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Daniel Hauer

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Daniel Hauer. Problèmes d'évolution associés au p-laplacien: comportement asymptotique et non existence. Equations aux dérivées partielles [math.AP]. Soutenance, 2012. Français. NNT: . tel-01331203

HAL Id: tel-01331203

<https://hal.science/tel-01331203>

Submitted on 13 Jun 2016

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DISSERTATION

ZUR ERLANGUNG
DES DOKTORGRADES DR. RER. NAT.
DER FAKULTÄT FÜR MATHEMATIK UND WIRTSCHAFTS-
WISSENSCHAFTEN DER UNIVERSITÄT ULM
UND DES GRADES DOCTEUR EN MATHÉMATIQUES
DER UNIVERSITÄT LOTHRINGEN
VORGELEGT VON

Daniel Hauer

TITEL DER DISSERTATION:

**Evolutionsprobleme für den p -Laplace Operator:
asymptotisches Verhalten und Nichtexistenz**

/

**Problèmes d'évolution associés au p -laplacien:
comportement asymptotique et non existence**

TAG DER PROMOTION 18. DEZEMBER 2012 IN ULM VOR DER KOMMISSION:

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THÈSE DE DOCTORAT

SPÉCIALITÉ: MATHÉMATIQUES

PRÉSENTÉE PAR

Daniel Hauer

POUR OBTENIR LE TITRE DE

DOCTEUR DE L'UNIVERSITÉ DE LORRAINE

ET

DOCTEUR DR. RER. NAT. DE L'UNIVERSITÉ D'ULM

SUJET DE LA THÈSE:

**Problèmes d'évolution associés au p -laplacien:
comportement asymptotique et non existence**

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**Evolutionsprobleme für den p -Laplace Operator:
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SOUTENUE LE 18 DÉCEMBRE 2012 À ULM DEVANT LE JURY COMPOSÉ DE:

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Meinen Freunden Andi, Philippe, Magga, Domenico, Matze und Musti,
sowie meinem Lehrer Ralph gewidmet.

Abstract

This thesis is dedicated to the study of two subjects in the field of evolution problems associated with the p -Laplace operator. The first subject is concerned with the study of long time behavior of bounded solutions and the second subject is devoted to the study of nonexistence of positive nontrivial solutions.

The first chapter of this thesis is devoted to a general introduction to the p -Laplace operator and a résumé of this thesis. The first chapter is written in French.

Chapter 2 is dedicated to the study of convergence as the time $t \rightarrow +\infty$ of bounded solutions of evolution problems associated with the p -Laplace operator on a bounded interval. In this chapter, we prove that for every $1 < p \leq 2$ and for every continuous function f mapping from $[0, 1] \times \mathbb{R}$ to \mathbb{R} , which is Lipschitz continuous in the second variable, uniformly with respect to the first one, each bounded solution of the one-dimensional heat equation

$$u_t - \{|u_x|^{p-2}u_x\}_x + f(x, u) = 0 \quad \text{in } (0, 1) \times (0, +\infty)$$

equipped with homogeneous Dirichlet, Neumann, or Robin boundary conditions converges as $t \rightarrow +\infty$ to a stationary solution. The proof follows an idea of H. Matano [61] which, in particular, is based on a comparison principle. Thus, one key step is to prove a comparison principle on non-cylindrical open sets, for Dirichlet, Neumann and Robin boundary conditions, which may be of independent interest. The results of Chapter 2 are contained in article [46], which was published in the journal *Nonlinear Differential Equations and Applications NoDea*.

Chapter 3 is devoted to the study of nonexistence of positive nontrivial weak solutions of equation

$$u_t - K_p u = \frac{\lambda}{|x|^p} |u|^{p-2} u \quad \text{in } \Omega \times (0, T).$$

Thereby, we assume that $1 < p < +\infty$, Ω is a domain of \mathbb{R}^d containing $x = 0$ if $d \geq 2$ and $\Omega = (0, +\infty)$ if $d = 1$. Here, K_p denotes the p -Kolmogorov operator defined by

$$K_p u = \Delta_p u + \langle |\nabla u|^{p-2} \nabla u, \frac{\nabla \rho}{\rho} \rangle \quad \text{for all } u \in W_{loc}^{1,p}(\Omega),$$

where $\rho > 0$ is a smooth function. The proof follows an idea of X. Cabré and Y. Martel [16], which is based on a Hardy inequality and on the optimality of the Hardy constant. Thus, it is our first task to establish that the following new weighted Hardy inequality

$$\left(\frac{|d-p|}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} d\mu \leq \int_{\Omega} |\nabla u|^p d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \text{sgn}(d-p) \int_{\Omega} |u|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu,$$

holds for all $u \in W_{\mu,0}^{1,p}(\Omega)$ if the dimension $d \geq 2$, $1 < p < d$, and for all $u \in W_{\mu,0}^{1,p}(\Omega \setminus \{0\})$ if $p > d \geq 1$. Furthermore, we give a sufficient condition, when the Hardy constant $\left(\frac{|d-p|}{p}\right)^p$ in this inequality is optimal. The results of Section 3.2 and Section 3.4.1 of Chapter 3 are contained in article [47], which was accepted for publication in the journal *Archiv der Mathematik*. The results of Section 3.4.2 of Chapter 3 are not yet published.

Acknowledgements

I would like to thank my supervisors, Professor Wolfgang Arendt and Professor Ralph Chill, for their continued support and guidance. They shared with me some of their insight into the World of nonlinear evolution equations, and spent time on helpful discussions that contributed greatly to this thesis. Besides, they allowed me to work independently and participate in many conferences, as well as spring- and summer-schools.

I am very grateful to Professor Abdelaziz Rhandi who introduced me to the theory of Hardy's inequality and showed me how to connect to the phenomenon of nonexistence of positive solutions. He provided guidance throughout my time at the University of Salerno. Since 2010, we have been collaborating on several interesting projects.

In particular, I would like to thank Professor Jerome Goldstein, one of my referee, for his kind and important comments on my thesis and many helpful discussions by email or at the University of Tübingen.

I greatly appreciate the committee members Professor Karsten Urban, Professor Werner Lütkebohmert, Professor Mourad Choulli, and Professor Philippe Souplet for their time and interest in this work. Especially, I would like to thank Professor Mourad Choulli of the University of Lorraine in Metz and Professor Philippe Souplet of the University of Paris 13 because they made both the long journey from France to Ulm the day of my thesis defence.

I am indebted to the *Laboratoire de Mathématiques et Applications de Metz et CNRS* at the University of Lorraine in Metz and the *Institute of Applied Analysis* at the University of Ulm for their support during my studies. In addition, I would like to thank the scholarship *Mathematische Analyse von Evolution, Information und Komplexität* of the Landesgraduiertenförderung Baden-Württemberg for the financial support during the last three years.

Finally, I would like to thank my brother Sebastian, who supported me with many good books, as well as my mum Vesna and my girlfriend Hanna, who both were always there for me.

Contents

1	Introduction générale et résumé	1
1.1	Le p -laplacien	2
1.2	Introduction et résumé du Chapitre 2	5
1.2.1	Convergence des solutions bornées en dimension finie	7
1.2.2	Convergence des solutions bornées en dimension infinie : cas $p = 2$	9
1.2.3	Convergence des solutions bornées en dimension infinie : cas $p \neq 2$	12
1.2.4	Le résultat principal du Chapitre 2	13
1.3	Introduction et résumé du Chapitre 3	15
1.3.1	L'inégalité de Hardy et non existence des solutions positives non triviales	15
1.3.2	Les résultats principaux du Chapitre 3	17
2	Convergence of bounded solutions of nonlinear parabolic problems	19
2.1	Introduction	20
2.2	Preliminaries	21
2.2.1	Abstract dynamical systems	21
2.2.2	Subgradient systems	25
2.3	Main result	27
2.4	Proof of Theorem 2.3.1	31
2.5	A comparison principle for Dirichlet boundary conditions	32
2.6	A comparison principle for Neumann or Robin boundary conditions	37
2.7	Proof of Theorem 2.3.1 (Continued)	39
3	A weighted Hardy inequality and nonexistence of positive nontrivial solutions	43
3.1	Introduction and notation	44
3.2	A weighted Hardy and Poincaré inequality	47
3.3	Preliminaries	51
3.3.1	Realization of the Dirichlet p -Kolmogorov operator	51
3.3.2	Solvability of the first initial boundary value problem	53
3.3.3	Definition of weak solutions	54
3.3.4	Comparison principles	55
3.3.5	Steklov averages and an integration by parts	58
3.3.6	Some geometric properties of the boundary	62
3.3.7	Linear second-order differential operator of parabolic type	71
3.3.8	The strong maximum principle of linear parabolic equations	73
3.4	Main results: Nonexistence of positive nontrivial solutions	74
3.4.1	Nonexistence of positive solutions in dimension $d = 1$	74
3.4.2	Nonexistence of positive solutions for $d \geq 2$	79
	Bibliography	87

1

Introduction générale et résumé

Cette thèse s'inscrit dans le cadre de l'étude de deux sujets concernant les problèmes d'évolution liés au p -laplacien. Le premier sujet concerne l'étude du comportement asymptotique des solutions bornées lorsque le temps $t \rightarrow +\infty$. Quant au deuxième sujet, il porte sur l'étude de la non existence des solutions positives non triviales.

Cette thèse se répartit en trois chapitres. Le premier chapitre est consacré à une introduction générale. Le deuxième chapitre porte sur l'étude de la convergence, lorsque $t \rightarrow +\infty$, des solutions bornées d'une équation parabolique associée au p -laplacien dans un intervalle borné avec des conditions aux limites du type soit Dirichlet, Neumann ou Robin. Ce travail était l'objet d'un article [46] accepté pour publication dans *Nonlinear Differential Equations and Applications NoDea*. Le dernier chapitre concerne l'étude de la non existence des solutions positives des équations paraboliques associées au p -laplacien avec un terme de convection et un potentiel singulier. La deuxième et quatrième section du Chapitre 3 reprennent un article [47] accepté pour publication dans le journal *Archiv der Mathematik*. La deuxième sous-section de la Section 4 du Chapitre 3 contient un résultat qui améliore le travail [34] de G. Goldstein, J. Goldstein et A. Rhandi. Ce résultat n'est pas encore publié.

Dans la suite nous introduirons le p -laplacien, présenterons quelques applications en physique, chimie, et en traitement d'image. Dans la deuxième et la troisième section de cette introduction, nous rappellerons les travaux concernant l'étude de la convergence en temps des solutions bornées et de la non existence des solutions positives. Nous présenterons également un résumé du Chapitre 2 et 3, mettrons en lumière les résultats principaux de cette thèse et décrirons les points essentiels qui sont utilisés dans les preuves pour résoudre les problèmes en question.

Contents

1.1	Le p-laplacien	2
1.2	Introduction et résumé du Chapitre 2	5
1.2.1	Convergence des solutions bornées en dimension finie	7
1.2.2	Convergence des solutions bornées en dimension infinie : cas $p = 2$	9
1.2.3	Convergence des solutions bornées en dimension infinie : cas $p \neq 2$	12
1.2.4	Le résultat principal du Chapitre 2	13
1.3	Introduction et résumé du Chapitre 3	15
1.3.1	L'inégalité de Hardy et non existence des solutions positives non triviales	15
1.3.2	Les résultats principaux du Chapitre 3	17

1.1. Le p -laplacien

Soit $1 < p < +\infty$ et Ω un ouvert de \mathbb{R}^d . Le p -laplacien Δ_p est l'opérateur différentiel non linéaire d'ordre 2 défini par

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{pour tout } u \in W^{1,p}(\Omega).$$

Cet opérateur différentiel intervient dans différents domaines parmi lesquels on peut citer : la modélisation des phénomènes physiques, le traitement d'images et quelques problèmes géométriques. Dans ce qui suit on va donner quelques exemples.

En 1967, O. Ladyzhenskaya a proposé dans [55] des systèmes d'équations du type

$$\begin{cases} \vec{u} = (u_1, \dots, u_d), u_i \in L^\infty(0, T; L^1(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)), i = 1, \dots, d, \\ u_{it} - \operatorname{div}(|\nabla \vec{u}|^{p-2} \nabla u_i) = 0 \quad \text{au sens des distributions dans } \Omega \times]0, T[, \end{cases}$$

pour modéliser le mouvement de quelques fluides non newtonien.

Considérons maintenant $u = u(x, t)$ la concentration d'une composante des particules (ou la densité de la chaleur) dans un milieu Ω qui dépend de l'espace $x \in \Omega$ et du temps $t \geq 0$. L'équation suivante

$$u_t - \operatorname{div}(a(x, t, u, \nabla u) \nabla u) = b(x, t, \nabla u) + c(x, t, u) \quad \text{dans } \Omega \times]0, T[$$

décrit l'évolution temporelle de la concentration pendant la propagation des particules à travers le milieu Ω . Le terme $\operatorname{div}(a(x, t, u, \nabla u) \nabla u)$ décrit la diffusion des particules et $b(x, t, \nabla u)$ le transport des particules à travers le milieu. Quant au terme $c(x, t, u)$, il décrit une source s'il est positive ou une cuvette s'il est négative. On note $a(x, t, u, \nabla u)$ le coefficient de diffusion qui reflète la capacité intrinsèque de diffusion chez les particules à travers le milieu. Ce dernier est donné par $a(x, t, u, \nabla u) = |\nabla u|^{p-2}$ pour le p -laplacien. Si $p = 2$, ce coefficient $|\nabla u|^{p-2} \equiv 1$ et la diffusion est linéaire. Dans ce cas le p -laplacien se réduit au laplacien Δ .

Si $1 < p < 2$, le coefficient de diffusion du p -laplacien devient très grand dans des régions où la pente de la concentration est très petite et devient très petit dans des régions où la pente est très grande. Dans le traitement d'image ce comportement de la diffusion non linéaire apporte un avantage par rapport au cas linéaire ($p = 2$). En effet, dans la reconstitution d'une image brouillée en noir et blanc, on ne veut pas lisser l'image dans des régions où il y a une grande variation de concentration des particules noires et blanches pour garder les contours de l'image. Cette idée d'employer la diffusion non linéaire dans le traitement d'image était introduite en 1990 par P. Perona et J. Malik [67].

En 1973, M. C. Pélissier et L. Reynaud [66] ont proposé l'équation elliptique suivante

$$(1.1) \quad \Delta_p u = 0 \quad \text{dans } \Omega,$$

pour décrire l'écoulement des glaciers. Pour ce sujet nous renvoyons aussi à [65].

En 1977, K. Uhlenbeck [75] a souligné, du point de vue géométrique, l'importance de considérer le système d'équations stationnaires du type

$$\begin{cases} \vec{u} = (u_1, \dots, u_d), u_i \in W_{loc}^{1,p}(\Omega), i = 1, \dots, d, \\ \operatorname{div}(|\nabla \vec{u}|^{p-2} \nabla u_i) = 0 \quad \text{dans } \Omega. \end{cases}$$

On trouvera d'autres exemples et d'autres références dans E. DiBenedetto [20].

Pour analyser la structure du p -laplacien, on commence par énoncer la définition d'un opérateur quasi-linéaire uniformément elliptique comme établi dans D. Gilbarg et N. S. Trudinger [31].

DÉFINITION 1.1.1. Soit G un sous-ensemble de $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ et a une application de G à valeur dans \mathbb{R}^d . On dit qu'un opérateur quasi-linéaire du second ordre L sous forme divergence

$$Lu = \operatorname{div}(a(x, u, \nabla u)) + b(x, u, \nabla u)$$

est *elliptique* dans G si pour tout $(x, u, z) \in G$, il existe $\lambda(x, u, z), \Lambda(x, z, p) > 0$ telles que

$$(1.2) \quad \lambda(x, u, z) |\xi|^2 \leq \sum_{i,j=1}^d \frac{\partial a_i(x, u, z)}{\partial z_j} \xi_i \xi_j \leq \Lambda(x, z, p) |\xi|^2$$

pour tout $\xi = (\xi_1, \dots, \xi_d)^t \in \mathbb{R}^d$. Si, de plus, Λ/λ est borné dans G , alors on appelle L un opérateur *uniformément elliptique* dans G .

Pour le p -laplacien on a $a_i(x, u, z) = |z|^{p-2} z_i$ pour tout $(x, u, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$. On vérifie facilement que

$$(1.3) \quad \frac{\partial a_i(x, u, z)}{\partial z_j} = |z|^{p-2} \left\{ (p-2) |z|^{-2} z_i z_j + \delta_{ij} \right\}$$

pour tout $i, j = 1, \dots, d$ et $z \neq 0$, où δ_{ij} désigne le symbole de Kronecker définie par $\delta_{ii} = 1$ et $\delta_{ij} = 0$ si $i \neq j$. Dans le cas où $p = 2$, on peut déduire de (1.3) que l'inégalité (1.2) est satisfaite avec $\lambda(x, u, z) = \Lambda(x, u, z) = 1$. Si $1 < p < 2$, on a

$$(p-1) |z|^{p-2} |\xi|^2 \leq \sum_{i,j=1}^d \frac{\partial a_i(x, u, z)}{\partial z_j} \xi_i \xi_j \leq 2 |z|^{p-2} |\xi|^2$$

pour tout $z \in \mathbb{R}^d \setminus \{0\}$ et tout $\xi \in \mathbb{R}^d$, et si $p > 2$, on a

$$|z|^{p-2} |\xi|^2 \leq \sum_{i,j=1}^d \frac{\partial a_i(x, u, z)}{\partial z_j} \xi_i \xi_j \leq (p-1) |z|^{p-2} |\xi|^2$$

pour tout $z \in \mathbb{R}^d$ et tout $\xi \in \mathbb{R}^d$. Puisque pour $1 < p < 2$, la constante $\lambda(x, u, z) = (p-1) |z|^{p-2}$ tend vers l'infini lorsque $|z| \rightarrow 0+$, le p -laplacien admet une singularité en $z = 0$. Dans le cas où $p > 2$, la constante $\Lambda(x, u, z) = (p-1) |z|^{p-2}$ tend vers zéro lorsque $|z| \rightarrow 0+$. Donc dans le cas où $p \neq 2$, le p -laplacien n'est plus elliptique dans les sous-ensembles G de $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ contenant $(x, u, 0)$. Suite à cette caractéristique particulière du p -laplacien, l'équation (1.1) est nommée par E. DiBenedetto [20] *singulière* si $1 < p < 2$ et *dégénérée* si $p > 2$.

Néanmoins, le p -laplacien possède encore une structure satisfaisante pour développer une théorie des solutions faibles ou fortes afin de résoudre des problèmes elliptiques et paraboliques aux limites (cf. J.-L. Lions [58], H. Brezis [11] et les références citées). Le p -laplacien est un prototype d'opérateurs quasi-linéaires strictement monotones, en effet, si on pose $a(\xi) = |\xi|^{p-2} \xi$ pour tout $\xi \in \mathbb{R}^d$, alors pour tout $\xi, \zeta \in \mathbb{R}^d$, le produit scalaire

$$\langle a(\xi) - a(\zeta), \xi - \zeta \rangle_{\mathbb{R}^d} > 0 \quad \text{si } \xi \neq \zeta.$$

La généralisation suivante du p -laplacien aux opérateurs quasi-linéaires est dû à J. Leray et J.-L. Lions [57] en 1965.

DÉFINITION 1.1.2. Soit Ω un ouvert de \mathbb{R}^d , $1 < p < +\infty$ et on pose $p' = \frac{p}{p-1}$. Un opérateur $A : W^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ défini par

$$A(u) = -\operatorname{div}(a(x, u, \nabla u)) \quad \text{pour toute } u \in W^{1,p}(\Omega)$$

est dit *quasi-linéaire du type Leray-Lions* défini sur $W^{1,p}(\Omega)$ si $a : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ est une fonction de Carathéodory vérifiant les hypothèses classiques de Leray-Lions ([57]) :

$$\begin{aligned} (i) : & \quad |a(x, u, \xi)| \leq c(x) + k_1|u|^{p-1} + k_2|\xi|^{p-1} \\ (ii) : & \quad \langle a(x, u, \xi) - a(x, u, \zeta), \xi - \zeta \rangle_{\mathbb{R}^d} > 0 \\ (iii) : & \quad \frac{\langle a(x, u, \xi), \xi \rangle_{\mathbb{R}^d}}{|\xi| + |\zeta|^{p-1}} \rightarrow +\infty \quad \text{si } |\xi| \rightarrow +\infty \end{aligned}$$

pour tout $\xi, \zeta \in \mathbb{R}^d$ avec $\xi \neq \zeta$, pour tout $u \in \mathbb{R}$, pour presque toute $x \in \Omega$, où $c \in L^{p'}(\Omega)$ est positive, et k_1, k_2 sont des constantes positives.

Il y a aussi une version parabolique de cette définition (cf. e.g. L. Boccardo et F. Murat [8]). En outre, il est important de faire remarquer que le p -laplacien peut être réalisé comme le sous-gradient dans l'espace d'Hilbert $L^2(\Omega)$ d'une fonction \mathcal{E} définie sur un sous-ensemble de $L^2(\Omega)$ à valeurs dans $] -\infty, +\infty]$ semi-continue inférieurement, propre et convexe. Voir par exemple la fonction \mathcal{E} donnée par (1.10) et (1.11) dans la section suivante.

Le fait que le p -laplacien n'est pas un opérateur uniformément elliptique a plusieurs conséquences. Les problèmes associés au p -laplacien n'ont pas généralement des solutions classiques et donc il faut une théorie des solutions faibles, une des solutions fortes, et une théorie relative à la régularité des solutions. Il y a une perte de régularité des solutions des équations associées au p -laplacien si $p \neq 2$. De plus le principe du maximum fort pour les équations elliptiques ou paraboliques associées au p -laplacien n'est pas toujours assuré et ceci complique l'étude géométrique des équations associées au p -laplacien. D'un autre côté, quelques principes de comparaison restent valides. Nous renvoyons à J. L. Vázquez et V. A. Galaktionov [29] pour une liste exhaustive des propriétés des équations associées au p -laplacien.

La perte de régularité des solutions des équations aux dérivées partielles est une restriction énorme pour l'étude qualitative, notamment pour établir des propriétés sur la dérivée u_x d'une solution u de l'équation

$$u_t - u_{xx} + f(t, u) = 0 \quad \text{dans }]a, b[\times]0, T[,$$

on utilise souvent le fait que $w = u_x$ résoud

$$w_t - w_{xx} + \frac{\partial f}{\partial u}(t, u)w = 0 \quad \text{dans }]a, b[\times]0, T[$$

(cf. e.g., H. Matano [62]). Dans le cas du p -laplacien pour $p \neq 2$, il arrive souvent qu'on ne peut pas emprunter la même idée de la preuve du cas où $p = 2$.

Le principe du maximum fort est éminemment utile dans l'étude des équations aux dérivées partielles. En particulier, il intervient dans l'étude des équations paraboliques associées au laplacien en terme du comportement asymptotique en temps (cf. H. Matano [61] et aussi A. Haraux

[40]) mais aussi en terme de la non existence des solutions positives (cf. X. Cabré et Y. Martel [16]). Pour étudier les mêmes problèmes en remplaçant le laplacien par le p -laplacien avec $p \neq 2$, il faut souvent contourner les arguments où le principe du maximum fort était appliqué. Il y a des cas où on peut remplacer l'application du principe du maximum fort par un principe de comparaison. Dans les deux sujets de cette thèse le principe du maximum fort et le principe de comparaison sont des outils importants pour établir nos résultats.

1.2. Introduction et résumé du Chapitre 2 :

Convergence des solutions bornées des problèmes non linéaires sur un intervalle borné : le cas singulier

Dans le deuxième chapitre de cette thèse, nous étudions pour $1 < p < +\infty$ et Ω étant un intervalle borné, la convergence, lorsque $t \rightarrow +\infty$, des solutions bornées de l'équation parabolique

$$(1.4) \quad u_t - \Delta_p u + f(x, u) = 0 \quad \text{dans } \Omega \times]0, +\infty[$$

vers une solution stationnaire. Ici, nous supposons que la solution u de (1.4) vérifie une des trois conditions classiques au bord pour presque tout $t > 0$:

$$(1.5) \quad u = 0 \quad \text{sur } \partial\Omega \quad (\text{condition de Dirichlet homogène}),$$

$$(1.6) \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{sur } \partial\Omega \quad (\text{condition de Neumann homogène}),$$

$$(1.7) \quad |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + j(u) = 0 \quad \text{sur } \partial\Omega \quad (\text{condition de Robin homogène})$$

En plus, on suppose que $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ est une fonction continue telle que $f(x, \cdot)$ est lipschitzienne sur \mathbb{R} avec constante $L \geq 0$, uniformément pour tout $x \in \Omega$, et j une fonction continue et croissante sur \mathbb{R} qui vérifie $|j(r)| \leq c|r|^{p_2-1}$ pour tout $r \in \mathbb{R}$ avec $p_2 = \min\{p, 2\}$. Notons que la condition (1.7) contient la condition (1.6) comme un cas limite. Une solution stationnaire associée au problème parabolique ci-dessus est une solution de l'équation elliptique

$$(1.8) \quad -\Delta_p \varphi + f(x, \varphi) = 0 \quad \text{dans } \Omega$$

qui vérifie une des conditions au bord (1.5), (1.6) ou (1.7).

Pour mieux décrire le problème du comportement asymptotique en temps des solutions bornées, nous supposons dans ce qui suit que Ω est un ouvert borné de \mathbb{R}^d avec un bord régulier. Nous notons avec $W_{p,2}^1(\Omega)$ l'espace des fonctions $u \in L^2(\Omega)$ telles que $\nabla u \in L^p(\Omega)^d$ muni de la norme $\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(\Omega)^d}$, ($u \in W_{p,2}^1(\Omega)$). On trouvera plus d'informations concernant l'espace $W_{p,2}^1(\Omega)$ dans Maz'ja [63, p. 161]. Nous notons par $Tr : W_{p,2}^1(\Omega) \rightarrow L^{p_2}(\partial\Omega)$ la prolongation continue de l'opérateur $u \mapsto u|_{\partial\Omega}$ de $W_{p,2}^1(\Omega) \cap C(\overline{\Omega})$ dans $C(\partial\Omega)$. En utilisant la théorie des solutions fortes (cf. H. Brezis [9, Proposition 8] et d'après une application standard du lemme de Gronwall), le problème parabolique aux limites muni d'une donnée initiale $u(0) \in L^2(\Omega)$ est bien posé au sens de Hadamard. Plus précisément, pour toute donnée initiale $u_0 \in L^2(\Omega)$, il existe une fonction unique u vérifiant

$$(i) \quad u \in C([0, +\infty[; L^2(\Omega)) \cap W_{loc}^{1,\infty}([0, +\infty[; L^2(\Omega)) \cap C([0, +\infty[; W_{p,2}^1(\Omega)),$$

$$(ii) \quad u(0) = u_0 \text{ dans } L^2(\Omega), \text{ et pour tout } t > 0,$$

$$u(t) \in L^2(\Omega) \quad \text{telle que} \quad \nabla u(t) \in L^p(\Omega)^d \quad \text{et} \quad \Delta_p u(t) \in L^2(\Omega),$$

(iii) u admet en tout $t > 0$ une dérivée à droite, notée $\frac{du}{dt}_+$, en plus pour tout $t > 0$, on a

$$\frac{du}{dt}_+(t) - \Delta_p u(t) + f(x, u(t)) = 0 \quad \text{dans } L^2(\Omega)$$

et $u(t)$ satisfait une des conditions au bord (1.5), (1.6) ou (1.7),

(iv) si u (resp. v) est une solutions du problème de Cauchy avec donnée initiale u_0 (resp. v_0) dans $L^2(\Omega)$, alors

$$\|u(t) - v(t)\|_{L^2(\Omega)} \leq e^{Lt} \|u_0 - v_0\|_{L^2(\Omega)} \quad \text{pour tout } t \geq 0.$$

Si on pose

$$S(t)u_0 = u(t) \quad \text{pour tout } t \geq 0 \text{ et pour tout } u_0 \in L^2(\Omega),$$

où u est l'unique solution du problème parabolique aux limites avec la donnée initiale u_0 , alors on obtient que la famille $\{S(t)\}_{t \geq 0}$ est un système dynamique sur $L^2(\Omega)$. Il est donc naturel de tirer profit du concept des systèmes dynamiques et de la théorie adjacente pour étudier le comportement asymptotique en temps des solutions bornées du problème parabolique aux limites et par conséquent, on aboutit aux questions suivantes :

Si u est une solution bornée du problème parabolique aux limites,

1. quelle sont des conditions suffisantes pour que la limite $\lim_{t \rightarrow +\infty} u(t)$ existe ?

Une fois l'existence de la limite $\lim_{t \rightarrow +\infty} u(t)$ est établie,

2. est-elle une solution du problème stationnaire ?
3. finalement, quelle est la vitesse de convergence ?

Pour répondre à la question 2., nous définissons l'ensemble des points d'accumulation d'une solution u du problème parabolique aux limites lorsque $t \rightarrow +\infty$ par

$$\omega(u) = \left\{ \varphi \in W_{p,2}^1(\Omega) \mid \text{il existe une suite } t_n \uparrow +\infty \text{ telle que } \lim_{n \rightarrow +\infty} u(t_n) = \varphi \right\}$$

et on l'appelle l'ensemble ω -limite de u . On munit $\omega(u)$ avec la topologie induite par la norme dans $W_{p,2}^1(\Omega)$. Remarquons que suivant la régularité de la solution u du problème parabolique aux limites, $\omega(u)$ peut être muni d'une topologie plus forte. On déduit immédiatement de la définition de $\omega(u)$ que pour toute solution u du problème parabolique aux limites définie sur $]0, +\infty[$ et ayant l'image $\bigcup_{t \geq 0} \{u(t)\}$ relativement compacte dans $W_{p,2}^1(\Omega)$, la limite $\lim_{t \rightarrow +\infty} u(t)$ existe dans $W_{p,2}^1(\Omega)$ si et seulement si $\omega(u)$ est réduit à un seul élément. Le problème parabolique aux limites peut être réécrit sous la forme d'un système gradient abstrait

$$(1.9) \quad \dot{u} + \nabla_{L^2(\Omega)} \mathcal{E}(u) = 0 \quad \text{dans } L^2(\Omega) \text{ sur } [0, +\infty[$$

pour une fonction $\mathcal{E} : L^2(\Omega) \rightarrow]-\infty, +\infty]$ semi-continue inférieurement dont la restriction sur le domaine $D(\mathcal{E})$ muni de la norme de $W_{p,2}^1(\Omega)$ est différentiable au sens de Gâteaux. Dans le cas des conditions au bord de Dirichlet (1.5), la fonction \mathcal{E} est donnée pour toute $u \in L^2(\Omega)$ par

$$(1.10) \quad \mathcal{E}_D(u) = \begin{cases} \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} F(x, u) dx & \text{si } \nabla u \in L^p(\Omega)^d \text{ avec } \text{Tr}(u) = 0, \\ +\infty & \text{sinon,} \end{cases}$$

et dans le cas des conditions au bord de Neumann (1.6) ou de Robin (1.7), \mathcal{E} est donnée par

$$(1.11) \quad \mathcal{E}_{NR}(u) = \begin{cases} \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \int_{\partial\Omega} \beta(u) dx + \int_{\Omega} F(x, u) dx & \text{si } \nabla u \in L^p(\Omega)^d, \\ +\infty & \text{sinon,} \end{cases}$$

pour tout $u \in L^2(\Omega)$. Ici, nous posons $F(x, u) = \int_0^u f(x, s) ds$ et $\beta(u) = \int_0^u j(s) ds$ pour tout $u \in \mathbb{R}$. De plus, une fonction φ est une solution stationnaire associée au problème parabolique aux limites si est seulement si

$$\nabla_{L^2(\Omega)} \mathcal{E}(u) = 0,$$

pour la fonction \mathcal{E} donnée par (1.10) et (1.11) correspondante aux conditions (1.5) - (1.7) au bord. Autrement dit, une solution stationnaire associée au problème parabolique aux limites est un point d'équilibre de \mathcal{E} et vice versa. Toute solution u du système gradient (1.9) vérifie l'équation intégrale

$$(1.12) \quad \int_{t_1}^{t_2} \|\dot{u}\|_{L^2(\Omega)}^2 + \mathcal{E}(u(t_2)) = \mathcal{E}(u(t_1)) \quad \text{pour tout } 0 < t_1 < t_2.$$

D'où on déduit que pour toute solution u de (1.9) la fonction $t \mapsto \mathcal{E}(u(t))$ est décroissante sur $]0, +\infty[$ et donc \mathcal{E} est une fonction de Liapunov pour le système gradient (1.9). Pour cela, on dit aussi que le système gradient est dissipative. En outre, on déduit de (1.12) que pour toute solution u de (1.9) telle que $t \mapsto \mathcal{E}(u(t))$ est constante sur $]0, +\infty[$, alors $t \mapsto u(t)$ est constante et $u(t)$ est un point d'équilibre du système gradient (1.9). Par la suite, la fonction \mathcal{E} est une fonction de Liapunov stricte pour le système (1.9). D'après le principe d'invariance de LaSalle (cf. e.g. [40, Théorème 2.1.3]), pour toute solution u de (1.9) à image relativement compacte dans $W_{p,2}^1(\Omega)$, l'ensemble $\omega(u)$ est non vide, compact, connexe, et inclus dans l'ensemble des points d'équilibre. C'est une réponse à la question 2.. De plus, on en déduit facilement un premier résultat de la stabilisation.

THÉORÈME 1.2.1. ([40, Corollaire 2.1.9]) *Si u est une solution du système gradient (1.9) à image relativement compacte dans $W_{p,2}^1(\Omega)$ et si $\omega(u)$ est discret, alors il existe un point d'équilibre φ tel que*

$$\lim_{t \rightarrow +\infty} u(t) = \varphi \quad \text{dans } W_{p,2}^1(\Omega).$$

Il y a plusieurs facteurs qui interviennent dans l'étude des questions 1. et 3. :

- (i) la dimension d de la variable spatiale x ,
- (ii) les conditions sur la fonction $f(x, u)$,
- (iii) si $1 < p < 2$, si $p = 2$ ou si $p > 2$,
- (iv) les conditions sur l'ouvert Ω , et
- (v) le type de condition au bord $\partial\Omega$.

1.2.1 Convergence des solutions bornées en dimension finie

Considérons l'équation différentielle ordinaire suivante

$$(1.13) \quad u'(t) + \mathcal{E}'(u(t)) = 0 \quad \text{dans } [0, +\infty[.$$

où $\mathcal{E}(u) = \int_0^u f(s) ds$, $u \in \mathbb{R}$, et f est une fonction localement lipschitzienne sur \mathbb{R} . Puisque toute solution de (1.13) est monotone, alors toute solution définie et bornée sur $[0, +\infty[$ de cette équation, converge vers une limite $\varphi \in \mathbb{R}$ lorsque $t \rightarrow +\infty$. D'après le principe d'invariance de LaSalle, $\mathcal{E}'(\varphi) = 0$. Nous avons donc démontré la première partie du théorème suivant, qui est bien connu dans la théorie des équations différentielles ordinaires (cf. e.g. A. Haraux [40, Exemple 2.2.4, p. 20]).

THÉORÈME 1.2.2. *Toute solution de l'équation (1.13) définie et bornée sur $[0, +\infty[$ converge en temps vers une limite $\varphi \in \mathbb{R}$ telle que $\mathcal{E}'(\varphi) = 0$. Si de plus pour $k \geq 1$, \mathcal{E} est de classe C^k dans un voisinage de $\varphi \in \mathbb{R}$ vérifiant $\mathcal{E}'(\varphi) = \dots = \mathcal{E}^{(k-1)}(\varphi) = 0$ et $\mathcal{E}^{(k)}(\varphi) \neq 0$ et si u est une solution de l'équation (1.13) définie sur $[0, +\infty[$ qui converge en temps vers φ , alors on a*

$$(1.14) \quad |u(t) - \varphi| = \begin{cases} \mathcal{O}(e^{-ct}) & \text{pour une } c > 0, \text{ si } k = 2, \\ \mathcal{O}\left(t^{-\frac{1}{k-2}}\right) & \text{si } k \geq 3. \end{cases} \quad \text{lorsque } t \rightarrow +\infty.$$

Le point essentiel de la preuve de la deuxième assertion du Théorème 1.2.2 est l'inégalité de Łojasiewicz [60, Proposition 1, p. 92] (1965) et [59](1963).

LEMME 1.2.1. (Lemme de Łojasiewicz) *Soit U un ouvert de \mathbb{R}^d , soit $\mathcal{E} : U \rightarrow \mathbb{R}$ une fonction analytique, et soit $\varphi \in U$ telle que $\nabla \mathcal{E}(\varphi) = 0$. Alors il existe des constantes $\theta \in]0, \frac{1}{2}]$, $\sigma > 0$ et $C \geq 0$ telles que pour tout $u \in U$ vérifiant $|u - \varphi| < \sigma$ on a*

$$(1.15) \quad |\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\theta} \leq C \|\nabla \mathcal{E}(u)\|.$$

D'après ce lemme, toute fonction analytique vérifie l'inégalité (1.15) au voisinage d'un point d'équilibre. Mais lorsque la dimension $d = 1$, nous obtenons le résultat suivant dû à R. Chill [17, Proposition 2.3 (b)] (2003).

LEMME 1.2.2. *Soit U un ouvert de \mathbb{R} et soit φ un élément de U . Si pour $k \geq 2$, $\mathcal{E} \in C^k(U; \mathbb{R})$ vérifiant $\mathcal{E}'(\varphi) = \dots = \mathcal{E}^{(k-1)}(\varphi) = 0$ et $\mathcal{E}^{(k)}(\varphi) \neq 0$, alors il existe des constantes $\sigma > 0$ et $C \geq 0$ telles que pour toute $u \in U$ vérifiant $|u - \varphi| < \sigma$, \mathcal{E} satisfait l'inégalité (1.15) avec $\theta = \frac{1}{k}$.*

DÉMONSTRATION DU LEMME 1.2.2. Puisque $\mathcal{E}^{(k)}$ est continue et $\mathcal{E}^{(k)}(\varphi) \neq 0$, il existe des constantes $\sigma, c > 0$ telles que $\mathcal{E}^{(k)}(u) \geq c > 0$ pour toute $u \in U$ vérifiant $|u - \varphi| < \sigma$. Soit $B(\varphi, \sigma)$ la boule ouverte de centre φ et de rayon σ et soit $u \in U \cap B(\varphi, \sigma)$. On applique la formule de Taylor à \mathcal{E} et \mathcal{E}' au point $u = \varphi$. Donc il existe ξ_1 et ξ_2 entre u et φ tels que

$$\mathcal{E}(u) - \mathcal{E}(\varphi) = \frac{\mathcal{E}^{(k)}(\xi_1)}{k!} (u - \varphi)^k \quad \text{et} \quad \mathcal{E}'(u) = \frac{\mathcal{E}^{(k)}(\xi_2)}{(k-1)!} (u - \varphi)^{k-1}.$$

D'où on déduit que

$$|\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\frac{1}{k}} \leq \left[\frac{\|\mathcal{E}^{(k)}\|_{L^\infty(B(\varphi, \sigma))}}{k!} \right]^{(k-1)/k} \frac{(k-1)!}{c} \|\mathcal{E}'(u)\|.$$

Ceci achève la démonstration du lemme. □

La vitesse de convergence (1.14) était déduite de l'inégalité (1.15) la première fois par L. Simon [72] et puis par A. Haraux et Jendoubi [41] (voir aussi R. Chill et E. Fařangová [18]). L'idée de

l'application de l'inégalité de Łojasiewicz (1.15) pour étudier le comportement asymptotique du système gradient en dimension fini est due à S. Łojasiewicz lui même (cf. [59]). Plus précisément, il a montré le résultat suivant.

THÉORÈME 1.2.3. (Łojasiewicz) *Si $\mathcal{E} : \mathbb{R}^d \rightarrow \mathbb{R}$ est une fonction analytique, alors pour toute solution u du système*

$$(1.16) \quad \dot{u} + \nabla \mathcal{E}(u) = 0 \quad \text{dans } [0, +\infty[$$

définie et bornée sur $[0, +\infty[$ il existe une $\varphi \in \mathbb{R}^d$ telle que $\nabla \mathcal{E}(\varphi) = 0$ et $u(t) \rightarrow \varphi$ lorsque $t \rightarrow +\infty$.

Lorsque la dimension $d \geq 2$ et si f n'est pas analytique, ce résultat de Łojasiewicz ne reste pas vrai en général. En 1982, J. Palis et W. de Melo [52, p. 14] ont suggéré la fonction suivante de classe C^∞ :

$$\mathcal{E}(r \cos \theta, r \sin \theta) = \begin{cases} \exp\left(\frac{1}{r^2-1}\right) & \text{si } r < 1, \\ 0 & \text{si } r = 1, \\ \exp\left(\frac{1}{1-r^2}\right) \sin\left(\frac{1}{r-1} - \theta\right) & \text{si } r > 1. \end{cases}$$

Le graphe de cette fonction dans \mathbb{R}^3 forme un chapeau mexicain et le système gradient (1.16) associé admet une solution définie et bornée sur $[0, +\infty[$ dont l'ensemble ω -limite est toute la sphère unité dans \mathbb{R}^2 . P.-A. Absil, R. Mahony et B. Andrews [1] ont proposé une autre énergie \mathcal{E} de classe C^∞ pour laquelle il est plus facile de montrer l'existence d'une solution globale et bornée de (1.16) qui ne converge pas (cf. aussi [18] pour un calcul immédiat).

1.2.2 Convergence des solutions bornées en dimension infinie : cas $p = 2$

En 1983, L. Simon [72] a généralisé l'inégalité de Łojasiewicz dans l'espace de dimension infinie en utilisant le Lemme de Łojasiewicz (Lemme 1.2.1).

THÉORÈME 1.2.4. (Simon, Theorem 3., p. 537) *Soit Σ une variété riemannienne compacte et soit $\mathcal{E} : C^\infty(\Sigma) \rightarrow \mathbb{R}$ donnée par*

$$\mathcal{E}(u) = \int_{\Sigma} E(x, u, \nabla u) dx \quad \text{pour toute } u \in C^\infty(\Sigma),$$

où la fonction $E = E(x, u, p)$, ($x \in \Sigma$, $u \in \mathbb{R}$, $p \in T_x \Sigma$) vérifie les hypothèses suivantes :

- (i) *E est de classe C^∞ ,*
- (ii) *il existe une constante $\eta > 0$ telle que $\frac{d^2}{ds^2} E(x, 0, sp)|_{s=0} \geq \eta |p|^2$ pour tout $x \in \Sigma$, $p \in T_x \Sigma$,*
- (iii) *la fonction $(u, p) \mapsto E(x, u, p)$ est analytique uniformément en $x \in \Sigma$.*

Si $\varphi \in C^\infty(\Sigma)$ vérifie $\nabla \mathcal{E}(\varphi) = 0$, alors il existe des constantes $\theta \in]0, \frac{1}{2}[$, $\sigma > 0$ et $C \geq 0$ telles que pour tout $u \in C^\infty(\Sigma)$ avec $\|u - \varphi\| < \sigma$ l'inégalité (1.15) est satisfaite.

Puis L. Simon a utilisé le Théorème 1.2.4 pour établir un résultat analogue au Théorème 1.2.3 sur le comportement asymptotique en temps pour l'équation

$$\dot{u} + \nabla_{L^2(\Sigma)} \mathcal{E}(u) = f \quad \text{dans } \Sigma \times]0, +\infty[,$$

où $\nabla_{L^2(\Sigma)} \mathcal{E}(u)$ désigne le gradient en $L^2(\Sigma)$ et $f = f(x, t)$ est une fonction qui tend exponentiellement vers zéro lorsque $t \rightarrow +\infty$. En 1998, M. A. Jendoubi [48] a repris le travail [72] de L. Simon

et a simplifié l'idée de la preuve de L. Simon. En plus, il a appelé l'inégalité (1.15) *l'inégalité de Łojasiewicz-Simon*. En outre, M. A. Jendoubi a souligné que L. Simon a montré dans le cas $p = 2$, la convergence, lorsque $t \rightarrow +\infty$, des solutions globales et à image relativement compacte du problème (1.4) + (1.5) :

THÉORÈME 1.2.5. (Jendoubi, Theorem 1.1. & Example 4.1, p. 190) *Soit Ω un domaine borné de \mathbb{R}^d avec un bord de class C^∞ , $d \geq 1$, $q \geq 2$ telles que $q > \frac{d}{2}$, $X = W^{2,q}(\Omega) \cap H_0^1(\Omega)$. Soit F une fonction de $\Omega \times \mathbb{R}^d$ à valeurs réelles vérifiant*

- (i) $u \mapsto f(x, u)$ est analytique uniformément en $x \in \Omega$,
- (ii) $\frac{\partial f}{\partial u}$ et $\frac{\partial^2 f}{\partial u^2}$ sont bornées sur $\Omega \times]-K, K[^d$ pour toute $K \geq 0$.

Alors, pour toute solution u du problème (1.4) + (1.5) à image relativement compacte dans X , il existe une unique solution stationnaire $\varphi \in X$ du problème (1.8) + (1.5) telle que

$$\lim_{t \rightarrow +\infty} \|u(t) - \varphi\|_X = 0.$$

Dans un autre travail [41] (2001), A. Haraux et M. A. Jendoubi ont établi à l'aide de l'inégalité de Łojasiewicz-Simon (1.15) dans un espace de dimension infinie des estimations de la vitesse de convergence pour ce problème. Toutefois, si nous nous restreignons à la dimension $d = 1$ dans le cas $p = 2$, l'hypothèse d'analyticit  de f est trop forte pour montrer la convergence des solutions born es du probl me (1.4) + (1.5) lorsque $t \rightarrow +\infty$. En 1968, T. I. Zelenyak [79] a montr  le r sultat suivant :

TH OR ME 1.2.6. (Zelenyak-Matano) *Soit $\Omega =]a, b[$ un intervalle ouvert born  et $s \mapsto f(x, s)$ une fonction de classe C^1 uniform ment en $x \in \Omega$. Alors, pour toute solution globale et born e u de l' quation (1.4) v rifiant une des conditions (1.5)-(1.7) au bord de $]a, b[$, il existe une unique solution stationnaire $\varphi \in H^2(]a, b[)$ associ e telle que*

$$\lim_{t \rightarrow +\infty} \|u(t) - \varphi\|_{H^1(]a, b[)} = 0.$$

Pour d montrer l'assertion du Th or me 1.2.6, T. I. Zelenyak a conclu avec un joli raisonnement par l'absurde   travers lequel il d duit aussi l'in galit  de Łojasiewicz-Simon. Cependant, on ne sait pas si T. I. Zelenyak a eu connaissance de l'in galit  (1.15). Son id e du preuve  tait reprise par A. Haraux et M. A. Jendoubi [42]. Nous d crivons bri vement la preuve du Th or me 1.2.6.

ID E DE LA PREUVE DU TH OR ME 1.2.6. DE T. I. ZELENYAK [79]. Soit u une solution globale et born e du probl me (1.4) + (1.5), \mathcal{E} la fonction donn e par (1.10), et \mathcal{E}'' la deuxi me d riv e de la restriction de \mathcal{E} sur $H_0^1(]a, b[)$. Notons d'abord qu'  l'aide de l'injection compacte de $H_0^1(]a, b[)$ dans $C([a, b])$, on a $\mathcal{E}''(\varphi)$ est un op rateur de Fredholm d'indice z ro pour toute $\varphi \in H_0^1(]a, b[)$. En utilisant la th orie des  quations diff rentielles, il est facile de voir que la dimension du noyau $\ker(\mathcal{E}''(\varphi))$ est inf rieure ou  gale   1.

Supposons d'abord qu'il existe un $\varphi \in \omega(u)$ tel que le noyau $\ker(\mathcal{E}''(\varphi)) = \{0\}$. Alors l'op rateur inverse de $\mathcal{E}''(\varphi)$ est continu. Donc d'apr s [18, Theorem 12.3], l' nergie \mathcal{E} v rifie l'in galit  de Łojasiewicz-Simon au voisinage de φ avec $\theta = 1/2$. D'o , l'on d duit que $u(t) \rightarrow \varphi$ dans $H_0^1(]a, b[)$ lorsque $t \rightarrow +\infty$.

Maintenant, supposons que la solution u du probl me (1.4) + (1.5) ne converge pas. Alors l'ensemble $\omega(u)$ est un continuum compact. D'apr s la premi re partie, pour tout $\varphi \in \omega(u)$, la

dimension du noyau $\ker(\mathcal{E}''(\varphi))$ est égal à 1. Notons que la variété critique définie par

$$S = \{u \in H_0^1(]a, b[) \mid (I - P')\mathcal{E}'(u) = 0\}$$

est une variété différentiable, où P' est l'opérateur adjoint de la projection P sur $\ker(\mathcal{E}''(\varphi))$. Puisque S contient tout point d'équilibre de \mathcal{E} , elle contient aussi $\omega(u)$. Mais d'après [18, Lemma 12.6], la dimension de S est égale à la dimension du noyau $\ker(\mathcal{E}''(\varphi))$. Donc S contient un intervalle compact de points d'équilibres et qui n'est pas réduit à un seul point. D'où l'existence d'un point d'équilibre $\varphi \in S$ et d'un ouvert U de S tels que $\varphi \in U$ et \mathcal{E} est constante sur U . Donc $\mathcal{E}|_S$ vérifie l'inégalité de Łojasiewicz-Simon sur le voisinage U de φ avec $\theta = 1/2$. D'après [18, Theorem 12.7], \mathcal{E} vérifie donc l'inégalité de Łojasiewicz-Simon sur un voisinage de φ avec le même θ . D'où la solution $u(t)$ converge vers φ dans $H_0^1(]a, b[)$ lorsque $t \rightarrow +\infty$, ce qui est une contradiction à notre hypothèse. \square

En 1978, H. Matano [61] a donné une autre démonstration du Théorème 1.2.6, également par l'absurde, qui repose sur des arguments plus géométriques et qui n'utilise pas l'inégalité de Łojasiewicz-Simon. Dans ce qui suit nous en donnons brièvement la description :

IDÉE DE LA PREUVE DU THÉORÈME 1.2.6. DE H. MATANO [61]. Soit u une solution bornée du problème (1.4) + (1.5). Supposons que l'ensemble $\omega(u)$ n'est pas discret. Alors $\omega(u)$ admet une infinité d'éléments différents. D'où et à l'aide de la théorie des équations différentielles, on déduit que pour tout $\varphi \in \omega(u)$, il existe un $t_\varphi > 0$ telle que la fonction $x \mapsto u(x, t_\varphi) - \varphi(x)$ admet au plus un nombre fini de changements de signe sur $[a, b]$. D'où on déduit que la fonction $t \mapsto u_x(0, t) - \varphi_x(0)$ admet au plus un nombre fini de changements de signe sur $[t_\varphi, +\infty[$.

D'après la théorie des équations différentielles, si φ^1, φ^2 et φ^3 sont trois éléments différents de $\omega(u)$, alors on peut supposer sans perte de généralité que

$$(1.17) \quad \varphi_x^1(0) < \varphi_x^2(0) < \varphi_x^3(0).$$

Il existe un $t_{\varphi^2} > 0$ tel que $t \mapsto u_x(0, t) - \varphi_x^2(0)$ ne change pas de signe sur $[t_{\varphi^2}, +\infty[$. Si on suppose que $u_x(0, t) - \varphi_x^2(0) \geq 0$ sur $[t_{\varphi^2}, +\infty[$, alors on déduit de (1.17) que

$$0 < \varphi_x^2(0) - \varphi_x^1 \leq u_x(0, t) - \varphi_x^1(0) \leq \|u_x(t) - \varphi_x^1\|_{C[a,b]} \quad \text{sur } [t_{\varphi^2}, +\infty[$$

ce qui est en contradiction avec $\varphi^1 \in \omega(u)$. Si on suppose que $u_x(0, t) - \varphi_x^2(0) \leq 0$ sur $[t_{\varphi^2}, +\infty[$, on obtient la même contradiction avec $\varphi^3 \in \omega(u)$. Donc $\omega(u)$ contient au plus deux éléments ce qui le rend discret, d'où la contradiction. \square

Notons que dans l'argument :

"...il existe un $t_\varphi > 0$ telle que la fonction $x \mapsto u(x, t_\varphi) - \varphi(x)$ admet au plus un nombre fini de changements de signe sur $[a, b]$. D'où on déduit que la fonction $t \mapsto u_x(0, t) - \varphi_x(0)$ admet au plus un nombre fini de changements de signe sur $[t_\varphi, +\infty[$."

Matano a montré indirectement le premier théorème de C. Sturm [74] (1836) de la dénombrabilité du changement de signe appliqué à la fonction $u - \varphi$. On trouvera d'autres applications du théorème de Sturm, par exemple, dans V. A. Galaktionov [27] et en particulier dans P. Souplet [73]. Puisque l'idée du preuve dans [61] repose sur des arguments du principe du maximum, on pouvait généraliser cette idée [61] pour d'autres problèmes d'évolution aux limites non linéaires. Voir, par exemple, E. Feireisl et F. Simondon [25], M. Falliero et F. Simondon [23], ou aussi M.

Gokieli et F. Simondon [32].

Remarquons que le théorème de Zelenyak-Matano, ne donne pas une information sur la vitesse de convergence, mais il nous montre qu'en la dimension $d = 1$ et dans le cas $p = 2$, il est suffisant de supposer que la fonction $u \mapsto f(x, u)$ est lipschitzienne uniformément en $x \in \Omega$. Ce résultat ne reste pas vrai en général dans la dimension $d = 2$. En 1996, P. Poláčik et K. Rybakowski ont établi dans [69, Theorem 3.1] le résultat suivant de la non convergence des solutions bornées du problème (1.4) + (1.5) dans le cas $p = 2$:

THÉORÈME 1.2.7. (Poláčik-Rybakowski) *Soit Ω la boule unité ouverte de \mathbb{R}^2 . Pour tout $m \in \mathbb{N}_0$, soit Y_m l'ensemble de toute fonction $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ pour laquelle toutes dérivées partielles d'ordre inférieur ou égal à m existent, sont continues et bornées sur $\overline{\Omega} \times \mathbb{R}$. Alors, pour tout $m \in \mathbb{N}_0$, il existe une fonction $f_m \in Y_m$ et une solution u globale et bornée du problème (1.4) + (1.5) telle que l'ensemble $\omega(u)$ est isomorphe à la sphère unité en \mathbb{R}^2 . En particulier, u ne converge pas lorsque $t \rightarrow +\infty$.*

On trouvera d'autres exemples de la non convergence des solutions globales et bornées notamment dans P. Poláčik [68], J. K. Hale et G. Raugel [39] ou E. Fašangová et E. Feireisl [24].

1.2.3 Convergence des solutions bornées en dimension infinie : cas $p \neq 2$

Considérons maintenant le cas $p \neq 2$ pour le problème de convergence, lorsque $t \rightarrow +\infty$, des solutions bornées de l'équation (1.4) vérifiant une des conditions (1.5)-(1.7) au bord. Supposons que Ω est un ouvert borné avec un bord régulier, et $u \mapsto f(x, u)$ une fonction croissante uniformément en $x \in \Omega$ qui vérifie la condition de croissance

$$(1.18) \quad |f(x, u)| \leq c_1(x) + c_2|u| \quad \text{pour tout } u \in \mathbb{R} \text{ et pour presque tout } x \in \Omega,$$

où $c_1 \in L^2(\Omega)$ et $c_2 > 0$. Alors la fonction \mathcal{E} donnée par (1.10) ou (1.11) devient convexe et nous pouvons déduire du théorème classique [11, Théorème 3.11] dû à H. Brezis, le résultat suivant concernant la convergence en temps.

THÉORÈME 1.2.8. *Soit $1 < p < +\infty$, alors pour toute solution u à image relativement compacte dans $W_{p,2}^1(\Omega)$ du problème (1.4) vérifiant une des conditions au bord (1.5)-(1.7), il existe un unique $\varphi \in D(\nabla_{L^2(\Omega)} \mathcal{E})$ tel que $\nabla_{L^2(\Omega)} \mathcal{E}(\varphi) = 0$ et*

$$\lim_{t \rightarrow +\infty} \|u(t) - \varphi\|_{W_{p,2}^1(\Omega)} = 0.$$

En 1975, J. Simon [71] a établi la vitesse de convergence du problème (1.4) + (1.5) si $p > 2$, et $f(x, u) \equiv f(x)$ ($x \in \Omega$) pour une fonction $f \in L^{p'}(\Omega)$. On est en mesure d'énoncer l'inégalité suivante qui joue un rôle important dans la preuve du résultat de J. Simon.

LEMME 1.2.3. (J. Simon) *Soit I un ouvert de \mathbb{R} et ϕ une fonction réelle, continue, positive, presque partout dérivable sur I qui vérifie*

$$\phi'(t) + c(\phi(t))^{p/2} \leq k(t) \quad \text{presque partout sur } I,$$

où $c \geq 0$ et k est une fonction positive intégrable sur I . Alors pour tout $s, t \in I$ tels que $s \leq t$ on a

$$\phi(t) \leq \left[\frac{p-2}{2} c(t-s) \right]^{-2/(p-2)} + \int_s^t k(\sigma) d\sigma.$$

et si $\phi(s) > 0$,

$$\phi(t) \leq \left[(\phi(s))^{-(p-2)/2} + \frac{p-2}{2} c(t-s) \right]^{-2/(p-2)} + \int_s^t k(\sigma) d\sigma.$$

Il est facile de généraliser le résultat de J. Simon au cas où la fonction $u \mapsto f(x, u)$ est croissante uniformément en $x \in \Omega$. Donc nous avons le résultat suivant.

THÉORÈME 1.2.9. Soient $d \geq 1$, $p > 2$, Ω un ouvert borné avec un bord de classe C^1 , et $u \mapsto f(x, u)$ est une fonction croissante uniformément en $x \in \Omega$ qui vérifie (1.18). Si u est une solutions globale du problème (1.4) + (1.5) qui converge vers une solution stationnaire ϕ associée, alors on a

$$\|u(t) - \phi\|_{L^2(\Omega)} \leq \left[\frac{1}{\|u(0) - \phi\|_{L^2(\Omega)}^{p-2}} + \frac{p-2}{c} t \right]^{-1/(p-2)} \quad \text{pour tout } t > 0,$$

où c est une constante positive.

Nous voulons brièvement décrire la démonstration de ce résultat.

DÉMONSTRATION DU THÉORÈME 1.2.9. Notons d'abord que le p -laplacien est fortement monotone quand $p \geq 2$ (cf. E. DiBenedetto [20, Lemma 4.4]); on a

$$\int_{\Omega} [|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2] [\nabla u_1 - \nabla u_2] dx \geq \gamma \int_{\Omega} |\nabla u_1 - \nabla u_2|^p dx$$

pour tout $u_1, u_2 \in W_{p,2}^1(\Omega)$, où γ est une constante positive qui depend de p et d . Multiplions donc l'équation (1.4) avec $u - \phi$ et utilisons le fait que $W_{p,2}^1(\Omega)$ s'injecte continûment dans $L^2(\Omega)$. On a

$$(1.19) \quad \frac{d}{dt} \frac{1}{2} \|u(t) - \phi\|_{L^2(\Omega)}^2 + \frac{\gamma}{c} \|u(t) - \phi\|_{L^2(\Omega)}^{p/2} \leq -(f(x, u(t)) - f(x, \phi), u(t) - \phi)_{L^2(\Omega)} \leq 0$$

p.p. sur $]0, +\infty[$. Pour achever la démonstration, il suffit donc d'appliquer le Lemme 1.2.3. \square

Remarquons que dans le cas $p = 2$, l'inégalité (1.19) implique que, toute solution u du problème (1.4) + (1.5) qui converge vers une solution stationnaire ϕ associée, vérifie la vitesse de convergence classique

$$\|u(t) - \phi\|_{L^2(\Omega)} \leq \|u(0) - \phi\|_{L^2(\Omega)} e^{-\gamma/c t} \quad \text{pour tout } t > 0.$$

Récemment, R. Chill et A. Fiorenza [19] ont montré que l'énergie \mathcal{E} donné par (1.10) vérifie au voisinage de $\phi = 0$ une version généralisée de l'inégalité de Łojasiewicz-Simon sous l'hypothèse que $f(x, u) = f(u)$, $f(0) = 0$ et f vérifie des conditions de décroissance lorsque $u \rightarrow 0$.

1.2.4 Le résultat principal du Chapitre 2

Le but du deuxième chapitre de cette thèse est d'établir en dimension $d = 1$ et pour $p \neq 2$ que toute solution bornée de l'équation (1.4) vérifiant au bord

$$(1.20) \quad \begin{cases} \alpha_0 u(a, t) - \beta_0 |u_x(a, t)|^{p-2} u_x(a, t) = 0 & \text{pour tout } t > 0, \\ \alpha_1 u(b, t) + \beta_1 |u_x(b, t)|^{p-2} u_x(b, t) = 0 & \text{pour tout } t > 0, \end{cases}$$

converge vers une solution φ du problème stationnaire

$$(1.21) \quad \begin{cases} \varphi \in C^1[a, b] \text{ telle que } |\varphi_x|^{p-2}\varphi_x \in C^1[a, b], \\ -\Delta_p \varphi + f(x, \varphi(x)) = 0 \quad \text{pour tout } x \in [a, b], \\ \alpha_0 \varphi(a) - \beta_0 |\varphi_x(a)|^{p-2}\varphi_x(a) = 0, \\ \alpha_1 \varphi(b) + \beta_1 |\varphi_x(b)|^{p-2}\varphi_x(b) = 0. \end{cases}$$

Ici, on suppose que $\Omega =]a, b[$ est un intervalle borné, $u \mapsto f(x, u)$ est une fonction lipschitzienne uniformément en $x \in [a, b]$, et $\alpha_0, \beta_0, \alpha_1, \beta_1 \in [0, 1]$ sont des constantes données telles que

$$\alpha_0 + \beta_0 = 1 \quad \text{et} \quad \alpha_1 + \beta_1 = 1.$$

Le théorème suivant contient le résultat principal du Chapitre 2 (cf. Theorem 2.3.1).

THÉORÈME 1.2.10. *Si $1 < p \leq 2$, alors pour toute solution globale u du problème (1.4) + (1.20) qui est bornée à valeurs dans $L^2(\Omega)$, il existe une unique solution φ du problème stationnaire (1.21) telle que*

$$u(t) \rightarrow \varphi \quad \text{dans } C^1[0, 1] \quad \text{lorsque } t \rightarrow +\infty.$$

Pour établir ce résultat nous reprenons l'idée de H. Matano dans [61] et étendons cette méthode pour le p -laplacien dans le cas $1 < p < 2$. Il faut évoquer que sous les hypothèses du Théorème 1.2.10, nous ne savons pas étendre cette méthode pour le cas $p > 2$ à cause de l'argument suivant : Un des points essentiels de la démonstration original du Matano (cf. ci-dessus) est de différencier les solutions du problème stationnaire (1.21). Ceci est possible s'il y a unicité locale des solutions du problème de Cauchy de l'équation différentielle associée à (1.21). Soient $x_0 \in [a, b]$, $\varphi_1, \varphi_2 \in \mathbb{R}$ et considérons le problème de Cauchy suivant associé à l'équation (1.8) :

$$-\Delta_p \varphi(x) + f(x, \varphi(x)) \quad \text{pour } x \in [a, b], \quad \varphi(x_0) = \varphi_0, \quad \varphi_x(x_0) = \varphi_1.$$

En substituant les variables v_1, v_2 de la sorte

$$v_1 = \varphi \quad \text{et} \quad v_2 = |\varphi_x|^{p-2}\varphi_x,$$

on obtient le système des équations différentielles d'ordre 1 suivant

$$\begin{pmatrix} \frac{d}{dx} v_1 \\ \frac{d}{dx} v_2 \end{pmatrix} = \begin{pmatrix} |v_2|^{\frac{2-p}{p-1}} v_2 \\ f(x, v_1) \end{pmatrix}, \quad v_1(x_0) = \tilde{\varphi}_0, v_2(x_0) = \tilde{\varphi}_1.$$

Ce système admet pour tout $x_0 \in [a, b]$, $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathbb{R}$ une solution unique si $g(s) := |s|^{(2-p)/(p-1)}s$, ($s \in \mathbb{R}$), est localement lipschitzienne sur \mathbb{R} . Or $g(s)$ est localement lipschitzienne sur \mathbb{R} si et seulement si $1 < p \leq 2$.

Même pour le cas $1 < p < 2$, il n'est pas évident d'étendre la méthode de Matano [61]. Pour appliquer cette méthode, il est nécessaire que l'ensemble ω -limite soit muni de la topologie induite par la norm de $C^1[a, b]$. En plus, dans ce cas non linéaire, on ne peut pas appliquer le principe du maximum fort directement. Pour contourner ce point, nous avons établi un principe de comparaison qui est suffisamment fort (cf. Lemma 2.5.1).

1.3. Introduction et résumé du Chapitre 3 : Une inégalité de Hardy pondérée et non existence des solutions positives non triviales

Le troisième chapitre de cette thèse est dédié à l'étude de la non existence des solutions faibles positives non triviales du problème

$$(1.22) \quad \begin{cases} u_t - K_p u = \frac{\lambda}{x^p} |u|^{p-2} u & \text{dans } \Omega \times]0, T[, \\ u = 0 & \text{sur } \partial\Omega \times]0, T[, \\ u(0) = u_0 \geq 0 & \text{dans } \Omega, \end{cases}$$

sous l'hypothèse que Ω est un ouvert de \mathbb{R}^d tel que $0 \in \Omega$ si la dimension $d \geq 2$, $\Omega =]0, +\infty[$ si $d = 1$, et $1 < p < +\infty$. Si A est une matrice symétrique réelle semi-définie positive d'ordre d , $c > 0$, et

$$(1.23) \quad d\mu := \rho(x) dx \quad \text{avec la densité} \quad \rho(x) = c \cdot \exp(-\frac{1}{p}(x^t A x)^{p/2}), \quad x \in \Omega,$$

alors K_p est l'opérateur différentiel non linéaire défini par

$$(1.24) \quad K_p u = \Delta_p u + \langle |\nabla u|^{p-2} \nabla u, \frac{\nabla \rho}{\rho} \rangle \quad \text{pour tout } u \in W_{loc}^{1,p}(\Omega).$$

Nous désignons par $L_\mu^p(\Omega)$ l'espace des fonction u mesurable sur Ω à valeurs dans \mathbb{R} et telles que $\int_\Omega |u|^p d\mu$ est finie. Comme d'habitude, on identifie deux fonction de $L_\mu^p(\Omega)$ qui coïncident presque partout sur Ω . De plus, on désigne par $W_\mu^{1,p}(\Omega)$ l'espace de toute $u \in L_\mu^p(\Omega)$ telle que toute dérivée partielle $\frac{\partial u}{\partial x_i}$ ($i = 1, \dots, d$) au sens des distributions appartient à $L_\mu^p(\Omega)$ et $W_{\mu,0}^{1,p}(\Omega)$ la fermeture de $C_c^1(\Omega)$ dans $W_\mu^{1,p}(\Omega)$.

Pour aborder le problème de la non existence des solutions positives, il est essentiel d'établir une nouvelle inégalité de Hardy pondérée. Donc dans une première étape, nous prouvons que l'inégalité

$$(1.25) \quad \left(\frac{|d-p|}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} d\mu \leq \int_\Omega |\nabla u|^p d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \text{sgn}(d-p) \int_\Omega |u|^p \frac{(x^t A x)^{p/2}}{|x|^p} d\mu$$

est vérifiée avec la constante $\left(\frac{|d-p|}{p}\right)^p$ optimale pour tout $u \in W_{\mu,0}^{1,p}(\Omega)$ si la dimension $d \geq 2$, $1 < p < d$ et pour tout $u \in W_{\mu,0}^{1,p}(\Omega \setminus \{0\})$ si $p > d \geq 1$. Si $p > d \geq 1$ et si A est définie positive, nous pouvons déduire de l'inégalité (1.25), une inégalité pondérée de Poincaré sur $W_{\mu,0}^{1,p}(\Omega \setminus \{0\})$.

1.3.1 L'inégalité de Hardy et non existence des solutions positives non triviales

Dans le cas où $A \equiv 0$ et $c = 1$, la mesure de Borel μ définie en (1.23) se réduit à la mesure de Lebesgue sur Ω de telle façon que l'inégalité (1.25) deviennent l'inégalité classique de Hardy

$$(1.26) \quad \left(\frac{|d-p|}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \leq \int_\Omega |\nabla u|^p dx$$

pour tout $u \in W_0^{1,p}(\Omega)$ si $d \geq 2$, $1 < p < d$, et pour tout $u \in W_0^{1,p}(\Omega \setminus \{0\})$ si $p > d \geq 1$. L'inégalité (1.26) a été introduite pour la dimension $d = 1$ sur l'intervalle $\Omega =]0, +\infty[$ dans le

travail pionnier [44] de G. H. Hardy en 1920. Depuis, multiples généralisations de l'inégalité de Hardy ont été trouvées avec des applications dans des branches mathématiques diverses qu'on peut trouver par exemple dans Mitrinović, Pečarić et Fink [64, p. 175f]. Parmi ces applications, on essaiera tout au long de cette introduction générale de revoir et mettre en valeur la relation étroite entre l'inégalité de Hardy (1.26) ou (1.25) et la théorie de la non existence des solutions positives des équations paraboliques perturbées par un potentiel singulier.

En 1984, P. Baras et J. Goldstein ont prouvé dans [7] le résultat suivant.

THÉORÈME 1.3.1. (Baras-Goldstein, 1984) *Soit $\Omega =]0, R[$ pour $1 \leq R \leq +\infty$ si $d = 1$, et un domaine de \mathbb{R}^d qui contient la boule unité ouverte et à frontière régulière si $d \geq 2$. Si $\lambda > \left(\frac{d-2}{2}\right)^2$ et si $u_0 \in L^1(\Omega) \setminus \{0\}$ est positive, alors le problème*

$$(1.27) \quad \frac{\partial u}{\partial t} - \Delta u = \frac{\lambda}{|x|^2} u \quad \text{dans } \Omega \times]0, T[, \quad u = 0 \quad \text{sur } \partial\Omega \times]0, T[, \quad u(\cdot, 0) = u_0 \quad \text{dans } \Omega$$

n'admet pas de solution positive quelque soit $T > 0$.

Ici, et tout au long de cet article, une fonction mesurable $u : \Omega \times]0, T[\rightarrow \mathbb{R}$ sera dite positive si $u(x, t) \geq 0$ presque partout sur $\Omega \times]0, T[$. Il est clair que le phénomène de la non existence trouve son origine au niveau du potentiel singulier $\lambda|x|^{-p}$ qui est contrôlé par l'inégalité de Hardy (1.26) et avec sa constante optimale.

En 1999, une seconde preuve plus intuitive du Théorème 1.3.1 a été élaborée par X. Cabré et Y. Martel (cf. [16]). Ils ont montré dans [16] que l'existence et la non existence des solutions positives de l'équation (1.27) sont déterminées par la valeur propre généralisée de $-\Delta - \lambda|x|^{-2}$ définie par

$$\sigma(\lambda|x|^{-2}, \Omega) := \inf_{0 \neq \varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 d\mu - \int_\Omega \frac{\lambda}{|x|^2} |\varphi|^2 d\mu}{\int_\Omega |\varphi|^2 d\mu}.$$

En 2003, J. Goldstein et I. Kombe ont démontré dans [35] que la méthode due à X. Cabré et Y. Martel dans [16] peut être très utile afin d'établir la non existence locale en temps des solutions positives pour les équations de diffusion non linéaire singulières. Ils ont abouti dans [35] au résultat suivant.

THÉORÈME 1.3.2. (Goldstein-Kombe, 2003) *Si $d = 1$, on prend $\Omega =]0, +\infty[$ et $1 < p < 2$ et si $d \geq 2$, on prend Ω un domaine borné de \mathbb{R}^d à frontière régulière tel que $0 \in \Omega$ et $\frac{2d}{d+1} \leq p < 2$. Si $\lambda > \left(\frac{d-p}{p}\right)^p$ et si $u_0 \in L^1(\Omega) \setminus \{0\}$ est positive, alors le problème*

$$(1.28) \quad u_t - \Delta_p u = \frac{\lambda}{|x|^p} u^{p-1} \quad \text{dans } \Omega \times]0, T[, \quad u = 0 \quad \text{sur } \partial\Omega \times]0, T[, \quad u(\cdot, 0) = u_0 \quad \text{dans } \Omega$$

n'admet pas de solution positive quelque soit $T > 0$.

Dans les dernières années, l'existence et le comportement qualitative des solutions positives des problèmes (1.27) et (1.28) a été étudié intensivement, notamment par H. Brezis et J. L. Vázquez [14], J. A. Aguilar Crespo et I. Peral Alonso [2] et J. L. Vázquez et E. Zuazua [76]. En particulier, J. P. García Azorero et I. Peral Alonso ont réussi dans [30] à établir le résultat suivant en utilisant la méthode de la séparation des variables.

THÉORÈME 1.3.3. (García Azorero - Peral Alonso, 2003) *Soit Ω un domaine borné de \mathbb{R}^d tel que $0 \in \Omega$ et $2 < p < d$. Si $\lambda > \left(\frac{d-p}{p}\right)^p$ et si $u_0 \in L^\infty(\Omega)$ est positive de sorte que $u_0 \geq \delta > 0$ dans un voisinage de $x = 0$, alors le problème (1.28) n'admet pas de solution positive quelque soit $T > 0$.*

À ce stade, il est important de mentionner le travail [28] dû à V. Galaktionov. Dans cet article, l'auteur a employé le premier théorème de C. Sturm [74] de la dénombrabilité du changement de signe pour montrer que l'hypothèse de positivité dans les Théorèmes 1.3.1 et 1.3.2 peut être omise pourvu que la donnée initiale soit continue et $u_0(0) > 0$.

Récemment, G. R. Goldstein, J. A. Goldstein et A. Rhandi ont établi dans [34] l'inégalité pondérée de Hardy (1.25) pour le cas $p = 2$ et en utilisant l'approche dans [16] de X. Cabré et Y. Martel, ils ont démontré le résultat remarquable suivant.

THÉORÈME 1.3.4. (Goldstein-Goldstein-Rhandi, 2011) *Soient $d \geq 3$, $p = 2$, A une matrice symétrique réelle semi-définie positive d'ordre d , et μ la mesure de Borel sur \mathbb{R}^d définie par (1.23) avec $c = \left(\int \rho dx\right)^{-1}$. Si $\lambda > \left(\frac{d-2}{2}\right)^2$ et si $u_0 \in L^2_\mu(\mathbb{R}^d) \setminus \{0\}$ est positive, alors le problème*

$$(1.29) \quad \frac{\partial u}{\partial t} - \Delta u + \langle Ax, \nabla u \rangle = \frac{\lambda}{|x|^2} u \quad \text{dans } \mathbb{R}^d \times]0, +\infty[, \quad u(\cdot, 0) = u_0 \quad \text{dans } \mathbb{R}^d$$

n'admet pas de solution global u vérifiant

$$u \in C([0, +\infty[; L^2_\mu(\mathbb{R}^d)) \quad \text{et} \quad \|u(t)\|_{L^2_\mu(\mathbb{R}^d)} \leq M e^{\omega t} \|u_0\|_{L^2_\mu(\mathbb{R}^d)} \quad \text{pour tout } t \geq 0,$$

où $M \geq 1$ et $\omega \in \mathbb{R}$ sont des constantes.

En d'autres termes, si $\lambda > \left(\frac{d-2}{2}\right)^2$, l'opérateur $\Delta - \langle Ax, \nabla \cdot \rangle + \lambda |x|^{-2}$ n'engendre pas un semi-groupe fortement continu positive dans $L^2_\mu(\mathbb{R}^d)$. Si on compare le Théorème 1.3.4 avec le Théorème 1.3.1, alors il semble que le terme de convection non bornée Ax qui apparait dans l'opérateur symétrique de Ornstein-Uhlenbeck $L = \Delta - \langle Ax, \nabla \cdot \rangle$ a une influence forte concernant la non existence des solutions positives du problème (1.29). En analogie aux résultats cités ci-dessus, il est naturel de se demander si le phénomène de la non existence des solutions positives persiste dans (1.29), si la diffusion dévient singulière ou dégénérée. Plus précisément, est-ce qu'il est encore possible d'établir la non existence des solutions positives du problème (1.29) si on remplace l'opérateur linéaire L par l'opérateur non linéaire K_p défini par (1.24) et si on remplace $\lambda |x|^{-2}$ par $\lambda |x|^{-p}$. Notons que l'opérateur K_p a été introduit dans [33] par G. R. Goldstein, J. A. Goldstein and A. Rhandi. Ils ont appelé K_p le *p-kolmogorovien*.

1.3.2 Les résultats principaux du Chapitre 3

Le but du troisième chapitre est d'établir la non existence des solutions positives du problème (1.22). Dans notre étude nous traiterons séparément le cas de la dimension $d = 1$ et le cas $d \geq 2$.

En dimension $d = 1$, nous supposons que $\Omega =]0, +\infty[$, $1 < p < +\infty$, et la matrice $A = (a_{11})$ satisfait $a_{11} = a \geq 0$. Dans ce cas, la densité ρ de la mesure μ définie par (1.23) se réduit à

$$\rho(x) = c \cdot \exp\left(-\frac{1}{p} a^{p/2} |x|^p\right) \quad \text{quelque soit } x \in]0, +\infty[, \text{ et pour } c > 0 \text{ fixe.}$$

Le théorème suivant contient le résultat principal en dimension $d = 1$ (cf. Theorem 3.4.1).

THÉORÈME 1.3.5. *Sous les hypothèses ci-dessus, les assertions suivantes sont vraies :*

- (i) Si $1 < p \leq 2$, $\lambda > \left(\frac{p-1}{p}\right)^p$ et si $u_0 \in L^2_{\mu,loc}([0, +\infty[) \setminus \{0\}$ est positive, alors le problème (1.22) n'admet pas de solution faible positive quelque soit $T > 0$.
- (ii) Si $p > 2$, $\lambda > \left(\frac{p-1}{p}\right)^p$ et si $u_0 \in L^2_{\mu,loc}([0, +\infty[)$ est positive tel qu'il existe $r, \delta > 0$ pour lesquelles

$$(1.30) \quad \operatorname{ess\,inf}_{x \in]0, r[} u_0(x) \geq \delta > 0,$$

alors le problème (1.22) n'admet pas de solution faible positive quelque soit $T > 0$.

En dimension $d \geq 2$, nous supposons que Ω est un domaine de \mathbb{R}^d tel que $0 \in \Omega$, $1 < p < +\infty$ et A est une matrice symétrique réelle définie positive d'ordre d ou $A \equiv 0$. Le théorème suivant contient le résultat principal si $d \geq 2$ (cf. Theorem 3.4.4).

THÉORÈME 1.3.6. *Sous les hypothèses ci-dessus, les assertions suivantes sont vraies :*

- (i) Soit $1 < p < 2$ si $d = 2$ ou $\frac{2d}{d+2} \leq p \leq 2$ si $d \geq 3$. Si $\lambda > \left(\frac{d-p}{p}\right)^p$ et si $u_0 \in L^2_{\mu,loc}(\Omega) \setminus \{0\}$ est positive, alors le problème (1.22) n'admet pas de solution faible positive quelque soit $T > 0$.
- (ii) Soit $d \geq 2$, $p > 2$ et $p \neq d$. Si $\lambda > \left(\frac{|d-p|}{p}\right)^p$, et si $u_0 \in L^2_{\mu,loc}(\Omega)$ est positive tel qu'il existe $r, \delta > 0$ pour lesquelles u_0 vérifie (1.30), alors le problème (1.22) n'admet pas de solution faible positive quelque soit $T > 0$.

D'après le Théorème 1.3.5 et le Théorème 1.3.6, l'influence du terme de convection Ax qui apparaît dans $L = \Delta - \langle Ax, \nabla \cdot \rangle$ n'est pas assez forte pour empêcher la non existence des solutions positives du problème (1.29). Le Théorème 1.3.6 améliore le résultat du Théorème 1.3.4 dû à G. R. Goldstein, J. A. Goldstein et A. Rhandi.

Pour établir ces deux théorèmes, nous suivons la méthode dans [16] de X. Cabré et Y. Martel. Le point essentiel dans la démonstration des deux Théorème 1.3.5 et 1.3.6 est l'optimalité de la constante $\left(\frac{|d-p|}{p}\right)^p$ dans l'inégalité de Hardy pondérée (1.25). Le théorème suivant est aussi un résultat principal du Chapitre 3 (cf. Theorem 3.2.1).

THEOREM 1.3.1. *Soit Ω un ouvert de \mathbb{R}^d , et A est une matrice symétrique réelle positive semi-définie d'ordre d . Alors, l'inégalité (1.25) est vérifiée pour toute $u \in W^{1,p}_{\mu,0}(\Omega)$ si la dimension $d \geq 2$, $1 < p < d$ et pour tout $u \in W^{1,p}_{\mu,0}(\Omega \setminus \{0\})$ si $p > d \geq 1$. Si de plus $0 \in \Omega$, ou en dimension $d = 1$ il suffit que $x = 0$ est un point au bord de Ω , et si A est définie positive ou $A \equiv 0$, alors la constante $\left(\frac{|d-p|}{p}\right)^p$ est optimale.*

De ce théorème, on déduira facilement l'inégalité de Poincaré pondérée suivante (cf. Corollary 3.2.2).

COROLLAIRE 1.3.1. *Soit Ω un ouvert de \mathbb{R}^d , A une matrice symétrique réelle définie positive d'ordre d et $\lambda(A)$ la première valeur propre de la matrice A , et si $p > d \geq 1$, alors*

$$\left(\frac{p-d}{p}\right)^{p-1} \lambda^{p/2}(A) \int_{\Omega} |u|^p d\mu \leq \int_{\Omega} |\nabla u|^p d\mu \quad \text{pour tout } u \in W^{1,p}_{\mu,0}(\Omega \setminus \{0\}).$$

2

Convergence of bounded solutions of nonlinear parabolic problems on a bounded interval: the singular case

In this chapter, we prove that for every $1 < p \leq 2$ and for every continuous function f mapping from $[0, 1] \times \mathbb{R}$ to \mathbb{R} , which is Lipschitz continuous in the second variable, uniformly with respect to the first one, each bounded solution of the one-dimensional heat equation

$$(2.1) \quad u_t - \{|u_x|^{p-2}u_x\}_x + f(x, u) = 0 \quad \text{in } (0, 1) \times (0, +\infty)$$

with homogeneous Dirichlet boundary conditions converges as $t \rightarrow +\infty$ to a stationary solution. The proof follows an idea of Matano [61] which in particular is based on a comparison principle. Thus, one key step is to prove a comparison principle on non-cylindrical open sets.

This chapter is organized as follows. In the first section, we describe the problem of convergence of bounded solutions and provide an overview of the basic results to the convergence problem. The second section is devoted to a brief preliminary to abstract dynamical systems, introduces the definition of subgradients, and recalls the existence theorems of evolution problems associated to subgradients on Hilbert spaces. In Section 3, we introduce a slightly more general version of equation (2.1) and the associated parabolic boundary valued problem with Dirichlet, Neumann and Robin boundary conditions. We give the definition of solutions to the parabolic boundary value problem and study their regularity. We show that for every solution u of the parabolic problem, which is bounded with values in $L^2(0, 1)$, the set $\{u(t) \mid t \geq 1\}$ is relatively compact in $C^1[0, 1]$. At the end of Section 3, we state our main theorem (Theorem 2.3.1). Section 4 and 7 contain the proof of Theorem 2.3.1. In Section 5 and 6, we prove a comparison principle on non-cylindrical open sets for Dirichlet, Neumann and Robin boundary conditions, which may be of independent interest.

Contents

2.1	Introduction	20
2.2	Preliminaries	21
2.2.1	Abstract dynamical systems	21
2.2.2	Subgradient systems	25
2.3	Main result	27
2.4	Proof of Theorem 2.3.1	31
2.5	A comparison principle for Dirichlet boundary conditions	32
2.6	A comparison principle for Neumann or Robin boundary conditions	37
2.7	Proof of Theorem 2.3.1 (Continued)	39

2.1. Introduction

The aim of this chapter is to study the convergence as $t \rightarrow +\infty$ of bounded solutions of the parabolic boundary value problem

$$(2.2) \quad \begin{cases} u_t(x, t) - \{|u_x(x, t)|^{p-2}u_x(x, t)\}_x + f(x, u(x, t)) = 0 & \text{for } x \in (0, 1), t > 0, \\ u(0, t) = u(1, t) = 0 & \text{for } t > 0, \end{cases}$$

to a stationary solution; i.e., a solution of the elliptic problem

$$(2.3) \quad \begin{cases} -\{|\varphi_x(x)|^{p-2}\varphi_x(x)\}_x + f(x, \varphi(x)) = 0 & \text{for } x \in (0, 1), \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is any continuous function such that $f(x, \cdot)$ is Lipschitz continuous on \mathbb{R} , uniformly for $x \in [0, 1]$, and $1 < p \leq 2$.

In the non-degenerate case $p = 2$, Zelenyak [79] (1968) and Matano [42] (1978) have independently shown that every bounded solution of (2.2) converges to a unique stationary solution as $t \rightarrow +\infty$. The method of Zelenyak in [79] is based on a gradient inequality. It turns out that the therein used gradient inequality is in fact the so-called Łojasiewicz-Simon gradient inequality ([59, 72]), which later has been further studied and developed by several authors in order to prove convergence results and the rate of convergence of bounded solutions for many different infinite dimensional gradient or gradient-like systems (see, for example, Haraux & Jendoubi [41], Haraux & Jendoubi & Kavian [43], or Chill [17]). The idea of Zelenyak has been revisited and extended by Haraux & Jendoubi [42, Cor. 3.2]. On the other hand, Matano's method in [42] is based on the convergence in $C^1[0, 1]$ of the solution of (2.2) to an ω -limit point, a maximum principle on non-cylindrical open sets, and the unique solvability of the ordinary differential equation satisfied by stationary solutions but equipped with initial values instead of boundary values. Both methods in [79] and [42], can not directly be applied to problem (2.2) if the divergence part in (2.2) is singular (i.e. $1 < p < 2$) or degenerate (i.e., $p > 2$) owing to the loss of regularity of the solutions and the unknown maximum principle on non-cylindrical open sets. This has already been mentioned by several authors (see Feireisl & Simondon [25], and Falliero & Simondon [23]) who therein have extended the idea of Matano to obtain convergence of bounded non-negative solutions of

$$(2.4) \quad u_t - \phi(u)_{xx} + g(u)_x + f(u) = 0 \quad \text{in } (-1, 1) \times (0, +\infty)$$

with Dirichlet boundary conditions. The convergence of bounded nonnegative solutions of the equation (2.4) with Neumann or mixed boundary conditions has been studied by Gokiel & Simondon in [32].

We emphasize that the results of [23, 25] and [32] can not be applied to problem (2.2) due to the different character of singularity or degeneracy. Moreover, we study arbitrary solutions which are not necessarily nonnegative and we do not impose monotonicity assumptions on f . In particular, the set of stationary solutions can be infinite. It is not clear whether the set of stationary solutions is always discrete or can be a continuum.

The approach developed by Matano in [42] seems to be a good approach in order to establish asymptotic stability results for solutions of a large class of singular and degenerate nonlinear diffusion problems since the therein used arguments can be circumvented by regularization techniques or by using an appropriate maximum principle on non-cylindrical open sets. We note that all the above mentioned authors studying the convergence of solutions of the equation (2.4) followed closely the approach in [42]. In this chapter, we also follow closely the idea in [42] in all

but the application of the maximum principle on non-cylindrical open sets. Since for both singular and degenerate equations such as (2.2) and (2.4) it is not clear that a maximum principle holds true, the authors in [23, 25] and [32] circumvent this direct application of the maximum principle by approximating the solution u of equation (2.4) by solutions u_n solving a regularized equation for which the maximum principle on non-cylindrical open sets is known to hold. Our idea is different to the one of [23, 25], and [32]: In this chapter, we prove a comparison principle for the equation (2.2) on non-cylindrical open sets and thus we avoid the regularization step from [23, 25, 32]. For the rest we adapt the approach used in [42] to the problem (2.2) in the singular case $1 < p < 2$. We outline that for every solution u of problem (2.2), which is bounded with values in $L^2(0, 1)$, the set $\{u(t) \mid t \geq 1\}$ is relatively compact in $C^1[0, 1]$ and obtain thereby that the ω -limit set is contained in the set of stationary solutions, and it is non-empty, compact, and connected in $C^1[0, 1]$. Our first main result of this chapter is the following.

THEOREM 2.1.1. *Let $1 < p \leq 2$. Then for every global solution u of (2.2), which is bounded with values in $L^2(0, 1)$, there is a solution φ of the stationary problem (2.3) such that*

$$u(t) \rightarrow \varphi \quad \text{in } C^1[0, 1] \quad \text{as } t \rightarrow +\infty.$$

2.2. Preliminaries

This preliminary section is devoted to recall briefly some fundamental definitions and basic results concerning abstract dynamical systems and abstract subgradient systems in Hilbert spaces. These introduced notions and results will be used in the subsequent sections of this chapter.

2.2.1 Abstract dynamical systems

In this subsection, we follow closely the classical book [40] by A. Haraux and the lecture notes [18] by R. Chill and E. Fařangová. Throughout this subsection, let (Z, d) be a metric space consisting of a set Z and a metric d defined on Z .

DEFINITION 2.2.1. We call a family $\{S(t)\}_{t \geq 0}$ of mappings $S(t)$ from Z to Z a *dynamical system* on (Z, d) provided it satisfies the following four conditions:

- (i): For all $t \geq 0$, $S(t)$ is continuous from Z to Z .
- (ii): $S(0) = \text{id}_Z$, where id_Z denotes the identity on Z .
- (iii): For all $s, t \geq 0$, $S(t + s) = S(t) \circ S(s)$.
- (iv): For all $z \in Z$, $S(\cdot)z$ is continuous from $[0, +\infty)$ to Z .

Until the end of this subsection, let $\{S(t)\}_{t \geq 0}$ be a dynamical system on a metric space (Z, d) .

REMARK 2.2.1. It is not hard to see that if F is a subset of Z such that

$$S(t)F \subseteq F \quad \text{for all } t \geq 0,$$

then the family $\{S(t)|_F\}_{t \geq 0}$ of restrictions of $S(t)$ on F defines a dynamical system on (F, d) .

DEFINITION 2.2.2. For every $z \in Z$, we call the continuous mapping $t \mapsto S(t)z$ from $[0, +\infty)$ to Z the *trajectory* of z and the set $\bigcup_{t \geq 0} \{S(t)z\}$ the *orbit* of z .

DEFINITION 2.2.3. For every $z \in Z$, we call the set

$$\omega(z) := \{ \varphi \in Z \mid \text{there exists } (t_n)_{n \geq 1} \subseteq [0, +\infty) \text{ s.t. } t_n \nearrow +\infty \text{ and } \lim_{n \rightarrow +\infty} S(t_n)z = \varphi \text{ in } Z \}$$

the ω -limit set of the trajectory $t \mapsto S(t)z$.

PROPOSITION 2.2.4. For every $z \in Z$, we have that

$$(2.5) \quad \omega(z) = \bigcap_{s \geq 0} \overline{\{S(t)z \mid t \geq s\}}^Z$$

and hence the ω -limit set of the trajectory $t \mapsto S(t)z$ is a closed subset of Z .

PROOF. Fix $z \in Z$. If $\varphi \in \omega(z)$, then by definition of $\omega(z)$, there is a sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $S(t_n)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Since for every $s \geq 0$, there is an $n_s \geq 1$ such that $t_n \geq s$ for all $n \geq n_s$, φ belongs to the closure of the set $\{S(t)z \mid t \geq s\}$ in Z for all $s \geq 0$. Thus, $\omega(z)$ is contained in the set on the right-hand side of (2.5).

Now, let φ be an element of the set on the right-hand side of (2.5). Then, for every positive integer $n \geq 1$, there is a sequence $(t_{n_k})_{k \geq 1} \subseteq [n, +\infty)$ such that $S(t_{n_k})z \rightarrow \varphi$ in Z as $k \rightarrow +\infty$. Thus, in particular, for every $n \geq 1$, we find a $t_n \geq n$ such that $d(S(t_n)z, \varphi) \leq \frac{1}{n}$. Therefore $\varphi \in \omega(z)$ and hence the equality in (2.5) holds true.

Since the arbitrary intersection of closed subsets of a topological space is again closed, it is a direct consequence of the characterization (2.5) that $\omega(z)$ is closed. This completes the proof of this proposition. \square

PROPOSITION 2.2.5. If for $z \in Z$, the trajectory $t \mapsto S(t)z$ has a relatively compact image in Z , then $\omega(z)$ is nonempty, compact, and connected.

PROOF. First, let $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ be defined by $t_n = n$ for every integer $n \geq 1$. Since by hypothesis, the set $\{S(t_n)z \mid n \geq 1\}$ is relatively compact in Z , there is a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ and some $\varphi \in Z$ such that $S(t_{k_n})z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Thus $\varphi \in \omega(z)$ and therefore, $\omega(z)$ is nonempty.

Owing to Proposition 2.2.4, $\omega(z)$ is closed in Z and contained in the compact set $\overline{\cup_{t \geq 0} \{S(t)z\}}$. Thus $\omega(z)$ is compact.

In order to prove that $\omega(z)$ is connected, we assume the contrary and shall reach to a contradiction. Then, there are two open disjoint subsets $U, V \subseteq Z$ such that $\omega(z) \subseteq U \cup V$, $\omega(z) \cap U \neq \emptyset$, and $\omega(z) \cap V \neq \emptyset$. The last conditions imply that there are $\varphi_1 \in \omega(z) \cap U$ and $\varphi_2 \in \omega(z) \cap V$, sequences $(t_n^{(1)})_{n \geq 1}, (t_n^{(2)})_{n \geq 1}$ such that $t_n^{(1)} \nearrow +\infty, t_n^{(2)} \nearrow +\infty, S(t_n^{(1)})z \rightarrow \varphi_1$ in Z and $S(t_n^{(2)})z \rightarrow \varphi_2$ in Z as $n \rightarrow +\infty$. Since U and V are both open and since both sequences $(t_n^{(1)})_{n \geq 1}$ and $(t_n^{(2)})_{n \geq 1}$ are strictly increasing and unbounded, we can extract a subsequence $(t_{k_n}^{(1)})_{n \geq 1}$ of $(t_n^{(1)})_{n \geq 1}$ and $(t_{k_n}^{(2)})_{n \geq 1}$ of $(t_n^{(2)})_{n \geq 1}$ such that for every $n \geq 1$, $t_{k_n}^{(2)} > t_{k_n}^{(1)} \geq n$, $S(t_{k_n}^{(1)})z \in U$, and $S(t_{k_n}^{(2)})z \in V$. Then, for every $n \geq 1$, the set

$$K_n := S([t_{k_n}^{(1)}, t_{k_n}^{(2)}])z = \left\{ S(t)z \mid t \in [t_{k_n}^{(1)}, t_{k_n}^{(2)}] \right\}$$

is connected as the continuous image of the connected set $[t_{k_n}^{(1)}, t_{k_n}^{(2)}]$, $K_n \cap U \neq \emptyset$, and $K_n \cap V \neq \emptyset$. Thus, the set K_n can not be contained in the union $U \cup V$ of the open and disjoint sets U and V .

Hence for every $n \geq 1$, there is a $\tau_n \in (t_{k_n}^{(1)}, t_{k_n}^{(2)})$ such that $S(\tau_n)z \in (U \cup V)^C$. Since the set $\{S(\tau_n)z \mid n \geq 1\}$ is contained in a relatively compact set, there are $\varphi \in Z$ and a subsequence $(\tau_{k_n})_{n \geq 1}$ of $(\tau_n)_{n \geq 1}$ such that $S(\tau_{k_n})z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. By construction of the sequence $(\tau_{k_n})_{n \geq 1}$, $\varphi \in \omega(z)$ and $S(\tau_{k_n})z \in (U \cup V)^C$ for all $n \geq 1$. Thus and since the set $(U \cup V)^C$ is closed, $\varphi \in (U \cup V)^C$. But this obviously contradicts to the assumption that $\omega(u) \subseteq U \cup V$. \square

PROPOSITION 2.2.6. *For every $z \in Z$ and every $t \geq 0$, we have that*

$$\omega(S(t)z) = \omega(z) \quad \text{and} \quad S(t)(\omega(z)) \subseteq \omega(z).$$

PROOF. First, let $\varphi \in \omega(S(t)z)$ for an arbitrary but fixed $t \geq 0$. Then, there is a sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $S(t_n + t)z = S(t_n)S(t)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Since $t_n + t \nearrow +\infty$, we have thereby shown that $\varphi \in \omega(z)$. On the other hand, if $\varphi \in \omega(z)$, then there is sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $S(t_n)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Since for each $t \geq 0$, there is a $n_0 \geq 1$ such that $t_n - t \in [0, +\infty)$ for all $n \geq n_0$ and $S(t_n - t)S(t)z = S(t_n)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$, we obtain that φ belongs to $\omega(S(t)z)$ for every $t \geq 0$. Thus $\omega(z)$ is contained in $\omega(S(t)z)$ and therefore we have shown that $\omega(S(t)z) = \omega(z)$.

Now, let $\varphi \in \omega(z)$ and take $(t_n)_n \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $S(t_n)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Since for every $t \geq 0$, $S(t)$ is continuous from Z to Z ,

$$S(t + t_n)z = S(t)S(t_n)z \rightarrow S(t)\varphi \quad \text{in } Z \quad \text{as } n \rightarrow +\infty.$$

Thus $S(t)\varphi \in \omega(z)$ and since $\varphi \in \omega(z)$ has been arbitrary, we have thereby shown that the set $S(t)(\omega(z))$ is contained in $\omega(z)$. \square

PROPOSITION 2.2.7. *If for $z \in Z$, the trajectory $t \mapsto S(t)z$ has a relatively compact image in Z , then the following assertions hold true:*

(1) *For all $t \geq 0$, $S(t)(\omega(z)) = \omega(z) \neq \emptyset$.*

(2) *We have that*

$$d(S(t)z, \omega(z)) := \inf_{\varphi \in \omega(z)} d(S(t)z, \varphi) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

(3) *The limit $\lim_{t \rightarrow +\infty} S(t)z$ exists in Z if and only if $\omega(z)$ contains exactly one element.*

PROOF. To see that claim (1) holds, we fix $t \geq 0$ and $\varphi \in \omega(z)$. Then, there is a sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $S(t_n)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Further, there is a $n_0 \geq 1$ such that $\tau_n := t_n - t \geq 0$ for all $n \geq n_0$ and $\tau_n \nearrow +\infty$ as $n \rightarrow +\infty$. Since the trajectory $t \mapsto S(t)z$ has a relatively compact image, there is a subsequence $(\tau_{k_n})_{n \geq 1}$ of $(\tau_n)_{n \geq 1}$ and there is a $w \in Z$ such that $\tau_{k_n} \nearrow +\infty$ and $S(\tau_{k_n})z \rightarrow w$ in Z as $n \rightarrow +\infty$. Thus and by the continuity of $S(t)$ from Z to Z , we obtain that

$$S(t)w = \lim_{n \rightarrow +\infty} S(t)S(\tau_{k_n})z = \lim_{n \rightarrow +\infty} S(t_{k_n})z = \varphi.$$

Thus $\varphi = S(t)w \in S(t)(\omega(z))$ and so by Proposition 2.2.6, claim (1) holds true.

In order to verify that claim (2) holds, we assume the contrary. Then, there is some $\varepsilon > 0$

and a sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $d(S(t_n), \omega(z)) \geq \varepsilon$ for all $n \geq 1$. By hypothesis, the trajectory $t \mapsto S(t)z$ has a relatively compact image in Z . Thus, there is a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ and a $\varphi \in Z$ such that $d(S(t_{k_n})z, \varphi) \rightarrow 0$ as $n \rightarrow +\infty$. Since

$$0 < \varepsilon \leq d(S(t_{k_n})z, \omega(z)) \leq d(S(t_{k_n})z, \varphi) \rightarrow 0 \quad \text{for all } n \geq 1,$$

sending $n \rightarrow +\infty$ in these inequalities, obviously leads to a contradiction.

Now, we show that claim (3) holds. To do so, we assume first that the limit $\lim_{t \rightarrow +\infty} S(t)z =: \varphi$ exists in Z and fix $\tilde{\varphi} \in \omega(z)$. Then there is a sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \nearrow +\infty$ and $S(t_n)z \rightarrow \tilde{\varphi}$ in Z and hence by the uniqueness of the limit, we have that $\varphi = \tilde{\varphi}$. Therefore $\omega(z)$ contains exactly one element. Now, we assume that $\omega(z) = \{\varphi\}$ for some $\varphi \in Z$. Since the trajectory $t \mapsto S(t)z$ has a relatively compact image in Z , for every sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ such that $t_n \rightarrow +\infty$, there is some $\tilde{\varphi} \in Z$ and we can extract a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ such that $S(t_{k_n})z \rightarrow \tilde{\varphi}$ in Z as $n \rightarrow +\infty$. Since every sequence in \mathbb{R} admits a monotone subsequence, we can assume without loss of generality that the sequence $(t_{k_n})_{n \geq 1}$ is increasing. Hence $\tilde{\varphi} \in \omega(z)$ and so by assumption, $\tilde{\varphi} = \varphi$. Thereby we have shown that from every sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ satisfying $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$, one can extract a subsequence $(t_{k_n})_{n \geq 1}$ such that $S(t_{k_n})z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Thus the limit $\lim_{t \rightarrow +\infty} S(t)z$ exists in Z . \square

DEFINITION 2.2.8. We call a continuous function $\Phi : Z \rightarrow \mathbb{R}$ a *Liapunov function* or an *energy* for the dynamical system $\{S(t)\}_{t \geq 0}$ if for every $z \in Z$, the function Φ is decreasing along the trajectory $t \mapsto S(t)z$.

THEOREM 2.2.1. (LaSalle's invariance principle) *Let Φ be a Liapunov function for the dynamical system $\{S(t)\}_{t \geq 0}$. If $z \in Z$ such that the trajectory $t \mapsto S(t)z$ has a relatively compact image in Z , then the following assertions are true:*

- (1) *The limit $\lim_{t \rightarrow +\infty} \Phi(S(t)z) =: c$ exists in \mathbb{R} ,*
- (2) *For every $\varphi \in \omega(z)$, we have $\Phi(\varphi) = c$; i.e., the energy $\Phi \equiv c$ on $\omega(z)$.*
- (3) *In particular, for every $\varphi \in \omega(z)$, we have that $\Phi(S(t)\varphi) \equiv \Phi(\varphi) = c$ on $[0, +\infty)$.*

PROOF. By hypothesis, the set $\{S(t)z \mid t \geq 0\}$ is relatively compact in Z . Thus and since Φ is continuous on Z , the function Φ is bounded on the set $\{S(t)z \mid t \geq 0\}$. Thus and since Φ is a Liapunov function for the dynamical system $\{S(t)\}_{t \geq 0}$, the function $t \mapsto \Phi(S(t)z)$ is decreasing and bounded from below on $[0, +\infty)$. Thus by the monotone convergence theorem, we have that

$$\lim_{t \rightarrow +\infty} \Phi(S(t)z) = \inf_{t \in \mathbb{R}_+} \Phi(S(t)z) =: c \quad \text{exists in } \mathbb{R} \text{ and therefore claim (1) holds.}$$

Now, we fix $\varphi \in \omega(z)$. Then, there is a sequence $(t_n)_{n \geq 1} \subseteq [0, +\infty)$ satisfying $t_n \nearrow +\infty$ and $S(t_n)z \rightarrow \varphi$ in Z as $n \rightarrow +\infty$. Thus by the continuity of Φ on Z , the limit

$$\lim_{n \rightarrow +\infty} \Phi(S(t_n)z) = \Phi(\varphi).$$

On the other hand, by claim (1), the function $t \mapsto \Phi(S(t)z)$ admits a limit as $t \rightarrow +\infty$. Thus $\Phi(\varphi) = c$. Since $\varphi \in \omega(z)$ has been arbitrary, we have thereby proved that the claim (2) holds.

According to Proposition 2.2.6, the ω -limit set $\omega(z)$ is an invariant set for the dynamical system $\{S(t)\}_{t \geq 0}$; in other words, $S(t)(\omega(z)) \subseteq \omega(z)$ for all $t \geq 0$. Thus for all $t \geq 0$ and for each

$\varphi \in \omega(z)$, $S(t)\varphi \in \omega(z)$ and so by claim (2), we obtain that $\Phi(S(t)\varphi) = c = \Phi(\varphi)$. This completes the proof of this Theorem. \square

DEFINITION 2.2.9. We call an element $z \in Z$ an *equilibrium point* or a *steady state* of the dynamical system $\{S(t)\}_{t \geq 0}$ provided z is a fix point for the family $\{S(t)\}_{t \geq 0}$; i.e., $S(t)z = z$ for all $t \geq 0$. We denote by \mathbb{E} the set of all equilibrium points of $\{S(t)\}_{t \geq 0}$.

DEFINITION 2.2.10. We call a Liapunov function Φ for $\{S(t)\}_{t \geq 0}$ a *strict Liapunov function* if Φ satisfies the following condition: if for $z \in Z$, the function $t \mapsto \Phi(S(t)z)$ is constant on $[0, +\infty)$, then the trajectory $t \mapsto S(t)z$ is constant on $[0, +\infty)$.

COROLLARY 2.2.2. (LaSalle's invariance principle, Part II) *If Φ is a strict Liapunov function for $\{S(t)\}_{t \geq 0}$ and if $z \in Z$ such that the trajectory $t \mapsto S(t)z$ has a relatively compact image in Z , then $\omega(z)$ is contained in the set of equilibrium points \mathbb{E} .*

PROOF. By LaSalle's invariance principle (Theorem 2.2.1), for every $\varphi \in \omega(z)$,

$$\Phi(S(t)\varphi) = \Phi(\varphi) = c \quad \text{for all } t \geq 0$$

and since Φ is a strict Liapunov function for $\{S(t)\}_{t \geq 0}$, $S(t)\varphi = \varphi$ for all $t \geq 0$. Therefore the claim of this corollary holds. \square

2.2.2 Subgradient systems

In this subsection, we recall some important fact concerning the existence and uniqueness theory of (nonlinear) evolution problems associated to subgradients on Hilbert spaces. These evolution problems generate a dynamical system and have the particular property that there is a natural Liapunov function. Here we closely follow the standard literature [12, §I.3] and [11] by H. Brezis. Throughout this subsection, let H be a real Hilbert space with inner product $(\cdot, \cdot)_H$.

DEFINITION 2.2.11. A function $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *lower semicontinuous* if for all $\alpha \in \mathbb{R}$, the set $\{u \in H \mid \mathcal{E}(u) \leq \alpha\}$ is closed.

DEFINITION 2.2.12. A function $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *convex* if for all $u, v \in H$, and for all $\lambda \in [0, 1]$,

$$\mathcal{E}(\lambda u + (1 - \lambda)v) \leq \lambda \mathcal{E}(u) + (1 - \lambda)\mathcal{E}(v).$$

If the function \mathcal{E} is not identically $+\infty$, then we call \mathcal{E} *proper*. With other words, \mathcal{E} is proper if its *effective domain* $D(\mathcal{E}) := \{u \in H \mid \mathcal{E}(u) < +\infty\}$ is not empty.

DEFINITION 2.2.13. If $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex, proper, function, then for every $u \in D(\mathcal{E})$, we call the set

$$\partial_H \mathcal{E}(u) := \left\{ h \in H \mid (h, v - u)_H \leq \mathcal{E}(v) - \mathcal{E}(u) \text{ for all } v \in H \right\}$$

the *subgradient* of \mathcal{E} at u . Further, we call the possible set-valued mapping $\partial_H \mathcal{E}$ from H to the power set of H , the *subgradient* of \mathcal{E} . We say that $\partial_H \mathcal{E}$ is *single-valued* provided the set $\partial_H \mathcal{E}(u)$

consists of one element. The *effective domain* of the subgradient is the set

$$D(\partial_H \mathcal{E}) := \{u \in H \mid \partial_H \mathcal{E}(u) \neq \emptyset\}.$$

DEFINITION 2.2.14. (Strong solution) Let $A : H \rightarrow 2^H$ be a set-valued mapping with effective domain $D(A) := \{u \in H \mid Au \neq \emptyset\}$, and let $f \in L^1(0, T; H)$ for some $T > 0$. Then, we call every function $u \in C([0, T]; H)$ a *strong solution* of equation $\dot{u} + Au = f$ if $u : (0, T) \rightarrow H$ is locally absolutely continuous and for almost every $t \in (0, T)$, one has

$$u(t) \in D(A) \quad \text{and} \quad f(t) - \dot{u}(t) \in Au(t).$$

LEMMA 2.2.3. Let $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function on H with subgradient $\partial_H \mathcal{E}$. Suppose that $u \in L^2(0, T; H)$ satisfying $\dot{u} \in L^2(0, T; H)$ and $u(t) \in D(\partial_H \mathcal{E})$ for almost every $t \in (0, T)$. If there exists a function $g \in L^2(0, T; H)$ such that $g(t) \in \partial_H \mathcal{E}(u(t))$ for almost every $t \in (0, T)$, then the function $t \mapsto \mathcal{E}(u(t))$ is absolutely continuous on $[0, T]$. In particular, if \mathcal{L} denotes the set of all point $t \in (0, T)$ such that $u(t) \in D(\partial_H \mathcal{E})$ and if the functions $s \mapsto u(s)$ and $s \mapsto \mathcal{E}(u(s))$ are differentiable at $s = t$, then one has that

$$\frac{d}{dt} \mathcal{E}(u(t)) = (h, \dot{u}(t))_H \quad \text{for all } t \in \mathcal{L} \text{ and all } h \in \partial_H \mathcal{E}(u(t)).$$

THEOREM 2.2.4. (Brezis [9, Proposition 5]) Let $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function on H . We denote by $\overline{D(\mathcal{E})}$ the closure of $D(\mathcal{E})$ in H . Then, for every $u_0 \in \overline{D(\mathcal{E})}$ and every $f \in L^2(0, T; H)$, there is a unique strong solution u of the abstract Cauchy problem

$$\begin{cases} \dot{u} + \partial_H \mathcal{E}(u) \ni f & \text{on } (0, T), \\ u(0) = u_0 \end{cases}$$

satisfying

$$(2.6) \quad u(t) \in D(\partial_H \mathcal{E}) \quad \text{for a. e. } t \in (0, T),$$

$$(2.7) \quad u(t) \text{ is differentiable from the right at a. e. } t \in (0, T), \quad \dot{u} \in L^2(\delta, T; H) \text{ for all } \delta \in (0, T),$$

$$(2.8) \quad \mathcal{E} \circ u \in L^1(0, T), \text{ and } \mathcal{E} \circ u \text{ is absolutely continuous on } [\delta, T] \text{ for all } \delta \in (0, T).$$

If, in addition, $u_0 \in D(\mathcal{E})$, then $\dot{u} \in L^2(0, T; H)$ and $\mathcal{E} \circ u \in C[0, T]$.

THEOREM 2.2.5. (Brezis [11, Proposition 3.12]) Let $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function on H and g be a mapping from $[0, T] \times \overline{D(\mathcal{E})}$ to H satisfying

(i) there is a $L \geq 0$ such that

$$\|g(t, u_1) - g(t, u_2)\|_H \leq L \|u_1 - u_2\|_H \quad \text{for all } u_1, u_2 \in \overline{D(\mathcal{E})}, \text{ and a.e. } t \in (0, T),$$

(ii) for every $u \in \overline{D(\mathcal{E})}$, the mapping $t \mapsto g(t, u)$ belongs to $L^2(0, T; H)$.

Then, for every $u_0 \in \overline{D(\mathcal{E})}$, there is a unique strong solution $u \in C([0, T]; H)$ of Cauchy problem

$$(2.9) \quad \begin{cases} \dot{u} + \partial_H \mathcal{E}(u) + g(t, u) \ni 0 & \text{on } (0, T), \\ u(0) = u_0 \end{cases}$$

satisfying (2.6), (2.7), and (2.8). If, in addition, $u_0 \in D(\mathcal{E})$, then $\dot{u} \in L^2(0, T; H)$ and $\mathcal{E} \circ u \in C[0, T]$.

THEOREM 2.2.6. (Brezis [9, Proposition 8]) *Let $\mathcal{E} : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function on H and let $g : \overline{D(\mathcal{E})} \rightarrow H$ be a Lipschitz continuous mapping. Then, for every $u_0 \in \overline{D(\mathcal{E})}$, there is a unique strong solution $u \in C([0, +\infty); H)$ of Cauchy problem (2.9) with $g(u)$ in stead of $g(t, u)$ satisfying*

- (i) $u(t) \in D(\partial_H \mathcal{E})$ for all $t > 0$,
- (ii) $u(t)$ is differentiable from the right at every $t > 0$
- (iii) $\dot{u} \in L^\infty(\delta, T; H)$ for all $0 < \delta < T < +\infty$,
- (iv) $t \mapsto \mathcal{E}(u(t))$ belongs to $L^1(0, T)$ and is absolutely continuous on $[\delta, T]$ for all $0 < \delta < T < +\infty$.

If, in addition, $u_0 \in D(\mathcal{E})$, then $\dot{u} \in L_{loc}^\infty([0, +\infty); H)$ and $\mathcal{E} \circ u \in C[0, +\infty)$.

2.3. Main result

In fact, we study the convergence as $t \rightarrow +\infty$ of bounded solutions of the following more general problem

$$(2.10) \quad \begin{cases} u_t(x, t) - a(x, u_x(x, t))_x + f(x, u(x, t)) = 0 & \text{for } x \in (0, 1), t > 0, \\ \alpha_0 u(0, t) - \beta_0 a(0, u_x(0, t)) = 0 & \text{for } t > 0, \\ \alpha_1 u(1, t) + \beta_1 a(1, u_x(1, t)) = 0 & \text{for } t > 0, \end{cases}$$

where f is given as above, $\alpha_0, \beta_0, \alpha_1, \beta_1 \in [0, 1]$ are such that $\alpha_0 + \beta_0 = 1$ and $\alpha_1 + \beta_1 = 1$, and the function $a : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following suitable regularity, growth, and coercivity assumptions:

$$(H \text{ i}) \quad \begin{cases} (x, u) \mapsto a(x, u) & \text{is continuous on } [0, 1] \times \mathbb{R}, \\ u \mapsto a(x, u) & \text{is strictly increasing on } \mathbb{R}, \end{cases}$$

$$(H \text{ ii}) \quad \begin{cases} \text{there are } p \in (1, +\infty), c_1 \in L^p(0, 1), c_2 \geq 0, \text{ and } \eta > 0 \text{ such that} \\ |a(x, u)| \leq c_1(x) + c_2|u|^{p-1} & \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in (0, 1) \\ a(x, u)u \geq \eta|u|^p & \text{for all } u \in \mathbb{R} \text{ and all } x \in [0, 1] \end{cases}$$

and

$$(H \text{ iii}) \quad \begin{cases} \text{there is a } b \in C([0, 1] \times \mathbb{R}) \text{ such that} \\ b(x, \cdot) \text{ is locally Lipschitz continuous, uniformly for } x \in [0, 1], \\ a(x, b(x, u)) = u, \quad b(x, a(x, u)) = u & \text{for all } u \in \mathbb{R} \text{ and all } x \in [0, 1], \end{cases}$$

Obviously, the prototype example of a function a satisfying the hypotheses (H i) and (H ii) is $a(x, u) := |u|^{p-1} \text{sgn}(u)$ for all $(x, u) \in [0, 1] \times \mathbb{R}$ ($1 < p < +\infty$). For this choice of a , problem (2.10) reduces to problem (2.2) from the Introduction. However, it is easily verified for this example that the inverse function $b(x, u) := |u|^{\frac{1}{p-1}} \text{sgn}(u)$ is locally Lipschitz continuous if and only if $1 < p \leq 2$.

It is well-known that problem (2.10) can be rewritten as an abstract gradient system on $L^2(0, 1)$ associated with the energy $\mathcal{E} : W_0^{1,p}(0, 1) \rightarrow \mathbb{R}$ given by

$$(2.11) \quad \mathcal{E}(u) := \int_0^1 A(x, u_x(x))dx + \int_0^1 F(x, u(x))dx \quad \text{for all } u \in W_0^{1,p}(0, 1)$$

(for Dirichlet boundary conditions), or with the energy $\mathcal{E} : W^{1,p}(0, 1) \rightarrow \mathbb{R}$ given by

$$(2.12) \quad \mathcal{E}(u) := \int_0^1 A(x, u_x(x))dx + \int_0^1 F(x, u(x))dx + \frac{\alpha_0}{2\beta_0}u^2(0) + \frac{\alpha_1}{2\beta_1}u^2(1)$$

for all $u \in W^{1,p}(0, 1)$ (for Neumann or Robin boundary conditions, $\beta_i > 0$). Here A , and F are primitives of a , and f , respectively, with respect to the second variable. Note that in order to keep this chapter readable we do not treat all possibilities of homogeneous mixed boundary conditions. It follows from the hypotheses (H i), (H ii), the continuity of f on $[0, 1] \times \mathbb{R}$, the Lipschitz continuity of f in the second variable, uniformly with respect to the first one, and the theory of subdifferential operators on Hilbert spaces (see Lemma 2.2.3 and Theorem 2.2.6) that for every $u_0 \in L^2(0, 1)$ there exists a unique function

$$(2.13) \quad u \in C(\mathbb{R}_+; L^2(0, 1)) \cap W_{loc}^{1,\infty}((0, +\infty); L^2(0, 1))$$

such that

- (i) the function u is differentiable from the right in every $t > 0$,
- (ii) we have for all $t > 0$,

$$(2.14) \quad u(\cdot, t) \in W^{1,p}(0, 1) \quad \text{and} \quad a(\cdot, u_x(\cdot, t)) \in W^{1,2}(0, 1),$$

- (iii) we have

$$(2.15) \quad \begin{cases} \frac{du}{dt_+}(\cdot, t) - a(\cdot, u_x(\cdot, t))_x + f(\cdot, u(\cdot, t)) = 0 & \text{in } L^2(0, 1) \quad \text{for all } t > 0, \\ \alpha_0 u(0, t) - \beta_0 a(0, u_x(0, t)) = 0 & \text{for all } t > 0, \\ \alpha_1 u(1, t) + \beta_1 a(1, u_x(1, t)) = 0 & \text{for all } t > 0, \\ u(\cdot, 0) = u_0(\cdot), \end{cases}$$

- (iv) the map $t \mapsto \mathcal{E}(u(t))$ is locally absolutely continuous on $(0, +\infty)$, and

$$(2.16) \quad \int_{t_1}^{t_2} \|\dot{u}(t)\|_{L^2(0,1)}^2 dt + \mathcal{E}(u(t_2)) = \mathcal{E}(u(t_1)) \quad \text{for all } 0 < t_1 < t_2.$$

We call every function u satisfying the above properties a solution of (2.10).

PROPOSITION 2.3.1. *Suppose that a satisfies the hypotheses (H i) and (H ii). Then, every solution u of problem (2.10) belongs to*

$$C((0, +\infty); C^1[0, 1]).$$

PROOF. Let u be a solution of problem (2.10). We first note that by (2.14) and by the Sobolev embedding of $W^{1,2}(0, 1)$ into $C[0, 1]$, we have $a(\cdot, u_x(\cdot, t)) \in C[0, 1]$ for all $t > 0$. By hypothesis (H i), there is a function $b \in C([0, 1] \times \mathbb{R})$ such that $u_x(x, t) = b(x, a(x, u_x(x, t)))$ for all

$(x, t) \in [0, 1] \times (0, +\infty)$. Thus, $u(\cdot, t) \in C^1[0, 1]$ for all $t > 0$. In order to prove the claim of this proposition, we show that for every $t_0 \in (0, +\infty)$ and every sequence $(t_n)_{n \geq 1} \subseteq (0, +\infty)$ such that $t_n \rightarrow t_0$ as $n \rightarrow +\infty$, there exists a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ such that

$$(2.17) \quad u(\cdot, t_{k_n}) \rightarrow u(\cdot, t_0) \quad \text{in } C^1[0, 1] \quad \text{as } n \rightarrow +\infty.$$

So fix $t_0 \in (0, +\infty)$ and let $(t_n)_{n \geq 1} \subseteq (0, +\infty)$ be a sequence such that $t_n \rightarrow t_0$ as $n \rightarrow +\infty$. Since $u : \mathbb{R}_+ \rightarrow L^2(0, 1)$ is continuous and since $u \mapsto \int_0^1 F(x, u(x)) dx$ maps bounded sets of $L^2(0, 1)$ into bounded sets of \mathbb{R} , the sequence $(\int_0^1 F(x, u(x, t_n)) dx)_{n \geq 1}$ is bounded. The map $t \mapsto \mathcal{E}(u(t))$ is continuous on $(0, +\infty)$ and in particular, $t \mapsto \mathcal{E}(u(t))$ is locally bounded. Hence $(\mathcal{E}(u(t_n)))_{n \geq 1}$ is bounded, too. By the coercivity assumption in hypothesis (H ii), for every $t > 0$,

$$(2.18) \quad \frac{\eta}{p} \int_0^1 |u_x(t)|^p dx \leq \int_0^1 A(x, u_x(t)) dx \leq \mathcal{E}(u(t)) - \int_0^1 F(x, u(x, t)) dx.$$

Hence $(u(t_n))_{n \geq 1}$ is bounded in $W^{1,p}(0, 1)$. By the reflexivity of $W^{1,p}(0, 1)$, the compact embedding of $W^{1,p}(0, 1)$ into $C([0, 1])$ and since $u(t_n) \rightarrow u(t_0)$ in $L^2(0, 1)$ as $n \rightarrow +\infty$, there is a subsequence of $(t_n)_{n \geq 1}$, which for simplicity we denote again by $(t_n)_{n \geq 1}$, such that

$$(2.19) \quad u(t_n) \rightharpoonup u(t_0) \quad \text{weakly in } W^{1,p}(0, 1) \text{ as } n \rightarrow +\infty,$$

$$(2.20) \quad u(t_n) \rightarrow u(t_0) \quad \text{in } C[0, 1] \text{ as } n \rightarrow +\infty.$$

By the growth condition in hypothesis (H ii), $(a(\cdot, u_x(\cdot, t_n)))_{n \geq 1}$ is bounded in $L^{p'}(0, 1)$. We set

$$g_n := f(\cdot, u(t_n)) + \frac{du}{dt_+}(t_n) \quad \text{for all } n \geq 1.$$

Then, by (2.20), by the continuity of f and by (2.13), $(g_n)_{n \geq 1}$ is bounded in $L^2(0, 1)$. Thus, and since for all $n \geq 1$, the distributional derivative $\frac{d}{dx} a(\cdot, u_x(\cdot, t_n)) = g_n$, we have that the sequence $(a(\cdot, u_x(\cdot, t_n)))_{n \geq 1}$ is bounded in $W^{1,2}(0, 1)$. Thus, by the compact Sobolev embedding of $W^{1,2}(0, 1)$ into $C[0, 1]$, there is a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ and some $\varphi \in C[0, 1]$ such that

$$a(\cdot, u_x(\cdot, t_{k_n})) \rightarrow \varphi \quad \text{in } C[0, 1] \quad \text{as } n \rightarrow +\infty.$$

By hypothesis (H i), the map $u \mapsto b(\cdot, u)$ is continuous from $C[0, 1]$ to $C[0, 1]$ and $b(x, \cdot)$ is the inverse of $a(x, \cdot)$, for every $x \in [0, 1]$. Thus

$$u_x(\cdot, t_{k_n}) \rightarrow b(\cdot, \varphi) \quad \text{in } C[0, 1] \quad \text{as } n \rightarrow +\infty$$

and so by the limit (2.19), it follows that (2.17) holds. \square

PROPOSITION 2.3.2. *Suppose that a satisfies the hypotheses (H i) and (H ii). If u is a solution of problem (2.10) and if u is bounded with values in $L^2(0, 1)$, then for every $\delta > 0$, the set $\{u(t) \mid t \geq \delta\}$ is relatively compact in $C^1[0, 1]$.*

PROOF. Fix $\delta > 0$. Then, obviously, it suffices to show that for every sequence $(t_n)_{n \geq 1} \subseteq [\delta, +\infty)$ there is a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ such that the sequence $(u(t_{k_n}))_{n \geq 1}$ converges in $C^1[0, 1]$. If $(t_n)_{n \geq 1}$ is bounded, then the claim is true owing to Proposition 2.3.1. Now, let $(t_n)_{n \geq 1}$ be unbounded. First, let $t, h > 0$ and let $L > 0$ be the Lipschitz constant of f in the second

variable, uniformly with respect to the first one. Then, by the monotonicity of $u \mapsto a(x, u)$ and the Lipschitz continuity of f ,

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|u(t) - u(t+h)\|_{L^2(0,1)}^2 &= \left(\frac{du}{dt_+}(t) - \frac{du}{dt_+}(t+h), u(t) - u(t+h) \right)_{L^2(0,1)} \\ &= \left(a(\cdot, u_x(t))_x - a(\cdot, u_x(t+h))_x, u(t) - u(t+h) \right)_{L^2(0,1)} \\ &\quad - \left(f(\cdot, u(t)) - f(\cdot, u(t+h)), u(t) - u(t+h) \right)_{L^2(0,1)} \\ &\leq L \|u(t) - u(t+h)\|_{L^2(0,1)}^2. \end{aligned}$$

We integrate this inequality over (s, t) for $0 < s < t$. Then,

$$\frac{1}{2} \|u(t) - u(t+h)\|_{L^2(0,1)}^2 \leq \frac{1}{2} \|u(s) - u(s+h)\|_{L^2(0,1)}^2 + L \int_s^t \|u(r) - u(r+h)\|_{L^2(0,1)}^2 dr.$$

By (2.13), $\frac{du}{dt_+} \in L^\infty(s, t; L^2(0, 1))$. Thus, dividing this inequality by h^2 , letting $h \rightarrow 0+$, and using Lebesgue's dominated convergence theorem, yields to

$$(2.21) \quad \frac{1}{2} \left\| \frac{du}{dt_+}(t) \right\|_{L^2(0,1)}^2 \leq \frac{1}{2} \left\| \frac{du}{dt_+}(s) \right\|_{L^2(0,1)}^2 + L \int_s^t \left\| \frac{du}{dt_+}(r) \right\|_{L^2(0,1)}^2 dr.$$

Since the set $\{u(t) \mid t \geq 0\}$ is bounded in $L^2(0, 1)$ and since $u \mapsto \int_0^1 F(x, u(x)) dx$ maps bounded sets of $L^2(0, 1)$ into bounded sets of \mathbb{R} , the estimate (2.18) implies that the map $t \mapsto \mathcal{E}(u(t))$ is bounded from below. Thus, by the energy equality (2.16), $\frac{du}{dt_+} \in L^2(1, +\infty; L^2(0, 1))$ and so, by inequality (2.21),

$$\frac{du}{dt_+} \in L^\infty(1, +\infty; L^2(0, 1)).$$

Replacing $(t_n)_{n \geq 1}$ by a strictly increasing subsequence, we may assume without loss of generality that $(t_n)_{n \geq 1}$ is strictly increasing. Since, by hypothesis, $(u(t_n))_{n \geq 1}$ is bounded in $L^2(0, 1)$, the sequence $(\int_0^1 F(x, u(x, t_n)) dx)_{n \geq 1}$ is bounded. By the energy equality (2.16), the map $t \mapsto \mathcal{E}(u(t))$ is non-increasing along $(0, +\infty)$ and hence, by the estimate (2.18), $(u(t_n))_{n \geq 1}$ is bounded in $W^{1,p}(0, 1)$. By the compact Sobolev embedding of $W^{1,p}(0, 1)$ into $C[0, 1]$, and by the reflexivity of $W^{1,p}(0, 1)$, there is a subsequence $(t_{k_n})_{n \geq 1}$ of $(t_n)_{n \geq 1}$ and there is a $\xi \in W^{1,p}(0, 1)$ such that

$$\begin{aligned} u(t_{k_n}) &\rightharpoonup \xi && \text{weakly in } W^{1,p}(0, 1) \text{ as } n \rightarrow +\infty, \text{ and} \\ u(t_{k_n}) &\rightarrow \xi && \text{in } C[0, 1] \text{ as } n \rightarrow +\infty. \end{aligned}$$

For the rest of this proof, we can now proceed analogously to the proof of Proposition 2.3.1. \square

Now, let u be a solution of (2.10), which is bounded with values in $L^2(0, 1)$. Then, by Proposition 2.3.2, the set $\{u(t) \mid t \geq 1\}$ is relatively compact in $C^1[0, 1]$. Therefore, the ω -limit set of the solution u of (2.10) given by

$$\omega(u) := \left\{ \varphi \in C^1[0, 1] \mid \exists t_n \nearrow +\infty \text{ s.t. } \lim_{n \rightarrow +\infty} u(t_n) = \varphi \text{ in } C^1[0, 1] \right\}$$

is nonempty and connected. Moreover, $\omega(u)$ contains exactly one element if and only if the limit $\lim_{t \rightarrow +\infty} u(t)$ exists in $C^1[0, 1]$. By the energy equality (2.16) and by LaSalle's invariance principle (see, Theorem 2.2.1 and Corollary 2.2.2), $\omega(u)$ is contained in the set of stationary solutions

of (2.10), that is, the set of solutions φ of the elliptic problem

$$(2.22) \quad \begin{cases} \varphi \in C^1[0, 1] \text{ such that } a(\cdot, \varphi_x(\cdot)) \in C^1[0, 1], \\ -a(x, \varphi_x(x))_x + f(x, \varphi(x)) = 0 & \text{for all } x \in [0, 1], \\ \alpha_0 \varphi(0) - \beta_0 a(0, \varphi_x(0)) = 0, \\ \alpha_1 \varphi(1) + \beta_1 a(1, \varphi_x(1)) = 0. \end{cases}$$

We are ready to formulate our main result which includes, as a special case, Theorem 2.1.1.

THEOREM 2.3.1. *Assume that the hypotheses (H i)-(H iii) are fulfilled. Then for every global solution u of problem (2.10) which is bounded with values in $L^2(0, 1)$ there is a unique solution φ of the stationary problem (2.22) such that*

$$u(t) \rightarrow \varphi \quad \text{in } C^1[0, 1] \quad \text{as } t \rightarrow +\infty.$$

2.4. Proof of Theorem 2.3.1

Throughout this section we assume that the hypotheses (H i)-(H iii) are fulfilled and let u be a global solution of problem (2.10) which is bounded with values in $L^2(0, 1)$.

LEMMA 2.4.1. *If the ω -limit set of u contains more than one element, then for every $\varphi \in \omega(u)$ there is a $t_0 > 0$ such that the function $x \mapsto u(x, t_0) - \varphi(x)$ has at most a finite number of sign changes in $[0, 1]$.*

PROOF. We suppose that $\omega(u)$ contains two distinct elements φ and ψ . We show that there is a $\delta > 0$ such that

$$(2.23) \quad |\varphi(x) - \psi(x)| + |\varphi_x(x) - \psi_x(x)| \geq \delta \quad \text{for all } x \in [0, 1].$$

To this end, let $\tilde{\varphi} \in C^1[0, 1]$ be a stationary solution of problem (2.10). By substituting

$$v_1 = \tilde{\varphi} \quad \text{and} \quad v_2 = a(\cdot, \tilde{\varphi}_x),$$

the differential equation

$$-a(x, \tilde{\varphi}_x(x))_x + f(x, \tilde{\varphi}(x)) = 0$$

is equivalent to the two-dimensional first order system

$$(2.24) \quad v_1' = b(\cdot, v_2), \quad v_2' = f(\cdot, v_1).$$

Due to the hypotheses, both maps b and f are locally Lipschitz continuous in the second variable, uniformly with respect to the first one. Thus, for every $x_0 \in [0, 1]$, $v_{0,1}, v_{0,2} \in \mathbb{R}$, the first order system (2.24) with initial values

$$v_1(x_0) = v_{0,1}, v_2(x_0) = v_{0,2}$$

has a unique solution. Therefore, if we assume for two different $\varphi, \psi \in \omega(u)$ that there is an $x_0 \in [0, 1]$ such that

$$|\varphi(x_0) - \psi(x_0)| + |\varphi_x(x_0) - \psi_x(x_0)| = 0,$$

then the above stated local uniqueness of the initial value problem associated with the first order system (2.24) implies that $\varphi = \psi$. But this obviously contradicts to our hypothesis that φ and ψ are distinct elements of $\omega(u)$.

Since $\psi \in \omega(u)$, there is sequence $(t_n)_{n \geq 1} \subseteq (0, +\infty)$ such that $t_n \nearrow +\infty$ and $u(t_n) \rightarrow \psi$ in $C^1[0, 1]$ as $n \rightarrow +\infty$. Hence, and by using the inequality (2.23), there is a $t_0 > 0$ such that

$$(2.25) \quad |u(x, t_0) - \varphi(x)| + |u_x(x, t_0) - \varphi_x(x)| \geq \frac{\delta}{2} \quad \text{for all } x \in [0, 1].$$

Now, we suppose that the function $x \mapsto u(x, t_0) - \varphi(x)$ admits infinitely many sign changes in the interval $[0, 1]$ and we shall reach a contradiction. In fact, by the intermediate value theorem, there is a sequence $(x_n)_{n \geq 1} \subseteq [0, 1]$ of pairwise different zeros. We can extract a monotone subsequence $(x_{k_n})_{n \geq 1} \subseteq (x_n)_{n \geq 1}$ and we find a zero $x_0 \in [0, 1]$ of the function $u(\cdot, t_0) - \varphi(\cdot)$ such that $x_{k_n} \rightarrow x_0$ as $n \rightarrow +\infty$. By Rolle's theorem, there is a second sequence $(\xi_{k_n})_{n \geq 1} \subseteq [0, 1]$ of pairwise different zeros of the derivative $u_x(\cdot, t_0) - \varphi_x(\cdot)$ such that $x_{k_n} < \xi_{k_n} < x_{k_{n+1}}$ (respectively, $x_{k_n} > \xi_{k_n} > x_{k_{n+1}}$) for every $n \geq 1$. Thus $\xi_{k_n} \rightarrow x_0$ as $n \rightarrow +\infty$, too, and it turns out that the point x_0 is also a zero of the derivative $u_x(\cdot, t_0) - \varphi_x(\cdot)$ in $[0, 1]$. But this contradicts to inequality (2.25). \square

2.5. A comparison principle for Dirichlet boundary conditions

The following notation is similar to the one used by Brown & Hu & Lieberman in [15]. For given $(x_0, t_0) \in \mathbb{R}^2$ and $\rho > 0$ we denote by $Q((x_0, t_0), \rho)$ the cylinder

$$\{(x, t) \in \mathbb{R}^2 \mid |x - x_0| < \rho, \quad t_0 - \rho < t < t_0\}.$$

Let \mathcal{C} be an open subset of \mathbb{R}^2 . If $\partial\mathcal{C}$ denotes its topological boundary and \mathcal{C}^c the complement of \mathcal{C} with respect to \mathbb{R}^2 , then we define by

$$\mathcal{PC} := \{(x, t) \in \partial\mathcal{C} \mid Q((x, t), \rho) \cap \mathcal{C}^c \neq \emptyset \text{ for all } \rho > 0\}$$

the parabolic boundary of \mathcal{C} . We note that if the set \mathcal{C} is a bounded cylinder, i.e., $\mathcal{C} = (a_0, b_0) \times (t_0, t_1)$, then

$$\mathcal{PC} = (\{a_0, b_0\} \times [t_0, t_1]) \cup ([a_0, b_0] \times \{t_0\}).$$

For the following, let $a : \mathcal{C} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy

$$\begin{cases} (x, t) \mapsto a(x, t, u) & \text{is measurable on } \mathcal{C} \text{ for all } u \in \mathbb{R}, \\ u \mapsto a(x, t, u) & \text{is continuous and increasing for every } (x, t) \in \mathcal{C}, \\ \text{there are } p \in (1, +\infty), c_1 \in L^{p'}(\mathcal{C}), \text{ and } c_2 \geq 0 \text{ such that} \\ |a(x, t, u)| \leq c_1(x, t) + c_2|u|^{p-1} & \text{for all } u \in \mathbb{R} \text{ and a.e. } (x, t) \in \mathcal{C}, \end{cases}$$

and let $f : \mathcal{C} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable in the first two variables, Lipschitz continuous in the third variable with constant $L > 0$, uniformly with respect to the first two, and $f(\cdot, \cdot, 0) \in L^2(\mathcal{C})$.

For a continuous function $u : \bar{\mathcal{C}} \rightarrow \mathbb{R}$, we denote by $u^+ := \max\{0, u\}$ its positive part and for every $T \in \mathbb{R}$, we define $\mathcal{C}_T := \{(x, t) \in \mathcal{C} \mid t < T\}$. If the set \mathcal{C} is bounded from below, then we denote by t_{bot} the infimum of all $t \in \mathbb{R}$ for which there exists an $x \in \mathbb{R}$ such that $(x, t) \in \mathcal{C}$.

DEFINITION 2.5.1. Let $\mathcal{C} \subseteq \mathbb{R}^2$ be an open subset. We say that two continuous functions u and v on $\bar{\mathcal{C}}$ satisfy the inequality

$$(2.26) \quad \begin{aligned} u_t(x, t) - a(x, t, u_x(x, t))_x + f(x, t, u(x, t)) \\ \leq v_t(x, t) - a(x, t, v_x(x, t))_x + f(x, t, v(x, t)) \end{aligned}$$

on \mathcal{C} if for all bounded sub-cylinders $Q = (a_0, b_0) \times (t_0, t_1) \subseteq \mathcal{C}$,

$$u, v \in W^{1,2}(t_0, t_1; L^2(a_0, b_0)) \cap C([t_0, t_1]; W^{1,p}(a_0, b_0))$$

and for all non-negative $\zeta \in C_c^1(\mathcal{C})$,

$$(2.27) \quad \begin{aligned} \int_{\mathcal{C}} [u_t - v_t] \zeta \, d(x, t) + \int_{\mathcal{C}} [a(x, t, u_x) - a(x, t, v_x)] \zeta_x \, d(x, t) \\ + \int_{\mathcal{C}} [f(x, t, u) - f(x, t, v)] \zeta \, d(x, t) \leq 0. \end{aligned}$$

LEMMA 2.5.1 (Comparison Principle for Dirichlet boundary conditions). Let $\mathcal{C} \subseteq \mathbb{R}^2$ be an open subset such that for all $T \in \mathbb{R}$, \mathcal{C}_T is bounded and topological regular, that is, the interior $\text{int}(\bar{\mathcal{C}}_T) = \mathcal{C}_T$. If u and v are two continuous functions on $\bar{\mathcal{C}}$ satisfying the inequality (2.26) on \mathcal{C} , then

$$(2.28) \quad \sup_{(x,t) \in \bar{\mathcal{C}}} e^{-L(t-t_{\text{bot}})} (u - v)(x, t) \leq \sup_{(x,t) \in \mathcal{P}\mathcal{C}} e^{-L(t-t_{\text{bot}})} [u - v]^+(x, t).$$

REMARK 2.5.1. We note that for Lemma 2.5.1, we have been inspired by a comparison principle of a linear unilateral problem without Lipschitz-perturbation given by Brezis in [10, Proposition II.5]. See also the maximum principle for classical solutions of the linear inhomogeneous heat equation in Ladyženskaja & Solonnikov & Ural'ceva [53, Theorem 2.1]. We point out that Lemma 2.5.1 remains true if $f : \mathcal{C} \times \mathbb{R} \rightarrow \mathbb{R}$ is only locally Lipschitz continuous in the third variable, uniformly with respect to the first two, and if u and v are uniformly bounded. In this case, the constant L in (2.28) depends on bounds of u and v .

PROOF. Part I: First we consider the case when \mathcal{C} is the bounded cylinder $(a_0, b_0) \times (t_0, t_1)$. Obviously, then $t_0 = t_{\text{bot}}$. Let u and v satisfy the inequality (2.26) on \mathcal{C} . We set

$$k := \sup_{(x,t) \in \mathcal{P}\mathcal{C}} e^{-L(t-t_0)} [u - v]^+(x, t) \quad \text{and} \quad w(x, t) := u(x, t) - v(x, t) - ke^{L(t-t_0)}$$

for all $(x, t) \in \bar{\mathcal{C}}$. Then

$$w^+ \in W^{1,2}(t_0, t_1; L^2(a_0, b_0)) \cap C([t_0, t_1]; W^{1,p}(a_0, b_0))$$

with the weak derivatives

$$\frac{d}{dt} w^+(x, t) = [u_t(x, t) - v_t(x, t) - kL e^{L(t-t_0)}] \cdot \mathbb{1}_{\{u-v > ke^{L(t-t_0)}\}}$$

and

$$\frac{d}{dx} w^+(x, t) = [u_x(x, t) - v_x(x, t)] \cdot \mathbb{1}_{\{u-v > ke^{L(t-t_0)}\}}$$

for a.e. $(x, t) \in \mathcal{C}$. Here we denote by $\mathbb{1}_{\{u-v > ke^{L(t-t_0)}\}}$ the characteristic function of the set

$$\{u - v > ke^{L(t-t_0)}\} := \{(x, t) \in \mathcal{C} \mid u(x, t) - v(x, t) > ke^{L(t-t_0)}\}.$$

Moreover, $w^+(x, t) = 0$ for all $(x, t) \in \mathcal{PC}$. Recall that for every $t \in (t_0, t_1)$, \mathcal{C}_t denotes the cylinder $(a_0, b_0) \times (t_0, t)$. Note that for cylinders \mathcal{C} , an approximation argument shows that inequality (2.27) remains true for non-negative function

$$\xi \in W^{1,2}(t_0, t_1; L^2(a_0, b_0)) \cap \mathcal{C}([t_0, t_1]; W^{1,p}(a_0, b_0))$$

such that $\xi = 0$ on \mathcal{PC} . Thus, by the monotonicity of $u \mapsto a(x, u)$ (hypotheses (H i)), and by the Lipschitz continuity of f , we obtain that for all $t \in (t_0, t_1)$,

$$\begin{aligned} & \frac{1}{2} \int_{a_0}^{b_0} w^+(x, t)^2 dx \\ &= \int_{a_0}^{b_0} \int_{t_0}^t \frac{d}{ds} \frac{1}{2} (w^+(x, s))^2 ds dx \\ &= \int_{\mathcal{C}_t} \frac{d}{ds} w^+(x, s) w^+(x, s) d(x, s) \\ &= \int_{\mathcal{C}_t} [u_s(x, s) - v_s(x, s)] w^+(x, s) d(x, s) - \int_{\mathcal{C}_t} k L e^{L(s-t_0)} w^+(x, s) d(x, s) \\ &\leq - \int_{\mathcal{C}_t} [a(x, s, u_x(x, s)) - a(x, s, v_x(x, s))] [u_x(x, s) - v_x(x, s)] \mathbb{1}_{\{u-v > ke^{L(s-t_0)}\}} d(x, s) \\ &\quad - \int_{\mathcal{C}_t} [f(x, s, u(x, s)) - f(x, s, v(x, s))] w^+(x, s) d(x, s) \\ &\quad - \int_{\mathcal{C}_t} k L e^{L(s-t_0)} w^+(x, s) d(x, s) \\ &\leq L \int_{\mathcal{C}_t} [u(x, s) - v(x, s)] w^+(x, s) d(x, s) - \int_{\mathcal{C}_t} k L e^{L(s-t_0)} w^+(x, s) d(x, s) \\ &= L \int_{t_0}^t \int_{a_0}^{b_0} w^+(x, s)^2 dx ds. \end{aligned}$$

Thus, by Gronwall's inequality (see Gronwall [38]) and by the continuity of w on $\bar{\mathcal{C}}$, we deduce from the above estimates that $w^+(x, t) = 0$ for every $(x, t) \in \bar{\mathcal{C}}$. This shows that the inequality (2.28) holds when \mathcal{C} is a cylinder.

Part II: Now, we assume that the open nonempty set $\mathcal{C} \subseteq \mathbb{R}^2$ is as in the assumption. Let u and v satisfy the inequality (2.26) on \mathcal{C} and let $T > t_{bot}$. We prove the inequality (2.28) for $\mathcal{C} = \mathcal{C}_T$ and for unbounded $\bar{\mathcal{C}}$ let T tend to $+\infty$. We set

$$k := \sup_{(x,t) \in \mathcal{PC}} e^{-L(t-t_{bot})} [u - v]^+(x, t),$$

and

$$w(x, t) := u(x, t) - v(x, t) - ke^{L(t-t_{bot})}, \quad \text{for all } (x, t) \in \bar{\mathcal{C}}.$$

Let $\varepsilon > 0$. Since the function w^+ is uniformly continuous on $\bar{\mathcal{C}}$ and $w^+ = 0$ on \mathcal{PC} , there exists a $\delta > 0$ such that

$$w^+(x, t) < \varepsilon \quad \text{for all } (x, t) \in \Sigma_\delta := \{(x, t) \in \bar{\mathcal{C}} \mid d((x, t), \mathcal{PC}) \leq \delta\}.$$

Here we denote by $d((x, t), A) := \inf_{(y,s) \in A} \max\{|x - y|, |t - s|\}$ the distance of a bounded set $A \subseteq \mathbb{R} \times (0, +\infty)$ and the point (x, t) with respect to the supremum norm. Furthermore, we set

$$\mathcal{C}_\delta := \{(x, t) \in \bar{\mathcal{C}} \mid d((x, t), \mathcal{PC}) > \delta/2\}.$$

1st step: We set

$$t_{top} := \sup \{t \in \mathbb{R} \mid \exists x \in \mathbb{R} \text{ s.t. } (x, t) \in \mathcal{C}\}.$$

Since the set \mathcal{C} is bounded, open and nonempty, $-\infty < t_{bot} < t_{top} < +\infty$. We may always assume that $t_{top} - t_{bot} > \delta$; otherwise we choose $\delta > 0$ smaller. We choose an integer $n_0 \geq 3$ such that $\frac{t_{top} - t_{bot} - \delta}{n_0} \leq \frac{\delta}{2}$ and set

$$t_0 := t_{bot} + \frac{\delta}{2}, \quad t_1 := t_0 + \frac{t_{top} - t_{bot} - \delta}{n_0}.$$

Since the points in $\overline{\mathcal{C}_\delta} \subseteq \overline{\mathcal{C}}$ have positive distance to the parabolic boundary, and by definition of the parabolic boundary,

$$(2.29) \quad \text{for every } (x, t) \in \overline{\mathcal{C}_\delta} \text{ there exists an } \rho > 0 \text{ such that } Q((x, t), \rho) \subseteq \mathcal{C}.$$

As a consequence, there exists an open set $\ell_1 \subseteq \mathbb{R}$ such that

$$\mathcal{C}_\delta \cap \{t = t_1\} = \ell_1 \times \{t_1\}.$$

Since ℓ_1 is open, there is a family $((a_i^{(1)}, b_i^{(1)}))_{i \in I_1}$ of open and pairwise disjoint intervals such that $\ell_1 = \bigcup_{i \in I_1} (a_i^{(1)}, b_i^{(1)})$. Let $Q_i^{(1)} := (a_i^{(1)}, b_i^{(1)}) \times (t_0, t_1)$. We first show that $Q_i^{(1)} \subseteq \mathcal{C}$. By (2.29) and a compactness argument, there exists a $\rho > 0$ such that $(a_i^{(1)}, b_i^{(1)}) \times (t_1 - \rho, t_1) \subseteq \mathcal{C}$. Hence, if we define

$$t' := \inf \{t \in (t_0, t_1) \mid (a_i^{(1)}, b_i^{(1)}) \times (t, t_1) \subseteq \mathcal{C}\}$$

then $t_0 \leq t' < t_1$ and

$$Q_1 := (a_i^{(1)}, b_i^{(1)}) \times (t', t_1) \subseteq \mathcal{C}.$$

Assume that $t' > t_0$. Note that for every $x \in [a_i^{(1)}, b_i^{(1)}]$, one has $(x, t') \in \overline{Q_i^{(1)}} \subseteq \overline{\mathcal{C}}$ and

$$d((x, t'), \mathcal{PC}) \geq d((x, t_1), \mathcal{PC}) - (t_1 - t') \geq \delta/2 - (t_1 - t') > 0,$$

so that $[a_i^{(1)}, b_i^{(1)}) \times \{t'\} \subseteq \overline{\mathcal{C}}$ has no intersection with the parabolic boundary. A compactness argument again implies that there exists a $\rho' > 0$ such that

$$Q_2 := (a_i^{(1)}, b_i^{(1)}) \times (t' - \rho', t') \subseteq \mathcal{C}.$$

As a consequence,

$$(a_i^{(1)}, b_i^{(1)}) \times (t' - \rho', t_1) = \text{int}(\overline{Q_1 \cup Q_2}) \subseteq \text{int}(\overline{\mathcal{C}}) = \mathcal{C}.$$

But this is a contradiction to the definition of t' if $t' > t_0$. Hence, $Q_i^{(1)} \subseteq \mathcal{C}$. It follows from (2.29) and the definition of \mathcal{C}_δ that

$$d((a_i^{(1)}, t_1), \mathcal{PC}) \leq \delta/2 \quad \text{and} \quad d((b_i^{(1)}, t_1), \mathcal{PC}) \leq \delta/2.$$

By the triangle inequality, for every $t \in [t_0, t_1]$,

$$d((a_i^{(1)}, t), \mathcal{PC}) \leq d((a_i^{(1)}, t_1), \mathcal{PC}) + (t_1 - t) \leq \delta/2 + \delta/2 = \delta$$

and similarly, $d((b_i^{(1)}, t), \mathcal{PC}) \leq \delta$ so that $\{a_i^{(1)}, b_i^{(1)}\} \times [t_0, t_1] \subseteq \Sigma_\delta$. Moreover, by the definition of t_0 , $[a_i^{(1)}, b_i^{(1)}] \times \{t_0\} \subseteq \Sigma_\delta$, so that all together $\mathcal{P}Q_i^{(1)} \subseteq \Sigma_\delta$.

Since for $\tilde{u} := u - k e^{L(t-t_{bot})}$,

$$(f(x, \tilde{u}) - f(x, u)) - k L e^{L(t-t_{bot})} \leq 0,$$

we find that \tilde{u} and v verify on \mathcal{C} that

$$\begin{aligned} \tilde{u}_t - a(x, \tilde{u}_x)_x + f(x, \tilde{u}) &= u_t - k L e^{L(t-t_{bot})} - a(x, u_x)_x + f(x, \tilde{u}) \\ &= u_t - a(x, u_x)_x + f(x, u) + (f(x, \tilde{u}) - f(x, u)) - k L e^{L(t-t_{bot})} \\ &\leq v_t - a(x, v_x)_x + f(x, v). \end{aligned}$$

Thus, it follows by inequality (2.28) applied to \tilde{u} and v on the cylinder $Q_i^{(1)}$ (proved in part I) that

$$(2.30) \quad w^+(x, t) \leq e^{L \cdot \frac{t_{top} - t_{bot} - \delta}{n_0}} \varepsilon$$

for all $(x, t) \in \overline{Q_i^{(1)}}$. Since this estimate is independent of the index i , we have that w^+ satisfies (2.30) in $\bigcup_{i \in I_1} Q_i^{(1)}$.

2nd Step: Let

$$t_2 := t_{bot} + \frac{\delta}{2} + 2 \frac{t_{top} - t_{bot} - \delta}{n_0}.$$

There exists again a family $((a_i^{(2)}, b_i^{(2)}))_{i \in I_2}$ of open and pairwise disjoint intervals $(a_i^{(2)}, b_i^{(2)})$ such that

$$\mathcal{C}_\delta \cap \{t = t_2\} = \bigcup_{i \in I_2} (a_i^{(2)}, b_i^{(2)}) \times \{t_2\}.$$

Let $Q_i^{(2)} = (a_i^{(2)}, b_i^{(2)}) \times (t_1, t_2)$. Similarly as before, one shows that $Q_i^{(2)} \subseteq \mathcal{C}$. Moreover, the lateral boundary $\{a_i^{(2)}, b_i^{(2)}\} \times [t_1, t_2] \subseteq \Sigma_\delta$, and the bottom $[a_i^{(2)}, b_i^{(2)}] \times \{t_1\}$ is contained in $(\ell_1 \times \{t_1\}) \cup \Sigma_\delta$. Thus, we have that for every $i \in I_2$, w^+ satisfies the estimate (2.30) on the parabolic boundary $\mathcal{P}Q_i^{(2)}$ and so

$$w^+(x, t) \leq e^{L \cdot 2 \cdot \frac{t_{top} - t_{bot} - \delta}{n_0}} \varepsilon$$

for all $(x, t) \in \bigcup_{i \in I_2} \overline{Q_i^{(2)}} \cup \bigcup_{i \in I_1} \overline{Q_i^{(1)}}$. Now, for every $3 \leq n \leq n_0$, let

$$t_n := t_{bot} + \frac{\delta}{2} + n \frac{t_{top} - t_{bot} - \delta}{n_0}.$$

We find again a family $((a_i^{(n)}, b_i^{(n)}))_{i \in I_n}$ of open and pairwise disjoint intervals $(a_i^{(n)}, b_i^{(n)})$ such that

$$\mathcal{C}_\delta \cap \{t = t_n\} = \bigcup_{i \in I_n} (a_i^{(n)}, b_i^{(n)}) \times \{t_n\}.$$

We set

$$\ell_n := \bigcup_{i \in I_n} (a_i^{(n)}, b_i^{(n)}) \quad \text{and} \quad Q_i^{(n)} = (a_i^{(n)}, b_i^{(n)}) \times (t_{n-1}, t_n).$$

By the same reasoning as given for $n = 2$, we obtain that the cylinder $Q_i^{(n)} \subseteq \mathcal{C}$, the lateral boundary $\{a_i^{(n)}, b_i^{(n)}\} \times [t_{n-1}, t_n] \subseteq \Sigma_\delta$, and the bottom $[a_i^{(n)}, b_i^{(n)}] \times \{t_{n-1}\}$ is contained in $\ell_{n-1} \cup \Sigma_\delta$. Thus for each $3 \leq n \leq n_0$, we obtain that

$$w^+(x, t) \leq e^{L \cdot n \cdot \frac{t_{top} - t_{bot} - \delta}{n_0}} \varepsilon \quad \text{for every } (x, t) \in \bigcup_{i=1}^n \bigcup_{j \in I_i} \overline{Q_j^{(i)}}$$

and in particular for $n = n_0$, we have that

$$(2.31) \quad w^+(x, t) \leq e^{L(t_{top}-t_{bot}-\delta)\varepsilon} \leq e^{L(t_{top}-t_{bot})\varepsilon} \quad \text{for all } (x, t) \in \mathcal{Q}_{\delta, n_0} := \bigcup_{i=1}^{n_0} \bigcup_{j \in I_i} \mathcal{Q}_j^{(i)}.$$

Since $\mathcal{C}_\delta \subseteq \mathcal{Q}_{\delta, n_0}$, this estimate in particular holds for all $(x, t) \in \mathcal{C}_\delta$. By construction,

$$w^+(x, t) \leq \varepsilon \leq e^{L(t_{top}-t_{bot})\varepsilon} \quad \text{for all } (x, t) \in \Sigma_\delta$$

and $\mathcal{C}_\delta \cup \Sigma_\delta = \mathcal{C}$. Thus

$$w^+(x, t) \leq e^{L(t_{top}-t_{bot})\varepsilon} \quad \text{for all } (x, t) \in \mathcal{C}$$

and so by letting $\varepsilon \rightarrow 0+$ in this inequality, the claim of this lemma holds. \square

2.6. A comparison principle for Neumann or Robin boundary conditions

For the sequel we need the following notation. If \mathcal{C} is an open subset of $[0, 1] \times \mathbb{R}$, then we denote by

$$\mathcal{S}_0\mathcal{C} := \{(0, t) \in \mathcal{P}\mathcal{C} \mid \exists \rho > 0 \text{ s.t. } \mathcal{Q}((\rho, t + \rho), \rho) \subseteq \overline{\mathcal{C}}\}$$

the left lateral boundary of \mathcal{C} touching the vertical line $\{0\} \times \mathbb{R}$, and by

$$\mathcal{S}_1\mathcal{C} := \{(1, t) \in \mathcal{P}\mathcal{C} \mid \exists \rho > 0 \text{ s.t. } \mathcal{Q}((1 - \rho, t + \rho), \rho) \subseteq \overline{\mathcal{C}}\}$$

the right lateral boundary of \mathcal{C} touching the vertical line $\{1\} \times \mathbb{R}$.

We suppose that f satisfies the same hypotheses as for the Dirichlet boundary conditions and that a satisfies

$$\left\{ \begin{array}{l} (x, t) \mapsto a(x, t, u) \quad \text{is continuous on } \mathcal{C} \text{ for all } u \in \mathbb{R}, \\ u \mapsto a(x, t, u) \quad \text{is continuous and increasing for every } (x, t) \in \mathcal{C}, \\ \text{there are } p \in (1, +\infty), c_1 \in L^{p'}(\mathcal{C}), \text{ and } c_2 \geq 0 \text{ such that} \\ |a(x, t, u)| \leq c_1(x, t) + c_2|u|^{p-1} \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } (x, t) \in \mathcal{C}. \end{array} \right.$$

DEFINITION 2.6.1. Let $\mathcal{C} \subseteq [0, 1] \times \mathbb{R}$ be an open subset, $\alpha_0, \alpha_1 \in [0, 1)$, $I_0, I_1 \subseteq \mathbb{R}$ be two open subsets such that $\{0\} \times I_0 \subseteq \mathcal{S}_0\mathcal{C}$ and $\{1\} \times I_1 \subseteq \mathcal{S}_1\mathcal{C}$, and let $\varphi_0, \psi_0 \in C(I_0)$ and $\varphi_1, \psi_1 \in C(I_1)$. We say that two continuous functions u and v on $\overline{\mathcal{C}}$ satisfy the inequality (2.26) together with the Neumann (if $\alpha_0 = \alpha_1 = 0$) or Robin boundary conditions

$$(2.32) \quad \begin{cases} \alpha_0 u(0, t) - a(0, t, u_x(0, t)) = \varphi_0(t) & \text{for all } t \in I_0, \\ \alpha_1 u(1, t) + a(1, t, u_x(1, t)) = \varphi_1(t) & \text{for all } t \in I_1, \end{cases}$$

and

$$(2.33) \quad \begin{cases} \alpha_0 v(0, t) - a(0, t, v_x(0, t)) = \psi_0(t) & \text{for all } t \in I_0, \\ \alpha_1 v(1, t) + a(1, t, v_x(1, t)) = \psi_1(t) & \text{for all } t \in I_1 \end{cases}$$

if for every bounded sub-cylinder $Q = (a_0, b_0) \times (t_0, t_1) \subseteq \mathcal{C}$,

$$u, v \in W^{1,2}(t_0, t_1; L^2(a_0, b_0)) \cap C([t_0, t_1]; W^{1,p}(a_0, b_0))$$

and for all non-negative $\xi \in C_c^1(\bar{\mathcal{C}})$ with $\xi = 0$ on $\mathcal{P}\mathcal{C} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1))$, the inequality

$$(2.34) \quad \begin{aligned} & \int_{\mathcal{C}} (u_t - v_t) \xi \, d(x, t) + \int_{\mathcal{C}} [a(x, t, u_x) - a(x, t, v_x)] \xi_x \, d(x, t) \\ & - \left\{ \int_{I_1} [\varphi_1(t) - \psi_1(t) + \alpha_1(v(1, t) - u(1, t))] \xi(1, t) \, dt \right. \\ & \quad \left. - \int_{I_0} [\alpha_0(u(0, t) - v(0, t)) + (\psi_0(t) - \varphi_0(t))] \xi(0, t) \, dt \right\} \\ & + \int_{\mathcal{C}} [f(x, t, u) - f(x, t, v)] \xi \, d(x, t) \leq 0 \end{aligned}$$

holds.

LEMMA 2.6.1 (Comparison principle for Neumann or Robin boundary conditions). *Let \mathcal{C} be an open subset of $[0, 1] \times \mathbb{R}$ such that for all $T \in \mathbb{R}$, \mathcal{C}_T is bounded and topological regular. Let $L \geq 0$ be the Lipschitz-constant of f , $\alpha_0, \alpha_1 \in [0, 1)$, $I_0, I_1 \subseteq \mathbb{R}$ be open subsets such that $\{0\} \times I_0 \subseteq \mathcal{S}_0\mathcal{C}$, $\{1\} \times I_1 \subseteq \mathcal{S}_1\mathcal{C}$, $\varphi_0, \psi_0 \in C(I_0)$, and $\varphi_1, \psi_1 \in C(I_1)$. If u and v are two continuous function on $\bar{\mathcal{C}}$ satisfying the inequality (2.26) on \mathcal{C} together with Neumann or Robin boundary conditions (2.32) and (2.33) and if*

$$(2.35) \quad \begin{cases} \varphi_0(t) \leq \psi_0(t) & \text{for all } t \in I_0, \\ \varphi_1(t) \leq \psi_1(t) & \text{for all } t \in I_1, \end{cases}$$

then

$$(2.36) \quad \sup_{(x,t) \in \bar{\mathcal{C}}} e^{-L(t-t_{\text{bot}})} (u - v)(x, t) \leq \sup_{(x,t) \in \mathcal{P}\mathcal{C} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1))} e^{-L(t-t_{\text{bot}})} [u - v]^+(x, t).$$

PROOF. Part I: First, we show that the inequality (2.36) holds when \mathcal{C} is the bounded cylinder $(a_0, b_0) \times (t_0, t_1)$, I_0 and I_1 are open subsets of (t_0, t_1) , and u and v satisfy the inequality (2.26) on \mathcal{C} in connection with the boundary conditions (2.32) on $\{a_0\} \times I_0$ and (2.33) on $\{b_0\} \times I_1$. We set

$$k := \sup_{(x,t) \in \mathcal{P}\mathcal{C} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1))} e^{-L(t-t_0)} [u - v]^+(x, t)$$

and

$$w(x, t) := u(x, t) - v(x, t) - ke^{L(t-t_0)}$$

for all $(x, t) \in \bar{\mathcal{C}}$. Then $w^+ \in W^{1,2}(t_0, t_1; L^2(a_0, b_0)) \cap C([t_1, t_2]; W^{1,p}(a_0, b_0))$ and $w^+ = 0$ on $\mathcal{P}\mathcal{C} \setminus ((\{a_0\} \times I_0) \cup (\{b_0\} \times I_1))$. By the condition (2.35) on $(\{a_0\} \times I_0) \cup (\{b_0\} \times I_1)$, we have that

$$\begin{aligned} & \left\{ \int_{\{b_0\} \times I_1} [\varphi_1(t) - \psi_1(t) + \alpha_1(v(b_0, t) - u(b_0, t))] w^+(b_0, t) \, dt \right. \\ & \quad \left. - \int_{\{a_0\} \times I_0} [\alpha_0(u(a_0, t) - v(a_0, t)) + (\psi_0(t) - \varphi_0(t))] w^+(a_0, t) \, dt \right\} \leq 0. \end{aligned}$$

Thus, we can use the same arguments as in the part I of the proof of Lemma 2.5.1 when we replace the therein used inequality (2.27) by the inequality (2.34).

Part II: Now, let $\mathcal{C} \subseteq [0, 1] \times \mathbb{R}$ be a general open and subset as stated in the hypothesis of this lemma. We fix $T > t_{bot}$ and we first assume that $\mathcal{C} = \mathcal{C}_T$. We set

$$k := \sup_{(x,t) \in \mathcal{PC} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1))} e^{-L(t-t_0)} [u - v]^+(x, t)$$

and

$$w(x, t) := u(x, t) - v(x, t) - ke^{L(t-t_{bot})} \quad \text{for all } (x, t) \in \overline{\mathcal{C} \cap \{t < T\}}.$$

Let $\varepsilon > 0$. Then, by the uniform continuity of w^+ on $\overline{\mathcal{C}}$ and since

$$w^+ = 0 \quad \text{on } \mathcal{PC} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1)),$$

there is a $\delta > 0$ such that

$$w^+(x, t) < \varepsilon \quad \text{for all } (x, t) \in \Sigma_{\delta(\varepsilon)}$$

where

$$\Sigma_{\delta(\varepsilon)} := \{(x, t) \in \overline{\mathcal{C}} \mid d((x, t), \mathcal{PC} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1))) \leq \delta\}.$$

Next, we set

$$\mathcal{C}_\delta := \{(x, t) \in \overline{\mathcal{C}} \mid d((x, t), \mathcal{PC} \setminus ((\{0\} \times I_0) \cup (\{1\} \times I_1))) > \delta/2\}.$$

Now, we can proceed analogously to the part II of the proof of Lemma 2.5.1 where for the estimates on the cylinders $Q_i^{(j)}$ we use inequality (2.36) (proved in part I) instead of inequality (2.28). \square

2.7. Proof of Theorem 2.3.1 (Continued)

We define for every $\varphi \in \omega(u)$ the sets

$$A^+(\varphi) := \{(x, t) \in [0, 1] \times (0, +\infty) \mid u(x, t) - \varphi(x) > 0\}$$

and

$$A^-(\varphi) := \{(x, t) \in [0, 1] \times (0, +\infty) \mid u(x, t) - \varphi(x) < 0\}.$$

Furthermore, we call a connected component \mathcal{C} of $A^+(\varphi)$ (resp., of $A^-(\varphi)$) a maximal open and connected subset of $A^+(\varphi)$ (resp., of $A^-(\varphi)$).

LEMMA 2.7.1. *Let $\varphi \in \omega(u)$ and let $t_0 > 0$. Then for every nonempty connected component \mathcal{C} of the set*

$$(2.37) \quad A^+(\varphi) \cap ([0, 1] \times [t_0, +\infty))$$

there is an $x_0 \in [0, 1]$ such that $(t_0, x_0) \in \mathcal{C}$. The same assertion also holds for the set

$$A^-(\varphi) \cap ([0, 1] \times [t_0, +\infty)).$$

PROOF. We only prove the first claim since the second claim is shown similarly. To do so, we assume there is a nonempty connected component \mathcal{C} of the set (2.37) and there is no element $x \in [0, 1]$ such that $(x, t_0) \in \mathcal{C}$. We replace \mathcal{C} by its topological regularization $\text{int}(\overline{\mathcal{C}})$. Let $S_0 := \{0\} \times (0, +\infty)$ and $S_1 := \{1\} \times (0, +\infty)$. Since $u = \varphi$ on $\mathcal{PC} \setminus (S_0 \cup S_1)$ and since u and φ satisfy on $\mathcal{PC} \cap (S_0 \cup S_1)$, either again $u = \varphi$, or the inequalities (2.35) for the boundary conditions (2.32) and (2.33), we obtain by the comparison principle with respect to the appropriate boundary conditions (Lemma 2.5.1, or Lemma 2.6.1, respectively) that $u = \varphi$ on $\overline{\mathcal{C}}$. But this contradicts that \mathcal{C} is a subset of $A^+(\varphi)$ and therefore the claim of this lemma holds true. \square

LEMMA 2.7.2. *Suppose the ω -limit set $\omega(u)$ contains more than one element. Then, for every $\varphi \in \omega(u)$, there exists a $t_0 > 0$ such that the functions $t \mapsto u(0, t) - \varphi(0)$ and $t \mapsto u_x(0, t) - \varphi_x(0)$ do not change the sign along $[t_0, +\infty)$.*

PROOF. We take $\varphi \in \omega(u)$ and chose $t_0 > 0$ such that the function $u(\cdot, t_0) - \varphi(\cdot)$ has only a finite number of sign changes in $[0, 1]$, according to Lemma 2.4.1.

First, we show that the function $t \mapsto u(0, t) - \varphi(0)$ has only a finite number of sign changes on $[t_0, +\infty)$. If $\beta_0 = 0$, then there is nothing to show. So we assume that $\beta_0 > 0$. We suppose that the function $u(0, \cdot) - \varphi(0)$ has infinitely many sign changes along $[t_0, +\infty)$. Then there is a strictly increasing sequence $(t_n)_{n \geq 1}$ such that for all $n \geq 1$, $u(0, t_{2n-1}) - \varphi(0) > 0$ and $u(0, t_{2n}) - \varphi(0) < 0$, that is, for all $n \geq 1$,

$$(0, t_{2n-1}) \in A^+(\varphi) \cap ([0, 1] \times [t_0, +\infty)) \quad \text{and} \quad (0, t_{2n}) \in A^-(\varphi) \cap ([0, 1] \times [t_0, +\infty)).$$

Let for each $n \geq 1$,

$$\mathcal{C}_{2n-1} \subseteq A^+(\varphi) \cap ([0, 1] \times [t_0, +\infty)) \quad \text{and} \quad \mathcal{C}_{2n} \subseteq A^-(\varphi) \cap ([0, 1] \times [t_0, +\infty))$$

denote the connected component such that $(0, t_n) \in \mathcal{C}_n$. Then, due to Lemma 2.7.1, there is a continuous curve $\gamma_n \subseteq \mathcal{C}_n$ and there is a point $x_n \in [0, 1]$ such that γ_n connects $(0, t_n)$ with (x_n, t_0) . Since for all $n \geq 1$, γ_{2n-1} is contained in $A^+(\varphi)$ and γ_{2n} in $A^-(\varphi)$, the sequence $(x_n)_{n \geq 1}$ is strictly increasing and by construction, $u(x_{2n-1}, t_0) - \varphi(x_{2n-1}) > 0$ and $u(x_{2n}, t_0) - \varphi(x_{2n}) < 0$ for all $n \geq 1$. But this obviously contradicts to our choice of t_0 and hence the first claim of this lemma holds.

Next, we show that the function $t \mapsto u_x(0, t) - \varphi_x(0)$ has only a finite number of sign changes on $[t_0, +\infty)$. Here, due to the monotonicity of the function $u \mapsto a(x, u)$, it obviously suffices to treat the case when $\beta_0 < 1$. We take $\varphi \in \omega(u)$, choose $t_0 > 0$ as given by Lemma 2.4.1, and assume again there is a strictly increasing sequence $(t_n)_{n \geq 1} \subseteq (t_0, +\infty)$ such that for every $n \geq 1$,

$$(2.38) \quad u_x(0, t_{2n-1}) - \varphi_x(0) > 0 \quad \text{and} \quad u_x(0, t_{2n}) - \varphi_x(0) < 0.$$

If $\beta_0 = 0$, then for all $n \geq 1$, $u(0, t_n) - \varphi(0) = 0$. Since $u(\cdot, t_n) \in C^1[0, 1]$, we find a sequence $(x_n)_{n \geq 1}$ of points $x_n \in (0, 1)$ such that in view of (2.38), for every $n \geq 1$, the segments

$$(2.39) \quad (0, x_{2n-1}] \times \{t_{2n-1}\} \subseteq A^+(\varphi) \cap ([0, 1] \times [t_0, +\infty))$$

and

$$(2.40) \quad (0, x_{2n}] \times \{t_{2n}\} \subseteq A^-(\varphi) \cap ([0, 1] \times [t_0, +\infty)).$$

If $\beta_0 \in (0, 1)$, then by the monotonicity of $a(x, \cdot)$ and in view to (2.38), for all $n \geq 1$,

$$u(0, t_{2n-1}) - \varphi(0) = \frac{\beta_0}{\alpha_0} (a(0, u_x(0, t_{2n-1})) - a(0, \varphi_x(0))) > 0 \quad \text{and} \quad u(0, t_{2n}) - \varphi(0) < 0.$$

By the continuity of $u(\cdot, t_n) - \varphi(\cdot)$ on $[0, 1]$ for each $n \geq 1$, we find again a sequence $(x_n)_{n \geq 1} \subseteq (0, 1)$ such that for all $n \geq 1$, (2.39) and (2.40) hold.

Now, for $\beta_0 \in [0, 1)$ and for every $n \geq 1$, let \mathcal{C}_n be the connected component such that

$$(x_{2n-1}, t_{2n-1}) \in \mathcal{C}_{2n-1} \subseteq A^+(\varphi) \cap ([0, 1] \times [t_0, +\infty))$$

and

$$(x_{2n}, t_{2n}) \in \mathcal{C}_{2n} \subseteq A^-(\varphi) \cap ([0, 1] \times [t_0, +\infty)).$$

Then by Lemma 2.7.1, for every $n \geq 1$, there is a point $y_n \in [0, 1]$ such that $(y_n, t_0) \in \mathcal{C}_n$ and so we find a continuous path γ_n in \mathcal{C}_n connecting (x_n, t_n) and (y_n, t_0) . Now, for every $n \geq 1$, we denote by κ_n the path from $(0, t_n)$ to (y_n, t_0) by running along both paths $[0, x_n] \times \{t_n\}$ and γ_n . Due to this construction we have that the path $\kappa_{2n-1} \setminus \{(0, t_{2n-1})\} \subseteq A^+(\varphi)$ and $\kappa_{2n} \setminus \{(0, t_{2n})\} \subseteq A^-(\varphi)$ for every $n \geq 1$. Thus the curves $(\kappa_n)_n$ cannot intersect each other and hence the sequence $(y_n)_{n \geq 1}$ is strictly increasing and for all $n \geq 1$, $u(y_{2n-1}, t_0) - \varphi(y_{2n-1}) > 0$ and $u(y_{2n}, t_0) - \varphi(y_{2n}) < 0$. But this is again a contradiction to the choice of $t_0 > 0$. \square

PROOF OF THEOREM 2.3.1. We assume that the set $\omega(u)$ is not discrete. Since $\omega(u)$ is non-empty and connected, it then contains infinitely many elements. We take three distinct elements of $\omega(u)$ and denote them φ^1, φ^2 , and φ^3 . Because of the different boundary conditions in the stationary problem (2.22), we need to treat two cases:

The case $\alpha_0 = 1$: Owing to the hypotheses (H i), and (H iii), the initial value problem

$$\begin{cases} -a(x, v_x(x))_x + f(x, v(x)) = 0 & \text{for } x \in [0, 1], \\ v(0) = 0, \quad v_x(0) = v_0 \end{cases}$$

admits for every $v_0 \in \mathbb{R}$ a unique solution v on $[0, 1]$. Thus, we may assume without loss of generality that

$$(2.41) \quad \varphi_x^1(0) < \varphi_x^2(0) < \varphi_x^3(0).$$

According to Lemma 2.7.2, there is a $t_0 > 0$ such that the function $t \mapsto u_x(0, t) - \varphi_x^2(0)$ has constant sign along the interval $[t_0, +\infty)$. First, we assume that $u_x(0, t) \geq \varphi_x^2(0)$ for every $t \geq t_0$. Since $\varphi^1 \in \omega(u)$, there is an unbounded strictly increasing sequence $(t_n)_n \subseteq (0, +\infty)$ such that

$$(2.42) \quad u_x(t_n) \rightarrow \varphi_x^1 \quad \text{in } \mathcal{C}[0, 1] \text{ as } n \rightarrow +\infty.$$

But on the other hand, due to inequality (2.41), we have

$$0 < \varphi_x^2(0) - \varphi_x^1(0) \leq u_x(0, t) - \varphi_x^1(0) \leq \|u_x(t) - \varphi_x^1\|_{\mathcal{C}[0,1]} \quad \text{for all } t \geq t_0,$$

which contradicts to the convergence in (2.42) and hence φ^1 can not be an element of $\omega(u)$. Similarly, we obtain that φ^3 can not belong to $\omega(u)$ if $u_x(0, t) \leq \varphi_x^2(0)$ for all $t \geq t_0$. Thus, in this case the set $\omega(u)$ contains at most two different elements, which contradicts to our assumption that $\omega(u)$ is not discrete.

The case $\alpha_0 \in [0, 1)$: Since under the hypotheses (H i), and (H iii), the initial value problem

$$\begin{cases} -a(x, v_x(x))_x + f(x, v(x)) = 0 & \text{for } x \in [0, 1], \\ v(0) = v_0, \quad a(0, v_x(0)) = \frac{\alpha_0}{\beta_0} v(0) \end{cases}$$

possesses for every $v_0 \in \mathbb{R}$ a unique solution v on $[0, 1]$, the values $\varphi^1(0)$, $\varphi^2(0)$, and $\varphi^3(0)$ have to be different from one another. Now, arguing as in the case $\alpha_0 = 1$ yields that also in this case the set $\omega(u)$ contains at most two different elements and therefore also in this case our assumption is false.

This shows that the claim of Theorem 2.3.1 holds. □

3

A weighted Hardy inequality and nonexistence of positive nontrivial solutions

This chapter contains the joint work with Abdelaziz Rhandi¹.

This chapter is dedicated to the study of nonexistence of positive nontrivial weak solutions of

$$(3.1) \quad u_t - K_p u = \frac{\lambda}{|x|^p} |u|^{p-2} u \quad \text{in } \Omega \times (0, T),$$

where we assume throughout this chapter that $1 < p < +\infty$, Ω is a domain of \mathbb{R}^d containing $x = 0$ if $d \geq 2$ and $\Omega = (0, +\infty)$ if $d = 1$. Here, K_p denotes the p -Kolmogorov operator defined by $K_p u = \Delta_p u + \langle |\nabla u|^{p-2} \nabla u, \frac{\nabla \rho}{\rho} \rangle$ for all $u \in W_{loc}^{1,p}(\Omega)$, where $\rho > 0$ is a smooth function. To establish our results we follow an idea of X. Cabré and Y. Martel [16], which is based on a Hardy inequality and on the optimality of the constant $(\frac{|d-p|}{p})^p$. Thus, it is our first task to establish a new weighted Hardy inequality with optimal constant in Hardy's inequality.

The organization of this chapter is as follows. In the first section, we introduce the problem of nonexistence of positive nontrivial weak solutions of (3.1) and give a review of the most worthy contributions concerning this problem. In Section 2, we state a new weighted Hardy inequality and give a sufficient condition, when the Hardy constant is optimal. We deduce from the weighted Hardy inequality a weighted Poincaré inequality. Section 3 is concerned with an extensive preliminary to prepare our study in the subsequent section. In Section 4, we present our main results concerning the nonexistence of positive nontrivial weak solutions of equation (3.1).

Contents

3.1	Introduction and notation	44
3.2	A weighted Hardy and Poincaré inequality	47
3.3	Preliminaries	51
3.3.1	Realization of the Dirichlet p -Kolmogorov operator	51
3.3.2	Solvability of the first initial boundary value problem	53
3.3.3	Definition of weak solutions	54
3.3.4	Comparison principles	55
3.3.5	Steklov averages and an integration by parts	58
3.3.6	Some geometric properties of the boundary	62
3.3.7	Linear second-order differential operator of parabolic type	71
3.3.8	The strong maximum principle of linear parabolic equations	73
3.4	Main results: Nonexistence of positive nontrivial solutions	74
3.4.1	Nonexistence of positive solutions in dimension $d = 1$	74
3.4.2	Nonexistence of positive solutions for $d \geq 2$	79

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3.1. Introduction and notation

Let Ω be the open interval $(0, +\infty)$ if $d = 1$, and let Ω be a domain of \mathbb{R}^d such that $0 \in \Omega$ if $d \geq 2$. Then, it is the main task of this chapter to establish that for $1 < p < +\infty$ and if $\lambda > \left(\frac{|d-p|}{p}\right)^p$, then for every $T > 0$ and every positive initial value $u_0 \in L^2_{\mu,loc}(\Omega) \setminus \{0\}$, there is no positive weak solution of problem

$$(3.2) \quad \begin{cases} u_t - K_p u = \frac{\lambda}{x^p} |u|^{p-2} u & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

Here and throughout this chapter, we call a measurable function $u : \Omega \times (0, T) \rightarrow \mathbb{R}$ positive if $u(x, t) \geq 0$ for a.e. $(x, t) \in \Omega \times (0, T)$. Let A be a real symmetric positive semi-definite $d \times d$ -matrix, $c > 0$, and let μ be the Borel measure on Ω defined by

$$(3.3) \quad d\mu := \rho(x) dx \quad \text{with density} \quad \rho(x) = c \cdot \exp\left(-\frac{1}{p}(x^t A x)^{p/2}\right), \quad x \in \Omega.$$

Then, we denote by K_p the quasi-linear differential operator of second order defined by

$$(3.4) \quad K_p u = \Delta_p u + \left\langle |\nabla u|^{p-2} \nabla u, \frac{\nabla \rho}{\rho} \right\rangle \quad \text{for all } u \in W^{1,p}_{loc}(\Omega).$$

The operator K_p was first introduced in [33] by G. R. Goldstein, J. A. Goldstein, and A. Rhandi. These authors called K_p the *p-Kolmogorov operator*.

NOTATION 3.1.1. In order to keep this chapter readable, we need to introduce briefly the following weighted Lebesgue and weighted Sobolev spaces with respect to the measure μ defined in (3.3). For $1 \leq p < +\infty$, for any open subset Ω of \mathbb{R}^d , we denote by $L^p_\mu(\Omega)$ the space of all real valued measurable functions u on Ω with finite integral $\int_\Omega |u|^p d\mu$. Thereby we identify two functions of $L^p_\mu(\Omega)$ if they are equal almost everywhere on Ω . We denote by $W^{1,p}_\mu(\Omega)$ the set of all $u \in L^p_\mu(\Omega)$ having all distributional partial derivatives $\frac{\partial u}{\partial x_i} \in L^p_\mu(\Omega)$ ($i = 1, \dots, d$) and by $W^{1,p}_{\mu,0}(\Omega)$ the closure in $W^{1,p}_\mu(\Omega)$ of the set $C^1_c(\Omega)$ of continuously differentiable functions with compact support contained in Ω .

The phenomenon of nonexistence of positive nontrivial solutions of (3.2) is caused by the potential $\lambda |x|^{-p}$. It is well-known in the case $p = 2$ (see, for instance [34]) that this potential can be controlled by a weighted Hardy inequality. It is our first task to establish that for all $1 < p < +\infty$ and $p \neq d$, the following weighted Hardy inequality

$$(3.5) \quad \left(\frac{|d-p|}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} d\mu \leq \int_\Omega |\nabla u|^p d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \operatorname{sgn}(d-p) \int_\Omega |u|^p \frac{(x^t A x)^{p/2}}{|x|^p} d\mu,$$

holds for all $u \in W^{1,p}_{\mu,0}(\Omega)$ if the dimension $d \geq 2$, $1 < p < d$, and for all $u \in W^{1,p}_{\mu,0}(\Omega \setminus \{0\})$ if $p > d \geq 1$. In particular, the constant $\left(\frac{|d-p|}{p}\right)^p$ is optimal provided A is a real symmetric positive definite $d \times d$ -matrix or $A \equiv 0$ and $0 \in \Omega$ if $d \geq 1$, where it is sufficient in $d = 1$ that $x = 0$ is a boundary point of Ω .

In the case, when $A \equiv 0$ and $c = 1$, the Borel measure μ defined in (3.3) reduces to the Lebesgue

measure on Ω and so inequality (3.5) becomes the well-known Hardy inequality

$$(3.6) \quad \left(\frac{|d-p|}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx \leq \int_{\Omega} |\nabla u|^p dx$$

for all $u \in W_0^{1,p}(\Omega)$ if $d \geq 2$, $1 < p < d$, and for all $u \in W_0^{1,p}(\Omega \setminus \{0\})$ if $p > d \geq 1$. Inequality (3.6) was first stated in dimension $d = 1$ for $\Omega =]0, +\infty[$ in the pioneering work [44] due to G. H. Hardy in 1920. Various generalization of Hardy's inequality have been found since then with applications in various branches of mathematics. For instance see H. Brezis [12, p.194], V. G. Maz'ja [63, p. 96], and for a more detailed list of applications, we refer the interested reader, to D. S. Mitrinović, J. E. Pečarić, and A. M. Fink [64, p. 175f] and the reference therein.

In this chapter, we intend to revisit the intimate relation between Hardy's inequality (3.6) or (3.5) and the nonexistence results of positive solutions of parabolic equations with a singular potential.

In 1984, P. Baras and J. Goldstein proved in [7] the following noteworthy result.

THEOREM 3.1.1. (Baras-Goldstein) *Let $\Omega =]0, R[$ for some $1 \leq R \leq +\infty$ if $d = 1$ and if $d \geq 2$, let Ω be a domain in \mathbb{R}^d with $B(0, 1) \subseteq \Omega$ and a smooth boundary $\partial\Omega$. If $\lambda > \left(\frac{d-2}{2}\right)^2$ and if $u_0 \in L^1(\Omega) \setminus \{0\}$ is positive, then for any $T > 0$ problem*

$$(3.7) \quad \frac{\partial u}{\partial t} - \Delta u = \frac{\lambda}{|x|^2} u \quad \text{in } \Omega \times (0, T), \quad u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega$$

has no positive solution.

In 1999, X. Cabré and Y. Martel discovered in [16] a second and more intuitive proof of Theorem A. 3.1.1. They proved in [16] that indeed the existence and nonexistence of positive solutions of problem (3.7) is largely determined by the generalized eigenvalue of $-\Delta - \lambda|x|^{-2}$ defined by

$$\sigma(\lambda|x|^{-2}, \Omega) := \inf_{0 \neq \varphi \in C_c^\infty(\Omega)} \frac{\int_{\Omega} |\nabla \varphi|^2 d\mu - \int_{\Omega} \frac{\lambda}{|x|^2} |\varphi|^2 d\mu}{\int_{\Omega} |\varphi|^2 d\mu}.$$

In 2003, J. Goldstein and I. Kombe showed in [35] that the method in [16] due to Cabré and Martel can be very useful to establish nonexistence (locally in time) of positive solutions of singular nonlinear diffusion equations. They proved in [35] the following result.

THEOREM 3.1.2. (Goldstein-Kombe, 2003) *If $d = 1$, let $\Omega = (0, +\infty)$ and $1 < p < 2$, if $d \geq 2$, let $\frac{2d}{d+1} \leq p < 2$ and Ω be a bounded domain in \mathbb{R}^d such that $0 \in \Omega$ and with a smooth boundary $\partial\Omega$. If $\lambda > \left(\frac{d-p}{p}\right)^p$ and if $u_0 \in L^1(\Omega) \setminus \{0\}$ is positive, then for any $T > 0$, problem*

$$(3.8) \quad u_t - \Delta_p u = \frac{\lambda}{|x|^p} u^{p-1} \quad \text{in } \Omega \times (0, T), \quad u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega$$

has no positive solution.

In (3.8), $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denotes the well-known p-Laplace operator. The existence and the qualitative behavior of positive solution of problem (3.8) has been intensively studied, for instance, in the articles [30] and [2]. By using a separation of variables method, Garcia Azorero and Peral Alonso proved in [30] the following result.

THEOREM 3.1.3. (Garcia Azorero-Peral Alonso, 2003) *Let Ω be a bounded domain in \mathbb{R}^d containing the origin, and let $2 < p < d$. If $\lambda > \left(\frac{d-p}{p}\right)^p$ and if $u_0 \in L^\infty(\Omega)$ is positive such that $u_0 \geq \delta > 0$ in a neighborhood of the origin, then for any $T > 0$, problem (3.8) has no positive solution.*

At this point, it is worth mentioning the work [5] due to W. Arendt, G. R. Goldstein and J. A. Goldstein (2006). The authors proved in [5] by using form methods and the spectral theorem that if $d \geq 5$ and if $\lambda > \left(\frac{d-2}{2}\right)^2$, then the minimal operator $A_{\lambda, \min}$ on $L^2(\mathbb{R}^d)$ given by

$$A_{\lambda, \min} u = \Delta u + \frac{\lambda}{|x|^2} u, \quad u \in D(A_{\lambda, \min}) = C_c^\infty(\mathbb{R}^d \setminus \{0\})$$

admits no extension that generates a positive (C_0) semigroup on $L^2(\mathbb{R}^d)$. Another noteworthy contribution is the article [28] due to Galaktionov (2008), wherein the author employed the zero counting method (Sturm's first theorem [74]) to show that the positivity assumption in Theorem A.3.1.1 and in Theorem A.3.1.2 can be omitted but the initial datum u_0 is assumed to be continuous and $u_0(0) > 0$.

Recently, G. R. Goldstein, J. A. Goldstein and A. Rhandi established in [34] first the weighted Hardy inequality (3.5) in the cases $p = 2$ and then by employing the Cabré-Martel approach ([16]), they proved the following noteworthy result.

THEOREM 3.1.4. (Goldstein-Goldstein-Rhandi, 2011) *Let $d \geq 3$, A a real symmetric positive definite $d \times d$ -matrix, μ the Borel measure on \mathbb{R}^d defined by (3.3) for $p = 2$ and $c = \left(\int \rho dx\right)^{-1}$. If $\lambda > \left(\frac{d-2}{2}\right)^2$ and if $u_0 \in L_\mu^2(\mathbb{R}^d) \setminus \{0\}$ is positive, then problem*

$$(3.9) \quad \frac{\partial u}{\partial t} - \Delta u + \langle Ax, \nabla u \rangle = \frac{\lambda}{|x|^2} u \quad \text{in } \mathbb{R}^d \times (0, +\infty), \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d$$

has no positive solution $u \in C([0, +\infty); L_\mu^2(\mathbb{R}^d))$ satisfying $\|u(t)\|_{L_\mu^2(\mathbb{R}^d)} \leq M e^{\omega t} \|u_0\|_{L_\mu^2(\mathbb{R}^d)}$ for all $t \geq 0$, for some $M \geq 1$, and $\omega \in \mathbb{R}$.

In other words, Theorem 3.1.4 says that for $\lambda > \left(\frac{d-2}{2}\right)^2$, the operator $\Delta - \langle Ax, \nabla \cdot \rangle + \lambda |x|^{-2}$ does not generate a positive strongly continuous semigroup in $L_\mu^2(\mathbb{R}^d)$. By comparing Theorem 3.1.4 with Theorem 3.1.1, it seems that the unbounded drift term Ax appearing in the symmetric Ornstein-Uhlenbeck operator $L = \Delta - \langle Ax, \nabla \cdot \rangle$ has a strong influence on the qualitative behavior of the solutions of problem (3.9). In analogs to the above stated results, it is natural to ask if there is still a nonexistence phenomenon of positive solutions, when we replace in problem (3.9) the linear operator L by the nonlinear p -Kolmogorov operator K_p defined in (3.4). Note that in the case $p = 2$, K_p coincides with the symmetric Ornstein-Uhlenbeck operator L .

It is the main task of this chapter to give a satisfying answer to these questions. In Section 3.4 of this chapter, we state our main results concerning the nonexistence of positive nontrivial weak solutions of problem (3.2). Thereby our proofs are based on the Cabré-Martel ([16]) approach.

At the end of this introduction we note that the result [7] due to P. Baras and J. Goldstein stimulated many interesting results in the study of the elliptic and parabolic equations with singular potentials, which we still have not mentioned above. See, for instance, [21, 22] by L. Dupaigne or [13] by H. Brezis and X. Cabré for a study of linear elliptic equations with a singular potential, [30] by J. P. García Azorero and I. Peral Alonso for a study of elliptic and parabolic p -Laplace equations with a singular potential, [37, 36] by J. A. Goldstein and Q. S. Zhang for a study of more general linear parabolic equations with a singular potential, and [49, 50, 51] by I. Kombe for nonlinear parabolic equations with a singular potential.

3.2. A weighted Hardy and Poincaré inequality

The content of this subsection is part of the article [47] together with A. Rhandi, which was accepted for publication in *Archiv der Mathematik*.

The following theorem is one of our main results of this chapter.

THEOREM 3.2.1. (Weighted Hardy inequality) *Let Ω be an open subset of \mathbb{R}^d , and A be a real symmetric positive semi-definite $d \times d$ -matrix. Then,*

$$(3.5) \quad \left(\frac{|d-p|}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} d\mu \leq \int_{\Omega} |\nabla u|^p d\mu + \left(\frac{|d-p|}{p}\right)^{p-1} \operatorname{sgn}(d-p) \int_{\Omega} |u|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu$$

for all $u \in W_{\mu,0}^{1,p}(\Omega)$ provided $d \geq 2$ and $1 < p < d$ or for all $u \in W_{\mu,0}^{1,p}(\Omega \setminus \{0\})$ if $p > d \geq 1$. If in addition, $0 \in \Omega$, where in dimension $d = 1$ it suffices that $x = 0$ is a boundary point of Ω , and if A is positive definite, then the constant $\left(\frac{|d-p|}{p}\right)^p$ is optimal (including the case $p = d$ for $d \geq 2$).

REMARK 3.2.1. If Ω is an open subset of \mathbb{R}^d , and if A is a real symmetric positive semi-definite $d \times d$ -matrix, then by the elementary inequality $(x^t Ax) \leq |x|^2 |A|$, ($x \in \mathbb{R}^d$), the second term on the right hand-side in (3.5) satisfies

$$\operatorname{sgn}(d-p) \int_{\Omega} |u|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu \leq |A|^{p/2} \int_{\Omega} |u|^p d\mu \quad \text{for all } u \in L_{\mu}^p(\Omega).$$

Thus if $1 < p < d$ then Hardy's inequality (3.5) implies that $W_{\mu,0}^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p} d\mu)$ and if $p > d \geq 1$, then $W_{\mu,0}^{1,p}(\Omega \setminus \{0\}) \hookrightarrow L^p(\Omega \setminus \{0\}, |x|^{-p} d\mu)$ each by a continuous injection.

COROLLARY 3.2.2. (Weighted Poincaré inequality) *Let Ω be an open subset of \mathbb{R}^d and let $\lambda(A)$ denote the lowest eigenvalue of the real symmetric positive semi-definite $d \times d$ -matrix A . If A is positive definite and if $p > d \geq 1$, then*

$$(3.10) \quad \left(\frac{p-d}{p}\right)^{p-1} \lambda^{p/2}(A) \int_{\Omega} |u|^p d\mu \leq \int_{\Omega} |\nabla u|^p d\mu \quad \text{for all } u \in W_{\mu,0}^{1,p}(\Omega \setminus \{0\}).$$

PROOF OF COROLLARY 3.2.2. If the real symmetric $d \times d$ -matrix A is positive definite, then the lowest eigenvalue $\lambda(A) > 0$ and hence $x^t Ax \geq \lambda(A) |x|^2$ for all $x \in \mathbb{R}^d$. Therefore we can deduce inequality (3.10) from inequality (3.5) provided $p > d$. \square

To prove Theorem 3.2.1, we make use of the following Lemma, which in the case $\rho \equiv 1$ is a well-known result in the theory of Sobolev spaces, see, for instance, Remarque 18. in [12, § IX.4, p. 171], and see also Remark 2.6 in [4, p. 1019].

LEMMA 3.2.3. *Let $d \geq 2$, Ω be an open subset of \mathbb{R}^d , and let $1 \leq p < d$. Then,*

$$W_{\mu,0}^{1,p}(\Omega) = W_{\mu,0}^{1,p}(\Omega \setminus \{a\}) \quad \text{for every } a \in \Omega.$$

PROOF. In this proof, we follow the idea of Lemma 2.4 and Remark 2.6 in [4]. Let $(\rho_n)_{n \geq 1}$ denote a standard mollifier (see [12, p.70]): i.e., for every $n \geq 1$, $\rho_n \in C_c^\infty(\mathbb{R}^d)$, the support

$\text{supp}(\rho_n) \subseteq \bar{B}(0, \frac{1}{n})$, $\rho_n \geq 0$, and $\int_{\mathbb{R}^d} \rho_n dx = 1$. We fix $a \in \Omega$ and set

$$\psi(x) = \int_{\mathbb{R}^d} \rho_5(y) \mathbb{1}_{B(a, 5/4)}(x-y) dy \quad \text{for all } x \in \mathbb{R}^d,$$

where $\mathbb{1}_{B(a, 5/4)}$ denote the indicator function over the open ball $B(a, \frac{5}{4})$ of center $x = a$ and radius $r = \frac{5}{4}$. Then, it is not hard to verify that $\psi \in C_c^\infty(\mathbb{R}^d)$, $\psi \geq 0$, $\psi \equiv 1$ on $B(a, 1)$ and $\psi \equiv 0$ on $\mathbb{R}^d \setminus B(a, 2)$. For every $n \geq 1$, we set $\psi_n(x) = \psi(nx)$ for every $x \in \mathbb{R}^d$. Then, $\psi_n \in C_c^\infty(\mathbb{R}^d)$, $\psi_n \geq 0$, $\psi_n \equiv 1$ on $B(a, \frac{1}{n})$, and $\psi_n \equiv 0$ on $\mathbb{R}^d \setminus B(a, \frac{2}{n})$. Since

$$\|\psi_n\|_{L^p(\mathbb{R}^d)} = n^{-\frac{d}{p}} \|\psi\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad \left\| \frac{\partial \psi_n}{\partial x_i} \right\|_{L^p(\mathbb{R}^d)} = n^{-\frac{d-p}{p}} \left\| \frac{\partial \psi}{\partial x_i} \right\|_{L^p(\mathbb{R}^d)}, \quad \text{for every } i = 1, \dots, d,$$

and since by hypothesis, $p < d$, we have that $\psi_n \rightarrow 0$ in $W^{1,p}(\mathbb{R}^d)$ as $n \rightarrow +\infty$.

Obviously, it suffices to show that $W_{\mu,0}^{1,p}(\Omega) \subseteq W_{\mu,0}^{1,p}(\Omega \setminus \{a\})$ since the other implication is clear. Since $W_{\mu,0}^{1,p}(\Omega)$ is the closure of the set $C_c^1(\Omega)$ in $W_{\mu}^{1,p}(\Omega)$, we need to show that for every $\varphi \in C_c^1(\Omega)$ and for every $\varepsilon > 0$, there is $\theta \in C_c^1(\Omega \setminus \{a\})$ such that $\|\varphi - \theta\|_{W_{\mu}^{1,p}(\Omega)} \leq \varepsilon$. To do so, we fix $\varphi \in C_c^1(\Omega)$ and $\varepsilon > 0$. Since Ω is open and $a \in \Omega$, there is an $r > 0$ such that the open ball $B(a, r)$ of center $x = a$ and radius r is contained in Ω . By the first step of this proof, there is an index $n_r \geq 1$ such that $\psi_n \in C_c^\infty(B(a, r))$ for every $n \geq n_r$. Then, for every $n \geq n_r$, $\theta_n := \varphi(1 - \psi_n) \in C_c^1(\Omega \setminus \{a\})$ and

$$\|\varphi - \theta_n\|_{W_{\mu}^{1,p}(\Omega)} = \|\varphi \psi_n\|_{W_{\mu}^{1,p}(\Omega)} \leq C \|\psi_n\|_{W^{1,p}(\mathbb{R}^d)},$$

where the constant $C \geq 0$ depends on φ and $\|\rho\|_{L^\infty(\text{supp}(\varphi))}$ but is independent of ψ_n . Since we can choose $n \geq n_r$ large enough such that $\|\psi_n\|_{W^{1,p}(\mathbb{R}^d)} \leq \frac{\varepsilon}{C+1}$, the claim of this Lemma holds. \square

PROOF OF THEOREM 3.2.1. It is not hard to see that in the case $p = d$ for $d \geq 2$, inequality (3.5) holds true with optimal constant 0. Thus it is sufficient to establish inequality (3.5) if when $p \neq d$, $d \geq 1$, and Ω is an open subset of \mathbb{R}^d . Suppose that A is a real symmetric positive semi-definite $d \times d$ -matrix. We fix $\lambda \geq 0$, which will be chosen later, and set

$$F(x) = \lambda \text{sgn}(d-p) \frac{x}{|x|^p} \rho(x) \quad \text{for every } x \in \Omega \setminus \{0\}.$$

Here we recall that ρ is defined as in (3.3). Then, one easily verifies that

$$\text{div}(F(x)) = \sum_{i=1}^d \frac{\partial F_i}{\partial x_i}(x) = \lambda \text{sgn}(d-p) \left[\frac{d-p}{|x|^p} - \frac{(x^t A x)^{p/2}}{|x|^p} \right] \rho(x) \quad \text{for } x \in \Omega \setminus \{0\}.$$

We multiply the last equation by $u \in C_c^1(\Omega \setminus \{0\})$ and integrate over Ω . Then, by integration by parts and by Young's inequality,

$$\begin{aligned} \int_{\Omega} |u|^p \lambda \text{sgn}(d-p) \left[\frac{d-p}{|x|^p} - \frac{(x^t A x)^{p/2}}{|x|^p} \right] d\mu &= -p \int_{\Omega} |u|^{p-1} \text{sgn}(u) \langle \nabla u, F \rangle_{\mathbb{R}^d} dx \\ &= -p \lambda \text{sgn}(d-p) \int_{\Omega} |u|^{p-1} \text{sgn}(u) \langle \nabla u, \frac{x}{|x|^p} \rangle_{\mathbb{R}^d} d\mu \\ &\leq \int_{\Omega} |\nabla u|^p d\mu + (p-1) \lambda^{p'} \int_{\Omega} \frac{|u|^p}{|x|^p} d\mu. \end{aligned}$$

And hence

$$\int_{\Omega} [\lambda |d-p| - (p-1) \lambda^{p'}] \frac{|u|^p}{|x|^p} d\mu \leq \int_{\Omega} |\nabla u|^p d\mu + \lambda \operatorname{sgn}(d-p) \int_{\Omega} |u|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu.$$

Now, if we choose $\lambda = \left(\frac{|d-p|}{p}\right)^{p-1}$ in the last inequality, which is, in fact, the maximum of the function $\lambda \mapsto [\lambda |d-p| - (p-1) \lambda^{p'}]$ on the half line $[0, +\infty[$, then it follows that inequality (3.5) holds for all $u \in C_c^1(\Omega \setminus \{0\})$. If $1 < p < d$, then by Lemma 3.2.3, we see that (3.5) holds for all $u \in W_{\mu,0}^{1,p}(\Omega)$. If $p > d$, then (3.5) holds for all $u \in W_{\mu,0}^{1,p}(\Omega \setminus \{0\})$.

Next, we show the optimality of the constant $\left(\frac{|d-p|}{p}\right)^p$ provided $d \neq p$, and the matrix A is positive definite. Thereby, we restrict ourselves to the case that $0 \in \Omega$. The proof of optimality in dimension $d = 1$ and when 0 is a boundary point of Ω proceeds analogously. Then, Ω contains the open ball $B(0, r)$ for some $r > 0$. To do so, we fix $\lambda > \left(\frac{|d-p|}{p}\right)^p$, choose a test function $\zeta \in C_c^\infty(B(0, r))$ such that $0 \leq \zeta \leq 1$ on $B(0, r)$ and $\zeta \equiv 1$ on $B(0, \frac{r}{2})$, and for γ such that

$$(3.11) \quad 1 - \frac{d}{p} < \gamma < 0 \quad \text{if } p < d \quad \text{and} \quad 1 - \frac{d}{p} < \gamma < 1 \quad \text{if } p > d,$$

we set $\varphi(x) = |x|^\gamma \zeta$. Then, $\nabla \varphi(x) = \gamma |x|^{\gamma-2} x$ for all $x \in B(0, \frac{r}{2}) \setminus \{0\}$, and so

$$\int_{\Omega} |\varphi|^p d\mu = \int_{B(0, \frac{r}{2})} |x|^{\gamma p} d\mu + \mathcal{I}_1(\gamma) \quad \text{and} \quad \int_{\Omega} |\nabla \varphi|^p d\mu \leq \int_{B(0, \frac{r}{2})} |\gamma|^p |x|^{p(\gamma-1)} d\mu + \mathcal{I}_2(\gamma)$$

where

$$\mathcal{I}_1(\gamma) := \int_{\{\frac{r}{2} \leq |x| < r\}} |x|^{\gamma p} \zeta^p d\mu, \quad \mathcal{I}_2(\gamma) := C_p \int_{\{\frac{r}{2} \leq |x| < r\}} \left[|\nabla \zeta|^p |x|^{\gamma p} + |\gamma|^p |x|^{p(\gamma-1)} \zeta^p \right] d\mu.$$

By the assumption, there are $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 |x|^2 \leq x^t Ax \leq \alpha_2 |x|^2$ for all $x \in \mathbb{R}^d$. Hence for every $\beta \in \mathbb{R}$, we have that

$$(3.12) \quad \int_{B(0, \frac{r}{2})} |x|^{p\beta} d\mu \leq c \int_{B(0, \frac{r}{2})} |x|^{p\beta} e^{-\frac{\alpha_1^{p/2} |x|^p}{p}} dx, \quad \int_{B(0, \frac{r}{2})} |x|^{p\beta} d\mu \geq c \int_{B(0, \frac{r}{2})} |x|^{p\beta} e^{-\frac{\alpha_2^{p/2} |x|^p}{p}} dx.$$

For every $i = 1, 2$, we set $R_i = \frac{\alpha_i^{p/2} r^p}{p 2^p}$. Then, for every $i = 1, 2$ and every $\beta \in \mathbb{R}$,

$$(3.13) \quad \begin{aligned} \int_{B(0, \frac{r}{2})} |x|^{p\beta} e^{-\frac{\alpha_i^{p/2} |x|^p}{p}} dx &= \sigma(S_{d-1}) \int_0^{\frac{r}{2}} s^{p\beta} e^{-\frac{\alpha_i^{p/2} s^p}{p}} s^{d-1} ds \\ &= \sigma(S_{d-1}) p^{-\frac{p\beta+d-p}{p}} \alpha_i^{-\frac{p\beta+d}{2}} \int_0^{R_i} t^{\beta+\frac{d}{p}-1} e^{-t} dt, \end{aligned}$$

where $\sigma(S_{d-1})$ denotes the total surface measure of the unite sphere $S_{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$ with respect to the surface measure σ on S_{d-1} . We note that

$$0 \leq \int_0^{R_i} t^{\beta+\frac{d}{p}-1} e^{-t} dt \leq \int_0^{+\infty} t^{\beta+\frac{d}{p}-1} e^{-t} dt = \Gamma\left(\beta + \frac{d}{p}\right)$$

and that $\Gamma\left(\beta + \frac{d}{p}\right)$ is finite for every $\beta > -\frac{d}{p}$, and in particular for $\beta = \gamma$ or $\beta = \gamma - 1$ when γ is chosen as in (3.11). Thus and since the both integrals $\mathcal{I}_1(\gamma)$ and $\mathcal{I}_2(\gamma)$ are finite for all given

γ from (3.11), $\varphi(x) = |x|^\gamma \zeta$ belongs to $W_\mu^{1,p}(\Omega)$. Since φ has a compact support in Ω and since $\varphi(0) = 0$ if $p > d$, one easily verifies that $\varphi \in W_{\mu,0}^{1,p}(\Omega)$ if $p < d$ and $\varphi \in W_{\mu,0}^{1,p}(\Omega \setminus \{0\})$ if $p > d$.

On the other hand, since for all $x \in \mathbb{R}^d$, $x^t Ax \leq |A| |x|^2$, we have that

$$(3.14) \quad \int_{\Omega} \left[|\nabla \varphi|^p + \lambda^{1/p'} \operatorname{sgn}(d-p) |\varphi|^p \frac{(x^t Ax)^{\frac{p}{2}}}{|x|^p} - \frac{\lambda}{|x|^p} |\varphi|^p \right] d\mu \\ \leq [|\gamma|^p - \lambda] \int_{B(0, \frac{r}{2})} |x|^{p(\gamma-1)} d\mu + |A|^{\frac{p}{2}} \lambda^{\frac{1}{p'}} \int_{B(0, \frac{r}{2})} |x|^{\gamma p} d\mu + \mathcal{I}_3(\gamma),$$

where

$$\mathcal{I}_3(\gamma) = \int_{\{\frac{r}{2} \leq |x| < r\}} \left[|\nabla \varphi|^p + \lambda^{1/p'} |A|^{\frac{p}{2}} |\varphi|^p - \frac{\lambda}{|x|^p} |\varphi|^p \right] d\mu \quad \text{is finite for all } \gamma \text{ in (3.11).}$$

Since $\int_{\Omega} |\varphi|^p d\mu \geq \int_{B(0, \frac{r}{2})} |x|^{\gamma p} d\mu$ and in view of (3.12)-(3.14),

$$\frac{\int_{\Omega} \left[|\nabla \varphi|^p + \lambda^{\frac{1}{p'}} \operatorname{sgn}(d-p) |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} - \frac{\lambda}{|x|^p} |\varphi|^p \right] d\mu}{\int_{\Omega} |\varphi|^p d\mu} \\ \leq \frac{[|\gamma|^p - \lambda] \int_{B(0, \frac{r}{2})} |x|^{p(\gamma-1)} d\mu + |A|^{\frac{p}{2}} \lambda^{\frac{1}{p'}} \int_{B(0, \frac{r}{2})} |x|^{\gamma p} d\mu + \mathcal{I}_3(\gamma)}{\int_{B(0, \frac{r}{2})} |x|^{\gamma p} d\mu} \\ \leq \frac{[|\gamma|^p - \lambda] p^{\frac{p(\gamma-1)+d-p}{p}} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} \int_0^{R_1} t^{(\gamma-1)+\frac{d}{p}-1} e^{-t} dt}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} \int_0^{R_2} t^{\gamma+\frac{d}{p}-1} e^{-t} dt} \\ + \frac{|A|^{\frac{p}{2}} \lambda^{\frac{1}{p'}} p^{\frac{p\gamma+d-p}{p}} \alpha_1^{-\frac{p\gamma+d}{2}} \sigma(S_{d-1}) \int_0^{R_1} t^{\gamma+\frac{d}{p}-1} e^{-t} dt + \mathcal{I}_3(\gamma)}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} \sigma(S_{d-1}) \int_0^{R_2} t^{\gamma+\frac{d}{p}-1} e^{-t} dt}.$$

Since,

$$\lim_{\gamma \rightarrow (1-\frac{d}{p})^+} p^{\frac{p(\gamma-1)+d-p}{p}} = p^{-1}, \quad \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} p^{\frac{p\gamma+d-p}{p}} = 1, \quad \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} = 1, \\ \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} \alpha_i^{-\frac{p\gamma+d}{2}} = \alpha_i^{-\frac{p}{2}} \quad (i = 1, 2), \quad \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} \int_0^{R_i} t^{\gamma+\frac{d}{p}-1} e^{-t} dt = 1 - e^{-R_i} > 0 \quad (i = 1, 2), \\ \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} \mathcal{I}_3(\gamma) = \mathcal{I}_3(1 - \frac{d}{p}) < +\infty, \quad \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} \int_0^{R_1} t^{(\gamma-1)+\frac{d}{p}-1} e^{-t} dt = +\infty$$

and since

$$\lim_{\gamma \rightarrow (1-\frac{d}{p})^+} [|\gamma|^p - \lambda] = \left(\frac{|d-p|}{p}\right)^p - \lambda < 0,$$

we have that

$$(3.15) \quad \lim_{\gamma \rightarrow (1-\frac{d}{p})^+} \frac{[|\gamma|^p - \lambda] p^{\frac{p(\gamma-1)+d-p}{p}} \alpha_1^{-\frac{p(\gamma-1)+d}{2}} \int_0^{R_1} t^{(\gamma-1)+\frac{d}{p}-1} e^{-t} dt}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} \int_0^{R_2} t^{\gamma+\frac{d}{p}-1} e^{-t} dt} = -\infty,$$

and

$$\lim_{\gamma \rightarrow (1 - \frac{d}{p})^+} \frac{|A|^{\frac{p}{2}} \lambda^{\frac{1}{p'}} p^{\frac{p\gamma+d-p}{p}} \alpha_1^{-\frac{p\gamma+d}{2}} \sigma(S_{d-1}) \int_0^{R_1} t^{\gamma + \frac{d}{p} - 1} e^{-t} dt + \mathcal{I}_3(\gamma)}{p^{\frac{p\gamma+d-p}{p}} \alpha_2^{-\frac{p\gamma+d}{2}} \sigma(S_{d-1}) \int_0^{R_2} t^{\gamma + \frac{d}{p} - 1} e^{-t} dt} < +\infty.$$

Since for every γ in (3.11), $\varphi(x) = |x|^\gamma \zeta \in X := \begin{cases} W_{\mu,0}^{1,p}(\Omega) & \text{if } p < d \\ W_{\mu,0}^{1,p}(\Omega \setminus \{0\}) & \text{if } p > d \end{cases}$, we can deduce from (3.15) that

$$\inf_{\varphi \in X: \|\varphi\|_{L_\mu^p(\Omega)} > 0} \frac{\int_\Omega |\nabla \varphi|^p d\mu + \lambda^{\frac{1}{p'}} \operatorname{sgn}(d-p) \int_\Omega |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu - \int_\Omega \frac{\lambda}{|x|^p} |\varphi|^p d\mu}{\int_\Omega |\varphi|^p d\mu} = -\infty.$$

Hence for every $M > 0$, there is a $\varphi \in X$, such that $\|\varphi\|_{L_\mu^p(\Omega)} > 0$ and

$$\int_\Omega |\nabla \varphi|^p d\mu + \lambda^{\frac{1}{p'}} \operatorname{sgn}(d-p) \int_\Omega |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu - \int_\Omega \frac{\lambda}{|x|^p} |\varphi|^p d\mu < (-M) \int_\Omega |\varphi|^p d\mu < 0.$$

Therefore the constant $(\frac{|d-p|}{p})^p$ in inequality (3.5) is optimal. \square

3.3. Preliminaries

To provide a self-contained proof of nonexistence of positive nontrivial weak solutions of inequality (3.1), we need to recall the following definitions, notations and lemmata.

3.3.1 Realization of the Dirichlet p -Kolmogorov operator

We start this preliminary section to introduce a realization of the Dirichlet p -Kolmogorov operator in the Hilbert space $L_\mu^2(\Omega)$ on an open subset $\Omega \subseteq \mathbb{R}^d$.

Let $d \geq 1$, $1 < p < +\infty$, let A be a real symmetric positive semi-definite $d \times d$ -matrix, $c > 0$, and μ the Borel measure on Ω defined in (3.3). We denote by V_μ the set of all $u \in L_\mu^2(\Omega)$ such that the distributional derivatives $\frac{\partial u}{\partial x_i}$ belong to $L_\mu^p(\Omega)$. Then, V_μ equipped with the norm

$$\|u\|_{V_\mu} := \|\nabla u\|_{L_\mu^p(\Omega)^d} + \|u\|_{L_\mu^2(\Omega)} \quad \text{for all } u \in V_\mu$$

defines a reflexive and separable Banach space. We denote by $V_{\mu,0}$ the closure in V_μ of the set $C_c^\infty(\Omega)$ of infinitely differentiable function with compact support in Ω . Then, $V_{\mu,0}$ endowed with the induced norm of V_μ is a closed subspace and hence (see, for instance Proposition III.17 and Proposition III.22 in [12]) is a reflexive and separable Banach space. We denote by $V_{\mu,0}'$ the dual space of $V_{\mu,0}$.

With these notations, we define the function $\mathcal{E} : L_\mu^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{E}(u) := \frac{1}{p} \int_\Omega |\nabla u|^p d\mu \quad \text{if } u \in V_{\mu,0}, \quad \text{and} \quad \mathcal{E}(u) := +\infty \quad \text{if } u \in L_\mu^2(\Omega) \setminus V_{\mu,0}.$$

It is not hard to verify that the function \mathcal{E} is proper, convex, and lower semicontinuous. To ensure that the last claim holds, take $\alpha \in \mathbb{R}$ and take a sequence $(u_n)_{n \geq 1}$ in $L_\mu^2(\Omega)$, which converges

to $u \in L^2_\mu(\Omega)$ as $n \rightarrow +\infty$ and satisfies $\mathcal{E}(u_n) \leq \alpha$ for all $n \geq 1$. Then, the sequence $(u_n)_{n \geq 1}$ is bounded in $V_{\mu,0}$. Since $V_{\mu,0}$ is reflexive and since $V_{\mu,0}$ is continuously embedded into $L^2_\mu(\Omega)$, by the theorem of Banach-Alaoglu-Bourbaki (see Theorem III.15 in [12]), we obtain that $u \in V_{\mu,0}$ and we can extract a subsequence of $(u_n)_{n \geq 1}$, which we denote, for simplicity, again by $(u_n)_{n \geq 1}$ such that $u_n \rightharpoonup u$ weakly in $V_{\mu,0}$ as $n \rightarrow +\infty$. Since \mathcal{E} is convex, the sublevel set $E_\alpha := \{v \in V_{\mu,0} \mid \mathcal{E}(v) \leq \alpha\}$ is a convex subset of $V_{\mu,0}$. One easily verifies that E_α is strongly closed in $V_{\mu,0}$. But by Mazur's Theorem (cf. Théorème III.7 in [12]), a convex subset C of a Banach space X is strongly closed in X if and only if the set C is weakly closed in X . Thus E_α is, in particular, weakly closed in $V_{\mu,0}$ and hence $u \in E_\alpha$. This shows that \mathcal{E} is lower semicontinuous on $L^2_\mu(\Omega)$.

Further, one does not need hard efforts to check (see [45]) that the restriction $\mathcal{E}|_{V_{\mu,0}} : V_{\mu,0} \rightarrow \mathbb{R}$ is continuously differentiable and the derivative $\mathcal{E}'|_{V_{\mu,0}} : V_{\mu,0} \rightarrow V'_{\mu,0}$ is given by

$$\langle \mathcal{E}'|_{V_{\mu,0}}(u), v \rangle_{V'_{\mu,0}} = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v \, d\mu \quad \text{for all } u, v \in V_{\mu,0}.$$

We note that by construction, $V_{\mu,0}$ is contained in $L^2_\mu(\Omega)$ by a continuous injection, say j , with a dense image. The adjoint operator j^* of j is a continuous injection from the dual space $(L^2_\mu(\Omega))'$ into the dual space $V'_{\mu,0}$. By Riesz-Fréchet's Theorem, the dual space $(L^2_\mu(\Omega))'$ is isometrically isomorphic to $L^2_\mu(\Omega)$. We identify $L^2_\mu(\Omega)$ and $(L^2_\mu(\Omega))'$. Then we achieve to the situation that

$$V_{\mu,0} \hookrightarrow L^2_\mu(\Omega) \equiv (L^2_\mu(\Omega))' \hookrightarrow V'_{\mu,0}$$

each with continuous injection and thus the following Proposition makes clear that the subgradient $\partial_{L^2_\mu(\Omega)} \mathcal{E}$ of \mathcal{E} is the restriction operator on $L^2_\mu(\Omega)$ of the derivative $\mathcal{E}'|_{V_{\mu,0}}$. In particular, it shows that the subgradient $\partial_{L^2_\mu(\Omega)} \mathcal{E}$ is single-valued and characterizes its domain $D(\partial_{L^2_\mu(\Omega)} \mathcal{E})$.

PROPOSITION 3.3.1. *For every $u \in D(\mathcal{E}) = V_{\mu,0}$, we have that u belongs to $D(\partial_{L^2_\mu(\Omega)} \mathcal{E})$ if and only if there is a unique $h \in L^2_\mu(\Omega)$ such that*

$$(3.16) \quad \langle \mathcal{E}'|_{V_{\mu,0}}(u), v \rangle_{V'_{\mu,0}} = (h, v)_{L^2_\mu(\Omega)} \quad \text{for all } v \in V_{\mu,0}.$$

In particular, for every $u \in D(\partial_{L^2_\mu(\Omega)} \mathcal{E})$, $\partial_{L^2_\mu(\Omega)} \mathcal{E}(u) = \{h\}$ and $h \in L^2_\mu(\Omega)$ satisfies equation (3.16).

PROOF. First, we note that every $h \in L^2_\mu(\Omega)$ is uniquely determined by equation (3.16) since $V_{\mu,0}$ lies dense in $L^2_\mu(\Omega)$. Fix $u \in D(\mathcal{E})$ and suppose first that $u \in D(\partial_{L^2_\mu(\Omega)} \mathcal{E})$. Then by Definition 2.2.13, there is an $h \in L^2_\mu(\Omega)$ such that

$$(h, w - u)_{L^2_\mu(\Omega)} \leq \mathcal{E}(w) - \mathcal{E}(u) \quad \text{for all } w \in L^2_\mu(\Omega).$$

We take $w = u + tv$ in this inequality for fixed $t > 0$, and fixed $v \in V_{\mu,0}$, and divide the resulting inequality by t . Then sending $t \rightarrow 0+$ and using that \mathcal{E} is differentiable, we obtain that

$$(h, v)_{L^2_\mu(\Omega)} \leq \langle \mathcal{E}'|_{V_{\mu,0}}(u), v \rangle_{V'_{\mu,0}}.$$

Since $v \in V_{\mu,0}$ has been arbitrary, the last inequality holds in particular for $-v$ instead of v . Hence h satisfies equation (3.16). Now, we suppose that there is an $h \in L^2_\mu(\Omega)$ satisfying equation (3.16).

Since $\mathcal{E}'_{|V_{\mu,0}}$ is convex, the mapping $t \mapsto \frac{\mathcal{E}_{|V_{\mu,0}}(u+tw) - \mathcal{E}_{|V_{\mu,0}}(u)}{t}$ is monotonically decreasing as $t \rightarrow 0+$ for every $w \in V_{\mu,0}$. Thus, by equation (3.16), and since $\mathcal{E}_{|V_{\mu,0}}$ is differentiable, we have that

$$(h, w)_{L^2_{\mu}(\Omega)} = \inf_{t>0} \frac{\mathcal{E}_{|V_{\mu,0}}(u+tw) - \mathcal{E}_{|V_{\mu,0}}(u)}{t} \leq \mathcal{E}'_{|V_{\mu,0}}(u + w) - \mathcal{E}_{|V_{\mu,0}}(u).$$

Taking $w = v - u$ for any $v \in V_{\mu,0}$ in this inequality shows that h belongs to $\partial_{L^2_{\mu}(\Omega)} \mathcal{E}(u)$. \square

NOTATION 3.3.1. For every $f \in W^{1,1}_{loc}(\Omega)$ and every $i = 1, \dots, d$, an integration by parts with respect to dx shows that

$$\int_{\Omega} f \frac{\partial \bar{\xi}}{\partial x_i} d\mu = - \int_{\Omega} \left\{ \frac{\partial f}{\partial x_i} + f \frac{\partial \rho}{\partial x_i} \rho^{-1} \right\} \bar{\xi} d\mu \quad \text{for all } \bar{\xi} \in C_c^{\infty}(\Omega).$$

Note that if $f \in W^{1,1}_{loc}(\Omega)$ and there are $g_1, g_2 \in L^1_{loc}(\Omega)$ such that for $i = 1, 2$,

$$(3.17) \quad \int_{\Omega} f \frac{\partial \bar{\xi}}{\partial x_i} d\mu = - \int_{\Omega} g_i \bar{\xi} d\mu \quad \text{for all } \bar{\xi} \in C_c^{\infty}(\Omega),$$

then by [12, Lemme IV.2] and since $\rho > 0$, $g_1 = g_2$ a.e. on Ω . Thus, it is natural to set

$$\frac{\partial^{\mu} f}{\partial x_i} = \frac{\partial f}{\partial x_i} + f \frac{\partial \rho}{\partial x_i} \rho^{-1} \quad \text{and} \quad \operatorname{div}^{\mu}(f_1, \dots, f_d) = \sum_{i=1}^d \frac{\partial f_i}{\partial x_i} + f_i \frac{\partial \rho}{\partial x_i} \rho^{-1}.$$

Then $\frac{\partial^{\mu} f}{\partial x_i}$ is uniquely defined by equation (3.17). We call $\frac{\partial^{\mu} f}{\partial x_i}$ the *ith weak partial derivative of f with respect to the measure μ* and $\operatorname{div}^{\mu}(f_1, \dots, f_d)$ the *weak divergence of the vector (f_1, \dots, f_d) with respect to μ* for $f, f_1, \dots, f_d \in W^{1,1}_{\mu,loc}(\Omega)$.

In view of Notation 3.3.1 and by equation (3.16), for every $u \in D(\partial_{L^2_{\mu}(\Omega)} \mathcal{E})$, $\partial_{L^2_{\mu}(\Omega)} \mathcal{E}(u)$ is the unique extension on $L^2_{\mu}(\Omega)$ of the distribution

$$- \operatorname{div}^{\mu}(|\nabla u|^{p-2} \nabla u) = - \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) - |\nabla u|^{p-2} \left\langle \nabla u, \frac{\nabla \rho}{\rho} \right\rangle.$$

This justifies the following definition.

DEFINITION 3.3.2. (The Dirichlet p -Kolmogorov operator in $L^2_{\mu}(\Omega)$) Under the above fixed assumptions and settings, we define the operator $K_p : L^2_{\mu}(\Omega) \supseteq D(K_p) \rightarrow L^2_{\mu}(\Omega)$ by

$$D(K_p) = D(\partial_{L^2_{\mu}(\Omega)} \mathcal{E}) \quad \text{and} \quad K_p u = -\partial_{L^2_{\mu}(\Omega)} \mathcal{E}(u) \quad \text{for every } u \in D(K_p).$$

We call K_p the *Dirichlet p -Kolmogorov operator in $L^2_{\mu}(\Omega)$* .

3.3.2 Solvability of the first initial boundary value problem

Due to Theorem 2.2.4 and Theorem 2.2.5 of Chapter 2, we can state the following existence and uniqueness theorems of strong solutions as introduced in Definition 2.2.14.

THEOREM 3.3.1. For every $u_0 \in L^2_\mu(\Omega)$ and for every $f \in L^2(0, T; L^2_\mu(\Omega))$, the initial boundary-value problem

$$\begin{cases} u_t - K_p u = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

has a unique strong solution $u \in W^{1,2}(\delta, T; L^2(\Omega)) \cap L^p(0, T; V_{\mu,0}) \cap C((0, T]; V_{\mu,0})$ for all $0 < \delta < T$. If, in addition $u_0 \in V_{\mu,0}$, then $u \in W^{1,2}(0, T; L^2(\Omega)) \cap C([0, T]; V_{\mu,0})$.

THEOREM 3.3.2. Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function, in the sense that the function $x \mapsto f(x, u)$ is measurable on Ω for all $u \in \mathbb{R}$ and the function $u \mapsto f(x, u)$ is continuous on \mathbb{R} for a.e. $x \in \Omega$. Suppose that $f(x, 0) = 0$ for a.e. $x \in \Omega$, and there is a constant $L > 0$ such that $|f(x, u_1) - f(x, u_2)| \leq L|u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, and a.e. $x \in \Omega$. Then, for every $u_0 \in L^2_\mu(\Omega)$, the initial boundary-value problem

$$\begin{cases} u_t - K_p u = f(u) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

has a unique strong solution $u \in W^{1,\infty}(\delta, T; L^2(\Omega)) \cap L^p(0, T; V_{\mu,0})$ for all $0 < \delta < T$. If, in addition $u_0 \in V_{\mu,0}$, then $u \in W^{1,\infty}(0, T; L^2(\Omega)) \cap C([0, T]; V_{\mu,0})$.

3.3.3 Definition of weak solutions

In this subsection, we introduce the notion of weak solutions of

$$(3.18) \quad \begin{cases} u_t - K_p u = \Phi(x) |u|^{p-2} u & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where Φ is a given positive potential of class $L^\infty_{loc}(\Omega \setminus \{0\})$, Ω is a domain of \mathbb{R}^d such that $0 \in \Omega$ if $d \geq 2$, $\Omega = [0, +\infty)$ if $d = 1$, $u_0 \in L^2_{\mu,loc}(\Omega)$ is positive, $1 < p < +\infty$, and $T > 0$. For convenience, we set $\dot{\Omega} = \Omega \setminus \{0\}$.

In the case $p = 2$, our definition of weak solution has been motivated by the one given in [16] by X. Cabré and Y. Martel, and in [34] by G. R. Goldstein, J. A. Goldstein and A. Rhandi. In the case $p \neq 2$, the author has been motivated by the definition of weak solution given in [20] by E. DiBenedetto.

DEFINITION 3.3.3. If $p \neq 2$, then for given $u_0 \in L^2_{\mu,loc}(\Omega)$ (respectively, $u_0 \in L^2_{\mu,loc}[0, +\infty)$ if $d = 1$), we call u a weak solution of problem (3.18) provided

$$u \in C([0, T]; L^2_{\mu,loc}(\Omega)) \cap L^p(0, T; W^{1,p}_{\mu,loc}(\Omega \setminus \{0\})),$$

(respectively, $u \in C([0, T]; L^2_{\mu,loc}[0, +\infty))$ if $d = 1$) for all open set \mathcal{K} with compact closure contained in Ω , (abbreviated, $\mathcal{K} \Subset \Omega$)

$$(3.19) \quad u(t) \rightarrow u_0 \quad \text{in } L^2_\mu(\mathcal{K}) \text{ as } t \rightarrow 0+,$$

for all $0 \leq t_1 < t_2 < T$, for each $\mathcal{K} \Subset \Omega \setminus \{0\}$, and all $\varphi \in W^{1,2}(t_1, t_2; L^2_\mu(\mathcal{K})) \cap L^p(t_1, t_2; W^{1,p}_{\mu,0}(\mathcal{K}))$,

$$(3.20) \quad (u, \varphi)_{L^2_\mu(\mathcal{K})} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} \left\{ -u \varphi_t + |\nabla u|^{p-2} \nabla u \nabla \varphi \right\} d\mu dt = \int_{t_1}^{t_2} \int_{\mathcal{K}} \Phi(x) |u|^{p-2} u \varphi d\mu dt.$$

If $p = 2$, then for given $u_0 \in L^2_{\mu,loc}(\Omega)$, we call u a *weak solution* of problem (3.18) provided $u \in C([0, T]; L^2_{\mu,loc}(\Omega))$ (respectively, $u \in C([0, T]; L^2_{\mu,loc}[0, +\infty))$ if $d = 1$) satisfies for every $\mathcal{K} \Subset \Omega$ the initial condition (3.19), for every open ball $B(0, r) \subseteq \Omega$ centered at $x = 0$ with radius $r > 0$ and every $0 \leq t_1 < t_2 < T$, one has that

$$\Phi u \in L^1(t_1, t_2, L^1_\mu(B(0, r))) \quad (\text{respectively, } \Phi u \in L^1(t_1, t_2, L^1_\mu(0, r)) \text{ if } d = 1)$$

and

$$(u, \varphi)_{L^2_\mu(\Omega)} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} u \left\{ -\varphi_t - K_2 \varphi \right\} d\mu dt = \int_{t_1}^{t_2} \int_{\Omega} \Phi(x) u \varphi d\mu dt$$

for all $\varphi \in W^{1,2}(t_1, t_2; L^2_\mu(\Omega)) \cap L^2(t_1, t_2; W^{2,2}(\Omega))$ with $\varphi(\cdot, t)$ having compact support in Ω .

We call a weak solution u of problem (3.18) *positive* if $u \geq 0$ almost everywhere on $\Omega \times (0, T)$

3.3.4 Comparison principles

In this subsection, we recall two standard comparison principles, which we employ in the subsequent section.

A weak comparison principle for strong solutions

LEMMA 3.3.3. (Comparison principle for strong solutions) *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open subset with a Lipschitz continuous boundary, and suppose there is a function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the same hypotheses as in Theorem 3.3.2. If u and v belong to $W^{1,2}(\delta, T; L^2_\mu(\Omega)) \cap L^p(0, T; V_{\mu,0})$ for any $0 < \delta < T$ and satisfy for a.e. $t \in (0, T)$, and for all positive $\varphi \in V_{\mu,0}$,*

$$(3.21) \quad \int_{\Omega} [u_t(t) - v_t(t)] \varphi d\mu + \int_{\Omega} [|\nabla u(t)|^{p-2} \nabla u(t) - |\nabla v(t)|^{p-2} \nabla v(t)] \nabla \varphi d\mu \\ + \int_{\Omega} [f(x, u(t)) - f(x, v(t))] \varphi d\mu \leq 0,$$

then

$$(3.22) \quad \operatorname{ess\,sup}_{(x,t) \in \Omega_T} e^{-Lt} (u - v)(x, t) \leq \operatorname{ess\,sup}_{(x,t) \in \mathcal{P}\Omega_T} e^{-Lt} [u - v]^+(x, t).$$

PROOF. To prove the assertion of this lemma, we employ the truncation method of Stampacchia (see also the proof of Théorème X.3 in [12, p.211]). Suppose that

$$k = \operatorname{ess\,sup}_{(x,t) \in \mathcal{P}\Omega_T} e^{-Lt} [u - v]^+(x, t) \quad \text{is finite.}$$

We denote by $[v]^+ := \max\{0, v(x, t)\}$ the positive part of a function v defined on Ω_T , and set

$$w(x, t) = u(x, t) - v(x, t) - ke^{Lt} \quad \text{for a.e. } (x, t) \in \Omega_T.$$

Since the function $s \mapsto [s]^+$ is Lipschitz-continuous on \mathbb{R} , we have by Corollaire VIII.10 in [12] that $w^+ \in W^{1,2}(\delta, T; L_\mu^2(\Omega)) \cap L^p(0, T, V)$. And since by hypothesis, $w^+ = 0$ on $\partial\Omega \times (0, T)$, we have by Lemma 3.3 in [54], that $w^+ \in L^p(0, T; V_{\mu,0})$. We denote by $\mathbb{1}_{\{u-v > ke^{Lt}\}}$ the characteristic function of the set $\{(x, t) \in \Omega_T \mid u(x, t) - v(x, t) > ke^{Lt}\}$, and we set $\varphi(t) = \frac{1}{2} \|w^+(t)\|_{L_\mu^2(\Omega)}^2$ for all $t \in [0, T]$. Then, $\varphi \in W^{1,2}(\delta, T)$ for all $0 < \delta < T$, $\varphi \in C[0, T]$, $\varphi(0) = 0$, $\varphi \geq 0$ on $[0, T]$, and for a.e. $t \in (0, T)$,

$$\begin{aligned} \varphi'(t) &= \int_{\Omega} [u_t(t) - v_t(t)] w^+(t) d\mu - \int_{\Omega} k L e^{Lt} w^+(t) d\mu \\ &\leq - \int_{\Omega} [|\nabla u(t)|^{p-2} \nabla u(t) - |\nabla v(t)|^{p-2} \nabla v(t)] [\nabla u(t) - \nabla v(t)] \mathbb{1}_{\{u-v > ke^{Lt}\}} d\mu \\ &\quad - \int_{\Omega} [f(x, u(t)) - f(x, v(t))] w^+(t) d\mu - \int_{\Omega} k L e^{Lt} w^+(t) d\mu \\ &\leq L \int_{\{u-v > ke^{Lt}\}} [u(t) - v(t)] w^+(t) d\mu - \int_{\Omega} k L e^{Lt} w^+(t) d\mu \\ &\leq 0. \end{aligned}$$

Thus, $\varphi(t) \equiv 0$, and hence inequality (3.22) holds. \square

A weak comparison principle for positive weak solutions

The following weak comparison principle for positive weak solution is a more subtle version of Proposition 4.1 in [34].

PROPOSITION 3.3.4. (Weak comparison principle of positive weak solutions) *Let $u_0 \in L_\mu^2(\Omega)$ and $\Phi \in L_{loc}^\infty(\Omega \setminus \{0\})$ be both positive, $p = 2$, D be an open and bounded subset of Ω with a Lipschitz-continuous boundary, and $g \in L^\infty(D)$ be positive and satisfying*

$$(3.23) \quad 0 \leq g(x) \leq \Phi(x) \quad \text{for a.e. } x \in D.$$

If $v \in W^{1,2}(0, T; L_\mu^2(D)) \cap L^2(0, T; W_{\mu,0}^{1,2}(D))$ denotes the unique strong solution of problem

$$(3.24) \quad \begin{cases} v_t - K_2 v = g(x)v & \text{in } D \times (0, T), \\ v = 0 & \text{on } \partial D \times (0, T), \\ v(0) = \min\{n, u_0\} & \text{in } D, \end{cases}$$

where $n \geq 1$ is an arbitrary integer, and if u is a positive weak solution of problem (3.18) for $p = 2$, then $0 \leq v \leq u$ almost everywhere on $D \times (0, T)$.

PROOF. Let $\varphi \in W^{1,2}(0, T; L_\mu^2(D)) \cap L^2(0, T; W_{\mu,0}^{1,2}(D))$ be positive such that $\varphi(\cdot, T - \varepsilon) \equiv 0$ for any fixed $0 < \varepsilon < T$. If we extend φ by zero on $(\Omega \setminus D) \times (0, T)$, the extension has compact support in Ω for almost every $t \in (0, T)$ and belongs to $W^{1,2}(0, T; L_\mu^2(\Omega)) \cap L^2(0, T; W_{\mu,0}^{1,2}(\Omega))$. Thus every weak solution u of problem (3.18) on $\Omega \times (0, T)$ is, in particular, a weak solution of problem (3.18) on $D \times (0, T)$. Thus

$$(3.25) \quad \int_0^{T-\varepsilon} \int_D u \{ -\varphi_s - K_2 \varphi \} d\mu ds = (u(0), \varphi)_{L_\mu^2(D)} + \int_0^{T-\varepsilon} \int_D \Phi(x) u \varphi d\mu ds.$$

First, we note that by Theorem 3.3.2, problem (3.24) has a unique strong solution, say v , belonging to $W^{1,\infty}(\delta, T; L^2_\mu) \cap C([\delta, T]; W^{1,2}_{\mu,0}(\Omega))$ for any $0 < \delta < T$. Since $W \equiv 0$ and v satisfy inequality (3.21) in $D \times (0, T)$ and since $W \leq v$ on the parabolic boundary of $D \times (0, T)$, the comparison principle for strong solutions (Lemma 3.3.3) implies that v is positive. Further, for almost every $t \in (0, T)$,

$$(v_t(t), \phi)_{L^2_\mu(D)} = (K_2 v(t), \phi)_{L^2_\mu(D)} + (g(x)v(t), \phi)_{L^2_\mu(D)} \quad \text{for all } \phi \in L^2_\mu(D).$$

If we take $\phi = \varphi$ in this equation, integrate over $(0, T - \varepsilon)$, and apply integration by parts once with respect to dt and twice with respect to dx , then

$$(3.26) \quad \int_0^{T-\varepsilon} \int_D v \{ -\varphi_s - K_2 \varphi \} d\mu ds = (v(0), \varphi)_{L^2_\mu(D)} + \int_0^{T-\varepsilon} \int_D g(x) v \varphi d\mu ds.$$

Using both equations (3.25) and (3.26), we obtain for $(v - u)$ that

$$\begin{aligned} \int_0^{T-\varepsilon} \int_D (v - u) \{ -\varphi_s - K_2 \varphi - g(x)\varphi \} d\mu ds \\ = (v(0) - u(0), \varphi)_{L^2_\mu(D)} + \int_0^{T-\varepsilon} \int_D [g(x) - \Phi(x)] u \varphi d\mu ds. \end{aligned}$$

By hypothesis, $v(0) = \min\{n, u_0\} \leq u_0 = u(0)$, since we assume that $g(x)$ satisfies condition (3.23), and since u is positive, we can deduce from the last equation that

$$(3.27) \quad \int_0^{T-\varepsilon} \int_D (v - u) \{ -\varphi_s - K_2 \varphi - g(x)\varphi \} d\mu ds \leq 0$$

for all positive $\varphi \in W^{1,2}(0, T - \varepsilon; L^2_\mu(D)) \cap L^2(0, T - \varepsilon; W^{1,2}_{\mu,0}(D))$.

Let $\psi \in C([0, T - \varepsilon]; C_c^\infty(D))$ be positive and consider the parabolic boundary value problem

$$(3.28) \quad \begin{cases} z_t - K_2 z = g(x)z + \psi & \text{in } D \times (0, T - \varepsilon), \\ z = 0 & \text{on } \partial D \times (0, T - \varepsilon), \\ z(0) = 0 & \text{in } D. \end{cases}$$

By Theorem 2.2.5, problem (3.28) has a unique strong solution

$$z \in W^{1,2}(0, T - \varepsilon; L^2_\mu(D)) \cap C([0, T - \varepsilon]; W^{1,2}_{\mu,0}(D))$$

Owing to Lemma 3.3.3, and since $W \equiv 0$ and z satisfy inequality (3.21) in $D \times (0, T - \varepsilon)$ and since $W \leq z$ on the parabolic boundary of the cylinder $D \times (0, T - \varepsilon)$, the strong solution z of problem (3.28) is positive almost everywhere on $D \times (0, T - \varepsilon)$.

We set $\varphi_0(x, s) = z(x, t - s)$ for a.e. $x \in D$ and all $s \in [0, T - \varepsilon]$. Then, φ_0 is a strong solution of

$$\begin{cases} -\varphi_{0t} - K_2 \varphi_0 - g(x)\varphi_0 & = \psi & \text{in } D \times (0, T - \varepsilon), \\ \varphi_0 & = 0 & \text{on } \partial D \times (0, T - \varepsilon), \\ \varphi_0(T - \varepsilon) & = 0 & \text{in } D. \end{cases}$$

Inserting φ_0 into inequality (3.27) shows that for all positive $\psi \in C([0, T - \varepsilon]; C_c^\infty(D))$,

$$\int_0^{T-\varepsilon} \int_D (v - u) \psi d\mu dt \leq 0.$$

The set $C([0, T - \varepsilon]; C_c^\infty(D))$ lies dense in $L^2(0, T - \varepsilon; L^2_\mu(D))$. Thus by an approximation argument, we can take $\psi = [v - u]^+$ in the last inequality and hence we obtain that $v \leq u$ a.e. on $D \times (0, T - \varepsilon)$. Since $\varepsilon > 0$ has been arbitrary, the claim of this lemma holds. \square

3.3.5 Steklov averages and an integration by parts

The following notation, definition and subsequent auxiliary proposition are standard in the theory of quasi-linear parabolic equations. See, for instance [53, 20] or [78].

NOTATION 3.3.2. For any open subset D of \mathbb{R}^d and any $T > 0$, we denote by D_T the cylinder $D \times (0, T)$. For $q, r \geq 1$, we denote by $L^{q,r}(D_T)$ the *parabolic Lebesgue space* $L^r(0, T; L^q(D))$. The space $L^{q,r}(D_T)$ is equipped by the norm

$$\|u\|_{L^{q,r}(D_T)} := \left(\int_0^T \left(\int_D |u(x)|^q dx \right)^{\frac{r}{q}} dt \right)^{1/r} \quad \text{for all } u \in L^{q,r}(D_T).$$

DEFINITION 3.3.5. Let D an open subset of \mathbb{R}^d and let $T > 0$. Then, for $v \in L^1(D_T)$, $h > 0$, $t \in (0, T)$, and for a.e. $x \in D$, we define the *Steklov mean values* of v (also called *Steklov averages*) by

$$v_h(x, t) := \begin{cases} \frac{1}{h} \int_t^{t+h} v(x, s) ds & \text{if } t \in (0, T - h), \text{ and} \\ 0 & \text{if } t > T - h. \end{cases}$$

The following Lemma is a more detailed version of Lemma 4.7 in [53, p.85].

LEMMA 3.3.4. Let $D \subseteq \mathbb{R}^d$ be an open set. Then the following assertions hold true.

(i) For $v \in L^{q,r}(D_T)$, let $\tilde{v}(t) := v(t)$ if $t \in [0, T]$ and $\tilde{v}(t) = 0$ if $t \in \mathbb{R} \setminus [0, T]$. Further for every $0 < h < \delta < T$, let $\rho_h(t) = h^{-1} \mathbf{1}_{[-1,0]}(h^{-1}t)$ for every $t \in \mathbb{R}$. Then,

$$(3.29) \quad \int_{\mathbb{R}} \rho_h(t) dt = 1, \quad \rho_h \geq 0, \quad \int_{\mathbb{R} \setminus (-r,r)} \rho_h dt \rightarrow 0 \quad \text{as } h \rightarrow 0+, \text{ for all } 0 < r < T,$$

and

$$(3.30) \quad (\tilde{v} * \rho_h)(t) = \int_{\mathbb{R}} \tilde{v}(s) \rho_h(t - s) ds = v_h(t) \quad \text{for every } t \in (0, T - \delta).$$

(ii) If $v \in L^{q,r}(D_T)$, then for every $0 < \delta < T$,

$$v_h \rightarrow v \quad \text{in } L^{q,r}(D_{T-\delta}) \quad \text{as } h \rightarrow 0+.$$

(iii) If $v \in C([0, T]; L^q(D))$, then $v_h(t)$ can be defined in $t = 0$ by $v_h(x, 0) = \frac{1}{h} \int_0^{0+h} v(x, s) ds$ ($x \in D$) for all $h > 0$, and for every $0 < \varepsilon < T$ and every $t \in [0, T - \varepsilon]$,

$$v_h \rightarrow v \quad \text{in } C([0, T - \delta]; L^q(D)) \quad \text{as } h \rightarrow 0+.$$

(iv) If $v \in V^p(D_T) := C([0, T]; L^2(D)) \cap L^p(0, T; W^{1,p}(D))$, then for every $0 < h < \delta < T$,

$$v_h \in W^{1,2}([0, T - \delta]; L^2(D)), \quad \frac{\partial v_h}{\partial t}(t) = h^{-1}(v(t+h) - v(t)) \quad \text{for every } t \in (0, T - \delta).$$

REMARK 3.3.1. Any sequence $(\rho_h)_{h>0}$ in $L^1(\mathbb{R})$ having the same properties as in (3.29) is in literature also called *approximate identity*, *approximate unit*, or *Dirac sequence* (cf., for instance, [3, Definition 2.13] by H. W. Alt).

In the standard references (as, e.g., [53, 20]) the second and third claim of Lemma 3.3.4 are often employed, but in general without any proof.

PROOF. First, we proof that the sequence $(\rho_h)_{h>0}$ defined in (i) satisfies the properties (3.29). Since $\mathbb{1}_{[-1,0]}(h^{-1}t) = \mathbb{1}_{[-h,0]}(t)$ for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} \rho_h(t) dt = h^{-1} \int_{-h}^0 \mathbb{1} dt = 1.$$

Further, for every $0 < r < T$, we have that $(\mathbb{R} \setminus (-r, r)) \cap [-h, 0] = \emptyset$ for every $0 < h < r$. Hence

$$\int_{\mathbb{R} \setminus]-r, r[} \rho_h(t) dt = h^{-1} \int_{\mathbb{R} \setminus]-r, r[} \mathbb{1}_{[-h,0]} dt = 0.$$

To see that formula (3.30) holds, we fix $t \in]0, T - \delta[$, and note that $\mathbb{1}_{[-1,0]}(\frac{t-s}{h}) = \mathbb{1}_{[t, t+h]}(s)$ for every $s \in \mathbb{R}$. Hence,

$$(\tilde{v} * \rho_h)(t) = h^{-1} \int_{\mathbb{R}} \tilde{v}(s) \mathbb{1}_{[-1,0]}(\frac{t-s}{h}) ds = h^{-1} \int_{\mathbb{R}} v(s) \mathbb{1}_{[t, t+h]}(s) ds = v_h(t).$$

Now, let $v \in L^{q,r}(D_T)$ and fix $0 < h < \delta < T$. Then, for every $t \in]0, T - \delta[$,

$$v_h(t) = h^{-1} \int_t^{t+h} v(s) ds = \int_{\frac{t}{h}}^{\frac{t+h}{h}} v(hr) dr = \int_0^1 v(h(s + \frac{t}{h})) ds,$$

where we applied in the first equality the substitution $s \mapsto r(s) = h^{-1}s$ and in the second equality the substitution $r \mapsto s(r) = r - \frac{t}{h}$. Hence for every $t \in (0, T - \delta)$,

$$(3.31) \quad \begin{aligned} \|v_h(t) - v(t)\|_{L^q(D)} &= \left\| \int_0^1 (v(h(s + \frac{t}{h})) - v(t)) ds \right\|_{L^q(D)} \\ &\leq \int_0^1 \|v(h(s + \frac{t}{h})) - v(t)\|_{L^q(D)} ds. \end{aligned}$$

If $v \in C([0, T]; L^q(D))$, then v is uniformly continuous on $[0, T - \delta]$ with values in $L^q(D)$. Thus for every $\varepsilon > 0$, there is a $\tilde{\delta} > 0$ such that for all $0 < h < \tilde{\delta}$, and all $s \in [0, 1]$, all $t \in [0, T - \delta]$,

$$\|v(h(s + \frac{t}{h})) - v(t)\|_{L^q(D)} < \varepsilon,$$

and so by estimate (3.31), for all $0 < h < \tilde{\delta}$,

$$\sup_{t \in [0, T - \delta]} \|v_h(t) - v(t)\|_{L^q(D)} < \varepsilon.$$

This shows that claim (iii) holds. On the other hand, by estimate (3.31) and by Hölder's inequality,

$$\left(\int_0^{T-\delta} \|v_h(t) - v(t)\|_{L^q(D)}^r dt \right)^{1/r} \leq \left(\int_0^{T-\delta} \left(\int_0^1 \|v(h \cdot s + t) - v(t)\|_{L^q(D)} ds \right)^r dt \right)^{1/r}$$

$$\leq \left(\int_0^{T-\delta} \int_0^1 \|v(h \cdot s + t) - v(t)\|_{L^q(D)}^r ds dt \right)^{1/r}.$$

By Tonelli's theorem and again Hölder's inequality,

$$\begin{aligned} \left(\int_0^{T-\delta} \int_0^1 \|v(h \cdot s + t) - v(t)\|_{L^q(D)}^r ds dt \right)^{1/r} &= \left(\int_0^1 \int_0^{T-\delta} \|v(h \cdot s + t) - v(t)\|_{L^q(D)}^r dt ds \right)^{1/r} \\ &\leq \int_0^1 \left(\int_0^{T-\delta} \|v(h \cdot s + t) - v(t)\|_{L^q(D)}^r dt \right)^{1/r} ds. \end{aligned}$$

If $v \in L^{q,r}(D_T)$, then for a.e. $s \in (0, 1)$,

$$\left(\int_0^{T-\delta} \|v(h \cdot s + t) - v(t)\|_{L^q(D)}^r dt \right)^{1/r} \rightarrow 0 \quad \text{as } h \rightarrow 0+,$$

and by Minkowski's inequality, for a.e. $s \in (0, 1)$ and all $0 < h < \delta$,

$$\begin{aligned} \left(\int_0^{T-\delta} \|v(h \cdot s + t) - v(t)\|_{L^q(D)}^r dt \right)^{1/r} &\leq \left(\int_0^{T-\delta} \left(\|v(h \cdot s + t)\|_{L^q(D)} + \|v(t)\|_{L^q(D)} \right)^r dt \right)^{1/r} \\ &\leq \left(\int_0^{T-\delta} \|v(h \cdot s + t)\|_{L^q(D)}^r dt \right)^{1/r} + \|v\|_{L^{q,r}(D_{T-\delta})} \\ &= \left(\int_{h \cdot s}^{h \cdot s + T - \delta} \|v(\tau)\|_{L^q(D)}^r d\tau \right)^{1/r} + \|v\|_{L^{q,r}(D_{T-\delta})} \\ &\leq 2 \cdot \|v\|_{L^{q,r}(D_T)}. \end{aligned}$$

Therefore, by Lebesgue's dominated convergence theorem, we obtain that claim (ii) holds. To see that claim (iv) holds, let $0 < h < \delta$ and fix $\xi \in C_c^1(0, T - \delta)$. Then by Fubini's theorem,

$$\begin{aligned} \int_0^{T-\delta} h^{-1} \int_t^{t+h} v(s) \frac{d\xi}{dt}(t) ds dt &= h^{-1} \int_0^{T+h-\delta} \int_{\max\{0, s-h\}}^{\min\{s, T-\delta\}} v(s) \frac{d\xi}{dt}(t) dt ds \\ &= h^{-1} \int_0^h \int_0^s v(s) \frac{d\xi}{dt}(t) dt ds + h^{-1} \int_h^{T-\delta} \int_h^s v(s) \frac{d\xi}{dt}(t) dt ds \\ &\quad + h^{-1} \int_{T-\delta}^{T+h-\delta} \int_{s-h}^{T-\delta} v(s) \frac{d\xi}{dt}(t) dt ds. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^{T-\delta} v_h(t) \frac{d\xi}{dt}(t) dt &= \int_0^{T-\delta} h^{-1} \int_t^{t+h} v(s) \frac{d\xi}{dt}(t) ds dt \\ &= h^{-1} \int_0^h \int_0^s v(s) \frac{d\xi}{dt}(t) dt ds + h^{-1} \int_h^{T-\delta} \int_h^s v(s) \frac{d\xi}{dt}(t) dt ds \\ &\quad + h^{-1} \int_{T-\delta}^{T+h-\delta} \int_{s-h}^{T-\delta} v(s) \frac{d\xi}{dt}(t) dt ds \end{aligned}$$

$$\begin{aligned}
&= h^{-1} \int_0^h v(s)(\xi(s) - \xi(0)) ds + h^{-1} \int_h^{T-\delta} v(s)(\xi(s) - \xi(s-h)) ds \\
&\quad + h^{-1} \int_{T-\delta}^{T+h-\delta} v(s)(\xi(T-\delta) - \xi(s-h)) ds \\
&= h^{-1} \int_0^{T-\delta} v(s)\xi(s) ds - h^{-1} \int_h^{T-\delta} v(s)\xi(s-h) ds \\
&\quad - h^{-1} \int_{T-\delta}^{T+h-\delta} v(s)\xi(s-h) ds \\
&= h^{-1} \int_0^{T-\delta} v(s)\xi(s) ds - h^{-1} \int_h^{T+h-\delta} v(s)\xi(s-h) ds \\
&= h^{-1} \int_0^{T-\delta} v(s)\xi(s) ds - h^{-1} \int_0^{T-\delta} v(r+h)\xi(r) dr \\
&= - \int_0^{T-\delta} h^{-1}(v(s+h) - v(s)) \xi(s) ds.
\end{aligned}$$

Therefore, claim (iv) holds true and this completes the proof of Lemma 3.3.4. \square

The following Lemma generalizes Remark 2.1.4 in [78, p.158].

LEMMA 3.3.5. *Let $p \neq 2$, and let $\Phi \in L_{loc}^\infty(\Omega \setminus \{0\})$ be positive. If u is a weak solution of equation (3.18) and if $g : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz-continuous, then for every $\phi \in C_c^1(\Omega \setminus \{0\})$ and every $0 \leq t_1 < t_2 < T$,*

$$\begin{aligned}
(3.32) \quad &\int_{\Omega} \int_0^{u(t_2)} g(s) ds \phi d\mu - \int_{\Omega} \int_0^{u(t_1)} g(s) ds \phi d\mu + \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^p g'(u) \phi d\mu dt \\
&+ \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi g(u) d\mu dt = \int_{t_1}^{t_2} \int_{\Omega} \Phi(x) |u|^{p-2} u g(u) \phi d\mu dt.
\end{aligned}$$

PROOF. We fix $\phi \in C_c^1(\Omega \setminus \{0\})$, and for fixed $0 < t < t+h < T$, we take $t_1 = t$, $t_2 = t+h$, and multiply equation (3.20) by h^{-1} . Then

$$\begin{aligned}
&\int_{\Omega} h^{-1} (u(t+h) - u(t)) \phi d\mu + \int_t^{t+h} \int_{\Omega} h^{-1} (|\nabla u|^{p-2} \nabla u) \nabla \phi d\mu dt \\
&= \int_t^{t+h} \int_{\Omega} h^{-1} (\Phi(x) |u|^{p-2} u + f) \phi d\mu dt.
\end{aligned}$$

Due to Fubini's theorem, since $\frac{\partial u_h}{\partial t}(t) = h^{-1} (u(t+h) - u(t))$, and by the definition of Steklov averages, the last equation can be rewritten as

$$\int_{\Omega} \frac{\partial u_h}{\partial t}(t) \phi d\mu + \int_{\Omega} (|\nabla u|^{p-2} \nabla u)_h(t) \nabla \phi d\mu = \int_{\Omega} (\Phi(x) |u|^{p-2} u + f)_h(t) \phi d\mu.$$

By the hypothesis and by Theorem 2.1.11 in [80], for any $t \in (0, T)$, $g(u_h(t)) \phi \in W_{\mu,0}^{1,p}(\mathcal{K})$ with distributional partial derivatives

$$\frac{\partial}{\partial x_i} (g(u_h(t)) \phi) = g'(u_h(t)) \left(\frac{\partial u}{\partial x_i} \right)_h(t) \phi + g(u_h(t)) \frac{\partial \phi}{\partial x_i},$$

where $\mathcal{K} \Subset \Omega \setminus \{0\}$ is chosen such that the support of ϕ is contained in \mathcal{K} . Thus, we can replace ϕ by $g(u_h(t))\phi$ in the last equation. Now, we integrate the resulting equation over $]t_1, t_2[$ for fixed $0 \leq t_1 < t_2 < T$ and apply first Fubini's theorem and then the fundamental theorem of calculus with respect to dt . We obtain that

$$\begin{aligned} & \int_{\Omega} \left(\int_0^{u_h(t_2)} g(s) ds \right) \phi d\mu - \int_{\Omega} \left(\int_0^{u_h(t_1)} g(s) ds \right) \phi d\mu \\ & + \int_{t_1}^{t_2} \int_{\Omega} \left\{ \left(|\nabla u|^{p-2} \nabla u \right)_h (t) g'(u_h(t)) (\nabla u)_h (t) \phi + \left(|\nabla u|^{p-2} \nabla u \right)_h (t) g(u_h(t)) \nabla \phi \right\} d\mu dt \\ & = \int_{t_1}^{t_2} \int_{\Omega} \left(\Phi(x) |u|^{p-2} u + f \right)_h (t) g(u_h(t)) \phi d\mu dt. \end{aligned}$$

Sending $h \rightarrow 0+$ in the last equation and using Lemma 3.3.4 leads to equation (3.32). \square

3.3.6 Some geometric properties of the boundary

The aim of this subsection is to prove that if D is an open subset of \mathbb{R}^d with a boundary of class C^2 and if $d(x)$ denotes the distance function on D to the boundary of D , then the function $x \mapsto \log d(x)$ belongs to $L^p(D)$ for all $p > 1$ (cf. Lemma 3.3.9). For this, we recall the definition and some basic results concerning the regularity of the boundary of a bounded open set. Here, we employ the same setting as in the book [6] by W. Arendt and K. Urban.

DEFINITION 3.3.6. (∂D of class C^k) Let D be an open subsets of \mathbb{R}^d and denote by ∂D the boundary of D . Let U be another open subset of \mathbb{R}^d and let $k \geq 1$.

- (1) We call $U \cap \partial D$ a *normal graph of class C^k* (with respect to D) if there is a $g \in C^k(\mathbb{R}^{d-1})$ and there are $r, h > 0$ such that

$$(3.33) \quad U = \{(y, g(y) + s)^t \in \mathbb{R}^d \mid y \in \mathbb{R}^{d-1}, |y| < r, s \in \mathbb{R}, |s| < h\}$$

and every $x = (y, g(y) + s)^t \in U$ satisfies

$$(3.34) \quad x \in D \quad \text{if and only if} \quad s > 0,$$

$$(3.35) \quad x \in \partial D \quad \text{if and only if} \quad s = 0,$$

$$(3.36) \quad x \notin \overline{D} \quad \text{if and only if} \quad s < 0.$$

Roughly speaking, this means that the boundary ∂D in U is a graph of class C^k and D lies only at one side of the graph.

- (2) We say that the set $U \cap \partial D$ is a *graph of class C^k* (with respect to D) if there is an orthogonal $d \times d$ -matrix A and a vector $b \in \mathbb{R}^d$ such that for the affine function $\Phi(x) := Ax + b$, ($x \in \mathbb{R}^d$), the set $\Phi(U \cap \partial D) = \Phi(U) \cap \partial \Phi(D)$ is a normal graph of class C^k (with respect to $\Phi(D)$). Since Φ is a diffeomorphism, $\Phi(U)$ is an open neighborhood of $\Phi(z)$ for all $z \in U \cap \partial D$. With other words, $U \cap \partial D$ is a graph of class C^k if after eventual rotation and translation, $U \cap \partial D$ is a normal graph of class C^k .
- (3) We say that D has a boundary ∂D of class C^k and we write $\partial D \in C^k$ if for every $z \in \partial D$, there is a open neighborhood $U \subseteq \mathbb{R}^d$ of z such that $U \cap \partial D$ is a graph of class C^k .

THEOREM 3.3.6. (The tangent space T_z to ∂D at z) *Let D be an open and bounded subset with $\partial D \in C^1$. For a boundary point $z \in \partial D$, we call a vector $v \in \mathbb{R}^d$ tangential to ∂D at z if there is an $\varepsilon > 0$ and there is a function $\psi \in C^1((-\varepsilon, \varepsilon); \mathbb{R}^d)$ satisfying*

$$\psi(0) = z, \quad \psi'(0) = v, \quad \psi(t) \in \partial D \cap U \text{ for all } t \in (-\varepsilon, \varepsilon),$$

where $U \subseteq \mathbb{R}^d$ is an open neighborhood of z such that $U \cap \partial D$ is a normal graph of class C^1 . The set T_z given by

$$T_z = \{v \in \mathbb{R}^d \mid v \text{ is tangential to } \partial D \text{ at } z\}$$

is a $(d-1)$ -dimensional subspace of \mathbb{R}^d . We call T_z the tangent space to ∂D at z .

PROOF. We fix $z \in \partial D$. Then by definition, there is an open neighborhood $U \subseteq \mathbb{R}^d$ of z such that $U \cap \partial D$ is a graph of class C^1 (with respect to D). Further, we may assume without loss of generality that $U \cap \partial D$ is a normal graph of class C^1 with respect to D , otherwise we rotate and translate the set $U \cap \partial D$ by an affine mapping $\Phi(x) = Ax + b$, $x \in \mathbb{R}^d$, such that $\Phi(U) \cap \partial\Phi(D)$ is a normal graph. Then, by Definition 3.3.6 (1), there is a function $g \in C^1(\mathbb{R}^{d-1})$ and there are $r, h > 0$ such that U satisfies (3.33) and if for $y \in \mathbb{R}^d$, $y' := (y_1, \dots, y_{d-1})^t \in \mathbb{R}^{d-1}$, then every $x = (y', g(y') + s)^t$ that belongs to U satisfies (3.34)-(3.36). In particular, since $z \in \partial D$, we have that

$$(3.37) \quad z' := (z_1, \dots, z_{d-1})^t \in B'(0, r) \quad \text{and} \quad z_{d-1} = g(z'),$$

where we denote by $B'(0, r)$ the $(d-1)$ -dimensional open ball with center $x = 0$ and radius $r > 0$. For $i = 1, \dots, d-1$, we denote by e'_i the i th-unit vector in \mathbb{R}^{d-1} and we set $u_i = (e'_i, \frac{\partial g}{\partial x_i}(z'))^t \in \mathbb{R}^d$. Since the $d \times (d-1)$ -matrix

$$(u_1, u_2, \dots, u_{d-1}) = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \ddots & \dots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 1 \\ \frac{\partial g}{\partial x_1}(z') & \frac{\partial g}{\partial x_2}(z') & \dots & \frac{\partial g}{\partial x_{d-3}}(z') & \frac{\partial g}{\partial x_{d-2}}(z') & \frac{\partial g}{\partial x_{d-1}}(z') \end{pmatrix}$$

has range $d-1$, the vectors $\{u_1, \dots, u_{d-1}\}$ are linearly independent in \mathbb{R}^d and hence the linear span $\text{span}_{\mathbb{R}}\{u_1, \dots, u_{d-1}\}$ forms a $(d-1)$ -dimensional linear subspace of \mathbb{R}^d . We show that

$$T_z = \text{span}_{\mathbb{R}}\{u_1, \dots, u_{d-1}\}.$$

To see this, take $v \in \text{span}_{\mathbb{R}}\{u_1, \dots, u_{d-1}\}$. Then, there are $\lambda_1, \dots, \lambda_{d-1} \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{d-1} \lambda_i u_i = \left(\lambda_1, \dots, \lambda_{d-1}, \sum_{i=1}^{d-1} \lambda_i \frac{\partial g}{\partial x_i}(z') \right)^t.$$

For $\lambda' := (\lambda_1, \dots, \lambda_{d-1})^t \in \mathbb{R}^{d-1}$, we define the function $\psi \in C^1((-\varepsilon, \varepsilon); \mathbb{R}^d)$ by

$$\psi(t) = (z' + t\lambda', g(z' + t\lambda'))^t \in \mathbb{R}^d \quad \text{for all } t \in \mathbb{R} \text{ with } |t| < \varepsilon,$$

where $\varepsilon > 0$ is still to be chosen. Then $\psi \in C^1(\mathbb{R}; \mathbb{R}^d)$ and by (3.37), we see that $\psi(0) = z$. Since $z \in U \cap D$ and $U \cap D$ is open, there is an $\varepsilon > 0$ such that for all $|t| < \varepsilon$, $\psi(t) \in U$ and so by the properties (3.33) and (3.35) in Definition 3.3.6 (1), $\psi(t) \in \partial D \cap U$ for all $t \in \mathbb{R}$ such that $|z' + t\lambda'| < r$. By the chain rule and since $g \in C^1$,

$$\psi'(t)|_{t=0} = (\lambda', \nabla g(z')\lambda')^t = \left(\lambda_1, \dots, \lambda_{d-1}, \sum_{i=1}^{d-1} \lambda_i \frac{\partial g}{\partial x_i}(z') \right)^t = v.$$

Thus $v \in T_z$ and thereby $\text{span}_{\mathbb{R}}\{u_1, \dots, u_{d-1}\} \subseteq T_z$. Now, take $v \in T_z$. Then, there is an $\varepsilon > 0$ and there is a function $\psi \in C^1((-\varepsilon, \varepsilon); \mathbb{R}^d)$ such that $\psi(0) = z$, $\psi'(0) = v$, and $\psi(t) \in \partial D \cap U$ for all $|t| < \varepsilon$. Hence and by the properties (3.33) and (3.35) in Definition 3.3.6 (1), the d th component $\psi_d(t)$ of $\psi(t)$ satisfies

$$(3.38) \quad \psi_d(t) = g(\psi_1(t), \dots, \psi_{d-1}(t)) \quad \text{for all } |t| < \varepsilon.$$

Since $z = (z', z_d)^t = \psi(0)$ and since $v_i = \psi'_i(0)$ for all $i = 1, \dots, d$, we have that

$$\psi'_d(0) = \nabla g(\psi_1(0), \dots, \psi_{d-1}(0))(\psi'_1(0), \dots, \psi'_{d-1}(0))^t = \sum_{i=1}^{d-1} v_i \frac{\partial g}{\partial x_i}(z').$$

Thus and by $u_i = (e'_i, \frac{\partial g}{\partial x_i}(z'))^t$,

$$v = \left(v_1, \dots, v_{d-1}, \sum_{i=1}^{d-1} v_i \frac{\partial g}{\partial x_i}(z') \right)^t = \sum_{i=1}^{d-1} v_i u_i \in \text{span}_{\mathbb{R}}\{u_1, \dots, u_{d-1}\}.$$

This completes the proof of this theorem. \square

THEOREM 3.3.7. (The unit outer normal vector $v(z)$ to ∂D at z) *Let D be an open and bounded subset with $\partial D \in C^1$. Then, for every boundary point $z \in \partial D$, there is a unique vector $v(z) \in \mathbb{R}^d$ satisfying*

$$(3.39) \quad v(z) \in T_z^\perp := \{u \in \mathbb{R}^d \mid \langle u, v \rangle_{\mathbb{R}^d} = 0 \text{ for all } v \in T_z\}$$

$$(3.40) \quad |v(z)| = 1$$

$$(3.41) \quad z + tv(z) \notin \bar{D} \quad \text{for all } t \in (0, \varepsilon) \quad \text{and} \quad z + tv(z) \in D \quad \text{for all } t \in (-\varepsilon, 0).$$

We call the vector $v(z)$ the unit outer normal vector $v(z)$ to ∂D at z . Further, the mapping $z \mapsto v(z)$ from ∂D to \mathbb{R}^d is continuous.

PROOF. We fix $z \in \partial D$ and employ the same notation as in the proof of Theorem 3.3.6. Then by the properties (3.33) and (3.35) in Definition 3.3.6 (1), for $z = (z', z_d)$, we have $g(z') = z_d$. We set

$$w = (\nabla g(z'), -1)^t \in \mathbb{R}^d \quad \text{and} \quad v(z) := \frac{(\nabla g(z'), -1)^t}{\sqrt{|\nabla g(z')|^2 + 1}}.$$

First, we show that $T_z = (\text{span}_{\mathbb{R}}\{w\})^\perp$. To see this, take $v \in T_z$. Then, there is an $\varepsilon > 0$ and there is a function $\psi \in C^1((-\varepsilon, \varepsilon); \mathbb{R}^d)$ such that $\psi(0) = z$, $\psi'(0) = v$, $\psi(t) \in \partial D \cap U$ for all $|t| < \varepsilon$,

and in hence, in particular, $\psi'_d(0)$ satisfies (3.38). Thus and since $v_i = \psi'_i(0)$ for all $i = 1, \dots, d$,

$$v_d = \sum_{i=1}^{d-1} v_i \frac{\partial g}{\partial x_i}(z') \quad \text{and hence} \quad \langle v, w \rangle_{\mathbb{R}^d} = \sum_{i=1}^{d-1} v_i \frac{\partial g}{\partial x_i}(z') - v_d \mathbf{1} = 0.$$

Therefore $T_z \subseteq (\text{span}_{\mathbb{R}}\{w\})^\perp$. By Theorem 3.3.6 and since $\text{span}_{\mathbb{R}}\{w\}$ has dimension one, T_z and $(\text{span}_{\mathbb{R}}\{w\})^\perp$ are both $(d-1)$ -dimensional linear subspace of \mathbb{R}^d . Thus $T_z = (\text{span}_{\mathbb{R}}\{w\})^\perp$ and $v(z)$ satisfies the properties (3.39) and (3.40).

Next, we show that $v(z)$ satisfies property (3.41). Let $v'(z) := (v_1(z), \dots, v_{d-1}(z))^t \in \mathbb{R}^{d-1}$ such that $v(z) = (v'(z), v_d(z))^t$. Since $g \in C^1(\mathbb{R}^{d-1})$, we can calculate the directional derivative of g at z' in direction $v'(z)$. Then, and since $v'(z) = \frac{\nabla g(z')}{\sqrt{|\nabla g(z')|^2+1}}$, we obtain that for all t in a neighborhood of $t = 0$,

$$(3.42) \quad g(z' + t v'(z)) = g(z') + \langle \nabla g(z'), v'(z) \rangle_{\mathbb{R}^{d-1}} t + o(t) = g(z') + \frac{|\nabla g(z')|^2}{\sqrt{|\nabla g(z')|^2+1}} t + o(t),$$

where $\frac{o(t)}{t} \rightarrow 0$ as $t \rightarrow 0$. Hence and since $g(z') = z_d$, for all sufficiently small t ,

$$(z + t v(z))_d = z_d - t \frac{1}{\sqrt{|\nabla g(z')|^2+1}} = g(z' + t v'(z)) - \|w\|_2 t - o(t).$$

For every $\delta > 0$, there is an $\varepsilon > 0$ such that for all $0 < |t| < \varepsilon$, equation (3.42) holds and

$$-o(t) < \delta t \quad \text{for all } t \in (0, \varepsilon) \quad \text{and} \quad -o(t) > \delta t \quad \text{for all } t \in (-\varepsilon, 0).$$

Therefore, if we choose $\delta = \|w\|_2 > 0$, then

$$\alpha(t) := (z + t v(z))_d - g(z' + t v'(z)) < 0 \quad \text{if } t \in (0, \varepsilon), \quad \text{and} \quad \alpha(t) > 0 \quad \text{if } t \in (-\varepsilon, 0)$$

and so by the properties (3.34) and (3.36) in Definition 3.3.6 (1),

$$(z' + t v'(z), (z + t v(z))_d) = (z' + t v'(z), g(z' + t v'(z)) + \alpha(t)) \in D \quad \text{if } t \in (-\varepsilon, 0)$$

and

$$(z' + t v'(z), (z + t v(z))_d) \notin \overline{\Omega} \quad \text{if } t \in (0, \varepsilon).$$

□

DEFINITION 3.3.7. (Uniform interior sphere condition) For an open set D in \mathbb{R}^d , we say that ∂D satisfies the *uniform interior sphere condition* if for each $z \in \partial D$ there is an open ball $B_z \subseteq D$ depending on z such that $\overline{B_z} \cap D^c = \{z\}$, and the radii of the balls $\{B_z\}_{z \in \partial D}$ are bounded from below by a positive constant.

LEMMA 3.3.8. If D is an open and bounded set in \mathbb{R}^d with a boundary $\partial D \in C^2$, then ∂D satisfies the uniform interior sphere condition. More precisely, there is an $\delta \in (0, 1)$ such that for every $z \in \partial D$, the open ball $B_{z, \delta} := B(z - \nu(z) \frac{\delta}{2}, \frac{\delta}{2}) \subseteq D_\delta := \{x \in D \mid d(x) < \delta\}$ and $\partial B_{z, \delta} \cap \partial D = \{z\}$, where $\nu(z)$ denotes the unit outward normal vector to ∂D at z , and $d(x) := \text{dist}(x, \partial D)$.

PROOF. By the hypothesis, for every $z \in \partial D$, there is an open neighborhood $U_z \subseteq \mathbb{R}^d$ of z such that $U_z \cap \partial D$ is graph of class C^2 (with respect to D). Thus and since ∂D is compact, there are finitely many open subsets $U_1, \dots, U_m \subseteq \mathbb{R}^d$ such that $\partial D \subseteq \bigcup_{i=1}^m U_i$, and for each $i = 1, \dots, m$, $U_i \cap \partial D$ is graph of class C^2 (with respect to D). Further, there is an $0 < \varepsilon < 1$ such that the set $D_\varepsilon := \{x \in D \mid d(x) < \varepsilon\}$ satisfies

$$(3.43) \quad D_\varepsilon \subseteq \bigcup_{i=1}^m U_i.$$

Indeed, otherwise for every $n \geq 1$, there is an $x_n \in D_{\frac{1}{n}} \cap (\bigcup_{i=1}^m U_i)^c$. Since $\overline{D_1} \cap (\bigcup_{i=1}^m U_i)^c$ is compact, we can extract a subsequence of $(x_n)_{n \geq 1}$, which we denote, for simplicity again by $(x_n)_{n \geq 1}$ and there is an element $x \in \overline{D_1} \cap (\bigcup_{i=1}^m U_i)^c$ such that $x_n \rightarrow x$ as $n \rightarrow +\infty$. On the other hand, $\text{dist}(x_n, \partial D) < \frac{1}{n} \rightarrow 0$ as $n \rightarrow +\infty$. Thus $x \in \partial D \cap (\bigcup_{i=1}^m U_i)^c$, but this contradicts the hypothesis that $(U_i)_{i=1}^m$ is an open covering of ∂D .

We may assume without loss of generality that every $U_i \cap \partial D$ is a normal graph with respect to D . Thus, by Definition 3.3.6 (1), for each $i = 1, \dots, m$, there is a function $g_i \in C^1(\mathbb{R}^{d-1})$ and there are $r_i, h_i > 0$ such that

$$U_i = \{(y, g_i(y) + s) \in \mathbb{R}^d \mid y \in \mathbb{R}^{d-1}, |y| < r_i, s \in \mathbb{R}, |s| < h_i\}$$

and every $x = (y, g_i(y) + s)^t \in U_i$ satisfies the properties (3.34)-(3.36). We fix $i = 1, \dots, m$. For every $j = 1, \dots, d-1$, let e'_j denote the j th-unit vector in \mathbb{R}^{d-1} and set $u_j^{(i)} = (e'_j, \frac{\partial g_i}{\partial x_j}(z'))^t \in \mathbb{R}^d$.

Then, as outlined in the proof of Theorem 3.3.6, the vectors $\{u_1^{(j)}, \dots, u_{d-1}^{(j)}\}$ are linearly independent in \mathbb{R}^d and the tangent space

$$T_z = \text{span}_{\mathbb{R}}\{u_1, \dots, u_{d-1}\} \quad \text{for every } z \in U_i \cap \partial D.$$

Take $z \in U_i \cap \partial D$ and take $v \in T_z$. Then, there are unique $\lambda_1, \dots, \lambda_{d-1} \in \mathbb{R}$ such that

$$v = \sum_{j=1}^{d-1} \lambda_j u_j^{(i)} = \left(\lambda_1, \dots, \lambda_{d-1}, \sum_{j=1}^{d-1} \lambda_j \frac{\partial g_i}{\partial x_j}(z') \right)^t.$$

Let $\lambda' := (\lambda_1, \dots, \lambda_{d-1})^t \in \mathbb{R}^{d-1}$. There is an $\varepsilon_i > 0$ such that the function $\psi_i \in C^2((-\varepsilon_i, \varepsilon_i); \mathbb{R}^d)$ defined by

$$\psi_i(t) = (z' + t\lambda', g_i(z' + t\lambda'))^t \in \mathbb{R}^d \quad \text{for all } t \in \mathbb{R} \text{ with } |t| < \varepsilon_i.$$

Then, $\psi_i(t) \in U_i \cap \partial D$ for every $|t| < \varepsilon_i$ and $\psi_i(0) = z$. We calculate an upper bound of the curvature $\kappa_i(t)$ of $\psi_i(t)$ for $t \in (-\varepsilon_i, \varepsilon_i)$, which is given by the formula (see [77, p. 173])

$$\kappa_i(t) = \frac{\sqrt{|\psi_i''(t)|^2 |\psi_i'(t)|^2 - \langle \psi_i'(t), \psi_i''(t) \rangle_{\mathbb{R}^{d-1}}}}{|\psi_i'(t)|^3}.$$

For this, we denote by $H_{g_i}(y') \in \mathbb{R}^{(d-1) \times (d-1)}$ the Hessian matrix of $g_i(y')$, ($y' \in \mathbb{R}^{d-1}$) and set

$$(3.44) \quad \sup_{i=1, \dots, m} \sum_{k, l=1}^{d-1} \left\| \frac{\partial^2 g_i}{\partial x_k \partial x_l} \right\|_{L^\infty(U_i)} + 1 =: C$$

Since $g_1, \dots, g_m \in C^2(\mathbb{R}^{d-1})$, the constant C is finite. Further, we have that

$$\psi_i'(t) = (\lambda', \langle \nabla g_i(z' + t\lambda'), \lambda' \rangle_{\mathbb{R}^{d-1}})^t \in \mathbb{R}^d, \quad \psi_i''(t) = (0', \langle \lambda', H_{g_i}(z' + t\lambda') \lambda' \rangle_{\mathbb{R}^{d-1}})^t \in \mathbb{R}^d,$$

$$|\psi'_i(t)|^2 = |\lambda'|^2 + |\langle \nabla g_i(z' + t\lambda'), \lambda' \rangle_{\mathbb{R}^{d-1}}|^2, \quad |\psi''_i(t)|^2 = |\langle \lambda', H_{g_i}(z' + t\lambda') \lambda' \rangle_{\mathbb{R}^{d-1}}|^2,$$

and

$$\langle \psi'_i(t), \psi''_i(t) \rangle_{\mathbb{R}^d} = \langle \nabla g_i(z' + t\lambda'), \lambda' \rangle_{\mathbb{R}^{d-1}} \cdot \langle \lambda', H_{g_i}(z' + t\lambda') \lambda' \rangle_{\mathbb{R}^{d-1}}.$$

Thus, for every $i = 1, \dots, m$, and every $t \in (-\varepsilon_i, \varepsilon_i)$,

$$(3.45) \quad \kappa_i(t) = |\langle \lambda', H_{g_i}(z' + t\lambda') \lambda' \rangle_{\mathbb{R}^{d-1}}| \frac{|\lambda'|}{(|\lambda'|^2 + |\langle \nabla g_i(z' + t\lambda'), \lambda' \rangle_{\mathbb{R}^{d-1}}|^2)^{3/2}} \leq |H_{g_i}(z' + t\lambda')| \leq C.$$

On the other hand, the unit outer normal vector $\nu : \partial D \rightarrow \mathbb{R}^d$ is given for every $z \in \partial D$ with $z' = (z_1, \dots, z_{d-1})^t \in \mathbb{R}^{d-1}$ by

$$(3.46) \quad \nu(z) = \frac{(\nabla g_i(z'), -1)^t}{\sqrt{|\nabla g_i(z')|^2 + 1}} \quad \text{if } z = (z', z_d)^t \in U_i \cap \partial D \text{ for one } i \in \{1, \dots, m\}.$$

For fixed $i \in \{1, \dots, m\}$ and fixed $z = (z', z_d)^t \in U_i \cap \partial D$, we calculate the directional derivative of g_i at z' in direction $-\nu'(z)$ by Taylor expansion at $t = 0$. Then for every $t \in \mathbb{R}$ with $|t| > 0$, there is a $\theta(t) \in (0, 1)$ such that

$$g_i(z' - t\nu'(z)) = g_i(z') - \langle \nabla g_i(z'), \nu'(z) \rangle_{\mathbb{R}^{d-1}} t + \nu'(z)^t H_{g_i}(z' - \theta(t)t\nu'(z)) \nu'(z) \frac{t^2}{2}$$

and so by (3.46),

$$(3.47) \quad g_i(z' - t\nu'(z)) = g_i(z') - \frac{|\nabla g_i(z')|^2}{\sqrt{|\nabla g_i(z')|^2 + 1}} t + \frac{1}{2} \nu'(z)^t H_{g_i}(z' - \theta(t)t\nu'(z)) \nu'(z) t^2.$$

We set $w_i = (\nabla g_i(z'), -1)^t$ and $R_i(t) = \frac{1}{2} \nu'(z)^t H_{g_i}(z' - \theta(t)t\nu'(z)) \nu'(z)$. By the properties (3.33) and (3.35) in Definition 3.3.6 (1), for every $z = (z', z_d)^t \in U_i \cap \partial D$, $z_d = g_i(z')$. Thus, by equation (3.47), and by (3.46), for $|t| > 0$,

$$(z - t\nu(z))_d = z_d + t \frac{1}{\sqrt{|\nabla g_i(z')|^2 + 1}} = g_i(z' - t\nu'(z)) + |w_i| t + R_i(t) t^2$$

Since $|w_i| \geq 1$, and by (3.44), we have that $R_i(t) \geq -C$. Thus, if we choose $0 < \delta \leq \frac{2}{C}$, then

$$\alpha_i(t) := (z - t\nu(z))_d - g_i(z' - t\nu'(z)) \geq \frac{1}{C} t - C \frac{t^2}{2} > 0 \quad \text{if } t \in (0, \delta).$$

Therefore and by the properties (3.34) and (3.36) in Definition 3.3.6 (1),

$$(z' - t\nu'(z), (z - t\nu(z))_d) = (z' - t\nu'(z), g_i(z' - t\nu'(z)) + \alpha_i(t)) \in D \quad \text{for all } t \in (0, \delta).$$

Since $z \in U_i \cap \partial D$ and $i = 1, \dots, m$ have been arbitrary, we have thereby shown that

$$(3.48) \quad (z' - t\nu'(z), (z - t\nu(z))_d) \in D \quad \text{for all } t \in (0, \delta) \text{ and for all } z \in \partial D.$$

Now, we take $\delta := \min\{\frac{3\varepsilon}{4}, \frac{2}{C}\}$ and let $z \in \partial D$ be arbitrary but fixed. Since by (3.48), the point $z - \nu(z) \frac{\delta}{2} \in D_{\frac{\varepsilon}{2}}$, we find that the open ball $B := B(z + \nu(z) \frac{\delta}{4}, \frac{\delta}{4})$ is contained in D_ε . In fact, $z \in \partial D$ and B has the constant curvature $\kappa_B = \frac{4}{\delta} \geq 2C$. Thus, and since by (3.45), the curvature of ∂D is everywhere smaller or equal to C , the graph of ∂D can never touch the boundary of B again. Thus and since by construction the set D lies only at one side of the graph of ∂D , the ball $B \subseteq D$. Furthermore, for every $x \in B$ there is a $t \in]-\frac{\delta}{4}, \frac{\delta}{4}[$ and a vector $e \in \mathbb{R}^d$ with $|e| = 1$ such that $x = z - \nu(z) \frac{\delta}{4} + te$ and hence $\text{dist}(x, \partial D) \leq |(z - \nu(z) \frac{\delta}{4} + te) - z| \leq \frac{\delta}{2} < \varepsilon$. Therefore $B \subseteq D_\varepsilon$. Since $z \in \partial D$ has been arbitrary, the above construction shows that ∂D has the property of the uniform interior sphere condition. \square

Claim (2) of the following Lemma, is used in [16] by X. Cabré and Y. Martel but without proof and without further references.

LEMMA 3.3.9. *Let D be an open and bounded set in \mathbb{R}^d with a boundary $\partial D \in C^2$ and let $\delta \in (0, 1)$ (given by Lemma 3.3.8) such that for every $z \in \partial D$ the open ball $B_{z,\delta} := B(z - \nu(z)\frac{\delta}{2}, \frac{\delta}{2}) \subseteq D_\delta$ and $\partial B_{z,\delta} \cap \partial D = \{z\}$. Then,*

(1) *for every $x \in D_\delta := \{x \in D \mid d(x) < \delta\}$, there exists a unique $z = z(x) \in \partial D$ such that*

$$(3.49) \quad |x - z(x)| = d(x),$$

(2) *for every $p > 1$, the function $x \mapsto \log d(x)$ belongs to $L^p(D)$.*

PROOF. First, we show that for every $x \in D_\delta$, there is a $z \in \partial D$ satisfying equation (3.49). Indeed, we note that for $x \in D_\delta$, $d(x)$ is bounded. Thus there is sequence $(z_n)_{n \geq 1} \subseteq \partial D$ such that $d(x) + \frac{1}{n} > |x - z_n| \geq d(x)$ for all $n \geq 1$. Hence $|x - z_n| \rightarrow d(x)$ as $n \rightarrow +\infty$, and since ∂D is compact, there is a $z \in \partial D$ and we can extract a subsequence of $(z_n)_{n \geq 1}$, which we denote, for simplicity, again by $(z_n)_{n \geq 1}$ such that $z_n \rightarrow z$ in \mathbb{R}^d . Thereby, we obtain that $|x - z| = d(x)$.

To see that for $x \in D_\delta$ the element $z \in \partial D$ satisfying equation (3.49) is unique, we note that only the line segment between two distinct vectors has the euclidean distance as arc length. Thus, since $x \in D$, since $|x - z| = d(x) < \delta$, and since D lies on one side of the boundary ∂D , we have that $x \in B(z - \nu(z)\frac{\delta}{2}, \frac{\delta}{2})$ and in particular there is a $t \in [0, \delta[$ such that $x = z - \nu(z)t$. But then $t = |\nu(z)t| = |x - z| = d(x)$ and so $x = z - \nu(z)d(x)$. This shows that x is unique.

Now, we prove that claim (2) of this lemma holds and for this let $p > 1$. Since ∂D is compact, there are open subsets $U_1, \dots, U_m \subseteq \mathbb{R}^d$ such that $\partial D \subseteq \bigcup_{i=1}^m U_i$ and for every $i = 1, \dots, m$, $U_i \cap \partial D$ is a normal graph of class C^2 (with respect to D). According to Definition 3.3.6 (1), for each $i = 1, \dots, m$, there is a function $g_i \in C^2(\mathbb{R}^{d-1})$ and there are $r_i, h_i > 0$ such that

$$U_i = \{(y, g_i(y) + s)^t \in \mathbb{R}^d \mid y \in \mathbb{R}^{d-1}, |y| < r_i, s \in \mathbb{R}, |s| < h_i\}$$

and every $x = (y, g_i(y) + s)^t \in U_i$ satisfies the properties (3.34)-(3.36). Now, for each $i = 1, \dots, m$, let $B'_i := \{y' \in \mathbb{R}^{d-1} \mid |y'| < r_i\}$ and let $\phi_i : B'_i \times (-h_i, h_i) \rightarrow U_i$ be defined by

$$\phi_i(y', s) = (y', g(y') + s) \quad \text{for every } (y', s) \in B'_i \times (-h_i, h_i).$$

It is not hard to verify that ϕ_i is a diffeomorphism with $\det D\phi_i(y', s) = 1$, $(y', s) \in B'_i \times (-h_i, h_i)$, and by the properties (3.34)-(3.36) in Definition 3.3.6 (1), we may assume that $0 < h_i \leq \delta$ otherwise we replace h_i by δ and U_i by $\phi_i^{-1}(B'_i \times (-\delta, \delta))$. Therefore, we may assume that

$$\bigcup_{i=1}^m U_i \cap D \subseteq D_\delta.$$

We set $U_0 = D \setminus \bigcup_{i=1}^m \overline{U_i \cap D}$ and choose for the family $\{U_0, U_1 \cap D, \dots, U_m \cap D\}$ a partition of unity $(\varphi_i)_{i=0}^m \subseteq C_c^\infty(\mathbb{R}^d)$ such that for every $i = 0, \dots, m$,

$$0 \leq \varphi(x) \leq 1 \text{ for } x \in \mathbb{R}^d, \quad \text{supp}(\varphi_i) \subseteq U_i \quad \text{and} \quad \sum_{i=0}^m \varphi_i(x) = 1 \text{ for all } x \in \overline{D}.$$

Then,

$$(3.50) \quad \int_D |\log d(x)|^p dx = \int_{U_0} |\log d(x)|^p \varphi_0(x) dx + \sum_{i=1}^m \int_{U_i \cap D} |\log d(x)|^p \varphi_i(x) dx.$$

We claim that the distance function $d(x)$ on U_0 is bounded from below by a constant $d_0 > 0$. Indeed, otherwise there is a sequence $(x_n)_{n \geq 1} \subseteq U_0$ such that $d(x_n) \rightarrow 0$ as $n \rightarrow +\infty$. Since U_0 is a relative compact subset of \mathbb{R}^d , we can extract a subsequence, which we denote again by $(x_n)_{n \geq 1}$ such that $x_n \rightarrow x$ as $n \rightarrow +\infty$ for some $x \in \overline{D}$. Then $d(x) = 0$ and hence $x \in \partial D$. But on the other hand, $\partial D \subseteq \bigcup_{i=1}^m U_i$ and hence $x \in U_i$ for some $i \in \{1, \dots, m\}$. Since $x_n \rightarrow x$ as $n \rightarrow +\infty$ and since U_i is open, there is some $n_0 \geq 1$ such that $x_n \in U_i$ for all $n \geq n_0$, but this contradicts that $x_n \in U_0$ for all $n \geq 1$. Thus

$$\int_{U_0} |\log d(x)|^p \varphi_0(x) dx \leq |\log d_0|^p \int_{U_0 \cap \{d(x) < 1\}} \varphi_0(x) dx + \left(\sup_{x, z \in \overline{D}} |x - z| \right)^p \int_{U_0 \cap \{d(x) \geq 1\}} \varphi_0(x) dx$$

and hence the first integral in the sum of integrals on the right hand-side of equation (3.50) is finite. By substitution, and since by claim (1) of this lemma, for every $x \in D_\delta$, there exists a unique $z =: z(x) \in \partial D$ such that $|x - z(x)| = d(x)$, we have that for every $i = 1, \dots, m$,

$$\begin{aligned} \int_{U_i \cap D} |\log d(x)|^p \varphi_i(x) dx &= \int_{B'_i} \int_0^{h_i} |\log d(y', g(y') + s)|^p \varphi_i(y', g(y') + s) ds dy' \\ &\leq \int_{B'_i} \int_0^{h_i} |\log s|^p ds dy' \\ &= |B'_i| \int_0^{h_i} |\log s|^p ds. \end{aligned}$$

Therefore the second term of the sum of integrals on the right hand-side of equation (3.50) is finite and thereby the second claim of this lemma holds, if we can show that for every $r > 0$, the function $s \mapsto \log s$ belongs to $L^p(0, r)$. To see that $\log s$ is in $L^p(0, r)$, we recall that

$$\lim_{x \rightarrow 0^+} x^\alpha |\log x|^\beta = 0 \quad \text{for every } \alpha > 0 \text{ and every } \beta \in \mathbb{R}.$$

Thus, and since there is an integer $k \geq 1$ such that $k \leq p < k + 1$, it follows by $(k + 1)$ -times employing of integration by parts that

$$\begin{aligned} \int_0^r |\log s|^p dx &= \int_0^r (-\log s)^p dx \\ &= \left[s (-\log s)^p \right]_0^r + p \int_0^r (-\log s)^{p-1} dx \\ &= r (-\log r)^p + \left[p s (-\log s)^{p-1} \right]_0^r + p(p-1) \int_0^r (-\log s)^{p-2} dx \\ &\quad \vdots \\ &= \sum_{i=1}^{k+1} \prod_{\nu=1}^i (p - \nu + 1) r (-\log r)^{p-i+1} + \prod_{\nu=1}^{k+1} (p - \nu + 1) \int_0^r (-\log s)^{p-k-1} dx. \end{aligned}$$

Since for $0 < s \leq r$, $-\log s > -\log r > 0$, we have that

$$\int_0^r (-\log s)^{p-k-1} dx \leq r (-\log r)^{p-k-1} < +\infty,$$

and hence $\log s \in L^p(0, r)$. □

DEFINITION 3.3.8. (Property of positive geometric density) We say that the boundary ∂D of an open subset D of \mathbb{R}^d has the property of *positive geometric density* if there exists an $\alpha \in (0, 1)$ and there exists a $\rho > 0$ such that

$$|D \cap B(z, r)| \leq (1 - \alpha) |B(z, r)| \quad \text{for all } z \in \partial D \text{ and for all } r \in (0, \rho].$$

PROPOSITION 3.3.9. *If D is an open and bounded set in \mathbb{R}^d with a boundary $\partial D \in C^2$, then ∂D has the property of positive geometric density.*

PROOF. We employ the same notation as in the proof of Lemma 3.3.8. Then, there is a $\delta \in (0, 1)$ such that for every $z \in \partial D$ the open ball $B(z - \nu(z) \frac{\delta}{2}, \frac{\delta}{2}) \subseteq D_\delta$ and $\overline{B}(z - \nu(z) \frac{\delta}{2}, \frac{\delta}{2}) \cap \partial D = \{z\}$. According to (3.47), for every $i = 1, \dots, m$, and every $t \in \mathbb{R}$ with $|t| > 0$, there is a $\theta_i(t) \in (0, 1)$ such that

$$g_i(z' - t\nu'(z)) = g_i(z') - \frac{|\nabla g_i(z')|^2}{\sqrt{|\nabla g_i(z')|^2 + 1}} t + \frac{1}{2} \nu'(z)^t H_{g_i}(z' - \theta_i(t) t \nu'(z)) \nu'(z)^t t^2.$$

We recall that $w_i := (\nabla g_i(z'), -1)^t$ and $R_i(t) := \frac{-1}{2} \nu'(z)^t H_{g_i}(z' - \theta_i(t) t \nu'(z)) \nu'(z)$, where H_{g_i} denotes the Hessian matrix of g_i . By the properties (3.33) and (3.35) in Definition 3.3.6 (1), for every $z = (z', z_d)^t \in U_i \cap \partial D$, $z_d = g_i(z')$. Thus and by (3.46), for $|t| > 0$,

$$(z - t\nu(z))_d = z_d + t \frac{1}{\sqrt{|\nabla g_i(z')|^2 + 1}} = g_i(z' - t\nu'(z)) + |w_i| t + R_i(t) t^2 \leq |w_i| t + C t^2.$$

We have that $|w_i| \geq 1$ and by (3.44), $R_i(t) \leq C$. Thus, if we take $t \in (-\frac{1}{C}, 0)$, then $-\frac{|w_i|}{C} < t < 0$ and so

$$\alpha_i(t) := (z - t\nu(z))_d - g_i(z' - t\nu'(z)) \leq |w_i| t + C t^2 < 0 \quad \text{if } t \in (-\frac{1}{C}, 0).$$

Therefore and by the properties (3.34) and (3.36) in Definition 3.3.6 (1),

$$(z' - t\nu'(z), (z - t\nu(z))_d) = (z' - t\nu'(z), g_i(z' - t\nu'(z)) + \alpha_i(t)) \notin D \quad \text{for all } t \in (-\frac{1}{C}, 0).$$

Since $z \in U_i \cap \partial D$ and $i = 1, \dots, m$ have been arbitrary, we have thereby shown that

$$(z' - t\nu'(z), (z - t\nu(z))_d) \in D \quad \text{for all } t \in (-\frac{1}{C}, 0) \text{ and for all } z \in \partial D.$$

According to the proof of Lemma 3.3.8, $\delta \in (0, 1)$ is chosen such that $\delta \leq \frac{2}{C}$. Thus, if we take $0 < \delta < \min\{1, \frac{1}{C}\}$, where C is given by (3.44), then by the same arguments as used at the end of the proof of Lemma 3.3.8, one sees that for every $z \in \partial D$ and for every $0 < r \leq \delta$,

$$B(z + \nu(z) \frac{r}{2}, \frac{r}{2}) \subseteq D^c, \quad B(z - \nu(z) \frac{r}{2}, \frac{r}{2}) \subseteq D, \quad \overline{B}(z + \nu(z) \frac{r}{2}, \frac{r}{2}) \cap \overline{B}(z - \nu(z) \frac{r}{2}, \frac{r}{2}) \cap \partial D = \{z\}.$$

Now, for such δ , we set $\rho = \delta$, take $r \in (0, \rho]$ and fix $z \in \partial D$. The curvature $\kappa_{z,r}$ of the sphere of the open balls $B(z - t\nu(z) \frac{r}{2}, \frac{r}{2})$ and $B(z + t\nu(z) \frac{r}{2}, \frac{r}{2})$ given by $\kappa_{z,r} = \frac{2}{r}$ is strictly larger than the curvature of the sphere of the ball $B(z, \rho)$ given by $\kappa_z = \frac{1}{\delta}$. Thus and by the particular choice of the center of the balls $B(z - t\nu(z) \frac{r}{2}, \frac{r}{2})$ and $B(z + t\nu(z) \frac{r}{2}, \frac{r}{2})$, they are both contained in $B(z, \rho)$. Thus, since by the above fixed δ , the curvature $\kappa_{z,r}$ is larger than the curvature of ∂D , and since the d -dimensional volume of an open ball in \mathbb{R}^d of radius $R > 0$ is given by $\frac{\pi^{d/2} R^d}{\Gamma(\frac{d}{2} + 1)}$, we find that

$$0 < \frac{1}{2^d} = \frac{|B(z + \nu(z) \frac{r}{2}, \frac{r}{2})|}{|B(z,r)|} \leq \frac{|B(z,r) \cap D^c \cap B_{z,r}^c|}{|B(z,r)|} \leq \frac{|B(z,r) \cap B_{z,r}^c|}{|B(z,r)|} = \frac{|B(z,r)| - |B_{z,r}^c|}{|B(z,r)|} = \frac{2^d - 1}{2^d} < 1.$$

Therefore, if we take $\alpha = \frac{1}{2^d}$, then it is not hard to see that for such ρ and α , ∂D admits the property of the positive geometric density. \square

3.3.7 Linear second-order differential operator of parabolic type

In this subsection, we review the definition of linear second-order differential operators of parabolic type and state the dependence of the smoothness of weak solutions on smoothness of the data of the considered boundary problem. Here we recall the definition and results from the classical book [53] by O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva.

DEFINITION 3.3.10. We say that an operator $\frac{\partial}{\partial t} - L$ is a *linear second-order differential operator of parabolic type* if either L is given by

$$(3.51) \quad Lu = \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_i} \right) - \sum_{i=1}^d b_i(x, t) \frac{\partial u}{\partial x_i} - c(x, t)u$$

of L is given by

$$(3.52) \quad Lu = \sum_{i,j=1}^d a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^d b_i(x, t) \frac{\partial u}{\partial x_i} - c(x, t)u.$$

If L is given by formula (3.51), then we say that $\frac{\partial}{\partial t} - L$ is in *divergence form* and if L is given by (3.52), then we say that $\frac{\partial}{\partial t} - L$ is in *nondivergence form*. In both cases, we call the operator $\frac{\partial}{\partial t} - L$ *uniformly parabolic* in a domain $D \subseteq \mathbb{R}^{d+1}$ if there is constant $\mu > 0$ such that

$$(3.53) \quad \sum_{i,j=1}^d a_{ij}(x, t) \zeta_i \zeta_j \geq \mu |\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^d, \text{ and every } (x, t) \in D.$$

REMARK 3.3.2. It is not hard to see that if the coefficients a_{ij} of the principle part in L are continuously differentiable, then an operator $\frac{\partial}{\partial t} - L$ with principle part in divergence form can be written in nondivergence form and vice versa. But there is an advantage to study both structures of $\frac{\partial}{\partial t} - L$ separately. The divergence form has the advantage to obtain existence of solutions of equation

$$\frac{\partial u}{\partial t} - Lu = f \quad \text{in } \Omega \times (0, T)$$

under very weak assumptions on the coefficients a_{ij}, b_i, c and f , while the nondivergence form is more appropriate for arguments using the (strong) maximum principle.

DEFINITION 3.3.11. Let D be an open subset, $T > 0$, and L be the operator given by (3.51) having coefficients $a_{ij}, b_i, c \in L^\infty(D \times (0, T))$. We call a function u a *weak solution* of equation

$$(3.54) \quad \frac{\partial u}{\partial t} - Lu = 0 \quad \text{in } D \times (0, T),$$

if

$$u \in C([0, T]; L^2(D)) \cap L^2(0, T; W^{1,2}(D))$$

and for every $0 \leq t_1 < t_2 \leq T$,

$$\int_D u \zeta dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_D \left\{ -u \frac{\partial \zeta}{\partial t} + \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \zeta}{\partial x_j} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} \zeta + c \zeta \right\} dx dt = 0$$

for all $\zeta \in W^{1,2}(0, T; L^2(D)) \cap L^2(0, T; W_0^{1,2}(D))$. We call a weak solution u of equation (3.52) bounded if u belongs to $L^\infty(D \times (0, T))$.

NOTATION 3.3.3. In order to state properly the dependence of the smoothness of weak solutions on the smoothness of the data of the problem, we need to introduce the following Hölder spaces. For this, let D be an open subset of \mathbb{R}^d with a continuous boundary and $T > 0$. Then, we denote by Q_T the cylinder $D \times (0, T)$ and by $S_T = \partial D \times (0, T)$ the lateral boundary of Q_T . We say that a function $u : D \rightarrow \mathbb{R}$ satisfies a Hölder condition in x with exponent $\alpha \in (0, 1)$, if

$$[u]_D^{(\alpha)} := \sup_{y_1, y_2 \in D} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|^\alpha} \text{ is finite.}$$

We set $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and for $l = (l_1, \dots, l_d)^t \in \mathbb{N}_0^d$, let $|l| := l_1 + \dots + l_d$. For every $l \in \mathbb{N}_0^d$, we denote by

$$D_x^{|l|} u := \frac{\partial^{|l|} u}{\partial x_1^{l_1} \dots \partial x_d^{l_d}}$$

an $|l|$ th order partial derivative of u with respect to the variables x_1, \dots, x_d . For any $\ell \in \mathbb{R}$, we denote by $[\ell]$ the largest integer smaller or equal to ℓ . Let $\ell > 0$ be a nonintegral number. Then, we set

$$\|u\|_{C(D)} = \sup_{y \in D} |u(y)|, \quad \langle u \rangle_D^{(j)} = \sum_{l \in \mathbb{N}_0^d: |l|=j} \|D_x^{|l|} u\|_{C(D)} \text{ for every } j = 0, 1, \dots, [\ell],$$

and

$$\langle u \rangle_D^{(\ell)} = \sum_{l \in \mathbb{N}_0^d: |l|=[\ell]} \left[D_x^{|l|} u \right]_D^{(\ell-[\ell])}.$$

Then, we denote by $H^\ell(\bar{D})$ the Banach space of $[\ell]$ -times continuously differentiable function on \bar{D} and with finite norm

$$\|u\|_{H^\ell(\bar{D})} = \sum_{j=0}^{[\ell]} \langle u \rangle_D^{(j)} + \langle u \rangle_D^{(\ell)}.$$

Further, we denote by $H_{loc}^\ell(D)$ the set of functions belonging to $H^\ell(\bar{D}')$ for every open subset D' having compact closure contained in D . For $\alpha \in (0, 1)$, we set

$$\begin{aligned} [u]_{x, Q_T}^{(\alpha)} &:= \sup_{(y_1, t), (y_2, t) \in Q_T} \frac{|u(y_1, t) - u(y_2, t)|}{|y_1 - y_2|^\alpha}, \\ [u]_{t, Q_T}^{(\alpha)} &:= \sup_{(y, t_1), (y, t_2) \in Q_T} \frac{|u(y, t_1) - u(y, t_2)|}{|t_1 - t_2|^\alpha}, \\ \|u\|_{C(Q_T)} &= \sup_{y \in Q_T} |u(y)|. \end{aligned}$$

For every integer $k \geq 0$, we set $D_t^k = \frac{\partial^k}{\partial t^k}$. Then, for every nonintegral number $\ell > 0$, let

$$\begin{aligned} \langle u \rangle_{Q_T}^{(j)} &= \sum_{k \geq 0, l \in \mathbb{N}_0^d: 2k+|l|=j} \|D_t^k D_x^l u\|_{C(Q_T)} \text{ for every } j = 0, \dots, [\ell], \\ \langle u \rangle_{x, Q_t}^{(\ell)} &= \sum_{k \geq 0, l \in \mathbb{N}_0^d: 2k+|l|=[\ell]} \left[D_t^k D_x^l u \right]_{x, Q_T}^{(\ell-[\ell])}, \\ \langle u \rangle_{t, Q_t}^{(\ell/2)} &= \sum_{k \geq 0, l \in \mathbb{N}_0^d: 0 < \ell - 2k - |l| < 2} \left[D_t^k D_x^l u \right]_{t, Q_T}^{(\frac{\ell-2k-l}{2})}. \end{aligned}$$

Then, we denote for every nonintegral $\ell > 0$ by $H^{\ell, \ell/2}(\overline{Q_T})$ the Banach space of continuous functions $u : Q_T \rightarrow \mathbb{R}$ having partial derivatives $D_t^k D_x^l u$ for all integers k, l with $2k + l < \ell$ and having finite norm

$$\|u\|_{H^{\ell, \ell/2}(\overline{Q_T})} = \sum_{j=0}^{[\ell]} \langle u \rangle_{Q_T}^{(j)} + \langle u \rangle_{x, Q_t}^{(\ell)} + \langle u \rangle_{t, Q_t}^{(\ell/2)}.$$

We denote by $H_{loc}^{\ell, \ell/2}(Q_T)$ the set of functions belonging to $H^{\ell, \ell/2}(Q')$ for every open subseteq Q' having compact closure contained in Q_T .

THEOREM 3.3.10. (Theorem 12.1 [53, Chap. III] & Theorem 1.1 [53, Chap. V]) *Let $T > 0$, D be a bounded domain of \mathbb{R}^d such that the boundary has the property of the positive geometric density, and for L given by (3.51) with $a_{ij}, b_i, c \in L^\infty(Q_T)$ let $\frac{\partial}{\partial t} - L$ be uniformly parabolic in Q_T . Then, every bounded weak solution $u \in L^2(0, T; W_0^{1,2}(D))$ of equation (3.54) belongs to $H_{loc}^{\alpha, \alpha/2}(Q_T \cup S_T)$ for some $\alpha \in (0, 1)$ depending on d , the constant μ given by (3.53), $\|a_{ij}\|_{L^\infty(Q_T)}$, $\|b_i\|_{L^\infty(Q_T)}$, $\|c\|_{L^\infty(Q_T)}$, and $\|u\|_{L^\infty(Q_T)}$. If for any integer $m \geq 0$ and $\alpha \in (0, 1)$, all coefficients a_{ij}, b_i , and c belong to $H_{loc}^{2m+\alpha, m+\alpha/2}(Q_T)$, then every weak solution u of equation (3.54) belongs to $H_{loc}^{2m+2+\alpha, m+1+\alpha/2}(Q_T)$. If, in addition, some part Γ' of the lateral boundary S_T belongs to $H^{2+2m+\alpha}$ and if $u \equiv 0$ on Γ' , then $u \in H_{loc}^{2m+2+\alpha, m+1+\alpha/2}(Q_T \cup \Gamma')$.*

3.3.8 The strong maximum principle of linear parabolic equations

At the end of this preliminary section, we recall the strong maximum principle for uniformly parabolic operators $\frac{\partial}{\partial t} - L$ in nondivergence form (3.52). For a more detailed discussion we refer the interested reader to standard text books, as, for instance, [70] by M. H. Protter and H. F. Weinberger, [26] by A. Friedman, or [56] by E. M. Landis.

THEOREM 3.3.11. ([70, Theorem 5, Chapter III], The strong maximum principle) *Let $\frac{\partial}{\partial t} - L$ be uniformly parabolic in the domain D of \mathbb{R}^{d+1} with L in nondivergence form (3.52) having continuous and bounded coefficients in D , and $c \equiv 0$. Let u be a real-valued function on D such that $u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial u}{\partial t}$ ($i, j = 1, \dots, d$), are continuous functions on D and u satisfies inequality*

$$(3.55) \quad Lu - \frac{\partial u}{\partial t} \geq 0$$

in a region $D_{\bar{t}} := \{(x, t) \in D \mid t \leq \bar{t}\}$. Suppose that u attains a maximum in $D_{\bar{t}}$ at a point P of $D_{\bar{t}}$ and let $M = u(P)$. Then for every point $Q \in D_{\bar{t}}$, which can be connected to P by a path in D consisting only of horizontal segments and upward vertical segments, one has that $u(Q) = M$.

THEOREM 3.3.12. ([70, Theorem 6, Chapter III]) *Let D be a domain of \mathbb{R}^{d+1} , and let $\frac{\partial}{\partial t} - L$ and u satisfy in D the same hypotheses as in Theorem 3.3.11. Further, suppose that u satisfies inequality (3.55) in D and that the maximum M of u is attained at some point $P = (x_0, t_0)$ on the boundary ∂D . Assume that $u < M$ in the set $D_{t_0} = \{(x, t) \in D \mid t \leq t_0\}$ and that P lies on the boundary of an open ball $B((x_1, t_1), r)$ centered at (x_1, t_1) with radius $r > 0$ such that $B((x_1, t_1), r)$ is tangent to ∂D , $x_0 \neq x_1$, and the part $B((x_1, t_1), r) \cap \{t \leq t_0\}$ lies in D_{t_0} . If $\frac{\partial}{\partial \nu}$ denotes any directional derivative in an outward direction from D_{t_0} , then we have that*

$$\frac{\partial u}{\partial \nu}(P) > 0.$$

3.4. Main results: Nonexistence of positive nontrivial solutions

In this section, we turn to the main problem of this chapter: the nonexistence (locally in time) of positive nontrivial weak solutions of problem

$$(3.2) \quad \begin{cases} u_t - K_p u = \frac{\lambda}{x^p} |u|^{p-2} u & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

In order to investigate this problem in all details, we study the cases of dimension $d = 1$ and dimension $d \geq 2$ separately.

3.4.1 Nonexistence of positive solutions in dimension $d = 1$

The content of this subsection is part of the article [47] joint with A. Rhandi. The article [47] was accepted for publication in *Archiv der Mathematik*.

In the case of dimension $d = 1$, we study the nonexistence of positive solutions of problem (3.2) provided $\Omega = (0, +\infty)$, $1 < p < +\infty$, and the real symmetric matrix $A = (a_{11})$ satisfies $a_{11} = a \geq 0$. Then, the density ρ of the Borel measure μ defined in (3.3) reduces to

$$\rho(x) = c \cdot \exp\left(-\frac{1}{p} a^{p/2} |x|^p\right) \quad \text{for } x \in [0, +\infty), \text{ and fixed } c > 0.$$

THEOREM 3.4.1. (Nonexistence of positive solutions, $d = 1$) *Under the hypotheses of this subsection, the following assertions are true:*

- (i) *If $1 < p \leq 2$, if $\lambda > \left(\frac{p-1}{p}\right)^p$, and if $u_0 \in L^2_{\mu,loc}[0, +\infty)$ is positive and $u_0 \neq 0$, then for any $T > 0$, problem (3.2) has no positive weak solution.*
- (ii) *If $p > 2$, if $\lambda > \left(\frac{p-1}{p}\right)^p$, and if $u_0 \in L^2_{\mu,loc}[0, +\infty)$ is positive and for some $r > 0$,*

$$(3.56) \quad \operatorname{ess\,inf}_{x \in (0,r)} u_0(x) \geq \delta > 0,$$

then for any $T > 0$, problem (3.2) has no positive weak solution.

In order to prove Theorem 3.4.1, we need the following two lemmata. Lemma 3.4.2 has already been used for $\rho \equiv 1$ in [35].

LEMMA 3.4.2. *We suppose that all conditions stated in the beginning of this subsection hold and let $r > 0$. If $M \in L^1_{\mu}(0, r)$, then for every $\varepsilon \in (0, 1)$, there is a constant $C(\varepsilon) > 0$ such that*

$$(3.57) \quad \int_0^r M |\phi|^p d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_0^r |\phi'|^p d\mu + C(\varepsilon) \int_0^r |\phi|^p d\mu \quad \text{for all } \phi \in W^{1,p}_{\mu,0}(0, r).$$

PROOF. Since for every $\phi \in C^1_c(0, r)$, and every $x \in [0, r]$,

$$|\phi(x)|^p = \int_0^x \frac{d}{ds} |\phi(s)|^p ds = p \int_0^x |\phi(s)|^{p-2} \phi(s) \phi'(s) ds$$

and since $\max_{[0,r]} \rho^{-1} = \rho^{-1}(r)$, it follows by Young's inequality that

$$|\phi(x)|^p \leq p \int_0^r |\phi|^{p-1} |\phi'| dx \leq \varepsilon_0 \int_0^r |\phi'| dx + \frac{(1-p)}{\rho^{p'}(r)\varepsilon_0^{1/p-1}} \int_0^r |\phi|^p dx$$

for all $x \in [0, r]$ and for every $\varepsilon_0 > 0$. We take $\varepsilon_0 = \frac{\varepsilon}{(1-\varepsilon)(\|M\|_{L_\mu^1(0,r)}+1)}$ for any given $\varepsilon \in (0, 1)$.

Then

$$\begin{aligned} \int_0^r M |\phi|^p d\mu &\leq \|M\|_{L_\mu^1(0,r)} \left(\varepsilon_0 \int_0^r |\phi'| dx + \frac{(1-p)}{\rho^{p'}(r)\varepsilon_0^{1/p-1}} \int_0^r |\phi|^p dx \right) \\ &\leq \frac{\varepsilon}{1-\varepsilon} \int_0^r |\phi'|^p d\mu + C(\varepsilon) \int_0^r |\phi|^p d\mu \quad \text{with } C(\varepsilon) = \|M\|_{L_\mu^1(0,r)} \frac{(1-p)}{\rho^{p'}(r)\varepsilon_0^{1/p-1}}. \end{aligned}$$

Thus, inequality (3.67) holds for all $\phi \in C_c^1(0, r)$. By a standard approximation argument, we obtain that (3.67) holds for all functions $\phi \in W_{\mu,0}^{1,p}(0, r)$. \square

LEMMA 3.4.3. *For fixed $r > 0$, let $d(x) = \min\{x, r-x\}$ for all $x \in [0, r]$. If $u \in L_\mu^1(0, r)$ and if there is a constant $c > 0$ such that $u(x) \geq c d(x)$ for a.e. $x \in (0, r)$, then we have that*

$$\log\left(\frac{u}{cd}\right) \in L_\mu^1(0, r).$$

PROOF. By hypothesis, $\frac{u}{cd} \geq 1$ a.e. on $(0, r)$, and the density ρ of measure μ defined in (3.3) satisfies $0 < \rho \leq \rho(0)$ and $0 < \rho^{-1} \leq \rho^{-1}(r)$ on $[0, r]$. Thus and by the elementary inequality $\log y \leq y - 1$ for all $y > 0$, we have that

$$\begin{aligned} (3.58) \quad \int_0^r \left| \log\left(\frac{u}{cd}\right) \right| d\mu &\leq \rho(0) \int_0^r \log u dx - \rho(0) (\log c) r - \rho(0) \int_0^r \log d(x) dx \\ &\leq \rho(0)\rho^{-1}(r) \int_0^r u d\mu + \rho(0)|\log c| r - \rho(0) \int_0^r \log d(x) dx. \end{aligned}$$

Note that

$$\int_0^{r/2} \log d(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^{r/2-\varepsilon} \log x dx = \frac{r}{2} \log \frac{r}{2} - \frac{r}{2}$$

and by substitution,

$$\int_{r/2}^r \log d(x) dx = \int_{r/2}^r \log(r-x) dx = \int_0^{r/2} \log x dx = \frac{r}{2} \log \frac{r}{2} - \frac{r}{2}.$$

Therefore, the integral on the left hand-side of estimate (3.58) is finite and hence the claim of this lemma holds. \square

PROOF OF THEOREM 3.4.1. Let $1 < p < 2$ and let $\lambda > \left(\frac{p-1}{p}\right)^p$. We suppose that there is a positive $u_0 \in L_{\mu,loc}^2[0, +\infty)$, $u_0 \neq 0$, such that there is a $T > 0$ and a positive weak solution u of problem (3.2). Then, we shall reach to a contradiction by refining the arguments given in [35]. By assumption, there is an $r > 0$ such that $u_0 \neq 0$ on $(0, r)$ and $u \neq 0$ on $(0, r) \times (0, t)$ for some

$t \in (0, T)$. We fix $\varphi \in C_c^1(0, r)$, and for every integer $k \geq 1$ and every $s \in \mathbb{R}$, let $g_k(s) = (s + \frac{1}{k})^{1-p}$ if $s \geq 0$ and $g_k(s) = k^{p-1}$ if $s < 0$. Then, by Lemma 3.3.5 for $g = g_k$, $\phi = |\varphi|^p$, $t_1 = 0$, and $t_2 = t$,

$$(3.59) \quad \int_0^r \frac{(u(t) + \frac{1}{k})^{2-p}}{2-p} |\varphi|^p d\mu - \int_0^r \frac{(u(0) + \frac{1}{k})^{2-p}}{2-p} |\varphi|^p d\mu + (1-p) \int_0^t \int_0^r \frac{|u_x(s)|^p |\varphi|^p}{(u(s) + \frac{1}{k})^p} d\mu ds \\ + p \int_0^t \int_0^r \frac{|u_x(s)|^{p-2} u_x(s) |\varphi|^{p-2} \varphi}{(u(s) + \frac{1}{k})^{p-1}} \varphi_x d\mu ds = \int_0^t \int_0^r \frac{\lambda}{x^p} \frac{u^{p-1}(s)}{(u(s) + \frac{1}{k})^{p-1}} |\varphi|^p d\mu ds.$$

Since by Young's inequality,

$$p \int_0^t \int_0^r |u_x(s)|^{p-2} u_x(s) (u(s) + \frac{1}{k})^{1-p} \varphi_x |\varphi|^{p-2} \varphi d\mu ds \\ \leq (p-1) \int_0^t \int_0^r |u_x(s)|^p (u(s) + \frac{1}{k})^{-p} |\varphi|^p d\mu ds + t \int_0^r |\varphi_x|^p d\mu,$$

and since $(u(0) + \frac{1}{k})^{2-p} |\varphi|^p$ is positive a.e. on $(0, r)$, we can deduce from equality (3.59) that

$$(3.60) \quad \int_0^t \int_0^r \frac{\lambda}{x^p} u^{p-1}(s) (u(s) + \frac{1}{k})^{1-p} |\varphi|^p d\mu ds \leq t \int_0^r |\varphi_x|^p d\mu + \frac{1}{2-p} \int_0^r (u(t) + \frac{1}{k})^{2-p} |\varphi|^p d\mu.$$

For almost every $(x, s) \in (0, r) \times]0, t[$,

$$0 \leq \frac{\lambda}{x^p} \frac{u^{p-1}(x, s) |\varphi(x)|^p}{(u(x, s) + \frac{1}{k})^{p-1}} \nearrow \frac{\lambda}{x^p} |\varphi(x)|^p \quad \text{and} \quad (u(x, s) + \frac{1}{k})^{2-p} |\varphi(x)|^p \searrow u^{2-p}(x, s) |\varphi(x)|^p$$

as $k \rightarrow +\infty$. Thus, by Beppo-Levi's theorem, sending $k \rightarrow +\infty$ in inequality (3.60) gives

$$t \int_0^r \frac{\lambda}{x^p} |\varphi|^p d\mu - t \int_0^r |\varphi_x|^p d\mu \leq \frac{1}{2-p} \int_0^r u^{2-p}(t) |\varphi|^p d\mu.$$

Since $1 < p < 2$, and since $u(t) \in L_{\mu}^2(0, r)$, we obtain by Hölder's inequality that $u^{2-p} \in L_{\mu}^1(0, r)$. Thus by Lemma 3.4.2, for any $\varepsilon \in (0, 1)$, there is constant $C(\varepsilon) > 0$ such that

$$\int_0^r \frac{\lambda}{x^p} |\varphi|^p d\mu - \int_0^r |\varphi_x|^p d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_0^r |\varphi_x|^p d\mu + C(\varepsilon) \int_0^r |\varphi|^p d\mu.$$

Since $\varphi \in C_c^1(0, r)$ has been arbitrary, we have thereby shown that

$$\inf \frac{\int_0^r |\varphi_x|^p d\mu - [(1-\varepsilon)\lambda]^{\frac{p-1}{p}} a^{p/2} \int_0^r |\varphi|^p d\mu - (1-\varepsilon) \int_0^r \frac{\lambda}{x^p} |\varphi|^p d\mu}{\int_0^r |\varphi|^p d\mu} \\ \geq -a^{p/2} [(1-\varepsilon)\lambda]^{\frac{p-1}{p}} - C(\varepsilon) (1-\varepsilon) > -\infty,$$

where the infimum is taken over all $\varphi \in W_{\mu,0}^{1,p}(0, r)$ with $\|\varphi\|_{L_{\mu}^p(0,r)} > 0$. But for every

$$0 < \varepsilon < 1 - \lambda^{-1} \left(\frac{p-1}{p}\right)^p \quad \text{we have that} \quad (1-\varepsilon)\lambda > \left(\frac{p-1}{p}\right)^p$$

and hence this obviously contradicts to the optimality of the constant $\left(\frac{p-1}{p}\right)^p$ in Hardy's inequality (3.5).

Now, we study the case $p = 2$. In this case, we follow closely the idea described in [16] due to X. Cabré and Y. Martel. Let $\lambda > \frac{1}{4}$ and let $u_0 \in L^2_{\mu,loc}[0, +\infty)$ be positive and $u_0 \neq 0$. We suppose that there is a $T > 0$ and there is a positive weak solution u of problem (3.2) and then we shall reach to a contradiction. For this, take $r > 0$ such that $u_0 \neq 0$ in $(0, r)$, and fix $\varphi \in C^1_c(0, r)$. For every integer $n \geq 1$, let u_n be the unique positive strong solutions of

$$\begin{cases} u_{nt} - K_2 u_n = \min\left\{n, \frac{\lambda}{x^2}\right\} u_n & \text{in } (0, r) \times (0, T), \\ u_n(0, t) = u_n(r, t) = 0 & \text{for } t \in (0, T), \\ u_n(x, 0) = \min\{n, u_0\} & \text{for } x \in (0, r), \end{cases}$$

and v_n the unique positive strong solution of

$$(3.61) \quad \begin{cases} v_{nt} - K_2 v_n = 0 & \text{in } (0, r) \times (0, T), \\ v_n(0, t) = v_n(r, t) = 0 & \text{for } t \in (0, T), \\ v_n(x, 0) = \min\{n, u_0\} & \text{for } x \in (0, r). \end{cases}$$

By the weak comparison principle (see Lemma 3.3.3 and Proposition 3.3.4), for all $n \geq 1$,

$$(3.62) \quad 0 \leq v_1 \leq v_n \leq u_n \leq u \quad \text{a.e. on } (0, r) \times (0, T).$$

We note that for every $t \in (0, T)$, $v_1(\cdot, t) \in C^2(0, r)$ such that $v_1(\cdot, t)$, $v'_1(\cdot, t)$ and $v''_1(\cdot, t)$ are continuous on $[0, r]$. In particular, $v_1(\cdot, t) > 0$ on $\text{supp}(\varphi)$. Indeed, if $v_1(x_0, t) = 0$ for some $x_0 \in \text{supp}(\varphi)$, then by the strong maximum principle (cf. Theorem 3.3.11), $v_1 \equiv 0$ on $[0, r] \times (0, t)$. But this contradicts to the fact that $v_1(\cdot, 0) = \min\{1, u_0\} \neq 0$ almost everywhere on $(0, r)$. Thus for all $t \in (0, T)$, there is a constant $C_0(t) > 0$ such that

$$u_n(\cdot, t) \geq v_n(\cdot, t) \geq v_1(\cdot, t) \geq C_0(t) > 0 \quad \text{on } \text{supp}(\varphi).$$

Therefore, we may multiply the parabolic equation in problem (3.61) by $u_n^{-1} |\varphi|^2$ with respect to the $L^2_\mu(0, r)$ inner product and subsequently integrate with respect to ds over the interval (t_0, t) for any fixed $0 < t_0 < t < T$. Then, we obtain for every $n \geq 1$,

$$(3.63) \quad \begin{aligned} & \int_0^r \log\left(\frac{u_n(t)}{u_n(t_0)}\right) |\varphi|^2 d\mu - \int_{t_0}^t \int_0^r |u_{nx}(s)|^2 (u_n(s))^{-2} |\varphi|^2 d\mu ds \\ & + 2 \int_{t_0}^t \int_0^r u_{nx}(s) (u_n(s))^{-1} \varphi_x \varphi d\mu ds = (t - t_0) \int_0^r \min\left\{n, \frac{\lambda}{x^2}\right\} |\varphi|^2 d\mu. \end{aligned}$$

Applying Young's inequality as in the case $1 < p < 2$, we can deduce from equality (3.80) that

$$(3.64) \quad \int_0^r \min\left\{n, \frac{\lambda}{x^2}\right\} |\varphi|^2 d\mu - \int_0^r |\varphi_x|^2 d\mu \leq \frac{1}{t-t_0} \int_0^r \log\left(\frac{u_n(t)}{u_n(t_0)}\right) |\varphi|^2 d\mu.$$

For every $t \in (0, T)$, $v_1(\cdot, t)$ attains its minimum at the boundary points $x = 0$ and $x = r$. Thus according to Theorem 3.3.12, the normal derivatives $-v_{1,x}(0, t) < 0$ and $v_{1,x}(r, t) < 0$. By continuity of $v_{1,x}(\cdot, t)$, we can choose $0 < r_1 < r_2 < r$ such that $v_{1,x}(\cdot, t) \geq c_2(t) > 0$ on $[0, r_1]$ and $-v_{1,x}(\cdot, t) \geq c_3(t) > 0$ on $[r_2, r]$. Further, by the strong maximum principle, $v_1(t) \geq c_4(t) > 0$ on $[r_1, r_2]$. We set $\delta(x) := \min\{x, r - x\}$ for every $x \in [0, r]$ and may suppose without loss of generality that $\delta(x) = x$ on $[0, r_1]$ and $\delta(x) = r - x$ on $[r_2, r]$. Then, by the mean value theorem, we obtain that for every $t \in (0, T)$ and every $x \in [0, r]$,

$$v_1(x, t) \geq c_4(t) \cdot \inf_{[r_1, r_2]} (\delta^{-1}) \delta(x) \mathbb{1}_{[r_1, r_2]}(x) + c_2(t) x \mathbb{1}_{[0, r_1]}(x) + c_3(t) (r - x) \mathbb{1}_{(r_2, r]}(x).$$

Thus and by (3.62), for every $t \in (0, T)$, there is $C_1(t) > 0$ such that for all $n \geq 1$,

$$u(x, t) \geq u_n(x, t) \geq v_1(x, t) \geq C_1(t) \delta(x) \quad \text{almost everywhere on } (0, r).$$

Hence and by monotonicity of $\log x$, we obtain that for all $n \geq 1$

$$\log \left(\frac{u_n(t)}{u_n(t_0)} \right) \leq \log \left(\frac{u(t)}{C_1(t_0)\delta} \right) \quad \text{almost everywhere on } (0, r).$$

Then, applying this estimate to inequality (3.81) and sending $n \rightarrow +\infty$ yields to

$$\int_0^r \frac{\lambda}{x^2} |\varphi|^2 d\mu - \int_0^r |\varphi_x|^2 d\mu \leq \frac{1}{t-t_0} \int_0^r \log \left(\frac{u(t)}{C_1(t_0)\delta} \right) |\varphi|^2 d\mu.$$

Since $\log \left(\frac{u(t)}{C_1(t_0)\delta} \right) \in L^1_\mu(0, r)$, we obtain by Lemma 3.4.2 that for every $\varepsilon \in (0, 1)$, there is $C(\varepsilon) > 0$ such that

$$\int_0^r \frac{\lambda}{x^2} |\varphi|^2 d\mu - \int_0^r |\varphi_x|^2 d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_0^r |\varphi_x|^2 d\mu + C(\varepsilon) \int_0^r |\varphi|^2 d\mu.$$

Since $\varphi \in C_c^1(0, r)$ has been arbitrary, the last inequality holds for all $\varphi \in C_c^1(0, r)$. Therefore by proceeding as in the case $1 < p < 2$, we reach a contradiction to the optimality of the Hardy constant $\frac{1}{4}$ in Hardy's inequality (3.5).

We turn to the case $p > 2$. Let $\lambda > \left(\frac{p-1}{p}\right)^p$ and suppose there is a positive nontrivial $u_0 \in L^2_{\mu,loc}[0, +\infty)$ satisfying (3.56) for some $r > 0$ such that there is a $T > 0$ and positive weak solution of problem (3.2). Then we shall reach to a contradiction. To see this, we fix $\varphi \in C_c^1(0, r)$ and for every integer $k \geq 1$ and every $s \in \mathbb{R}$, we set $g_k(s) = (s + \frac{1}{k})^{1-p}$ if $s \geq 0$ and $g_k(s) = k^{p-1}$ if $s < 0$. Then, by Lemma 3.3.5 for $t_1 = 0$ and $t_2 = t > 0$, by using Young's inequality as in the case $1 < p < 2$, and since $(u(t) + \frac{1}{k})^{2-p} |\varphi|^p$ is positive almost everywhere on $(0, r) \times (0, T)$, we obtain that

$$\int_0^t \int_0^r \frac{\lambda}{x^p} u^{p-1}(s) (u(s) + \frac{1}{k})^{1-p} |\varphi|^p d\mu ds \leq t \int_0^r |\varphi_x|^p d\mu + \int_0^r \frac{(u_0 + \frac{1}{k})^{2-p}}{p-2} |\varphi|^p d\mu.$$

Sending $k \rightarrow +\infty$ in this last estimate and using Beppo-Levi's convergence theorem gives

$$(3.65) \quad \int_0^r \frac{\lambda}{x^p} |\varphi|^p d\mu - \int_0^r |\varphi_x|^p d\mu \leq \frac{1}{t^{p-2}} \int_0^r u_0^{2-p} |\varphi|^p d\mu.$$

Thus, since $\text{supp}(\varphi) \subseteq (0, r)$, and by (3.56), we have that

$$\int_0^r \frac{\lambda}{x^p} |\varphi|^p d\mu - \int_0^r |\varphi_x|^p d\mu \leq \frac{\delta^{2-p}}{t^{p-2}} \int_0^r |\varphi|^p d\mu$$

and hence

$$\inf_{\varphi \in W_{\mu,0}^{1,p}(0,r): \|\varphi\|_{L^p_\mu} > 0} \frac{\int_0^r |\varphi_x|^p d\mu - \lambda \frac{p-1}{p} a^{p/2} \int_0^r |\varphi|^p d\mu - \int_0^r \frac{\lambda}{x^p} |\varphi|^p d\mu}{\int_0^r |\varphi|^p d\mu} \geq -\frac{\delta^{2-p}}{t^{2-p}} - a^{p/2} \lambda \frac{p-1}{p}.$$

But this obviously contradicts the optimality of the constant $\left(\frac{p-1}{p}\right)^p$ in Hardy's inequality (3.5). \square

3.4.2 Nonexistence of positive solutions for $d \geq 2$

The content of this subsection is not yet published.

In this section, we study the nonexistence of positive solutions of problem (3.2) in dimension $d \geq 2$ provided $\Omega \subseteq \mathbb{R}^d$ is a domain with $0 \in \Omega$, $1 < p < +\infty$, and A is either a real symmetric positive definite $d \times d$ -matrix or $A \equiv 0$.

THEOREM 3.4.4. (Nonexistence of positive solutions, $d \geq 2$) *Under the hypotheses of this subsection, the following assertions are true:*

(i) For $d = 2$ let $1 < p < 2$, and for $d \geq 3$ let $\frac{2d}{d+2} \leq p \leq 2$. If $\lambda > \left(\frac{d-p}{p}\right)^p$ and if u_0 is a positive nontrivial element of $L^2_{\mu,loc}(\Omega)$, then for any $T > 0$, problem (3.2) has no positive weak solution.

(ii) Let $d \geq 2$, $p > 2$, and $p \neq d$. If $\lambda > \left(\frac{d-p}{p}\right)^p$, and if $u_0 \in L^2_{\mu,loc}(\Omega)$ is positive and for some $r > 0$,

$$(3.66) \quad \operatorname{ess\,inf}_{x \in B(0,r)} u_0(x) \geq \delta > 0,$$

then for any $T > 0$, problem (3.2) has no positive weak solution.

For the proof of Theorem 3.4.4, we need the following three lemmata. Lemma 3.4.5 is in the case $\rho \equiv 1$ the well-known Proposition A.1 in [35].

LEMMA 3.4.5. Let $D \subseteq \mathbb{R}^d$ be a bounded domain and let $1 \leq p < d$. If the function $M \in L^{d/p}_{\mu}(D)$, then for every $\varepsilon \in (0, 1)$, there is a constant $C(\varepsilon) > 0$ such that

$$(3.67) \quad \int_D M |\phi|^p d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_D |\nabla \phi|^p d\mu + C(\varepsilon) \int_D |\phi|^p d\mu \quad \text{for all } \phi \in W^{1,p}_{\mu,0}(D).$$

PROOF. Let $(M_n)_{n \geq 1}$ be the sequence defined by $M_n(x) := \min\{M(x), n\}$ for a.e. $x \in D$, and every $n \geq 1$. Then, $M_n(x) \rightarrow M(x)$ as $n \rightarrow +\infty$ for a.e. $x \in D$ and $|M_n| \leq |M(x)|$ for a.e. $x \in D$ and all $n \geq 1$. Thus and since by hypothesis, $M \in L^{d/p}_{\mu}(D)$, we have by Lebesgue's dominated convergence theorem (see Théorème IV.2 in [12]) that

$$(3.68) \quad M_n \rightarrow M \quad \text{in } L^{d/p}_{\mu}(D) \text{ as } n \rightarrow +\infty.$$

We fix $\phi \in C^1_c(D)$. Then by Hölder's inequality with $l = \frac{d}{p}$ and $l' = \frac{d}{d-p}$, for every $n \geq 1$,

$$(3.69) \quad \begin{aligned} \int_D M |\phi|^p d\mu &\leq \int_D |M - M_n| |\phi|^p d\mu + n \int_D |\phi|^p d\mu \\ &\leq \left(\int_D |M - M_n|^{\frac{d}{p}} d\mu \right)^{\frac{p}{d}} \|\rho\|_{L^\infty(D)}^{\frac{d-p}{d}} \left(\int_D |\phi|^{\frac{dp}{d-p}} dx \right)^{\frac{d-p}{d}} + n \int_D |\phi|^p d\mu. \end{aligned}$$

Since ϕ is assumed to have a compact support, the function ϕ belongs in particular to $C^1_c(\mathbb{R}^d)$. Thus by the Sobolev-Gagliardo-Nirenberg inequality (see Théorème IX.9 in [12]), there is a constant $C = C(p, d) > 0$ such that

$$\left(\int_D |\phi|^{\frac{dp}{d-p}} dx \right)^{\frac{(d-p)p}{d}} \leq C \int_D |\nabla \phi|^p dx \leq C \|\rho^{-1}\|_{L^\infty(D)} \int_D |\nabla \phi|^p d\mu.$$

Inserting this inequality into estimate (3.69), gives

$$\int_D M |\phi|^p d\mu \leq \left(\int_D |M - M_n|^{\frac{d}{p}} d\mu \right)^{\frac{p}{d}} \|\rho\|_{L^\infty(D)}^{\frac{d-p}{d}} C \|\rho^{-1}\|_{L^\infty(D)} \int_D |\nabla \phi|^p d\mu + n \int_D |\phi|^p d\mu.$$

Due to the limit (3.68), for every given $\varepsilon \in (0, 1)$, there is a $n(\varepsilon) \geq 1$ such that

$$\left(\int_D |M - M_{n(\varepsilon)}|^{\frac{d}{p}} d\mu \right)^{\frac{p}{d}} \leq \frac{\varepsilon}{(1-\varepsilon)C} \|\rho\|_{L^\infty(D)}^{\frac{d-p}{d}} \|\rho^{-1}\|_{L^\infty(D)}^{-1},$$

and hence

$$\int_D M |\phi|^p d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_D |\nabla \phi|^p d\mu + n(\varepsilon) \int_D |\phi|^p d\mu.$$

Thus and since $C_c^1(D)$ lies dense in $W_{\mu,0}^{1,p}(D)$, the claim of this lemma holds with $C(\varepsilon) = n(\varepsilon)$. \square

LEMMA 3.4.6. *Let $D \subseteq \mathbb{R}^d$ be an open and bounded set with $\partial D \in C^2$, and let $d(x)$ denote the distance of a point $x \in D$ to the boundary ∂D . If $u \in L_\mu^1(D)$ and if there is a constant $c > 0$ such that $u(x) \geq c d(x)$ for a.e. $x \in D$, then*

$$(3.70) \quad \log(u d) \in L_\mu^p(D) \quad \text{for all } p > 1.$$

For the proof of Lemma 3.4.6, we have been partially inspired by the proof of Theorem 1.1 in [37] due to J. A. Goldstein and Q. S. Zhang.

PROOF. We set $D = D_1 \cup D_2$ with $D_1 = \{x \in D \mid ud < 1\}$ and $D_2 = \{x \in D \mid ud \geq 1\}$. Since D is bounded, the diameter $\text{diam}(D) := \sup\{|x - y| \mid x, y \in D\}$ of D is bounded. Then,

$$(3.71) \quad \int_D |\log(u d)|^p d\mu = \int_{D_1} |\log(u d)|^p d\mu + \int_{D_2} |\log(u d)|^p d\mu.$$

Furthermore, for $D_2 = D_{2,1} \cup D_{2,2}$ with $D_{2,1} = D_2 \cap \{ud \geq e^{p-1}\}$ and $D_{2,2} = D_2 \cap \{ud < e^{p-1}\}$,

$$\int_{D_2} |\log(u d)|^p d\mu = \mathcal{I}_{2,1} + \mathcal{I}_{2,2} \quad \text{with} \quad \mathcal{I}_{2,1} = \int_{D_{2,1}} |\log(u d)|^p d\mu \quad \text{and} \quad \mathcal{I}_{2,2} = \int_{D_{2,2}} |\log(u d)|^p d\mu.$$

We show that $\mathcal{I}_{2,1}$ and $\mathcal{I}_{2,2}$ are both finite. Indeed,

$$\mathcal{I}_{2,2} = \int_{D_{2,2}} (\log(u d))^p d\mu \leq \mu(D) (p-1)^p \quad \text{is finite.}$$

To see that $\mathcal{I}_{2,1}$ is also finite, we assume that $\mu(D_{2,1}) > 0$, since otherwise there would be nothing to show. The mapping $s \mapsto (\log s)^p$ is concave on the interval $]e^{p-1}, +\infty[$. Thus by Jensen's inequality for concave functions and since $d(x) \leq \text{diam}(D)$ for every $x \in D$, we obtain that

$$\mathcal{I}_{2,1} = \int_{D_{2,1}} (\log(u d))^p d\mu \leq \mu(D) \left(\log \frac{\text{diam}(D)}{\mu(D_{2,1})} \int_{D_{2,1}} u d\mu \right)^p.$$

Since $u \in L^1_\mu(D)$, the right hand-side of this inequality is finite and so the second integral on the right hand-side in equation (3.71) is finite. It remains to verify that also the first integral on the right hand-side in equation (3.71) is finite. For this, we note that $\log(ud) < 0$ on D_1 . Thus and since by hypothesis, there is a $c > 0$ such that $u(x) \geq c d(x)$ for a.e. $x \in D_1$, we have that

$$\begin{aligned} \int_{D_1} |\log(ud)|^p d\mu &= \int_{D_1} (-\log(ud))^p d\mu \\ &\leq \int_{D_1} (-\log(cd^2))^p d\mu \\ &\leq C_p \left(|\log c|^p \mu(D_1) + 2^p \int_D |\log d|^p d\mu \right). \end{aligned}$$

Due to claim (2) of Lemma 3.3.9, for all $p > 1$, $\log d(x)$ belongs to $L^p(D)$ and for the density $\rho(x)$ of the measure μ , one has that $\rho(x) \leq c\mathbb{1}(x)$ on D . Thus, we obtain that $\log d \in L^p_\mu(D)$ and so by the last estimate, the claim of this lemma holds. \square

LEMMA 3.4.7. *Let $D \subseteq \mathbb{R}^d$ be a bounded domain with $\partial D \in C^2$. If $v \in C^{2,1}(\overline{D} \times (0, T))$ is a positive nontrivial solution of the boundary value problem*

$$\begin{cases} v_t - K_2 v = 0 & \text{in } D \times (0, T), \\ v = 0 & \text{on } \partial D \times (0, T), \end{cases}$$

then for every $t \in (0, T)$, there is a constant $C(t) > 0$ such that

$$(3.72) \quad v(x, t) \geq C(t) d(x) \quad \text{for all } x \in D.$$

PROOF. Since v attains its minimum at the boundary $\partial D \times (0, T)$, and since by Lemma 3.3.8, at every point $z \in \partial D$ there is an open ball $B(y, r)$ centered at some $y \in D$ with some radius $r > 0$ such that $B(y, r) \subseteq D$ and $\overline{B}(y, r) \cap \partial D = \{z\}$, the outer normal derivative $\frac{\partial v}{\partial \nu}(\cdot, t) < 0$ on ∂D for every $t \in (0, T)$ according to Theorem 3.3.12. Since for every $t \in (0, T)$, $\frac{\partial v}{\partial \nu}(\cdot, t)$ is continuous on the compact set ∂D , $\frac{\partial v}{\partial \nu}(\cdot, t)$ attains a maximum on ∂D and so $v_0(t) := \max \{ \frac{\partial v}{\partial \nu}(x, t) \mid x \in \partial D \} < 0$. For the rest of this proof, we fix $t \in (0, T)$.

Due to Lemma 3.3.8, there is a $\delta \in (0, 1)$ such that for every $z \in \partial D$, the open ball $B_{z, \delta} := B(z - \nu(z) \frac{\delta}{2}, \frac{\delta}{2}) \subseteq D_\delta := \{x \in D \mid d(x) < \delta\}$ and $\partial B_{z, \delta} \cap \partial D = \{z\}$. Here $\nu(z)$ denotes the unit outward normal vector to ∂D at z . Furthermore, by claim (1) of Lemma 3.3.9, for every $x \in D_\delta := \{x \in D \mid d(x) < \delta\}$, there exists a unique $z = z(x) \in \partial D$ such that $|x - z(x)| = d(x)$ holds. Thus every $x \in D_\delta$ can be written as $x = z(x) - \nu(z(x))d(x)$ for a unique $d(x) \in (0, \delta)$. By hypothesis,

$$\sum_{i,j=1}^d \left\| \frac{\partial^2 v(\cdot, t)}{\partial x_i \partial x_j} \right\|_{C(\overline{D})} + 1 =: C(t) \quad \text{is finite.}$$

We calculate the Taylor expansion of $v(\cdot, t)$ at $z(x) \in \partial D$ for every $x \in D_\delta$. Then for every $x \in D_\delta$,

there is a $\theta(x) \in (0, 1)$ such that

$$\begin{aligned} v(x, t) &= v(z(x) - v(z(x))d(x), t) \\ &= -\langle \nabla v(z(x), t), v(z(x)) \rangle_{\mathbb{R}^d} d(x) + \frac{1}{2} \langle v(z(x))^t H_v(\theta(x)v(z(x))), v(z(x)) \rangle_{\mathbb{R}^d} d(x)^2 \\ &= -\frac{\partial v}{\partial v}(z(x), t) d(x) + \frac{1}{2} \langle v(z(x))^t H_v(\theta(x)v(z(x))), v(z(x)) \rangle_{\mathbb{R}^d} d(x)^2 \\ &\geq (-v_0(t)) d(x) - C(t) d^2(x). \end{aligned}$$

We take $\delta_0 = \min \left\{ \delta, \frac{(-v_0(t))}{2C(t)} \right\}$. Then, for all $x \in D_{\delta_0}$, $(-v_0(t)) - C(t) d(x) > \frac{(-v_0(t))}{2}$. Therefore

$$v(x, t) \geq \frac{(-v_0(t))}{2} d(x) \quad \text{for all } x \in D_{\delta_0}.$$

On the other hand, by the strong maximum principle (see Theorem 3.3.11), $v(t) > 0$ on the compact set $D \setminus D_{\delta_0}$. Thus and since $D \setminus D_{\delta_0}$ has a positive and uniform distance to the boundary of D , there is a constant $C_2(t) > 0$ such that

$$v(x, t) \geq C_2(t) d(x) \quad \text{for all } x \in D \setminus D_{\delta_0}.$$

Therefore, if we set $C_3(t) = \min \left\{ C_2(t), \frac{(-v_0(t))}{2} \right\}$, then $C_3(t) > 0$ and for this constant $v(t)$ satisfies inequality (3.72) on D . \square

PROOF OF THEOREM 3.4.4. First, we study the case $1 < p < 2$ if $d = 2$, and $\frac{2d}{d+2} \leq p < 2$ if $d \geq 3$. For this part of the proof, the author has been inspired by the pioneering work [35] by J. A. Goldstein and I. Kombe. Let $\lambda > \left(\frac{d-p}{p}\right)^p$. We argue by contradiction and hence we suppose that there is a positive nontrivial $u_0 \in L^2_{\mu, loc}(\Omega)$ such that there is a $T > 0$ and a positive weak solution u of problem (3.2) on $\Omega \times (0, T)$. Let for every $l \geq 1$, Ω_l denote the set $\{x \in \Omega \mid d(x) > 1/l, |x| < l\}$, then $\Omega_l \subseteq \Omega_{l+1} \subseteq \Omega$, Ω_l is open, bounded, and $\bigcup_{l \geq 1} \Omega_l = \Omega$. Moreover, since by assumption, $u_0 \neq 0$ and $0 \in \Omega$, there is a $l_0 \geq 1$ such that $u_0 \neq 0$ on Ω_{l_0} , $0 \in \Omega_{l_0}$, and by continuity of the solution u , $u \neq 0$ on $\Omega_{l_0} \times (0, t)$ for some $t \in (0, T)$. We set $D = \Omega_{l_0}$, fix $\varphi \in C^1_c(D \setminus \{0\})$, and for every integer $k \geq 1$ and every $s \in \mathbb{R}$, we set $g_k(s) = (s + \frac{1}{k})^{1-p}$ if $s \geq 0$ and $g_k(s) = k^{p-1}$ if $s < 0$. Then, by Lemma 3.3.5 for $g = g_k$, $\phi = |\varphi|^p$, $t_1 = 0$, and $t_2 = t$,

$$\begin{aligned} (3.73) \quad & \int_D \frac{(u(t) + \frac{1}{k})^{2-p}}{2-p} |\varphi|^p d\mu - \int_D \frac{(u(0) + \frac{1}{k})^{2-p}}{2-p} |\varphi|^p d\mu + (1-p) \int_0^t \int_D \frac{|\nabla u(s)|^p |\varphi|^p}{(u(s) + \frac{1}{k})^p} d\mu ds \\ & + p \int_0^t \int_D \frac{|\nabla u(s)|^{p-2} \nabla u(s) |\varphi|^{p-2} \varphi}{(u(s) + \frac{1}{k})^{p-1}} \nabla \varphi d\mu ds = \int_0^t \int_D \frac{\lambda}{|x|^p} \frac{u^{p-1}(s)}{(u(s) + \frac{1}{k})^{p-1}} |\varphi|^p d\mu ds. \end{aligned}$$

By Young's inequality,

$$\begin{aligned} & p \int_0^t \int_D |\nabla u(s)|^{p-2} \nabla u(s) (u(s) + \frac{1}{k})^{1-p} \nabla \varphi |\varphi|^{p-2} \varphi d\mu ds \\ & \leq (p-1) \int_0^t \int_D |\nabla u(s)|^p (u(s) + \frac{1}{k})^{-p} |\varphi|^p d\mu ds + t \int_D |\nabla \varphi|^p d\mu, \end{aligned}$$

and since $(u(0) + \frac{1}{k})^{2-p} |\varphi|^p \geq 0$ a.e. on D , we can deduce from equality (3.73) that

$$(3.74) \quad \int_0^t \int_D \frac{\lambda}{|x|^p} u^{p-1}(s) (u(s) + \frac{1}{k})^{1-p} |\varphi|^p d\mu ds \leq t \int_D |\nabla \varphi|^p d\mu + \frac{1}{2-p} \int_D (u(t) + \frac{1}{k})^{2-p} |\varphi|^p d\mu.$$

For almost every $(x, s) \in D \times (0, t)$, we have that

$$0 \leq \frac{\lambda}{|x|^p} \frac{u^{p-1}(x,s)|\varphi(x)|^p}{(u(x,s) + \frac{1}{k})^{p-1}} \nearrow \frac{\lambda}{|x|^p} |\varphi(x)|^p \quad \text{and} \quad (u(x,s) + \frac{1}{k})^{2-p} |\varphi(x)|^p \searrow u^{2-p}(x,s) |\varphi(x)|^p$$

as $k \rightarrow +\infty$. Thus, by Beppo-Levi's theorem, sending $k \rightarrow +\infty$ in inequality (3.74) gives

$$t \int_D \frac{\lambda}{|x|^p} |\varphi|^p d\mu - t \int_D |\nabla \varphi|^p d\mu \leq \frac{1}{2-p} \int_D u^{2-p}(t) |\varphi|^p d\mu.$$

Now, first, we consider the case $d = 2$ and $1 < p < 2$. Then, $(2-p)\frac{2}{p} < 2$, and so $\frac{p}{(2-p)} > 1$. Thus by Hölder's inequality,

$$\int_D u(t)^{\frac{(2-p)2}{p}} d\mu \leq \mu(D)^{\frac{2p-2}{p}} \|u\|_{L_\mu^{\frac{2-p}{p}}(D)}^{\frac{p}{2-p}}.$$

Since $u(t) \in L_\mu^2(D)$, the last estimate implies that $u^{2-p} \in L_\mu^{2/p}(D)$.

Now, we consider the case $d \geq 3$ and let $\frac{2d}{d+2} \leq p < 2$. Then, $(2-p)\frac{d}{p} \leq 2$ and since $u(t) \in L_\mu^2(D)$, we obtain again by Hölder's inequality that $u^{2-p} \in L_\mu^{d/p}(D)$.

Therefore in both cases, $d = 2$ and $1 < p < 2$ or $d \geq 3$ and $\frac{2d}{d+2} \leq p < 2$, Lemma 3.4.5 implies that for any $\varepsilon \in (0, 1)$, there is constant $C(\varepsilon) > 0$ such that

$$(3.75) \quad \int_D \frac{\lambda}{|x|^p} |\varphi|^p d\mu - \int_D |\nabla \varphi|^p d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_D |\nabla \varphi|^p d\mu + C(\varepsilon) \int_D |\varphi|^p d\mu.$$

If the matrix A is positive definite, then there are

$$(3.76) \quad \alpha_1, \alpha_2 > 0 \quad \text{such that} \quad \alpha_1 |x|^2 \leq x^t A x \leq \alpha_2 |x|^2 \quad \text{for all } x \in \mathbb{R}^d.$$

Furthermore, by Lemma 3.2.3 and since $1 < p < 2 \leq d$, the set $C_c^1(D \setminus \{0\})$ lies dense in $W_{\mu,0}^{1,p}(D)$. Therefore, by estimate (3.75), and since $\varphi \in C_c^1(D \setminus \{0\})$ has been arbitrary, we obtain that

$$(3.77) \quad \inf \frac{\int_D |\nabla \varphi|^p d\mu - [(1-\varepsilon)\lambda]^{\frac{p-1}{p}} \int_D |\varphi|^p \frac{(x^t A x)^{p/2}}{|x|^p} d\mu - (1-\varepsilon) \int_D \frac{\lambda}{|x|^p} |\varphi|^p d\mu}{\int_D |\varphi|^p d\mu} \geq -\alpha_2^{p/2} [(1-\varepsilon)\lambda]^{\frac{p-1}{p}} - C(\varepsilon)(1-\varepsilon) > -\infty,$$

where the infimum is taken over all $\varphi \in W_{\mu,0}^{1,p}(D)$ with $\|\varphi\|_{L_\mu^p(D)} > 0$.

We note that if $A \equiv 0$, then $\alpha_2 = 0$ and the integral $\int_D |\varphi|^p \frac{(x^t A x)^{p/2}}{|x|^p} d\mu = 0$. In this case, we reach to a similar inequality as (3.77) with the measure $d\mu = c dx$ on D .

But for every

$$0 < \varepsilon < 1 - \lambda^{-1} \left(\frac{d-p}{p}\right)^p \quad \text{we have that} \quad (1-\varepsilon)\lambda > \left(\frac{d-p}{p}\right)^p.$$

Thus and since $0 \in D$, (3.77) obviously contradicts to the optimality of the constant $\left(\frac{d-p}{p}\right)^p$ in Hardy's inequality (3.5). Therefore the assumption is false and hence claim (i) of Theorem 3.4.4 is

true for $1 < p < 2$ if $d = 2$ and $\frac{2d}{d+2} \leq p < 2$ if $d \geq 3$.

We turn to the case $p = 2$ and $d \geq 3$. In this case, we follow very closely the idea described in [16] by X. Cabré and Y. Martel. Again, we argue by contradiction. Let $\lambda > \left(\frac{d-2}{2}\right)^2$ and let u_0 be a positive nontrivial element of $L^2_\mu(\Omega)$. We suppose that there is a $T > 0$ and there is a positive weak solution u of problem (3.2). Let $(\Omega_l)_{l \geq 1}$ be a sequence of bounded domains Ω_l of \mathbb{R}^d satisfying $\Omega_l \subseteq \Omega_{l+1}$, $\overline{\Omega_l} \subseteq \Omega$, with boundary $\partial\Omega_l$ of class C^∞ , and $\bigcup_{l \geq 1} \Omega_l = \Omega$. We note that for every domain Ω of \mathbb{R}^d such a sequence $(\Omega_l)_{l \geq 1}$ exists. Then by hypothesis, there is an $\Omega_{l_0} \subseteq \Omega$ with $0 \in \Omega_{l_0}$, and $u_0 \neq 0$ almost everywhere on Ω_{l_0} . We set $D = \Omega_{l_0}$ and fix $\varphi \in C^1_c(D)$. For every integer $n \geq 1$, let u_n be the unique positive strong solutions of

$$(3.78) \quad \begin{cases} u_{nt} - K_2 u_n = \min\left\{n, \frac{\lambda}{|x|^2}\right\} u_n & \text{in } D \times (0, T), \\ u_n = 0 & \text{on } \partial D \times (0, T), \\ u_n(0) = \min\{n, u_0\} & \text{in } D \end{cases}$$

and let v_n be the unique positive strong solutions of

$$\begin{cases} v_{nt} - K_2 v_n = 0 & \text{in } D \times (0, T), \\ v_n = 0 & \text{on } \partial D \times (0, T), \\ v_n(0) = \min\{n, u_0\} & \text{in } D. \end{cases}$$

By the weak comparison principle (see Lemma 3.3.3 and Proposition 3.3.4), for all $n \geq 1$,

$$(3.79) \quad 0 \leq v_1 \leq v_n \leq u_n \leq u \quad \text{almost everywhere on } D \times (0, T).$$

Since the boundary of D is smooth, Theorem 3.3.10 implies that v_1 is infinitely differentiable in $D \times (0, T)$ and v_1 , $\frac{\partial v_1}{\partial x_i}$ and $\frac{\partial^2 v_1}{\partial x_i \partial x_j}$ are continuous up to the boundary of D . Since v_1 is positive and since $v_1(0) \neq 0$ a.e. on D , the strong maximum principle (see Theorem 3.3.11) implies that for every $t \in (0, T)$, $v_1(t) > 0$ on the compact subset $\text{supp}(\varphi)$ of D . Thus for all $t \in (0, T)$, there is a constant $C_0(t) > 0$ such that for all $n \geq 1$,

$$u(t) \geq u_n(t) \geq v_n(t) \geq v_1(t) \geq C_0(t) > 0 \quad \text{on } \text{supp}(\varphi).$$

Thus, for every $n \geq 1$, we may multiply the equation in problem (3.78) by $u_n^{-1} |\varphi|^2$ with respect to the $L^2_\mu(D)$ inner product, and subsequently integrate over the interval (t_0, t) with respect to ds for any fixed $0 < t_0 < t < T$. Then, we obtain that

$$(3.80) \quad \begin{aligned} & \int_D \log\left(\frac{u_n(t)}{u_n(t_0)}\right) |\varphi|^2 d\mu - \int_{t_0}^t \int_D |\nabla u_n(s)|^2 (u_n(s))^{-2} |\varphi|^2 d\mu ds \\ & + 2 \int_{t_0}^t \int_D \nabla u_n(s) \nabla \varphi (u_n(s))^{-1} \varphi d\mu ds = (t - t_0) \int_D \min\left\{n, \frac{\lambda}{|x|^2}\right\} |\varphi|^2 d\mu. \end{aligned}$$

By Young's inequality,

$$\begin{aligned} & 2 \int_{t_0}^t \int_D \nabla u_n(s) \nabla \varphi (u_n(s))^{-1} \varphi d\mu ds \\ & \leq (t - t_0) \int_D |\nabla \varphi|^2 d\mu + \int_{t_0}^t \int_D |\nabla u_n(s)|^p (u_n(s))^{-2} |\varphi|^2 d\mu ds. \end{aligned}$$

Thus, we can deduce from equality (3.80) that

$$(3.81) \quad \int_D \min \left\{ n, \frac{\lambda}{|x|^2} \right\} |\varphi|^2 d\mu - \int_D |\nabla \varphi|^2 d\mu \leq \frac{1}{t-t_0} \int_D \log \left(\frac{u_n(t)}{u_n(t_0)} \right) |\varphi|^2 d\mu.$$

Furthermore, by Lemma 3.4.7, for every $t \in (0, T)$, there is another constant $C_1(t) > 0$ such that $v_1(t) \geq C_1(t) d(x)$ for all $x \in D$ and so by (3.79), we have that for all $n \geq 1$,

$$u(x, t) \geq u_n(x, t) \geq C_1(t) d(x) \quad \text{for almost every } x \in D.$$

Thus

$$\int_D \log \left(\frac{u_n(t)}{u_n(t_0)} \right) |\varphi|^2 d\mu \leq \int_D \log \left(\frac{u(t)}{C_1(t_0) d} \right) |\varphi|^2 d\mu = \int_D \log \left(\frac{u(t) d}{C_1(t_0) d^2} \right) |\varphi|^2 d\mu$$

and hence sending $n \rightarrow +\infty$ in inequality (3.81) yields to

$$\int_D \frac{\lambda}{|x|^2} |\varphi|^2 d\mu - \int_D |\nabla \varphi|^2 d\mu \leq \frac{1}{t-t_0} \int_D \log \left(\frac{u(t) d}{C_1(t_0) d^2} \right) |\varphi|^2 d\mu.$$

By Lemma 3.4.6, $\log(u d) \in L^p(D)$ for all $p > 1$ and by claim (2) of Lemma 3.3.9,

$$\log(C_1(t_0) d^2) = \log(C_1(t_0)) + 2 \log d \in L^p_\mu(D) \quad \text{for all } p > 1.$$

Thus $\log \left(\frac{u(t) d}{C_1(t_0) d^2} \right) \in L^{d/2}_\mu(D)$. By Lemma 3.4.5, for every $\varepsilon \in (0, 1)$ there is a $C(\varepsilon) > 0$ such that

$$\int_D \frac{\lambda}{|x|^2} |\varphi|^2 d\mu - \int_D |\nabla \varphi|^2 d\mu \leq \frac{\varepsilon}{1-\varepsilon} \int_D |\nabla \varphi|^2 d\mu + C(\varepsilon) \int_D |\varphi|^2 d\mu.$$

Now, we proceed as in the proof of claim (i) and reach a contradiction to the optimality of $\left(\frac{d-2}{2}\right)^2$ in Hardy's inequality (3.5).

Let $p > 2$, $d \neq p$, and let $\lambda > \left(\frac{|d-p|}{p}\right)^p$ and let $u_0 \in L^2_{\mu,loc}(\Omega)$ be positive and satisfying (3.66) for some $r > 0$. We argue by contradiction and so we assume, there is a $T > 0$ such that (3.2) admits a positive weak solution u . For the above given $r > 0$, we fix $\varphi \in C^1_c(B(0, r) \setminus \{0\})$ and for every integer $k \geq 1$ and every $s \in \mathbb{R}$, we set $g_k(s) = (s + \frac{1}{k})^{1-p}$ if $s \geq 0$ and $g_k(s) = k^{p-1}$ if $s < 0$. Then, by Lemma 3.3.5 for $t_1 = 0$, $t_2 = t > 0$, and $\varphi = |\varphi|^p$, we obtain that

$$(3.82) \quad \int_{B(0,r)} \frac{(u(0) + \frac{1}{k})^{2-p}}{p-2} |\varphi|^p d\mu - \int_{B(0,r)} \frac{(u(t) + \frac{1}{k})^{2-p}}{p-2} |\varphi|^p d\mu + (1-p) \int_0^t \int_{B(0,r)} \frac{|\nabla u(s)|^p |\varphi|^p}{(u(s) + \frac{1}{k})^p} d\mu ds \\ + p \int_0^t \int_{B(0,r)} \frac{|\nabla u(s)|^{p-2} \nabla u(s) |\varphi|^{p-2} \varphi}{(u(s) + \frac{1}{k})^{p-1}} \nabla \varphi d\mu ds = \int_0^t \int_{B(0,r)} \frac{\lambda}{|x|^p} \frac{u^{p-1}(s)}{(u(s) + \frac{1}{k})^{p-1}} |\varphi|^p d\mu ds.$$

By Young's inequality

$$p \int_0^t \int_{B(0,r)} |\nabla u(s)|^{p-2} \nabla u(s) (u(s) + \frac{1}{k})^{1-p} \nabla \varphi |\varphi|^{p-2} \varphi d\mu ds \\ \leq (p-1) \int_0^t \int_{B(0,r)} |\nabla u(s)|^p (u(s) + \frac{1}{k})^{-p} |\varphi|^p d\mu ds + t \int_{B(0,r)} |\nabla \varphi|^p d\mu,$$

and since $(u(t) + \frac{1}{k})^{2-p} |\varphi|^p$ is positive a.e. on $B(0, r) \times (0, T)$, we can deduce from (3.82) that

$$\int_0^t \int_{B(0,r)} \frac{\lambda}{|x|^p} u^{p-1}(s) (u(s) + \frac{1}{k})^{1-p} |\varphi|^p d\mu ds \leq t \int_{B(0,r)} |\nabla \varphi|^p d\mu + \int_{B(0,r)} \frac{(u(0) + \frac{1}{k})^{2-p}}{p-2} |\varphi|^p d\mu.$$

We send $k \rightarrow +\infty$ in this inequality. Then, by Beppo-Levi's convergence theorem, we obtain that

$$(3.83) \quad \int_{B(0,r)} \frac{\lambda}{|x|^p} |\varphi|^p d\mu - \int_{B(0,r)} |\nabla \varphi|^p d\mu \leq \frac{1}{t^{(p-2)}} \int_{B(0,r)} u^{2-p}(0) |\varphi|^p d\mu.$$

Thus, since $\text{supp}(\varphi) \subseteq B(0,r) \setminus \{0\}$, and by hypothesis (3.66), we have that

$$(3.84) \quad \int_{B(0,r)} \frac{\lambda}{|x|^p} |\varphi|^p d\mu - \int_{B(0,r)} |\nabla \varphi|^p d\mu \leq \frac{\delta^{2-p}}{t^{(p-2)}} \int_{B(0,r)} |\varphi|^p d\mu.$$

First, we suppose that the matrix A is positive definite. Then by (3.76) and since inequality (3.84) holds for all $\varphi \in C_c^1(B(0,r) \setminus \{0\})$, we have thereby shown that

$$\inf \frac{\int_{B(0,r)} |\nabla \varphi|^p d\mu - \lambda \frac{p-1}{p} \int_{B(0,r)} |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu - \int_{B(0,r)} \frac{\lambda}{|x|^p} |\varphi|^p d\mu}{\int_{B(0,r)} |\varphi|^p d\mu} \geq -\frac{\delta^{2-p}}{t^{(2-p)}} - \alpha_2^{p/2} \lambda \frac{p-1}{p},$$

where the infimum is taken over all $\varphi \in C_c^1(B(0,r) \setminus \{0\})$ with $\|\varphi\|_{L_\mu^p(B(0,r))} > 0$. Again, we note that if the matrix $A \equiv 0$, then $\alpha_2 = 0$ and $\int_{B(0,r)} |\varphi|^p \frac{(x^t Ax)^{p/2}}{|x|^p} d\mu$ vanishes in the quotient of the last estimate.

If $2 < p < d$, then by Lemma 3.2.3, the set $C_c^1(B(0,r) \setminus \{0\})$ lies dense in $W_{\mu,0}^{1,p}(B(0,r))$ and if $p > d \geq 2$, then the set $C_c^1(B(0,r) \setminus \{0\})$ lies dense in $W_{\mu,0}^{1,p}(B(0,r) \setminus \{0\})$. Therefore in both cases, the last inequality contradicts to the optimality of the constant $(\frac{|d-p|}{p})^p$ in Hardy's inequality (3.5). \square

REMARK 3.4.1. In view of Theorem 3.4.1 and Theorem 3.4.4, one sees that the influence of convection term Ax appearing the symmetric Ornstein-Uhlenbeck operator $L = \Delta - \langle Ax, \nabla \cdot \rangle$ is not strong enough to stop the phenomenon of nonexistence of positive nontrivial weak solutions of problem (3.2). Furthermore, we stress that Theorem 3.4.4 improves the result of Theorem 3.1.4 by G. R. Goldstein, J. A. Goldstein et A. Rhandi and of Theoreme 3.1.3 by J. P. García Azorero et I. Peral Alonso.

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Ulm, den 19. Dezember 2012