+1) \times (n+1)}$ be the matrix

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad (2.12)$$

Then, the following observer is designed:

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{h}} \end{bmatrix} = \hat{\Lambda}_{ext} \hat{X} + f_{ext}(\hat{X}) + g_{ext}(y)u(t) - \lambda^{-1}(\hat{\xi})S_\rho^{-1}C_{ext}^T(C_{ext}\hat{X} - y) \\ \rho S_\rho + \bar{A}^T S_\rho + S_\rho \bar{A} + C_{ext}^T C_{ext} = 0 \\ \dot{\hat{h}}_p = \text{proj}_{[h_{min}, h_{max}]}(\dot{\hat{h}}) \end{cases} \quad (2.13)$$

where $\hat{X} = [\hat{x}^T, \hat{h}]^T \in \mathbb{R}^{n+1}$ and $\hat{\Lambda}_{ext} = \Lambda_{ext}(\hat{\xi}(t)) \in \mathbb{R}^{(n+1) \times (n+1)}$ with

$$\hat{\xi}(t) = G(y) \sum_{i=1}^m \frac{(-1)^i \hat{h}^{i-1}}{i!} u^{(i)}(t) \quad (2.14)$$

and \hat{h} is the estimated delay. It is reminded that $y = x_1$. Note that $\dot{\hat{h}}$ is computed from the first equation of system (2.13) and its projection, denoted $\dot{\hat{h}}_p$, is defined by the last equation of (2.13). The *proj* operator is given by

$$\text{proj}_{[h_{min}, h_{max}]}(\dot{\hat{h}}) = \begin{cases} 0, & \hat{h} = h_{max} \text{ and } \dot{\hat{h}} > 0 \\ 0, & \hat{h} = h_{min} \text{ and } \dot{\hat{h}} < 0 \\ \dot{\hat{h}} & \text{else.} \end{cases} \quad (2.15)$$

This projection guarantees that:

Lemma 2.2.1. *If the initial condition is such that $\hat{h}(0) \in [h_{min}, h_{max}]$ and $\dot{\hat{h}}_p = \text{proj}_{[h_{min}, h_{max}]}(\dot{\hat{h}})$ then, for all $t > 0$, one has*

$$\hat{h}_p(t) \in [h_{min}, h_{max}]. \quad (2.16)$$

The proof is given in Lavretsky et al. [2011]. As a result, it will be assumed in the rest of the chapter that

2. See the works of Gauthier et al. [1992], Busawon et al. [1998], Busawon and De Léon Morales [2000] for further details on high gain observers.

Assumption 2.2.1. *The initial condition of the estimated delay is such that*

$$\hat{h}(0) \in [h_{\min}, h_{\max}]. \quad (2.17)$$

Remark 2.2.1. *With a slight abuse of notation, the projected delay \hat{h}_p will be denoted \hat{h} in the sequel. It was shown in [Ioannou and Sun \[2012\]](#) that the projection retains the properties that are established in absence of the projection and guarantees that (2.16) is verified. Thus, the error $\hat{h}(t) - h(t)$ is bounded for all $t > 0$.*

The following notations will be used in the next computations

$$\begin{aligned} \hat{\Lambda}_{ext} &= \Lambda_{ext}(\hat{\xi}(t)), \\ \Lambda_{ext} &= \Lambda_{ext}(\xi(t)). \end{aligned} \quad (2.18)$$

Denoting $e = \hat{X} - X$ the estimation error, its dynamics is governed by

$$\dot{e} = \hat{\Lambda}_{ext}\hat{X} - \Lambda_{ext}X + f_{ext}(\hat{X}) - f_{ext}(X) - \lambda^{-1}(\hat{\xi})S_{\rho}^{-1}C_{ext}^T C_{ext}e - \Gamma(t, y, h). \quad (2.19)$$

For the proof of the main result, the next assumption is required:

Assumption 2.2.2. *There are time intervals I where the time derivative of $\hat{\xi}$ is bounded so for all $t \in I$*

$$|\dot{\hat{\xi}}(t)| \leq \nu \quad (2.20)$$

with $\nu > 0$ and $\hat{\xi}$ verifies

$$\alpha \leq |\hat{\xi}(t)| \quad (2.21)$$

with $\alpha > 0$.

Remark 2.2.2. *Equation (2.20) is a technical condition that will be used in the next section to prove the convergence of the estimation error; this condition arises because of the choice of the high gain observer. Equation (2.21) appears because of the form of the extended system (2.5) and is linked to the intrinsic observability conditions $G(y) \neq 0$ and $\dot{u} \neq 0$ presented in [Theorem 1.1](#).*

It is now possible to state the main result of this chapter. It is reminded that U_{m+1} is the bound of the m time derivative of u ([Assumption 2.0.5](#)) and that $\bar{\eta}$ is the upper bound of the delay dynamics ([Assumption 1.4.3](#)).

Theorem 2.1 ([Léchappé et al. \[2015b\]](#)). *Let $I = [t_0, t]$ with $t_0 > 0$ such that [Assumptions 1.4.1, 1.4.2, 1.4.3, 2.0.5, 2.2.1](#) and [2.2.2](#) hold on I . Then, provided that ρ is sufficiently large, there exist $k, \sigma, r > 0$ such that for all $t \in I$*

$$\|e(t)\| \leq k\|e(t_0)\|e^{-\sigma(t-t_0)} + r. \quad (2.22)$$

Proof. *The proof use similar argument as in [Busawon et al. \[1998\]](#). First note that $\Lambda_{ext} = \lambda^{-1}(\xi)\bar{A}\lambda(\xi)$ and $C_{ext}\lambda(\xi) = C_{ext}$. Furthermore,*

$$\hat{\Lambda}_{ext}\hat{X} - \Lambda_{ext}X = (\hat{\Lambda}_{ext} - \Lambda_{ext})\hat{X} + \Lambda_{ext}e \quad (2.23)$$

and the only non zero term of $\left[\hat{\Lambda}_{ext} - \Lambda_{ext} \right] \hat{X}$ is

$$\begin{aligned} \left(\left[\hat{\Lambda}_{ext} - \Lambda_{ext} \right] \hat{X} \right)_{(n, n+1)} &= G(y) \left[\sum_{i=1}^m \frac{(-1)^i \hat{h}^{i-1}}{i!} u^{(i)}(t) - \sum_{i=1}^m \frac{(-1)^i h^{i-1}}{i!} u^{(i)}(t) \right] \hat{h} \\ &= \hat{h}G(y) \sum_{i=1}^m \frac{(-1)^i}{i!} u^{(i)}(t) \left[\hat{h}^{i-1} - h^{i-1} \right]. \end{aligned} \quad (2.24)$$

Since all the terms on the right hand side of (2.24) are bounded then

$$\left\| \left(\left[\hat{\Lambda}_{ext} - \Lambda_{ext} \right] \hat{X} \right)_{(n,n+1)} \right\| \leq c_0 \quad (2.25)$$

with $c_0 > 0$. In addition, since f is C^1 (Assumption 1.4.1) then f_{ext} is also C^1 and from the mean value theorem one gets

$$f_{ext}(\hat{X}) - f_{ext}(X) = \left. \frac{df_{ext}(X)}{dX} \right|_{\kappa} e \quad (2.26)$$

for some κ lying between the segment with end points X and \hat{X} . Besides, the Jacobian matrix of f_{ext} is lower triangular from the definition of f in (1.42). Substituting (2.23) and (2.26) in (2.19)

$$\dot{e} = (\Lambda_{ext} - \lambda^{-1}(\hat{\xi})S_{\rho}^{-1}C_{ext}^T C_{ext})e + (\hat{\Lambda}_{ext} - \Lambda_{ext})\hat{X} + \left. \frac{df_{ext}(X)}{dX} \right|_{\kappa} e - \Gamma(t, y, h). \quad (2.27)$$

Let Δ_{ρ} be the diagonal matrix

$$\Delta_{\rho} = \text{diag} \left(1, \frac{1}{\rho}, \dots, \frac{1}{\rho^n} \right) \in \mathbb{R}^{(n+1) \times (n+1)} \quad (2.28)$$

then it is shown in Busawon et al. [1998] that

$$S_{\rho} = \frac{1}{\rho} \Delta_{\rho} S_1 \Delta_{\rho} \quad (2.29)$$

where S_1 is the solution of the Lyapunov equation in (2.13) for $\rho = 1$. Now, set $\bar{e} = \lambda(\hat{\xi})\Delta_{\rho}e$, then

$$\begin{aligned} \dot{\bar{e}} &= \rho(\bar{A} - S_1^{-1}C_{ext}^T C_{ext})\bar{e} + \lambda(\hat{\xi})\Delta_{\rho}(\hat{\Lambda}_{ext} - \Lambda_{ext})\hat{X} + \lambda(\hat{\xi})\Delta_{\rho} \left. \frac{df_{ext}(X)}{dX} \right|_{\kappa} \Delta_{\rho}^{-1}\lambda^{-1}(\hat{\xi})\bar{e} \\ &\quad + \dot{\lambda}(\hat{\xi})\lambda^{-1}(\hat{\xi})\bar{e} - \lambda(\hat{\xi})\Delta_{\rho}\Gamma(t, y, h). \end{aligned} \quad (2.30)$$

Furthermore, $\lambda(\hat{\xi})\Delta_{\rho} \left. \frac{df_{ext}(x)}{dX} \right|_{\kappa} \Delta_{\rho}^{-1}\lambda^{-1}(\hat{\xi}) = \Delta_{\rho} \left. \frac{df_{ext}(x)}{dX} \right|_{\kappa} \Delta_{\rho}^{-1}$ and

$$\Delta_{\rho} \left. \frac{df_{ext}(x)}{dX} \right|_{\kappa} \Delta_{\rho}^{-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & \dots & 0 & 0 \\ \frac{1}{\rho} \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \frac{1}{\rho^{n-1}} \frac{\partial f_n}{\partial x_1} & \dots & \frac{1}{\rho} \frac{\partial f_n}{\partial x_{n-1}} & \frac{\partial f_n}{\partial x_n} & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (2.31)$$

so since the partial derivative of the f_i are bounded and choosing $\rho \geq 1$, there exists a constant c_0 independent from ρ such that

$$\left\| \lambda(\hat{\xi})\Delta_{\rho} \left. \frac{df_{ext}(X)}{dX} \right|_{\kappa} \lambda^{-1}(\hat{\xi})\Delta_{\rho}^{-1} \right\| \leq c_1. \quad (2.32)$$

Besides, the product $\lambda(\hat{\xi})\Delta_{\rho}(\hat{\Lambda}_{ext} - \Lambda_{ext})\hat{X}$ results in a matrix whose coefficients are equal to zero except one that depends on $\left(\left[\hat{\Lambda}_{ext} - \Lambda_{ext} \right] \hat{X} \right)_{(n,n+1)}$ so one gets

$$\left\| \lambda(\hat{\xi})\Delta_{\rho}(\hat{\Lambda}_{ext} - \Lambda_{ext})\hat{X} \right\| \leq \frac{c_0}{\rho}. \quad (2.33)$$

From Assumption 2.2.2, one derives that

$$\left\| \dot{\lambda}(\hat{\xi})\lambda^{-1}(\hat{\xi}) \right\| \leq \frac{\nu}{\alpha} \quad (2.34)$$

for all $t \in I$. Define the following Lyapunov candidate function V as follows

$$V(t) = \bar{e}(t)^T S_1 \bar{e}(t). \quad (2.35)$$

The matrix S_1 verifies

$$\underline{\delta}_1 \|\bar{e}\|^2 \leq \bar{e}^T S_1 \bar{e} \leq \bar{\delta}_1 \|\bar{e}\|^2 \quad (2.36)$$

with $\underline{\delta}_1, \bar{\delta}_1 > 0$ because it is the solution of the Lyapunov equation for $\rho = 1$ (see Gauthier et al. [1992]) so the Lyapunov function (2.35) is well defined. Note that $\left\| \lambda(\hat{\xi})\Delta_\rho \right\| \leq 1$ for $\rho \geq 1$. Differentiating V along the trajectories of (2.30) and using above properties leads to

$$\dot{V} \leq -\rho \bar{e}^T S_1 \bar{e} + 2\frac{c_0}{\rho} \|S_1\| \|\bar{e}\| + 2c_1 \|S_1\| \|\bar{e}\|^2 + 2\frac{\nu}{\alpha} \|S_1\| \|\bar{e}\|^2 + 2\|S_1\| \|\Gamma(t, y, h)\| \|\bar{e}\|. \quad (2.37)$$

Thus, the following maximization of (2.37) is obtained

$$\dot{V} \leq -(\rho \underline{\delta}_1 - 2c_1 \|S_1\| - 2\frac{\nu}{\alpha} \|S_1\|) \|\bar{e}\|^2 + 2\|S_1\| \left(\frac{c_0}{\rho} + \|\Gamma(t, y, h)\| \right) \|\bar{e}\|. \quad (2.38)$$

In addition, from equation (2.4) and Assumptions 1.4.2, 1.4.3 and 2.0.5, it can be deduced that Γ is bounded and verifies

$$\sup_{t \geq 0} \|\Gamma(t, y, h)\| < \bar{\Gamma} \quad (2.39)$$

with $\bar{\Gamma} = \bar{G} \frac{h_{max}^{m+1}}{(m+1)!} U_{m+1} + \bar{\eta}$. Consequently, the gain ρ can be chosen sufficiently large such that the above inequality becomes

$$\dot{V}(t) \leq -c_3 V(t) + c_4 \sqrt{V(t)} \quad (2.40)$$

with $c_3 = (\rho \underline{\delta}_1 - 2c_1 \|S_1\| - 2\frac{\nu}{\alpha} \|S_1\|) / \bar{\delta}_1 > 0$ and $c_4 = 2\|S_1\| (\frac{c_0}{\rho} + \bar{\Gamma}) / \sqrt{\bar{\delta}_1}$. The comparison lemma given in [Khalil, 2002, Lemma 9.4] and the relations $\bar{e} = \lambda(\hat{\xi})\Delta_\rho e$ and $e = \Delta_\rho^{-1} \lambda^{-1}(\hat{\xi}) \bar{e}$ ensure that equation (2.22) is satisfied. \square

Note that the radius r depends on $\bar{\Gamma}$. Then for constant delays ($\bar{\eta} = 0$) and some particular inputs such that the bound of the m -time derivatives is equal to zero³ ($U_m = 0$), the convergence radius is minimum. For time-varying delays and input such that $\gamma_m(t, -h(t)) \neq 0$, the observation error converges to a neighborhood of the origin whose size depends on the accuracy of the Taylor approximation and on the dynamics of the delay $\eta(t)$. Note that increasing the m -order of the Taylor expansion will decrease the size of r because $\lim_{m \rightarrow \infty} \frac{h_{max}^{(m+1)}}{(m+1)!} = 0$ where

$h_{max}^{(m+1)}$ denotes the upper bound of the $m+1$ time derivative of h . However, it requires the computation of input time derivatives so a trade off has to be made between estimation precision and computation complexity.

Remark 2.2.3. A similar result has been obtained for linear systems with a Kalman-like observer in the particular case with $m = 1$ in Léchappé et al. [2015a]. In the general case of nonlinear systems, the convergence with the kalman-like observer is not ensured by choosing a sufficiently high gain because of the nonlinear part $f(x)$.

3. An example of such an input is a ramp because $\ddot{u} = 0$

Remark 2.2.4. *Theorem 2.1 can be extended to systems such that G depends on x provided the extra assumption G Lipschitz is made: $|G(x) - G(z)| \leq K_g \|x - z\|$ with $K_g > 0$.*

Remark 2.2.5. *In open-loop, the time derivatives of the input are analytically computable if the input is appropriately chosen. On the contrary, in closed-loop, even if the analytic expression of the input is known, it is not so obvious; especially if there is additive noise or perturbation. Let us take an academic example to illustrate this statement. Assume that the system is defined by $\dot{x} = Ax + Bu(t - h) + d(t)$ with d an unknown external perturbation. Choosing the controller $u(t) = Kx(t)$ implies that its time derivative reads as $\dot{u}(t) = KAx(t) + KBu(t - h) + Kd(t)$. The terms $KAx(t)$ and $KBu(t - h)$ may be perfectly known if the exact model of the system is available. However, $Kd(t)$ is unknown so \dot{u} cannot be computed analytically in presence of external perturbation. As a result, it can be necessary to use numerical differentiator to compute the successive time derivatives of the input. In this case, inevitable errors are introduced by the differentiator. Denoting $\hat{u}^{(i)}$ the estimated time derivatives and $e_{u_i} = u^{(i)} - \hat{u}^{(i)}$, one obtains that*

$$u^{(i)} = e_{u_i} + \hat{u}^{(i)} \quad (2.41)$$

so the Taylor approximation (2.3) becomes

$$u(t - h(t)) = \sum_{i=0}^m \frac{(-1)^i h(t)^i}{i!} \hat{u}^{(i)}(t) + \gamma'_m(t, -h(t)) \quad (2.42)$$

with

$$\gamma'_m(t, -h(t)) = \gamma_m(t, -h(t)) + \sum_{i=0}^m \frac{(-1)^i h(t)^i}{i!} e_{u_i}(t). \quad (2.43)$$

If the e_{u_i} are bounded and sufficiently small then γ'_m is bounded and the extra error added by the numerical differentiation will only decrease the observation accuracy (by increasing the radius r defined in (2.22)).

2.3 An extension to obtain a more accurate estimation

The original idea in the method presented above is to consider the delay as a state and to use the Taylor's theorem in order to take the delay out of the input. To reduce the convergence radius in (2.22), it can be interesting to estimate the perturbation Γ in (Σ_{ext}) in (2.5). The term $g(y)\gamma_m(t, -h(t))$ can be minimized by a wise choice of input⁴ but it is not possible to act on $\eta(t)$. Consequently, the objective is to observe η to try to reduce its influence. Various methods are used to reconstruct perturbations as discussed in Radke and Zhiqiang [2006]: Disturbance Observer (DOB), Unknown Input Observer (UIO) and Extended State Observer (ESO). The ESO technique consists in turning the original system into an extended one with the disturbance as part of the new state. In Miklosovic et al. [2006], the ESO method is modified to the GESO (Global ESO) method by including time derivatives of the perturbation. Indeed, this allows to improve the estimation of time-varying disturbances. Applying the GESO technique to (Σ_{ext}) , the augmented state becomes $X = [x^T, h, \dot{h}, \dots, h^{(q)}]^T \in \mathbb{R}^{n+1+q}$ and then the extended system reads as:

$$\begin{cases} \dot{X} = \Lambda_{ext} X + f_{ext}(X) + g_{ext}(y)u(t) + \Gamma(t, y, h) \\ y = C_{ext} X = x_1 \end{cases} \quad (2.44)$$

with

4. The term $\gamma_m(t, -h(t))$ is the remainder of the Taylor's approximation of the retarded input $u(t - h(t))$.

$$\Lambda_{ext} = \begin{bmatrix} 0 & I_{n-1} & 0_{n \times 1} & \dots & 0_{n \times 1} & 0_{n \times 1} \\ 0 & 0 & G(y) \sum_{i=1}^m \frac{(-1)^i h^{i-1}}{i!} u^{(i)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+1+q) \times (n+1+q)}, \quad (2.45)$$

$$f_{ext}(X) = [f(x)^T, 0, \dots, 0]^T \in \mathbb{R}^{(n+1+q)}, \quad (2.46)$$

$$g_{ext}(y) = [g(y)^T, 0, \dots, 0]^T \in \mathbb{R}^{(n+1+q)}, \quad (2.47)$$

and

$$\Gamma(t, y, h) = [g(y)^T \gamma_m(t, -h(t)), 0, \dots, 0, \eta'(t)]^T \in \mathbb{R}^{(n+1+q)} \quad (2.48)$$

where $h^{(q+1)}(t) = \eta'(t)$ is unknown.

For system (2.44), an observer similar to (2.13) can be designed and the convergence proof will be analogous to the one of Theorem 2.1 provided that η' is bounded.

The observer proposed here has three parameters: the gain ρ , the degree m of the Taylor approximation and the number of delay dynamics q included in the extended system. To tune them, one has to take into account various factors: the computation time, the delay, the input dynamics and the precision needed. For example, if a slow-varying input (with respect to the delay magnitude) can be applied to the system, m can be small because the Taylor approximation will be accurate (even for small m). In the same way, q can be equal to zero if the delay varies slowly.

2.4 Illustrative example

2.4.1 Model presentation

A second order system has been chosen to illustrate previous results (see Section 1.2.1 in Khalil [2002]). It models the behavior of a pendulum and reads as

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m_p} x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_p l^2} \end{bmatrix} u(t - h(t)) \\ y(t) = Cx(t) = x_1(t) \end{cases} \quad (2.49)$$

with g the acceleration due to gravity, m_p the mass of the bob, l the length of the rod and k the friction coefficient. Consequently, system (2.49) has the form of (Σ) in (1.40) with $\Lambda = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f(x) = [0, -\frac{g}{l} \sin x_1 - \frac{k}{m_p} x_2]^T$ and $G(y) = \frac{1}{m_p l^2}$. Functions f and G comply with Assumptions 1.4.1 and 1.4.2. The delay used in the sequel are defined by

$$h(t) = \begin{cases} 0.6 & \text{for } 0 \leq t < 20 \\ 1.2 & \text{for } 20 \leq t < 40 \\ 0.3 & \text{for } 40 \leq t < 60 \\ 0.8 + 0.45 \sin(0.2t) & \text{for } t \geq 60. \end{cases} \quad (2.50)$$

The delay $h(t)$ is bounded in $[h_{min}, h_{max}]$ with $h_{min} = 0$ s and $h_{max} = 1.5$ s. The delay is differentiable on the time interval $[60, 120]$ seconds: Assumption 1.4.3 is verified. When the

delay is piecewise constant the differentiability is not required (see Assumption 1.4.3). Note that these delays are very large in comparison with the dynamics of the pendulum: the time constant associated to the linearized system is less than 1 s. Finally, the input signals applied to system (2.49) are defined below

$$u(t) = \begin{cases} u_1(t) = 0.009t \\ \text{or} \\ u_2(t) = \sin(0.15t). \end{cases} \quad (2.51)$$

Both inputs are indefinitely differentiable and their time derivatives are bounded: Assumption 2.0.5 is satisfied. Furthermore, the observability condition (1.48) is always true since G is constant. Condition (1.49) is always verified for u_1 but almost everywhere for u_2 . However, u_2 cancels only in isolated points so there exist time intervals where Assumption 1.4.4 is true and therefore Assumption 2.2.2 as well. The design of the observer is given in the next subsection. All system parameters are sum up in Table 2.1.

g (m.s ⁻²)	m_p (kg)	l (m)	k (kg.s ⁻¹)	h (s)
9.81	0.2	0.5	0.3	[0, 1.5]

Table 2.1 – System parameters

2.4.2 Observer design

In above sections, the general case of the Taylor expansion of order m has been studied. In practice, the choice of m can be made according to the precision desired on \hat{h} and the available time derivatives of u . In this section, three observers are going to be tested with $m = 1$, $m = 2$ and $m = 3$ in order to study the impact of the choice of m on the estimation result. For the three observers, the gain ρ has been fixed to 5. The observer equations are given below:

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{h}} \end{bmatrix} = \hat{\Lambda}_{ext} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{h} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{l} \sin \hat{x}_1 - \frac{k}{m_p} \hat{x}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_p l^2} \\ 0 \end{bmatrix} u(t) - \lambda^{-1}(\hat{\xi}) S_\rho^{-1} C_{ext}^T \begin{bmatrix} \hat{x}_1 - x_1 \\ 0 \\ 0 \end{bmatrix} \\ 0 = \rho S_\rho + \bar{A}^T S_\rho + S_\rho \bar{A} + C_{ext}^T C_{ext} \\ \dot{\hat{h}}_p = \text{proj}_{[h_{min}, h_{max}]}(\dot{\hat{h}}) \end{cases} \quad (2.52)$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \hat{\Lambda}_{ext} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \hat{\xi} \\ 0 & 0 & 0 \end{bmatrix}, \hat{\lambda}(\hat{\xi}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \hat{\xi} \end{bmatrix} \quad (2.53)$$

with

$$\hat{\xi} = \begin{cases} -\frac{1}{ml^2} \dot{u} & \text{for } m = 1, \\ \frac{1}{ml^2} \left(-\dot{u} + \frac{1}{2} \ddot{u} \hat{h} \right) & \text{for } m = 2, \\ \frac{1}{ml^2} \left(-\dot{u} + \frac{1}{2} \ddot{u} \hat{h} - \frac{1}{6} u^{(3)} \hat{h}^2 \right) & \text{for } m = 3, \end{cases} \quad (2.54)$$

and $C_{ext} = [1, 0, 0]$. The initial conditions of the system and the observer are gathered in Table 2.2.

$x(0)$ [rad,rad.s ⁻¹]	$\hat{x}(0)$ [rad,rad.s ⁻¹]	$\hat{h}(0)$ (s)	ρ
$[0.87, 0]^T$	$[0, 0]^T$	0.3	5

Table 2.2 – Observer parameter and initial conditions

2.4.3 Results

Four simulations have been carried out:

- Simulation 1: system (2.49) and observer (2.13) with $u = u_1$,
- Simulation 2: system (2.49) and observer (2.13) with $u = u_2$,
- Simulation 3: system (2.49) and observer (2.13) with $u = u_2$ and noisy measurement and input,
- Simulation 4: system (2.49) and extended observer of Section 2.3 with $q = 1$ and $u = u_2$.

The following notations are used in the sequel:

$$e_x = \begin{bmatrix} \hat{x}_1 - x_1 \\ \hat{x}_2 - x_2 \end{bmatrix} \quad \text{and} \quad e_h = \hat{h} - h. \quad (2.55)$$

The norm of the relative error is given by

$$\|e_x\|_r = \frac{\|e_x\|}{\|x\|}. \quad (2.56)$$

Simulation 1 On Figure 2.1a, the input does not have any observation singularity. Besides, when $u = u_1$, the three observers have the same trajectories since $\dot{u} = 0$, and $\ddot{u} = 0$ that is why all the curves are exactly identical. The dynamics of the delay is equal to zero from $t = 0$ s to $t = 60$ s so $\eta(t) = 0$. For a ramp, $\gamma_1(t, -h(t)) = 0$: Γ is equal to zero and system (Σ_{ext}) in (2.5) is disturbance-free. The convergence radius r in (2.22) is minimum, we even see that the estimation error converges exponentially to zero in simulation⁵. This is confirmed by Figure 2.1a-Bottom. Note that the fast delay variations have an effect on the state observation error as shown by the peaks at 20 s and 40s on Figure 2.1a-Top. After 60 s, the delay is time-varying. One observes on Figure 2.1a-Top that the state observation error converges to a small neighborhood around the origin. However, the exponential convergence to zero is not achieved anymore because of an additional error due to the dynamics η . The delay observation error also stays in a ball around the exact value (Figure 2.1a-Bottom). The radius of the ball can be reduced by increasing the observer gain ρ .

Simulation 2 With the input u_2 , there is an observation singularity when $\dot{u}_2 = 0$. This observability loss is clear on Figure 2.1b-Bottom when, for $m = 1$, some peaks precisely occur when \dot{u}_2 cancels. Remark that the peaks at $t = 32$ s and $t = 74$ s are truncated because of the delay projection in the interval $[0, 1.5]$. It can be observed that the magnitude of the peaks are related to the delay size: the larger the delay is, the larger the peaks are. For $m = 2$ and $m = 3$, these peaks are very small that they almost not affect the estimation anymore and the observation error stays in a small neighborhood of the origin. This analysis confirms the advantage to

5. The convergence radius r has been overestimated because even if it is minimum when $\Gamma = 0$, it is not equal to zero.

include higher order time-derivatives of the input in the term ξ by attenuating the effect of the observability loss⁶. The size of this neighborhood is directly related to m and the delay. The larger m is, the more accurate the Taylor approximation is and then the smaller γ_m is. Similarly the smaller the delay is, the smaller γ_m is. These two remarks are illustrated on Figure 2.2. It can be seen that the convergence radius for $m = 2$ is larger than for $m = 3$. However, this difference is not significant and the extra computation cost to compute the third time derivative may not worth the small accuracy improvement. Since the observation singularity is very detrimental for $m = 1$, the observer with $m = 2$ seems to be a good tradeoff between precision and computation cost. As a result, in the next simulations, only the results for $m = 2$ will be shown. Finally, note that the state observation is very accurate between the transient caused by the fast delay variations (Figure 2.1b-Top).

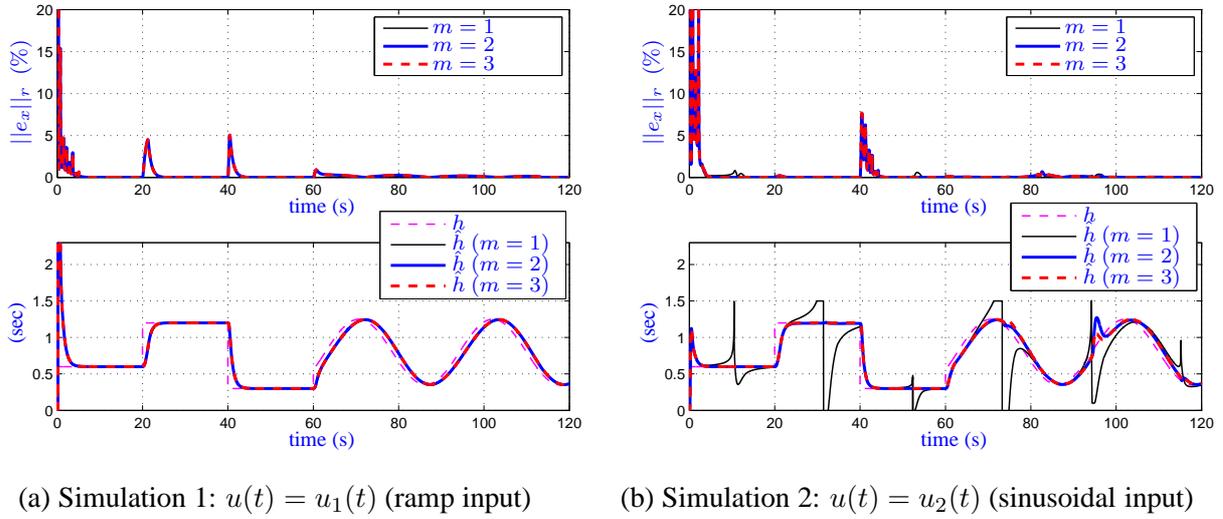


Figure 2.1 – Comparison of the observer performances for different orders of the Taylor approximation.

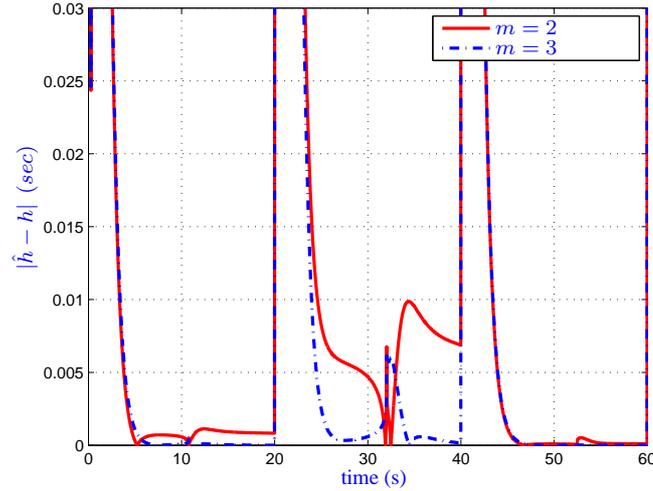
Simulation 3 In practice, noise can affect the measurement and the input. In this simulation, white noise has been added to the measured signal x_1 and the input signal u . The variance has been chosen to obtain a noise of about 5% of the amplitude of the original signals. The time derivatives \dot{u} , \ddot{u} have been computed thanks to a Levant differentiator [Levant \[1993\]](#) whose equations are given below:

$$\begin{cases} \dot{z}_0 = v_0 \\ v_0 = -\lambda_0 L^{\frac{1}{3}} |z_0 - u(t)|^{\frac{2}{3}} \text{sign}(z_0 - u(t)) + z_1 \\ \dot{z}_1 = v_1 \\ v_1 = -\lambda_1 L^{\frac{1}{2}} |z_1 - v_0|^{\frac{1}{2}} \text{sign}(z_1 - v_0) + z_2 \\ \dot{z}_2 = -\lambda_2 \text{sign}(z_2 - v_1). \end{cases} \quad (2.57)$$

If the parameters λ_i and L are properly chosen, then the result in [\[Levant, 1993, Theorem 5\]](#) guarantees that the following equalities are true in the absence of input noises and after a finite time:

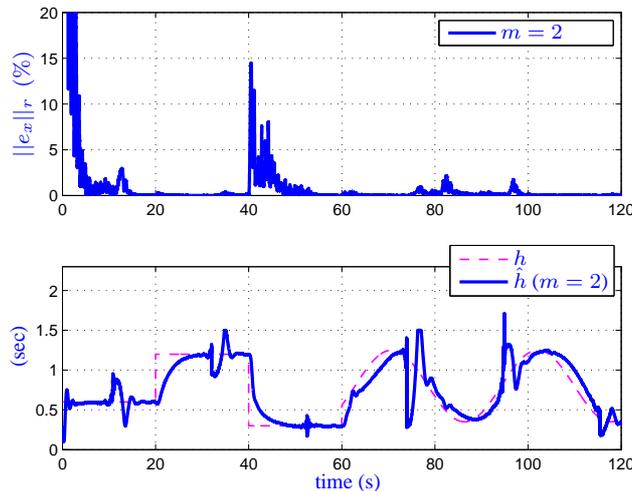
$$z_0(t) = u(t) \quad \text{and} \quad z_i(t) = u^{(i)}(t) \quad \text{for } i = 1, 2. \quad (2.58)$$

6. It has been proposed in [Léchappé et al. \[2015a\]](#) to turned off the observer (only the part concerning the delay) when getting closer to the singularity. This solution is interesting only if the observer has already converged and if the delay is constant otherwise this solution is not appropriate.

Figure 2.2 – Simulation 2: $u(t) = u_2(t)$ (sinusoidal input)

If the input u is noisy then the estimation error converges in a ball around the origin whose radius is proportional to the noise value [Levant, 1993, Theorem 6].

Besides, a matrix $R = \epsilon I_3$ with $\epsilon = 0.09$ has been introduced in the observer to filter the noise: the gain of observer (2.13) becomes $\lambda^{-1}(z\hat{\epsilon}ta)S_\rho^{-1}RC_{ext}^T$. The differentiator parameters are $\lambda_0 = 3$, $\lambda_1 = 1.5$, $\lambda_2 = 1$ and $L = 0.11$. Figure 2.3-Bottom shows that the singularity are magnified because of the noise. However, between these observation singularities, the error estimation tends to a small neighborhood of the origin. On Figure 2.3-Top, one can see that the state observation error still converges to a ball around the origin. However, its radius is slightly amplified by the noise in comparison to Figure 2.1b.

Figure 2.3 – Simulation 3: $u(t) = u_2(t)$ (sinusoidal input) with noise

Simulation 4 Finally, the extension presented in Section 2.3 with $q = 1$ is compared to the above results ($q = 0$). The simulation is carried out on system (2.49). In both case ($q = 0$ and $q = 1$), the second order Taylor approximation is used ($m = 2$). All the parameters are the same as in the previous simulations. On Figure 2.4, the convergence radius for the state observation error has slightly decreased when the dynamics of the delay is added in the extended system

($q = 1$). The accuracy of the delay observation is largely improved: the convergence radius is about three times smaller for $q = 1$.

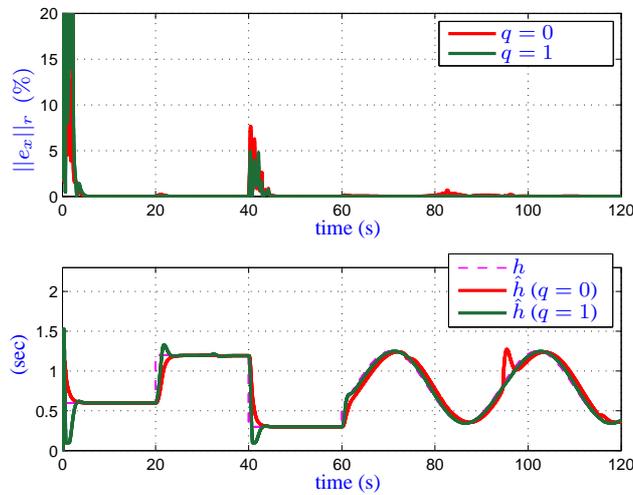


Figure 2.4 – Extended observer: $u(t) = u_2(t)$ (sinusoidal input), $m = 2$

Summary of the simulation results Above simulations have confirmed theoretical results. They illustrate the efficiency of the observation technique to

- reconstruct the state of a system with an unknown and possibly time-varying delay in the input,
- estimate the delay value.

Furthermore, the following properties are of particular interest:

- for $m > 1$, the observation singularity has almost no effect on the observation for noiseless signals,
- the convergence radius can be reduced using large values of m . This is particularly interesting in open-loop when the time-derivatives of the input are perfectly known.

The quality of delay estimation highly depends on the nature of the input. The input u should not be too fast-varying in order to have a good Taylor approximation (and a small γ_m).

2.5 Summary

Contributions

- ✓ Estimation of both the state and the delay of a class of nonlinear SISO systems.
- ✓ The method works for time-varying delays.
- ✓ Use of a standard observation technique (high gain observer).

Interconnected delay-state observers

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A solution to observe both the state and the delay has been presented in the previous chapter. A drawback of the previous technique is that the dynamics of the state and delay estimations are governed by the same gain. An alternative solution that allows a separate tuning of the delay and state observers is proposed in this chapter. First, a delay estimator based on an optimization method is presented. Then, a new extended state observer (ESO) is designed to estimate the delayed input. Finally, an interconnected observers scheme is introduced to observe both the state and the delay using standard state observers.

3.1 Gradient based delay observer

In this section, a delay estimation method based on a gradient descent algorithm¹ is studied. The idea is to compare the delayed signal $u(t - h(t))$ to a virtually retarded signal $u(t - \hat{h}(t))$ and find the value of $\hat{h}(t)$ that minimizes the criterion²

$$J(\hat{h}(t)) = [u(t - \hat{h}(t)) - u(t - h(t))]^2 \quad (3.1)$$

1. See the work of [Liberzon and Tempo \[2004\]](#) for details about gradient descent algorithms.

2. In [Bresch-Pietri et al. \[2012\]](#), the authors use a similar technique but the criterion involved a state prediction.

on $[h_{min}, h_{max}]$. Defining the dynamics of the delay estimation \hat{h} thanks to the gradient descent algorithm, one has

$$\dot{\hat{h}}(t) = -\frac{\rho_h}{2} \frac{dJ(\hat{h})}{d\hat{h}} \Big|_{\hat{h}(t)} = -\rho_h [u(t - \hat{h}(t)) - u(t - h(t))] \frac{\partial u(t - \hat{h})}{\partial \hat{h}} \Big|_{\hat{h}(t)} \quad (3.2)$$

with $\rho_h > 0$. Since

$$\frac{\partial u(t - \hat{h})}{\partial \hat{h}} = -\frac{\partial u(t - \hat{h})}{\partial t}, \quad (3.3)$$

it results that the dynamics of \hat{h} is

$$\dot{\hat{h}}(t) = \rho_h [u(t - \hat{h}(t)) - u(t - h(t))] \dot{u}(t - \hat{h}(t)). \quad (3.4)$$

In order to compute (3.4), the knowledge of \dot{u} and $u(t - h(t))$ is necessary. In some practical cases, this is not always possible so some methods have to be used to estimate these quantities. In the next sections, three cases are going to be analyzed:

- $u(t - h)$ and \dot{u} known in Section 3.1.1,
- $u(t - h)$ known and \dot{u} unknown in Section 3.1.2,
- $u(t - h)$ and \dot{u} unknown in Section 3.1.3.

3.1.1 $u(t - h)$ and \dot{u} known

The ideal case is when $u(t - h)$ and \dot{u} are perfectly known. In the control case, the assumption $u(t - h)$ known may be reasonable. In practice, if the packets are timestamped then the system that receives the delayed input can send it to the delay estimator as shown on Figure 3.1. As far as \dot{u} is concerned, adding an integrator preprocessing the input (dynamic feedback), as in Example 3.1.1, allows to perfectly know the input time derivative.

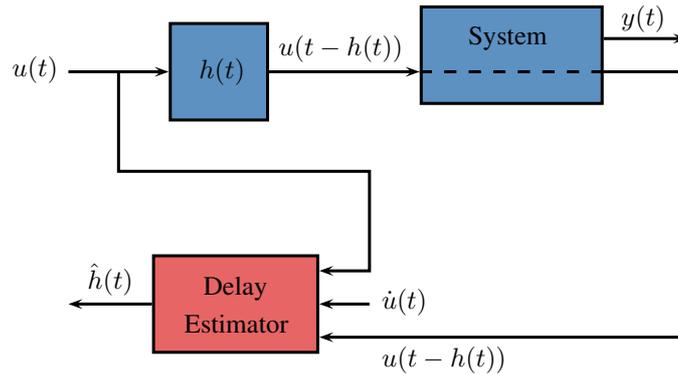


Figure 3.1 – Delay estimator with $u(t - h(t))$ and \dot{u} known

Example 3.1.1. Consider the scalar system

$$\dot{x}(t) = ax(t) + bu(t - h(t)). \quad (3.5)$$

System (3.5) can be rewritten as follows

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t - h(t)) \\ u(t - h(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \quad (3.6)$$

with $v(t) = \dot{u}(t)$ as a new input. Any kind of controllers v can be designed to stabilize extended system (3.6); then the control u can be deduced by $u(t) = \int_0^t v(s)ds$.

The stability property of estimator (3.4) when $u(t-h)$ and \dot{u} are perfectly known is given in Theorem 3.1.

Theorem 3.1. *Let $I = [t_0, t]$ with $t_0 > 0$ be such that $t - \hat{h}(t) \in I$ and $t - h(t) \in I$, if $|\dot{u}(s)| > \alpha$ for all $s \in I$, then the delay estimation error resulting from estimator (3.4) verifies*

$$\|e(t)\| \leq \|e(t_0)\|e^{-\rho_h \alpha^2 (t-t_0)} + \frac{1}{\rho_h \alpha^2} \sup_{s \geq t_0} |\dot{h}(s)| \quad (3.7)$$

for all $t \in I$ and with $e(t) = \hat{h}(t) - h(t)$.

Proof. For the sake of clarity the argument t will be omitted when no confusion is possible. Let

$$V = \frac{1}{2}(\hat{h} - h)^2 \quad (3.8)$$

be a Lyapunov candidate function. Taking the time derivative of V along the trajectories of (3.4) gives

$$\dot{V} = \rho_h(\hat{h} - h)[u(t - \hat{h}) - u(t - h)]\dot{u}(t - \hat{h}) - (\hat{h} - h)\dot{h}. \quad (3.9)$$

Reminding that from the mean value theorem there exists $\phi_t \in [\min(t-h, t-\hat{h}), \max(t-h, t-\hat{h})]$ such that

$$\dot{u}(\phi_t) = \frac{u(t - \hat{h}) - u(t - h)}{h - \hat{h}}, \quad (3.10)$$

one gets

$$\dot{V} = -\rho_h(\hat{h} - h)^2 \dot{u}(t - \hat{h})\dot{u}(\phi_t) - (\hat{h} - h)\dot{h}. \quad (3.11)$$

Since $t - \hat{h}$ and ϕ_t are in the interval I , the terms $\dot{u}(t - \hat{h})$ and $\dot{u}(\phi_t)$ have the same sign. It follows that

$$\dot{V} \leq -\rho_h \alpha^2 (\hat{h} - h)^2 - (\hat{h} - h)\dot{h} \quad (3.12)$$

and then that

$$\dot{V}(t) \leq -\rho_h \alpha^2 \|\hat{h} - h\|^2 + \|\hat{h} - h\| \sup_{s \leq t} |\dot{h}(s)|. \quad (3.13)$$

Applying Lemma 9.4 in Khalil [2002] leads to equation (3.7). \square

When the delay is constant, $\dot{h} = 0$, the delay estimation error converges exponentially to zero. In addition, note that the convergence rate is proportional to the square of the time derivative lower-bound α .

3.1.2 $u(t-h)$ known and \dot{u} unknown

In practice, \dot{u} may not be perfectly known. In that case, one has to compute an estimation $\hat{\dot{u}}$. An estimation based on an interpolation strategy is chosen in Diop et al. [2001], a Levant differentiator Levant [2003] or any numerical differentiator can be used. The only condition to have an efficient delay estimator is that $\dot{u}(t - \hat{h}(t))$ and its estimate $\hat{\dot{u}}(t - \hat{h})$ have the same sign

(the exact estimation is not required). That is why a simpler solution has been chosen³: the time derivative of \hat{u} is approximated by the finite difference as follows:

$$\mu(t) = \frac{u(t - \hat{h}(t)) - u(t - \hat{h}(t) - h_1)}{h_1} \quad (3.14)$$

with $h_1 > 0$ a virtual delay. Then, a similar estimator to (3.4) is designed but using (3.14) instead of $\dot{u}(t - \hat{h}(t))$:

$$\dot{\hat{h}}(t) = \rho_h [u(t - \hat{h}(t)) - u(t - h(t))] \mu(t). \quad (3.15)$$

The stability property of estimator (3.15) is given in Theorem 3.2.

Theorem 3.2. *Let $I = [t_0, t]$ with $t_0 > 0$ be such that $t - \hat{h}(t) - h_1 \in I$ and $t - h(t) \in I$, if $|\dot{u}(s)| > \alpha$ for all $s \in I$, then the delay estimation error resulting from estimator (3.15) verifies*

$$\|e(t)\| \leq \|e(t_0)\| e^{-\rho_h \alpha^2 (t-t_0)} + \frac{1}{\rho_h \alpha^2} \sup_{s \geq t_0} |\dot{h}(s)| \quad (3.16)$$

for all $t \in I$ and with $e(t) = \hat{h}(t) - h(t)$.

Noting that the mean value theorem ensures the existence of $\theta_t \in [t - h_1, t]$ such that

$$\mu(t) = \dot{u}(\theta_t) = \frac{u(t - \hat{h}) - u(t - \hat{h} - h_1)}{h_1}, \quad (3.17)$$

the result is deduced by similar computations as in Theorem 3.1.

If h_1 is sufficiently small then the sign of $\dot{u}(t - \hat{h})$ and $\mu(t)$ is the same then the same stability property is obtained (see Figure 3.2).

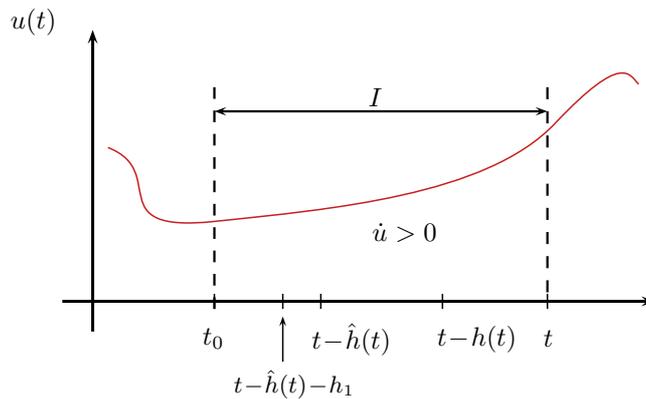


Figure 3.2 – Delay estimator with $u(t - h(t))$ and \dot{u} known

3.1.3 $u(t - h)$ and \dot{u} unknown

If the delayed input is not available, the objective is to compute an estimation of $u(t - h)$. An ESO method is going to be used to estimate it⁴. First, a standard ESO will be reminded in order to compare its performance with respect to a new one presented thereafter.

3. Numerical differentiators are often complex to implement and to tune.

4. See Madonski and Herman [2015] for references on this method.

Since the input only appears in the last equation of (Σ) in (1.40), other equations are deliberately left out not to overload the presentation. Furthermore, the full state knowledge is considered since the focus is on delay estimation in this subsection. In the next sections, the goal is to design an extended observer to estimate the unknown term $u(t-h(t))$ in the following equation

$$\dot{x}_n(t) = f_n(x(t)) + G(y(t))u(t-h(t)). \quad (3.18)$$

As in Chapter 2, the assumption below is required

Assumption 3.1.1. *Let $p \in \mathbb{N}^*$, the input u is in $C^{p+1}([-h_{max}, +\infty[, \mathbb{R})$ and its derivatives are known and bounded, i.e. for all $i = 1, \dots, p+1$, there exist constants $U_i > 0$ such that for all $t \geq -h_{max}$*

$$|u^{(i)}(t)| \leq U_i. \quad (3.19)$$

Before dealing with the design of ESO observers, the concept of persistence is recalled. Consider a state affine system of the form

$$\begin{cases} \dot{x}(t) = A(s(t))x(t) \\ y(t) = Cx(t) \end{cases} \quad (3.20)$$

with s a known signal, $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^n$. The observability Grammian reads as

$$W(t, T, s(t)) = \int_t^{t+T} \Phi_s(\theta, t) C^T C \Phi_s(\theta, t) d\theta \quad (3.21)$$

where $\Phi_s(\theta, t)$ is the transition matrix of system (3.20)

$$\frac{d}{d\theta} \Phi_s(\theta, t) = A(s(\theta)) \Phi_s(\theta, t). \quad (3.22)$$

Thus, the universality index is given by

$$w(t, T, s) = \min_i \lambda_i(W(t, T, s)). \quad (3.23)$$

Definition 3.1.1 (Besançon et al. [1996]). *The signal s is said to be regularly persistent for system (3.20) if there exist $t_0, T, \alpha > 0$ such that $w(t, T, s) > \alpha$ for all $t \geq t_0$.*

This condition usually allows to relax the observability condition by allowing pointwise observability losses.

Standard Extended State Observer

The idea of the ESO method is to consider the unknown term and its time derivatives as new state variables. For instance, in the present case, the extended system reads as

$$\begin{cases} \dot{z} = \Lambda_z z + f_z(x) + \Gamma_z(t) \\ y = C_z z = x_n \end{cases} \quad (3.24)$$

with $z = [x_n, u(t-h), \dots, u^{(p-1)}(t-h)]^T \in \mathbb{R}^{p+1}$ for $p \in \mathbb{N}$ and $p \geq 1$,

$$\Lambda_z = \begin{bmatrix} 0 & G(y) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(p+1) \times (p+1)}, \quad (3.25)$$

$$f_z(x) = [f(x)^T, 0, \dots, 0]^T \in \mathbb{R}^{p+1}, \quad (3.26)$$

$$\Gamma_z(t) = [0, -\dot{h}u(t-h), \dots, (1-\dot{h})u^{(p)}(t-h)]^T \in \mathbb{R}^{p+1} \quad (3.27)$$

and $C_z = [1, 0, \dots, 0] \in \mathbb{R}^{p+1}$. The term Γ_z is bounded from Assumptions 1.4.3 and 3.1.1 and can be considered as a perturbation. An observability analysis shows that the pair (C_z, Λ_z) is observable if $G(y) \neq 0$. It has been decided to use a Kalman-like observer⁵. Indeed, provided that $G(y)$ is persistent for system (3.24), the Kalman-like observer can achieve exponential convergence in the disturbance free case. The equations of the observer are given below

$$\begin{cases} \dot{\hat{z}} = \Lambda_z \hat{z} + f_z(x) - S_z^{-1} C_z^T C_z [\hat{z} - z] \\ \dot{S}_z = -\rho_z S_z - \Lambda_z^T S_z - S_z \Lambda_z + C_z^T C_z \\ S_z(0) = S_z(0)^T > 0 \end{cases} \quad (3.28)$$

with $\hat{z} = [\hat{x}_n, \hat{u}(t-h), \dots, \hat{u}^{(p-1)}(t-h)]^T$. Consequently, the error dynamics, denoted $e_z(t) = \hat{z}(t) - z(t)$, reads as

$$\dot{e}_z(t) = (\Lambda_z - S_z^{-1} C_z^T C_z) e_z(t) - \Gamma_z(t) \quad (3.29)$$

and its stability property is established in Theorem 3.3.

Theorem 3.3 (Léchappé et al. [2016]). *Suppose that Assumptions 1.4.3 and 3.1.1 hold and that $G(y)$ is persistent for system (3.24). Then, there exist $k, \sigma, r > 0$ such that*

$$\|e_z(t)\| \leq k \|e_z(0)\| e^{-\sigma t} + r \sup_{s \geq 0} \|\Gamma_z(s)\| \quad (3.30)$$

with $\Gamma_z(t) = [0, -\dot{h}u(t-h), \dots, (1-\dot{h})u^{(p)}(t-h)]^T$.

The convergence proof of Theorem 3.3 is similar to the one by Hammouri and de León Morales [1990] since $f_z(x)$ is a known signal that cancels when computing the observation error.

New Extended State Observer

The major drawback of the previous method is that the time derivative of the last term is approximated by $u^{(p)}(t-h(t)) \approx 0$. Here, a different approximation is proposed by introducing a virtual and constant delay $h_2 > 0$:

$$u^{(p)}(t-h(t)) \approx \frac{u^{(p-1)}(t-h(t)+h_2) - u^{(p-1)}(t-h(t)-h_2)}{2h_2}. \quad (3.31)$$

Choosing h_2 sufficiently small will make the approximation of $u^{(p)}(t-h(t))$ more accurate hence the estimation of $u(t-h(t))$ will be better and finally the delay estimation will be improved. As a consequence, the terms $u^{(p-1)}(t-h+h_2)$ and $u^{(p-1)}(t-h-h_2)$ have to be added to the extended state and their time derivatives are approximated by

$$\dot{u}^{(p-1)}(t-h(t)+h_2) \approx \frac{u^{(p-1)}(t-h(t)+h_2) - u^{(p-1)}(t-h(t))}{h_2} \quad (3.32)$$

and

$$\dot{u}^{(p-1)}(t-h(t)-h_2) \approx \frac{u^{(p-1)}(t-h(t)) - u^{(p-1)}(t-h(t)-h_2)}{h_2}. \quad (3.33)$$

5. See the works of [Bornard et al. \[1989\]](#) and [Hammouri and de León Morales \[1990\]](#) for further details on Kalman-like observers.

As a result, the extended system is

$$\begin{cases} \dot{z} = \Lambda_z z + f_z(x) + \Gamma_z(t) \\ y = C_z z = x_n \end{cases} \quad (3.34)$$

with $z = [x_n, u(t-h), \dots, u^{(p-1)}(t-h), u^{(p-1)}(t-h+h_2), u^{(p-1)}(t-h-h_2)]^T \in \mathbb{R}^{p+3}$ for $p \in \mathbb{N}, p \geq 1$,

$$\Lambda_z = \begin{bmatrix} 0 & G(y) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2h_2} & -\frac{1}{2h_2} \\ 0 & 0 & 0 & 0 & -\frac{1}{h_2} & \frac{1}{h_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{h_2} & 0 & -\frac{1}{h_2} \end{bmatrix} \in \mathbb{R}^{(p+3) \times (p+3)}, \quad (3.35)$$

$$f_z(x) = [f(x)^T, 0, \dots, 0]^T \in \mathbb{R}^{p+3}, \quad (3.36)$$

$$C_z = [1, 0, \dots, 0] \in \mathbb{R}^{p+3} \quad (3.37)$$

and

$$\Gamma_z(t) = [0, -\dot{h}u(t-h), \dots, -\dot{h}u^{(p)}(t-h), -\dot{h}u^{(p)}(t-h) + \epsilon_1(h_2), -\dot{h}u^{(p)}(t-h) + \epsilon_2(h_2), -\dot{h}u^{(p)}(t-h) + \epsilon_3(h_2)]^T \in \mathbb{R}^{p+3} \quad (3.38)$$

where ϵ_i are functions such that $\lim_{h_2 \rightarrow 0} \epsilon_i(h_2) = 0$ for $i = 1, 2, 3$. The term Γ_z can be seen as a perturbation and is bounded from Assumptions 1.4.3 and 3.1.1. An observability analysis shows that the pair (C_z, Λ_z) is observable provided that $G(y) \neq 0$. Similarly to the previous case, it has been decided to use a Kalman-like observer. Indeed, provided that $G(y)$ is persistent for system (3.34), the Kalman-like observer can achieve exponential convergence in the disturbance free case. The equations of the observer are given below

$$\begin{cases} \dot{\hat{z}} = \Lambda_z \hat{z} + f_z(x) - S_z^{-1} C_z^T C_z [\hat{z} - z] \\ \dot{S}_z = -\rho_z S_z - \Lambda_z^T S_z - S_z \Lambda_z + C_z^T C_z \\ S_z(0) = S_z(0)^T > 0 \end{cases} \quad (3.39)$$

with $\hat{z} = [\hat{x}_n, \hat{u}(t-h), \dots, \hat{u}^{(p-1)}(t-h), \hat{u}^{(p-1)}(t-h+h_2), \hat{u}^{(p-1)}(t-h-h_2)]^T$. Consequently, the error dynamics, denoted $e_z(t) = \hat{z}(t) - z(t)$, reads as

$$\dot{e}_z(t) = (\Lambda_z - S_z^{-1} C_z^T C_z) e_z(t) - \Gamma_z(t) \quad (3.40)$$

and its stability property is established in Theorem 3.4.

Theorem 3.4. *Suppose that Assumptions 1.4.3 and 3.1.1 hold and that $G(y)$ is persistent for system (3.34). Then, there exist $k', \sigma', r' > 0$ such that*

$$\|e_z(t)\| \leq k' \|e_z(0)\| e^{-\sigma' t} + r' \sup_{s \geq 0} \|\Gamma_z(s)\| \quad (3.41)$$

with $\Gamma_z(t) = [0, -\dot{h}u(t-h), \dots, -\dot{h}u^{(p)}(t-h), -\dot{h}u^{(p)}(t-h) + \epsilon_1(h_2), -\dot{h}u^{(p)}(t-h) + \epsilon_2(h_2), -\dot{h}u^{(p)}(t-h) + \epsilon_3(h_2)]^T$ where ϵ_i are functions such that $\lim_{h_2 \rightarrow 0} \epsilon_i(h_2) = 0$ for $i = 1, 2, 3$.

The convergence proof of Theorem 3.4 is similar to the one by Hammouri and de León Morales [1990] since $f_z(x)$ is a known signal that cancels when computing the observation error.

The main difference with the standard ESO is the size of the radius Γ_z . It will be illustrated in simulation that the new ESO (3.39) achieves a more accurate estimation of $u(t - h)$ than observer (3.28). Note that when the delay is constant, $\dot{h} = 0$, and Γ_z in Theorem 3.4 becomes $\Gamma_z(t) = [0, 0, \dots, \epsilon_1(h_2), \epsilon_2(h_2), \epsilon_3(h_2)]^T$. Thus, the radius of the convergence ball can be made arbitrarily small choosing a small h_2 . This is not possible with $\Gamma_z = [0, \dots, 0, u^{(p)}(t - h)]$ in Theorem 3.3. However, in presence of noise, h_2 cannot be reduced arbitrarily so the delay estimation error will remain in a neighborhood of the origin.

Application to delay estimation

The estimation of $u(t - h)$ obtained from the new ESO can now be substituted in the delay estimator dynamics. Thus, estimator (3.15) becomes

$$\dot{\hat{h}}(t) = \rho_h [u(t - \hat{h}(t)) - \hat{u}(t - h(t))] \mu(t) \quad (3.42)$$

where $\mu(t)$ is the same time derivative approximation as in (3.14) and $\hat{u}(t - h)$ is obtained from observer (3.39). The stability property of estimator (3.42) is given in Theorem 3.5.

Theorem 3.5. *Let $I = [t_0, t]$ with $t_0 > 0$ be such that $t - \hat{h}(t) - h_1 \in I$ and $t - h(t) \in I$, if $|\dot{u}(s)| > \alpha$ for all $s \in I$, then the delay estimation error resulting from estimator (3.15) verifies*

$$\|e(t)\| \leq \|e(t_0)\| e^{-\rho_h \alpha^2 (t - t_0)} + \frac{1}{\rho_h \alpha^2} \sup_{s \geq t_0} [\rho_h U_1 \|e_z(s)\| + |\dot{h}(s)|] \quad (3.43)$$

for all $t \in I$ and with $e(t) = \hat{h}(t) - h(t)$, e_z verifying (3.43) and U_1 defined in Assumption 3.1.1.

First, the existence of $\theta_t \in [t - h_1, t]$ such that

$$\mu(t) = \dot{u}(\theta_t) = \frac{u(t - \hat{h}) - u(t - \hat{h} - h_1)}{h_1}. \quad (3.44)$$

is guaranteed by the mean value theorem. Similarly, there exists $\phi_t \in [t - h_1, t]$ such that $u(t - \hat{h}) - \hat{u}(t - h) = (h - \hat{h})\dot{u}(\phi_t) - C_u e_z(t)$ with

$$C_u = [0, 1, 0, \dots, 0]. \quad (3.45)$$

The result is then deduced by similar computations as in Theorem 3.1.

From equation (3.43), it can be noted that the error e_z due to the observation of $u(t - h)$ directly affects the accuracy of the delay estimation \hat{h} via the term $\sup_{s \geq t_0} [\rho_h U_1 \|e_z(s)\| + |\dot{h}(s)|]$.

As a consequence, the error on the estimation of $u(t - h)$, $C_u e_z$, directly affects the accuracy of the delay estimation.

Remark 3.1.1. *For noisy signal, the alternative dynamics can be used*

$$\dot{\hat{h}}(t) = \frac{\rho_h}{T} \int_{t-T}^t [u(s - \hat{h}(s)) - \hat{u}(s - h(s))]^T \mu(s) ds \quad (3.46)$$

with $T > 0$ the window length that can be adjusted with respect to the noise level: the larger the noise is, the larger T should be. Note that it is a filtered version of the dynamics defined in (3.42). The integral acts as a low pass filter whose time-constant is tuned by the parameter T . Similarly, if the input signal is fast varying, (3.46) can be normalized as follows

$$\tau_h(t) = \frac{\rho_h}{T} \frac{I(t)}{\epsilon + |I(t)|} \quad (3.47)$$

with $I(t) = \int_{t-T}^t [u(s - \hat{h}(s)) - u(s - h(s))]^T \mu(s) ds$ and $\epsilon > 0$. This normalization allows to make the convergence speed almost independent from the input dynamics.

3.2 Interconnected delay-state observers scheme

3.2.1 General presentation

In Chapter 2, an observer that estimates both the delay and the state has been presented. However, the dynamics of the delay and the dynamics of the system can be very different so it should be possible to tune the observer dynamics independently. This is not possible with observer (2.13): this drawback can be circumvented by using an other observer scheme proposed in the sequel. It is based on the interconnection of a delay observer with a state observer: the scheme is displayed on Figure 3.3. Especially, all the state observers existing for delay free systems can be used. In this section, a general framework is considered and some particular observers will be studied in Section 3.2.2.

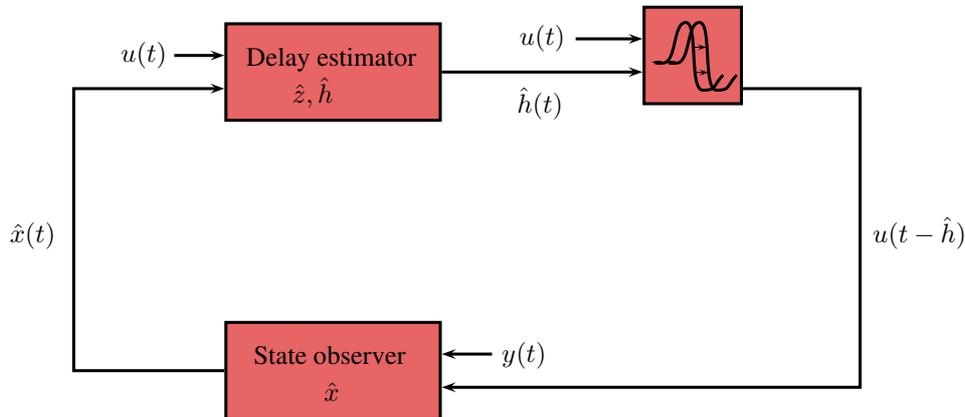


Figure 3.3 – Delay estimator interconnected with a standard state observer

Remark 3.2.1. The words “estimator” and “observer” will be used without distinction here.

Following the notation of Figure 3.3, the state estimation error is denoted by

$$e_x = \hat{x} - x \quad (3.48)$$

and the errors of the delay estimator are

$$e_z = \hat{z} - z \quad \text{and} \quad e_h = \hat{h} - h. \quad (3.49)$$

The variable \hat{z} is an auxiliary variable that will be useful to compute an estimation of the delay. The z variable can contain for example the input $u(t - h)$ and its time derivative as shown

in observer (3.39). In the sequel, V_x (respectively V_{zh}) will refer to a Lyapunov candidate function for the state observer (respectively for the delay estimator). These two estimators are connected by the estimated delay \hat{h} that enters the state observer through $u(t - \hat{h})$ and by the state estimation \hat{x} that enters the delay estimator. To study the convergence of the whole system, the interconnected subsystems approach is going to be used. The next theorem states a general result for interconnected delay and state observers.

Theorem 3.6 (Léchappé et al. [2016]). *For the disturbance free system, suppose that there exist a state observer such that*

$$\dot{V}_x \leq -c_x \|e_x\|^2 + c_{xh} \|e_x\| \|e_h\| \quad (3.50)$$

and a delay observer such that

$$\begin{aligned} \dot{V}_{zh} \leq & -c_z \|e_z\|^2 - c_h \|e_h\|^2 \\ & + c_{zh} \|e_z\| \|e_h\| + c_{zx} \|e_z\| \|e_x\| \end{aligned} \quad (3.51)$$

with $c_x, c_z, c_h, c_{xh}, c_{zx}, c_{zh} > 0$. An appropriate choice of c_x, c_z, c_h , guarantees the existence of a class \mathcal{KL} function β , a finite time T and an initial time t_0 such that the error $e = [e_x^T, e_z^T, e_h^T]^T$ of the disturbed system is uniformly bounded

$$\|e(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad (3.52)$$

for all $t_0 \leq t \leq t_0 + T$ and

$$\|e(t)\| \leq \delta \quad (3.53)$$

for all $t > t_0 + T$.

Remark 3.2.2. *The constants $c_x, c_z, c_\tau, c_{x\tau}, c_{zx}, c_{z\tau}$ will depend on the observer parameters. In particular, c_x, c_z, c_τ will be directly proportional to the observer gains as it will be shown in the next section on particular examples.*

Proof. *First, the convergence of the disturbance free system is worked out. Let the sum of the Lyapunov functions of the isolated systems:*

$$V = V_x + V_{zh} \quad (3.54)$$

be a candidate Lyapunov function for the interconnected system. Consequently, from (3.50) and (3.51), the time derivative of V satisfies

$$\dot{V} \leq -\left[\|e_x\|, \|e_z\|, \|e_h\| \right] \underbrace{\begin{bmatrix} c_x & -\frac{1}{2}c_{zx} & -\frac{c_{xh}}{2} \\ -\frac{1}{2}c_{zx} & c_z & -\frac{c_{zh}}{2} \\ -\frac{c_{xh}}{2} & -\frac{c_{zh}}{2} & c_h \end{bmatrix}}_S \begin{bmatrix} \|e_x\| \\ \|e_z\| \\ \|e_h\| \end{bmatrix}. \quad (3.55)$$

In order to have $\dot{V} \leq 0$, S has to be positive definite: then, the leading principal minors of S must be positive:

$$\begin{cases} c_x > 0, \\ c_x c_z - \frac{1}{4} c_{zx}^2 > 0, \\ c_x c_z c_h - \frac{1}{4} c_{zh}^2 c_x - \frac{1}{4} c_{xh}^2 c_z - \frac{1}{4} c_{zx} c_{xh} c_{zh} - \frac{1}{4} c_h c_{zx}^2 > 0. \end{cases} \quad (3.56)$$

The terms c_x, c_z, c_h are proportional to the observers gains as mentioned in Remark 3.2.2 so they can be chosen sufficiently large in order to make S definite positive.

The uniform boundedness of the disturbed system directly results from the application of Lemma 9.3 in Khalil [2002].

3.2.2 Particular example

In this section, some observation solutions are exhibited to illustrate the general consideration of the above section. First, a state observer is designed and it is shown that it complies with (3.50). Then, a delay observer is presented and condition (3.51) is checked.

State Observer

A high gain observer⁶ is chosen to observe the state of system (Σ) in (1.40). Observer equations are

$$\dot{\hat{x}} = \Lambda \hat{x} + f(\hat{x}) + g(y)u(t - \hat{h}) - \Delta_{\rho_x}^{-1} L(C\hat{x} - y), \quad (3.57)$$

where $\Delta_{\rho_x} = \text{diag} \left[\frac{1}{\rho_x}, \frac{1}{\rho_x^2}, \dots, \frac{1}{\rho_x^n} \right]$ and L such that

$$(\Lambda - LC)^T P + P(\Lambda - LC) = -I_n \quad (3.58)$$

with $P > 0$. It is important to note that the input u is affected by the delay estimation \hat{h} and not the real delay h . The dynamics of the state observation error is given by

$$\dot{e}_x = (\Lambda - \Delta_{\rho_x}^{-1} LC)e_x + f(\hat{x}) - f(x) + g(y)[u(t - \hat{h}) - u(t - h)]. \quad (3.59)$$

Since f is C^1 (Assumption 1.4.1), from the mean value theorem one gets

$$f(\hat{x}) - f(x) = \left. \frac{df(x)}{dx} \right|_{\gamma} e_x \quad (3.60)$$

for some γ lying between the segment with end points x and \hat{x} . Besides, the Jacobian matrix is lower triangular from the definition of f in (1.42). Defining $\tilde{e}_x = \Delta_{\rho_x} e_x$, one obtains

$$\dot{\tilde{e}}_x = \rho_x (\Lambda - LC) \tilde{e}_x + \Delta_{\rho_x} \left. \frac{df(x)}{dx} \right|_{\gamma} \Delta_{\rho_x}^{-1} \tilde{e}_x + \Delta_{\rho_x} g(y)[u(t - \hat{h}) - u(t - h)]. \quad (3.61)$$

since $\Delta_{\rho_x} \Lambda \Delta_{\rho_x}^{-1} = \rho_x \Lambda$ and $C \Delta_{\rho_x}^{-1} = \rho_x C$. Furthermore,

$$\Delta_{\rho_x} \frac{df(x)}{dx} \Delta_{\rho_x}^{-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & \dots & 0 \\ \frac{1}{\rho_x} \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{\rho_x^{n-1}} \frac{\partial f_n}{\partial x_1} & \dots & \frac{1}{\rho_x} \frac{\partial f_n}{\partial x_{n-1}} & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (3.62)$$

so for $\rho_x \geq 1$ and since the partial derivative of the f_i are bounded, there exists a constant c_0 independent from ρ_x such that

$$\left\| \Delta_{\rho_x} \left. \frac{df(x)}{dx} \right|_{\gamma} \Delta_{\rho_x}^{-1} \right\| \leq c_0 \quad (3.63)$$

6. See the works of Gauthier et al. [1992], Busawon et al. [1998], Busawon and De Léon Morales [2000] for further details on high gain observers.

Remark 3.2.3. In (3.59), it is clear from the continuity of u that the error term $u(t-\hat{h})-u(t-h)$ will tend to zero as soon as \hat{h} tends to h .

To prove that condition (3.50) holds for state observer (3.57), the following candidate Lyapunov function is defined

$$V_x = \tilde{e}_x^T P \tilde{e}_x. \quad (3.64)$$

Taking the time-derivatives of V_x along the trajectories of (3.61) and using (3.58) and (3.63) leads to

$$\dot{V}_x \leq -\rho_x \|\tilde{e}_x\|^2 + 2\|P\|c_0 \|\tilde{e}_x\|^2 + 2\tilde{e}_x^T P \Delta_{\rho_x} G(y)[u(t-\tilde{h}) - u(t-h)]. \quad (3.65)$$

Then, from Assumptions 1.4.1, 1.4.2 and 3.1.1, one derives the inequality

$$\dot{V}_x \leq -(\rho_x - 2\|P\|c_0) \|\tilde{e}_x\|^2 + 2\bar{G}U_1 \|\Delta_{\rho_x}\| \|P\| \|\tilde{e}_x\| |\hat{h} - h|. \quad (3.66)$$

Since $\|\tilde{e}_x\| \leq \|\Delta_{\rho_x}\| \|e_x\|$, one gets

$$\dot{V}_x \leq -(\rho_x - 2\|P\|c_0) \|\Delta_{\rho_x}\|^2 \|e_x\|^2 + 2\bar{G}U_1 \|\Delta_{\rho_x}\| \|P\| \|\Delta_{\rho_x}\| \|e_x\| \|e_h\| \quad (3.67)$$

Taking ρ_x sufficiently large, the above equation can be rewritten as follows

$$\dot{V}_x \leq -c_x \|e_x\|^2 + c_{xh} \|e_x\| \|e_h\| \quad (3.68)$$

with $c_x, c_{xh} > 0$. Consequently, this state observer satisfies (3.50).

Gradient based delay observer

Here it is assumed that $u(t-h)$ and \dot{u} are not available so the estimator (3.42) is used. However, the output $y = C_z z = x_n$ of observer (3.39) is not available so $y = \hat{x}_n$ is going to be used as the measurement of the new extended observer (3.39) as shown on Figure 3.4. Thus,

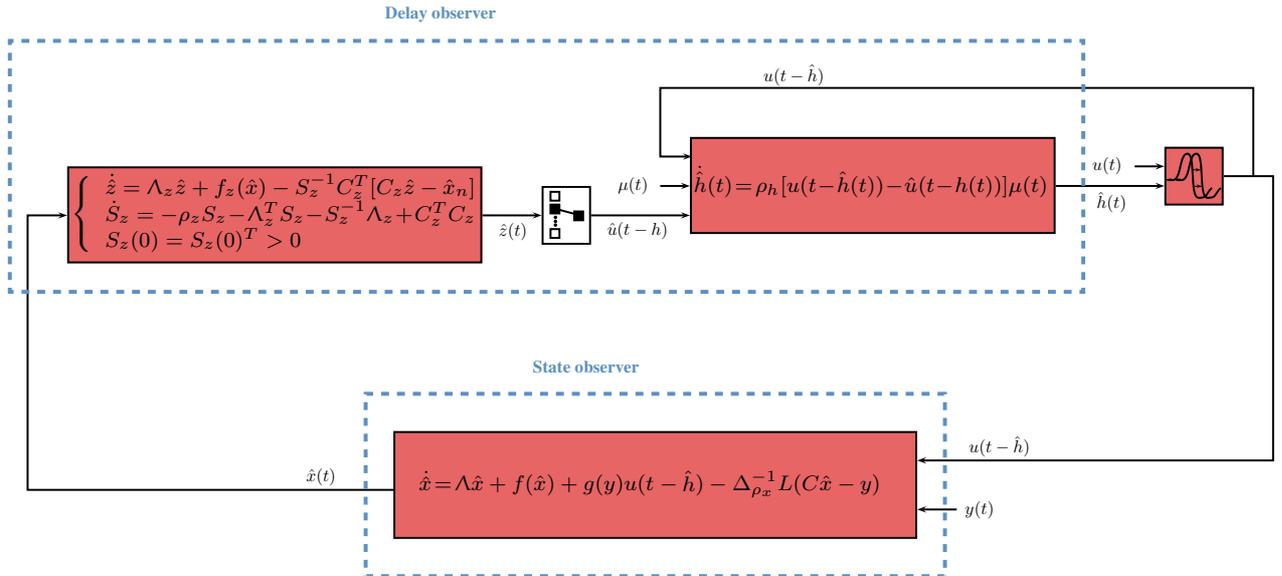


Figure 3.4 – Delay and state interconnected observers: a particular example

error equation (3.40) becomes

$$\dot{e}_z = [\Lambda_z - S_z^{-1} C_z^T C_z] e_z + f_z(\hat{x}) - f_z(x) + S_z^{-1} C_z^T C_n e_x - \Gamma_z(t) \quad (3.69)$$

with

$$C_n = [0, \dots, 0, 1] \in \mathbb{R}^n \quad (3.70)$$

and $\Lambda_z, C_z, S_z, \Gamma_z, f_z$ defined in 3.1.3. The term $f_z(\hat{x}) - f_z(x)$ arises because here all the state x is not measured anymore. As for, $S_z^{-1}C_z^T C_n e_x$, it arises because the output is \hat{x}_n instead of x_n so the correction term becomes $S_z^{-1}C_z^T C_z(C_z \hat{z} - C_n \hat{x})$ and since

$$\begin{aligned} C_z \hat{z} - C_n \hat{x} &= \hat{z}_n - \hat{x}_n \\ &= \hat{z}_n - x_n - (\hat{x}_n - x_n) \\ &= C_z e_z - C_n e_x. \end{aligned} \quad (3.71)$$

The delay estimator dynamics is

$$\dot{\hat{h}}(t) = \rho_h [u(t - \hat{h}(t)) - \hat{u}(t - h(t))] \mu(t) \quad (3.72)$$

where $\mu(t)$ is the same time derivative approximation as in (3.14) and $\hat{u}(t - h)$ is obtained from observer (3.39) with $y = \hat{x}_n$. Thus, delay estimation error reads as

$$\dot{e}_h = \rho_h [u(t - \hat{h}) - \hat{u}(t - h)] \dot{u}(t - \hat{h}) - \dot{h}(t). \quad (3.73)$$

Note that the dynamics Γ_z and \dot{h} are unknown and will be considered as a perturbation of the error dynamics. To analyze the stability of this observer, the following Lyapunov candidate function is defined

$$V_{zh} = V_z + V_h \quad (3.74)$$

with $V_z = e_z^T S_z e_z$ and $V_h = \frac{1}{2}(\hat{h} - h)^2$. After some computations similar to those in the proof of Theorem 3.1, one has

$$\dot{V}_h = -\rho_h (\hat{h} - h)^2 \dot{u}(\phi_t) \dot{u}(\theta_t) - (\hat{h} - h) \rho_h \dot{u}(\theta_t) C_u e_z \quad (3.75)$$

for $\dot{h} = 0$ and C_u defined in (3.45). So the maximization

$$\dot{V}_h \leq -\rho_h \alpha^2 \|e_h\|^2 + \rho_h U_1 \|e_z\| \|e_h\| \quad (3.76)$$

can be deduced using Assumption 3.1.1. Taking the time derivative of V_z along the trajectories of (3.69) leads to

$$\dot{V}_z = -\rho_z e_z^T S_z e_z - e_z^T C_z C_z e_z + 2e_z^T S_z [f(\hat{x}) - f(x)] + 2e_z^T C_z^T C_n e_x \quad (3.77)$$

for $\Gamma_z = 0$. Besides, there exists $t_0 > 0$, such that for all $t \geq t_0$, S_z verifies

$$\underline{\delta}_z I_n \leq S_z(t) \leq \bar{\delta}_z I_n \quad (3.78)$$

for ρ_z sufficiently large and with $\underline{\delta}_z$ and $\bar{\delta}_z$ are positive constants (see Besançon et al. [1996]). Since f is globally Lipschitz (Assumption 1.4.1) and from equations (3.77) and (3.78), one obtains

$$\dot{V}_z \leq -\rho_z \underline{\delta}_z \|e_z\|^2 + c_{zx} \|e_z\| \|e_x\| \quad (3.79)$$

with $c_{xz} > 0$. Then, from (3.76) and (3.79), one gets

$$\dot{V}_{zh} \leq -c_z \|e_z\|^2 + c_{zx} \|e_z\| \|e_x\| - c_h \|e_h\|^2 + c_{zh} \|e_z\| \|e_h\| \quad (3.80)$$

with $c_z = \rho_z \underline{\delta}_z$, $c_h = \rho_h \alpha^2$ and $c_{zh} = \rho_h U_1$. Therefore, V_{zh} satisfies condition (3.51).

It has been seen that the conditions of Theorem 3.6 are verified for a particular choice of state and delay estimators. In the next section, this observation scheme is going to be illustrated by simulation.

3.3 Illustrative example

3.3.1 A nonlinear example: the pendulum

Simulations are carried out on the same system as in Section 2.4. The pendulum equations are reminded below

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l} \sin x_1(t) - \frac{k}{m_p} x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_p l^2} \end{bmatrix} u(t - h(t)) \\ y(t) = Cx(t) = x_1(t). \end{cases} \quad (3.81)$$

The delay tested in the simulation is defined by (2.50) (see Section 2.4). To illustrate the results stated above, the following interconnected observer is simulated:

The state observer reads as

$$\dot{\hat{x}}(t) = \Lambda \hat{x}(t) + \begin{bmatrix} 0 \\ -\frac{g}{l} \sin \hat{x}_1(t) - \frac{k}{m_p} \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_p l^2} \end{bmatrix} u(t - \hat{h}(t)) - \Delta_{\rho_x}^{-1} L \begin{bmatrix} \hat{x}_1(t) - x_1(t) \\ 0 \end{bmatrix} \quad (3.82)$$

where $\Lambda = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\Delta_{\rho_x} = \text{diag} \left[\frac{1}{\rho_x}, \frac{1}{\rho_x^2} \right]$, L is such that $\Lambda - LC$ is Hurwitz and the delay observer is defined by

$$\begin{cases} \dot{\hat{z}} = \Lambda_z \hat{z} + f_z(\hat{x}) - S_z^{-1} C_z^T [C_z \hat{z} - C_2 \hat{x}] \\ \dot{S}_z = -\rho_z S_z - \Lambda_z^T S_z - S_z \Lambda_z + C_z^T C_z \\ S_z(0) = S_z(0)^T > 0 \\ \dot{\hat{h}}(t) = \rho_h [u(t - \hat{h}(t)) - C_u \hat{z}] \mu(t) \\ \mu(t) = \frac{u(t - \hat{h}) - u(t - \hat{h} - h_1)}{h_1} \end{cases} \quad (3.83)$$

with Λ_z defined in (3.35), C_z in (3.34), $C_2 = [0, 1]$ and C_u such that $C_u \hat{z} = \hat{u}(t - h)$. Note that the definition of Λ_z , C_z and C_u depends on the choice of p , the size of the new extended observer. Their exact expression is not given here since two different observer sizes will be compared $p = 1$ and $p = 2$. The vector C_2 has to be introduced because only \hat{x}_2 is available and not x_2 . It is reminded that the general definitions of C_u and C_n are given respectively in (3.45) and (3.70). The gains of this interconnected observer are tuned as follows

- state observer (3.82): $\rho_x = 10$,
- delay observer (3.83): $\rho_z = 10$, $\rho_h = 5$.

First, the extended state observers (3.28) and (3.39) are compared on Figures 3.5 and 3.6. In the noise free case, it is clear that the estimation error is smaller for the new extended observer. The difference is particularly significant on Figure 3.5a ($p = 1$). For $p = 2$, the estimation error for the standard ESO has been largely reduced but the new ESO still performs better and almost cancel the observation error. The conclusions for a noisy signal $u(t - h)$ are the same (Figure 3.6) so the new extend observer is more efficient to estimate unknown signals. This better accuracy will be very useful in the delay estimation since a small error on the observation of $u(t - h)$ leads to a large error on the delay estimation.

The estimation of $u(t - h)$ obtained above is now used to estimate the delay thanks to estimator (3.42). The results are presented on Figures 3.7 and 3.8. On Figure 3.7a, the delay estimation is similar for both estimation $\hat{u}(t - h)$ from the standard ESO or from the new ESO. However, when the input varies faster (Figure 3.7b), the delay estimation is better for the estimation $\hat{u}(t - h)$ coming from the new ESO. Indeed, the approximation $\dot{u}^{(p)} = 0$ is not appropriate for fast input so the estimation of $u(t - h)$ is not accurate and as a consequence

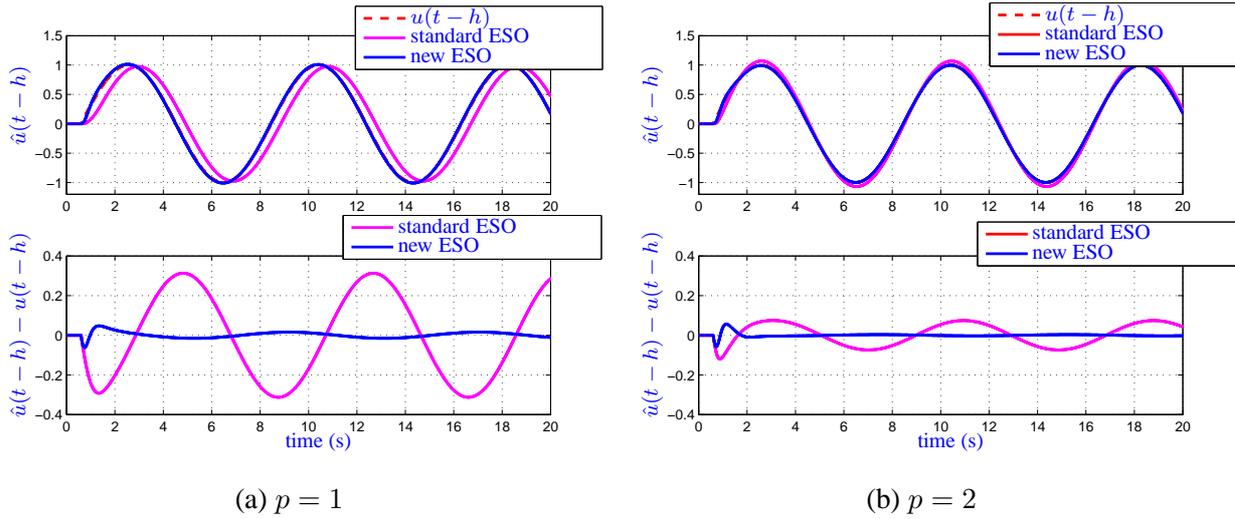


Figure 3.5 – Comparison of Extended State Observers (ESO) with $h_2 = 0.1$ s and $u(t) = \sin(0.8t)$ without noise.

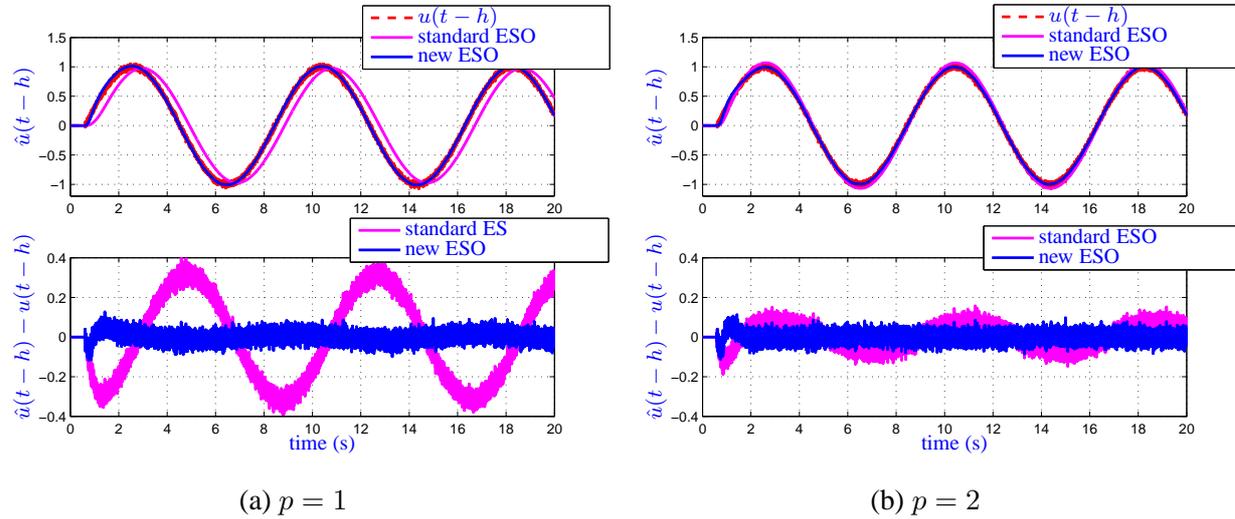


Figure 3.6 – Comparison of Extended State Observers (ESO) with $h_2 = 0.1$ s and $u(t) = \sin(0.8t)$ with a white noise on the signal $u(t-h)$.

the delay estimation is degraded. On the contrary, the delay estimation using the estimation of $u(t-h)$ from the new ESO are similar for both input signals; which means that $u(t-h)$ is properly estimated in both cases. As a comparison, the case with $u(t-h)$ known is presented on Figure 3.8. One can observe that the result from estimator (3.42) with new ESO ($u(t-h)$ and \dot{u} unknown) are very close to the “ideal” case ($u(t-h)$ and \dot{u} known). Further analysis shows that asymptotic convergence is achieved for a constant delay whereas a small error remains when the delay is time varying. This is in accordance with Theorems 3.1 and 3.2.

3.3.2 Comparison with the high gain observer

Finally, interconnected scheme (3.82)-(3.83) is compared to the high gain observer (2.52) presented in Chapter 2. The relative norm, defined by (2.56), is reminded here: $\|e_x\|_r = \frac{\|e_x\|}{\|x\|}$. For a constant delay, the interconnected scheme achieves a very accurate delay estimation, so the state estimation is also very accurate (Figure 3.9a). Observer (2.52) achieves an accurate convergence for small delays (Figure 3.9b). For a time-varying delay, the delay estimation is

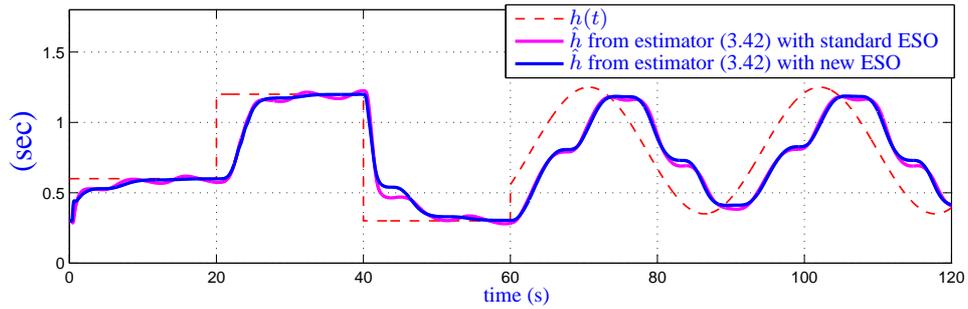
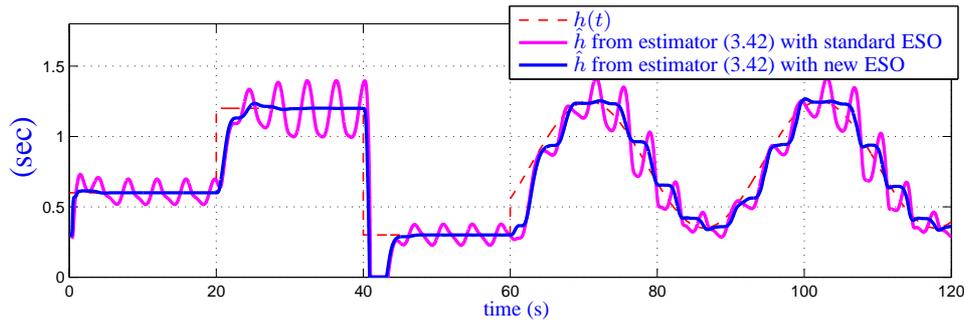
(a) $u(t) = \sin(0.4t)$ (b) $u(t) = \sin(0.8t)$

Figure 3.7 – Comparison of delay estimators performance for $u(t - h)$ and \dot{u} unknown ($p = 2$ for both ESO).

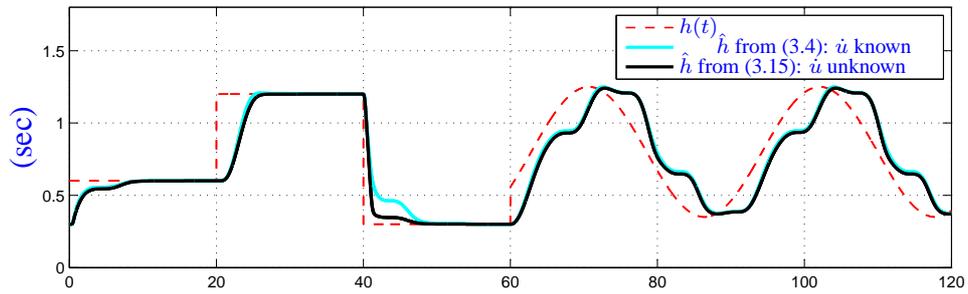
more accurate for interconnected observers (3.82)–(3.83) and especially when the delay is large.

Remark 3.3.1. *The main difference between these two methods come from the sensitivity of the high gain observer to the observation loss in $\dot{u} = 0$. Indeed, the input u presents observation singularities because its time derivative cancels in some isolated points. However, the delay estimator (3.83) does not shoot up during the observability loss.*

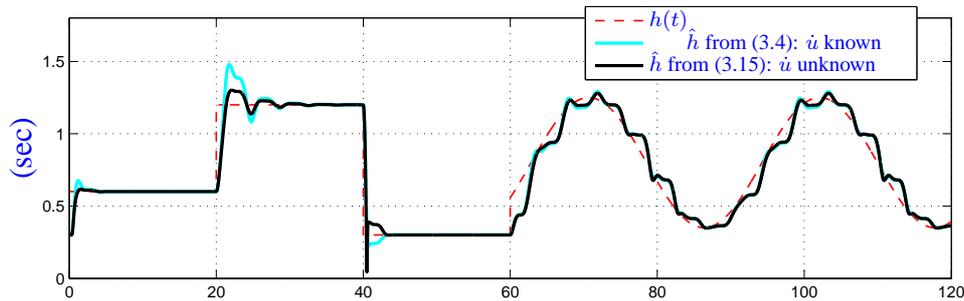
3.4 Summary

Contributions

- ✓ Presentation of a new Extended State Observer.
- ✓ Estimation of both the state and the delay of a class of nonlinear SISO systems.
- ✓ The method works for time-varying delays.
- ✓ Presentation of a delay observer that does not require the knowledge of the input time derivative.
- ✓ Use of an interconnected scheme that allows a separate tuning of the state and the delay observation dynamics.

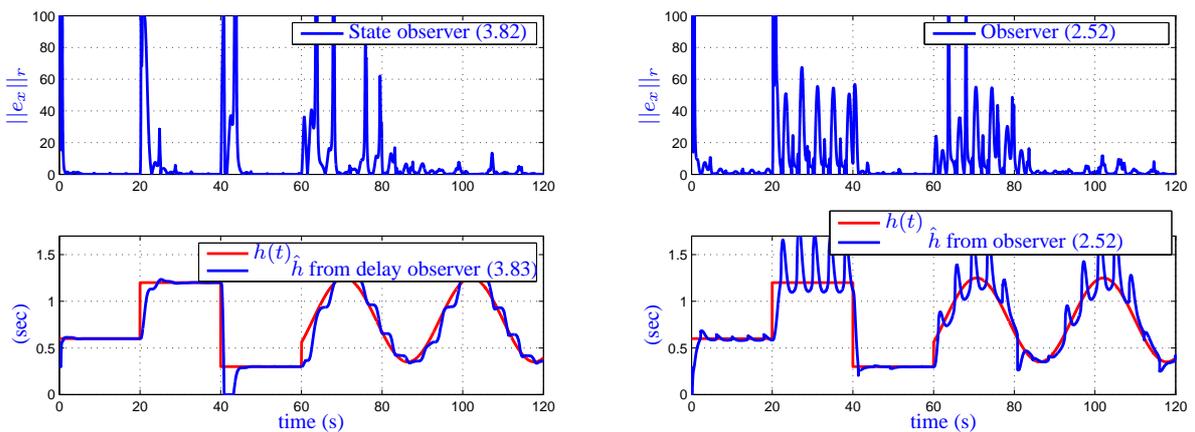


(a) $u(t) = \sin(0.4t)$



(b) $u(t) = \sin(0.8t)$

Figure 3.8 – Comparison of delay estimators performance for $u(t - h)$ known.



(a) Interconnected observers (3.82)-(3.83)

(b) High gain observer (2.52) ($m=2, \rho=2$)

Figure 3.9 – Comparison of the interconnected observer scheme and the high gain observer of chapter 2 for $u(t) = \sin(0.8t)$

Conclusion

Two solutions have been provided to solve the problem of delay and state observation for a class of nonlinear systems with a time-varying delay in the input. The first one is based on the Taylor approximation of the retarded input. Then a new system is constructed with the delay as a part of the extended state and finally a high gain observer is designed to estimate the augmented state. The second solution shows that it is possible to connect standard delay free state observers with a delay observer; the convergence is ensured under some conditions. Some simulations have shown the efficiency and the limit of each methods. The reconstructed state and the delay are useful to design predictive controllers: it will be studied in details in the following part.

A comparison of the different strong points of each estimation method is drawn in Table 3.1. Note that the comparison is based on the high gain observer of Chapter 2 and the interconnected observer Chapter 3 (with the delay estimator (3.42)). Some explanations about the criteria are given here:

- the “Complexity” criterion refers to the implementation complexity to compute the estimation,
- the “Singularity sensibility” criterion refers to the behavior of the estimator when going through an observation singularity in $\dot{u}(t) = 0$,
- the “Tuning” criterion refers to the complexity of the tuning process.

The classification arguments are detailed below. The computation of the input time derivatives is a weak point of the high gain observer. On the contrary, the delay estimator (3.42) does not need the input time derivative. It has been seen in the simulation section (Section 3.3.2) that the delay estimator proposed in the interconnected scheme only slows down but do not shoot up as the high gain observer when crossing a singularity in $\dot{u}(t) = 0$. Finally, the high gain observer is easy to tune since it only has one gain to adjust. However, it may be a weak point since the dynamics of the delays can be very different from the dynamics of the system. The advantage of the interconnected scheme is that the two gains can be tuned separately. However, the delay estimator gain should be adjusted with respect to the dynamics of the input.

Delay-state observer	Complexity	Singularity sensibility	Tuning
High gain	★	★	★★
Interconnected observers	★★★★	★★★★	★★

Table 3.1 – Comparison of the high gain observer and the interconnected observers presented respectively in Chapters 2 and 3. Criteria are explained above.



Prediction-based control of input delay systems

Introduction

Prediction-based control is the only technique able to guarantee a good level of performances in presence of large input delays. However, some limits prevent its use in practice. The computation complexity, the exact knowledge of the delay or the sensibility to unmodeled dynamics are some of its weak points. In this chapter, some solutions are presented in order to reduce these limitations and widen the application range of prediction-based control. Throughout this part, the stability analysis will be performed thanks to the construction of systematic Lyapunov-Krasovskii functionals inspired by the recent works of [Ahmed-Ali et al. \[2012\]](#) and [Mazenc et al. \[2014\]](#).

Organization

Three different predictions are going to be studied and compared for known delays (Chapter 4) and unknown delays (Chapter 5):

- the standard prediction,
- the dynamic prediction,
- the new prediction.

Remark 3.4.1. *The term “prediction” refers to a way to predict the future of the state. There are various methods to compute a predictive value of the state so there are various “predictions”. However, there is only one **exact** prediction⁷.*

The “standard” prediction is the one introduced by [Mayne \[1968\]](#) and [Manitius and Olbrot \[1979\]](#) and is commonly used in predictive control of systems with a constant input delay. The computation of this prediction requires the discretization of an integral term. Recently, another predictive method has been presented in [Najafi et al. \[2013\]](#). They use a dynamic system to compute a prediction. That is why this prediction will be called “dynamic” prediction⁸. Finally, a novel prediction that is able to deal with parameter uncertainties and external disturbances has been proposed in [Léchappé et al. \[2015c\]](#). It will be referred to as the “new” prediction.

Remark 3.4.2. *Even if the concept of dynamic prediction is not new, it has been widely extended in this thesis and constitutes a contribution in the same way as the new prediction for example.*

The organization is reminded on Figure 3.10.

7. The exact prediction can be computed only in the ideal case when the system and the delay are perfectly known.

8. In the work by [Besançon et al. \[2007\]](#) (which we have been aware of only at the end of the redaction), the authors call this method “asymptotic prediction”.

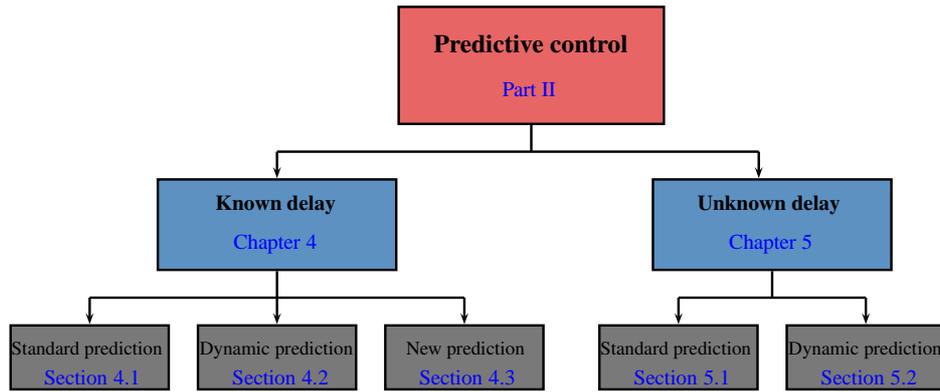


Figure 3.10 – Organization of the part on predictive control

General assumptions

Thereafter, some general assumptions that will be used throughout Part II are introduced. The systems considered have the following form

$$(\mathcal{E}_h) \quad \dot{x}(t) = Ax(t) + Bu(t - h(t)) \quad (3.84)$$

where $h(t)$ is the delay. The delay free systems associated to system (\mathcal{E}_h) in (3.84) is

$$(\mathcal{E}) \quad \dot{x}(t) = Ax(t) + Bu(t) \quad (3.85)$$

In this part, the objective is to design predictive controllers based on classic delay free methods and study the influence of the delay on the stability. That is to say, the controller is designed on the delay free system and then the prediction is “plugged” in. As a consequence, throughout this part it is assumed that

Assumption 3.4.1. *There exists a time differentiable and globally Lipschitz controller $u : x \mapsto u(x)$, that guarantees the existence a function $V_u : \mathbb{R}^n \rightarrow \mathbb{R}$*

$$V_u(x) = x^T Px \quad (3.86)$$

with P a $n \times n$ symmetric matrix that satisfies the inequalities

$$\begin{cases} \underline{c}\|x\|^2 \leq V_u(x) \leq \bar{c}\|x\|^2, \\ \dot{V}_u(x(t)) \leq -c_u\|x(t)\|^2, \\ \left\| \frac{dV_u}{dx} \right\| \leq c\|x\|, \end{cases} \quad (3.87)$$

with $\underline{c}, \bar{c}, c_u, c > 0$.

Remark 3.4.3. *Note that the argument “ t ” in $u(x(t))$ will be omitted for clarity. Furthermore, the notations $u(t)$ or $u(x)$ will be used indifferently when no confusion is possible.*

Assumption 3.4.1 implies that controller $u(x)$ globally exponentially stabilizes delay free system (3.85). Note that Theorem 4.14 by Khalil [2002] guarantees the existence of a Lyapunov function that verifies (3.87) for a general class of exponentially stable nonlinear systems. However, here a special form (but classic) of the Lyapunov function ($V_u = x^T Px$) is assumed. Remark that differentiating V_u along the trajectories of (3.85) and using equations (3.87), one gets

$$x^T(t)[A^T P + PA]x(t) + u^T(t)B^T Px(t) + x^T(t)PBu(t) \leq -c_u\|x(t)\|^2. \quad (3.88)$$

In addition, since $u : x \mapsto u(x)$ is globally Lipschitz, there exists $l_u > 0$ such that

$$\|u(x)\| \leq l_u \|x\| \quad (3.89)$$

for all $x \in \mathbb{R}^n$. When only a part of the state is available, the output is denoted $y(t) = Cx(t)$ with $C \in \mathbb{R}^{p \times n}$ and p the number of measured states. Then an estimation \hat{x} of the state x can be computed thanks to an observer:

$$(\mathcal{O}) \quad \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t - h(t)) + g(C\hat{x}(t) - y(t)) \quad (3.90)$$

where the correction term g is such that

Assumption 3.4.2. *There exists a globally Lipschitz function g that guarantees the existence of a function $V_g : \mathbb{R}^n \rightarrow \mathbb{R}$*

$$V_g(e) = e^T Q e \quad (3.91)$$

with $e = \hat{x} - x$ and Q a $n \times n$ symmetric matrix that satisfies the inequalities

$$\begin{cases} \underline{m} \|e\|^2 \leq V_g(e) \leq \bar{m} \|e\|^2, \\ \dot{V}_g(e(t)) \leq -m_g \|e(t)\|^2, \\ \left\| \frac{dV_g}{de} \right\| \leq m \|e\|, \end{cases} \quad (3.92)$$

with $\underline{m}, \bar{m}, m_u, m > 0$.

Remark 3.4.4. *The notation g to denote the correction term has nothing to do with the function g defined by (1.40) in the previous part. Furthermore, by a slight abuse of notation, $g(Ce)$ will sometimes denoted $\underline{g}(e)$ to underline that it depends on the observation error e .*

Assumption 3.4.2 implies that the function g globally exponentially stabilizes the observation error

$$\dot{e}(t) = Ae(t) + g(Ce(t)). \quad (3.93)$$

Similarly to (3.88), one gets

$$e^T(t)[A^T Q + QA]e(t) + g^T(Ce(t))Qe(t) + e^T(t)Qg(Ce(t)) \leq -m_g \|e(t)\|^2. \quad (3.94)$$

In addition, since g is globally Lipschitz, there exists $l_g > 0$ such that

$$\|g(Ce)\| \leq l_g \|Ce\| \quad (3.95)$$

for all $e \in \mathbb{R}^n$. Some assumptions are also made to characterize the time-varying delay $h(t)$:

Assumption 3.4.3. *The time-varying delay $h(t)$ is bounded, i.e. there exist $h_{min}, h_{max} > 0$ such that*

$$h_{min} \leq h(t) \leq h_{max}. \quad (3.96)$$

Assumption 3.4.4. *The delay dynamics is bounded, i.e. there exists $\delta > 0$ such that*

$$|\dot{h}(t)| \leq \delta < 1. \quad (3.97)$$

When the constant delay h (respectively the time-varying delay $h(t)$) is not known, it is supposed that a constant approximation of the delay denoted \hat{h} (respectively time-varying approximation $\hat{h}(t)$) is available. In addition, the time-varying estimation $\hat{h}(t)$ complies with the assumptions below

Assumption 3.4.5. *The estimated delay is bounded, i.e.*

$$h_{min} \leq \hat{h}(t) \leq h_{max} \quad (3.98)$$

where h_{min} and h_{max} are the same as in Assumption 3.4.3.

Assumption 3.4.6. *The estimated delay dynamics is bounded, i.e. there exists $\hat{\delta} > 0$*

$$|\dot{\hat{h}}(t)| \leq \hat{\delta} < 1. \quad (3.99)$$

In the following chapters of Part II, all these assumptions are fulfilled.

Some mathematical reminders

Some useful inequalities that will be used in the following chapters are reminded below.

- Young's inequality (see [Young \[1912\]](#)):

Let $\alpha, \beta \geq 0$, then one has

$$\alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2}. \quad (3.100)$$

- Jensen's inequality (see [Fridman \[2014b\]](#)):

Let $u : [0, +\infty[\rightarrow \mathbb{R}^n$ be a continuous function and $h(t) > 0$, then one has

$$-\int_{t-h(t)}^t \|u(s)\|^2 ds \leq -\frac{1}{h(t)} \left(\int_{t-h(t)}^t \|u(s)\| ds \right)^2. \quad (3.101)$$

- “Completing the square” method (see [Narasimhan \[2009\]](#)):

Let $\alpha, \beta \in \mathbb{R}$ then, for any $o, p > 0$ and $q \in \mathbb{R}$, one has

$$-o\alpha^2 + q\alpha\beta - p\beta^2 = -\left(\sqrt{o}\alpha - \frac{q}{2\sqrt{o}}\beta\right)^2 + \left(\frac{q^2}{4o} - p\right)\beta^2. \quad (3.102)$$

- Leibniz differentiation rule (see [Flanders \[1973\]](#)):

Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that g and $\frac{\partial g}{\partial t}$ are continuous on \mathbb{R}^2 and let λ_1 and λ_2 be differentiable functions, then one has

$$\frac{d}{dt} \left(\int_{\lambda_1(t)}^{\lambda_2(t)} g(t, s) ds \right) = \dot{\lambda}_2(t)g(t, \lambda_2(t)) - \dot{\lambda}_1(t)g(t, \lambda_1(t)) + \int_{\lambda_1(t)}^{\lambda_2(t)} \frac{\partial}{\partial t} g(t, s) ds. \quad (3.103)$$

- Particular case of Hölder's inequality (see [Kuptsov \[2001\]](#)):

Let $x_1, \dots, x_n \in \mathbb{R}$, then one has

$$\left(\sum_{i=1}^n |x_i| \right)^2 \leq n \sum_{i=1}^n |x_i|^2. \quad (3.104)$$

Prediction with a known delay

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Predictive control is often studied for perfectly known plants with constant delays. However, these assumptions are very restrictive in practice. Furthermore, the computation effort to discretize the integral term makes its application very limited. That is why, some new predictive techniques are going to be presented in this chapter. The constant and time-varying delay cases are considered as well as the state feedback and output feedback cases. Closed-loop stability conditions will be given based on the systematic construction of Lyapunov-Krasovskii functionals. **All the developments presented in this chapter are for known delays.**

In Section 4.1, the closed-loop stability with the standard prediction is worked out for constant and time-varying delays (state feedback and output feedback). A recent method based on the computation of a prediction thanks to a dynamic system is extended in Section 4.2. The main advantage of this technique is that it does not require the integral discretization anymore. Finally, a new prediction is introduced in Section 4.3 and its robustness properties are compared with the standard prediction in presence of parameter uncertainties and an external disturbance.

4.1 Standard prediction with a known delay

In this section, the work of [Bresch-Pietri, 2012, chap 10] is revisited using a recent Lyapunov-Krasovskii functional. This analysis allows to obtain a more explicit bound for stability condition and to extend the result to a broader class of controllers. As it has been mentioned in Section 1.2.2, it is usually not possible to compensate perfectly a time-varying delay because its exact expression has to be known in advance. The straightforward idea is to use the same prediction as for a constant delay and to substitute the constant delay by the time-varying one. In this case, it is shown in Bresch-Pietri [2012] that there exists an upper bound on the delay derivative such that the solutions of the closed-loop system globally exponentially converges to the origin. The proof relies on the construction of a Lyapunov-Krasovskii functional via the backstepping PDE method similar to the one in Section 1.2.1. Here, a simpler upper bound on the delay time derivative is provided and its expression explicitly involves the system parameters, the gain values and the maximum delay value. In addition, the condition is extended to the partial measurement case.

The section is organized as follows. The simple case of a constant and known delay is reminded for state and output feedback in Section 4.1.1. Then, these results are extended to a time-varying delay in Section 4.1.2.

4.1.1 Constant delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \quad \dot{x}(t) = Ax(t) + Bu(t-h)$$

where h is a constant and known delay. Since Assumption 3.4.1 holds, there exists a feedback $u(x)$ that exponentially stabilizes system (\mathcal{E}_h) when $h = 0$.

Full state knowledge

In the case of a constant and known delay h and when the whole state is available, the exact prediction reads as

$$z(t) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds \quad (4.1)$$

for all $t \geq 0$. Then a predictive feedback $u(z)$ is applied to stabilize system (\mathcal{E}_h) .

Remark 4.1.1. *On top of the initial condition on x , an initial condition $u(s) = \phi_u(s)$ for $s \in [-h, 0]$, with ϕ_u a continuous function,¹ is needed to compute $z(t)$.*

The stability property of the closed-loop system is given in the next theorem.

Theorem 4.1. *Consider system (\mathcal{E}_h) , with a known and constant delay h , and assume that there exists a controller $u(x)$ satisfying Assumption 3.4.1. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (4.1) and define*

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-h, t]} \|u(s)\|^2. \quad (4.2)$$

1. The extra condition $\phi_u(0) = u(z(0))$ may be required to guarantee the continuity of z at $t = h$ (and consequently of u at $t = h$).

Then, there exist $\varsigma, \varrho > 0$ such that one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (4.3)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. By differentiating (4.1) thanks to Leibniz's rule (3.103) and considering (\mathcal{E}_h) , it can be verified that $z(t)$ is the solution of the following equation

$$\dot{z}(t) = Az(t) + Bu(z(t)). \quad (4.4)$$

Let

$$V(t) = z^T(t)Pz(t), \quad (4.5)$$

where P is defined in Assumption 3.4.1, be a Lyapunov candidate function. Differentiating V along the trajectories of (4.4) and using (3.88) leads to

$$\dot{V}(t) \leq -c_u \|z(t)\|^2 \leq -c_1 V(t) \quad (4.6)$$

with $c_1 = \frac{c_u}{\bar{c}}$. It follows from (4.6) that

$$V(t) \leq V(0) e^{-c_1 t} \quad (4.7)$$

and since $\|z(t)\|^2 \leq \frac{1}{\bar{c}} V(t)$ (Assumption 3.4.1), one obtains

$$\|z(t)\|^2 \leq \frac{1}{\bar{c}} V(0) e^{-c_1 t} \quad (4.8)$$

From the definition of $V(t)$ in (4.5), one has

$$V(0) \leq \bar{c} \|z(0)\|^2. \quad (4.9)$$

In addition, from (4.1) and by using Hölder's inequality (3.104) and Jensen's inequality (3.101), the following maximization can be deduced

$$\|z(0)\|^2 \leq 2 \|e^{Ah}\|^2 \|x(0)\|^2 + 2e^{2\|A\|h} \|B\|^2 h \int_{-h}^0 \|u(s)\|^2 ds. \quad (4.10)$$

Thus, maximizing the integral term gives

$$\|z(0)\|^2 \leq c_2 \|x(0)\|^2 + c_3 \sup_{s \in [-h, 0]} \|u(s)\|^2 \quad (4.11)$$

with $c_2 = 2 \|e^{Ah}\|^2$ and $c_3 = 2e^{2\|A\|h} \|B\|^2 h^2$. Combining (4.9) with (4.11) leads to

$$V(0) \leq c_4 \Upsilon(0) \quad (4.12)$$

with $c_4 = \bar{c} \max(c_2, c_3)$. Then, from (4.8) and (4.12), one deduces

$$\|z(t)\|^2 \leq c_5 \Upsilon(0) e^{-c_1 t} \quad (4.13)$$

for all $t \geq 0$ and with $c_5 = \frac{c_4}{\bar{c}}$. Since $u(z)$ is Lipschitz with respect to z for all $t \geq 0$, one has

$$\sup_{s \in [t-h, t]} \|u(s)\|^2 \leq \begin{cases} \sup_{s \in [-h, 0]} \|u(s)\|^2 + l_u^2 \sup_{s \in [0, t]} \|z(s)\|^2 & t < h \\ l_u^2 \sup_{s \in [t-h, t]} \|z(s)\|^2 & t \geq h \end{cases} \quad (4.14)$$

so

$$\sup_{s \in [t-h, t]} \|u(s)\|^2 \leq \begin{cases} \Upsilon(0) + l_u^2 c_5 \Upsilon(0) & \text{if } t < h \\ c_6 \Upsilon(0) e^{-c_1 t} & \text{if } t \geq h \end{cases} \quad (4.15)$$

with $c_6 = (l_u^2 c_5 + 1)e^{c_1 h}$. Noting that $c_6 \Upsilon(0) e^{-c_1 t} \geq \Upsilon(0) + l_u^2 c_5 \Upsilon(0)$ for all $t \in [0, h]$, one can state that

$$\sup_{s \in [t-h, t]} \|u(s)\|^2 \leq c_6 \Upsilon(0) e^{-c_1 t} \quad (4.16)$$

for all $t \geq 0$. Moreover, rearranging (4.1) gives

$$x(t) = e^{-Ah} z(t) - \int_{t-h}^t e^{A(t-h-s)} B u(s) ds \quad (4.17)$$

so by the same steps as in (4.11), one gets

$$\|x(t)\|^2 \leq c_7 \|z(t)\|^2 + c_3 \sup_{s \in [t-h, t]} \|u(s)\|^2 \quad (4.18)$$

for all $t \geq 0$ with $c_7 = 2\|e^{-Ah}\|^2$. Finally, from (4.13), (4.16) and (4.18), one obtains

$$\Upsilon(t) \leq c_8 \Upsilon(0) e^{-c_1 t} \quad (4.19)$$

for all $t \geq 0$ with $c_8 = c_7 c_5 + c_6(1 + c_3)$. \square

Remark 4.1.2. The exponential convergence of $\|x(t)\|$ to zero can be directly deduced from (4.18) and using (4.13) and (4.16):

$$\|x(t)\|^2 \leq c_9 e^{-c_1 t} \quad (4.20)$$

with $c_9 = (c_7 c_5 + c_3 c_6) \Upsilon(0)$. However, the constant c_9 does not depend only on $\|x(0)\|$ that is why the norm Υ in (4.3) has to be introduced to deal with the initial conditions. This is a standard method² that allows to conclude on exponential stability (in the sense of the norm Υ) of input delay systems with predictive control. Note that the term $\sup_{s \in [t-h, t]} \|u(s)\|^2$ can be substituted by $\int_{t-h}^t \|u(s)\|^2 ds$ as in Krstic [2008a].

Partial state knowledge

The full state knowledge is required to compute prediction (4.1). However, when only the output $y(t) = Cx(t)$ is available, a state observer (O)(3.90) has to be designed to estimate the state of the system. In this case, it is assumed that the function g complies with Assumption 3.4.2 and the prediction is computed from the estimated state \hat{x} as follows³

$$z(t) = e^{Ah} \hat{x}(t) + \int_{t-h}^t e^{A(t-s)} B u(s) ds. \quad (4.21)$$

The closed-loop stability property is given in Theorem 4.2. It is reminded that the estimation error $\hat{x}(t) - x(t)$ is denoted $e(t)$.

2. See for example Karafyllis and Krstic [2013a], Bresch-Pietri and Krstic [2010] and Mazenc et al. [2014].

3. See Remark 4.1.1 for the definition of initial conditions and add the initial condition $\hat{x}(0)$.

Theorem 4.2. Consider system (\mathcal{E}_h) and observer (\mathcal{O}) , with a known and constant delay h , and assume that there exist a controller $u(x)$ and a correction term $g(e)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (4.21) and define

$$\Upsilon(t) = \|x(t)\|^2 + \|e(t)\|^2 + \sup_{s \in [t-h, t]} \|u(s)\|^2. \quad (4.22)$$

Then, there exist $\varsigma, \varrho > 0$ such that one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (4.23)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$, $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ and the convergence is exponential.

Proof. By differentiating (4.21) thanks to Leibniz's rule (3.103), one obtains that

$$\dot{z}(t) = Az(t) + Bu(z(t)) + e^{Ah}g(Ce(t)) \quad (4.24)$$

where $e = \hat{x} - x$. Let

$$V(t) = V_1(t) + \gamma V_2(t) \quad (4.25)$$

be a Lyapunov candidate function with $V_1(t) = z^T(t)Pz(t)$, $V_2(t) = e^T(t)Qe(t)$ where P and Q are defined in Assumptions 3.4.1 and 3.4.2 and $\gamma > 0$. Differentiating V along the trajectories of (4.24)-(3.93) and using (3.88), (3.94), (3.95), it follows that

$$\dot{V}(t) \leq -c_u \|z(t)\|^2 + 2l_g \|P\| \|e^{Ah}\| \|C\| \|z(t)\| \|e(t)\| - \gamma m_g \|e(t)\|^2. \quad (4.26)$$

Applying equality (3.102) allows to get rid of the cross term " $\|z(t)\| \|e(t)\|$ " and one gets

$$\dot{V}(t) \leq -\frac{c_u}{2} \|z(t)\|^2 + \left(\frac{R^2}{2c_u} - \gamma m_g \right) \|e(t)\|^2 \quad (4.27)$$

where $R = 2l_g \|P\| \|C\| \|e^{Ah}\|$. Then, it is clear that choosing γ sufficiently large guarantees that

$$\dot{V}(t) \leq -\frac{c_u}{2} \|z(t)\|^2 - c_1 \|e(t)\|^2 \quad (4.28)$$

with $c_1 = \gamma m_g - \frac{R^2}{2c_u} > 0$. From Assumptions 3.4.1 and 3.4.2, it can be deduced that

$$\dot{V}(t) \leq -\frac{c_u}{2\bar{c}} V_1(t) - \frac{c_1}{\bar{m}} V_2(t) \quad (4.29)$$

so finally

$$\dot{V}(t) \leq -c_2 V(t) \quad (4.30)$$

with $c_2 = \min\left(\frac{c_u}{2\bar{c}}, \frac{c_1}{\bar{m}}\right) > 0$. Equation (4.23) can be deduced from (4.30) by similar computations as in Theorem 4.1. \square

For a constant and known delay, the predictive controller can stabilize every controllable and observable systems with an arbitrary long delay. In the next section, the time-varying delay case is investigated.

4.1.2 Time-varying delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \quad \dot{x}(t) = Ax(t) + Bu(t - h(t))$$

where $h(t)$ is a known and time-varying delay. Since Assumption 3.4.1 holds, there exists a feedback $u(x)$ that exponentially stabilizes system (\mathcal{E}_h) when $h(t) = 0$.

Full state knowledge

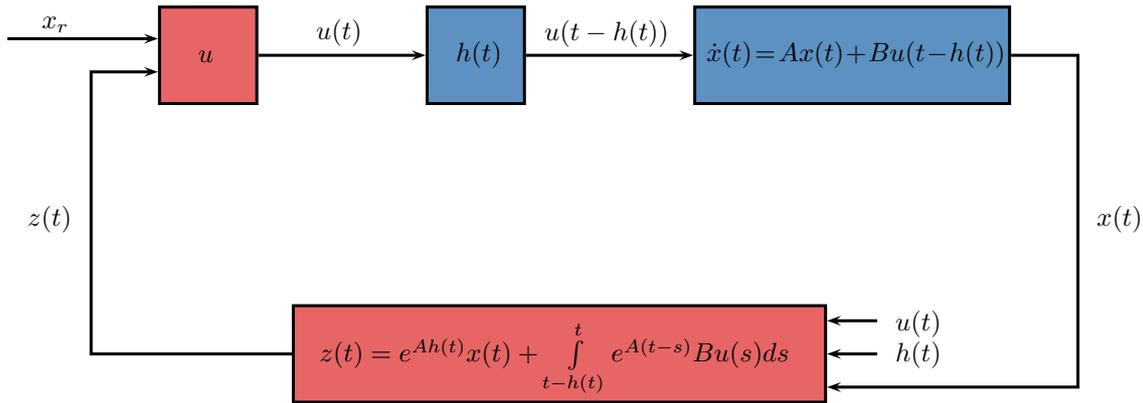


Figure 4.1 – Closed-loop scheme with standard prediction for time-varying delay and full state measurement

In this part, the delay $h(t)$ and the state $x(t)$ are known so the prediction is computed as follows

$$z(t) = e^{Ah(t)}x(t) + \int_{t-h(t)}^t e^{A(t-s)}Bu(s)ds. \quad (4.31)$$

Remark 4.1.3. On top of the initial condition on x , an initial condition $u(s) = \phi_u(s)$ for $s \in [-h_{max}, 0]$, with ϕ_u a continuous function,⁴ is needed to compute $z(t)$.

The stability of the closed-loop system is given in Theorem 4.3. It is reminded that δ stands for the upper bound of the delay time derivative: $|\dot{h}(t)| < \delta$ (Assumption 3.4.4).

Theorem 4.3. Consider system (\mathcal{E}_h) , where $h(t)$ is known and complies with Assumptions 3.4.3 and 3.4.4 and assume that there exists a controller $u(x)$ satisfying Assumption 3.4.1. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (4.31) and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-h_{max}, t]} \|u(s)\|^2. \quad (4.32)$$

Then, there exist $\varsigma, \varrho, \delta^* > 0$ such that, provided $\delta < \delta^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (4.33)$$

4. The extra condition $\phi_u(0) = u(z(0))$ may be required to guarantee the continuity of z at $t = h(t)$ (and consequently of u at $t = h(t)$).

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. By differentiating (4.31) thanks to Leibniz's rule (3.103) and using (\mathcal{E}_h) , it can be verified that the prediction $z(t)$ is solution of the following equation

$$\dot{z}(t) = Az(t) + Bu(t) + \dot{h}Az(t) + \dot{h}e^{Ah}Bu(t-h) - \dot{h}A \int_{t-h}^t e^{A(t-s)}Bu(s)ds. \quad (4.34)$$

for $t \geq h$. Remark that the delay has not disappeared in the above equation because the prediction is not exact anymore. The following Lyapunov-Krasovskii functional candidate is chosen

$$V(t) = V_1(t) + \gamma V_2(t) \quad (4.35)$$

where

$$V_1(t) = z^T(t)Pz(t) \quad (4.36)$$

and

$$V_2(t) = \int_{t-h}^t (h_{max} + h_{min} + s - t) \|u(s)\|^2 ds \quad (4.37)$$

and $\gamma > 0$. Note that P is defined in Assumption 3.4.1 and that V_2 is necessary to deal with delayed terms of (4.34) that arise because of the time-varying delay. First, note that the inequality

$$2|\dot{h}| \|z\| \|P\| \|A\| \left\| \int_{t-h}^t e^{A(t-s)}Bu(s)ds \right\| \leq 2\delta \|P\| \|A\| \|B\| e^{\|A\|h_{max}} \|z\| \int_{t-h}^t \|u(s)\| ds \quad (4.38)$$

holds. For details on matrix analysis, one can refer to [Horn and Johnson, 2012, pp. 501]. Then, by using Young's inequality (3.100), one has

$$2\delta M \|z\| \int_{t-h}^t \|u(s)\| ds \leq \delta M \left[\|z\|^2 + \left(\int_{t-h}^t \|u(s)\| ds \right)^2 \right] \quad (4.39)$$

with $M = \|P\| \|A\| \|B\| e^{\|A\|h_{max}}$. Furthermore, the Jensen's inequality (3.101) guarantees that

$$2|\dot{h}| \|z\| \|P\| \|A\| \left\| \int_{t-h}^t e^{A(t-s)}Bu(s)ds \right\| \leq \delta M [\|z\|^2 + h\|q\|^2] \quad (4.40)$$

with

$$\|q(t)\|^2 = \int_{t-h}^t \|u(s)\|^2 ds. \quad (4.41)$$

As a result, taking the time derivative of V_1 along the trajectories of (4.34) and using equations (3.88) and (4.40), one obtains

$$\begin{aligned} \dot{V}_1(t) \leq & -c_u \|z(t)\|^2 + 2\delta \|P\| \|A\| \|z(t)\|^2 + 2\delta N \|z(t)\| \|u(t-h)\| \\ & + \delta M \|z\|^2 + h\delta M \|q(t)\|^2 \end{aligned} \quad (4.42)$$

with $N = \|P\| \|B\| e^{\|A\| h_{max}}$. Besides, Leibniz's rule (3.103) leads to

$$\dot{V}_2(t) = (h_{max} + h_{min}) \|u(t)\|^2 - (h_{max} + h_{min} - h)(1 - \dot{h}) \|u(t-h)\|^2 - \int_{t-h}^t \|u(s)\|^2 ds. \quad (4.43)$$

Furthermore, from (3.97) and since $u(z)$ is globally Lipschitz (Assumption 3.4.1), it follows that

$$\dot{V}_2(t) \leq 2h_{max} l_u^2 \|z(t)\|^2 - h_{min}(1 - \delta) \|u(t-h)\|^2 - \|q(t)\|^2. \quad (4.44)$$

Applying Young's inequality (3.100) leads to

$$\begin{aligned} \dot{V}(t) \leq & -[c_u - 2\delta\|P\| \|A\| - 2\gamma h_{max} l_u^2 - \delta(M + N)] \|z(t)\|^2 \\ & - [(1 - \delta)\gamma h_{min} - \delta N] \|u(t-h)\|^2 - [\gamma - h_{max}\delta M] \|q(t)\|^2. \end{aligned} \quad (4.45)$$

It is sufficient to choose the three coefficients of the square terms positive to guarantee the negativeness of \dot{V} . Thus, one gets

$$\begin{cases} c_u - 2\gamma h_{max} l_u^2 - \delta(2\|P\| \|A\| + M + N) > 0, & (4.46) \\ \gamma h_{min} - \delta(\gamma h_{min} + N) > 0, & (4.47) \\ \gamma - h_{max}\delta M > 0. & (4.48) \end{cases}$$

Defining

$$\delta_1 = \frac{c_u - 2\gamma h_{max} l_u^2}{(2\|P\| \|A\| + M + N)}, \quad (4.49)$$

$$\delta_2 = \frac{\gamma h_{min}}{\gamma h_{min} + N} \quad (4.50)$$

and

$$\delta_3 = \frac{\gamma}{h_{max} M} \quad (4.51)$$

and choosing γ sufficiently small guarantees that δ_1 is positive. In this case, taking

$$\delta < \delta^* = \min(\delta_1, \delta_2, \delta_3) \quad (4.52)$$

ensures that

$$\dot{V}(t) \leq -c_1 \|z(t)\|^2 - c_2 \|q(t)\|^2 \quad (4.53)$$

with $c_1, c_2 > 0$. Finally, from

$$- \|z(t)\|^2 \leq -\frac{1}{\bar{c}} V_1(t) \quad (4.54)$$

and since

$$- \|q(t)\|^2 \leq -\frac{1}{h_{max} + h_{min}} V_2(t), \quad (4.55)$$

one has

$$\dot{V}(t) \leq -c_3 V(t) \quad (4.56)$$

with $c_3 = \min\left(\frac{c_1}{\bar{c}}, \frac{c_2}{h_{max} + h_{min}}\right) > 0$. So equation (4.33) can be deduced from (4.56) by similar computations as in Theorem 4.1. \square

Qualitative analysis: Theorem 4.3 guarantees the exponential convergence of x to zero provided that the delay dynamics $\dot{h}(t)$ is sufficiently slow. Even if the bound (4.52) is conservative, because it is obtained via a Lyapunov analysis, it retains interesting features. An important fact is that the bounds h_{max} and h_{min} appear explicitly in the expression of δ^* and that δ_2 and δ_3 are independent from the controller u . A qualitative analysis is given below.

- For a large value of h_{max} , then δ_1 and δ_3 decrease which means that the larger the upper bound of the delay is, the slower its variation has to be.
- For a large value of h_{min} , then δ_2 increases which means that the smaller the delay interval is, the faster the variation can be. For example, even if the delay displayed on Figure 4.2b varies faster than the delay on Figure 4.2a, the closed-loop system may be stable for h_2 and not for h_1 .
- For a large value of l_u , then δ_1 decreases which means that the delay variations have to be smaller for a fast controller.

These conclusions seem to capture the physical behavior of the system, that was not the case for the bounds in Bresch-Pietri [2012].

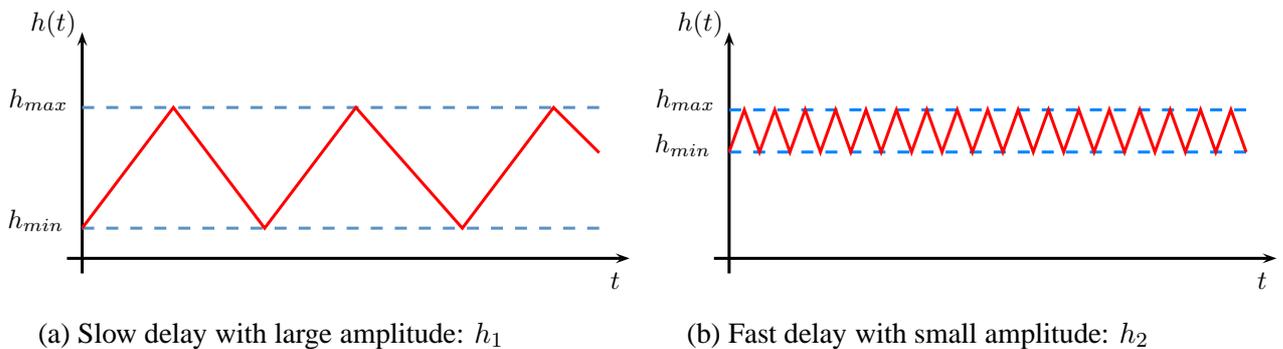


Figure 4.2 – Different types of time-varying delays

The methodology used in the proof of Theorem 4.3 will be similar for the next proofs so it is detailed below.

When the prediction is not exact, the reduced system still involves delays. For example, when the exact prediction x_p is approximated by prediction z in (4.31), the delayed input $u(t - h(t))$ still appears in reduced system (4.34). As a result, Lyapunov-Krasovskii functionals have to be designed to analyze the stability of the reduced system. The steps of the stability proof are the following

- Compute the reduced system from the prediction z in order to make appear $Az(t) + B(z(t))$,
- Use the Lyapunov function given for the delay free part (Assumption 3.4.1) and add a Lyapunov-Krasovskii functionals to deal with pointwise delayed terms $u(t - h)$ and the distributed delay terms $\int_{t-h}^t e^{A(t-s)} Bu(s) ds$.
- Conclude on the stability of the original system in x by using the prediction expression z .

Partial state knowledge

When only the output $y = Cx$ is measurable, prediction (4.31) cannot be computed directly. Therefore, an observer (\mathcal{O}) in (3.90) can be designed to estimate the state of the system. In this

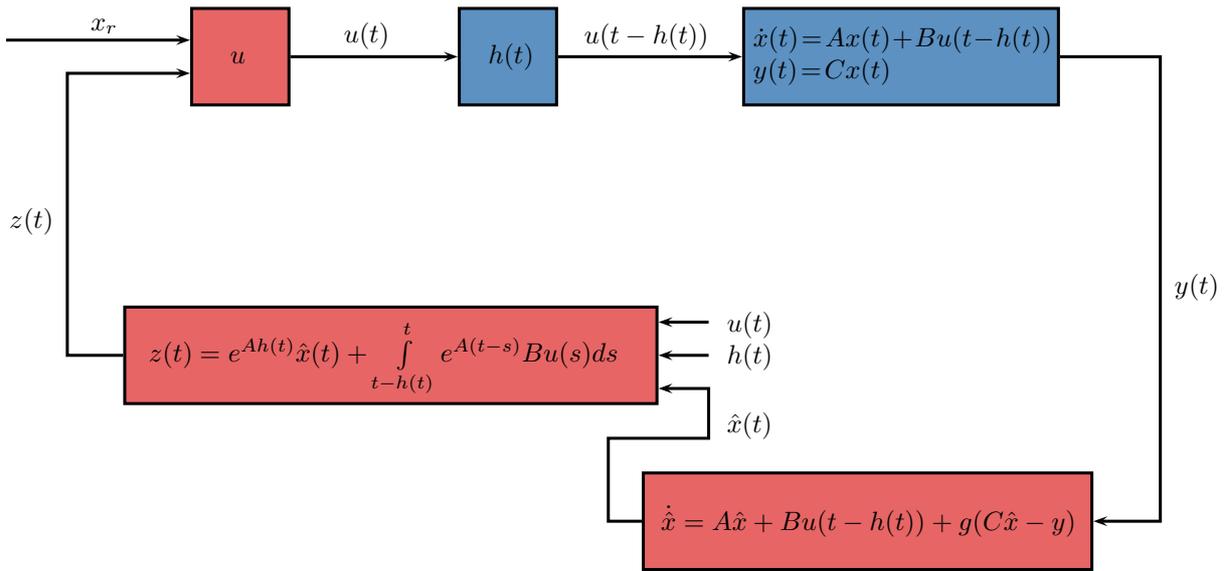


Figure 4.3 – Closed-loop scheme with standard prediction for time-varying delay and partial state measurement

case, it is assumed that the correction term $g(e)$ complies with Assumption 3.4.2. Thus, the “reconstructed prediction” is computed thanks to the observed state \hat{x} as follows⁵

$$z(t) = e^{Ah(t)}\hat{x}(t) + \int_{t-h(t)}^t e^{A(t-s)}Bu(s)ds. \quad (4.57)$$

The condition to guarantee the stability of the closed-loop system is given in Theorem 4.4. It is reminded that δ stands for the upper bound of the delay time derivative, $|\dot{h}(t)| < \delta$ (Assumption 3.4.4), and that the estimation error $\hat{x}(t) - x(t)$ is denoted $e(t)$.

Theorem 4.4. Consider system (\mathcal{E}_h) and observer (\mathcal{O}) , where $h(t)$ is known and complies with Assumptions 3.4.3 and 3.4.4 and assume that there exist a controller $u(x)$ and a correction term $g(e)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (4.57) in (3.90) and define

$$\Upsilon(t) = \|x(t)\|^2 + \|e(t)\|^2 + \sup_{s \in [t-h_{max}, t]} \|u(s)\|^2. \quad (4.58)$$

Then, there exist $\varsigma, \varrho, \delta^* > 0$ such that, provided $\delta < \delta^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (4.59)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$, $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ and the convergence is exponential.

Proof. The proof is similar to the one for full measurement state case so the reader can refer to the proof of Theorem 4.3 for more details.

By differentiating (4.57) thanks to Leibniz’s rule (3.103) and using the expression of the

5. See Remark 4.1.1 for the definition of initial conditions and add the initial condition $\hat{x}(0)$.

original system (\mathcal{E}_h), one deduces that prediction z is solution of the following equation

$$\dot{z} = Az + Bu(z) + e^{Ah}g(Ce) + \dot{h}Az - \dot{h}A \int_{t-h}^t e^{A(t-s)} Bu(s)ds + \dot{h}e^{Ah}Bu(t-h). \quad (4.60)$$

In this case, the dynamics of z also depends on the observation error $e(t)$. Obviously if $e = 0$, equation (4.60) is equivalent to the one in the full state knowledge case (4.34). In addition, if the delay is constant then (4.60) is equal to the constant delay case (4.24). Consider the following Lyapunov-Krasovskii functional candidate

$$V(t) = V_1(t) + \gamma V_2(t) + \theta V_3(t) \quad (4.61)$$

where

$$V_1(t) = z^T(t)Pz(t), \quad (4.62)$$

$$V_2(t) = \int_{t-h}^t (h_{max} + h_{min} + s - t) \|u(s)\|^2 ds, \quad (4.63)$$

$$V_3(t) = e^T(t)Qe(t) \quad (4.64)$$

and $\gamma, \theta > 0$. Note that V_1 and V_2 are the same as in the full state measurement case and V_3 is introduced to deal with the observer stability. Taking the time derivative of V_1 along the trajectories of (4.60) and using the same arguments as in Theorem 4.3, one gets

$$\begin{aligned} \dot{V}_1(t) \leq & -c_u \|z\|^2 + R \|e\| \|z\| + 2\delta \|P\| \|A\| \|z\|^2 \\ & + 2\delta N \|z\| \|u(t-h)\| + \delta M \|z\|^2 + h_{max} \delta M \|q(t)\|^2 \end{aligned} \quad (4.65)$$

with $q(t) = \int_{t-h}^t \|u(s)\|^2 ds$, $M = \|P\| \|A\| \|B\| e^{\|A\| h_{max}}$, $N = \|P\| \|B\| e^{\|A\| h_{max}}$ and $R = 2l_g e^{\|A\| h_{max}} \|P\| \|C\|$. Besides, Leibniz's rule (3.103) leads to

$$\dot{V}_2(t) = (h_{max} + h_{min}) \|u(t)\|^2 - (h_{max} + h_{min} - h)(1 - \dot{h}) \|u(t-h)\|^2 - \int_{t-h}^t \|u(s)\|^2 ds. \quad (4.66)$$

and since u is globally Lipschitz (3.89) then

$$\dot{V}_2(t) \leq 2h_{max} l_u^2 \|z(t)\|^2 - h_{min}(1 - \delta) \|u(t-h)\|^2 - \|q(t)\|^2. \quad (4.67)$$

Finally, taking the time derivative of V_3 along the trajectories of (3.93), one gets

$$\dot{V}_3(t) = -m_g \|e(t)\|^2. \quad (4.68)$$

Using equation (3.102) allows to get rid of the cross term " $\|z\| \|e\|$ ", it follows that

$$\begin{aligned} \dot{V}(t) \leq & -\left[\frac{c_u}{2} - \delta(2\|P\| \|A\| + M + N) - 2\gamma h_{max} l_u^2\right] \|z(t)\|^2 \\ & - [(1 - \delta)\gamma h_{min} - \delta N] \|u(t-h)\|^2 \\ & - [\gamma - h_{max} \delta M] \|q(t)\|^2 - \left(\theta m_g - \frac{R^2}{2c_u}\right) \|e(t)\|^2. \end{aligned} \quad (4.69)$$

To have \dot{V} negative, one has the following conditions

$$\begin{cases} \frac{c_u}{2} - 2\gamma h_{max} l_u^2 - \delta(2\|P\| \|A\| + M + N) > 0, & (4.70) \\ \gamma h_{min} - \delta(\gamma h_{min} + N) > 0, & (4.71) \\ \gamma - h_{max} \delta M > 0, & (4.72) \\ \theta m_g - \frac{R^2}{2c_u} > 0. & (4.73) \end{cases}$$

Defining

$$\delta_1 = \frac{c_u/2 - 2\gamma h_{max} l_u^2}{(2\|P\| \|A\| + M + N)}, \quad (4.74)$$

$$\delta_2 = \frac{\gamma h_{min}}{\gamma h_{min} + N} \quad (4.75)$$

and

$$\delta_3 = \frac{\gamma}{h_{max} M} \quad (4.76)$$

and choosing θ sufficiently large and γ sufficiently small guarantees that (4.73) is true and that δ_1 is positive. In this case, taking

$$\delta < \delta^* = \min(\delta_1, \delta_2, \delta_3) \quad (4.77)$$

ensures that

$$\dot{V}(t) \leq -c_1 \|z(t)\|^2 - c_2 \|q(t)\|^2 - c_3 \|e(t)\|^2. \quad (4.78)$$

with $c_1, c_2, c_3 > 0$. Finally, from Assumptions 3.4.1 and 3.4.2, it can be concluded that

$$\dot{V}(t) \leq -c_4 V(t) \quad (4.79)$$

with $c_4 > 0$. Equation (4.59) can be deduced from (4.79) by similar computations as in Theorem 4.1. \square

Remark 4.1.4. Condition (4.77) is very similar to the one in the full state measurement case so the quantitative analysis of the bound is the same.

4.1.3 Illustrative example

Consider the scalar system

$$\dot{x}(t) = ax(t) + bu(t - h(t)) \quad (4.80)$$

with $a = 1$, $b = 1$, $h(t) \in [0.5, 2]$ and the static feedback

$$u(t) = kx(t). \quad (4.81)$$

Remark that this system is unstable when $u = 0$ since $a > 0$. It is clear that controller (4.81) complies with Assumption 3.4.1 for any scalar $P > 0$ when $k < -a/b$. In this case, $l_u = |k|$ and $c_u = -2P(a + bk)$ so for $k = -2$, $h_{min} = 0.5$ s, $h_{max} = 2$ s, $P = 1$ and $\gamma = 0.1$, from equation (4.77), one obtains $\delta^* = 0.0067$. Remark that it is possible to optimize the choice of P and γ to increase δ^* . In order to illustrate the effect of delay variations on closed-loop stability, two simulations have been carried out:

- In Simulation 1, the upper bound on the delay variation is illustrated.

- In Simulation 2, it is shown that the stability property also depends on the size of the delay and not only on its time derivative.

The delays in Simulation 1 are

- $h_1(t) = 1.2 + 0.75 \sin(0.1t)$ so $\dot{h}_1(t) = 0.075 \cos(0.1t)$ and $\max_{t \geq 0} \dot{h}_1(t) = 0.075 < 1$,
- $h_2(t) = 1.2 + 0.75 \sin(0.8t)$ so $\dot{h}_2(t) = 0.6 \cos(0.8t)$ and $\max_{t \geq 0} \dot{h}_2(t) = 0.6 < 1$,
- $h_3(t) = 1.2 + 0.75 \sin(1.3t)$ so $\dot{h}_3(t) = 0.975 \cos(0.1t)$ and $\max_{t \geq 0} \dot{h}_3(t) = 0.975 < 1$.

The delays in Simulation 2 are

- $h_4(t) = 0.5 + 0.4 \sin(t)$ so $\dot{h}_4(t) = 0.4 \cos(t)$ and $\max_{t \geq 0} \dot{h}_4(t) = 0.4 < 1$,
- $h_5(t) = 1.5 + 0.4 \sin(t)$ so $\dot{h}_5(t) = 0.4 \cos(t)$ and $\max_{t \geq 0} \dot{h}_5(t) = 0.4 < 1$.

Note that the delay values are given in seconds in all simulations and the delay derivatives are much larger than δ^* . The computation of δ^* has been done to provide an order of magnitude of the conservativeness. Consequently, no numerical value will be provided in the next examples but only some quantitative analysis will be made.

Remark 4.1.5. *In this example, the variable x has no particular physical meaning so it will be displayed without unit in the next plots.*

The results of Simulation 1 are plotted on Figure 4.4. Delays h_1 , h_2 and h_3 have been chosen such that they have an increasing delay dynamics: the maximum values of the delay dynamics are $\max_{t \geq 0} \dot{h}_1(t) < \max_{t \geq 0} \dot{h}_2(t) < \max_{t \geq 0} \dot{h}_3(t)$. Remark, that the maximum value h_{max} is the same for h_1 , h_2 and h_3 . It is clear that the closed-loop response is degraded for faster delays and it is even unstable for h_3 . This confirms the existence of an upper bound on the delays dynamics to preserve stability.

The results of Simulation 2 are displayed on Figure 4.5. In this case, h_4 and h_5 have the same time derivative but $h_4(t) < h_5(t)$ for all t . It is interesting to note that the closed-loop is stable for h_4 and unstable for h_5 . It can be deduced that it would be less conservative to find a condition that involves both the delay and its dynamics such as for instance $h_{max} + \delta < C^*$.

4.1.4 Summary

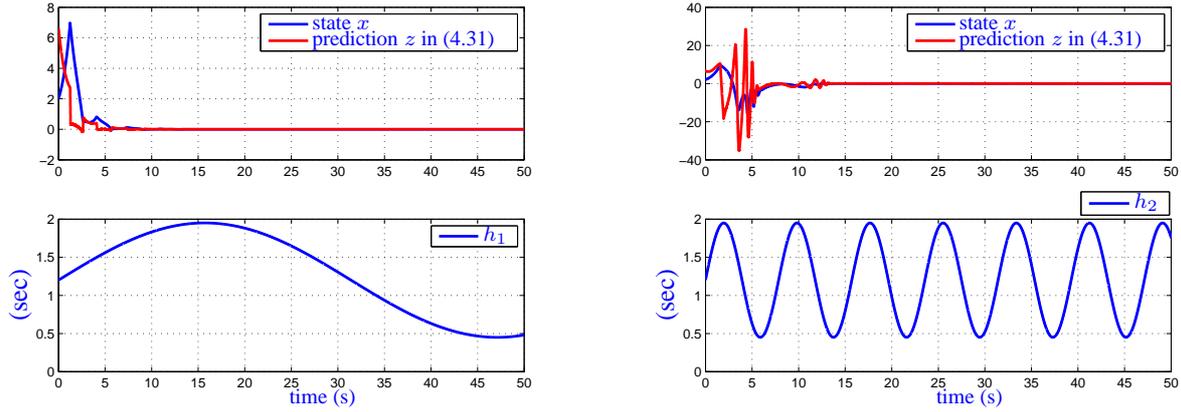
Contributions

- ✓ Explicit bounds on the delay dynamics that capture the physical behavior of the system and preserve the closed-loop stability are provided.
- ✓ Recent Lyapunov-Krasovskii functionals are used in the proofs of stability.
- ✓ Extension of the result of [Bresch-Pietri \[2012\]](#) to a larger class of controllers and to the output feedback case.

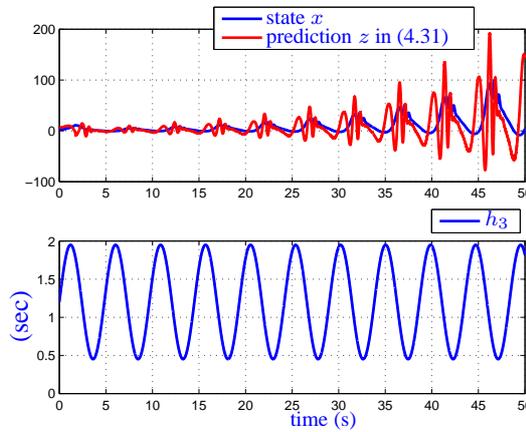
Table 4.1 sums up the conclusions of this section. Note that the bound δ^* may have different values for each case. In the next chapter, an extension to unknown delays will be studied.

4.2 Dynamic observation-prediction with a known delay

In this section, a recent predictive method for a constant delay and full state knowledge by [Najafi et al. \[2013\]](#) is extended to systems with a time-varying delay and to a broader class of



(a) Prediction (4.31) with controller (4.81) and input delay $h_1(t)$ (b) Prediction (4.31) with controller (4.81) and input delay $h_2(t)$



(c) Prediction (4.31) with controller (4.81) and input delay $h_3(t)$

Figure 4.4 – Simulation 1: faster and faster delays are tested to see the influence on the closed-loop stability.

controllers. Furthermore, the partial measurement case is also worked out. The major advantage of this technique is that it does not require an integral discretization. Indeed, a prediction is computed from a delay free copy of the original system.

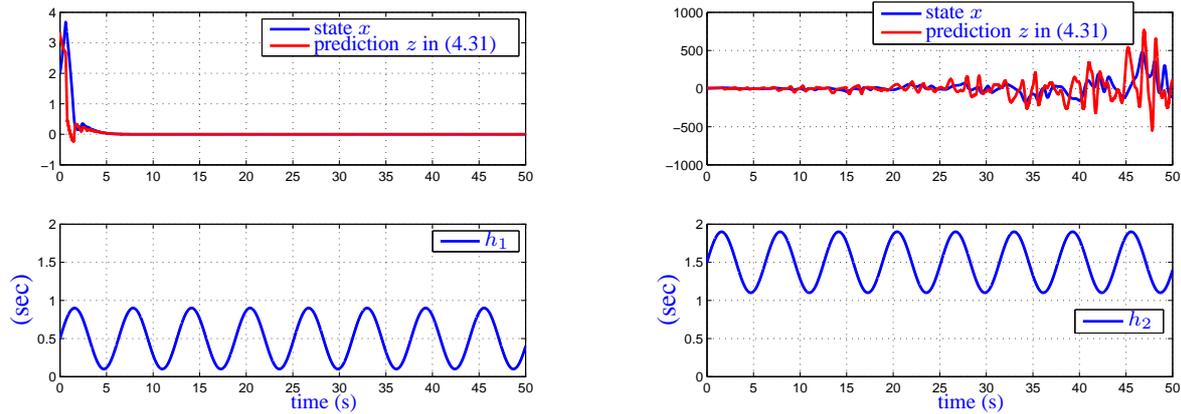
Remark 4.2.1. *We have been aware very recently that the idea of such a prediction has been reported before in Besançon et al. [2007] under the name of “asymptotic prediction”. In Besançon et al. [2009], they mention it as “direct dynamical computation scheme”.*

First, the original work of Najafi et al. [2013] is recalled in Section 4.2.1. Then, an extension is proposed for a single dynamic observer-predictor in Section 4.2.2 and for dynamic sequential sub observers-predictors in Section 4.2.3.

4.2.1 Presentation of existing results

Single predictor with a constant and known delay

In Najafi et al. [2013], a new predictive control method is proposed. The technique is based on the computation of a prediction thanks to a dynamic system. Their result is described there-



(a) Prediction (4.31) with controller (4.81) and input delay $h_4(t)$ (b) Prediction (4.31) with controller (4.81) and input delay $h_5(t)$

Figure 4.5 – Simulation 2: two delays, with the same time-derivative but different maximum values, are compared.

Prediction	Delay	Convergence	Condition	Feedback	Theorem	
Standard	Known	h	gl. & exp.	no condition	state	4.1
					output	4.2
	$h(t)$	gl. & exp.	$\delta < \delta^*$	state	4.3	
				output	4.4	

Table 4.1 – Sum up of the convergence results for standard prediction with a known delay.

- gl.: global – exp.: exponential
- h : constant delay
- $h(t) \in [h_{min}, h_{max}]$ and $|\dot{h}(t)| \leq \delta$: time-varying delay

after. The same class of system (\mathcal{E}_h) in (3.84) is considered in their work and is reminded below

$$(\mathcal{E}_h) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t - h) \\ x(0) = x_0, \end{cases}$$

with h a known and constant input delay. The variable z computed from the delay free system

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ z(0) = x(h), \end{cases} \quad (4.82)$$

is the exact prediction of (\mathcal{E}_h), that is to say $z(t) = x(t + h)$. However, this is an “open-loop” predictor and if $x(h)$ is not perfectly known, then $z(t) \neq x(t + h)$. Consequently, the idea is to “close the loop” by adding a feedback term in (4.82). Since $x(t + h)$ is not known, it has been decided to use $x(t)$. As a result, (4.82) becomes

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) + L[z(t - h) - x(t)] \\ z(t) = z_0(s) \quad \forall s \in [-h, 0]. \end{cases} \quad (4.83)$$

Assuming that L can be adequately chosen in order to have $z(t) \rightarrow x(t + h)$, then a predictive feedback

$$u(t) = Kz(t) \quad (4.84)$$

can be designed to stabilize system (\mathcal{E}_h) . This control scheme is displayed on Figure 4.6. The

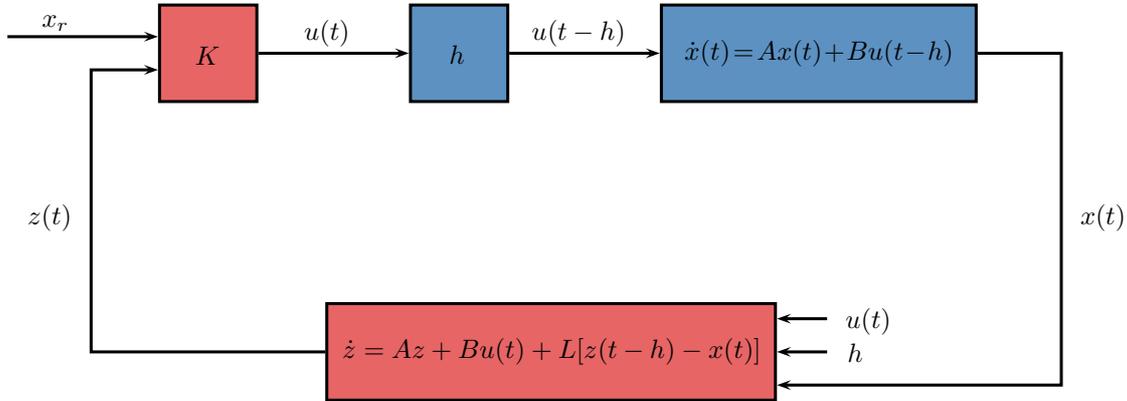


Figure 4.6 – Closed-loop scheme with dynamic predictor (4.83) for known and constant delay h

dynamic equation of the prediction error is

$$\dot{e}(t) = Ae(t) + Le(t - h) \quad (4.85)$$

where $e(t) = z(t) - x(t + h)$. The result of Najafi et al. [2013] states that system (\mathcal{E}_h) with feedback (4.83)-(4.84) is asymptotically stable under a LMI condition⁶ involving the gains K and L . The result is based on the following proposition:

Proposition 4.2.1 (Dugard and Verriest [1997]). *Consider system (4.85). If $A + L$ is Hurwitz, then there exists a sufficiently small $h_m > 0$ such that (4.85) is asymptotically stable for all $h \in [0, h_m]$.*

It can be seen from (4.85) that for a stable matrix A , $e(t)$ will asymptotically tend to zero if $L = 0$ for any time delay h . For an unstable matrix A , Proposition 4.2.1 shows that asymptotic stability can be achieved for a sufficiently small delay.

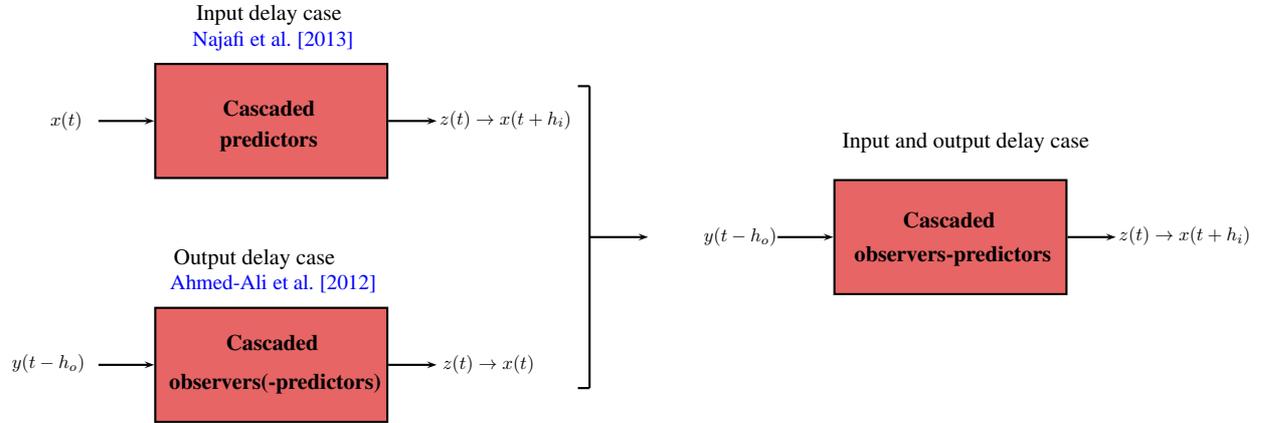
In Najafi et al. [2013], the argument is inverted since the delay is fixed but the tuning variables are the gains K and L that have to satisfy an explicit LMI condition. However, this LMI condition is hardly ever satisfied for a large delay or an unstable matrix A . As a consequence, Najafi et al. [2013], proposed a second result based on a sequential sub predictors structure.

Sequential sub predictors with a constant and known delay

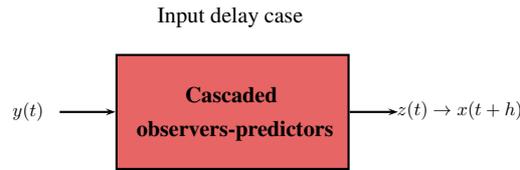
The idea is to split the time-delay and use cascaded predictors. The idea has been previously proposed to estimate the state from a delayed measurement. The works of Germani et al. [2002] and Kazantzis and Wright [2005] consider a class of nonlinear systems with a constant delay in the output whereas Ahmed-Ali et al. [2012] deals with time-varying delays. The concept of these three works can be combined with the work of Najafi et al. [2013] as shown on Figure 4.7a. Remark that the term “predictor” has been quoted because even if there is a prediction in the work of Ahmed-Ali et al. [2012], the word is generally reserved for the prediction of the

6. This condition is not copied here for sake of clarity.

future state at $t + h$ and not at instant t . In this thesis, the focus is on input delay system; so, the problem addressed thereafter is represented on Figure 4.7b that is a particular case of the more general input and output delay case. The principle of cascaded predictors is reminded thereafter.



(a) From cascaded observers and cascaded predictors to cascaded observers-predictors. h_i : input delay, h_o : output delay



(b) Cascade observers-predictors scheme considered in this section

Figure 4.7 – Cascaded observers and predictors schemes

Each of the r predictors predicts the state for \bar{h} seconds where

$$\bar{h} = \frac{h}{r}, \quad r \in \mathbb{N}^*. \quad (4.86)$$

Sequential sub predictors equations are as follows:

$$\begin{cases} \dot{z}_1(t) = Az_1(t) + Bu(t) + L_1[z_1(t - \bar{h}) - z_2(t)] \\ \vdots \\ \dot{z}_i(t) = Az_i(t) + Bu(t - (i-1)\bar{h}) + L_i[z_i(t - \bar{h}) - z_{i+1}(t)] \\ \vdots \\ \dot{z}_r(t) = Az_r(t) + Bu(t - (r-1)\bar{h}) + L_r[z_r(t - \bar{h}) - x(t)] \end{cases} \quad (4.87)$$

where $z_i \in \mathbb{R}^n$, $i = 1, \dots, r$. Assuming an appropriate choice of the gains L_i , it follows that $z_1(t)$ approaches $x(t + h)$. Thus, the predictive feedback

$$u(t) = Kz_1(t) \quad (4.88)$$

can be applied to stabilize (\mathcal{E}_h) . This control scheme is depicted on Figure 4.8 for $r = 2$. The advantage of splitting the prediction into r sub predictions is to allow to deal with arbitrarily large delays. Theorem 5.1 in Najafi et al. [2013] guarantees that feedback (4.88) with prediction z_1 computed by sequential sub predictors (4.87) asymptotically stabilizes (\mathcal{E}_h) provided that K

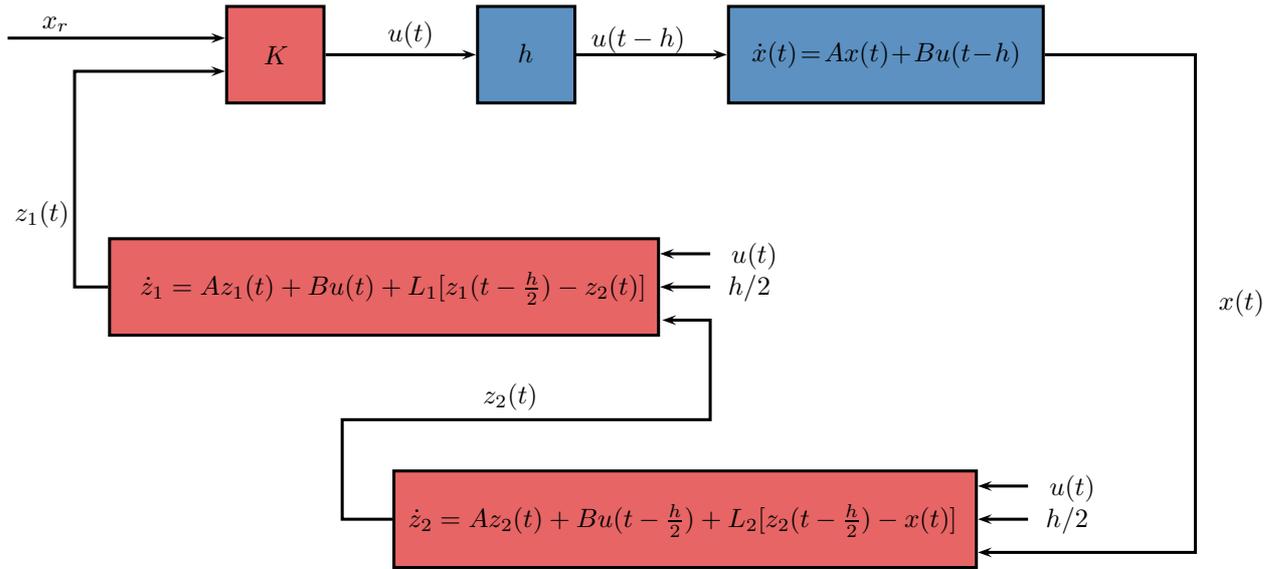


Figure 4.8 – Closed-loop scheme with two sequential sub predictors (4.87) for known and constant delay h

and the L_i satisfy a LMI condition⁷. From Proposition 4.2.1, it can be deduced that it is possible to choose r sufficiently large such that the closed-loop system is stable for an arbitrarily large delay.

Example 4.2.1. Consider the double integrator

$$\ddot{y}(t) = u(t - h) \quad (4.89)$$

and its state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t - h) \quad (4.90)$$

with $x = [y, \dot{y}]^T$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The control has the form $u(t) = Kz(t)$ (respectively $u(t) = Kz_1(t)$) with z given by (4.83) (respectively (4.87)) and $K = [-2, -3]^T$ chosen in order to place the eigenvalues of the system to -1 and -2 . The delay h is equal to 1 s. The influence of the gain L is studied by drawing a comparison for 3 different values of L . The gain L is computed with a pole placement approach by considering that the whole state is measured. Three combination of eigenvalues are tested:

- Simulation 1 (slowest case): $\lambda(A - LC) = \{-0.1, -0.2\}$ so

$$L = \begin{bmatrix} -0.1 & -1 \\ 0 & -0.2 \end{bmatrix}. \quad (4.91)$$

The closed-loop dynamics of the dynamic predictor imposed by L is slower than the delay value.

- Simulation 2 (mean case): $\lambda(A - LC) = \{-0.5, -1\}$ so

$$L = \begin{bmatrix} -0.5 & -1 \\ 0 & -1 \end{bmatrix}. \quad (4.92)$$

The closed-loop dynamics of the dynamic predictor imposed by L is of the same order of magnitude as the delay value.

7. This condition is not copied here for sake of clarity.

- *Simulation 3 (fastest case):* $\lambda(A - LC) = \{-1, -1.4\}$ so

$$L = \begin{bmatrix} -1 & -1 \\ 0 & -1.4 \end{bmatrix}. \quad (4.93)$$

The closed-loop dynamics of the dynamic predictor imposed by L is faster than the delay value.

Remark 4.2.2. In this example, the variable y has no particular physical meaning so it will be displayed without unit in the next plots.

On Figure 4.9, it can be seen that the closed-loop system is stabilized to zero for each choice of L . The convergence rate of the predictor, characterized by the choice of L , has to be carefully chosen. If L is large, some undesired oscillations appear as shown on Figure 4.9c. A too large value for this gain would end up destabilizing the system. If L is small, there is no problem for stable systems (A Hurwitz) but unstable systems require a sufficiently large value of L to counterweight the instability caused by “Ae” in (4.85).

A sine reference has been added to the input (Figure 4.10) in order to study more precisely the influence of L on the accuracy of the prediction z . For a small L , it can be observed that $z(t)$ slowly approaches the exact prediction (Figure 4.10a). Increasing L leads to a faster convergence of the approximated prediction to the exact one (Figure 4.10b). However, a large gain L can deteriorate the performances of the closed-loop system as shown on Figure 4.10c.

When increasing the delay ($h = 1.7$ s) and keeping L defined by (4.92), the system is unstable (Figure 4.11a). For a larger delay, there may not always exist L and K satisfying the LMI condition of Theorem 3.1 in Najafi et al. [2013]. In that case, one can design sequential sub predictors with r sufficiently large. On Figure 4.11b, two sequential sub predictors have been designed with $L_1 = L_2 = L$ from (4.92). It is clear that the sequential sub prediction technique allows to deal with a larger delay.

On Figure 4.12, the feedback computed from the dynamic prediction (4.87) is compared to the feedback computed from the exact prediction (that requires the integral discretization). It can be observed that the response is slower but it can be improved increasing the value of the gains L_i . Note that the value of K is the same for all the Figures 4.9, 4.10, 4.16 and 4.12.

Remark 4.2.3. In Najafi et al. [2013], only feedback controllers $u(t) = Kz(t)$ or $u(t) = Kz_1(t)$ are used. However, it will be shown in the next sections that these results hold for a larger class of controllers.

In the next sections, this valuable method will be extended to a time-varying delay and a partial measurement knowledge. Schemes that matched the different cases are indexed in Table 4.2.

Type of observer-predictor	Constant delay	Time-varying delay
Single observer-predictor	Figure 4.6	Figure 4.13
Sequential sub observers-predictors	Figure 4.8	Figure 4.14

Table 4.2 – Dynamic prediction for a known delay

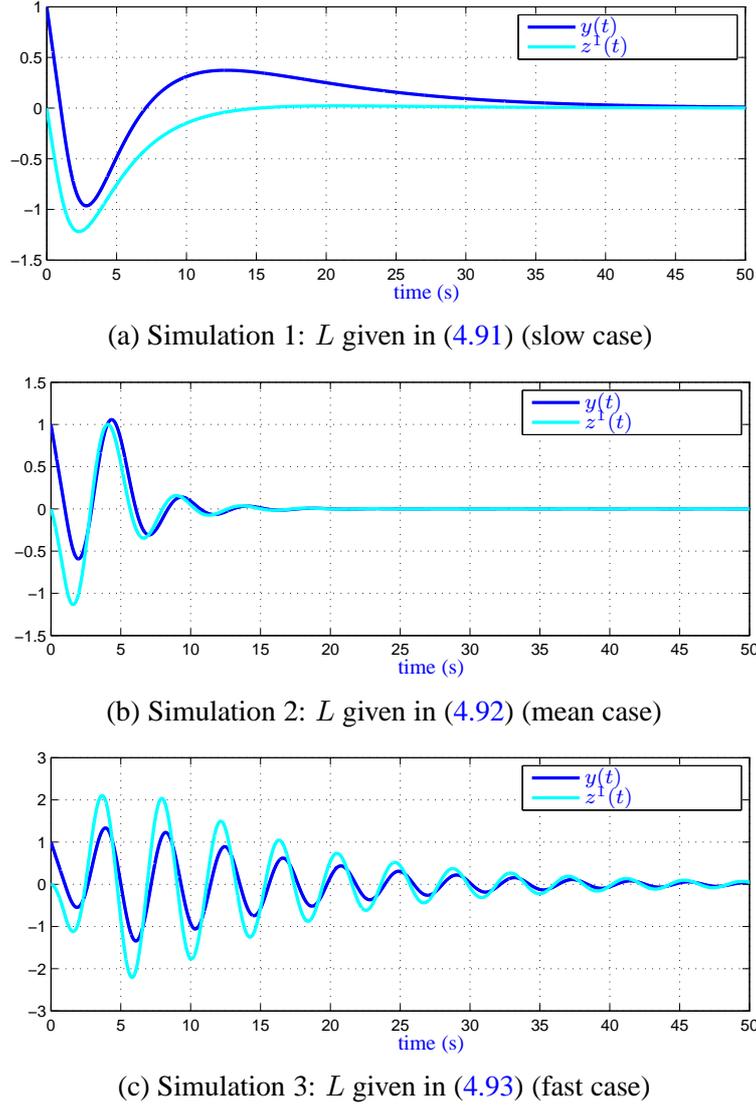


Figure 4.9 – Comparison of control scheme (4.83)-(4.84) for different values of L and $h = 1$ s. The notation z^1 denotes the first component of the vector $z(t)$.

4.2.2 Single observer-predictor with a known delay

In this section, an extension of the method described in Section 4.2.1 is presented. The work of Najafi et al. [2013] considers full state measurement and deals only with a constant delay. It is shown in the sequel that their results can be extended to a known time-varying delay and partial state knowledge. Before tackling the time-varying delay case, the simpler case of a constant delay with partial state measurement is studied in order to facilitate the exposition.

Constant delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t-h) \\ y(t) = Cx(t). \end{cases}$$

The loop is closed with a feedback u that verifies Assumption 3.4.1. Since “predictor” (4.83) has an observer structure, the prediction and the observation can be made by a single dynamic

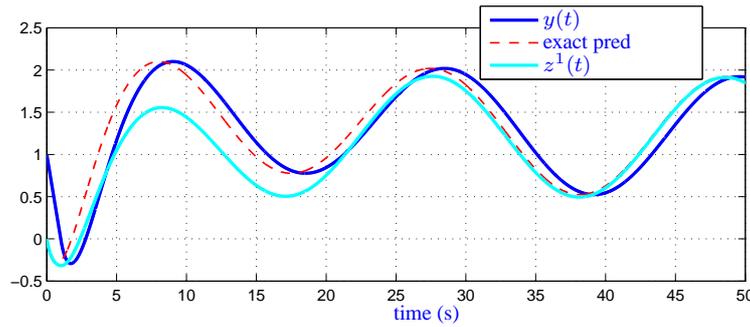
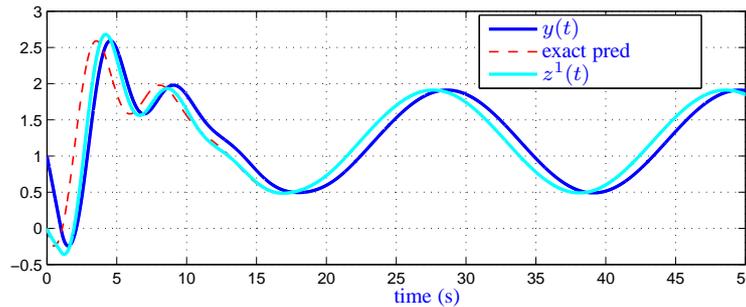
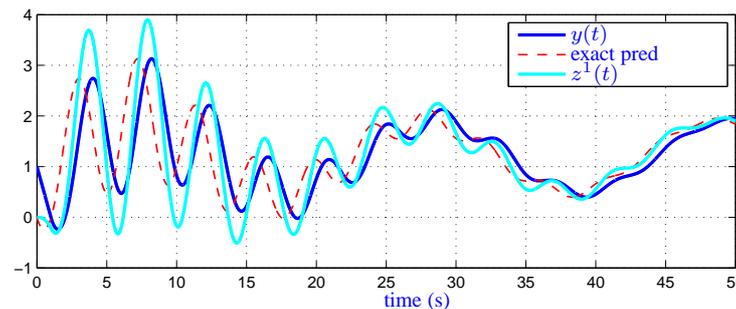
(a) Simulation 1: L given in (4.91) (slow case) with a sine trajectory reference(b) Simulation 2: L given in (4.92) (mean case) with a sine trajectory reference(c) Simulation 3: L given in (4.93) (fast case) with a sine trajectory reference

Figure 4.10 – Comparison of control scheme (4.83)-(4.84) for different values of L , $h = 1$ s and a sine trajectory. The notation z^1 denotes the first component of the vector $z(t)$.

system:

$$\dot{z}(t) = Az(t) + Bu(t) + g[Cz(t-h) - y(t)] \quad (4.94)$$

where g satisfies Assumption 3.4.2.

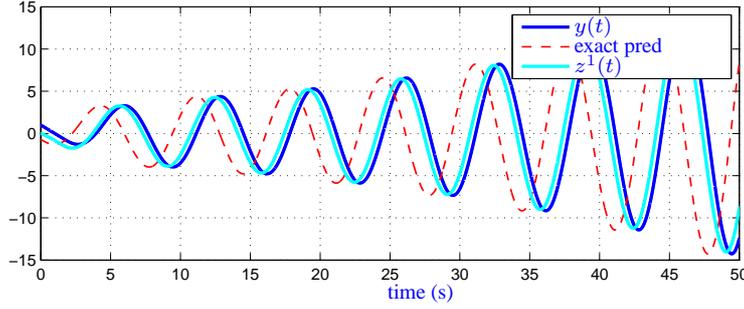
Remark 4.2.4. Note that to compute $z(t)$, the initial condition $z(t) = \phi_z(t)$ for $t \in [-h, 0]$ with ϕ_z a continuous function, is required. The prediction z will be continuous in $t = h$.

Denoting

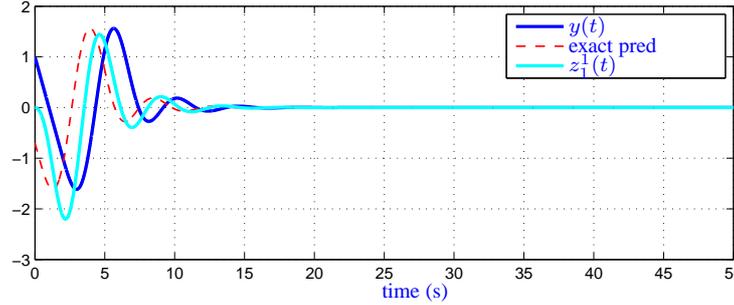
$$e_p(t) = z(t-h) - x(t) \quad (4.95)$$

the prediction error, the following result holds when the feedback $u(z)$ is applied to stabilize system (\mathcal{E}_h) .

Theorem 4.5. Consider system (\mathcal{E}_h) , with a known and constant delay h , and assume that there exist a controller $u(x)$ and a correction term $g(e_p)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z computed by



(a) Prediction with one predictor (gain L in (4.92)): control scheme displayed on Figure 4.6



(b) Prediction with two sequential sub predictors (gain $L_1 = L_2 = L$ in (4.92)): control scheme displayed on Figure 4.8

Figure 4.11 – Comparison of control schemes (4.83)-(4.84) and (4.87)-(4.88) (for $r = 2$) with $h = 1.7$ s. The notation z^1 (respectively z_1^1) denotes the first component of the vector $z(t)$ (respectively $z_1(t)$).

observer-predictor (4.94) and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-h, t]} \|z(s)\|^2 + \sup_{s \in [t-h, t]} \|\dot{e}_p(s)\|^2. \quad (4.96)$$

Then, there exist $\varsigma, \varrho, h^* > 0$ such that, provided $h < h^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (4.97)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. First, with $e_p(t) = z(t-h) - x(t)$, equation (4.94) becomes

$$\dot{z}(t) = Az(t) + Bu(z(t)) + g(Ce_p(t)). \quad (4.98)$$

The dynamics of $e_p(t)$ reads as

$$\dot{e}_p(t) = Ae_p(t) + g(Ce_p(t-h)) \quad (4.99)$$

and can be rewritten as follows

$$\dot{e}_p(t) = Ae_p(t) + g(Ce_p(t)) - \int_{Ce_p(t-h)}^{Ce_p(t)} \frac{dg}{ds}(s) ds. \quad (4.100)$$

The following Lyapunov-Krasovskii functional candidate is proposed

$$V(t) = \gamma V_1(t) + V_2(t) + V_3(t) \quad (4.101)$$

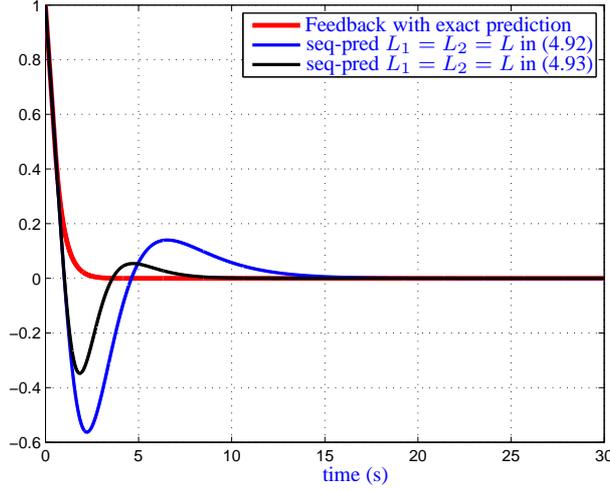


Figure 4.12 – Comparison of the feedback computed from the exact prediction (integral discretization) and the sequential sub predictors (4.87) for $h = 0.5$ s.

where

$$V_1(t) = z^T(t)Pz(t), \quad (4.102)$$

with P defined in Assumption 3.4.1,

$$V_2(t) = e_p^T(t)Qe_p(t), \quad (4.103)$$

with Q defined in Assumption 3.4.2,

$$V_3(t) = \int_{t-h}^t (h+s-t) \|\dot{e}_p(s)\|^2 ds. \quad (4.104)$$

and $\gamma > 0$. Remark that functions V_1 and V_2 are usually used for delay free systems. The functional V_3 is added to cope with the integral term in (4.100). From equation (3.88) and since g is globally Lipschitz, the time derivatives of V_1 satisfies

$$\dot{V}_1(t) \leq -c_u \|z(t)\|^2 + R \|z(t)\| \|e_p(t)\|, \quad (4.105)$$

with $R = 2l_g \|P\| \|C\|$. Furthermore, by substitution one gets

$$\left\| \int_{Ce_p(t-h)}^{Ce_p(t)} \frac{dg}{ds}(s) ds \right\| \leq l_g \|C\| \int_{t-h}^t \|\dot{e}_p(s)\| ds, \quad (4.106)$$

so using (3.94), it follows that

$$\dot{V}_2(t) \leq -m_g \|e_p(t)\|^2 + R' \|e_p(t)\| \|m(t)\| \quad (4.107)$$

with $R' = 2l_g \|Q\| \|C\|$ and $\|m(t)\| = \int_{t-h}^t \|\dot{e}_p(s)\| ds$. Finally,

$$\dot{V}_3(t) = h \|\dot{e}_p(t)\|^2 - \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds \quad (4.108)$$

so, reminding that

$$-\int_{t-h}^t \|\dot{e}_p(s)\|^2 ds \leq -\frac{1}{h} \left(\int_{t-h}^t \|\dot{e}_p(s)\| ds \right)^2, \quad (4.109)$$

it follows that

$$\dot{V}_3(t) \leq h\|\dot{e}_p(t)\|^2 - \frac{1}{2h}\|m(t)\|^2 - \frac{1}{2} \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds. \quad (4.110)$$

In addition, it can be deduced from (4.100) and from Hölder's inequality (3.104) that

$$\|\dot{e}_p\|^2 \leq c_1\|e_p\|^2 + c_2\|m\|^2 \quad (4.111)$$

with $c_1 = 2(\|A\| + l_g\|C\|)^2$ and $c_2 = 2l_g^2\|C\|^2$. As a result, from (4.110) and (4.111), it can be derived that

$$\dot{V}_3(t) \leq hc_1\|e_p(t)\|^2 + hc_2\|m(t)\|^2 - \frac{1}{2h}\|m(t)\|^2 - \frac{1}{2} \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds. \quad (4.112)$$

Thus, the following maximization is obtained

$$\begin{aligned} \dot{V}(t) \leq & -\gamma c_u \|z(t)\|^2 + \gamma R \|z(t)\| \|e_p(t)\| - m_g \|e_p(t)\|^2 + R' \|e_p(t)\| \|m(t)\| \\ & + hc_1 \|e_p(t)\|^2 + hc_2 \|m(t)\|^2 - \frac{1}{2h} \|m(t)\|^2 - \frac{1}{2} \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds. \end{aligned} \quad (4.113)$$

Moreover, equation (3.102) allows to get rid of the cross terms " $\|z\| \|e_p\|$ " and " $\|e_p\| \|m\|$ " so

$$\begin{aligned} \dot{V}(t) \leq & -\frac{\gamma c_u}{2} \|z(t)\|^2 - \left[\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - hc_1 \right] \|e_p(t)\|^2 - \left[\frac{1}{2h} - \frac{R'^2}{2m_g} - hc_2 \right] \|m(t)\|^2 \\ & - \frac{1}{2} \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds. \end{aligned} \quad (4.114)$$

To prove the exponential stability of (4.98)-(4.99), it is sufficient to find conditions which guarantee the following inequality

$$\dot{V}(t) + \varepsilon V(t) \leq 0 \quad (4.115)$$

with $\varepsilon > 0$. From equation (4.114), Assumptions 3.4.1 and 3.4.2 and since $0 \leq s - t + h < h$ for all $s \in [t - h, t]$, one has

$$\begin{aligned} \dot{V}(t) + \varepsilon V(t) \leq & -\frac{\gamma c_u}{2} \|z(t)\|^2 - \left[\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - hc_1 \right] \|e_p(t)\|^2 - \left[\frac{1}{2h} - \frac{R'^2}{2m_g} - hc_2 \right] \|m(t)\|^2 \\ & - \frac{1}{2} \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds + \varepsilon \bar{c} \gamma \|z(t)\|^2 + \varepsilon \bar{m} \|e_p(t)\|^2 + \varepsilon h \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds. \end{aligned} \quad (4.116)$$

It follows that

$$\begin{aligned} \dot{V}(t) + \varepsilon V(t) \leq & -\gamma \left[\frac{c_u}{2} - \varepsilon \bar{c} \right] \|z(t)\|^2 - \left[\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - hc_1 - \varepsilon \bar{m} \right] \|e_p(t)\|^2 \\ & - \left[\frac{1}{2h} - \frac{R'^2}{2m_g} - hc_2 \right] \|m(t)\|^2 - \left[\frac{1}{2} - \varepsilon h \right] \int_{t-h}^t \|\dot{e}_p(s)\|^2 ds. \end{aligned} \quad (4.117)$$

In order to have $\dot{V} + \varepsilon V$ negative, the following conditions have to be verified

$$\left\{ \begin{array}{l} \frac{c_u}{2} - \varepsilon \bar{c} > 0, \end{array} \right. \quad (4.118)$$

$$\left\{ \begin{array}{l} \frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - hc_1 - \varepsilon \bar{m} > 0, \end{array} \right. \quad (4.119)$$

$$\left\{ \begin{array}{l} \frac{1}{2h} - \frac{R'^2}{2m_g} - hc_2 > 0, \end{array} \right. \quad (4.120)$$

$$\left\{ \begin{array}{l} \frac{1}{2} - \varepsilon h > 0. \end{array} \right. \quad (4.121)$$

From (4.120), one has

$$h < \frac{\sqrt{\frac{R'^4}{4m_g^2} + 2c_2} - \frac{R'^2}{2m_g}}{2c_2} < \frac{1}{\sqrt{2c_2}}. \quad (4.122)$$

In addition, conditions (4.119) and (4.120) can be reformulated as follows

$$h < h^* = \min(h_1, h_2) \quad (4.123)$$

with

$$h_1 = \frac{1}{c_1} \left(\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - \varepsilon \bar{m} \right) \quad (4.124)$$

and

$$h_2 = \frac{1}{\sqrt{2c_2}}. \quad (4.125)$$

Choosing γ and ε sufficiently small guarantees that (4.118) and (4.121) are true and that h_1 is positive. In this case, taking $h < h^*$ ensures that

$$\dot{V} \leq -\varepsilon V(t). \quad (4.126)$$

It follows from (4.126) that

$$V(t) \leq V(0)e^{-\varepsilon t} \quad (4.127)$$

and since $\|z(t)\|^2 + \|e_p(t)\|^2 \leq \frac{1}{c_1}V(t)$ with $c_1 = \min(\gamma \underline{c}, \underline{m})$, one obtains

$$\|z(t)\|^2 + \|e_p(t)\|^2 \leq \frac{1}{c_1}V(0)e^{-\varepsilon t}. \quad (4.128)$$

From the definitions of $V(t)$ in (4.101), one has

$$V(0) \leq \gamma \bar{c} \|z(0)\|^2 + \bar{m} \|e_p(0)\|^2 + h^2 \sup_{s \in [-h, 0]} \|\dot{e}_p(s)\|^2. \quad (4.129)$$

In addition, from (4.95) and by using Hölder's inequality (3.104), the following inequality can be deduced

$$\|e_p(0)\|^2 \leq 2\|z(-h)\|^2 + 2\|x(0)\|^2 \leq 2 \sup_{s \in [-h, 0]} \|z(s)\|^2 + 2\|x(0)\|^2 \quad (4.130)$$

and then it follows that

$$\gamma \bar{c} \|z(0)\|^2 + \bar{m} \|e_p(0)\|^2 \leq c_2 \left(\sup_{s \in [-h, 0]} \|z(s)\|^2 + \|x(0)\|^2 \right) \quad (4.131)$$

with $c_2 = \gamma\bar{c} + 2\bar{m}$. Combining (4.129) with (4.131) leads to

$$V(0) \leq c_3\Upsilon(0) \quad (4.132)$$

with $c_3 = \max(c_2, h^2)$. Then, from (4.128) and (4.132) one deduces

$$\|z(t)\|^2 + \|e_p(t)\|^2 \leq c_5\Upsilon(0)e^{-\varepsilon t} \quad (4.133)$$

for all $t \geq 0$ and with $c_5 = \frac{c_3}{c_1}$. Since $\|x(t)\|^2 \leq 2\|z(t-h)\|^2 + 2\|e_p(t)\|^2$, one gets

$$\|x(t)\|^2 \leq c_6\Upsilon(0)e^{-\varepsilon t} \quad (4.134)$$

with $c_6 = 2c_5(e^{\varepsilon h} + 1)$. Besides

$$\sup_{s \in [t-h, t]} \|z(s)\|^2 \leq c_7\Upsilon(0)e^{-\varepsilon t} \quad (4.135)$$

with $c_7 = c_5e^{\varepsilon h}$. From equation (4.99) and using Hölder's inequality (3.104), one can deduce that

$$\sup_{s \in [t-h, t]} \|\dot{e}_p(s)\|^2 \leq c_8\Upsilon(0)e^{-\varepsilon t} \quad (4.136)$$

with $c_8 = 2c_5(\|A\|^2 + l_g^2\|C\|^2e^{\varepsilon h})e^{\varepsilon h}$. Finally, from (4.134), (4.135) and (4.136), one obtains

$$\Upsilon(t) \leq c_9\Upsilon(0)e^{-\varepsilon t} \quad (4.137)$$

for all $t \geq 0$ with $c_9 = c_6 + c_7 + c_8$. □

Qualitative analysis: This theorem shows that a predictive feedback computed from the observer-predictor (4.94) can always stabilize system (\mathcal{E}_h) provided that h is sufficiently small. In addition, a qualitative analysis of the behavior of h_1 and h_2 (respectively in (4.124) and (4.125)) shows that:

- For a small value of ε , then h_1 increases which means that a slow convergence rate of the closed-loop system allows a larger value of the delay.
- For a large value of l_g ⁸ (that appears through c_1 and c_2), then h_1 and h_2 decrease which means that the admissible delay is smaller for a fast observer.

This result presents the first improvement with respect to the paper of Najafi et al. [2013] since it works out the partial state measurement case and achieves exponential stability. Another improvement is to consider a time-varying delay in the input. This result is presented thereafter.

Remark 4.2.5. A similar qualitative analysis is given in Besançon et al. [2007]. Similarly, the extension of this method to linear systems with Lipschitz nonlinearities and to partial state measurement is mentioned in Besançon et al. [2007].

Remark 4.2.6. A similar result for nonlinear globally Lipschitz systems with sampled measurements and hold control has been published very recently by Ahmed-Ali et al. [2016].

8. l_g is the Lipschitz constant associated to the function g and is defined in (3.95).

Time-varying delay

The class of systems considered is defined by (\mathcal{E}_h) and is reminded below

$$(\mathcal{E}_h) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t - h(t)) \\ y(t) = Cx(t) \end{cases}$$

with $h(t)$ a known and time-varying input delay. The observation and the prediction are computed by a single dynamic system

$$\dot{z}(t) = Az(t) + Bu(t) + g(Cz(t - h(t)) - y(t)) \quad (4.138)$$

with g a correction term verifying Assumption 3.4.2.

Remark 4.2.7. Note that to compute $z(t)$, the initial condition $z(t) = \phi_z(t)$ for $t \in [-h_{max}, 0]$ with ϕ_z a continuous function, is required. The prediction z will be continuous in $t = h(t)$.

Then, a predictive feedback that verifies Assumption 3.4.1 can be applied to stabilize (\mathcal{E}_h) . The closed-loop scheme is given on Figure 4.13. Denoting the prediction error

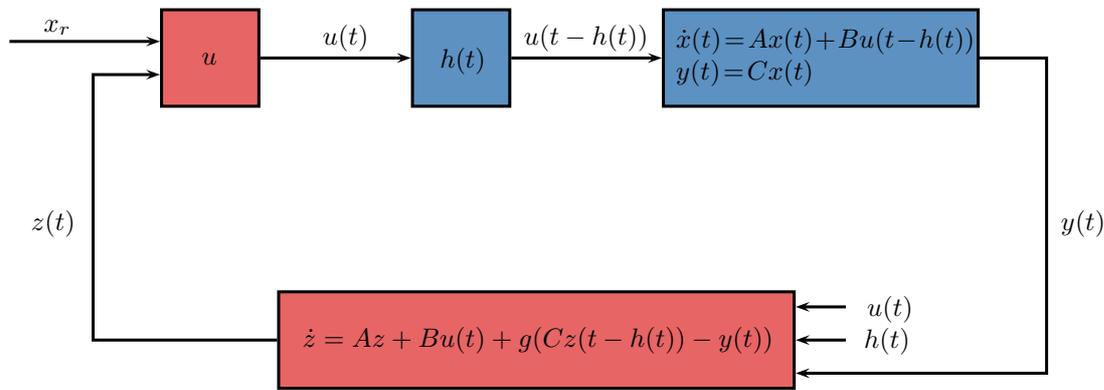


Figure 4.13 – Closed-loop scheme with observer-predictor for a known time-varying delay and partial state measurement

$$e_p(t) = z(t - h(t)) - x(t) \quad (4.139)$$

and reminding that δ stands for the upper bound of the delay time derivative: $|\dot{h}(t)| < \delta$ (Assumption 3.4.4), it is now possible to introduce the following theorem.

Theorem 4.6. Consider system (\mathcal{E}_h) , where $h(t)$ is known and complies with Assumptions 3.4.3 and 3.4.4, and assume that there exist a controller $u(x)$ and the correction term $g(e_p)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z computed by observer-predictor (4.138) and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-h_{max}, t]} \|z(s)\|^2 + \sup_{s \in [t-h_{max}, t]} \|\dot{e}_p(s)\|^2. \quad (4.140)$$

Then, there exist $\varsigma, \varrho, h^*, \delta^* > 0$ such that, provided $h_{max} < h^*$ and $\delta < \delta^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (4.141)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. Note that the delay $h(t)$ (respectively $\dot{h}(t)$) will be denoted h (respectively \dot{h}) in the proof to facilitate the reading. First, denoting $e_p(t) = z(t - h(t)) - x(t)$, equation (4.138) becomes

$$\dot{z}(t) = Az(t) + Bu(t) + g(Ce_p(t)). \quad (4.142)$$

The dynamics of $e_p(t)$ reads as

$$\dot{e}_p(t) = Ae_p(t) + g(Ce_p(t-h)) - \dot{h}[Az(t-h) + Bu(t-h) + g(Ce_p(t-h))] \quad (4.143)$$

and can be rewritten as follows

$$\begin{aligned} \dot{e}_p(t) = & Ae_p(t) + g(Ce_p(t)) - \int_{Ce_p(t-h)}^{Ce_p(t)} \frac{dg}{ds}(s)ds \\ & - \dot{h}[Az(t-h) + Bu(t-h) + g(Ce_p(t-h))]. \end{aligned} \quad (4.144)$$

The following Lyapunov-Krasovskii functional candidate is proposed

$$V(t) = \gamma V_1(t) + V_2(t) + V_3(t) + V_4(t) \quad (4.145)$$

where

$$V_1(t) = z^T(t)Pz(t), \quad (4.146)$$

with P defined in (3.86),

$$V_2(t) = e_p^T(t)Qe_p(t), \quad (4.147)$$

with Q defined in (3.91),

$$V_3(t) = \int_{t-h_{max}}^t (h_{max} + s - t) \|\dot{e}_p(s)\|^2 ds, \quad (4.148)$$

$$V_4(t) = \int_{t-h}^t (h_{max} + h_{min} + s - t) (\|z(s)\|^2 + \|e_p(s)\|^2) ds \quad (4.149)$$

and $\gamma > 0$. Remark that V_1, V_2 are the same as in the constant delay case. Functionals V_3 and V_4 are introduced to deal respectively with the integral term of (4.144) and with the time-varying delay. Since g is globally Lipschitz and from (3.88), the time derivatives of V_1 satisfies

$$\dot{V}_1(t) \leq -c_u \|z(t)\|^2 + R \|z(t)\| \|e_p(t)\|, \quad (4.150)$$

with $R = 2l_g \|P\| \|C\|$. Moreover, by substitution one gets

$$\left\| \int_{Ce_p(t-h(t))}^{Ce_p(t)} \frac{dg}{ds}(s)ds \right\| \leq l_g \|C\| \int_{t-h(t)}^t \|\dot{e}_p(s)\| ds, \quad (4.151)$$

so using (3.94), it follows that

$$\begin{aligned} \dot{V}_2(t) \leq & -m_g \|e_p(t)\|^2 + R' \|e_p(t)\| \|m(t)\| + \delta M \|e_p(t)\| \|z(t-h)\| \\ & + \delta R' \|e_p(t)\| \|e_p(t-h)\| \end{aligned} \quad (4.152)$$

with $R' = 2l_g\|Q\| \|C\|$, $M = 2\|P\| (\|A\| + l_u\|B\|)$ and $\|m(t)\| = \int_{t-h}^t \|\dot{e}_p(s)\| ds$. Furthermore,

$$\dot{V}_3(t) = h_{max}\|\dot{e}_p(t)\|^2 - \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds \quad (4.153)$$

so, reminding that

$$- \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds \leq - \int_{t-h(t)}^t \|\dot{e}_p(s)\|^2 ds \leq -\frac{1}{h_{max}} \left(\int_{t-h(t)}^t \|\dot{e}_p(s)\| ds \right)^2, \quad (4.154)$$

it follows that

$$\dot{V}_3(t) \leq h_{max}\|\dot{e}_p(t)\|^2 - \frac{1}{2h_{max}}\|m(t)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds. \quad (4.155)$$

In addition, since u and g are globally Lipschitz and using (4.151) and Hölder's inequality (3.104), it can be deduced from (4.144) that

$$\|\dot{e}_p\|^2 \leq c_1\|e_p\|^2 + c_2\|m\|^2 + c_3\delta^2\|z(t-h)\|^2 + c_2\delta^2\|e_p(t-h)\|^2 \quad (4.156)$$

with $c_1 = 4(\|A\| + l_g\|C\|)^2$, $c_2 = 4l_g^2\|C\|^2$, $c_3 = 4(\|A\| + l_u\|B\|)^2$ so

$$\begin{aligned} \dot{V}_3(t) &\leq h_{max}[c_1\|e_p\|^2 + c_2\|m\|^2 + c_3\delta^2\|z(t-h)\|^2 + c_2\delta^2\|e_p(t-h)\|^2] \\ &\quad - \frac{1}{2h_{max}}\|m(t)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds. \end{aligned} \quad (4.157)$$

Finally,

$$\begin{aligned} \dot{V}_4(t) &\leq (h_{max} + h_{min})(\|z(t)\|^2 + \|e_p(t)\|^2) - h_{min}(1 - \delta)(\|z(t-h)\|^2 + \|e_p(t-h)\|^2) \\ &\quad - \int_{t-h}^t (\|z(s)\|^2 + \|e(s)\|^2) ds. \end{aligned} \quad (4.158)$$

Thus, the following maximization is obtained

$$\begin{aligned} \dot{V}(t) &\leq -\gamma c_u\|z(t)\|^2 + \gamma R\|z(t)\| \|e_p(t)\| - m_g\|e_p(t)\|^2 + R'\|e_p(t)\| \|m(t)\| \\ &\quad + \delta M\|e_p(t)\| \|z(t-h)\| + \delta R'\|e_p(t)\| \|e_p(t-h)\| + h_{max}[c_1\|e_p\|^2 + c_2\|m\|^2 \\ &\quad + c_3\delta^2\|z(t-h)\|^2 + c_2\delta^2\|e_p(t-h)\|^2] - \frac{1}{2h_{max}}\|m(t)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds \\ &\quad + 2h_{max}(\|z(t)\|^2 + \|e_p(t)\|^2) - h_{min}(1 - \delta)(\|z(t-h)\|^2 + \|e_p(t-h)\|^2) \\ &\quad - \int_{t-h}^t (\|z(s)\|^2 + \|e(s)\|^2) ds. \end{aligned} \quad (4.159)$$

The cross terms can be maximized by (3.102) and Young's inequality (3.100); it gives

$$\begin{aligned}
\dot{V}(t) \leq & - \left[\frac{\gamma c_u}{2} - 2h_{max} \right] \|z(t)\|^2 - \left[\frac{1}{2h_{max}} - \frac{R'^2}{2m_g} - h_{max}c_2 \right] \|m(t)\|^2 \\
& - \left[\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - h_{max}(2 + c_1) - \delta(M/2 + R'/2) \right] \|e_p(t)\|^2 \\
& - [h_{min} - \delta(h_{min} + \delta h_{max}c_3 + M/2)] \|z(t-h)\|^2 \\
& - [h_{min} - \delta(h_{min} + \delta h_{max}c_2 + R'/2)] \|e_p(t-h)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds \\
& - \int_{t-h}^t (\|z(s)\|^2 + \|e(s)\|^2) ds.
\end{aligned} \tag{4.160}$$

To prove the exponential stability of (4.142)-(4.99), it is sufficient to find conditions which guarantee the following inequality

$$\dot{V}(t) + \varepsilon V(t) \leq 0 \tag{4.161}$$

with $\varepsilon > 0$. From equation (4.160), Assumptions 3.4.1 and 3.4.2, one has

$$\begin{aligned}
\dot{V}(t) + \varepsilon V(t) \leq & - \left[\frac{\gamma c_u}{2} - 2h_{max} \right] \|z(t)\|^2 - \left[\frac{1}{2h_{max}} - \frac{R'^2}{2m_g} - h_{max}c_2 \right] \|m(t)\|^2 \\
& - \left[\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - h_{max}(2 + c_1) - \delta(M/2 + R'/2) \right] \|e_p(t)\|^2 \\
& - [h_{min} - \delta(h_{min} + \delta h_{max}c_3 + M/2)] \|z(t-h)\|^2 \\
& - [h_{min} - \delta(h_{min} + \delta h_{max}c_2 + R'/2)] \|e_p(t-h)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds \\
& \gamma \varepsilon \bar{c} \|z\|^2 + \varepsilon \bar{m} \|e_p\|^2 + \varepsilon \int_{t-h_{max}}^t h_{max} \|\dot{e}_p(s)\|^2 ds \\
& - [1 - \varepsilon(h_{max} + h_{min})] \int_{t-h}^t (\|z(s)\|^2 + \|e_p(s)\|^2) ds.
\end{aligned} \tag{4.162}$$

It follows that

$$\begin{aligned}
\dot{V} + \varepsilon V \leq & - \left[\frac{\gamma c_u}{2} - 2h_{max} - \gamma \varepsilon \bar{c} \right] \|z(t)\|^2 - \left[\frac{1}{2h_{max}} - \frac{R'^2}{2m_g} - h_{max}c_2 \right] \|m(t)\|^2 \\
& - \left[\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - h_{max}(2 + c_1) - \delta(M/2 + R'/2) - \varepsilon \bar{m} \right] \|e_p(t)\|^2 \\
& - [h_{min} - \delta(h_{min} + \delta h_{max}c_3 + M/2)] \|z(t-h)\|^2 \\
& - [h_{min} - \delta(h_{min} + \delta h_{max}c_2 + R'/2)] \|e_p(t-h)\|^2 \\
& - \left[\frac{1}{2} - \varepsilon h_{max} \right] \int_{t-h_{max}}^t \|\dot{e}_p(s)\|^2 ds \\
& - [1 - 2\varepsilon h_{max}] \int_{t-h}^t (\|z(s)\|^2 + \|e_p(s)\|^2) ds.
\end{aligned} \tag{4.163}$$

To make $\dot{V}(t) + \varepsilon V(t) \leq 0$, it is sufficient to have the coefficients pre multiplying the quadratic

terms negative. It leads to the following conditions

$$\left\{ \begin{array}{l} \frac{\gamma c_u}{2} - 2h_{max} - \gamma \varepsilon \bar{c} > 0, \end{array} \right. \quad (4.164)$$

$$\left\{ \begin{array}{l} \frac{1}{2h_{max}} - \frac{R'^2}{2m_g} - h_{max}c_2 > 0, \end{array} \right. \quad (4.165)$$

$$\left\{ \begin{array}{l} \frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - h_{max}(2 + c_1) - \delta(M/2 + R'/2) - \varepsilon \bar{m} > 0, \end{array} \right. \quad (4.166)$$

$$\left\{ \begin{array}{l} h_{min} - \delta(h_{min} + \delta h_{max}c_3 + M/2) > 0, \end{array} \right. \quad (4.167)$$

$$\left\{ \begin{array}{l} h_{min} - \delta(h_{min} + \delta h_{max}c_2 + R'/2) > 0, \end{array} \right. \quad (4.168)$$

$$\left\{ \begin{array}{l} \frac{1}{2} - \varepsilon h_{max} > 0, \end{array} \right. \quad (4.169)$$

$$\left\{ \begin{array}{l} 1 - 2\varepsilon h_{max} > 0. \end{array} \right. \quad (4.170)$$

First, (4.164), (4.169) and (4.170) can be reformulated as follows

$$h_{max} < h_1 \quad (4.171)$$

and

$$h_{max} < h_2 \quad (4.172)$$

and

$$h_{max} < h_3 \quad (4.173)$$

with

$$h_1 = \frac{\gamma}{2} \left[\frac{c_u}{2} - \varepsilon \bar{c} \right] \quad (4.174)$$

and

$$h_2 = \frac{1}{2\varepsilon}. \quad (4.175)$$

Besides, (4.165) can be rewritten in the following form

$$h_{max} < \frac{\sqrt{\left(\frac{R'^2}{2m_g}\right)^2 + 2c_2} - \left(\frac{R'^2}{2m_g}\right)}{2c_2} < h_3 \quad (4.176)$$

with

$$h_3 = \frac{1}{\sqrt{2c_2}}. \quad (4.177)$$

Conditions (4.167)-(4.168) depend simultaneously on h_{max} and δ . Rearranging the terms gives

$$h_{max} < h_4 \quad (4.178)$$

and

$$h_{max} < h_5 \quad (4.179)$$

with

$$h_4 = \frac{1}{c_3 \delta^2} [h_{min} - \delta(h_{min} + M/2)] \quad (4.180)$$

and

$$h_5 = \frac{1}{c_2 \delta^2} [h_{min} - \delta(h_{min} + R'/2)]. \quad (4.181)$$

Similarly condition (4.166) depends simultaneously on h_{max} and δ and can be reformulated as follows

$$h_{max} < h_6 \quad (4.182)$$

with

$$h_6 = \frac{1}{2 + c_1} \left(\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - \delta(M/2 + R'/2) - \varepsilon \bar{m} \right). \quad (4.183)$$

In order to have h_4 , h_5 and h_6 positive, δ has to comply with the condition below

$$\delta < \delta^* = \min(\delta_1, \delta_2, \delta_3) \quad (4.184)$$

with

$$\delta_1 = \frac{h_{min}}{h_{min} + M/2}, \quad (4.185)$$

$$\delta_2 = \frac{h_{min}}{h_{min} + R'/2}, \quad (4.186)$$

$$\delta_3 = \frac{2}{(M + R')} \left(\frac{m_g}{2} - \frac{\gamma R^2}{2c_u} - \varepsilon \bar{m} \right). \quad (4.187)$$

Choosing ε and γ sufficiently small guarantees that h_1 and δ_3 are positive. In that case, taking

$$h_{max} < h^* = \min(h_1, h_2, h_3, h_4, h_5, h_6) \quad (4.188)$$

and $\delta < \delta^*$ ensures that

$$\dot{V}(t) \leq -\varepsilon V(t). \quad (4.189)$$

So equation (4.141) can be deduced from (4.189) by similar computations as in Theorem 4.5. \square

Qualitative analysis: This theorem shows that a predictive feedback computed from observer-predictor (4.94) can stabilize system (\mathcal{E}_h) provided that the delay and its variation are sufficiently small. In addition, a qualitative analysis of the bounds h^* and δ^* is given below.

- For a small value of ε , then h_1 , h_2 and h_6 increase which means that a slow convergence rate of the closed-loop system allows larger value of the delay.
- Similarly, for a small value of ε , then δ_3 increases which means that the delay dynamics can be faster for a slow convergence rate of the close-loop system.
- For a large value of l_g ⁹ (that appears through c_1 and c_2), then h_3 , h_5 and h_6 decrease which means that the admissible delay is smaller for a fast observer.
- For a large value of l_u ¹⁰ (that appears through c_3), then h_4 decreases which means that the admissible delay is smaller for a fast controller.
- For a large value of δ , then δ_1 and δ_2 increase which means that the delay dynamics can be faster for a small delay interval (see Figure 4.2).
- For a small value of δ , then h_4 , h_5 and h_6 increase which means that a slow-varying delay can have a larger amplitude.

Moreover, some simulation tests show that for stable systems (A Hurwitz), the real values h^* and δ^* can be large. On the contrary, for unstable systems, these bounds become very small. However, the use of sequential sub observers-predictors as in Najafi et al. [2013] may be a solution to relax this constraint.

9. l_g is the Lipschitz constant associated to the function g and is defined in (3.95).

10. l_u is the Lipschitz constant associated to the function u and is defined in (3.89).

4.2.3 Sequential sub observers-predictors with a known delay

As it was shown in Section 4.2.1, the sequential sub predictors allow to cope with a larger delay when the full state is known and when the delay is constant. This idea can be extended to partial state measurement and to a time-varying delay. First, the sequential sub predictors idea is applied to the partial state measurement case with a constant delay.

Constant delay

Each of the r sub systems estimates and predicts the state for \bar{h} seconds where

$$\bar{h} = \frac{h}{r}, \quad r \in \mathbb{N}^*. \quad (4.190)$$

Sequential sub observer-predictor equations read as:

$$\begin{cases} \dot{z}_1(t) = Az_1(t) + Bu(t - (r-1)\bar{h}) + g_1([Cz_1(t - \bar{h}) - y(t)]) \\ \vdots \\ \dot{z}_i(t) = Az_i(t) + Bu(t - (r-i)\bar{h}) + g_i(C[z_i(t - \bar{h}) - z_{i-1}(t)]) \\ \vdots \\ \dot{z}_r(t) = Az_r(t) + Bu(t) + g_r(C[z_r(t - \bar{h}) - z_{r-1}(t)]) \end{cases} \quad (4.191)$$

where $z_i \in \mathbb{R}^n$, $i = 1, \dots, r$.

Remark 4.2.8. *The name of the variables has been changed with respect to the work of Najafi et al. [2013]. Indeed, here $z_r(t) \rightarrow x(t+h)$ whereas it is $z_1(t)$ that tends to $x(t+h)$ in their work. The choice has been made in order to facilitate the reading of the proof. Note that the same convention is adopted in Ahmed-Ali et al. [2012], Germani et al. [2002] and Kazantzis and Wright [2005].*

Remark 4.2.9. *To compute $z_r(t)$, the initial conditions $z_i(t) = \phi_{z_i}(t)$ for $t \in [-\bar{h}, 0]$ with ϕ_{z_i} continuous functions, are required for all $i = 1, \dots, r$.*

Assuming an appropriate choice of the gains g_i , it follows that $z_r(t)$ tends to $x(t+h)$. Thus, a predictive feedback $u(z_r)$ that verifies Assumption 3.4.1 can be applied to stabilize (\mathcal{E}_h) . Reminding that r is the number of sub observers-predictors and defining the sub prediction errors

$$e_{p_1} = z_1(t - \bar{h}) - x(t) \quad (4.192)$$

and

$$e_{p_i}(t) = z_i(t - \bar{h}) - z_{i-1}(t) \quad (4.193)$$

for all $i = 2, \dots, r$, the following theorem holds.

Theorem 4.7. *Consider system (\mathcal{E}_h) , with a known and constant delay h , and assume that there exist a controller $u(x)$ and gains $g_i(e_{p_i})$ satisfying Assumptions 3.4.1 and 3.4.1 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z_r)$ with z_r computed by sequential observers-predictors (4.191). Then, there exists $r^* \in \mathbb{N}^*$ such that, provided $r > r^*$, $\|x(t)\|$ exponentially converges to zero.*

Proof. *The proof is divided into three steps:*

- *It will be shown that for all $i = 1, \dots, r$ the errors $\|e_{p_i}(t)\|$ exponentially converge to zero,*

- then it will be proved that $\|z_r(t)\|$ exponentially tends to zero
- and finally that $\|x(t)\|$ exponentially converges to zero.

Step 1:

The dynamics of the sub prediction errors \dot{e}_{p_i} can be rewritten as follows

$$\left\{ \begin{array}{l} \dot{e}_{p_1}(t) = Ae_{p_1}(t) + g_1(Ce_{p_1}(t)) - \int_{Ce_{p_1}(t-\bar{h})}^{Ce_{p_1}(t)} \frac{dg_1}{ds}(s)ds \\ \vdots \\ \dot{e}_{p_i}(t) = Ae_{p_i}(t) + g_i(Ce_{p_i}(t)) - \int_{Ce_{p_i}(t-\bar{h})}^{Ce_{p_i}(t)} \frac{dg_i}{ds}(s)ds - g_{i-1}(Ce_{p_{i-1}}(t)) \\ \vdots \\ \dot{e}_{p_r}(t) = Ae_{p_r}(t) + g_r(Ce_{p_r}(t)) - \int_{Ce_{p_r}(t-\bar{h})}^{Ce_{p_r}(t)} \frac{dg_r}{ds}(s)ds - g_{r-1}(Ce_{p_{r-1}}(t)). \end{array} \right. \quad (4.194)$$

The dynamics of e_{p_1} in (4.194) is similar to (4.100). Consequently, it can be shown by using the following Lyapunov-Krasovskii candidate function

$$V_1(t) = e_{p_1}^T(t)Qe_{p_1}(t) + \int_{t-\bar{h}}^t (\bar{h} + s - t) \|\dot{e}_{p_1}(s)\|^2 ds \quad (4.195)$$

that $\|e_{p_1}\|$ exponentially converges to zero provided that \bar{h} is sufficiently small

$$\bar{h} < h_1^* \text{ }^{11}. \quad (4.196)$$

Consequently, the recursive process is initialized. In order to prove by induction the convergence of the error $\|e_{p_i}\|$ to zero, it is supposed that $\|e_{p_{i-1}}\|$ converges exponentially to zero. Then, let

$$V_i = e_{p_i}^T(t)Qe_{p_i}(t) + \int_{t-\bar{h}}^t (\bar{h} + s - t) \|\dot{e}_{p_i}(s)\|^2 ds \quad (4.197)$$

be a Lyapunov-Krasovskii functional candidate. Similarly to (4.107) and (4.110) in the proof of Theorem 4.5, one gets

$$\begin{aligned} \dot{V}_i(t) \leq & -m_g \|e_{p_i}(t)\|^2 + R' \|e_{p_i}(t)\| \|m_i(t)\| + R' \|e_{p_i}(t)\| \|e_{p_{i-1}}(t)\| \\ & + \bar{h}c_1 \|e_{p_i}(t)\|^2 + \bar{h}c_2 \|m_i(t)\|^2 + \bar{h}c_2 \|e_{p_{i-1}}(t)\|^2 - \frac{1}{2h} \|m_i(t)\|^2 - \frac{1}{2} \int_{t-\bar{h}}^t \|\dot{e}_{p_i}(s)\|^2 ds \end{aligned} \quad (4.198)$$

with $R' = 2l_g \|Q\| \|C\|$, $\|m_i(t)\| = \int_{t-\bar{h}}^t \|\dot{e}_{p_i}(s)\| ds$, $c_1 = 3(\|A\| + l_g \|C\|)^2$ and $c_2 = 3l_g^2 \|C\|^2$.

In addition, inequality (3.102) allows to get rid of the cross terms " $\|e_{p_i}(t)\| \|m_i(t)\|$ " so

$$\begin{aligned} \dot{V}_i(t) \leq & - \left[\frac{m_g}{2} - \bar{h}c_1 \right] \|e_{p_i}(t)\|^2 + R' \|e_{p_i}(t)\| \|e_{p_{i-1}}(t)\| + \bar{h}c_2 \|e_{p_{i-1}}(t)\|^2 \\ & - \left[\frac{1}{2h} - \frac{R'^2}{2m_g} - \bar{h}c_2 \right] \|m_i(t)\|^2 - \frac{1}{2} \int_{t-\bar{h}}^t \|\dot{e}_{p_i}(s)\|^2 ds. \end{aligned} \quad (4.199)$$

11. An explicit value can be computed following similar steps as in Theorem 4.5.

Reminding that

$$-\frac{m_g}{4} \|e_{p_i}(t)\|^2 + R' \|e_{p_i}(t)\| \|e_{p_{i-1}}(t)\| = -\left(\sqrt{\frac{m_g}{4}} \|e_{p_i}(t)\| - \frac{R'}{\sqrt{m_g}}\right)^2 + \frac{R'^2}{m_g} \|e_{p_{i-1}}(t)\|^2, \quad (4.200)$$

one deduces that

$$\begin{aligned} \dot{V}_i(t) \leq & -\left[\frac{m_g}{4} - \bar{h}c_1\right] \|e_{p_i}(t)\|^2 + \left[\bar{h}c_2 + \frac{R'^2}{m_g}\right] \|e_{p_{i-1}}(t)\|^2 \\ & - \left[\frac{1}{2h} - \frac{R'^2}{2m_g} - \bar{h}c_2\right] \|m_i(t)\|^2 - \frac{1}{2} \int_{t-\bar{h}}^t \|\dot{e}_{p_i}(s)\|^2 ds. \end{aligned} \quad (4.201)$$

To prove the exponential convergence, it is sufficient to find conditions which guarantee the following inequality

$$\dot{V}_i(t) + \varepsilon V_i(t) \leq 0 \quad (4.202)$$

with $\varepsilon > 0$. From equation (4.199), Assumption 3.4.2, one has

$$\begin{aligned} \dot{V}_i(t) + \varepsilon V_i(t) \leq & -\left[\frac{m_g}{2} - \bar{h}c_1 - \varepsilon \bar{m}\right] \|e_{p_i}(t)\|^2 + \left[\bar{h}c_2 + \frac{R'^2}{2m_g}\right] \|e_{p_{i-1}}(t)\|^2 \\ & - \left[\frac{1}{2h} - \frac{R'^2}{2m_g} - \bar{h}c_2\right] \|m_i(t)\|^2 - \left[\frac{1}{2} - \varepsilon \bar{h}\right] \int_{t-\bar{h}}^t \|\dot{e}_{p_i}(s)\|^2 ds. \end{aligned} \quad (4.203)$$

It is possible to compute h_i^* such that for

$$\bar{h} < h_i^* \quad (4.204)$$

and for γ and ε sufficiently small, the coefficients of the terms $\|e_{p_i}(t)\|^2$, $\|m_i(t)\|^2$ and $\int_{t-\bar{h}}^t \|\dot{e}_{p_i}(s)\|^2 ds$ are positive (see proof of Theorem 4.5). Thus, one has

$$\dot{V}_i(t) \leq -\varepsilon V_i(t) + \left[\bar{h}c_2 + \frac{R'^2}{m_g}\right] \|e_{p_{i-1}}(t)\|^2. \quad (4.205)$$

Since $\|e_{p_{i-1}}\|$ exponentially converges to zero, the comparison lemma of Khalil [2002] guarantees the exponential convergence of $\|e_{p_i}\|$ to zero. The induction process ensures that $\|e_{p_i}\|$ for all $i = 1, \dots, r$ exponentially converges to zero provided that

$$\bar{h} < h^* = \min_{i=1, \dots, r} (h_i^*). \quad (4.206)$$

This ends the first part of the proof.

Step 2:

In order to prove the exponential convergence of $\|z_r\|$ to zero, the following Lyapunov candidate function is defined

$$V(t) = z_r^T(t) P z_r(t). \quad (4.207)$$

Taking the time derivative of $V(t)$ along the trajectories of z_r in (4.191) gives

$$\dot{V}(t) \leq -c_u \|z_r(t)\|^2 + R \|z_r(t)\| \|e_{p_r}(t)\| \quad (4.208)$$

with $R = 2l_g \|P\| \|C\|$. One can rewrite (4.208) as follows

$$\dot{V}(t) \leq -\frac{c_u}{2} \|z_r(t)\|^2 + \frac{R^2}{2c_u} \|e_{p_r}(t)\|^2 \quad (4.209)$$

and since $\|e_{p_r}(t)\|$ exponentially converges to zero, the comparison lemma from Khalil [2002] ensures the exponential convergence of $\|z_r(t)\|$ to zero.

Step 3:

The last step consists in showing the exponential convergence of $\|x(t)\|$ to zero. Let

$$e_p(t) = z_r(t - h) - x(t) \quad (4.210)$$

be the prediction error, then using the sub prediction errors e_{p_i} one gets

$$e_p(t) = \sum_{i=1}^r e_{p_i}(t - h + (r - i + 1)\bar{h}). \quad (4.211)$$

so $\|e_p(t)\|$ exponentially converges since for all $i = 1, \dots, r$ $\|e_{p_i}(t)\|$ exponentially tends to zero from step 1. The exponential convergence of $\|x(t)\|$ to zero is deduced from the result of step 2.

Finally, taking $r^* = \lfloor \frac{h}{h^*} \rfloor$ with h^* defined by (4.206) and taking $r > r^*$ lead to the exponential convergence of $\|x(t)\|$ to zero. \square

Remark 4.2.10. In previous theorems, the exponential stability in the sense of the norm Υ was shown. Here, only the exponential convergence of $\|x(t)\|$ is proven because of the induction proof and especially to the form of (4.205).

Time varying delay

An example for the case of two cascaded sub observers-predictors is given on Figure 4.14. The idea is similar to the one for the constant delay case presented before. Each of the r observers-predictors has a time-varying prediction horizon

$$\bar{h}(t) = \frac{h(t)}{r}, \quad r \in \mathbb{N}^*. \quad (4.212)$$

The equations of the sequential sub observers-predictors are given below

$$\begin{cases} \dot{z}_1(t) &= Az_1(t) + Bu(t - (r - 1)\bar{h}(t)) + g_1([Cz_1(t - \bar{h}(t)) - y(t)]) \\ \vdots & \\ \dot{z}_i(t) &= Az_i(t) + Bu(t - (r - i)\bar{h}(t)) + g_i(C[z_i(t - \bar{h}(t)) - z_{i-1}(t)]) \\ \vdots & \\ \dot{z}_r(t) &= Az_r(t) + Bu(t) + g_r(C[z_r(t - \bar{h}(t)) - z_{r-1}(t)]) \end{cases} \quad (4.213)$$

where $z_i \in \mathbb{R}^n$, $i = 1, \dots, r$.

Remark 4.2.11. Note that to compute $z_r(t)$, the initial conditions $z_i(t) = \phi_{z_i}(t)$ for $t \in [-h_{max}/r, 0]$ with ϕ_{z_i} continuous functions are required for all $i = 1, \dots, r$.

A predictive output feedback $u(z_r)$ that verifies Assumption 3.4.1 can be applied to stabilize system (\mathcal{E}_h) . We define the sub prediction errors

$$e_{p_1} = z_1(t - \bar{h}(t)) - x(t) \quad (4.214)$$

and

$$e_{p_i}(t) = z_i(t - \bar{h}(t)) - z_{i-1}(t) \quad (4.215)$$

for all $i = 2, \dots, r$. The advantage of the sequential structure is to relax the condition on the maximum delay value as it is stated in Theorem 4.8. It is reminded that δ stands for the upper bound of the delay time derivative: $|\dot{h}(t)| < \delta$ (Assumption 3.4.4).

Theorem 4.8. Consider system (\mathcal{E}_h) , where $h(t)$ is known and complies with Assumptions 3.4.3 and 3.4.4, and assume that there exist a controller $u(x)$ and gains $g_i(e_{p_i})$ satisfying Assumptions 3.4.1 and 3.4.1 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z_r)$ with z_r computed by sequential observers-predictors (4.213). Then, there exist $r^* \in \mathbb{N}^*$ and $\delta^* > 0$ such that, provided $r > r^*$ and $\delta < \delta^*$, $\|x(t)\|$ exponentially converges to zero.

The proof follows the same steps as the one of Theorem 4.7 and the computations for the first step of the proof are similar to the those of the proof of Theorem 4.6.

The results of Theorems 4.5 and 4.6 hold for a given controller u and a given correction term g . It has been shown that the stability is preserved for sufficiently small delays $h < h^*$. However, some choices of u and g can lead to very small values of h^* and then to very large values of r^* . In practice, choosing a “slower” controller and a “slower” correction term allows to reduce the number of sub observers-predictors.

Remark 4.2.12. Note that from a practical point of view, h^* can be obtained from simulation results in order not to overestimate the number of sub observers-predictors.

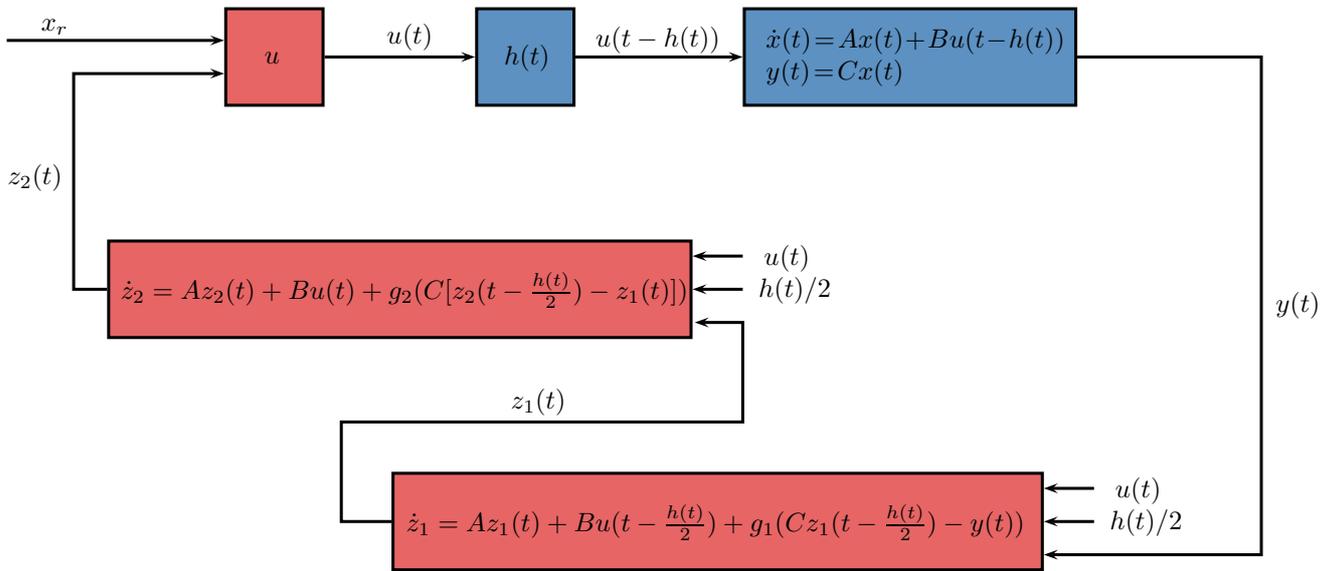


Figure 4.14 – Closed-loop scheme with two sequential sub observers-predictors (4.213) for a known and time-varying delay and partial state measurement

4.2.4 Illustrative example

The same double integrator as in Example 4.2.1 is used to illustrate above results

$$\ddot{y}(t) = u(t - h(t)) \tag{4.216}$$

with the state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t - h(t)) \tag{4.217}$$

where $x = [y, \dot{y}]^T$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The same controller as in Example 4.2.1 is chosen here

$$u(t) = Kz(t) \tag{4.218}$$

where the gain $K = [-2, -3]^T$ is tuned in order to place the eigenvalues of the closed-loop system to -1 and -2 . The Luenberger observer-predictor reads as

$$\dot{z} = Az + Bu(t) + L(Cz(t - h(t)) - y(t)) \quad (4.219)$$

with the observer-predictor gain

$$L = [-1.5, 0.5]^T \quad (4.220)$$

so the eigenvalues of $A+LC$ are -0.5 and -1 . Remark that controller (4.218) and the correction term LCE comply with Assumptions 3.4.1 and 3.4.2 respectively.

Remark 4.2.13. *In this example, the variable y has no particular physical meaning so it will be displayed without unit in the next plots.*

On Figure 4.15, the observer coupled with the standard predictor (4.57) is compared with the observer-predictor (4.219). For a fair comparison, the gain L is the same for both schemes. It can be seen that the two methods have a similar behavior. In this case the advantage of using the observer-predictor structure over the standard one is that it does not require an integral discretization so the computations are easier.

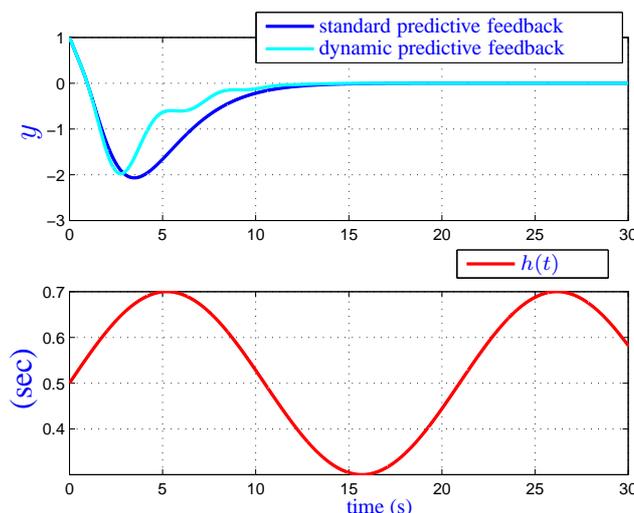
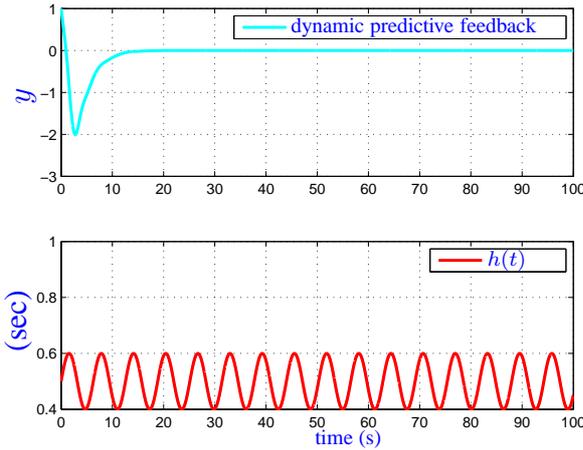


Figure 4.15 – Comparison of the standard predictive feedback (4.218)-(4.57) with the dynamic predictive feedback (4.218)-(4.219) for $h(t) = 0.5 + 0.2 \sin(0.3t)$.

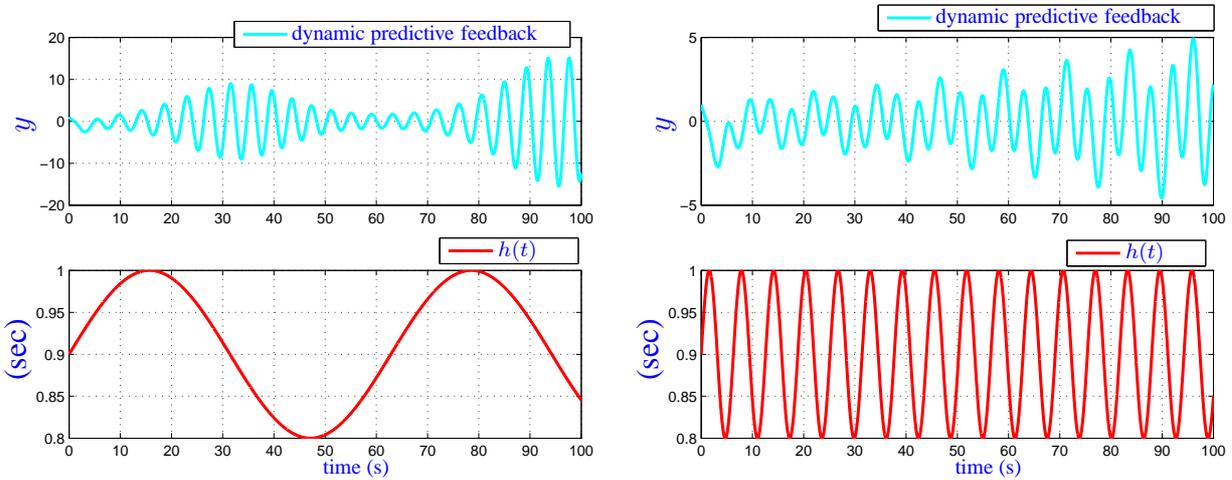
On Figure 4.16, the performances of (4.218)-(4.219) have been tested for different time-varying delays:

- $h_1(t) = 0.5 + 0.1 \sin(t)$ so $\dot{h}_1(t) = 0.1 \cos(t)$ and $\max_{t \geq 0} \dot{h}_1(t) = 0.1$ and $h_{1_{max}} = 0.6$,
- $h_2(t) = 0.9 + 0.1 \sin(0.1t)$ so $\dot{h}_2(t) = 0.01 \cos(0.1t)$ and $\max_{t \geq 0} \dot{h}_2(t) = 0.01$ and $h_{2_{max}} = 1$,
- $h_3(t) = 0.9 + 0.1 \sin(t)$ so $\dot{h}_3(t) = 0.1 \cos(t)$ and $\max_{t \geq 0} \dot{h}_3(t) = 0.1$ and $h_{3_{max}} = 1$.

On Figure 4.16a, one can observe that the system is stabilized in spite of the fast-varying delay. Figure 4.16b illustrates that the stability is particularly sensitive to large delays because it can be seen that y tends to zero only when the delay is sufficiently small. On Figure 4.16c, the system is unstable because the delay is fast-varying and add a large maximum value. As mentioned before, the solution to stabilize systems with a larger time-varying delay is to use sequential sub observers-predictors (4.213)-(4.218). On Figure 4.17, the same delay as in Figure 4.16c



(a) $h_1(t) = 0.5 + 0.1 \sin(t)$: fast-varying but small maximum value



(b) $h_2(t) = 0.9 + 0.1 \sin(0.1t)$: slow-varying but large maximum value

(c) $h_3(t) = 0.9 + 0.1 \sin(t)$: fast-varying and large maximum value

Figure 4.16 – Performance comparison of dynamic predictive feedback with a single observer-predictor (4.218)-(4.219) for different time-varying delays.

has been used. In this case, it is clear that the sequential sub observers-predictors with $r = 2$ and $L_1 = L_2 = L$ are able to stabilize the system. However, the number of sub observers-predictors cannot be increased arbitrarily since the perturbation attenuation property worsens with the number of sub structures as it is illustrated on Figure 4.18.

Indeed, a constant external perturbation d has been added such that system (4.216) becomes $\dot{y}(t) = u(t - h(t)) + d$. It can be observed that the perturbation is less attenuated for the dynamic predictive feedback with $r = 2$ than with $r = 1$. This is due to an accumulation of the observation errors in each sub observer-predictor. As a result in real applications, the number of sub observers-predictors cannot be increased arbitrarily. A new prediction that is more robust to external disturbances is presented in the next section.

4.2.5 Summary

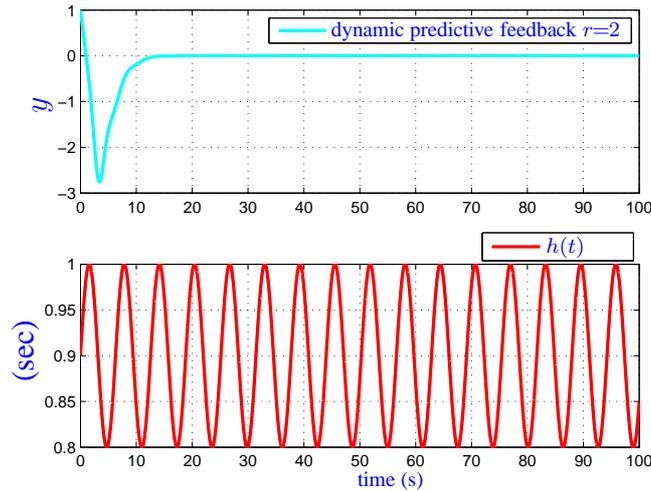


Figure 4.17 – Dynamic predictive feedback with sequential sub observers-predictors (4.213) with $r = 2$ with the same delay $h_3(t) = 0.9 + 0.1 \sin(t)$ as in Figure 4.16c.

Contributions

- ✓ Extension of the work of Najafi et al. [2013] to
 - a time-varying delay,
 - a larger class of controllers,
 - partial measurement knowledge.
- ✓ The exponential stability is proved whereas only asymptotic convergence is given in the original work.
- ✓ Explicit stability conditions are provided.

Table 4.3 sums up previous results. Note that the bounds δ^* , h^* and r^* may have different values for each case.

Prediction	Delay	Conv.	Condition	Theorem	
Dynamic (1 obs/pred)	Known	h	gl. & exp.	$h_{max} < h^*$	4.5
		$h(t)$	gl. & exp.	$h_{max} < h^* \ \& \ \delta < \delta^*$	4.6
Dynamic (r obs/pred)		h	gl. & exp.	$r > r^*$	4.7
		$h(t)$	gl. & exp.	$r > r^* \ \& \ \delta < \delta^*$	4.8

Table 4.3 – Sum up of the convergence results for dynamic prediction with a known delay

- gl.: global – exp.: exponential
- h : constant delay
- $h(t) \in [h_{min}, h_{max}]$ and $|\dot{h}(t)| \leq \delta$: time-varying delay

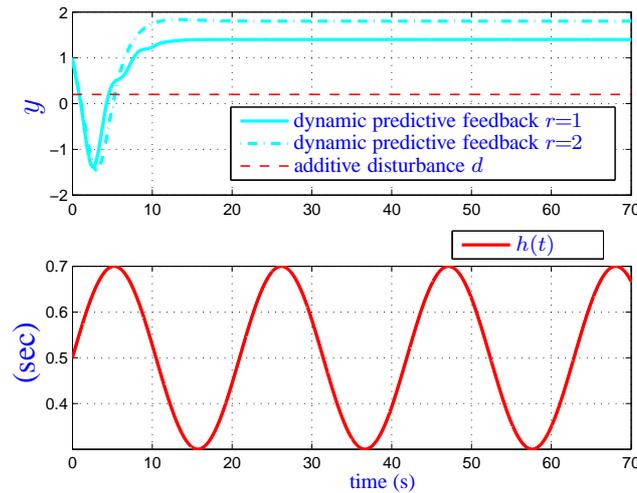


Figure 4.18 – Comparison of the disturbance attenuation of the dynamic prediction for $h(t) = 0.5 + 0.2 \sin(0.3t)$.

4.3 New prediction with a known delay

It has been shown in Section 1.2.3 that predictive control strongly relies on the model accuracy. If the model is not perfectly known or if an external disturbance appears then the standard prediction is not accurate anymore and the controller based on this prediction is less efficient. In this section, a new prediction that compensates for model uncertainties and external disturbances is presented. It has been introduced for state feedback by Léchappé et al. [2015c] and extended to output feedback by Léchappé et al. [2015d]. The accuracy of this prediction will be studied in the case of a constant delay in Section 4.3.1 and for a time-varying delay in Section 4.3.1.

In the above section, perfectly known systems (\mathcal{E}_h) defined in (3.84) have been considered. However, in many practical cases, it is very difficult to know exactly the values of A and B . Consequently, some parameter uncertainties can be taken into account as follows:

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t - h(t)). \quad (4.221)$$

Another source of model mismatch is the presence of an unknown external disturbance such as

$$\dot{x}(t) = Ax(t) + Bu(t - h(t)) + d(t). \quad (4.222)$$

Parameter uncertainties, represented by unknown matrices ΔA and ΔB are supposed to comply with the assumption below

Assumption 4.3.1. *The matrices ΔA and ΔB do not affect the controllability property of the system and*

$$\|\Delta A\| \leq c_A \quad (4.223)$$

and

$$\|\Delta B\| \leq c_B. \quad (4.224)$$

with $c_A, c_B > 0$.

The external disturbance, represented by the vector $d(t)$, is assumed to verify the following assumption

Assumption 4.3.2. d is an unknown locally integrable function such that for all $t \geq 0$,

$$\|d(t)\| \leq d_{max} < +\infty \quad (4.225)$$

and for all $t \geq h$ (for a constant delay h),

$$\|d(t) - d(t - h)\| \leq h\mathcal{D}_{max} < +\infty. \quad (4.226)$$

If the delay is time-varying, then (4.226) is replaced by

$$\|d(t) - d(t - h)\| \leq h_{max}\mathcal{D}_{max} < +\infty. \quad (4.227)$$

4.3.1 Constant delay

Since ΔA , ΔB and $d(t)$ defined in (4.221) and (4.222) are unknown, they cannot be included in the computation of the prediction. In both cases, the so-called “standard prediction” at time “ t ”, of the state of the system at time “ $t + h$ ”, reads as

$$x_{\hat{p}}(t) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds. \quad (4.228)$$

There is an inevitable prediction error because $x_{\hat{p}} \neq x(t + h)$. In order to improve the prediction accuracy even in presence of unknown terms, a so-called “new prediction” is going to be defined. The basic idea is to compare the “true” state of the system at time t , namely $x(t)$, with its approximated prediction made at time $t - h$ which is $x_{\hat{p}}(t - h)$:

Definition 4.3.1 (Léchappé et al. [2015c]). *The new prediction is defined by*

$$z(t) = x_{\hat{p}}(t) + \underbrace{x(t) - x_{\hat{p}}(t - h)}_{\text{correction term}} \quad (4.229)$$

where the standard prediction $x_{\hat{p}}$ is

$$x_{\hat{p}}(t) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds. \quad (4.230)$$

Note that when the system is perfectly known, $x_{\hat{p}}(t) = x(t + h)$ for all $t > h$, the correction term is equal to zero. When there are some model uncertainties or disturbances, this correction allows to indirectly introduce some information about unmodeled dynamics in the prediction. Figure 4.19 illustrates the method to obtain this new prediction. Note that all the controllers that use this new prediction will be referred at “new predictive controllers”. Similarly, all the controllers that use the standard prediction will be called “standard predictive controllers”.

The full state knowledge is required to compute the standard prediction (4.230) and then to compute new prediction (4.229). However, a “reconstructed new prediction” can be defined by

Definition 4.3.2 (Léchappé et al. [2015d]). *Let \hat{x} be an estimation of x . The new reconstructed prediction is defined by*

$$\hat{z}(t) = \hat{x}_{\hat{p}}(t) + \hat{x}(t) - \hat{x}_{\hat{p}}(t - h) \quad (4.231)$$

where the reconstructed standard prediction $\hat{x}_{\hat{p}}$ is

$$\hat{x}_{\hat{p}}(t) = e^{Ah}\hat{x}(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds. \quad (4.232)$$

In the next section, the advantage of these new predictions z (respectively \hat{z}) over the standard prediction $x_{\hat{p}}$ (respectively $\hat{x}_{\hat{p}}$) is illustrated.

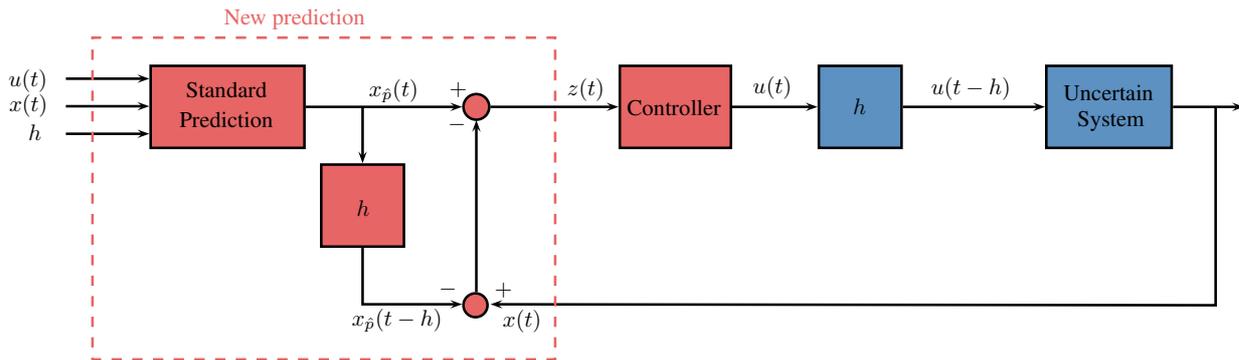


Figure 4.19 – New prediction design

Disturbance attenuation

In this section, the improvement of the new prediction in terms of attenuation of the disturbance effect is studied. The systems under consideration have the form

$$\dot{x}(t) = Ax(t) + Bu(t - h) + d(t). \quad (4.233)$$

In this case, the exact prediction is

$$x_p(t) = x(t + h) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}[Bu(s) + d(t + h)]ds \quad (4.234)$$

but cannot be computed in practice since it requires the knowledge of the disturbance. Remark that computing the exact prediction would require the knowledge of the future values of d (prediction of the perturbation). As a consequence, only the standard prediction (4.230) can be computed and the prediction error is equal to

$$x(t + h) - x_{\hat{p}}(t) = \int_{t-h}^t e^{A(t-s)}d(s + h)ds. \quad (4.235)$$

The interest of the prediction is to design a predictive controller based on the reduced system

$$\dot{x}_{\hat{p}}(t) = Ax_{\hat{p}}(t) + Bu(t) + e^{Ah}d(t). \quad (4.236)$$

Indeed, the reduced system is input delay free so standard control methods can be applied to stabilize it. However, one has the following proposition:

Proposition 4.3.1 (Léchappé et al. [2015c]). *For a constant disturbance d , the convergence of $x_{\hat{p}}$ to zero implies the convergence of x to $\left[\int_{t-h}^t e^{A(t-s)}ds \right] d$.*

Proof. *If $x_{\hat{p}}$ tends to zero, it can be deduced from (4.236) that $Bu(t)$ tends to $e^{Ah}d(t)$. As a result, substituting $Bu(t)$ by $e^{Ah}d(t)$ in (4.230) concludes the proof. \square*

In the general case, $\left[\int_{t-h}^t e^{A(t-s)}ds \right] d$ is different from zero so Proposition 4.3.1 states that even if the prediction $x_{\hat{p}}$ converges to zero, the state x will not converge to zero in presence of an external disturbance. This is due to the inaccuracy of the standard prediction that does not take into account the perturbation in its computation. Therefore, it is necessary to design

another prediction that includes some disturbance information. The new prediction computed from (4.229) can be turned into the integral form

$$z(t) = \underbrace{x_{\hat{p}}(t) + x(t) - x_{\hat{p}}(t-h)}_{\text{computation method}} = \underbrace{e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)} [Bu(s) + d(s)] ds}_{\text{integral form}}. \quad (4.237)$$

It is important to note that the disturbance appears in the integral form (4.237). The addition of the correction term in (4.229) allows to add some disturbance information in the prediction. The only difference between (4.237) and the exact prediction (4.234) is that “ $d(s)$ ” appears instead of “ $d(s+h)$ ” in the integral so $z(t) \neq x(t+h)$ but it is generally more accurate than $x_{\hat{p}}(t)$ and the prediction error is equal to

$$x(t+h) - z(t) = \int_{t-h}^t e^{A(t-s)} [d(s+h) - d(s)] ds. \quad (4.238)$$

A substitution can be made with this new prediction and the reduced system

$$\dot{z}(t) = Az(t) + Bu(t) + d(t) + e^{Ah} [d(t) - d(t-h)] \quad (4.239)$$

is obtained. Applying a predictive controller to stabilize $z(t)$ in (4.239) leads to the following proposition

Proposition 4.3.2 (Léchappé et al. [2015c]). *For a constant disturbance d , the convergence of z to zero implies the convergence of x to zero.*

Proof. *If $x_{\hat{p}}$ tends to zero, it can be deduced from (4.239) that $Bu(t)$ tends to $d(t) + e^{Ah}[d(t) - d(t-h)]$. Since d is constant $d(t) - d(t-h) = 0$, substituting $Bu(t)$ by $d(t)$ in (4.237) concludes the proof. \square*

For constant disturbances, the convergence of the new prediction to zero implies the convergence of x to zero. In the case of time-varying disturbances, the exact convergence to zero is not possible anymore (even without delay) but Theorem 4.9, that will be given in the sequel, shows that designing a controller from the new prediction induces a better attenuation than designing it from the standard prediction. The details are given thereafter.

Assume that $u(x_{\hat{p}})$ verifies Assumption 3.4.1 thus system (4.236) with $d(t) = 0$ is globally exponentially stable. In addition, the perturbation of system (4.236) is bounded and the following maximization holds

$$\|e^{Ah}d(t)\| \leq \|e^{Ah}\|d_{max}, \quad \forall t \geq 0. \quad (4.240)$$

Therefore, the assumptions of Lemma 9.4 in Khalil [2002] are fulfilled (equations (3.88) and (4.240)) so one deduces that for all $t \geq 0$

$$\|x_{\hat{p}}(t)\| \leq \beta \|x(0)\| e^{-\alpha t} + l_1 \|e^{Ah}\| d_{max} \quad (4.241)$$

with α , β and l_1 positive constants. Since (4.236) and (4.239) have the same form when $d(t) = 0$, the same controller $u(z)$ guarantees that $z = 0$ is a globally exponentially stable equilibrium point of the nominal system (4.239) with $d(t) = 0$. From Assumption 4.3.2, the inequality

$$\left\| d(t) + e^{Ah} [d(t) - d(t-h)] \right\| \leq d_{max} + h \|e^{Ah}\| \mathcal{D}_{max} \quad (4.242)$$

is verified for all $t \geq h$. Similarly to (4.241), the following inequality is obtained

$$\|z(t)\| \leq \beta \|x(0)\| e^{-\alpha t} + l_1 \left[d_{max} + h \|e^{Ah}\| \mathcal{D}_{max} \right] \quad (4.243)$$

for all $t \geq h$. The constants α , β and l_1 are the same as in (4.241) because they only depend on the form of the undisturbed system. As it has been mentioned before, systems (4.236) and (4.239) have the same representation $\dot{\chi} = A\chi + Bu(t)$ when there is no perturbation. From the evaluation of (4.235) and (4.238) in “ $t - h$ ”, it follows that

$$\|x(t)\| \leq \|x_{\hat{p}}(t - h)\| + l_2 d_{max} \quad (4.244)$$

and

$$\|x(t)\| \leq \|z(t - h)\| + l_2 h \mathcal{D}_{max} \quad (4.245)$$

with $l_2 = \left\| \int_{-h}^0 e^{As} ds \right\|$. As a result, if system (4.233) is controlled by the feedback $u(x_{\hat{p}})$, the inequality

$$\|x(t)\| \leq \beta \|x(0)\| e^{\alpha h} e^{-\alpha t} + \underbrace{\left[l_2 + l_1 \|e^{Ah}\| \right]}_{r_1} d_{max} \quad (4.246)$$

holds and, if system (4.233) is controlled by the feedback $u(z)$, the inequality

$$\|x(t)\| \leq \beta \|x(0)\| e^{\alpha h} e^{-\alpha t} + \underbrace{l_1 d_{max} + \left[l_2 + l_1 \|e^{Ah}\| \right] h \mathcal{D}_{max}}_{r_2} \quad (4.247)$$

is verified. This proves the following theorem.

Theorem 4.9 (Léchappé et al. [2015c]). *Consider system (4.233) and assume that there exists a controller $u(x)$ satisfying Assumption 3.4.1. Suppose that predictor-controllers $u(z)$ with z defined by (4.229) and $u(x_{\hat{p}})$ with $x_{\hat{p}}$ defined by (4.230) result in uniform ultimate bounds respectively r_1 in (4.246) and r_2 in (4.247). Then, there exists \mathcal{D}^* such that, provided $\mathcal{D}_{max} < \mathcal{D}^*$, the uniform ultimate bounds verify $r_2 < r_1$.*

Theorem 4.9 shows that a feedback controller with the new prediction leads to a smaller uniform ultimate bound than designing a controller with the standard prediction for a sufficiently slow-varying disturbance. Consequently, the new predictive scheme is said to better attenuate the slow-varying disturbances than the standard one. In this case of a constant delay, for systems such that $\|e^{Ah}\| > 1$, an explicit value of \mathcal{D}^* is given by

$$\mathcal{D}^* = \frac{d_{max} (\|e^{Ah}\| - 1)}{h \|e^{Ah}\|}. \quad (4.248)$$

Note that some additional assumptions on the eigenvalues of A would be necessary to determine whether or not, equation (4.248) is fulfilled for a given h . Besides, unstable systems are usually such that $\|e^{Ah}\| \gg 1$ so (4.248) can be approximated by

$$\frac{d_{max}}{\mathcal{D}_{max}} > h. \quad (4.249)$$

In this case, the attenuation only depends on the dynamics of the perturbation with respect to the delay size. However, the converse of Theorem 4.9 is not true; even if the perturbation does not comply with (4.248), a controller using the new prediction can better attenuate the perturbation but it is not guaranteed. This result can be extended to the partial state knowledge case by using the reconstructed predictions (Definition 4.3.2) to compute feedbacks $u(\hat{x}_{\hat{p}}(t))$ and $u(\hat{z}(t))$. Then, Theorem 4.9 can be rewritten as follows with $e(t) = \hat{x}(t) - x(t)$.

Theorem 4.10 (Léchappé et al. [2015d]). Consider system (4.233), observer (\mathcal{O}) in (3.90) and assume that there exist a controller $u(x)$ and a correction term $g(e)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that predictor-controllers $u(\hat{z})$ with \hat{z} defined by (4.231) and $u(\hat{x}_{\hat{p}})$ with $\hat{x}_{\hat{p}}$ defined by (4.232) result in uniform ultimate bounds respectively \hat{r}_1 and \hat{r}_2 . Then, there exists \mathcal{D}^* such that provided $\mathcal{D}_{max} < \mathcal{D}^*$, the uniform ultimate bounds verify $\hat{r}_2 < \hat{r}_1$.

The proof is very similar to the one of Theorem 4.9 and is given in Léchappé et al. [2015d] for observer (\mathcal{O}) in (3.90) being a Luenberger observer. \square

The results of this section are gathered in Table 4.4 for a particular controller $u(x) = Kx$. In order to compare the influence of the delay on attenuation property, the delay free case has been added too.

Remark 4.3.1. The assumption u globally Lipschitz is not used in this section, the local Lipschitz property is sufficient here. As a consequence, the results hold for a large class of controllers. This gives the possibility to robustly stabilize z at zero. However, even if z converges to zero, there is an inevitable error, $\eta h \mathcal{D}_{max}$ in (4.245), independent from the control and that cannot be reduced for time-varying disturbances.

Control	Without delay	With a constant and known delay	
		Standard prediction:	New prediction:
State Feedback	$\begin{cases} \dot{x} = Ax + Bu(t) + d(t) \\ \text{with } u(t) = Kx(t) \Rightarrow \\ \begin{cases} \dot{x} = (A+BK)x + d(t) \\ r_0 = d_{max}l_1 \end{cases} \end{cases}$	$\begin{cases} \dot{x} = Ax + Bu(t-h) + d(t) \\ \dot{\hat{x}}_p = A\hat{x}_p + Bu(t) + e^{Ah}d(t) \\ \text{with } u(t) = K\hat{x}_p(t) \Rightarrow \\ \begin{cases} \dot{\hat{x}}_p = (A+BK)\hat{x}_p + e^{Ah}d(t) \\ r_1 = d_{max}(l_1 e^{Ah} + l_2) \end{cases} \end{cases}$	$\begin{cases} \dot{x} = Ax + Bu(t-h) + d(t) \\ \dot{z} = Az + Bu(t) + d(t) + e^{Ah}[d(t) - d(t-h)] \\ \text{with } u(t) = Kz(t) \Rightarrow \\ \begin{cases} \dot{z} = (A+BK)z + d(t) + e^{Ah}[d(t) - d(t-h)] \\ r_2 = d_{max}l_1 + \mathcal{D}_{max}h(l_1 e^{Ah} + l_2) \end{cases} \end{cases}$
Output Feedback	$\begin{cases} \dot{x} = Ax + Bu(t) + d(t) \\ \dot{\hat{x}} = A\hat{x} + Bu(t) + L(C\hat{x} - y) \\ \text{with } u(t) = K\hat{x}(t) \Rightarrow \\ \begin{cases} \dot{e} = (A+LC)e - d(t) \\ \dot{x} = (A+BK)x + BK e(t) + d(t) \\ \hat{r}_0 = d_{max}l_1(1 + BK l_3) \end{cases} \end{cases}$	$\begin{cases} \dot{x} = Ax + Bu(t-h) + d(t) \\ \dot{\hat{x}} = A\hat{x} + Bu(t-h) + L(C\hat{x} - y) \\ \dot{\hat{x}}_p = A\hat{x}_p + Bu(t) + e^{Ah}L C e(t) \\ \text{with } u(t) = K\hat{x}_p(t) \Rightarrow \\ \begin{cases} \dot{e}(t) = (A+LC)e(t) - d(t) \\ \dot{\hat{x}}_p(t) = (A+BK)\hat{x}_p(t) + e^{Ah}L C e(t) \\ \hat{r}_1 = d_{max}(l_3l_4 e^{Ah} + l_2) \end{cases} \end{cases}$	$\begin{cases} \dot{x} = Ax + Bu(t-h) + d(t) \\ \dot{\hat{x}} = A\hat{x} + Bu(t-h) + L(C\hat{x} - y) \\ \dot{\hat{z}} = A\hat{z} + Bu(t) + L C e(t) + e^{Ah}L C [e(t) - e(t-h)] \\ \text{with } u(t) = K\hat{z}(t) \Rightarrow \\ \begin{cases} \dot{e}(t) = (A+LC)e(t) - d(t) \\ \dot{\hat{z}} = (A+BK)\hat{z} + L C e(t) + e^{Ah}L C [e(t) - e(t-h)] \\ \hat{r}_2 = d_{max}l_3l_4 + h\mathcal{D}_{max}(l_3l_4 e^{Ah} + l_2) \end{cases} \end{cases}$

Table 4.4 – Disturbance attenuation of the standard and new predictive schemes. The delay free case is added as a comparison reference.

- $e(t) = \hat{x}(t) - x(t)$
- $l_1, l_2, l_3, l_4 > 0$ only depends on matrices A, B, C, K, L
- r_i and \hat{r}_i are the convergence radius

Parameter uncertainties

In this section, the improvement of the new prediction with respect to the presence of parameter uncertainties is worked out for constant reference tracking. In this section, it will be shown that a controller using the standard prediction ($x_{\hat{p}}$) is not able to guarantee that x converges to x_r in presence of parameter uncertainties whereas it is possible with a controller using the new prediction (z). The systems under consideration have the form

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t - h) \quad (4.250)$$

and the constant reference x_r verifies

$$0 = Ax_r + Bu_r. \quad (4.251)$$

The control u_r is the constant input required to maintain x to the equilibrium point x_r . In this case, the exact prediction is

$$x_p(t) = x(t + h) = e^{(A + \Delta A)h}x(t) + \int_{t-h}^t e^{(A + \Delta A)(t-s)}(B + \Delta B)u(s)ds \quad (4.252)$$

but cannot be computed in practice since it requires the knowledge of ΔA and ΔB . As a consequence, only the standard prediction $x_{\hat{p}}$

$$x_{\hat{p}}(t) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds \quad (4.253)$$

can be computed in practice. The reduced system obtained from $x_{\hat{p}}$ reads as

$$\dot{x}_{\hat{p}}(t) = (A + \Delta A)x_{\hat{p}}(t) + Bu(t) + \phi_1(t) \quad (4.254)$$

with

$$\phi_1(t) = -\Delta A \int_{t-h}^t e^{A(t-s)}Bu(s)ds + e^{Ah}\Delta Bu(t - h). \quad (4.255)$$

The term ϕ_1 can be seen as a perturbation of the delay free system

$$\dot{x}_{\hat{p}}(t) = (A + \Delta A)x_{\hat{p}}(t) + Bu(t). \quad (4.256)$$

Since system (4.256) is not retarded anymore, all the controllers available for delay free systems can be used to stabilize $x_{\hat{p}}$ around x_r . In particular, PID or sliding mode algorithms can be designed to reject the ‘‘perturbation’’ ϕ_1 and make $x_{\hat{p}}(t)$ tend to x_r . With $x_{\hat{p}}$ converging to x_r , it is expected that $x(t)$ converges to x_r . However, $x_{\hat{p}}$ is not the exact prediction of x because it does not take into account the perturbation; so there is an unavoidable error when $\Delta A \neq 0$ or $\Delta B \neq 0$. The consequence of this error is stated in the next proposition where $\tilde{x} = x_{\hat{p}} - x_r$:

Proposition 4.3.3 (Léchappé et al. [2015e]). *Consider system (4.250) and suppose that there exists a predictive controller $u(\tilde{x})$ continuous at the origin such that $\tilde{x}(t)$ and its time-derivative tend to zero. Then, the convergence of the state $x(t)$ to the reference x_r is not ensured.*

Proof. The dynamics of \tilde{x} is governed by

$$\dot{\tilde{x}}(t) = (A + \Delta A)\tilde{x}(t) + \Delta Ax_r + Bu(t) - Bu_r + \phi_1(t). \quad (4.257)$$

Assuming that there exists a controller continuous at the origin¹² such that \tilde{x} converges to zero, then, one has $\lim_{t \rightarrow +\infty} u(\tilde{x}(t)) = u(\lim_{t \rightarrow +\infty} \tilde{x}(t)) = u(\tilde{x} = 0) = u_\infty$. Passing to the limit in (4.257), it can be deduced that

$$\begin{aligned} Bu_\infty &= -\Delta Ax_r + Bu_r - e^{Ah}\Delta Bu_\infty + \left[\int_{-h}^0 e^{A(t-s)} ds \right] Bu_\infty \\ &= Bu_r + \Phi \end{aligned} \quad (4.258)$$

with $\lim_{t \rightarrow +\infty} \|u(t) - u_\infty\| = 0$ and $\Phi = -\Delta Ax_r - e^{Ah}\Delta Bu_\infty + \int_{-h}^0 e^{A(t-s)} ds Bu_\infty$. Reminding that $Ax_r = -Bu_r$, equality (4.258) becomes

$$Bu_\infty = -Ax_r + \Phi. \quad (4.259)$$

Besides, from (4.253), it follows that

$$x_r = e^{Ah}x_\infty + \left[\int_{-h}^0 e^{-As} ds \right] Bu_\infty \quad (4.260)$$

with $\lim_{t \rightarrow +\infty} \|x(t) - x_\infty\| = 0$. Combining (4.259) and (4.260) and noting that A and e^{-As} commute and that $-\int_{-h}^0 Ae^{-As} ds = I_n - e^{Ah}$, one gets

$$0 = e^{Ah}(x_\infty - x_r) + \left[\int_{-h}^0 e^{-As} ds \right] \Phi. \quad (4.261)$$

For the systems such that $-\left[\int_{-h}^0 e^{-As} ds \right] \Phi \neq 0$ for all ΔA and ΔB , one obtains $x_\infty \neq x_r$.

This ends the proof. \square

When a predictive controller is designed using the standard prediction $x_{\hat{p}}$, it is not possible to drive the state x to the reference x_r due to modeling errors. On the contrary, using the new prediction (4.229), one has the reduced system

$$\dot{z}(t) = (A + \Delta A)z(t) + Bu(t) + \phi_2(t) \quad (4.262)$$

with

$$\phi_2(t) = \Delta Bu(t-h) + \phi_1(t) - \phi_1(t-h). \quad (4.263)$$

Similarly to (4.254), the term ϕ_2 can be seen as a perturbation of the delay free system

$$\dot{z}(t) = (A + \Delta A)z(t) + Bu(t). \quad (4.264)$$

The consequence of this new prediction on the tracking error accuracy is given in Proposition 4.3.4 where $\tilde{X} = X_{\hat{p}} - x_r$.

12. If u is not continuous at the origin but one has $\tilde{x}(t) = 0$ for all $t > T$, then it follows that $u(\tilde{x}(t)) = u(\tilde{x} = 0)$ for all $t > T$.

Proposition 4.3.4 (Léchappé et al. [2015e]). *Consider system (4.250) and suppose that there exists a predictive controller $u(\tilde{X})$ continuous at the origin such that $\tilde{X}(t)$ and its time-derivative tend to zero. Then, the state $x(t)$ converges to the reference x_r .*

Proof. *The tracking error dynamics is governed by*

$$\dot{\tilde{z}}(t) = (A + \Delta A)\tilde{z}(t) + \Delta Ax_r + Bu(t) - Bu_r + \phi_2(t). \quad (4.265)$$

Assuming that there exists a controller such that \tilde{z} converges to zero, it can be deduced from (4.265) that

$$(B + \Delta B)u_\infty = Bu_r - \Delta Ax_r \quad (4.266)$$

with $\lim_{t \rightarrow +\infty} \|u(t) - u_\infty\| = 0$ (see the proof of proposition 4.3.3 for the existence of the limit). Reminding that $Ax_r = -Bu_r$, equality (4.266) becomes

$$(B + \Delta B)u_\infty = -(A + \Delta A)x_r. \quad (4.267)$$

Besides, from the definition of the new prediction (4.229), it follows that

$$\begin{aligned} z(t) &= e^{Ah}[x(t) - x(t-h)] + e^{(A+\Delta A)h}x(t-h) \\ &\quad + \int_{t-h}^t e^{A(t-s)}B[u(s) - u(s-h)]ds \\ &\quad + \int_{-h}^0 e^{-(A+\Delta A)s}(B + \Delta B)u(t+s-h)ds \end{aligned} \quad (4.268)$$

so if \tilde{z} converges to zero, one has

$$x_r = e^{(A+\Delta A)h}x_\infty + \left[\int_{-h}^0 e^{-(A+\Delta A)s}ds \right] (B + \Delta B)u_\infty \quad (4.269)$$

with $\lim_{t \rightarrow +\infty} \|x(t) - x_\infty\| = 0$. Combining (4.267) and (4.269) and noting that $A + \Delta A$ and $e^{-(A+\Delta A)s}$ commute and that $-\int_{-h}^0 (A + \Delta A)e^{-(A+\Delta A)s}ds = I_n - e^{(A+\Delta A)h}$ leads to

$$0 = e^{(A+\Delta A)h}[x_\infty - x_r]. \quad (4.270)$$

Since $e^{(A+\Delta A)h} \neq 0$ for all ΔA and ΔB , $x_\infty = x_r$. This ends the proof. \square

The consequence of this proposition is that, if it is possible to find a robust controller that makes z tend to x_r , then x will tend to x_r in spite of parameter uncertainties. As a result, the problem of robustness with respect to model uncertainties for input delay systems is reduced to a problem of robustness versus model uncertainties for delay free systems. Note that the uncertainties ΔA and ΔB should be sufficiently small¹³ to be able to find a predictive controller ensuring that $z(t)$ tends to x_r .

As in the disturbance attenuation case, this result can be easily extended to the partial measurement case using the reconstructed standard prediction and the reconstructed new prediction (Definition 4.3.2). Similarly to Propositions 4.3.3 and 4.3.4, the following results hold with $e(t) = \hat{x}(t) - x(t)$.

13. c_A and c_B in Assumption 4.3.1 has to be sufficiently small.

Proposition 4.3.5. Consider system (4.250) and observer (O) in (3.90), with a correction term $g(e)$ that verifies Assumption 3.4.2, in closed-loop with a predictive controller such that the standard prediction $\hat{x}_{\hat{p}}$ converges to the reference x_r . Then, the convergence of the state x to the reference x_r is not ensured.

The proof is similar to the one of Proposition 4.3.3.

Proposition 4.3.6. Consider system (4.250) and observer (O) in (3.90), with a correction term $g(e)$ that verifies Assumption 3.4.2, in closed-loop with a predictive controller such that the new prediction \hat{z} converges to the reference x_r . Then, the state x converges to the reference x_r .

The proof is similar to the one of Proposition 4.3.4.

To summarize, in this section, some properties of the new prediction have been highlighted for a constant and known delay. In the next section, the performance of the new prediction in presence of a known and time-varying delay are analyzed.

4.3.2 Time-varying delay

Similarly to Definitions 4.3.1 and 4.3.2, the new prediction can also be defined for a time-varying delay.

Definition 4.3.3. The new prediction is defined by

$$z(t) = x_{\hat{p}}(t) + \underbrace{x(t) - x_{\hat{p}}(t - h(t))}_{\text{correction term}} \quad (4.271)$$

where the standard prediction $x_{\hat{p}}$ is

$$x_{\hat{p}}(t) = e^{Ah(t)}x(t) + \int_{t-h(t)}^t e^{A(t-s)}Bu(s)ds. \quad (4.272)$$

Note that even if the system is perfectly known, the correction term is never equal to zero because of the time-varying delay (Section 4.1.2). For partial state knowledge, one has the following definition.

Definition 4.3.4. The reconstructed new prediction reads as

$$\hat{z}(t) = \hat{x}_{\hat{p}}(t) + \hat{x}(t) - \hat{x}_{\hat{p}}(t - h(t)) \quad (4.273)$$

where the reconstructed standard prediction $\hat{x}_{\hat{p}}$ is

$$\hat{x}_{\hat{p}}(t) = e^{Ah(t)}\hat{x}(t) + \int_{t-h(t)}^t e^{A(t-s)}Bu(s)ds. \quad (4.274)$$

Disturbance attenuation

Let us consider disturbed LTI systems with a known time-varying delay as follows

$$\dot{x}(t) = Ax(t) + Bu(t - h(t)) + d(t). \quad (4.275)$$

In this case, the reduced system with prediction (4.274) reads as

$$\dot{\hat{x}}_{\hat{p}}(t) = A\hat{x}_{\hat{p}}(t) + Bu(t) + e^{Ah}d(t) + \dot{h}(t)\psi_1(t) \quad (4.276)$$

with $\psi_1(t) = Ax_{\hat{p}}(t) + e^{Ah}Bu(t-h) - A \int_{t-h}^t e^{A(t-s)}Bu(s)ds$. For a time-varying delay, Proposition 4.3.2 does not hold anymore. Indeed, the time-varying delay prevents the exact convergence of $x_{\hat{p}}(t)$ to zero; only the convergence in a ball around zero is achievable even for a constant disturbance. However, Theorem 4.9 can be extended to the time-varying case. Let z be the new prediction defined in (4.271), thus the reduced system is

$$\begin{aligned} \dot{z}(t) = & Az(t) + Bu(t) + \underbrace{d(t) + e^{Ah(t)}d(t) - e^{Ah(t-h)}d(t-h) + \dot{h}e^{Ah(t-h)}d(t-h)}_{\text{due to the external disturbance}} \\ & + \underbrace{\dot{h}\psi_1(t) - (1-\dot{h})\dot{h}(t-h)\psi_1(t-h) + \dot{h}Bu(t-h) + \dot{h}Ax_{\hat{p}}(t-h)}_{\text{due to inexact prediction for a time-varying delay}}. \end{aligned} \quad (4.277)$$

It can be observed that in the slow-varying delay case $|\dot{h}(t)| \ll 1$, the reduced system (4.276) (respectively (4.277)) tends to the reduced system for constant a delay (4.236) (respectively (4.239)). Then, similarly to Theorem 4.9, the following result holds

Theorem 4.11. *Consider system (4.275) and assume that there exists a controller $u(x)$ satisfying Assumption 3.4.1. Suppose that predictor-controllers $u(z)$ with z defined by (4.271) and $u(x_{\hat{p}})$ with $x_{\hat{p}}$ defined by (4.272) result in uniform ultimate bounds respectively r_1 and r_2 . Then, there exist δ^*, \mathcal{D}^* such that, provided $\delta < \delta^*$ and $\mathcal{D}_{max} < \mathcal{D}^*$, the uniform ultimate bounds verify $r_2 < r_1$.*

Proof. Only a sketch of proof is given because it is similar to those before. The proof is divided into two steps. First, it is shown that system (\mathcal{E}_h) is globally exponentially stable with the new predictive feedback $u(z)$ when there is no disturbance ($d(t) = 0$) and provided that $\delta < \delta^*$. The proof of this part is similar to the one of Theorem 4.3. Then, in presence of an external disturbance, the ultimate bounds are compared as in Theorem 4.9. \square

Theorem 4.11 shows that a feedback controller with the new prediction leads to a smaller uniform ultimate bound than a controller with the standard prediction for a sufficiently slow-varying disturbance and a slow-varying delay. The condition on \mathcal{D}_{max} is related to the delay dynamics \dot{h} and no simple expression can be computed explicitly.

Remark 4.3.2. *The global Lipschitz assumption of u is needed to show the exponential stability of system (\mathcal{E}_h) with the new predictive feedback $u(z)$.*

This result can be extended to the partial state knowledge case by using the reconstructed predictions $\hat{x}_{\hat{p}}$ and \hat{z} (Definition 4.3.4) to compute design the controllers. Then, Theorem 4.11 can be rewritten as follows with $e(t) = \hat{x}(t) - x(t)$.

Theorem 4.12. *Consider system (4.275), observer (\mathcal{O}) in (3.90) and assume that there exist a controller $u(x)$ and a correction term $g(e)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that predictor-controllers $u(\hat{z})$ with \hat{z} defined by (4.273) and $u(\hat{x}_{\hat{p}})$ with $\hat{x}_{\hat{p}}$ defined by (4.274) result in uniform ultimate bounds respectively \hat{r}_1 and \hat{r}_2 . Then, there exist δ^*, \mathcal{D}^* such that provided $\delta < \delta^*$ and $\mathcal{D}_{max} < \mathcal{D}^*$, the uniform ultimate bounds verify $\hat{r}_2 < \hat{r}_1$.*

The proof is divided into two steps as in Theorem 4.11 and uses the same argument as in Theorems 4.4 and 4.10.

Parameter uncertainties

The systems under consideration have the form

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t - h(t)) \quad (4.278)$$

and the constant tracking reference x_r verifies

$$0 = Ax_r + Bu_r. \quad (4.279)$$

In this case, the exact prediction is

$$x_p(t) = x(t+h) = e^{(A+\Delta A)h(t)}x(t) + \int_t^{t+h(t)} e^{(A+\Delta A)(t+h(t)-s)}(B + \Delta B)u(s-h(s))ds. \quad (4.280)$$

but cannot be computed without the knowledge of ΔA and ΔB . As a consequence, the standard prediction $x_{\hat{p}}$ is often used in practice

$$x_{\hat{p}}(t) = e^{Ah}x(t) + \int_{t-h}^t e^{A(t-s)}Bu(s)ds. \quad (4.281)$$

The new prediction computed from (4.271) has the following integral expression

$$\begin{aligned} z(t) &= e^{Ah(t)}x(t) + \int_{t-h(t)}^t e^{A(t-s)}Bu(s)ds + [e^{(A+\Delta A)h(t)} - e^{Ah(t)}]x(t-h(t)) \\ &\quad + \int_{t-h(t)}^t [e^{(A+\Delta A)(t-s)}(B + \Delta B) - e^{A(t-s)}B]u(s-h(s))ds. \end{aligned} \quad (4.282)$$

It is clear that it includes some information about the uncertainties ΔA and ΔB . However, for a time-varying delay, asymptotic tracking is not possible anymore even with the new prediction: Proposition 4.3.4 is not satisfied anymore. Only the convergence in a ball around the constant reference is possible in presence of parameter uncertainties (and a non zero reference). It has been noted in simulation that the radius of the ball is smaller for controllers of the form $u(z(t))$ than for $u(x_{\hat{p}}(t))$. Nevertheless, no formal results is available.

4.3.3 Illustrative example

The same system as in Example 4.2.1 is used to illustrate above results

$$\ddot{y}(t) = u(t - h(t)) \quad (4.283)$$

and its state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t - h(t)) \quad (4.284)$$

with $x = [y, \dot{y}]^T$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Remark 4.3.3. *In this example, the whole state x is assumed to be measurable.*

Note that the components of a vector are denoted by the subscript “i”. For instance, one has $x_{\hat{p}} = [x_{\hat{p}1}, x_{\hat{p}2}]^T$. Two PID controllers have been designed. The first one, based on the standard prediction $x_{\hat{p}}(t)$

$$u(t) = k_p x_{\hat{p}1}(t) + k_d \dot{x}_{\hat{p}1}(t) + k_i \int_0^t x_{\hat{p}1}(s) ds, \quad (4.285)$$

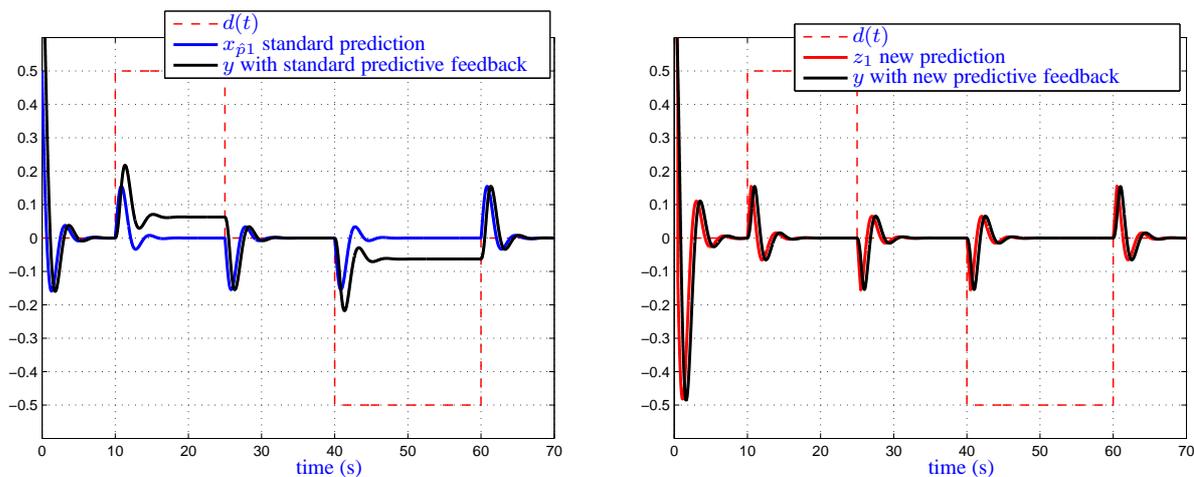
and the second one based on the new prediction $z(t)$

$$u(t) = k_p z_1(t) + k_d \dot{z}_1(t) + k_i \int_0^t z_1(s) ds. \quad (4.286)$$

The gains for both controllers and for all the simulations are $k_p = -8$, $k_d = -4.5$, $k_i = -10$. On Figure 4.20, a piecewise constant disturbance and a constant delay are applied to system (4.284).

Remark 4.3.4. *In this example, the variable y has no particular physical meaning so it will be displayed without unit.*

On both figures, the predictions $x_{\hat{p}}$ and z are able to perfectly reject the disturbance. However, only the state from the system controlled by the new predictive scheme is stabilized to 0 (Figure 4.20b). This illustrates Propositions 4.3.1 and 4.3.2. Indeed, $x_{\hat{p}} \rightarrow 0$ does not imply $x \rightarrow 0$ for disturbed systems. On the contrary, in the case of a constant disturbance, $z \rightarrow 0$ implies $x \rightarrow 0$.



(a) Standard predictive controller (4.285)

(b) New predictive controller (4.286)

Figure 4.20 – Comparison of predictive schemes for a piecewise constant disturbance and a constant delay $h = 0.5$ s

Figure 4.21 confirms the result of Theorem 4.9 since it can be observed that the time-varying perturbation is more attenuated for the system controlled by the PID with new prediction (4.286).

The influence of a time-varying delay on the disturbance attenuation is shown on Figure 4.22 for piecewise constant and time-varying disturbances. On both Figures 4.22a and 4.22b, it is clear that the new predictive controller achieves a better disturbance attenuation than the standard predictive controller. In addition, note that the exact rejection is not possible anymore even for a constant disturbance as on Figure 4.20b. Also remark that the delay size influences the magnitude of the attenuation.

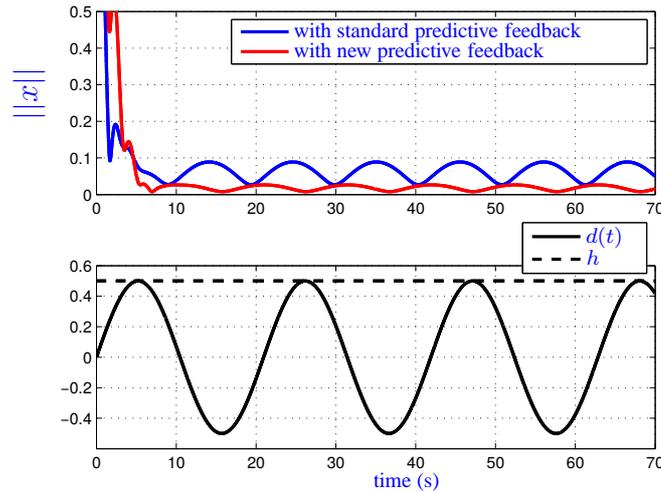
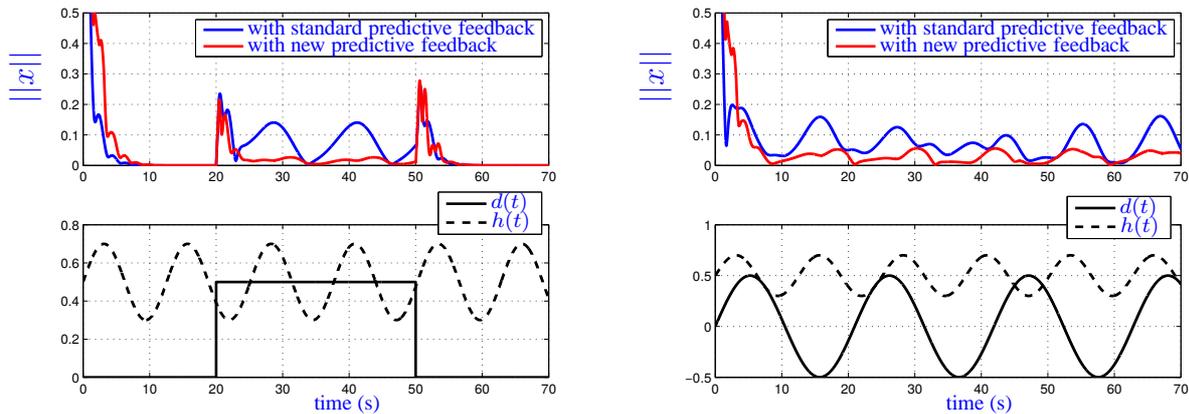


Figure 4.21 – Comparison of predictive schemes for a time-varying disturbance $d(t) = 0.5 \sin(0.3)$ and a constant delay $h = 0.5$ s.



(a) Piecewise constant perturbation d

(b) Time-varying perturbation $d(t) = 0.5 \sin(0.3)$

Figure 4.22 – Comparison of predictive schemes for a time-varying delay $h(t) = 0.5 + 0.2 \sin(0.5t)$ s

In order to test the robustness of the different methods to model accuracy, some parameter uncertainties ΔA and ΔB have been added to model (4.284):

$$\Delta A = \begin{bmatrix} 0 & 0.1 \\ 0.15 & 0 \end{bmatrix} \tag{4.287}$$

and

$$\Delta B = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}. \tag{4.288}$$

In the sequel, a constant reference $x_r = [3, 0]^T$ has to be tracked. On Figures 4.23a and 4.23b, it can be seen that both predictions track perfectly the reference. However, only the use of the new predictive scheme makes the state of the system converge to the desired trajectory (Figure 4.23b). When the standard predictive scheme is used, a constant tracking error appears (Figure 4.23a). This phenomenon is in accordance with Propositions 4.3.3 and 4.3.4.

Figure 4.24 shows the tracking performance of the two controllers in presence of a time-varying delay and parameter uncertainties. As above results, the new predictive controller

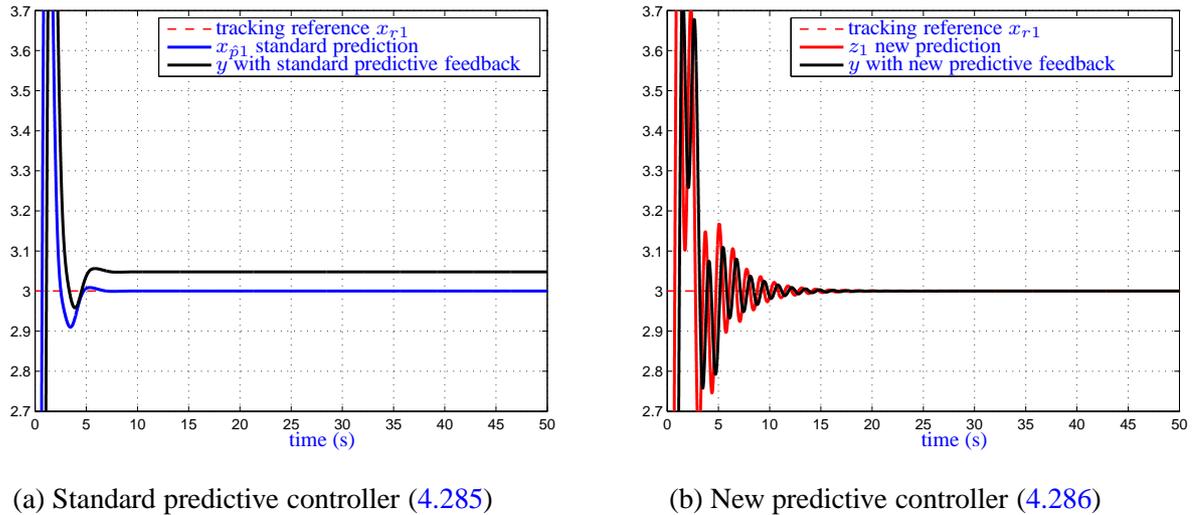


Figure 4.23 – Comparison of the predictive schemes for trajectory tracking with a constant and known delay $h = 0.5$ s and some parameter uncertainties ΔA and ΔB defined in (4.287) and (4.288)

(4.286) achieves better tracking performance than the controller based on the standard prediction (4.285). However, this result has not been demonstrated yet and can only be noticed numerically.

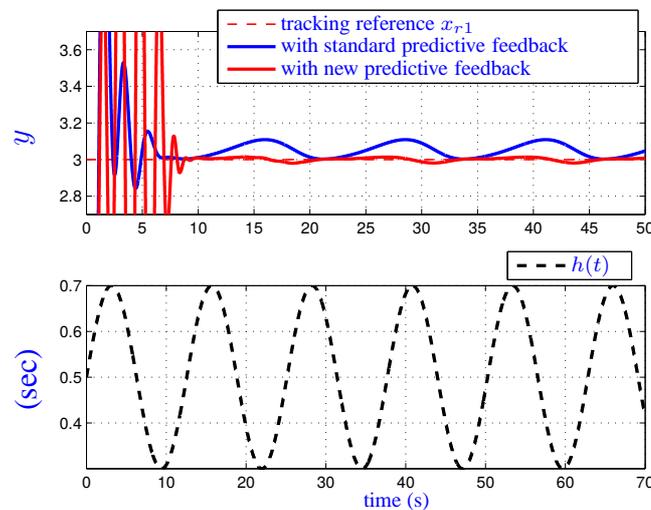


Figure 4.24 – Comparison of the predictive schemes for trajectory tracking with a time-varying and known delay $h = 0.5 + 0.2 \sin(0.5t)$ s and some parameter uncertainties ΔA and ΔB defined in (4.287) and (4.288)

Finally, Figure 4.25 displays a comparison of the three predictions for disturbance attenuation. The PID controller fed by the dynamic prediction of Section 4.2 is called a dynamic predictive controller. Note that PID gains are the same for the three cases. It can be observed that the dynamic predictive feedback rejects perfectly the constant disturbance (Figure 4.25a). It is the only controller that asymptotically stabilizes the system in presence of a constant disturbance, the other only achieves stability in a ball around the origin. However, the attenuation is strongly worsen when the perturbation becomes time-varying as shown on Figure 4.25b. The new predictive controller performs the best attenuation of the time-varying disturbance.

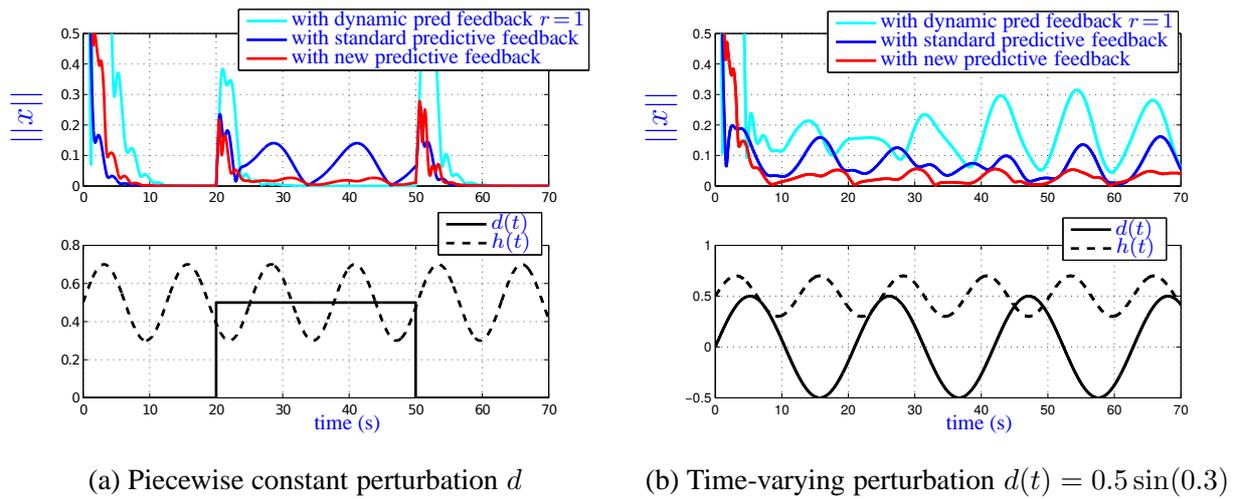


Figure 4.25 – Comparison of predictive schemes for a time-varying delay $h(t) = 0.5 + 0.2 \sin(0.5t)$ s

4.3.4 Summary

Contributions

- ✓ Design of a new prediction (for a constant delay) that improves the robustness of predictive control to unmodeled dynamics:
 - better attenuation of a large class of external disturbances,
 - more accurate trajectory tracking in presence of parameter uncertainties.
- ✓ Extension of the new prediction to a time-varying delay.

The results for a constant delay are gathered in Table 4.5.

Control objective		Convergence	
		to the origin	to a ball around the origin
Stabilization or Trajectory tracking without parameter uncertainties		✓✓	
Trajectory tracking with parameter uncertainties		✓	✓
Disturbance attenuation	Constant	✓	✓
	Time-varying		✓✓

Table 4.5 – Comparison of the best convergence results achievable for the [standard predictive](#) methods (✓) and the [new predictive](#) method (✓) for a known delay, a slow-varying disturbance, and a constant reference

Prediction with an unknown delay

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In the previous chapter, the delay value was used to compute the different predictions. However, the exact value of h is rarely available in real applications. An alternative could be to use memoryless controllers that do not need the value of the delay¹. Nevertheless, these techniques only achieve fast performances for a small delay. When this latter becomes larger, predictive control is necessary to achieve fast behaviour. Sometimes, the mean value of the delay is known so an approximated prediction can be computed thanks to this value and then the robustness to delay variation is studied. In this chapter, it is assumed that the exact value of the delay is not known but that an estimation is available. This estimation can be obtained via the estimation techniques of Part I for example. The predictive techniques of Chapter 4 are going to be revisited by considering the delay estimation instead of the exact delay. The cases of a constant and a time-varying delay estimation are going to be studied.

In Section 5.1, the standard prediction is computed from a delay estimation and some new conditions for the closed-loop stability are established. In Section 5.2, the stability of the dynamic observation-prediction method, introduced in Section 4.2 for a known delay, is analyzed in presence of an estimated delay. The new prediction, presented in Section 4.3, is not extended to the unknown delay case since its design is based on the knowledge of an accurate delay value.

1. See the beginning of Section 1.2.1 for references on memoryless controllers.

5.1 Standard prediction with unknown delay

When the exact delay is not available, the exact prediction cannot be computed. However, an approximated prediction can be designed from an estimated value of the delay. In this case, the delay will not be perfectly compensated. The objective of this section is to find stability conditions for feedbacks that are based on this approximated prediction. First the reduction method will be revisited and then a Lyapunov analysis will be performed using a Lyapunov-Krasovskii functional.

Stability conditions are given in Section 5.1.1 for an unknown and constant delay and in Section 5.1.2 for an unknown and time-varying delay. State and output feedbacks are treated in both sections.

5.1.1 Constant delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \quad \dot{x}(t) = Ax(t) + Bu(t-h)$$

where h is an unknown and constant delay. Reminding that Assumption 3.4.1 holds, there exists a feedback $u(x)$ that exponentially stabilizes system (\mathcal{E}_h) when $h = 0$. In addition, it is supposed that a delay estimation \hat{h} is available².

Full state knowledge

Since the whole state is known, the following prediction is computed thanks to the constant estimation \hat{h} :

$$z(t) = e^{A\hat{h}}x(t) + \int_{t-\hat{h}}^t e^{A(t-s)}Bu(s)ds. \quad (5.1)$$

Remark 5.1.1. On top of the initial condition on x , an initial condition $u(s) = \phi_u(s)$ for $s \in [-\hat{h}, 0]$, with ϕ_u a continuous function,³ is needed to compute $z(t)$.

The following theorem describes the condition for the closed-loop stability of the above system.

Theorem 5.1. Consider system (\mathcal{E}_h) , with an unknown and constant delay h , and assume that there exists a controller $u(x)$ satisfying Assumption 3.4.1. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (5.1), where \hat{h} is a constant estimation of h , and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-\hat{h}, t]} \|u(s)\|^2 + \sup_{s \in [t-\max(h, \hat{h}), t]} \|\dot{u}(s)\|^2. \quad (5.2)$$

Then, there exist $\varsigma, \varrho, D^* > 0$ such that, provided $D = |\hat{h} - h| < D^*$, for any initial condition one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (5.3)$$

2. See Chapters 2 and 3 for delay estimation techniques.

3. The extra condition $\phi_u(0) = u(z(0))$ may be required to guarantee the continuity of z at $t = \hat{h}$ (and consequently of u at $t = \hat{h}$).

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. Without loss of generality, the case $\hat{h} > h$ is considered here. By using Leibniz's formula for integral differentiation (3.103), it can be deduced that prediction (5.1) is solution of the following equation

$$\dot{z}(t) = Az(t) + Bu(t) + e^{A\hat{h}}B \int_{t-\hat{h}}^{t-h} \dot{u}(s)ds. \quad (5.4)$$

Remark that if $\hat{h} = h$ then the delay is perfectly compensated and $\dot{z}(t) = Az(t) + Bu(t)$. The following Lyapunov-Krasovskii functional candidate is proposed

$$V(t) = V_1(t) + \gamma V_2(t) \quad (5.5)$$

where

$$V_1(t) = z^T(t)Pz(t) \quad (5.6)$$

with P defined in Assumption 3.4.1,

$$V_2(t) = \int_{t-\hat{h}}^t (\hat{h} + s - t) \|\dot{u}(s)\|^2 ds \quad (5.7)$$

and $\gamma > 0$. Note that V_1 is the standard Lyapunov function used in the delay free case and V_2 is a functional that is introduced to deal with the integral $\int_{t-\hat{h}}^{t-h} \dot{u}(s)ds$ that arises from the delay mismatch. From equation (3.88) and taking the time derivative of V_1 along the trajectories of system (5.4), one gets

$$\dot{V}_1(t) \leq -c_u \|z(t)\|^2 + N \|z(t)\| \|w(t)\| \quad (5.8)$$

with $N = 2\|P\| \|B\| \|e^{A\hat{h}}\|$ and $\|w(t)\| = \int_{t-\hat{h}}^{t-h} \|\dot{u}(s)\| ds$. In addition, the time derivative of V_2 is given by

$$\dot{V}_2(t) = \hat{h} \|\dot{u}(t)\|^2 - \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \quad (5.9)$$

Reminding that

$$- \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds \leq - \int_{t-\hat{h}}^{t-h} \|\dot{u}(s)\|^2 ds \quad (5.10)$$

and using Jensen's inequality (3.101), one has

$$\dot{V}_2(t) \leq \hat{h} \|\dot{u}(t)\|^2 - \frac{1}{2D} \|w(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds \quad (5.11)$$

with $D = |\hat{h} - h|$. Furthermore, since u is globally Lipschitz and applying Hölder's inequality (3.104) to (5.4), it can be deduced that

$$\|\dot{z}(t)\|^2 \leq c_1 \|z(t)\|^2 + c_2 \|w(t)\|^2. \quad (5.12)$$

with $c_1 = 2(\|A\| + l_u \|B\|)^2$ and $c_2 = 2\|e^{A\hat{h}}B\|^2$. From $\|\dot{u}(t)\| \leq l_u \|\dot{z}(t)\|$ where l_u is the Lipschitz constant associated to the function u (see (3.89)), one has

$$\dot{V}_2(t) \leq \hat{h}l_u^2 c_1 \|z(t)\|^2 + \hat{h}l_u^2 c_2 \|w(t)\|^2 - \frac{1}{2D} \|w(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \quad (5.13)$$

From (5.8) and (5.13), it follows that

$$\begin{aligned} \dot{V}(t) \leq & - \left[c_u - \gamma \hat{h} l_u^2 c_1 \right] \|z(t)\|^2 - \left[\frac{\gamma}{2D} - \gamma \hat{h} l_u^2 c_2 \right] \|w(t)\|^2 \\ & + N \|z(t)\| \|w(t)\| - \frac{\gamma}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.14)$$

Maximizing the cross term by (3.102), above equation becomes

$$\dot{V}(t) \leq - \left[\frac{c_u}{2} - \gamma \hat{h} l_u^2 c_1 \right] \|z(t)\|^2 - \left[\frac{\gamma}{2D} - \frac{N^2}{2c_u} - \gamma \hat{h} l_u^2 c_2 \right] \|w(t)\|^2 - \frac{\gamma}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \quad (5.15)$$

To prove the exponential stability of (5.4), it is sufficient to find conditions which guarantee the following inequality

$$\dot{V}(t) + \varepsilon V(t) \leq 0 \quad (5.16)$$

with $\varepsilon > 0$. From equation (5.15), Assumptions 3.4.1 and 3.4.2, one has

$$\begin{aligned} \dot{V}(t) + \varepsilon V(t) \leq & - \left[\frac{c_u}{2} - \gamma \hat{h} l_u^2 c_1 \right] \|z(t)\|^2 - \left[\frac{\gamma}{2D} - \frac{N^2}{2c_u} - \gamma \hat{h} l_u^2 c_2 \right] \|w(t)\|^2 \\ & - \frac{\gamma}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds + \varepsilon \bar{c} \|z(t)\|^2 + \gamma \varepsilon \int_{t-\hat{h}}^t \hat{h} \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.17)$$

It follows that

$$\begin{aligned} \dot{V}(t) + \varepsilon V(t) \leq & - \left[\frac{c_u}{2} - \gamma \hat{h} l_u^2 c_1 - \varepsilon \bar{c} \right] \|z(t)\|^2 - \left[\frac{\gamma}{2D} - \frac{N^2}{2c_u} - \gamma \hat{h} l_u^2 c_2 \right] \|w(t)\|^2 \\ & - \left[\frac{\gamma}{2} - \gamma \varepsilon \hat{h} \right] \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.18)$$

In order to make $\dot{V}(t) + \varepsilon V(t) \leq 0$, it is sufficient to choose the coefficients multiplying the quadratic terms negative so

$$\left\{ \begin{array}{l} \frac{c_u}{2} - \gamma \hat{h} l_u^2 c_1 - \varepsilon \bar{c} > 0, \end{array} \right. \quad (5.19)$$

$$\left\{ \begin{array}{l} D < D^* = \frac{\gamma/2}{\frac{N^2}{2c_u} + \gamma \hat{h} l_u^2 c_2}, \end{array} \right. \quad (5.20)$$

$$\left\{ \begin{array}{l} \frac{1}{2} - \varepsilon \hat{h} > 0. \end{array} \right. \quad (5.21)$$

Choosing γ and ε sufficiently small guarantees that (5.19) and (5.21) are true. In this case, taking $D < D^*$ ensures that

$$\dot{V}(t) \leq -\varepsilon V(t). \quad (5.22)$$

Finally, inequality (5.3) can be deduced from (5.22) by similar computations as in Theorem 4.1. \square

Qualitative analysis: Condition (5.20) is intuitive because it means that the approximated delay should be sufficiently close to the real delay. An exact value of D^* is available but because of the conservatism of the Lyapunov approach, this value will be smaller than the maximum admissible one. However, a qualitative analysis of the bound is possible and is given below.

- For a large value of l_u ⁴ then D^* becomes smaller which means that the error estimation has to be smaller for a fast controller.
- For a large estimated delay \hat{h} then D^* becomes smaller which means that the estimation error is more prejudicial for a large delay.

Partial measurement knowledge

In this part, the state is partially measured so an estimated state \hat{x} is computed from observer (\mathcal{O}) in (3.90) with a correction term g that verifies Assumption 3.4.2. Since the delay h is unknown, the estimated delay \hat{h} is used in the observer (\mathcal{O}) in (3.90), then the dynamics of the estimation error (3.93), denoted $e = \hat{x} - x$, becomes

$$\dot{e}(t) = Ae(t) + g(Ce(t)) + B[u(t - \hat{h}) - u(t - h)] \quad (5.23)$$

and the prediction⁵ reads as

$$z(t) = e^{A\hat{h}}\hat{x}(t) + \int_{t-\hat{h}}^t e^{A(t-s)}Bu(s)ds. \quad (5.24)$$

The following theorem describes the necessary conditions for closed-loop stability of the above system. It is reminded that $e(t) = \hat{x}(t) - x(t)$.

Theorem 5.2. Consider system (\mathcal{E}_h) and observer (\mathcal{O}), with an unknown and constant delay h , and assume that there exist a controller $u(x)$ and a correction term $g(e)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (5.24), where \hat{h} is a constant estimation of h , and define

$$\Upsilon(t) = \|x(t)\|^2 + \|e(t)\|^2 + \sup_{s \in [t-\hat{h}, t]} \|u(s)\|^2 + \sup_{s \in [t-\max(h, \hat{h}), t]} \|\dot{u}(s)\|^2. \quad (5.25)$$

Then, there exist $\varsigma, \varrho, D^* > 0$ such that, provided $D = |\hat{h} - h| < D^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (5.26)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$, $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ and the convergence is exponential.

Proof. Without loss of generality, the case $\hat{h} > h$ is considered here. The prediction (5.24) is solution of the following equation:

$$\dot{z}(t) = Az(t) + Bu(t) + e^{A\hat{h}}g(Ce(t)). \quad (5.27)$$

4. l_u is the Lipschitz constant associated to the function u and is defined in (3.89).

5. See Remark 5.1.1 for the definition of initial conditions and add the initial condition $\hat{x}(0)$.

System (5.27) is coupled with the observation error whose dynamics is given by

$$\dot{e}(t) = Ae(t) + g(Ce(t)) + B \int_{t-h}^{t-\hat{h}} \dot{u}(s) ds. \quad (5.28)$$

Let

$$V(t) = V_1(t) + \gamma V_2(t) + \theta V_3(t) \quad (5.29)$$

be a Lyapunov-Krasovskii functional candidate with

$$V_1(t) = z^T(t)Pz(t), \quad (5.30)$$

$$V_2(t) = \int_{t-\hat{h}}^t (\hat{h} + s - t) \|\dot{u}(s)\|^2 ds, \quad (5.31)$$

$$V_3(t) = e^T(t)Qe(t), \quad (5.32)$$

and $\gamma, \theta > 0$. By similar computations to those of Theorem 5.1, the asymptotic stability of system (\mathcal{E}_h) to the origin can be derived provided that $D = \hat{h} - h$ is sufficiently small. Note that V_1 and V_2 are the same as in Theorem 5.1 and that V_3 is added to deal with the observer dynamics. \square

5.1.2 Time-varying delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \quad \dot{x}(t) = Ax(t) + Bu(t - h(t))$$

where $h(t)$ is an unknown time-varying delay.

Full state knowledge

Since the whole state is known, the following prediction is computed thanks to the estimated value \hat{h}

$$z(t) = e^{A\hat{h}(t)}x(t) + \int_{t-\hat{h}(t)}^t e^{A(t-s)}Bu(s)ds. \quad (5.33)$$

Remark 5.1.2. On top of the initial condition on x , an initial condition $u(s) = \phi_u(s)$ for $s \in [-h_{max}, 0]$, with ϕ_u a continuous function, ⁶ is needed to compute $z(t)$.

The following theorem describes the necessary conditions for closed-loop stability. It is reminded that $\hat{\delta}$ stands for the upper bound of the estimated delay time derivative: $|\dot{\hat{h}}(t)| < \hat{\delta}$ (Assumption 3.4.6).

Theorem 5.3. Consider system (\mathcal{E}_h) , where $h(t)$ is unknown and complies with Assumptions 3.4.3 and 3.4.4, and assume that there exists a controller $u(x)$ satisfying Assumption 3.4.1. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (5.33), where $\hat{h}(t)$

6. The extra condition $\phi_u(0) = u(z(0))$ may be required to guarantee the continuity of z at $t = \hat{h}(t)$ (and consequently of u at $t = \hat{h}(t)$).

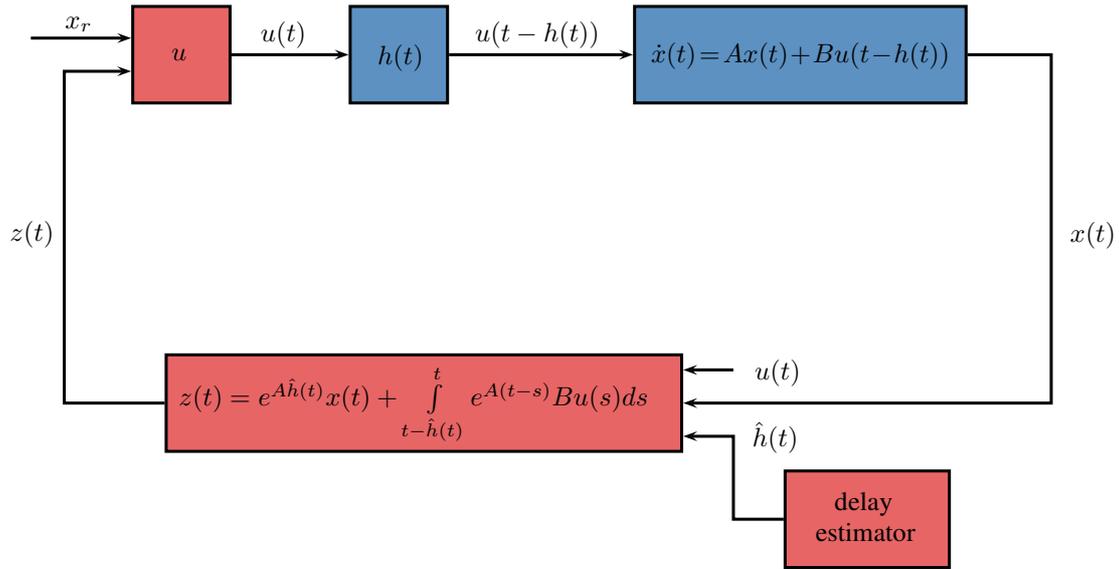


Figure 5.1 – Closed-loop scheme with standard prediction computed from an estimated delay and the full state.

is an estimation of $h(t)$ that complies with Assumptions 3.4.5 and 3.4.6, and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-h_{max}, t]} \|u(s)\|^2 + \sup_{s \in [t-h_{max}, t]} \|\dot{u}(s)\|^2. \quad (5.34)$$

Then, there exist $\varsigma, \varrho, \hat{\delta}^*, D^* > 0$ such that, provided $\hat{\delta} < \hat{\delta}^*$ and $D = \max_{t \geq 0} |\hat{h}(t) - h(t)| < D^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (5.35)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. By using Leibniz's formula for integral differentiation (3.103), the prediction z defined by (5.33) is solution of the following equation:

$$\dot{z} = Az + Bu(t) + \dot{\hat{h}}Az + e^{A\hat{h}}B \int_{t-\hat{h}}^{t-h} \dot{u}(s)ds + \dot{\hat{h}}e^{A\hat{h}}Bu(t-\hat{h}) - \dot{\hat{h}}A \int_{t-\hat{h}}^t e^{A(t-s)}Bu(s)ds. \quad (5.36)$$

Note that the argument “ t ” in $h(t)$, $\hat{h}(t)$, and $\dot{\hat{h}}(t)$ may be omitted for clarity. Remark that if $\hat{h} = h$ then (5.36) is equal to the ideal case (4.34). The following Lyapunov-Krasovskii functional candidate is proposed

$$V(t) = V_1(t) + \gamma V_2(t) + \theta V_3(t) \quad (5.37)$$

with $\theta, \gamma > 0$ and where

$$V_1(t) = z^T(t)Pz(t), \quad (5.38)$$

$$V_2(t) = \int_{t-h_{max}}^t (h_{max} + s - t) \|\dot{u}(s)\|^2 ds, \quad (5.39)$$

$$V_3(t) = \int_{t-\hat{h}}^t (h_{max} + h_{min} + s - t) \|u(s)\|^2 ds. \quad (5.40)$$

The function V_1 is the standard Lyapunov function candidate used in the delay free case (Assumption 3.4.1) and V_2 and V_3 are functionals used to deal with the delay mismatch and the time-varying delay respectively. From equation (3.88) and taking the time derivative of V_1 along the trajectories of system (5.36), one obtains

$$\begin{aligned} \dot{V}_1(t) \leq & -[c_u - 2\hat{\delta}\|P\| \|A\|] \|z(t)\|^2 + N \|z(t)\| \|w(t)\| + \hat{\delta}N \|z(t)\| \|u(t - \hat{h})\| \\ & + \hat{\delta}M \|z(t)\| \|v(t)\| \end{aligned} \quad (5.41)$$

with $M = 2\|P\| \|A\| \|B\| e^{\|A\|h_{max}}$, $N = 2\|P\| \|B\| e^{\|A\|h_{max}}$,

$$\|v(t)\| = \int_{t-\hat{h}}^t \|u(s)\| ds \quad (5.42)$$

and

$$\|w(t)\| = \int_{\min(t-h, t-\hat{h})}^{\max(t-h, t-\hat{h})} \|\dot{u}(s)\| ds. \quad (5.43)$$

The time derivative of V_2 is given by

$$\dot{V}_2(t) = h_{max} \|\dot{u}(t)\|^2 - \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds. \quad (5.44)$$

Since

$$- \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds \leq - \int_{\min(t-h, t-\hat{h})}^{\max(t-h, t-\hat{h})} \|\dot{u}(s)\|^2 ds \quad (5.45)$$

and from Jensen's inequality (3.101)

$$- \int_{\min(t-h, t-\hat{h})}^{\max(t-h, t-\hat{h})} \|\dot{u}(s)\|^2 ds \leq -\frac{1}{D} \left(\int_{\min(t-h, t-\hat{h})}^{\max(t-h, t-\hat{h})} \|\dot{u}(s)\| ds \right)^2, \quad (5.46)$$

with $D = \max_{t \geq 0} |\hat{h}(t) - h(t)|$. It follows that

$$\dot{V}_2(t) \leq h_{max} \|\dot{u}(t)\|^2 - \frac{1}{2D} \|w(t)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds. \quad (5.47)$$

Furthermore, from (5.36) and Hölder's inequality (3.104), it can be deduced that

$$\|\dot{z}(t)\|^2 \leq c_1 \|z(t)\|^2 + c_2 \|w(t)\|^2 + c_2 \hat{\delta}^2 \|u(t - \hat{h})\|^2 + c_4 \hat{\delta}^2 \|v(t)\|^2 \quad (5.48)$$

with $c_1 = 4(\|A\|(1 + \delta) + l_u\|B\|)^2$, $c_2 = 4e^{2\|A\|h_{max}}\|B\|^2$ and $c_4 = e^{2\|A\|h_{max}}\|A\|^2\|B\|^2$. As a consequence, (5.47) becomes

$$\begin{aligned} \dot{V}_2(t) \leq & h_{max}l_u^2[c_1\|z(t)\|^2 + c_2\|w(t)\|^2 + c_2\hat{\delta}^2\|u(t - \hat{h})\|^2 + c_4\hat{\delta}^2\|v(t)\|^2] \\ & - \frac{1}{2D}\|w(t)\|^2 - \frac{1}{2} \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.49)$$

The time derivative of V_3 verifies

$$\dot{V}_3(t) \leq (h_{max} + h_{min})\|u(t)\|^2 - h_{min}\|u(t - \hat{h})\|^2 - \frac{1}{2\hat{h}}\|v(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|u(s)\|^2 ds. \quad (5.50)$$

From the above maximizations, it follows that

$$\begin{aligned} \dot{V}(t) \leq & -[c_u - 2\hat{\delta}\|P\|\|A\|]\|z(t)\|^2 + N\|z(t)\|\|w(t)\| + \hat{\delta}N\|z(t)\|\|u(t - \hat{h})\| \\ & + \hat{\delta}M\|z(t)\|\|v(t)\| + \gamma h_{max}l_u^2c_1\|z(t)\|^2 - \gamma \left[\frac{1}{2D} - h_{max}l_u^2c_2\right]\|w(t)\|^2 \\ & + \gamma h_{max}l_u^2c_2\hat{\delta}^2\|u(t - \hat{h})\|^2 - \left[\frac{\theta}{2\hat{h}} - \gamma h_{max}l_u^2c_4\hat{\delta}^2\right]\|v(t)\|^2 \\ & - \frac{\gamma}{2} \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds + 2\theta h_{max}\|u(t)\|^2 - \theta h_{min}\|u(t - \hat{h})\|^2 - \frac{\theta}{2} \int_{t-\hat{h}}^t \|u(s)\|^2 ds. \end{aligned} \quad (5.51)$$

Applying (3.102) and Young's inequality (3.100) to get rid of the cross terms leads to

$$\begin{aligned} \dot{V}(t) \leq & -[c_u - 2\hat{\delta}\|P\|\|A\| - D\frac{N^2}{\gamma} - 2\theta h_{max}l_u^2 - \gamma h_{max}l_u^2c_1 - \hat{\delta}(N + M)/2]\|z(t)\|^2 \\ & - \left[\theta h_{min} - \gamma h_{max}l_u^2c_2\hat{\delta}^2 - \hat{\delta}N/2\right]\|u(t - \hat{h})\|^2 - \gamma \left[\frac{1}{4D} - h_{max}l_u^2c_2\right]\|w(t)\|^2 \\ & - \left[\frac{\theta}{2\hat{h}} - \gamma h_{max}l_u^2c_4\hat{\delta}^2 - \hat{\delta}M/2\right]\|v(t)\|^2 - \frac{\gamma}{2} \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds - \frac{\theta}{2} \int_{t-\hat{h}}^t \|u(s)\|^2 ds. \end{aligned} \quad (5.52)$$

To prove the exponential stability of (5.36), it is sufficient to find conditions which guarantee the following inequality

$$\dot{V}(t) + \varepsilon V(t) \leq 0 \quad (5.53)$$

with $\varepsilon > 0$. From equation (5.52) and Assumptions 3.4.1, one has

$$\begin{aligned} \dot{V} - \varepsilon V \leq & -[c_u - 2\hat{\delta}\|P\|\|A\| - D\frac{N^2}{\gamma} - 2\theta h_{max}l_u^2 - \gamma h_{max}l_u^2c_1 - \hat{\delta}(N + M)/2]\|z(t)\|^2 \\ & - \left[\theta h_{min} - \gamma h_{max}l_u^2c_2\hat{\delta}^2 - \hat{\delta}N/2\right]\|u(t - \hat{h})\|^2 - \gamma \left[\frac{1}{4D} - h_{max}l_u^2c_2\right]\|w(t)\|^2 \\ & - \left[\frac{\theta}{2\hat{h}} - \gamma h_{max}l_u^2c_4\hat{\delta}^2 - \hat{\delta}M/2\right]\|v(t)\|^2 - \frac{\gamma}{2} \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds - \frac{\theta}{2} \int_{t-\hat{h}}^t \|u(s)\|^2 ds \\ & \varepsilon \bar{c}\|z(t)\|^2 + \varepsilon \gamma \int_{t-h_{max}}^t h_{max}\|\dot{u}(s)\|^2 ds + \varepsilon \theta \int_{t-\hat{h}}^t (h_{max} + h_{min})\|u(s)\|^2 ds. \end{aligned} \quad (5.54)$$

It follows that

$$\begin{aligned} \dot{V} - \varepsilon V \leq & -[c_u - 2\hat{\delta}\|P\|\|A\| - D\frac{N^2}{\gamma} - 2\theta h_{max}l_u^2 - \gamma h_{max}l_u^2c_1 - \hat{\delta}(N + M)/2 - \varepsilon \bar{c}]\|z(t)\|^2 \\ & - \left[\theta h_{min} - \gamma h_{max}l_u^2c_2\hat{\delta}^2 - \hat{\delta}N/2\right]\|u(t - \hat{h})\|^2 - \gamma \left[\frac{1}{4D} - h_{max}l_u^2c_2\right]\|w(t)\|^2 \\ & - \left[\frac{\theta}{2\hat{h}} - \gamma h_{max}l_u^2c_4\hat{\delta}^2 - \hat{\delta}M/2\right]\|v(t)\|^2 \\ & - \gamma \left[\frac{1}{2} - \varepsilon h_{max}\right] \int_{t-h_{max}}^t \|\dot{u}(s)\|^2 ds - \theta \left[\frac{1}{2} - 2\varepsilon h_{max}\right] \int_{t-\hat{h}}^t \|u(s)\|^2 ds. \end{aligned} \quad (5.55)$$

In order to make $\dot{V}(t) + \varepsilon V(t) \leq 0$, it is sufficient to choose the coefficients multiplying the quadratic terms negative. Thus, the following conditions have to be verified

$$\left\{ \begin{array}{l} c_u - 2\hat{\delta} \|P\| \|A\| - D \frac{N^2}{\gamma} - 2\theta h_{max} l_u^2 - \gamma h_{max} l_u^2 c_1 - \hat{\delta}(N + M)/2 - \varepsilon \bar{c} > 0, \end{array} \right. \quad (5.56)$$

$$\theta h_{min} - \gamma h_{max} l_u^2 c_2 \hat{\delta}^2 - \hat{\delta} N/2 > 0, \quad (5.57)$$

$$\frac{1}{4D} - h_{max} l_u^2 c_2 > 0, \quad (5.58)$$

$$\left\{ \begin{array}{l} \frac{\theta}{2\hat{h}} - \gamma h_{max} l_u^2 c_4 \hat{\delta}^2 - \hat{\delta} M/2 > 0, \end{array} \right. \quad (5.59)$$

$$\frac{1}{2} - \varepsilon h_{max} > 0, \quad (5.60)$$

$$\frac{1}{2} - 2\varepsilon h_{max} > 0. \quad (5.61)$$

Rearranging the terms of condition (5.56) and (5.58) gives

$$D < D_1 \quad (5.62)$$

and

$$D < D_2 \quad (5.63)$$

with

$$D_1 = \frac{\gamma}{N^2} \left[c_u - \hat{\delta}(2\|P\| \|A\| + (N + M)/2) - h_{max} l_u^2 (\gamma c_1 + 2\theta) - \varepsilon \bar{c} \right], \quad (5.64)$$

$$D_2 = \frac{1}{4h_{max} l_u^2 c_2}. \quad (5.65)$$

Solving the binomial inequations (5.57) and (5.59) in $\hat{\delta}$ leads to

$$\hat{\delta} < \delta_1 \quad (5.66)$$

and

$$\hat{\delta} < \delta_2 \quad (5.67)$$

with

$$\delta_1 = \sqrt{\frac{\theta h_{min}}{\gamma h_{max} l_u^2 c_2}}, \quad (5.68)$$

$$\delta_2 = \sqrt{\frac{\theta}{2\gamma \hat{h} h_{max}^2 l_u^2 c_4}}. \quad (5.69)$$

In order to guarantee that D_1 is positive, $\hat{\delta}$ has to satisfy the following condition

$$\delta < \delta_3 \quad (5.70)$$

with

$$\delta_3 = \frac{c_u - h_{max} l_u^2 (\gamma c_1 + 2\theta) - \varepsilon \bar{c}}{2\|P\| \|A\| + (N + M)/2}. \quad (5.71)$$

Choosing γ , θ and ε sufficiently small guarantees that (5.60) and (5.61) are true and that δ_3 is positive. In this case, taking

$$D < D^* = \min(D_1, D_2) \quad (5.72)$$

and

$$\hat{\delta} < \hat{\delta}^* = \min(\delta_1, \delta_2, \delta_3) \quad (5.73)$$

ensures that

$$\dot{V}(t) \leq -\varepsilon V(t). \quad (5.74)$$

Finally, inequality (5.35) can be deduced from (5.74) by similar computations as in Theorem 4.1. \square

Qualitative analysis: The conditions obtained are intuitive because it means that the approximated delay should be sufficiently close to the real delay and that it should be slow-varying. It is interesting to note that this latter condition ($\hat{\delta}(t) < \hat{\delta}^*$) is similar to the one in the known delay case⁷. Further qualitative comments on the bounds D^* and $\hat{\delta}^*$ are given below.

- For a large value of l_u ⁸ then D_1 and D_2 become smaller which means that the error estimation has to be smaller for a fast controller.
- Similarly for a large value of l_u then δ_1 , δ_2 and δ_3 decrease which means that the admissible delay variations are slower for a fast controller.
- For a “large” value of ε then D_1 becomes smaller which means that the convergence rate of the closed-loop system can be larger for a small error estimation.
- For “large” value of ε then δ_3 decrease which means that the convergence rate of the closed-loop system can be larger for slow delay variations.
- For a large value of h_{max} then all the D_i and δ_i decrease which means that the estimation error and the delay variations are more prejudicial for a large delay.

Partial measurement knowledge

In this part, the state is partially measured so observer (\mathcal{O}) in (3.90) is introduced to estimate the state with a correction term g that verifies Assumption 3.4.2. Since the delay $h(t)$ is unknown, the estimated delay $\hat{h}(t)$ is used in the observer and the observation error, denoted by $e = \hat{x} - x$, has the following dynamics

$$\dot{e}(t) = Ae(t) + g(Ce(t)) + B[u(t - \hat{h}(t)) - u(t - h(t))] \quad (5.75)$$

and the prediction⁹ is

$$z(t) = e^{A\hat{h}(t)}\hat{x}(t) + \int_{t-\hat{h}(t)}^t e^{A(t-s)}Bu(s)ds. \quad (5.76)$$

The closed-loop scheme is given on Figure 5.2. The following theorem describes the necessary conditions for closed-loop stability of the above system. It is reminded that $\hat{\delta}$ stands for the upper bound of the delay time derivative, $|\dot{\hat{h}}(t)| < \hat{\delta}$ (Assumption 3.4.6), and that $e(t) = \hat{x}(t) - x(t)$.

Theorem 5.4. Consider system (\mathcal{E}_h) and observer (\mathcal{O}), where $h(t)$ is unknown and complies with Assumptions 3.4.3 and 3.4.4, and assume that there exist a controller $u(x)$ and a correction term $g(e)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that

7. In Section 4.1.2, the real delay $h(t)$ had to be slow-varying, here it is the estimated delay $\hat{h}(t)$.

8. l_u is the Lipschitz constant associated to the function u and is defined in (3.89).

9. See Remark 5.1.2 for the definition of initial conditions and add the initial condition $\hat{x}(0)$.

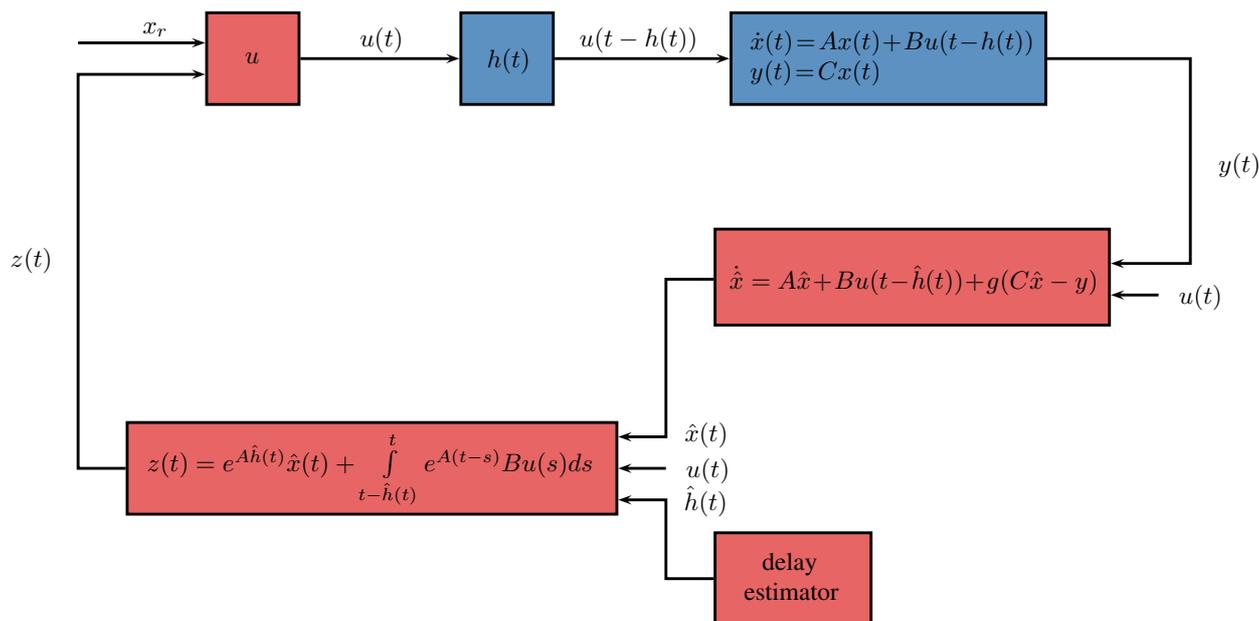


Figure 5.2 – Closed-loop scheme with the standard prediction computed from an estimated delay and the estimated state.

system (\mathcal{E}_h) is controlled by $u(z)$ with z defined by (5.76), where $\hat{h}(t)$ is an estimation of $h(t)$ that complies with Assumptions 3.4.5 and 3.4.6, and define

$$\Upsilon(t) = \|x(t)\|^2 + \|e(t)\|^2 + \sup_{s \in [t-h_{max}, t]} \|u(s)\|^2 + \sup_{s \in [t-h_{max}, t]} \|\dot{u}(s)\|^2. \quad (5.77)$$

Then, there exist $\varsigma, \varrho, \hat{\delta}^*, D^* > 0$ such that, provided $\hat{\delta} < \hat{\delta}^*$ and $D = \max_{t \geq 0} |\hat{h}(t) - h(t)| < D^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (5.78)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$, $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ and the convergence is exponential.

Proof. The prediction z in (5.76) is solution of the following equation

$$\dot{z} = Az + Bu(z) + e^{A\hat{h}}g(Ce) + \dot{\hat{h}}e^{A\hat{h}}Bu(t - \hat{h}) + \dot{\hat{h}}Az - \dot{\hat{h}}A \int_{t-\hat{h}(t)}^t e^{A(t-s)}Bu(s)ds. \quad (5.79)$$

Let

$$V(t) = V_1(t) + \gamma V_2(t) + \theta V_3(t) + \zeta V_4(t) \quad (5.80)$$

be a Lyapunov-Krasovskii functional candidate with

$$V_1(t) = z^T(t)Pz(t), \quad (5.81)$$

$$V_2(t) = \int_{t-h_{max}}^t (h_{max} + s - t) \|\dot{z}(s)\|^2 ds, \quad (5.82)$$

$$V_3(t) = \int_{t-\hat{h}}^t (h_{max} + h_{min} + s - t) \|z(s)\|^2 ds, \quad (5.83)$$

$$V_4(t) = e^T(t)Qe(t), \tag{5.84}$$

and $\gamma, \theta, \zeta > 0$. By similar computations to those of Theorems 5.3 and 5.2, the global exponential stability of system (\mathcal{E}_h) to the origin can be derived. Note that V_1, V_2 and V_3 are the same as in Theorem 5.3 and V_4 is added to deal with the observer dynamics. \square

5.1.3 Illustrative example

Consider the same scalar system as in Section 4.1.3:

$$\dot{x}(t) = ax(t) + bu(t - h(t)) \tag{5.85}$$

with $a = 1$ and $b = 1$. The same controller $u(t) = kz(t)$ is going to be tested with the feedback gain $k = -2$. Note that k is chosen to have $\lambda(a + bk) = -1$. A time-varying delay $h(t) = 1 + 0.3 \sin(0.3t)$ is applied to the system.

Remark 5.1.3. *In this example, the variable x has no particular physical meaning so it will be displayed without unit in the next plots.*

On Figure 5.3, constant estimations \hat{h} have been used in order to study the influence of the estimation error $|h - \hat{h}|$ on the convergence of the system. On Figure 5.3a, the delay is underestimated and it is clear that the system is unstable. On Figure 5.3b, the mean value of $h(t)$ is used to compute the prediction. In this case, the delay estimation error is small and the closed-loop is stable. The analysis of Figure 5.3c shows that the system tends to stabilize when $|h - \hat{h}|$ is small whereas it diverges when this error increases. These simulations are in accordance with the result of Theorem 5.3.

Finally, the delay estimator presented in Section 3.2.2 is used to obtain an estimation of the delay. On Figure 5.4, three phases can be distinguished. Between 0 and 9 seconds the delay estimation is far from the real value so the prediction is not accurate and the system diverges. Between 9 and 20 seconds, the delay estimation is close to the real value of the delay so the predictive feedback is able to stabilize the system. From 20 seconds to the end, the system does not move anymore so the input is constant and the delay has no influence on the system. During this period, the delay cannot be estimated since u is constant.

5.1.4 Summary

Contributions

- ✓ Extension of the results of Section 4.1 to the unknown delay case.
- ✓ Explicit stability conditions are provided.

Table 5.1 sums up the results of this section. Note that the bounds δ^* and D^* may have different values for each case.

5.2 Dynamic observation-prediction with an unknown delay

The dynamic observation-prediction presented in Section 4.2 is particularly interesting since it is easy to compute. However, the exact delay knowledge is necessary. In this section, the goal is to find a condition on the delay accuracy that preserves the result of the known delay case. The stability analysis is carried out thanks to an original Lyapunov-Krasovskii functional.

The single observer-predictor with an unknown delay is worked out in Section 5.2.1 and Section 5.2.2 extends the result to sequential sub observers-predictors.

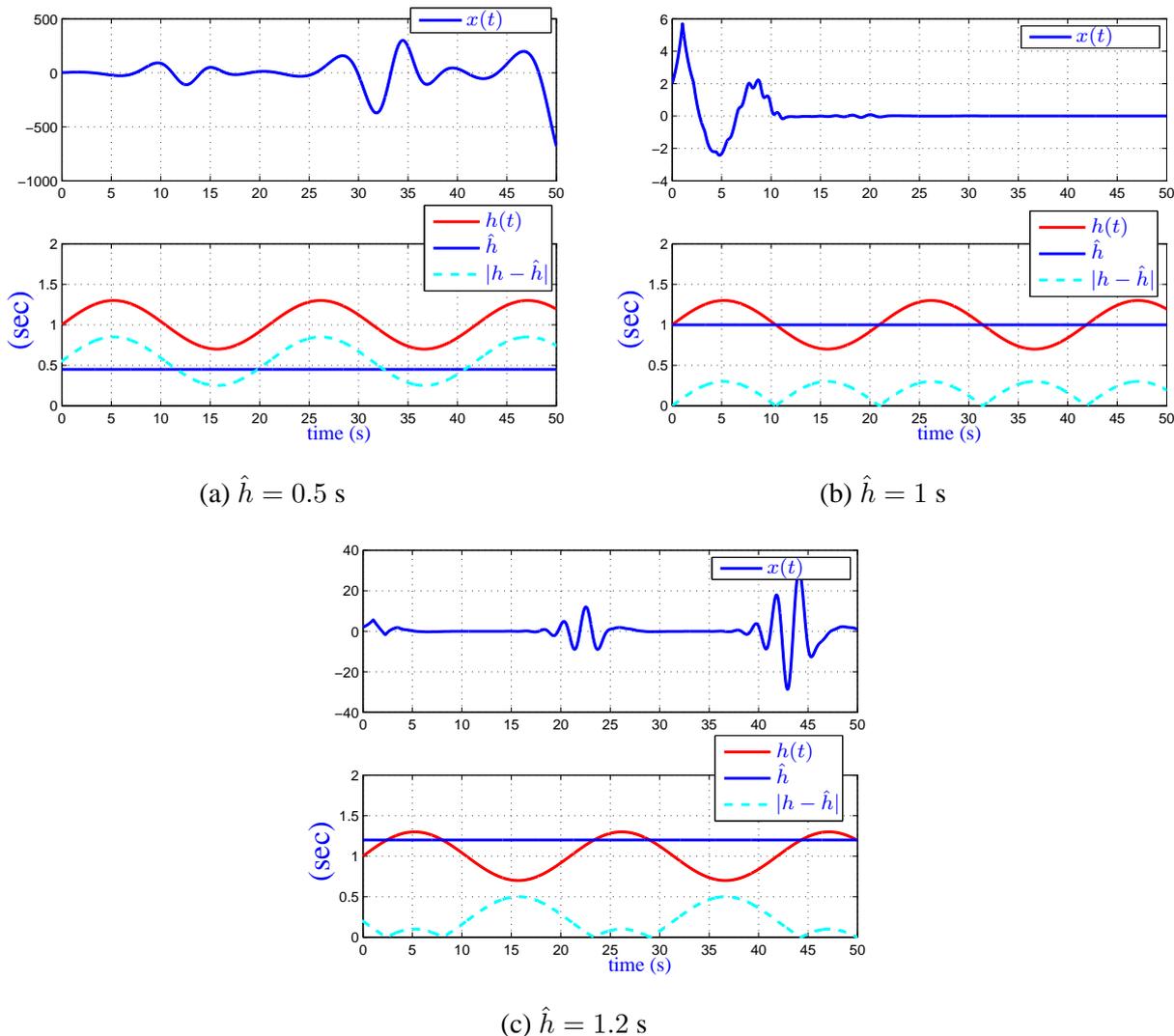


Figure 5.3 – Comparison of the closed-loop stability for different constant values of the estimated delay \hat{h} and for an unknown time-varying delay $h(t) = 1 + 0.3 \sin(0.3t)$ s.

5.2.1 Single observer-predictor with an unknown delay

In this section, an extension of the method described in 4.2 for a known delay is extended to the case of an unknown delay. Before dealing with the time-varying delay case, the case of a constant delay is worked out in order to introduce the basic concept of the proof.

Constant delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t - h) \\ y(t) = Cx(t). \end{cases}$$

The prediction z is computed from an estimated delay \hat{h} by the observer-predictor

$$\dot{z}(t) = Az(t) + Bu(t) + g(Cz(t - \hat{h}) - y(t)) \tag{5.86}$$

where the correction term g satisfies Assumption 3.4.2.

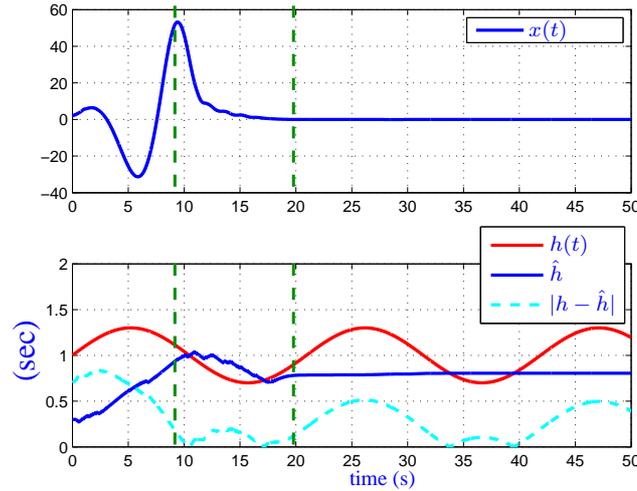


Figure 5.4 – Closed-loop performance for a predictive feedback with a delay estimation computed by the delay observer of Section 3.2.2. The unknown delay is equal to $h(t) = 1 + 0.3 \sin(0.3t)$ s.

Prediction	Delay		Convergence	Condition	Feedback	Thm
Standard	Unknown	h	gl. & exp.	$ \hat{h} - h < D^*$	state	5.1
					output	5.2
	$h(t)$		gl. & exp.	$\hat{\delta} < \hat{\delta}^*$ $\max_{t \geq 0} \hat{h}(t) - h(t) < D^*$	state	5.3
					output	5.4

Table 5.1 – Sum up of the convergence results for standard prediction with an unknown delay

- gl.: global – exp.: exponential
- h : constant delay
- \hat{h} : constant delay estimation
- $h(t) \in [h_{min}, h_{max}]$ and $|\dot{h}(t)| \leq \delta$: time-varying delay
- $\hat{h}(t) \in [h_{min}, h_{max}]$ and $|\dot{\hat{h}}(t)| \leq \hat{\delta}$: time-varying delay estimation

Remark 5.2.1. Note that to compute $z(t)$, the initial condition $z(t) = \phi_z(t)$ for $t \in [-\hat{h}, 0]$ with ϕ_z a continuous function is required. The prediction z will be continuous in $t = \hat{h}$.

Since Assumption 3.4.1 holds, there exists a feedback $u(x)$ that exponentially stabilizes system (\mathcal{E}_h) when $h = 0$. Then a predictive feedback $u(z)$ is going to be applied to stabilize $(\mathcal{E}_{\hat{h}})$. Denoting

$$e_p(t) = z(t - \hat{h}) - x(t) \quad (5.87)$$

the prediction error, the following result holds.

Theorem 5.5. Consider system (\mathcal{E}_h) , with an unknown and constant delay h , and assume that there exist a controller $u(x)$ and a correction term $g(e_p)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z computed by observer-predictor (5.86), where \hat{h} is a constant estimation of h , and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-\hat{h}, t]} \|z(s)\|^2 + \sup_{s \in [t-\hat{h}, t]} \|\dot{e}_p(s)\|^2 + \sup_{s \in [t-\hat{h}, t]} \|\dot{u}(s)\|^2. \quad (5.88)$$

Then, there exist $\varsigma, \varrho, \hat{h}^*, D^* > 0$ such that, provided $\hat{h} < \hat{h}^*$ and $D = |\hat{h} - h| < D^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\varrho t}. \quad (5.89)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. Without loss of generality the case $\hat{h} > h$ is considered here. First, combining (5.86) with (\mathcal{E}_h) leads to

$$\dot{z}(t) = Az(t) + Bu(t) + g(Ce_p(t)). \quad (5.90)$$

The dynamics of $e_p(t)$ reads as

$$\dot{e}_p(t) = Ae_p(t) + B[u(t - \hat{h}) - u(t - h)] + g(Ce_p(t - \hat{h})) \quad (5.91)$$

and can be rewritten as follows

$$\dot{e}_p(t) = Ae_p(t) + g(Ce_p(t)) - B \int_{t-\hat{h}}^{t-h} \dot{u}(s) ds - \int_{Ce_p(t-\hat{h})}^{Ce_p(t)} \frac{dg}{ds}(s) ds. \quad (5.92)$$

The following Lyapunov-Krasovskii functional candidate is proposed

$$V(t) = \gamma V_1(t) + V_2(t) + V_3(t) + V_4(t) \quad (5.93)$$

where

$$V_1(t) = z^T(t) P z(t), \quad (5.94)$$

with P defined in Assumption 3.4.1,

$$V_2(t) = e_p^T(t) Q e_p(t), \quad (5.95)$$

with Q defined in Assumption 3.4.2,

$$V_3(t) = \int_{t-\hat{h}}^t (\hat{h} + s - t) \|\dot{e}_p(s)\|^2 ds, \quad (5.96)$$

$$V_4(t) = \int_{t-\hat{h}}^t (\hat{h} + s - t) \|\dot{u}(s)\|^2 ds \quad (5.97)$$

and $\gamma > 0$. Remark that V_1 and V_2 are the standard Lyapunov function candidates used in the delay free case whereas V_3 and V_4 are added to deal with the integral terms $\int_{Ce_p(t-h)}^{Ce_p(t)} \frac{dg}{ds}(s) ds$ and

$\int_{t-\hat{h}}^{t-\hat{h}} \dot{u}(s)ds$ respectively. Using (3.88) and taking the time derivative of V_1 along the trajectories of (5.90), one obtains

$$\dot{V}_1(t) \leq -c_u \|z(t)\|^2 + R \|z(t)\| \|e_p(t)\| \tag{5.98}$$

with $R = 2l_g \|P\| \|C\|$. By substitution one gets

$$\left\| \int_{C e_p(t-\hat{h})}^{C e_p(t)} \frac{dg}{ds}(s) ds \right\| \leq l_g \|C\| \int_{t-\hat{h}}^t \|\dot{e}_p(s)\| ds. \tag{5.99}$$

So using (3.94), it follows that

$$\dot{V}_2(t) \leq -m_g \|e_p(t)\|^2 + R' \|e_p(t)\| \|m(t)\| + N \|e_p(t)\| \|w(t)\| \tag{5.100}$$

with $R' = 2l_g \|Q\| \|C\|$, $N = 2\|Q\| \|B\|$, $\|m(t)\| = \int_{t-\hat{h}}^t \|\dot{e}_p(s)\| ds$ and $\|w(t)\| = \int_{t-\hat{h}}^{t-h} \|\dot{u}(s)\| ds$.

In addition,

$$\dot{V}_3(t) = \hat{h} \|\dot{e}_p(t)\|^2 - \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds \tag{5.101}$$

so, reminding that

$$- \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds \leq -\frac{1}{\hat{h}} \left(\int_{t-\hat{h}}^t \|\dot{e}_p(s)\| ds \right)^2, \tag{5.102}$$

it follows that

$$\dot{V}_3(t) \leq \hat{h} \|\dot{e}_p(t)\|^2 - \frac{1}{2\hat{h}} \|m(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds. \tag{5.103}$$

From system (5.92) and using the Hölder's inequality (3.104), the maximization

$$\|\dot{e}_p\|^2 \leq c_1 \|e_p\|^2 + c_2 \|m\|^2 + c_3 \|w\|^2 \tag{5.104}$$

is obtained with $c_1 = 3(\|A\| + l_g \|C\|)^2$, $c_2 = 3\|B\|^2$, $c_3 = 3l_g^2 \|C\|^2$ so

$$\dot{V}_3(t) \leq \hat{h} c_1 \|e_p\|^2 + \hat{h} c_2 \|m\|^2 + \hat{h} c_3 \|w\|^2 - \frac{1}{2\hat{h}} \|m(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds. \tag{5.105}$$

Finally,

$$\dot{V}_4(t) = \hat{h} \|\dot{u}(t)\|^2 - \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds \tag{5.106}$$

and since

$$- \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds \leq - \int_{t-\hat{h}}^{t-h} \|\dot{u}(s)\|^2 ds, \tag{5.107}$$

from Jensen's inequality (3.101), the following maximization is deduced

$$\dot{V}_4(t) \leq \hat{h} \|\dot{u}(t)\|^2 - \frac{1}{2D} \|w(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds \quad (5.108)$$

with $D = \hat{h} - h$. From system (5.90), the maximization

$$\|\dot{z}\|^2 \leq c_4 \|z\|^2 + c_5 \|e_p\|^2 \quad (5.109)$$

is obtained with $c_4 = 2(\|A\| + l_u \|B\|)^2$, $c_5 = 2l_g^2 \|C\|^2$ so reminding that $\|\dot{u}(z)\| \leq l_u \|\dot{z}\|$, one gets

$$\dot{V}_4(t) \leq \hat{h} l_u^2 c_4 \|z(t)\|^2 + \hat{h} l_u^2 c_5 \|e_p(t)\|^2 - \frac{1}{2D} \|w(t)\|^2 - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \quad (5.110)$$

Maximizing the cross terms thanks to (3.102) and Young's inequality (3.100), it can be derived that

$$\begin{aligned} \dot{V}(t) \leq & - \left[\frac{\gamma c_u}{2} - \hat{h} l_u^2 c_4 \right] \|z(t)\|^2 - \left[\frac{m_g}{3} - \frac{\gamma R^2}{2c_u} - \hat{h}(c_1 + l_u^2 c_5) \right] \|e_p(t)\|^2 \\ & - \left[\frac{1}{2\hat{h}} - \frac{3R^2}{4m_g} - \hat{h} c_2 \right] \|m(t)\|^2 - \left[\frac{1}{2D} - \frac{3N^2}{4m_g} - \hat{h} c_5 \right] \|w(t)\|^2 \\ & - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.111)$$

To prove the exponential stability of (5.90) and (5.91), it is sufficient to find conditions which guarantee the following inequality

$$\dot{V}(t) + \varepsilon V(t) \leq 0 \quad (5.112)$$

with $\varepsilon > 0$. From equation (5.111) and Assumptions 3.4.1 and 3.4.2, one has

$$\begin{aligned} \dot{V} + \varepsilon V \leq & - \left[\frac{\gamma c_u}{2} - \hat{h} l_u^2 c_4 \right] \|z(t)\|^2 - \left[\frac{m_g}{3} - \frac{\gamma R^2}{2c_u} - \hat{h}(c_1 + l_u^2 c_5) \right] \|e_p(t)\|^2 \\ & - \left[\frac{1}{2\hat{h}} - \frac{3R^2}{4m_g} - \hat{h} c_2 \right] \|m(t)\|^2 - \left[\frac{1}{2D} - \frac{3N^2}{4m_g} - \hat{h} c_5 \right] \|w(t)\|^2 \\ & - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds - \frac{1}{2} \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds + \gamma \varepsilon \bar{c} \|z\|^2 + \varepsilon \bar{m} \|e_p\|^2 \\ & + \varepsilon \int_{t-\hat{h}}^t \hat{h} \|\dot{e}_p(s)\|^2 ds + \varepsilon \int_{t-\hat{h}}^t \hat{h} \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.113)$$

It follows that

$$\begin{aligned} \dot{V} + \varepsilon V \leq & - \left[\frac{\gamma c_u}{2} - \hat{h} l_u^2 c_4 - \gamma \varepsilon \bar{c} \right] \|z(t)\|^2 - \left[\frac{m_g}{3} - \frac{\gamma R^2}{2c_u} - \hat{h}(c_1 + l_u^2 c_5) \varepsilon - \bar{m} \right] \|e_p(t)\|^2 \\ & - \left[\frac{1}{2\hat{h}} - \frac{3R^2}{4m_g} - \hat{h} c_2 \right] \|m(t)\|^2 - \left[\frac{1}{2D} - \frac{3N^2}{4m_g} - \hat{h} c_5 \right] \|w(t)\|^2 \\ & - \left[\frac{1}{2} - \varepsilon \hat{h} \right] \int_{t-\hat{h}}^t \|\dot{e}_p(s)\|^2 ds - \left[\frac{1}{2} - \varepsilon \hat{h} \right] \int_{t-\hat{h}}^t \|\dot{u}(s)\|^2 ds. \end{aligned} \quad (5.114)$$

In order to have $\dot{V} - \varepsilon V$ negative, the following conditions have to be verified

$$\left\{ \begin{array}{l} \frac{\gamma c_u}{2} - \hat{h} l_u^2 c_4 - \gamma \varepsilon \bar{c} > 0, \end{array} \right. \quad (5.115)$$

$$\left\{ \begin{array}{l} \frac{m_g}{3} - \frac{\gamma R^2}{2c_u} - \hat{h}(c_1 + l_u^2 c_5) - \varepsilon \bar{m} > 0, \end{array} \right. \quad (5.116)$$

$$\left\{ \begin{array}{l} \frac{1}{2\hat{h}} - \frac{3R^2}{4m_g} - \hat{h}c_2 > 0, \end{array} \right. \quad (5.117)$$

$$\left\{ \begin{array}{l} \frac{1}{2D} - \frac{3N^2}{4m_g} - \hat{h}c_5 > 0, \end{array} \right. \quad (5.118)$$

$$\left\{ \begin{array}{l} \frac{1}{2} - \varepsilon \hat{h} > 0. \end{array} \right. \quad (5.119)$$

Conditions (5.115), (5.116) and (5.117) can be rewritten as follows

$$\hat{h} < \hat{h}^* = \min(h_1, h_2, h_3, h_4) \quad (5.120)$$

with

$$h_1 = \frac{\gamma}{l_u^2 c_4} \left(\frac{c_u}{2} - \varepsilon \bar{c} \right), \quad (5.121)$$

$$h_2 = \frac{1}{c_1 + l_u^2 c_5} \left(\frac{m_g}{3} - \frac{\gamma R^2}{2c_u} - \varepsilon \bar{m} \right), \quad (5.122)$$

$$h_3 = \frac{1}{\sqrt{2}c_2}, \quad (5.123)$$

$$h_4 = \frac{1}{c_5} \left(\frac{1}{2D} - \frac{3N^2}{4m_g} \right). \quad (5.124)$$

In order to have h_4 positive, D has to comply with the following condition

$$D < D^* = \frac{2m_g}{3N^2}. \quad (5.125)$$

Choosing γ and ε sufficiently small guarantees that h_1, h_2 are positive. In this case, taking $\hat{h} < \hat{h}^*$ and $D < D^*$ ensures that

$$\dot{V}(t) \leq -\varepsilon V(t). \quad (5.126)$$

So equation (5.89) can be deduced from (5.126) by similar computations as in Theorem 4.5. \square

Qualitative analysis: The condition $\hat{h} < \hat{h}^*$ is similar to the condition for the known delay case (Section 4.2.2). The condition D sufficiently small means that the delay estimation has to be close enough to the real value of the delay. Some remarks about the qualitative behavior of the bounds \hat{h}^* and D^* are given below.

- For a large value of l_u ¹⁰ then h_1, h_2 become smaller which means that the admissible delay is smaller for a fast controller.
- For a large value of l_g ¹¹ then h_2, h_3 become smaller which means that the admissible delay is smaller for a fast observer-predictor.
- For a large value of ε then h_1, h_2 become smaller which means that, to achieve a fast convergence rate of the closed-loop system, the delay has to be small enough.

10. l_u is the Lipschitz constant associated to the function u and is defined in (3.89).

11. l_g is the Lipschitz constant associated to the function g and is defined in (3.95).

Time-varying delay

The class of systems considered is defined by (\mathcal{E}_h) in (3.84) and is reminded below

$$(\mathcal{E}_h) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t - h(t)) \\ y(t) = Cx(t) \end{cases}$$

with $h(t)$ an unknown time-varying input delay. Since the delay is unknown, the prediction z is computed from an estimated delay $\hat{h}(t)$ by the observer-predictor

$$\dot{z}(t) = Az(t) + Bu(t) + g(Cz(t - \hat{h}(t)) - y(t)) \quad (5.127)$$

where the correction term g satisfies Assumption 3.4.2.

Remark 5.2.2. Note that to compute $z(t)$, the initial condition $z(t) = \phi_z(t)$ for $t \in [-h_{max}, 0]$ with ϕ_z a continuous function is required. The prediction z will be continuous in $t = \hat{h}(t)$.

Since Assumption 3.4.1 holds, there exists a feedback $u(x)$ that exponentially stabilizes system (\mathcal{E}_h) when $h(t) = 0$. Then a predictive feedback $u(z)$ is going to be applied to stabilize (\mathcal{E}_h) . Denoting the prediction error

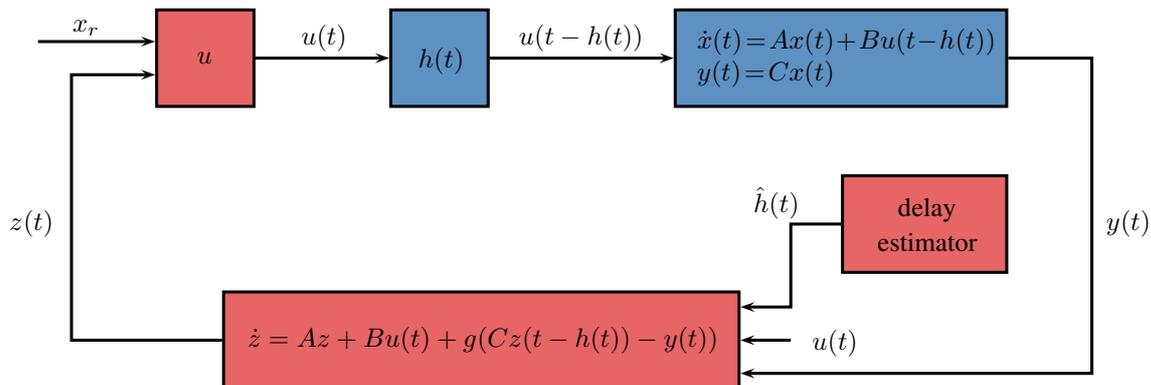


Figure 5.5 – Closed-loop scheme with observer-predictor for unknown time-varying delay and partial state measurement

$$e_p(t) = z(t - \hat{h}(t)) - x(t) \quad (5.128)$$

and reminding that $\hat{\delta}$ stands for the upper bound of the delay time derivative: $|\dot{\hat{h}}(t)| < \hat{\delta}$ (Assumption 3.4.6), it is now possible to introduce the theorem that extends the work of Najafi et al. [2013] to unknown and time-varying delays, to partial state knowledge and to a larger class of controllers.

Theorem 5.6. Consider system (\mathcal{E}_h) , where $h(t)$ is unknown and complies with Assumptions 3.4.3 and 3.4.4, and assume that there exist a controller $u(x)$ and a correction term $g(e_p)$ satisfying Assumptions 3.4.1 and 3.4.2 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z)$ with z computed by observer-predictor (5.127), where $\hat{h}(t)$ is an estimation of $h(t)$ that complies with Assumptions 3.4.5 and 3.4.6, and define

$$\Upsilon(t) = \|x(t)\|^2 + \sup_{s \in [t-h_{max}, t]} \|z(s)\|^2 + \sup_{s \in [t-h_{max}, t]} \|\dot{e}_p(s)\|^2 + \sup_{s \in [t-h_{max}, t]} \|\dot{u}(s)\|^2. \quad (5.129)$$

Then, there exist $\varsigma, \varrho, h^*, \hat{\delta}^*, D^* > 0$ such that, provided $h_{max} < h^*$, $\hat{\delta} < \hat{\delta}^*$ and $D =$

$\max_{t \geq 0} |\hat{h}(t) - h(t)| < D^*$, $\hat{\delta} < \hat{\delta}^*$, one has

$$\forall t \geq 0 \quad \Upsilon(t) \leq \varsigma \Upsilon(0) e^{-\alpha t}. \quad (5.130)$$

Therefore $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ and the convergence is exponential.

Proof. Note that the argument “ t ” may be omitted to facilitate the reading of the proof. First, combining (5.127) with (\mathcal{E}_h) leads to

$$\dot{z}(t) = Az(t) + Bu(t) + g(Ce_p(t)). \quad (5.131)$$

The dynamics of $e_p(t)$ reads as

$$\dot{e}_p(t) = Ae_p(t) + B[u(t - \hat{h}) - u(t - h)] + g(Ce_p(t - \hat{h})) - \dot{\hat{h}}[Az(t - \hat{h}) + Bu(t - \hat{h}) + g(Ce_p(t - \hat{h}))] \quad (5.132)$$

and can be rewritten as follows

$$\begin{aligned} \dot{e}_p(t) = & Ae_p(t) + g(Ce_p(t)) - B \int_{t-\hat{h}}^{t-h} \dot{u}(s) ds - \int_{Ce_p(t-\hat{h})}^{Ce_p(t)} \frac{dg}{ds}(s) ds \\ & - \dot{\hat{h}}[Az(t - \hat{h}) + Bu(t - \hat{h}) + g(Ce_p(t - \hat{h}))]. \end{aligned} \quad (5.133)$$

The following Lyapunov-Krasovskii functional candidate is proposed

$$V(t) = \gamma V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t) \quad (5.134)$$

where

$$V_1(t) = z^T(t) P z(t), \quad (5.135)$$

with P defined in Assumption 3.4.1,

$$V_2(t) = e_p^T(t) Q e_p(t), \quad (5.136)$$

with Q defined in Assumption 3.4.2,

$$V_3(t) = \int_{t-\hat{h}}^t (\hat{h} + s - t) \|\dot{e}_p(s)\|^2 ds, \quad (5.137)$$

$$V_4(t) = \int_{t-\hat{h}}^t (\hat{h} + s - t) \|\dot{u}(s)\|^2 ds, \quad (5.138)$$

$$V_5(t) = \int_{t-\hat{h}}^t (h_{\max} + h_{\min} + s - t) \|z(s)\|^2 ds, \quad (5.139)$$

$$V_6(t) = \int_{t-\hat{h}}^t (h_{\max} + h_{\min} + s - t) \|e_p(s)\|^2 ds, \quad (5.140)$$

and $\gamma > 0$. Remark that V_1 , V_2 , V_3 and V_4 are the same as in the constant delay case and V_5 and V_6 are added to deal with the term “ $\dot{\hat{h}}[Az(t - \hat{h}) + Bu(t - \hat{h}) + g(Ce_p(t - \hat{h}))]$ ” that arises because of the time-varying delay. The end of the proof is obtained applying the same arguments as in Theorems 5.3 and 5.5. \square

5.2.2 Sequential sub observers-predictors with an unknown delay

As it was shown in Section 4.2, the sequential sub predictors allows to cope with a large and known delay. This idea can be extended to an unknown delay.

Constant delay

First, the sequential sub predictors idea is applied to the partial state measurement case with a constant unknown delay. Each of the sub system estimates and predicts the state for \bar{h} seconds where

$$\bar{h} = \frac{\hat{h}}{r}, \quad r \in \mathbb{N}^*. \quad (5.141)$$

with \hat{h} a constant delay estimation. Sequential sub observers-predictors equations are as follows:

$$\begin{cases} \dot{z}_1(t) = Az_1(t) + Bu(t - (r-1)\bar{h}) + g_1([Cz_1(t - \bar{h}) - y(t)]) \\ \vdots \\ \dot{z}_i(t) = Az_i(t) + Bu(t - (r-i)\bar{h}) + g_i(C[z_i(t - \bar{h}) - z_{i-1}(t)]) \\ \vdots \\ \dot{z}_r(t) = Az_r(t) + Bu(t) + g_r(C[z_r(t - \bar{h}) - z_{r-1}(t)]) \end{cases} \quad (5.142)$$

where $z_i \in \mathbb{R}^n$, $i = 1, \dots, r$.

Remark 5.2.3. Note that to compute $z_r(t)$, the initial conditions $z_i(t) = \phi_{z_i}(t)$ for $t \in [-\bar{h}, 0]$ with ϕ_{z_i} continuous functions are required for all $i = 1, \dots, r$.

Assuming an appropriate choice of the correction terms g_i , it follows that $z_r(t)$ tends to $x(t + \hat{h})$. Thus, a predictive controller $u(z_r)$ that verifies Assumption 3.4.1 can be applied to stabilize (\mathcal{E}_h) . Reminding that r is the number of sub observer-predictors and defining the sub prediction errors

$$e_{p1} = z_1(t - \bar{h}) - x(t) \quad (5.143)$$

and

$$e_{pi}(t) = z_i(t - \bar{h}) - z_{i-1}(t) \quad (5.144)$$

for all $i = 2, \dots, r$, the following theorem holds.

Theorem 5.7. Consider system (\mathcal{E}_h) , with an unknown and constant delay h , and assume that there exist a controller $u(x)$ and correction terms $g_i(e_{pi})$ satisfying Assumptions 3.4.1 and 3.4.1 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z_r)$ with z_r computed by sequential observers-predictors (5.142) where \hat{h} is a constant estimation of h . Then, there exist $r^* \in \mathbb{N}$ and $D^* > 0$, such that provided $r > r^*$ and $|\hat{h} - h| < D^*$, $\|x(t)\|$ exponentially converges to zero.

The proof follows the same steps as those of Theorem 4.7 and the computations for the first step of the proof are similar to those of Theorem 5.5.

Time-varying delay

An example for the case of two cascaded sub observers-predictors is given on Figure 5.6. The idea is similar to the constant delay case presented before.

Each of the r observer-predictor has a time-varying prediction horizon that depends on the estimated delay \hat{h}

$$\bar{h}(t) = \frac{\hat{h}(t)}{r}, \quad r \in \mathbb{N}^*. \tag{5.145}$$

The equations of the sequential sub observers-predictors are given below

$$\begin{cases} \dot{z}_1(t) = Az_1(t) + Bu(t - (r - 1)\bar{h}(t)) + g_1([Cz_1(t - \bar{h}(t)) - y(t)]) \\ \vdots \\ \dot{z}_i(t) = Az_i(t) + Bu(t - (r - i)\bar{h}(t)) + g_i(C[z_i(t - \bar{h}(t)) - z_{i-1}(t)]) \\ \vdots \\ \dot{z}_r(t) = Az_r(t) + Bu(t) + g_r(C[z_r(t - \bar{h}(t)) - z_{r-1}(t)]) \end{cases} \tag{5.146}$$

where $z_i \in \mathbb{R}^n, i = 1, \dots, r$.

Remark 5.2.4. Note that to compute $z_r(t)$, the initial conditions $z_i(t) = \phi_{z_i}(t)$ for $t \in [-h_{max}/r, 0]$ with ϕ_{z_i} continuous functions are required for all $i = 1, \dots, r$.

Assuming an appropriate choice of the correction terms g_i , it follows that $z_r(t)$ tends to $x(t + \hat{h}(t))$. Thus, a predictive $u(z_r)$ that verifies Assumption 3.4.1 can be applied to stabilize (\mathcal{E}_h) . We define the sub prediction errors

$$e_{p_1} = z_1(t - \bar{h}(t)) - x(t) \tag{5.147}$$

and

$$e_{p_i}(t) = z_i(t - \bar{h}(t)) - z_{i-1}(t) \tag{5.148}$$

for all $i = 2, \dots, r$. The advantage of the sequential structure is to remove the condition on the maximum delay value as it is stated in Theorem 5.8. It is reminded that $\hat{\delta}$ stands for the upper bound of the delay time derivative: $|\dot{\hat{h}}(t)| < \hat{\delta}$ (Assumption 3.4.6).

Theorem 5.8. Consider system (\mathcal{E}_h) , where $h(t)$ is unknown and complies with Assumptions 3.4.3 and 3.4.4, and assume that there exist a controller $u(x)$ and correction terms $g_i(e_{p_i})$ satisfying Assumptions 3.4.1 and 3.4.1 respectively. Suppose that system (\mathcal{E}_h) is controlled by $u(z_r)$ with z_r computed by sequential observers-predictors (5.146) where $\hat{h}(t)$ is an estimation of $h(t)$ that complies with Assumptions 3.4.5 and 3.4.6. Then, there exist $r^* \in \mathbb{N}, \hat{\delta}^* > 0$ and $D^* > 0$, such that provided $r > r^*, \hat{\delta} < \hat{\delta}^*$ and $\max_{t \geq 0} |\hat{h}(t) - h(t)| < D^*, \|x(t)\|$ exponentially converges to zero.

The proof follows the same steps as those of Theorem 4.7 and the computations for the first step of the proof are similar to those of Theorem 5.6.

5.2.3 Illustrative example

The same system as in Example 4.2.1 is used to illustrate above results. This is a double integrator

$$\ddot{y}(t) = u(t - h(t)) \tag{5.149}$$

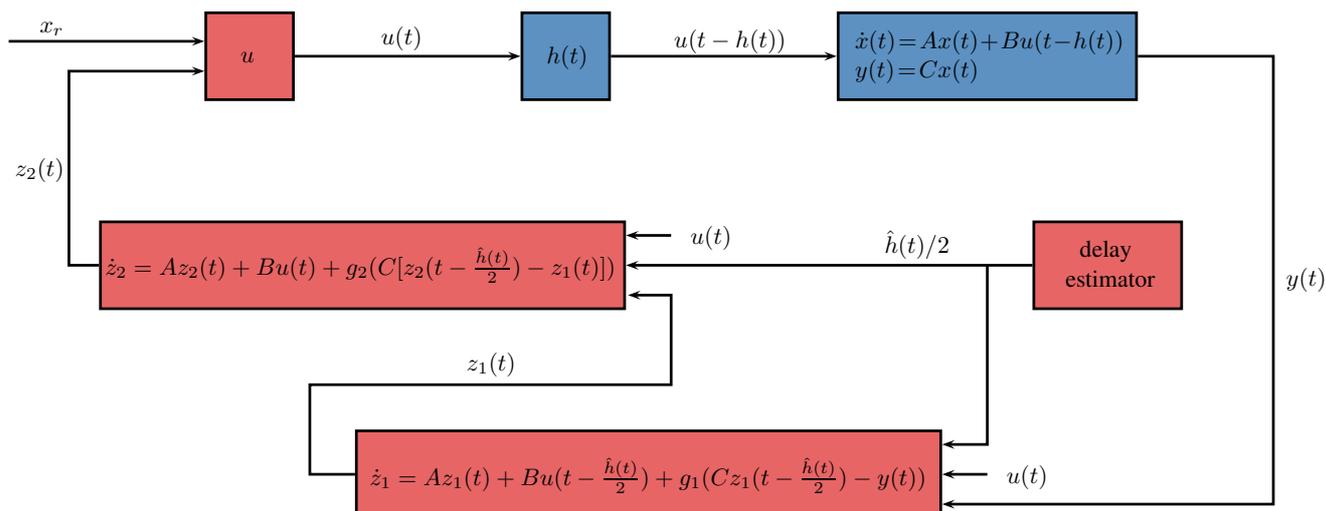


Figure 5.6 – Closed-loop scheme with two sequential sub observers-predictors (4.213) for an unknown and time-varying delay and partial state measurement.

with the state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t - h(t)) \quad (5.150)$$

with $x = [y, \dot{y}]^T$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The Luenberger observer-predictor reads as

$$\dot{\hat{z}} = A\hat{z} + Bu(t) + L(Cz(t - \hat{h}(t)) - y(t)) \quad (5.151)$$

with

$$L = [-1.5, 0.5]^T. \quad (5.152)$$

The eigenvalues of $A + LC$ are placed to -0.5 and -1 . Then, the static feedback

$$u(t) = Kz(t) \quad (5.153)$$

where the gain $K = [-2, -3]^T$ is chosen in order to place the eigenvalues of the system to -1 and -2 .

Remark 5.2.5. *In this example, the variable y has no particular physical meaning so it will be displayed without unit in the next plots.*

On Figure 5.7, the dynamic predictive feedback with a single observer-predictor ($r = 1$) is compared to the dynamic predictive feedback with sequential observers-predictors $r = 2$. The delay estimation error is the same for Figures 5.7a and 5.7b. However, it can be observed that when the delay $h(t)$ is small (Figure 5.7a) then the results are similar for both controllers. On the contrary, when the delay is larger (Figure 5.7b), the controller with $r = 1$ diverges. This illustrates that increasing the number of sub observers-predictors allows to deal with a larger unknown delay.

Figure 5.8 shows a comparison between standard and dynamic predictive feedbacks. A Luenberger state observer, with the same gain L , defined in (5.152), is designed to estimate the state and compute prediction (5.76). The performances of both feedbacks are very similar so it confirms the interest of using a dynamic predictive control since it is much easier to implement. Indeed, the dynamic prediction is computed thanks to a dynamic system whereas the standard predictive control requires an integral discretization and a state observer.

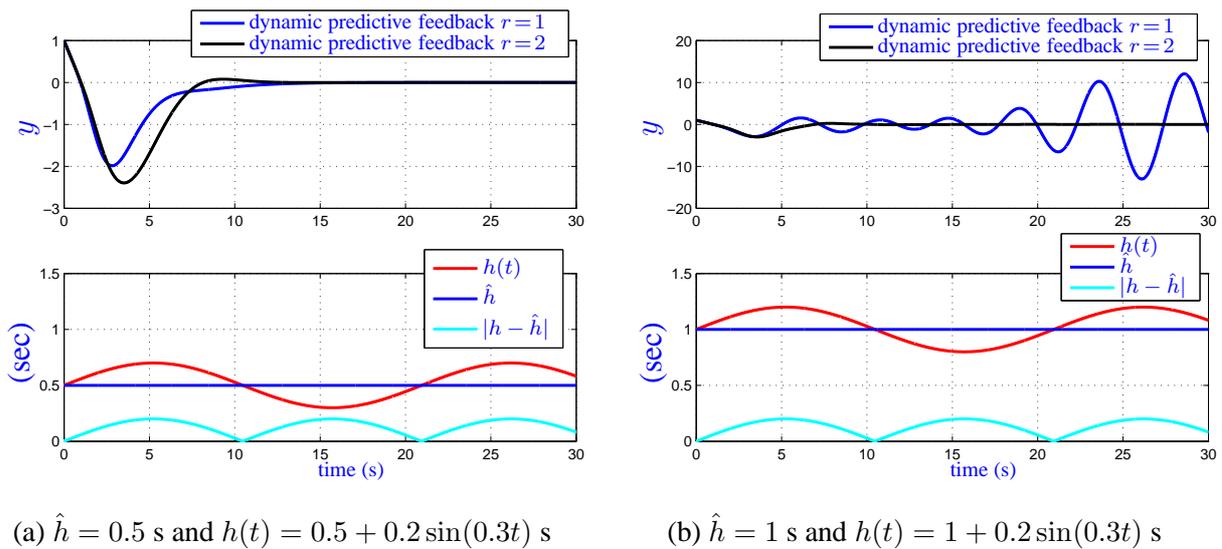


Figure 5.7 – Comparison of dynamic predictive feedbacks with a single observer-predictor $r = 1$ (5.127) and with sequential observers-predictors $r = 2$ (5.146).

Finally, the influence of the delay estimation accuracy is illustrated on Figure 5.9. On the one hand, a constant delay estimation \hat{h}_1 is used to compute the dynamic prediction (5.146) with $r = 2$. On the other hand, a time-varying delay estimation $\hat{h}_2(t)$ computed thanks to the delay observer (3.15) of Section 3.1 is used. Note that the initial condition of the delay estimator is equal to \hat{h}_1 . It follows that the closed-loop system is unstable for the constant estimation \hat{h}_1 whereas the dynamic predictive feedback using \hat{h}_2 stabilizes the system. It confirms that the delay estimation has to be sufficiently accurate. As soon as the system is stabilized, the delay estimation \hat{h}_2 does not move anymore and the estimation error becomes larger but it has no influence anymore because the input is constant.

5.2.4 Summary

Contributions

- ✓ Extension of the results of Section 4.2 to the unknown delay case.
- ✓ Explicit stability conditions are provided.

Table 5.2 sums up the results of this section. Note that the bounds δ^* , h^* and r^* may have different values for each case.

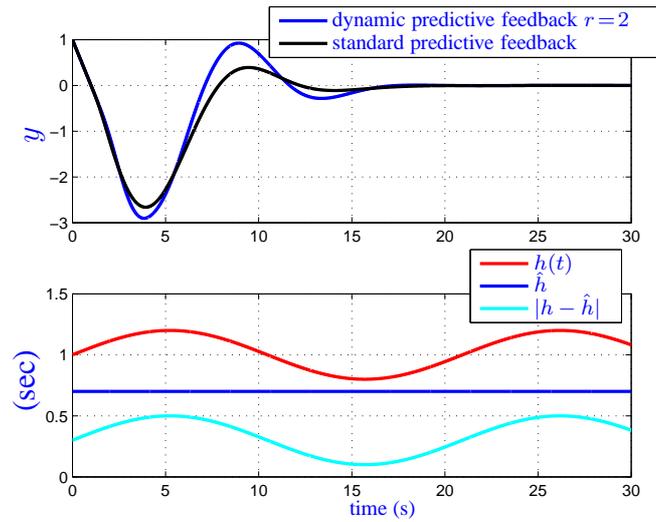


Figure 5.8 – Comparison of the standard predictive feedback (5.76)-(5.151) with the dynamic predictive feedback (5.146)-(5.151) with $r = 2$.

Prediction	Delay	Convergence	Condition	Theorem	
Dynamic (1 obs/pred)	Unknown	h	gl. & exp.	$\hat{h} < \hat{h}^* \ \& \ \hat{h} - h < D^*$	5.5
		$h(t)$	gl. & exp.	$h_{max} < h^* \ \& \ \hat{\delta} < \hat{\delta}^*$ $\max_{t \geq 0} \hat{h}(t) - h(t) < D^*$	5.6
h		gl. & exp.	$r > r^* \ \& \ \hat{h} - h < D^*$	5.8	
$h(t)$		gl. & exp.	$r > r^* \ \& \ \hat{\delta} < \hat{\delta}^*$ $\max_{t \geq 0} \hat{h}(t) - h(t) < D^*$	5.8	

Table 5.2 – Sum up of the convergence results for dynamic prediction with an unknown delay

- gl.: global – exp.: exponential
- h : constant delay
- \hat{h} : constant delay estimation
- $h(t) \in [h_{min}, h_{max}]$ and $|\dot{h}(t)| \leq \delta$: time-varying delay
- $\hat{h}(t) \in [h_{min}, h_{max}]$ and $|\dot{\hat{h}}(t)| \leq \hat{\delta}$: time-varying delay estimation

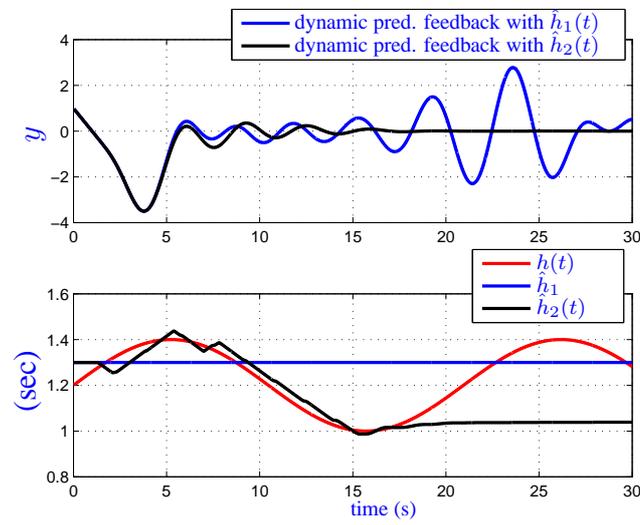


Figure 5.9 – Influence of the delay estimation accuracy with the dynamic prediction computed by sub observers-predictors $r = 2$.

Conclusion

Three predictive techniques have been presented in this part. The standard predictive method has been extended to an unknown and time-varying delay. A recent method based on a dynamic system to compute a prediction has been revisited to deal with partial state knowledge and with an estimated delay. Finally, a new prediction has been proposed to include information about unmodeled dynamics into the prediction. Explicit closed-loop stability conditions have been provided in the case of known and unknown delays thanks to a Lyapunov-Krasovskii analysis. It has been proved that if the delay estimation is sufficiently accurate then the properties of the known delay case are still true. As a consequence, observation techniques developed in Part I can be coupled to the different predictive techniques seen in this part in order to compute an approximated prediction. In addition, the technical argument of global Lipschitz continuity may be relaxed since, in practical applications, systems evolve in bounded domains. As a result, the locally Lipschitz property is sufficient. In this case, the results presented in this part can be applied to a large class of controllers. All the results have been illustrated and compared by numerous simulations. In the next part, a validation of these techniques on an experimental setup is done in order to evaluate their efficiency in real conditions.

A comparison of the different strong points of each prediction is drawn in Table 5.3. Some explanations about the criteria are given here:

- the “Accuracy” criterion refers to the prediction error when the system is perfectly known (no disturbance, no parameter uncertainty);
- the “Complexity” criterion refers to the implementation complexity to compute the prediction;
- the “Robustness” criterion refers to the performance of a predictive feedback associated with the prediction to attenuate an external disturbance or to guarantee an accurate trajectory tracking in presence of parameter uncertainties.

The classification arguments are detailed below. The standard prediction and the new prediction are exact for a perfectly known system with a constant delay. When the delay is time-varying, the correction term of the new prediction can slightly degrade the accuracy of the prediction. The dynamic prediction only converges asymptotically to the exact prediction for both constant and time-varying delays. As far as the complexity, the implementation of the dynamic prediction is reduced to the integration of a dynamic system whereas the standard prediction requires a more complex integral discretization. The new prediction is computed from the standard prediction and its delayed values so it requires a large computation effort. Finally, it has been proved that the new prediction better attenuates the effect of external disturbances and guarantees a more accurate trajectory tracking in presence of parameter uncertainties than the standard prediction. No theoretical argument has been provided to quantify the robustness of the dynamic prediction to unmodeled dynamics but the simulations provided at the end of Section 4.3 seem to exhibit a good robustness property at least for a constant disturbance.

Table 5.4 sums up previous results. Note that the limits δ^* , D^* , $\hat{\delta}^*$, h^* and R^* do not have the same value in the different cases.

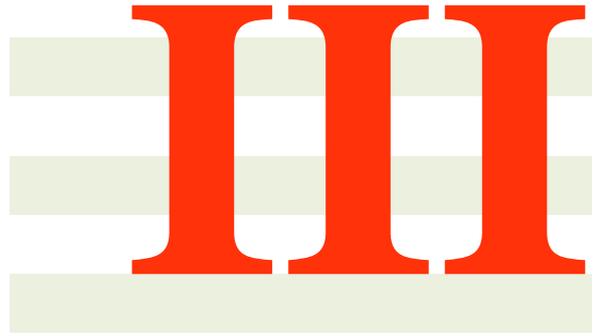
Prediction	Accuracy	Complexity	Robustness
Standard	★★★	★★	★
Dynamic	★	★★★	★★
New	★★	★	★★★★

Table 5.3 – Comparison of the different predictive methods for a known delay. Criterion are explained above.

Prediction	Delay	Conv.	Condition	Feedback	Thm	
Standard	Known	h	gl. & exp.	no condition	state	4.1
					output	4.2
		$h(t)$	gl. & exp.	$\delta < \delta^*$	state	4.3
					output	4.4
	Unknown	h	gl. & exp.	$ \hat{h} - h < D^*$	state	5.1
					output	5.2
		$h(t)$	gl. & exp.	$\hat{\delta} < \hat{\delta}^*$ $\max_{t \geq 0} \hat{h}(t) - h(t) < D^*$	state	5.3
					output	5.4
Dynamic (1 obs/pred)	Known	h	gl. & exp.	$h < h^*$	output	4.5
		$h(t)$	gl. & exp.	$h_{max} < h^* \ \& \ \delta < \delta^*$	output	4.6
	Unknown	h	gl. & exp.	$\hat{h} < \hat{h}^* \ \& \ \hat{h} - h < D^*$	output	5.5
		$h(t)$	gl. & exp.	$h_{max} < h^* \ \& \ \hat{\delta} < \hat{\delta}^*$ $\max_{t \geq 0} \hat{h}(t) - h(t) < D^*$	output	5.6
Dynamic (r obs/pred)	Known	h	gl. & exp.	$r > r^*$	output	4.7
		$h(t)$	gl. & exp.	$r > r^* \ \& \ \delta < \delta^*$	output	4.8
	Unknown	h	gl. & exp.	$r > r^* \ \& \ \hat{h} - h < D^*$	output	5.7
		$h(t)$	gl. & exp.	$r > r^* \ \& \ \hat{\delta} < \hat{\delta}^*$ $\max_{t \geq 0} \hat{h}(t) - h(t) < D^*$	output	5.8

Table 5.4 – Sum up of the convergence result with a known or an unknown delay

- gl.: global – exp.: exponential
- h : constant delay
- \hat{h} : constant delay estimation
- $h(t) \in [h_{min}, h_{max}]$ and $|\dot{h}(t)| \leq \delta$: time-varying delay
- $\hat{h}(t) \in [h_{min}, h_{max}]$ and $|\dot{\hat{h}}(t)| \leq \hat{\delta}$: time-varying delay estimation



Experimental validation

Experimental Setup: DC motor

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6.1 Introduction

DC motors are commonly used in many areas such as robotics or industry. Furthermore, the simple modeling facilitates its use as a benchmark system for the evaluation of new control laws. Numerous techniques have been applied to control DC motors; for example, sliding mode control in [Utkin \[1993\]](#), optimal control in [Pelczewski and Kunz \[1990\]](#). However, only few works consider DC motors with a delayed input. One can cite for example [Matsuo et al. \[2006\]](#), which shows the influence of the delay time distribution on the stability of a DC motor with a PI controller. In [Luck and Ray \[1994\]](#), an observer-based delay compensator, associated with buffers, is designed to reduce the unknown delay variations. An adaptive controller is used

in [Tipsuwan and Chow \[2003\]](#) to follow the "Quality-of-Service" (QoS) variations of the network. All these works consider very small and known delays and a perfect model (no parameter uncertainty, no external perturbation).

Recently, numerous works on the control of DC motor through communication networks have been published. They generally take into account random delays as in [Liu et al. \[2007\]](#) and often also both delay and packet dropout as in [Chai et al. \[2008\]](#), [Zhang et al. \[2011\]](#), [Ahmadi et al. \[2014\]](#), [Sun et al. \[2014\]](#), [Li et al. \[2009\]](#). Memoryless controllers are used in [Zhang et al. \[2011\]](#) and [Ahmadi et al. \[2014\]](#) because the delays are small (< 100 ms). When the delay becomes larger, predictive techniques are used as in [Chai et al. \[2008\]](#), [Liu et al. \[2007\]](#) and [Sun et al. \[2014\]](#). The work by [Li et al. \[2009\]](#) does not use predictive control, that is why the performances are clearly degraded when the delay increases. Among the works that use predictive techniques, none considers an external disturbance or parameter uncertainties.

In the sequel, the estimation and control strategies presented in Parts [I](#) and [II](#) are used to control the speed of a DC motor in presence of a large delay (with respect to the motor dynamics) and unmodeled dynamics (external disturbance and parameter uncertainties).

6.2 Experimental setup and model presentation

6.2.1 Experimental setup

The experimental setup¹ is composed by a DC motor, a synchronous motor, a torque sensor, an asynchronous motor and an encoder. They are coupled by flexible joints. The platform is represented on [Figure 6.1](#). The plant to control is the DC motor and the synchronous motor has been used to introduce an external disturbance. The torque sensor and the asynchronous motor are not used in these experimentations. The control is computed thanks to a dSPACE® board (DS1104). The optical encoder gives the angular position (1000 pulses per revolution) and the speed is derived from a Kalman filter. The dSPACE® board uses the real-time model produced via Matlab®/Simulink®. The delay is artificially introduced in the loop by adding a delay block in the Simulink® model². The armature voltage of the DC motor is controlled by a PWM signal generated by the dSPACE® from the scaled input voltage $u(t) = \frac{v(t)}{v_{max}}$ with $v_{max} = 54V$.

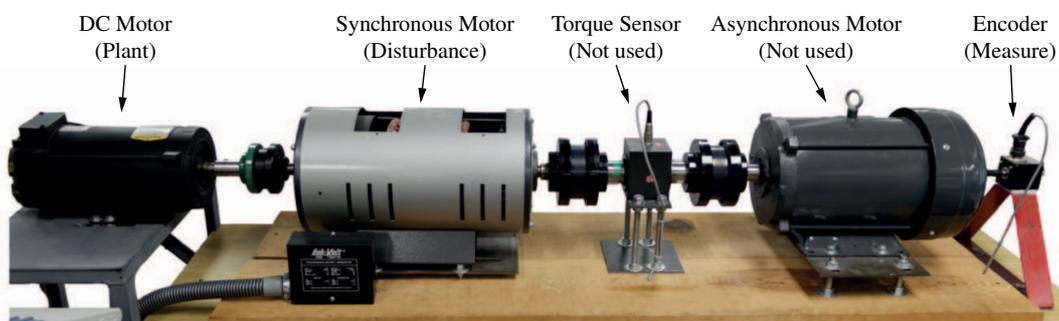


Figure 6.1 – Experimental setup for the control of a DC motor

1. The experiments have been carried out in the FIME laboratory of the University of Nuevo León (Mexico).
 2. This delay can represent, for example, the communication latency between the motor and the remote controller.

6.2.2 Modeling

The transfer function of the DC motor is:

$$\frac{\Omega(s)}{U(s)} = \frac{K_{DC}}{1 + s\tau} \quad (6.1)$$

with Ω and U the Laplace transforms of the angular velocity ω and the input voltage u respectively. The steady-state gain K_{DC} and the time constant τ have been identified and are given in Table 6.1. The transfer function $\frac{K_{DC}}{1+s\tau}$ is a classical simplified model for DC motors where

K_{DC} (min ⁻¹)	τ (s)
894	1.10

Table 6.1 – DC Motor model parameters

the inductance term is neglected. The model is assumed to be linear and then does not take into account any nonlinear phenomenon such as dry friction. However, the identification algorithm gives a fitting rate of about 95%; so the model precisely describes the motor behavior. To apply above results, system (6.1) has to be turned into its state-space representation

$$\dot{\omega}(t) = a\omega(t) + bu(t) \quad (6.2)$$

with $a = -1/\tau$ and $b = K_{DC}/\tau$. The reference trajectory is given by $\omega_r(t)$ so from the equilibrium equation $\dot{\omega}_r(t) = a\omega_r(t) + bu_r(t)$, one has $u_r(t) = \frac{1}{b}(\dot{\omega}_r(t) - a\omega_r(t))$. It is assumed that

Assumption 6.2.1. *The expression of the trajectory $\omega_r(t)$ is known for all $t \geq 0$.*

Note that the argument “ t ” will be omitted in ω_r and u_r when the reference is constant. In the next sections, various tests are performed on the DC motor to show the efficiency of the different techniques presented before. Only experimental plots will be displayed.

6.3 Experimental results without delay

In this first section, some controllers have been tested on the delay free system

$$\dot{\omega}(t) = a\omega(t) + bu(t) + d(t) \quad (6.3)$$

to assess the “ideal” response time and to have a comparison point for the next sections when an input delay will be added. In order to drive the motor speed ω to a desired position ω_r , three controllers have been designed. The objective is not to compare the controllers themselves but to illustrate that the predictive schemes can be applied to any controller that is able to stabilize the delay free system. That is why some standard controllers have been chosen: a static state feedback, a PI controller and a Super Twisting algorithm.

Memoryless static feedback controller The static feedback controller, called “K” in the sequel, reads as

$$u(t) = -k(\omega(t) - \omega_r(t)) + u_r(t). \quad (6.4)$$

The gain k chosen here is given in Table 6.2.

Memoryless PI controller The PI controller, called “PI” in the sequel, reads as

$$u(t) = u_r(t) - k_p(\omega(t) - \omega_r(t)) - k_i \int_0^t (\omega(s) - \omega_r(s)) ds. \quad (6.5)$$

The gains k_p and k_i chosen here are given in Table 6.2.

Memoryless Super Twisting algorithm Levant [1993] The Super Twisting algorithm, called “ST” in the sequel, reads as

$$u(t) = u_r(t) + \frac{1}{b} \left[-a(\omega(t) - \omega_r(t)) - k_1 \sqrt{|\omega(t) - \omega_r(t)|} \text{sign}(\omega(t) - \omega_r(t)) + \nu(t) \right] \quad (6.6)$$

with the dynamics of ν governed by the following equation

$$\dot{\nu}(t) = -k_2 \text{sign}(\omega(t) - \omega_r(t)). \quad (6.7)$$

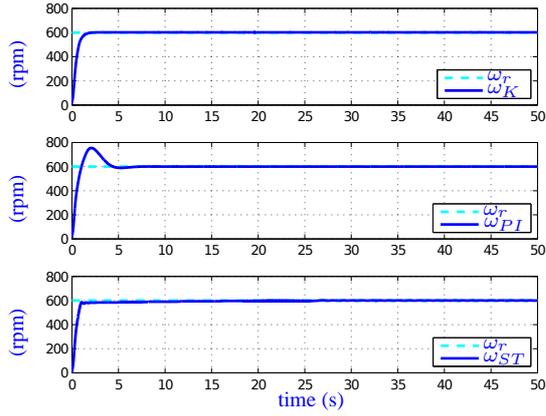
The gains should satisfy $k_2 > 0$ and $k_1 > \sqrt{2k_2}$. Their expressions for the control of the DC motor are given in Table 6.2. This controller is a second order sliding mode controller so it does not require the acceleration values and has a robust finite-time convergence.

Controller				
K	PI		ST	
k	k_p	k_i	k_1	k_2
0.002	0.001	0.002	0.25	0.025

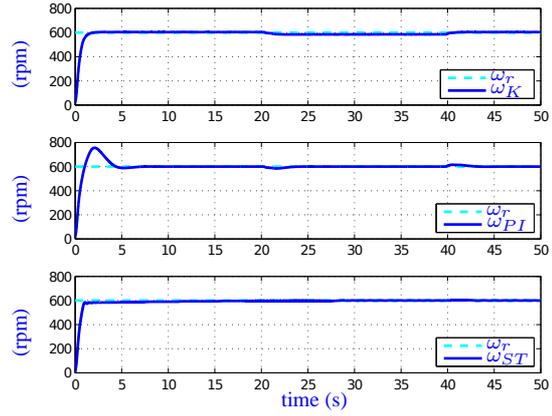
Table 6.2 – Controller gains for the control of the DC motor

Remark 6.3.1. *Controller gains will be the same in all experimental results presented thereafter.*

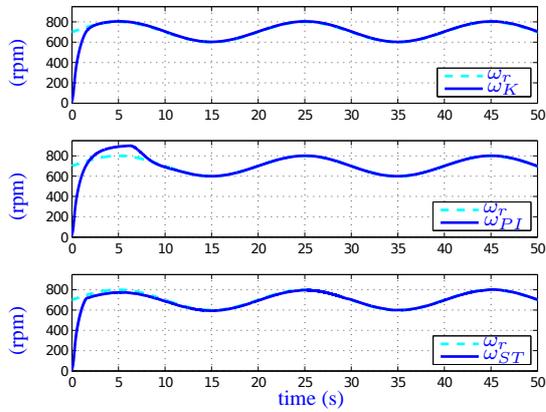
A comparison of the performance of these controllers is drawn on Figure 6.2. The notations ω_K , ω_{PI} and ω_{ST} hold for the state of system (6.3) when the loop is closed respectively by controllers K (6.4), PI (6.5) and ST (6.6). It can be observed on Figure 6.2a that controllers K and ST have the same response time of about 3 seconds whereas controller PI is slower (6 seconds response time). In the disturbance free case, the motor speed accurately converges to the reference (Figures 6.2a and 6.2b). On Figures 6.2c and 6.2d, it can be observed that the constant disturbance is perfectly rejected by controllers PI and ST whereas a small error remains for controller K. To sum up, Figure 6.2 validates the tuning of the different controllers in the delay free case. In presence of a delay in the loop, the performances will be necessarily worsen.



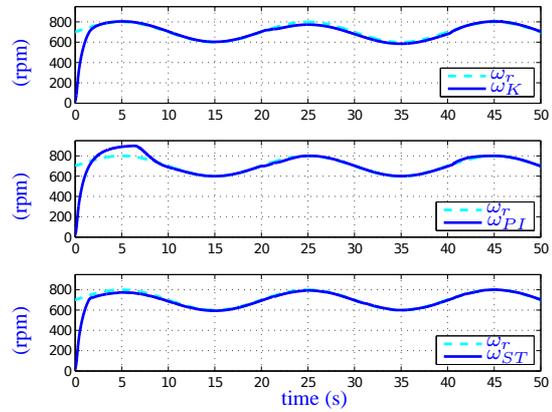
(a) Trajectory tracking with $\omega_r = 600$ rpm and without disturbance ($d = 0$)



(b) Trajectory tracking with $\omega_r = 600$ rpm with a constant disturbance d between 20 s and 40 s



(c) Trajectory tracking with $\omega_r = 700 + 100 \sin(0.1\pi t)$ rpm and without disturbance ($d = 0$)



(d) Trajectory tracking with $\omega_r = 700 + 100 \sin(0.1\pi t)$ rpm with a constant disturbance d between 20 s and 40 s

Figure 6.2 – Comparison of controllers K (6.4), PI (6.5) and ST (6.6), with and without disturbance and for two different trajectories w_r .

6.4 Experimental results with a known input delay

6.4.1 Memoryless/Memory control

In this section, the input is retarded by a piecewise constant delay h so system (6.2) becomes

$$\dot{\omega}(t) = a\omega(t) + bu(t-h) \quad (6.8)$$

The delay h is not internal to the motor model but is supposed to be introduced by the control input, as for example, by the remote control over a network. In order to design a memory controller, the standard prediction is introduced

$$\omega_{\hat{p}}(t) = e^{ah}\omega(t) + \int_{t-h}^t e^{a(t-s)}bu(s)ds. \quad (6.9)$$

The computation of $\omega_{\hat{p}}$ from (6.9) requires an integration. Usually, the integral has to be discretized in a finite number of points. However, this method can lead to instability for some

systems [Mondié and Michiels \[2003\]](#). Since the DC motor is open-loop stable, it is possible to compute the integral term of (6.9) without discretization, by using the structure given in [Watanabe and Ito \[1981b\]](#). The method is reminded thereafter. The prediction $\omega_{\hat{p}}(t)$ given by (6.9) can be rewritten as follows

$$\omega_{\hat{p}}(t) = e^{ah}\omega(t) + \xi_h(t) \quad (6.10)$$

with

$$\xi_h(t) = \xi(t) - e^{ah}\xi(t-h); \quad (6.11)$$

$\xi(t)$ being the solution of the equation

$$\dot{\xi}(t) = a\xi(t) + bu(t). \quad (6.12)$$

Remark 6.4.1. *Computing $\omega_{\hat{p}}$ from (6.10) gives the exact prediction.*

In the rest of the section, the method of [Watanabe and Ito \[1981b\]](#) will be used to compute the prediction. From this prediction, the predictive controller can be obtained just by “plugging” $\omega_{\hat{p}}$ instead of ω in the “delay free” controller as follows

Predictive static feedback controller Controller (6.4) becomes

$$u(t) = -k(\omega_{\hat{p}}(t) - \omega_r) + u_r \quad (6.13)$$

with $\omega_{\hat{p}}(t)$ defined by (6.9).

Predictive PI controller Controller (6.5) becomes

$$u(t) = u_r - k_p(\omega_{\hat{p}}(t) - \omega_r) - k_i \int_0^t (\omega_{\hat{p}}(s) - \omega_r) ds \quad (6.14)$$

with $\omega_{\hat{p}}(t)$ defined by (6.9).

Predictive Super Twisting algorithm Controller (6.6) becomes

$$u(t) = u_r + \frac{1}{b} \left[-a(\omega_{\hat{p}}(t) - \omega_r) - k_1 \sqrt{|\omega_{\hat{p}}(t) - \omega_r|} \text{sign}(\omega_{\hat{p}}(t) - \omega_r) + \nu(t) \right] \quad (6.15)$$

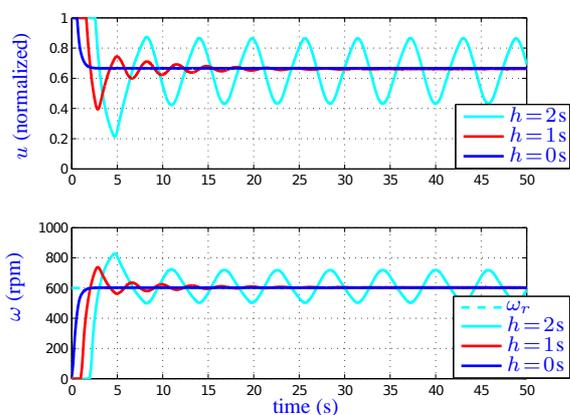
with the dynamics of ν governed by the following equation

$$\dot{\nu}(t) = -k_2 \text{sign}(\omega_{\hat{p}}(t) - \omega_r) \quad (6.16)$$

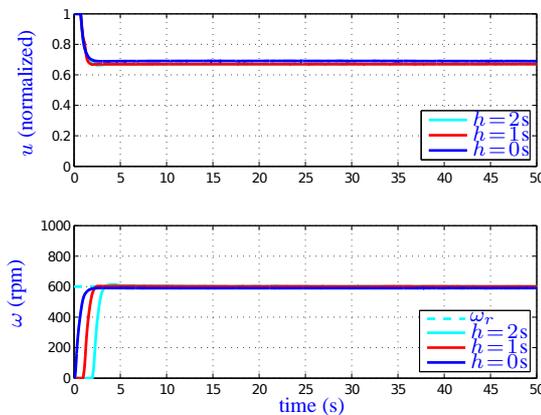
where $\omega_{\hat{p}}(t)$ is defined by (6.9).

Figures 6.3a, 6.3c and 6.3e show the results of memoryless controllers (6.4), (6.5) and (6.6) applied to delayed system (6.8) for different values of h . It can be seen that the performances are worsen in comparison with the delay free case and that all the systems have large oscillations. PI and ST controllers even lead to closed-loop instability (not clear because of input saturation) for a two seconds delay and their performances drastically decrease for $h = 1$ s. Static feedback (6.4) seems to be less affected by the delay because it does not diverge even for $h = 2$ s. Predictive controllers (6.13), (6.14) and (6.15) allow to accurately compensate the delay and guarantee an oscillation free stabilization as shown on Figures 6.3b, 6.3d and 6.3f.

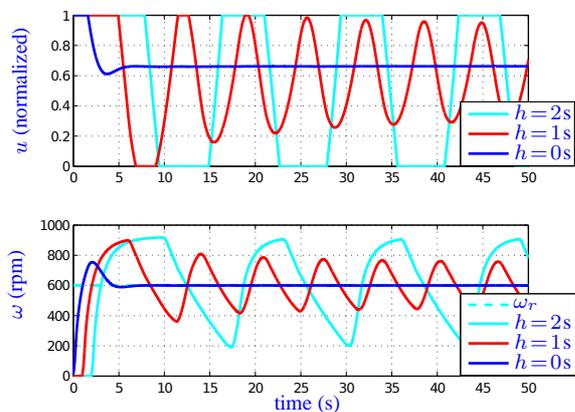
On Figure 6.4, a large piecewise constant delay has been applied to the DC motor. It can be seen that some overshoot affects the system during a time interval of the delay size just after the delay jump. However, in steady state, the tracking error is not degraded even for a very large delay. As a conclusion, memory controllers are very efficient to control the DC motor in presence of a large delay.



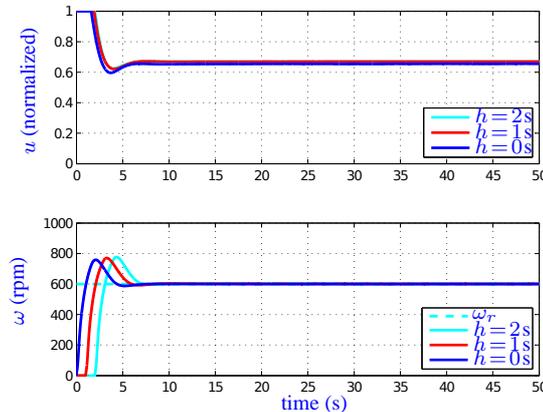
(a) Memoryless controller K (6.4)



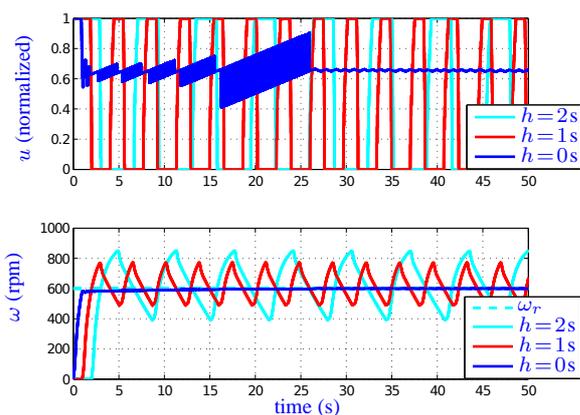
(b) Memory controller K (6.13)



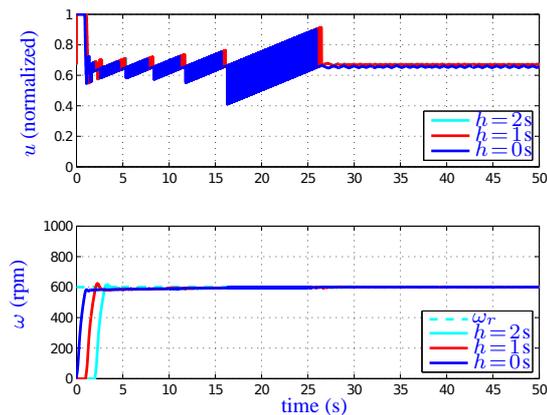
(c) Memoryless controller PI (6.5)



(d) Memory controller PI (6.14)



(e) Memoryless controller ST (6.6)



(f) Memory controller ST (6.15)

Figure 6.3 – Memoryless and memory controllers K, PI and ST applied to delayed system (6.8) for different delays $h = 0$ s, $h = 1$ s and $h = 2$ s.

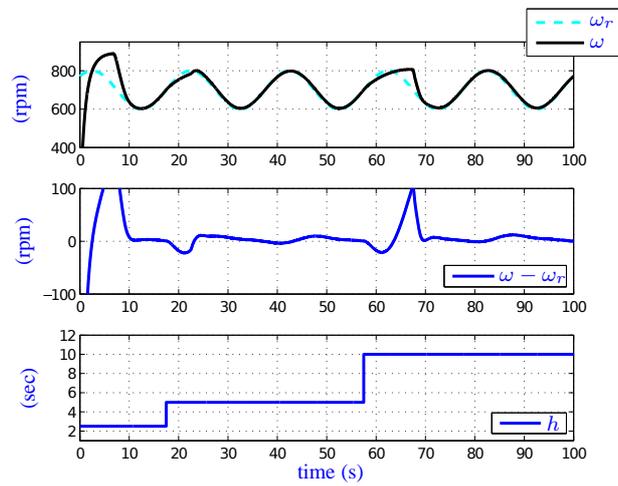


Figure 6.4 – Memory controller (6.14) applied to delayed system (6.8) with large delays ($h = 2.5$ s, $h = 5$ s, $h = 10$ s)

6.4.2 Standard prediction with a time-varying delay

In this section, the input delay is time-varying and system (6.8) becomes

$$\dot{\omega}(t) = a\omega(t) + bu(t - h(t)) \quad (6.17)$$

Thus, standard prediction (6.9) becomes

$$\omega_{\hat{p}}(t) = e^{ah(t)}\omega(t) + \int_{t-h(t)}^t e^{a(t-s)}bu(s)ds \quad (6.18)$$

and the predictive feedback reads as follows.

Predictive static feedback controller Controller (6.13) becomes

$$u(t) = -k[\omega_{\hat{p}}(t) - \omega_r(t + h(t))] + u_r(t + h(t)). \quad (6.19)$$

with $\omega_{\hat{p}}(t)$ defined by (6.18).

Predictive PI controller Controller (6.14) becomes

$$u(t) = u_r(t + h(t)) - k_p[\omega_{\hat{p}}(t) - \omega_r(t + h(t))] - k_i \int_0^t [\omega_{\hat{p}}(s) - \omega_r(s + h(s))]ds. \quad (6.20)$$

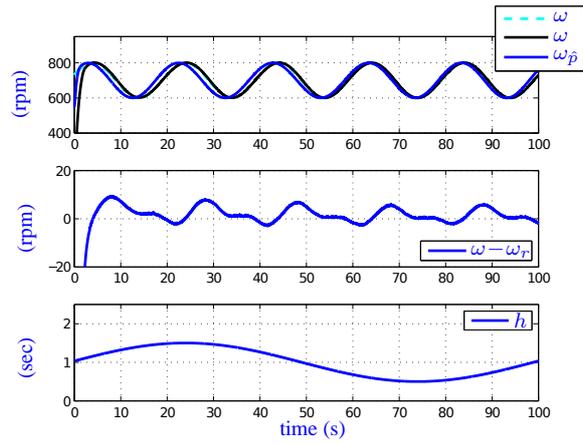
with $\omega_{\hat{p}}(t)$ defined by (6.18).

Remark 6.4.2. Note that it is “ $\omega_r(t + h)$ ” and not “ $\omega_r(t)$ ” in (6.19) and (6.20) because the objective is to have $\omega(t) \rightarrow \omega_r(t)$. The term $\omega_{\hat{p}}$ being the h seconds ahead prediction of $\omega(t)$, $\omega_{\hat{p}}(t)$ must tend to $\omega_r(t + h)$. Moreover, the trajectory $\omega_r(t)$ is known for all $t \geq 0$ (Assumption 6.2.1) then $\omega_r(t + h)$ and $u_r(t + h)$ can be computed at each instant t .

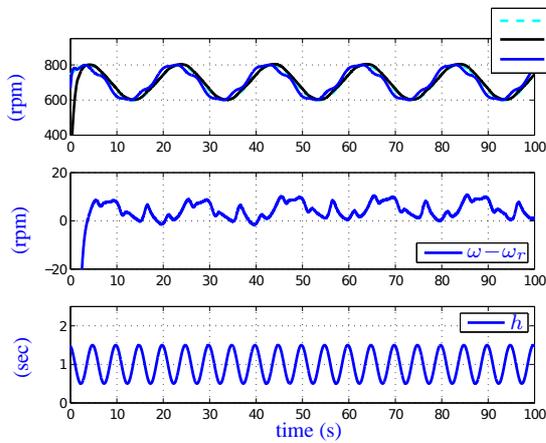
Figure 6.5 displays experimental results when controller (6.19) is applied to system (6.17) for different time-varying delays $h(t)$. It can be observed that the tracking error slightly increases when the delay dynamics is faster. This is consistent with the theoretical result of Section 4.1.2. Indeed, Theorem 4.3 states that there exists an upper bound on the delay dynamics that preserves the closed-loop stability; so it is expected that the behavior of the system worsens when the delay dynamics increases. Since the DC motor is stable and has a relatively “slow”³ dynamics, it may not be destabilized with a “small”⁴ delay even if it is fast-varying. On Figure 6.6, the PI feedback, which is more sensitive to a large delay than the static state feedback (see Section 6.4.1), is applied to system (6.17) with a large and fast delay ($\max_{t \geq 0} |\dot{h}(t)| > 1$). In this case, one can observe that the prediction and the trajectory tracking are not accurate anymore. However, the system does not diverge. A possible reason is that the DC motor is a stable system.

3. It has to be understood as “slow” with respect to the input delay.

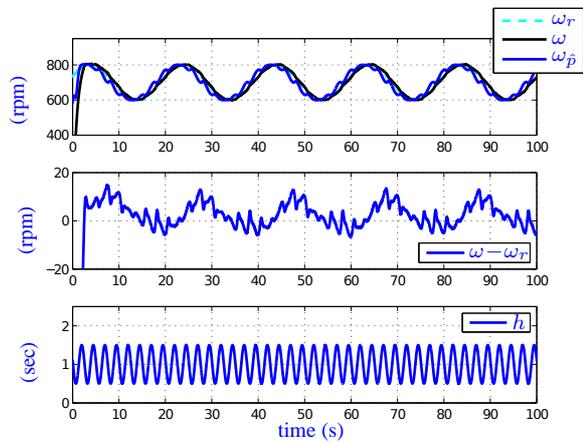
4. It has to be understood as “small” with respect to the motor dynamics.



(a) $h(t) = 1 + 0.5 \sin(0.02\pi t)$



(b) $h(t) = 1 + 0.5 \sin(0.4\pi t)$



(c) $h(t) = 1 + 0.5 \sin(0.8\pi t)$

Figure 6.5 – Static predictive feedback (6.19) with different time-varying delays

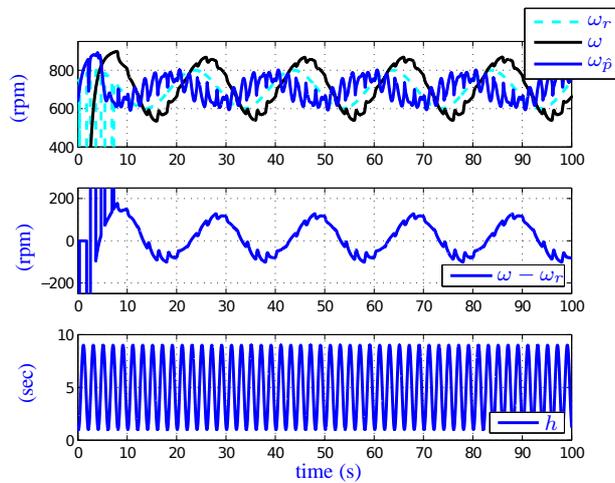


Figure 6.6 – PI predictive feedback (6.20) with large and fast time varying delay $h(t) = 5 + 4 \sin(\pi t)$

6.4.3 Standard/Dynamic prediction

In this section, the dynamic prediction introduced in Section 4.2 is tested. A time-varying delay affects the control input as follows

$$\begin{cases} \dot{\omega}(t) = a\omega(t) + bu(t - h(t)) \\ y(t) = C\omega(t) \end{cases} \quad (6.21)$$

In the case of the DC motor, all the state variables are measured so $C = 1$ and $y(t) = \omega(t)$. As a result, observer-predictor (4.138) becomes

$$\dot{z}(t) = az(t) + bu(t) + L[z(t - h(t)) - \omega(t)]. \quad (6.22)$$

Remark that choosing $L = 0$ would lead to a similar method as the one of [Watanabe and Ito \[1981b\]](#) (see Section 6.4.1). However, this technique can only be applied to stable systems so the gain L is intentionally taken different from zero to test the more general method proposed in Section 4.2.

Predictive PI controller The dynamic predictive PI controller is

$$u(t) = u_r - k_p[z(t) - \omega_r] - k_i \int_0^t [z(s) - \omega_r] ds. \quad (6.23)$$

with $z(t)$ computed by (6.22). Feedback (6.23) is going to be compared to the predictive controller computed from the standard prediction for time-varying delays given by

$$u(t) = u_r - k_p[\omega_p(t) - \omega_r] - k_i \int_0^t [\omega_p(s) - \omega_r] ds. \quad (6.24)$$

with

$$\omega_{\hat{p}}(t) = e^{ah(t)}\omega(t) + \int_{t-h(t)}^t e^{a(t-s)}bu(s)ds. \quad (6.25)$$

The comparison is drawn on Figures 6.7 and 6.8 for different constant delays and on Figure 6.9 for a time-varying one. On Figure 6.7a, it is clear that predictive controllers (6.23) and (6.24) lead to the same behavior. When h is larger, the standard prediction reacts similarly as when $h = 1$ s whereas some oscillations appear for the dynamic prediction (Figure 6.7b). It is mentioned in Section 4.2.2 that there exists an upper bound on the delay value to preserve closed loop stability: increasing h again will increase the oscillations magnitude and at the end, the system will become unstable for h larger than 10 s. In order to circumvent this problem, the sub predictor structure presented in Section 4.2.3 can be used. On Figure 6.8, the performance of a predictive controller with one, two or four sub predictors⁵ is displayed. It is clear that the oscillations are reduced for $r = 2$ and $r = 4$ with respect to the $r = 1$ case (especially on the second step). It seems in accordance with the fact that the maximum delay to preserve stability will also be larger for multiple sub predictors.

Finally, similar conclusions hold for the time-varying delay case, provided that the delay is slow-varying, as it can be observed on Figure 6.8.

5. The number of sub predictors is denoted “ r ”.

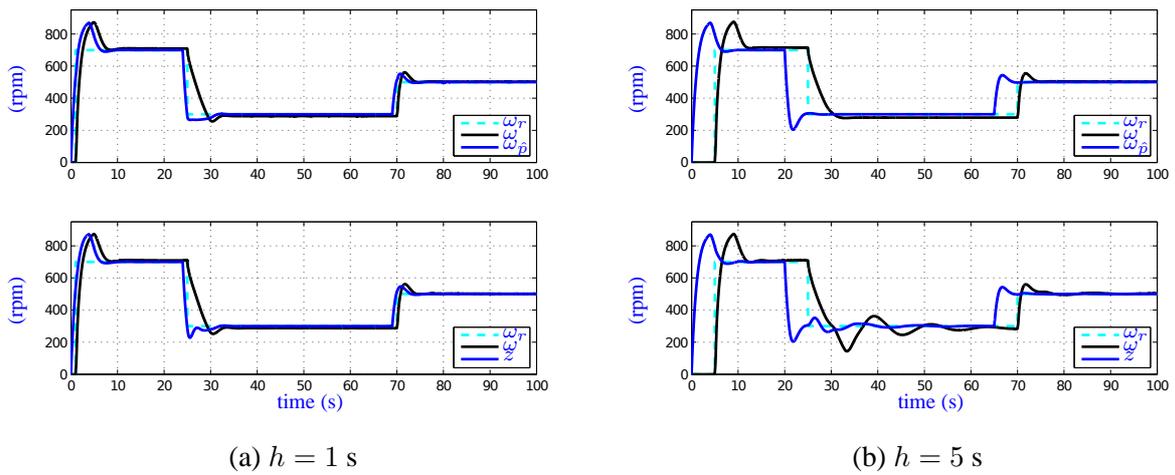


Figure 6.7 – Comparison of the the dynamic predictive feedback (6.23) and the standard predictive feedback (6.24) with $L = 0.5$.

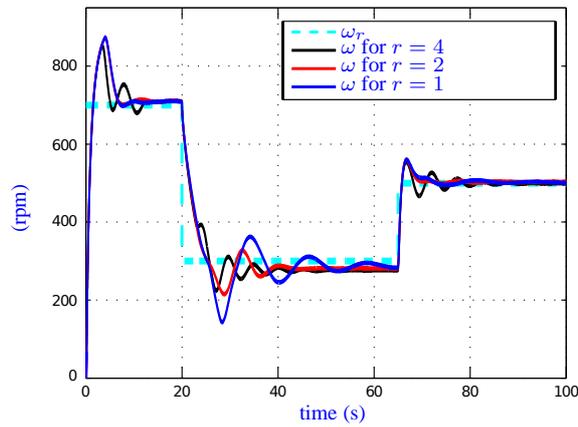


Figure 6.8 – Dynamic prediction with different numbers of sub predictors $r = 1$, $r = 2$ and $r = 4$.

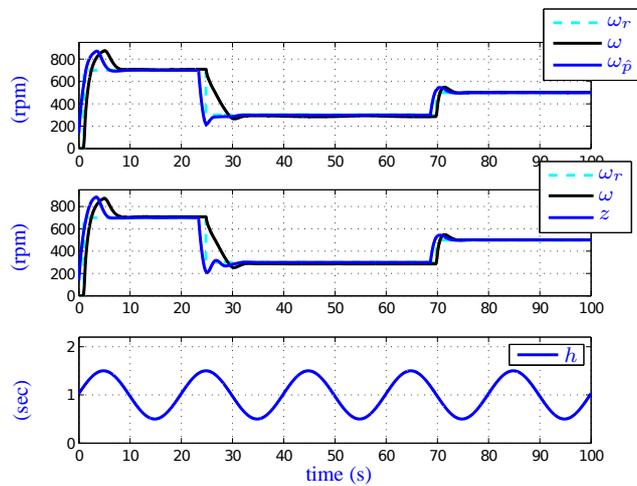


Figure 6.9 – Dynamique prediction

6.4.4 External disturbance and parameter uncertainties

In this section, the robustness of the new prediction with respect to parameter uncertainties and an external disturbance is studied. The following system is considered

$$\dot{\omega}(t) = (a + \Delta a)\omega(t) + (b + \Delta b)u(t - h) + d \quad (6.26)$$

with a and b known nominal terms, Δa and Δb unknown constant modeling errors and d a constant unknown external disturbance.

Remark 6.4.3. *Only the constant disturbance case is studied because, for practical reasons, it is not possible to apply a repeatable time-varying disturbance with the experimental setup at our disposal. Indeed, the external disturbance was created by connecting the synchronous motor of Figure 6.1 to deliver a current into a resistor. The disturbance is then proportional to the value of the resistor but this latter can not be modified.*

The new prediction introduced in Section 4.3 is defined as follows

$$z(t) = \omega_{\hat{p}}(t) + \omega(t) - \omega_{\hat{p}}(t - h) \quad (6.27)$$

where

$$\omega_{\hat{p}}(t) = e^{ah}\omega(t) + \int_{t-h}^t e^{a(t-s)}bu(s)ds. \quad (6.28)$$

Remark 6.4.4. *As it has been explained in Remark 6.4.3, the external disturbance is generated by turning on the asynchronous motor. Therefore, the exact value of perturbation d is not known but it is the same value for each experiment.*

Remark 6.4.5. *The model of the motor is very accurate so some additional uncertainties have been intentionally added to test the new prediction. However, the parameters of the DC motor cannot be modified easily so the uncertainties are added to the computation of the prediction*

$$\omega_{\hat{p}}(t) = e^{(a+\Delta a)h}\omega(t) + \int_{t-h}^t e^{(a+\Delta a)(t-s)}(b + \Delta b)u(s)ds. \quad (6.29)$$

with $\Delta a = -0.2a$ and $\Delta b = 0.2b$ ($\pm 20\%$ error).

The advantage of the predictive schemes is that it is possible to use whatever controllers available for delay free systems and just "plug" the prediction $w_{\hat{p}}$ or z instead of the state ω . To test and compare the schemes, a PI controller and a Super Twisting (ST) controller will be designed. It is important to keep in mind that the objective is to compare both predictions and to illustrate that they can be used with any kind of controllers. Consequently, no comparison between the PI controller and the ST will be drawn.

Predictive PI controller The PI controller reads as

$$u(t) = u_r - k_p(\chi(t) - \omega_r) - k_i \int_0^t (\chi(s) - \omega_r)ds \quad (6.30)$$

with $\chi = \omega_{\hat{p}}$ or $\chi = z$.

Predictive Super Twisting algorithm The Super Twisting algorithm (ST) reads as

$$u(t) = u_r + \frac{1}{b} \left[-a(\chi(t) - \omega_r) - k_1 \sqrt{|\chi(t) - \omega_r|} \text{sign}(\chi(t) - \omega_r) + \dot{\nu}(t) \right] \quad (6.31)$$

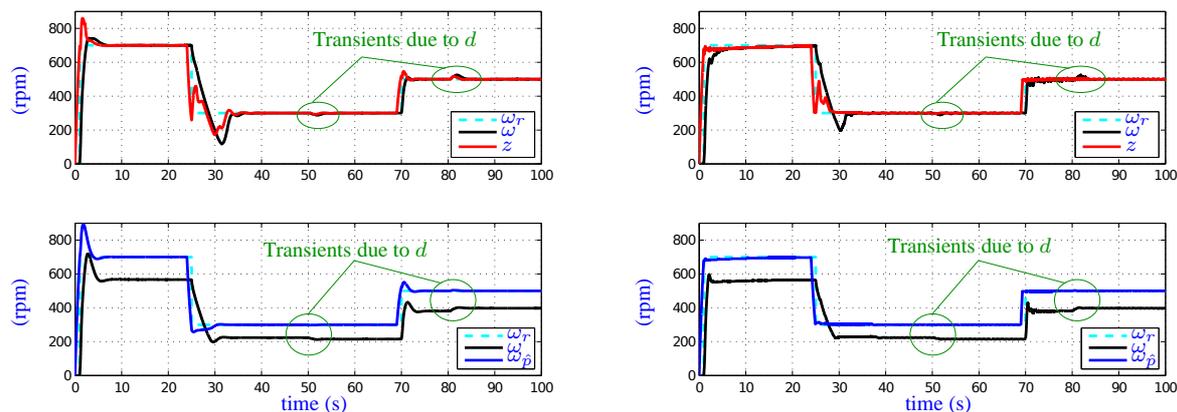
with the dynamics of ν governed by the following equation

$$\dot{\nu}(t) = -k_2 \text{sign}(\chi(t) - \omega_r(t)) \quad (6.32)$$

and $\chi = \omega_{\hat{p}}$ or $\chi = z$.

Controllers based on the standard prediction, $\chi = \omega_{\hat{p}}$, are called standard predictive controllers whereas controllers that use the new prediction, $\chi = z$, are called new predictive controllers. The comparison between standard predictive controllers and new predictive controllers is illustrated on Figure 6.10 for a constant delay and on Figure 6.11 for a time-varying one.

On Figure 6.10, it can be seen that both controllers (PI and ST) achieve a better tracking accuracy in presence of an external disturbance and of parameter uncertainties by using the new prediction $z(t)$. Indeed, the constant disturbance between $t = 50$ s and $t = 80$ s is perfectly rejected as stated in Proposition 4.3.2. In addition, exact trajectory tracking is obtained even in presence of parameter uncertainties. It is in accordance with Proposition 4.3.4. On the contrary, there always exist constant tracking errors for the controllers using the standard prediction as stated in Propositions 4.3.1 and 4.3.3. The conclusions are the same for the time-varying case



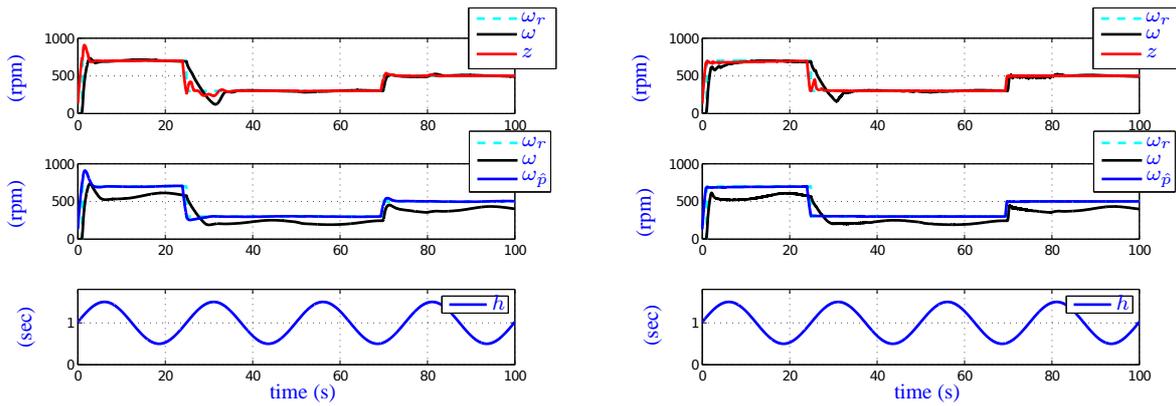
(a) PI controller (6.30) with $\chi = z$ (Top) and $\chi = \omega_{\hat{p}}$ (Bottom) (b) ST controller (6.31) with $\chi = z$ (Top) and $\chi = \omega_{\hat{p}}$ (Bottom)

Figure 6.10 – Comparison of the standard ($\chi = \omega_{\hat{p}}$) and the new predictive controllers ($\chi = z$) for a constant and known delay $h = 1$ s. A constant perturbation is applied between $t = 50$ s and $t = 80$ s parameter uncertainties are given in Remark 6.4.4.

presented on Figure 6.11. Both controllers are able to perfectly reject the external perturbation and the parameter uncertainties so the predictions are stabilized at the desired value. However, only the speed controlled by the new predictive controller tracks accurately the constant reference. Moreover, it can be noted that the tracking error for standard predictive feedbacks depends on the delay amplitude (middle plot on Figures 6.11a and 6.11b). Note that the transient due to the external disturbance is very small and almost not visible at that scale.

An important improvement that would be studied in future works is to compute the new prediction thanks to the dynamic prediction as follows:

$$z_2(t) = z_1(t) + \omega(t) - z_1(t - h) \quad (6.33)$$



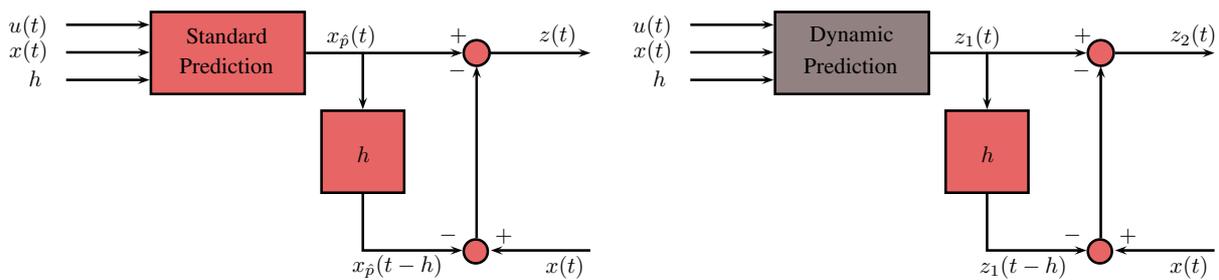
(a) PI controller (6.30) with $\chi = z$ (Top) and $\chi = \omega_{\hat{p}}$ (Bottom) (b) ST controller (6.31) with $\chi = z$ (Top) and $\chi = \omega_{\hat{p}}$ (Bottom)

Figure 6.11 – Comparison of the standard ($\chi = \omega_{\hat{p}}$) and the new predictive controllers ($\chi = z$) for a time-varying delay $h = 1 + 0.5 \sin(0.08\pi t)$. A constant perturbation is applied between $t = 50$ s and $t = 80$ s and parameter uncertainties are given in Remark 6.4.4.

where z_1 is computed by

$$\dot{z}_1(t) = az_1(t) + bu(t) + L[z_1(t - h) - \omega(t)]. \tag{6.34}$$

New predictions z in (6.27) and z_2 in (6.33) are different but the design idea is similar as shown on Figure 6.12. It is expected that the robustness properties of the new prediction (6.33) are preserved and in this case no integral discretization is required since z_1 is computed by (6.34). Some preliminary tests have been done on the DC motor and the results are displayed on Figure 6.13. In both case the PI controller (6.30) is used but with $\chi = z$ or $\chi = z_2$. Figure 6.13 shows that the results are similar for both computation methods which confirms the expectations.



(a) New prediction from standard prediction (b) New prediction from dynamic prediction

Figure 6.12 – Two (slightly) different new predictions

These experimental tests are in accordance with the theoretical results of Section 4.3.1 and thus clearly show the advantage of the new prediction over the standard prediction in presence of unmodeled dynamics.

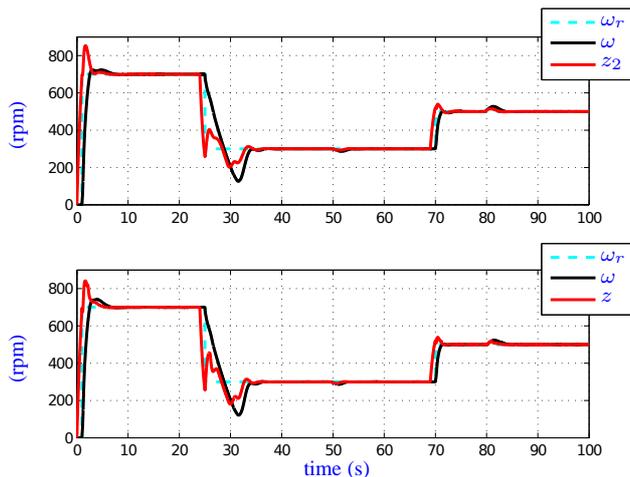


Figure 6.13 – Comparison of the predictions z and z_2 computed respectively from $\omega_{\hat{p}}$ (6.28) and z_1 (6.34) with parameter uncertainties (Remark 6.4.4) and an external perturbation between $t = 50$ s and $t = 80$ s.

6.5 Experimental results with an unknown input delay

As previously mentioned, the exact value of the delay is not always available. In this case, two solutions are possible. A constant mean value of the delay can be used to compute the prediction or a delay estimator⁶ can be designed to get a delay approximation. This two solutions are compared on Figure 6.15. Note that no disturbance nor parameter uncertainty affects the system. The standard predictive feedback given by

$$u(t) = -k[\omega_{\hat{p}}(t) - \omega_r(t + \hat{h}(t))] + u_r(t + \hat{h}(t)) \quad (6.35)$$

with

$$\omega_{\hat{p}}(t) = e^{a\hat{h}(t)}\omega(t) + \int_{t-\hat{h}(t)}^t e^{a(t-s)}bu(s)ds \quad (6.36)$$

is applied to control the DC motor.

In the next experimentations, the delay estimation \hat{h} is

- the mean value of the delay on Figure 6.15a,
- the output of the delay estimator presented in Section 3 on Figures 6.15b, 6.16a and 6.16b:

$$\dot{\hat{h}}(t) = \rho_h[u(t - \hat{h}(t)) - u(t - h(t))]\mu(t). \quad (6.37)$$

with μ an approximation of $\dot{u}(t - h)$ computed by the finite difference method (3.14)

$$\mu(t) = \frac{u(t - \hat{h}) - u(t - \hat{h} - h_1)}{h_1} \quad (6.38)$$

with $h_1 = 0.1$ s.

This estimated value enters into the predictor block to compute the prediction (6.36) as shown on Figure 6.14.

6. See Part I for examples of delay estimators.

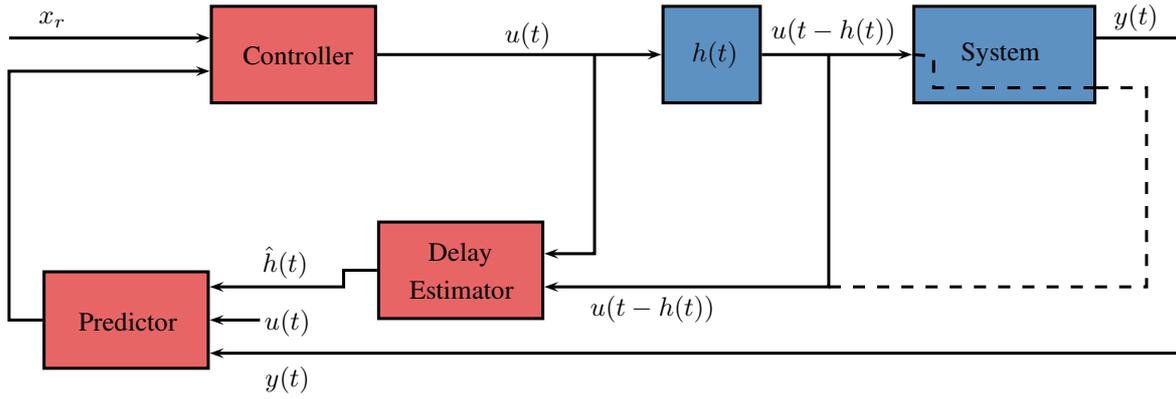
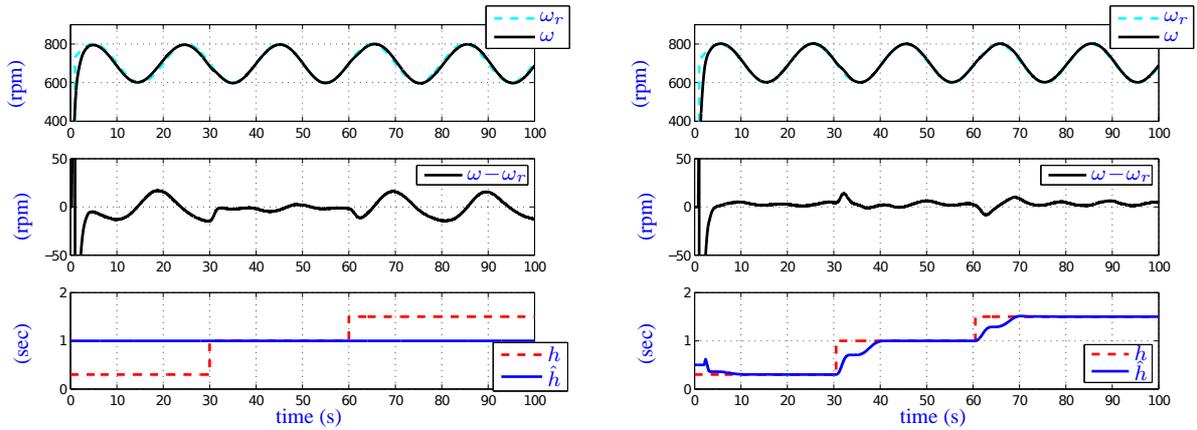


Figure 6.14 – Closed-loop scheme of the predictive control with a delay estimator (see Remark 6.5.1 for the explanation about the dashed line).

Remark 6.5.1. From an applicative point of view, the retarded input $u(t - h)$ can be sent by the system along with the measurements (dashed line on Figure 6.14).

It can be seen on Figure 6.15a that the tracking error is smaller when the delay estimation is exact. On Figure 6.15b, it is clear that the same level of tracking error can be achieved with a delay estimator but for a larger range of delay values. This illustrates the advantage of the delay estimator. When an external disturbance and parameter uncertainties affect the system, the new



(a) Constant estimation \hat{h} (b) Time-varying estimation $\hat{h}(t)$

Figure 6.15 – Influence of the delay estimate on the tracking performance

prediction can be constructed based on the estimated delay as follows

$$z(t) = \omega_{\hat{p}}(t) + \omega(t) - \omega_{\hat{p}}(t - \hat{h}(t)) \quad (6.39)$$

where $\omega_{\hat{p}}$ is given by (6.36). The new predictive feedback

$$u(t) = u_r(t + \hat{h}(t)) - k_p[z(t) - \omega_r(t + \hat{h}(t))] - k_i \int_0^t [z(s) - \omega_r(s + \hat{h}(s))] ds \quad (6.40)$$

with z given by (6.39) and the standard predictive feedback

$$u(t) = u_r(t + \hat{h}(t)) - k_p[\omega_{\hat{p}}(t) - \omega_r(t + \hat{h}(t))] - k_i \int_0^t [\omega_{\hat{p}}(s) - \omega_r(s + \hat{h}(s))] ds \quad (6.41)$$

with $\omega_{\hat{p}}$ given by (6.36) are compared on Figure 6.16.

On Figure 6.16a, one can see that the piecewise constant delay is correctly estimated for both predictive schemes. It can be noted that the delay estimation is not destabilized by the disturbance: this is a very important advantage of this estimation method. As in the above section, only the new predictive controller (6.40) allows an effective tracking of the velocity reference in spite of the uncertainties and the external disturbance: ω_1 correctly tracks ω_r whereas ω_2 displays a large tracking error.

Figure 6.16b shows that when the delay is time-varying, exponential convergence of the delay estimator cannot be achieved anymore. The small oscillations that can be observed on the delay estimate come from the periodic cancellation of the input derivative. Indeed, when \dot{u} gets closer to zero, the dynamics of the estimator tends to zero. However, the estimation is sufficiently accurate to guarantee an efficient tracking with the new predictive controller. As for the previous cases, the standard predictive controller (6.41) exhibits a large tracking error due to modeling errors Δa , Δb and external disturbance d .

These two graphs show that the new predictive scheme combined with the delay estimator (6.37) provides an efficient trajectory tracking in presence of a time-varying and unknown delay and in spite of parameter uncertainties and an external disturbance.

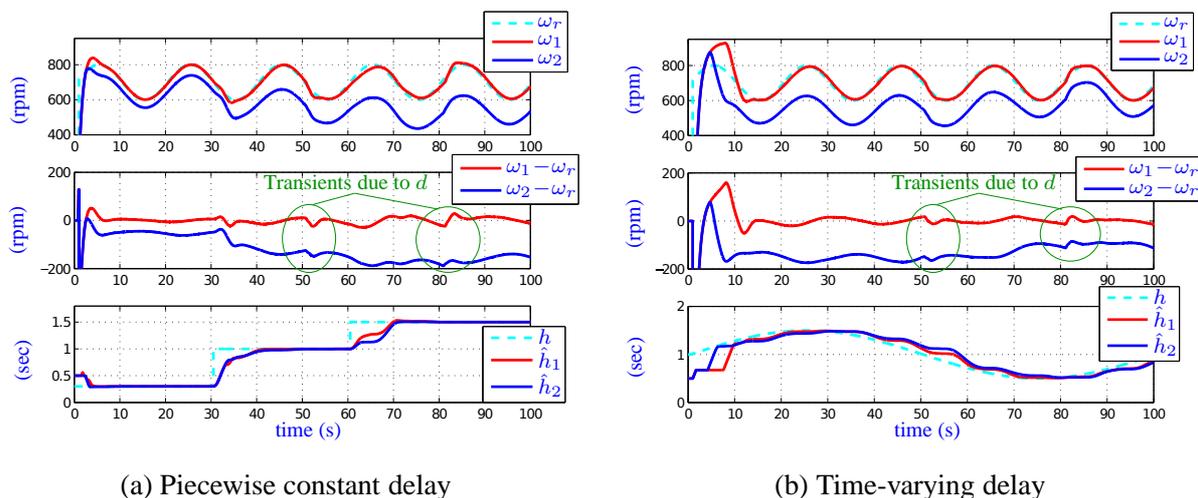


Figure 6.16 – Comparison of the standard and the new predictive PI controllers for different unknown delays in presence of parameter uncertainties (Remark 6.4.4) and an external disturbance between $t = 50$ s and $t = 80$ s. The speed ω_1 and the delay estimation \hat{h}_1 are obtained from the **new predictive scheme** whereas ω_2 and \hat{h}_2 are obtained from the **standard predictive scheme**

On Figure 6.17, the persistence condition is illustrated. Various phases are distinguishable. First, when the input saturates then $\dot{u} = 0$ and the delay is unobservable. This explains the two stages between $t = 0$ and $t = t_1$ and between $t = t_4$ and $t = t_5$. Remark that the delay estimator does not diverge reaching this singularity; this is a crucial advantage over the estimator presented in Chapter 2. Similarly, when ω has converged to ω_r , the input becomes constant so $\dot{u} = 0$. Consequently, the delay estimation cannot converge toward the exact value: see for example between $t = t_2$ and $t = t_3$ and between $t = t_8$ and $t = t_9$. The delay estimator can only give a good estimation when the input varies. The origin of input variation can be the change of the reference such as between $t = t_3$ and $t = t_4$ and between $t = t_9$ and $t = t_{10}$ or it can be due to a disturbance such as between $t = t_7$ and $t = t_8$ and between $t = t_{11}$ and $t = t_{12}$ s. This illustrates that the input should move sufficiently in order to estimate accurately the delay. This constraint is inherent to the input delay and the quality of the estimation will

always depend on the richness the input signal.

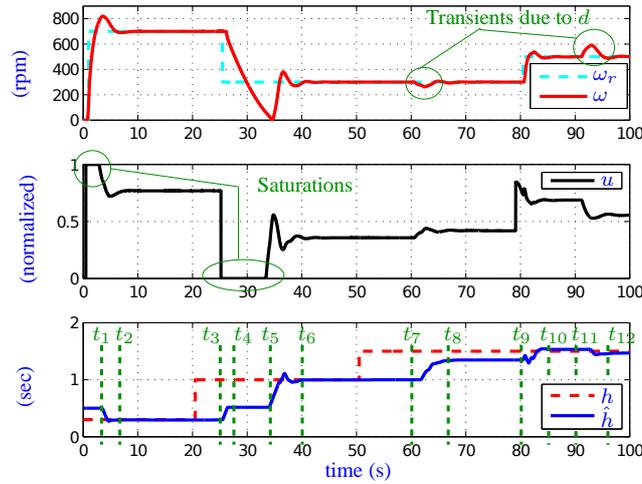


Figure 6.17 – Experimental results with a PI controller and piecewise constant and unknown delay $h(t)$, with parameter uncertainties (Remark 6.4.4) and a constant disturbance d between 60 s and 90 s. The speed ω and the delay estimate \hat{h} are obtained from the **new predictive scheme** – Top: velocity trajectory – Middle: control input – Bottom: exact delay $h(t)$ and estimated delay $\hat{h}(t)$.

6.6 Experimental results with input and output delays

Some experiments have been carried out to illustrate that the methods studied before may be adapted to the case of an input delay (h_i) and an output delay (h_o). This scenario is more realistic than the single input delay case in particular for remote controlled devices. The DC motor is now modeled as follows

$$\begin{cases} \dot{\omega}(t) = (a + \Delta a)\omega(t) + (b + \Delta b)u(t - h_i(t)) + d \\ y(t) = \omega(t - h_o(t)). \end{cases} \quad (6.42)$$

6.6.1 External disturbance and parameter uncertainties with a known delay

Similarly to the input delay case, the new prediction and the standard prediction are

$$z(t) = \omega_{\hat{p}}(t) + \omega(t - h_o(t)) - \omega_{\hat{p}}(t - h(t)) \quad (6.43)$$

with

$$\omega_{\hat{p}}(t) = e^{ah(t)}\omega(t - h_o(t)) + \int_{t-h(t)}^t e^{a(t-s)}bu(s)ds \quad (6.44)$$

where

$$h(t) = h_i(t) + h_o(t). \quad (6.45)$$

Then, standard and new predictive controllers can be designed.

The PI controller reads as

$$u(t) = u_r - k_p(\chi(t) - \omega_r) - k_i \int_0^t (\chi(s) - \omega_r) ds \quad (6.46)$$

with χ equal to $\omega_{\hat{p}}$ in (6.44) for the standard predictive controller and χ equal to z in (6.43) for the new predictive controller. In the same way, the Super Twisting algorithm (ST) reads as

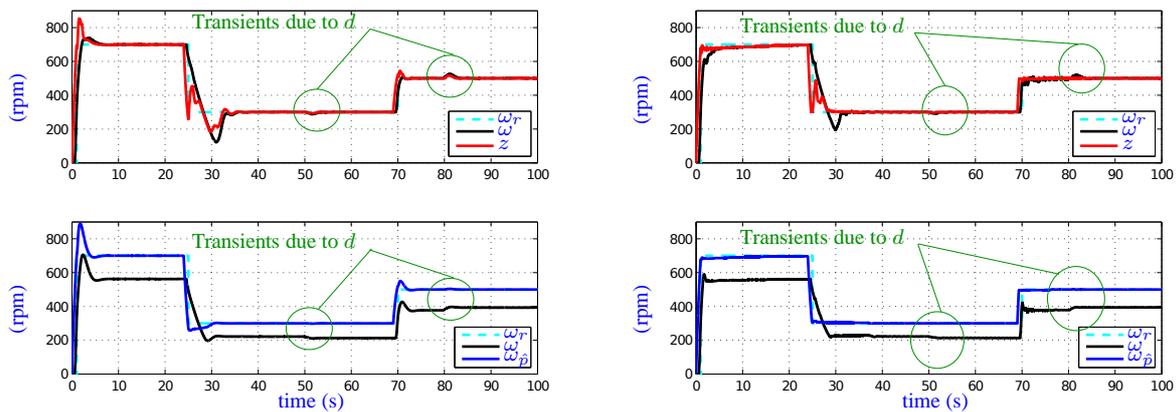
$$u(t) = u_r + \frac{1}{b} \left[-a(\chi(t) - \omega_r) - k_1 \sqrt{|\chi(t) - \omega_r|} \text{sign}(\chi(t) - \omega_r) + \nu(t) \right] \quad (6.47)$$

with the dynamics of ν governed by the following equation

$$\dot{\nu}(t) = -k_2 \text{sign}(\chi(t) - \omega_r(t)) \quad (6.48)$$

and χ equal to $\omega_{\hat{p}}$ in (6.44) or z in (6.43).

The results are shown on Figure 6.18. As it was expected, the results are very similar to the case of a single input delay (Figure 6.10): new predictive controllers lead to a more accurate trajectory tracking than standard predictive controllers.



(a) PI controller (6.51) with $\chi = z$ (Top) and $\chi = \omega_{\hat{p}}$ (Bottom) (b) ST controller (6.47) with $\chi = z$ (Top) and $\chi = \omega_{\hat{p}}$ (Bottom)

Figure 6.18 – Comparison of the standard and new predictive controllers for known input and output delays, $h_i = 0.6$ s, $h_o = 0.4$ s, with parameter uncertainties and an external disturbance.

6.6.2 External disturbance and parameter uncertainties with an unknown delay

To compute the predictions (6.44) and (6.43), only the sum of the delays h_i and h_o is required. Assuming that the retarded input $u(t - h_i - h_o)$ is known, a single delay estimator (6.37) can be used (as in the input delay case). In this case, the estimated delay denoted \hat{h} will be the sum of the input and output delays. Note that the assumption “ $u(t - h_i - h_o)$ known” is not very restrictive in practice since the input value can be sent along with the measurements and then it will undergo the same output delay. The method is illustrated on Figure 6.19. As a result, the new prediction is computed as follows

$$z(t) = \omega_{\hat{p}}(t) + \underbrace{\omega(t - h_o)}_{\text{measurement}} - \omega_{\hat{p}}(t - \hat{h}(t)) \quad (6.49)$$

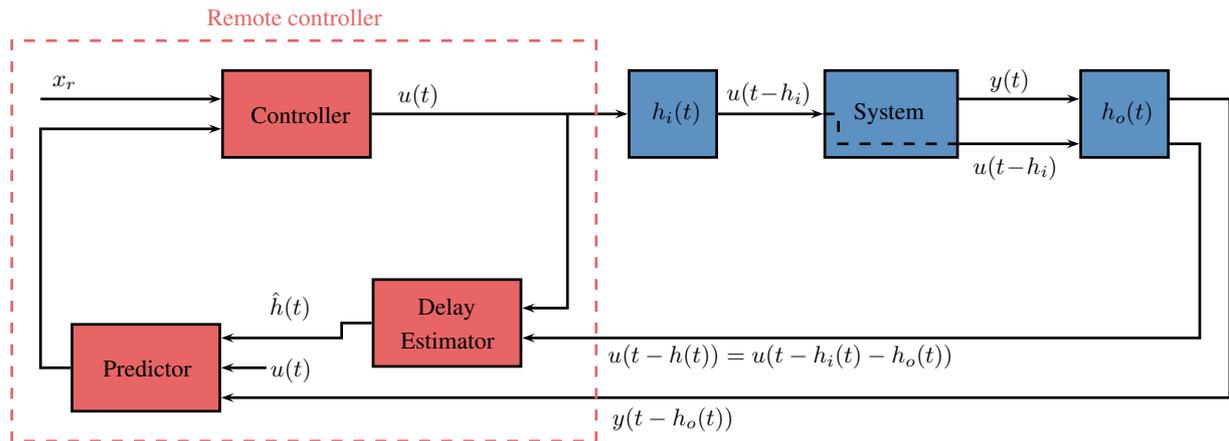


Figure 6.19 – Predictor and delay estimator for input and output delays

with

$$\omega_{\hat{p}}(t) = e^{a\hat{h}(t)}\omega(t - h_o) + \int_{t-\hat{h}(t)}^t e^{a(t-s)}bu(s)ds \quad (6.50)$$

where \hat{h} is an estimation of $h_i + h_o$. Standard and new PI predictive controllers can be computed from above predictions:

$$u(t) = u_r(t + \hat{h}(t)) - k_p[\chi(t) - \omega_r(t + \hat{h}(t))] - k_i \int_0^t [\chi(s) - \omega_r(s + \hat{h}(s))]ds \quad (6.51)$$

with χ equal to $\omega_{\hat{p}}$ in (6.50) for the standard predictive controller and χ equal to z in (6.49) for the new predictive controller.

On Figure 6.20, the same conclusion as in the input delay case can be drawn. The only difference is that the sum of the delays is estimated instead of the input delay. Consequently, the experiment shows that it is possible to achieve an accurate trajectory tracking for unknown input and output time-varying delays in presence of parameter uncertainties and an external disturbance.

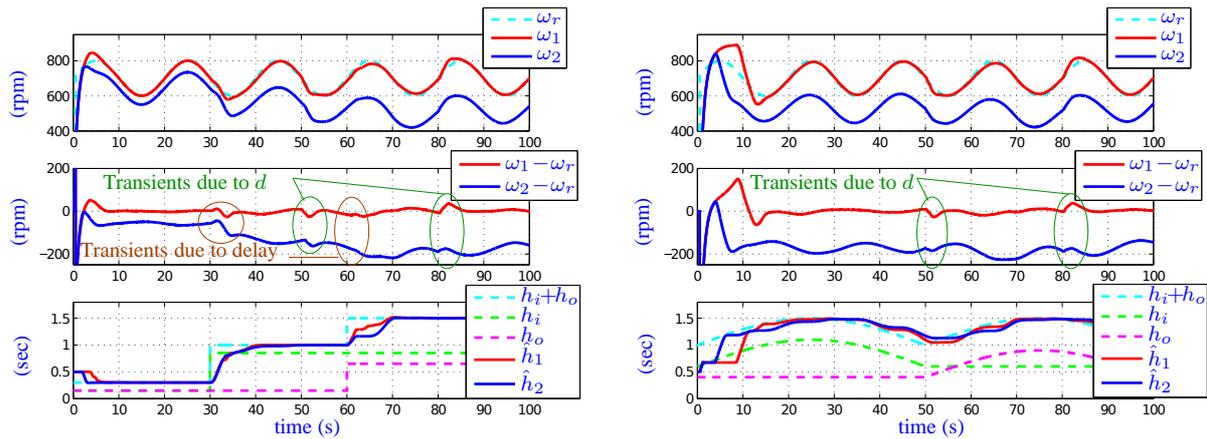
(a) Piecewise constants delays h_i and h_o (b) Time-varying delays h_i and h_o

Figure 6.20 – Comparison of the standard and the new predictive PI controllers for unknown input and output delays in presence of parameter uncertainties and an external disturbance between $t = 50$ s and $t = 80$ s. The speed ω_1 and the delay estimation \hat{h}_1 are obtained from the **new predictive scheme** whereas ω_2 and \hat{h}_2 are obtained from the **standard predictive scheme**

6.7 Summary

Contributions

- ✓ Application of theoretical results on a practical setup.
- ✓ Experimental results and theoretical ones are very consistent which shows the potential of the methods presented in Parts I and II to control real devices.
- ✓ The tracking performance of the DC motor with a large, unknown and time-varying input delay, is maintained in presence of parameter uncertainties and an external disturbance.

Concluding Remarks and Future Works

In this thesis, some tools have been developed to control uncertain systems with an unknown time-varying delay in the input. This work has been especially focused on two main topics:

- the design of estimation techniques to reconstruct the state and the delay,
- the conception of new predictive control laws.

The main contributions of the thesis are recalled below along with some future works opportunities.

Part I: In Chapter 2, a delay-state observation method based on the construction of an extended system has been presented. This method relies on the approximation of the delayed input by its Taylor approximation. As a consequence, this method requires the knowledge of the time derivatives of the input. **This is a major limitation of this technique that may be relaxed by substituting the time derivatives by some approximations. Keeping in mind that the exact knowledge of the delay is not always necessary, some basic finite difference approximation can be enough to achieve a satisfactory estimation accuracy.** In Chapter 3, another technique to estimate both the delay and the state has been proposed. The main idea is to connect a standard state observer with a delay estimator. A particular delay estimator based on an optimization method and a new Extended State Observer (ESO) has been designed. A comparison of the new ESO with a standard ESO has been done in simulation but no formal comparison has been given. **In order to compare formally both ESO, a precise quantification of the estimation error has to be done. In addition, for both delay-state observation techniques, an important direction is to study the influence of external disturbances and parameter uncertainties on the estimation accuracy. Furthermore, future developments will aim at including a controller that will compute a prediction using delay and state estimations.**

Part II: In Chapter 4, some predictive techniques have been analyzed in presence of a known delay. For each methods, stability conditions have been provided thanks to a systematic Lyapunov analysis. **The conclusion of Section 4.1 has been drawn for LTI systems but similar conclusions may probably be derived for nonlinear systems under some conditions. Moreover, reminding that the integral discretization is very complex in the case of nonlinear systems, the extension of the dynamic prediction method (Section 4.2) to nonlinear systems is very promising⁷. Furthermore, a possible extension of the dynamic predictive scheme to input and output delays, as presented on Figure 4.7a, is worth considering for future developments.** In Section 4.3, a new prediction has been designed by comparing the state and the retarded standard prediction. **A similar prediction can be computed for nonlinear systems because the underlying idea is independent from the form of the system. Further analysis is necessary to show that the robustness properties will be preserved in**

7. The main advantage of this method is to avoid the integral discretization.

the nonlinear case. It would also be interesting to study the robustness properties of the dynamic prediction and compare with the new prediction. If the dynamic prediction does not exhibit good robustness properties then a solution to combine the attractive features of both methods (easy implementation and robustness properties) is to compute the new prediction from the dynamic one (see Figure 6.12) as tested at the end of Section 6.4.4.

In Chapter 5, the techniques developed for a known delay have been extended to the unknown delay case except the one based on the new prediction. The reason is that from an intuitive point of view, the delay mismatched could be seen as the effect of an external disturbance: this solution is maybe not well suited for the unknown delay case. **The size of the tolerable delay uncertainty has to be precisely quantified and some other robust methods have to be thought about.** The delay observer of Chapter 3 has been combined with the predictive techniques of Chapter 4 in simulation and in experimentation; the results are convincing but have not been proven yet. **A formal proof of the convergence of the coupled systems (estimator/predictor) has to be provided**⁸.

In addition, some new analysis tools have to be developed to get rid of the assumptions on the delay rate ($\delta < 1$) in order to deal with non causal delays. Indeed, in NCS, delays are usually unknown and also non causal which means that control values are not always received in chronological order. **In future works, it may be more realistic to consider stochastic delays. Furthermore, the extension of these methods to input and output delays has been illustrated in the experimental section and seems to be transposable directly from the input delay case. Nevertheless, the theory behind this assumption has to be deeply investigated. Results about input and output delays are particularly attractive for practical applications.**

Part III: Finally, previous theoretical developments are tested and confirmed on a DC motor in Chapter 6. DC motor is a benchmark system since it is used in various fields. It is also quite “easy” to control since it is open-loop stable and the available model is very accurate. **As a result, it will be relevant to test the algorithms on a more complex system.** Furthermore, since the predictions are computed numerically, the integral terms have to be discretized and the control laws sampled. **As it has been mentioned in Chapter 1, the discretization may cause some instability so it would be interesting to take into account these phenomena theoretically.**

8. A convergence proof is given for each system separately.

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Thèse de Doctorat

Vincent LÉCHAPPÉ

Commande prédictive et estimation de systèmes incertains avec entrée retardée

Predictive control and estimation of uncertain systems with an input delay

Résumé

Cette thèse est dédiée à la commande prédictive de systèmes linéaires incertains en présence de retards variants et inconnus affectant l'entrée de commande. Des solutions d'observation sont proposées pour estimer à la fois l'état et le retard. De plus, de nouvelles techniques de prédiction permettant de s'affranchir des contraintes usuelles liées à la commande prédictive sont présentées. La validité de ces nouvelles méthodes est testée expérimentalement sur un moteur à courant continu.

Mots clés

commande à base de prédiction, systèmes à entrée retardée, estimation de retard.

Abstract

This thesis deals with the predictive control of uncertain linear systems in presence of unknown and time-varying input delays. Some observation solutions are proposed to estimate the state and the delay. In addition, new predictive techniques that allows to overcome usual weak points of predictive control are presented. The efficiency of these new methods is tested experimentally on a DC motor.

Key Words

prediction-based control, input delay systems, delay estimation.