Elements of Observation and Estimation for Networked Control Systems

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Chapter 1

Introduction

1.1 General element of introduction

Network control systems constitute an active field of study, where interacting components spatially distributed try to achieve a global goal [54].

Teleoperation, the internet of things, the increasing demand of electronics in the automotive industry are relevant examples for a theoretical investigation of this field.

Network control systems constitute an essential subject in the general field of the Cyber–Physical Systems [75], [66], where systems with embedded computational capabilities interact with the physical world.

The rapidly growing demand for decentralised and communicating control systems has an important economical impact in areas as diverse as smart grids, electronic in transportation systems, healthcare devices. The economical interests coupled to challenging theoretical and technical problems call for a great activities with respect to CPS in research and development. This new paradigm introduces both new opportunities and new challenge in term of control synthesis, and requires adaptation of the existing embedded computing methods where real time and communication constraints are a crucial matter.

To model and study such complex systems, the hybrids dynamical systems (HDS) paradigm is well suited, since it provides a powerful modelling framework of systems involving both discrete and continuous dynamics. Notably, the HDS framework allows dealing with continuous systems (e.g. physical processes) which undergo transitions due to discrete events (e.g. computer decisions or external events).

Although networked control systems constitute a powerful tool to study large scale of communicating systems. In several domain as for exemple teleoperation via internet, they raise theoretical problems that need to be addressed, such as their technical limits in terms of time delays, packet drops, and limited bandwidth available for communication and real time scheduling. Classical methodologies of automatic control, used in the context of network control systems, need to take these constraints into consideration.

In this thesis, one focus on the classical problems of observation and estimation [73], [18], [49].This problems are reconsidered in the light of the network characteristics and constraint, investigating some of the cases where these constraints of real time and limited communication capabilities due to the network call for adapted observation and estimation schemes. General observation problematic arises from limitations in the number of sensors, due to several reasons such as cost reduction and technological constraints, therefore not all the variables of interest (state variables and system parameters) are

measured. Then the observability of the system accounts for the ability to determine their values from measurements. Clearly, Network Control Systems and observability are two vast fields, and the aim of this thesis is not to cover in detail all the aspects of interaction between them. For this reason, hereafter the attention is focused on some cases that appeared more meaningful.

Nevertheless linear and nonlinear systems are considered. Whereas the observability property for a linear system is global (i.e. if a system is observable, then it is observable for all states and all inputs), this is not the case for nonlinear system, where singularities of observation could emerge, meaning that a nonlinear system could be observable locally for some inputs and outputs and not for others. Moreover, even in the linear case, it should be distinguished the notion of observability from that of detectability. Avoiding for the moment the precise definition, the paradigm of observability used here is, roughly speaking, the ability to reconstruct the system state. In the body of the thesis, where needed, the notion of observability used will be precisely defined.

The contributions presented in the next chapters will be divided into three independent parts.

In Chapter 2 the problem of observation and observer–based feedback stabilization of linear systems by event–triggered sampling is considered. The event triggering is a recent sampling paradigm, arising from the development of networked systems. Concepts relevant to event–triggered systems will be recalled for both sensors and actuators. The fundamental idea of event triggered sampling is to use information on the state to sample only when "needed" while using an appropriate criterion to specify what "needed" means. It will be shown that under some sufficient conditions the event triggering policy can be used for linear systems with time–varying uncertainties. In the last part of Chapter 2, an impulsive observer will be designed along with a triggering condition to observe and stabilize the system. A notable feature of this impulsive observer is its ability to estimate the state even in the absence of a stabilizing control, which can be interpreted as a separation principle.

In Chapter 3 the problem of observation and observer–based feedback stabilization are considered for a class of nonlinear systems, under the paradigm of the event triggered sampling policy. Sufficient conditions in term of input–to–state stability will be given, and from those conditions an event–triggered sampling mechanism will be derived for the observation and stabilization of the class of nonlinear systems under consideration. In the second part of Chapter 3, an impulsive observer will be designed which has the ability to estimate the state with or without a stabilizing control. Furthermore, an observer–based event–triggered control stabilizing the nonlinear system will be proposed.

In Chapter 4 the problem of average consensus where a set of agents try find their barycentre, without absolute knowledge of their position, will be considered. It will be shown how a decentralised observer can be used to solve the average consensus problem, in the presence of range connectivity constraints for the communicating mobile agents.

This result demonstrates that using communication in a decentralised fashion allows new and simpler way to solve a range of problems with respect to multi-agents systems.

Finally, considering a more accurate vehicle modelling, and in view of the future trend towards networks of collaborative vehicles (agents), in Chapter 5 it is presented an estimation procedure for the tire–road friction coefficient, making use of a high order sliding–mode observer. In order to asses the relevance of the proposed estimators and highlight their performance, the results are verified and compared with classical estimators in CarSim, which gives a high fidelity simulation scenario.

1.2 Scientific publications conducted during the Ph.D

Conference Publications

2013 L. Etienne, S. Di Gennaro, J–P Barbot, *Active ground vehicle control with use of a super-twisting algorithm in the presence of parameter variations* in 3rd International Conference on Systems and Control in Alger

2014 L. Etienne, S. Di Gennaro, J–P Barbot, J. Zerad, *Observer synthesis for some classes of switched dynamical systems* In the European Control Conference in Strasbourg

2015 L. Etienne, S. Di Gennaro, J–P Barbot, *Event Triggered Observer–Based Control for Linear Systems with time varying Uncertainties* In American Control Conferencein Chicago

2015 L. Etienne, S. Di Gennaro, J–P Barbot, *Event–Triggered Observers and Observer–Based Controllers for a Class of Nonlinear Systems* In American Control Conference Chicago

Submitted publications for international Journals

L. Etienne, S. Di Gennaro, J–P Barbot, *A Super Twisting Controller for Active Control of Ground Vehicles with Tire Road Friction Estimation and CarSim Validation* ISA Transactions

L. Etienne, S. Di Gennaro, J–P Barbot, *Asynchronous Event–Triggered Observation and Control of Linear Systems via Impulsive Observers* Journal of the Franklin Institute

L. Etienne, S. Di Gennaro and J.P. Barbot *Asynchronous Event–Triggered Observation and Control of Nonlinear Lipschitz Systems via Impulsive Observers* International Journal of Robust and Nonlinear control,

Chapter 2

Event triggering policy for observation and output based feedback for linear systems

2.1 General elements on event trigger for linear systems

Digital technology is commonly used in modern control systems, where the control task consists of sampling the outputs of the plant, computing the control law, and implementing the actuators signals. The classic way to proceed is to use a periodic sampling, thus allowing the closed–loop system to be analyzed on the basis of sampled–data formalism, see e.g. [12]. In such time–triggered control the sensing, control, and actuation are driven by a clock, and it can be seen as an "open–loop" sampling. Recent years have seen the development of a different technique [11], where the periodic sampling is substituted by an event–triggered policy, see for instance [53] for an introduction to the topic and [13], [95], [101], [74], [52] for further details. In this case the sampling is performed in order to ensure that some desired properties established in the control design, primarily stability, can be maintained under sampling. Conceptually this means to introduce a feedback in the sampling process, since an event–triggered control requires the constant monitoring of the system state to determine if the desired properties is ensured.

Event–triggered techniques allow the execution of control tasks as rarely as possible, so minimizing energy consumption and load in network control system, and/or leaving the digital processor available for other tasks. To analyse the stability properties of a system the sampling induced error can be seen as a delay [42][47], via impulsive modelling [53] or as a perturbation acting on a system with continuous feedback [95]. This later case will be considered throughout our work.

However some issues appear naturally, such as that of the implementation of dynamical feedbacks. Indeed introducing a feedback in the sampling mechanism introduce a coupling between the dynamic of the plant and the sampling instant, therefore it is important to know under what condition a pathological sampling could arise.

In this chapter we will introduce some classic notion and results of event triggering policy for control then we will present the problem of observer synthesis for different class of systems. We first show some robustness properties of a classic Luenberger like observer

subject to event triggering sampling policy then we establish a weak separation principle for linear system where observation can be performed with or without control.

Notation: $\|\cdot\|$ denotes the euclidean norm when applied to a vector, and the norm induced by the euclidean norm when applied to a matrix; $\|\cdot\|_\infty$ is the component with the biggest absolute value. Moreover, $\lambda^P_{\min}, \lambda^P_{\max}$ are the smallest and the biggest eigenvalue of a square matrix P, respectively. Furthermore, $\mathbb{R}^+$, $\mathbb{R}^+_0$ will denote respectively the set of positive real numbers and the set of positive real numbers including zero, and $\mathbb{N} = \{0, 1, 2, \cdots\}$ the set of natural numbers including zero. With $I_{p\times p}$ we denote the $p \times p$ identity matrix. When there is no ambiguity, x, $x_{t^s_k}$, $y_{t^s_k}$ will denote $x(t)$, $x(t^s_k)$, $y(t^s_k)$. Finally, $t^+_k = \lim_{|h|\to 0}(t_k + |h|)$, $k \in \mathbb{N}$.

2.1.1 Event trigger with full state information

A growing amount of result is available to achieve desired performance in terms of stability while reducing *in average* the total amount of communication we will now recall important results when considering full state feedback. In [13] a one dimensional process is considered consisting of a stochastic drift and a control action

$$dx = udt + dv$$

where v is a Brownian motion and u is an impulsive controller triggered each time the x cross a specified threshold. As a consequence the control impulse restarts x at the origin. In this simple case it is shown that event trigger control allows better performance in terms of number of sampling and variance if the threshold is not too small. This simple case already exhibits intrinsic features of event triggering sampling. Namely by definition of Brownian motion it is not possible to ensure a minimum time between two samplings. However the event triggered sampling (also called Lebesgue sampling) performs better in term of stability and average communication.

A bad event trigger sampling policy could lead to something worst than absence of minimum inter–event time, the Zeno behaviour where an infinite number of sampling occur on a finite time interval. The simple model of a bouncing ball is classic example of Zeno behaviour when the transition between bouncing and steady state occur.

Another example of Zeno behaviour occurs considering a simple dynamical system

$$\dot{x}(t) = u(t)$$

$$u(t) = -sign(x(t_k)), \forall t \in [t_k, t_{k+1}[$$

$$t_{k+1} = \min_t\{t > t_k, |x - x(t_k)| \geq \frac{|x(t)|}{2}\}$$

The sampling sequence is $t_{k+1} = \dfrac{2^k - 1}{2^k}$ therefore in the time interval $[0, 1]$ there is an infinite number of samples.

This example highlights the need of two definitions

Definition *2.1.1 (Zeno behaviour) [61] Considering $(t_k)_{k\in\mathbb{N}}$ the series of sampling instant. The system exhibits Zeno behaviour if there exists a finite $a > 0$ such that $\forall k \in \mathbb{N}, t_k < a$ (i.e. there is an infinite number of sampling on some finite time interval*

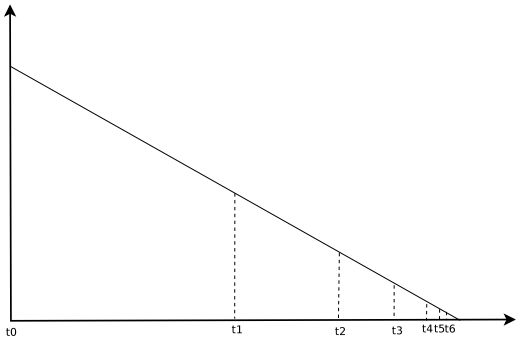

Figure 2.1: Control scheme with sampled output and zero order holder.

Definition *2.1.2 (Minimum inter–event time) [53] The difference $\delta_k := t_{k+1} - t_k$ is usually called the inter–event time, or sampling interval, if there exist $\tau_{min} > 0$ such that for all $k \in \mathbb{N}$, $\delta_k \geq \tau_{min}$ then the system is said to posses a minimum inter–event time*

In the scope of Event–trigger control, to avoid Zeno behaviours [61][8], it is important that the chosen sampling policy ensures that $\lim_{k\to\infty} t_k = \infty$ for all $k \in \mathbb{N}$, possibly under additional conditions.

2.1.2 Event Triggering Policies for linear system

We will first recall some known facts and terminologies about event–triggered systems. Consider a system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= x \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$. Due to the *communication constraints*, there is no continuous communication between the controller and the plant. The value u_{t_k} is applied to the system, through a classic zero order holder H_0. The transmissions are assumed instantaneous. The transmission instants between the controller and the plant are denoted by t_k, $k \in \mathbb{N}$.

Let us consider first a simple case in which the state x is available for measuring, and let us assume that there exists a state–feedback $u = Kx$, $K \in \mathbb{R}^{m\times n}$, rendering system (2.1) asymptotically stable at the origin, i.e. $A + BK$ Hurwitz. When the controller is implemented making use of the sampled value x_{t_k} of the state,

$$t_{k+1} = \min_t \{t \mid t > t_k + \delta\}.$$

This is clearly a time–triggered policy. The event–triggered paradigm replaces this condition with one on the state values $x(t), x_{t_k}$. A simple condition of this kind is, for instance, the ε–crossing policy, which is of the form

$$t_{k+1} = \min_t \{t > t_k \mid \|x(t) - x_{t_k}\| > \varepsilon\}$$

viz. $x(t)$ is sampled when $\|x(t) - x_{t_k}\|$ is greater then a certain threshold value $\varepsilon \in \mathbb{R}^+$. When this condition is verified, an event is triggered, which determines the sampling time t_{k+1}.

Alternatively more complex sampling policies can be found for instance

$$t_{k+1} = \min_t \left\{ t > t_k \mid \|x(t) - x_{t_k}\| > \gamma \|x\| + \varepsilon \right\}$$

with , $\gamma \in \mathbb{R}^+$, or a mixed triggering policy

$$t_{k+1} = \min_t \left\{ t > t_k + \delta_{\min} \mid \|x(t) - x_{t_k}\| > \varepsilon \right\}$$

with $\varepsilon, \delta_{\min} \in \mathbb{R}^+$. Different triggering policy can be considered depending on different choice of convergence or robustness [95][**?**]

using a Lyapunov argument we will show different properties of convergence of the triggering policy.

Practical stability

Definition *2.1.3 (Practical stability) [64] The origin of a system*

$$\dot{\xi} = f(\xi, u, d), \qquad \xi \in \mathbb{R}^{\bar{n}},\ u \in \mathbb{R}^{\bar{p}}\, d \in \mathbb{R}^{\bar{m}} \tag{2.2}$$

is globally ultimately bounded if there is a time $T_{\xi(0),\varepsilon}$ such that

$$\|\xi(t)\| \leq \varepsilon \quad \forall t > T_{\xi(0),\varepsilon},\ \forall \xi(0) \tag{2.3}$$

for some $\varepsilon > 0$. *The origin of* (2.2) *is* practically stable *if there is a time $T_{\xi(0),\varepsilon}$ such that* (2.3) *holds* for any $\varepsilon > 0$. ◇

Given the event trigger policy

$$t_{k+1} = \min_t \left\{ t > t_k \mid \|x(t) - x_{t_k}\| > \varepsilon \right\} \tag{2.4}$$

And assuming $A + BK$ is Hurwitz then there exist P a positive symmetric matrix such that $(A + BK)^T P(A + BK) = -I$ and for all $t \in [t_k, t_{k+1}[$ $u = Bx(t_k)$ therefore $\dot{x} = Ax + BKx(t_k) = Ax + BKx - BKx + BKx(t_k)$ with $||-BKx + BKx(t_k)|| \leq ||BK||\varepsilon$ denoting $e = x(t_k) - x(t)$ therefore considering the candidate Lyapunov function $V(x) = x^T P x$ one has

$$\dot{V}(x) = ((A + BK)x + BKe)^T P((A + BK)x + BKe)$$

$$\dot{V}(x) = x^T[(A + BK)^T P + P(A + BK)]x + 2x^T PBKe$$

$\forall t$,

$$\dot{V}(x) \leq -||x||^2 + 2||x||||PBK||\varepsilon$$

considering ε as a non vanishing perturbation it is known that x will converge to the ball $\mathcal{B}(0, ||PBK||\varepsilon)$ To show that for each initial condition there exist a minimal inter-event time (i.e. a semi global inter-event time) it is sufficient to see that because $e(t_k) = 0$ with $\dot{e} = -\dot{x}$ and since given the initial conditions $\dot{x}$ is always bounded this imply the existence of a semi global minimal inter-event time

Asymptotic stability

Given the same candidate Lyapunov function as before

$$t_{k+1} = \min_t \left\{ t > t_k \mid 2x^T PBKe > \sigma x^T Ix \right\} \tag{2.5}$$

with $\sigma \in]0,1[$

since $V(x) = x^T Px$ one has

$$\dot{V}(x) = ((A+BK)x + BKe)^T P((A+BK)x + BKe)$$

$$\dot{V}(x) = x^T[(A+BK)^T P + P(A+BK)]x + 2x^T PBKe$$

$\forall t$,

$$\dot{V}(x) \leq -||x||^2 + \sigma||x||^2$$

Therefore asymptotic (exponential) stability is preserved

To show that there exist a minimal inter-event time a bit more demanding. Since $e(t_k) = 0$ and $\dot{e} = -\dot{x}$

clearly $||e|| \leq \sigma||PBK||^{-1}||x||$ implies $2x^T PBKe \leq \sigma x^T Ix$.
We want to show that there exist a minimal time for which $||e|| \leq \sigma||PBK||^{-1}||x||$ holds. Therefore we compute

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} = \frac{2e^T \dot{e} x^T x - 2x^T \dot{x} e^T e}{x^T x x^T x}$$

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} \leq \frac{-2e^T((A+BK)x + BKe)x^T x - 2x^T((A+BK)x + BKe)e^T e}{x^T x x^T x}$$

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} \leq a\frac{||e||}{||x||} + b\frac{||e||^2}{||x||^2} + c\frac{||e||^3}{||x||^3}$$

using the comparison lemma with $(\dot{y^2}) = ay + by^2 + cy^3, y(0) = 0$ the existence of a minimal time between sampling is ensured. Note that asymptotic stability can be ensure by a more conservative version of (2.5) that is

$$t_{k+1} = \min_t \left\{ t > t_k \mid ||e|| > \gamma||x|| \right\}$$

Example

Considering a linear plant with full state measurement

$$\dot{x} = Ax + Bu$$

with $u = Kx$ and the triggering policy $t_{k+1} = \min \left\{ t > t_k \mid ||e|| > \gamma||x|| \right\}$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.179 & 7.73 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.52 & 51.57 & 0 \end{pmatrix}$$

,

$$B = \begin{pmatrix} 0 \\ 1.79 \\ 0 \\ 5.26 \end{pmatrix}, K = (0.3102, 0.7204, -14.4655, -1.7309)$$

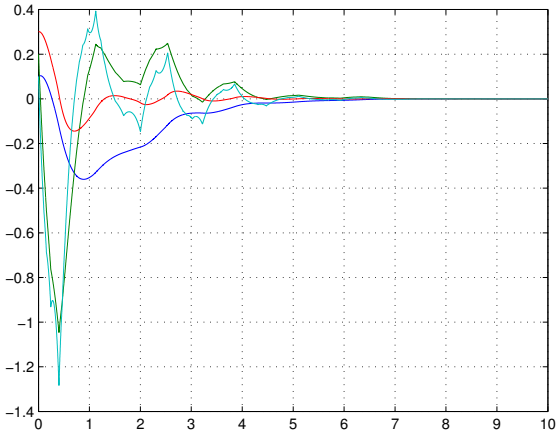

Figure 2.2: State of the plant with $\gamma = 0.3$

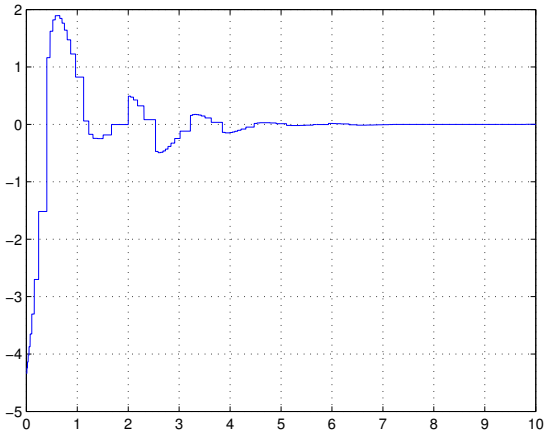

Figure 2.3: input of of the plant with $\gamma = 0.3$

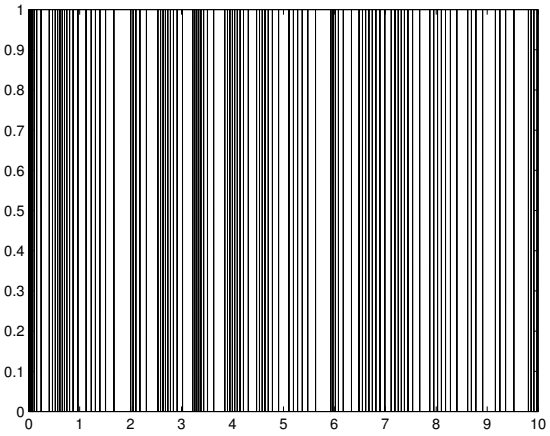

Figure 2.4: Sampling instant of the input

The system is asymptotically stabilised while the triggering instants are not periodic. One can see more frequent sampling at the during the transient.

2.1.3 Observation and control using event triggering

Observer-based control of linear systems submitted to event–triggered transmission has been a recent and growing fields of research [35],[67],[97]. We will now briefly introduce generic ideas of output based event triggered observation and control.

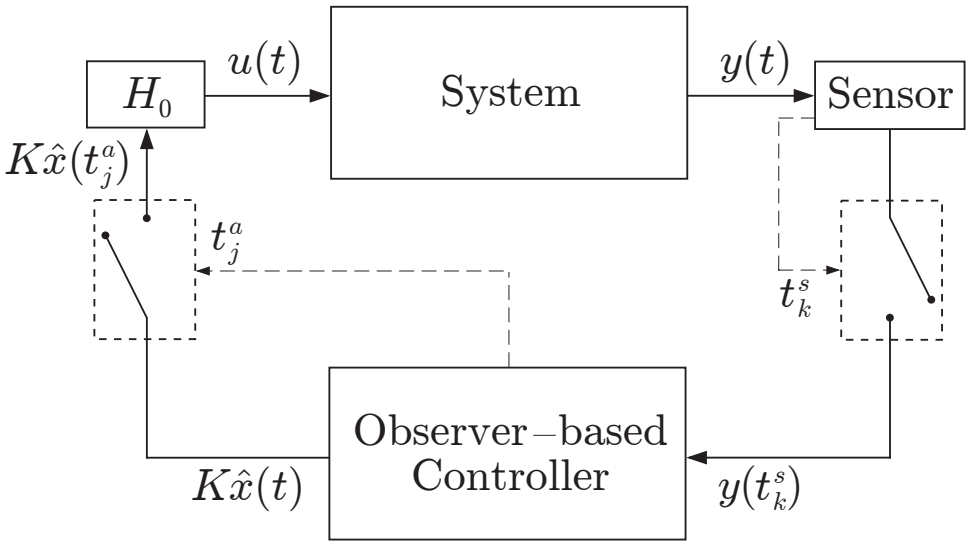

Figure 2.5: The communications of the output and of the input are performed in an event triggered manner. The command delivered by the computer is assumed to be continuous (i.e. its period and small delays are neglected)

Consider a linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.6}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^p$ is the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$. We assume that the pair (A, B) is stabilizable and that the pair (A, C) is detectable. The natural question occurring is: Is it possible to apply the event–trigger methodology in order to reduce the communication between sensor and controller and between controller and the plant ?

Considering the following classic Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + LC(x(t_k) - \hat{x}(t_k)). \tag{2.7}$$

A key difference when considering partial knowledge of the state is the fact that the output can tend to (or cross) 0 when the state is not zero. Therefore a triggering condition of the form

$$t_{k+1}^s = \min_t \left\{ t > t_k \mid ||y(t_k) - y|| > \gamma ||y|| \right\}$$

can lead to Zeno behaviour. A simple way to prevent those Zeno behaviour from happening is to enforce a minimal inter-event time leading to

$$\begin{aligned} t_{k+1}^s &= \min_t \left\{ t > t_k + \tau \mid ||y(t_k) - y|| > \gamma ||y|| \right\} \\ t_{k+1}^a &= \min_t \left\{ t > t_k + \tau \mid ||\hat{x}(t_k) - \hat{x}|| > \gamma ||x|| \right\} \end{aligned} \tag{2.8}$$

where τ is the built-in minimum inter-event time. Clearly there exists by construction a minimal inter-event time, the following question is then whether this τ can provoke a loss of stability.
We will next for simplicity consider synchronous sampling of the input and the output meaning $t_{k+1} = min(t^a_{k+1}, t^s_{k+1})$.

Considering $z = x - \hat{x}$ the closed loop system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + LC(x(t_k) - \hat{x}(t_k)) \\ u &= K\hat{x}(t_k) \end{aligned} \tag{2.9}$$

writing the system in terms of x and z gives

$$\begin{aligned} \dot{x}(t) &= (A + BK)x + BKz + BK(\hat{x}(t_k) - \hat{x}) \\ \dot{z}(t) &= (A - LC)z + LC(x - x(t_k)) \end{aligned} \tag{2.10}$$

When considering synchronous sampling of the output and the controller, the extended state $X = (\hat{x}^T, z^T)^T$ is of the following form

$$\begin{aligned} \dot{X} &= MX + W\bar{X} \\ \dot{\bar{X}} &= M'X + W'\bar{X} \\ \bar{X}^+ &= 0 \end{aligned} \tag{2.11}$$

With $\bar{X} = X - X(t_k)$

$$M = \begin{pmatrix} A + BK & LC \\ 0 & A - LC \end{pmatrix}, W = \begin{pmatrix} -BK & 0 \\ LC & LC \end{pmatrix}$$

From the previous result on Event triggering control for system with full state measurement (it is possible to select τ such that $\forall t < \tau \ ||\bar{X}|| < \gamma'||X||$. Since $\bar{X}$ is not known we introduce an over estimation. We have that

$$||y(t_k) - y|| \leq \gamma||y||$$

and $||\hat{x}(t_k) - \hat{x}|| \leq \gamma||x||$ imply $||\bar{X}|| \leq \gamma'||X||$.

therefore the triggering condition (2.8) leads to asymptotic stability with minimum inter-event time.

Remark 2.1.4 *For $\gamma \in (0, \gamma_{\max})$ the system will converge (2.11) will converge asymptotically, and the value gamma represents a trade off between sampling frequency and convergence rate, however for $\gamma \geq \gamma_{\max}$ the system is potentially unstable.*

2.2 Observation and control of uncertain linear systems using event triggered sampling policy

2.2.1 Problem Formulation

Now, we want to consider the robustness issues of the event triggered observation and control with respect to time varying modeling uncertainties. The results obtained subsequently allow us to state that if the continuous closed loop system is robust with respect

to parameter uncertainties, then the uncertain system is stabilizable with the event triggered policy. Moreover, asymptotic stabilization of an uncertain system using an adapted event triggered policy can be obtained.

Consider a linear system subject to time varying uncertainties

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{2.12}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^p$ is the output, $\forall t\, A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{m\times n}$, $C(t) \in \mathbb{R}^{p\times n}$, and where $(A(t), B(t), C(t))$ are subject to parametric uncertainties.

Considering A_0, B_0, C_0 the nominal matrices, and $\Delta(t) = (\Delta_1(t), \Delta_2(t), \Delta_3(t))$ represent the time varying uncertainty

System (2.12) can be rewritten in the following form

$$\begin{aligned} \dot{x}(t) &= (A_0 + \Delta_1(t))x(t) + (B_0 + \Delta_2(t))u(t) \\ y(t) &= (C_0 + \Delta_3(t))x(t) \end{aligned} \tag{2.13}$$

. The pair (A_0, B_0) is stabilizable, and the pair (A_0, C_0) is detectable. In the following, the indication of the time instant t is dropped if there are no ambiguities.

The control scheme is shown in Fig. 2.1. Due to the *communication constraints*, there is no continuous communication either between the sensor and the controller, or between the controller and the plant. The value $y(t_k) = Cx(t_k)$ is available for the controller to implement the control, and the value $u(t_k)$ is applied to the system, through a classic zero order holder H_0. It is worth noting that this means that the output y and the input u are sampled synchronously, as assumed in this section for the sake of simplicity, even though generalizations can be done in the case of asynchronous sampling.

Let us consider the linear system (2.12). When the state x is not measurable, under the condition of detectability of the pair (A_0, C_0) it is possible to build a state observer [73] of the following form. In view of an implementation with a triggered policy, the observer will have the structure

$$\dot{\hat{x}} = A_0\hat{x} + B_0u + GCx(t_k) - GC_0\hat{x}. \tag{2.14}$$

A feedback controller based on $\hat{x}$ will be used in the following to stabilize the system (2.12). The input applied to the system, after sampling, is

$$u(t) = K\hat{x}(t_k), \qquad \forall\, t \in [t_k, t_{k+1}) \tag{2.15}$$

so obtaining the controlled dynamics

$$\dot{x} = (A_0 + \Delta_1(t))x + (B_0 + \Delta_2(t))K\hat{x}(t_k). \tag{2.16}$$

Moreover, considering the sampled value of the output, one gets the following closed–loop system

$$\begin{aligned} \dot{x} &= (A_0 + \Delta_1(t))x + (B_0 + \Delta_2(t))K\hat{x}(t_k) \\ \dot{\hat{x}} &= A_0\hat{x} + B_0K\hat{x}(t_k) + G(C_0 + \Delta)x(t_k) - GC_0\hat{x}. \end{aligned} \tag{2.17}$$

In the following sections we will address the problems of *output–based practical and asymptotic stabilization*, namely the problem of practical/asymptotic stabilization of the origin of (2.13) by means of an observed–based controller (2.14), (2.15), when appropriate event triggered policies are adopted.

2.2.2 Event Triggered Policies for Output–Based Stabilization

When the state x of (2.1) is not available, the triggered policies reviewed in Section 2.1.2 cannot be implemented directly. In the following, we introduce the triggered policy that will be used when the system state is not available. The following definition of mixed triggering for practical stabilization of an observer–based controllers will be used.

Lets introduce the following triggering policies

Definition *2.2.1 (Mixed Triggering for Practical Stability) The next sampling time for the control is*

$$t_{c,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|\hat{x} - \hat{x}(t_k)\| \geq \varepsilon_1\}\} \tag{2.18}$$

and for the observation is

$$t_{o,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|y - y(t_k)\| \geq \varepsilon_2\}\} \tag{2.19}$$

where $\varepsilon_1, \varepsilon_2, \tau_{\min} > 0$*, and*

$$t_{k+1} = \min\{t_{c,k+1}, t_{o,k+1}\} \tag{2.20}$$

is the next sampling time for the closed–loop system, with $t_0 = 0$ *and* $k \in \mathbb{N}$. ◇

The proposed triggering condition prevents pathological sampling to appear. In particular, the condition $t \geq t_k + \tau_{\min}$ in (2.18), (2.19) will ensure the absence of Zeno behaviour. The triggered policy (2.20) has obviously a global minimum inter–event time. In the following it will be shown that implementing (2.20) will ensure the practical solution of the observed–based control problem. However, this condition cannot in general lead to asymptotic stability. For, the following definition of mixed triggering for asymptotic stabilization of an observer–based controller has to be considered.

In order to achieve asymptotic stability we introduce the following triggering policies

Definition *2.2.2 (Mixed Triggering for Asymptotic Stability) The next sampling time for the control is*

$$t_{c,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|\hat{x} - \hat{x}(t_k)\| \geq \sigma_1\|\hat{x}\|\}\} \tag{2.21}$$

and for the observation is

$$t_{o,k+1} = \max\{t_k + \tau_{\min}, \min_t\{t > t_k \mid \|y - y(t_k)\| \geq \sigma_2\|y\|\}\} \tag{2.22}$$

where $\sigma_1, \sigma_2, \tau_{\min} > 0$*, and*

$$t_{k+1} = \min\{t_{c,k+1}, t_{o,k+1}\} \tag{2.23}$$

is the next sampling time for the closed–loop system, with $t_0 = 0$ *and* $k \in \mathbb{N}$. ◇

Remark 2.2.3 *The existence of* $\tau_{\min} > 0$ *can be seen both as a physical constraint and as a requirement in the triggering condition.* ◇

Remark 2.2.4 *The synchronization of the triggering condition on the observer and on the output are,* a priori*, not required in order to demonstrate the proposed result. However they allow a much simpler analysis.* ◇

Assumption on the Parameter Uncertainties $\Delta(t)$

Introducing the estimation error $e = x - \hat{x}$, from (2.17) we obtain the following

$$\begin{aligned} \dot{x} &= (A_0 + \Delta_1(t))x + BK(x(t_k) - e(t_k)) \\ \dot{e} &= (A_0 - GC_0)e - GC_0(x(t_k) - x) + \Delta_1(t)x \\ &\quad + \Delta_2(t)K(x(t_k) - e(t_k)) - G\Delta_3(t)x(t_k). \end{aligned} \tag{2.24}$$

Hence, introducing $\bar{x} = x - x(t_k), \bar{e} = e - e(t_k)$, we can rewrite the system in the following form

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = H(\Delta) \begin{pmatrix} x \\ e \end{pmatrix} + w_p \tag{2.25}$$

where

$$\begin{aligned} H(\Delta(t)) &= \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \\ H_{11} &= A_0 + \Delta_1(t) + (B_0 + \Delta_2(t))K \\ H_{12} &= -(B_0 + \Delta_2(t))K \\ H_{21} &= \Delta_1(t) + \Delta_2(t)K - G\Delta_3(t) \\ H_{22} &= A_0 - GC_0 - \Delta_2(t)K \\ w_p &= \begin{pmatrix} -(B_0 + \Delta_2(t))K(\bar{x} - \bar{e}) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) + G(C_0 + \Delta_3(t))\bar{x} \end{pmatrix}. \end{aligned}$$

We denote $h_{ij}(t)$ the elements of $H(\Delta(t))$.

Since $\Delta(t)$ represents the parametric uncertainty, it is natural to assume that it is small in a certain sense. K, G are chosen to render $H(0)$ Hurwitz. We make the following hypothesis on $H(\Delta(t))$.

Assumption 2.2.5 *$H(\Delta(t)) \in \mathcal{D}$, a convex compact set given by $h_{ij}^{\min} \leq h_{ij}(t) \leq h_{ij}^{\max}$, $\forall\, i, j$, $\forall\, t$, and there exist $P = P^T > 0$, $\gamma > 0$ such that*

$$H(\Delta(t))^T P + PH(\Delta(t)) + \gamma I_d < 0. \qquad \diamond$$

Remark 2.2.6 *Given $\mathcal{D}$, the hypothesis of a common quadratic Lyapunov function can be checked by a finite set of Linear Matrix Inequality. Using the convexity principle [23], checking the infinite LMIs in Assumption 2.2.5 reduces to check the LMI on the extremal point of $\mathcal{D}$.*

Remark 2.2.7 *By a continuity argument around $\Delta(t) = 0$, If $H(0)$ is detectable there exist a $\mathcal{D}$ is small enough (i.e. for the chosen K, G) then Assumption 2.2.5 is verified.* $\diamond$

Remark 2.2.8 *Assumption 2.2.5 implies a bound on Δ_1, $\Delta_2 K$,$G\Delta_3$. This assumption is less restrictive than a bound on Δ.*

According to Remark 2.2.7, one can get sufficient conditions for the stability of the perturbed system from the properties of the nominal system and of the perturbation size.

Practical Stability

In this section we show that if assumption 2.2.5, is verified then we can stabilize the system in an arbitrary small (dependant on the triggering parameters) neighbourhood of the origin. We show that the triggering paradigm can be applied to a system subject to the considered time varying parametric uncertainties.

Theorem 2.2.9 *Under Assumption 2.2.5, it is possible to choose $\varepsilon_1, \varepsilon_2, \tau_{\min} > 0$ so that the observed–based controller (2.14), (2.15), with the triggering condition (2.20), solves the output–based stabilization problem for system (2.6).* ◇

Proof 2.2.10 *We show that, for any $\varepsilon > 0$, it is possible to choose $\varepsilon_1, \varepsilon_2 > 0$ in (2.18), (2.19), respectively, such that the triggering condition (2.20) implies the asymptotic stability toward the ball of radius ε. The triggering condition ensures that $t_{k+1} \geq t_k + \tau_{\min}$.*

From assumption 2.2.5 it is possible to determine K, G such that, system (2.25) is an asymptotically stable system forced by a non vanishing perturbation w_p due to the difference between y and its sampled value $y(t_k)$, and between $\hat{x}$ and its sampled value $\hat{x}(t_k)$.

First case $t_{k+1} > t_k + \tau_{\min}$. *Since condition (2.20) ensures $\|\bar{x} - \bar{e}\| < \varepsilon_1$ and $\|(C_0 + \Delta_3(t))\bar{x}\| < \varepsilon_2$, therefore*

$$\|w_p\| \leq \|(B_0 + 2\Delta_2(t))K\|\varepsilon_1 + \|G\|\varepsilon_2.$$

Since w_p is non–vanishing, the trajectories of (2.25) converge in finite time, depending on the initial condition $(x(0), e(0))$, to a ball of the origin bounded by

$$b = \sqrt{\frac{\lambda^P_{\max}}{\lambda^P_{\min}}} \frac{\|P\|}{\vartheta\gamma} \Big[(\|B_0\| + 2\Gamma)\|K\|\varepsilon_1 + \|G\|\varepsilon_2 \Big]$$

(see[64]) where $P = P^T > 0, \gamma$ are defined in assumption 2.2.5 and Γ is an upper–bound of Δ_2. It is clear that it is always possible to choose $\varepsilon_1, \varepsilon_2$ such that $b \leq \varepsilon$.

Second case $t_{k+1} = t_k + \tau_{\min}$, *let us consider the system*

$$\begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\bar{x}} \\ \dot{\bar{e}} \end{pmatrix} = \bar{A}(t) \begin{pmatrix} x \\ e \\ \bar{x} \\ \bar{e} \end{pmatrix} = \begin{pmatrix} H(\Delta)(t) & \bar{A}_{12} \\ H(\Delta)(t) & \bar{A}_{12} \end{pmatrix} \begin{pmatrix} X \\ \bar{X} \end{pmatrix} \tag{2.26}$$

with $X = (x^T, e^T)^T, \bar{X} = (\bar{x}^T, \bar{e}^T)^T$, $\bar{x} = x - x(t_k), \bar{e} = e - e(t_k)$, and

$$\bar{A}_{12} = \begin{pmatrix} -(B_0 + \Delta_2)K & +(B_0 + \Delta_2)K \\ -G(C_0 + \Delta_3) - \Delta_2 K & \Delta_2 K \end{pmatrix}.$$

Setting $z = (X^T, \bar{X}^T)^T$, At the sampling instants we have $z(t_k) = (X^T(t_k), 0^T)^T$. Let $\mathcal{T}$ be the projection of z on $\bar{X}$ so that $\mathcal{T}z = \bar{X}$. Introducing the $\Phi(t, t_0)$ resolvent of (2.26) we denote $\|\Phi(t, t_0)\|_{\sup} = \max_{\|x\|_{\sup}=1} \|\Phi(t, t_0)x\|_{\sup}$.

We can write

$$z(t) = \Phi(t, t_k)z(t_k) = \Phi(t, t_k) \begin{pmatrix} X(t_k) \\ 0 \end{pmatrix}, \ t \in [t_k, t_{k+1})$$

Between two sampling instants we have

$$\begin{aligned}\bar{X}(t_{k+1}^-) &= \bar{X}(t_{k+1}^-) - \bar{X}(t_k^+) \\ &= \mathcal{T}\big(\Phi(t,t_k) - I_d\big)\begin{pmatrix} X(t_k) \\ 0 \end{pmatrix}.\end{aligned}$$

Furthermore $\Phi(t_0,t_0) = I_d$ *and from assumption 2.2.5 where* $\mathcal{D}$ *is a compact set, between two sampling*

$$e^{-M_\Phi(t_2-t_1)} \leq \|\Phi(t_2,t_1)\|_{\sup} \leq e^{M_\Phi(t_2-t_1)} \tag{2.27}$$

with M_Φ *a positive constant. Hence,*

$$\|\bar{X}(t)\|_{\sup} \leq \|\mathcal{T}\|\|\Phi(t,t_k) - I_d\|_{\sup}\|X(t_k)\|_{\sup}, \quad t \in [t_k, t_{k+1}).$$

then (2.27) implies

$$\|\bar{X}(t)\|_{\sup} \leq \|\mathcal{T}\|_{\sup}(e^{M_\Phi(t-t_k)} - 1)\|X(t_k)\|_{\sup}$$

with $\|\mathcal{T}\|_{\sup} = 1$ *since it is a projection. Furthermore from (2.27)*

$$\|X(t_k)\|_{\sup} \leq \|X(t)\|_{\sup} e^{M_\Phi(t-t_k)}.$$

Therefore, for all $t \in [t_k, t_{k+1})$

$$\|\bar{X}(t)\|_{\sup} \leq (e^{M_\Phi(t-t_k)} - 1)e^{M_\Phi(t-t_k)}\|X(t)\|_{\sup}$$

and hence

$$\|\bar{X}(t)\|_{\sup} \leq (e^{M_\Phi\tau_{\min}} - 1)e^{M_\Phi(\tau_{\min})}\|X(t)\|_{\sup}, \ t \in [t_k, t_{k+1}).$$

Considering

$$\rho(\tau_{\min}) = (e^{M_\Phi\tau_{\min}} - 1)e^{M_\Phi\tau_{\min}}$$

for any $\epsilon > 0$ *there exists* $\tau_{\min}$ *small enough so that* $\rho(\tau_{\min}) \leq \epsilon$*, since between two sampling instants* $\|\bar{X}\|_{\sup} \leq \rho\|X\|_{\sup}$ *and (2.2.5) is verified. From norm equivalence we have* $\rho' = C\rho$ *such that* $\|\bar{X}\| \leq \rho\|X\|$ *In conclusion, considering the Lyapunov candidate* $V = X^T P X$*, with*

$$\dot{V} \leq -\gamma\|X\|^2 + \rho'(\tau_{min})\|P\bar{A}_{12}\|\|X\|^2$$

One can always choose τ_{min} *such that*

$$-\gamma + \rho'(\tau_{min})\|P\bar{A}_{12}\| \leq -\sigma$$

for any $\sigma < \gamma$ *implying that* $(x^T, e^T)^T$ *converges to zero. i.e.* $\bar{X}$ *can be seen as a vanishing perturbation affecting* X*.*

So as long as $t_{k+1} - t_k = \tau_{\min}$*,* $(x^T, e^T)^T$ *converges to zero exponentially, and when* $t_{k+1} - t_k > \tau_{\min}$ *the system goes toward a ball of radius* ε*. Therefore the proposed triggering condition leads to practical stability.*

Asymptotic Stability

Theorem 2.2.9 ensures only practical stability, despite the fact that asymptotic stability of the continuous closed loop system is assumed. Nevertheless, asymptotic stability can be recovered changing the triggering condition (2.20) with (2.23), as stated by the following.

Theorem 2.2.11 *Under assumption 2.2.5, it is possible to choose $\sigma_1, \sigma_2, \tau_{\min} > 0$ so that the observed–based controller (2.14), (2.15), with the triggering condition (2.23), solves the output–based asymptotic stabilization problem for system (2.6).* ◇

Proof 2.2.12 *It was shown in the proof of Theorem 2.2.9 the existence of a $\tau_{\min}$ ensuring the asymptotic stability of (2.17) as long as $t_{k+1} - t_k = \tau_{min}$. Know assume $t_{k+1} - t_k > \tau_{\min}$, where t_k is the last triggering time, and let us introduce the extended state $X = (x^T, e^T)^T$. One writes*

$$\dot{X} = H(\Delta(t))X + \begin{pmatrix} -(B_0 + \Delta_2(t))K\big(\bar{x} - \bar{e}\big) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t))\bar{x} \end{pmatrix}$$

where the definitions of $\bar{x}$, $\bar{e}$ are as in the proof of Theorem 2.2.9. There exists $P = P^T > 0$ such that $= H(\Delta(t))^T P + PH(\Delta(t)) + \gamma Id < 0$, $\forall\, t$. Considering $V = X^T P X$

$$\begin{aligned} \dot{V} &= X^T\big(H(\Delta(t))^T P + PH(\Delta(t))\big)X \\ &\quad + 2X^T P \begin{pmatrix} -(B_0\Delta_2)K(\bar{x} - \bar{e}) \\ -\Delta_2 K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3)\bar{x} \end{pmatrix} \\ &\leq -\gamma\|X\|^2 \\ &\quad + 2\|X\|\|P\| \left\| \begin{pmatrix} -(B + \Delta_2(t))K(\bar{x} - \bar{e}) \\ -\Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t)\bar{x} \end{pmatrix} \right\|. \end{aligned}$$

The triggering condition (2.23) implies that

$$\|\bar{x} - \bar{e}\| \leq \sigma_1\|x - e\|, \quad \|C\bar{x}\| \leq \sigma_2\|(C_0 + \Delta_3(t))x\|.$$

Since

$$\left\| \begin{pmatrix} -(B + \Delta_2(t))K(\bar{x} - \bar{e}) \\ \Delta_2(t)K(\bar{x} - \bar{e}) - G(C_0 + \Delta_3(t)(\bar{x}) \end{pmatrix} \right\| \leq 2\max\{a_1, a_2\}$$

$$a_1 = \sigma_1\|(B + \Delta_2(t))K\|\|x - e\|$$

$$a_2 = \sigma_2\|G\|\|e\| + \Delta_2(t)K\|\|x - e\|$$

and $\|e\| \leq \|X\|, \|x - e\| \leq 2\|X\|$, one finally works out

$$\dot{V} \leq \|X\|^2\Big(-\gamma + 4\|P\|s(\sigma_1, \sigma_2)\Big)$$

with

$$s(\sigma_1, \sigma_2) = \max\Big\{\|(B + \Delta_2(t))K\|\sigma_1, 2\|G\|\sigma_2 + \Delta_2(t)K\|\sigma_1\Big\}.$$

It is always possible to choose σ_1, σ_2 such that

$$-\gamma + 4\|P\|s(\sigma_1, \sigma_2) < 0.$$

This guarantees the exponential stability of the extended system.

Remark 2.2.13 *The proof of Theorem 2.2.11 shows that the triggered policy (2.23), while enforcing asymptotic stability, can ensure slower convergence rates.* ◇

2.2.3 Simulation and comments

Let us consider system (2.6), with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(I+ml^2)b}{p} & \frac{m^2gl^2}{p} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mlb}{p} & \frac{mgl(M+m)}{p} & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \frac{I+ml^2}{p} & 0 & \frac{ml}{p} \end{pmatrix}^T, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

representing by the linearization of the inverted pendulum on a cart, with x_1 the cart position, x_2 its velocity, x_3 the pendulum angle, and x_4 its angular velocity. Clearly, $\Delta_3 = 0$. The nominal parameter values are

$$l_0 = 0.3 \text{ m}, \qquad m_0 = 0.5 \text{ Kg}, \qquad M_0 = 0.5 \text{ Kg}$$
$$I_0 = 0.006 \text{ Kg m}^2, \quad b_0 = 0.1 \text{ Kg/s}, \quad g_0 = 9.8 \text{ m/s}^2$$

and $p_0 = I_0(M_0 + m_0) + M_0 m_0 l_0^2$, while the real parameters are

$$l \in [0.27, 0.33] \text{ m} \qquad m \in [0.45, 0.55] \text{ Kg}$$
$$M \in [0.45, 0.55] \text{ Kg} \quad I \in [0.0056, 0.0064] \text{ Kg m}^2$$
$$b = b_0 \text{ Kg/s}, \qquad g = 9.8 \text{m/s}^2$$

and $p = I(M + m) + Mml^2$. The matrices K and L are chosen such that the nominal system has its biggest eigenvalue equal to -2.

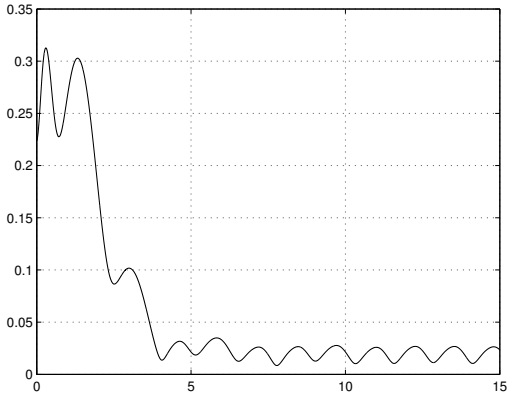

Figure 2.6: Euclidean norm of the state vs time.

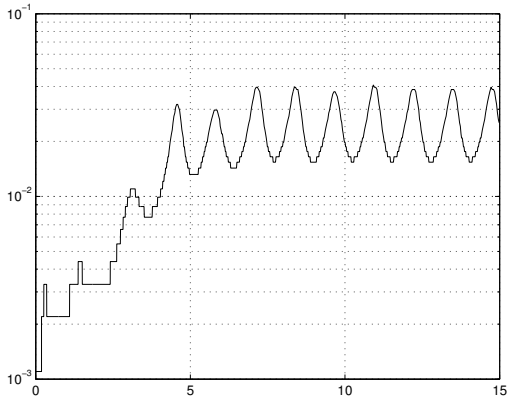

Figure 2.7: Inter–event time $t_{k+1} - t_k$ vs time.

The simulations results are shown in Fig. 2.6. The initial conditions are $x(0) = (0.1, 0, 0.2, 0)$, and $\hat{x}(0) = 0$. Fig. 2.6, 2.7 refers to a simulation in which the triggering condition (2.20) is used, with $\tau_{\min} = 10^{-3}$ s, $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. The practical stability can be observed in Fig. 2.6 and the inter–event time is shown in Fig. 2.7 while during the transient phase the inter–event times are $\tau_{\min}$, the successive inter–event times are determined

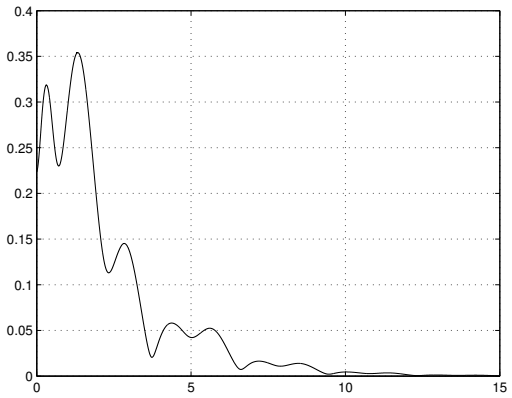

Figure 2.8: Euclidean norm of the state vs time.

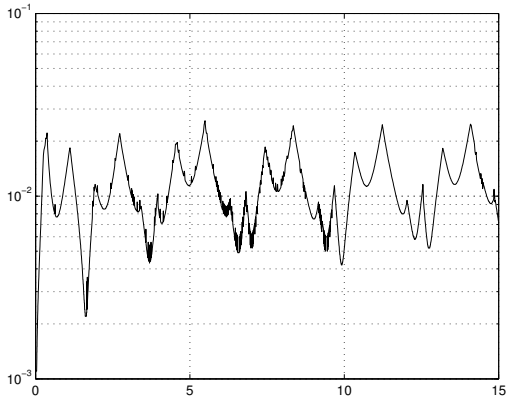

Figure 2.9: Inter–event time $t_{k+1}-t_k$ vs time.

by $\varepsilon_1, \varepsilon_2$. Fig. 2.8, 2.9 refers with the triggering condition (2.23), with $\tau_{\min} = 10^{-3}$ s, $\sigma_1 = \sigma_2 = 0.05$. This time, one observes asymptotic stability (Fig. 2.8, with a shorter average inter–event time (Fig. 2.9). The imposed minimum inter-event time does not turn to be useful in this instance of the problem since the output norm and the observer norm never cross 0.

2.2.4 Conclusions on uncertain linear systems using event triggered observation and control

For linear systems with time varying parametric uncertainties, and in the case of partial state knowledge, the event triggered paradigm can successfully be applied for stabilization via observer–based controllers. Practical and asymptotic stability are shown when considering model uncertainties. The proposed results recover full state feedback as a special case. The applicability of the proposed approach considering the linearized dynamics of the inverted pendulum on a cart as been considered. Further studies should include practical method of calculating an optimal choice of the triggering parameters, robustness with respect to disturbance and measurement noise.

2.3 Impulsive observer for Event triggered observation and control

In the previous section we described a classic Luenberger observer with a linear output based feedback the proposed methodology enables the observation and control of linear system while implementing event trigger sampling policy. It is shown under some condition that the control and observation gains can be computed offline and provided that the matrix of the extended linear system is Hurwitz, then the resulting closed loop system with event triggered sampling is stable (practically or asymptotically). This recovers somehow the classic idea of the separation principle. However the separation principle is somewhat stronger in stating the observation is possible without control. The classical Luenberger observer with event triggered sampling policy is able to conduct successful observation only when introducing a stabilizing dynamic controller. In the next section we

design an impulsive observer in order to recover the separation principle (i.e. Observation in absence of control).

2.3.1 Problem Formulation

Let us consider the linear system (2.6). When the state x is not measurable, under the condition of detectability of the pair (A, C) it is possible to build a Luenberger observer. In view of an implementation with a triggering policy, generically the observer has the structure

$$\dot{\hat{x}} = F\big(\hat{x}, y_{t_k^s}, u\big) \tag{2.28}$$

with possibly non smooth $F : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$. A feedback controller based on $\hat{x}$ will be used in the following in order to stabilize the system (2.6). After the input transmission at time t_j^a, the input applied to the system is

$$u(t) = K\hat{x}_{t_j^a}, \quad t \in [t_j^a, t_{j+1}^a) \tag{2.29}$$

with $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is Hurwitz, and the controlled dynamics are

$$\dot{x} = Ax + BK\hat{x}_{t_j^a}, \quad t \in [t_j^a, t_{j+1}^a). \tag{2.30}$$

Therefore, between two sampling instants, the closed–loop system dynamics are

$$\begin{aligned} \dot{x} &= Ax + BK\hat{x}_{t_j^a} \\ \dot{\hat{x}} &= F\big(\hat{x}, Cx_{t_k^s}, K\hat{x}_{t_j^a}\big) \end{aligned} \tag{2.31}$$

for $t \in [t_j^a, t_{j+1}^a) \cap [t_k^s, t_{k+1}^s) \neq \emptyset$. And the observer can have jumps at specific times $\hat{x}(t^+) = G(\hat{x}(t), Cx_{t_k^s}, u)$ These observer jumping times are to be determined later.

The following definition will be used.

Definition *2.3.1 Given the controlled system (2.30), the Practical Observer Design Problem (PODP) consists of finding an observer (2.28) and an event triggering policy such that the origin of the dynamics of the estimation error $e = x - \hat{x}$ is practically stable,*

$$\dot{e} = Ax + Bu - F\big(\hat{x}, Cx_{t_k^s}, u\big)$$

$$e^+ = x - G(\hat{x}, Cx_{t_k^s}, u)$$

and there exists a minimal inter–event time $\delta_{s,\min} > 0$ such that $\delta_{s,k} := t_{k+1}^s - t_k^s > \delta_{s,\min}$, $\forall\, k \in \mathbb{N}$.

Remark 2.3.2 *Note that in general the PODP does not required that u stabilizes the system. In fact, the PODP aims at designing an observer for the system (2.6) even if $x(t)$ becomes unbounded.* □

Definition *2.3.3 Given the system (2.31), the Practical Observer–Based Control Problem (POBCP) consists of finding an observer–based controller (2.28), (2.29) such that (i)the origin of the dynamics*

$$\begin{aligned} \dot{x} &= Ax + BK(x_{t_j^a} - e_{t_j^a}) \\ \dot{e} &= Ax + BK\hat{x}_{t_j^a} - F\big(\hat{x}, Cx_{t_k^s}, K\hat{x}_{t_j^a}\big) \end{aligned} \tag{2.32}$$

with $t \in [t_j^a, t_{j+1}^a) \cap [t_k^s, t_{k+1}^s) \neq \emptyset$ and $e = x - \hat{x}$ the estimation error, is practically stable, and (ii) there exists a minimal inter–event time $\delta_{\min} > 0$ such that $\delta_{s,k} = t_{k+1}^s - t_k^s > \delta_{\min}$ and $\delta_{a,j} = t_{j+1}^a - t_j^a > \delta_{\min}$, $\forall\, k, j \in \mathbb{N}$.

Definition *2.3.4 (Weak Separation Principle) Given system (2.31), the so called weak separation principle is verified if the PODP can be solved independently from the POBCP (i.e. it is possible to observe the system is without stabilising it.*

Remark 2.3.5 *The weak separation principle recovers the classic notion of separation principle, where the error dynamics do not depend on the system state, and the observer trajectories converge to those of the system state with no assumptions on the properties of the system dynamics. On the contrary, to the best of the authors' knowledge, the available results in the literature deal only with POBCP, and its resolution allows determining independently the dynamics of the first and of the second of (2.31), but nothing can be ensured for the observer convergence when the state x does not converge towards a compact containing the origin.* □

Event Triggering Policies for the resolution of the PODP and the POBCP

In the following, we introduce a triggering policy which can be used when the system state is not available, and which takes into account the communication constraints for the output and the input over the network.

Definition *2.3.6 (Mixed Triggering for Observer–Based Controllers) In the so called mixed triggering policy, the next sampling time for the actuation updating is*

$$\begin{aligned} t_{j+1}^a &= \min\left(\max\left\{t_{j+1}^{a1}, t_{j+1}^{a2}\right\}, t_{j+1}^{a3}\right) \\ t_{j+1}^{a1} &= \min_t \left\{t > t_j^a \mid \|\hat{x}(t) - \hat{x}(t_j^a)\| \geq \varepsilon_a\right\} \\ t_{j+1}^{a2} &= \min_t \left\{t \mid t \geq t_j^a + \tau_{\min}^a\right\} \\ t_{j+1}^{a3} &= \min_t \left\{t \mid t \geq t_j^a + \tau_{\max}^a\right\} \end{aligned} \tag{2.33}$$

and for the sensor sampling is

$$\begin{aligned} t_{k+1}^s &= \min\left(\max\left\{t_{k+1}^{s1}, t_{k+1}^{s2}\right\}, t_{k+1}^{s3}\right) \\ t_{k+1}^{s1} &= \min_t \left\{t > t_k^s \mid \|y(t) - y(t_k^s)\| \geq \varepsilon_s\right\} \\ t_{k+1}^{s2} &= \min_t \left\{t \mid t \geq t_k^s + \tau_{\min}^s\right\} \\ t_{k+1}^{s3} &= \min_t \left\{t \mid t \geq t_k^s + \tau_{\max}^s\right\} \end{aligned} \tag{2.34}$$

where $\varepsilon_a, \varepsilon_s, \tau_{\min}^a, \tau_{\min}^s \in \mathbb{R}^+$, $\tau_{\max}^s, \tau_{\max}^a \in \mathbb{R}^+$ are parameters, and $\tau_{\max}^s > \tau_{\min}^s, \tau_{\max}^a > \tau_{\min}^a$.

Remark 2.3.7 *For the sensor (resp the actuator) ,* $t^s_{k+1} - t^s_k \in [tau^s_{\min}, tau^s_{\max}]$*,(resp* $t^a_{k+1} - t^a_k \in [tau^a_{\min}, tau^a_{\max}]$*) in this time interval the sampling time is given by the triggering policy.*

Remark 2.3.8 *The values* t^{a2}_{j+1} *in* (2.33) *and* t^{s2}_{j+1} *in* (2.34) *will ensure the existence of a global minimum inter–event time, and this avoids Zeno phenomena. The saturation parameters* $\tau^s_{\max}, \tau^a_{\max}$ *are used to impose a maximum value to the inter–event time. This is a quite natural condition, imposed in practical cases. When* $\tau^s_{max} = \tau^a_{\max} = \infty$*, no saturation is applied to the maximal inter–event time. In the next section it will be shown that* (2.34) *solves the PODP, and that* (2.33)*, combined with* (2.34)*, solves the POBCP.*

Remark 2.3.9 *Note that* (2.33) *and* (2.34) *allow independent sampling of the output and of the controller, while ensuring minimum inter–event times for both the controller and the output samplings.* □

2.3.2 Impulsive Observers and Impulsive Observer–Based Controllers

The use of the sampled state, changing impulsively the information the observer possesses at the sampling instants, suggests the use of an observer in which the estimated state is changed impulsively. We will refer to such observers as impulsive observers [63]. To introduce such observers, we first recall that, as well known, it is possible to consider an appropriate state change T such that system (2.6) is rewritten in the form [62]

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u \\ y &= x_1 \end{aligned} \tag{2.35}$$

with $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{n-p}$, $\begin{pmatrix} x_1^T & x_2^T \end{pmatrix}^T = Tx$, and

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad TB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$
$$CT^{-1} = \begin{pmatrix} I_{p\times p} & 0 \end{pmatrix}.$$

For system (2.35), we can consider the following *impulsive observer* for $[t^s_k, t^s_{k+1})$

$$\begin{aligned} \dot{\hat{x}}_1 &= A_{11}\hat{x}_1 + A_{12}\hat{x}_2 + B_1u \\ \dot{\hat{x}}_2 &= A_{21}\hat{x}_1 + A_{22}\hat{x}_2 + G_0(\hat{x}_2 - z_2) + B_2u \\ \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 + G_1(\hat{x}_1 - z_1) + B_1u \\ \dot{z}_2 &= A_{21}z_1 + A_{22}z_2 + G_2(\hat{x}_1 - z_1) + B_2u \end{aligned} \tag{2.36a}$$

with $\hat{x}_1, z_1 \in \mathbb{R}^p$, $\hat{x}_2, z_2 \in \mathbb{R}^{n-p}$, $G_0 \in \mathbb{R}^{(n-p)\times(n-p)}$, $G_1 \in \mathbb{R}^{p\times p}$, $G_2 \in \mathbb{R}^{(n-p)\times p}$, and

$$\begin{aligned} \hat{x}_1(t^{s+}_k) = y(t^s_k) \text{ whenever } \|\hat{x}_1(t) - y(t^s_k)\| \geq \varepsilon_o \\ \text{and } t \geq t^s_k + \tau^s_{\min}. \end{aligned} \tag{2.36b}$$

for a $\varepsilon_o > 0$.

The dynamics (2.36a) are discontinuous, due to the reset conditions (2.36b), where one needs to consider the right–value of $\hat{x}_1$ at $t = t_k^s$. Note that $y(t_k^s) = x_1(t_k^s)$ is a continuous signal.

The dynamic controller is given by (2.29), (2.36) for $t \in [t_j^a, t_{j+1}^a) \cap [t_k^s, t_{k+1}^s) \neq \emptyset$. It is worth noticing that the reset $\hat{x}_1(t_k^{s+}) = x_1(t_k^s) = y(t_k^s)$ in (2.36b) occurs when the triggering condition occurs, while the reset $\hat{x}_1(t) = x_1(t_k^s) = y(t_k^s)$ is internal to the observer dynamics, and is imposed to bound the transient estimation error.

It is clear that (2.36) contains two coupled observers. In particular, that described by the variables z_1, z_2 is a Luenberger observer, forcing the dynamics of $\hat{x}_2$.

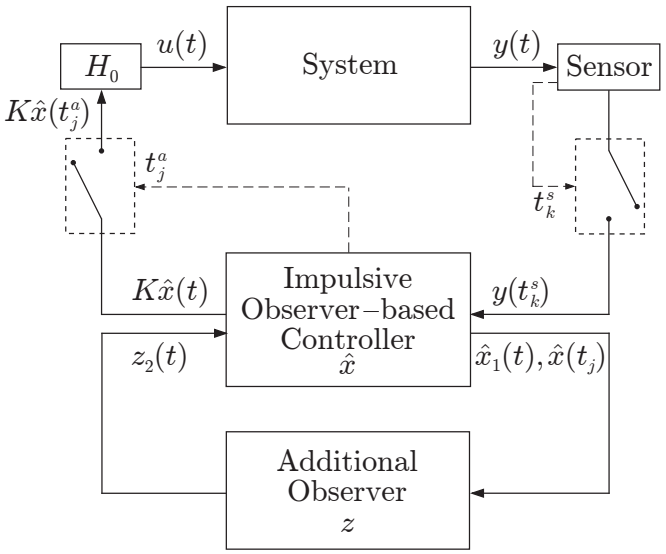

Figure 2.10: Structure of the impulsive observer.

Theorem 2.3.10 *Let (A, C) be detectable for system (2.6). Then, using the triggering condition (2.34) and the impulsive observer (2.36), it is possible to choose $\tau_{\min}^s, \varepsilon_s > 0$ such that the PODP is solved.*

Proof 2.3.11 *Considering the appropriate state change T such that system (2.6) is transformed into system (2.35), for $t \in [t_k^s, t_{k+1}^s)$ the dynamics of the errors $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$, $\bar{e}_1 = x_1 - z_1$, $\bar{e}_2 = x_2 - z_2$ are*

$$\begin{pmatrix} \dot{e}_1 \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ \eta \end{pmatrix} = \bar{A} \begin{pmatrix} e_1 \\ \eta \end{pmatrix} \tag{2.37}$$

with $\eta = \left(e_2^T,\ \bar{e}_1^T,\ \bar{e}_2^T\right)^T$, and the reset conditions are

$$\begin{aligned} e_1(t_k^{s+}) = x_1(t_k^s) - \hat{x}_1(t_k^{s+}) = y(t_k^s) - y(t_k^s) = 0 \\ \hat{x}_1(t) = y(t_k^s) \ \textit{whenever} \ \|\hat{x}_1(t) - x_1(t_k^s)\| \geq \varepsilon_o \\ \textit{and} \ t \geq t_k^s + \tau_{\min}^s \end{aligned} \tag{2.38}$$

where

$$\bar{A}_{12} = \begin{pmatrix} A_{12} & 0 & 0 \end{pmatrix}, \qquad \bar{A}_{21} = \begin{pmatrix} A_{21} \\ G_1 \\ G_2 \end{pmatrix}$$

$$\bar{A}_{22} = \begin{pmatrix} A_{22} + G_0 & 0 & -G_0 \\ 0 & A_{11} - G_1 & A_{12} \\ 0 & A_{21} - G_2 & A_{22} \end{pmatrix}.$$

The matrix G_0 can be chosen so that the matrix $A_{22} + G_0$ is Hurwitz. Moreover, since (A, C) is detectable, it is possible to determine G_1, G_2 such that the matrix

$$\begin{pmatrix} A_{11} - G_1 & A_{12} \\ A_{21} - G_2 & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix}$$

is Hurwitz as well. Therefore, $\bar{A}_{22}$ can be rendered Hurwitz, so that there exists $P = P^T > 0$, solution of $P\bar{A}_{22} + \bar{A}_{22}^T P = -2Q$ for any fixed $Q = Q^T > 0$. We can distinguish two cases.

a) If $t_{k+1}^s = t_{k+1}^{s1} > t_{k+1}^{s2} = t_k^s + \tau_{\min}^s$, the triggering condition (2.34) and the reset condition (2.36b) ensure that

$$\|e_1\| \leq \|y - y_{t_k^s}\| + \|\hat{x}_1 - y_{t_k^s}\| \leq \varepsilon_s + \varepsilon_o = \varepsilon_1.$$

As far as η is concerned, considering the Lyapunov candidate $V = \eta^T P \eta / 2$, one gets

$$\begin{aligned} \dot{V} &= \eta^T P \bar{A}_{21} e_1 - \eta^T Q \eta \leq -\lambda_{\min}^Q \Big(\|\eta\| - \Delta_2 \Big) \|\eta\| \\ \Delta_2 &= \frac{\|P\bar{A}_{21}\|}{\lambda_{\min}^Q} \varepsilon_1 \end{aligned} \tag{2.39}$$

so that, for any initial conditions $\eta(0)$, the ball $\mathcal{B}_{\varepsilon_2}$, with

$$\varepsilon_2 = \sqrt{\frac{\lambda_{\max}^P}{\lambda_{\min}^P}} \frac{\|P\bar{A}_{21}\|}{\lambda_{\min}^Q} \varepsilon_1$$

centred in the origin is attractive for η, i.e. the trajectories of η converge inside $\mathcal{B}_{\varepsilon_2}$ in finite time. Therefore, one can conclude that the ball $\mathcal{B}_\epsilon$, $\epsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}$, is attractive for $(e_1^T, \eta^T)^T$.

b) If $t_{k+1}^s = t_{k+1}^{s2} = t_k^s + \tau_{\min}^s \geq t_{k+1}^{s1}$ in (2.34), the solution of (2.37) over $[t_k^{s+}, t) = [t_k^s, t)$ is

$$\begin{pmatrix} e_1(t) \\ \eta(t) \end{pmatrix} = e^{\bar{A}(t - t_k^s)} \begin{pmatrix} e_1(t_k^{s+}) \\ \eta(t_k^s) \end{pmatrix} \tag{2.40}$$

since $\eta(t_k^{s+}) = \eta(t_k^s)$ (the discontinuity is only on $\hat{x}_1$ due to the reset), so that

$$\begin{pmatrix} e_1(t) - e_1(t_k^{s+}) \\ \eta(t) - \eta(t_k^s) \end{pmatrix} = \Big(e^{\bar{A}(t - t_k^s)} - I_{2n \times 2n} \Big) \begin{pmatrix} e_1(t_k^{s+}) \\ \eta(t_k^s) \end{pmatrix}.$$

Considering (2.38), and premultiplying by $\begin{pmatrix} I_{p \times p} & 0 \end{pmatrix}$, one gets

$$e_1(t) = \begin{pmatrix} I_{p \times p} & 0 \end{pmatrix} \Big(e^{\bar{A}(t - t_k^s)} - I_{2n \times 2n} \Big) \begin{pmatrix} 0 \\ \eta(t_k^s) \end{pmatrix}.$$

Therefore,

$$\|e_1(t)\| \leq \left\|e^{\bar{A}(t-t_k^s)} - I_{2n\times 2n}\right\| \left\|\begin{pmatrix} 0 \\ \eta(t_k^s)\end{pmatrix}\right\|$$

and since

$$\begin{aligned}\left\|e^{\bar{A}(t-t_k^s)} - I_{2n\times 2n}\right\| &= \left\|\sum_{j=1}^{\infty} \frac{(t_k^s)^j}{j!}\bar{A}^j\right\| \\ &\leq \sum_{j=1}^{\infty} \frac{(t_k^s)^j}{j!}\|\bar{A}\|^j = e^{\|\bar{A}\|(t-t_k^s)} - 1\end{aligned}$$

one finally gets

$$\|e_1(t)\| \leq \left(e^{\|\bar{A}\|(t-t_k)} - 1\right)\|\eta(t_k^s)\|. \tag{2.41}$$

Considering again (2.40) *and the first of* (2.38),

$$\begin{pmatrix} e_1(t_k^{s+}) \\ \eta(t_k^s)\end{pmatrix} = \begin{pmatrix} 0 \\ \eta(t_k^s)\end{pmatrix} = e^{-\bar{A}(t-t_k^s)}\begin{pmatrix} e_1(t) \\ \eta(t)\end{pmatrix}$$

so that

$$\|\eta(t_k^s)\| \leq e^{\|\bar{A}\|(t-t_k^s)}\left\|\begin{pmatrix} e_1(t) \\ \eta(t)\end{pmatrix}\right\|$$

since $\|e^{-\bar{A}(t-t_k^s)}\| \leq e^{\|\bar{A}\|(t-t_k^s)}$, *and*

$$\|\eta(t_k^s)\|^2 \leq e^{2\|\bar{A}\|(t-t_k^s)}\Big(\|e_1(t)\|^2 + \|\eta(t)\|^2\Big). \tag{2.42}$$

Therefore, from (2.41), (2.42), *one works out*

$$\begin{aligned}\|e_1(t)\|^2 &\leq \left(e^{\|\bar{A}\|(t-t_k^s)} - 1\right)^2\|\eta(t_k^s)\|^2 \\ &\leq \left(e^{\|\bar{A}\|(t-t_k^s)} - 1\right)^2 e^{2\|\bar{A}\|(t-t_k^s)}\Big(\|e_1(t)\|^2 \\ &\qquad + \|\eta(t)\|^2\Big)\end{aligned}$$

and finally

$$\|e_1(t)\| \leq \varrho(\|\bar{A}\|\tau)\|\eta(t)\| < \varrho(\|\bar{A}\|\tau_{\min}^s)\|\eta(t)\| \tag{2.43}$$

with $\tau = t - t_k^s \in [0, \tau_{\min}^s)$, *and*

$$\varrho(s) = \left[\left(e^s - 1\right)^{-2}e^{-2s} - 1\right]^{-1/2}$$

which is such that $\varrho(0) = 0$, *is strictly increasing and goes to infinity as* s *tends to* $\bar{s} = (1+\sqrt{5})/2$. *Hence,* ϱ *is uniquely invertible for* $s \in [0, \bar{s})$. *Considering the Lyapunov candidate* $V = \eta^T P\eta/2$, *as in the case a), and repeating the same passages, one gets*

$$\dot{V} < -\Big(\lambda_{\min}^Q - \|P\bar{A}_{21}\|\varrho(\|\bar{A}\|\tau_{\min}^s)\Big)\|\eta\|^2.$$

Hence, $\dot{V} < 0$ *choosing* Q *and* $\tau_{\min}^s$ *so that*

$$\frac{\lambda_{\min}^Q}{\|P\bar{A}_{21}\|} < \bar{s}, \quad \tau_{\min}^s < \frac{1}{\|\bar{A}\|}\varrho^{-1}\left(\frac{\lambda_{\min}^Q}{\|P\bar{A}_{21}\|}\right).$$

Therefore, η *will converge asymptotically to zero, as well as* e_1 *thanks to* (2.43).

Considering the cases a) (practical stability) and b) (asymptotic stability), and since the triggering condition (2.34) considers the maximum between τ_{k+1}^{s1} and τ_{k+1}^{s2}, one concludes that the origin of the dynamics of (e_1, e_2) is practically stable. Moreover, the inter-event time is at least $\tau_{\min}^{s} > 0$. Therefore, one can can conclude that the PODP is solved.

Theorem 2.3.12 *Let (A, B), (A, C) be stabilizable and detectable, respectively, for system (2.6). Then, using the triggering conditions (2.33), (2.34) and the observed–based controller (2.29), (2.36), it is possible to choose $\tau_{\min}^{s}, \varepsilon_s, \tau_{\min}^{a}, \varepsilon_a > 0$ such that the POBCP is solved.*

Proof 2.3.13 *Denoting $\bar{x} = x - x_{t_j^a}$ and $w = x_{t_j^a} - \hat{x}_{t_j^a}$ From (2.3.12) $\forall \varepsilon$ there exist ε_s and T such that $\forall t > T$, $\|w\| < \varepsilon$*

$$\begin{pmatrix} \dot{x} \\ \dot{\bar{x}} \end{pmatrix} = \begin{pmatrix} A+BK & -BK \\ A+BK & -BK \end{pmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} - \begin{pmatrix} BKw \\ BKw \end{pmatrix}$$

a) If $t_{j+1}^{a} = t_{j+1}^{a1} \geq t_{j+1}^{a2} = t_j^a + \tau_{\min}^{a}$, the triggering condition (2.33) ensures $\|x - x_{t_j^a}\| \leq \varepsilon_a$ Therefore choosing the dynamic of $\dot{x}$ is just linear convergent dynamic given by $A + BK$ perturbed with non vanishing perturbation.

Therefore there is a quadratic Lyapunov function $V = x^t \bar{P} x$ ensuring practical stability in a ball $\mathcal{B}_b$.

b) If $t_{j+1}^{a} = t_{j+1}^{a2} = t_j^a + \tau_{\min}^{a} > t_{j+1}^{a1}$ in (2.33), let us denote $X = (x^T, \bar{x}^T)^T$ and $\bar{A} = \begin{pmatrix} A+BK & -BK \\ A+BK & -BK \end{pmatrix}$.

Since $\bar{x}(t_j^{a+}) = 0$, if we consider $\mathcal{T}$ the projection of $(x^T, \bar{x}^T)^T$ on $\bar{x}$ ($\mathcal{T}X = \bar{x}$), one has

$$\begin{aligned} X(t) &= \mathrm{e}^{\bar{A}(t-t_j^a)} X_{t_j^a} + \int_{t_j^a}^{t} e^{\bar{A}(\tau - t)} \begin{pmatrix} BKw(\tau) \\ BKw(\tau) \end{pmatrix} d\tau \\ &= \mathrm{e}^{\bar{A}(t-t_j^a)} \begin{pmatrix} x_{t_j^a} \\ 0 \end{pmatrix} + \int_{t_j^a}^{t} e^{\bar{A}(\tau - t)} \begin{pmatrix} BKw(\tau) \\ BKw(\tau) \end{pmatrix} d\tau \end{aligned}$$

and since between two samplings the system is linear, on obtain

$$\begin{aligned} \bar{x}(t_{j+1}^{a-}) &= \bar{x}(t_{j+1}^{a-}) - \bar{x}(t_j^{a+}) \\ &= \mathcal{T}\big(\mathrm{e}^{\bar{A}(t-t_j^a)} - I_{2n\times 2n}\big) \begin{pmatrix} x_{t_j^a} \\ 0 \end{pmatrix} \\ &+ \mathcal{T} \int_{t_j^a}^{t} e^{\bar{A}(\tau - t)} \begin{pmatrix} BKw(\tau) \\ BKw(\tau) \end{pmatrix} d\tau. \end{aligned}$$

So for each $t \in [t_j^a, t_{j+1}^a)$ the inequality $\|BKw\| < \|BK\|\varepsilon$ gives

$$\begin{aligned} \|\bar{x}(t)\| \leq \|\mathcal{T}\| \big[&\| \mathrm{e}^{\bar{A}(t-t_j^a)} - I_{2n\times 2n}\| \|X_{t_j^a}\| \\ &+ 2\|BK\|\varepsilon \int_{t_j^a}^{t_{j+1}^a} e^{\|\bar{A}\|_\infty (t_{j+1}^a - \tau)} d\tau \big]. \end{aligned}$$

So

$$\|\bar{x}(t)\| \leq \|\mathcal{T}\|(e^{\|\bar{A}\|_\infty(t-t_j^a)}-1)\|x(t_j^a)\| + \frac{2\|BK\|\varepsilon(e^{\|\bar{A}\|_\infty\tau_{\min}^a}-1)}{\|\bar{A}\|_\infty}$$

where $\|\mathcal{T}\|$*=1. Since the system is affine, the following inequality also holds.*

$$\|x(t_j^a)\| \leq \|X(t)\|e^{\|\bar{A}\|_\infty(t-t_j^a)} + \frac{2\|BK\|\varepsilon(e^{\|\bar{A}\|_\infty(t-t_j^a)}-1)}{\|\bar{A}\|_\infty}.$$

we have $\|X\| \leq \|x\| + \|\bar{x}\|$ *Therefore, for all* $t \in [t_j^a, t_{j+1}^a)$

$$\begin{aligned}\|\bar{x}(t)\| \leq{}& \frac{2\|BK\|\varepsilon(e^{\|\bar{A}\|_\infty\tau_{\min}^a}-1)}{\|\bar{A}\|_\infty}(e^{\|\bar{A}\|_\infty\tau_{\min}^a}-1) \\ &+ e^{\|\bar{A}\|_\infty\tau_{\min}^a}(e^{\|\bar{A}\|_\infty\tau_{\min}^a}-1)\big((\|\bar{x}(t)\| + \|x(t)\|\big) \\ &+ \frac{2\|BK\|\varepsilon(e^{\|\bar{A}\|_\infty\tau_{\min}^a}-1)}{\|\bar{A}\|_\infty}\end{aligned}$$

$$\|\bar{x}(t)\| \leq \rho(\|\bar{A}\|_\infty\tau_{\min}^a)\|x\| + \frac{2\|BK\|\varepsilon}{\|\bar{A}\|_\infty}\theta(\|\bar{A}\|_\infty\tau_{\min}^a) \tag{2.44}$$

where

$$\rho(s) = \frac{(e^s-1)e^s}{1-(e^s-1)e^s}$$

$$\theta(s) = \frac{(e^s-1)}{(1-(e^s-1)e^s)}.$$

For any $\epsilon > 0$ *and any* $\|\bar{A}\|_\infty$ *there exists a* $\tau_{\min}^a$ *small enough so that* $\rho(\|\bar{A}\|_\infty\tau_{\min}^a) \leq \epsilon$*. We define* $V = x^T\bar{P}x$ *the same candidate Lyapunov function as in the case* $t_{j+1}^a > t_j^a + \tau_{\min}^a$ *from* (2.44) *and* $\|w\| < \varepsilon$ *we have the following inequality*

$$\begin{aligned}\dot{V} \leq{}& \Big[-\lambda_{\max}\Big((A+BK)^T\bar{P} + \bar{P}(A+BK)\Big) \\ &+ 2\rho(\tau_{\min}^a)\|\bar{P}BK\|\Big]\|x\|^2 \\ &+ 2\|\bar{P}BK\|\varepsilon\big(\theta(\tau_{\min}^a)\frac{\|BK\|}{\|\bar{A}\|_\infty} + 1\big)\|x\|.\end{aligned}$$

Since $A+BK$ *is Hurwitz we can consider the term* $\rho(\tau_{\min}^a)$ *as a vanishing perturbation and* θ *a non vanishing one. We can choose* $\tau_{\min}^a$ *such that* $-\lambda_{\max}(A^TP + PA) + 2\rho\|PBK\| < 0$*. From known result on vanishing perturbations [?], the ball* $\mathcal{B}_r$

$$r = \frac{2\|\bar{P}BK\|\varepsilon\big(\theta(\tau_{\min}^a)\|BK\| + 1\big)}{\lambda_{\max}(A^TP + PA) - 2\rho(\tau_{\min}^a)\|PBK\|}$$

is attractive,

In conclusion, for both cases the same lyapunov function decrease as long as it is not in a neighbourhood of zero. Therefore, the proposed triggering condition leads to practical stability to a ball and the ultimate bound is given by $\{||x|| \leq b + r\}$*.*

2.3.3 Simulation and comments

In this section the proposed impulsive observer is used for the stabilization of the linearized model of an inverted pendulum on a cart. To better show the advantages of the impulsive observer, a comparison is presented with the case in which the following observer

$$\dot{\hat{x}} = Ax + BK\hat{x}_{t_j^a} + LC(x_{t_k^s} - \hat{x}) \tag{2.45}$$

is considered, which represents the obvious implementation of the classic Luenberger observer in the event–triggering setting.

The dynamics of a linearized inverted pendulum on a cart are in the form (2.6), with $z = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix}^T$ and z_1 the cart position, z_2 the cart velocity, z_3 the pendulum angle with respect to the vertical, z_4 the pendulum angular velocity. Introducing a coordinate change : $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} z_1 & z_3 & z_2 & z_4 \end{pmatrix}^T$ The system is in the form (2.35), with the position x_1 and the angle x_2 measured variables, and

$$A_{11} = \begin{pmatrix} 0 & 1 \\ 0 & -\dfrac{(J+Ml^2)b}{p} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ \dfrac{M^2gl^2}{p} & 0 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{Mlb}{p} \end{pmatrix}, \qquad A_{22} = \begin{pmatrix} 0 & 1 \\ \dfrac{Mgl(m+M)}{p} & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 \\ \dfrac{J+Ml^2}{p} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 \\ \dfrac{Ml}{p} \end{pmatrix}$$

where the parameters have the following values

$$l = 0.3 \text{ m}, \qquad m = 0.5 \text{ Kg}, \qquad M = 0.5 \text{ Kg}$$
$$J = 0.006 \text{ Kg m}^2, \quad b = 0.1 \text{ Kg/s}, \quad g = 9.8 \text{ m/s}^2$$

and $p = J(M+m) + Mml^2$. The observer (2.36a) is given by

$$G_0 = \begin{pmatrix} -119.8211 & 0.5263 \\ 0 & -80.0000 \end{pmatrix}$$
$$G_1 = \begin{pmatrix} -22.8712 & 1.0085 \\ 1.6639 & -20.9498 \end{pmatrix}$$
$$G_2 = \begin{pmatrix} -126.1812 & 2.6591 \\ 24.0036 & -160.3360 \end{pmatrix}$$

corresponding to fix the spectrum $\sigma(A_{22})$ in $\{-119.9931, -80.0069, -13.5, -9, -10, -11.5\}$, while the applied control (2.29) is given by

$$K = \begin{pmatrix} 0.3102 & -14.4655 & 0.7204 & -1.7309 \end{pmatrix}$$

and corresponds to fix all the eigenvalues of $A + BK$ in -2.

In Fig. 3.3 the performance of the controllers based on the impulsive observer (2.36) and on the Luenberger–like observer (2.45) is shown, where the values

$$\varepsilon_a = 0.01, \quad \varepsilon_s = 0.01, \quad \tau^a_{\min} = \tau^s_{\min} = 10^{-3} \text{ s}$$
$$\tau^a_{\max} = \tau^a_\ell = \tau^s_{\max} = \tau^s_\ell = \infty$$

have been used in (2.33), (2.34). The initial conditions are chosen randomly in the unit hypercube.

To show the controller performance for different values of ε_a, ε_s, these parameters are let to vary.The values of ε_a, ε_s are varied to change the radius of the ball where the error variables converge.

Choosing randomly the initial conditions in the unit hypercube, and then taking the mean value of the communication and stabilization errors, one gets the graphics of Figs. 2.12, 2.13. Considering the maximal stabilization error norm $\|x\|_{ss}$ during the steady state, defined as

$$\|x\|_{ss} := \max_{t \in T_{ss}} \|x(t)\|$$

with $T_{ss} = [10, 15]$ s in the case considered, in these figures the number of sensor and actuator communications N_s, N_a are given as function of $\|x\|_{ss}$, to provide the stabilization performance obtained with a certain number of communications. Either in the case of the Luenberger–like observer (2.45), or in the case of the impulsive observer (2.36), the simulations show good results in term of observation and control performance, during both the transient and the steady state. However, these two observers have different behaviours in terms of samplings. In fact, Fig. 2.12 shows that, for comparable performance in terms of stabilization, the impulsive observer (2.36) needs more sensors communications (Fig. 2.12.a), but provides a state estimation which determines less updates of the control value (Fig. 2.12.b). Moreover, the sum of the number $N_s + N_a$ of sensor and controller communications is smaller for the impulsive observer–based controller, i.e. it is less demanding in terms of communication resources (Fig. 2.12.c).

A quite interesting property shown by the simulations is the following. If a saturation function is used in (2.34), with $\tau^s_{\max} = \tau^s_\ell = 0.3$ s, i.e. if a maximum inter–event time $\tau^s_{\max}$ is imposed, as in [11], the performance of the impulsive observer–based controller improves sensibly, surprisingly not generating more communications from the sensor. On the contrary, the introduction of a maximum inter–event time $\tau^a_{\max}$ does not influence sensibly the performance of the controller. Note that a periodic sampling of 0.3 s would lead to instability. Hence, while the triggering parameters $\tau^s_{\max}, \tau^a_{\max}$ in (2.33), (2.34) should (intuitively) induce a more frequent sampling, in practice they determine a reduction of the number of samplings. Intuitively, this is due to the fact that they ensure that the system trajectory does not diverge abruptly, in which case a faster sampling should be necessary to force it again close to the origin. The comparison of the performance shown in Fig. 2.13 with that of Fig. 2.12, shows that $\tau^s_{\max}$ (but also $\tau^a_{\max}$) can be used as *control parameter*. For instance, for $\|x\|_{ss} = 10^{-4}$, for the impulsive observer (2.36) the total number of sampling is 3500 when $\tau^a_{\max} = \tau^s_{\max} = \infty$, while it drops to 2400 when $\tau^s_{\max} = 0.3$ s. On the contrary, $\tau^s_{\max}$ has no visible impact on the Luenberger–like observer (2.45), with which the number of sampling remains always about 3900.

Figure 2.14 shows that when no saturation is implemented on the maximal inter-sampling time instability close to zero can occur. While this instability is benign in term of observation and control it has an impact on sampling. Namely they will occur later to correct the instability.
Figure 2.15 highlight the fact that a saturation on the maximal inter-sampling time can prevent this phenomenon. There is a trade off since a saturation on maximal inter-event time could also lead to more communication. This saturation must not be seen as a periodic sampling since this maximum time alone would lead to instability if used as the period of a periodic sampling. Comparing Figure 2.12 and Figure 2.13 along with

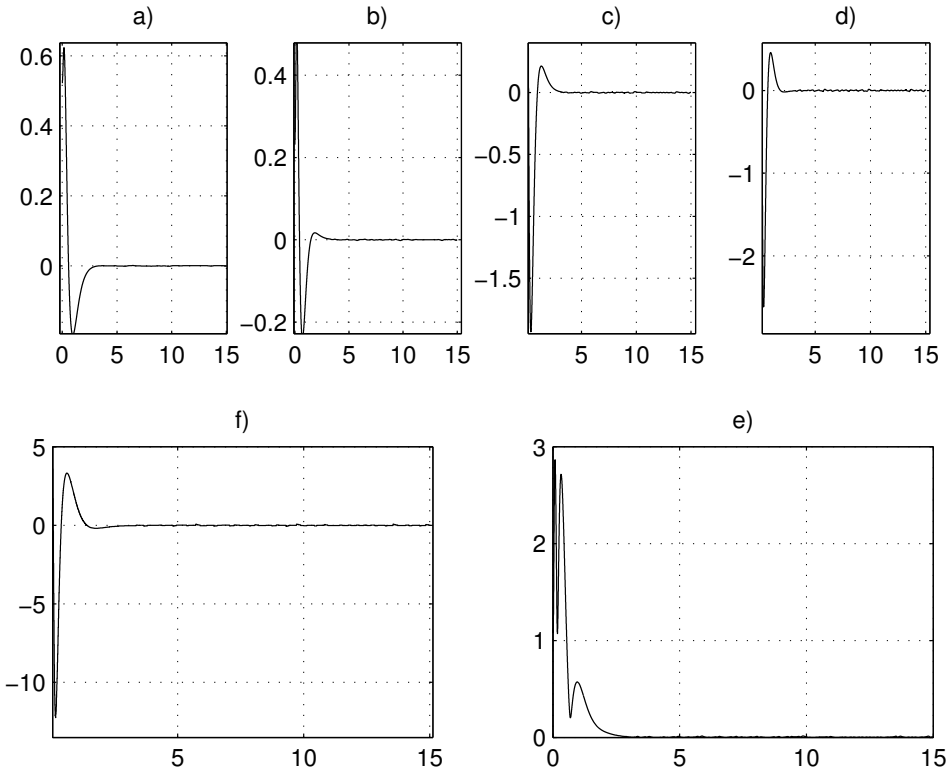

Figure 2.11: Controller based on the impulsive observer (2.36) (circle) and Luenberger–like observer (2.45) (star) with $\tau^a_{\max} = \tau^a_\ell = \tau^s_{\max} = \tau^s_\ell = \infty$: a) x_1 [m vs s]; b) x_2 [m/s vs s]; c) x_3 [rad vs s]; d) x_4 [rad/s vs s]; e) $\|x\|$ [dimensionless vs s]; f) u [N vs s].

Figure 2.14 and Figure 2.15 show that on average the overall communication load is not increased with a saturation on the maximum inter-event time.

2.3.4 Conclusions on impulsive observer for event-triggered linear systems

In this section a weak separation principle has been introduced for linear systems stabilized by observer–based controllers, making use of an event–triggered technique.

An impulsive observer has been designed along with the event–triggered policy, to ensure practical convergence of the estimate to the system state, and nonzero inter–event times, avoiding Zeno behaviors. The motivation of the impulsive observer (as opposed to a discrete time Luenberger observer) in this context being to make the most of the information transmitted at the moment where it is most relevant. The practical stability of the observation error can be achieved without stability assumptions on the system dynamics, which can be stable or unstable. The proposed event–triggered policy allows asynchronous communications sensor–controller and controller–actuator. Finally, it has been shown in the case of a linearized inverted pendulum that it is possible to tune the various parameters of the event–triggered control scheme in order to reduce the overall communications among the different components. The proposed observer has been compared to a classical Luenberger observer, subject to the same triggering policy.

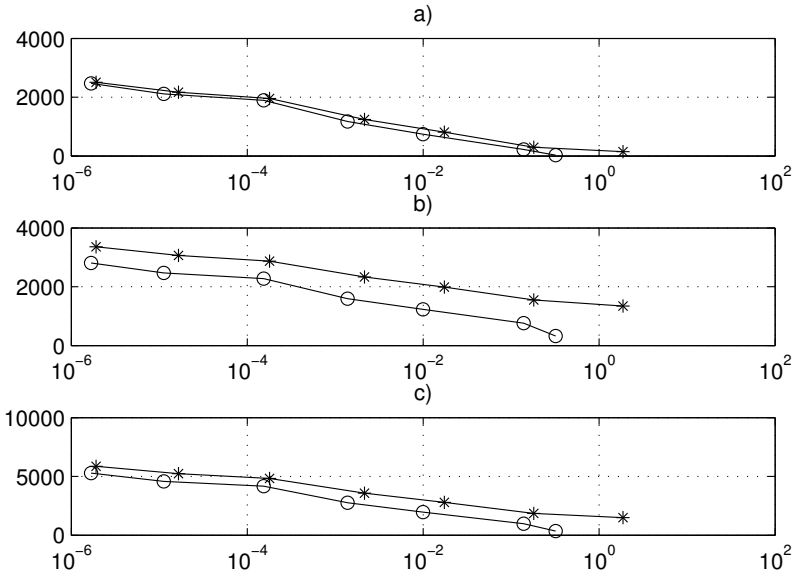

Figure 2.12: Controller based on the impulsive observer (2.36) (circle) and Luenberger–like observer (2.45) (star) with $\tau^a_{\max} = \tau^a_\ell = \tau^s_{\max} = \tau^s_\ell = \infty$: a) Number N_s of sensor communications versus stabilization error norm $\|x\|_{ss}$; b) Number N_a of controller communications versus stabilization error norm $\|x\|_{ss}$; c) Number $N_s + N_a$ of sensor and controller communications versus stabilization error norm $\|x\|_{ss}$.

The simulation results suggest the implementation of a maximum inter–event time may bring some benefit in terms of the reduction of the communications.

2.4 Conclusion on event triggering policy for observation and output based feedback of linear systems

In this chapter we introduced key concepts of event triggered sampling for observation and control of linear systems. We demonstrated that for some classes of systems (Linear Systems, Uncertain Linear Systems) it is possible to adapt classic observer to allow dynamic feedback under event triggered sampling. The main objective of event triggering being the reduction of the communication among node in a network control systems. To this end special problems were adressed (i.e adaptated event triggered sampling policy, Special observer structure). However some issues remain open and will be investigate in future work. In this chapter only linear systems have been considered and allow global features such as global stability and global obsrvability. It as been shown that under the assumption of detectability and stabilisability one can always derive adaptated event triggered policy that allows (practical)convergence of the state to 0

A natural question arising is the extension of the results for linear systems to the case of non linear systems. This will be the object of the next chapter

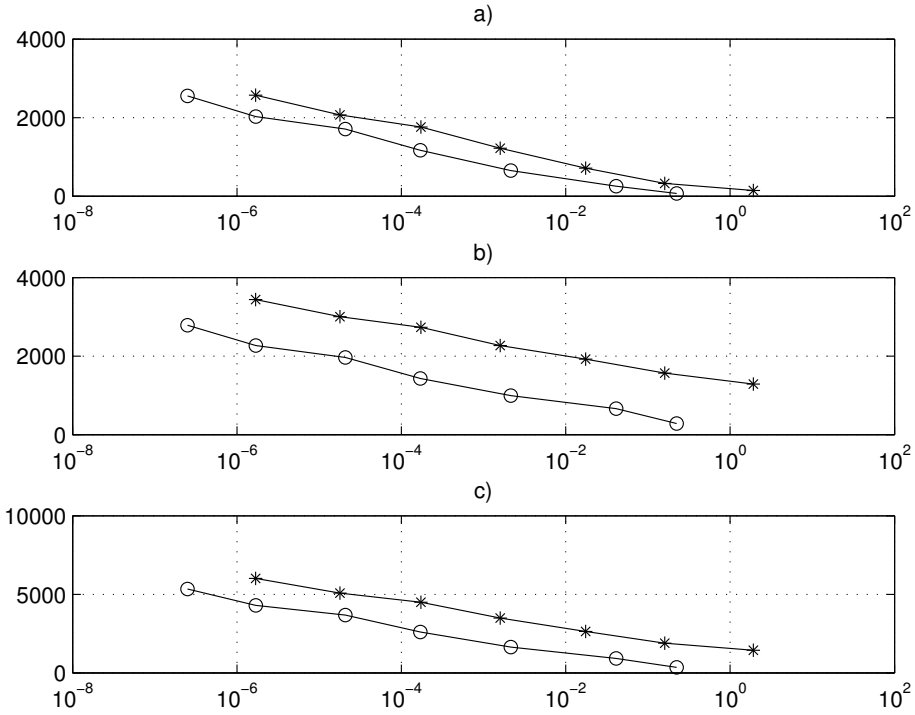

Figure 2.13: Controller based on the impulsive observer (2.36) (circle) and Luenberger–like observer (2.45) (star) with $\tau^s_{\max} = \tau^s_\ell = 0.3$ s and $\tau^a_{\max} = \tau^a_\ell = \infty$: a) Number N_s of sensor communications versus stabilization error norm $\|x\|_{ss}$; b) Number N_a of controller communications versus stabilization error norm $\|x\|_{ss}$; c) Number $N_s + N_a$ of sensor and controller communications versus stabilization error norm $\|x\|_{ss}$.

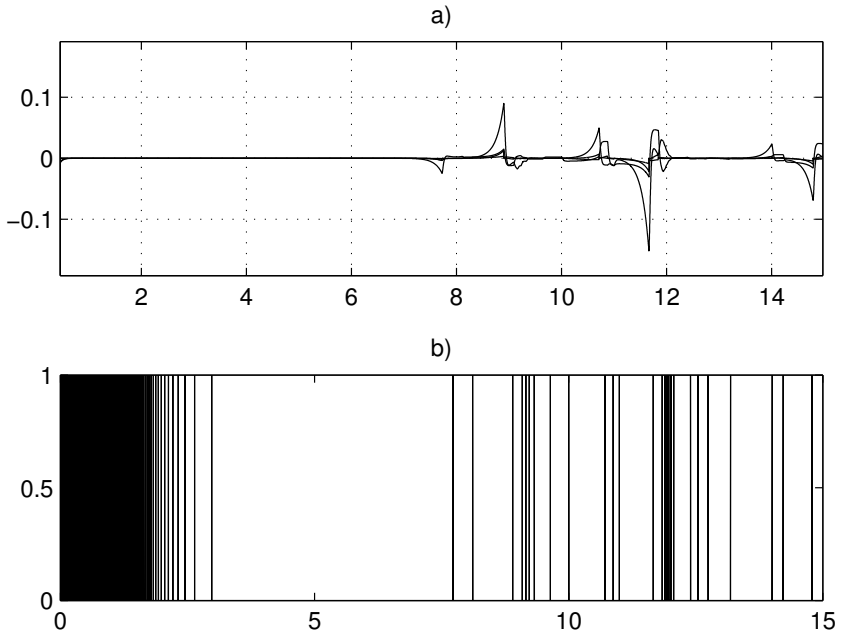

Figure 2.14: $\tau^s_{\max} = \tau_\ell = \infty$: a) Observation error; b) Sensor sampling instants t^s_k (305 samplings).

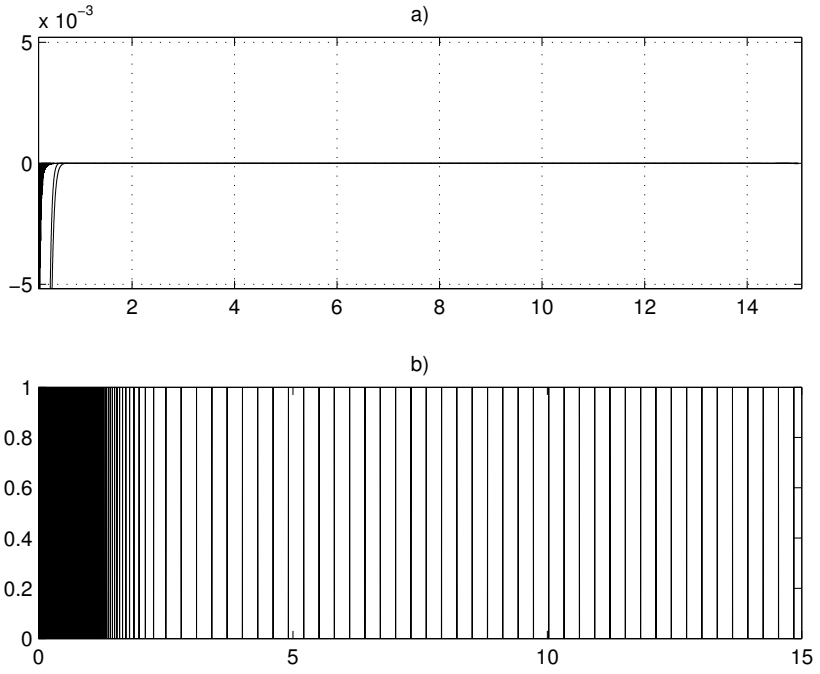

Figure 2.15: $\tau^s_{\max} = \tau_\ell = 0.3$ s: a) Observation error; b) Sensor sampling instants t^s_k (293 samplings).

Chapter 3

Event triggering policy for observation and output based feedback for some classes of non linear systems

In the previous chapter we considered event triggering observation and control of Linear systems, in this second chapter we will extend our previous results to non linear systems subject to communication constraints. Observability/observation and controllability/control of non linear systems can be a challenging task especially when considering communication constraints, however throughout this chapter systems under consideration possesses some features that make them tractable. The details of those features will be stated in the problem formulation, however since they might appear as embedded in the hypothesis we want to state some generic properties. The systems we consider are Lipschitz. Furthermore embedded in the observability assumption is the notion of instantaneous observability (i.e. knowing an arbitrary small time interval of the output allow reconstruction of the initial condition of the system). Those properties hold for linear system however they restrict the classes of non linear system under consideration.

3.1 General elements on event triggering for non linear systems

In order to established necessary condition for observer based stabilisation of a non linear system subject to some event triggered mechanism, we will first recall some useful result of event triggering for non linear systems when the full state is available and no observer is needed.

It is remarkable that, when considering non linear systems with full state available for control results similar to those established for linear systems hold (see for instance [95]). This results can be established, provided hypothesis on the stability of the system when there exists a robust continuous feedback with respect to sampling error are verified. A key notion when considering event trigger policy for non linear system is the notion of input to state stability. Considering

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t)) \\ u(t) &= \rho(x(t_k)) \end{aligned} \tag{3.1}$$

Definition *3.1.1 (Input–to–state stability–ISS [93]) System (3.1) is said to be locally ISS if there exist a class $\mathcal{KL}$ function β, a $\mathcal{K}$ function α, and some constants $k_1, k_2 > 0$ such that*

$$|x(t)| \leq \beta(|x_0|, t) + \alpha(|u|), \quad \forall\, t \geq 0$$

for all $x_0 \in D$, $u \in D_u$ satisfying $|x_0| < k_1$, $|u| < k_2$. System (3.1) is said (globally) input–to–state stable if $D = \mathbb{R}^n$, $D_u = \mathbb{R}^m$, and the above inequalities are satisfied for any initial state and any bounded input.

Definition *3.1.2 (ISS Lyapunov function) A continuous function $V : D \to R$ is an ISS Lyapunov function on D for system (3.1) if there exist class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3, \beta$ such that the following two conditions are satisfied*

$$\begin{aligned} \alpha_1(|x|) \leq V(x(t)) \leq \alpha_2(|x|) \quad \forall\, x \in D, t \geq 0 \\ \frac{\partial V(x)}{\partial x} g(x,u) \leq -\alpha_3(|x|) + \beta(|u|) \quad \forall\, x \in D, u \in D_u. \end{aligned}$$

Moreover, V is a global ISS Lyapunov function if $D = R^n, D_u = R^m$, and $\alpha_1, \alpha_2, \alpha_3, \beta, \in \mathcal{K}_\infty$.

Consider the closed loop system (3.1) with $e = x(t_k) - x(t)$ and write

$$\dot{x} = g(x, \gamma(x + e))$$

we write a new system

$$\dot{x} = f(x, e)$$

Making the assumption that there exists an ISS Lyapunov function with e as the new input

$$\begin{aligned} \exists V, \alpha_1(|x|) \leq V(x(t)) \leq \alpha_2(|x|) \quad \forall\, x \in D, t \geq 0 \\ , \frac{\partial V(x)}{\partial x} f(x, \rho(x + e)) \leq -\alpha_3(|x|) + \beta(|e|) \quad \forall\, x \in D, u \in D_u. \end{aligned}$$

Then it is possible to choose an event trigger mechanism rendering the closed loop system stable.

$$t_{k+1} = \min_t \{t > t_k \mid -\alpha_3(|x|) + \beta(|e|) < 0\} \tag{3.2}$$

To ensure a minimal inter-event time hypothesise on α_3, β, f and γ are sufficient. Assume f,γ, β and α_3^{-1} are Lipschitz with for simplicity the same Lipschitz constant L Assuming $||e|| \leq \frac{1}{L^2}||x||$ then

$$\begin{aligned} L^2||e|| \geq L\beta(||e||) \geq \alpha_3^{-1}(\beta(||e||)) \\ \alpha_3^{-1}(\beta(||e||)) \leq L^2||e|| \leq ||x|| \\ (\beta(||e||) - \alpha_3(||x||) \leq 0 \end{aligned}$$

therefore for all time where $||e|| \leq \frac{1}{L^2}||x||$ holds $-\alpha_3(|x|) + \beta(|e|) < 0$ holds as well.

Furthermore $|f(x, \gamma(x + e)| < L(||x| + |e||)$ As in the linear case we have

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} = \frac{2e^T\dot{e}x^Tx - 2x^T\dot{x}e^Te}{x^Txx^Tx}$$

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} \leq \frac{L(||x| + |e||)||e||||x||^2 + L(||x| + |e||)||e|^2|||x||}{x^Txx^Tx}$$

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} \leq L\frac{||x||}{||e||} + 2L\frac{||x||^2}{||e||^2} + L\frac{||x||^3}{||e||^3}$$

Since

$$\frac{d(\frac{||e||^2}{||x||^2})}{dt} = 2\frac{||e||}{||x||}\frac{d(\frac{||e||}{||x||})}{dt}$$

then $\frac{d(\frac{||e||}{||x||})}{dt} \leq \frac{L}{2}(1 + \frac{||e||}{||x||})^2$ Therefore using a comparison lemma with $\dot{y} = \frac{L}{2}(1 + y)^2, y(0) = 0$ There is a minimum inter event time.

3.2 Observation and control of decentralized Non Linear systems

The main objective of this section is to address the problem of the event–triggered Observer–based feedback for non linear systems. That is to say when full state is not available for measurement but is needed for stabilization, giving sufficient conditions for the dynamic feedback control of non linear plants to converge when subject to network constraints, using an event–triggered strategy.

3.2.1 Problem formulation and definitions

Consider the system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{3.3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^p$ is the output. The time instant t is omitted if there are no ambiguities. The functions f and h are assumed sufficiently smooth. We also assume the existence of a continuous state-based controller which renders the origin asymptotically stable.

The control scheme is shown in Fig. 3.1. Due to the *communication constraints*, there is no continuous communication either between sensors and observer, or between observer and actuators. The inputs and the outputs are partitioned into actuator/sensor nodes $u = (u_1^T, \cdots, u_q^T)^T$, $y = (y_1^T, \cdots, y_r^T)^T = (h_1^T(x), \cdots, h_r^T(x))^T$, with $u_1, \cdots, u_q$, $y_1, \cdots, y_r$, not necessarily scalars.

The value $y_i(t_{k_i}) = h_i(x(t_{k_i}))$, $i = 1, \cdots, r$, is the last sampled value at the i^{th} sensor node, available for the controller to implement the control, while the value $u_i(t_{j_i})$,

$i = 1, \cdots, q$, is applied to the system at the i^{th} actuator node, through a classic zero–order holder H_0. It is worth noting that this means that the different outputs $\{y_i\}_{i=1,\cdots,r}$ and the different inputs $\{u_i\}_{i=1,\cdots,q}$ are not sampled synchronously. For this reason, at time t the latest output available is

$$\bar{y}(t) = \left(y_1^T(t_{k_1}), y_2^T(t_{k_2}) \cdots, y_p^T(t_{k_p})\right)^T$$

while the control is

$$\bar{u}(t) = \left(u_1^T(t_{j_1}), u_2^T(t_{j_2}) \cdots, u_q^T(t_{j_q})\right)^T.$$

Denoting by $e_u = u - \bar{u}$ and $e_y = y - \bar{y}$ the difference vectors between the continuous and sampled values, one considers the vector $E = (e_u^T, e_y^T)^T$ of the error due to the sampling.

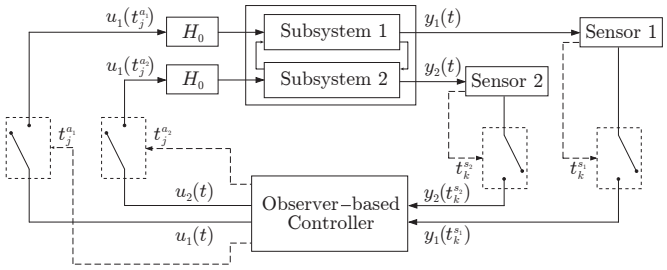

Figure 3.1: Control scheme with sampled output and zero order holder

When the state x of (3.3) is not measurable, classical event triggering policies cannot be implemented. In the following, we will introduce the triggering policy that will be used in this case, taking into account the constraints on the communication of outputs and inputs. A natural assumption is that it is possible to design an observer that converges asymptotically to x, of the form

$$\dot{\hat{x}} = f_o(\hat{x}, y, u),$$

where $f_o : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ is not smooth, in general. In view of an implementation via a triggering policy, and since the observer has no continuous access to $y(t)$, one can use the vector $\bar{y}$, so considering the observer

$$\dot{\hat{x}} = f_o(\hat{x}, u, \bar{y}). \tag{3.4}$$

A feedback controller based on $\hat{x}$ given by (3.4) will be used in the following to stabilize the system (3.3) at the origin. The input applied to the system, due to the communication channel, is $\bar{u} = \bar{\gamma}(\hat{x})$, so obtaining the controlled dynamics

$$\dot{x} = f(x, \bar{\gamma}(\hat{x})).$$

Eventually, one gets the following closed–loop system

$$\begin{aligned} \dot{x} &= f(x, \bar{\gamma}(\hat{x})) \\ \dot{\hat{x}} &= f_o(\hat{x}, \bar{\gamma}(\hat{x}), \bar{y}). \end{aligned}$$

The observation error is $z =: x - \hat{x}$. We assume that the observation error dynamics can be written in the form

$$\dot{z} := f(x, \bar{\gamma}(\hat{x})) - f_o(\hat{x}, \bar{\gamma}(\hat{x}), \bar{y}) = g(z, \theta_1(e_u), \theta_2(e_y), \hat{x})$$

where θ_1, θ_2 give the dependence on the input and the output errors e_u, e_y, due to the sampling.

3.2.2 Hypothesis on the Dynamics of the State Observer and of the Observation Error

Since the observer state is available, in the following we consider the observer dynamics, along with the observation error dynamics, allowing us to impose on $\hat{x}$ a triggering condition.

$$\dot{\hat{x}} = f_o(\hat{x}, \bar{\gamma}(\hat{x}), \bar{h}(\hat{x} + z)) \tag{3.5a}$$
$$\dot{z} = g(z, \theta_1(e_u), \theta_2(e_y), \hat{x}) \tag{3.5b}$$

where $y = h(\hat{x} + z)$ and $\bar{y} = \bar{h}(\hat{x} + z)$, or equivalently

$$\dot{X} = G(X, E) \tag{3.6}$$

where $X = (\hat{x}^T, z^T)^T$ is an extended state vector,$E = (e_u^T, e_y^T)^T$ is the total error induced by the triggering policies and $G = (f_o^T, g^T)^T$. In the following we consider the following assumptions.

(A_1) There exists an ISS Lyapunov V_c function for (3.5a) such that $\forall\, \hat{x}, z \in \mathbb{R}^n, E \in \mathbb{R}^{m+p}$, $\forall\, t \geq 0$

$$\alpha_{c,1}(|\hat{x}|) \leq V_c(\hat{x}(t)) \leq \alpha_{c,2}(|\hat{x}|)$$
$$\frac{\partial V_c(\hat{x})}{\partial \hat{x}} f_o(\hat{x}, \bar{\gamma}(\hat{x}), \bar{h}(\hat{x} + z)) \leq -\alpha_{c,3}(|\hat{x}|) + \beta_c(|(z, E)|)$$

with $\alpha_{c,1}, \alpha_{c,2}, \alpha_{c,3}, \beta_c \in \mathcal{K}$, and $\beta_c, \alpha_{c,3}^{-1}$ Lipschitz;

(A_2) There is an ISS Lyapunov V_o function for (3.5b) such that $\forall\, z \in \mathbb{R}^n, E \in \mathbb{R}^{m+p}$, $\forall\, t \geq 0$

$$\alpha_{o,1}(|z|) \leq V_o(z(t)) \leq \alpha_{o,2}(|z|)$$
$$\frac{\partial V_o(z)}{\partial z} g(z, \theta_1(e_u), \theta_2(e_y), \hat{x}) \leq -\alpha_{o,3}(|z|) + \beta_o(|E|)$$

with $\alpha_{o,1}, \alpha_{o,2}, \alpha_{o,3}, \beta_o \in \mathcal{K}$, and $\beta_o, \alpha_{o,3}^{-1}$ Lipschitz;

(A_3) f_o, h and γ are Lipschitz;

(A_4) g is Lipschitz with respect to $(z, \theta_1(e_u), \theta_2(e_y))$, uniformly in $\hat{x}$, and θ_1, θ_2 are Lipschitz.

Remark 3.2.1 (A_1) *ensures the asymptotic convergence to the origin of the observer, in absence of sampling errors and observation error, and an ISS property with respect to* z, e_u, e_y.

(A_2) *ensures the asymptotic convergence to zero of the observation error in absence of sampling errors, and an ISS property with respect to* e_u, e_y. *Those two assumptions suppose a separation principle between state estimation and control.*

Since we are interested in the stabilisation of the observer state $\hat{x}$ and of the observation error z, in the following we will assume that $X(0) \neq 0$.

Lemma 3.2.2 *Under the Assumptions $(A_1), (A_2), (A_3), (A_4)$, the extended system $X = (\hat{x}^T, z^T)^T$ admits an ISS Lyapunov function $V(X)$ such that $\forall\, X \in \mathbb{R}^{2n}, \forall E \in R^{n+p}$, $\forall\, t \geq 0$*

$$a_1(|X|) \leq V(X) \leq a_2(|X|)$$
$$\frac{\partial V(X)}{\partial X} G(X, E) \leq -a_3(|X|) + b(|E|)$$

with $a_1, a_2, a_3, b \in \mathcal{K}$, b, a_3^{-1}, G Lipschitz.

Proof 3.2.3 *Let us consider the candidate ISS Lyapunov function*

$$V(X) = \lambda_c V_c(\hat{x}) + V_o(z).$$

From $(A_1), (A_2)$,

$$\begin{aligned} a_1(|X|) &= \min_{|(\hat{x},z)|=|X|} \lambda_c \alpha_{c,1}(|\hat{x}|) + \alpha_{o,1}(|z|) \\ &\leq \lambda_c \alpha_{c,1}(|\hat{x}|) + \alpha_{o,1}(|z|) \\ a_2(|X|) &= \max_{|(\hat{x}^T,z^T)|=|X|} \lambda_c \alpha_{c,2}(|\hat{x}|) + \alpha_{o,2}(|z|) \\ &\geq \lambda_c \alpha_{c,2}(|\hat{x}|) + \alpha_{o,2}(|z|) \end{aligned}$$

with $a_1, a_2 \in \mathcal{K}$. Furthermore,

$$\begin{aligned} \frac{\partial V(X)}{\partial X} G(X, E) &= \begin{pmatrix} \frac{\partial V(X)}{\partial \hat{x}} & \frac{\partial V(X)}{\partial z} \end{pmatrix} \begin{pmatrix} f_o \\ g \end{pmatrix} \\ &\leq \lambda_c \Big(- \alpha_{c,3}(|\hat{x}|) + \beta_c(|(z^T, E^T)^T|) \Big) \\ &\quad - \alpha_{o,3}(|z|) + \beta_o(|E^T|) \\ &\leq -\Big(\lambda_c \alpha_{c,3}(|\hat{x}|) + \alpha_{o,3}(|z|) - \lambda_c L_{\beta_c} |z| \Big) \\ &\quad + \lambda_c L_{\beta_c} |E| + \beta_o(|E|), \end{aligned}$$

where we have used the fact that

$$\beta_c(|(z^T, E^T)^T|) \leq L_{\beta_c} |(z^T, E^T)^T| \leq L_{\beta_c} |E| + L_{\beta_c} |z|.$$

It is always possible to choose λ_c sufficiently small such that $\alpha_{o,3}(|z|) - \lambda_c L_{\beta_c} |z|$ is a class $\mathcal{K}$ function. Therefore

$$a_3(|X|) = \min_{|X|} \Big\{ \lambda_c \alpha_{c,3}(|X|), \alpha_{o,3}(X) - \lambda_c L_{\beta_c} |X| \Big\}.$$

is a class $\mathcal{K} function$ To show that a_3^{-1} is Lipschitz, note first that since $\alpha_{o,3}^{-1}$ is Lipschitz

$$\alpha_{o,3}(|z|) \geq \frac{1}{L_{\alpha_{o,3}^{-1}}} |z|.$$

Moreover, one can compute an upper bound of the derivative of $(\alpha_{o,3}(|\cdot|) - \lambda_c L_{\beta_c} |\cdot|)^{-1}$, since

$$\frac{d}{d|z|} (\alpha_{o,3}(|z|) - \lambda_c L_{\beta_c} |z|)^{-1} \leq \frac{L_{\alpha_{o,3}^{-1}}}{1 - \lambda_c L_{\alpha_{o,3}^{-1}} L_{\beta_c}}$$

Hence it is always possible to choose λ_c sufficiently small such that $(\alpha_{o,3}(|\cdot|) - \lambda_c L_{\beta_c}|\cdot|)^{-1}$ is a class $\mathcal{K}$ function with the Lipschitz constant

$$L_{a_3^{-1}} = \max\left\{\frac{\lambda_c}{L_{\alpha_{c,3}}}, \frac{L_{\alpha_{o,3}^{-1}}}{1 - \lambda_c L_{\alpha_{o,3}^{-1}} L_{\beta_c}}\right\}.$$

Furthermore,

$$b(|E|) = L_{\beta_c}|E| + \beta_o(|E|)$$

which is Lipschitz with constant $L_b = L_{\beta_c} + L_{\beta_o}$. Finally, thanks to $(A_3), (A_4)$, $G(X, E)$ is Lipschitz.

In the following, we are interested in providing sufficient conditions on the stabilisation of a non linear system using the event trigger paradigm. The key concept will be the ISS of both the closed–loop system and of the observer dynamics. For, we introduce the following lemmas.

Lemma 3.2.4 *If the observer and the error dynamics verify $(A_1), (A_2), (A_3), (A_4)$, then there exist a $\sigma > 0$ such that any sampling policy ensuring $|E| \leq \sigma|X|$, leads to asymptotic convergence of the overall system to the origin.*

Proof 3.2.5 *From Lemma 3.2.2, the existence of an ISS Lyapunov function V is ensured. Since a_3^{-1} and b are Lipschitz, $a_3(|X|) \geq \frac{1}{L_{a_3^{-1}}}|X|$ and*

$$-a_3(|X|) + b(\sigma|X|) \leq -\left(\frac{1}{L_{a_3^{-1}}} - L_b\sigma\right)|X|.$$

Therefore for all $\sigma \in \left(0, \frac{1}{L_{a_3^{-1}} L_b}\right)$ the system (3.5) converges asymptotically to the origin.

Remark 3.2.6 *Under the hypothesis that a_2, a_1^{-1} are Lipschitz, one can prove exponential convergence of (3.5). In fact, since*

$$\frac{1}{L_{a_1^{-1}}}|X| < a_1(|X|) < V(|X|),$$

one has that

$$\dot{V}(|X|) \leq -\left(\frac{1}{L_{a_3^{-1}}} - L_b\sigma\right)|X|.$$

Therefore,

$$\dot{V}(|X|) \leq -\left(\frac{1}{L_{a_3^{-1}}} - L_b\sigma\right) L_{a_1^{-1}} V(|X|).$$

Remark 3.2.7 *The choice* $\sigma \in \left(0, \dfrac{1}{L_{a_3^{-1}} L_b}\right)$ *represents a trade–off between the sampling rate and the convergence rate.*

Since $|E| < \sigma|X|$, using the norm equivalence in $\mathbb{R}^n$ there exists a $\sigma' > 0$ such that $\|E\| < \sigma'\|X\|$ implies $|E| < \sigma|X|$.

Lemma 3.2.8 *For every* $\kappa_i > 0$ *there is a minimal time* $\tau_{\min} > 0$ *such that if* $|E| \leq \sigma|X|$, *then* $\forall\, t_k$, $\forall\, t \in [t_k, t_k + \tau_{\min})$ *the following inequalities are verified*

$$\begin{aligned} \|\gamma_i(\hat{x}(t)) - \gamma_i(\hat{x}(t_k))\| &\leq \kappa_i \|X\| \\ \|h_j(\hat{x}(t) + z(t)) - h_j(\hat{x}(t_k) + z(t_k)\| &\leq \kappa_j \|X\|. \end{aligned}$$

Proof 3.2.9 *In the following we assume* $X \neq 0$. *The argument follows the proof of Theorem 1 in [95]. Denoting* $e_{u_i} = \gamma_i(\hat{x}(t)) - \gamma_i(\hat{x}(t_k))$, *one works out*

$$\begin{aligned} \frac{d}{dt}\frac{\|e_{u_i}\|}{\|X\|} &= \frac{e_{u_i}^T \dot{e}_{u_i}}{\|e_{u_i}\|\|X\|} - \frac{X^T \dot{X}\|e_i\|}{\|X\|^3} \\ &\leq \frac{\|e_{u_i}\|\|\dot{e}_{u_i}\|}{\|e_{u_i}\|\|X\|} + \frac{\|\dot{X}\|\|e_i\|}{\|X\|^2}. \end{aligned}$$

Since $\|\dot{e}_{u_i}\| \leq L_{\gamma_i}\|\dot{\hat{x}}\| \leq L_{\gamma_i}\|\dot{X}\|$,

$$\frac{d}{dt}\frac{\|e_{u_i}\|}{\|X\|} \leq \frac{\|\dot{X}\|}{\|X\|}\left(L_{\gamma_i} + \frac{\|e_i\|}{\|X\|}\right).$$

Moreover, G *is Lipschitz, so that*

$$\frac{d}{dt}\frac{\|e_{u_i}\|}{\|X\|} \leq \frac{L_G(\|X\| + \|E\|)}{\|X\|}\left(L_{\gamma_i} + \frac{\|e_i\|}{\|X\|}\right)$$

Since $\|E\| < \sigma'\|X\|$,

$$\frac{d}{dt}\frac{\|e_{u_i}\|}{\|X\|} \leq L_G(1+\sigma')\left(L_{\gamma_i} + \frac{\|e_i\|}{\|X\|}\right).$$

At each reset time one has $e_{u_i} = 0$. *Using the comparison lemma with the differential equation*

$$\dot{y} = L_G(1+\sigma')\big(L_{\gamma_i} + y\big), \quad y(0) = 0$$

one has

$$\frac{\|e_{u_i}(t)\|}{\|X(t)\|} \leq \big(e^{L_G(1+\sigma'(t-t_k)} - 1\big)L_{\gamma_i}.$$

Therefore the inequality

$$\|\gamma_i(\hat{x}(t)) - \gamma_i(\hat{x}(t_k))\| > \kappa_i\|X\|$$

can not be true before time

$$\tau_{\min}^i = \frac{1}{L_G(1+\sigma')}\ln\big(1 + \frac{\kappa_i}{L_{\gamma_i}}\big).$$

Analogously, the inequality

$$\|h_j(\hat{x}(t)+z(t))-h_j(\hat{x}(t_k)+z(t_k)\|>\kappa_j\|X\|$$

gives for the sensors

$$\tau^j_{\min}=\frac{1}{L_G(1+\sigma')}\ln\big(1+\frac{\kappa_j}{L_{h_i}}\big).$$

Let us define the triggering function at each node

$$t^i_{k+1}=\min_t\{t\geq t^i_k+\tau^i_{\min},|\ \|u_i(t)-u_i(t^i_k)\|>\kappa_i\|X\|\} \tag{3.7}$$

$$t^j_{k+1}=\min_t\{t\geq t^j_k+\tau^j_{\min},|\ \|y_j(t)-y_j(t^i_k)\|>\kappa_j\|X\|\}. \tag{3.8}$$

Remark 3.2.10 *From Lemma 3.2.8,* $t^i_{k+1}=\min_t\{t\geq t^i_k+\tau^i_{\min},|\ \|u_i(t)-u_i(t^i_k)\|>\kappa_i\|X\|\}=t^i_{k+1}=\min_t\{t\geq t^i_k,|\ \|u_i(t)-u_i(t^i_k)\|>\kappa_i\|X\|\}$.

Lemma 3.2.11 *If* $\sum_{\{1,\cdots,r\}\cup\{1,\cdots,q\}}\kappa_i\leq\sigma'$ *then (3.7) and (3.8) ensure* $\|E\|\leq\sigma\|X\|$.

Proof 3.2.12 *from (3.7) and (3.8)*

$$\|E\|\leq\sum_{\{1,\cdots,r\}\cup\{1,\cdots,q\}}\kappa_i\|X\|\leq\sigma'\|X\|.$$

The proposed triggering conditions allow asymptotic convergence with a nonzero minimum inter–event time. Unfortunately, they are not implementable on a network for two reasons. The first is that X is not available, since the observation error is not known. The second is that sensors do not communicate among them nor receive information from the observer–based controller. Nevertheless, considering the following modified triggering conditions

$$t^i_{k+1}=\min_t\{t\geq t^i_k+\tau^i_{\min},|\ \|u_i(t)-u_i(t^i_k)\|>\frac{\kappa_i}{L_{\gamma_i}}\|\gamma_i(\hat{x})\|\} \tag{3.9}$$

$$t^j_{k+1}=\min_t\{t\geq t^j_k+\tau^j_{\min},|\ \|y_j(t)-y_j(t^i_k)\|>\frac{\kappa_j}{2L_h}\|y_j\|\} \tag{3.10}$$

this approach can be used on a network, allowing asymptotic convergence and a nonzero minimal inter–event time, using only information available at each node.

Theorem 3.2.13 *If* $(A_1),(A_2),(A_3),(A_4)$ *are verified, and the sampling instants are defined by (3.9), (3.10), then the origin of the closed–loop system (3.5) is asymptotically stable and there exists a nonzero minimum inter–event time for each node.*

Proof 3.2.14 *Under the hypotheses of the theorem, Lemma 3.2.2 holds. Since* $\frac{\|y_j\|}{2L_h}<\|X\|$ *and* $\frac{\|u_i\|}{L_{\gamma_i}}<\|X\|$ *and* $\forall i\in\{1,\cdots,r\}$ *from Lemma 3.2.8 one can state that between* t_{k_i} *and* $t_{k_i}+\tau^i_{\min}$, $\|u_i(t)-u_i(t^i_k)\|>\kappa_i\|X\|$, *while* $\forall j\in\{1,\cdots,q\}$ *between* t_{k_j} *and* $t_{k_j}+\tau^j_{\min}$, *one has* $\|y_j(t)-y_j(t_{k_j})\|>\kappa_j\|X\|$.

Therefore, $\|E\|<\sigma'\|X\|$. *Using Lemma 3.2.4, there is asymptotic convergence of (3.5) to the origin.*

3.2.3 Examples of Systems Fitting into the Proposed Framework

The proposed results are quit generic in the sense that they give necessary conditions in term of input to state stability for non linear systems that allow an event triggered sampling policy. In the next sections we will give example of system verifying assumptions $(A_1), (A_2), (A_3), (A_4)$.

Linear Systems

Let us consider a detectable and stabilizable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= x \end{aligned} \tag{3.11}$$

with

$$\dot{\hat{x}} = A\hat{x} + Bu + LC(\bar{x} - \hat{x}) \tag{3.12}$$

a Luenberger observer where $C\bar{x} = \bar{y}(t) = \left(y_1^T(t_{k_1}), y_2^T(t_{k_2}) \cdots, y_p^T(t_{k_p})\right)^T$, with control $K\bar{\hat{x}}$, with triggering conditions (3.7) (3.8) one gets

$$\begin{aligned} \dot{\hat{x}} &= (A + BK)\hat{x} + BK(\bar{\hat{x}} - \hat{x}) + LCz - LC(x - \bar{x}) \\ \dot{z} &= (A - LC)z + LC(x - \bar{x}). \end{aligned}$$

Since $A + BK$ and $A - LC$ are Hurwitz, it is possible to find an ISS Lyapunov function for the extended system.

Non linear Lipschitz Systems

Let us consider a non linear Lipschitz system

$$\begin{aligned} \dot{x} &= Ax + Bu + \phi(x, u) \\ y &= Cx. \end{aligned} \tag{3.13}$$

Several results are available for the observer synthesis of non linear Lipschitz systems when the control and the output are implemented in a continuous fashion. We consider an observer of the form

$$\dot{\hat{x}} = Ax + BK\bar{\hat{x}} + \phi(\bar{x}, K\bar{\hat{x}}). \tag{3.14}$$

Hence, the extended closed–loop system is

$$\dot{\hat{x}} = A\hat{x} + BK\bar{\hat{x}} + \phi(\hat{x}, K\bar{\hat{x}}) + LCz \tag{3.15a}$$

$$\dot{z} = (A - LC)z + \phi(x, K\hat{x}) - \phi(\hat{x}, K\bar{\hat{x}}). \tag{3.15b}$$

To implement an event–triggered control strategy, we need to consider the following structural properties.

(H_1) $\|\phi(x_1, u) - \phi(x_2, u)\| \leq \rho\|x_1 - x_2\|$, $\forall u \in \mathbb{R}^p$, $x_1, x_2 \in \mathbb{R}^n$;

(H_2) $\|\phi(x, u)\| \leq \rho\|x\|, \forall u \in \mathbb{R}^p$;

(H_3) There exists a gain K such that $u = Kx$ for the system (3.14) and there exist a quadratic Lyapunov function

$$V_c(x) = x^T P_c x, \quad \dot{V}_c(x) \leq -\eta_c x^T x \tag{3.16}$$

with $P_c = P_c^T > 0$, $\eta_c > 0$;

(H_4) There exists a gain L such that for (3.15b) there exist quadratic Lyapunov function for the z dynamics

$$V_o(z) = z^T P_o z, \quad \dot{V}_o(z) \leq -\eta_o z^T z \tag{3.17}$$

with $P_o = P_o^T > 0$, $\eta_o > 0$.

In (H_2), for $\rho = 0$ we have a linear system, and the existence of V_c, V_o derive from the stabilizability and the detectability. Moreover, there always exists a $\rho_{\max} > 0$ small enough such that the proposed Lyapunov function exist forall $\rho \in [0, \rho_{\max}]$. For other (more complex) conditions of existence of V_c, V_o verifying (3.16), (3.17), see for instance [80].

Lemma 3.2.15 *If $(H_1), (H_2), (H_3), (H_4)$ are verified, then the proposed observer and the observation error verify $(A_1), (A_2), (A_3), (A_4)$.*

Proof 3.2.16 *When subject to the trigger conditions, the observer has the following dynamics*

$$\dot{\hat{x}} = (A + BK)\hat{x} + \phi(\hat{x}, K\bar{\hat{x}}) + BK(\bar{\hat{x}} - \hat{x}) + LCz + LC(\bar{z} - z).$$

Let us consider the candidate ISS Lyapunov function $2\sqrt{V_c}$ which verifies

$$2\sqrt{\lambda_{\min}(P_c)}\|\hat{x}\| \leq 2\sqrt{V_c(\hat{x})} \leq 2\sqrt{\lambda_{\max}(P_c)}\|\hat{x}\|$$

and having derivative

$$\begin{aligned}\frac{d}{dt}2\sqrt{\hat{x}^T P_c \hat{x}} &= \frac{1}{\sqrt{\hat{x}^T P_c \hat{x}}}\Big(- \hat{x}^T Q \hat{x} + 2\hat{x}^T P \phi(\hat{x}, u)\Big) \\ &+ \frac{1}{\sqrt{\hat{x}^T P_c \hat{x}}}(2\hat{x}^T P_c \Big(BK(x - \bar{x}) + LCz - LC(z - \bar{z})\Big)\end{aligned}$$

where $Q = -(A + BK)^T P + P(A + BK)$. In virtue of (H_1), one can write

$$\begin{aligned}\frac{d}{dt}2\sqrt{\hat{x}^T P_c \hat{x}} &\leq -\frac{\eta_c \|\hat{x}\|^2}{\sqrt{\hat{x}^T P_c \hat{x}}} \\ &+ \frac{1}{\sqrt{\hat{x}^T P_c \hat{x}}}\Bigg[\|P\|\|\hat{x}\|\big(\|BK\|\|(x - \bar{x})\| \\ &\qquad + \|LC\|\|z\| + \|LC\|\|(z - \bar{z})\|\big)\Bigg] \\ &\leq \frac{-\eta_c}{\sqrt{\lambda_{\max}(P_c)}}\|\hat{x}\| \\ &+ \frac{\big(\|BK\|\|(x - \bar{x})\| + \|LC\|\|z\| + \|LC\|\|(z - \bar{z})\|\big)}{\sqrt{\lambda_{\min}(P_c)}}\end{aligned}$$

which verifies assumption (A_1). *Analogously, using the candidate ISS Lyapunov function* $2\sqrt{V_o}$, *one can prove that* (A_2) *holds. Furthermore, it is trivial to show that* $(H_1), (H_2)$ *imply* $(A_3), (A_4)$.

Therefore, applying Lemma 3.2.2 to the system (3.13), and using Theorem 3.2.13, on the event–triggered observer–based controller ensures asymptotic convergence to the origin.

Corollary 3.2.17 *If* $(H_1), (H_2), (H_3), (H_4)$ *are verified, the event–triggered control policy* (3.9), (3.10) *and the control* $u = K\bar{\hat{x}}$ *ensure the asymptotic stability of the closed–loop system* (3.15).

Remark 3.2.18 *This corollary of Theorem 3.2.13 uses assumption that are easier to verify in the specific context of Lipschitz systems ([80]). Therefore as a practical contribution it is more useful than Theorem 3.2.13 This class of systems will be tested in simulation.*

Proof 3.2.19 *Lemma 3.2.15 ensures that* $(A_1), (A_2), (A_3), (A_4)$ *are verified. Then one applies Theorem 3.2.13 to the system* (3.15).

3.2.4 Simulations and comments

The proposed methodology will be applied to a robot with a flexible link, used as a benchmark example in several papers dealing with Lipschitz observers (see for instance [86], [4], [80]). The dynamics are in the form (3.13), with

$$\begin{aligned} \dot{x} &= Ax + \phi(x,u) + BK\bar{\hat{x}} \\ \dot{\hat{x}} &= A\hat{x} + \phi(\hat{x},u) + BK\bar{\hat{x}} + LC\bar{z} \\ y &= Cx \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 21.6 & 0 & 0 \end{pmatrix}^T$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 & 3.3\sin x_3 \end{pmatrix}^T.$$

One considers the control $u = K\bar{\hat{x}}$, with

$$K = \begin{pmatrix} 7.8428 & 1.1212 & -4.3666 & 1.1243 \end{pmatrix}$$

and the observer (3.14), with

$$L = \begin{pmatrix} 9.3334 & 1.0001 \\ -48.7804 & 22.3665 \\ -0.0524 & 3.3194 \\ 19.4066 & -0.3167 \end{pmatrix}.$$

The closed–loop equations are in the form (3.15). The simulations have been performed considering the initial states

$$x(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T, \quad \hat{x}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

The theoretical values obtained on the triggering policy can be used but are too restrictive, due to the over–approximation on the convergence rate of the non linear observer and on the triggering parameter estimations. Via simulations it is possible to better tune the triggering parameters. It is worth noting that there is an order of magnitude of 100 between the theoretical value and the practical ones. We compared the result of a system controlled using triggering policy

$$t_{k_i+1} = \min_t \{t \geq t_{k_i} + 0.01, | \ \|u_i(t) - u_i(t_{k_i})\| > 0.2\|u_i(t)\|\}$$

$$t_{k_j+1} = \min_t \{t \geq t_{k_j} + 0.01, | \ \|y_j(t) - y_j(t_{k_j})\| > 0.2\|y_j(t)\|\}$$

with the case in which $t_{k_i+1} = t_{k_i} + 0.05$. The simulations show that for $t \in [0, 2]$ second the system and observer are closed to the equilibrium, while at $t = 2$ s an impulse drives the system away from equilibrium. Then, for $t \in [2, 15]$ s, the system is stabilized at the origin by the proposed observer–based controller.

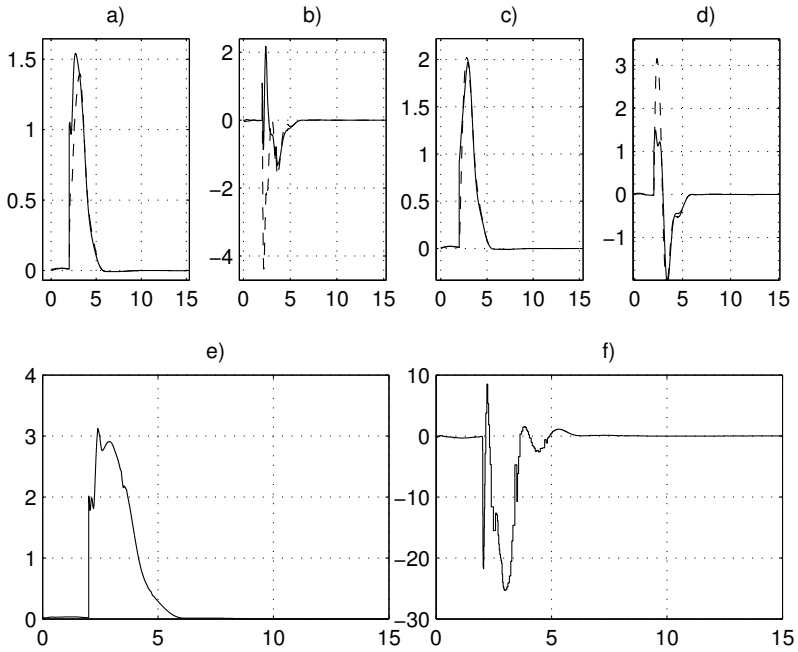

Figure 3.2: System and observer state with the event–triggering: a) x_1, $\hat{x}_1$; b) x_2, $\hat{x}_2$; c) x_3, $\hat{x}_3$; d) x_4, $\hat{x}_4$; e) $\|x\|$; f) u.

Figs. 3.2 and 3.3 show the convergence of the observer and the stabilization at the origin of the overall system. We can note that the event triggering is relatively slower with respect to the periodic sampling, but introduces a lower peaking. When confronting the number of triggers in Fig.s 3.4.a, 3.4.b, it is clear that the number of communications is greater when considering the periodic sampling, so justifying the interest of the proposed event–triggering scheme. It is worth noting that the advantage of the method appears

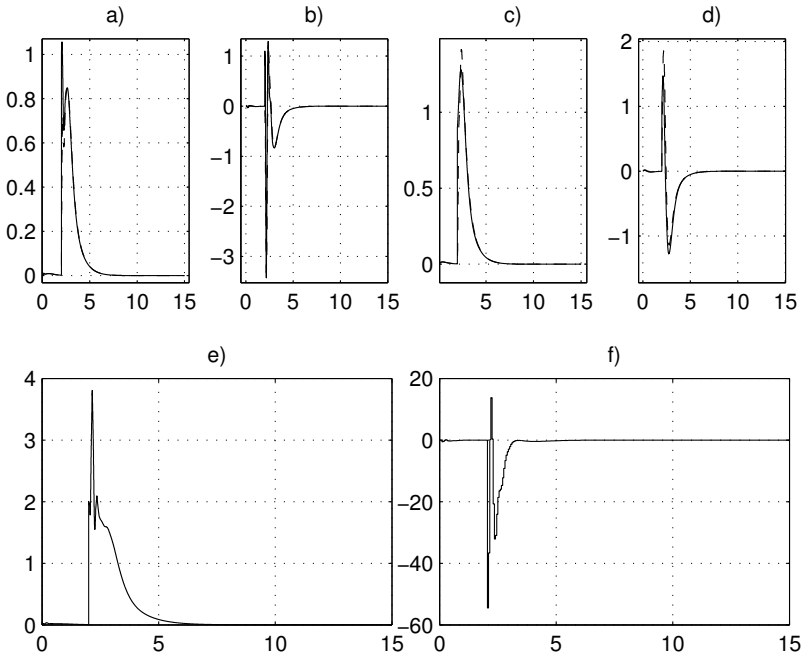

Figure 3.3: System and observer state with periodic sampling: a) x_1, $\hat{x}_1$; b) x_2, $\hat{x}_2$; c) x_3, $\hat{x}_3$; d) x_4, $\hat{x}_4$; e) $\|x\|$; f) u.

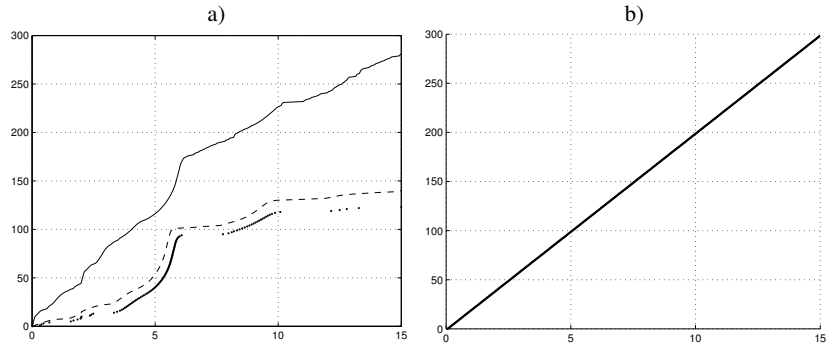

Figure 3.4: Number of triggers with the a) Proposed event triggered policy; b) Periodic sampling. u_1 (solid), y_1 (dashed), y_2 (dotted).

more clearly for output communications. As already noted, this is due to the fact that the observation and the control communications are done only when it is necessary. The comparison of Figs. 3.4.a and

3.2.5 Conclusions

In this first part we presented necessary conditions for networked non linear system to be observed and stabilised using event triggering sampling mechanism. The result proposed are based on input to state stability properties of the observation error and the observer dynamic. We discussed the generality of our result by giving example of class of system encompassed by our theorem. We illustrated our result on simulation of a flexible link.

The proposed observation and control scheme require the control of the system to achieve observation. In the next section and following the spirit of chapter one, we design an impulsive observer along with an adapted triggering policy to ensure observation of a system in absence of a stabilising control action. A major difference to our previous result we adopt for the next section the "periodic" event trigger paradigm.

3.3 Asynchronous Event–Triggered Observation and Control of Non Linear Lipschitz Systems via Impulsive Observers

Revoir la redaction de ce paragraphe In the previous chapter we defined an impulsive observer for linear systems , we also gave necessary condition in terms of input to state stability for a non linear systems to be observed and controlled in an event triggered manner next we will introduce an impulsive observer for Lipschitz non linear system with a periodic event triggering scheme. In the following both the sensors and actuators sample periodically, not necessarily at the same period and decide whether or not to send the information based on an event triggered sampling policy.

The proposed methodology aims at practically observe a non linear system using event triggered sampling policy at the level of the sensors in order to reduce the amount of communication between sensor and plant. And to subsequently use an observer based control scheme to stabilize the plant, under communication constraints. The proposed observer does not require stabilisation of the plant in order to converge and the size of the attractive set around the real state is not dependant on the actual state. The fact that the observer does not require the system to be stable reminds the definition of the separation principle made in section (Impulsive observer).

The novelty of this section resides in the extension of the event trigger paradigm to some class of non linear systems. Results of periodic sampling are used to generate an periodic event triggered sampling policy, furthermore we show in this work that an observer based stabilizing control policy can be implemented using event triggered sampling at the level of both the observer and of the actuators. The communication does not need to be synchronous, the fact that observation can be performed independently of observation ensures a separation principle.

3.4 Problem Statement

The class of systems under study is characterized by the following equation

$$\begin{aligned} \dot{x} &= Ax + Bu + D\phi(Hx) \\ y &= Cx \end{aligned} \tag{3.18}$$

with $x \in \mathcal{D}_x \subseteq \mathbb{R}^n$ the state, $u \in \mathcal{D}_u \subseteq \mathbb{R}^m$ the input, $y \in \mathcal{D}_y \subseteq \mathbb{R}^q$ the output, and $\mathcal{D}_x$, $\mathcal{D}_u$, $\mathcal{D}_y$ the domains. Moreover, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{n\times \nu}$, $H \in \mathbb{R}^{p\times n}$ are constant matrices. In (3.18), $D\phi(Hx)$ gives the structure of the nonlinearity acting on the system, with $\phi : \mathbb{R}^p \to \mathbb{R}^\nu$ a nonlinear function satisfying the following condition

$$\big\|\phi(\chi_1) - \phi(\chi_2)\big\| \leq \gamma \|\chi_1 - \chi_2\|, \quad \forall (\chi_1, \chi_2) \in \mathcal{D}_\chi \times \mathcal{D}_\chi \tag{3.19}$$

with $\mathcal{D}_\chi = \{\chi \colon \chi = Hx \mid x \in \mathcal{D}_x\}$, for some $\gamma > 0$. It is clear that (3.19) implies that (3.18) is Lipschitz with respect to x.

In what follows we will state global result, so that $\mathcal{D}_x = \mathbb{R}^n$, $\mathcal{D}_u = \mathbb{R}^m$, $\mathcal{D}_y = \mathbb{R}^q$. The pair (A, B) is assumed controllable, while the pair (A, C) is assumed observable. Note that milder results could be easily expressed in terms of stabilizability and detectability.

Both sensors and actuator are sampled periodically, not necessarily synchronously, but the transmission instants are a subset of these sampling instants. More precisely, while $\mathcal{S}_s = \{t_k = k\delta\}_{k\in\mathbb{N}}$ is the periodic sampling sequence for the sensor, with $\delta = t_{k+1} - t_k > 0$ the sensor sampling period, the sensor transmission sequence is $\{t^s_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathcal{S}_s$. Analogously, while the actuator sampling sequence is $\mathcal{S}_a = \{t_j = j\tau\}_{j\in\mathbb{N}}$, with $\tau = t_{j+1} - t_j > 0$ the actuator sampling period, the sensor transmission sequence is $\{t^a_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_a$. The situation is depicted in Fig. 3.5. The data communications between the plant and the controller, assumed instantaneous, take place only at the discrete time instants t^s_ℓ (from the system to the controller) and t^a_n (from the controller to the system). The value $u(t^a_n)$ is applied to the system, through a classic zero-order holder.

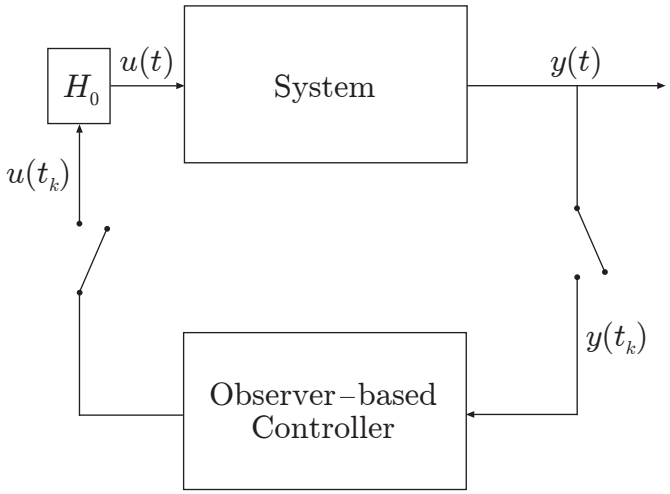

Figure 3.5: Sampling and transmission time instants for sensor and controller.

3.5 An Event–Triggered State Feedback Controller

In this section we first introduce a controller, assuming piecewise constant values over the actuator sampling period. This controller ensures the exponential stabilization of the system to the origin, under the assumption of full state information and further sufficient conditions expressed in terms of sampling period. Then, we prove the robustness of this controller with respect to bounded perturbations, and finally we establish a practical

stability result, showing that the event–triggered transmission mechanisms of both the sampled output and the calculated input can be be considered as bounded perturbations.

We assume that the system state is available for feedback, and we consider the controller

$$u = Kx(j\tau), \qquad \forall t \in [j\tau, (j+1)\tau) \tag{3.20}$$

piecewise constant over the actuator sampling period $\tau > 0$.

Theorem 3.5.1 *Let us consider the system (3.18), with (A, B) controllable, ϕ satisfying (3.19), and a matrix K such that $A + BK$ is Hurwitz. If there exist a matrix $P_c = P_c^T > 0$, an $\varepsilon_c > 0$, and a sampling period τ such that*

$$\begin{aligned} M &= \begin{pmatrix} (A+BK)^T P_c + P_c(A+BK) + \varepsilon_c P_c + \gamma^2 H^T H & P_c D \\ D^T P_c & -I \end{pmatrix} \leq 0 \\ \tau < \tau_{\max} &= \frac{1}{\max\{\|A+BK\| + \gamma\|D\|\|H\|, \|BK\|\}} \frac{1}{1 + 2\frac{\|BK\|}{\varepsilon_c}\frac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}}} \end{aligned} \tag{3.21}$$

then the feedback system (3.18), (3.20) is exponentially stable to the origin.

Proof 3.5.2 *The dynamics of the feedback system (3.18), (3.20) can be written*

$$\dot{x} = (A+BK)x + D\phi(Hx) + BKd_1$$

where $d_1 = x(t) - x(t_j)$, $t_j = j\tau$, $j \in \mathbb{N}$. Note that, tanks to (3.19),

$$\|\dot{x}\| \leq \|(A+BK)x + D\phi(Hx)\| + \|BKd_1\| \leq \|A+BK\|\gamma\|D\|\|H\|\|x\| + \|BK\|\|d_1\| \leq L(\|x\| + \|d_1\|)$$

with $L = \max\{\|A+BK\| + \gamma\|D\|\|H\|, \|BK\|\}$ Hence, considering $V(x) = x^T P_c x$ as Lyapunov candidate, one gets

$$\begin{aligned} \dot{V}(x) &= x^T \Big((A+BK)^T P_c + P_c(A+BK) \Big) x + 2\phi^T D^T P_c x + 2x^T P_c BK d_1 \\ &\leq -\varepsilon_c V(x) - x^T S_2 x + 2x^T P_c BK d_1 \end{aligned}$$

Since

$$2\phi^T D^T P_c x \leq \phi^T \phi + x^T P_c D D^T P_c x \leq \gamma^2 x^T H^T H x + x^T P_c D D^T P_c x$$

we have $S_2 = (A+BK)^T P_c + P_c(A+BK) + \varepsilon_c P_c + \gamma^2 H^T H + P_c D D^T P_c$ is the Schur complement of the element (2,2) of the matrix $-M$ of (3.21). Since $-M \geq 0$ and $I > 0$, then $S_2 \geq 0$

$$\dot{V}(x) \leq -\varepsilon_c V(x) + 2x^T P_c BK d_1$$

now, considering a $\theta \in (0, 1)$,

$$\begin{aligned} \dot{V}(x) &\leq -\varepsilon_c(1-\theta)V(x) + 2\|x\|\lambda_{\max}^{P_c}\|BK\|\|d_1\| - \varepsilon_c\theta V(x) \\ &\leq -\varepsilon_c(1-\theta)V(x) + \|x\|\Big(2\lambda_{\max}^{P_c}\|BK\|\|d_1\| - \varepsilon_c\theta\lambda_{\min}^{P_c}\|x\|\Big) \leq -\varepsilon_c(1-\theta)V(x) \end{aligned}$$

for

$$\varrho = \frac{\|d_1\|}{\|x\|} \leq \varrho_{\max} = \frac{\theta \lambda_{\min}^{P_c}}{2\|BK\|\lambda_{\max}^{P_c}} \varepsilon_c.$$

Following [95], let us determine a bound for the minimal time for ϱ, starting from zero, to reach the value $\varrho_{\max}$. $\dot{d}_1 = \dot{x}$

$$\dot{\varrho} = \frac{\|x\| \dfrac{d_1^T \dot{x}}{\|d_1\|} - \|d_1\| \dfrac{x^T \dot{x}}{\|x\|}}{\|x\|^2} \leq \frac{1+\varrho}{\|x\|} \|\dot{x}\| \leq L(1+\varrho)^2$$

with $\varrho(t_j) = 0$, for all $j \in \mathbb{N}$, since $d_1(t_j) = 0$. Using the comparison lemma [64], and considering $\dot{\varrho}_a = L(1+\varrho_a)^2$, with $\varrho_a(t_j) = 0$, $j \in \mathbb{N}$, one obtains

$$\varrho(t) \leq \varrho_a(t) = \frac{1}{1 - L(t - t_j)} - 1.$$

Therefore, the bound for the minimal time is obtained imposing

$$\varrho_a(t) = \frac{1}{1 - L(t - t_j)} - 1 = \frac{\theta \lambda_{\min}^{P_c}}{2\|BK\|\lambda_{\max}^{P_c}} \varepsilon_c < \frac{\lambda_{\min}^{P_c}}{2\|BK\|\lambda_{\max}^{P_c}} \varepsilon_c$$

obtaining the time instant $t = t_j + \tau_{\max}$. Since the sampling period is chosen such that $\tau < \tau_{\max}$, then $\dot{V}(x) \leq -\varepsilon_c(1-\theta)V(x)$ and (3.18), (3.20) is exponentially stable to the origin.

When the controller (3.20) is implemented via an event–triggered mechanism, one obtains an event–triggered controller

$$u = Kx(t_n^a) \tag{3.22}$$

where the triggering mechanism is given by

$$t_{n+1}^a = \min_j \Big\{ j\delta \geq t_n^a \mid \|u(j\delta) - u(t_n^a)\| \geq \varepsilon_a \Big\}. \tag{3.23}$$

Treating the event–triggering mechanism as a perturbation, and using the fact that, as well–known, a system exponentially stable at the origin has the property of robustness with respect to bonded perturbations, which can be interpreted in term of input–to–state stability [64], one can show the practical stability of (3.18), and (3.22) with (3.23). This result can be considered as the first step towards the statement of a separation principle for the class of nonlinear system considered, namely of the possibility of designing controller and state observer independently.

Theorem 3.5.3 *Let us consider the system (3.18), with (A, B) controllable, ϕ satisfying (3.19), and a matrix K such that $A + BK$ is Hurwitz. If there exist a matrix $P_c = P_c^T > 0$, an $\varepsilon_c > 0$, and a sampling time τ such that (3.21) are verified, the feedback system (3.18), (3.23) with the event–triggering condition (3.23) is practically exponentially stable, with attractive set*

$$\mathcal{I}_{\varepsilon_a} = \Big\{ \|x\| \leq \rho_a \Big\}, \quad \rho_a = 2 \left(\frac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}} \right)^{3/2} \frac{\|B\|}{(1-\theta)\theta_1} \frac{\varepsilon_a}{\varepsilon_c}$$

where $\theta, \theta_1 \in (0,1)$.

Proof 3.5.4 *Using the control* (3.23) *one obtains the following feedback system dynamics*

$$\dot{x} = Ax + BKx(t_n^a) + D\phi(Hx) = Ax + BKx(j\delta) + BK\Big(x(t_n^a) - x(j\delta)\Big) + D\phi(Hx).$$

Considering $V(x) = x^T P_c x$ *as Lyapunov candidate and the event–triggering policy* (3.23)*, one has*

$$\begin{aligned}\dot{V}(x) &\leq -\varepsilon_c(1-\theta)V(x) + 2x^T P_c BK\Big(x(t_n^a) - x(j\delta)\Big)\\ &\leq -\varepsilon_c(1-\theta)\lambda_{\min}^{P_c}\|x\|^2 + 2\lambda_{\max}^{P_c}\|B\|\|x\|\varepsilon_a\\ &\leq -\varepsilon_c(1-\theta)(1-\theta_1)\lambda_{\min}^{P_c}\|x\|^2\end{aligned}$$

for $\|x\| \geq 2\dfrac{\|B\|}{(1-\theta)\theta_1}\dfrac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}}\dfrac{\varepsilon_a}{\varepsilon_c}$*. This implies that [64]*

$$\|x\| \leq \sqrt{\frac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}}}\,\|x_0\|e^{-\lambda t} + \rho_a, \quad \lambda = \frac{\varepsilon_c(1-\theta)(1-\theta_1)\lambda_{\min}^{P_c}}{2\lambda_{\max}^{P_c}} > 0$$

so that the attractive set is $\mathcal{I}_{\varepsilon_a}$*.*

3.6 An Event–Triggered Impulsive Observer

In order to reconstruct the state vector, in the following an impulsive observer will be considered, having the following structure

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + D\phi(H\hat{x})\\ \hat{x}(k\delta^+) &= \hat{x}(k\delta) + \delta G\big(y(t_\ell^s) - \hat{y}(k\delta)\big) = (I - \delta GC)\hat{x}(k\delta) + \delta GCx(t_\ell^s)\end{aligned} \tag{3.24}$$

where $\hat{x}(k\delta^+)$ is the left limit of $\hat{x}(t)$, and $G \in \mathbb{R}^{n\times q}$ is the observer gain matrix. Note that the right limit is $\hat{x}(k\delta^-) = \hat{x}(k\delta)$. The observer dynamics correspond to a copy of the system dynamics between sampling instants $k\delta$, $(k+1)\delta$, while it undergoes a jump in the state at the sampling instants.

The impulsive observer (3.24) will be implemented making use of a triggering mechanism, determining when the sensor transmits the sampled data to the plant. More precisely, these data are sent to the plant at the time instants $t = t_\ell^s$, $\ell \in \mathbb{N}$, such that the following event–triggering condition is satisfied

$$t_{\ell+1}^s = \min_k \Big\{k\delta > t_\ell^s \mid \|y(k\delta) - y(t_\ell^s)\| \geq \varepsilon_s\Big\} \tag{3.25}$$

where $\varepsilon_s > 0$ is a threshold value on the output error $y(k\delta) - y(t_\ell^s)$.

Given the system and observer dynamics (3.18), (3.24), one has to consider either the continuous dynamics of the observation error $e(t) = x(t) - \hat{x}(t)$, given by

$$\dot{e} = Ae + D\big(\phi(Hx) - \phi(H\hat{x})\big) \tag{3.26}$$

or the error discrete dynamics, due to the impulses on the observer state. These latter have the expression

$$e(k\delta^+) = x(k\delta^+) - \hat{x}(k\delta^+) = (I - \delta GC)e(k\delta) + \delta GC\Big(x(k\delta) - x(t_\ell^s)\Big) \tag{3.27}$$

since $x(k\delta^+) = x(k\delta^-) = x(k\delta)$. It is worth noting that, at the triggering instants t_ℓ^s in which (3.25) is satisfied and the system output sensor sends the sampled data to the controller, this expression reduces to

$$e(t_\ell^{s+}) = e(k\delta^+) = (I - \delta GC)e(k\delta) = (I - \delta GC)e(t_\ell^s)$$

since $x(k\delta) = x(t_\ell^s)$. In all the other discrete time instants, the term $\delta GC\big(x(k\delta) - x(t_\ell^s)\big) = \delta G\big(y(k\delta) - y(t_\ell^s)\big)$ appears, which can be seen as a perturbation induced by the absence of communications from the sensor.

As stated by the following result, the event triggering condition (3.25) along with an appropriate choice of the observer gain ensure the exponential practical stability of the observation error [65]. It is worth noting that this is true also when the system dynamics (3.18) are not stable. This result represents the second step towards the separation principle for the class of non linear systems taken into account.

Theorem 3.6.1 *Let us consider the system (3.18), with (A, C) observable and ϕ satisfying (3.19). If, for a fixed sampling time $\delta > 0$, the following LMIs*

$$N_1 = \begin{pmatrix} P_1A + A^TP_1 + \gamma^2H^TH + \dfrac{P_2 - P_1}{\delta} & P_1D \\ P_1D & -I \end{pmatrix} \leq -\varepsilon I \tag{3.28a}$$

$$N_2 = \begin{pmatrix} P_2A + A^TP_2 + \gamma^2H^TH + \dfrac{P_2 - P_1}{\delta} & P_2D \\ P_2D & -I \end{pmatrix} \leq -\varepsilon I \tag{3.28b}$$

$$N_3 = \begin{pmatrix} -P_2 & P_1 - \delta P_3C \\ (P_1 - \delta P_3C)^T & -P_1 \end{pmatrix} \leq 0 \tag{3.28c}$$

have solutions P_1, P_2, P_3, with $P_i = P_i^T > 0$, $i = 1, 2$, for an $\varepsilon > 0$, then the observer (3.24), with the event–triggering condition (3.25), and the gain $G = P_1^{-1}P_3$, ensures that the origin of error dynamics (3.26), (3.27) is globally practically exponentially stable, with attractive set

$$\mathcal{I}_{\varepsilon_s} = \Big\{\|e\| \leq \rho_s\Big\}, \quad \rho_s = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\frac{\delta\|G\|}{1 - e^{-\delta\bar{\varepsilon}/2}}\varepsilon_s \tag{3.29}$$

where $\bar{\varepsilon} = \varepsilon/\lambda_{\max}$, and $\lambda_{\min} = \min\left\{\lambda_{\min}^{P_1}, \lambda_{\min}^{P_2}\right\}$, $\lambda_{\max} = \max\left\{\lambda_{\max}^{P_1}, \lambda_{\max}^{P_2}\right\}$.

Proof 3.6.2 *The error dynamics can be rewritten as follows*

$$\begin{aligned} \dot{e} &= Ae + Dd_2 \\ e(k\delta^+) &= (I - \delta GC)e(k\delta) + \delta GCd_3 \end{aligned} \tag{3.30}$$

with $d_2 = \phi(Hx) - \phi(H\hat{x})$ and $d_3 = x(k\delta) - x(t_\ell^s)$. Following, [84], let us consider the Lyapunov candidate $V_o = \|e\|_{P(t)}^2$, where $P(t)$ is a time–varying matrix given by the convex combination of the matrices P_1, P_2

$$P(t) = P_1 + \frac{t - k\delta}{\delta}(P_2 - P_1) = \lambda P_1 + (1 - \lambda)P_2, \qquad \lambda = \frac{(k+1)\delta - t}{\delta} \in [0, 1)$$

defined for $t \in (k\delta, (k+1)\delta]$. *Note that* $P(k\delta^+) = P_1$ *and* $P((k+1)\delta) = P_2$. *As explained in Remark 3.6.6, one considers* $P(t)$ *since .....* *Considering that* $P(t)$ *is periodic with period* δ, *its definition can be extended for all* $t \geq 0$. *Note that* $\lambda_{\min}\|e\|^2 \leq V_o \leq \lambda_{\max}\|e\|^2$. *Using (3.30), one works out*

$$\dot{V}_o = e^T \left(PA + A^T P + \frac{P_2 - P_1}{\delta} \right) e + e^T P D d_2 + d_2^T D^T P e + d_2^T d_2 - d_2^T d_2$$

where $\dot{P} = (P_2 - P_1)/\delta$ *has been taken into account. Using (3.19),* $\|d_2\|^2 \leq \gamma^2 e^T H^T H e$ *and the definition of* $P(t)$, *one gets*

$$\begin{aligned} \dot{V}_o &\leq e^T \left(PA + A^T P + \frac{P_2 - P_1}{\delta} + \gamma^2 H^T H \right) e + e^T P D d_2 + d_2^T D^T P e - d_2^T d_2 \\ &= \xi^T N_1 \xi + \frac{t - k\delta}{\delta} \xi^T \bar{N} \xi = \frac{(k+1)\delta - t}{\delta} \xi^T N_1 \xi + \frac{t - k\delta}{\delta} \xi^T N_2 \xi \end{aligned}$$

for $t \in (k\delta, (k+1)\delta]$, *where* $\xi = \begin{pmatrix} e^T & d_2^T \end{pmatrix}^T$ *and* $\bar{N} = (N_2 - N_1)/\delta$. *Hence, using (3.28a), (3.28b), one finally obtains*

$$\dot{V}_o \leq -\varepsilon \frac{(k+1)\delta - t}{\delta} \|\xi\|^2 - \varepsilon \frac{t - k\delta}{\delta} \|\xi\|^2 = -\varepsilon \|\xi\|^2 \leq -\varepsilon \|e\|^2 \leq -\bar{\varepsilon} V_o$$

i.e. $\dot{V}_o$ *is bounded by a negative definite function for all* $t \in (k\delta, (k+1)\delta]$. *Therefore,*

$$V_o(t) = e^{-\bar{\varepsilon}(t - t_0)} V_o(t_0), \quad \forall\, t_0, t \in (k\delta, (k+1)\delta],\ t_0 \leq t \tag{3.31}$$

and, in particular,

$$V_o\big((k+1)\delta\big) \leq e^{-\delta\bar{\varepsilon}} V_o(k\delta^+). \tag{3.32}$$

Note also that in the intersampling

$$\|e(t)\| = e^{-\bar{\varepsilon}(t - t_0)/2} \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \|e(t_0)\|, \quad \forall\, t_0, t \in (k\delta, (k+1)\delta],\ t_0 \leq t. \tag{3.33}$$

Let us now analyze the stability of the discrete error dynamics, i.e. the error dynamics in the discontinuity, considering the same Lyapunov candidate $V_o(t) = \|e(t)\|^2_{P(t)}$ *and recalling that* $P(k\delta^+) = P_1$ *and* $P(k\delta) = P((k+1)\delta) = P_2$, *as already noted. Using (3.30), one gets*

$$\begin{aligned} \Delta V_o &= V_o(k\delta^+) - V_o(k\delta) \\ &= \Big((I - \delta GC) e(k\delta) + \delta GC d_3 \Big)^T P_1 \Big((I - \delta GC) e(k\delta) + \delta GC d_3 \Big) - e^T(k\delta) P_2 e(k\delta) \\ &= -e^T(k\delta) S_2 e(k\delta) + 2\delta \zeta^T (P_1 - \delta P_3 C) e(k\delta) + \delta^2 \zeta^T P_1 \zeta \end{aligned}$$

with $G = P_1^{-1} P_3$, $\zeta = GCd_3 = G\big(y(k\delta) - y(t_\ell^s)\big)$, *and* $S_2 = P_2 - (P_1 - \delta P_3 C)^T P_1^{-1} (P_1 - \delta P_3 C)$ *the Schur complement of the element (2,2) of the matrix* $-N_3$. *It is well–known that for Hermitian matrices,* $-N_3 \geq 0$ *is equivalent to* $P_2 > 0$ *and* $S_2 \geq 0$. *Hence, since* $-N_3 \geq 0$ *and* $P_2 > 0$, *then* $S_2 \geq 0$. *Therefore,*

$$-e^T(k\delta) S_2 e(k\delta) \leq 0.$$

Furthermore, note that

$$e^T(k\delta)(I-\delta GC)^T P_1(I-\delta GC)e(k\delta) = e^T(k\delta)(P_1-\delta P_3 C)^T P_1^{-1}(P_1-\delta P_3 C)e(k\delta)$$
$$= e^T(k\delta)(P_2-S_2)e(k\delta) \leq e^T(k\delta)P_2 e(k\delta) = V_o(k\delta)$$

This last observation allows writing

$$\zeta^T(P_1-\delta P_3 C)e(k\delta) = \zeta^T P_1(I-\delta GC)e(k\delta) \leq \sqrt{\zeta^T P_1 \zeta}\sqrt{e^T(k\delta)(I-\delta GC)^T P_1(I-\delta GC)e(k\delta)}$$
$$\leq \sqrt{\lambda_{\max}^{P_1}}\|\zeta\|\sqrt{V_o(k\delta)} \leq \sqrt{\lambda_{\max}}\|G\|\varepsilon_s\sqrt{V_o(k\delta)}$$

where (3.25) *has been used. Finally,*

$$\zeta^T P_1 \zeta = \|G\big(y(k\delta)-y(t_\ell^s)\big)\|_{P_1}^2 \leq \lambda_{\max}^{P_1}\|G\|^2\varepsilon_s^2 \leq \lambda_{\max}\|G\|^2\varepsilon_s^2.$$

Therefore,

$$V_o(k\delta^+) \leq V_o(k\delta) + 2\delta\sqrt{\lambda_{\max}}\|G\|\varepsilon_s\sqrt{V_o(k\delta)} + \delta^2\lambda_{\max}\|G\|^2\varepsilon_s^2 = \Big(\sqrt{V_o(k\delta)}+c\Big)^2$$
$$\leq \Big(a\sqrt{V_o((k-1)\delta^+)}+c\Big)^2$$

$a = e^{-\delta\bar{\varepsilon}/2}$, $c = \sqrt{\lambda_{\max}}\,\delta\|G\|\varepsilon_s$, *where* (3.32) *has been used, so that*

$$\sqrt{V_o(k\delta^+)} \leq a\sqrt{V_o((k-1)\delta^+)}+c.$$

This linear discrete–time dynamics is exponentially stable to the origin since the dynamic matrix is Schur. Moreover, its solution is given by

$$\sqrt{V_o(k\delta^+)} \leq a\sqrt{V_o(0^+)} + c\sum_{j=0}^{k-1} a^{k-j-1} \tag{3.34}$$

and for $k\to\infty$

$$\lim_{k\to\infty}\sqrt{\lambda_{\min}}\|e(k\delta^+)\| \leq \lim_{k\to\infty}\sqrt{V_o(k\delta^+)} \leq \frac{c}{1-a} = \frac{c}{1-e^{-\delta\bar{\varepsilon}/2}} = \sqrt{\lambda_{\max}}\frac{\delta\|G\|}{1-e^{-\delta\bar{\varepsilon}/2}}\varepsilon_s$$

where the series sum exists since $a<1$. *Hence,* $\lim_{k\to\infty}\|e(k\delta^+)\| \leq \rho_s$. *Considering that* (3.33) *ensures that* $\|e(t)\|$ *decreases exponentially between* $k\delta^+$ *and* $(k+1)\delta$, *one can conclude that the error dynamics* (3.26), (3.27) *converges exponentially to the attractive set* $\mathcal{I}_{\varepsilon_s}$. *The convergence is global since all the passages do not depend on the initial state.*

Remark 3.6.3 *One can also interpret the result of Theorem 3.6.1 as an input–to–state stability property of the observer with respect to a bounded perturbation, due to the event–triggered transmission mechanism.*

The convergence ensured by Theorem 3.6.1 to $\mathcal{I}_{\varepsilon_s}$ is asymptotic. If one requires a finite–time convergence one needs to enlarge $\mathcal{I}_{\varepsilon_s}$. This is stated in the following result.

Corollary 3.6.4 *Let us consider the system (3.18), with (A, C) observable and ϕ satisfying (3.19). Under the same hypotheses and notations of Theorem 3.6.1, the observer (3.24), with the event–triggering condition (3.25), and the gain $G = P_1^{-1}P_3$, ensures that the origin of error dynamics (3.26), (3.27) globally practically converge to the set*

$$\mathcal{I}_{\varepsilon_b} = \left\{ \|e\| \leq (1+\varepsilon_b)\rho_s \right\}$$

ρ_s given by (3.29), in a time

$$T \leq T_{\varrho,\varepsilon_b} = \frac{2}{\bar{\varepsilon}} \ln \frac{d - (1+d)e^{-\delta\bar{\varepsilon}/2}}{\varepsilon_b}, \qquad d = \frac{1}{\delta \|G\| \varepsilon_s} \varrho$$

for any fixed $\varepsilon_b, \varrho > 0$, and for any observer initial condition such that $\|e(0)\| \leq \varrho$.

Proof 3.6.5 *Since*

$$V_o(0) = \|e(0)\|_{P_2}^2 \leq \lambda_{\max}^{P_2} \varrho^2 \leq \lambda_{\max} \varrho^2$$

one has

$$\sqrt{V_o(0^+)} \leq \sqrt{V_o(0)} + c \leq \sqrt{\lambda_{\max}}\, \varrho + c$$

and, from (3.34),

$$\sqrt{\lambda_{\min}} \|e(k\delta^+)\| \leq \sqrt{V_o(k\delta^+)} \leq a\sqrt{V_o(0^+)} + c\sum_{i=0}^{k-1} a^i \leq a^k \left(\sqrt{\lambda_{\max}}\, \varrho + c \right) + c\frac{1-a^k}{1-a}$$

$a = e^{-\delta\bar{\varepsilon}/2}$, $c = \sqrt{\lambda_{\max}}\, \delta \|G\| \varepsilon_s$. Dividing by $\sqrt{\lambda_{\min}}$ and imposing

$$\|e(k\delta^+)\| \leq a^k \left(\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\, \varrho + \frac{c}{\sqrt{\lambda_{\min}}} \right) + (1-a^k)\rho_s = (1+\varepsilon_b)\rho_s, \quad \frac{c}{\sqrt{\lambda_{\min}}} \frac{1}{1-a} = \rho_s$$

one gets the bound of $k\delta$ for the time T in which the error trajectory enters $\mathcal{I}_{\varepsilon_b}$.

Remark 3.6.6 *It is interesting to note the use of a periodic time varying Lyapunov function combination of quadratic Lyapunov function such a choice is technical. An interesting question would be to know whether or not it is possible to find a quadratic Lyapunov function as a simple condition to solve this problem. As a contribution to this question we want to breafly discuss the result on theorem 1 of [84].*

3.6.1 Problem arising when considering discrete and continuous dynamics independently

The paradigm used for the event trigger observation is based on previous results on impulsive observer. However we want to demonstrate that some results cannot be used even in a periodic paradigm namely the theorem One of [84] (even in the linear case)

Considering

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \tag{3.35}$$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + GC(x - \hat{x}) \\ \hat{x}(k\delta^+) &= \hat{x}(k\delta) + \delta K C(x(t_\ell^s) - \hat{x}(k\delta)) \end{aligned} \tag{3.36}$$

according to [84] if there exist $P > 0$ symmetric ,$\tau > 0$,$\epsilon_1, \epsilon_2 > 0, \epsilon_3 > 0$ such that

$$\begin{aligned} P(A - GC) + (A - GC)^T P - \epsilon_1 \leq 0 \\ (I - KC)^T P (I - KC) - \epsilon_2 P \leq 0 \\ \epsilon_1 \tau + ln\epsilon_2 + \epsilon_3 \leq 0 \end{aligned} \tag{3.37}$$

Then the observation error converge exponentially to zero. This theorem is true however it successfully apply to very restricted classes of systems. Considering the second inequality $(I - KC)^T P(I - KC) - \epsilon_2 \leq 0$

and $0 \leq \epsilon_2 \leq 1$ is equivalent to find a gain K such that the discrete time system $\tilde{z}_{k+1} = \tilde{z}_k - KC\tilde{z}_k$ is exponentially stable. the pole placement is possible if the system $z_{k+1} = z_k$, $y_k = Cz_k$ is observable (or detectable) , in this case that means $rank(C) = n$ i.e. The entire system is measured. If $rank(C) \leq n$ it is not possible to render the discrete time observation error stable and P does not exist. If $\epsilon_2 \geq 1$ the LMI can be solver. but it implies that the impulsive part of the observer cannot ensure any improvement over purely continuous observation.

From the previous argument and assuming $G = 0$ (i.e. purely impulsive observation) then if $rank(C) \leq n$ then the LMI is never feasible. If $rankC = n$ a trivial observer can be designed (i.e. $\hat{x}(t_k^+) = C^{-1}y(t_k)$ which trivially converge after one impulse).

The question of the existence of a Quadratic common Lyapunov function allowing the computation of matrix gain for Liepschitz non linear system is not negatively answered by the limitation of this theorem. However implicitly in its methodology a separation has been made between continuous dynamics and impulse at the level of the observer. If such a separation is made in the analysis then it is not possible to have an observer using only smpled measurement by solving LMI (3.37) and one need to take into account the interplay of continuous and discrete dynamics. If want to compute the flow between sampling it is possible to find a quadratic (common) Lyapunov function as it has been proposed in [9] but other computational difficulties arise.

3.7 An Event–Triggered Observer–Based Controller

In this section we study the conditions under which the event–triggered observer (3.24) and the event–triggered controller (3.22), with the triggering mechanisms (3.25), (3.23), can ensure practical exponential stability to the origin of the closed–loop system (3.18), (3.24), (3.22). This is the last step towards the statement of a separation principle for the class of non linear system under study.

Theorem 3.7.1 *Let us consider the system (3.18), with (A, B) controllable, (A, C) observable, ϕ satisfying (3.19), and a matrix K such that $A + BK$ is Hurwitz. Under the following conditions*

1. *There exist a matrix $P_c = P_c^T > 0$, an $\varepsilon_c > 0$, and a sampling time τ such that (3.21) are verified;*

2. *For a fixed sampling time $\delta > 0$, the LMIs (3.28) have solutions P_1, P_2, P_3, with $P_i = P_i^T > 0$, $i = 1, 2$, for an $\varepsilon > 0$;*

then the feedback system (3.18), (3.24), *with*

$$u = K\hat{x}(t_n^a) \tag{3.38}$$

and with the triggering conditions (3.23), (3.25), *is practically exponentially stable, with attractive set*

$$\mathcal{I}_{\varepsilon_d} = \Big\{\|x\| \le \rho_d\Big\}, \quad \rho_d = \frac{2}{(1-\theta)\theta_1}\left(\frac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}}\right)^{3/2}\frac{\varepsilon_d}{\varepsilon_c}, \quad \varepsilon_d = \|BK\|(1+\varepsilon_b)\rho_s + \|B\|\varepsilon_a \tag{3.39}$$

where ρ_s *is given by* (3.29), *for* $\theta, \theta_1 \in (0,1)$, *and for any fixed* $\varepsilon_b, \varrho > 0$, *and for any observer initial condition such that* $\|e(0)\| \le \varrho$.

Proof 3.7.2 *Using* (3.38) *in* (3.18) *one has*

$$\dot{x} = Ax + BK\hat{x}(t_n^a) + D\phi(Hx) = Ax + BKx(j\delta) + D\phi(Hx) + BKd_4$$

where

$$d_4 = \hat{x}(t_n^a) - x(j\delta) = \Big(\hat{x}(j\delta) - x(j\delta)\Big) + \Big(\hat{x}(t_n^a) - \hat{x}(j\delta)\Big)$$

can be treated as a perturbation acting on the feedback system. From Corollary 3.6.4, for any fixed $\varepsilon_b, \varrho > 0$, *and for any observer initial condition such that* $\|e(0)\| \le \varrho$, *there exist a finite time* $T_{\varrho,\varepsilon_b}$ *such that* $\|e(t)\| \le (1+\varepsilon_b)\rho_s$ *for all* $t \ge T_{\varrho,\varepsilon_b}$. *Hence,*

$$\|\hat{x}(j\delta) - x(j\delta)\| \le (1+\varepsilon_b)\rho_s$$

while, due to the event–triggering policy

$$\left\|K\Big(\hat{x}(t_n^a) - \hat{x}(j\delta)\Big)\right\| \le \varepsilon_a.$$

Proceeding as in the proof of Theorem 3.5.3, one obtains the dynamics

$$\dot{x} = (A+BK)x + D\phi(Hx) + BK(d_1 + d_4)$$

which are practically stable, with practical stability region given by the set $\mathcal{I}_{\varepsilon_d}$. *In fact, considering as in the proof of Theorem 3.5.3* $V(x) = x^T P_c x$ *as Lyapunov candidate, for* $\|x\| \ge b$ *one has*

$$\begin{aligned}\dot{V}(x) &\le -\varepsilon_c(1-\theta)V(x) + 2x^T P_c BKd_4 \le -\varepsilon_c(1-\theta)\lambda_{\min}^{P_c}\|x\|^2 + 2\lambda_{\max}^{P_c}\|x\|\varepsilon_d \\ &\le -\varepsilon_c(1-\theta)(1-\theta_1)\lambda_{\min}^{P_c}\|x\|^2\end{aligned}$$

for $\|x\| \ge \dfrac{2}{(1-\theta)\theta_1}\dfrac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}}\dfrac{\varepsilon_d}{\varepsilon_c}$. *This implies that [64]*

$$\|x\| \le \sqrt{\frac{\lambda_{\max}^{P_c}}{\lambda_{\min}^{P_c}}}\,\|x_0\|e^{-\lambda t} + \rho_d, \quad \lambda = \frac{\varepsilon_c(1-\theta)(1-\theta_1)\lambda_{\min}^{P_c}}{2\lambda_{\max}^{P_c}} > 0$$

so that the attractive set is $\mathcal{I}_{\varepsilon_d}$.

The results proposed in this second section can be seen as continuation of the results exposed at the beginning of the chapter. Indeed we have provided an observer able to estimate the state of the system that is input to state stable with respect to sampling error , and used an observer based controller that is also input to state stable with respect to disturbances. We want to point out a few difference between the two part. First we are able to state a form of separation principle, that is the event triggered observer can perform his task in the absence of control action whereas in

3.7.1 Simulations and comments

Considering

$$\begin{aligned}
\dot{x} &= Ax + G\phi(Hx) + BK\hat{x}(t_n^c) \\
\dot{\hat{x}} &= A\hat{x} + G\phi(H\hat{x}) + BK\hat{x}(t_n^c) \\
\hat{x}(\delta k^+) &= \hat{x}(\delta k^-) + \delta LC(\hat{x}(\delta k^-) - x(t_\ell^s)) \\
y(\delta k) &= Cx(\delta k)
\end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 21.6 & 0 & 0 \end{pmatrix}^T$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 & 3.3\sin(x_3) \end{pmatrix}^T.$$

One considers the control $u = K\bar{\hat{x}}$, with

$$K = \begin{pmatrix} 7.8428 & 1.1212 & -4.3666 & 1.1243 \end{pmatrix}$$

and the observer gain is

$$L = \begin{pmatrix} 9.3334 & 1.0001 \\ -48.7804 & 22.3665 \\ -0.0524 & 3.3194 \\ 19.4066 & -0.3167 \end{pmatrix}.$$

with $\delta = .15s$ $\epsilon_s = 10^{-7}s, h = 8 * 10^{-6}s; \epsilon_c = 10^{-4}$ The following parameter guaranty that the observation error $||e||$ will be smaller than 0.001 after convergence and that $||x||$ will be smaller than 0.05

$$P_c = \begin{pmatrix} 0.1592 & -0.3627 & 0.0807 & -0.4019 \\ -0.3627 & 3.7540 & 0.1179 & 0.0810 \\ 0.0807 & 0.1179 & 0.0993 & -0.2477 \\ -0.4019 & 0.0810 & -0.2477 & 1.6406 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 28.6269 & -0.3675 & -11.6820 & 1.0816 \\ -0.3675 & 0.5296 & -1.5176 & 0.0455 \\ -11.6820 & -1.5176 & 18.8721 & -1.7201 \\ 1.0816 & 0.0455 & -1.7201 & 0.7153 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 22.9349 & -0.5313 & -11.6047 & 1.0195 \\ -0.5313 & 0.4136 & -1.9257 & 0.0810 \\ -11.6047 & -1.9257 & 20.9671 & -1.8715 \\ 1.0195 & 0.0810 & -1.8715 & 0.7393 \end{pmatrix}$$

In figure 3.7.1 the practical stability of the considered system is shown. The parameter $\varepsilon_s, \varepsilon_o$ where chosen very small, this show that the bound given on the observation and control error is not thight since the attractive set (of both the observer and the controller) is smaller than the worst case predicted.

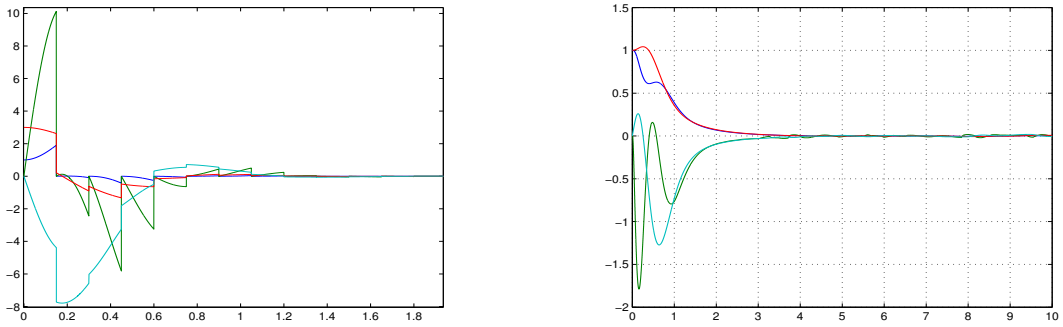

Figure 3.6: Observation error and state of the system.

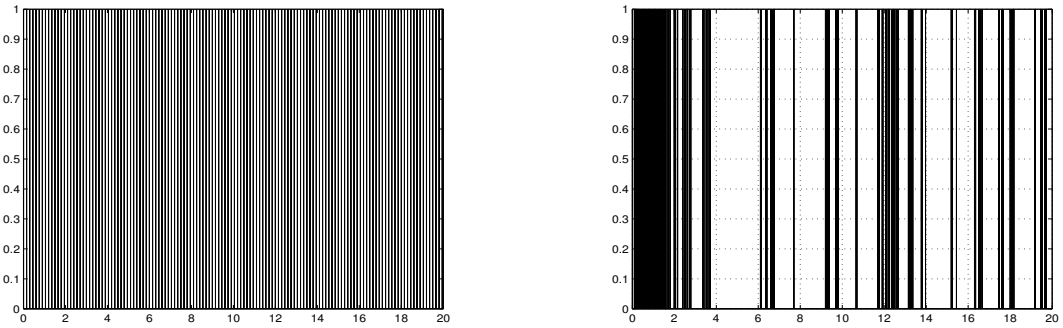

Figure 3.7: Transmission instant of the sensors (left) transmission instant of the actuator

whereas in figure 3.7.1 one see that close to equilibrium the sampling frequency is substantially reduced for the controller, the sampling frequency for the sensor is always given by the sampling period, since the sampling period is very large with respect to the parameter ε_s.

Since a practical separation principle as been introduce one can consider the system without control (i.e. $u(t) = 0$).

3.7.2 Conclusion on the impulsive observer for Lipschitz non linear systems

In this section we presented an impulsive observer coupled with a control aiming at stabilising a Lipschitz non linear system with periodic even triggered sensor and control action. In the proposed scheme control and observation gain can be computed independently, the observer converge whether or not the system is stable furthermore the sampling instant of the controller and of the sensor do not need to be synchronous (or at the same frequency), thus ensuring a week separation principle . The triggering parameters of both the sensor and the actuator can be computed in order the ensure an upper bound on the size of the attractive set of the observer and of the plant.

Simulation results suggest however that the upper bound is not tight, ways to circumvent this over-approximation should be the object of future research.

3.8 Conclusion on observation and control for some classes of non linear systems

In this chapter we presented result on observer synthesis adapted to observation of non linear plant subject to event triggered sampling. We first gave necessary condition in term of input to state stability of the observation error in order to be robustly observable then we gave conditions for a stabilizing controller to asymptotically stabilize a system subject to output feedback in the presence of communication constraints however the results holds when a control action stabilize the system. So an impulsive non linear observer was introduce to solve the observation problem without requirement of stabilization, as for the linear case when no stabilization controller is introduce, asymptotic stability is switched for practical stability. Then output feedback is still possible. In the light of the work presented in this chapter and as for system without communication constraint the linear systems can be recovered as special instance of non linear systems however a drawback of considering non linear (i.e. a more general classes of system than linear systems) is that the tightness of the triggering parameter is worst than the one obtain for linear system. It is a general features of the work presented in the last two chapter than the bound on the triggering parameter is not tight. This fact can be understood as a consequences of the nature observation based event trigger paradigm. Indeed the fundamental motivation of event triggering sampling is the use of the knowledge of additional information (namely knowledge of the state) in order to reduce the number of communication. In the case of observer based event trigger. The state is not known but only a sub part of it, this in return imply some conservatism with respect to better performance in terms of event triggered sampling with full state. Therefore while it has been shown that the results and insights of even trigger with full state information can be extended (at least qualitatively) to systems with partial measurement of the state, the quantitative effectiveness of the proposed paradigm require further investigations.

Chapter 4

Decentralised estimator for consensus

In the previous section we investigated several problems of observation and control when considering event triggered sampling policy. Such a scheme is relevant to network control system when aiming at reducing the communication burden a problem naturally arising in the framework of Cyber-physical systems. In this chapter we will consider a different kind of networked system: Multi agent system trying to reach consensus. Consensus for multi agent systems appears in a broad variety of application ranging from robotic to wireless sensor network and is relevant to many kind of networked and decentralysed system.

In this chapter we will focus on a particular class of consensus problems for multi agents systems: the problem of consensus while maintaining range connectivity. A key underlying assumption of this chapter is that agents are assumed to be able to communicate with each other. This communication capacity is used to estimate the consensus. An interesting fact to notice is that communication among agent make the system under consideration cyber-physical. To make this statement clear We will first recall some classic result on consensus among multi-agent system and present notion of graph theory, we will then introduce a new decentralized consensus estimation scheme making use of the aforementioned ability to communicate and show some tracking policy in order to solve the problem under consideration.

4.1 Generality on consensus among multi-agent

The problem of consensus where a set of communicating agents needs to reach an agreement (i.e. converging to a given point or a final formation) while not possessing the ability to communicate to every agent of the set has been studied intensively in the recent past. An important part of this problem is that none of the agents posses a common global positioning system or a centralized controller. Therefore the control strategy needs to take into account this information constraint using a decentralized controller.

The classes of systems under consideration as well as the range of application (informatics robotics , synchronization) has given rise to a considerable amount of result exist on consensus both in the continuous and discrete cases. [77] provides a general presentation with an extensive bibliography, in [29] a more recent overview is provided on the subject of consensus.

A key idea when considering a set of communicating agent with whether fixed or dynamic communication topologies is the notion of Graph [76]. In the existing literature both event trigger sampling (see for instance)[91] and observer for consensus [72][103] are considered.

In the mentioned work on observation for consensus the observation scheme is performed via single agent to reconstruct local informations in our work estimation is done by information exchange to reconstruct the Average Consensus (AC).

In the following we will introduce necessary concept on Graph theory in order to formulate our control problem

4.1.1 Graph theoretic notion

The aim of this sub-part is to introduce only the necessary concept of graph theory.

The static communication graph is given by the undirected graph: $G(V, E)$, where V denotes the set of M agents, E denotes the set of (undirected) edges. $(i, j) \in E$ means agent i and j are able to communicate information. In particular i can send information to j and j can send information to i. For convenience we assume $(i, i) \notin E$ and denote the set of neighbourhood of i is $N_i := \{j \in V | (i, j) \in E\}$.

We now want to introduce some important Graph related matrix.
$\Delta(G) = diag(|N_i|)_{i \in V}$ is the degree matrix.

$\mathcal{A}(G)_{ij} = \begin{cases} 1 \text{ if } e_{ij} \in E \\ 0 \text{ if } e_{ij} \notin E \end{cases}$ is the adjacency matrix.
$\mathcal{L}(G) = \Delta(G) - A(G)$ is the Laplacian matrix
Furthermore building any orientation on the graph $G(V, E)$ giving rise $toG(V, E')$ (i.e. giving an arbitrary direction to any edge in E)

$$\mathcal{I}(G)_{ij} = \begin{cases} 1 \text{ if } e_{ij} \in E' \\ -1 \text{ if } e_{ji} \in E' \\ 0 \end{cases}$$

The value of $\mathcal{I}(G)$ is orientation independent and $I^T I = \mathcal{L}$

with $\Delta(G), \mathcal{A}(G), \mathcal{L}(G) \in \mathbb{R}^{M \times M}$. The Laplacian is positive semi-definite and hence we consider the ordering of his eigenvalues

$$0 = \lambda_1(\mathcal{L}(G)) \leq \lambda_2(\mathcal{L}(G)) \leq \ldots \leq \lambda_M(\mathcal{L}(G)).$$

We want to recall a classical theorem of spectral graph theory

Theorem 4.1.1 *G is connected if and only if $\lambda_2(\mathcal{L}(G)) > 0$*

Furthermore from the construction of $\mathcal{L}$ one has $\mathcal{L}(1, \ldots, 1)^T = 0$

4.1.2 Definition and classic results

Considering that each agent is modelled as of simple integrator we have the following:

$$\dot{X}_i = U_i,\ i \in V, X_i \in \mathbb{R} \tag{4.1}$$

$|V|$ is equal to M.

Definition *4.1.2 (Consensus of a multi agent system) A control U is said to solve the Consensus problem if U controlling the process $\dot{X} = U$ leads to*

$$\forall (i,j) \in V \times V, \lim_{t\to\infty} \|X_i(t) - X_j(t)\| = 0$$

By this previous definition it is not required that the agent meet at a special point.

A stronger requirement in terms of consensus would be for the agent to meet at the barycentre of their initial coordinate. That is the object of the following definition.

Definition *4.1.3 (AC of a multi agent system) A control U is said to solve the AC problem if U controlling the process $\dot{X} = U$ leads to*

$$\forall i \in V, \lim_{t\to\infty} X_i(t) = X_{eq}$$

In the following $X_{eq} := \frac{1}{M}\sum_{i\in V} X_i(0)$.

Since we will later consider communication it is interesting to consider the exact discretization of the previous model

In the next part the constant sampling period will be noted

$$h := t_{k+1} - t_k, \forall k \in \mathbb{N}$$

$$U_i(t) = U_i(t_k), \forall t \in [t_k, t_{k+1}[$$
$$X_i^k = X_i(t_k)$$
$$X_{k+1}^i = X_k^i + h * U_k^i$$

A classic control to solve the average consensus is

$$U_k^i = \epsilon/h \sum_{j\in N_i} (X_j^k - X_i^k)$$

where ϵ/h is a control gain

$$X_i^{k+1} = X_i^k + \epsilon \sum_{j\in N_i} (X_j^k - X_i^k) \tag{4.2}$$

It is clear that

$$\hat{X}^{k+1} = P\hat{X}^k$$

where P is the Perron matrix of graph G with parameter ϵ.

Theorem 4.1.4 *if G is connected and if $0 < \epsilon < \frac{1}{\max_{i\in V} |N_i|}$ and X dynamic's is given by (4.2) then X converge exponentially to the AC.*

Since G is a connected symmetric graph P is a Perron matrix and since $0 < \epsilon < \frac{1}{\max_{i\in V} N_i}$ then

$$\sum_{i\in V} |\hat{X}_i^k - (\frac{1}{M}\sum_{i\in V} \hat{X}_i^0)| \leq \sum_{i\in V} |\hat{X}_i^0 - (\frac{1}{M}\sum_{i\in V} \hat{X}_i^0)|(1 - \epsilon\lambda_2(\mathcal{L}))^k$$

by assumption of connectedness of the original graph. Furthermore

$$\sum_{i\in V} X_i^{k+1} = \sum_{i\in V} X_i^k$$

Therefore (4.2) Solve the AC problem.

However the control policy can lead to an overshoot in the transient leading to increase of the distance between two neighbouring agents. In the next section we will precisely define this problem and propose some new solutions to it.

4.2 Average consensus among agent with decentralised estimators using a waiting time

In the following chapter we want to address the problem of average consensus (AC) while preserving connectivity for a set of agent i.e. the agents will reach a common point that is the barycentre of their initial position and the distance between two neighbouring agents do not at any time cross a given threshold. The problem of connectivity preserving consensus has been solved using nonlinear decentralized controller at the level of each agent [34],[94],[59] in the continuous case. And in [6] when considering discrete time.

However for the control strategy proposed in [34],[94],[59][6] the consensus reached is not the average of the vehicle initial position.

In [44] the problem of average consensus with constrained range connectivity is solved using model predictive control however the result is valid for scalar agent however the tools used in this work don't allow for extension to non scalar agent.

First we introduce an estimation scheme for the AC and a reference tracking allowing to solve the problem of AC while preserving connectivity when inter agents and control actualisation are discrete with a guaranteed convergence rate. The second result is an estimation scheme to solve the formation problem (i.e. each agent reach a desired position with respect to the average consensus) provided the formation respect the range connectivity constrain.

The problem of estimators/observers for consensus has been considered for instance in [28]. Estimators for consensus allow in general to perform a task not possible for a direct control scheme.

The decentralised estimator use only locally available information and is then used to solve the problem of average consensus with range connectivity constrain. For simplicity of notation the case of scalar agent is considered but the methodology is valid for any finite dimensional single integrator[1].

In this estimator based controller the communication enables each vehicle to find the AC and the controller (and sensors) ensure that connectivity is preserved at all time. Thus the Multi-agent system shall be seen as a cyber-physical system.

4.2.1 Problem statement and definition

Due to constrain in term of sensor range as well as communication devices another problem to be considered is the following: Is it possible to achieve consensus when considering

[1]The choice of $dim(X_i) = 1$ is motivated only for simplicity of notation. The extension only need to perform the consensus estimation for each component

a supplementary constraint on the distance between mobile agent i.e.

$$\forall t \geq 0, \forall i \in V, \forall j \in N_i,\ \|X_i(t) - X_j(t)\| \leq \Delta$$

provided

$$\forall i \in V, \forall j \in N_i \|X_i(0) - X_j(0)\| \leq \Delta.$$

This constraint give rise to the following definition.

Definition *4.2.1 (Δ-Connectivity Preserving* AC of a multi agent system*):*
A control U is said to solve the AC problem if U controlling the process $\dot{X} = U$ leads to

$$\forall i\ \in V, \lim_{t\to\infty} X_i(t) = X_{eq}$$

and

$$\forall t \geq 0, \forall (i,j) \in E,\ \|X_i(t) - X_j(t)\| \leq \Delta$$

provided $\|X_i(0) - X_j(0)\| \leq \Delta$.

In this next section we will give our communication/control scheme to AC problem while maintaining connectivity. To do so, we want to explicitly mention our structural hypothesis:

H1 At the initial time each vehicle is in a radius Δ of his neighbour.

H2 the communication graph is static, undirected, and connected.

H3 The sample time for control actualisation, state measurement and communication among agent is synchronized. and the sampling period h is fixed.

Next we will give our control scheme end prove that it will lead to AC while maintaining Δ-connectivity.

From the previous assumptions each vehicle is able by communicating with his neighbour to make an estimation of the AC. Therefore the extended dynamic of vehicle i is given by the following state:

We will first give our control algorithm to be implemented in a vehicle and we will show next that the dynamical system resulting from the proposed algorithm leads to average consensus while maintaining range connectivity.

where u_i^k is the effective control applied to vehicle i and ϵ_i^k and ϵ will be defined later. This algorithm use only locally available data since $X_j^0 - X_i^0$ is known to each agent. To perform the analysis of his property we will rewrite it in an equivalent way.

Since $\hat{X}_i = \tilde{X}_i + X_i^0$ and $X_i^k = X_i^0 + \sum_{l=0}^{k-1} u_i^l$

$$\hat{X}_i^{k+1} = \hat{X}_i^k + \epsilon \sum_{j\in N_i} (\hat{X}_j^k - \hat{X}_i^k) \tag{4.3a}$$

$$X_i^{k+1} = X_i^k + \epsilon(\hat{X}_i^k - X_i^k) \text{ if } k > F(G) \tag{4.3b}$$

$$X_i^{k+1} = X_i^{k+1} \text{ if } k \leq F(G)$$

In the following $\hat{X} = (\hat{X}^T, \cdots, \hat{X}_M^T)^T$ and $\hat{X}^0 = X^0$

Algorithm 1 Observation and control scheme for agent i

Initialization of estimator
$k = 0$
$\tilde{X}_i^0 = 0$
Initialization of the control
$u_i^0 = 0$
End of initialization
while 1=1 **do**
    Dynamic of estimator
    $\tilde{X}_i^{k+1} = \tilde{X}_i^k + \epsilon(\sum_{j\in N_i} \tilde{X}_j^k - \tilde{X}_i^k + X_j^0 - X_i^0)$
    Dynamic of Control
    $u_i^k = \epsilon(\tilde{X}_i^k - \sum_{l=0}^{k-1} u_i^l)$ if$k > \dfrac{-ln(2M)}{ln(1-\epsilon\lambda_2(\mathcal{L}))}$
    $u_i^k = 0 if k < \dfrac{-ln(2M)}{ln(1-\epsilon\lambda_2(\mathcal{L}))}$
    $k = k + 1$

Remark 4.2.2 *In the previous algorithm we present the discrete version of the control input. In continuous time the dynamics is given by:* $\dot{X}_i = \alpha(\hat{X}_i - X_i)$ *with* $\epsilon = \alpha h$ *when* $t > (k+1)hF(G)$ *and* $\dot{X}_i = 0$ *when* $t \leq (k+1)hF(G)$

Remark 4.2.3 *The relation* $\epsilon = \alpha h$ *highlight a trade-off between sampling rates and maximal acceptable control. Indeed in the following it is needed to have* ϵ *bounded by a given constant therefore for fixed sampling period the control action cannot be arbitrary high. Furthermore the waiting time* $F(G)$ *is given by a discrete number of communication exchange, an increasing communication rate leads in continuous time to a shorter wait.*

It is clear that

$$\hat{X}^{k+1} = P\hat{X}^k$$

where P is the Perron matrix of graph G with parameter ϵ. and $F(G)$ is given by

$$F(G) := \frac{-ln(2M)}{ln(1-\epsilon\lambda_2(\mathcal{L}))}.$$

The observation and control scheme can be decomposed in two parts. The first part ($k \leq F(G)$) correspond to a mode where all agents are immobile and wait for the estimators to have a sufficiently good estimation of the average consensus while the second part ($k > F(G)$) correspond to a mode where every vehicle converge to their consensus estimate. The connectivity is not lost due to the fact that the difference of average consensus estimation for each agent is small.

It will be shown in the following section waiting for the agents to have a good estimate of the consensus allow for a simple control to maintain connectivity even if the control is piecewise continuous and the observation of the state is sampled.

4.3 Average consensus using varying control gain

In this section we propose a second method to solve the problem of average consensus using decentralised observer. In the previous section we used a waiting time to ensure

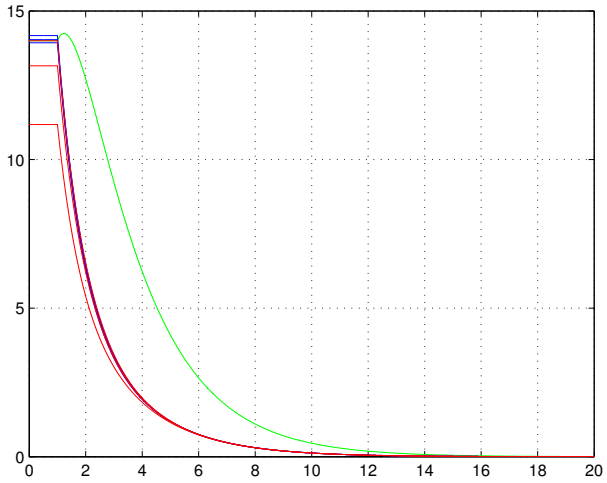

Figure 4.1: After convergence of the estimator in a neighbourhood of the average consensus the agents converge

convergence of the average consensus estimator , the waiting time require information on the multi agent system (namely an lower bound on $\lambda_2(G)$ the algebraic connectivity of the graph G and on M the number of agents. In order to mitigate this problem of information we will introduce in the next section non linear control gain taking into account the estimator state at the level of each agent, the proposed scheme only use locally available information.

4.3.1 Problem statement

As in the previous section each agent is modelled as a continuous-time integrator of the form:

$$\dot{X}_i = U_i,\ i \in V, X_i \in \mathbb{R}^d, \tag{4.4}$$

where V denotes the agent's index set with M elements. The agents' controls are updated discretely at sampling times $t_k := kh$, $\forall k \in \mathbb{N}$, where h denotes the sampling period, and the control inputs are kept constant between sampling times, i.e., $U_i(t) = U_i(t_k)$, $\forall t \in [t_k, t_{k+1}[$.

4.3.2 Average Consensus

Next we describe the control scheme end prove that it will lead to AC while Preserving Δ-connectivity.

From the previous assumptions each vehicle is able by communicating with his neighbour to make an estimation of the AC. Therefore the extended dynamic of vehicle i is given by the following state. Therefore the observation and control algorithm at each vehicle is

where u_i^k is the effective control applied to vehicle i and ϵ_i^k and ϵ will be defined later. This algorithm use only locally available data since $X_j^0 - X_i^0$ is known to each agent. To perform the analysis of his property we will rewrite it in an equivalent way.

Algorithm 2 Observation and control scheme for agent i

Initialization of estimator
$k = 0$
$\tilde{X}_i^0 = 0$
Initialization of the control
$u_i^0 = 0$
End of initialization
while 1=1 **do**
    Dynamic of estimator
    $\tilde{X}_i^{k+1} = \tilde{X}_i^k + \epsilon(\sum_{j\in N_i} \tilde{X}_j^k - \tilde{X}_i^k + X_j^0 - X_i^0)$
    Dynamic of Control
    $\mu_{i,j}^k = |X_i^k - X_j^k|^2$
    if $(X_i^k - X_j^k)^T(\hat{X}_i^k - X_i^k) > -\dfrac{|X_i^k - X_j^k|^2}{4}$ **then** $\mu_{i,j}^k = 0$
    $\rho_{ij}^k = \Delta - |X_i^k - X_j^k|$
    $\epsilon_i^k = \dfrac{1}{(2M\Delta^2)^2} \min_{j\in N_i}\{\max(\mu_{i,j}^k, \rho_{ij}^k)\}$
    $u_i^k = \dfrac{\epsilon_i^k}{h}(\tilde{X}_i^k - \sum_{l=0}^{k-1} u_i^l)$
    $k = k + 1$

Since $\hat{X}_i = \tilde{X}_i + X_i^0$ and $X_i^k = X_i^0 + \sum_{l=0}^{k-1} u_i^l$ we have the following dynamic:

$$\hat{X}_i^{k+1} = \hat{X}_i^k + \epsilon \sum_{j\in N_i} (\hat{X}_j^k - \hat{X}_i^k) \tag{4.5a}$$

$$X_i^{k+1} = X_i^k + \epsilon_i^k(\hat{X}_i^k - X_i^k) \tag{4.5b}$$

$$\epsilon_i^k = \frac{1}{(2M\Delta^2)^2} \min_{j\in N_i}\{\max(\mu_{i,j}^k, \rho_{ij}^k)\} \tag{4.5c}$$

$$\rho_{ij}^k = \Delta - |X_i^k - X_j^k| \tag{4.6}$$

$$\mu_{i,j}^k = \begin{cases} |X_i^k - X_j^k|^2 \text{ if } (X_i^k - X_j^k)^T(\hat{X}_i^k - X_i^k) \leq -\dfrac{|X_i^k - X_j^k|^2}{4} \\ 0 \text{ if } (X_i^k - X_j^k)^T(\hat{X}_i^k - X_i^k) > -\dfrac{|X_i^k - X_j^k|^2}{4} \end{cases} \tag{4.7}$$

In the following $\hat{X} = (\hat{X}_1^T, \cdots, \hat{X}_M^T)^T$ and $\hat{X}^0 = X^0$ $\hat{X} = (\hat{X}_{11}, \ldots, \hat{X}_{1d}, \ldots \hat{X}_{M1}, \ldots, \hat{X}_{Md})^T$ For convenience we reorder $\hat{X}$ $Z = (\hat{X}_{11}, \ldots, \hat{X}_{M1}, \ldots \hat{X}_{1d}, \ldots, \hat{X}_{Md})^T$

The Gain ϵ will be selected below.

In (4.5c) $\mu_{i,j}^k$ and ρ_{ij}^k account for two different possibility. When the average consensus can cause loss of connectivity $\rho_{ij}^k = 0$ we choose a gain $(\mu_{i,j}^k)$ that goes to zero when $|X_i^k - X_j^k|$ goes to Δ. It will be showed that by appropriately choosing connectivity is maintained. But When the direction given by the estimate consensus is good (i.e reduce inter vehicular distance) the gain does not consider $\Delta - |X_i^k - X_j^k|$

Theorem 4.3.1 *if* $0 < \epsilon < \dfrac{1}{\max_{i\in V} |N_i|}$, $\hat{X}^0 = X^0$ *and* $\max_{i\in V, j\in N_i} ||X_i^0 - X_j^0|| \leq \Delta$ *then* (4.3a) *converge to the AC while preserving* Δ*-connectivity. Furthermore for all*

$k > \dfrac{-2ln(4M)}{ln(1-\epsilon\lambda_2(G))}$ *the convergence speed is exponential.*

In order to prove 4.3.1 we introduce two lemmas.

Lemma 4.3.2 *choosing* $0 < \epsilon < \dfrac{1}{\max_{i\in V}|N_i|}$ *then* (4.5a) *leads to*

$$\forall i \in V \lim_{k\to\infty} \hat{X}_i^k = \frac{1}{M}\sum_{i\in V} X_i^0$$

and

$$\max_{i\in V, j\in N_i} |\hat{X}_i^k - \hat{X}_j^k| \leq (1-\epsilon\lambda_2(\mathcal{L}))^k 4M^2\Delta$$

Proof 4.3.3 *Since* G *is a connected symmetric the matrix* P *is a Perron matrix and since* $0 < \epsilon < \dfrac{1}{\max_{i\in V} N_i}$ *then*

$$\sum_{i\in V} |\hat{X}_i^k - (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0)| \leq \sum_{i\in V} |\hat{X}_i^0 - (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0)|(1-\epsilon\lambda_2(\mathcal{L}))^k$$

see [77] for more details. By assumption of connectedness of the original graph and by definition of $\hat{X}$, $\hat{X}^0 := X^0$ *it holds* $\forall i \in V$ *that* $|\hat{X}_i^0| < M\Delta$ *Considering that*

$$A := \sum_{i\in V} |\hat{X}_i^0 - (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0)| < 2M^2\Delta$$

we have $\max_{i\in V, j\in N_i} |\hat{X}_i^k - \hat{X}_j^k| = \max_{i\in V, j\in N_i} |\hat{X}_i^k - (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0) + (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0) - \hat{X}_j^k|$ *Therefore*

$$\max_{i\in V, j\in N_i} |\hat{X}_i^k - \hat{X}_j^k| \leq 2\sum_{i\in V} |\hat{X}_i^k - (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0)|$$

since $2\sum_{i\in V} |\hat{X}_i^k - (\frac{1}{M}\sum_{i\in V}\hat{X}_i^0)| \leq 2A(1-\epsilon\lambda_2(\mathcal{L}))^k$ *So*

$$\max_{i\in V, j\in N_i} |\hat{X}_i^k - \hat{X}_j^k| \leq (1-\epsilon\lambda_2(\mathcal{L}))^k 4M^2\Delta$$

This conclude the lemma.

Lemma 4.3.4 *for all* $i^* \in argmax|X_i^k - X_{eq}|$ $\forall k, \forall j \in N_{i^*}$,
$(X_{i^*}^k - X_j^k)^T(\hat{X}_{i^*}^k - X_{i^*}^k) \leq -|X_{i^*}^k - X_j^k|(\dfrac{|X_{i^*}^k - X_j^k|}{2} - |\hat{X}_{i^*}^k - X_{eq}^k|)$

This lemma ensure that a subset of agent will have "good" control law.

Proof 4.3.5 *Considering* (4.5a) *for agent in the set* i^* *we are interested in* $(X_{i^*}^k - X_j^k)^T(\hat{X}_{i^*}^k - X_{i^*}^k)$.
$(X_{i^*}^k - X_j^k)^T(\hat{X}_{i^*}^k - X_{i^*}^k) = (X_{i^*}^k - X_j^k)^T(X_{eq}^k - X_{i^*}^k) + (X_{i^*}^k - X_j^k)^T(\hat{X}_{i^*}^k - X_{eq}^k)$

$$(X_{i^*}^k - X_j^k)^T(X_{eq}^k - X_{i^*}^k) = -|X_{i^*}^k - X_j^k||X_{i^*}^k - X_{eq}^k|cos(\theta)$$

with $\theta = \widehat{X_{eq}X_{i^*}^k X_j^k}$

From hypothesis on i^* $cos(\theta) \in [\frac{|X_{i^*}^k - X_j^k|}{2|X_{i^*}^k - X_{eq}^k|}, 1]$ *And* $\frac{|X_{i^*}^k - X_j^k|}{2|X_{i^*}^k - X_{eq}^k|} \leq 1$ *since* $|X_{i^*}^k - X_j^k| \leq |X_{i^*}^k - X_{eq}^k| + |X_j^k - X_{eq}^k|$ *so* $|X_{i^*}^k - X_j^k| < 2|X_{i^*}^k - X_{eq}^k|$ *This conclude the lemma*

We can now prove the Theorem 4.3.1.

Proof 4.3.6 *We subdivide the proof in two parts. The first part will deal with connectivity preservation and the second part with convergences.*

1) Connectivity preservation

Considering two neighbour agent i *and* j*. Assume for all* $i \in V, j \in N_i$ $|X_i^k - X_j^k| < \Delta$

$$\begin{aligned}|X_i^{k+1} - X_j^{k+1}|^2 &= |X_i^k - X_j^k|^2 \\ &+ 2(X_i^k - X_j^k)[\epsilon_i^k(\hat{X}_i - X_i) + \epsilon_i^k(\hat{X}_j - X_j)] \\ &+ |\epsilon_i^k(\hat{X}_i - X_i) - \epsilon_i^k(\hat{X}_j - X_j)|^2\end{aligned}$$

Assume $\frac{\rho_{i,j}^k}{(2M\Delta^2)^2} < \epsilon_i^k \leq \frac{\mu_{i,j}^k}{(2M\Delta^2)^2}, \frac{\rho_{j,i}^k}{(2M\Delta^2)^2} < \epsilon_j^k \leq \frac{\mu_{j,i}^k}{(2M\Delta^2)^2}$ *from definition of* $\mu_{i,j}^k$ *it follows That the distance between agent* i *and* j *decrease.*

Considering $Q := |X_i^{k+1} - X_j^{k+1}|^2 - |X_i^k - X_j^k|^2$

$$\begin{aligned}Q &= 2(X_i^k - X_j^k)^T[\epsilon_i^k(\hat{X}_i - X_i) + \epsilon_j^k(\hat{X}_j - X_j)] \\ &+ |\epsilon_i^k(\hat{X}_i - X_i) - \epsilon_j^k(\hat{X}_j - X_j))|^2\end{aligned}$$

By construction of $\mu_{i,j}$, $-2(X_i^k - X_j^k)^T[\epsilon_i^k(\hat{X}_i - X_i) + \epsilon_j^k(\hat{X}_j - X_j)] \geq |\epsilon_i^k(\hat{X}_i - X_i) - \epsilon_j^k(\hat{X}_j - X_j))|^2$

Now $\epsilon_i^k \leq \frac{\mu_{i,j}^k}{\Delta}, \epsilon_j^k \leq \frac{\rho_{j,i}^k}{\Delta}$

$$\begin{aligned}|X_i^{k+1} - X_j^{k+1}| &= |X_i^k - X_j^k + \epsilon_i^k(\hat{X}_i^k - X_i^k)| \\ &+ \epsilon_j^k|\hat{X}_j^k - X_j^k|\end{aligned}$$

from (4.7) *we have* $|X_i^k - X_j^k| \geq |X_i^k - X_j^k + \epsilon_i^k(\hat{X}_i^k - X_i^k)|$ *and from* (4.6)

$$|X_i^{k+1} - X_j^{k+1}| \leq |X_i^k - X_j^k| + (\Delta - |X_i^k - X_j^k|)$$

$|X_i^{k+1} - X_j^{k+1}| \leq \Delta$

The same apply assuming $\epsilon_i^k \leq \frac{\rho_{i,j}^k}{\Delta}$ $\epsilon_j^k \leq \frac{\rho_{j,i}^k}{\Delta}$ *Furthermore during between sampling instants* $\forall i$, $X_i(t) = \lambda X_i^k + (1-\lambda)X_i^{k+1}$ *with* $\lambda \in [0,1]$ *Therefore*

$$|X_i(t) - X_j(t)|^2 - |X_i^k - X_j^k|^2 \leq (\lambda^2 - 1)|X_i^k - X_j^k|^2 + (1-\lambda)^2|X_i^{k+1} - X_j^{k+1}|^2$$

$$|X_i(t) - X_j(t)|^2 - |X_i^k - X_j^k|^2 \leq (1-\lambda)|X_i^{k+1} - X_j^{k+1}|^2 - |X_i^k - X_j^k|^2$$

Thus the range connectivity is not lost between sampling instant.

2) Convergence to average consensus

We consider the subset of the agents

$$i^* = argmax_{i\in V}|X_i^k - X_{eq}|$$

that are the farthest from the average consensus at time k For those agents from 4.3.4 $(X_{i^*}^k - X_j^k)^T(\hat{X}_{i^*}^k - X_{i^*}^k) \leq -|X_{i^*}^k - X_j^k|(\frac{|X_{i^*}^k - X_j^k|}{2} - |\hat{X}_{i^*}^k - X_{eq}^k|)$.
Using lemma 4.3.2 it is clear that $|\hat{X}_{i^*}^k - X_{eq}^k|$ *goes to* 0 *and there is* $k^* \geq \frac{-2ln(4M)}{ln(1-\epsilon\lambda_2(G))}$ *such that*

$$\forall k > k^*; \forall j \in N_{i^*}\ |\hat{X}_{i^*}^k - X_{eq}^k| < \Delta/4$$

So considering $2(|X_i^k - \hat{X}_i^k| + 1) < 2(M\Delta + 1)$
if $|X_{i^*}^k - X_j^k| < \Delta/2$ *then* $\rho_{i^*,j}^k \geq \frac{\Delta}{2}, \forall j \in N_{i^*}$ *if* $|X_{i^*}^k - X_j^k| \geq \Delta/2$ *then* $\mu_{i^*,j}^k \geq \frac{\Delta^2}{4}, \forall j \in N_{i^*}$

$$\epsilon_{i^*}^k = \frac{1}{(2M\Delta)^2}\min_{j\in N_{i^*}}\{\max(\mu_{i^*,j}^k, \rho_{i^*j}^k)\}.$$

Therefore

$$\frac{1}{2} \geq \epsilon_{i^*}^k \geq c$$

with $c = \min\frac{1}{(2M\Delta)^2}\{\frac{\Delta}{2}, \frac{\Delta^2}{4}\}$ *Since for all agents*
$|X_i^{k+1} - X_{eq}| < |X_i^k - X_{eq}|(1 - \epsilon_i^k) + \varepsilon_i^k|X_{eq} - \hat{X}_i^k|$
and for agents i^*, ϵ^i *is lower bounded since the subset of the agents*

$$i^* = argmax_{i\in V}|X_i^k - X_{eq}|$$

is never empty convergence follows. Furthermore for all $k > k^*$
Considering the worst case (i.e. agent $i*$ *is active at time k and does not move for* $M-1$ *time steps)*

$$|X_i^{k+M} - X_{eq}| < |X_i^k - X_{eq}|(1-c) + 2M^2\Delta(1 - \epsilon\lambda_2(\mathcal{L}))^k \tag{4.12}$$

denoting $\mu = \max(1 - \epsilon\lambda_2(\mathcal{L})^M, 1-c)$
From (4.12) one has an upper bound on

$$||(X_i^{k^*+M} - X_{eq}, \hat{X}_i^{k^*+M} - X_{eq}|| \leq ||(X_i^{k^*+jM} - X_{eq}, \hat{X}_i^{k^*+jM} - \hat{X}_{eq}||C\mu^j$$

leading to $|X_i^{k^*+jM} - X_{eq}| < 2CM\Delta(1-\mu)^j$

Theorem 4.3.7 *if* $0 < \epsilon < \frac{1}{\max_{i\in V}|N_i|}$, $\hat{X}^0 = X^0$ *and* $\max_{i\in V, j\in N_i}||X_i^0 - X_j^0|| \leq \Delta$ *then (4.3a) converge to the AC while preserving* Δ*-connectivity. Furthermore for all* $k > \frac{-2ln(4M)}{ln(1-\epsilon\lambda_2(G))}$ *the convergence speed is exponential*

4.3.3 Control formation

The proposed scheme where a consensus is reached on consensus estimator serve as a reference trajectory for agents in the network can be adapted to reach a final formation were at equilibrium $\lim_{k\to\infty} X_i^k = X_{eq} + \delta_i$ Therefore in order to have a feasible final configuration one require that

We also require global knowledge on the minimal margin between final distance and loss of range connectivity

$$\forall i \in V, \forall\, j \in\ N_i : \Delta - |\delta_i - \delta_j| > \beta_{ij} > 0 \tag{4.13}$$

with

$$\delta = \max_{i\in V} |\delta_i|; \beta = \min_{i\in V,\ j\in N_i} \beta_{ij}$$

known.

Definition *4.3.8 (Δ-Connectivity Preserving* Formation of a multi agent system*):* *Given a formation vector $\delta = (\delta_1 \dots \delta_M)^T$ a control U is said to solve the formation problem if U controlling the process $\dot{X} = U$ leads to*

$$\forall i\ \in V, \lim_{t\to\infty} X_i(t) = X_{eq} + \delta_i$$

and

$$\forall t \geq 0, \forall (i,j) \in E,\ \|X_i(t) - X_j(t)\| \leq \Delta$$

provided $\|X_i(0) - X_j(0)\| \leq \Delta, \forall (i,j) \in E$.

$$\forall i \in V, \forall\, j \in\ N_i\Delta - |\delta_i - \delta_j| > \beta_{ij} > 0 \tag{4.14}$$

with

$$\delta = \max_{i\in V} |\delta_i|; \beta = \min_{i\in V,\ j\in N_i} \beta_{ij}$$

known.

Modifying the previous observation and control scheme to take into account the final position requirement we have the following discrete dynamical system

$$\hat{X}_i^{k+1} = \hat{X}_i^k + \epsilon \sum_{j\in N_i} (\hat{X}_j^k - \hat{X}_i^k) \tag{4.15a}$$

$$X_i^{k+1} = X_i^k + \epsilon_i^k (\hat{X}_i^k - X_i^k + \delta_i) \tag{4.15b}$$

$$\epsilon_i^k = \min_{j\in N_i} \{\rho_{ij}^k\} \tag{4.15c}$$

$$\rho_{ij}^k = \frac{\Delta - |X_i^k - X_j^k|}{4(M\Delta + 1 + \delta)}$$

In the following $\hat{X} = (\hat{X}^T, \cdots, \hat{X}_M^T)^T$ and $\hat{X}^0 = X^0$

Theorem 4.3.9 *if $0 < \epsilon < \dfrac{1}{\max_{i\in V} |N_i|}$, and $\exists \gamma > 0, \max_{i\in V, j\in N_i} ||X_i^0 - X_j^0|| \leq \Delta - \gamma$ then (4.15a) ensure that $\lim_{k\to\infty} X_i = X_{eq} + \delta_i$ while preserving Δ-connectivity*

From lemma 4.3.2 $\lim_{k\to\infty} \hat{X}_i^k = X_{eq}$
We can then prove theorem 4.3.9.

Proof 4.3.10 *The proof of connectedness is identical as the previous one.*

$$|X_i^{k+1} - X_j^{k+1}| < |X_i^k - X_j^k| + 2\Delta/4 - \frac{|X_i^k - X_j^k|}{2} \; |X_i^{k+1} - X_j^{k+1}| < \Delta/2 + |X_i^k - X_j^k|/2$$

This prove connectedness and imply the relation $\varepsilon_i^{k+1} > \varepsilon_i^k/2$

To prove the convergence we first note that there is k^**. such that* $\forall k > k^*$ *given than for all*

$$\forall i \in V, \forall\, j \in \; N_i\Delta - |\hat{X}_i^k - \hat{X}_j^k + \delta_i - \delta_j| > \rho > 0$$

we also have the following relation

$$\begin{aligned}(X_i^{k+1} - \delta_i - X_{eq}) =& (X_i^{k+1} - \delta_i - X_{eq})(1 - \epsilon_i^k) \\ &+ \epsilon_i^k(\hat{X}_i - X_{eq})\end{aligned} \tag{4.16a}$$

We need to prove that ϵ_i *does not converge to zero.*

To do so let's consider

$X_i^{k+1} - X_j^{k+1} = X_i^k - X_j^k + \varepsilon_i^k(\hat{X}_i^k - X_i^k + \delta_i) - \varepsilon_j^k(\hat{X}_j^k - X_j^k + \delta_j)$

$X_i^{k+1} - X_j^{k+1} = (X_i^k - X_j^k)(1 - \varepsilon_i) + \varepsilon_i^k(\hat{X}_i^k - \hat{X}_j^k + \delta_i - \delta_j) - (\varepsilon_j^k - \varepsilon_i^k)(\hat{X}_j^k - X_j^k + \delta_j)$

Considering $\Delta - \max_{l\in N_i} |X_i^k - X_l^k| = \bar{\varepsilon}_i^k$

$\Delta - \max_{l\in N_j} |X_j^k - X_l^k| = \bar{\varepsilon}_j^k$ *Without loss of generality assume* $\bar{\varepsilon}_i^k \leq \bar{\varepsilon}_j^k$

For $k > k^*$ *combining the previous equations we have* $|X_i^{k+1} - X_j^{k+1}| \leq |X_i^k - X_j^k|(1 - \varepsilon_i) + \varepsilon_i^k(\Delta - \rho) + \bar{\varepsilon}_i^k - \bar{\varepsilon}_j^k$

$$|X_i^{k+1} - X_j^{k+1}| \leq (\Delta - \bar{\varepsilon}_i^k)(1 - \varepsilon_i) + \varepsilon_i^k(\Delta - \rho) - \bar{\varepsilon}_i^k + \bar{\varepsilon}_j^k$$

$$|X_i^{k+1} - X_j^{k+1}| \leq \Delta - \bar{\varepsilon}_j^k + \bar{\varepsilon}_i^k\varepsilon_i^k - \varepsilon_i^k\rho$$

Then considering $\varepsilon^k = \min_{i\in V} \varepsilon_i^k$

if $\bar{\varepsilon}_i^k < \rho$ *then* $\bar{\varepsilon}_j^{k+1} > \bar{\varepsilon}_i^k$ *and* $\varepsilon_j^{k+1} > \varepsilon_i^k$ *or* $\bar{\varepsilon}_j^k \geq \rho$ *and then* $\bar{\varepsilon}_j^{k+1} \geq \bar{\varepsilon}_j^k/2$ *then* $\varepsilon_j^{k+1} > \frac{\rho}{2 * (4M(\Delta+1))}$ *Therefore for all* $k > k^*$, $\forall i \in V$ $\varepsilon^{k+1} \geq min(\varepsilon^{k*}, \frac{\rho}{2 * (4M(\Delta+1))})$

It follows from (4.16a) *the convergence of the control formation*

4.3.4 Simulation

In this section we will present simulation results of the proposed controller and compare it with two different control protocols. For all the controller the agreement protocol is decentralized. The proof we provide were made considering one dimension point for ease of notation however, we show our result for simple integrator of dimension 2. The classical linear control protocol [77] in discrete time and the Δ-connectivity preserving control protocol from [34] (continuous time)

The communication topology is given on Fig 4.2 The initial conditions are the following $X1 = (-31, 1)^T, X2 = (30, 5)^T, X3 = (-31, 2.2)^T, X4 = (-17, 0)^T, X5 = (-3, 0)^T, X6 = (10, -2)^T, X7 = (11, 0)^T, X8 = (8, 2)^T$ The barycentre is $b = (-10.375, 1.025)^T$. $\Delta = 15$, the simulation stop at time 25

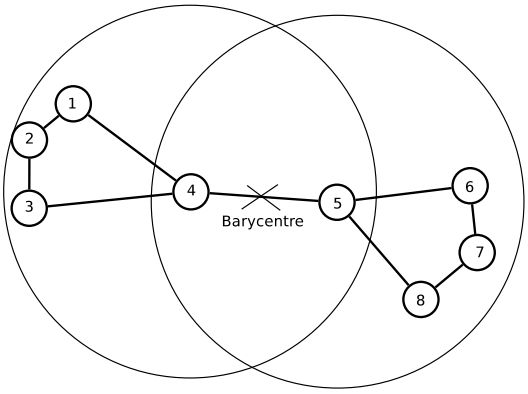

Figure 4.2: Communication topology and sensing radius.The lines represent the communication graph of the simulation and we represent the sensing/communication range of agent 4 and 5.

The Laplacian matrix of the example is :

$$\mathcal{L}(G) = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

with eigenvalues algebraic multiplicity $\lambda_2(G) = 0.29$

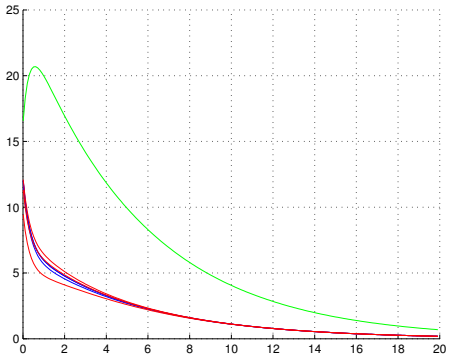

Figure 4.3: Evolution of distance between agent with controller [77]. Blue curve distance between agent 4 and agent 1, 2 and 3. Green distance between agent 4 and agent 5. Red distance between agent 5 and agent 6, 7 and 8

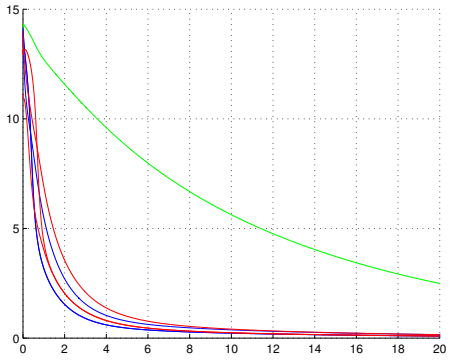

Figure 4.4: Evolution of distance between agent with controller [34] with $K = 100$. Blue curve distance between agent 4 and agent 1,2 and 3. Green distance between agent 4 and agent 5. Red distance between agent 5 and agent 6, 7 and 8

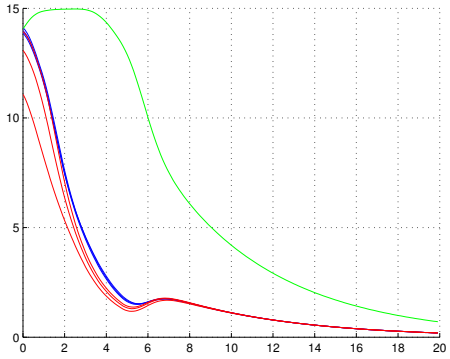

Figure 4.5: Evolution of distance between agent with the proposed controller . Blue curve distance between agent 4 and agent 1,2 and 3. Green distance between agent 4 and agent 5. Red distance between agent 5 and agent 6, 7 and 8

The graphic Fig 4.3 shows classical linear controller [77], the convergence rate is exponential but Δ-connectivity is not maintained as the first curve cross $\Delta = 15$.

We can see on Fig 4.4 applying the controller described in [34] that the connectivity is preserved. For a chosen gain the convergence rate is smaller Furthermore the convergence to a consensus is obtained, it is however not the AC.

Fig 4.5 shows the proposed control given by (4.3). The connectivity is preserved. Defining the control effort of each controller defined as

$$\int_0^{25} \sum_{i \in V} u_i^2(\tau) d\tau$$

The following chart will present the properties of the different control law tested in simulation. The proposed controller exhibit good properties in term of velocity and control effort, however there is a need of communication among agents that is not required for the other controllers.

	[77]	[34]	Our Controller
AC	Yes	No	Yes
Δ Connectivity	No	Yes	Yes
Need of communication	No	No	Yes
Controle effort	670	800	450
Error	0.01	2	0.03

We can simulate the formation of the multi-agents system following the algorithm we describe with $\delta_i = (0, 20)$ for $i = 1, 2, 3$, $\delta_4 = (010], \delta_5 = (00)$ $\delta_j = (0 - 10)$ for $j = 6, 7, 8$

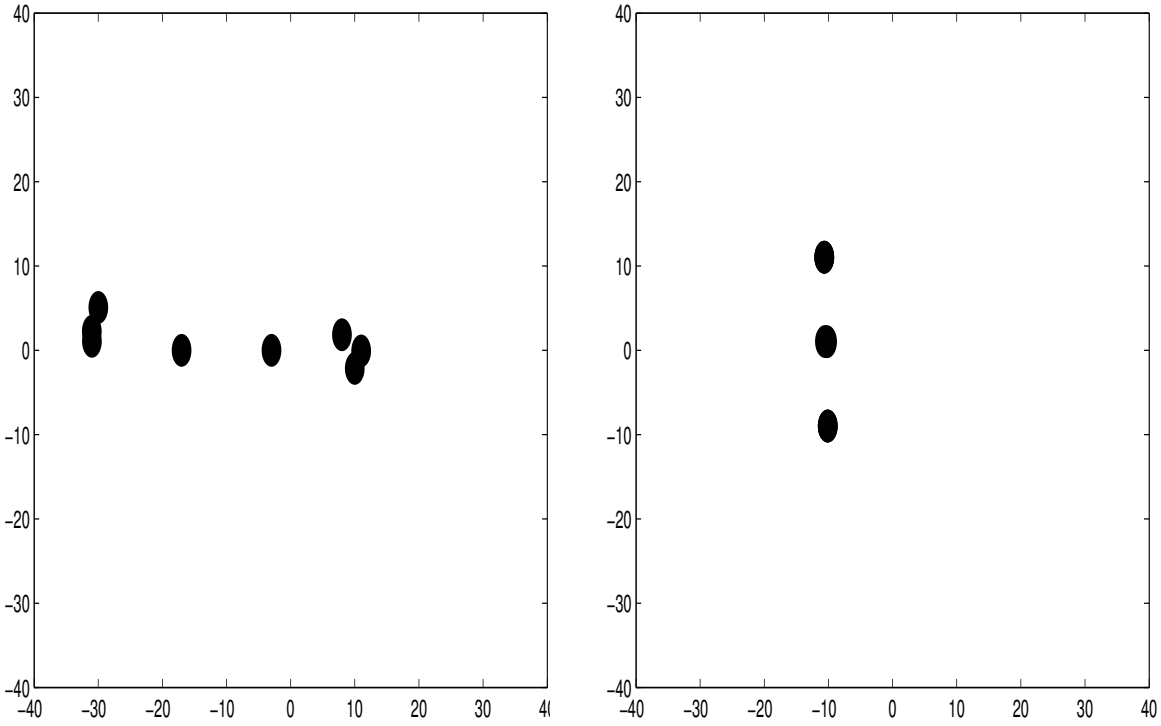

Figure 4.6: Initial and final configuration

4.3.5 Conclusions

We introduced a control policy based on an estimate of the AC performed online by agents able to communicate with their neighbours. The control action is designed to track this estimate. To ensure range connectivity the tracking policy tack into account either the distance to loss of range connectivity or the direction given by the tracking policy. The control input is piecewise constant and allows the agent to reach the consensus while preserving connectivity. The proposed method is easily implementable for single integrator of any finite dimension. Simulation are presented that illustrate our results, furthermore they suggest improvement in term of convergence speed and control effort when compared to existing policy. The method can also be applied to the problem

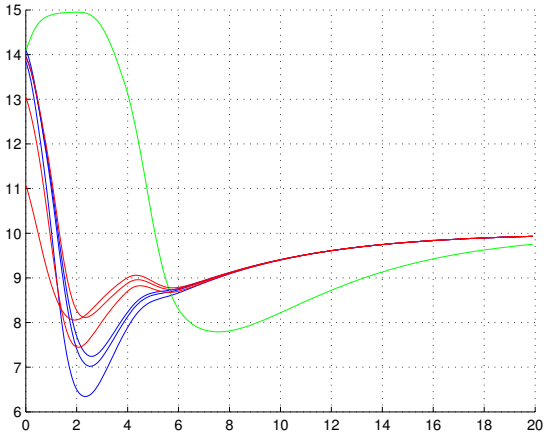

Figure 4.7: Evolution of distance between agent with the proposed controller for the formation with. Blue curve distance between agent 4 and agent 1,2 and 3. Green distance between agent 4 and agent 5. Red distance between agent 5 and agent 6, 7 and 8

of formation control provided agents have knowledge of the final position they want to achieve with respect to the consensus. An important fact is that even thought the proposed scheme was used to maintain Δ-conectivity it could be used to generate better trajectory in terms of control effort or robustness to perturbation. However the proposed protocol do not take into account network uncertainties and future work should focus on a protocol robustness with respect to such network imperfections.

Chapter 5

Active vehicle control

5.1 Introduction

In modern automobiles, the wide use of electronics allows the control of a number of functions. When considering safety, one of the aims is the reduction of the number of accidents. Active control actions represent an important tool for increasing safety. These actions impose forces/torques to the vehicle in order to track a (feasible and safe) reference behavior. These forces/torques have to be applied by means of tires, and the friction coefficient represents a crucial parameter in this respect. In fact, road condition is one of the most relevant parameter causing losses of driving control.

A precise evaluation of the tire–road friction coefficient would increase considerably the efficiency of control systems. There is a wide literature regarding either modeling [27]–[98] or estimation [15]–[89] of the tire–road friction coefficient. In particular, [16] proposes an estimation procedure to calculate lateral tire forces, vehicle sideslip angle, and road friction. In [20], an adaptive control is presented. In [81], an estimation scheme of the coefficient of road surface adhesion is presented, and a sliding mode observer driven by wheel speed measurements is proposed. A comparative study of a sliding–mode observer–based scheme and an adaptive observer–based scheme is presented in [83], where the effect of imperfect measurements are also taken into account.

To the best of our knowledge, High Order Sliding Mode (HOSM) was introduced in [39]. Since this seminal work, many papers have dealt with such technique [38]–[71]. HOSM techniques have been used first for smooth control systems and, more recently, for finite time differentiators. In [7] an interesting comparison of HOSM differentiators and high gain observers, with respect to measurement noise, has been done. Finally, in [58] HOSM is used to solve vehicle control problems. The main object of this section is to show that high order sliding mode methods can be successfully applied to a relevant industrial applications, in order to propose an improved solution to the estimation problem of the tire–road friction coefficient.

To this aim, in this work a rear–wheel drive vehicle equipped with Active Front Steering (AFS) and Rear Torque Vectoring (RTV) devices is considered. The AFS provides an additional steering angle over the driver steering angle, while the RTV gives an asymmetric left/right wheel torque on the rear axle. First, the estimation of the tire–road friction coefficient and the tire stiffness is addressed: A second order sliding mode algorithm, the well–known super–twisting algorithm, is proposed for estimating the tire–road friction coefficient in finite time as well as to track a desired trajectory for the vehicle. This esti-

mation technique does not depend on the type of model used to simulate the lateral force, and the parameter estimations will be used either in the reference generator, and in the control law. Then, a high order sliding mode controller is designed, in order to achieve the tracking of the desired lateral and yaw velocity references. This controller provides both good estimations and good tracking. The robustness properties of the controller with respect to model uncertainties is finally tested with simulations, making use of CarSim and a double steer maneuver. CarSim provides a vehicle dynamic behavior very close to a real automobile, and it is proved to represent accurately the vehicle dynamics. Moreover, CarSim is extensively validated with experimental testing and supported by automotive enterprises such as Ford, Chrysler, etc. The simplicity and the tested robustness render the proposed solution a good alternative for tire–road friction estimation in automotive applications, where the success of a control strategy has also to be judged on the basis of the feasibility and simplicity of its onboard implementation.

5.2 Mathematical model of a ground vehicle and problem formulation

We consider a vehicle equipped with AFS, which add an incremental steer angle δ_c on top of the driver's input δ_d, and RTV, which imposes a torque M_z by means of the rear axle. Since we consider a rear–wheel drive vehicle, these two actions are decoupled. In fact, the AFS control is actuated through the front tires, while the RTV is actuated through the rear tires.

The vehicle dynamics is very complex. A resulting model, having 6 degrees of freedom, is not suitable for designing a control action. Hence, it is usual to consider a simpler model, to be used to design the controller, whose validity and performance have to be then checked on more accurate models. For, the single track model is considered, which represents the essential dynamics of interest [56],

$$\begin{aligned} m(\dot{v}_x - v_y\omega_z) &= 0 \\ m(\dot{v}_y + v_x\omega_z) &= \mu(F_{yf} + F_{yr}) \\ J_z\dot{\omega}_z &= \mu(F_{yf}l_f - F_{yr}l_r) + M_z \end{aligned} \tag{5.1}$$

where m, J_z are the vehicle mass and inertia momentum, l_f, l_r are the front and rear vehicle length, v_x, v_y are the longitudinal, lateral velocities of the vehicle center of mass, and ω_z is the yaw rate. Moreover, μ is the maximum tire–road friction coefficient, M_z is the RTV moment, F_{yf}, F_{yr} are the tire front and rear lateral forces, normalized with respect to μ. This model, nonlinear due to the forces F_{yf}, F_{yr}, is very used in the literature and in applications since, despite its simplicity, it well captures the major characteristics of a real vehicle, such as the steady state and dynamic responses of the yaw rate, lateral acceleration, lateral velocity [56]. The validity of this model, used to design the control law, is shown by the simulation results of the resulting controller to CarSim, which provides a vehicle dynamic behavior very close to a real automobile.

The front/rear lateral forces

$$F_{yf} = F_{yf}(\alpha_f), \quad F_{yr} = F_{yr}(\alpha_r)$$

depend on the front/rear tire slip angles (rad)

$$\alpha_f = \delta - \frac{v_y + l_f\omega_z}{v_x}, \quad \alpha_r = -\frac{v_y - l_r\omega_z}{v_x}$$

with $\delta = \delta_d + \delta_c$ the road wheel angle (rad), sum of the driver angle δ_d (rad) and the AFS angle δ_c (rad). The drive angle δ_d is assumed at least continuously differentiable with respect to time.

Popular tire models considered in the literature is the Pacejka's model [78], in which some experimental parameters appear. Also interesting is the Burckardt model [27], which presents less parameters, and hence it is appealing when parameter estimation algorithms in the controller are considered. In this section we will not consider any particular tire model, since the same approach can be adapted to models in which the lateral tire forces can be written as

$$F_{yf}(\alpha_f) = D_f\varphi_f(\alpha_f), \qquad F_{yr}(\alpha_r) = D_r\varphi_r(\alpha_r) \tag{5.2}$$

with D_f, D_r the tire stiffness coefficients, and where φ_f, φ_r are the normalized tire characteristics, which usually increase linearly with the slip angles α_f, α_r until a maximum value, and then decrease reaching asymptotic values for high slip angles, as shown in Fig. 5.1.a. They are odd functions, namely φ_f, φ_r are symmetric with respect to the origin. Hence, φ_f, φ_r reach a minimum for negative values of α_f, α_r, and then increase to an asymptotic value for high negative slip angles.

Since between the minimum and the maximum, the functions φ_f describing the automobile tire lateral force as function of the slip angle are invertible (see Fig. 5.1.a), and to avoid mathematical difficulties arising from the fact that the input δ_c appears inside a (possibly) quite complicated function, it is usual to consider as AFS control the difference

$$\Delta_c = \varphi_f(\alpha_f) - \varphi_f(\alpha_{f0}), \qquad \alpha_{f0} = \delta_d - \frac{v_y + l_f\omega_z}{v_x}. \tag{5.3}$$

For a given value φ°, the real input δ_c can be determined as follows

$$\delta_c = \begin{cases} -\delta_d + \dfrac{v_y + l_f\omega_z}{v_x} + \varphi_f^{-1}(\bar{\Delta}_c) & \text{if } |\bar{\Delta}_c| \le \varphi_{f,\max} \\ -\delta_d + \dfrac{v_y + l_f\omega_z}{v_x} + \alpha_{f,\max} & \text{otherwise} \end{cases} \tag{5.4}$$

namely inverting the function φ_f up to the tire maximum point $\alpha_{f,\max}$, and saturating the inverse function elsewhere.

With the convention (5.4), equations (5.1) can be rewritten in the form

$$\begin{aligned} \dot{v}_x &= v_y\omega_z \\ \dot{v}_y &= -v_x\omega_z + \frac{1}{m}\Big(\theta_f\varphi_f(\alpha_{f0}) + \theta_r\varphi_r(\alpha_r)\Big) + \frac{\theta_f}{m}\Delta_c \\ \dot{\omega}_z &= \frac{1}{J_z}\Big(\theta_f\varphi_f(\alpha_{f0})l_f - \theta_r\varphi_r(\alpha_r)l_r\Big) + \frac{\theta_f l_f}{J_z}\Delta_c + \frac{1}{J_z}M_z \end{aligned} \tag{5.5}$$

where

$$\theta_f = \mu D_f, \qquad \theta_r = \mu D_r \tag{5.6}$$

are the products of the tire–road friction coefficient with the tire stiffness coefficients.

The control aim is to track some physically acceptable signals $v_{y,\text{ref}}$, $\omega_{z,\text{ref}}$ of lateral and yaw velocities. Hence, the *control problem* is to determine Δ_c, M_z so that the tracking errors

$$e_{v_y} = v_y - v_{y,\text{ref}}, \qquad e_{\omega_z} = \omega_z - \omega_{z,\text{ref}} \tag{5.7}$$

tend to zero asymptotically and uniformly.

In the following, we will assume that $v_{y,\text{ref}}$, $\omega_{z,\text{ref}}$ are bounded signals with bounded derivatives $\dot{v}_{y,\text{ref}}, \dot{\omega}_{z,\text{ref}}$. To this aim, it is possible to introduce the dynamics of a "reference vehicle", with $v_{x,\text{ref}} = v_x$ and [21], [22]

$$\begin{aligned}
\dot{v}_{y,\text{ref}} &= -v_x\omega_{z,\text{ref}} + \frac{\mu}{m}\Big(F_{yf,\text{ref}}(\alpha_{f0,\text{ref}}) + F_{yr,\text{ref}}(\alpha_{r,\text{ref}})\Big) \\
\dot{\omega}_{z,\text{ref}} &= \frac{\mu}{J_z}\Big(F_{yf,\text{ref}}(\alpha_{f0,\text{ref}})l_f - F_{yr,\text{ref}}(\alpha_{r,\text{ref}})l_r\Big) \\
\alpha_{f0,\text{ref}} &= \delta_d - \frac{v_{y,\text{ref}} + l_f\omega_{z,\text{ref}}}{v_x} \\
\alpha_{r,\text{ref}} &= -\frac{v_{y,\text{ref}} - l_r\omega_{z,\text{ref}}}{v_x}.
\end{aligned}$$

Usually $F_{yf,\text{ref}}$, $F_{yr,\text{ref}}$ can be taken to resemble the real characteristics, "shaping" the curves to avoid unwanted behaviors of the reference vehicle, e.g. a spin. This is essential, since otherwise the controller would impose an unwanted behavior, which is clearly undesirable for the driver. Hence, they can be chosen as

$$\begin{aligned}
F_{yf,\text{ref}}(\alpha_{f0,\text{ref}}) &= D_j\varphi_{f,\text{ref}}(\alpha_{f0,\text{ref}}) \\
F_{yr,\text{ref}}(\alpha_{r,\text{ref}}) &= D_r\varphi_{r,\text{ref}}(\alpha_{r,\text{ref}})
\end{aligned}$$

with $\varphi_{j,\text{ref}}$ strictly increasing functions. Therefore, the reference generator is given by

$$\begin{aligned}
\dot{v}_{y,\text{ref}} &= -v_x\omega_{z,\text{ref}} + \frac{1}{m}\Big(\theta_f\varphi_{f,\text{ref}}(\alpha_{f0,\text{ref}}) + \theta_r\varphi_{r,\text{ref}}(\alpha_{r,\text{ref}})\Big) \\
\dot{\omega}_{z,\text{ref}} &= \frac{1}{J_z}\Big(\theta_f\varphi_{f,\text{ref}}(\alpha_{f0,\text{ref}})l_f - \theta_r\varphi_{r,\text{ref}}(\alpha_{r,\text{ref}})l_r\Big)
\end{aligned} \tag{5.8}$$

with initial values $v_{y,\text{ref}}(0) = 0$, $\omega_{z,\text{ref}}(0) = 0$, corresponding to the case of a reference vehicle going straight at the initial time, i.e. when the maneuver has not started yet. Note that the approach presented hereinafter can be used also with different reference generators, e.g. static reference generators. Note also in (5.8) the presence of the tire–road friction coefficient μ, through θ_f, θ_r, also used in (5.5). In fact, to avoid to impose behaviors which could results to be impossible to track, due to the finite lateral force that can be exerted by the tires, the "reference" coefficients θ_f, θ_r should be equal to the real one. This poses the problem of their estimation.

5.3 A Super–Twisting Controller for Active Control of a Ground Vehicle

The Super–Twisting Controller (STC) introduced in [70] is widely used for control, observation and robust exact differentiation. The STC represents a second order sliding

mode controller. To briefly recall the basic aspects of STC, let us consider the system $\dot{x}_1 = u$, $x_1, u \in \mathbb{R}$. One can consider the following dynamic feedback

$$\begin{aligned} u &= -\lambda_1 |x_1|^{1/2} \operatorname{sign}(x_1) + x_2 \\ \dot{x}_2 &= -\lambda_2 \operatorname{sign}(x_1) \end{aligned} \tag{5.9}$$

where $x_2 \in \mathbb{R}$ is the controller state, and $\lambda_1, \lambda_2 > 0$ are gains. The feedback system is eventually given by

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 |x_1|^{1/2} \operatorname{sign}(x_1) + x_2 \\ \dot{x}_2 &= -\lambda_2 \operatorname{sign}(x_1). \end{aligned} \tag{5.10}$$

An important property of (5.10), proved in [70], is that the origin $(x_1, x_2) = (0, 0)$ is finite–time stable, namely it is reached in finite time.

5.3.1 The Case of Perfect Parameter Knowledge

In the following we will use the STC (5.9) to solve the control problem, under the hypothesis of perfect knowledge of the parameters appearing in the vehicle model. From (5.7), (5.5), (5.8) the tracking error dynamics are

$$\begin{aligned} \dot{e}_{v_y} &= -v_x e_{\omega_z} + \frac{1}{m}(\theta_f e_f + \theta_r e_r) + \frac{\theta_f}{m}\Delta_c \\ \dot{e}_{\omega_z} &= \frac{1}{J_z}(\theta_f l_f e_f - \theta_r l_r e_r) + \frac{\theta_f l_f}{J_z}\Delta_c + \frac{1}{J_z} M_z \end{aligned} \tag{5.11}$$

with

$$\begin{aligned} e_f(\alpha_{f0}, \alpha_{f0,\mathrm{ref}}) &= \varphi_{yf}(\alpha_{f0}) - \varphi_{yf,\mathrm{ref}}(\alpha_{f0,\mathrm{ref}}) \\ e_r(\alpha_r, \alpha_{r,\mathrm{ref}}) &= \varphi_{yr}(\alpha_r) - \varphi_{yr,\mathrm{ref}}(\alpha_{r,\mathrm{ref}}). \end{aligned} \tag{5.12}$$

Making use of (5.9), one can consider the following STC

$$\begin{aligned} \Delta_c &= \frac{m}{\theta_f}\Big(-\lambda_{11} |e_{v_y}|^{1/2} \operatorname{sign}(e_{v_y}) + \chi_1 \Big) + \frac{m v_x}{\theta_f} e_{\omega_z} - e_f - \frac{\theta_r}{\theta_f} e_r \\ \dot{\chi}_1 &= -\lambda_{12} \operatorname{sign}(e_{v_y}) \\ M_z &= J_z \Big(-\lambda_{21} |e_{\omega_z}|^{1/2} \operatorname{sign}(e_{\omega_z}) + \chi_2 \Big) - (\theta_f l_f e_f - \theta_r l_r e_r) - \theta_f l_f \Delta_c \\ &= J_z \Big(-\lambda_{21} |e_{\omega_z}|^{1/2} \operatorname{sign}(e_{\omega_z}) + \chi_2 \Big) - m l_f \Big(-\lambda_{11} |e_{v_y}|^{1/2} \operatorname{sign}(e_{v_y}) + \chi_1 \Big) \\ &\quad - m v_x l_f e_{\omega_z} + \theta_r (l_f + l_r) e_r \\ \dot{\chi}_2 &= -\lambda_{22} \operatorname{sign}(e_{\omega_z}) \end{aligned} \tag{5.13}$$

with $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22} > 0$ chosen to ensure finite time convergence to the origin of the resulting closed–loop error vehicle dynamics (5.11), (5.13)

$$\begin{aligned} \dot{e}_{v_y} &= -\lambda_{11} |e_{v_y}|^{1/2} \operatorname{sign}(e_{v_y}) + \chi_1 \\ \dot{\chi}_1 &= -\lambda_{12} \operatorname{sign}(e_{v_y}) \\ \dot{e}_{\omega_z} &= -\lambda_{21} |e_{\omega_z}|^{1/2} \operatorname{sign}(e_{\omega_z}) + \chi_2 \\ \dot{\chi}_2 &= -\lambda_{22} \operatorname{sign}(e_{\omega_z}). \end{aligned}$$

Note that its structure is that of equations (5.10). This ensures that the error is finite–time stable [70].

Remark 5.3.1 *The terms in* (5.13)

$$\dot{\chi}_1 = -\lambda_{12}\,\mathrm{sign}(e_{v_y}), \quad \dot{\chi}_2 = -\lambda_{22}\,\mathrm{sign}(e_{\omega_z})$$

are the integral actions on the sign of the errors e_{v_y}, e_{ω_z}, *respectively.* ◇

Remark 5.3.2 *The performance offered by the STC* (5.13) *can be compared with the following PI–like controller*

$$\begin{aligned}
\dot{I}_v &= e_{v_y} \quad \dot{I}_\omega = e_{\omega_z} \\
\Delta_c^\circ &= -\frac{m}{\theta_f}(k_{11}e_{v_y} + k_{10}I_v) + \frac{mv_x}{\theta_f}e_{\omega_z} - e_f - \frac{\theta_r}{\theta_f}e_r \\
M_z^\circ &= -J_z(k_{21}e_{\omega_z} + k_{20}I_\omega) - (\theta_f l_f e_f - \theta_r l_r e_r) - \theta_f l_f \Delta_c^\circ \\
&= ml_f(k_{11}e_{v_y} + k_{10}I_v) - J_z(k_{21}e_{\omega_z} + k_{20}I_\omega) - mv_x l_f e_{\omega_z} + \theta_r(l_f + l_r)e_r
\end{aligned} \tag{5.14}$$

$\theta_f \neq 0$, *where positive gains* $k_{10}, k_{11}, k_{20}, k_{21}$ *ensure the global exponential closed–loop stability of the error dynamics*

$$\begin{aligned}
\ddot{e}_{v_y} + k_{11}\dot{e}_{v_y} + k_{10}e_{v_y} &= 0 \\
\ddot{e}_{\omega_z} + k_{21}\dot{e}_{\omega_z} + k_{20}e_{\omega_z} &= 0.
\end{aligned} \tag{5.15}$$

Note that (5.14) *has the same dimension of the STC* (5.13). *PI controllers are usually preferred in industrial applications due to theirs good performances in terms of asymptotic stability and precision in steady–state. They are also easier to tune with respect to a PIDs. In the present case, it is easy to see that derivative actions* $k_{12}\dot{e}_{v_y}$, $k_{22}\dot{e}_{\omega_z}$ *for* e_{v_y} *and* e_{ω_z} *would not change the result, since one would get again the error dynamics* (5.15) *with gains* $\bar{k}_{10} = k_{10}/(1+k_{12})$, $\bar{k}_{11} = k_{11}/(1+k_{12})$, $\bar{k}_{20} = k_{20}/(1+k_{22})$ $\bar{k}_{21} = k_{21}/(1+k_{22})$, *at the place of* k_{10}, k_{11}, k_{20}, k_{21}. ◇

5.3.2 The Case of Parameter Uncertainties

In this section one considers the fact that, in practice, the model parameters are known up to a certain precision. In particular, the tire–road friction coefficient may vary suddenly due to a change of road conditions. More in general, also the stiffness coefficients D_f, D_r in (5.6) may vary, due to changes in the tire conditions, as well as the mass m and the inertia J_z with respect to nominal design values. Finally, parameters appearing in the expressions of the normalized tire characteristics φ_f, φ_r may vary as well.

In order to estimate the parameters θ_f, θ_r in (5.6) a super–twisting estimator [45] is used hereinafter, while the perturbations induced by the variations of the further parameters will be dominated by the robustness characteristic of the STC (5.13). As already pointed out for the STC (5.13), the main advantage of the super–twisting technique is that it ensures finite time convergence.

To illustrate the basic aspects of the sliding super–twisting sliding mode estimator, let us consider the system

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + \theta \\
y &= x_1
\end{aligned}$$

with x_1 the state, $f(x_1)$ and $\dot{f}(x_1)$ bounded, θ the unknown parameter [70], [7]. One assumes that θ and $\dot{\theta}$ are bounded. Under these hypotheses, it is possible to design an estimator of θ using the super–twisting estimator

$$\begin{aligned}\dot{\xi}_1 &= f(y) - \gamma_1|\xi_1 - y|^{1/2}\,\mathrm{sign}(\xi_1 - y) + \xi_2\\ \dot{\xi}_2 &= -\gamma_2\,\mathrm{sign}(\xi_1 - y)\end{aligned} \tag{5.16}$$

with ξ_1, ξ_2 the estimates of x_1, θ, and $\gamma_1, \gamma_2 > 0$. Let us define the observation errors $e = \xi_1 - y$ and $e_\theta = \xi_2 - \theta$. The observer gains γ_1, γ_2 are defined in function of the bounds for $f(x), \dot{f}(x), \theta$ and $\dot{\theta}$. Note that, since $f(x)$ is function of measured state only, which will be ensured bounded by the control, the hypothesis of boundedness of $f(x)$ could be removed. The observation error dynamics are given by

$$\begin{aligned}\dot{e} &= -\gamma_1|e|^{1/2}\,\mathrm{sign}(e) + e_\theta\\ \dot{e}_\theta &= -\gamma_2\,\mathrm{sign}(e) - \dot{\theta}\end{aligned}$$

so that ξ_2 is equal to θ after a finite times. It is worth noticing that in (5.16) the cancellation of $f(y)$ is performed. Hence, (5.16) is not a simple differentiator, and consequently the gains γ_1, γ_2 can be sensibly lower that in the case of simple differentiator.

In the present case, assuming ω_z measurable, a finite–time estimate of $\dot{\omega}_z$ is given by

$$\begin{aligned}\dot{\xi}_1 &= -\gamma_1|\xi_1 - \omega_z|^{1/2}\,\mathrm{sign}(\xi_1 - \omega_z) + \xi_2\\ \dot{\xi}_2 &= -\gamma_2\,\mathrm{sign}(\xi_1 - \omega_z)\\ \widehat{\dot{\omega}}_z &= \xi_2.\end{aligned} \tag{5.17}$$

The estimate $\widehat{\dot{\omega}}_z = \xi_2$ in (5.17) can be used along with the measurement of the lateral acceleration a_y, commonly available for in modern automobiles, to obtain the desired parameter estimation. Since $a_y = \dot{v}_y + v_x\omega_z$, from (5.5) one gets

$$\begin{aligned}ma_y &= \theta_f\varphi_f(\alpha_f) + \theta_r\varphi_r(\alpha_r)\\ J_z\dot{\omega}_z - M_z &= \theta_f\varphi_f(\alpha_f)l_f - \theta_r\varphi_r(\alpha_r)l_r\end{aligned} \tag{5.18}$$

with $\varphi_f(\alpha_f) = \varphi_f(\alpha_{f0}) + \Delta_c$, from (5.3). Making use of the estimate (5.17), converging to $\dot{\omega}_z$ in finite time, one finally obtains the parameter estimation

$$\begin{pmatrix}\hat{\theta}_f\\ \hat{\theta}_r\end{pmatrix} = \frac{1}{\mathcal{D}}\begin{pmatrix}\varphi_f(\alpha_f) & \varphi_r(\alpha_r)\\ l_f\varphi_f(\alpha_f) & -l_r\varphi_r(\alpha_r)\end{pmatrix}\begin{pmatrix}ma_{y,m}\\ J_z\xi_2 - M_z\end{pmatrix} \tag{5.19}$$

where $\mathcal{D} = (l_f + l_r)\varphi_f(\alpha_f)\varphi_r(\alpha_r)$ is nonzero for $\varphi_f(\alpha_f), \varphi_r(\alpha_r) \neq 0$. Physically, this means that the estimate can be carried out only when the (front, rear) lateral forces excerpted by the tires are nonzero. This is typically satisfied (except in some time instants, i.e. in zero measure time sets) during a maneuver. The analysis of these special cases will be the object of future work, while here an approximated solution with a threshold on $|\mathcal{D}|$ has been used. Hence, this threshold determines when (5.19) is applied, avoiding excessively high peaks on the estimates. Finally, due to the possible measurement noise on the measurements of a_y, ω_z, in practice it is convenient to filter $\hat{\theta}_f, \hat{\theta}_r$ obtained with (5.19), by means of a low pass filter. The introduction of low–pass filters cancels the finite–time

convergence because the output information arrives through the filter, which introduces some delays. Hence, as common trade–off, the filter bandwidth must be chosen small enough to ensure noise attenuation, and large enough to ensure the closed loop stability. The reader is referred to [7] for a useful discussion about the relation between gains and noise in the context of high order differentiators.

The estimations $\hat{\theta}_f, \hat{\theta}_r$ can be used, along with the nominal parameters of the other parameters (denoted with a "zero" as in m_0, J_{z0}, etc), either in the reference generator (5.8), so obtaining

$$\begin{aligned} \dot{v}_{y,\mathrm{ref}} &= -v_x\omega_{z,\mathrm{ref}} + \frac{1}{m_0}\Big(\hat{\theta}_f\varphi_{f,\mathrm{ref}}(\alpha_{f0,\mathrm{ref}}) + \hat{\theta}_r\varphi_{r,\mathrm{ref}}(\alpha_{r,\mathrm{ref}})\Big) \\ \dot{\omega}_{z,\mathrm{ref}} &= \frac{1}{J_{z0}}\Big(\hat{\theta}_f\varphi_{f,\mathrm{ref}}(\alpha_{f0,\mathrm{ref}})l_f - \hat{\theta}_r\varphi_{r,\mathrm{ref}}(\alpha_{r,\mathrm{ref}})l_r\Big) \end{aligned} \tag{5.20}$$

or in the control law (5.13) yielding, along with (5.16), to the STC with estimated parameters

$$\begin{aligned} \dot{\xi}_1 &= -\gamma_1|\xi_1 - \omega_z|^{1/2}\,\mathrm{sign}(\xi_1 - \omega_z) + \xi_2 \\ \dot{\xi}_2 &= -\gamma_2\,\mathrm{sign}(\xi_1 - \omega_z) \\ \dot{\chi}_1 &= -\lambda_{12}\,\mathrm{sign}(e_{v_y}) \\ \dot{\chi}_2 &= -\lambda_{22}\,\mathrm{sign}(e_{\omega_z}) \\ \hat{\Delta}_c &= \frac{m_0}{\hat{\theta}_f}\Big(-\lambda_{11}|e_{v_y}|^{1/2}\,\mathrm{sign}(e_{v_y}) + \chi_1\Big) + \frac{m_0 v_x}{\hat{\theta}_f}e_{\omega_z} - e_{f0} - \frac{\hat{\theta}_r}{\hat{\theta}_f}e_{r0} \\ \hat{M}_z &= J_{z0}\Big(-\lambda_{21}|e_{\omega_z}|^{1/2}\,\mathrm{sign}(e_{\omega_z}) + \chi_2\Big) - m_0 l_f\Big(-\lambda_{11}|e_{v_y}|^{1/2}\,\mathrm{sign}(e_{v_y}) + \chi_1\Big) \\ &\quad - m_0 v_x l_f e_{\omega_z} + \hat{\theta}_r(l_f + l_r)e_r \end{aligned} \tag{5.21}$$

with e_{v_y}, e_{ω_z} as in (5.7), and e_{f0}, e_{r0} defined as

$$\begin{aligned} e_{f0} &= \varphi_{f0}(\alpha_{f0}) - \varphi_{f,\mathrm{ref}}(\alpha_{f0,\mathrm{ref}}) \\ e_{r0} &= \varphi_{r0}(\alpha_r) - \varphi_{r,\mathrm{ref}}(\alpha_{r,\mathrm{ref}}) \end{aligned} \tag{5.22}$$

where $\varphi_{f0}, \varphi_{r0}$ are the normalized tire characteristic with nominal parameters, namely the same functions φ_f, φ_r considered in (5.2) but with the nominal values of the parameters appearing in them.

Remark 5.3.3 *The STC (5.21) can be compared with the PI–based controller (5.14), with the estimates (5.19) and the constraint $\hat{\theta}_f \neq 0$*

$$\begin{aligned} \dot{I}_v &= e_{v_y} \quad \dot{I}_\omega = e_{\omega_z} \\ \hat{\Delta}_c^\circ &= -\frac{m_0}{\hat{\theta}_f}(k_{11}e_{v_y} + k_{10}I_v) + \frac{m_0 v_x}{\hat{\theta}_f}e_{\omega_z} - e_{f0} - \frac{\hat{\theta}_r}{\hat{\theta}_f}e_{r0} \\ \hat{M}_z^\circ &= m_0 l_f(k_{11}e_{v_y} + k_{10}I_v) - J_{z0}(k_{21}e_{\omega_z} + k_{20}I_\omega) - m_0 v_x l_f e_{\omega_z} + \hat{\theta}_r(l_f + l_r)e_{r0} \end{aligned} \tag{5.23}$$

with e_{f0}, e_{r0} as in (5.22). ◇

The remaining of this section will be devoted to highlight some difficulties that could arise during the transient, due to the fact that the estimation of θ_f, θ_r can not be instantaneous and/or perfect. For, let us rewrite the error dynamics considering (5.5), (5.20) and the control (5.21)

$$
\begin{aligned}
\dot{e}_{v_y} &= \frac{\theta_f}{\theta_f - \tilde{\theta}_f}\frac{m_0}{m}\Big(- \lambda_{11}|e_{v_y}|^{1/2}\,\mathrm{sign}(e_{v_y}) + \chi_1\Big) + E_1 \\
\dot{\chi}_1 &= -\lambda_{12}\,\mathrm{sign}(e_{v_y}) \\
\dot{e}_{\omega_z} &= \frac{J_{z0}}{J_z}\Big(- \lambda_{21}|e_{\omega_z}|^{1/2}\,\mathrm{sign}(e_{\omega_z}) + \chi_2\Big) + E_2 \\
\dot{\chi}_2 &= -\lambda_{22}\,\mathrm{sign}(e_{\omega_z}) \\
E_1 &= -v_x e_{\omega_z} + \frac{1}{m}\Big(\theta_f\varphi_f(\alpha_{f0}) + \theta_r\varphi_r(\alpha_r)\Big) + \frac{\theta_f}{m}\left(\frac{m_0 v_x}{\theta_f - \tilde{\theta}_f}e_{\omega_z} - e_{f0} - \frac{\theta_r - \tilde{\theta}_r}{\theta_f - \tilde{\theta}_f}e_{r0}\right) \\
&\quad - \frac{1}{m_0}\Big((\theta_f - \tilde{\theta}_f)\varphi_{f,\mathrm{ref}}(\alpha_{f0,\mathrm{ref}}) + (\theta_r - \tilde{\theta}_r)\varphi_{r,\mathrm{ref}}(\alpha_{r,\mathrm{ref}})\Big) \\
E_2 &= \frac{1}{J_z}\Big(\theta_f\varphi_f(\alpha_{f0})l_f - \theta_r\varphi_r(\alpha_r)l_r\Big) - \frac{1}{J_z}\Big((\theta_f - \tilde{\theta}_f)l_f e_{f0} - (\theta_r - \tilde{\theta}_r)l_r e_{r0}\Big) \\
&\quad + \frac{\tilde{\theta}_f l_f}{J_z}\hat{\Delta}_c - \frac{1}{J_{z0}}\Big((\theta_f - \tilde{\theta}_f)\varphi_{f,\mathrm{ref}}(\alpha_{f0,\mathrm{ref}})l_f - (\theta_r - \tilde{\theta}_r)\varphi_{r,\mathrm{ref}}(\alpha_{r,\mathrm{ref}})l_r\Big)
\end{aligned}
$$

where $\tilde{\theta}_f = \theta_f - \hat{\theta}_f$, $\tilde{\theta}_r = \theta_r - \hat{\theta}_r$. A sufficient condition ensuring a finite time convergence of the error dynamics is the existence of $b_{11}, b_{12}, b_{21}, b_{22}, c_1, c_2 > 0$ such that [45]

$$
\begin{aligned}
&0 < b_{11} \le \frac{\theta_f}{\theta_f - \tilde{\theta}_f}\frac{m_0}{m} \le b_{12} \\
&0 < b_{21} \le \frac{J_{z0}}{J_z}\frac{m_0}{m} \le b_{22} \\
&|\dot{E}_1| < c_1, \quad |\dot{E}_2| < c_2.
\end{aligned}
\tag{5.24}
$$

Assuming that the model parameters have bounded variations, with bounded derivatives, conditions (5.24) are clearly verified.

Since E_1, E_2 depend on $\tilde{\theta}_f$, $\tilde{\theta}_r$, it is clear that the errors E_1, E_2 influence the vehicle's controlled dynamics. As already noted, in practice the signals $\hat{\theta}_f, \hat{\theta}_r$ are filtered in order to obtain smoother signals. This smoothing introduces a delay in the estimation. Since the controller gains could be high, this estimation could determine a deterioration of the transient in the tracking. Therefore, attention has to be posed for the correct choice of these gains and those used in the filter.

5.4 Simulation results

The control law (5.21), (5.20), (5.19) has been applied to a vehicle characterized by $m = 1480$ kg, $J_z = 2386$ kg m^2, $l_f = 1.17$ m, $l_r = 1.43$ m. A challenging test maneuvers has been considered, given by a step steer of $\delta_{d,sw} = +100°$ of the steering wheel at $t = 1$ s, followed by a step steer of $\delta_{d,sw} = -100°$ at $t = 3$ s, and finally $\delta_{d,sw} = 0$ at $t = 5$ s (see Fig. 5.1.b). The ratio between the steering wheel angle $\delta_{d,sw}$ and δ_d is 16.

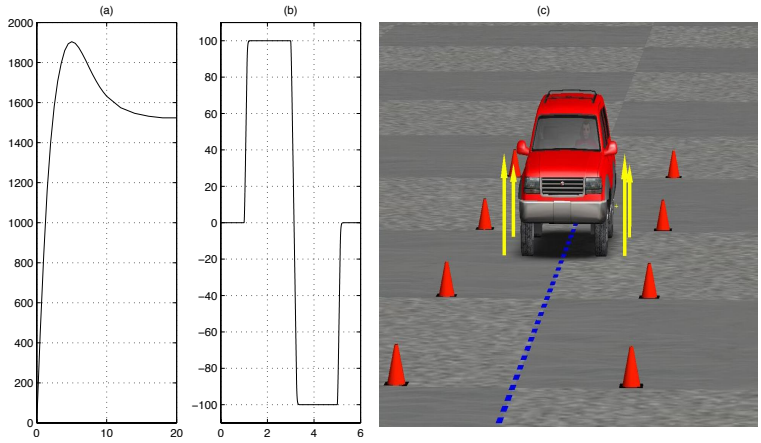

Figure 5.1: a) Typical characteristic of the lateral force F_y as function of the slip angle [N vs s]; b) Steering wheel angle δ_d in the double lane change maneuver [deg vs s]; c) CarSim Simulator.

This maneuver has been performed with an initial longitudinal velocity $v_x(0) = 28$ m/s. To make the maneuver more challenging, at $t = 3.5$ s there a change of the friction from $\mu = 0.9$ (dry road) to $\mu = 0.4$ (wet road) has been considered. A random variation has been superimposed to these values. The simulation has been performed using the CarSim software, with a nonlinear car model C–Class Hatchback provided by CarSim (see Fig. 5.1.c). The tire selected in CarSim has been the 245/40–R17, with a normal force of 1725 N. Independent suspensions on both axles have been used. The variables measured from CarSim were v_x, v_y, α_z, ω_z, a_x, and a_y. As far as the actuators are concerned, to have a more realistic situation, a saturation of 3° for δ_c, and a saturation of 8000 N m for M_z have been considered.

5.4.1 Super–Twisting Controller with Estimated Parameters

The controller gains have been set equal to $\lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 250$. A threshold $b = 0.1$ is considered for φ_f, φ_r to apply (5.19). A simple filter with $G(s) = 1/(1 + 10s)$ has been used to filter $\hat{\theta}_f, \hat{\theta}_r$.

Figs. 5.2–5.6 show the importance of parametric estimation. In fact, in the case of controller (5.13) with nominal parameter values (obtained from CarSim) $D_{f0} = D_f = 1904.02$ N, $D_{r0} = D_r = 1904.02$ N, $\mu_0 = 0.9$ the reference trajectories can not be correctly tracked (Figs. 5.2.a, 5.2.b, 5.3.a, 5.3.b). On the contrary, in the case of the controller (5.13) with parametric estimation (5.19) one obtains a satisfactory tracking, see Figs. 5.2.c, 5.2.d, 5.3.c, 5.3.d. Fig. 5.4 shows the parameters θ_f, θ_r, varying abruptly at $t = 3.5$ s, along with their estimation $\hat{\theta}_f, \hat{\theta}_r$.

5.4.2 STC versus PI–based Controller

In this section we compare the STC (5.13) with the PI–based controller (5.14). In order to have a good comparison, the control gains of the STC (5.13) have been fixed equal to $\lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 250$, while the controller gains for the PI–based controller (5.14) have been chosen equal to $k_{10} = k_{20} = 22.5$, $k_{11} = k_{21} = 18$. These values ensure

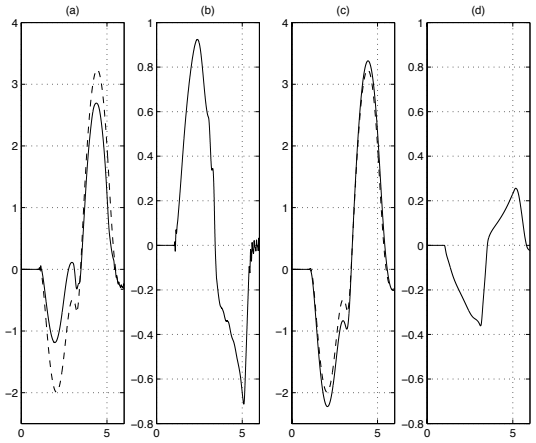

Figure 5.2: STC (5.13) with nominal parameters: a) v_y (solid) and reference $v_{y,\text{ref}}$ (dashed) [m/s vs s]; b) Tracking error $e = v_y - v_{y,\text{ref}}$. STC (5.13) with parametric estimation (5.19): c) v_y (solid) and reference $v_{y,\text{ref}}$ (dashed) [m/s vs s]; d) Tracking error $e = v_y - v_{y,\text{ref}}$.

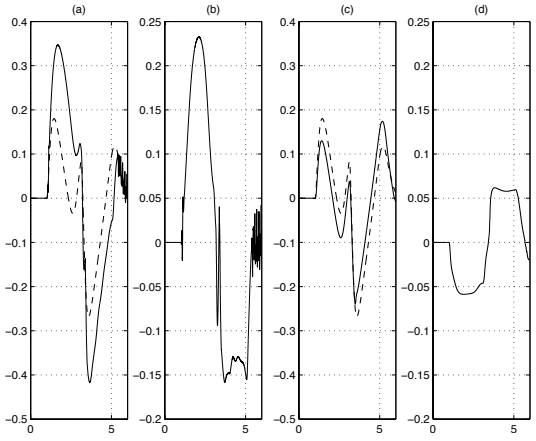

Figure 5.3: STC (5.13) with nominal parameters: a) ω_z (solid) and $\omega_{z,\text{ref}}$ (dashed) [rad/s vs s]; b) Tracking error $e = \omega_z - \omega_{z,\text{ref}}$. STC (5.13) with parametric estimation (5.19): c) ω_z [rad/s vs s]; d) Tracking error $e = \omega_z - \omega_{z,\text{ref}}$.

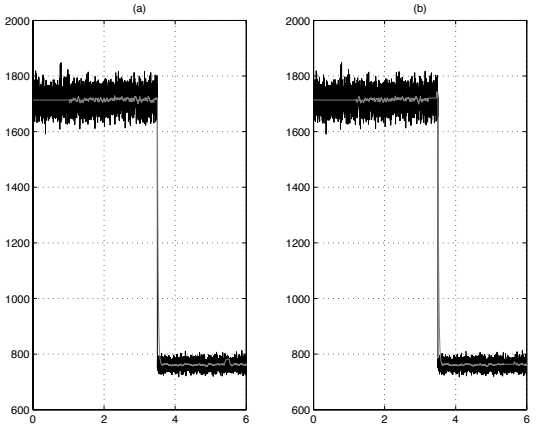

Figure 5.4: Vehicle parameters: a) Real θ_f (back) and estimated $\hat{\theta}_f$ (gray) [N vs s]; b) Real θ_r (black) and estimated $\hat{\theta}_r$ (gray) [N vs s].

comparable results for the ST and PI–based controllers in the case of absence of parameter variations.

First, we consider the case in which m, J_z are equal to the nominal values, while $\theta_f = \theta_{f0} = \mu_0 D_{f0}$, $\theta_r = \theta_{r0} = \mu_0 D_{r0}$. With both the ST controller (5.13) and the PI–based controller (5.14) the references are well tracked, see Figs. 5.5.a, 5.5.b and 5.6.a, 5.6.b. As can be noted, the tracking errors are small for both the controllers.

To test the robustness properties of these controllers, we consider a parameter variation with respect to the nominal ones, setting $m_0 = 0.81\, m$, $J_{z0} = 0.92\, J_z$, $\theta_f = \mu D_f$, $\theta_r = \mu D_r$, $D_f = 0.95\, D_{f0}$, $D_r = 0.97\, D_{r0}$. Moreover, since the function $\text{sign}(\cdot)$ is discontinuous in zero, and therefore very sensible to noise and quantization errors, the approximation $\text{sign}(x) \approx 2\arctan(100x)/\pi$ has been considered. It is worth noticing that a consequence of this approximation is the loss of the finite time convergence property. In fact, the approximation of the sign function by an arctan function replaces locally (i.e. around the sliding surface) the sliding mode behaviour by a high gain behaviour. However, this doesn't change the stability domain.

When dealing with parameter variations, the tracking behavior is better with the STC, due to its robustness with respect to matching perturbations. As shown in Figs. 5.5.c, 5.5.d and 5.6.c, 5.6.d, both controllers show good results but, nevertheless, we can note that the STC ensures a smaller tracking error.

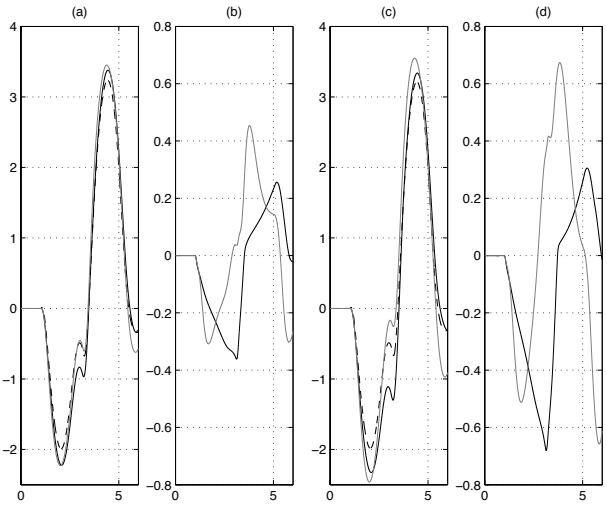

Figure 5.5: Case of nominal parameters in the controllers: a) v_y with the STC (5.13) (solid black) and with the PI–based controller (5.14) (solid gray), and reference $v_{y,\text{ref}}$ (dashed) [m/s vs s]; b) Tracking error $e = v_y - v_{y,\text{ref}}$ with the STC (5.13) (black) and with the PI–based controller (5.14) (gray). Case of real parameters in the controllers: c) v_y with the STC (5.13) (solid black) and with the PI–based controller (5.14) (solid gray), and reference $v_{y,\text{ref}}$ (dashed) [m/s vs s]; d) Tracking error $e = v_y - v_{y,\text{ref}}$ with the STC (5.13) (black) and with the PI–based controller (5.14) (gray).

5.5 Conclusions

In this chapter a nonlinear control law using estimated parameters has been designed. The parameter estimation and the controller design problems have been solved using the

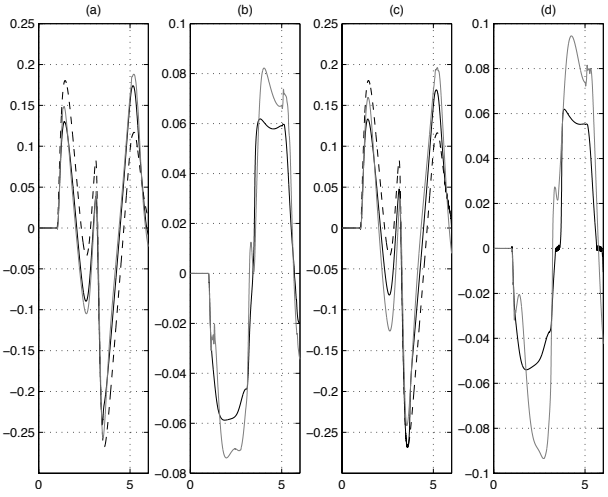

Figure 5.6: Case of nominal parameters in the controllers: a) ω_z with the STC (5.13) (solid black) and with the PI–based controller (5.14) (solid gray), and $\omega_{z,\text{ref}}$ (dashed) [rad/s vs s]; b) Tracking error $e = \omega_z - \omega_{z,\text{ref}}$ with the STC (5.13) (black) and with the PI–based controller (5.14) (gray). Case of real parameters in the controllers: c) ω_z with the STC (5.13) (solid black) and with the PI–based controller (5.14) (solid gray), and $\omega_{z,\text{ref}}$ (dashed) [rad/s vs s]; d) Tracking error $e = \omega_z - \omega_{z,\text{ref}}$ with the STC (5.13) (black) and with the PI–based controller (5.14) (gray).

high order sliding mode technique. The simulation results highlight the efficiency of the proposed approach. In particular, the control scheme takes particular advantage of the finite time parameter estimation. It is worth noting that simulation shows the robustness of the controller with respect to measurement noise and parameter uncertainties. In the future, the real time implementation of this controller will be investigated.

Chapter 6

Conclusions

6.1 General element of conclusion

In this work we have investigated various aspects regarding observation and estimation applied to Network Control Systems. The observation and estimation schemes are then used for control. Motivated by the ever greater integration of physical and computer process introducing both new opportunities and constraints. Those considerations has led us to study three related topics. First, an event–triggering scheme, for both sensors and actuators, has been considered for the observation and control of linear systems. The results show that event–triggering schemes are robust with respect to time–varying uncertainties. Moreover An impulsive observer has been designed, and a "separation principle" has been proved, stating that one can design separately an event–triggered impulsive observer, even for unstable systems, and an event–triggered controller relying on the availability of the full state. The resulting observer–based controller guarantees practical stability to the origin.

The observer–based stabilization of a class of non linear systems has also been studied. Sufficient conditions have been determined in terms of input–to–state stability, in order the ensure the existence of an event–triggering policy. It has been shown how to design an impulsive observer along with an event–triggering sampling policy to ensure the stabilization of the given class of non linear systems, and the validity of a separation principle.

In the second part of this thesis we have considered a network of autonomous agents, and we have showed how a decentralized estimator can solve the coordination problems. These results assumes communication among agents. The results exposed could be the object of further studies considering more complex class of systems and more challenging control and estimation task.

Considering a complex vehicle model, the tire–road friction coefficient is estimated using a super–twisting observer. The control action, for tracking of a given signal, is given by a super–twisting controller, and a comparison between the proposed control scheme and a classical PI controller is made using CarSim, a high–fidelity simulation environment. This work results could be extended to a network of collaborative agents exchanging information.

The topics considered here are relevant in the framework of Cyber-Physical Systems and it could be interesting to consider their junction as future work, where vehicles with complex dynamics and complex tasks would have to exchange information to enhance

their performance while minimizing the communication load. Indeed intelligent cars, eco-driving and platooning are already being consider by the industry yet many problems of theoretical nature remains.

6.2 Broader research direction

In a more general setting, the investigation effectuated and the ideas developed during this thesis could be useful in considering other classes of systems. Indeed, both network control systems and observation are ubiquitous. In neural network where massively parallel computation is performed a huge amount of resources is devoted to information exchange and event triggered paradigm could be a way of reducing this burden. Another interesting perspective is the application of the concept developed for network control systems in the area of biology, either for modelling or for estimation and control. In the automatic control community a growing amount of work is dedicated to biological system, to both observe and control biological systems and to design bio-inspired controller. An interesting area of investigation would then be event triggered modelling and control for biological systems. Finally, another very ambitious topic is the synthesis of bio-inspired even triggered policies for network control systems.

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