



HAL
open science

Quelques problèmes de modélisation et d'estimation en traitement du signal aléatoire

Thierry Chonavel

► **To cite this version:**

Thierry Chonavel. Quelques problèmes de modélisation et d'estimation en traitement du signal aléatoire. Traitement du signal et de l'image [eess.SP]. Télécom Bretagne, 2000. tel-01295210

HAL Id: tel-01295210

<https://hal.science/tel-01295210>

Submitted on 10 Feb 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Habilitation à diriger les recherches

Université de rennes I

Présentée par

Thierry CHONAVEL

Quelques problèmes de modélisation et d'estimation en traitement du signal aléatoire

Présentée le 20 décembre 2000

Membres du jury :

M. FUCHS, Jean-Jacques	PU, Université de rennes I, FR	Président
M. CHAMPAGNE, Benoît	Associate Professor, Université de mac Gill, CA	Rapporteur
M. DROUCHE, Karim	MdC, HdR, Université d'Orsay, FR	Rapporteur
M. SEGHIER, Abdelatif	MdC, HdR qualifié PR, Université d'Orsay, FR	Rapporteur
M. BOUCHER, Jean Marc	Professeur, ENST Bretagne, FR	Examineur
M. SAOUDI, Samir	Professeur, ENST Bretagne, FR	Examineur

Table des matières

1	Introduction.	1
I	Curriculum Vitae	3
1	Informations générales	5
1.1	Cursus et diplômes	5
1.1.1	Cursus	5
1.1.2	Diplômes	5
1.2	Expérience professionnelle	6
1.2.1	Mars à juin 1987: stage de fin d'études	6
1.2.2	Août 1987 à juillet 1988: service national	6
1.2.3	Octobre 1988 à décembre 1989: ingénieur à ALCATEL	6
1.2.4	Janvier 1990 à décembre 92: doctorat	7
1.2.5	Depuis juillet 1993: enseignement et recherche	7
2	Publications	13
2.1	Thèse de doctorat	13
2.2	Rapports de recherche et d'étude	13

2.3	Polycopiés de cours	14
2.4	Contributions à des livres et document informatique	14
2.5	Articles de revue	15
2.6	Articles de revue soumis	15
2.7	Articles de revue en préparation	15
2.8	Articles de conférences en français (avec comité de lecture)	16
2.9	Articles de conférences internationales (avec comité de lecture)	16
 II Quelques problèmes de modélisation et d'estimation		19
 1 Modélisation spectrale à bande limitée		21
1.1	Les coefficients de réflexion des spectres à bande limitée	22
1.2	Travaux en cours	24
 2 Estimation de temps de retard		29
2.1	Introduction	29
2.2	Les méthodes pénalisées	31
2.3	Détection quadratique	31
2.4	Approches MCMC pour l'estimation de temps de retard	32
2.4.1	Une première approche	33
2.4.2	Travaux en cours : prise en compte de la corrélation du bruit	34
 3 Décomposition adaptative en valeurs propres		39
3.1	Introduction	39

	iii
3.2 L'algorithme MALASE	41
3.3 Travaux en cours	42
4 Estimation de supports de lois	45
4.1 Introduction : les modèles markoviens en télétrafic	45
4.2 Une nouvelle approche	46
4.3 L'estimation d'un mélange de lois décalées	47
5 Livre de traitement du signal aléatoire	51
III Recueil d'articles	53
1 Coefficients de réflexion des spectres à bande limitée	
2 Estimation des corrélations des spectres à bande limitée	
3 Décomposition en valeurs propres adaptative	
4 Calcul de performances de l'algorithme MALASE	
5 Estimation de paramètres de décalage dans un mélange de lois	
6 Estimation de temps de retard au moyen de SBLM	
7 Livre de traitement du signal aléatoire	

Chapitre 1

Introduction.

Les progrès récents du traitement du signal dans divers domaines d'application sont dus à une meilleure prise en compte des informations *a priori* dont on dispose concernant les phénomènes étudiés. Bien souvent, ces *a priori* étaient depuis longtemps connus des spécialistes des domaines concernés, mais leur intégration aux méthodes de traitement du signal n'avait pas toujours été envisagée.

Une raison essentielle de cette attitude est bien entendu liée aux limitations de l'outil informatique qui ont, pendant longtemps, imposé de restreindre les traitements à l'emploi de méthodes simples. Les ambitions que l'on pouvait fonder sur le traitement des signaux s'en trouvaient naturellement réduites. Ceci conduisait alors à privilégier le développement des capacités de l'expertise humaine par rapport au raffinement des techniques d'acquisition des données et à l'automatisation de leur traitement.

L'augmentation des capacités de traitement informatique ont permis d'enrichir l'éventail des techniques de traitement du signal. Cet enrichissement a été d'autant plus rapide que depuis une trentaine d'années de nombreux outils sont devenus familiers des spécialistes du traitement de signal. On peut citer en particulier les statistiques d'ordres supérieurs, les techniques temps-fréquence, les méthodes bayésiennes et de Monte-Carlo ou encore la théorie des systèmes. L'introduction de ces outils a souvent permis des progrès remarquables qui ont largement contribué à faire du traitement du signal une discipline à part entière.

Malgré cela, des techniques très anciennes telles que les méthodes de moindres carrés, très robustes vis-à-vis des erreurs de modélisation, restent d'un grand intérêt en traitement du signal. C'est là un des aspects marquants et aussi un des charmes de cette discipline où les tentatives de modélisation peuvent rapidement conduire à des problèmes relativement complexes. Toutefois il est souvent possible d'obtenir des résultats d'un grand intérêt pratique par l'emploi de méthodes qui n'intègrent que partiellement les connaissances *a priori* sur les signaux étudiés. La méthode la plus simple n'est pas toujours la moins

bonne, tant pour ce qui concerne la robustesse aux erreurs de modélisation, que pour la stabilité numérique ou le coût de calcul, qui reste souvent un problème particulièrement sensible pour les applications en temps réel.

Cependant, l'expérience montre que l'on a souvent intérêt à affiner autant que possible la modélisation des problèmes étudiés. Cela permet d'acquérir une compréhension plus précise des phénomènes, ce qui permet souvent de mettre en évidence des critères statistiques pertinents en vue de l'estimation de paramètres d'intérêt. Si on est amené à effectuer certaines hypothèses simplificatrices dans le cadre du traitement des données, on le fait alors en connaissance de cause.

La démarche suivie dans les travaux présentés dans ce document s'inspire des remarques précédentes. On s'attache à y apporter un nouvel éclairage sur quelques domaines du traitement du signal, grâce à une prise en compte affinée de la nature des signaux étudiés. Les signaux et les problèmes envisagés concernent plus particulièrement :

- les signaux à bande limitée et leur modélisation spectrale,
- les signaux acoustiques sous-marins et le problème de l'estimation des paramètres d'un canal à trajets multiples,
- les signaux multicateurs et les techniques de décomposition adaptative en valeurs propres d'une matrice de covariance,
- les signaux de télétrafic et leur modélisation markovienne,
- les signaux audio stéréophoniques et leur codage,
- les signaux de communications et la réception de signaux à spectre étalé.

Parmi ces sujets, les trois premiers ont fait l'objet d'une étude relativement approfondie, tandis que la contribution aux trois derniers concerne des aspects plus ponctuels ou plus récents de mes recherches.

La première partie du rapport est consacrée au curriculum vitae qui présente des informations générales et la liste des travaux réalisés. La seconde partie présente une synthèse des thèmes de recherche les plus significatifs. Les détails n'y sont pas abordés permettant ainsi de mieux mettre en évidence les idées d'ensemble et la démarche adoptée. La troisième partie rassemble des travaux dans lesquels certains résultats techniques sont plus longuement développés.

Première partie
Curriculum Vitae

Chapitre 1

Informations générales

Thierry CHONAVEL,
né le 30 janvier 1963 à Neuilly sur Seine,
vie maritale, deux enfants,
10, impasse Guynemer, 29290 Saint Renan,
tel. 02 98 00 13 74,
thierry.chonavel@enst-bretagne.fr.

1.1 Cursus et diplômes

1.1.1 Cursus

- 1981 à 1984 : Mathématiques supérieures et mathématiques spéciales.
- 1984 : Concours commun Mines-Ponts, admission à l'ENST Bretagne.
- 1984 à 1987 : Elève ingénieur à l'ENST Bretagne.
- 1987 à 1988 : Service national.
- 1988 à 1989 : Ingénieur à ALCATEL.
- 1990 à 1992 : Thèse de doctorat à l'ENST.
- 1993 à 1995 : Maître de Conférences à l'Ecole Franco-Polonaise de Poznan.
- Depuis juillet 1995 : Maître de Conférences au département Signal et Communications de l'ENST Bretagne. Spécialité : traitement du signal (section 61).

1.1.2 Diplômes

- Doctorat de l'ENST, mention Traitement du Signal et des Images (décembre 1992).

- Diplôme d'ingénieur de l'ENST Bretagne (option micro-ondes, juin 1997).
- Baccalauréat série C (juin 1981).

1.2 Expérience professionnelle

1.2.1 Mars à juin 1987 : stage de fin d'études

Stage d'ingénieur au laboratoire de recherche de la compagnie TELEFONICA (Madrid) : conception et réalisation d'oscillateurs hyperfréquences à 30GHz.

1.2.2 Août 1987 à juillet 1988 : service national

- Service national en tant qu'assistant en mathématiques et traitement du signal à l'Ecole Navale : cours de mathématiques, travaux dirigés de mathématiques et de traitement du signal, travaux pratiques d'électronique.
- Vacances à l'UBO¹ : travaux dirigés de transmissions en maîtrise EEA².
- Vacances à l'Ecole Supérieure de Commerce de Brest : travaux dirigés de mathématiques en classes préparatoires aux Ecoles de commerce et en première année de l'Ecole Supérieure de Commerce.

1.2.3 Octobre 1988 à décembre 1989 : ingénieur à ALCATEL

Ingénieur d'études à ATFH³, responsable de projet pour le développement d'un faisceau hertzien à grande capacité (140Mbit/s) utilisant la modulation MAQ 16. Principaux travaux effectués :

- réalisation d'un logiciel de simulation de faisceaux hertziens (simulation de la modulation, des filtres analogiques et numériques, des non-linéarités d'amplification, des trajets multiples, de l'égaliseur numérique adaptatif ...),
- rédaction du cahier des charges du système,
- suivi de la réalisation et du test des maquettes et planification des activités entre les différents laboratoires,
- coordination du projet avec la filiale allemande du groupe, Standard Electronic Lorenz.

1. Université de Bretagne Occidentale

2. Electronique et Automatique

3. ALCATEL Transmissions par Faisceaux Hertziens

1.2.4 Janvier 1990 à décembre 92 : doctorat

Thèse de doctorat à l'ENST.

- Spécialité : Signal et Images.
- Titre : Estimation Spectrale à Bande Limitée et Traitement d'Antennes.
- Soutenance : le 14/12/92, à l'ENST.
- Mention : très honorable, avec les félicitations du jury.
- Encadrant : Philippe LOUBATON.

Les principaux aboutissements de ce travail sont la résolution complète du problème des moments trigonométriques sur un intervalle, la mise en oeuvre de techniques performantes d'estimation des coefficients de corrélation pour les signaux à bande limitée et leur application au problème de traitement d'antenne pour la localisation de sources.

1.2.5 Depuis juillet 1993 : enseignement et recherche

De juillet 1993 à juin 1995, Maître de Conférences à l'EFP⁴ de Poznan (Pologne), avec pour mission de mettre en place les activités d'enseignement et de recherche en traitement du signal; depuis juillet 1995, Maître de Conférences à l'ENST Bretagne.

Activités d'enseignement

- Responsabilité de modules d'enseignement :
 - traitement des images et de la parole (à l'EFP, 1993 à 1995),
 - définition du contenu des enseignements de traitement du signal (EFP, 1993 à 1995),
 - probabilités (en première année, depuis 1997),
 - optimisation (en troisième année, depuis 1997),
 - projets d'ingénierie (en troisième année, depuis 1998),
 - méthodes avancées en traitement de signal (en troisième année, depuis 1999).
- Participation aux enseignements :
 - cours de traitement de la parole (EFP, 1993 à 1995),
 - travaux dirigés de traitement du signal analogique et numérique (en première année, 1995 à 1998),
 - cours et travaux dirigés de probabilités (en première année et pour les élèves admis sur titres, depuis 1995),
 - travaux dirigés de processus aléatoires (en deuxième année, 1999),

4. Ecole Franco-Polonoise des Nouvelles Technologies de l'Information et de la Communication

- cours d'analyse numérique matricielle (en deuxième année, depuis 1998),
- cours de traitement du signal aléatoire (pour les élèves de mastère, depuis 1997),
- cours et travaux dirigés d'optimisation (en troisième année, depuis 1998),
- cours de traitement du signal (en troisième année, depuis 1994),
- cours de formation continue en signal et en communications,
- encadrement de projets d'élèves dans les trois années de la formation,
 - programmation algorithmique (en première année),
 - projets expérimentaux (en deuxième année, simulation de chaînes de communications) ,
 - projets bibliographiques et de programmation, projets de recherche et d'ingénierie (en troisième année),
- participation aux soutenances de stages,
- encadrement de stage IAESTE⁵.
- Interventions extérieures :
 - cours de communications numériques à l'Ecole Navale (1995 à 1998),
 - cours de traitement du signal aléatoire à l'ENSAI (depuis 1997),
 - travaux dirigés de communications en DEA d'électronique à l'UBO (1995 à 1998),
 - participation à des jurys de soutenance de stage de l'Ecole Navale, de l'ENSAI, de l'ENSIETA et de l'UBO,
 - cours d'estimation spectrale en DEA d'électronique à l'UBO (à partir de l'année 2000-2001),
 - cours de formation doctorale à l'université espagnole de Valence (en langue espagnole, programme européen SOCRATES, année 2000-2001).

Activités de recherche

- Encadrement de jeunes chercheurs.

Encadrement annuel de deux à quatre étudiants de l'ENST Bretagne en stage de fin d'études dans l'industrie. Encadrement des étudiants suivants en stage de fin d'études à l'école⁶

1. Julien TCHERNIA, "Décomposition des matrices de Toeplitz en somme de matrices de rang un", *stage de fin d'études de l'ENST Bretagne à l'EFP de Poznan*, avril-sept. 1994.
2. Skander BENCHICOU "Estimation de temps d'arrivée de signaux tomographiques", *stage de fin d'études de l'ENST Bretagne*, avril-août 1997.
3. Maite BOYERO EGIDO, "Application du traitement multi-capteurs à l'estimation de temps d'arrivée", *stage de fin d'études de mastère de l'ENST Bretagne*, avril-sept. 1997.

5. International Association for the Exchange of Students for Technical Experience

6. par défaut, l'encadrement des stagiaires est de 100%

4. Christophe TOUPIN, "Egalisation multicapteurs en communications sous-marines", *stage long*, sept. 1997-sept. 1998. Encadrement : 50%.
5. Dominique BARRIA, "Détection d'émission simultanée de balises sous-marines", *stage de fin d'études de l'ENST Bretagne*, avril-sept. 1998. Encadrement : 50%.
6. Cesareo PALOMINO LOPEZ, "Comparaison de méthodes d'estimation de temps de retard", *stage de fin d'études du CINVESTAV⁷ de Mexico*, sept. 1997 - février 1998.
7. Hector CRUZ OTERO "Comparaison de méthodes d'estimation de temps de retard", *stage de fin d'études du CINVESTAV de Mexico*, sept. 1998 - février 1999.
8. Sébastien-Yves FLATRESSE, "Détection multi-utilisateurs pour les systèmes de communication à spectre étalé", *stage du DEA STIR⁸, de l'Université de Rennes I*, mars-août 2000.
9. Florent AVERTY, "Systèmes de communications à spectre étalé multicapteurs", *stage de fin d'études de l'ENST Bretagne*, avril-juillet 2000. Encadrement : 50%.
10. Ainhoa VALERIO VICENTE, "Stratégies de décision en suppression successive d'interférences pour les systèmes DS-CDMA⁹", *stage de fin d'études de l'école de télécommunications de Gandia, Universidad Politecnica de Valencia*, oct. 2000 - mars 2001.

– **Encadrement de thèses.**

1. Thèse de Christian RIOU,
 - titre : *Estimation adaptative de sous-espaces et applications*,
 - université : Université de Rennes I,
 - soutenance : 30 janvier 1997,
 - directeur : Alain HILLION,
 - encadrement : 100%.
2. Thèse de Sandrine VATON,
 - titre : *Modélisation statistique de trafic sur réseau local : application au contrôle dynamique de bande passante*,
 - université : ENST,
 - soutenance : 11 décembre 1998,
 - directeur : Eric MOULINES,
 - encadrement : thèse en co-encadrement avec l'ENST, réalisée à 70% à l'ENST Bretagne.
3. Thèse de Fabienne POREE,
 - titre : *Estimation des paramètres des canaux à trajets multiples : application à la tomographie acoustique sous-marine*,
 - université : Université de Rennes I,

7. Centro de INVestigacion y de ESTudios AVanzados

8. Signal, Télécommunications, Images et Radar

9. Direct Sequence - Code Division Multiple Access

- soutenance : prévue pour la début 2001,
- directeur : Jean Marc BOUCHER,
- encadrement : 100%.
- 4. Thèse de Mahmoud AMMAR
 - titre: *Techniques de réception en communications à spectre étalé pour la norme UMTS*¹⁰,
 - université: UBO,
 - soutenance : prévue pour début 2002,
 - directeur : Alain GLAVIEUX,
 - encadrement : 75%.
- **Participation à des jurys de thèse** (hors thèses encadrées)
 1. Thèse de Philippe PANGO (en tant que rapporteur) : *Algorithmes rapides de suivi de sous-espaces basés sur la SVD et applications en traitement d'antenne et d'images*, INRS, Montréal, 18 février 2000.
 2. Thèse d'Olivier ROSEC : *Déconvolution aveugle multicateurs en sismique réflexion marine à très haute résolution*, Brest, 26 avril 2000.
- **Participation à des contrats de recherche** (montant global : 3,2MF)
 1. Bilan et perspectives en tomographie acoustique par petits fonds (responsabilité du contrat), 1997.
 2. Positionnement acoustique sous marin (participation au contrat), 1998.
 3. Tomographie acoustique océanique par grands fonds (responsabilité du contrat), 1998.
 4. Tomographie acoustique par petits fonds (responsabilité du contrat), 1999.
 5. Contrat CNET sur les communications UMTS (participation au contrat), débuté fin 1999.
 6. Contrat incitatif GET¹¹ : détection multi-utilisateurs pour le système UMTS, aspects mono et multi-capteurs (responsabilité du contrat), 2000.
 7. Sismique réflexion océanique (participation au contrat), débuté fin 2000 .
 8. Détection opérationnelle des surcourants océaniques (responsabilité du contrat), débuté fin 2000.
 9. Contrat incitatif GET : localisation du mobile dans un système cellulaire, cas du système UMTS (responsabilité du contrat), débuté fin 2000.
- **Participation à des comités de lecture de revues**
 - Signal Processing Journal (3 articles).
 - IEEE Trans. on Signal Processing (4 articles).
 - Designs Codes and Cryptography (1 article).
- **Animation**
 - Organisation des séminaires de recherche du département Signal et Communications de l'ENST Bretagne.

10. Universal Mobile Telecommunication System

11. Groupe des Ecoles de Télécommunications

- Représentation du département Signal et Communications auprès du GDR-PRC ISIS¹².
- Président de séance de la session tomographie acoustique, 5^{èmes} journées d'acoustique, Brest, déc. 2000.

¹². Groupes De Recherches - Programme de Recherches Coordonnées en Information Signal, Image et viSion

Chapitre 2

Publications

2.1 Thèse de doctorat

Th. CHONAVEL, *Estimation Spectrale à Bande Limitée. Application au Traitement d'Antennes*, Thèse de Doctorat de l'ENST, Paris, déc. 1992.

2.2 Rapports de recherche et d'étude

[1] Th. CHONAVEL, "Conception et réalisation d'oscillateurs hyperfréquences à 30GHz", Telefonica, Madrid, juin 1987. (150 p.)

[2] Th. CHONAVEL, "16QAM-2G, 140 Mbit/s radio relay system in the 6.4-7.1 frequency band. Technical report", ALCATEL ATFH, Levallois-Perret, sept. 1989. (132 p.)

[3] Th. CHONAVEL, J. M. BOUCHER, "Tomographie acoustique sous marine par petits fonds: résultats et perspectives", convention ENSTBr-IFREMER, déc. 1997. (21 p.)

[4] S. SAOUDI, Th. CHONAVEL, "Détection simultanée en positionnement acoustique", convention ENST Bretagne-IFREMER, Juil. 1998. (36 p.)

[5] Th. CHONAVEL, F. POREE, "Estimation de temps de retard en tomographie acoustique océanique", rapport intermédiaire, convention ENST Bretagne-IFREMER, Juil. 1998. (30 p.)

[6] F. POREE, Th. CHONAVEL, "Estimation de temps de retard en tomographie acoustique océanique", rapport final, convention ENST Bretagne-IFREMER, déc. 1998. (57p.)

[7] F. POREE, Th. CHONAVEL, "Estimation de temps de retard en tomographie acoustique sous marine: une approche bayésienne", rapport intermédiaire, convention ENST Bretagne-IFREMER, juil. 1999. (42 p.)

[8] F. POREE, Th. CHONAVEL, "Méthodes bayésiennes pour l'estimation de temps de retard en tomographie acoustique sous marine", rapport final, convention ENST Bretagne-IFREMER, déc. 1999. (39 p.)

2.3 Polycopiés de cours

[1] Th. CHONAVEL, *Traitement d'Antennes et Statistiques d'Ordres Supérieurs*, janv. 1995. (76 p.)

[2] Th. CHONAVEL, *Traitement de la Parole*, mai 1995. (25 p.)

[3] Th. CHONAVEL, *Optimisation Continue, Méthodes de Simulation et Optimisation Stochastique*, déc. 1998. (86 p.)

[4] Th. CHONAVEL, *Notes de cours de probabilités*, mai 1999. (28 p.)

[5] Th. CHONAVEL, S. VATON, *Exercices de probabilités. STP¹ de probabilités: simulation de phénomènes aléatoires*, avril 2000. (35 p.)

2.4 Contributions à des livres et document informatique

[1] S. VATON, H. KOREZLIOGLU, Th. CHONAVEL, *Is the Apparent Self-Similarity of the Broadband Traffic due to Non Stationarity?*, in *Performance Analysis of ATM Networks*, D. Kouvatsos ed., Kluwer Academic Publishers, The Netherlands, pp. 48-76, 1999.

[2] Th. CHONAVEL, *Traitement du Signal Aléatoire*, Springer, collection Sciences et Techniques des Télécommunications², avril 2000³. (300 p., voir l'annexe 7 pour la table des matières et l'introduction).

1. Suivi de Travail Personnel

2. collection publiée sous l'égide du CNET (centre de recherche et développement de France Télécom)

3. une version augmentée et en anglais de ce livre, cosignée avec S. VATON, sera diffusée par Springer début 2001

[3] Th. CHONAVEL, *Traitement du Signal Aléatoire : Exemples de Simulations avec MATLAB*, CD-ROM d'accompagnement au livre [2], document HTML. (77 p.)

2.5 Articles de revue

[1] Th. CHONAVEL, Ph. LOUBATON, "Le problème des moments trigonométriques sur un intervalle", *Compte Rendu à l'Académie des Sciences*, Paris, t.314, série I, pp.41-44, 1992.

[2] Th. CHONAVEL, Ph. LOUBATON, "On the realization of band-limited power spectra from partial covariance sequence", *IEEE Trans. on Inf. Theory*, pp. 1395-1397, juil. 1993.

[3] Th. CHONAVEL, S. DEGERINE, "Reflection coefficients of band-limited spectra, and band-limited spectrum modeling", *IEEE Trans. on Inf. Theory*, pp.2030-2038, sept 1999. (voir annexe 1)

[4] Th. CHONAVEL, B. CHAMPAGNE, C. RIOU "Fast adaptive eigenvalue decomposition: a maximum likelihood approach", à paraître dans *Signal Processing*. (voir annexe 3)

2.6 Articles de revue soumis

[1] S. VATON, Th. CHONAVEL, "Estimating the offset parameters of a mixture in the Fourier Domain", *rapport de recherche ENST-Br RR-200001-SC*, soumis à *Statistical Inference for Stochastic Processes*, Kluwer. (voir annexe 5)

[2] M. AMMAR, Th. CHONAVEL, S. SAOUDI, "Iterative Successive Interference Cancellation for Multi-user DS-CDMA detectors in Multipath Channels", soumis au *Journal of Communications and networks*.

2.7 Articles de revue en préparation

[1] Th. CHONAVEL, Ph. LOUBATON, "Band-limited extrapolation of partial correlation sequence: a trigonometric moment approach".

[2] Th. CHONAVEL, S. DEGERINE, Ph. LOUBATON, "Estimation of the covariance coefficients of a band-limited process". (voir annexe 2)

[3] Th. CHONAVEL, B. CHAMPAGNE, C. RIOU "Convergence properties of MALASE algorithm". (voir annexe 4)

[4] F. POREE, Th. CHONAVEL, "A bayesian approach for multipath time delay estimation at the Output of a square-law detector".

2.8 Articles de conférences en français (avec comité de lecture)

[1] Th. CHONAVEL, Ph. LOUBATON, "Extension à bande limitée d'une suite de corrélations contigues", *GRETSI'91*, pp.381-384, sept. 1991.

[2] Th. CHONAVEL, "Construction d'estimateurs spectraux absolument continus à partir d'estimateurs ponctuels particuliers, application aux signaux à bande limitée", *GRETSI'93*, pp.73-76, Août 1993.

[3] Th. CHONAVEL, S. DEGERINE, Ph. LOUBATON, "Coefficients de réflexion des spectres à bande limitée et estimation spectrale", *GRETSI'95*, pp.25-28, sept. 1995.

[4] S. VATON, Th CHONAVEL, "Estimation d'un mélange de lois décalées dans le domaine de Fourier", *XXXèmes journées de la statistique*, pp. 559-562, Rennes, mai 1998.

[5] F. POREE, Th. CHONAVEL, Th. TERRE, "Une nouvelle technique d'estimation des temps de retard en tomographie acoustique océanique", *Acoustique Sous-Marine*⁴, IFREMER, Brest, pp. 145-153, nov. 1998.

[6] F. POREE, Th. CHONAVEL, Th. TERRE, "Détection, estimation et suivi de trajets multiples en tomographie acoustique océanique", *GRETSI'99*, pp. 375-378, sept. 1999.

2.9 Articles de conférences internationales (avec comité de lecture)

[1] Th. CHONAVEL, Ph. LOUBATON, "Estimation of the autocorrelation coefficients of band-limited signals. application to the sources localisation problem", *EUSIPCO'92*, pp. 1825-1828, 1992.

[2] Th. CHONAVEL, Ph. LOUBATON, "Band limited spectral power estimation from

4. conférence SEE (Société des Electriciens et electroniciens), SFA (Société Française d'acoustique)

partial correlation sequences", *Workshop on Design Methodologies for Microelectronics and Signal Processing*, Cracowie, pp. 389-396, oct.1993.

[3] Th. CHONAVEL, "A maximum likelihood estimation algorithm for autocovariance matrices of bandlimited processes", *EUSIPCO'94*, pp. 1825-1828, 1994.

[4] C. RIOU, Th. CHONAVEL, P.Y. COCHET, "Adaptive techniques for moving sources localization ", actes de la conférence *KST*, Bidgoszcz, sept. 1995. (10 p.)

[5] Th. CHONAVEL, "Estimation of a density probability function : a trigonometric moment approach", actes de la conférence *KST*, Bidgoszcz, sept. 1995. (10 p.)

[6] Th. CHONAVEL, S. SAOUDI, "Multi-Channel Linear predictive coding of audio signals", *EUROSPEECH'95*, pp. 53-56, sept. 1995.

[7] Th. CHONAVEL, S. DEGERINE, Ph. LOUBATON, "Estimation of band-limited spectra, a reflection coefficients approach", *ICASSP'96*, pp. 2531-2534, avril 1996.

[8] C. RIOU, Th. CHONAVEL, P.Y. COCHET, "Adaptive subspace estimation - application to moving sources localization and blind channel identification", *ICASSP'96*, pp. 1649-1652, avril 1996.

[9] C. RIOU, Th. CHONAVEL, "Fast adaptive eigenvalue decomposition: A maximum likelihood approach", *ICASSP'97*, pp. 3565-3568, avril 1997.

[10] S. VATON, Th. CHONAVEL, "Estimating the offset parameters of a mixture in the Fourier domain", *ICASSP'99*, pp. 152-155, mars 1999.

[11] F. POREE, Th. CHONAVEL, Th. TERRE, "Multipath time delay estimation for ocean acoustic tomography, a bayesian approach", *OCEAN'99*, Seattle, pp. 1587-1590, sept. 1999.

[12] M. AMMAR, Th. CHONAVEL, S. SAOUDI, "Performance of DS-CDMA receivers with multiuser detection", *ISIVC⁵*, Rabat, avril 2000.

[13] F. POREE, O. ROSEC, Th. CHONAVEL, J.M. BOUCHER, "Two bayesian methods for multipath propagation parameters estimation", *ICASSP'2000*, Istanbul, juin 2000. (voir l'annexe 6)

[14] O. ROSEC, F. POREE, Th. CHONAVEL, J.M. BOUCHER, "A bayesian method for ocean acoustic tomography in the presence of transducers distortions", *ECUA'2000⁶*, Lyon, juillet 2000.

5. International Symposium on Image and Video Communications over fixed and mobile networks (conférence ISIS, IEEE).

6. European Conference on Underwater Acoustics, conférence EAA (European Acoustics Association)

- [15] F. POREE, Th. CHONAVEL, "Déconvolution en tomographie acoustique océanique. Application aux données Thetis et Intimate", 5èmes journées d'acoustique, Brest, déc. 2000.
- [16] M. AMMAR, Th. CHONAVEL, S. SAOUDI, "A new successive interference cancellation structure for multipath asynchronous DS-CDMA channels", accepté pour la conférence SPAWC'01, Taiwan, mars 2001.
- [17] M. AMMAR, S. SAOUDI, Th. CHONAVEL, "A new SIC structure for multipath asynchronous Rayleigh channels", soumis à la conférence VTC'01, Tel'Aviv, 2001.
- [18] M. AMMAR, Th. CHONAVEL, S. SAOUDI, "A New Multi-Stage Successive Interference Cancellation Structure for Multipath Asynchronous DS-CDMA Signals", soumis à la conférence ICASSP'01, Salt Lake City, 2001.

Deuxième partie

Quelques problèmes de modélisation et d'estimation

Chapitre 1

Modélisation spectrale à bande limitée

Dans un certain nombre d'applications du traitement du signal, l'observation est constituée par des signaux discrets stationnaires au second ordre dont le spectre est porté par un sous intervalle de la bande de fréquences sélectionnée par l'échantillonnage, i.e de la bande $[-f_e/2, f_e/2]$ si f_e désigne la fréquence d'échantillonnage. C'est en particulier ce qui se produit en traitement d'antenne lorsqu'on considère une antenne linéaire de capteurs équidistants et que l'on étudie le spectre spatial de signaux de nature propagative dont la fréquence reste inférieure à $c/2d$, où c est la vitesse de propagation des ondes dans le milieu et d l'espacement entre les capteurs. Par ailleurs, dans le contexte du traitement d'antenne, on est souvent amené à étudier les propriétés spectrales des signaux à partir de la connaissance d'un nombre restreint de coefficients de corrélations estimés du spectre spatial. Ce genre de situation nous a conduit à envisager le problème plus général de l'estimation spectrale des processus à bande limitée.

Ce travail s'inscrit dans la continuité de mes travaux de thèse, avec une approche de la modélisation spectrale à bande limitée basée sur les propriétés des coefficients de réflexion. Cette étude a été réalisée avec Serge DEGERINE, Professeur au Laboratoire de Modélisation et Calcul (LMC) de l'Université de Grenoble. De plus, des travaux sur l'estimation de la bande passante du spectre d'un processus ainsi que sur la construction d'estimateurs spectraux à bande limitée sont menés en collaboration avec Philippe LOUBATON, Professeur à l'Université de Marne la Vallée.

1.1 Les coefficients de réflexion des spectres à bande limitée

L'étude des propriétés des coefficients de réflexion des spectres à bande limitée montre qu'ils appartiennent nécessairement à certains sous ensembles du disque unité. Ceci permet de considérer le problème de l'estimation spectrale à bande limitée de façon plus appropriée qu'en utilisant des approches basées sur l'extension directe d'une séquence de corrélations sous la contrainte de support du spectre. En particulier, on obtient une nouvelle technique d'extension spectrale à bande limitée à partir de la connaissance des $N + 1$ premiers coefficients de corrélation d'un processus à bande limitée. Cette extension peut être interprétée comme l'analogie à bande limitée du spectre du maximum d'entropie associé à une séquence de covariances.

Plus précisément, rappelons qu'une approche possible pour estimer le spectre d'un processus stationnaire au second ordre à valeurs complexes $X = (X_n)_{n \in \mathbb{Z}}$ consiste à rechercher un spectre dont les $N + 1$ premiers coefficients de corrélation correspondent à ceux de X , notés $r_n = \mathbb{E}[X_{n+k} X_k^*]$ ($n = 0, N$), ou d'estimées de ces grandeurs. Cela revient à caractériser une extension r_{N+1}, r_{N+2}, \dots de la séquence $(r_n)_{n=0, N}$. Mais on sait que la connaissance de $(r_n)_{n=0, N}$ est équivalente à celle de la séquence $(\beta_n)_{n=0, N}$ des $N + 1$ premiers coefficients de réflexion correspondants, avec la notation $\beta_0 = r_0$. Aussi, une façon équivalente d'envisager le problème consiste à chercher à étendre la séquence $(\beta_n)_{n=0, N}$ par des coefficients $\beta_{N+1}, \beta_{N+2}, \dots$. En particulier, l'extension du maximum d'entropie est obtenue pour $\beta_{N+k} = 0, \forall k \geq 1$.

On rencontre souvent les coefficients de réflexion dans la littérature. Ils sont également connus sous le nom de coefficients de Schur [6]. En particulier, il apparaît que les coefficients de Schur paramètrent une famille de polynômes orthogonaux sur le cercle unité [10]. Le terme coefficient de réflexion fait son apparition dans les années 1960 avec la redécouverte des coefficients de Schur par les géophysiciens dans le cadre de l'étude des interfaces entre les différentes strates du sous-sol. En statistiques, ils sont encore connus sous le nom de coefficients de corrélation partielle et sont largement utilisés. En traitement du signal, ils apparaissent notamment dans le cadre de l'algorithme de prédiction linéaire de Levinson [13] et jouent un rôle essentiel en estimation spectrale [11] et dans des applications telles que le traitement de la parole, en particulier pour les communications mobiles (système GSM¹). Ils ont également été mis en oeuvre pour l'estimation des matrices de Toeplitz [7][8]. Les travaux présentés ici se placent dans la continuité de l'étude du problème des moments trigonométriques sur un intervalle [2][3][4], et suivent l'approche initialement développée dans [9].

Rappelons tout d'abord que la séquence $(r_n)_{n=0, N}$ représente les premiers coefficients de corrélation d'un certain spectre porté par un sous intervalle $[-\Omega, \Omega]$ de $[-\pi, \pi[$ si et seulement si les matrices de Toeplitz Γ_N et Γ'_{N-1} , de tailles respectives $N + 1$ et N définies

1. Groupe Spécial Mobile, Global System for Mobile communications

par $(\Gamma_N)_{i,j} = r_{i-j}$ et $(\Gamma'_{N-1})_{i,j} = r'_{i-j}$, avec :

$$r'_n = r_{n+1} + r_{n-1} - 2r_n \cos(\Omega) \quad (n = 0, \pm 1, \dots, \pm(N-1)) \quad (1.1)$$

sont toutes deux positives [12].

L'ensemble des coefficients r_{N+1} tels que $(r_n)_{n=0,N+1}$ possède une extension à bande limitée est alors facile à caractériser [1]. Par une technique itérative, on pourrait théoriquement envisager de construire une extension à bande limitée de $(r_n)_{n=0,N}$. Malheureusement cela ne peut pas être fait en pratique à cause du mauvais conditionnement des matrices Γ_{N+k} et Γ'_{N+k-1} ($k \geq 1$) associées aux séquences $(r_n)_{n=0,N+k}$ [14]. De plus, la valeur de r_{N+k+1} ne peut pas être choisie indépendamment de $(r_n)_{n=0,N+k}$, ce qui empêche d'utiliser un procédé systématique d'extension de $(r_n)_{n=0,N}$.

Une autre approche consiste à étudier les coefficients de réflexion β_{N+k} ($k > 0$) tels que les séquences $(r_n)_{n=0,N+k}$ de coefficients de corrélation correspondantes possèdent des extensions à bande limitée. Afin d'y parvenir, on caractérise la condition de positivité des matrices Γ_N et Γ'_{N-1} en termes de séquences de coefficients de réflexion associées à $(r_n)_{n=0,N}$ et à $(r'_n)_{n=0,N-1}$, et on met en évidence des relations simples entre ces séquences. Il est alors facile de caractériser les coefficients β_{N+1} tels que la séquence $(r_n)_{n=0,N+1}$ correspondante admette une extension à bande limitée. On montre en particulier que l'ensemble des coefficients β_{N+1} admissibles coïncide avec l'intersection du disque unité avec un autre disque dont le rayon et le centre sont caractérisés de façon explicite par les paramètres de l'algorithme de Levinson associé à la séquence $(r'_n)_{n=0,N-1}$. Il est ainsi possible d'étendre itérativement la séquence des coefficients de réflexion en satisfaisant à la contrainte de support du spectre, tout en évitant les problèmes numériques évoqués dans le cas de l'extension directe d'une séquence de corrélations.

Rappelons maintenant que la caractérisation de l'ensemble des spectres de support $[-\Omega, \Omega]$ fixé dont les premiers coefficients de Fourier $(r_n)_{n=0,N}$ sont connus a été présentée dans [3]. Cependant, le problème du choix d'une solution particulière n'y avait pas été abordé. Lorsqu'on travaille avec les coefficients de réflexion, on peut construire une suite de coefficients $(\beta_{N+k})_{k>0}$ sans avoir à mettre en oeuvre d'algorithme itératif.

Ainsi, pour un choix particulier de β_{N+1} , la séquence $(\beta_{N+k})_{k>1}$ définie par :

$$\beta_{N+k} = (-1)^k e^{i\phi} \cos(\Omega/2) \quad (1.2)$$

(ϕ dépend de β_{N+1}), génère une extension positive à bande limitée. On peut alors calculer l'expression analytique de la densité spectrale de puissance $g(\omega)$ associée à cette extension. En notant $Q_{N+1}(z)$ le polynôme de Szegő de degré $N+1$ associé à la séquence $(\beta_n)_{n=0,N+1}$ fournie par l'algorithme de Levinson, σ_{N+1}^2 la variance correspondante de l'erreur de prédiction et $u(\omega) = (\cos(\omega/2))^{-1} \cos(\Omega/2)$, il vient que :

$$g(\omega) = \frac{\sigma_{N+1}^2 (1 - [u(\omega) - \sqrt{u(\omega)^2 - 1}]^2)}{|\tilde{Q}_{N+1}(e^{i\omega}) - e^{i\omega/2} e^{-i\phi} [u(\omega) - \sqrt{u(\omega)^2 - 1}] Q_{N+1}(e^{i\omega})|^2} \quad (\omega \in [-\Omega, \Omega]), \quad (1.3)$$

où $\tilde{Q}_{N+1}(z) = z^{N+1}[Q_{N+1}(1/z^*)]^*$.

Comme on a montré par ailleurs qu'aucune extension constante de $(\beta_n)_{n=0,N}$ n'est susceptible de représenter un spectre à bande limitée, il semble que l'approche proposée représente la façon la plus simple possible pour construire une extension à bande limitée de $(\beta_n)_{n=0,N}$. De plus, pour $\Omega = \pi$, $g(\omega)$ coïncide avec l'estimateur classique du maximum d'entropie ($\beta_{N+k} = 0$; $k \geq 1$). Aussi, l'extension proposée peut être vue comme une généralisation possible de la notion d'estimateur du maximum d'entropie au cas de spectres à bande limitée. Cette extension est d'autant plus intéressante que l'estimateur AR classique du maximum d'entropie ne converge pas pour les spectres à bande limitée puisqu'ils ne correspondent pas à des processus réguliers.

On a aussi montré la possibilité d'étendre simplement l'approche proposée au cas de spectres ayant des supports définis par une réunion finie d'intervalles quelconques.

Le problème de l'estimation spectrale à partir d'un processus X observé conduit à travailler avec des coefficients de corrélation estimés. Ce problème n'est pas traité dans cette étude. Il a fait l'objet de [2] et [4], et de l'article en cours de préparation présenté en annexe 2.

1.2 Travaux en cours

Outre le modèle de spectre particulier envisagé au paragraphe précédent, les travaux réalisés en fin de thèse et ultérieurement ont conduit à la mise en évidence de solutions particulières du problème des moments trigonométriques. Ces estimateurs ayant généralement de fortes oscillations dans la bande passante, on a envisagé d'autres approches telles que la généralisation de l'estimateur spectral de Capon au cas bande-limitée [1].

Par ailleurs, une idée simple pour effectuer la modélisation d'une densité spectrale à bande limitée consiste à régulariser le problème en ajoutant une constante strictement positive à r_0 . On peut alors construire simplement un estimateur qui satisfait aux contraintes de corrélations, vérifie avec une relativement bonne précision la contrainte de support et converge asymptotiquement vers le vrai spectre lorsque N tend vers l'infini. Pour N fini, il se peut cependant que le spectre obtenu ne soit pas positif en tout point. Des travaux en cours visent à prendre en compte ce problème. D'autres approches basées sur l'utilisation des coefficients de réflexion conjointement avec la contrainte de support sont à l'étude.

Dans le contexte de l'estimation spectrale à bande limitée le support du spectre du processus étudié peut constituer un des paramètres inconnus du problème. Partant du fait que si le spectre est à bande limitée de support Ω la matrice de Toeplitz $\Gamma'_{N-1}(\omega)$ de taille N et de terme général :

$$r'_n(\omega) = r_{n+1} + r_{n-1} - 2r_n \cos(\omega) \quad (n = 0, \pm 1, \dots, \pm(N-1)) \quad (1.4)$$

est positive pour $\omega \geq \Omega$ et non positive pour $\omega < \Omega$, on peut envisager d'effectuer un test de positivité de la matrice $\Gamma'_{N-1}(\omega)$ pour diverses valeurs de ω afin d'estimer le support du spectre. Notons que sous certaines hypothèses on peut estimer la matrice de covariance asymptotique de l'estimateur empirique de $\Gamma'_{N-1}(\omega)$ et obtenir un théorème central limite pour cet estimateur. Cela permet d'envisager la mise en oeuvre de tests statistiques classiques pour tester la positivité des matrices $\Gamma'_{N-1}(\omega)$.

Bibliographie

- [1] Th. CHONAVEL, *Estimation Spectrale à Bande Limitée. Application au Traitement d'Antennes*, thèse de doctorat, Paris, déc. 1992.
- [2] Th. CHONAVEL, Ph. LOUBATON, "Estimation of the Autocorrelation Coefficients of Band-Limited Signals, Application to the Sources Localization Problem", *EUSIPCO'92*, pp. 1825-1828, sept. 1992.
- [3] Th. CHONAVEL, Ph. LOUBATON, "Band-Limited Spectrum Estimation from Partial Covariance Sequence", *IEEE Trans. on Inf. Theory*, 39(4):1395-1397, juil. 1993.
- [4] Th. CHONAVEL, Ph. LOUBATON, "A Maximum Likelihood Estimation Algorithm for Autocovariance Matrices of Band-Limited Processes", *EUSIPCO'94*, sept. 1994.
- [5] Th. CHONAVEL, S. DEGERINE, "Reflection coefficients of Band-Limited spectra, and Band-Limited Spectrum estimation", *IEEE Trans. on Inf. Theory*, 45(6):2030-2038, sept 1999.
- [6] C. FOIAS, A.E. FRAZO, *The Commutant Lifting Approach to Interpolation Problems*, Birkhauser Verlag Berlin, 1990.
- [7] S. DEGERINE, "Maximum Likelihood Estimation of Autocovariance Matrices from Replicated Short Time Series", *J. Time Series Analysis*, 8(2):135-146, 1987.
- [8] S. DEGERINE, "On Local Maxima of the Likelihood Function for Toeplitz Matrix Estimation", *IEEE Trans. SP*, 40(6):1563-1565, june 1992.
- [9] S. DEGERINE, "Autocorrélations Partielles et Spectres à Bande Limitée", Technical Report n. RT 120, *LMC Institut d'Informatiques et de Mathématiques Appliquées de Grenoble*, jul.1994.
- [10] Y.L. GERONIMUS *Polynomials Orthogonal on the Unit Circle and Interval*, Pergamon Press, 1960.
- [11] S.M. KAY, *Modern Spectral Estimation*, Prentice Hall, Englewood Cliffs, N.J, 1988.
- [12] M.G. KREIN, A.A. NUDELMAN, *The Markov Moment Problem and Extremal Problems*, Transl. Math. Monographs, 50, A.M.S., Providence, Rhodes Island, 1977.
- [13] N. LEVINSON, "The Wiener RMS (root-mean-square) Error Criterion in Filter Design and Prediction", *J. Math. Phys.*, 25:261-278, 1947.
- [14] D. SLEPIAN, H.O. POLLAK, H.J. LANDAU, "Prolate Spheroidal Wave Functions", *Bell Syst. Tech. Journ.*, 40:43-84, janv. 1961.

Chapitre 2

Estimation de temps de retard

La caractérisation de canaux de propagation décrits par un modèle à trajets multiples constitue un problème très classique de traitement du signal. L'originalité du travail proposé réside dans la prise en compte d'une information *a priori* pour décrire les amplitudes des différents trajets et dans la mise en oeuvre de l'optimisation des critères bayésiens obtenus pour diverses structures de récepteurs.

Ce travail entre dans le cadre d'un programme régional qui s'est concrétisé à travers un contrat G2RA¹ réalisé en collaboration avec l'IFREMER² et l'EPSHOM³. L'étude de l'estimation de temps de retard en tomographie acoustique océanique fait actuellement l'objet de la thèse de Fabienne POREE. Compte tenu du fait que le problème de l'estimation de temps de retard dépasse largement le contexte de la tomographie acoustique océanique, on va le présenter dans un cadre général. L'approche MCMC a plus particulièrement été développée avec Olivier ROSEC dans le cadre de l'application de ses travaux de thèse au domaine de l'estimation de canaux multi-trajets au cours d'un travail post-doctoral.

2.1 Introduction

Le problème de l'estimation de temps de retard est très classique en traitement du signal. En effet, de nombreux problèmes peuvent être modélisés comme la sortie d'un filtre de réponse impulsionnelle $s(t)$ excité par une séquence $h(t)$ d'impulsions d'amplitudes et de positions (temporelles ou spatiales) inconnues, même si physiquement c'est plutôt le signal $s(t)$ qui est transmis à travers un filtre de réponse impulsionnelle $h(t)$. $h(t)$ peut s'écrire

1. Groupe de Recherche Régional en Acoustique

2. Institut Français de Recherche pour l'Exploitation de la MER

3. Etablissement Principal du service Hydrographique et Océanographique de la Marine

sous la forme :

$$h(t) = \sum_{k=1,p} \alpha_k \delta(t - \tau_k), \quad (2.1)$$

où les α_k et les τ_k représentent respectivement l'atténuation et le retard associés à la $k^{\text{ème}}$ impulsion. Le nombre p de composantes de $h(t)$ est généralement également inconnu.

Suivant le domaine d'application, la fonction $h(t)$ peut être, par exemple, une séquence de réflectivité que l'on cherche à estimer à partir de la connaissance d'une trace sismique [12] ou la réponse impulsionnelle d'un canal à trajets multiples en communications hertziennes [17] ou en communications sous-marines [22]. Dans le premier cas, la fonction $s(t)$ est généralement inconnue, tandis qu'en communications on est souvent en mesure d'utiliser une fonction $s(t)$ connue du récepteur. Dans ce cas, le problème de l'estimation des paramètres $(\alpha_k, \tau_k)_{k=1,p}$ et éventuellement du nombre p de trajets peut être envisagé avec des techniques classiques. La méthode du corrélateur qui consiste à effectuer un filtrage adapté à $s(t)$ au niveau du récepteur n'est généralement intéressante que dans la mesure où $s(t)$ possède une fonction d'autocorrélation étroite, ce qui permet une meilleure discrimination de pics de corrélation voisins. Un aspect souvent négligé du corrélateur est qu'il introduit généralement une corrélation entre les échantillons du bruit en sortie du filtrage adapté, même lorsque le bruit d'entrée est blanc.

Les critères de maximum de vraisemblance ont l'inconvénient de souvent présenter des maxima locaux et imposent une connaissance *a priori* sur le nombre de trajets observés. La complexité de l'optimisation de la vraisemblance peut être réduite par l'emploi de l'algorithme EM (Expectation-Maximisation, [4]). Lorsque le nombre de trajets n'est pas connu, on peut envisager l'emploi d'approches pénalisées pour lesquelles le maximum de la vraisemblance n'est plus une fonction croissante du nombre p de composantes de $h(t)$ [1][18]. Notons que les méthodes de Monte Carlo à sauts réversibles [7] permettent également de traiter ce type de problème.

Notons que pour le problème de suivi de paramètres de retard, on peut envisager des approches bayésiennes [13] qui conduisent à pénaliser la vraisemblance par un terme de distance entre les valeurs des retards estimés sur les traces précédentes et les valeurs de ces retards sur la trace courante. Cependant, cette méthode n'est performante que si le nombre de trajets est connu, si l'estimation initiale des retards est assez précise et si l'évolution des paramètres du canal reste assez régulière. Dans le cas de la tomographie acoustique océanique, les paramètres du canal sont souvent estimés à des intervalles de temps de plusieurs heures et le choix du facteur de pénalisation ne possède pas d'interprétation physique simple. La valeur de cette pénalisation influence la dispersion des retards estimés. Or, cette dispersion, qui est particulièrement informative pour les physiciens qui étudient la tomographie acoustique océanique, ne doit pas être lissée de façon arbitraire.

2.2 Les méthodes pénalisées

Une technique possible pour prendre en compte le caractère parcimonieux de la réponse $h(t)$, c'est-à-dire le fait que $h(t)$ est nulle pour la plupart des valeurs de t , consiste à rajouter un terme de pénalisation portant sur la norme de $h(t)$. Ainsi, dans la littérature, certains auteurs ont proposé des critères de la forme [21][14]

$$\phi(h) = \| r(t) - (h * s)(t) \|_a + \lambda \| h(t) \|_b, \quad (2.2)$$

où $\| \cdot \|_a$ et $\| \cdot \|_b$ sont des normes choisies de façon adéquate. Il est à noter que ce type de critère est généralement réécrit sous forme vectorielle : après discrétisation des signaux observés et de l'échelle des temps t , on obtient un critère à optimiser de la forme

$$\Phi = \| \mathbf{y} - \mathbf{S}\mathbf{g} \|_a + \lambda \| \mathbf{g} \|_b, \quad (2.3)$$

où \mathbf{S} est la matrice de convolution par le signal $s(t)$, \mathbf{y} le vecteur des observations échantillonnées et \mathbf{g} le vecteur des amplitudes des trajets. Les temps de retard correspondent aux indices des composantes non nulles du vecteur \mathbf{g} qui minimise le critère et les valeurs de ces composantes fournissent les amplitudes des trajets correspondants.

2.3 Détection quadratique

Une technique classiquement employée en pratique pour estimer les paramètres d'un canal de communications consiste à transmettre une séquence pseudo-aléatoire $s(t)$, encore appelée SBLM (Séquence Binaire de Longueur Maximale, voir par exemple [17]) avec une modulation MDP2 (Modulation De Phase à deux états) et à utiliser un détecteur quadratique constitué d'un démodulateur en phase et en quadrature suivi de filtres adaptés de réponse impulsionnelle $\tilde{s}(t) = s(-t)$ et d'une somme quadratique des signaux (cf figure 2.1). Cette procédure permet d'éliminer l'ambiguïté concernant la phase des signaux reçus.

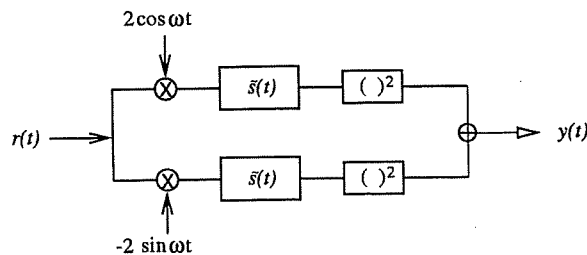


FIG. 2.1 - Détecteur quadratique.

Lorsque cette phase évolue de façon significative au cours de la transmission du signal, il faudrait envisager un traitement approprié. En présence d'effet Doppler, une technique

simple consisterait à utiliser une batterie de filtres adaptés aux signaux distordus pour différentes valeurs de la fréquence Doppler et de retrouver des pics de corrélation dans le plan temps-fréquence Doppler. Une approche pénalisée pour la résolution de ce type de problème [6] pourrait être adaptée au contexte de notre étude.

En négligeant les termes d'interférence liés aux transformations non linéaires dans le détecteur quadratique et en faisant une hypothèse gaussienne sur l'amplitude du bruit, ainsi que sur celles des trajets sur les voies en phase et en quadrature, on obtient en sortie du détecteur quadratique des contributions relatives au bruit et aux amplitudes des trajets retardés qui suivent des lois exponentielles. En négligeant les termes d'interférence entre le signal et le bruit et les termes d'interférences entre pics proches, on est alors conduit à un critère à optimiser de la forme (2.2) avec $a = b = 1$. Notons qu'ici le paramètre de pénalisation λ s'interprète comme l'inverse d'un rapport signal à bruit dont la valeur peut souvent être déterminée simplement. L'interférence entre le signal utile et le bruit peut être négligée lorsque le rapport signal à bruit est élevé, et les termes d'interférences entre trajets distincts peuvent être négligés lorsqu'on travaille avec des SBLM et que l'écart minimum entre pics de corrélation reste de l'ordre de la durée symbole.

La première approximation est assez bien vérifiée en pratique par le fait que l'autocorrélation très étroite de la SBLM conduit à un rapport signal à bruit élevé en sortie des filtres adaptés. La deuxième hypothèse peut ne pas être vérifiée pour certains trajets. Dans ce cas, la méthode proposée pourra conduire à des erreurs de détection et d'estimation des temps de retard. Elle n'en fournit pas moins des résultats pratiques relativement cohérents, à la différence des méthodes de maximum de vraisemblance pour lesquelles une mauvaise initialisation peut facilement conduire à des résultats absurdes. Par contre, une bonne initialisation des méthodes de maximum de vraisemblance fournira des estimateurs performants en terme de biais et de variance. Aussi, la sortie du détecteur quadratique pourra être utilisée avec la méthode bayésienne proposée pour une initialisation rapide et relativement précise (et même très précise pour des pics suffisamment espacés) d'une méthode de type maximum de vraisemblance. Elle pourra également être utilisée pour initialiser une technique d'estimation au sens du maximum *a posteriori* telle que celle décrite dans le paragraphe suivant.

2.4 Approches MCMC pour l'estimation de temps de retard

Afin de s'affranchir des limitations liées à une estimation des paramètres de canal en sortie du détecteur quadratique, et qui sont dues à l'approximation linéaire effectuée, on peut chercher à estimer les paramètres en sortie des filtres adaptés, lorsque ces sorties sont accessibles à l'observateur. On doit alors traiter une information à valeurs complexes dont les parties réelle et imaginaire sont fournies par les sorties des voies en phase et en quadrature respectivement. Ce traitement peut s'effectuer par des approches de type

maximum de vraisemblance, mais comme on l'a déjà indiqué cela pose des problèmes du fait de l'existence d'optima locaux de la vraisemblance et de la nécessité de connaître *a priori* le nombre de trajets.

Un critère d'optimisation à notre avis plus souple et qui permet d'incorporer de l'information *a priori* relativement aux dynamiques respectives des amplitudes des trajets du canal et du bruit serait fourni par une approche bayésienne [19]. Mais il est bien connu que les méthodes bayésiennes conduisent souvent à des critères difficiles à optimiser, pour les estimateurs de type MAP (Maximum A Posteriori), ou à des expressions intégrales difficiles à évaluer, pour des estimateurs du type espérance *a posteriori*.

Dans ces situations les méthodes de simulation par Monte Carlo s'avèrent être particulièrement bien adaptées à la résolution de problèmes complexes d'optimisation ou d'intégration, grâce à la simulation de variables aléatoires. Les potentialités de ce type de technique en estimation bayésienne et en particulier pour certains problèmes importants de traitement du signal tels que le filtrage dans les modèles d'état ou l'estimation de fonctions de transfert (voir par exemple la thèse de A. Doucet [3]) se sont trouvées considérablement renforcées avec l'utilisation des algorithmes de Hastings-Metropolis et de Gibbs, utilisation rendue possible par la diffusion d'outils informatiques puissants. Ces algorithmes de type MCMC (Monte Carlo Markov Chain) [20] conduisent à la génération de variables aléatoires suivant les lois d'intérêt. Ces simulations ne pourraient que difficilement être envisagées avec les techniques de génération de variables aléatoires indépendantes telles que la méthode d'acceptation-rejet.

2.4.1 Une première approche

En faisant appel aux outils précédents, on peut envisager le problème de l'estimation bayésienne des paramètres du canal à partir de l'observation des sorties des filtres adaptés et avec un modèle gaussien pour les amplitudes des trajets en sortie de ces filtres. L'observation est alors modélisée suivant une approche classique en déconvolution aveugle (en particulier en traitement des données sismiques) et qui conduit ici à voir l'observation comme la sortie d'un filtre dont la réponse impulsionnelle est l'autocorrélation du signal émis, excité par un bruit blanc dont la loi à chaque instant, de densité notée g , est un mélange de deux lois gaussiennes centrées :

$$g \sim \nu \mathcal{N}(0, \sigma_1^2) + (1 - \nu) \mathcal{N}(0, \sigma_2^2). \quad (2.4)$$

Le signal d'excitation modélise le fait qu'à chaque instant l'entrée correspond soit à l'amplitude de la composante retardée du signal transmis arrivant à cet instant, soit à du bruit. Dans ce modèle, la puissance moyenne des amplitudes du signal utile (σ_1^2) et du bruit (σ_2^2) peuvent ou non être connues *a priori*. Dans ce dernier cas, on se donne en général une loi pour ces paramètres de type inverse gamma. De même, le paramètre ν qui représente la fréquence d'apparition de trajets retardés sur la durée de l'observation peut

être vu comme un paramètre aléatoire lorsqu'il est inconnu ; on utilise alors en général une loi béta pour le modéliser.

La modélisation précédente conduit à décrire le bruit d'excitation du modèle comme une chaîne de Markov cachée à deux états dont l'état à un instant donné indique si oui ou non un trajet arrive à l'instant considéré. La suite des états de la chaîne constitue un paramètre dont l'estimation représente une décision quant aux instants d'arrivée des signaux retardés.

L'utilisation d'un algorithme de Gibbs est souvent indiquée pour les modèles en données cachées. Une estimation des paramètres réalisée avec l'utilisation de cet algorithme a mis en évidence la possibilité d'un traitement efficace en sortie des filtres adaptés et les améliorations apportées par rapport à l'étude de la sortie du détecteur quadratique.

Notons que lorsque seule la sortie du détecteur quadratique est accessible à l'observateur, on pourrait envisager de décrire l'observation au moyen d'un modèle d'état non linéaire excité par un bruit blanc dont la loi est un mélange de lois gaussiennes et chercher à estimer les paramètres du modèle au moyen d'une approche de type MCMC.

2.4.2 Travaux en cours : prise en compte de la corrélation du bruit

Dans cette section, on présente rapidement les travaux actuels visant à enrichir le modèle précédent. La plupart des techniques de déconvolution utilisant un filtrage adapté ne prennent pas en compte le fait qu'un bruit blanc devient corrélé en sortie d'un filtre adapté.

Notons $\tilde{\mathbf{s}}$ et \mathbf{g} respectivement la réponse impulsionnelle du filtre adapté échantillonné et son autocorrélation échantillonnée. Les signaux utiles et le bruit peuvent être décrits simplement au moyen de modèles d'états de la forme

$$\left\{ \begin{array}{l} \mathbf{X}_{s,n} = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 1 & 0 & 0 \\ 0 & \cdot & \cdot & 1 & 0 \end{pmatrix} \mathbf{X}_{s,n-1} + \begin{pmatrix} V_n \\ 0 \\ \cdot \\ 0 \end{pmatrix} \\ Y_{s,n} = \mathbf{z}^T \mathbf{X}_{s,n} \end{array} \right. \quad (2.5)$$

pour le signal, où $V = (V_n)_{n=1,T}$ est un processus Bernoulli-Gaussien, et

$$\begin{cases} \mathbf{X}_{b,n} = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 1 & 0 & 0 \\ 0 & \cdot & \cdot & 1 & 0 \end{pmatrix} \mathbf{X}_{s,n-1} + \begin{pmatrix} W_n \\ 0 \\ \cdot \\ 0 \end{pmatrix} \\ Y_{s,n} = \tilde{\mathbf{s}}^T \mathbf{X}_{b,n} \end{cases} \quad (2.6)$$

pour le bruit, où W est un bruit blanc gaussien. La taille des vecteurs d'état \mathbf{X}_s et \mathbf{X}_b est égale respectivement à la longueur de l'autocorrélation de la SBLM échantillonnée et au nombre d'échantillons par symbole. En concaténant ces deux modèles d'état, on obtient un modèle d'état global qui intègre à la fois les paramètres du signal utile et ceux du bruit. Ce modèle simple pourra ensuite être estimé suivant une approche MCMC.

Notons que lorsque les signaux utilisés sont des SBLM, le bruit en sortie de filtre adapté est décorrélé si on effectue un échantillonnage à la cadence symbole, notée T_S . Lorsque les signaux sont échantillonnés avec une période T_S/k , la longueur de corrélation est égale au facteur de suréchantillonnage k . Comme l'autocorrélation d'une SBLM est une fonction triangulaire de support égal à deux fois la durée symbole, il apparaît que cette autocorrélation est aussi celle d'un signal constant de durée T_S . Ceci permet de remplacer $\tilde{\mathbf{s}}^T$ dans (2.6) par un vecteur de la forme $[1, \dots, 1]$ et de longueur k , ce qui permet de réduire de façon très significative la taille du vecteur d'état $\mathbf{X}_{b,n}$.

De plus, il apparaît dans de nombreuses applications, et en particulier en tomographie, que le filtrage adapté utilisé n'intègre pas les distorsions linéaires supplémentaires introduites par les transducteurs d'émission et de réception des dispositifs utilisés. On se rapproche alors d'un problème de déconvolution aveugle que l'on pourra encore résoudre par une approche MCMC au prix d'une augmentation de complexité du traitement. Il faudra alors voir le signal utile \mathbf{s} comme un signal inconnu dont les coefficients seront modélisés dans l'approche bayésienne comme des grandeurs aléatoires qui interviendront dans l'équation d'observation du modèle d'état. Lorsque les distorsions introduites restent faibles, on pourra cependant espérer une convergence rapide des algorithmes en initialisant les paramètres du modèle d'état par les valeurs qu'ils prennent en l'absence de distorsion.

Bibliographie

- [1] H. AKAIKE, "A new look at the statistical model identification", *IEEE Trans. AC*, 19:716:723, 1974.
- [2] I. BARRODALE, F.D.K. ROBERTS, "An improved algorithm for discrete l_1 linear approximations". *SIAM Numer. Anal.*, 10(5):839:848, Oct. 1973.
- [3] A. DOUCET, *Algorithmes de Monte-Carlo pour l'Estimation Bayésienne de Modèles Markovien Cachés. Application au traitement des Signaux de Rayonnement*, Thèse de Doctorat, n. 97PA112347, Paris 11, déc. 1997.
- [4] M. FEDER, E. WENSTEIN, "Parameters estimation of superimposed signals using the EM algorithm", *IEEE Trans. on ASSP*, 36(4):477-489, 1988.
- [5] J.J. FUCHS, "Multipath Time Delay Detection and Estimation", *IEEE Trans. SP*, 47(1):237-243, Janv. 1999.
- [6] J.J. FUCHS, "Détection d'une cible mobile en présence de fouillis à l'aide d'un radar très large bande", *GRETSI'99* pp. 531-534, sept. 1999.
- [7] P.J. GREEN, *Reversible Jump Markov Chain Monte Carlo Computation and Bayesian Model Determination*, *Biometrika*, 82(4):711-732, 1995.
- [8] M. LAVIELLE, "Bayesian deconvolution of Bernoulli-Gaussian processes", *Signal Processing*, 33:67-79, 1993.
- [9] M. LAVIELLE, "A stochastic algorithm for parametric and non-parametric estimation in the case of incomplete data", *Signal Processing*, 42:3-17, 1995.
- [10] TZE FEN LI, "Multipath Time delay Estimation Using regression Stepwise Procedure", *IEEE Trans. SP*, 46(1):191-195, janv. 1998.
- [11] The Numerical Algorithms Group Limited, *The Man Fortran Library Manual*, Mark 14, 1^{ère} édition, avril 1990.
- [12] J. MENDEL, *Maximum likelihood deconvolution : a journey into model-based signal processing*, Springer-Verlag, 1990.
- [13] V. NIMIER, G. JOURDAIN, "Active estimation of a multipath propagation channel with a bayesian strategy", *Traitement du Signal*, 10(3):201-213, 1993.
- [14] M. O'BRIEN, A.N. SINCLAIR, S. KRAMER, "Recovery of a sparse spike train series by l_1 norm deconvolution", *IEEE Trans. SP*, 42(12):3353-3365, déc. 1994.
- [15] F. POREE, T. CHONAVEL, T. TERRE, "Multipath time delay detection and estimation for ocean acoustic tomography : a bayesian approach", *Oceans'99 MTS/IEEE Proceedings*, pp. 1587-1590, 1999.

- [16] F. POREE, O. ROSEC, T. CHONAVEL, J.M. BOUCHER, "Two bayesian methods for multipath propagation parameters estimation", *ICASSP'2000*.
- [17] J.G. PROAKIS, *Digital Communications*, Mc Graw-hill, 1989.
- [18] J.RISSANEN, "Modeling by the shortest data description", *Automatica*, 14:465-471, 1978.
- [19] C.P. ROBERT, *The bayesian choice*, Springer-Verlag, 1994.
- [20] C.P. ROBERT, *Les méthodes de Monte-Carlo par chaînes de Markov*, Economica, 1996.
- [21] H. TAYLOR, S. BANKS, F.Mc COY, "deconvolution with the l_1 norm", *Geophys.*, 44(1):39-52, 1979.
- [22] C. WUNSH, W. MUNK, P. WORCESTER, *Ocean acoustic tomography*, Cambridge Univ. press, 1995.

Chapitre 3

Décomposition adaptative en valeurs propres

L'estimation de la position de sources en traitement d'antenne et plus généralement les techniques de traitement du signal qui font appel à la décomposition en valeurs propres de matrices de covariance ont longtemps été limitées pour les applications en temps réel, du fait des faibles performances en termes de coût de calcul des algorithmes adaptatifs existants pour effectuer cette décomposition. Au cours des dix dernières années de nombreux travaux ont contribué à remédier à ce problème, parmi lesquels s'inscrivent les résultats développés dans ce chapitre. Une originalité de ce travail a certainement été d'envisager l'estimation adaptative de la décomposition en valeurs propres des matrices de covariance au moyen d'une optimisation stochastique du critère de maximum de vraisemblance.

Cette étude, en partie réalisée dans le cadre de la thèse de Christian RIOU, s'est poursuivie en collaboration avec Benoît CHAMPAGNE, Professeur à l'Université de Mac Gill de Montréal.

3.1 Introduction

Soit $X = (X_n)_{n \in \mathbb{Z}}$ un processus aléatoire gaussien complexe circulaire. On suppose que X_n est à valeurs dans \mathbb{C}^N . La décomposition en valeurs propres de la matrice de covariance $R_{X,n} = \mathbb{E}[X_n X_n^H]$, ou du moins la recherche de certains de ses sous espaces propres est un problème qui intervient de façon classique dans de nombreuses applications de traitement des signaux multivariés.

L'espace vectoriel engendré par les vecteurs propres associés aux plus grandes valeurs propres et celui correspondant aux plus petites valeurs propres de la matrice $R_{X,n}$ sont connus sous le nom de sous espace signal et de sous espace bruit. La caractérisation de ces

espaces a largement été envisagée dans le cadre d'applications telles que la localisation de sources en traitement d'antenne (algorithme MUSIC, [5] [18]), la quantification vectorielle [12], l'identification aveugle de réponses impulsionnelles de canaux de transmission [15], ou encore le suivi de courbes dans des images (algorithmes SLIDE [1] et SALE [16]).

En pratique, les sous espaces propres d'intérêt sont calculés par une décomposition en valeurs propres d'un estimateur de la matrice de covariance des observations, ou au moyen d'une décomposition en valeurs singulières appliquée à une matrice dans laquelle sont stockés les vecteurs d'observation.

Lorsque le processus X est stationnaire au second ordre, la matrice de covariance $R_{X,n}$ est constante ($R_{X,n} = R_X$) et elle peut être estimée facilement au moyen de n'importe quelle technique classique telle que l'estimation empirique, qui fournit l'estimateur $\hat{R}_{X,n} = L^{-1} \sum_{k=1,L} X_k X_k^H$. La décomposition en valeurs propres peut alors être obtenue par l'emploi d'algorithmes standards tels que la méthode de l'itération orthogonale (e.g. [13]).

Mais, dans de nombreuses applications telles que le suivi de sources mobiles ou l'identification de canaux de propagation non stationnaires, la matrice de covariance $R_{X,n}$ varie au cours du temps et les algorithmes évoqués précédemment ne peuvent plus être utilisés. Dans ce cas, différentes approches ont été suggérées pour suivre la décomposition en valeurs propres instantanée de $R_{X,n}$. A ce sujet, on pourra voir [8] pour une très bonne synthèse des méthodes d'estimation adaptative de sous espaces antérieures à 1990, et [10] pour la description d'algorithmes plus récents.

Dans ce domaine, un premier type d'approche consiste à adapter des techniques classiques au contexte non stationnaire [11]. En général, on se contente de réactualiser de façon adaptative la matrice de covariance des observations par

$$\hat{R}_{X,n} = (1 - \mu)\hat{R}_{X,n-1} + \mu \mathbf{x}_n \mathbf{x}_n^H, \quad (3.1)$$

où $0 < \mu \ll 1$ et où \mathbf{x}_n représente le vecteur d'observation à l'instant n , puis on effectue une seule itération d'une technique classique de décomposition en valeurs propres. Cette itération met souvent en oeuvre une décomposition QR, avec un coût de calcul en $O(NK^2)$ opérations, où K est la dimension du sous espace recherché. Ce genre de technique a également été proposé pour la réalisation de décompositions en valeurs singulières adaptatives dans [14].

Une autre approche consiste à remplacer directement dans l'équation (3.1) l'estimée de $R_{X,n}$ par sa décomposition en valeurs propres ([9],[6]). Les sous espaces recherchés sont alors actualisés au moyen de techniques qui mettent en oeuvre des matrices de réflexion de Householder [9] ou de rotations de Givens [6]. Avec de telles paramétrisations des termes d'actualisation des matrices de vecteurs propres, on parvient généralement à assurer l'orthonormalité de la suite des bases de sous espaces estimées.

Notons que dans [9] les sous espace signal et bruit sont moyennés. On est alors conduit

à des bases des sous espaces signal et bruit, mais pas à la connaissance exacte de la décomposition propre, tandis que dans [6] on peut accéder à la décomposition propre complète. Ces méthodes requièrent environ $O(NK)$ opérations.

Un troisième type de méthodes est basé sur l'estimation adaptative d'un critère, éventuellement d'un critère contraint, au moyen d'un algorithme du type gradient stochastique. A ce sujet, on pourra voir [17], ou [20] [3] pour des approches de type RLS. Le coût de calcul de ces méthodes varie en général de $O(NK)$ à $O(NK^2)$, ou $O(N^2K)$.

3.2 L'algorithme MALASE

Certaines des méthodes précédentes fournissent une base de l'espace signal ou de l'espace bruit tandis que d'autres fournissent une décomposition propre complète. De nombreuses applications utilisent un projecteur sur l'espace signal ou sur l'espace bruit, et il importe souvent qu'une orthonormalité parfaite de la base de vecteurs obtenue soit assurée. En particulier, des déviations par rapport à l'orthonormalité conduisent à une dégradation des performances de l'algorithme root-MUSIC en terme de capacité de détection et de performance d'estimation [4].

Un autre intérêt, à notre avis essentiel, à la prise en compte de la contrainte de l'orthonormalité réside dans le fait que certains critères ne peuvent pas conduire de façon directe à des algorithmes stochastiques qui offrent des propriétés de convergence satisfaisante. C'est en particulier le cas du critère de maximum de vraisemblance. On a pu montrer que la prise en compte de la contrainte d'orthonormalité permet d'obtenir un algorithme adaptatif pour la décomposition en valeurs propres d'une matrice de covariance basé sur le critère du maximum de vraisemblance.

Cette approche conduit à une décomposition en valeurs propres complète, et en particulier à la connaissance des valeurs propres. Ceci est intéressant en particulier lorsqu'on souhaite suivre l'évolution du rang du sous espace signal ou du sous espace bruit par la mise en oeuvre de critères statistiques adaptés à la résolution de ce problème (e.g. [1], [19]).

Dans l'article présenté en annexe on montre que l'algorithme que nous proposons, et que nous avons nommé MALASE (MAXimum Likelihood Adaptive Subspace Estimation), possède un certain nombre de propriétés intéressantes. A la différence des travaux antérieurs, l'approche que l'on envisage est basée sur le critère statistique pertinent du maximum de vraisemblance gaussien. Ce type de critère est en effet justifié dans un contexte d'estimation statistique, même si les propriétés de variance minimale des estimateurs du maximum de vraisemblance sont des propriétés asymptotiques et que l'on est ici dans un contexte d'estimation adaptative. Le critère proposé permet d'effectuer une décomposition propre complète avec un coût de calcul en $O(N^2)$ à chaque itération pour une décomposition en valeurs propres complète. Si on se limite à la recherche d'un sous espace de taille K , on

peut réduire ce coût de calcul à $O(NK)$, en effectuant au besoin un lissage des valeurs propres du sous espace supplémentaire de dimension $N - K$.

On a mis en évidence les liens très étroits de l'algorithme MALASE avec la famille des algorithmes PROTEUS qui sont basés sur l'utilisation de rotations de Givens pour réactualiser les vecteurs propres. Enfin, on a montré que les seuls points stationnaires stables de l'algorithme MALASE représentent les décompositions en valeurs propres de la matrice de covariance des observations et on a pu montrer que la variance asymptotique des différents paramètres était identique, et égale à $\mu/2$.

La présentation générale de l'algorithme MALASE est donnée en annexe 3, et ses propriétés de convergence sont plus particulièrement détaillées en annexe 4.

3.3 Travaux en cours

La convergence des algorithmes PROTEUS a déjà été envisagée dans [7]. Comme MALASE peut être réécrit sous une forme assez proche de celle utilisée pour ces algorithmes, on cherchera à adapter l'étude de la variance asymptotique de MALASE pour étudier celle des algorithmes PROTEUS.

L'obtention de méthodes performantes en $O(NK)$ a constitué une avancée significative dans le domaine des techniques de décomposition adaptative en valeurs propres (ou en valeurs singulières) au cours de ces dernières années. Cependant, dans ce domaine, il demeure important de chercher à encore améliorer ces résultats.

La thèse récente de Philippe PANGO [16] fournit une première réponse à cette question. Il montre que les rotations de Givens mises en oeuvre pour effectuer une décomposition adaptative en valeurs singulières par des méthodes de type QR-Jacobi n'ont pas toutes la même utilité pour la réactualisation des sous espaces propres d'intérêt. Partant de ce constat, il élabore des stratégies pour limiter le nombre de rotations de Givens à mettre en oeuvre et parvient à réduire de façon très significative leur nombre à chaque itération.

Outre les nombreuses perspectives qu'offrent ce travail, mes travaux actuels avec Benoît CHAMPAGNE, Professeur à l'Université Mac Gill de Montréal, visent à exploiter la structure Toeplitz des matrices de covariances construites à partir de processus stationnaires en vue d'une réduction significative du coût de calcul, en exploitant en particulier le fait que la base des vecteurs propres tend à correspondre à la base de Fourier lorsque la taille de la matrice de covariance croît.

Bibliographie

- [1] H.K. AGHAJAN, T. KAILATH, "Sensor array processing techniques for super resolution multi-line fitting and straight-edge detection", *IEEE trans. on Image Processing*, vol.2, N.4, pp.454-465, oct. 1993.
- [2] P.A. PANGO, B. CHAMPAGNE, "Subspace-based line and curve extraction from noisy images", *Proc. Canadian Conf. on Electrical and Comp. Eng.*, Edmonton, Alberta, May 1999.
- [3] K. ABED-MERAIM, A. CHKEIF, Y. HUA, "Un algorithme PAST orthogonal rapide pour l'estimation et la poursuite de sous espaces", *GRETSI'99*, pp. 231-234, sept. 1999.
- [4] A. BARBARELL, "Improving the resolution performance of eigenstructure-based direction finding algorithms", *ICASSP'83*, pp.336:339, 1983.
- [5] G. BIENVENU, L. KOPP, "Optimality of high resolution array using the eigensystem approach", *IEEE Trans. on ASSP*, 31:1235-1247, oct. 1983.
- [6] B. CHAMPAGNE, Q.G. LIU, "Plane rotation based EVD updating schemes for efficient subspace tracking", *IEEE Trans. on SP*, 46(7):1886-1900, juil. 1998.
- [7] B. CHAMPAGNE, "On the asymptotic convergence and numerical stability of the PROTEUS EVD TRACKERS", *IEEE Trans. on SP*, 48(1):242-246, jan. 2000.
- [8] P. COMMON, G.H. GOLUB, "Tracking a few extreme singular values and vectors in signal processing", *proc. IEEE*, 78:1327:1343, août 1990.
- [9] R.D. DeGroat, "Noniterative subspace tracking", *IEEE Trans. on SP*, 40(3):571:577, mars 1992.
- [10] R.D. DeGroat, E.M. DOWLING, D.A. LINEBARGER, *Subspace tracking, in : The Digital Signal Processing Handbook*, CRC Press, V.K. Madisetti, D.B. Williams Eds., 1998.
- [11] S. ERLICH, R. YAO, "Convergence of adaptive block simultaneous iteration method for eigenstructure decomposition", *Signal Processing*, 37:1-13, janv. 1994.
- [12] A. GERSHO, R.M. gray, *Vector Quantization and Signal Compression*, Kluwer academic Publishers, 1995.
- [13] G.H. GOLUB, C.F. VAN LOAN, *Matrix Computation - Second Edition*, John Hopkins Series in Mathematical Science, John Hopkins Univ. Press, 1990.
- [14] M. MOONEN, VAN DOOREN, J. VANDEWALLE, "A singular value decomposition algorithm for subspace tracking", *SIAM Matrix Anal. Appl.*, 13:1015-1038, oct. 1992.

- [15] E. MOULINES, P. DUHAMEL, J.F. CARDOSO, S. MAYRARGUE, "Subspace methods for the blind identification of multichannel FIR filters", *IEEE Trans. on Signal Processing*, 43(2):516-525, fev. 1995.
- [16] P.A. PANGO, *Algorithmes rapides de suivi de sous-espaces basés sur la SVD et applications en traitement d'antenne et d'images*, Thèse de doctorat, INRS, Montréal, février 2000.
- [17] P.A. Regalia, "An adaptive unit norm filter with application to signal analysis and Karhunen-Loeve transformations", *IEEE Trans. on CAS*, 37(5):646-649, mai 1990.
- [18] R.O. SCHMIDT, "Multiple emitter location and signal parameter estimation", *IEEE Trans. on Antenna Propagation*, AP-34:276-280, mars 1986.
- [19] "Modelling by the shortest data description", *Automatica*, 14:465:471, 1978.
- [20] B. YANG, "Subspace tracking based on the projection approach and the recursive least square method", *ICASSP'93*, pp. 143-148, 1993.

Chapitre 4

Estimation de supports de lois de probabilité dans le domaine de Fourier

La modélisation du trafic dans les réseaux de communications est essentielle en vue de permettre une allocation satisfaisante du débit disponible entre les différents utilisateurs. Un des mérites du travail dans lequel s'inscrit l'étude présentée ci dessous est d'aboutir à une modélisation réaliste et exploitable du trafic.

Plus précisément, ce travail entre dans le cadre d'une thèse relative à la modélisation du trafic dans les réseaux de communications à large bande. Cette modélisation des réseaux a été menée par Sandrine VATON dans le cadre de sa thèse de doctorat, réalisée en co-encadrement avec le département Signal de l'ENST. Ma contribution à ce travail concerne essentiellement la participation à l'étude du problème statistique de l'estimation de paramètres de support pour un mélange de lois, dans le cadre d'une modélisation markovienne du trafic. Après un bref rappel de la problématique générale de la thèse, on présentera plus particulièrement ce problème.

4.1 Introduction : les modèles markoviens en télétrafic

Une des évolutions marquantes de ces dernières décennies est certainement le développement des possibilités d'échange d'informations entre les différents agents de la société contemporaine. Ce flux croissant d'informations est supporté par un réseau de télécommunications qui, s'il est bien dimensionné, est quasiment invisible à l'utilisateur.

La science du télétrafic a précisément pour objectif l'étude et le dimensionnement des réseaux de télécommunications. Cette discipline a connu son premier développement au début du siècle avec le déploiement du réseau téléphonique. Erlang [1] a eu l'intuition de la modélisation poissonnienne du trafic téléphonique et a établi les premières formules de dimensionnement pour le réseau téléphonique.

Lors de l'apparition des réseaux informatiques le modèle de Poisson utilisé jusqu'alors s'est avéré caduque, car il est apparu que le trafic informatique était beaucoup plus sporadique que le trafic téléphonique.

Au début des années 1990 la science du télétrafic a connu une nouvelle évolution : deux équipes ont montré (l'une des Bell Lawrence Laboratories [2] [4][5] et l'autre de l'université de Berkeley [6]) que le trafic circulant sur les réseaux large-bande est à dépendance longue, c'est à dire qu'il est corrélé sur des horizons de temps très long.

4.2 Une nouvelle approche

La théorie des files d'attente a été très largement développée dans le cadre des modèles markoviens ; en revanche très peu de résultats ont été établis en ce qui concerne le comportement des trafics à dépendance longue dans une file d'attente. Il est assez simple d'envisager une politique d'allocation des ressources d'un réseau quand le trafic peut être décrit par un modèle markovien. De plus la littérature concernant l'estimation des paramètres de modèles markoviens est abondante. Partant de ces deux constats on peut se demander s'il ne serait pas envisageable d'adapter les modèles markoviens au trafic large-bande.

Les techniques de caractérisation de trafic développées dans [10] reposent sur une modélisation par chaîne de Markov cachée quasi-décomposable, c'est-à-dire dont le comportement commute de façon markovienne sur les échelles de temps longues entre différents types de comportements markoviens cachés, valides sur des horizons de temps plus courts.

La structure quasi-décomposable du modèle proposé permet d'expliquer le comportement particulier du trafic, qui réside dans une invariance des propriétés statistiques du signal de trafic lors de changements de l'échelle à laquelle est effectuée l'observation. Notons que cette propriété est connue sous le nom d'autosimilarité ; et les processus autosimilaires sont à dépendance longue [3]. Les non stationnarités détectées dans le trafic sont une seconde explication pour les phénomènes observés par de nombreux auteurs sur les indices visuels de dépendance longue.

Une analyse qualitative des données de trafic étudiées amène à proposer de nouveaux modèles markoviens pour décrire la suite des temps entre arrivées de paquets. En particulier, un nouveau modèle, baptisé SEHMM (Shifted Exponential Hidden Markov Model),

permet d'intégrer la présence de délais entre les arrivées des paquets d'information.

4.3 L'estimation d'un mélange de lois décalées

Dans le contexte précédent, l'intérêt de l'estimation des paramètres de décalage d'un mélange de lois est apparu avec la mise en évidence du modèle SEHMM. En fait ce type de modèle représente une amélioration par rapport aux modèles MMPP (Markov Modulated Poisson Process) par l'incorporation de décalages dans la description des lois des inter-arrivées des paquets d'information. Cela permet de considérer des lois à décroissance exponentielle dont la moyenne est supérieure à l'écart type. En fait, les paramètres de décalages permettent de modéliser des délais incompressibles entre instants d'arrivée des paquets dus à la réémission des paquets lorsque l'émetteur ne reçoit pas de message de confirmation de réception après une certaine durée prédéterminée. La connaissance des paramètres du modèle est très importante en télétrafic. Dans le cas des modèles SEHMM, les supports des lois conditionnelles du modèle Markovien doivent être estimés. Comme on va le voir, cette particularité conduit à certaines difficultés que l'on a étudiées et pour lesquelles on a proposé des solutions [9].

L'algorithme EM constitue une approche essentielle pour l'estimation de paramètres dans le cadre de problèmes en données incomplètes. Cependant, l'algorithme EM souffre de quelques limitations sérieuses telles que la convergence possible vers des optima locaux de la fonction de vraisemblance. Afin d'éviter ce genre de problème, des versions stochastiques de l'algorithme EM ont été proposées.

Il nous est apparu que cet algorithme possède une autre limitation, que ses versions stochastiques ne peuvent pas non plus prendre en compte de façon satisfaisante. Pour illustrer ce problème, considérons une chaîne de Markov cachée dont les lois des observations conditionnellement aux états sont des lois exponentielles décalées, et que ces paramètres de décalage sont inconnus. La surface de vraisemblance associée à un tel problème présente une discontinuité en toute valeur de décalage égale à une des valeurs observées, ce qui rend la surface de vraisemblance extrêmement irrégulière. Mais le problème principal réside ici dans le fait que ni l'algorithme EM ni ses versions stochastiques (SEM, SAEM ou MCEM) ne permettent de résoudre correctement ce problème. En fait, le problème provient ici du fait qu'une observation qui n'appartient pas au support d'une composante donnée du mélange à une itération de l'algorithme ne pourra pas être affectée à cette composante du mélange dans la suite de l'algorithme. Ainsi, même les versions stochastiques de l'algorithme génèrent nécessairement des suites croissantes de valeurs de décalages estimées. En fait, les résultats établis pour ces algorithmes concernent essentiellement les modèles exponentiels ; mais ici, le modèle n'est pas de type exponentiel vis à vis des paramètres de décalage.

Afin de s'affranchir de ces problèmes, on propose une approche de l'estimation des para-

mètres de décalage dans le domaine transformé de Fourier. Plus précisément, on utilise une méthode de moments pour estimer les paramètres à partir de l'estimation empirique d'un certain nombre de moments de Fourier. Cette méthode consiste à minimiser une certaine distance quadratique entre ces estimateurs empiriques et les grandeurs paramétrées correspondantes. Un intérêt de l'utilisation de moments de Fourier réside dans le fait que les paramètres de décalages sont convertis en paramètres de pulsation qui introduisent une modulation d'amplitude des fonctions caractéristiques des composantes du mélange.

Le fait de travailler avec une méthode de moments permet d'énoncer certains résultats de convergence concernant les estimateurs obtenus. En fait, ces résultats imposent de vérifier une condition d'identifiabilité classique de la forme :

$$m(\theta_1) = m(\theta_2) \Rightarrow \theta_1 = \theta_2, \quad (4.1)$$

où $m(\theta)$ est le vecteur de moments paramétré par θ . Dans le problème considéré ici, certaines conditions d'identifiabilité que nous avons mises en évidence présentent des analogies avec des conditions d'identifiabilité rencontrées en traitement d'antenne et portant sur le nombre de sources identifiables.

Si on note $(X_t)_{t=1,T}$ les observations, et $(Y_t)_{t=1,T}$ les vecteurs transformés,

$$Y_t = [e^{i\omega_1 X_t}, \dots, e^{i\omega_L X_t}]^T, \quad (4.2)$$

où $(\omega_l)_{l=1,L}$ sont les pulsations auxquelles on considère la transformation de Fourier (on pourra choisir par exemple ω_l de la forme $\omega_l = l * \omega_1$), il apparaît que $Y^{(T)} = T^{-1} \sum_{t=1,T} Y_t$ est un vecteur asymptotiquement gaussien, ce qui permet d'envisager l'estimation des paramètres au sens du maximum de vraisemblance dans le domaine de Fourier.

De plus, $Y^{(T)}$ peut être vu comme une somme de contributions de vecteurs gaussiens, chacun des vecteurs étant généré à partir des observations qui correspondent à une des composantes du mélange étudié. Dans ces conditions, il apparaît que l'estimation des paramètres associés aux diverses composantes du modèle peut être découplée au moyen d'un algorithme EM. Ce problème est envisagé de façon détaillée dans l'article présenté au chapitre 5 de la troisième partie du document.

Bibliographie

- [1] A.K. ERLANG "Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges", *Elektroteknikeren*, 13:5-13, 1917.
- [2] W.E. LELAND, M.S. TAQQU, W. WILLINGER, D.V. WILSON, "On the self similar nature of ethernet traffic (extended version)", *IEEE ACM Trans. on Networking*, 2(1):1-14, feb.1994.
- [3] J. BERAN, *Statistics for long memory processes*, Chapman and Hall, 1994.
- [4] W.WILLINGER, M.S. TAQQU, R. SHERMANN, D.V. WILSON, "Self similarity through high variability: statistical analysis of ethernet lan traffic at source level", *SIGCOMM'95*, Cambridge, USA, jan. 1995.
- [5] W. WILLINGER, M.S. TAQQU, W.E. LELAND, "Self similarity in high speed packet traffic: analysis and modelling of ethernet traffic measurements", *Statistical Science*, 10(1):67-85, 1995.
- [6] V.PAXON, S.FLOYD, "Wide-area traffic: the failure of Poisson modeling", *SIGCOMM'94*, London, England, aug. 1994, pp.257-268.
- [7] S. ROBERT, *Modélisation Markovienne du trafic dans les Réseaux de Communication*, thèse de doctorat, Ecole Polytechnique fédérale de Lausanne, Switzerland, 1996.
- [8] L.B.WHITE, "Multiscale markov point process with application to the analysis of discrete event data", *IEEE SSAP'96*, Corfu, juin 1996.
- [9] S. VATON, Th CHONAVEL, "Estimating the offset parameters of a mixture in the Fourier domain", *ICASSP'99*, pp. 152-155, mars 1999.
- [10] S.VATON, *Modélisation statistique de trafic sur réseau local, et contrôle dynamique de bande passante*, thèse de doctorat, ENST de Paris, 1998.

Chapitre 5

Livre de traitement du signal aléatoire

Il s'agit d'un livre¹ qui témoigne avant tout d'un souci de formalisation de certaines notions de base en traitement du signal aléatoire. En effet, la justification de nombreux résultats classiques est souvent dispersée à travers différents ouvrages et dans certains cas ces résultats se trouvent dans des ouvrages à caractère plus mathématique. Ce livre suit sur le fond un schéma assez classique pour la première partie : partant de la notion de spectre et de représentation spectrale, on s'intéresse ensuite aux problèmes du filtrage et de l'estimation spectrale. Les derniers chapitres étendent les possibilités de traitement en ne limitant plus l'étude à celle des processus stationnaires au second ordre ni à des traitements basés sur les seules statistiques du second ordre, ce qui permet d'inscrire l'ouvrage dans le cadre des évolutions récentes en traitement du signal.

Ce livre contient les éléments nécessaires à la compréhension mathématique des résultats énoncés, pour des étudiants ayant atteint un niveau de fin de cours de maîtrise. Chaque chapitre est illustré par des exemples de simulations fournis sur le CD-ROM d'accompagnement. Il ne prétend pas fournir un large éventail de techniques existantes ni remplacer des ouvrages qui développent plus particulièrement certains domaines d'application. Ce livre est plutôt destiné à ceux qui veulent accéder, à travers une présentation relativement synthétique, à une compréhension précise des bases scientifiques du traitement du signal aléatoire et acquérir suffisamment d'autonomie pour pouvoir adapter ou développer par eux même les outils algorithmiques dont ils pourraient avoir besoin.

L'intérêt manifesté par l'éditeur SPRINGER pour cet ouvrage s'est trouvé confirmé par une proposition de traduction de la version française augmentée du livre en langue anglaise. Cette traduction sera fournie à l'éditeur pour la fin 2000 et réalisée en collaboration avec Sandrine VATON, enseignant-chercheur à l'ENST Bretagne.

1. Th. CHONAVEL, *Traitement du Signal Aléatoire*, Springer, collection Sciences et Techniques des Télécommunications, avril 2000.

On trouvera en annexe 7 la table des matières et le premier chapitre du livre.

Troisième partie
Recueil d'articles

Chapitre 1

Coefficients de réflexion des spectres à bande limitée

Reflection Coefficients of Band-Limited Spectra, and Band-Limited Spectrum Modeling.

Thierry CHONAVEL, Serge DEGERINE*

ENSTBr, Dpt SC, BP 832, 29285 Brest cedex, France,

* LMC-IMAG, BP 53, 38041, Grenoble, cedex 9, France,

(e.mail: thierry.chonavel@enst-bretagne.fr, serge.degerine@imag.fr)

Abstract

We study the properties of the reflection coefficients of band-limited spectra. We show that they must belong to simple subsets of the unit disk. Then, the problem of band-limited spectrum modeling is addressed in a more appropriate way than with approaches based on correlation coefficients. In particular, we derive a new band-limited spectrum model for band-limited signals with first correlation coefficients matching the signal ones. The practical behavior of the model is illustrated with an example.

1 Introduction

A possible approach to modeling the spectrum density of a stationary, complex-valued, discrete signal $X = (X_n)_{n=-\infty, \infty}$ lies in searching for a spectrum with first $N + 1$ correlation coefficients matching those of X , given by $r_n = E(X_{n+k} \overline{X_k})$ ($n = 0, N$). This problem is equivalent to that of finding an infinite positive extension r_{N+1}, r_{N+2}, \dots of the sequence $(r_n)_{n=0, N}$. As is well known, the sequence $(r_n)_{n=0, N}$ is characterized equivalently by a set of coefficients, called reflection coefficients, and denoted $(\beta_n)_{n=0, N}$, and the above-mentioned problem is also equivalent to that of finding an extension $\beta_{N+1}, \beta_{N+2}, \dots$ of $(\beta_n)_{n=0, N}$. In particular, the famous maximum entropy extension is associated with the reflection coefficients $\beta_{N+k} = 0, \forall k \geq 1$.

Reflection coefficients are often met in the literature. They are also called Schur coefficients [12]. In particular, these coefficients are the parameters of a family of polynomials that are orthogonal on the unit circle [13].

In statistics, reflection coefficients are also used in many situations, and are often referred to as partial autocorrelation coefficients. They are used for the Quenouille test (1949, [20]), for the Levinson (1947 [17]) and the Durbin (1960 [11]) algorithms, and play an important role in modern spectrum analysis ([14], [18]). More recently, they have made it possible to derive new results about the maximum likelihood estimation of Toeplitz matrices ([7], [8]), and autoregressive models ([9], [19]). Here, we consider the advantage of using this parametrization to study the trigonometric moment problem on an interval ([4], [5], [6]), following the approach of ([10]).

To be exact, we address the problem of band-limited positive extensions $(r_n)_{n > N}$, for which the spectrum μ defined by

$$r_n = \frac{1}{2\pi} \int_{]-\pi, \pi]} e^{in\omega} d\mu(\omega) \quad (n = 0, \pm 1, \pm 2, \dots) \quad (1)$$

is carried by a fixed sub-interval of $]-\pi, \pi]$, denoted $[-\Omega, \Omega]$. This problem is of practical

interest when one knows *a priori* that the spectrum is carried by $[-\Omega, \Omega]$, and is known as the trigonometric moment problem on the interval $[-\Omega, \Omega]$. We intend to look for the characterization of such spectra through their reflection coefficients $(\beta_{N+k})_{k \geq 1}$. The reason for choosing this particular approach will be presented in what follows.

First, we recall a necessary and sufficient condition for a sequence $(r_n)_{n=0, N}$ to represent the first correlation coefficients of some band-limited spectrum [15]. Let us denote

$$r'_n = r_{n+1} + r_{n-1} - 2r_n \cos(\Omega) \quad (n = 0, \pm 1, \dots, \pm(N-1)). \quad (2)$$

This necessary and sufficient condition is expressed by the fact that both Toeplitz matrices Γ_N and Γ'_{N-1} , with respective size $N+1$ and N , and defined by $(\Gamma_N)_{i,j} = r_{i-j}$ and $(\Gamma'_{N-1})_{i,j} = r'_{i-j}$, must be positive.

Thus, it is possible to characterize the set of all the coefficients r_{N+1} such that $(r_n)_{n=0, N+1}$ possesses band-limited positive extensions. Iterating this procedure, one could theoretically obtain a positive band-limited extension of $(r_n)_{n=0, N}$. Unfortunately, this technique fails in practice. This is due to numerical problems related to the ill-conditioning of the matrices Γ_{N+k} and Γ'_{N+k-1} ($k \geq 1$) associated with the finite band-limited positive extensions $(r_n)_{n=0, N+k}$ ([21], [4]). Furthermore, a one step band-limited extension r_{N+k+1} cannot be chosen independently from the $(r_n)_{n=0, N+k}$, which prevents us from being able to define a systematic way of extending $(r_n)_{n=0, N}$.

Another approach that we develop in this paper, consists of characterizing the reflection coefficients β_{N+k} ($k > 0$) such that the corresponding sequences $(r_n)_{n=0, N+k}$ admit positive band-limited extensions. In order to do this, we first express the positivity of the matrices Γ_N and Γ'_{N-1} in terms of the two corresponding sequences of reflection coefficients, and we point out simple relations between these sequences. Thus, it is easy to characterize the coefficients β_{N+1} that correspond to a coefficient r_{N+1} such that the sequence $(r_n)_{n=0, N+1}$ possesses band-

limited positive extensions. In particular, we prove that the set of all such coefficients β_{N+1} coincides with the intersection of the unit disk with another disk. Iterating this procedure, an infinite extension $(\beta_{N+k})_{k>0}$ can be obtained that defines a band-limited spectrum. Just as for correlation coefficients, extensions $(\beta_{N+k})_{k>0}$ that define a band-limited spectrum will be referred to as band-limited positive extensions. The approach that involves the reflection coefficients overcomes the above-mentioned numerical problems that appear when operating with correlation coefficients.

Furthermore, let us recall that the characterization of the set of all positive measures carried by $[-\Omega, \Omega]$ with first $N + 1$ Fourier coefficients matching the $(r_n)_{n=0,N}$, and denoted $M(\Omega)$, was derived in [5]. However, in that paper, we did not address the problem of selecting a particular element in $M(\Omega)$ that would be a suitable spectrum model (in a certain sense). Using the reflection coefficient approach, we show that it is possible to find such a spectrum for which no iterative algorithm is necessary to calculate the coefficients β_{N+k} . Furthermore, we shall see that it yields a particularly interesting band-limited spectrum model.

Let us briefly describe this particular extension. For an appropriate choice of β_{N+1} , the sequence β_{N+k} defined by $\beta_{N+k} = (-1)^k e^{i\phi} \cos(\Omega/2)$ ($k > 1$, and ϕ depends on β_{N+1}), generates a band-limited positive extension of the sequence $(\beta_n)_{n=0,N}$.

We derived the analytic expression of the power spectrum density model associated with this extension, that we denote $f_0(\omega)$. To express it, let $Q_{N+1}(z)$ be the first order Szegő polynomial of degree $N + 1$ associated with the sequence $(\beta_n)_{n=0,N+1}$ (in practical terms, it can be obtained from the Levinson recursion), σ_{N+1}^2 the corresponding error of prediction, and $u(\omega) = \cos(\frac{\omega}{2})[\cos(\frac{\Omega}{2})]^{-1}$. Then, we get

$$f_0(\omega) = \frac{\sigma_{N+1}^2 (1 - [u(\omega) - \sqrt{u(\omega)^2 - 1}]^2)}{|Q_{N+1}^*(e^{i\omega}) - e^{i\omega/2} e^{-i\phi} [u(\omega) - \sqrt{u(\omega)^2 - 1}] Q_{N+1}(e^{i\omega})|^2} \quad (\omega \in [-\Omega, \Omega]), \quad (3)$$

where $P^*(z) = z^{\deg(P)} \overline{P}(z^{-1})$ ($\overline{}$ represents the complex conjugation of the coefficients, and

$\deg(P)$ the degree of the polynomial $P(z)$.

In the next section, we are going to recall some results about the trigonometric moment problem, and introduce some notations. In section 3, we present the properties of the reflection coefficients that define band-limited positive sequences. In particular, we express the condition of the existence of positive band-limited extensions in terms of the corresponding reflection coefficient sequences, and discuss how to build such extensions step by step. In section 4, we derive the particular extension given by (3), and we discuss some of its properties. In section 5, we illustrate these results with an example.

To end this introduction, let us note that in order to address the practical problem of band-limited spectrum estimation from a set of observed data, the estimation of the correlation coefficients under the band-limited constraint must be investigated. Solutions to this problem based on a maximum likelihood estimation scheme have been proposed in [4] and [6], and this statistical issue will not be considered here.

2 Correlation coefficients of band-limited spectra

2.1 Characterization of positive band-limited correlation sequences

First, let us consider a sequence $(r_n)_{n=0,N}$ and the associated sequence $(r'_n)_{n=0,N-1}$ defined by (2). Then, let us recall the following result due to Krein [15] (see also [2] for a direct proof) :

Proposition 1 *$(r_n)_{n=0,N}$ represents the first correlation coefficients of a spectrum carried by $[-\Omega, \Omega]$ if and only if (iff) both Toeplitz matrices Γ_N and Γ'_{N-1} , with respective size $(N+1) \times (N+1)$ and $N \times N$, and with general term $(\Gamma_N)_{i,j} = r_{i-j}$ and $(\Gamma'_{N-1})_{i,j} = r'_{i-j}$, are positive.*

Furthermore, if $\Gamma_N > 0$ and $\Gamma'_{N-1} > 0$, $M(\Omega)$ contains infinitely many elements, while if at least Γ_N or Γ'_{N-1} is positive singular, $M(\Omega)$ is reduced to a single measure, carried by at most

$N + 1$ points.

When $\Gamma_N > 0$ and $\Gamma'_{N-1} > 0$, the positive sequence $(r_n)_{n=0,N}$ will be referred to as strictly positive band-limited, and in what follows, we will only consider this kind of sequence. To give a better understanding of this result, let us note that if μ is a positive measure with first Fourier coefficients $(r_n)_{n=0,N}$, the $(r'_n)_{n=0,N-1}$ are the first Fourier coefficients of the measure μ' defined by

$$d\mu'(\omega) = 2(\cos \omega - \cos \Omega)d\mu(\omega). \quad (4)$$

Clearly, the positive measure μ is carried by $[-\Omega, \Omega]$ iff μ' is positive, and in this case, the matrix Γ'_{N-1} is positive.

In what follows, we denote by $Q_n(z)$, and $P_n(z)$ ($n \geq 0$) the first order and second order, degree n , Szegő polynomials associated with the sequence $(r_k)_{k=0,N}$. σ_n^2 will represent the corresponding errors of prediction at the n^{th} step of the Levinson algorithm. The Levinson recursion is given by

$$\begin{aligned} Q_{n+1}(z) &= zQ_n(z) - \beta_{n+1}Q_n^*(z) \\ P_{n+1}(z) &= zP_n(z) + \beta_{n+1}P_n^*(z), \end{aligned} \quad (5)$$

with $P_0(z) = Q_0(z) = 1$. The all pass transfer functions $zQ_n(z)[Q_n^*(z)]^{-1}$ that play an important role in the study of the trigonometric moment problem on an interval ([3], [5]) will be denoted $B_n(z)$. We will denote with the superscript $'$ the analog of the elements introduced for the coefficients r_n , but related to the coefficients r'_n (e.g. $Q'_n(z)$, $P'_n(z)$, σ'^2_n , ...).

2.2 Step-by-step extension of band-limited correlation sequences

Now, let us explain why step-by-step extension of a strictly positive band-limited sequence $(r_k)_{k=0,N}$ is numerically difficult. Let r_{N+1}, \dots, r_n ($n > N$) be a finite extension of $(r_k)_{k=0,N}$,

such that $(r_k)_{k=0,n}$ is strictly positive band-limited. Then, the one-step strictly positive band-limited extensions of $(r_k)_{k=0,n}$ are the points r_{n+1} that belong to the intersection of the open disks, denoted E_n and F_n , and defined by $E_n = \{r_{n+1}; \Gamma_{n+1} > 0\}$, and $F_n = \{r_{n+1}; \Gamma'_n > 0\}$. It is well known that the radius of E_n is given by σ_n^2 (see e.g [15] p.155). Similarly, the radius of F_n is given by $\sigma'_n{}^2$, since relation (2) yields $F_n = 2r_n \cos \Omega - r_{n-1} + E'_{n-1}$, where $E'_n = \{r'_n; \Gamma'_n > 0\}$.

Let us note that the boundary circles ∂E_n and ∂F_n intersect at two points [5] (for a given set A , ∂A will denote its boundary).

From what precedes, it does not seem to be possible to build extensions r_{N+k+1} independently from the values $(r_n)_{n=0,N+k}$. We will see in the next section that extending the sequences of reflection coefficients will allow us to overcome this problem and enable very simple strategies of extension to be defined. In fact, this remark already stands without any band-limited constraint : in this case, r_{N+k+1} must belong to E_{N+k} , while β_{N+k+1} must simply be in the open unit disk, denoted D , and thus can be chosen independently from the $(\beta_k)_{k=0,n}$.

Furthermore, let $f(\omega)$ be a power spectrum density. It is well known (see e.g. [13] p.15) that the prediction errors of $f(\omega)$ satisfy the asymptotic property

$$\lim_{n \rightarrow \infty} \sigma_n^2 = \exp \left[\frac{1}{2\pi} \int_{[-\pi, \pi]} \ln[f(\omega)] d\omega \right]. \quad (6)$$

When $f(\omega)$ is carried by $[-\Omega, \Omega]$, the integral in (6) does not converge, and in this case $\lim_{n \rightarrow \infty} \sigma_n^2 = 0$. Similarly $\sigma'_n{}^2$ converges to 0 when $n \rightarrow \infty$ since it represents the error of prediction associated with the band-limited power spectrum density $2(\cos \omega - \cos \Omega)f(\omega)$. This shows that when $n \rightarrow \infty$, the radius of E_n and F_n tend to 0, and this explains why the numerical approximations will yield $E_n \cap F_n = \emptyset$, and thus prevent building a long extension of $(r_k)_{k=0,N}$ step by step.

2.3 Solutions of the trigonometric moment problem on an interval

We now briefly recall the solution to the classical trigonometric moment problem, i.e. the characterization of the set M of all the positive measures on $]-\pi, \pi]$ with first correlation coefficients matching the $(r_n)_{n=0, N}$, and the solution to the trigonometric moment problem on the interval $[-\Omega, \Omega]$ that yields the set, denoted $M(\Omega)$, that contains the elements of M carried by $[-\Omega, \Omega]$.

Let us assume that $\Gamma_N > 0$. The characterization of M is equivalent to a problem of interpolation in the set of positive real functions (also called Caratheodory functions); i.e. the functions $F(z)$ that are holomorphic in D , with $\text{Re}[F(z)] \geq 0, \forall z \in D$. This problem consists in finding all positive real functions that can be written in the form

$$F(z) = r_0 + 2r_{-1}z + \dots + 2r_{-N}z^N + \dots \quad (7)$$

($r_{-k} = \overline{r_k}$). It can be shown that there exists a one-to-one correspondence between the positive real functions $F(z)$ that satisfy (7), and the elements of M . In particular, if $\sum_{n=-\infty}^{+\infty} |r_n| < \infty$, the corresponding measure μ is absolutely continuous (with respect to the Lebesgue measure), and its density, denoted $f(\omega)$, is given by

$$f(\omega) = \sum_{n=-\infty, \infty} r_n e^{in\omega} = \frac{F(e^{i\omega}) + \overline{F(e^{i\omega})}}{2} \quad (8)$$

(more generally, $f(\omega) = \lim_{r \rightarrow 1^-} \text{Re}[F(re^{i\omega})]$). The positive real functions that satisfy (7) are given by

$$F(z) = \frac{P_N^*(z) + zP_N(z)S_N(z)}{Q_N^*(z) - zQ_N(z)S_N(z)}. \quad (9)$$

Here $S_N(z)$ is any Schur function (i.e. a function holomorphic in D , with $|S_N(z)| \leq 1$ for $|z| < 1$). Thus, it appears that the set of all Schur functions parametrizes the elements of M . When the measure associated with a solution (9) has a continuous density $f(\omega)$, it can be

shown that (8) and (9) yield

$$f(\omega) = \frac{\sigma_N^2(1 - |S_N(e^{i\omega})|^2)}{|Q_N^*(e^{i\omega}) - e^{i\omega}Q_N(e^{i\omega})S_N(e^{i\omega})|^2}. \quad (10)$$

In [5], we derived the characterization of $M(\Omega)$ by specifying the sub-class of Schur functions that defines measures carried by $[-\Omega, \Omega]$. In order to recall its expression, let us introduce the following notation : if α and β are two points of the unit circle, we denote by $[\alpha, \beta]$ the positively oriented arc of the unit circle with starting point α and end point β . The set $M(\Omega)$ is parametrized by the set $S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$ of all Schur functions that are continuous on $D \cup]e^{i\Omega}, e^{-i\Omega}[$, with $S(]e^{i\Omega}, e^{-i\Omega}[) \subset [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}]$. Then, we have [5] :

Theorem 1 *A measure μ belongs to $M(\Omega)$ iff it is associated to a Schur function of $S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$.*

In the context of spectrum modeling, a particular power density function, denoted $f(\omega)$, should be chosen in accordance with (10), and with $S_N \in S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$. Unfortunately, it seems difficult to determine a pertinent criterium that would be easy to optimize over $S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$, or some subclass of it, and thus to get a convenient band-limited spectrum model. This remark, together with the restrictions mentioned in paragraph 2.2 justify the search for an alternative approach.

3 Reflection coefficients of band-limited spectra

In the previous section, we studied band-limited spectrum modeling based on step-by-step extension of the sequence of correlation coefficients. Also, we saw that selecting a suitable solution of the trigonometric moment problem on $[-\Omega, \Omega]$ seems to be a difficult task.

As an alternative, we propose here to consider band-limited spectrum modeling through the extension of the sequence defined by its first reflection coefficients, denoted $(\beta_n)_{n=0,N}$, where

we define $\beta_0 = r_0$, and the β_n are known to have a modulus not greater than 1 for $n > 0$. With a view to doing this, we first derive the relation between the $(\beta_n)_{n \geq 0}$ and $(\beta'_n)_{n \geq 0}$ associated with band-limited spectra. Then, we will use it to build step-by-step positive band-limited extensions of the initial sequence $(\beta_n)_{n=0, N}$. In particular, we will present strategies to build step-by-step extensions that are simple and yield an explicit formula for the power density function that they represent.

3.1 Relation between the β_n , and the β'_n

The relations between the $(\beta_n)_{n \geq 0}$ and the $(\beta'_n)_{n \geq 0}$ will be helpful in describing the domain in which the $(\beta_n)_{n \geq 0}$ must lie to represent the reflection coefficients of a band-limited spectrum. In order to obtain these relations, we prove that we have the following relation between $Q_n(z)$ and $Q'_n(z)$:

Proposition 2 For $n > 1$,

$$(z - e^{i\Omega})(z - e^{-i\Omega})Q'_{n-1}(z) = [z + \frac{\sigma_{n-1}'^2}{\sigma_n^2}]Q_n(z) - [\beta_{n+1} - \beta'_n \frac{\sigma_{n-1}'^2}{\sigma_n^2}]Q_n^*(z) \quad (n > 1). \quad (11)$$

Remark This result is a particular case of a more general one, related to the study of multivariate processes with band-limited spectra, and presented in [1]. The proof given here is a direct one for the scalar case.

Proof : First, let us note that

$$\int_{]-\pi, \pi]} Q'_{n-1}(e^{i\omega})e^{-ik\omega} d\mu'(\omega) = \int_{]-\pi, \pi]} Q'_{n-1}(e^{i\omega})e^{-ik\omega} [e^{-i\omega} + e^{i\omega} - 2 \cos \Omega] d\mu(\omega), \quad (12)$$

and this expression is equal to 0 for $k = 0, \dots, n-2$, and to $\sigma_{n-1}'^2$ for $k = n-1$. Let us denote $K_{n+1}(z)$ the following degree $n+1$ monic polynomial

$$K_{n+1}(z) = (z - e^{-i\Omega})(z - e^{i\Omega})Q'_{n-1}(z), \quad (13)$$

and $\langle \cdot, \cdot \rangle$ the scalar product defined by $\langle z^m, z^n \rangle = \int_{]-\pi, \pi[} e^{i(m-n)\omega} d\mu(\omega)$. Since for $z = e^{i\omega}$, $(z - e^{-i\Omega})(z - e^{i\Omega}) = e^{i\omega}[e^{i\omega} + e^{-i\omega} - 2 \cos \Omega]$, it is clear that $\langle K_{n+1}(z), z^k \rangle = 0$ for $k = 1, 2, \dots, n-1$. Furthermore, $\langle K_{n+1}(z), z^n \rangle = \sigma_{n-1}'^2$, and $\langle K_{n+1}(z), 1 \rangle = \beta_n' \sigma_{n-1}'^2$. Thus, it is easy to check that the degree $n+1$ monic polynomial

$$L_{n+1}(z) = K_{n+1}(z) - \frac{\sigma_{n-1}'^2}{\sigma_n^2} Q_n(z) - \beta_n' \frac{\sigma_{n-1}'^2}{\sigma_n^2} Q_n^*(z), \quad (14)$$

satisfies $\langle L_{n+1}(z), z^k \rangle = 0$, for $k = 0, \dots, n$. Thus,

$$L_{n+1}(z) = Q_{n+1}(z) = zQ_n(z) - \beta_{n+1} Q_n^*(z). \quad (15)$$

Considering the relations (13), (14), and (15), and putting all the pieces together yields the result stated by the proposition.

Now, let us recall that $Q_n(0) = -\beta_n$ and $Q_{n-1}'(0) = -\beta_{n-1}'$. Then, the relation (11) yields

$$\beta_n' = \beta_n + (\beta_{n+1} - \beta_{n-1}') \frac{\sigma_n^2}{\sigma_{n-1}'^2} \quad (n \geq 2). \quad (16)$$

With direct calculation for $n = 0, 1$, we finally get :

Proposition 3 *The sequences $(\beta_n)_{n \geq 0}$ and $(\beta_n')_{n \geq 0}$ are related through the relations*

$$\begin{aligned} \beta_0' &= \beta_0[\beta_1 + \bar{\beta}_1 - 2 \cos \Omega] \\ \beta_1' &= \beta_1 + (\beta_2 + 1) \frac{\sigma_1^2}{\sigma_0'^2} \\ \beta_n' &= \beta_n + (\beta_{n+1} - \beta_{n-1}') \frac{\sigma_n^2}{\sigma_{n-1}'^2} \end{aligned} \quad (17)$$

Let us note that it is easy to check the positive band-limited conditions $\Gamma_N \geq 0$, and $\Gamma_{N-1}' \geq 0$ by calculating the $(\beta_n)_{n=0, N}$ by means of the Levinson or the Schur algorithm (e.g. [12]) applied to the sequence $(r_n)_{n=0, N}$, and calculating $(\beta_n')_{n=0, N-1}$ from (17) and the relations $\sigma_n^2 = \sigma_{n-1}^2(1 - |\beta_n|^2)$ and $\sigma_{n-1}'^2 = \sigma_{n-2}'^2(1 - |\beta_{n-1}'|^2)$. This can be done by testing the conditions $\beta_0 > 0$, and $\beta_0' > 0$, and $|\beta_n| \leq 1$ and $|\beta_l'| \leq 1$ ($n = 1, N$ and $l = 1, N-1$). Furthermore, strict positivity of the sequence $(\beta_n)_{n=0, N}$ is guaranteed *iff* $|\beta_n| < 1$ and $|\beta_l'| < 1$.

3.2 Step-by-step band-limited extension of $(\beta_n)_{n=0,N}$

First, let us introduce the following notations. We denote

$$\begin{aligned} R_1 &= \frac{\sigma_0'^2}{\sigma_1^2} & C_1 &= -1 - \beta_1 R_1 \\ R_n &= \frac{\sigma_{n-1}'^2}{\sigma_n^2} & C_n &= \beta'_{n-1} - \beta_n R_n \quad (n > 1). \end{aligned} \quad (18)$$

Let β_{N+1} be an element of D . Then, from (17), the corresponding β'_N is given by $\beta'_N = \beta_N + (\beta_{N+1} - \beta'_{N-1})R_N^{-1}$. Using notation (18), this relation can be rewritten in the form

$$\beta_{N+1} = C_N + \beta'_N R_N. \quad (19)$$

Clearly, this shows that if $(\beta_n)_{n=0,N}$ is a strictly positive band-limited sequence of reflection coefficients, $(\beta_n)_{n=0,N+1}$ is positive band-limited iff β_{N+1} belongs to the intersection of D , and of the open disk with center C_N and radius R_N , denoted D_N . Iterating this procedure yields :

Proposition 4 *A strictly positive band-limited extension $\beta_{N+1}, \beta_{N+2}, \dots$ can be obtained iteratively by choosing β_{N+k+1} ($k \geq 0$) in $D \cap D_{N+k}$, where D_{N+k} is the open disk with radius $R_{N+k} = \sigma_{N+k-1}'^2 \sigma_{N+k}^{-2}$ and center $C_{N+k} = \beta'_{N+k-1} - \beta_{N+k} R_{N+k}$.*

Remarks The circles ∂D and ∂D_n intersect at points $\beta_{n+1}^\pm = B_n(e^{\pm i\Omega})$. To justify this fact, let us first recall that $\forall \omega, |B_n(e^{i\omega})| = 1$. On the other hand, let us denote $\beta_n'^\pm$ the values of β'_n that correspond with $\beta_{n+1} = \beta_{n+1}^\pm$. If we consider the relation (11) at points $z = e^{\pm i\Omega}$, together with (17) and (19), it results that

$$\beta_n'^\pm = \frac{\beta_{n+1}^\pm - C_n}{R_n} = -e^{-(\pm i\Omega)} \beta_{n+1}, \quad (20)$$

that also belongs to the unit circle. Thus, $\partial D \cap \partial D_n = \{\beta_{n+1}^+, \beta_{n+1}^-\}$. From (20), it also results that

$$\beta_{n+1}^\pm = \frac{C_n}{1 + R_n e^{-(\pm i\Omega)}}, \quad (21)$$

and that

$$|C_n| = (1 + 2R_n \cos \Omega + R_n^2)^{1/2}. \quad (22)$$

These relations will be useful to prove some of the results presented in what follows.

Now, in order to show that we can avoid the kind of numerical problems met in the step-by-step extension of the correlation coefficient sequence $(r_n)_{n=0,N}$, we study the behavior of the radius R_n of the disk D_n . Clearly,

$$R_{n+1} = R_n \frac{1 - |\beta'_n|^2}{1 - |\beta_{n+1}|^2}. \quad (23)$$

Thus, for an appropriate choice of β_{n+1} ($n \geq N$), in $D \cap D_n$, it is possible to assign to R_{n+1} any value in the interval $]0, +\infty[$. Indeed, from (23), it is clear that $R_{n+1} \rightarrow 0$ when $\beta_{n+1} \rightarrow \partial D$, and that $R_{n+1} \rightarrow \infty$ when $\beta_{n+1} \rightarrow \partial D_n$. Thus, the radius of D_n does not necessarily decrease to 0. In particular, this condition is satisfied when we constrain R_{n+1} ($n \geq N$) not to depend on n (i.e. $R_{n+1}R_n^{-1} = 1$ for all $n \geq N$). From (23), this condition can be written in the form $|\beta_{n+1}|^2 = |\beta'_n|^2$, and it is easy to show that it is satisfied when β_{n+1} belongs to the arc of the circle defined by

$$\Delta_1 = \left\{ u; u = \frac{C_n}{1 + R_n e^{-i\psi}}, \psi \in]-\Omega, \Omega[\right\}. \quad (24)$$

More generally, the condition $R_{n+1}R_n^{-1} = k$ ($k \in]0, \infty[$) is satisfied when β_{n+1} belongs to the intersection of $D \cap D_n$ with a circle, the center and the radius of which are given respectively by

$$\frac{C_n}{1 - R_n^2 k}, \quad \text{and} \quad \frac{[1 + 2kR_n \cos \Omega + k^2 R_n^2]^{1/2}}{|R_n^{-1} - kR_n|}, \quad (25)$$

when $k \neq R_n^{-2}$. When $k = R_n^{-2}$, we obtain the segment with end points β_{n+1}^- and β_{n+1}^+ .

3.3 Particular infinite extensions

We now look for simple methods to build extensions of the sequence $(\beta_n)_{n=0,N}$. First, let us note that we have the following result:

Proposition 5 *Any constant extension $\beta_{N+k} = \beta$ ($k > 0$), $|\beta| < 1$ of a positive band-limited sequence of reflection coefficients $(\beta_n)_{n=0,N}$ yields a spectrum that is not carried by $[-\Omega, \Omega]$.*

Proof : see appendix.

However, it is possible to build very simple extensions $(\beta_{N+k})_{k>0}$ that yield band-limited spectra. In particular, let us consider the following strategy : the relation (17) $\beta'_n = \beta_n + (\beta_{n+1} - \beta'_{n-1})R_N$ suggests choosing $\beta_{n+1} = \beta'_{n-1}$ for $n > N$. Then, $\beta'_n = \beta_n$, and thus, $|\beta'_n| < 1$. The so defined infinite extension has two alternate values : $\beta_{N+2k-1} = \beta'_{N-1}$ and $\beta_{N+2k} = \beta_N$ ($k > 0$), and yields a band-limited spectrum. In fact, a similar property can be obtained with some others simple extension strategies :

Proposition 6 *The following three strategies*

$$\begin{aligned}\beta_{n+1} &= \beta'_{n-1} \\ \beta_{n+1} &= \frac{1}{2}[\beta_{n+1}^- + \beta_{n+1}^+] \quad (n \geq N) \\ \beta_{n+1} &= -\beta'_n\end{aligned}\tag{26}$$

yield infinite extensions that have alternate values, given respectively by the pairs (β'_{N-1}, β_N) , $(\frac{1+R_N \cos \Omega}{C_N}, -\frac{1+R_N \cos \Omega}{C_N})$, and $(\frac{C_N}{1+R_N}, \frac{-C_N}{1+R_N})$.

Proof : The proof of this proposition and some further properties are presented in the appendix.

The advantage of these strategies is that no calculation is necessary to get β_n for $n \geq N+2$. The problem is then to characterize the power spectra associated with such alternate sequences. To do this, let us note that if $S_n(z)$ denotes the Schur function associated with the extension

$(\beta_k)_{k>n}$, then, it is easy to check from (5) and (9) that

$$S_{n+1}(z) = \frac{S_n(z) - \bar{\beta}_{n+1}}{z(1 - S_n(z)\beta_{n+1})}. \quad (27)$$

The spectra associated with alternate extensions are parametrized by the Schur functions $S_N(z)$ associated with infinite sequences of reflection coefficients in the form a, b, a, b, \dots . To get such Schur functions, let us note that they obviously satisfy $S_N(z) = S_{N+2}(z)$. From (27), we then have

$$S_{N+2}(z) = \frac{\frac{S_N(z) - \bar{a}}{z(1 - aS_N(z))} - \bar{b}}{z(1 - b\frac{S_N(z) - \bar{a}}{z(1 - aS_N(z))})} = S_N(z), \quad (28)$$

that leads to the degree 2 polynomial equation

$$z(az + b)S_N(z)^2 + [1 - z^2 + z(a\bar{b} - \bar{a}b)]S_N(z) - (\bar{a} + \bar{b}z) = 0. \quad (29)$$

The solution to this equation that remains in D for all z in D , represents $S_N(z)$.

We will not treat these solutions here. In the next paragraph, we shall present an alternate extension that leads to a band-limited spectrum of particular interest.

4 A particular band-limited spectrum model

In this section, we are looking for a band-limited spectrum model that could be regarded as a band-limited alternative to the classical AR maximum entropy spectrum model and to the band-limited maximum entropy model (see paragraph 4.3.2), in the sense that it also represents a very simple way to extend the reflection coefficients under the correlations and positivity constraints, but unlike the classical AR model it also matches the band-limiting constraint, and unlike the band-limited maximum entropy model it can be calculated analytically.

To do this, we search for an extension of $(\beta_n)_{n=0,N}$ that depends as little as possible on $(\beta_n)_{n=0,N}$ and that corresponds to the classical AR maximum entropy model when $\Omega = \pi$; i.e.

$\beta_n = 0$ for $k > N$ when $\Omega = \pi$. To get such an extension, let us note that if we choose β_{N+1} such that $R_{N+1} = 1$ (i.e. on the circle with center and radius defined by (25), with $k = R_n^{-1}$), then, $|C_{N+1}| = [1 + 2R_{N+1} \cos \Omega + R_{N+1}^2]^{1/2} = 2 \cos(\Omega/2)$, and from the proposition (6), it is clear that the sequence $\beta_{N+k} = (-1)^k \cos(\Omega/2) e^{j\phi_{N+1}}$ ($k \geq 2$), where ϕ_{N+1} is the argument of C_{N+1} , defines a positive band-limited extension.

We will discuss the choice of ϕ_{N+1} later. Now, let us simply write the extension $(\beta_{N+k})_{k \geq 2}$ in the form $\beta_{N+k} = (-1)^k \beta$, with $|\beta| = \cos(\Omega/2)$, and let us derive the expression of the power density of the spectrum associated with this extension.

4.1 Spectrum associated with the extension

Paralleling the discussion in section 3, we have

$$S_{N+3}(z) = \frac{\frac{S_{N+1}(z) - \bar{\beta}}{z(1 - \beta S_{N+1}(z))} + \bar{\beta}}{z(1 + \beta \frac{S_{N+1}(z) - \bar{\beta}}{z(1 - \beta S_{N+1}(z))})} = S_{N+1}(z), \quad (30)$$

that yields the equation $z\beta S_{N+1}^2 - (1+z)S_{N+1} + \bar{\beta} = 0$. The discriminant of this equation is given by

$$\Delta(z) = (1+z)^2 - 4z|\beta|^2 = z^2 + 2z(1 - 2|\beta|^2) + 1 = (z - e^{i\Omega})(z - e^{-i\Omega}). \quad (31)$$

As is well known, it is possible to determine a holomorphic function, denoted $\Phi(z)$, defined in the complex plane stripped of a curve with end points $e^{i\Omega}$ and $e^{-i\Omega}$, such that $\Phi(z)^2 = \Delta(z)$. We consider such a curve, denoted ζ , that satisfies $\zeta \cap \bar{D} = \emptyset$ (\bar{D} represents the closed unit disk). Then, with the condition $\Phi(0) = 1$, $\Phi(z)$ is uniquely determined on the complex plane stripped of ζ . In particular, the expression of Φ on ∂D is given by

$$\Phi(e^{i\omega}) = \begin{cases} 2e^{i\omega/2} [\cos(\frac{\omega}{2})^2 - \cos(\frac{\Omega}{2})^2]^{1/2} & \text{on } [e^{-i\Omega}, e^{i\Omega}] \\ -2ie^{i\omega/2} [\cos(\frac{\Omega}{2})^2 - \cos(\frac{\omega}{2})^2]^{1/2} & \text{on } [e^{i\Omega}, e^{-i\Omega}]. \end{cases} \quad (32)$$

Then, $S_{N+1}(z)$ is given by $S_{N+1}(z) = \frac{1+z\pm\Phi(z)}{2z\beta}$ and the constraint $|S_{N+1}(0)| \leq 1$ yields

$$S_{N+1}(z) = \frac{1+z-\Phi(z)}{2z\beta}. \quad (33)$$

In particular, for $\omega \in [-\Omega, \Omega]$, we have $S_{N+1}(e^{i\omega}) = e^{-i\omega/2} e^{-i \arg(\beta)} [u(\omega) - \sqrt{u(\omega)^2 - 1}]$, where $u(\omega) = \frac{\cos(\omega/2)}{|\beta|}$. Then, from (10), the power spectrum density associated with the extension $(-1)^k \beta$ ($k > 1$), denoted $f_0(\omega)$, is given by

$$f_0(e^{i\omega}) = \frac{\sigma_{N+1}^2 [1 - (u(\omega) - \sqrt{u(\omega)^2 - 1})^2]}{|Q_{N+1}^*(e^{i\omega}) - e^{i\omega/2} e^{-i \arg(\beta)} [u(\omega) - \sqrt{u(\omega)^2 - 1}] Q_{N+1}(e^{i\omega})|^2}. \quad (34)$$

4.2 Choice of β_{N+1}

From the study in paragraph 3.2, it is clear that the values β_{N+1} that yield $R_{N+1} = 1$ must be chosen on an arc of a circle. More precisely, we have $k = R_{N+1} R_N^{-1} = R_N^{-1}$, and from (25), it results that β_{N+1} must belong to the arc of the circle denoted Δ_2 and given by

$$\Delta_2 = \left\{ \beta_{N+1}(\alpha) = \frac{C_N + 2R_N \cos(\Omega/2) e^{i\alpha}}{1 - R_N}; \alpha \in]\phi_N - \alpha_0, \phi_N + \alpha_0[^c \right\}, \quad (35)$$

where $\alpha_0 = \arccos\left(\frac{-(1+R_N)\cos(\Omega/2)}{|C_N|}\right)$, and the superscript c denotes the complementary set in $] -\pi, \pi]$.

For $\beta_{N+1} = \beta_{N+1}(\alpha)$, we get $C_{N+1} = 2 \cos(\Omega/2) e^{i\alpha}$, and the band-limited extension $\beta_{N+k} = (-1)^k \cos(\Omega/2) e^{i\alpha}$ ($k > 1$). Now, in order to make an appropriate choice for α , let us note that the variance of the error of prediction satisfies

$$\sigma_{N+k}^2 = \beta(0) \left(\prod_{l=1, N+k} (1 - |\beta_l|^2) \right) = \beta(0) \left(\prod_{l=1, N} (1 - |\beta_l|^2) \right) (1 - |\beta_{N+1}|^2) \left(\sin \frac{\Omega}{2} \right)^{2(k-1)}. \quad (36)$$

Since we are looking for an extension that could be interpreted as a band-limited alternative to the maximum entropy spectrum model, it is natural to take for β_{N+1} the value $\beta_{N+1}(\alpha)$ of Δ_2 that maximizes σ_{N+k}^2 . Clearly, we must choose $\alpha = \phi_N + \pi$. Then, in (34), we have $\arg(\beta) = \phi_N$ and for $k > 1$, we get $\beta_{N+k} = (-1)^k \cos(\Omega/2) e^{i\phi_N + \pi}$. Let us note that when $\Omega \rightarrow \pi$, this model matches the classical AR maximum entropy spectrum model.

4.3 Remarks

4.3.1 Connection with Schur functions

Now, let us briefly give an interpretation of (34) in terms of the corresponding Schur functions associated with the proposed band-limited extensions. From theorem 1, $S_N(z)$ must be chosen in $S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$. Let $\beta_{N+1} = \beta_{N+1}(\alpha)$ be an element of Δ_2 . Then, from (21)

$$B_{N+1}(e^{\pm i\Omega}) = \beta_{N+2}^{\pm} = \frac{C_{N+1}}{1 + R_{N+1}e^{-(\pm i\Omega)}} = e^{i(\alpha \pm \frac{\Omega}{2})}. \quad (37)$$

Thus, from the theorem 1, $S_{N+1}(z)$ must be chosen in

$$S(\Omega, [\overline{e^{i(\alpha + \frac{\Omega}{2})}}, \overline{e^{i(\alpha - \frac{\Omega}{2})}}]) = e^{-i\alpha} S(\Omega, [e^{-i\frac{\Omega}{2}}, e^{i\frac{\Omega}{2}}]). \quad (38)$$

It appears that, up to the phase factor $e^{-i\alpha}$, the set of all the Schur functions $S_{N+1}(z)$ that define band-limited spectra is $S(\Omega, [e^{-i\frac{\Omega}{2}}, e^{i\frac{\Omega}{2}}])$. Thus this set does not depend on the data $(\beta_k)_{k=0, N}$, but only on the bandwidth Ω .

4.3.2 Maximum entropy as an optimization problem

Let us recall first that the spectrum density of the classical maximum entropy spectrum model is $h_N(\omega) = \sigma_N^2 |Q_N^*(e^{i\omega})|^{-2}$, and it is the solution to the following constrained maximization problem

$$\begin{cases} \max_f \int_{[-\pi, \pi]} \ln[f(\omega)] d\omega \\ r_k = \frac{1}{2\pi} \int_{[-\pi, \pi]} e^{ik\omega} f(\omega) d\omega \quad (k = 0, N). \end{cases} \quad (39)$$

Similarly, a direct calculation shows that the band-limited constrained problem

$$\begin{cases} \max_f \int_{[-\Omega, \Omega]} \ln[f(\omega)] d\omega \\ r_k = \frac{1}{2\pi} \int_{[-\Omega, \Omega]} e^{ik\omega} f(\omega) d\omega \quad (k = 0, N), \end{cases} \quad (40)$$

yields a spectrum carried by $[-\Omega, \Omega]$ that has spectrum density in the form $h_N^\Omega(\omega) = |\sum_{k=0, N} \lambda_k e^{ik\omega}|$ for $\omega \in [-\Omega, \Omega]$. However, it does not seem to be possible to calculate the coefficients $(\lambda_k)_{k=0, N}$

from any analytical formula, but only by means of numerical techniques [16]. Furthermore, $h_N^\Omega(\omega)$ is not equal to 0 at points $\pm\Omega$. This makes it less meaningful than model (34) in applications where a band-limited spectrum with density continuous on $[-\pi, \pi]$ is searched for (in (34) we have $f_0(-\Omega) = f_0(\Omega) = 0$).

4.3.3 A particular spectrum

We have seen that the extensions that alternate on $\pm \cos(\Omega/2)$ (up to a phase factor $e^{i\alpha}$) are of particular interest. This suggests studying the spectra defined by $\beta_0 = 1$, and $\beta_n = \pm(-1)^n \cos(\Omega/2)$, ($n > 0$).

Clearly, the spectrum defined by $\beta_n = (-1)^n \cos(\Omega/2)$ is not carried by $[-\Omega, \Omega]$: from proposition (17), we have $\beta'_0 = 2\beta_0[\text{Re}(\beta_1) - \cos\Omega]$. Thus, if $\beta_1 = -\cos(\Omega/2)$, we have $\beta'_0 < 0$, and the band-limiting constraint is not matched.

Now, let us study the spectrum defined by $\beta_n = (-1)^{n+1} \cos(\Omega/2)$. From proposition (17), it is clear that

$$\begin{aligned}\beta'_0 &= 2(1 - \beta)(1 + 2\beta) \\ \beta'_1 &= (3\beta^2 + 2\beta + 1)/(2 + 4\beta) \\ \beta'_n &= (-1)^{n+1}\beta \quad (n > 1),\end{aligned}\tag{41}$$

with $\beta = \cos(\Omega/2)$. It is easy to check that the conditions $\beta'_0 \geq 0$ and $|\beta'_k| < 1$ ($k > 0$) are satisfied. Thus, the spectrum considered here is carried by $[-\Omega, \Omega]$.

Paralleling the study of paragraph 4.1, it results that the corresponding spectrum density is given by

$$h(\omega) = \frac{\sqrt{u(\omega)^2 - 1}}{(1 - \beta)u(\omega)} \quad \text{for } \omega \in [-\Omega, \Omega].\tag{42}$$

From the definition that we proposed for the band-limited alternative to the maximum entropy spectrum model, it is clear that $h(\omega)$ can be seen as a possible band-limited alternative to

the concept of white noise. Indeed, up to a phase factor, the approach proposed in section 4.1 consists in looking for an extension that has reflection coefficients matching those of $h(\omega)$; similarly, in the non band-limited case, the maximum entropy model has reflection coefficients β_{N+k} ($k > 0$) equal to 0, thus matching those of the white noise. Furthermore, we check that when $\Omega \rightarrow \pi$, $h(\omega)$ tends to a constant spectrum density, equal to one for all ω . $h(\omega)$ is represented in figure 1 for $\Omega = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π .

4.3.4 Extension to more general support constraints

It is easy to check that if a positive measure μ is carried by a non-symmetric interval, denoted $[\Omega_1, \Omega_2]$, then any of the previous results apply to the positive measure μ_2 defined by $d\mu_2(\omega) = d\mu(\omega - \frac{\Omega_1 + \Omega_2}{2})$, that is carried by $[-\Omega, \Omega]$, with $\Omega = \frac{\Omega_2 - \Omega_1}{2}$.

Now, if μ is carried by a set of intervals of $[-\pi, \pi]$, it is clear that this set can be rewritten in the form

$$\Delta = \left\{ \phi; e^{i\phi} \in \bigcap_{k=1,p} [e^{i\Omega_1^k}, e^{i\Omega_2^k}] \right\}, \quad (43)$$

where the $\left\{ \psi; e^{i\psi} \in \bigcup_{k=1,p} [e^{i\Omega_1^k}, e^{i\Omega_2^k}]^c \right\}$ define the intervals outside the support of the spectrum. Since μ is carried by the intersection of the intervals $[e^{i\Omega_1^k}, e^{i\Omega_2^k}]$ ($k = 1, p$), the reflection coefficient β_n of μ must lie in the intersection of the unit disk with p other disks, and the parameters of these disks are easily characterized.

5 Example.

In Figure 2, we consider the spectrum with power density constant and equal to 1 on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and to 0 outside this interval. This kind of spectrum can be met in some practical problems like array processing, where it corresponds with the spectrum of what is called a spheric isotropic noise (e.g. [4]). The classical maximum entropy AR model is represented for $N = 5$. It appears

not to be very satisfactory, while the band limited model applies much better. An ARMA(2,3) model is represented too. It also shows good behavior, but of course it matches neither the band-limiting nor the correlation constraints. This example shows that it is possible to build models that fit an unknown band-limited spectrum quite well, and match both correlation and bandwidth constraints.

6 Conclusion

In this paper, we have addressed the problem of band-limited spectrum modeling through a reflection coefficient approach. We have shown that this is more appropriate than approaches based on step-by-step extension of the initial correlation coefficients, or the direct search for a suitable solution in the set of the solutions of the trigonometric moment problem on an interval. We have characterized a simple necessary and sufficient condition for reflection coefficients to be associated with band-limited spectra, and presented a simple procedure for performing a particular infinite extension. This extension yields a power spectrum that can be calculated easily and can be seen as a band-limited alternative to the classical maximum entropy spectrum model.

References

- [1] D. ALPAY, Ph. LOUBATON, "The Trigonometric Moment Problem on an Interval : the Matrix Case", *Linear Algebra and its Applications*, vol 225, pp.141-161, 1995.
- [2] K.S. ARUN, L.E. POTTER, "Existence and Uniqueness of Band-Limited, Positive Semi-Definite Extrapolations", *IEEE Trans. Acous. Speech Signal Processing*, vol. 38, n.3, pp.547-549, March 1990.

- [3] Th. CHONAVEL, *Estimation Spectrale à Bande Limitée et Traitement d'Antennes*, PhD dissertation, Télécom Paris, 92 E 018, 1992.
- [4] Th. CHONAVEL, Ph. LOUBATON, "Estimation of the Autocorrelation Coefficients of Band-Limited Signals, Application to the Sources Localization Problem", *Signal Processing VII, proc. EUSIPCO*, pp.1825-1828, sept. 1992.
- [5] Th. CHONAVEL, Ph. LOUBATON, "Band-Limited Spectrum Estimation from Partial Covariance Sequence", *IEEE trans. on Inf. Theory*, vol.39, n.4, pp.1395-1397, jul. 1993.
- [6] Th. CHONAVEL, Ph. LOUBATON, "A Maximum Likelihood Estimation Algorithm for Autocovariance Matrices of Band-Limited Processes", *Signal Processing VII, proc. EUSIPCO*, sept. 1994.
- [7] S. DEGERINE, "Maximum Likelihood Estimation of Autocovariance Matrices from Replicated Short Time Series", *J. Time Series Analysis*, vol 8, n. 2, pp. 135-146, 1987.
- [8] S. DEGERINE, "On Local Maxima of the Likelihood Function for Toeplitz Matrix Estimation", *IEEE Trans. Sig. Proc.*, vol. SP-40, n.6, june 1992.
- [9] S. DEGERINE, "A Necessary and Sufficient Condition for the Existence of the Maximum Likelihood Estimate in Autoregressive Models". *IEEE Trans. Sig. Proc.*, vol SP-41, n.2, feb. 1993, pp.988-990.
- [10] S. DEGERINE, "Autocorrélations Partielles et Spectres à Bande Limitée", Technical Report n. RT 120, *LMC Institut d'Informatique et de Mathématiques Appliquées de Grenoble*, jul.1994.
- [11] J. DURBIN, "The Fitting of Time Series Models", *Rev. Inst. Inter. Statist.*, 28, 1960, pp.233-234.

- [12] C. FOIAS, A.E. FRAZO, *The Commutant Lifting Approach to Interpolation Problems*, Birkhauser Verlag Berlin, 1990.
- [13] Y.L. GERONIMUS *Polynomials Orthogonal on the Unit Circle and Interval*, Pergamon Press, 1960.
- [14] S.M. KAY, *Modern Spectral Estimation*, Prentice Hall, Englewood Cliffs, N.J, 1988.
- [15] M.G. KREIN, A.A. NUDELMAN, *The Markov Moment Problem and Extremal Problems*, Transl. Math. Monographs, 50, A.M.S., Providence, Rhodes Island, 1977.
- [16] S.W. LANG, J.H. McCLELLAN, "Multidimensional MEM Estimation", *IEEE Trans. Acoust. Speech, Sig. Proc.*, pp. 880-886, dec. 1982.
- [17] N. LEVINSON, "The Wiener RMS (root-mean-square) Error Criterion in Filter Design and Prediction", *J. Math. Phys.*, 25, 1947, pp. 261-278.
- [18] S.L. MARPLE, *Digital Spectral Analysis*, Prentice Hall, Englewood Cliffs, N.J, 1987.
- [19] D.T. PHAM, "Maximum Likelihood Estimation of Autoregressive Model by Relaxation on the Reflection Coefficients", *IEEE Trans. Acous. Speech, Signal Processing*, vol. ASSP-36, aug.1988, pp.175-177.
- [20] M.H.QUENOUILLE, "Approximate Tests of Correlation in Time series", *J. Roy. Statist. Soc., Ser. B*, 11, 1949, pp.68-84.
- [21] D. SLEPIAN, H.O. POLLAK, H.J. LANDAU, "Prolate Spheroidal Wave Functions", *Bell Syst. Tech. Journ.*, vol. 40, pp.43-84, jan. 1961.

Appendix

A Proof of proposition 5

Let λ be a measure with reflection coefficients $\beta_{\lambda,0} = 1$, and $\beta_{\lambda,n} = \beta$ ($k > 0$, $|\beta| \leq 1$). From [13] p.134, λ is not carried by $[-\Omega, \Omega]$. Let $F_\lambda(z)$ be the Caratheodory function associated with λ .

Now, let $S_N(z)$ be the Schur function associated with the the extension $\beta_{N+k} = \beta$ ($k > 0$). From theorem 1, proposition 5 can be proved by showing that $S_N(z) \notin S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$.

Let us note that we have

$$F_\lambda(z) = \frac{1 - zS_N(z)}{1 + zS_N(z)}. \quad (44)$$

Then, it is easy to check that if $S_N(z)$ were in $S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$, $F_\lambda(z)$ would be continuous on $D \cup]e^{i\Omega}, e^{-i\Omega}[$, with $\text{Re}[F(e^{i\omega})] = 0$ for any ω on $] \Omega, \pi] \cup [-\pi, -\Omega [$. Then, the support of λ would be inside $[-\Omega, \Omega]$. Since this is false, $S_N(z)$ does not belong to $S(\Omega, [\overline{B_N(e^{i\Omega})}, \overline{B_N(e^{-i\Omega})}])$. Thus, a spectrum defined from a constant extension $\beta_{N+k} = \beta$ ($k > 0$) cannot correspond to a spectrum carried by $[-\Omega, \Omega]$.

B Proof of proposition 6

We have already proved that the extension given by $\beta_{n+1} = \beta'_{n-1}$ ($n > N$) has two alternate values, β'_{N-1} and β_N , and is band-limited.

Let us now consider the extension given by $\beta_{n+1} = \frac{1}{2}[\beta_{n+1}^+ + \beta_{n+1}^-]$. Obviously, it can be derived from simple geometric considerations that $\beta_{n+1} \in D \cap D_n$. Thus, this strategy defines band-limited positive extensions. From the results of section (3.2), it is clear that

$R_{n+1}R_n^{-1} = R_n^{-2}$, i.e. $R_{n+1} = R_n^{-1}$. Furthermore,

$$C_{n+1} = \beta'_n - \beta_{n+1}R_{n+1} = -\frac{1}{R_n}[\beta_{n+1} - \beta'_n R_n] = -\frac{C_n}{R_n}, \quad (45)$$

and $C_{n+2} = -\frac{C_{n+1}}{R_{n+1}} = \frac{C_n}{R_n R_{n+1}} = C_n$. Thus, $D_n = D_{n+2}$, and $\beta_{n+2} = \beta_n$ ($n > N$).

A direct calculation shows that

$$\beta_{N+1} = \frac{1}{2} \left(\frac{C_n}{1 + R_N e^{i\Omega}} + \frac{C_n}{1 + R_N e^{-i\Omega}} \right) = \frac{1 + R_N \cos \Omega}{C_N}, \quad (46)$$

and

$$\beta_{N+2} = \frac{1 + R_{N+1} \cos \Omega}{C_{N+1}} = -\frac{(R_N + \cos \Omega)}{C_N}. \quad (47)$$

Let us note that this strategy is equivalent to the strategy $\beta_{n+1} = \beta'_{n-1}$, but here it is applied from $n = N + 1$ instead of $n = N$.

The third strategy is obtained by considering the relation $\beta_{n+1} = C_n + \beta'_n R_n$: letting $\beta'_n = -\beta_{n+1}$ yields $\beta_{n+1} = \frac{C_n}{1+R_n}$ that belongs to D_n . Furthermore, the relation $|C_n| = (1 + 2R_n \cos \Omega + R_n^2)^{1/2}$ makes it possible to check that $\beta_{n+1} \in D$. Thus, the strategy $\beta'_n = -\beta_{n+1}$ yields a positive band-limited extension. Furthermore,

$$\begin{aligned} R_{n+1} &= R_n \frac{1-|\beta'_n|^2}{1-|\beta_{n+1}|^2} = R_n, \\ C_{n+1} &= \beta'_n - \beta_{n+1}R_{n+1} = -\beta_{n+1}(1 + R_n) = -C_n. \end{aligned} \quad (48)$$

This proves that $(\beta_n)_{n>N}$ has the two alternate values $\frac{C_n}{1+R_n}$ and $-\frac{C_n}{1+R_n}$.

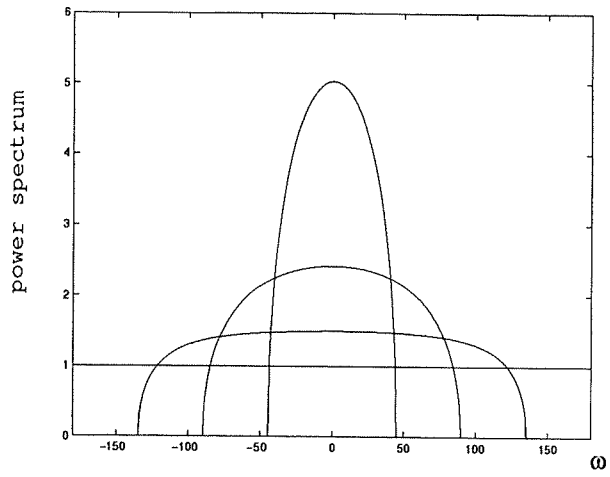


Figure 1: band-limited alternative to the white noise

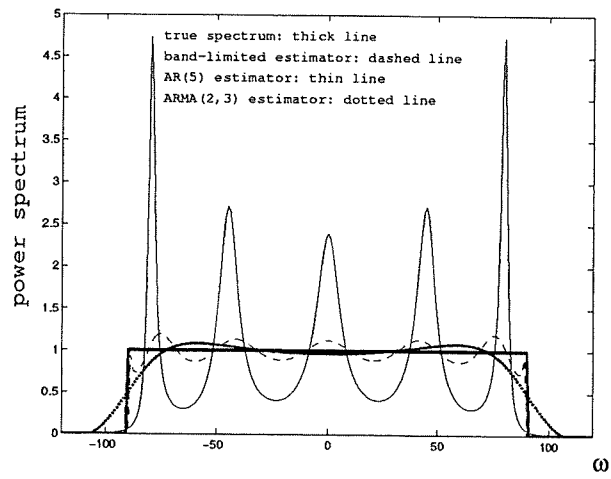


Figure 2: band-limited, AR, and ARMA models

Chapitre 2

Estimation des corrélations des spectres à bande limitée

Estimation of the Autocorrelation Coefficients of Band-Limited Processes

Thierry CHONAVEL*, Serge DEGERINE**, and Philippe LOUBATON***,

* ENSTBr, Dpt SC, BP 832, 29285 Brest cedex, France,

(e.mail: thierry.chonavel@enst-bretagne.fr)

** LMC IMAG, BP 53, F-38041 Grenoble cedex 9

*** Univ. de Marne la Vallée, section Sc. pour l'ing., F-93166 Noisy le Grand cedex

may 1995

Abstract

In this paper, we address the problem of estimating the first correlation coefficients $(r_n)_{n=0,N}$ of a band-limited discrete time stationary process with known bandwidth, in a way that accounts for this property. For this purpose, we estimate the associated Toeplitz matrix $(r_{m-n})_{(m,n)=0,N}$ from the data, under appropriate constraints. Two estimation schemes with asymptotical minimum variance are presented. Finally, we use the proposed estimators in the context of the localization of plane waves sources corrupted by spherical isotropic noise. The results illustrate the potential of our proposal.

Résumé

Dans cet article, on envisage le problème de l'estimation des premiers coefficients de corrélation $(r_n)_{n=0,N}$ d'un processus stationnaire à temps discret dont le spectre possède une bande limitée connue, par des méthodes qui prennent en compte cette information. Pour réaliser cette opération, nous estimons la matrice de Toeplitz définie par $(r_{m-n})_{(m,n)=0,N}$, à partir des observations et de contraintes appropriées. On présente deux techniques d'estimation qui ont une variance asymptotique minimale et on envisage leur mise en oeuvre dans le contexte de la localisation de sources en présence d'un bruit additif sphérique isotrope. Les résultats obtenus montrent l'intérêt des techniques proposées.

1 Introduction

Let us consider a zero mean, discrete time, complex valued, stationary process $(X_n)_{n=0,\pm 1,\pm 2,\dots}$. An important problem consists in estimating its first $N+1$ correlation coefficients $(r_n)_{n=0,N}$, defined by $r_n = \mathbb{E}[X_{n+k} X_k^*]$. Denoting R the $(N+1) \times (N+1)$ autocovariance matrix of the vector $Y = (X_n, \dots, X_{n-N})^T$, it follows that the general term of R is $[R]_{m,n} = r_{m-n}$, and that R is a Toeplitz matrix. If K samples of Y , denoted $(y_k)_{k=1,K}$ are available, a straightforward way to estimate R is to consider its empiric estimate denoted \hat{R} :

$$\hat{R} = \frac{1}{K} \sum_{k=1,K} y_k y_k^H. \quad (1)$$

\hat{R} is positive but a.s. it is not compatible with the stationarity of X , ie. \hat{R} is not a Toeplitz matrix. This drawback can be overcome by projecting \hat{R} on the set of hermitian Toeplitz matrices, denoted T . In practical terms, this consists of averaging the subdiagonals of \hat{R} . Unfortunately, this new estimate may no longer be positive, especially when \hat{R} is close to singularity.

In this case, one may use an alternate projection technique [5],[27], consisting in truncating the eigenvalue decomposition of the averaged matrix to its positive part, to project the new matrix on T and to iterate the procedure. However, the limit of the algorithm represents a positive singular Toeplitz matrix, thus corresponding to the correlation sequence of a purely harmonic signal with uniquely defined frequencies. Therefore, this estimation scheme is irrelevant in many applications.

Often, and we will assume that this is the case here, the process X is supposed to be Gaussian. This auxiliary information can be used to get an efficient estimate of R by maximizing the log-likelihood functional, or equivalently minimizing

$$\Phi(S, \hat{R}) = \text{Log}|S| + \text{Tr}(S^{-1} \hat{R}), \quad (2)$$

where $\text{Tr}(\cdot)$ and $|\cdot|$ represent the trace and the determinant operators respectively. Let us assume that \hat{R} is full rank, a condition which is fulfilled a.s. when $K > N$. Let T_+ be the

set of all $(N + 1) \times (N + 1)$ positive Toeplitz matrices, and denote by ∂T_+ the boundary of T_+ .

It was proved [12] that $\Phi(S, \widehat{R})$ is lower bounded over T_+ and that $\lim_{S \rightarrow \partial T_+, S \in T_+} \Phi(S, \widehat{R}) = +\infty$, where ∂T_+ denotes the boundary of T_+ . Thus, there exist matrices S that minimize $\Phi(S, \widehat{R})$ on T_+ and such matrices belong to the interior of T_+ , i.e. they are positive definite. Consequently, it is possible to get a minimum of $\Phi(S, \widehat{R})$ by means of a standard gradient-like descent algorithm performed over T_+ . As is well known, the main drawback of this method is the possible existence of local minima due to the non convexity of $\Phi(S, \widehat{R})$.

In some problems [10], the spectrum of X is band-limited, and it has a known bandwidth, denoted $[-\Omega, \Omega]$, which is a subinterval of the total sampling bandwidth $[-\pi, \pi]$ ($0 < \Omega < \pi$). The purpose of this paper is to take this additional information into account in order to estimate R , following the approach in [7].

First, we are going to recall some basic properties of the set of all Toeplitz covariance matrices $R = (r_{m-n})_{(m,n)=0,N}$ associated with processes whose spectrum are carried by $[-\Omega, \Omega]$. This set will be denoted T_+^Ω . In particular, we will point out the difficulties that arise in the estimation of R over T_+^Ω . Then, we derive two approaches to estimate R . The first one is based on the projection of \widehat{R} on a set approximating in some sense T_+^Ω . We derive the second one as a non trivial extension of a relaxation method developed to minimize $\Phi(S, \widehat{R})$ and presented in [11] for non band-limited real valued processes. Both methods use particular parametrizations of the elements of T_+^Ω in the form

$$S = \sum_{k=1, M} \lambda_k d(\omega_k) d(\omega_k)^H, \quad (3)$$

where $d(\omega) = [1, e^{i\omega}, \dots, e^{iN\omega}]^T$, $\lambda_k \geq 0$, and $\omega_k \in [-\Omega, \Omega]$. $S \in T_+^\Omega$, since the mass point spectrum $\sum_{k=1, M} \lambda_k \delta_{\omega_k}$ (δ_{ω_k} denotes the Dirac's delta measure with mass at point ω_k) is carried by $[-\Omega, \Omega]$ and its first $N + 1$ correlation coefficients match the elements of S .

The last section of this paper is devoted to the application of these algorithms to the sources localization problem of plane waves corrupted by a spherical isotropic noise.

2 Correlation matrices of band-limited processes

2.1 Structure of T_+^Ω

It is easy to check that T_+^Ω is a closed convex set. In addition, the structure of T_+^Ω can be derived from general results by Krein and Nudel'man [18] related to moment problems. A direct characterization of T_+^Ω is presented in [3] and in [8]-[9] it is presented in the context of band-limited spectrum estimation.

In view to describe T_+^Ω , let us consider an element of T , that we will denote R , with $[R]_{m,n} = r_{m-n}$ (R is a $(N+1) \times (N+1)$ matrix). Then, let us define the coefficients

$$r'_n = r_{n+1} + r_{n-1} - 2r_n \cos \Omega, \quad n = 0, N-1, \quad (4)$$

and the $N \times N$ matrix R' , with $[R']_{m,n} = r'_{m-n}$ ($r'_{-k} = \bar{r}'_k$). Then, we have the following result [18] :

Proposition 1 $R \in T_+^\Omega$ if and only if $R \geq 0$, and $R' \geq 0$.

Furthermore, if $R > 0$ and $R' > 0$, there exists an infinite number of spectra carried by $[-\Omega, \Omega]$, with first correlation coefficients matching the $(r_k)_{k=0,N}$. On the other hand, if R or R' is singular, the $(r_k)_{k=0,N}$ are the first correlation coefficients of only one such spectrum, and it is carried by at most $N+1$ points of $[-\Omega, \Omega]$. In other words, X is the sum of at most $N+1$ sinusoids with frequencies in $[-\Omega, \Omega]$.

It is clear that the boundary of T_+^Ω , denoted ∂T_+^Ω , is made of the elements $R \in T_+^\Omega$ such that R or R' is singular. From the study presented in [9], it appears that $\partial T_+^\Omega = \partial T_+^{\Omega,1} \cup \partial T_+^{\Omega,2}$, with

$$\begin{aligned}\partial T_+^{\Omega,1} &= \{R \in T/R \text{ is positive and singular, } R' \geq 0\} \\ \partial T_+^{\Omega,2} &= \{R \in T/R > 0, R' \text{ is positive and singular}\},\end{aligned}\tag{5}$$

and the corresponding mass point measures, denoted μ , are in the following form

$$\begin{aligned}\text{if } R \in \partial T_+^{\Omega,1}: \quad \mu(\omega) &= \sum_{n=1,m} \lambda_k \delta_{\omega_k}(\omega) \quad m \leq N \\ \text{if } R \in \partial T_+^{\Omega,2}: \quad \mu(\omega) &= \sum_{n=1,N+1} \lambda_k \delta_{\omega_k}(\omega) \quad -\Omega = \omega_1 < \dots < \omega_{N+1} = \Omega.\end{aligned}\tag{6}$$

2.2 Behaviour of $\Phi(S, \widehat{R})$ over T_+^Ω

It is clear that $\partial T_+^{\Omega,1} \subset \partial T_+^\Omega$. Consequently, since \widehat{R} is full rank, $\lim_{S \rightarrow \partial T_+^{\Omega,1}, S \in T_+^\Omega} \Phi(S, \widehat{R}) = +\infty$. This is no longer true on $\partial T_+^{\Omega,2}$ which lies in the interior of T_+^Ω , since $\Phi(S, \widehat{R})$ is bounded on any compact set inside T_+^Ω .

A possible consequence of this fact is that the minimum of $\Phi(S, \widehat{R})$ over T_+^Ω may not be reached by means of a standard descent algorithm, in particular if it is located on $\partial T_+^{\Omega,2}$, which occurs when $\nabla \Phi(S, \widehat{R}) \neq 0$ in T_+^Ω . Furthermore, as explained above the elements of $\partial T_+^{\Omega,2}$ correspond to very particular spectra.

To avoid these problems, a possible approach consists in adding a penalty factor that would prevent descent algorithms from reaching $\partial T_+^{\Omega,2}$. Then, the problem is to minimize the functional

$$\Psi(S, \widehat{R}) = \Phi(S, \widehat{R}) + \lambda g(S),\tag{7}$$

where $g(S)$ is a penalty function which tends toward $+\infty$ when $S \rightarrow \partial T_+^{\Omega,2}$. For instance, one can choose $g(S) = |S'|^{-1}$ (with $[S']_{m,n} = s_{m-n+1} + s_{m-n-1} - 2s_{m-n} \cos \Omega$), or $g(S) = -\log |S'|$. λ is a so-called penalty factor ($\lambda > 0$). The problem of making a good choice for λ in penalized problems is addressed in [25]. We would like to remark here that this choice requires much computation, and that from a physical point of view the additional function $\lambda g(S)$ is not very meaningful. For these reasons, we will not emphasize the penalized approaches.

2.3 Ill-conditioning of the elements of T_+^{Ω}

As it is well known, the covariance matrices associated with band-limited processes are ill-conditioned [24]. To explain this fact, let us consider the Capon spectrum model [6] which is defined by the spectral density

$$S(\omega) = \frac{N+1}{d(\omega)^H R^{-1} d(\omega)}, \quad (8)$$

where $d(\omega) = [1, e^{j\omega}, \dots, e^{jN\omega}]^T$. Some properties of this estimator can be found in [14]. Furthermore, adapting the proof of results given in [14] pp.40-42, it comes that when N increases, $S(\omega)$ tends to 0 outside $[-\Omega, \Omega]$, while it has significant values inside the spectrum bandwidth. This shows that for large enough N , the matrix R is ill-conditioned and that the vectors $\{d(\omega)/\omega \in [-\pi, -\Omega] \cup [\Omega, \pi]\}$ are close to the subspace spanned by the eigenvectors associated with the smallest eigenvalues of R . Similar conclusions would have been obtained considering the periodogram estimator (see for example [4]). However, the Capon estimator has been considered here since it will be referred to later.

To illustrate the above discussion, let us consider the spectrum with density $f(\omega) = \chi_{[-\Omega, \Omega]}(\omega)$, where χ_A is the characteristic function of the interval A . Table 1 presents the condition number of R , i.e. the ratio between its largest and its smallest eigenvalues, when N and Ω vary. As expected, it appears that ill-conditioning increases when N increases and when Ω decreases.

The remarks here above also apply to the matrix R' . Indeed, if $f(\omega)$ is a spectral density carried by $[-\Omega, \Omega]$, the first correlation coefficients of which are $(r_k)_{k=0,N}$, then $(r'_k)_{k=0,N-1}$ are the N first correlation coefficients of the spectrum defined by the spectrum density

$$f'(\omega) = 2\chi_{[-\Omega,\Omega]}(\omega)(\cos \omega - \cos \Omega)f(\omega). \quad (9)$$

Thus, R' is the covariance matrix associated with a band-limited process.

To illustrate the practical consequence of the ill-conditioning of R , we study the positivity of the projection, denoted \tilde{R} , of the empiric estimator \hat{R} defined by (1) on the set of Toeplitz matrices. In table 2, we represent the percentage of cases when \tilde{R} , \tilde{R}' , and both \tilde{R} and \tilde{R}' are positive for $\Omega = \pi/2$, obtained from 10^4 tests. We check that ill-conditioning yields a large amount of estimates outside T_+^Ω . In table 2, $N + 1$ represents the size of R , and K the number of samples used to calculate \hat{R} .

2.4 Minimization of $\Phi(S, \hat{R})$ over T_+^Ω

In addition to the problems discussed in section 2.2, another difficulty occurs when minimizing $\Phi(S, \hat{R})$ by means of a standard descent algorithm. Let us parametrize S by its coefficients s_0, \dots, s_N . Then, the gradient algorithm can be written as

$$S_{k+1} = S_k - \rho_{k+1}[\nabla\Phi(S, \hat{R})]_{S=S_k}. \quad (10)$$

We want all matrices S_k to remain inside T_+^Ω . But, since the elements of T_+^Ω are ill-conditioned, the steps ρ_k will generally have to be very small for this condition to be satisfied (otherwise, S_k or S'_k would not be positive). A consequence of this is a very slow convergence of the algorithm.

Thus, parametrizing S_k directly through its coefficients is certainly irrelevant. In section 3 and 4, we propose alternative approaches, which are much better adapted to our problem.

2.5 Local minima of $\Phi(S, \widehat{R})$

To avoid working with a criterium that may show local minima, it was proposed in [13] to replace $\Phi(S, \widehat{R})$ by the function

$$\Phi_2(S, \widehat{R}) = \left\| (S - \widehat{R})\widehat{R}^{-1} \right\|^2, \quad (11)$$

where $\|\cdot\|$ is the norm associated with the matrix scalar product defined by $\langle A, B \rangle = \text{Tr}(AB^H)$. $\Phi_2(S, \widehat{R})$ is a distance functional, and since it is a quadratic function of S , it admits a minimum on any closed convex set, and in particular on T_+^Ω . Furthermore, a main advantage of $\Phi_2(S, \widehat{R})$ is that it has the same asymptotic properties as $\Phi(S, \widehat{R})$, i.e. it yields an asymptotically efficient estimate of R . For this reason, we will use this functional instead of $\Phi(S, \widehat{R})$ in the next section.

One might think that the presence of the term \widehat{R}^{-1} advocates against the use of $\Phi_2(S, \widehat{R})$ since \widehat{R} is generally ill-conditioned in the context of this paper. However, a crux point of the parametrization of the elements of T_+^Ω that we are going to present is that it renders $\Phi_2(S, \widehat{R})$ very insensitive to the ill-conditioning of \widehat{R} .

3 Estimation of R over a convex approximation of T_+^Ω

The direct minimization of $\Phi_2(S, \widehat{R})$ over T_+^Ω seems to be a difficult task due to the rather complex structure of this set. Thus, we try to simplify the problem of the estimation of R by seeking a subset C of T_+^Ω that is a good approximation of T_+^Ω , and such that the constrained minimization problem

$$\min_{S \in C} \Phi_2(S, \widehat{R}) \quad (12)$$

can be solved easily.

In order to characterize such a set C , we introduce the following notations : let us consider M ($M > 2N$) equispaced points in $[-\Omega, \Omega]$, denoted $\omega_m = -\Omega + \frac{2m-1}{M}\Omega$ ($m = 1, M$), and the positive convex cone generated by the rank one matrices $F(\omega_m) = d(\omega_m)d(\omega_m)^H$, and denoted K_+^M :

$$K_+^M = \left\{ S = \sum_{m=1, M} \alpha_m F(\omega_m); \alpha_m \geq 0, \quad m = 1, M \right\}. \quad (13)$$

In view to show that K_+^M is a good approximation of T_+^Ω , we are going to prove that for any matrix S inside T_+^Ω , we have $S \in K_+^M$ for M large enough.

To do this, we consider the following isometric transformation

$$\begin{aligned} \tau : T &\rightarrow \mathbb{R}^{2N+1} \\ R &\rightarrow [r_0, Re(r_1), Im(r_1), \dots, Re(r_N), Im(r_N)] \end{aligned} \quad (14)$$

where $Re(\cdot)$ and $Im(\cdot)$ represent the real and the imaginary part. Furthermore, we denote $e(\omega) = \tau\{d(\omega)d(\omega)^H\} = [1, \cos \omega, \sin \omega, \dots, \cos N\omega, \sin N\omega]^T$. If (u_1, \dots, u_{2N}) is a basis of an hyperplane of \mathbb{R}^{2N+1} denoted Π , there exist two closed half spaces limited by Π , and they are in the form $E = \{v; u_i^T v \leq 0, i = 1, 2N\}$ and $E' = \{v; u_i^T v \geq 0, i = 1, 2N\}$. Then, we have the following lemma

Lemma 1 $\tau(K_+^M)$ is the cone defined by the intersections of half spaces limited by the hyperplanes

$$Sp\{e(\omega_{a_1}), e(\omega_{a_1+1}), e(\omega_{a_2}), e(\omega_{a_2+1}), \dots, e(\omega_{a_N}), e(\omega_{a_N+1})\},$$

$$\text{with } 1 \leq a_1 < a_2 < \dots < a_N < M$$

and

$$Sp\{e(\omega_1), e(\omega_{a_1}), e(\omega_{a_1+1}), \dots, e(\omega_{a_{N-1}}), e(\omega_{a_{N-1}+1}), e(\omega_M)\},$$

$$\text{with } 1 < a_1 < a_2 < \dots < a_{N-1} < M.$$

$Sp\{.\}$ represents the vector space spanned by a set of vectors. This lemma yields the proposition

Proposition 2 *for any $S \in \text{Int}(T_+^\Omega)$, there exists M_0 such that for any $M > M_0$ then $S \in K_+^M$.*

The proofs are given in Appendix A and in Appendix B. Thus, it appears that for sufficiently large values of M , K_+^M is a good approximation of T_+^Ω and that we can choose $C = K_+^M$.

3.1 Estimation of $\Phi_2(S, \widehat{R})$ over K_+^M

Letting $C = K_+^M$, the problem (12) can be reformulated as follows

$$\begin{cases} \min_{\alpha} \left\| \left[\sum_{m=1, M} \alpha_m F(\omega_m) - \widehat{R} \right] \widehat{R}^{-1} \right\|^2 \\ \alpha_m \geq 0, \quad m = 1, M. \end{cases} \quad (15)$$

Note that the parametrization of the covariance matrix proposed here has close connections with that chosen by Miller and Snyder in [20]. However, the set on which the optimization is performed, the criterium to be optimized and the way to optimize it are very different.

In (15), we have to minimize a quadratic function of $\alpha = (\alpha_1, \dots, \alpha_M)^T$, with linear constraints. Such a problem can be solved using Lagrange multipliers. Denoting $\mu = (\mu_1, \dots, \mu_M)^T$ the Lagrange multipliers, we look for the saddle point of the function

$$L(\alpha, \mu) = \alpha^T H \alpha + \alpha^T (b - \mu), \quad (16)$$

with $[H]_{m,n} = \left| d(\omega_m)^H \widehat{R}^{-1} d(\omega_n) \right|^2$, and $[b]_m = -2d(\omega_m)^H \widehat{R}^{-1} d(\omega_m)$.

The minimum of (15) can be obtained by means of the Arrow-Hurwicz algorithm [2]. In this iterative algorithm, the current values of vectors α and μ are computed according the following formulas

$$\begin{aligned}\alpha^{k+1} &= \alpha^k - \rho_1^k (2H\alpha^k + b - \mu^k) \\ \mu^{k+1} &= \Pi_{\mathbb{R}_+^M} (\mu^k - \rho_2^k \alpha^{k+1}),\end{aligned}\tag{17}$$

where $\Pi_{\mathbb{R}_+^M}(\cdot)$ is the projection on the set \mathbb{R}_+^M of M -dimensional real vectors with positive coefficients, i.e. the k^{th} component of $\Pi_{\mathbb{R}_+^M}(u)$ is $\max([u]_k, 0)$.

The initial value α^0 can be any point of \mathbb{R}_+^M . In order to have α^0 not too far from the optimum (and thus fast convergence), we can take, for instance, α_m^0 equal to the value of the Capon estimator at point ω_m , normalised by the factor $1/M$. In this case, $\alpha_m^0 = (N+1)[M \times d(\omega_m)^H \widehat{R}^{-1} d(\omega_m)]^{-1} = -2(N+1)/(M \times [b]_m)$.

It is also possible to optimize the choice of ρ_1^k at each step of the algorithm. To do this, we look for the minimum of $L(\alpha^k - \rho_1^k (2H\alpha^k + b - \mu^k), \mu^k)$. Thus

$$\rho_1^k = \frac{u^T u}{2u^T H u}, \text{ with } u = 2H\alpha^k + b - \mu^k.\tag{18}$$

On another hand, ρ_2^k can be chosen equal to $1/k \|\alpha^{k+1}\|$.

Of course, the algorithm (17) can easily be generalized to estimate the covariance matrices of signals that have more complicated spectrum support.

3.2 Influence of \widehat{R} ill-conditioning

From the discussion in section 2.3, it is clear that the vectors $d(\omega_m)$ are approximately orthogonal to the space spanned by the largest eigenvalues of \widehat{R}^{-1} . Therefore, the parameters $[H]_{m,n} = \left| d(\omega_m)^H \widehat{R}^{-1} d(\omega_n) \right|^2$, and $[b]_m = -2d(\omega_m)^H \widehat{R}^{-1} d(\omega_m)$ are little influenced by the ill-conditioning of \widehat{R} . Thus, $\Phi_2(S, \widehat{R})$ is not sensitive to the ill-conditioning of \widehat{R} for values of S chosen inside T_+^Ω .

3.3 Rank of matrix H

The parameter M has to be large enough to ensure that K_+^M is a good approximation of T_+^Ω . In the algorithm, the computation of $H\alpha$ requires M^2 multiplications. However, as is shown in appendix C, the rank of H is equal to $\min(M, 2N + 1)$. Thus, for $M > 2N + 1$, the computational cost increases linearly with M . But in practice, the gain in terms of bias and variance obtained when M increases is not so significant, and we can choose $M = 2N + 1$. Furthermore, for $M = 2N + 1$ the minimum for α can be reached at each iteration of the algorithm (17) since

$$\arg \min_{\alpha} L(\alpha, \mu^k) = H^{-1}(b - \mu^k), \quad (19)$$

and the update of α in (17) can be replaced by

$$\alpha^{k+1} = H^{-1}(b - \mu^k). \quad (20)$$

3.4 Mixing of a band-limited and a non band-limited process

Let us assume that the band-limited process X is observed together with an additional process which is not band-limited and is uncorrelated with X . Then, if the covariance matrix of this non band-limited process, denoted R_a , is definite and known up to a constant, it is possible to estimate the covariance matrix R of X by slightly modifying the previous algorithm : R can be estimated by solving

$$\left\{ \begin{array}{l} \min_{\alpha, \nu} \left\| \left[\sum_{m=1, M} \alpha_m F(\omega_m) + \nu R_a - \widehat{R} \right] \widehat{R}^{-1} \right\|^2 \\ \alpha_m \geq 0, \quad m = 1, M \\ \nu \geq 0 \end{array} \right. \quad (21)$$

The uniqueness of the solution (α, ν) is discussed in Appendix D. In the last section, we present an application of this method.

4 Minimization of $\Phi(S, \widehat{R})$ over T_+^Ω by means of a relaxation algorithm

In the previous approach, the estimate of R was constrained to be in a subset of T_+^Ω . Here, we propose an alternative technique which permits to look for an estimate of R over the whole set T_+^Ω . It is based on works by Degerine [11], developed in the context of non band-limited real valued processes. We have seen in section 2 that the direct parametrization of the matrix S through its coefficients is not relevant in view to perform the minimization of $\Phi(S, \widehat{R})$ on T_+^Ω . For this reason, we consider here an alternative parametrization of S in the form

$$S = \sum_{n=1, N+1} \lambda_n d(\omega_n) d(\omega_n)^H, \quad (22)$$

where the λ_n are positive, and the ω_n are in $] -\pi, \pi]$ ($n = 1, N + 1$). First, we are going to recall some basic facts about these parametrizations when S belongs to $Int(T_+)$ ($Int(A)$ represents the interior part of the set A). Then, we will present how they have been used in [11] to minimize $\Phi(S, \widehat{R})$ when dealing with real valued processes, and we will show how his approach can be adapted to the case of complex-valued, band-limited signals.

4.1 Parametrization of the elements of T_+

The representations of the elements S of $Int(T_+)$ in the form (22) are given by the following result that derives directly from [18] p. 79 :

Proposition 3 *Let S be an element of $Int(T_+)$, defined by the correlation sequence $(s_n)_{n=0, N}$. Then, for any fixed point α in $] -\pi, \pi]$, there exists a unique spectrum carried by $N + 1$ points with mass at point α and the first $N + 1$ correlation coefficients of which match the $(s_n)_{n=0, N}$.*

Let α be a point of $] - \pi, \pi]$, and $\sum_{n=1, N+1} \lambda_n \delta_{\omega_n}$, the corresponding spectrum defined as in the proposition 3 ($\alpha \in \{\omega_1, \dots, \omega_{N+1}\}$). Then, it is clear that $s_k = \sum_{n=1, N+1} \lambda_n e^{ik\omega_n}$ ($k = 0, N$), and thus $S = \sum_{n=1, N+1} \lambda_n d(\omega_n) d(\omega_n)^H$.

Let us recall how the value of the parameters $(\omega_n)_{n=1, N+1}$ and $(\lambda_n)_{n=1, N+1}$ can be calculated (see e.g. [18] p. 156). To do this, we denote by $Q_N(z)$ the degree N first order Szegö orthogonal polynomial¹ associated with the sequence $(s_n)_{n=0, N}$ [15]. Furthermore, let us denote $B_N(z) = zQ_N(z)/\tilde{Q}_N(z)$ where $\tilde{Q}_N(z) = z^N[Q_N(1/z^*)]^*$. Now, let $\alpha \in \{\omega_1, \dots, \omega_{N+1}\}$. Then, $\{\omega_1, \dots, \omega_{N+1}\} = \{\omega; B_N(e^{i\omega}) = B_N(e^{i\alpha})\}$. Furthermore, $B_N(e^{i\alpha})$ spans the unit circle ($N + 1$ times) when α spans $] - \pi, \pi]$. Together with the proposition 3, this yields

Proposition 4 *If $S \in \text{Int}(T_+)$, it admits representations in the form (22). To any such representation corresponds a point ε of the unit circle for which the $(\omega_n)_{n=1, N+1}$ are the solutions of the equation $B_N(e^{i\omega}) = \varepsilon$.*

On another hand, the corresponding λ_n are given by $\lambda_n = [d(\omega_n)^H S^{-1} d(\omega_n)]^{-1}$. It comes from (8) that up to a factor $N+1$, λ_n represents the value of the Capon spectrum estimator at point ω_n .

4.2 Optimization of $\Phi(S, \hat{R})$ in the real case

When dealing with a real process, S is a symmetric matrix with real coefficients $(s_n)_{n=0, N}$. In this case, a restricted version of the propositions 3 and 4 is available:

Proposition 5 *the coefficients $(s_n)_{n=0, N}$ of a given real covariance matrix S of $\text{Int}(T_+)$ represent the first correlation coefficients of exactly two spectra of real valued processes*

¹ $Q_N(z) = z^n A_n(z)$, where $A_n(z)$ is the classical prediction error filter

carried by $N + 1$ points. These spectra are carried by the points that are the solutions of the equations $B_N(e^{i\omega}) = -1$, and $B_N(e^{i\omega}) = +1$ respectively.

We will denote μ^- and μ^+ respectively the spectra obtained for $B_N(e^{i\omega}) = -1$, and $B_N(e^{i\omega}) = +1$. They are in the form

$$\mu^- = \sum_{n=1, N+1} \lambda_n^- \delta_{\omega_n^-}, \text{ and } \mu^+ = \sum_{n=1, N+1} \lambda_n^+ \delta_{\omega_n^+}. \quad (23)$$

Since $s_k = \sum_{n=1, N+1} \lambda_n^- e^{ik\omega_n^-} = \sum_{n=1, N+1} \lambda_n^+ e^{ik\omega_n^+}$ ($k = 0, N + 1$), S can be written

$$S = \sum_{n=1, N+1} \lambda_n^- d(\omega_n^-) d(\omega_n^-)^H = M^- D^- [M^-]^H \quad (24)$$

and $S = \sum_{n=1, N+1} \lambda_n^+ d(\omega_n^+) d(\omega_n^+)^H = M^+ D^+ [M^+]^H,$

where $M^\pm = (d(\omega_1^\pm), \dots, d(\omega_{N+1}^\pm))$, and $D^\pm = \text{diag}(\lambda_1^\pm, \dots, \lambda_N^\pm)$. These decompositions of S are well known in speech coding, and are named the Line Spectrum Pair (LSP) decompositions [17].

These decompositions are used in [11] to minimize $\Phi(S, \widehat{R})$. The principle is the following : first, let us consider a positive definite Toeplitz matrix, denoted S_0 , and one of its above defined decompositions, for instance that with superscript $+$, that we denote $M_0^+ D_0^+ [M_0^+]^H$. Now, let us consider the matrix

$$\widehat{D}_0 = \arg \left\{ \min_{D \in \Delta} \Phi(M_0^+ D [M_0^+]^H, \widehat{R}) \right\}, \quad (25)$$

where Δ is the set of all diagonal matrices. It is shown in [11] that \widehat{D}_0 is the diagonal part of $[M_0^+]^{-1} \widehat{R} [M_0^+]^{-H}$, and thus $\widehat{D}_0 > 0$. Then, we define $S_1 = M_0^+ \widehat{D}_0 [M_0^+]^H$. Clearly, $S_1 > 0$, and $\Phi(S_1, \widehat{R}) \leq \Phi(S_0, \widehat{R})$. The representations of S_1 in the form (24) will be denoted $M_1^- D_1^- [M_1^-]^H$ and $M_1^+ D_1^+ [M_1^+]^H$ ($M_1^+ = M_0^+$, and $D_1^+ = \widehat{D}_0$). Then it is possible to define S_2 by minimizing $\Phi(M_1^- D [M_1^-]^H, \widehat{R})$ over D :

$$S_2 = M_1^- \times \arg \left\{ \min_{D \in \Delta} \Phi(M_1^- D [M_1^-]^H, \widehat{R}) \right\} \times [M_1^-]^H. \quad (26)$$

Iterating this procedure yields a sequence $\{S_k\}_{k \geq 0}$ of positive definite Toeplitz matrices. It is clear that $\{\Phi(S_k, \widehat{R})\}_{k \geq 0}$ is a decreasing sequence. Furthermore, it was proved in [11] that its accumulation points satisfy $\nabla \Phi(S, \widehat{R}) = 0$. Also, $\lim_{k \rightarrow \infty} \|S_{k+1} - S_k\| = 0$, which ensures that either the sequence converges or it admits a continuous set of accumulation points.

4.3 Parametrization of the elements of T_+^Ω

It is clear that the approach described here above does not apply to the band-limited case since the sequence $\{S_k\}_{k \geq 0}$ will generally not belong to T_+^Ω .

In order to parallel the above discussion, and optimize $\Phi(S, \widehat{R})$ over T_+^Ω , we are going first to consider the decompositions of the elements of $Int(T_+^\Omega)$ in the form (22). More particularly, in order to be able to use these results to minimize $\Phi(S, \widehat{R})$ over T_+^Ω we will restrict our interest to the decompositions for which the $(\omega_n)_{n=1, N+1}$ are in $[-\Omega, \Omega]$. To get them, we use the following result which derives directly from [9]. Denoting $\varepsilon_- = B_N(e^{-i\Omega})$, $\varepsilon_+ = B_N(e^{i\Omega})$ and $[\varepsilon_-, \varepsilon_+]$ the positively oriented arc of the unit circle with initial point ε_- and end point ε_+ , we have

Proposition 6 *If $S \in T_+^\Omega$, it admits representations in the form (22) for which the $(\omega_n)_{n=1, N+1}$ lay inside $[-\Omega, \Omega]$. To any such representation corresponds a value $\varepsilon \in [\varepsilon_-, \varepsilon_+]$, for which the $(\omega_n)_{n=1, N+1}$ are the solutions of the equation $B_N(e^{i\omega}) = \varepsilon$.*

To explain the interest of the representations in the form (22) that are defined by $N + 1$ points of $[-\Omega, \Omega]$, let $S \in Int(T_+^\Omega)$ and

$$S = \sum_{n=1, N+1} \lambda_n d(\omega_n) d(\omega_n)^H = M D M^H \quad (27)$$

such a decomposition, with $D = \text{diag}(\lambda_1, \dots, \lambda_{N+1})$. Then, looking for $\min_{\Lambda \in \Delta} \Phi(M \Lambda M^H, \widehat{R})$ yields the matrix $S_1 = M D_1 M^H$, where $D_1 = \text{diag}(\lambda_1^1, \dots, \lambda_{N+1}^1)$ is the diagonal part of

$M^{-1}\widehat{R}M^{-H}$. Thus S_1 is positive definite. Consequently, $S_1 \in \text{Int}(T_+^\Omega)$ since its coefficients are the first correlation coefficients of the band-limited spectrum $\sum_{n=1, N+1} \lambda_n^1 \delta_{\omega_n}$. Furthermore we have $\Phi(S_1, \widehat{R}) \leq \Phi(S, \widehat{R})$.

In the real case, two representations corresponding with fixed values of ε ($\varepsilon = -1$ and $\varepsilon = +1$) were used to minimize $\Phi(S, \widehat{R})$. Here, we cannot specify two fixed values in $[\varepsilon_-, \varepsilon_+]$ since this arc depends on the matrix S under consideration, and thus will change at each step of the algorithm.

In fact, the only simple way to get decompositions for which the $(\omega_n)_{n=1, N+1}$ are always in $[-\Omega, \Omega]$ is to select the decompositions for which $\varepsilon = \varepsilon_-$ and $\varepsilon = \varepsilon_+$. They will be denoted respectively

$$S = \sum_{n=1, N+1} \lambda_n^{-\Omega} d(\omega_n^{-\Omega}) d(\omega_n^{-\Omega})^H = M^{-\Omega} D^{-\Omega} [M^{-\Omega}]^H \quad (28)$$

$$\text{and } S = \sum_{n=1, N+1} \lambda_n^{+\Omega} d(\omega_n^{+\Omega}) d(\omega_n^{+\Omega})^H = M^{+\Omega} D^{+\Omega} [M^{+\Omega}]^H.$$

Let us remark that $-\Omega \in \{\omega_n^{-\Omega}\}_{n=1, N+1}$, and that $+\Omega \in \{\omega_n^{+\Omega}\}_{n=1, N+1}$, because by definition $B_N(e^{\pm i\Omega}) = \varepsilon_{\pm}$, and the $\omega_n^{\pm\Omega}$ are the solutions of the equation $B_N(e^{i\omega}) = \varepsilon_{\pm}$.

4.4 Optimization of $\Phi(S, \widehat{R})$ over T_+^Ω

Let us now present the relaxation technique for minimizing $\Phi(S, \widehat{R})$ over T_+^Ω . Starting from a matrix S_0 in $\text{Int}(T_+^\Omega)$ and one of its decompositions in the form (28), for instance that with superscript $+\Omega$ ($S_0 = M_0^{+\Omega} D_0^{+\Omega} [M_0^{+\Omega}]^H$), we define

$$S_1 = M_0^{+\Omega} \times \arg \left\{ \min_{D \in \Delta} \Phi(M_0^{+\Omega} D [M_0^{+\Omega}]^H, \widehat{R}) \right\} \times [M_0^{+\Omega}]^H. \quad (29)$$

From the discussion in section 4.3, $S_1 \in \text{Int}(T_+^\Omega)$. Then, iteratively, we define the matrices

$$S_{k+1} = M_k \times \arg \left\{ \min_{D \in \Delta} \Phi(M_k D [M_k]^H, \widehat{R}) \right\} \times [M_k]^H, \quad (30)$$

where $M_k D_k [M_k]^H$ is the decomposition of S_k with superscript $(-1)^k \Omega$.

The sequence $\{S_k\}_{k \geq 0}$ belongs to $\text{Int}(T_+^\Omega)$. This ensures the existence of the decompositions in the form (28) for all S_k . Also, the sequence $\{\Phi(S_k, \widehat{R})\}_{k \geq 0}$ is decreasing and lower bounded, and thus converges. Consequently, $\{S_k\}_{k \geq 0}$ admits at least one accumulation point. If this accumulation point belongs to $\text{Int}(T_+^\Omega)$, then the gradient of $\Phi(S_k, \widehat{R})$ is equal to zero at this point. This can be checked by paralleling the proof given by Degerine for the real case on T_+ [11].

However, the algorithm presented here can have an accumulation point on the boundary $\partial T_+^{\Omega,2}$. Furthermore, it comes from (6) that for any matrix S on $\partial T_+^{\Omega,2}$ the decompositions $M^{+\Omega} D^{+\Omega} [M^{+\Omega}]^H$ and $M^{-\Omega} D^{-\Omega} [M^{-\Omega}]^H$ are the same and correspond to a mass point measure with mass at both points $-\Omega$, and $+\Omega$. Let us denote MDM^H this decomposition, and let $\text{diag}(A)$ be the diagonal matrix with same diagonal terms as A . Then, it is clear that $M \times \text{diag}(M^{-1} \widehat{R} M^{-H}) \times M^H$ belongs to $\partial T_+^{\Omega,2}$ and is a stationary point of the algorithm, even if it is not a local minimum of Φ .

Despite these theoretical restrictions, we have checked that the algorithm has a good behaviour in practice. Furthermore, in order to avoid the problem of local minima, it is possible to initialize the algorithm with a point located near the optimum. The projection algorithm proposed in section 3 can be used to get such an initial point. Applying then the relaxation technique should remove the bias of this initial estimate, that is due to the approximation of T_+^Ω by the subset K_+^M .

4.5 Practical computation of the parameters

Now, let us explain how the parameters of the representations $M^{-\Omega} D^{-\Omega} [M^{-\Omega}]^H$ and $M^{+\Omega} D^{+\Omega} [M^{+\Omega}]^H$ can be obtained. As we already mentioned it in proposition 6, the $\omega_n^{\pm\Omega}$ are the solutions of the equation $B_N(e^{i\omega}) = \varepsilon_{\pm}$. Also we have already mentioned that

the coefficients $\lambda_n^{\pm\Omega}$ are given by $\lambda_n^{\pm\Omega} = [d(\omega_n^{\pm\Omega})^H S^{-1} d(\omega_n^{\pm\Omega})]^{-1}$, but their computation is not necessary for the algorithm.

In order to avoid solving the equation $B_N(e^{i\omega}) = \varepsilon_{\pm}$ at each step of the algorithm, Degerine proposed a direct way to compute the optimum matrix $S_{l+1} = M_l \times \text{diag}(M_l^{-1} \widehat{R} M_l^{-H}) \times M_l^H$. Let us briefly recall his approach that can be extended directly to our algorithm. Since $\widehat{R} = K^{-1} \sum_{k=1, K} y_k y_k^H$, it comes that

$$S_{l+1} = \frac{1}{K} \sum_{k=1, K} M_l \times \text{diag}(M_l^{-1} y_k y_k^H M_l^{-H}) \times M_l^H. \quad (31)$$

It can be shown that the Toeplitz matrices $P_k = M_l \times \text{diag}(M_l^{-1} y_k y_k^H M_l^{-H}) \times M_l^H$ ($k = 1, K$) can be obtained without explicit calculation of the $e^{i\omega_n}$ nor inversion of M_l . Then, S_{l+1} is simply given by the relation $S_{l+1} = K^{-1} \sum_{k=1, K} P_k$. The calculation of the P_k is recalled in the Appendix E.

4.6 A relaxation algorithm applied to $\Phi_2(S, \widehat{R})$

Another way to account for the possible existence of local minima of $\Phi(S, \widehat{R})$, is to use the relaxation approach to find $\min_{S \in T_+^{\Omega}} \Phi_2(S, \widehat{R})$, where $\Phi_2(S, \widehat{R})$ is given by (11). In this case, at the k^{th} step of the algorithm, we have to minimize $\Phi_2(M_k D [M_k]^H, \widehat{R})$ with D spanning the set of diagonal matrices. Denoting $M_k = (d(\omega_1^k), \dots, d(\omega_{N+1}^k))$, the optimum value for D is the matrix \widehat{D} which diagonal terms are the coordinates of the vector $\widehat{d} = [H^k]^{-1} b^k$, with $[H^k]_{m,n} = \left| d(\omega_m^k)^H \widehat{R}^{-1} d(\omega_n^k) \right|^2$, and $[b^k]_m = d(\omega_m^k)^H \widehat{R}^{-1} d(\omega_m^k)$. Unfortunately, some coordinates of \widehat{d} may be negative. In this case, the decompositions in the form (28) would not exist for the matrix $M_k \widehat{D} [M_k]^H$, preventing thus from continuing the algorithm.

To solve this problem, it is possible to project the optimum vector \widehat{d} on the set $\{u = (u_1, \dots, u_{N+1})^T; u_n > \epsilon, n = 1, N + 1\}$ ($\epsilon > 0$), i.e. to constrain to ϵ the coordinates of \widehat{d} that are smaller than ϵ . This yields a vector \widehat{d}' , and a matrix $M_k \widehat{D}' [M_k]^H$ of $\text{Int}(T_+^{\Omega})$,

and then it is possible to continue the algorithm.

The convergence of this algorithm is not guaranteed. However, in practice, we obtained good results on simulations.

5 Application

First, we evaluate by simulation the performance of the estimates described here above. Then, we apply the preceding results to the problem of sources localization in the presence of a spherical-isotropic noise.

5.1 Performance of the estimators

Let us consider the spectrum with density function $f(\omega) = \chi_{[-\Omega, \Omega]}(\omega)$. On figure 1, we represent the function $F(R) = \{E[\frac{\|R - \tilde{R}\|^2}{\|R\|^2}]\}^{1/2}$, where \tilde{R} is an estimator of R . We give the results when \tilde{R} represents respectively the projection of the empirical estimator on the space of Toeplitz matrices, and the estimators obtained when accounting for the band-limiting constraint with the projection and the relaxation approaches. The results presented on Figure 1 are obtained for $N + 1 = 8$, $\Omega = \pi/2$, and $M = 2N + 1$. 10^4 tests have been performed to compute the empirical estimate of the mathematical expectation of $\|R - \tilde{R}\|^2 / \|R\|^2$. Let us remark that changing Ω or M does not change much these results. The projection technique yields worse results than the relaxation technique, partly because the projection technique yields a biased estimate.

5.2 The spherical isotropic noise

We address the problem of estimating the directions of arrival of p uncorrelated plane wave sources received by a linear array of $N + 1$ equispaced sensors ($p < N + 1$) corrupted by a spherical isotropic noise [1]. Classically a Fourier transform is performed behind each sensor, and the sources can be located using the information obtained at a given time frequency f . Of course, several techniques have been developed to account for the information obtained at all the time frequencies f under study (see e.g. [13]-[19]-[26]) in the sources localization scheme. These techniques will not be considered here, and we simply address the problem of sources localization from the observation at a given frequency f .

The spherical isotropic noise [1] is generated by a superposition of propagative phenomena with equal intensities in the set of all directions of the space. Let us precise the properties of this noise. For this purpose, we denote $n(f) = [n_1(f), \dots, n_{N+1}(f)]^T$ the random vector whose components represent the contributions of the noise to the signals received on the sensors of the array at the frequency f . Then, as usual, we assume that the distance d between two contiguous sensors is given by $d = c/2f_{\max}$, where f_{\max} is the maximum frequency of the received signal, and c the velocity of propagation. For $f \leq f_{\max}$, the covariance matrix of the noise is given by

$$R_N(f) = E[n(f)n(f)^H] = \sigma^2(f)B_N(f) \quad (32)$$

where $\sigma^2(f)$ is a positive scalar, and $B_N(f)$ is the $(N + 1) \times (N + 1)$ matrix whose general term is

$$[B_N]_{m,n} = \frac{\sin[(m - n)\pi f / f_{\max}]}{(m - n)\pi}. \quad (33)$$

The k^{th} correlation coefficient of the noise process is given by $r_k(f) = \sigma^2(f) \sin(k\pi f / f_{\max}) / k\pi$, which is equivalent to say that the spectral density of the noise at each frequency f is

given by

$$\begin{aligned} S_f(\omega) &= \sigma^2(f) \quad \text{if } \omega \in [-\pi f/f_{\max}, \pi f/f_{\max}] \\ S_f(\omega) &= 0 \quad \text{if } \omega \notin [-\pi f/f_{\max}, \pi f/f_{\max}], \end{aligned} \tag{34}$$

where ω is the spatial frequency. In particular, the spatial spectrum of the noise at each frequency f is carried by $[-\pi f/f_{\max}, \pi f/f_{\max}]$. More generally, this last property holds when the noise is originated by a superposition of propagative phenomena, i.e. when it satisfies the wave equation. Such a kind of noise is often met in practical cases (e.g. in underwater acoustics; see [1] for more details).

5.3 Sources localization

Let $(\theta_k)_{k=1,p}$ be the directions of arrival of the sources that are assumed to be uncorrelated. Then, their contribution to the spatial spectrum at the frequency $f \leq f_{\max}$, is a superposition of p complex sinusoids whose angular frequencies are $\varphi_k(f) = \pi f/f_{\max} \sin \theta_k$. Let us remark that $\varphi_k \in [-\pi f/f_{\max}, \pi f/f_{\max}]$.

When the sources are corrupted by a spherical isotropic noise, the spatial spectrum of the observation is carried by the interval $[-\pi f/f_{\max}, \pi f/f_{\max}]$. We are going to account for this a priori information to improve the performance of a standard MUSIC-like algorithm for estimating the $(\varphi_k(f))_{k=1,p}$ (see e.g.[23]). From now on, we fix the frequency f , with $f < f_{\max}$, and omit to mention that the spatial parameters depend on f . Also, we denote $\Omega = \pi f/f_{\max}$.

Let $\{\lambda_k^2\}_{k=1,p}$ be the power of the sources. The covariance matrix of the observed signal is

$$R = \sum_{k=1,p} \lambda_k^2 d(\varphi_k) d(\varphi_k)^H + \sigma^2 B_N, \tag{35}$$

and $R \in \text{Int}(T_+^\Omega)$.

Now, let us recall how the $(\varphi_k)_{k=1,p}$ can be estimated. It is clear that the space spanned by the vectors $(d(\varphi_k))_{k=1,p}$ is p -dimensional. This space is the orthogonal complement of the eigenspace corresponding to the $N + 1 - p$ smallest eigenvalues of the matrix pencil (R, B_N) . Let us denote $\{u_k\}_{k=p+1,N+1}$ the corresponding eigenvectors. Then, the angular frequencies of the sources are given by the p greatest maxima of the localization function $g(\varphi)$ defined by

$$g(\varphi) = \frac{1}{\sum_{k=p+1,N+1} |u_k^H d(\varphi)|^2}. \quad (36)$$

With true matrices R and B_N , the maxima of $g(\varphi)$ are infinite. In practice however, R is not known exactly but it has to be estimated by a certain matrix denoted \tilde{R} . In (36), the u_k are replaced by the eigenvectors associated with the $N + 1 - p$ smallest eigenvalues of the matrix pencil (\tilde{R}, B_N) . Then, the p greatest maxima of $g(\varphi)$ yield the estimates of the angular frequencies of the sources.

To estimate vectors $\{u_k\}_{k=p+1,N+1}$, we use the generalized eigenvalue decomposition technique [16]-[22] which applies to matrix pencils of hermitian matrices. Furthermore, it is robust when dealing with the ill-conditioned matrices \tilde{R} and B_N and it allows to compute directly a basis of the space spanned by the smallest eigenvalues.

5.4 Comparison of different estimators of R

We are going to present some simulations obtained when using the estimators of R presented in this paper.

The values of the parameters have been chosen as follows : $\Omega = \pi/2$, $N + 1 = 8$, $p = 3$, $\varphi_k \in \{-80^\circ, 45^\circ, 75^\circ\}$, $\lambda_k^2 = 1$ ($k = 1, 3$), and $\sigma^2 = 20$. Thus, the signal to noise ratio is $-10dB$ per source. The empiric covariance matrix \hat{R} is estimated from $K = 300$ independant data vectors.

On Figure 2, $g(\varphi)$ is obtained when the estimate of R is the projection of \widehat{R} on the set of Toeplitz matrices. 10 localization functions are plotted. It appears that close sources are not distinguished and that the localization is not precise.

Figure 3 represents the results obtained when minimizing $\Phi(S, \widehat{R})$ over K_+^M , with $M = 2N + 1$. Here, we achieve very good localization. Furthermore, on Figure 4, we account for an additional white noise with variance equal to 1. The sources are well located, which shows the capability of algorithm (21) to distinguish the respective contributions of band-limited and non band-limited signals.

Figure 5 is obtained with the relaxation approach. Results are about the same as those obtained with the projection technique.

6 Conclusion

In this paper, we proposed two algorithms to estimate the correlation coefficients of a band limited signal in a way that takes into account this property. By applying these algorithms to the problem of sources localization in the presence of a band-limited noise, we checked the improvement that they can bring, over other classical techniques.

References

- [1] P.Y. Arques, J.P. Le Cadre, Underwater Noise and their Identification for Array Processing Purpose, *Signal Processing IV, EURASIP*, 1988, pp. 15-21.
- [2] K.J. Harrow, L. Hurwitz, H. Uzawa, *Studies in Linear and Nonlinear Programming*, Stanford University Press, Stanford USA, 1958.

- [3] K.S. Arun, L.E. Potter, Existence and Uniqueness of Band-Limited, Positive Semi-Definite Extrapolations, *IEEE Trans. Acous., Speech Signal Proc.*, vol. 38, n. 3, march 1990, pp. 547-549.
- [4] P.J. Brockwell, R.A. Davis, *Time Series : Theory and Methods*, Springer Verlag, New York, 1991.
- [5] J.A. Cadzow, Signal Enhancement using Canonical Operators, *Proc. ICASSP*, 1987, pp. 673-676.
- [6] J. Capon, High Resolution Frequency-Wavenumber Spectrum Analysis, *Proc. IEEE*, vol. 57, Aug. 1969, pp. 1408-1418.
- [7] T. Chonavel, *Band-Limited Spectrum estimation. Application to Array Processing*, PhD dissertation, Paris, déc. 1992.
- [8] T. Chonavel, S. Degerine, "Reflection coefficients of band-limited spectra, and band-limited spectrum modeling", *IEEE Trans. on Inf. Theory*, pp.2030-2038 , sept 1999.
- [9] T. Chonavel, P. Loubaton, Band-Limited Spectral Estimation from Partial Covariance Sequences, *IEEE Trans. Inf. Theory*, vol. 39, n. 4, July 1993, pp. 1395-1397.
- [10] T. Chonavel, P. Loubaton, Estimation of the Autocorrelation Coefficients of Band-Limited Signals, Application to the Sources Localization Problem, *Signal processing VI, Proc. EUSIPCO* , 1992, pp. 1825-1828.
- [11] S. Degerine, Maximum Likelihood Estimation of Autocovariance Matrices from Replicated Short Time Series, *Jour. Time Series Analysis* , vol. 8, n. 2, pp. 135-146.
- [12] D.R. Fuhmann, M.I. Miller, On the Existence of Positive Definite Maximum Likelihood Estimates of Structured Covariance Matrices, *IEEE Trans. Inf. Theory*, vol. 34, n. 4, July 1988, pp. 722-729.
- [13] P. Forster, G. Vezzosi, Approximate ML After Spheroidal Beamforming in the Narrow Band and Wide Band Cases, *Signal Processing IV, EURASIP*, 1988, pp. 307-310.

- [14] Ya.L. Geronimus, *Polynomials Orthogonal on a Circle and Interval*, Pergamon Press, 1960.
- [15] U. Grenander, G. Szegö, *Toeplitz Forms and their Applications*, Univers. of California Press, Berkley and Los Angeles, 1958.
- [16] G.Golub, C. Van Loan, *Matrix Computation*, The Johns Opkins University Press, 1989.
- [17] F. Itakura, N. Sugumura, LSP Speech Synthesizer, its Principle and Implementation, *Proc. of Speech Study Group of the Acous. Soc. of Japan*, S79-46, Nov. 1979.
- [18] M.G. Krein, A.A. Nudel'man, *The Markov Moment Problem and Extremal Problems*, Transl. Math. Monographs 50, Amer. Math. Soc., Providence, Rhodes Island, 1977.
- [19] J.P. Le Cadre, P. Ravazolla, Model Reduction and Wideband Analysis, *Proc. ICASSP*, 1990, pp. 2427-2430.
- [20] M.I. Miller, D.L. Snyder, The Role of Likelihood and Entropy in Incomplete Data Problems : Application to Estimating Point-Process Intensities and Toeplitz Constrained Covariances, *Proc. IEEE*, 75, n. 7, July 1987, pp. 892-907.
- [21] P.M. Mullen, G.C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge Univ. Press, 1971.
- [22] C.C. Paige, M. Saunders, Towards a Generalized Singular Value Decomposition , *SIAM Jour. on Num. Analysis*, 18, 1981, pp. 398-405.
- [23] R.O. Schmidt, Multiple Emitter Location and Signal Parameter Estimation, *Proc. RADC Spectrum Est. Workshop*, Rome, NY, 1979, pp. 243-258.
- [24] D. Slepian, H.O. Pollak, H.J. Landau, Prolate Spheroidal Wave Functions, *Bell Syst. Tech. Jour.*, vol. 40, Janv. 1961, pp. 43-84.

- [25] A.M. Thomson, J.C. Brown, J.W. Kay, D.M. Titterington, A Study of Methods of Choosing the Smoothing Parameter in Image Restoration by Regularization, *IEEE Trans. on Pattern Analysis and Machine Intel.*, vol. 13, n. 4, April 1991, pp. 326-339.
- [26] H. Wang, M. Kaveh, Coherent Signal-Subspace Processing for the Detection and Estimation of Angles of Arrival of Multiple Wide-Band Sources, *IEEE Trans. Acous. Speech, Sig. Proc.*, vol. ASSP-33, August 1985, pp.823-831.
- [27] D.M. Wilkes, M.H. Hayes, Iterated Toeplitz Approximation of Covariance Matrices, *Proc. ICASSP*, 1988, pp. 1663-1666.

Appendix

A Proof of lemma 1

It is well known (e.g. [18], p. 16) that a convex set coincides with the intersection of all the half spaces that contain this convex set. Furthermore, a theorem proved by Weil (e.g. [21]) states that for a closed convex cone C in the form $C = \{\sum_{k=1,K} \alpha_k u_k; \alpha_k \geq 0\}$ where the u_k are fixed vectors of \mathbb{R}^l , C coincides with the intersection of the half spaces that contain C and that are limited by hyperplanes in the form $Sp\{u_{i_1}, \dots, u_{i_{l-1}}\}$ where i_1, \dots, i_{l-1} are distinct values in $\{1, 2, \dots, K\}$. Here, $Sp\{.\}$ represents the space spanned by a set of vectors.

From what precedes, it comes that $\tau(K_+^M)$ is the intersection of a set of half spaces that are bounded by hyperplanes in the form $Sp\{e(\omega_{i_1}), \dots, e(\omega_{i_{2N}})\}$, where $i_1, \dots, i_{2N} \in \{1, 2, \dots, M\}$ and $i_1 < \dots < i_{2N}$. Let Π be such an hyperplane and $v = (v_1, \dots, v_{2N+1})^T$ a vector of Π^\perp . Then, $\forall n = 1, 2N$, $v^T e(\omega_{i_n}) = 0$. Setting for instance $v_1 = 1$, the relations $v^T e(\omega_{i_n}) = 0$ yield

$$v = \begin{pmatrix} 1 \\ -A^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \end{pmatrix}, \text{ with } A = \begin{pmatrix} \cos \omega_{i_1} & \sin \omega_{i_1} & \cdots & \cos N\omega_{i_1} & \sin N\omega_{i_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos \omega_{i_{2N}} & \sin \omega_{i_{2N}} & \cdots & \cos N\omega_{i_{2N}} & \sin N\omega_{i_{2N}} \end{pmatrix}.$$

It is clear that Π defines a half space that contains $\tau(K_+^M)$ if and only if all the scalar products $v^T e(\omega_k)$ ($k = 1, M$) have the same sign. Then, from the matrix inversion lemma [16], we have

$$\begin{vmatrix} L_1 & L_3 \\ L_2 & L_4 \end{vmatrix} = |L_1| \cdot |L_4 - L_2 L_1^{-1} L_3|$$

where $|\cdot|$ is the determinant operator. Let $L_1 = A^T$, $L_2 = [1, \dots, 1]$, $L_3 = [\cos(\omega_k), \sin(\omega_k), \dots, \cos(N\omega_k), \sin(N\omega_k)]$ and $L_4 = 1$. It comes that

$$v^T e(\omega_k) = |A|^{-1} \left| [e(\omega_{i_1}), \dots, e(\omega_{i_{2N}}), e(\omega_k)] \right|$$

and the calculation of the vandermonde determinant $\left| [e(\omega_{i_1}), \dots, e(\omega_{i_{2N}}), e(\omega_k)] \right|$ yields

$$v^T e(\omega_k) = |A|^{-1} 4^{(2N+1)^2} \prod_{2N \geq l > m \geq 1} \sin \frac{\omega_{i_l} - \omega_{i_m}}{2} \prod_{2N \geq m \geq 1} \sin \frac{\omega_k - \omega_{i_m}}{2}.$$

Let us remark that the sign of $|A|^{-1}$ does not depend on ω_k .

If $k \in \{i_1, \dots, i_{2N}\}$, $v^T e(\omega_k) = 0$. Let us assume now that $k \notin \{i_1, \dots, i_{2N}\}$, and let m be the number such that $i_m < k < i_{m+1}$, if $1 < k < M$ and $m = 0$ otherwise. Then, it is clear that the sign of $|A|v^T e(\omega_k)$ is the sign of $(-1)^m$. This means that the sign of $v^T e(\omega_k)$ does not depend on k if and only if either $i_{m+1} = i_m + 1$ for all odd values of the index m , or $i_1 = 1$, $i_{2N} = M$ and $i_{m+1} = i_m + 1$ for all even values of the index m . The structure of the boundary hyperplanes derives directly from this fact.

B Proof of proposition 2

Here, we consider the $2N + 1$ -dimensional real space of size $N + 1$ hermitian Toeplitz matrices. The scalar product is defined by $\langle A, B \rangle = \text{Tr}(AB^H) = \text{Tr}(AB)$. The associated norm will be denoted $\|\cdot\|$. Let us remark that $\langle \cdot, \cdot \rangle$ is a real scalar product since if $\sum_{n=1, N+1} \lambda_k u_k u_k^H$ represents the eigenvalue decomposition of A , then $\langle A, B \rangle = \sum_{n=1, N+1} \lambda_k u_k^H B u_k$, which is real.

To prove the proposition 2, we establish a few intermediate results

First, we are going to show that when M is large, it is possible to approach any element of ∂K_+^M by an element of ∂T_+^Ω . In mathematical terms, this yields the following assumption

1) $\forall \varepsilon > 0, \exists M_1, \forall M > M_1, \forall L \in \partial K_+^M, \exists \tilde{L} \in \partial T_+^\Omega, \|L - \tilde{L}\| < \varepsilon \|L\|$.

From lemma 1, the condition $L \in \partial K_+^M$ can be written in the form

$$L = \sum_{k=1, N} [\lambda_{2k} F(\omega_{a_k}) + \lambda_{2k+1} F(\omega_{a_{k+1}})]$$

or in the form

$$L = \lambda_1 F(\omega_1) + \sum_{k=1, N-1} [\lambda_{2k} F(\omega_{a_k}) + \lambda_{2k+1} F(\omega_{a_{k+1}})] + \lambda_M F(\omega_M),$$

where the λ_n are positive, and $\omega_n = -\Omega + \Omega(2n-1)/M$. Let us denote \tilde{L} the corresponding matrix of ∂T_+^Ω defined by

$$\sum_{k=1, N} [\lambda_{2k} + \lambda_{2k+1}] F(\omega_{a_k}) \quad \text{or} \quad \lambda_1 F(-\Omega) + \sum_{k=1, N-1} [\lambda_{2k} + \lambda_{2k+1}] F(\omega_{a_k}) + \lambda_M F(\Omega)$$

respectively. Now, since $\lim_{\theta \rightarrow \omega} \|F(\theta) - F(\omega)\| = 0$, it comes that

$$\forall \alpha > 0, \exists M_1, \forall M > M_1, \forall k, \|F(\omega_k) - F(\omega_{k+1})\| < \alpha$$

Then, for $M > M_1$, $\|L - \tilde{L}\| < (\sum_k \lambda_k) \alpha < C \alpha \|L\|$, where C is a constant. Letting $\alpha = \varepsilon/C$ yields 1).

2) Let $S \in \text{Int}(T_+^\Omega)$, and denote a the distance from S to ∂T_+^Ω : $a = \min_{U \in \partial T_+^\Omega} \|S - U\|$. Then, the open ball with center S and radius $a/2$, denoted $B(S, a/2)$, is in $\text{Int}(T_+^\Omega)$. Let us show that, for M large enough, $B(S, a/2) \subset \text{Int}(K_+^M)$ or $B(S, a/2) \subset \text{Int}(T_+^\Omega) - K_+^M$.

Let L be any element of ∂K_+^M . From 1), for M large enough, $\exists \tilde{L} \in \partial T_+^\Omega, \|L - \tilde{L}\| < a/2$.

Then,

$$\|S - L\| \geq \left| \|S - \tilde{L}\| - \|\tilde{L} - L\| \right| > a - a/2.$$

Thus, $B(S, a/2)$ do not intersect with ∂K_+^M , and either $B(S, a/2) \subset \text{Int}(K_+^M)$ or $B(S, a/2) \subset \text{Int}(T_+^\Omega) - K_+^M$.

3) We now prove that for M large enough, $B(S, a/2) \subset \text{Int}(K_+^M)$, which will complete the proof of the proposition. From 2), it is sufficient to prove that $B(S, a/2)$ is not in $\text{Int}(T_+^\Omega) - K_+^M$.

From general results of [18] (p.16), it can be shown that since $S \in \text{Int}(T_+^\Omega)$, it can be written in the form

$$S = \sum_{k=1, 2N+1} \alpha_k F(\varphi_k),$$

with $\alpha_k > 0$ and the $\varphi_k \in [-\Omega, \Omega]$ ($k = 1, 2N + 1$). But, for a given $\varepsilon > 0$, and M large enough, we have

$$\forall k = 1, 2N + 1, \exists i_k \in \{1, \dots, M\}, \|F(\varphi_k) - F(\omega_{i_k})\| < \frac{\varepsilon}{\sum_{k=1, 2N+1} \alpha_k}.$$

Then, it comes that

$$\|S - \sum_{k=1, 2N+1} \alpha_k F(\omega_{i_k})\| < \varepsilon.$$

But, $\sum_{k=1, 2N+1} \alpha_k F(\omega_{i_k})$ belongs to K_+^M . Thus, taking $\varepsilon < a/2$, it comes that there are elements of K_+^M in $B(S, a/2)$, and consequently $B(S, a/2)$ is not in $\text{Int}(T_+^\Omega) - K_+^M$, which completes the proof.

C Rank of matrix H

As in the appendix B, we consider here the $2N + 1$ -dimensional real vector space of hermitian Toeplitz matrices with the scalar product $\langle A, B \rangle = \text{Tr}(AB^H) = \text{Tr}(AB)$. We denote $E_m = d(\omega_m)d(\omega_m)^H$, $E'_m = d'(\omega_m)d'(\omega_m)^H$, with $d'(\omega_m) = \widehat{R}^{-1/2}d(\omega_m)$, and $[H]_{m,n} = |d(\omega_m)^H \widehat{R}^{-1}d(\omega_n)|^2 = \langle E'_m, E'_n \rangle$. Also, we assume that the number of angles ω_m is $M \geq 2N + 1$.

Clearly $\{E_1, \dots, E_{2N+1}\}$ is a basis of the dimension $N + 1$ vector space of Toeplitz hermitian matrices and for any m in $1, M$, E_m is in the form

$$E_m = \sum_{n=1, 2N+1} \lambda_{m,n} E_n.$$

Of course, $\lambda_{m,n} = \delta_{m,n}$ (here δ represents the symbol of Kronecker) for $m \leq 2N + 1$.

Similarly, $\{E'_1, \dots, E'_{2N+1}\}$ is a basis of a subspace of hermitian matrices that also contains E'_{2N+2}, \dots, E'_M :

$$E'_m = \sum_{n=1, 2N+1} \lambda_{m,n} E'_n.$$

Now, let us denote Λ the matrix of size $M \times (2N+1)$ with general terms $[\Lambda]_{m,n} = \lambda_{m,n}$. The $2N+1$ first lines of Λ form the $2N+1$ -dimensional identity matrix, and thus Λ is full rank. Let A be the matrix of size $(2N+1) \times (2N+1)$ with general terms $[A]_{m,n} = Tr(E'_m E'_n)$. For any vector $u \neq 0$ we have $u^H A u = \|\sum_{m=1, M} [u]_m E'_m\|^2 \neq 0$, which shows that A is full rank.

Then, it is easy to check that

$$[H]_{m,n} = Tr(E'_m E'_n) = \sum_{(k,l)=1, 2N+1} \lambda_{m,k} \lambda_{n,l} A_{k,l} = [\Lambda A \Lambda^H]_{m,n} \quad (m, n = 1, M),$$

and $H = \Lambda A \Lambda$. Since Λ and A are full rank, the rank of H is $2N + 1$.

D Uniqueness of the solution of (21)

As in appendices B and C, we consider here the $2N + 1$ -dimensional real vector space T with the scalar product $\langle \cdot, \cdot \rangle$ defined by $\langle A, B \rangle = Tr(AB^H)$.

Let us denote $F_m = F(\omega_m)$ ($m = 1, M$), and $F_{M+1} = R_a$. Then, let us consider the closed convex cone $K = \{\sum_{m=1, M+1} \alpha_m F_m; \alpha_m \geq 0 (m = 1, M + 1)\}$. The problem

$\text{Min}_{S \in K} \|(S - \widehat{R})\widehat{R}^{-1}\|$ admits a unique solution denoted \widetilde{R} : \widetilde{R} is the projection of \widehat{R} on K , in the set of hermitian matrices associated with the norm $\|U\|_{\widehat{R}^{-1}} = [\text{Tr}(U\widehat{R}^{-1}U\widehat{R}^{-1})]^{1/2}$. Then, to know if the solution of (19) is unique, we must check whether the representation $\widetilde{R} = \sum_{m=1, M+1} \alpha_m F_m$, where the coefficients α_m ($m = 1, M + 1$) are positive, is unique.

First, let us remark that for any integers i_1, \dots, i_m in $\{1, 2, \dots, M\}$, with $m \leq 2N$, $S_p\{F_{i_1}, \dots, F_{i_m}\}$ is of dimension lower than the dimension of T . Thus, except for very particular cases, it will not contain $F_{M+1} = R_a$, and we will assume in the following that this condition is always satisfied. In particular, for $M \leq 2N$ the decomposition is unique.

For $M > 2N$, the uniqueness is no longer satisfied. However, Let us remark that when $\widehat{R} \notin K$, \widetilde{R} belongs to the boundary of K , because it represents the projection of \widehat{R} on K , for the norm $\|\cdot\|_{\widehat{R}^{-1}}$. In this case, we are going to show that the decomposition $\widetilde{R} = \sum_{m=1, M+1} \alpha_m F_m$ with $\alpha_m \geq 0$ ($m = 1, M + 1$) is unique. To show this, let us remark first that from the Theorem of Weil, recalled at the beginning of the Appendix A, when \widetilde{R} is ∂K it admits a decomposition in the form $\sum_{k=1, 2N} \lambda_k F_{i_k}$, with $i_1, \dots, i_{2N} \in \{1, 2, \dots, M + 1\}$. Now, let $\widetilde{R} = \sum_{m=1, M} \alpha_m F_m$ ($\alpha_m \geq 0$, $m = 1, M + 1$) be any decomposition of \widetilde{R} and $\widetilde{R}_1 = \sum_{m \in \{i_1, \dots, i_{2N}\}} \alpha_m F_m$ and $\widetilde{R}_2 = \sum_{m \notin \{i_1, \dots, i_{2N}\}} \alpha_m F_m$. Then, we are going to show that $\widetilde{R}_2 = 0$.

Let $L \neq 0$ be an element of the orthogonal of $S_p\{F_{i_1}, \dots, F_{i_{2N}}\}$ in T . More precisely, we have $S_p\{L\} = [S_p\{F_{i_1}, \dots, F_{i_{2N}}\}]^\perp$. Then, $\langle \widetilde{R}, L \rangle = 0 = \langle \widetilde{R}_1, L \rangle + \langle \widetilde{R}_2, L \rangle = 0 + \langle \widetilde{R}_2, L \rangle$. This yields

$$\langle \widetilde{R}_2, L \rangle = \sum_{m \notin \{i_1, \dots, i_{2N}\}} \alpha_m \langle F_m, L \rangle.$$

But for $m \notin \{i_1, \dots, i_{2N}\}$, $\langle F_m, L \rangle \neq 0$, and since $S_p\{F_{i_1}, \dots, F_{i_{2N}}\}$ is a boundary hyperplane of K , all the scalar products $\langle F_m, L \rangle$ have the same sign. From this and the conditions $\alpha_m \geq 0$ ($m = 1, M + 1$), it comes that the equality $\langle \widetilde{R}_2, L \rangle = 0$ is satisfied if and only if $\alpha_m = 0$ for $m \notin \{i_1, \dots, i_{2N}\}$. Then, $\widetilde{R} = \widetilde{R}_1$, and the representation is unique since the decomposition of any element of $S_p\{F_{i_1}, \dots, F_{i_{2N}}\}$ is unique in the basis $\{F_{i_1}, \dots, F_{i_{2N}}\}$.

E Calculation of P_k

We are going to explain how the matrices $P_k = M_l \times \text{diag}(M_l^{-1} y_k y_k^H M_l^{-H}) \times M_l^H$ can be computed in a simple way. First, let us remark that the columns $d(\omega_n)$ of M_l are defined by the roots $e^{i\omega_n}$ of the polynomial

$$\begin{aligned} zQ_n(z) - \varepsilon\tilde{Q}_n(z) &= \prod_{m=1, N+1} (z - e^{i\omega_n}) \\ &= \sum_{n=0, N+1} b_n z^{N+1-n} \end{aligned}$$

where $\varepsilon = \varepsilon_-$ or $\varepsilon = \varepsilon_+$. The coefficients b_k can be obtained from the Levinson recursion. Furthermore, let us denote $y_k = [y_k(0), \dots, y_k(N)]^T$, and let us extend the sequence $(y_k(m))_{m=0, N}$ with the relations

$$\sum_{n=0, N+1} b_n y_k(l-n) = 0 \quad , l \in \mathbb{Z}$$

that yield the sequence $(y_k(m))_{m \in \mathbb{Z}}$. Then, a classical result of analysis indicates that the terms of the serie $(y_k(n))_{n \in \mathbb{Z}}$ are in the form $y_k(n) = \sum_{m=1, N+1} \gamma_m e^{in\omega_m}$. The coefficients γ_m are given by the initial conditions on the serie $y_k(n) : [\gamma_1, \dots, \gamma_{N+1}]^T = M_l^{-1} y_k$. Then, one can easily check that

$$[y_k(n), \dots, y_k(n+N)]^T = [\gamma_1, \dots, \gamma_{N+1}] \times \text{diag}(e^{in\omega_1}, \dots, e^{in\omega_{N+1}}) \times M^T.$$

From this, it comes that

$$\begin{aligned} [P_k]_{1, n+1} &= y(0)y(n)^* \\ &= [1, \dots, 1] \times \text{diag}(M_l^{-1} y_k y_k^H M_l^{-H}) \times [e^{in\omega_1}, \dots, e^{in\omega_{N+1}}]^H \\ &= \sum_{m=1, N+1} |\gamma_m|^2 e^{-in\omega_m} = [\bar{\gamma}_1 e^{-in\omega_1}, \dots, \bar{\gamma}_{N+1} e^{-in\omega_{N+1}}] M_l^{-1} y_k \\ &= [\bar{y}_k(n), \dots, \bar{y}_k(n+N)] (M_l M_l^H)^{-1} y_k. \end{aligned}$$

These relations for $[P_k]_{1,1}, \dots, [P_k]_{1,N+1}$ completely define the Toeplitz hermitian matrix P_k .

The vector $[\bar{y}_k(n), \dots, \bar{y}_k(n+N)]$ can be computed directly from the recursion

$$\sum_{n=0, N+1} b_n y_k(l-n) = 0.$$

In order to compute $(M_l M_l^H)^{-1}$, let us remark that

$$[M_l M_l^H]_{a,b} = \rho_{a-b} = \sum_{n=1, N+1} e^{i(a-b)\omega_n}.$$

Since the $e^{i\omega_n}$ are the roots of the polynomial $\sum_{n=0, N+1} b_n z^{N+1-n}$, the coefficients ρ_l are calculated recursively from the relations

$$\rho_1 = -b_1 \text{ and } \rho_k = -kb_k - \sum_{l=1, k-1} b_l \rho_{k-l}.$$

Indeed, letting $z_k = e^{i\omega_k}$,

$$\begin{aligned} \rho_{k+1} &= (\sum_{i=1, N+1} z_i)(\sum_{i=1, N+1} z_i^k) - \sum_{i=1, N+1} z_i(\sum_{j \neq i} z_j^{k-1}) \\ &= -b_1 \rho_1 - \sum_{i \neq j} (z_i z_j) z_j^{k-1} \\ &= -b_1 \rho_k - b_2 \rho_{k-1} + \sum_{i < j} (z_i z_j) (\sum_{m \neq i, j} z_m^{k-1}) \\ &= \dots \\ &= -\sum_{l=1, k-1} b_l \rho_{k-l} + (-1)^{k-2} \sum_{c_1 < \dots < c_{k-1}} z_{c_1} \dots z_{c_{k-1}} (\sum_{m \neq c_1, \dots, c_{k-1}} z_m) \\ &= -\sum_{l=1, k-1} b_l \rho_{k-l} - kb_k. \end{aligned}$$

Finally, from the knowledge of the Toeplitz matrix $M_l M_l^H$, a Cholevski factorisation of its inverse is computed by means of the Levinson recursion, which completes the calculation of P_k .

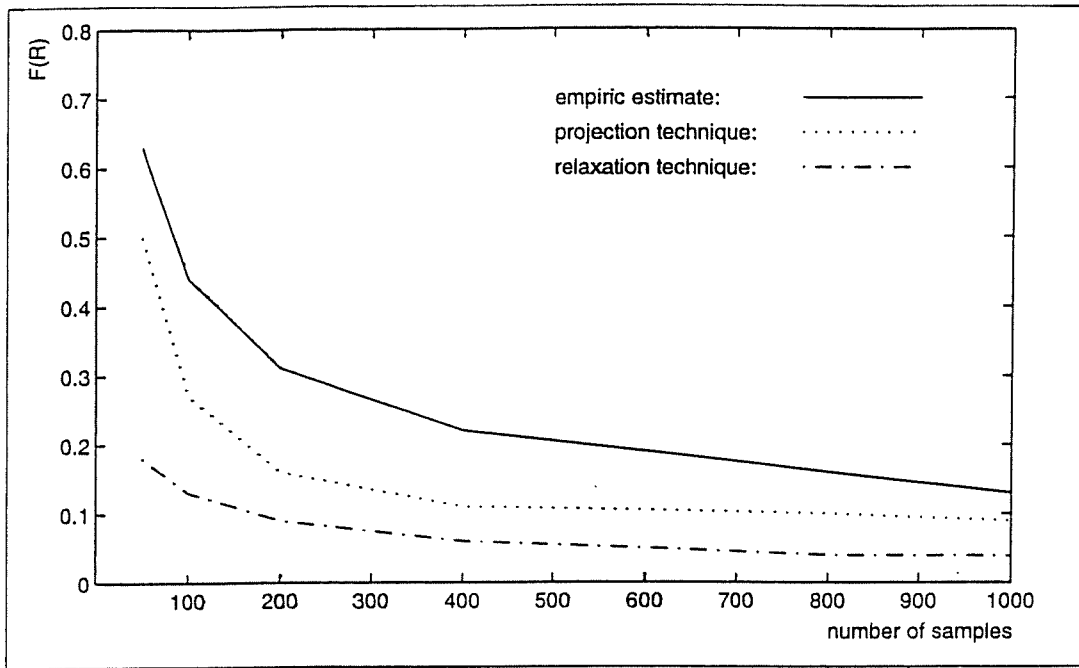


Figure 1 : performance of the estimators ($F(R) = E[\frac{\|R - \tilde{R}\|^2}{\|R\|^2}]^{1/2}$)

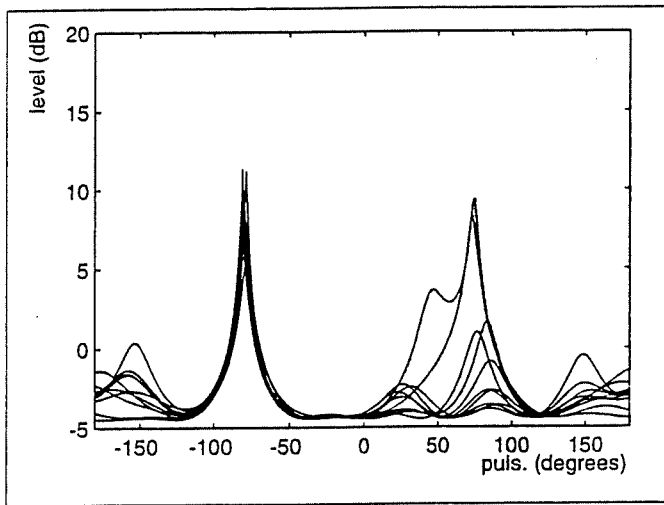


Figure 2 : classical technique

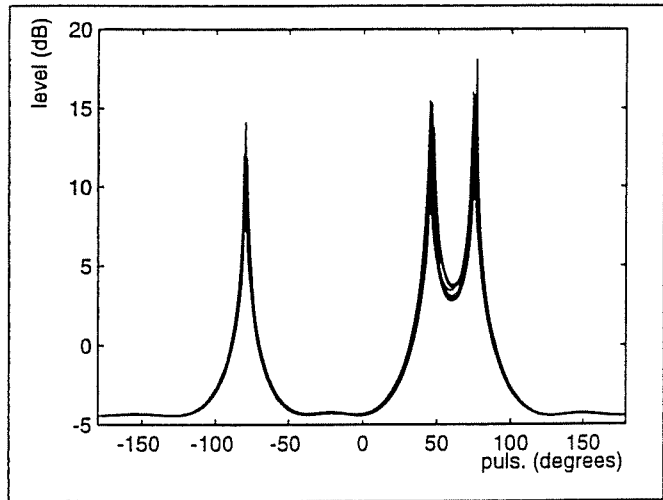


Figure 3 : projection technique

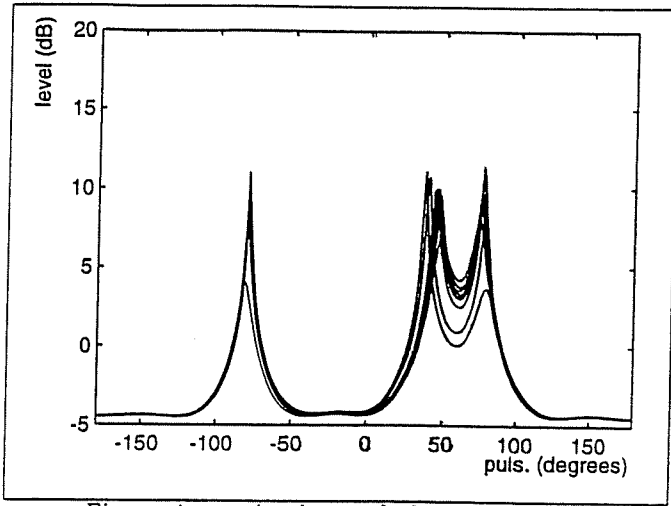


Figure 4 : projection technique with noise

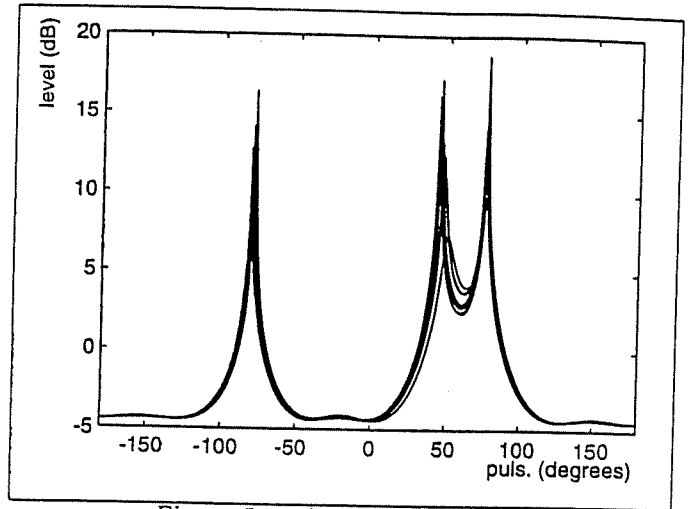


Figure 5 : relaxation technique

N	Ω	$3\pi/4$	$\pi/2$	$\pi/4$
4		4.8	87	$1.0 \cdot 10^4$
8		59	$6.2 \cdot 10^4$	$3.5 \cdot 10^9$
16		$2.3 \cdot 10^4$	$5.5 \cdot 10^{10}$	
32		$6.2 \cdot 10^9$		

Table 1 : ill-conditioning of R : $\lambda_{\max}/\lambda_{\min}$

<i>test</i>		% $\tilde{R} > 0$				% $\tilde{R}' > 0$				% $\tilde{R} > 0$ and $\tilde{R}' > 0$			
$N+1$	K	10	50	100	200	10	50	100	200	10	50	100	200
4		64.5	82.6	90.7	97	72.1	93.9	98.6	99.9	37.0	76.5	89.4	97.0
8		47.6	50.0	50.1	50.3	50.1	50.2	50.1	50.2	0	0.2	0.2	0.5
16		40.6	46.7	46.5	45.9	48.2	47.5	48.6	49.1	0	0	0	0

Table 2 : positivity of \tilde{R} and \tilde{R}'

Chapitre 3

Décomposition en valeurs propres adaptative

Fast Adaptive Eigenvalue Decomposition :

A Maximum Likelihood Approach.

Thierry CHONAVEL*, Benoît CHAMPAGNE**,
and Christian RIOU*

*ENSTBr, Dpt SC, BP 832, 29285 Brest cedex, France,
(e.mail: thierry.chonavel@enst-bretagne.fr)

**Mc Gill Univ., 3480 University St, Montreal, Quebec, Canada H3A 2A7
(e.mail: champagne@tsp.ece.mcgill.ca)

November 30, 2000

Abstract In this paper, we address the problem of adaptive EigenValue Decomposition (EVD). We propose a new approach, based on the optimization of the log-likelihood criterion. The parameters of the log-likelihood to be estimated are the eigenvectors and the eigenvalues of the data covariance matrix. They are actualized by means of a stochastic algorithm that requires little computational cost. Furthermore, the particular structure of the algorithm, that we named MALASE ensures the orthonormality of the estimated basis of eigenvectors at each step of the

algorithm. MALASE algorithm shows strong links with many Givens rotation based update algorithms that we discuss. We consider convergence issues for MALASE algorithm and give the expression of the asymptotic covariance matrix of the estimated parameters. The practical interest of the proposed method is shown on examples.

Résumé Cet article envisage le problème de la décomposition en valeurs propres d'une matrice de covariance au moyen d'un algorithme adaptatif. Nous proposons une nouvelle approche, basée sur l'optimisation du critère de maximum de vraisemblance. Les paramètres de la décomposition en valeurs propres sont actualisés au moyen d'un algorithme stochastique qui ne nécessite qu'un faible coût de calcul. La structure particulière de l'algorithme, que nous avons nommé MALASE, assure l'orthonormalité de la base des vecteurs propres estimés à chaque itération. L'algorithme MALASE présente des similitudes avec de nombreux algorithmes d'actualisation de sous espaces propres basés sur l'emploi de rotations de Givens. On indique ici les propriétés de convergence de l'algorithme et l'expression de la variance asymptotique des paramètres. L'intérêt pratique de l'algorithme est mis en évidence sur des simulations.

keywords maximum likelihood, eigenvalue decomposition, subspace tracking, unitary matrix, matrix manifold, Givens rotation.

1 Introduction

Let $X = (X_n)_{n \in \mathbb{Z}}$ denote a size- N , complex valued, process ($X_n \in \mathbb{C}^N$), with covariance matrix $R_{X,n} = E\{X_n X_n^H\}$. Subspace methods are based on the EigenValue Decomposition (EVD) of $R_{X,n}$, or at least on the estimation of the subspace spanned by the eigenvectors associated to the largest (resp. the smallest) eigenvalues, and named the signal (resp. the noise) subspace. These methods are classically used within several multivariate signal processing problems.

Applications such as sources localization (MUSIC algorithm, [5] [24]), vector quantization [15], or more recently blind channel equalization impulse response identification [19], often require estimation of such decompositions of the covariance matrices.

When X is a wide sense stationary process, the covariance matrix $R_{X,n}$ is constant ($R_{X,n} = R_X$), and it can be estimated easily by means of any classical technique, such as the empiric estimator (i.e. in the form $\frac{1}{L} \sum_{k=1,L} X_k X_k^H$, where L is the data length). Then, the corresponding eigenvalue decomposition can be obtained from standard algorithms such as the orthogonal iteration method (e.g. [16]).

But, in many applications, such as moving source localization and tracking, or time varying channel identification for mobile radio transmissions, the covariance matrix $R_{X,n}$ is time varying and the above methods no longer apply directly. In this case, several approaches have been proposed to track the instantaneous EVD of $R_{X,n}$ (see [10] for an excellent survey of adaptive subspace estimation prior to 1990; some more recent

algorithms are reviewed in [12]).

A first kind of techniques consists in adapting some classical algorithm for adaptive purposes [14]. They consist in a straightforward extension from the stationary case to the non-stationary case. Generally, the covariance matrix estimate is given by

$$\hat{R}_{X,n} = (1 - \mu)\hat{R}_{X,n-1} + \mu\mathbf{x}_n\mathbf{x}_n^H, \quad (1)$$

where $0 < \mu \ll 1$, and \mathbf{x}_n represents the data vector at time n . Then, the eigenvalue decomposition of $\hat{R}_{X,n}$ is performed. In fact, at each instant n , only one iteration of a classical EVD algorithm is performed. This iteration generally involves an orthonormalization procedure that requires a QR decomposition, at the expense of $O(NK^2)$ operations, where K is the dimension of the subspace to be estimated. Similar approaches have also been proposed to perform adaptive singular value decompositions. (see e.g. [18], or [25] for an algorithm requiring $O(NK)$ operations).

Another approach consists in directly replacing in equation (1) the estimate of $R_{X,n}$ by the parameters of its EVD ([11],[9]). Then, the subspaces are updated by techniques that involve Householder reflection matrices [11], or Givens rotation matrices [9]. With such parametrizations of the eigenvectors update, the orthonormality of the estimated basis is held at each iteration.

Let us remark that in [11] the signal and noise subspaces eigenvalues are averaged. Thus, only the estimation of a basis of both subspaces can be obtained. On the contrary, in [9] the complete EVD can be estimated. Both methods require $O(NK)$ operations.

A third kind of algorithms is designed to estimate the EVD (or a basis of the subspace of interest) by performing adaptive optimization of some possibly constrained criterion. This is done by means of some stochastic gradient algorithm (e.g. [20]-[22]), or as in [27] by means of an RLS approach. The computational cost of these methods ranges from $O(NK)$ to $O(NK^2)$, or $O(N^2K)$.

Some of the above methods simply intend to provide a basis of the signal or of the noise subspace, while some others supply an estimate of the complete EVD. Many applications make use of the projector onto the noise or signal subspace, and as pointed out in [9] perfect orthonormality is a desirable property for many subspace-based estimation techniques. In particular, for sources localization, deviations from the orthonormality lead to limited resolution capability and limited estimation performance of the root MUSIC algorithm [3]. Here, our main interest for considering the orthonormality issue stems from the fact that stochastic algorithms may sometimes become unstable if some constraints upon the parameters are not satisfied. In particular, we will see that we must ensure an orthonormality constraint upon the eigenbasis to derive a stable adaptive EVD based on the log-likelihood criterion.

Besides, it is often interesting to get the complete eigenvalue decomposition, involving in particular the estimation of the eigenvalues, to solve problems where the noise or signal subspace dimension has to be estimated. Indeed, the knowledge of the eigenvalues allows the use of statistical criteria to estimate their dimension (e.g. [2], [23]).

In this paper, we propose a new approach to adaptively estimate the eigenvalue decompo-

sition, that provides a number of nice features. First, unlike above mentioned techniques, it is based on adaptive optimization of the log-likelihood functional. This is quite a natural criterion for statistical estimation purposes, even if the minimum variance property of the log-likelihood functional is actually an asymptotic property. With this criterion, we could derive an algorithm that provides exact orthonormality of the estimated eigenvectors, with computational complexity in $O(N^2)$ for the calculation of the complete EVD.

This algorithm is closely related to algorithms such as PROTEUS_I and PROTEUS_II [9]. In particular, using the same preprocessing technique as in [9], it is possible to reduce the computational burden : when only $K < N$ eigenvalues of $R_{X,n}$ are distinct (as in many subspace tracking problems), the update of the eigendecomposition can be performed at the expense of $O(NK)$ operations. Furthermore, we have shown that the only stable stationary points of the algorithm represent the eigenvalue decomposition of the data covariance matrix. Also, we calculated the asymptotic covariance matrix for the parameters of the EVD. For the sake of conciseness, the calculations related to convergence and asymptotic covariance are omitted here, and will be published separately.

The paper is organized as follows: section II deals with the derivation of the algorithm, that we named **MALASE (MAXimum Likelihood Adaptive Subspace Estimation)**. Reduction of the computational burden and links with algorithms based on updates that involve Givens rotations are considered in section III, as well as the convergence and asymptotic performance. Section IV is devoted to examples that show the satisfactory behavior of the algorithm. Some concluding remarks are given in section V.

To end this section, let us introduce some notations. First, the superscript T , $*$, and H will respectively denote the transpose, conjugate, and tranpose-conjugate of a vector or a matrix. The trace, determinant, and expectation operators will be denoted by $Tr(\cdot)$, $|\cdot|$, and $E\{\cdot\}$ respectively.

In addition, $v = \text{diag}(A)$ will denote the vector v that contains the diagonal terms of the matrix A , while $A = \text{diag}(v)$ will denote the diagonal matrix with diagonal terms $[A]_{i,i} = v_i$ ($[A]_{i,j}$ denotes the coefficient of A at the intersection of the i^{th} line and j^{th} column). I_K will denote the size K identity matrix.

In general, the (estimated) EVD of $R_{X,n}$ will be denoted $U_n^H \Lambda_n U_n$, where U_n is the unitary matrix that contains the eigenvectors, denoted $(U_{i,n})_{i=1,N}$ ($U_n = [U_{1,n}, \dots, U_{N,n}]$), and $\Lambda_n = \text{diag}(\lambda_{1,n}, \dots, \lambda_{N,n})$ is the diagonal matrix that contains the eigenvalues $(\lambda_{i,n})_{i=1,N}$ at time n . When necessary, we will assume without loss of generality that the eigenvalues are classified in decreasing order.

2 Derivation of MALASE Algorithm.

In this section, we derive the proposed algorithm. It is based on iterative optimization of the log-likelihood functional, where the parameters to be estimated are those of the EVD of $R_{X,n}$. In particular, we show how it is possible to account for an orthonormality constraint upon U_n at each step of the algorithm.

2.1 The maximum-likelihood criterion.

Let f denote the probability density function of the random, complex circular, and zero mean gaussian vector, denoted X_n , with $X_n \in \mathbb{C}^N$. Under the hypothesis $R_{X,n} = R$, we have

$$f(\mathbf{x}_n; R) = \frac{1}{\pi^N |R|} \exp(-\mathbf{x}_n^H R^{-1} \mathbf{x}_n). \quad (2)$$

It is clear that up to an additive constant, the opposite of the log-likelihood is given by

$$\Psi(\mathbf{x}_n; R) = \log |R| + \mathbf{x}_n^H R^{-1} \mathbf{x}_n. \quad (3)$$

Letting $U\Lambda U^H$ represent the EVD of R , with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, $\Psi(\mathbf{x}_n, R)$ can be rewritten in the form

$$\Phi(\mathbf{x}_n; U, \Lambda) = \sum_{k=1, N} \log \lambda_k + \mathbf{x}_n^H U \Lambda^{-1} U^H \mathbf{x}_n. \quad (4)$$

It might seem natural to iteratively search a minimum of Φ by means of a gradient algorithm.

It is shown in Appendix A that a direct update of (U, Λ) by means of a stochastic gradient approach would not lead to a converging algorithm. In what follows, we are going to show that it is possible to overcome this problem by accounting for a unitary constraint upon U .

2.2 The proposed updating scheme.

In [13] matrix based algorithms are studied from the viewpoint of continuous-time systems theory and differential geometry. With a view to optimize $\Phi(\mathbf{x}_n; U, \Lambda)$ let us recall the following result :

Proposition 1 *Let $U(t)$ be a differentiable matrix function defined on the manifold of unitary matrices. Then, the gradient vector of the function $\text{Tr}[(UH_1U^H)H_2]$, where H_1 and H_2 are hermitian matrices and $\text{Tr}[\cdot]$ represents the trace operator, is given by*

$$\text{grad}_U \text{Tr}[(UH_1U^H)H_2] = (H_2(UH_1U^H) - (UH_1U^H)H_2)U, \quad (5)$$

where $\text{grad}_U f(U)$ represents the matrix with term (i, j) in the form $\partial f(U)/\partial U_{ij}$.

Proof: see [13] p.62-63.

Since $\Phi(\mathbf{x}_n; U, \Lambda)$ can be written as $\Phi(\mathbf{x}_n; U, \Lambda) = \sum_{k=1, N} \log \lambda_k + \text{Tr}[(U\Lambda^{-1}U^H)(\mathbf{x}_n\mathbf{x}_n^H)]$,

it comes from (5) that

$$\begin{aligned} \text{grad}_U \Phi(\mathbf{x}_n; U, \Lambda) &= (\mathbf{x}_n\mathbf{x}_n^H(U\Lambda^{-1}U^H) - (U\Lambda^{-1}U^H)\mathbf{x}_n\mathbf{x}_n^H)U \\ &= U(\mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1} - \Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H), \end{aligned} \quad (6)$$

where $\mathbf{y}_n = U_{n-1}^H \mathbf{x}_n$. Furthermore, recalling that the exponential of a square matrix A is defined by $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$, it is clear that $\text{grad}_U \Phi(\mathbf{x}_n; U, \Lambda)$ is tangent to the curve defined by $U(t) = U \exp[t(\mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1} - \Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H)]$ for $t = 0$. The reader will note that this curve lies in the manifold of unitary matrices. Indeed, a square matrix, say A , is a

skew-symmetric matrix (i.e. $A^H = -A$) if and only if $\exp(A)$ is a unitary matrix (see e.g. [9], [26]), and matrix $\mathbf{y}_n \mathbf{y}_n^H \Lambda_{n-1}^{-1} - \Lambda_{n-1}^{-1} \mathbf{y}_n \mathbf{y}_n^H$ is clearly skew-symmetric.

Moving on the curve $U(t)$ from point $t = 0$ in the direction of decreasing values of $\Phi(\mathbf{x}_n; U, \Lambda)$ amounts to let t decrease. Thus, a discretized version of the optimization of $\Phi(\mathbf{x}_n; U, \Lambda)$ as a continuous function of U is given by the following update scheme :

$$U_n = U_{n-1} \exp \left(\mu \left(\Lambda_{n-1}^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda_{n-1}^{-1} \right) \right), \quad (7)$$

where μ is the algorithm stepsize. Similarly Λ can be updated by means of a gradient technique, yielding the following update algorithm for (U_n, Λ_n) :

$$\mathbf{y}_n = U_{n-1}^H \mathbf{x}_n \quad (8a)$$

$$U_n = U_{n-1} \exp \left(\mu \left(\Lambda_{n-1}^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda_{n-1}^{-1} \right) \right) \quad (8b)$$

$$\Lambda_n = \Lambda_{n-1} + \mu' \left(\Lambda_{n-1}^{-2} \text{diag} \left(\text{diag} \left(\mathbf{y}_n \mathbf{y}_n^H \right) \right) - \Lambda_{n-1}^{-1} \right). \quad (8c)$$

We will call this algorithm as MALASE, for MAXimum Likelihood Adaptive Subspace Estimation. μ and μ' in (8) are possibly different stepsizes. The reason for choosing different stepsizes lies in the fact that the convergence rate may be different for U_n and Λ_n when $\mu = \mu'$.

For those readers who may not be familiar with derivation on matrix manifolds, an alternative direct derivation of MALASE is supplied in the appendix B.

2.3 Some properties of MALASE algorithm.

Let us briefly point out some nice features of MALASE algorithm.

2.3.1 Orthonormality.

As mentioned above the update factor $\exp(\mu(\Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1}))$ is a unitary matrix. This ensures that the orthonormality property is preserved by MALASE algorithm, provided that the algorithm is initialised with a unitary matrix U_0 . However, it is not necessary to have U_0 unitary to ensure the convergence since MALASE algorithm steers the matrix U towards the manifold of unitary matrices (see the simulation part).

2.3.2 Computational complexity.

MALASE algorithm may seem to involve high computational cost, due to the matrix exponential that appears in relation (8b). However, since $\exp(\mu(\Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1}))$ is the exponential of the sum of two rank one matrices, it is possible to show that the calculation of this matrix requires only $O(N^2)$ operations. To check this, let us denote $\mathbf{z}_n = \Lambda_{n-1}^{-1}\mathbf{y}_n$, and $[\mathbf{z}_n \ \mathbf{y}_n] = Q_n \times R_n$ the QR decomposition of the size- $N \times 2$ matrix $[\mathbf{z}_n \ \mathbf{y}_n]$. Then, letting $c_n = \sqrt{\|\mathbf{y}_n\|^2\|\mathbf{z}_n\|^2 - (\mathbf{y}_n^H\mathbf{z}_n)^2}$ it can be shown (see Appendix C) that

$$\exp(\mu(\Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1})) = I + Q_n \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} Q_n^H. \quad (9)$$

It is easy to check that the calculation of $U_{n-1} \exp(\mu(\Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1}))$ only requires $O(N^2)$ operations. We will explain further how this computational burden can be decreased when $R_{X,n}$ has less than N distinct eigenvalues. Finally, MALASE algorithm can be implemented as follows

MALASE algorithm

initialize U_0 unitary, $\lambda_{k,0}$ with distinct values ($k = 1, N$), and $0 < \mu, \mu' <$

1,

for each time step do

input \mathbf{x}_n

$$\begin{aligned}
 \mathbf{y}_n &= U_{n-1}^H \mathbf{x}_n \\
 \mathbf{z}_{k,n} &= \lambda_{k,n-1}^{-1} \mathbf{y}_{k,n} \quad (k = 1, N) \\
 c_n &= \sqrt{\|\mathbf{y}_n\|^2 \|\mathbf{z}_n\|^2 - (\mathbf{y}_n^H \mathbf{z}_n)^2} \\
 Q_n &= \left[\frac{\mathbf{z}_n}{\|\mathbf{z}_n\|} \frac{\mathbf{z}_n (\mathbf{z}_n^H \mathbf{y}_n) - \mathbf{y}_n \|\mathbf{z}_n\|^2}{c_n \|\mathbf{z}_n\|} \right] \\
 U_n &= U_{n-1} + \{U_{n-1} \times Q_n\} \times \left\{ \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} \times Q_n^H \right\} \\
 \lambda_{k,n} &= \lambda_{k,n-1} + \mu' (\lambda_{k,n-1}^{-2} |\mathbf{y}_{k,n}|^2 - \lambda_{k,n-1}^{-1}) \quad (k = 1, N).
 \end{aligned} \tag{10}$$

The $\lambda_{k,0}$ can be chosen for instance regularly spaced in the interval $[a, b]$, where a and b are rough estimates of the smallest eigenvalue and of the largest eigenvalue.

U_0 can be chosen equal to the identity matrix. Also, it is possible to perform an initial EVD, SVD, QR or GS decomposition from the first data samples in order to get a better initialization.

3 Reduction of the computational burden and convergence properties.

3.1 Reduction of the computational burden.

Now, let us assume that we are only interested in searching for the subspace associated to the p largest, or the p smallest, eigenvalues of $R_{X,n}$. We are going to show that in this case MALASE algorithm can be implemented in a way that requires updating only $p + 1 = K$ vectors at each iteration, and that this can be done at the expense of $O(NK)$ operations. This $O(NK)$ implementation will be referred to as **MALASE**(K).

We assume that the $N - p$ eigenvalues associated with the eigenvectors that we do not need are all equal. This situation arises in many applications. When this condition is not matched, it is possible to average these $N - p$ eigenvalues, as proposed in [11].

Let us assume for instance that we search for vectors $U_{1,n}, \dots, U_{p,n}$ and that the $N - p$ smallest eigenvalues are all equal : $\lambda_{p+1} = \dots = \lambda_N$. As in [9], we achieve computational complexity reduction by means of a convenient preprocessing of the vector \mathbf{x}_n , that replaces the calculation of $\mathbf{y}_n = U_n^H \mathbf{x}_n$.

The preprocessing is based on an isometric transform applied to \mathbf{x}_n . \mathbf{x}_n is transformed into a size $p + 1$ vector, denoted ξ_n , with first p components representing the projection of \mathbf{x}_n on the space spanned by $U_{1,n}, \dots, U_{p,n}$. Then, all but the first component of the projection of \mathbf{x}_n on the space spanned by $U_{p+1,n}, \dots, U_{N,n}$ are set to zero by means of

a bloc-Householder transform, denoted H_n (see eg [16] for more details). The non zero component defines $\xi_{p+1,n}$. These operations amount to calculate $H_n \mathbf{y}_n = [\xi_n^T \mathbf{0}^T]^T$, but we are going to see that this is done in a way that requires only $O(NK)$ operations. The following summary of the preprocessing shows this clearly.

Preprocessing :

for each time step do

input \mathbf{x}_n

$$\begin{aligned}
 U_S &= [U_{1,n-1}, \dots, U_{p,n-1}] \\
 a &= U_S^H \mathbf{x}_n \\
 b &= \mathbf{x}_n - U_S a \\
 \xi_n &= [a^T \ \|b\|]^T
 \end{aligned} \tag{11}$$

Let us remark that unlike in [9] no preprocessing in the form $a \rightarrow Da$, where $D = \text{diag}(a_1^*/|a_1|, \dots, a_p^*/|a_p|)$ is required.

Now, let us denote $\nu_n = \text{diag}(\lambda_{1,n}^{-1}, \dots, \lambda_{p+1,n}^{-1}) \xi_n$. Let us remark that $H_n \mathbf{z}_n = [\nu_n^T \mathbf{0}^T]^T$ since $\lambda_{p+1,n} = \dots = \lambda_{N,n}$. It is clear that the terms c_n and Q_n in MALASE algorithm can be rewritten as

$$\begin{aligned}
 c_n &= \sqrt{\|\xi_n\|^2 \|\nu_n\|^2 - (\xi_n^H \nu_n)^2} \\
 Q_n &= H_n^H \begin{pmatrix} Q_n^{(K)} \\ 0 \end{pmatrix} \\
 Q_n^{(K)} &= \begin{bmatrix} \frac{\nu_n}{\|\nu_n\|} & \frac{\nu_n(\nu_n^H \xi_n) - \xi_n \|\nu_n\|^2}{c_n \|\nu_n\|} \end{bmatrix}.
 \end{aligned} \tag{12}$$

Then, the update of U_n can be rewritten as

$$U_n = U_{n-1} + U_{n-1} H_n^H \begin{pmatrix} Q_n^{(K)} \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} Q_n^{(K)H} & 0 \\ 0 & 0 \end{pmatrix} H_n, \quad (13)$$

i.e.

$$U_n H_n^H = U_{n-1} H_n^H + U_{n-1} H_n^H \begin{pmatrix} Q_n^{(K)} \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} Q_n^{(K)H} & 0 \\ 0 & 0 \end{pmatrix}. \quad (14)$$

Let us denote $U_n^{(K)}$ the $N \times (p+1)$ matrix that contains the first $p+1$ vectors of $U_n H_n^H$.

It is clear that the first p columns of U_n , are left unchanged by the transform $U \rightarrow U H_n^H$.

The last column of $U_n^{(K)}$ is $U_{p+1,n}^{(K)} = \|b\|^{-1} b$, where b is defined as in the preprocessing.

Thus, $U_n^{(K)} = [U_{1,n}, \dots, U_{p,n}, \|b\|^{-1} b]$.

From these considerations, it appears that MALASE algorithm can be rewritten in a new way that we will name MALASE(K).

MALASE(K) implementation of MALASE algorithm

initialize U_0 unitary, initialize $\lambda_{k,0}$ with distinct values ($k = 1, K$), and

$0 < \mu, \mu' < 1$,

for each time step do

input \mathbf{x}_n

$$\begin{aligned}
U_S &= [U_{1,n-1}^{(K)}, \dots, U_{p,n-1}^{(K)}] \\
a &= U_S^H \mathbf{x}_n \\
b &= \mathbf{x}_n - U_S a \\
\xi_n &= [a^T \|b\|]^T \\
U_{p+1,n-1}^{(K)} &= \|b\|^{-1} b \\
\nu_{k,n} &= \lambda_{k,n-1}^{-1} \xi_{k,n}, \quad (k = 1, K) \\
c_n &= \sqrt{\|\xi_n\|^2 \|\nu_n\|^2 - (\xi_n^H \nu_n)^2} \\
Q_n^{(K)} &= \begin{bmatrix} \frac{\nu_n}{\|\nu_n\|} & \frac{\nu_n(\nu_n^H \xi_n) - \xi_n \|\nu_n\|^2}{c_n \|\nu_n\|} \end{bmatrix} \\
U_n^{(K)} &= U_{n-1}^{(K)} + \{U_{n-1}^{(K)} \times Q_n^{(K)}\} \times \left\{ \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} \times Q_n^{(K)H} \right\} \\
\lambda_{k,n} &= \lambda_{k,n-1} + \mu' (\lambda_{k,n-1}^{-2} |\xi_{k,n}|^2 - \lambda_{k,n}^{-1}) \quad (k = 1, K-1) \\
\lambda_{K,n} &= \lambda_{K,n-1} + \mu' (\lambda_{K,n-1}^{-2} \frac{|\xi_{K,n}|^2}{N-p} - \lambda_{K,n}^{-1}).
\end{aligned} \tag{15}$$

Since $Q_n^{(K)}$ is a size $K \times 2$ matrix, it is clear that MALASE(K) requires only $O(NK)$ operations.

3.2 EVD update and Givens rotations.

Several eigenvalue update techniques, such as MALASE or PROTEUS_I algorithm [9], rely on an update of U_n in the form

$$U_n = U_{n-1} \exp(\Theta_n), \quad (16)$$

where Θ_n is a skew symmetric matrix, or in the form

$$U_n = U_{n-1} \prod_{1 \leq i < j \leq N} G_{i,j}(\theta_{ij,n}), \quad (17)$$

where $G_{i,j}(\theta_{ij,n})$ is a Given's rotation that operates on columns i and j (see e.g. [13] and enclosed references). In general both approaches are equivalent up to a second order term in $O(\mu^2)$, letting $\theta_{ij,n} = [\Theta_n]_{ij}$. In particular, MALASE yields

$$\theta_{ij,n} = \mu \left(\frac{1}{\lambda_{i,n-1}} - \frac{1}{\lambda_{j,n-1}} \right) \xi_{i,n} \xi_{j,n}^*. \quad (18)$$

In [9], PROTEUS_I and PROTEUS_II algorithms involve rotations in the form

$$\theta_{ij,n} = \frac{\mu}{\lambda_{j,n-1} - \lambda_{i,n-1}} \xi_{i,n} \xi_{j,n}^*, \quad (19)$$

$$\text{and } \theta_{ij,n} = -\frac{\mu}{\lambda_{i,n-1}} \xi_{i,n} \xi_{j,n}^*, \text{ for } i < j, \quad \theta_{ji,n} = -\theta_{ij,n},$$

yielding computational costs in $O(NK^2)$ and $O(NK)$ operations per step respectively.

It is interesting to note that if $\forall i < j, \lambda_i \gg \lambda_j$, then for MALASE $\theta_{ij,n}$ can be approximated by $\theta_{ij,n} = -(\mu/\lambda_{j,n-1}) \xi_{i,n} \xi_{j,n}^*$. Then $|\theta_{ij,n}|$ is larger for MALASE than for PROTEUS_II. This is in accordance with the faster convergence and higher asymptotic variance often observed for MALASE compared to PROTEUS_II when μ is fixed (see examples in section 5).

Note also that PROTEUS algorithms are derived from the following update of $R_{X,n}$:

$$\hat{R}_{X,n} = (1 - \mu)\hat{R}_{X,n-1} + \mu\mathbf{x}_n\mathbf{x}_n^H. \quad (20)$$

From this, and above mentioned similarities between MALASE and PROTEUS algorithms, we get an interpretation of the update factor μ of MALASE algorithm : μ can be seen as the update factor of the stochastic gradient estimate of $R_{X,n}$.

3.3 Convergence and asymptotic covariance matrix.

As usual, the study of the convergence is considered for a stationary random process X . It has been possible to prove that stationary stable points of the algorithm correspond to the desired solution and to calculate the asymptotic variance of the parameters. However, the convergence analysis is rather involved and beyond the scope of this paper. For this reason we hope to present it separately, and only the main results are collected in this section.

3.3.1 Convergence of MALASE algorithm.

Let (U_*, Λ_*) represent a stationary point of MALASE algorithm. Then rewriting equations (8) in the form

$$\begin{aligned} U_{n+1} &= U_n + \mu H_1(X_n; \Lambda, U) \\ \Lambda_{n+1} &= \Lambda_n + \mu H_2(X_n; \Lambda, U), \end{aligned} \quad (21)$$

it can be shown that the stationary points correspond to unitary matrices U such that $U^H R_X U$ is a block diagonal matrix (the sizes of the different blocks may be different),

where the terms along the diagonal of a given block are all equal. Thus it appears that the eigenvalue decompositions are not the only stationary points of the algorithm. However, the following result ensures the desired convergence property :

Proposition 2 *The stationary stable points of MALASE algorithm correspond to the EVD of R_X .*

3.3.2 Asymptotic covariance matrix.

Using standard techniques involving linearization of the update of U_n for small μ and results of the ordinary differential equation theory [4], we have been able to derive the asymptotic covariance matrix for the parameters.

Let $(U_{1,n}, \dots, U_{N,n})$ and $(\lambda_{1,n}, \dots, \lambda_{N,n})$ denote the estimated eigenvectors and the estimated eigenvalues, and let us denote $V_n = [U_{1,n}^T, \dots, U_{N,n}^T, \lambda_{1,n}, \dots, \lambda_{N,n}]^T$ the size- $(N^2 + N)$ vector that contains these parameters. For small values of μ the asymptotic covariance matrix of V_n is equal to μP , where P is the only solution of a Lyapunov equation; this solution is in the form

$$\mu P = \frac{1}{2} \mu I_{N^2+N}. \quad (22)$$

This is a very interesting result since the asymptotic variance depends only on μ , and not on the parameters of the EVD of R_X , unlike other algorithms. Furthermore, any of the $N^2 + N$ parameters are asymptotically uncorrelated, and have the same variance, equal to $\mu/2$. For eigenvalues, this can be compared for instance to the usual update in the

form

$$\lambda_{k,n} = \lambda_{k,n-1} + \mu(|\xi_{k,n}|^2 - \lambda_{k,n}), \quad (23)$$

that yields theoretical asymptotic variance for λ_k equal to $(\mu/2)\lambda_k^2$.

3.3.3 Convergence rate.

Paralleling the discussion in [7] where it is shown that PROTEUS_II has similar convergence rates for all the parameters, it can be shown that this property may not be satisfied for MALASE, as already mentioned in the section 2.2. Using (23) for the update of the eigenvalues in MALASE would permit to change this. But then, the asymptotic variance of MALASE would no longer be similar for all the parameters.

4 Practical issues.

The above considerations upon the asymptotic covariance of the algorithm are derived for small values of μ . But in practice, reasonable values of μ , i.e. values of μ that ensure convergence at a speed about equivalent to that of other existing techniques, cannot be considered as small for MALASE algorithm. In particular, with X defined as in the following, we calculated the average variance of the estimated eigenvectors coefficients from 10^4 iterations of the algorithm. We obtained 6.10^{-5} for $\mu = 10^{-4}$, 0.0042 for $\mu = 0.005$, and 0.072 for $\mu = 0.05$. Thus, only for $\mu = 10^{-4}$ the variance of the parameters is about $\mu/2$. This shows that for large values of μ , one should better

consider values of the asymptotic covariance calculated from Monte Carlo simulations. However, the theoretical asymptotic variance is a useful index since to a certain extent the simulation results confirm the theoretical results.

4.1 Convergence of MALASE algorithm.

Now, let us consider the following exemple proposed in [9]. We assume that

$$X_n = \sum_{k=1,p} S_{k,n} d(\omega_k) + W_n, \quad (24)$$

where $d(\omega) = [1, e^{i\omega}, \dots, e^{i(N-1)\omega}]^T$. $(S_{k,n})_{n \in \mathbb{Z}}$ is a complex scalar, white gaussian source signal, and $(W_n)_{n \in \mathbb{Z}}$ a vector white gaussian noise with covariance matrix $\sigma_W^2 I_N$. The noise and the sources are uncorrelated. All sources have the same power, and the SNR (Signal to Noise Ratio) is defined as $E\{|S_{k,n}|^2\}/\sigma_W^2$.

We study the convergence of MALASE algorithm, implemented in the form MALASE(K), with $K = p + 1$. To this end, we consider the performance index defined by $\| P_S - U_{S,n} U_{S,n}^H \|$, where P_S is the true projector on the source subspace (i.e. on the subspace spanned by the set $\{d(\omega_1), \dots, d(\omega_p)\}$), $U_{S,n} U_{S,n}^H$ the estimated projector, and $\| A \| = [Tr(AA^H)]^{1/2}$. In other words, $U_{S,n}$ is a $N \times p$ matrix, and its columns are the eigenvectors associated with the p largest eigenvalues of R_X , estimated at time n .

We take $N = 16$ and $p = 4$. The values of the $(\omega_k)_{k=1,p}$ are 0.0, 0.25, 1, and 1.25. We plot the results obtained for MALASE, and for PROTEUS_I and PROTEUS_II algorithms. Also we plot the evolution of $\| P_S - U_n^{(S)} U_n^{(S)H} \|$ for Karasalo's method [17]. This method

shows high computational cost (in $O(NK^2)$), but it supplies an almost exact E.V.D., and thus is interesting as a reference.

Results for SNR=10dB and -5dB, and $\mu = 0.005$ and 0.02 are given on figure 1. The eigenbasis is initialized with identity matrix and the eigenvalues are initialized from a rough estimate of the largest eigenvalue : we choose them equal to $(200/(p + 1)) \times [p + 1, p, \dots, 1]$ for SNR=10dB, and $(10/(p + 1)) \times [p + 1, p, \dots, 1]$ for SNR=-5dB. It clearly appears that MALASE algorithm ensures fast convergence. At low SNR, it generally converges faster than PROTEUS_II algorithm but it shows higher asymptotic error. In fact, choosing a smaller μ for MALASE or a larger μ for PROTEUS_II would permit to get about the same performance for both algorithms. For $SNR = -5dB$, we get about the same convergence speed for MALASE and for PROTEUS_II when $\mu = 0.003$ and $\mu = 0.005$, or $\mu = 0.001$ and $\mu = 0.002$, respectively (see figure 1).

A popular way to check the robustness of a subspace tracking algorithm consists in changing the values $(\omega_k)_{k=1,p}$ by $(-\omega_k)_{k=1,p}$ after convergence. It seems that this technique was first proposed in [10]. At iteration 1000, we apply the change $(\omega_k)_{k=1,p} \rightarrow (-\omega_k)_{k=1,p}$ (figure 2). It appears that Karasalo's method converges at about the same speed as PROTEUS_I after the abrupt rotation of the eigenspaces, and that MALASE performs better than PROTEUS_II (although the asymptotic errors are not perfectly matched).

In order to check the influence of a lack of orthonormality of the estimated eigenvectors, a random perturbation is added to the matrix of the estimated eigenvectors after convergence. The ratio between the perturbation matrix norm and the initial matrix norm is

0.3. It appears (see figure 3) that the error is corrected efficiently by all the algorithms. In particular, this result shows that a lack of orthonormality is correctly suppressed by the MALASE algorithm, despite the fact that it has been derived under an orthonormality assumption upon the eigenvectors. This suggests that MALASE algorithm initial convergence should be little sensitive to a lack of orthonormality of U_0 . We checked that taking independent normal distributions for the coefficients of U_0 yields no change in the convergence speed of MALASE or PROTEUS_I algorithms. PROTEUS_II appears to be more sensitive to a lack of orthonormality : with U_0 main diagonal entries set to one and independent gaussian off diagonal terms with variance 0.1, PROTEUS_II converges twice more slowly.

In addition, in order to check the long term stability of the algorithm, we let the MALASE algorithm run for 10^5 iterations. For small μ ($\mu = 0.005$) no problem is encountered, but for large values of μ ($\mu = 0.02$) it appears that after some 10^4 iterations the algorithm sometimes diverges. A simple way to overcome this kind of problem consists in preventing the angle μc_n that appears in terms $\cos \mu c_n$ and $\sin \mu c_n$ of the update of the eigenbasis to become larger than some fixed value. Replacing μc_n by $\min\{\mu c_n, 0.1\}$ in the update of U_n ensures long term stability when $\mu = 0.02$, without slowing down significantly the convergence of the eigenvectors.

4.2 Asymptotic variance.

In order to compare the variance properties of the algorithms, we performed a Monte-Carlo simulation from 10^4 data with the true parameters as initial parameters. Tables 1 and 2 summarize the results for SNR=-5dB and SNR=10dB, and $\mu = 0.005$. These tables supply the mean value of the asymptotic variance for the eigenvectors and for the eigenvalues. At low SNR, MALASE and PROTEUS_II appear to work better than the two algorithms with higher computational complexity. At high SNR, PROTEUS_II shows very little variance for U . All but MALASE have high variance for the eigenvalues. It is easy to understand this since for these methods the eigenvalues are updated from (23), and for small μ the asymptotic variance for λ_k is $\sigma_{\lambda_k}^2 = (\mu/2)\lambda_k^2$ for (23), while it is $\mu/2$ for the MALASE update. Since for SNR=10dB $\lambda_1 = 205.5$, it is rather natural to obtain high variance from the update (23). Similarly, in accordance with theoretical variance calculations, we check that the asymptotic variance of distinct parameters do not differ too much for MALASE, especially for low SNR (i.e. when the eigenvalues are little scattered). On the contrary, for algorithms that use update (23), the variance of the estimated eigenvalues increases for increasing eigenvalues.

5 Conclusion.

We presented a new technique to perform adaptive eigenvalue decomposition of a time varying covariance matrix, based on the statistically meaningful maximum likelihood cri-

terion. We have shown that accounting for the orthonormality constraint upon the eigenvectors yields a simple iterative algorithm with nice features : desirable convergence properties are ensured, as well as orthonormality of the estimated eigenvectors, at the expense of small computational cost. We checked on examples the good practical behavior of the MALASE algorithm. To end this conclusion, note that RLS approaches have proved to be efficient for subspace tracking [1][28], in particular with respect to convergence rate and low sensitivity to the SNR. Thus, in further investigations RLS implementations of the maximum likelihood criterion for EVD tracking could be considered.

Acknowledgment The authors are grateful to the anonymous reviewer who supplied them with reference [13] that presents time matrix algorithms.

References

- [1] K. Abed-Meraim, A. Chkeif and Y. Hua, "Fast Orthogonal PAST Algorithm", IEEE Sig. Proc. Letters, mar. 2000, pp. 60-62.
- [2] Akaike, "A New Look at the Statistical Model Identification", IEEE Trans. on Automatic Control, vol. 19, dec. 1974, pp. 716-723.
- [3] A. Barabell, "Improving the Resolution Performance of Eigenstructure-Based Direction-Finding Algorithms", Proc. ICASSP'83, 1983, pp. 336-339.
- [4] A. Benveniste, M. Metivier and P. Priouret, Adaptive Algorithms and Stochastic Approximation, Springer Verlag, 1990.

- [5] G. Bienvenu and L. Kopp, "Optimality of High Resolution Array Using the Eigensystem Approach", IEEE Trans. on ASSP, vol. 31, oct. 1983, pp. 1235-1247.
- [6] J.F. Cardoso and B.H. Laheld, "Equivariant Adaptive Source Separation", IEEE Trans. on Signal Processing, vol. 44, dec. 1996, pp. 3017-3029.
- [7] B. Champagne, "Adaptive Eigendecomposition of Data Covariance Matrices Based on First-Order Perturbations", IEEE Trans. on Signal Processing, vol. 42, oct. 1994, pp. 2758-2770.
- [8] B. Champagne and Q.G. Liu, "A New Family of EVD Tracking Algorithms Using Givens Rotations", Proc. ICASSP'96, 1996, pp. 2539-1542.
- [9] B. Champagne and Q.G. Liu, "Plane Rotation-Based EVD Updating Schemes for Efficient Subspace Tracking", IEEE Trans. on Signal Processing, vol. 46, jul. 1998, pp. 1886-1900.
- [10] P. Comon and G.H. Golub, "Tracking a Few Extreme Singular Values and Vectors in Signal Processing", Proc. IEEE, vol. 1978, aug. 1990, pp. 1327-1343.
- [11] R.D. DeGroat, "Noniterative subspace tracking", IEEE Trans. on Signal Processing, vol. 40, mar. 1992, pp. 571-577.
- [12] R.D. DeGroat, E.M. Dowling and D.A. Linebarger Subspace Tracking, in : The Digital Signal Processing Handbook, CRC Press, V.K. Madisetti and D.B. Williams Eds., 1998.

- [13] J. Dehaene, Continuous Time Matrix Algorithms, Systolic Algorithms and Adaptive Neural Networks, PhD Thesis, Katholieke Universiteit Leuven, oct. 1995.
- [14] S. Erlich and R. Yao, "Convergences of Adaptive Block Simultaneous Iteration Method For Eigenstructure Decomposition", Signal Processing, vol. 37, jan. 1994, pp.1-13.
- [15] A. Gersho and R.M. Gray, Vector Quantization and Signal Compression, Kluwer Academic Publishers, 1995.
- [16] G. H. Golub and C. F. Van Loan, Matrix Computation - Second Edition, John Hopkins Series in Mathematical Sciences, John Hopkins University Press, 1990.
- [17] I. Karasalo, "Estimating the Covariance Matrix by Signal Subspace Averaging", IEEE Trans. on Signal Processing, vol. 37, jan 1986, pp. 8-12.
- [18] M. Moonen, Van Dooren and J. Vandewalle, "A Singular Value Decomposition Algorithm for Subspace Tracking", SIAM Matrix Anal. Appl., vol. 13, oct. 1992, pp. 1015-1038.
- [19] E. Moulines, P. Duhamel, J.F. Cardoso and S. Mayrargue, "Subspace Methods for the Blind Identification of Multichannel FIR Filters", IEEE Trans. on Signal Processing, vol. 43, feb. 1995, pp. 516-525.
- [20] P.A. Regalia, "An Adaptive Unit Norm Filter with Application to Signal Analysis and Karhunen–Loeve Transformations", IEEE Trans. on CAS, vol. 37, may 1990, pp. 646-649.

- [21] C. Riou and T. Chonavel and P.Y. Cochet, "Adaptive Subspace Estimation : Application to Moving Sources Localization and Blind Channel Identification", Proc. ICASSP'96, 1996, pp. 1649-1652.
- [22] C. Riou and T. Chonavel, "Fast Adaptive Eigenvalue Decomposition, a Maximum Likelihood Approach", Proc. ICASSP'97, 1997, pp. 3565-3568.
- [23] J. Rissanen, "Modeling by the Shortest Data Description", Automatica, vol. 14, 1978, pp. 465-471.
- [24] R.O. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation", IEEE Trans. on Antennas and Propagation, vol. 34, mar. 1986, pp. 276-280.
- [25] P. Strobach, "Bi-Iteration Recursive Instrumental Variable Subspace Tracking and Adaptive Filtering", IEEE Trans. on Signal Processing, vol. 46, oct. 1998, pp. 2708-2725.
- [26] H. Weyl, The Theory of Groups and Quantum Mechanics, Dover, 1950.
- [27] B. Yang, "Subspace Tracking Based on the Projection Approach and the Recursive Least Squares Method", Proc. ICASSP'93, 1993, pp. 145-148.
- [28] B. Yang, "Projection Approximation Subspace Tracking", IEEE Trans. on Signal Processing, vol. 43, jan. 1995, pp. 95-107.

APPENDIX

A Gradient based update of the ML criterion

For the likelihood criterion $\Phi(\mathbf{x}_n; U, \Lambda) = \sum_{k=1, N} \log \lambda_k + \mathbf{x}_n^H U \Lambda^{-1} U^H \mathbf{x}_n$, straightforward calculations show that the actualization of the EVD by means of a gradient algorithm is given by

$$\begin{aligned} \mathbf{y}_n &= U_{n-1}^H \mathbf{x}_n \\ U_n &= U_{n-1} - \mu \mathbf{x}_n \mathbf{y}_n^H \Lambda_{n-1}^{-1} \\ \Lambda_n &= \Lambda_{n-1} - \mu (\Lambda_{n-1}^{-1} - \Lambda_{n-1}^{-2} \text{diag}(\text{diag}(\mathbf{y}_n \mathbf{y}_n^H))), \end{aligned} \tag{25}$$

where μ is the step of the algorithm. Unfortunately, this algorithm is not satisfactory. Indeed, let X be a wide sense stationary process, and (U, Λ) denote a stationary point of algorithm (25). Then, the update term of U_n must satisfy

$$E\{X_n X_n^H U \Lambda^{-1}\} = R_X U \Lambda^{-1} = 0, \tag{26}$$

leading to $U \Lambda^{-1} = 0$. This proves that (U_n, Λ_n) does not converge to the parameters of the EVD of R_X .

B Direct derivation of the algorithm.

MALASE algorithm originates in the fact that the orthonormality of U_n is accounted for both in the computation of the gradient of the log-likelihood criterion and in the update

of U_n . To see it more clearly, let us consider the differentiation of $\Phi(\mathbf{x}_n; U, \Lambda)$ with respect to U , with the unitary matrix constraint $UU^H = I_N$ that yields $dU^H = -U^H dUU^H$. Up to a second order term, we have

$$\begin{aligned} d_U \Phi(\mathbf{x}_n; U, \Lambda) &= \Phi(\mathbf{x}_n; U + dU, \Lambda) - \Phi(\mathbf{x}_n; U, \Lambda) \\ &= \mathbf{x}_n^H (dU \Lambda^{-1} U^H + U \Lambda^{-1} dU^H) \mathbf{x}_n \\ &= -\mathbf{y}_n^H (\Lambda^{-1} U^H dU - U^H dU \Lambda^{-1}) \mathbf{y}_n, \end{aligned} \quad (27)$$

whith $\mathbf{y}_n = U^H \mathbf{x}_n$. Then, it comes that under the unitary constraint over U , the matrix $\frac{\partial \Phi}{\partial U}$ with general term $\left[\frac{\partial \Phi(\mathbf{x}_n; U, \Lambda)}{\partial U} \right]_{i,j} = \frac{\partial \Phi}{\partial [U]_{i,j}}$ is given by

$$\frac{\partial \Phi}{\partial U} = -U (\Lambda^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda^{-1}). \quad (28)$$

Then the stochastic gradient update of U yields

$$\begin{aligned} U_n &= U_{n-1} (I + \mu (\Lambda^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda^{-1})) \\ &= U_{n-1} \exp(\mu (\Lambda^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda^{-1})) + O(\mu^2). \end{aligned} \quad (29)$$

Forgetting the $O(\mu^2)$ term we get the proposed updating scheme that preserves orthonormality.

Let us remark that updating U_n in a way that involves the exponential of a skew symmetric matrix is a property shared with some other stochastic algorithms intended to update a unitary matrix (see e.g. [6][8]). This is quite natural since the tangent plane to the unitary matrices manifold at point U is generated by matrices $\{UA; A = -A^H\}$ and, as mentioned in section 2.2, UA is the tangent vector at point U for the matrix function $U(t) = U \exp(tA)$ defined on the unitary matrices manifold [13].

C Calculation of $\exp(\mu(\Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1}))$

Clearly,

$$[\mathbf{z}_n \ \mathbf{y}_n] = \left[\frac{\mathbf{z}_n}{\|\mathbf{z}_n\|} \frac{\mathbf{z}_n(\mathbf{z}_n^H\mathbf{y}_n) - \mathbf{y}_n\|\mathbf{z}_n\|^2}{\|\mathbf{z}_n\|c_n} \right] \times \begin{pmatrix} \|\mathbf{z}_n\| & \frac{\mathbf{z}_n^H\mathbf{y}_n}{\|\mathbf{z}_n\|} \\ 0 & \frac{-c_n}{\|\mathbf{z}_n\|} \end{pmatrix}, \quad (30)$$

with $c_n = \sqrt{(\mathbf{z}_n^H\mathbf{z}_n)(\mathbf{y}_n^H\mathbf{y}_n) - (\mathbf{y}_n^H\mathbf{z}_n)^2}$, and

$$\begin{aligned} \exp(\mu(\Lambda_{n-1}^{-1}\mathbf{y}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{y}_n^H\Lambda_{n-1}^{-1})) &= \exp(\mu(\mathbf{z}_n\mathbf{y}_n^H - \mathbf{y}_n\mathbf{z}_n^H)) \\ &= \exp(\mu[\mathbf{z}_n \ \mathbf{y}_n] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} [\mathbf{z}_n \ \mathbf{y}_n]^H) \\ &= I + Q_n \left[\sum_{k=1, \infty} \frac{\mu^k}{k!} (R_n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R_n^H)^k \right] Q_n^H \\ &= I + Q_n \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} Q_n^H. \end{aligned} \quad (31)$$

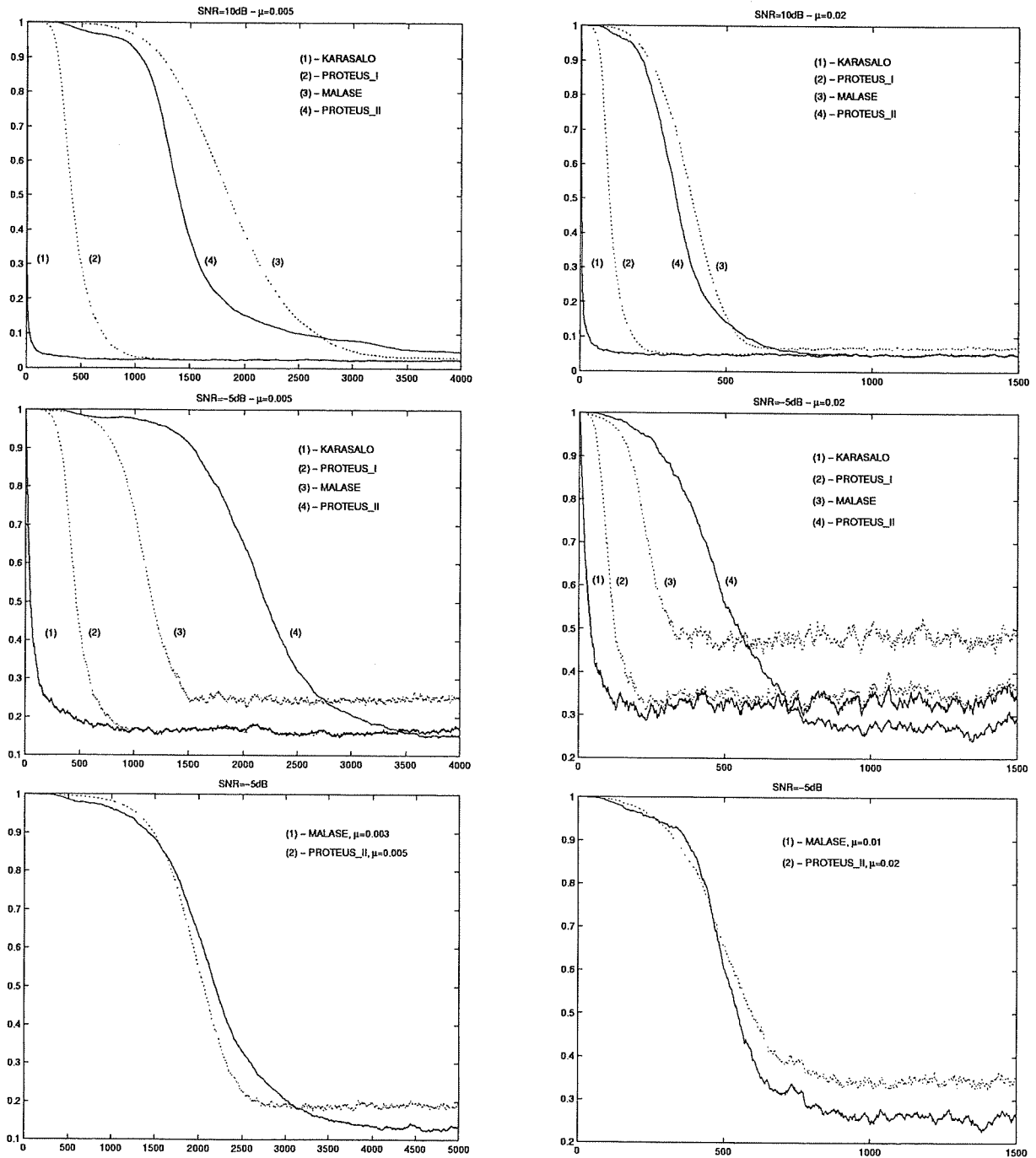


Figure 1: Convergence of source subspace eigenvectors for various scattering of the eigenvalues. Performance index : $\| P_S - U_{S,n} U_{S,n}^H \|$.

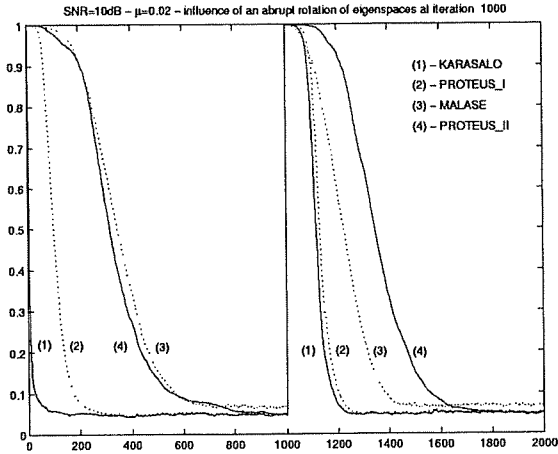


Figure 2: Influence of an abrupt rotation of the eigenvectors at iteration 1000. Performance index : $\| P_S - U_{S,n} U_{S,n}^H \|$.

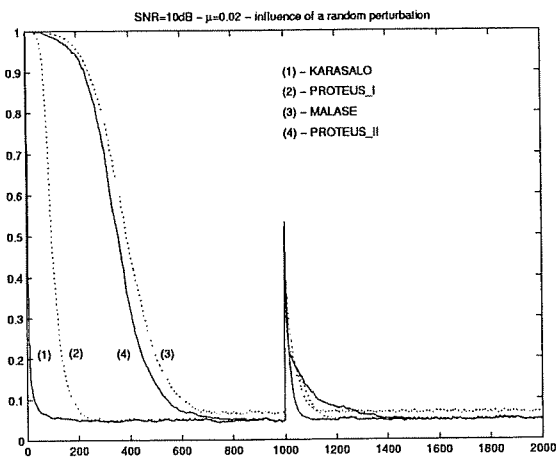


Figure 3: Influence of an additional random perturbation that affects the orthonormality of $U_{S,n}$ at iteration 1000. Performance index : $\| P_S - U_{S,n} U_{S,n}^H \|$.

average variance	mean var. U	mean var. λ
Karasalo	0.009	0.06
PROTEUS_I	0.009	0.07
PROTEUS_II	0.002	0.06
MALASE	0.004	0.008

Table 1: Asymptotic mean variance of the estimated parameters for SNR=-5dB, and $\mu = 0.005$.

average variance	mean var. U	mean var. λ
Karasalo	0.002	36
PROTEUS_I	0.003	38
PROTEUS_II	3×10^{-4}	36
MALASE	0.009	0.28

Table 2: Asymptotic mean variance of the estimated parameters for SNR=10dB, and $\mu = 0.005$.

Chapitre 4

Calcul de performances de l'algorithme MALASE

Convergence properties of MALASE algorithm.

Thierry CHONAVEL*, Benoît CHAMPAGNE**,

and Christian RIOU

*ENSTBr, Dpt SC, BP 832, 29285 Brest cedex, France,

(e.mail: thierry.chonavel@enst-bretagne.fr)

**Mc Gill Univ., 3480 University St, Montreal, Quebec, Canada H3A 2A7

(e.mail: champagne@tsp.ece.mcgill.ca)

April 12, 2000

Abstract

In this paper, we study the convergence properties of MALASE algorithm. This algorithm permits to solve the problem of adaptive eigenvalue decomposition and is based on the optimization of the log-likelihood criterion. We prove the convergence of MALASE algorithm and derive the expression of the asymptotic covariance matrix of the estimated parameters.

1 Introduction

During the last ten years new algorithms have been proposed to perform adaptive eigenvalue decomposition of a data covariance matrix at the expense of a low computational cost, with complexity ranging from $O(N^2K)$ or $O(NK^2)$ down to $O(NK)$, where N is the size of the matrix of interest and K the dimension of the tracked eigenspaces.

Among recently developed techniques, MALASE algorithm offers some nice features : it is based on an adaptive implementation of the maximum likelihood criterion by means of a stochastic gradient approach that incorporates the differentiation of the eigenbasis under the unitary constraint. Furthermore, this algorithm ensures the orthonormality of the estimated eigenvectors, which is a desirable property for many applications. In particular, for sources localization, deviations from the orthonormality lead to limited resolution capability and limited estimation performance of the root MUSIC algorithm [1].

In [8] and [10], the convergence properties of MALASE as well as its theoretical asymptotic covariance matrix are given without proof. In this paper, we present these results with detailed proofs, following the approach in [7]. This study is based on the Ordinary Differential Equation (ODE) method [2]. Note that in [6] the performance analysis of an LMS-type approximation algorithm proposed by Yang [11] has been presented using the ODE method. Unlike the approach that we propose, the study in [6] is based on considerations upon the projectors associated to the eigenspaces rather than on the eigenspaces themselves. This difference is discussed in the paper.

We recall some basic facts about MALASE algorithm and about the ODE method in sections 2 and 3 respectively. The study of the asymptotic properties of MALASE algorithm is presented in section 4.

2 MALASE algorithm

Let $X = (X_n)_{n \in \mathbb{Z}}$ denote a size- N , complex valued, process ($X_n \in \mathbb{C}^N$), with covariance matrix $R_{X,n} = \mathbb{E}[X_n X_n^H]$. Subspace methods are based on the EigenValue Decomposition (EVD) of $R_{X,n}$, or at least on the estimation of the subspace spanned by the eigenvectors associated to the greatest (resp. the smallest) eigenvalues, and named the signal (resp. the noise) subspace. These methods are classically used within several multivariate signal processing problems.

While many adaptive updates of the EVD are derived from the expression of the update of the empiric covariance matrix, MALASE is based on a stochastic gradient technique applied to the log-likelihood criterion. It is clear that if X is zero mean circular gaussian, up to an additive constant the opposite of the log-likelihood is given by

$$\Psi(\mathbf{x}_n; R) = \log |R_{X,n}| + \mathbf{x}_n^H R_{X,n}^{-1} \mathbf{x}_n, \quad (1)$$

where \mathbf{x}_n is the data vector observed at time n . Under the hypothesis $R_{X,n} = R$, we let $U \Lambda U^H$ represent the EVD of R and (U_n, Λ_n) the estimated parameters of the instantaneous eigenvalue decomposition of the data covariance matrix at time n . Then, accounting for the orthonormality of U_n in the calculation of dU_n^H ($dU_n^H = -U_n^H dU_n U_n^H$) and in the

update of U_n yields the following algorithm (see [10] for more details) :

$$\mathbf{y}_n = U_{n-1}^H \mathbf{x}_n \quad (2a)$$

$$U_n = U_{n-1} \exp(\mu (\Lambda_{n-1}^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda_{n-1}^{-1})) \quad (2b)$$

$$\Lambda_n = \Lambda_{n-1} + \mu' (\Lambda_{n-1}^{-2} \text{diag}(\mathbf{y}_n \mathbf{y}_n^H) - \Lambda_{n-1}^{-1}). \quad (2c)$$

Despite what it seems, it can be shown that this algorithm can be implemented with only $O(N^2)$ operations per iteration.

Using the same preprocessing technique as in [3], it is possible to reduce the computational burden : when only $K < N$ eigenvalues of $R_{X,n}$ are distinct (as in many subspace tracking problems), the update of the eigendecomposition can be performed at the expense of $O(NK)$ operations.

We assume that the $N - p$ eigenvalues associated with the eigenvectors that we do not need are all equal. This situation arises in many applications. When this condition is not matched, it is possible to average these $N - p$ eigenvalues, as proposed in [4].

Let us assume for instance that we search for vectors $U_{1,n}, \dots, U_{p,n}$ and that the $N - p$ smallest eigenvalues are all equal : $\lambda_{p+1} = \dots = \lambda_N$. The preprocessing is based on an isometric transform applied to \mathbf{x}_n . Finally, the version of MALASE algorithm designed to track a dimension $p = K - 1$ eigenspace, named MALASE(K), can be summerized as

MALASE(K) implementation of MALASE algorithm

initialize U_0 unitary, initialize $\lambda_{k,0}$ with distinct values ($k = 1, K$), and

$0 < \mu < 1$,

for each time step do

input \mathbf{x}_n

$$\begin{aligned}
U_S &= [U_{n,1}^{(K)}, \dots, U_{n,p}^{(K)}] \\
a &= U_S^H \mathbf{x}_n \\
b &= \mathbf{x}_n - U_S a \\
\xi_n &= [a^T \|b\|]^T \\
U_{n-1,p+1}^{(K)} &= \|b\|^{-1} b \\
\nu_{k,n} &= \lambda_{k,n-1}^{-1} \xi_{k,n}, \quad (k = 1, K) \\
c_n &= \sqrt{\|\xi_n\|^2 \|\nu_n\|^2 - (\xi_n^H \nu_n)^2} \\
Q_n^{(K)} &= \begin{bmatrix} \frac{\nu_n}{\|\nu_n\|} & \frac{\nu_n(\nu_n^H \xi_n) - \xi_n \|\nu_n\|^2}{c_n \|\nu_n\|} \end{bmatrix} \\
U_n^{(K)} &= U_{n-1}^{(K)} + \{U_{n-1}^{(K)} \times Q_n^{(K)}\} \times \left\{ \begin{pmatrix} \cos \mu c_n - 1 & -\sin \mu c_n \\ \sin \mu c_n & \cos \mu c_n - 1 \end{pmatrix} \times Q_n^{(K)H} \right\} \\
\lambda_{k,n} &= \lambda_{k,n-1} + \mu(\lambda_{k,n-1}^{-2} |\xi_{k,n}|^2 - \lambda_{k,n}^{-1}) \quad (k = 1, K-1) \\
\lambda_{K,n} &= \lambda_{K,n-1} + \mu(\lambda_{K,n-1}^{-2} \frac{|\xi_{K,n}|^2}{N-p} - \lambda_{K,n}^{-1}).
\end{aligned} \tag{3}$$

3 Convergence and asymptotic covariance matrix

As usual, the study of the convergence will be considered for a stationary random process X . First, we are going to show that the parameters of the eigenvalue decompositions of R_X represent stationary points for MALASE algorithm.

3.1 Stationary points of MALASE algorithm.

Let (U_*, Λ_*) represent a stationary point of MALASE algorithm, and let us rewrite equations (2) in the form

$$\begin{aligned} U_{n+1} &= U_n + \mu H_1(X_n; \Lambda, U) + O(\mu^2) \\ \Lambda_{n+1} &= \Lambda_n + \mu H_2(X_n; \Lambda, U). \end{aligned} \tag{4}$$

under the assumption of a small gain factor μ the term in $O(\mu^2)$ is classically neglected and a basic idea of the ODE method consists in seeing an update equation of a vector parameter θ in the form

$$\theta_{k+1} = \theta_k + \mu H(\theta_{k-1}, X_k) \tag{5}$$

as a stochastic and discretized version of the deterministic differential equation

$$\dot{\theta}(t) = h(\theta(t), X), \tag{6}$$

where $h(\theta(t), X) = \mathbb{E}[H(\theta_{k-1}, X_k)]$. The study of (6) is important to understand the behaviour of (5). Indeed, if θ^* denotes the stationary stable point of (6) it can be shown [2] (see also [6]) that in real stationary situations $\mu^{-1/2}(\theta_k - \theta^*)$ converges to a zero mean gaussian distribution as μ tends to 0 and k tends to infinity. In this section we study the stationary stable points of the ODE.

Letting $h_i(\Lambda, U) = \mathbb{E}[H_i(X_n; \Lambda, U)]$ ($i = 1, 2$), it is clear that the condition for the point (U_*, Λ_*) to be stationary is given by the relations $h_1(\Lambda_*, U_*) = 0$ and $h_2(\Lambda_*, U_*) = 0$:

$$\begin{aligned} h_1(\Lambda_*, U_*) &= U_* (\Lambda_*^{-1} U_*^H R_X U_* - U_*^H R_X U_* \Lambda_*^{-1}) = 0 \\ h_2(\Lambda_*, U_*) &= \Lambda_*^{-1} - \Lambda_*^{-2} \text{diag}(U_*^H R_X U_*) = 0. \end{aligned} \tag{7}$$

It is clear that if (U, Λ) represents the parameters of the EVD of R_X , then $h_1(\Lambda, U) = 0$ and $h_2(\Lambda, U) = 0$.

Accounting for the fact that U_* must be unitary, it comes that the constraint $h_1(\Lambda_*, U_*) = 0$ can be rewritten as $\Lambda_*^{-1} U_*^H R_X U_* = U_*^H R_X U_* \Lambda_*^{-1}$. Now, forgetting the particular cases where $\|\Lambda_*\| = +\infty$, constraints (7) yield

$$\begin{aligned} [\Lambda_*]_{kk} &= \lambda_{*k} = U_{*k}^H R_X U_{*k}, \quad (k = 1, N) \\ [\Lambda_*^{-1} U_*^H R_X U_*]_{kl} &= [U_*^H R_X U_* \Lambda_*^{-1}]_{kl}, \quad (k = 1, N, l = 1, N). \end{aligned} \quad (8)$$

From (8), we have

$$(U_{*k}^H R_X U_{*k})^{-1} U_{*k}^H R_X U_{*l} = U_{*k}^H R_X U_{*l} (U_{*l}^H R_X U_{*l})^{-1} \quad (9)$$

From this, it appears that the algorithm has other stable points than the parameters of the EVD of R_X . Indeed, let U be unitary. In that case, it is easy to check that the stationary points correspond to unitary matrices U such that $U^H R_X U$ is a block diagonal matrix (the sizes of the different blocks may be different), where the terms along the diagonal of a given block are all equal. Then the corresponding Λ is given by $[\Lambda]_{kk} = U_k^H R_X U_k$, and conditions $h_1(\Lambda_*, U_*) = 0$ and $h_2(\Lambda_*, U_*) = 0$ are satisfied. Thus it appears that the eigenvalue decompositions are not the only stationary points of the algorithm.

3.2 stationary stable points of MALASE algorithm.

First, let us recall that since R_X is positive definite, the functional $g(R) = \log |R| + \text{Tr}(R_X R^{-1})$ is lower bounded over the set of positive definite hermitian matrices, and $g(R)$ tends to infinity when R tends to a positive singular matrix. This shows that $g(R)$

admits a global minimum. Then, in order to show that only the EVD of R_X represents a stable point of the algorithm, it is sufficient to show that any other stationary point is not a stable point.

If (U_*, Λ_*) is a stationary point, it comes from the conditions (8) that

$$[\Lambda_*]_{k,k} = \lambda_{*,k} = U_{*,k}^H R_X U_{*,k} \quad k = 1, N. \quad (10)$$

This shows that stationary points (Λ_*, U_*) are completely characterized by the corresponding matrix U_* . In particular, if U_* represents a basis of eigenvectors, (U_*, Λ_*) corresponds to the eigenvalue decomposition of R_X . This suggests to consider the functional $K(U) = \mathbb{E}[\Psi(X_n; U \Lambda U^H)]$, where $\lambda_k = U_k^H R_X U_k$ ($k = 1, N$). Clearly, a necessary condition for (U_*, Λ_*) to be a stationary stable point, is that U_* must be a local minimum of $K(U)$. In appendix A, we prove the following result :

Lemma 1 *If the columns of the unitary matrix U_* are not eigenvectors, then U_* is not a local minimum of $K(U)$.*

Consequently, the only possible stationary stable points of the ODE correspond to the EVD of R_X , and since $g(R) = \log |R| + \text{Tr}(R_X R^{-1})$ admits a global minimum, it comes that

Proposition 1 *The stationary stable points of the ODE of MALASE algorithm correspond to the EVD of R_X .*

3.2.1 Remarks.

Let us assume first that the dimensions of the eigenspaces of R_X are equal to one. It is clear that the eigenvectors represent stationary stable points only up to a modulus one factor. Thus, we cannot rigorously say that the eigenvectors represent stable points. However, there are two ways to justify the fact that we regard eigenvalue decompositions of R_X as stationary points of MALASE algorithm.

A first way is to remark that if $U_{k,n}$ is multiplied by $e^{i\alpha_{k,n}}$ ($\alpha_{k,n} \in \mathbb{R}$), then the sequence $(U_{k,n+l})_{l \geq 0}$ supplied by MALASE algorithm will be replaced by the sequence $(U'_{k,n+l} = e^{i\alpha_{k,n}} U_{k,n+l})_{l \geq 0}$. Since up to a modulus one factor the convergence properties are not affected by this change, it appears that letting $U'_{k,n} = e^{-i\arg[U_{k,n}]l} U_{k,n}$, ensures the existence of a unique stationary stable point, denoted (U_*, Λ_*) , for the sequence $(U'_n, \Lambda_n)_{n \in \mathbb{N}}$, provided that l is chosen such that $[U_{k*}]_l \neq 0$ ($U_* = [U_{1*}, \dots, U_{N*}]$). Obviously, (U_*, Λ_*) represents the EVD of R_X .

Also, it is possible to replace the study of the convergence of the sequence $(U_{k,n})_{n \geq 0}$, by that of $(\Pi_{k,n})_{n \geq 0}$ where $\Pi_{k,n}$ denotes the projector associated to vector U_k at time n : $\Pi_{k,n} = U_{k,n} U_{k,n}^H$. To show this, let us rewrite the update of U_k as

$$U_{k,n} = U_{k,n-1} + C_{k,n} U_{k,n-1}, \quad (11)$$

where $C_{k,n}$ is the k^{th} column of the matrix C_n defined by

$$\begin{aligned}
C_n &= U_{n-1} [(\mathbf{z}\mathbf{z}^H + \mathbf{y}\mathbf{y}^H)(\cos \mu c_n - 1) + (\mathbf{y}\mathbf{z}^H - \mathbf{z}\mathbf{y}^H) \sin \mu c_n] U_{n-1}^H \\
&= [(U_{n-1}\Lambda_{n-1}^{-1}U_{n-1}^H)\mathbf{x}_n\mathbf{x}_n^H(U_{n-1}\Lambda_{n-1}^{-1}U_{n-1}^H) + \mathbf{x}_n\mathbf{x}_n^H] (\cos \mu c_n - 1) \\
&\quad + [\mathbf{x}_n\mathbf{x}_n^H(U_{n-1}\Lambda_{n-1}^{-1}U_{k,n-1}^H) - (U_{k,n-1}\Lambda_{n-1}^{-1}U_{n-1}^H)\mathbf{x}_n\mathbf{x}_n^H] \sin \mu c_n \\
&= [(\sum_{l=1,n} \lambda_{l,n}^{-1}\Pi_{l,n})\mathbf{x}\mathbf{x}^H(\sum_{l=1,n} \lambda_{l,n}^{-1}\Pi_{l,n}) + \mathbf{x}_n\mathbf{x}_n^H] (\cos \mu c_n - 1) \\
&\quad + [\mathbf{x}\mathbf{x}^H(\sum_{l=1,n} \lambda_{l,n}^{-1}\Pi_{l,n}) - (\sum_{l=1,n} \lambda_{l,n}^{-1}\Pi_{l,n})\mathbf{x}_n\mathbf{x}_n^H] \sin \mu c_n \\
&= \mu c_n [\mathbf{x}\mathbf{x}^H(\sum_{l=1,n} \lambda_{l,n}^{-1}\Pi_{l,n}) - (\sum_{l=1,n} \lambda_{l,n}^{-1}\Pi_{l,n})\mathbf{x}_n\mathbf{x}_n^H] + O(\mu^2).
\end{aligned} \tag{12}$$

It is clear that $\Pi_{k,n}$ and $C_{k,n}$ are left unchanged if $U_{k,n}$ is multiplied by a modulus one constant. Furthermore, the eigenspace update can be written in the form

$$\Pi_{k,n} = \Pi_{k,n-1} + [C_{k,n}\Pi_{k,n-1}C_{k,n}^H + C_{k,n}\Pi_{k,n-1} + \Pi_{k,n-1}C_{k,n}^H]. \tag{13}$$

Thus, it is clear that up to modulus one factors, there exists a one to one correspondence between the sequence of projectors $(\Pi_{k,n})_{n \geq 0}$ ($k = 1, N$) obtained from (13), and the sequence $(U_{k,n})_{n \geq 0}$ obtained from (2b). Thus, from the study of the convergence of MALASE algorithm, it appears that the sequence $(\Pi_{k,n})_{n \in \mathbb{N}}$ converges to the projector associated with the k^{th} eigenvector.

Also, let us remark that the convergence of $U_{k,n}$ to an eigenvector denoted U_k (or of $\Pi_{k,n}$ to the corresponding projector) cannot be ensured if it belongs to an eigenspace of dimension greater than one. In that case, if $U_{k,n}, \dots, U_{k+l-1,n}$ represent the estimated eigenvectors associated with a dimension l eigenspace, only the convergence of the estimated projector on that subspace, i.e. of $\sum_{i=0,l-1} U_{k+i,n}U_{k+i,n}^H$, can be checked. For the sake of simplicity, in the derivation of the asymptotic covariance matrix of MALASE algorithm, we will assume that all the eigenvalues are distinct.

3.3 Asymptotic covariance matrix.

Let us recall first a few basic facts about the derivation of the asymptotic covariance matrix for a stochastic algorithm in the form

$$\theta_{n+1} = \theta_n + \mu H(\theta_n, X_n), \quad (14)$$

where $\theta_n = [\theta_{1,n}, \dots, \theta_{L,n}]^T$ is the estimated parameter at time n . As explained previously $\mu^{1/2}(\theta_n - \theta_*)$ tends to a zero mean gaussian distribution when μ to 0 and n tends to infinity (see e.g. [2]), and the asymptotic covariance matrix of θ_n is equal to μP , where P is the only solution of the Lyapunov equation

$$GP + PG^T + K = 0, \quad (15)$$

with

$$P = \mathbb{E}[(\theta_n - \theta_*)(\theta_n - \theta_*)^H] \quad (16a)$$

$$[G]_{ij} = \left(\frac{\partial [h(\theta)]_i}{\partial \theta_j} \right)_{\theta=\theta_*} \quad (16b)$$

$$K = \sum_{n=-\infty}^{+\infty} \mathbb{E}[(H(\theta_*, X_n) - h(\theta_*))(H(\theta_*, X_n) - h(\theta_*))^H], \quad (16c)$$

with $h(\theta) = \mathbb{E}[H(\theta, X_n)]$. Let us remark that when the vectors $(X_n)_{n \in \mathbb{Z}}$ form an independent sequence of vectors with the same distribution, then K is simply the covariance matrix of $H(\theta_*, X_n)$. Since generally little prior information is available upon X , and since calculations are easier in this case, people often assume that vectors X_n are uncorelated when solving (15) (see e.g. [5] for studies in the correlated case). We will work with this simplifying assumption.

Now, let us remark that the update of U in MALASE is not in the form of (14). In order to be able to use formula (15), we will simply consider the first order development of the update of U with respect to μ :

$$U_n = U_{n-1} + \mu U_{n-1} (\Lambda_{n-1}^{-1} \mathbf{y}_n \mathbf{y}_n^H - \mathbf{y}_n \mathbf{y}_n^H \Lambda_{n-1}^{-1}).$$

Thanks to this expression, we will be able to derive an approximate expression of the asymptotic covariance matrix for MALASE algorithm.

Let $(U_{1,n}, \dots, U_{N,n})$ and $(\lambda_{1,n}, \dots, \lambda_{N,n})$ denote the estimated eigenvectors and the estimated eigenvalues, and let us denote $V_n = [U_{1,n}^T, \dots, U_{N,n}^T, \lambda_{1,n}, \dots, \lambda_{N,n}]^T$ the size- $(N^2 + N)$ vector that contains these parameters. It is clear that the update of V_n is in the form (14). The calculation of the asymptotic covariance matrix of V_n , denoted μP , is derived in the appendix B; it finally comes that

$$\mu P = \frac{1}{2} \mu I_{N^2+N} \quad (17)$$

This is a very interesting result since the asymptotic variance depends only on μ , and not on the parameters of the EVD of R_X , unlike other algorithms (e.g. [9]). Furthermore, any of the $N^2 + N$ parameters are asymptotically uncorrelated, and have the same variance, equal to $\mu/2$. For eigenvalues, this can be compared for instance to the classical update in the form

$$\lambda_{k,n} = \lambda_{k,n-1} - \mu(\lambda_{k,n-1} - |\mathbf{y}_{k,n}|^2), \quad k = 1, N. \quad (18)$$

used by many algorithms that yields theoretical asymptotic variance for λ_k equal to $(\mu/2)\lambda_k^2$.

References

- [1] A. Barabell. Improving the resolution performance of eigenstructure-based direction-finding algorithms. In *ICASSP'83*, pages 336–339, 1983.
- [2] A. Benveniste, M. Metivier, and P. Priouret. *Adaptive Algorithms and Stochastic Approximation*. Springer Verlag, 1990.
- [3] B. Champagne and Q.G. Liu. Plane rotation-based EVD updating schemes for efficient subspace tracking. *IEEE Trans. on Signal Processing*, 46(7):1886–1900, July 1998.
- [4] R.D. DeGroat. Noniterative subspace tracking. *IEEE Trans. on Signal Processing*, 40(3):571–577, March 1992.
- [5] J.P. Delmas. Performance analysis of parametrized adaptative eigen-subspace algorithms. In *ICASSP'95*, pages 2056–2059, 1995.
- [6] J.F. cardoso J.P. Delmas. Performance analysis of an adaptive algorithm for tracking dominant subspaces. *IEEE Trans. on SP*, 46(11):3045–3057, November 1998.
- [7] C. Riou. *Adaptive Subspace Estimation and Applications*. ENST Bretagne, Brest, 1997.
- [8] C. Riou, T. Chonavel, and P.Y. Cochet. Adaptative technique for moving sources localization. In *Krajowe Symposium Telekomunikacja*, September 1995.

- [9] C. Riou, T. Chonavel, and P.Y. Cochet. Adaptive subspace estimation : Application to moving sources localization and blind channel identification. In *ICASSP'96*, pages 1649–1652, 1996.
- [10] C. Riou T. Chonavel, B. Champagne. Fast adaptive eigenvalue decomposition, a maximum likelihood approach. *submitted to Signal processing*, 2000.
- [11] B. Yang. Subspace tracking based on the projection approach and the recursive least squares method. In *ICASSP'93*, pages 145–148, 1993.

APPENDIX

A Proof of lemma 1

Let us remark that

$$\begin{aligned}
 K(U) &= \sum_{k=1}^N \log(U_k^H R_X U_k) + \text{Tr}[R_X U (\text{diag}(U_1^H R_X U_1, \dots, U_N^H R_X U_N))^{-1} U^H] \quad (19a) \\
 &= \sum_{k=1}^N \log(U_k^H R_X U_k) + N, \quad (19b)
 \end{aligned}$$

and let us consider a perturbation of U in the form

$$U + dU = U G_{k,l}(\theta, \varphi) \quad (20)$$

where $G_{k,l}(\theta, \varphi)$ is the Given's rotation defined by

$$G_{k,l}(\theta, \varphi) = \begin{bmatrix} I_{k-1} & & & & & & \\ & \cos \theta & & & e^{j\varphi} \sin \theta & & \\ & & & I_{l-k-1} & & & \\ & & & & & & \\ & -e^{-j\varphi} \sin \theta & & & \cos \theta & & \\ & & & & & & I_{N-l} \end{bmatrix}.$$

The fact that $G_{k,l}(\theta, \varphi)$ may be close to the identity matrix (i.e. dU is close to 0) is expressed by the fact that θ is close to 0, but does not bring any constraint upon φ ; i.e. φ can have any value.

Let us denote $\Delta_{kl} = K(U + dU) - K(U)$. A necessary condition for U to represent a

stable point of K is expressed by the fact that $\Delta_{kl} > 0$ for $dU \neq 0$. We have

$$\begin{aligned}\Delta_{kl} &= \log \left([U_k \cos \theta + e^{j\varphi} U_l \sin \theta]^H R_X [U_k \cos \theta + e^{j\varphi} U_l \sin \theta] \right) \\ &\quad + \log \left([U_l \cos \theta - e^{-j\varphi} U_k \sin \theta]^H R_X [U_l \cos \theta - e^{-j\varphi} U_k \sin \theta] \right) \\ &\quad - \log (U_k^H R_X U_k) - \log (U_l^H R_X U_l). \end{aligned} \quad (21)$$

Letting $U_k^H R_X U_k = \lambda_k$, $U_l^H R_X U_l = \lambda_l$, $U_l^H R_X U_k = \lambda_{l,k}$ and $\text{Re}(\cdot)$ denote the real part, it comes that

$$\begin{aligned}[U_k \cos \theta + e^{j\varphi} U_l \sin \theta]^H R_X [U_k \cos \theta + e^{j\varphi} U_l \sin \theta] &= \\ \cos^2(\theta) \lambda_k + \sin(2\theta) \text{Re}(e^{j\varphi} \lambda_{k,l}) + \sin^2(\theta) \lambda_l, &\end{aligned} \quad (22a)$$

$$\begin{aligned}[U_l \cos \theta - e^{-j\varphi} U_k \sin \theta]^H R_X [U_l \cos \theta - e^{-j\varphi} U_k \sin \theta] &= \\ \cos^2(\theta) \lambda_l - \sin(2\theta) \text{Re}(e^{-j\varphi} \lambda_{l,k}) + \sin^2(\theta) \lambda_k. &\end{aligned} \quad (22b)$$

Then, letting $C = \text{Re}(e^{j\varphi} \lambda_{k,l}) = \text{Re}(e^{-j\varphi} \lambda_{l,k})$ yields

$$\begin{aligned}\exp \Delta_{kl} &= (\cos^2(\theta) + \sin(2\theta) \frac{C}{\lambda_k} + \sin^2(\theta) \frac{\lambda_l}{\lambda_k}) \times (\cos^2(\theta) - \sin(2\theta) \frac{C}{\lambda_l} + \sin^2(\theta) \frac{\lambda_k}{\lambda_l}) \\ &= \cos^4(\theta) + \sin^4(\theta) + \cos^2(\theta) \sin^2(\theta) \left(\frac{\lambda_l}{\lambda_k} + \frac{\lambda_k}{\lambda_l} \right) + \sin(2\theta) C \times \\ &\quad \left(-\frac{\cos^2(\theta)}{\lambda_l} - \frac{\sin^2(\theta)}{\lambda_k} + \frac{\cos^2(\theta)}{\lambda_k} + \frac{\sin^2(\theta)}{\lambda_l} - \sin(2\theta) \frac{C}{\lambda_k \lambda_l} \right) \\ &= (\cos^2(\theta) + \sin^2(\theta))^2 + \left(\frac{\sin(2\theta)}{2} \right)^2 \left(\frac{\lambda_l}{\lambda_k} + \frac{\lambda_k}{\lambda_l} - 2 \right) - \sin^2(2\theta) \frac{C^2}{\lambda_k \lambda_l} + \\ &\quad \sin(2\theta) C (\cos^2(\theta) - \sin^2(\theta)) \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_l} \right) \\ &= 1 + \sin^2(2\theta) \left[\frac{1}{4} \left(\frac{\lambda_l}{\lambda_k} + \frac{\lambda_k}{\lambda_l} - 2 \right) - \frac{C^2}{\lambda_k \lambda_l} \right] + \frac{\sin(4\theta)}{2} C \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_l} \right). \end{aligned}$$

Clearly, to ensure the stability condition, the coefficient of $\sin^2(\theta)$ must be strictly positive and the coefficient of $\sin(4\theta)$ must be equal to zero, for any φ . These conditions can be written in the form

$$\begin{cases} \operatorname{Re}(e^{j\varphi} U_l^H R_X U_k) \left(\frac{1}{U_k^H R_X U_k} - \frac{1}{U_l^H R_X U_l} \right) = 0, & \forall \varphi \\ \frac{1}{4} \left(\frac{U_l^H R_X U_l}{U_k^H R_X U_k} + \frac{U_k^H R_X U_k}{U_l^H R_X U_l} - 2 \right) - \frac{(\operatorname{Re}(e^{-j\varphi} U_l^H R_X U_k))^2}{(U_k^H R_X U_k)(U_l^H R_X U_l)} > 0, & \forall \varphi. \end{cases} \quad (23)$$

The first condition of (23) yields :

$$\operatorname{Re}(e^{-j\varphi} U_l^H R_X U_k) = 0, \quad \forall \varphi, \quad (24)$$

or

$$U_k^H R_X U_k = U_l^H R_X U_l. \quad (25)$$

Let us suppose first that (24) is satisfied, or equivalently that $U_l^H R_X U_k = 0$. Then, the inequality in (23) becomes

$$\frac{U_l^H R_X U_l}{U_k^H R_X U_k} + \frac{U_k^H R_X U_k}{U_l^H R_X U_l} - 2 > 0 \quad (26)$$

The variations of the function $f(x) = x + \frac{1}{x} - 2$ are given in the following table :

x	0	1	$+\infty$	(27)
$f(x) = x + \frac{1}{x} - 2$	$+\infty$	\searrow 0 \nearrow	$+\infty$	

Thus, strict inequality (26) is always satisfied provided that $U_k^H R_X U_k \neq U_l^H R_X U_l$ (equality is achieved only if $U_k^H R_X U_k = U_l^H R_X U_l$).

Now, if the condition $U_k^H R_X U_k = U_l^H R_X U_l$ were satisfied, the inequality in (23) would

become

$$\frac{(\operatorname{Re}(e^{-j\varphi}U_l^H R_X U_k))^2}{(U_k^H R_X U_k)(U_l^H R_X U_l)} < 0, \quad (28)$$

which is impossible.

If the columns of U do not represent an eigenbasis, then $U^H R_X U$ is not diagonal, and $U_k^H R_X U_l \neq 0$ for some (k, l) . In that case, the stability conditions (23) are not satisfied. This shows that if U is not a matrix of eigenvectors, it cannot be a stable point of K .

B Asymptotic covariance for MALASE algorithm.

Forgetting subscript index n , the update terms for the matrices U and Λ of the approximate MALASE algorithm obtained by accounting only for $O(\mu)$ terms of the update are respectively H_1 and H_2 with

$$H_1 = U\Lambda^{-1}U^H X X^H U - X X^H U \Lambda^{-1} \quad (29)$$

$$H_2 = -\Lambda^{-1} + \Lambda^{-2} \operatorname{diag}(U^H X X^H U) \quad (30)$$

Then, let us define

$$h_1 = \mathbb{E}[H_1] = U\Lambda^{-1}U^H R_X U - R_X U \Lambda^{-1} \quad (31a)$$

$$h_2 = \mathbb{E}[H_2] = -\Lambda^{-1} + \Lambda^{-2} \operatorname{diag}(U^H R_X U) \quad (31b)$$

Also, let us denote \tilde{H}_1 and \tilde{H}_2 (resp. \tilde{h}_1 and \tilde{h}_2) the vectors obtained by stacking the columns of H_1 and the diagonal terms of H_2 respectively, and $\tilde{h}_1 = \mathbb{E}[\tilde{H}_1]$, and $\tilde{h}_2 = \mathbb{E}[\tilde{H}_2]$.

The approximate asymptotic covariance matrix of the vector $V = [U_1^T, \dots, U_N^T, \lambda_1, \dots, \lambda_N]^T$, is μP , where P is the solution of the following Lyapounov equation :

$$GP + PG^H + K = 0, \quad (32)$$

and where the matrices G and K are given by :

$$\begin{aligned} G &= \begin{pmatrix} \frac{\partial h_1}{\partial U} & \frac{\partial h_1}{\partial \Lambda} \\ \frac{\partial h_2}{\partial U} & \frac{\partial h_2}{\partial \Lambda} \end{pmatrix}_{(U, \Lambda) = (U_*, \Lambda_*)} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \\ K &= \begin{pmatrix} \mathbb{E}[\tilde{H}_1 \tilde{H}_1^H] & \mathbb{E}[\tilde{H}_1 \tilde{H}_2^H] \\ \mathbb{E}[\tilde{H}_2 \tilde{H}_1^H] & \mathbb{E}[\tilde{H}_2 \tilde{H}_2^H] \end{pmatrix}_{(U, \Lambda) = (U_*, \Lambda_*)} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \end{aligned} \quad (33)$$

$\frac{\partial h_1}{\partial U}$ represents the matrix with term $(i + N(j - 1), k + N(l - 1))$ equal to $\frac{\partial [h_1]_{ij}}{\partial U_{k,i}}$, $\frac{\partial h_1}{\partial \Lambda}$ the matrix with term $(i + N(j - 1), k)$ equal to $\frac{\partial [h_1]_{ij}}{\partial \lambda_k}$, $\frac{\partial h_2}{\partial U}$ the matrix with term $(i, k + N(l - 1))$ equal to $\frac{\partial [h_2]_i}{\partial U_{k,i}}$, and $\frac{\partial h_2}{\partial \Lambda}$ the matrix with term (i, k) equal to $\frac{\partial [h_2]_i}{\partial \lambda_k}$. (U_*, Λ_*) represents the parameters of the stable point of the algorithm, i.e. the eigenvalue decomposition of R_X .

B.1 Derivation of G .

In order to get G_{11} , let us let us differentiate the matrix h_1 at point (U_*, Λ_*) with respect to U , under a unitary constraint upon U (i.e. $dU^H = -U^H dU U^H$) :

$$\begin{aligned} d_U h_1 &= dU \Lambda^{-1} U^H R_X U - U \Lambda^{-1} U^H dU U^H R_X U + U \Lambda^{-1} U^H R_X dU - R_X dU \Lambda^{-1} \\ &= 2dU - U \Lambda^{-1} U^H dU \Lambda - U \Lambda U^H dU \Lambda^{-1}. \end{aligned} \quad (34)$$

Let us denote Δ_{kl} the matrix with all terms equal to 0, except term (k, l) that is equal to

1. The matrix $\frac{\partial h_1}{\partial U_{kl}}$ is obtained by setting $dU = \Delta_{kl}$, which leads to :

$$\frac{\partial h_1}{\partial U_{kl}} = 2\Delta_{kl} - U\Lambda^{-1}U^H\Delta_{kl}\Lambda - U\Lambda U^H\Delta_{kl}\Lambda^{-1} \quad (35a)$$

$$= \left(2I - U\left(\lambda_l\Lambda^{-1} + \frac{\Lambda}{\lambda_l}\right)U^H\right)\Delta_{kl}. \quad (35b)$$

Let us remark that $\frac{\partial[h_1]_{ij}}{\partial U_{kl}}$ is the term (i, j) , of the matrix $\frac{\partial h_1}{\partial U_{kl}}$, and also the term $(i + N(j - 1), k + N(l - 1))$ of the matrix $\frac{\partial h_1}{\partial U}$. Then,

$$\frac{\partial[h_1]_{ij}}{\partial U_{kl}} = 2\delta_{ik}\delta_{jl} - \sum_{n=1}^N \left[U\left(\lambda_l\Lambda^{-1} + \frac{\Lambda}{\lambda_l}\right)U^H \right]_{in} [\Delta_{kl}]_{nj} \quad (36a)$$

$$= 2\delta_{ik}\delta_{jl} - \left[U\left(\lambda_l\Lambda^{-1} + \frac{\Lambda}{\lambda_l}\right)U^H \right]_{ik} \delta_{jl}, \quad (36b)$$

which leads to :

$$G_{11} = W_{11} \left[2I_{N^2} - \text{diag} \left(\lambda_1\Lambda^{-1} + \frac{\Lambda}{\lambda_1}, \dots, \lambda_N\Lambda^{-1} + \frac{\Lambda}{\lambda_N} \right) \right] W_{11}^H, \quad (37)$$

where W_{11} is the $N^2 \times N^2$ bloc diagonal matrix : $W_{11} = \text{diag}(U, \dots, U)$.

To get G_{22} , let us remark that

$$\frac{\partial h_2}{\partial \Lambda} = \Lambda^{-2} - 2\Lambda^{-3} \text{diag} (U^H R_X U). \quad (38)$$

Thus, $G_{22} = -\Lambda^{-2}$.

In order to get G_{12} , let us differentiate h_1 with respect to Λ

$$\begin{aligned} d_\Lambda h_1 &= -U\Lambda^{-1}d\Lambda\Lambda^{-1}U^H R_X U + R_X U\Lambda^{-1}d\Lambda\Lambda^{-1} \\ &= -U\Lambda^{-1}d\Lambda + U d\Lambda\Lambda^{-1} = -U(\Lambda^{-1}d\Lambda - d\Lambda\Lambda^{-1}). \end{aligned} \quad (39)$$

Since Λ and $d\Lambda$ are diagonal matrices, $d_\Lambda h_1 = 0$. Thus, $G_{12} = 0$.

In order to get G_{21} , let us calculate $d_U h_2$. Up to a second order term, and accounting for the orthonormality constraint upon U , it comes that

$$\begin{aligned} d_U h_2 &= \Lambda^{-2} \text{diag} \left((U + dU)^H R_X (U + dU) \right) - \Lambda^{-2} \text{diag} (U^H R_X U) \\ &= \Lambda^{-2} \text{diag} (\Lambda U^H dU - U^H dU \Lambda). \end{aligned} \quad (40)$$

But the diagonal terms of $\Lambda U^H dU - U^H dU \Lambda$ are equal to 0. Thus, $G_{21} = 0$.

Finally, we get :

$$G = W \begin{pmatrix} 2I_{N^2} - \text{diag} \left(\lambda_k \Lambda^{-1} + \frac{\Lambda}{\lambda_k} \right)_{k=1, \dots, N} & 0 \\ 0 & -\text{diag} \left(\frac{1}{\lambda_1^2}, \dots, \frac{1}{\lambda_N^2} \right), \end{pmatrix} W^H \quad (41)$$

where W is a $(N^2 + N) \times (N^2 + N)$ unitary bloc-diagonal matrix : $W = \text{diag} (U, \dots, U, I_N)$

B.2 Derivation of K .

In order to get K_{11} , let us remark that K_{11} is a bloc matrix with blocs of size $N \times N$. Let $[A]_{.k}$ denote the k^{th} column of the matrix A . Then, the bloc (i, j) of K_{11} is given by

$$\begin{aligned}
\mathbb{E}[[H_1]_{.i}([H_1]_{.j})^H] &= \mathbb{E}[(U\Lambda^{-1}U^HXX^HU - XX^HU\Lambda^{-1})_{.i}[(U\Lambda^{-1}U^H\lambda XX^H - XX^HU\Lambda^{-1})_{.j}]^H] \\
&= \mathbb{E}[(R_X^{-1}XX^H - XX^HR_X^{-1})U_iU_j^H[XX^HR_X^{-1} - R_X^{-1}XX^H]^H] \\
&= R_X^{-1}\mathbb{E}[XX^HU_iU_j^HXX^H]R_X^{-1} + \mathbb{E}[XX^HR_X^{-1}U_iU_j^HR_X^{-1}XX^H] \\
&\quad - R_X^{-1}\mathbb{E}[XX^HU_iU_j^HR_X^{-1}XX^H] - \mathbb{E}[XX^HR_X^{-1}U_iU_j^HXX^H]R_X^{-1} \\
&= R_X^{-1}(R_XU_iU_j^HR_X + R_X(U_j^HR_XU_i))R_X^{-1} \\
&\quad + (R_XR_X^{-1}U_iU_j^HR_X^{-1}R_X + R_X(U_j^HR_X^{-1}R_XR_X^{-1}U_i)) \\
&\quad - R_X^{-1}(R_XU_iU_j^HR_X^{-1}R_X + R_X(U_j^HR_X^{-1}R_XU_i)) \\
&\quad - (R_XR_X^{-1}U_iU_j^HR_X + R_X(U_j^HR_XR_X^{-1}U_i))R_X^{-1} \\
&= \delta_{ij} \left(\lambda_i R_X^{-1} + \frac{R_X}{\lambda_i} \right) - 2U_j^H U_i I \\
&= \delta_{ij} U \left(\lambda_i \Lambda^{-1} + \frac{1}{\lambda_i} \Lambda - 2I_N \right) U^H.
\end{aligned} \tag{42}$$

Finally, it comes that :

$$K_{11} = W_{11} \left[\text{diag} \left(\lambda_1 \Lambda^{-1} + \frac{\Lambda}{\lambda_1}, \dots, \lambda_N \Lambda^{-1} + \frac{\Lambda}{\lambda_N} \right) - 2I_{N^2} \right] W_{11}^H. \tag{43}$$

Let us calculate K_{22} :

$$\begin{aligned}
[K_{22}]_{ij} &= \mathbb{E}[[H_2]_{ii}[H_2]_{jj}] \\
&= \mathbb{E}\left[\left(-\frac{1}{\lambda_i} + \frac{U_i^H X X^H U_i}{\lambda_i^2}\right) \left(-\frac{1}{\lambda_j} + \frac{U_j^H X X^H U_j}{\lambda_j^2}\right)\right] \\
&= \frac{1}{\lambda_i \lambda_j} + \frac{1}{\lambda_i^2 \lambda_j^2} \mathbb{E}[U_i^H X X^H U_i U_j^H X X^H U_j] \\
&\quad - \frac{1}{\lambda_i^2 \lambda_j^2} \mathbb{E}[\lambda_i U_j^H X X^H U_j + \lambda_j U_i^H X X^H U_i] \\
&= -\frac{1}{\lambda_i \lambda_j} + \frac{1}{\lambda_i^2 \lambda_j^2} [(U_i^H R_X U_i)(U_j^H R_X U_j) + (U_i^H R_X U_j)(U_j^H R_X U_i)] \\
&= \frac{\delta_{ij}}{\lambda_i^2}.
\end{aligned} \tag{44}$$

Thus, $K_{22} = \Lambda^{-2}$.

Let us remark that $K_{12} = K_{21}^H$. K_{12} is a bloc vector, with blocs of size $N \times N$. The j^{th} column of the i^{th} block of K_{12} is given by

$$\begin{aligned}
[K_{12}]_{ij} &= \mathbb{E}[[H_1]_{.i}[H_2]_{jj}] = \mathbb{E}[(R_X^{-1} X X^H U_i - X X^H R_X^{-1} U_i) \left(-\frac{1}{\lambda_j} + \frac{U_j^H X X^H U_j}{\lambda_j^2}\right)] \\
&= -\frac{1}{\lambda_j} \mathbb{E}[(R_X^{-1} X X^H U_i - X X^H R_X^{-1} U_i)] + \frac{1}{\lambda_j^2} \mathbb{E}[(R_X^{-1} X X^H U_i)(U_j^H X X^H U_j)] \\
&\quad - \frac{1}{\lambda_j^2} \mathbb{E}[(X X^H R_X^{-1} U_i)(U_j^H X X^H U_j)] \\
&= 0 + \frac{1}{\lambda_j^2} [(R_X^{-1} R_X U_i)(U_j^H R_X U_j) + (R_X^{-1} R_X U_j)(U_j^H R_X U_i)] \\
&\quad - \frac{1}{\lambda_j^2} [(R_X R_X^{-1} U_i)(U_j^H R_X U_j) + (R_X U_j)(U_j^H R_X R_X^{-1} U_i)] \\
&= 0.
\end{aligned} \tag{45}$$

Consequently, $K_{12} = K_{21}^H = 0$.

From what precedes, it comes that

$$K = W \begin{pmatrix} -2I_{N^2} + \text{diag} \left(\lambda_k \Lambda^{-1} + \frac{\Lambda}{\lambda_k} \right)_{k=1, \dots, N} & 0 \\ 0 & \text{diag} \left(\frac{1}{\lambda_1^2}, \dots, \frac{1}{\lambda_N^2} \right) \end{pmatrix} W^H. \tag{46}$$

B.3 Solution of Lyapounov equation

Let us remark that $G = G^H = -K$. Thus P is the solution of the Lyapounov equation $KP + PK - K = 0$. Straightforward calculation yields

$$P = \frac{1}{2}I_{N^2+N}. \quad (47)$$

Chapitre 5

Estimation de paramètres de décalage dans un mélange de lois

Estimating the offset parameters of a mixture in the Fourier domain*

Sandrine Vaton and Thierry Chonavel

ENST Bretagne, BP 832, 29285 Brest Cedex, France.

e-mail: Sandrine.Vaton, Thierry.Chonavel@enst-bretagne.fr

April 12, 2000

Abstract

This paper concerns the estimation of the supports of probability density functions in a context of incomplete data (mixtures or Hidden Markov Models). These supports cannot be estimated with the EM algorithm or any stochastic version of the EM algorithm unlike more classical parameters (mean, variance...). Indeed the complete data model is not regular exponential with respect to the supports. In order to overcome this limitation of the EM algorithm and of its stochastic versions we develop the following approach : we minimize a quadratic distance between the Fourier transform of the data and its parameterized counterpart. Under very general conditions the Fourier transform of the data is approximately gaussian when the sample is large and our approach reduces to a maximum likelihood estimation. Unlike the problems encountered in the time domain, the asymptotic model is regular exponential with respect to the supports in the Fourier domain. We discuss the conditions under which the estimate is convergent and of minimum variance. Some simulation results are presented.

*Extended version; short version in ICASSP 1999.

Keywords: incomplete data models, mixture of distributions, Fourier transform, Maximum Likelihood Estimation, EM algorithm.

1 Introduction

The EM algorithm is a valuable approach for estimating some parameters in the case of incomplete data. The EM algorithm suffers yet from a number of serious limitations among which possible convergence on some local but not global extremum of the likelihood. In order to overcome these well known drawbacks of the EM algorithm some stochastic versions of this algorithm have been previously developed.

In this paper we bring to the fore an unknown limitation of the EM algorithm and of its stochastic versions. We treat the estimation of some parameters of offset in the case of incomplete data. To be more precise we consider hidden Markov chains, the distribution of the observations conditionally to the unobserved regime as being exponential distributions with unknown offsets¹. We make known that neither the EM algorithm nor its stochastic versions are valuable for this problem. Basically the problems come from the fact that the model is not exponential with respect to the offsets. The result of this is that the estimates of the offsets increase at each step of the EM algorithm or of any of the stochastic versions considered.

In order to overcome the difficulties encountered in the time domain with maximum likelihood techniques we transpose the estimation problem in the Fourier domain. The offsets are transposed into the pulsations of some complex exponentials that modulate the characteristic functions of the different components of the mixture. We minimize some quadratic distance between the sampled empirical Fourier transform and its parametric counterpart.

¹Although it applies to more general situations than the case of exponential distributions, we will show that it is of particular interest in this situation.

This approach can be formulated as a classical moment approach; this remark makes it possible to establish some convergence results. We demonstrate that the estimate is convergent under some conditions of identifiability of the sampled Fourier transform. We state some sufficient conditions for ensuring the identifiability. For a particular choice of the quadratic distance between Fourier transforms that we specify the estimate is what is more of minimum variance asymptotically.

1.1 Hidden Markov Models

Loosely speaking a Hidden Markov model (HMM) is a probabilistic function of a Markov chain : an unobserved sequence follows a Markov chain with finite state space and each observation depends on the current value of the underlying Markov chain only.

Practical situations in which hidden Markov models arise are numerous. In the telecommunications area the major application of hidden Markov models is certainly automatic speech recognition (Levinson, Rabiner and Sondhi, 1983 [21]; Juang and Rabiner, 1991 [19]). The theory of hidden Markov models has also found a major application in teletraffic science : Markov modulated processes have been proposed for modelling traffic on telecommunication networks (Meier Hellstern, 1993 [15]) and standard estimation techniques have been developed for these traffic models (Ryden, 1996 [30]). Another major field of their application is genetics and biochemistry. Hidden Markov models have proved useful for the analysis of DNA sequences with a special interest in the detection of homogeneous regions in DNA sequences (Churchill 1989 [8], 1992 [9] and 1995 [10]; Muri, 1997 [26]; Henderson, Salzberg and Fasman 1997 [18]). They have also been proposed in neurology for the analysis of the conductivity of an ion channel with inference on the state of the macromolecula (Ball and Rice, 1992 [14]; Fredkin and Rice, 1992 [16, 17]).

1.2 The Expectation Maximization algorithm

A problem of major interest when one deals with hidden Markov models is the estimation of the parameters of the model. The Expectation Maximization algorithm (EM) (Dempster, Laird and Rubin, 1977 [12]) is a popular and valuable approach for maximum likelihood estimation in hidden Markov models. Basically, the EM algorithm consists in iteratively maximizing a Q function defined as the expectation of the log-likelihood of the complete model given the observations and the current estimate of the parameters. Denote by $Y_{1:T}$ the observed sequence and by $X_{1:T}$ the underlying and unobserved Markov chain. θ denotes the set of parameters of the model. Each iteration of the EM algorithm can be decomposed into two steps :

Step E Compute $Q(\theta, \theta_k) = \mathbb{E}(L(X, Y; \theta) \mid Y = y, \theta_k)$.

Step M Maximize $Q(\theta, \theta_k)$ with respect to $\theta : \theta_{k+1} = \text{Arg max}_{\theta} Q(\theta, \theta_k)$.

where $L(X, Y; \theta)$ is the log-likelihood of the complete data $Z = (X, Y)$.

The EM algorithm has a number of advantages. The log-likelihood of the complete data $Z = (X, Y)$ has a very simple expression whereas the number of operations involved for the direct computation of the observed likelihood increases exponentially with the length of the sequence and is therefore not feasible in practice for reasonably long sequences. The EM algorithm has the basic property that it increases the likelihood of the observations at each iteration and that it converges to the set of stationary points of the likelihood of the observations (Wu, 1983 [39]; Redner and Walker, 1984 [28]). This basic property of the EM is not shared by standard optimization techniques such as the gradient or the Newton Raphson procedure.

But the EM algorithm suffers from a number of serious limitations. One of the best documented limitation of the EM algorithm is the possible convergence of the algorithm on some stationary but unstable points of the likelihood of the observations or some local but

not global extremum of the observed likelihood (Wu, 1983 [39]; Redner and Walker, 1984 [28]). Convergence thus depends crucially on the initialization point. Another problem is its speed : the convergence rate is governed by the fraction of missing information so that convergence can be slow (Dempster, Laird and Rubin, 1977 [12]). When the complete model is a regular exponential model with a complete data sufficient statistic, the step E is simply the computation of the expectation of the complete data sufficient statistics given the observations. However when the model is non exponential, the integral involved in the step E of the EM algorithm is not analytically tractable (eg. non censored mixtures of Weibullian distributions, Chauveau 1992 [6] and 1995 [7]). In some situations the maximization involved in the step M of the EM algorithm is not analytically tractable but requires iterative maximizations (log-linear models, Little and Rubin, 1987 [22]; mixtures of Weibullian distributions, Chauveau 1992 [6] and 1995 [7]).

1.3 Stochastic and non stochastic improvements of the EM algorithm

In order to overcome the above mentioned limitations of the EM algorithm various stochastic and non stochastic improvements of the EM algorithm have been proposed. The Generalized Expectation Maximization (GEM) algorithm (Dempster, Laird and Rubin, 1977 [12]) avoids iterative maximization by increasing the Q function rather than maximizing it at each step M. Like the EM algorithm this algorithm has the property of increasing the likelihood of the observations at each iteration but it does not share the convergence properties of the EM algorithm under general conditions. The Expectation/Conditional Maximization (ECM) algorithm (Meng and Rubin, 1993 [25]) is a particular case of a GEM algorithm that shares the attractive convergence properties of the EM algorithm.

Basically, stochastic versions of the EM algorithm (SEM [2], SAEM [5], MCEM [36]) are random perturbations of the EM algorithm. The random perturbations involved in the first iterations are likely to prevent these algorithms from remaining near saddle points or local maxima of the likelihood of the observations. The Stochastic Expectation

Maximization algorithm (SEM) (Celeux and Diebolt, 1985 [2]) replaces the calculation of the integral involved in the step E of the EM algorithm by random simulation of the missing data given the observations and the current estimate of the parameters. Under suitable conditions the sequence of estimates is a stationary and ergodic Markov chain and converges in distribution; the stationary distribution of this Markov chain is likely to favor the values of the parameters for which the proportion of missing data is low but it does not necessarily concentrate around a global extremum of the likelihood of the observations.

The Stochastic Annealing Expectation Maximization algorithm (SAEM) (Celeux and Diebolt, 1992 [5]) is a modification of the SEM algorithm. The SAEM converges almost surely instead of converging in distribution thus avoiding the possibly erratic behavior of the SEM algorithm in the case of small samples. Unlike the EM algorithm the SAEM avoids the stationary but unstable points of the likelihood of the observations; what is more the random perturbations involved in the first iterations of the algorithm prevents it from converging to the first local extremum encountered.

In the Gibbsian EM algorithm (Robert, Celeux and Diebolt, 1993 [29]) the simulation of the missing data involved in the SE step of the SEM algorithm is replaced by one step of a Gibbs sampler : non linear filtering is thus avoided and long sequences can be handled with relatively low computational burden. The Data Augmentation algorithm (Tanner and Wong, 1987 [32]) is nothing but a Bayesian version of the SEM algorithm : the parameters of the model are treated as random variables and simulated randomly given the observations. The Monte Carlo Expectation Maximization (MCEM) algorithm (Wei and Tanner, 1991 [36]) replaces analytic computation of the integral involved in the E step of the EM algorithm by Monte Carlo simulation.

1.4 Paper contents

In this paper we wish to draw the reader's attention to another, not documented limitation of the EM algorithm.

This paper concerns the estimation of some support parameters in the context of incomplete data. More specifically we explain how one can estimate the offsets θ_i when one observes an independent sequence distributed as a π weighted mixture of $\mathcal{E}(\lambda_i)$ exponential distributions with offsets θ_i : $g(x) = \sum \pi_i \lambda_i \exp(-\lambda_i(x - \theta_i)) \mathbb{1}_{\mathbb{R}^+}(x - \theta_i)$. The technique that we propose generalizes to dependent observations when one considers Hidden Markov Models. The EM algorithm and its stochastic counterparts are not suited to estimating some support parameters. The specificity of this problem is discussed in Section 2.

Section 3 is an overview of the estimators that are valuable for estimating a single offset in a distribution. These techniques do not generalize to a mixture of distributions with offsets.

To overcome these difficulties we propose in Section 4 to minimize different weighted distances between the Fourier transform of the observed sequence and its empirical counterpart. Unlike the problems encountered in the time domain the minimization of the distance between Fourier transforms can be performed with an EM algorithm.

2 The SEHMM model and the offset parameters estimation problem

Our interest in estimating the parameters of a partially observed model with unknown support has been prompted by the recent development of a new model for broadband traffic, the Shifted Exponential Hidden Markov Model (SEHMM) (Vaton, 1997 [35] [31])

and 1999 [34]). The necessity of new traffic models proved to be imperative after the work of Leland, Willinger, Taqqu et al. (Leland, Taqqu, Willinger and Wilson, 1994 [20] and 1995 [37]; Willinger, Taqqu, Sherman and Wilson, 1995 [38]) and of Paxson and Floyd (Paxson and Floyd, 1994 [27]). These authors showed a number of serious drawbacks of classical Poisson traffic models such as the Markov Modulated Poisson Process (MMPP) (Meier Hellstern, 1984 [24] and 1993 [15]); the striking conclusion of their studies is that classical traffic models greatly underestimate the resources necessary to guarantee Quality of Service on future broadband networks.

The SEHMM is an improvement of the MMPP that includes offsets $(\theta_i)_{1 \leq i \leq K}$ in the distribution of the interevent times given the unobserved regime. This makes it possible to model distributions with exponential decrease but with a mean strictly superior to standard deviation. The offsets θ_i model some incompressible delays between arrivals due to the reemission of packets when no acknowledgement has been received by the emitter after some predetermined period of time.

Knowing the parameters of the model of traffic is crucial for dimensioning. This problem is peculiar in the case of the SEHMM because the support of the distribution of the observations given the unobserved data has to be estimated. For simplicity we suppose in what follows that all but the support parameters are known; our algorithm generalizes easily to the joint estimation of all the parameters of the model. In the simulations we will consider mixtures of shifted exponential distributions, a particular case of SEHMM where the underlying Markov chain reduces to an independent and identically distributed sequence.

Let $(X_t)_t$ denote the observed process and $(Y_t)_t$ represent the underlying Markov chain. Thus, the model is described by the following equations

$$\begin{aligned}
 \mathbb{P}(Y_{t+1} = j \mid y_t, y_{t-1}, y_{t-2}, \dots) &= \mathbb{P}(Y_{t+1} = j \mid y_t) \\
 f(x_1, x_2, \dots, x_T \mid y_1, y_2, \dots, y_T) &= \prod_{t=1}^T f(x_t \mid y_t) \\
 f(x_t \mid y_t = i) &= \lambda_i \exp(-\lambda_i(x_t - \theta_i)) \mathbb{1}_{x_t \geq \theta_i}.
 \end{aligned} \tag{1}$$

2.1 Failure of the EM algorithm

Consider $(x_t)_{1 \leq t \leq T}$ T independent and identically distributed samples of a K component mixture of shifted exponentials $g(x) = \sum \pi_i \lambda_i \exp(-\lambda_i(x - \theta_i)) \mathbb{I}_{\mathbb{R}^+}(x - \theta_i)$. Any θ such that one of the θ_i is equal to one of the x_t is a point where $f(x_{1:T}; \theta)$ is discontinuous (see Fig. 1). The likelihood $f(x_{1:T}; \theta)$ thus does not satisfy to the assumptions of regularity under which the convergence properties of the EM algorithm have been established.

We are going to demonstrate that the sequence of estimates $(\theta_i^{(n)})_{n \in \mathbb{N}}$ produced by the EM algorithm suffers from $\theta_i^{(n+1)} \geq \theta_i^{(n)}$ so that a bad initialization cannot be recovered in the following iterations.

The EM algorithm goes iteratively through the two steps :

- Step E : Compute $Q(\theta, \theta^{(n)}) = \mathbb{E}(L(X, Y; \theta) \mid y, \theta^{(n)})$.
- Step M : Maximize $Q(\theta, \theta^{(n)})$ with respect to θ : $\theta^{(n+1)} = \text{Arg max}_{\theta} Q(\theta, \theta^{(n)})$.

In the case of the mixture of shifted exponential distributions the step M reduces to $\theta_i^{(n+1)} = \min\{x_t; p(y_t = i \mid x_t; \theta^{(n)}) \neq 0\}$. And $p(y_t = i \mid x_t; \theta_i^{(n)}) \neq 0$ if and only if $x_t \geq \theta_i^{(n)}$. It results that if $T(\bullet)$ denotes the operator associated with the EM algorithm and $\theta^{(n+1)} = T(\theta^{(n)})$ then $\theta_i^{(n+1)} \geq \theta_i^{(n)}$. In what follows we demonstrate that this inequality generalizes to the SEM, SAEM and MCEM algorithms.

2.2 Failure of the SEM, SAEM and MCEM algorithm

The SEM algorithm, the SAEM algorithm and the MCEM algorithm are random perturbations of the EM algorithm. They all rely on a stochastic imputation principle. Basically a random simulation of the unobserved regime given the observations and the current estimate of the parameters causes perturbation of the step E of the EM algorithm.

2.2.1 Failure of the SEM algorithm

The SEM algorithm was the first earliest stochastic version of the EM algorithm (Celeux and Diebolt, 1985 [2], 1986 [3] and 1987 [4]). The SEM algorithm goes iteratively through the following steps :

Step E: Compute the distribution of Y given the observations x and $\theta^{(n)}$.

Step S: Simulate $y^{(n+1)}$ given the observations x and $\theta^{(n)}$. If $\text{card}\{t; y_t^{(n+1)} = i\} < (d+1)/N$ then resume with one less component, else go to the step M.

Step M: Maximize $L(x, y^{(n+1)}; \theta)$ with respect to $\theta : \theta^{(n+1)} = \text{Arg max}_{\theta} L(x, y^{(n+1)}; \theta)$.

In the case of the mixture of shifted exponential distributions the step M reduces to : $\theta_i^{(n+1)} = \min\{x_t; y_t^{(n+1)} = i\}$ but the event $(y_t^{(n+1)} = i)$ has probability zero given the observations and $\theta^{(n)}$ if $(x_t < \theta_i^{(n)})$. This results in the same inequality as in the case of the EM algorithm : $\theta_i^{(n+1)} \geq \theta_i^{(n)}$.

The SEM algorithm produces a homogeneous Markov chain. But this Markov chain is not irreducible when the sample space over which the complete likelihood $f(z; \theta)$ is strictly positive depends on the parameter θ . The ergodicity and stationarity of the Markov chain produced by the SEM algorithm is consequently not verified in the case of the estimation of a mixture of shifted exponentials.

2.2.2 Failure of the SAEM algorithm

The SAEM algorithm is a simulated annealing version of the EM algorithm that goes from a pure SEM algorithm in the first iterations towards a pure EM algorithm at the end. It thus avoids the erratic behavior of the SEM algorithm without converging to the first local maximum of the likelihood of the observations it encounters as the EM algorithm is likely to do. In most examples encountered, the SAEM converges pointwise to the set

of stationary stable points of the likelihood of the observations. This property does not generalize to the estimation of the offsets in a mixture of shifted exponential distributions.

The SAEM algorithm is usually described under the assumption that the complete data model is regular exponential with a complete data sufficient statistic $t(z) : g(z; \theta) = a^{-1}(\theta)b(z) \exp(\langle t(z); \theta \rangle)$. The SAEM goes iteratively through the following steps :

Step E: Compute the distribution of Y given the observations x and $\theta^{(n)}$.

Step S: Simulate $z^{(n+1)} = (x, y^{(n+1)})$ given the observations x and $\theta^{(n)}$.

Step A: Stochastic averaging : $t^{(n+1)} = t^{(n)} + \gamma_{n+1}(t(z^{(n+1)}) - t^{(n)})$.

Step M: Maximize $-\log a(\theta) + \langle t^{(n+1)}; \theta \rangle$ with respect to θ :

$$\theta^{(n+1)} = \text{Arg max}_{\theta} -\log a(\theta) + \langle t^{(n+1)}; \theta \rangle.$$

These steps are equivalent to a random perturbation of the operator $T(\bullet)$ associated with the EM algorithm :

$$\theta^{(n+1)} = T(\theta^{(n)}) + \gamma_{n+1}(M(z^{(n+1)}) - T(\theta^{(n)})) \quad (2)$$

where $M(z)$ is the maximizer of the complete data likelihood $M(z) = \text{Arg max}_{\theta} g(z; \theta)$. Step γ_n is an equivalent of the temperature function in the classical simulated annealing algorithm. γ_1 should be equal to 1 and γ_n should decrease slowly in the first iterations in order to avoid the local extrema of the likelihood. Pointwise convergence of the SAEM algorithm is demonstrated under the condition that $\lim_{n \rightarrow \infty} \gamma_n = 0$ and that the series $\sum_n \gamma_n$ diverges.

If $\gamma_n = 1$ the SAEM algorithm reduces to a pure SEM algorithm; if $\gamma_n = 0$ the SAEM algorithm reduces to a pure EM algorithm. The SAEM algorithm thus realizes a balance between the SEM algorithm and the EM algorithm. It results from Eq. (2) that the SAEM algorithm suffers from the same limitation as the EM and SEM algorithms : $\theta_i^{(n+1)} \geq \theta_i^{(n)}$.

2.2.3 Failure of the MCEM algorithm

The MCEM algorithm replaces analytical computation of the integral involved in the step E of the EM algorithm by Monte-Carlo integration. The MCEM algorithm goes iteratively through the following steps :

Step E: Compute the distribution of Y given the observations x and $\theta^{(n)}$.

Step MC: Simulate m_T realizations $(z^{(n+1),j})_{1 \leq j \leq m_T} = (x, y^{(n+1),j})_{1 \leq j \leq m_T}$ of z given the observations x and $\theta^{(n)}$ and compute the average log-likelihood of the complete data

$$\bar{L}^{(n+1)}(\theta) = \frac{1}{m_T} \sum_j L(z^{(n+1),j}; \theta).$$

Step M: Maximize the average $\bar{L}^{(n+1)}(\theta)$ with respect to θ : $\theta^{(n+1)} = \text{Arg max}_{\theta} \bar{L}^{(n+1)}(\theta)$.

The step M reduces to $\theta_i^{(n+1)} = \min(x_t/y_t^{(n+1),j} = i)$. But $(y_t^{(n+1),j} = i)$ has probability zero if $\theta_i > x_t$ and consequently $\theta_i^{(n+1)} \geq \theta_i^{(n)}$.

2.3 Lack of Cramer-Rao Lower Bound

Maximum likelihood techniques are justified by the fact that the Maximum Likelihood Estimator is asymptotically unbiased and that its variance converges to the Cramer-Rao Lower Bound $CRLB = I^{-1}(\theta)$ where $I(\theta) = -\mathbb{E}_{\theta}(\frac{d^2 \log f(X; \theta)}{d\theta^2})$ is Fischer's information matrix of the model. For the problem under study the log-likelihood of the observations is not differentiable with respect to the offsets θ_i , and the conditions under which the Cramer-Rao Lower Bound is derived are not fulfilled. However, in the Fourier domain the normalized sampled Fourier transform converges to a gaussian distribution which belongs to the exponential family and the Maximum Likelihood Estimator is efficient in this case (Van der Waart, 1998 [33]).

3 Estimation of a single shift parameter

In this section we recall some techniques for estimating a single parameter of shift when the observations $(x_t)_{1 \leq t \leq T}$ are independent and identically distributed with probability density function $f_\theta(x) = f_0(x - \theta)$ and $f_0(x) = 0$ if $x < 0$.

3.1 Maximum Likelihood Estimation

The log-likelihood of $x_{1:T}$ is $L(x_{1:T}; \theta) = \sum_{t=1, T} \log f_0(x_t - \theta)$ so that $L(x_{1:T}; \theta) = -\infty$ if $x_t < \theta$ for some t . When $f_0(\bullet)$ is strictly decreasing over \mathbb{R}^+ then $\hat{\theta}_{ML} = X_{\min}$ where $X_{\min} \triangleq \min_{t=1, T} X_t$ is the minimum value of all the observations. This is the case in particular when $f_0(\bullet)$ is the exponential distribution : $f_0(x) = \lambda \exp(-\lambda x) \mathbb{I}_{\mathbb{R}^+}(x)$.

3.2 Bayesian estimation techniques

The choice of a prior distribution is usually the controversial point in Bayesian estimation. Denote by $g(\theta)$ the prior distribution of θ . Then $f(x_{1:T}) = (f_0 * g)(x_{1:T})$ where $*$ denotes convolution. $f(x_{1:T})$ is maximum for

$$g(\theta) = \frac{f_0(x_{1:T} - \theta)}{\int f_0(x_{1:T} - \theta) d\theta}. \quad (3)$$

Such a choice of $g(\theta)$ amounts to calculating $f(x_{1:T})$ by applying a matched filter to $f_0(x_{1:T})$. For this prior distribution the Maximum A Posteriori estimator is $\hat{\theta}_{MAP} = \text{Arg max}_\theta f(\theta | x_{1:T}) = X_{\min}$ when $f_0(\bullet)$ strictly decreases over \mathbb{R}^+ . The minimum mean square error estimate is $\hat{\theta}_{MMSE} = \mathbb{E}(\theta | X_{1:T}) = X_{\min} - (2\lambda T)^{-1}$ in the case of the shifted exponential distribution.

3.3 Pitman estimator

Consider the set of all equivariant estimators of θ that is to say of all the estimators S such that $S(x_{1:T} + \mu) = \mu + S(x_{1:T})$. It is relevant to look for an estimator that satisfies this equality when one is concerned with the estimation of a parameter of shift. The equivariant estimator with minimum quadratic risk function is the Pitman estimator [13]. In the case of the shifted exponential distribution $\hat{\theta}_{PIT} = X_{\min} - (\lambda T)^{-1}$.

3.4 Barankin estimator

The Barankin bound [23] is the greatest lower bound among minimum variance bounds for unbiased estimators. As a byproduct the calculation of this bound supplies a locally, and possibly globally, minimum variance unbiased estimator. In particular, for a shifted exponential distribution, we can get the minimum variance unbiased estimator $\hat{\theta}_B(X) = X_{\min} - (\lambda T)^{-1}$ [23].

Note that X_{\min} is an exhaustive statistic when θ is the only unknown parameter, so that all the estimators mentioned above can be expressed as a function of $X_{\min} = S(x_{1:T})$.

4 Estimation in the Fourier domain

There are numerous valuable approaches for estimating one single offset as listed in the previous section; but these techniques do not generalize to incomplete data models. On the other hand, the EM algorithm is a common technique for maximum likelihood estimation in incomplete data models. But neither this algorithm nor its random counterparts are valuable for estimating supports (see Section 2).

In order to overcome the above-mentioned limitations we propose to minimize a quadratic distance between the sampled Fourier transform and its parametric counterpart. Some

previous studies have proved that minimum distance estimation based on characteristic functions is valuable for estimating the mixing weights π_i of a distribution $g(x) = \sum \pi_i f_i(x)$ (see Bryant and Paulson 1979 [1] and the references therein). This technique is of particular interest for estimating supports in incomplete data models; unlike the mixing weights, the supports can not be estimated with standard approaches.

4.1 The estimation of the offsets in the Fourier domain

Note $f(x) = \sum_{k=1,K} \pi_k f_{\theta_k, \lambda_k}(x) = \sum_{k=1,K} \pi_k f_{0, \lambda_k}(x - \theta_k)$ a weighted mixture of probability density functions with shifts θ_k . $\Phi(\omega) = \int e^{i\omega x} f(x) dx$ is the characteristic function of $f(x)$. This transformation is a bijection. Knowing $\Phi(\omega)$ for any ω is equivalent to knowing $f(x)$ for any x .

The shift operator $T(x) = x + \theta$ is transposed into a modulation of the characteristic function by $e^{i\omega\theta} : \int e^{i\omega T(x)} f(x) dx = e^{i\omega\theta} \Phi(\omega)$. In the above mixture model it results that $\Phi(\omega) = \sum \pi_k e^{i\omega\theta_k} \Phi_{0, \lambda_k}(\omega)$ where $\Phi_{0, \lambda_k}(\bullet)$ is the characteristic function of the unshifted probability density function $f_{0, \lambda_k}(\bullet)$. Basically our procedure minimizes a quadratic distance between a sampled version of $\Phi(\omega)$ and a sampled version of its empirical counterpart $\hat{\Phi}(\omega) = \frac{1}{T} \sum_{t=1,T} e^{i\omega X_t}$.

4.2 Central Limit Theorems

Note $\tilde{m}(\theta) = (\Phi(\omega_1), \Phi(\omega_2), \dots, \Phi(\omega_L))^T$ the sampled characteristic function of $f(\bullet)$. Its empirical counterpart is the sampled Fourier transform of the data $\tilde{F}_T = T^{-1} \sum \tilde{A}_t$ with $\tilde{A}_t = (e^{i\omega_1 X_t}, \dots, e^{i\omega_L X_t})^T$.

In what follows we consider the real vectors whose components are the real part and the imaginary part of these vectors: $A_t = [Re(\tilde{A}_t)^T, Im(\tilde{A}_t)^T]^T$, $F_T = [Re(\tilde{F}_T)^T, Im(\tilde{F}_T)^T]^T$ and $m(\theta) = [Re(\tilde{m}(\theta))^T, Im(\tilde{m}(\theta))^T]^T$.

4.2.1 The Independent Identically Distributed Case

Assumption 1 $(X_t)_{t=1,T}$ are T independent and identically distributed random variables with probability distribution function $g(x) = \sum \pi_i \lambda_i \exp(-\lambda_i(x - \theta_i)) \mathbb{I}_{\mathbb{R}^+}(x - \theta_i)$.

Theorem 1 If Assumption 1 is fulfilled then $\sqrt{T}(F_T - m(\theta)) \sim \mathcal{AN}(0, \Gamma(\theta))$ where $\Gamma(\theta) = \mathbb{E}_\theta((A_t - m(\theta))(A_t - m(\theta))^T)$ is the variance covariance matrix of $(A_t)_t$.

4.2.2 The Hidden Markov Model Case

Assumption 2 $(Y_t)_t$ is a finite state irreducible and aperiodic hidden Markov chain; $(X_t)_t$ are independent given $(Y_t)_t$ and X_t depends on Y_t only with probability distribution function $p(y | X_t = i) = \lambda_i \exp(-\lambda_i(y - \theta_i)) \mathbb{I}_{\mathbb{R}^+}(y - \theta_i)$.

Theorem 2 If Assumption 2 is fulfilled then $\sqrt{T}(F_T - m(\theta)) \sim \mathcal{AN}(0, \Gamma(\theta))$ where $\Gamma(\theta) = \sum_{\tau \in \mathbb{Z}} \mathbb{E}_\theta((A_{t+\tau} - m)(A_t - m)^T)$.

We assume in what follows that Assumption 1 or Assumption 2 is fulfilled. Furthermore we assume that the sample is large so that the asymptotics is attained and that $\sqrt{T}(F_T - m(\theta))$ is approximately $\mathcal{N}(0, \Gamma(\theta))$.

4.3 Maximum Likelihood Estimation

A possible quadratic distance between the characteristic function and its empirical counterpart is the opposite log-likelihood of F_T when the asymptotics is attained. The maximization of the asymptotic log-likelihood of the empirical Fourier transform F_T with respect to θ is equivalent to the minimization of the distance

$$J(\theta) = \log |\Gamma_T(\theta)| + (F_T - m(\theta))^T \Gamma_T^{-1}(\theta) (F_T - m(\theta)) \quad (4)$$

where $\Gamma_T(\theta) = T^{-1}\Gamma(\theta)$.

4.4 Maximization of the Likelihood

Classical optimization techniques such as the gradient algorithm, the conjugate gradient algorithm or the Newton algorithm can be used to minimize $J(\theta)$. These techniques are iterative minimum search algorithms. One iteration of the gradient algorithm is

$$\theta_{n+1} = \theta_n - \mu_{n+1} \nabla J(\theta) |_{(\theta=\theta_n)}; \quad (5)$$

one iteration of the Newton algorithm is

$$\theta_{n+1} = \theta_n - H^{-1}(\theta) \nabla J(\theta) |_{(\theta=\theta_n)}. \quad (6)$$

where $\nabla J(\theta)$ is the gradient vector of the first order derivatives of $J(\theta)$ and $H(\theta)$ is the Hessian matrix of the second order derivatives of $J(\theta)$.

Note that the first order derivatives of $\log |\Gamma_T(\theta)|$ and of $\Gamma_T(\theta)^{-1}$ are :

$$\frac{\partial \log |\Gamma_T(\theta)|}{\partial \theta_k} = \text{Tr}(\Gamma_T(\theta)^{-1} \frac{\partial \Gamma_T(\theta)}{\partial \theta_k}) \quad (7)$$

$$\frac{\partial \Gamma_T^{-1}(\theta)}{\partial \theta_k} = -\Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_k} \Gamma_T^{-1}(\theta).$$

Then,

$$\begin{aligned} \frac{\partial J(\theta)}{\partial \theta_k} = & \text{Tr}(\Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_k}) - \frac{\partial m(\theta)^T}{\partial \theta_k} \Gamma_T^{-1}(\theta) [F_t - m(\theta)] - [F_t - m(\theta)]^T \Gamma_T^{-1}(\theta) \frac{\partial m(\theta)}{\partial \theta} \\ & - [F_t - m(\theta)]^T \Gamma_T(\theta)^{-1} \frac{\partial \Gamma_T(\theta)}{\partial \theta} \Gamma_T^{-1}(\theta) [F_t - m(\theta)]. \end{aligned} \quad (8)$$

The coefficient (i, j) of the Hessian matrix of $J(\theta)$ is

$$H(\theta)_{ij} = \frac{\partial^2 J(\theta)}{\partial \theta_i \partial \theta_j} = \text{Tr} \left(-\Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_i} \Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_j} + \Gamma_T^{-1}(\theta) \frac{\partial^2 \Gamma_T(\theta)}{\partial \theta_i \partial \theta_j} \right) \\ + S_{i,j} + S_{i,j}^* - (F_T - m(\theta))^T \Gamma_T^{-1}(\theta) \frac{\partial^2 \Gamma_T(\theta)}{\partial \theta_i \partial \theta_j} \Gamma_T^{-1}(\theta) (F_T - m(\theta)), \quad (9)$$

with

$$S_{i,j} = \frac{\partial m(\theta)^T}{\partial \theta_i} \Gamma_T^{-1}(\theta) \frac{\partial m(\theta)}{\partial \theta_j} + \frac{\partial m(\theta)^T}{\partial \theta_i} \Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_j} \Gamma_T^{-1}(\theta) [F_t - m(\theta)] \\ + \frac{\partial m(\theta)^T}{\partial \theta_j} \Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_i} \Gamma_T^{-1}(\theta) [F_t - m(\theta)] - \frac{\partial^2 m(\theta)^T}{\partial \theta_i \partial \theta_j} \Gamma_T^{-1}(\theta) [F_t - m(\theta)] \quad (10) \\ + [F_t - m]^T \Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_i} \Gamma_T^{-1}(\theta) \frac{\partial \Gamma_T(\theta)}{\partial \theta_j} \Gamma_T^{-1}(\theta) [F_t - m].$$

4.5 Approximate likelihood criterion

In practice $J(\theta)$ has a lot of local extrema so that classical optimization algorithms are likely to converge to the set of local but not global extrema of $J(\theta)$. Denote $\hat{\Gamma}_T$ an almost surely convergent estimate of $\Gamma(\theta)$. In practice $\hat{\Gamma}_T$ is the empirical variance covariance matrix of $(A_t)_t$ when $(A_t)_{1 \leq t \leq T}$ are independent random variables and $\hat{\Gamma}_T$ is a smoothed version of the periodogram of $(A_t)_{1 \leq t \leq T}$ at frequency zero when $(A_t)_{1 \leq t \leq T}$ follows a Hidden Markov Model. The minimum of

$$K(\theta) = [F_T - m(\theta)]^T \hat{\Gamma}_T^{-1} [F_T - m(\theta)], \quad (11)$$

shares the same asymptotic properties as the minimum of $J(\theta)$ but the criterion $K(\theta)$ has less local extrema than $J(\theta)$ so that global minimization is easier to perform.

The gradient and Hessian of $K(\theta)$ are simply given by the expressions

$$\frac{\partial K(\theta)}{\partial \theta_i} = -2 \frac{\partial m(\theta)^T}{\partial \theta_i} \hat{\Gamma}_T^{-1} [F_t - m] \\ \frac{\partial^2 K(\theta)}{\partial \theta_i \partial \theta_j} = 2 \left(-\frac{\partial m(\theta)^T}{\partial \theta_i} \hat{\Gamma}_T^{-1} \frac{\partial m(\theta)}{\partial \theta_j} + \frac{\partial^2 m(\theta)^T}{\partial \theta_i \partial \theta_j} \hat{\Gamma}_T^{-1} [F_t - m] \right). \quad (12)$$

4.6 Convergence properties

Let us denote $\|\cdot\|_R$ the norm defined by $\|u\|_R^2 = u^T R u$, where R is some symmetric positive definite matrix. Now let us consider the following result (see eg [11] pp.92-95), which is related to the convergence of estimators that contrast parameters, and which shows the possible interest of criteria $J(\theta)$ and $K(\theta)$:

Theorem 3 *Let $\alpha \rightarrow m(\alpha)$ be a continuous function defined on a compact set Θ , and assume that*

$$\forall \alpha, \alpha' \in \Theta, \quad m(\alpha) = m(\alpha') \Rightarrow \alpha = \alpha'. \quad (13)$$

Then, if the statistic $U_T(\alpha)$ defined by $U_T(\alpha) = \|F_T - m(\alpha)\|_R$ satisfies

$$\forall \alpha, \theta \in \Theta, \quad \forall \varepsilon > 0, \quad \lim_{T \rightarrow \infty} P_\theta(|U_T(\alpha) - \|m(\theta) - m(\alpha)\|_R| > \varepsilon) = 0, \quad (14)$$

where P_θ represents the distribution of F_T for the value θ of the parameter, then $\hat{\alpha}_T = \arg \min_\alpha U_T(\alpha)$ is a consistent estimator of θ .

Furthermore, if $R = \Gamma(\alpha)^{-1} = T \mathbb{E}_\alpha[(F_T - m(\alpha))(F_T - m(\alpha))^T]$, or if $U_T(\alpha) = \|F_T - m(\alpha)\|_{\hat{\Gamma}_T^{-1}}$, where $(\hat{\Gamma}_T)_{T \in \mathbb{N}^}$ has asymptotic minimum variance among the estimators that minimize criteria in the form $U_T(\alpha) = \|F_T - m(\alpha)\|_R$.*

Now, we note that the convergence in distribution of $U_T(\alpha)$ toward $\|m(\theta) - m(\alpha)\|_R$ is satisfied when the conditions of theorem 1 hold. On the other hand, it is generally possible to determine a compact set Θ in which the parameters are guaranteed to lie.

This shows that minimizing the criterion $J(\theta)$ or $K(\theta)$ yields a consistent estimator of θ with asymptotic minimum variance provided the identifiability condition $m(\alpha) = m(\alpha') \Rightarrow \alpha = \alpha'$ is satisfied. But, checking whether this identifiability condition is fulfilled may not be an easy task. Let us briefly discuss this problem : $m(\theta)$ can be rewritten in the form

$$m(\theta) = [u(\lambda_1, \theta_1), \dots, u(\lambda_K, \theta_K)] \times [\pi_1, \dots, \pi_K]^T, \quad (15)$$

where $u(\lambda_k, \theta_k) = [\phi_{0,\lambda_k}(\omega_1)e^{i\theta_k\omega_1}, \dots, \phi_{0,\lambda_k}(\omega_L)e^{i\theta_k\omega_L}]^T$. Then, it is easy to check that a condition for ensuring the identifiability of θ is given by the following result

Proposition 1 *A sufficient condition for ensuring the identifiability of $\theta = (\theta_k)_{k=1,K}$ from the knowledge of $m(\theta)$ is given by*

- $\forall (\theta_k)_{k=1,K+1} \in \theta^{K+1}$, $\{u(\lambda_1, \theta_1), \dots, u(\lambda_K, \theta_K), u(\lambda_{K+1}, \theta_{K+1})\}$ defines a linearly independent family of vectors whence $(\lambda_i, \theta_i) \neq (\lambda_j, \theta_j)$ for $i \neq j$
- $|\max_{k=1,K} \theta_k \times \min_{l=1,L} \omega_l| < \pi$ (or $< 2\pi$ if the θ_k and the ω_l are assumed to be positive).

In other words the first condition means that

$$u(a, b) \in \text{Span}\{u(\lambda_1, \theta_1), \dots, u(\lambda_K, \theta_K)\} \Rightarrow (a, b) \in \{(\lambda_1, \theta_1), \dots, (\lambda_K, \theta_K)\}, \quad (16)$$

i.e. that $m(\theta)$ is uniquely defined as a linear combination of p vectors in the form $u(a, b)$. The second condition ensures that θ_k can be recovered without ambiguity from the knowledge of $u(\lambda_k, \theta_k)$.

Now, let us consider the particular case where $\lambda_1 = \dots = \lambda_K$. In this case, it is clear that

$$m(\theta) = \text{diag}(\phi_{0,\lambda_1}(\omega_1), \dots, \phi_{0,\lambda_1}(\omega_L)) \times [d(\theta_1), \dots, d(\theta_L)] \times [\pi_1, \dots, \pi_K]^T, \quad (17)$$

where $d(\theta_k) = [e^{i\theta_k\omega_1}, \dots, e^{i\theta_k\omega_L}]^T$. Then, if the pulsations $(\omega_l)_{l=1,L}$ are regularly spaced, $[d(\theta_1), \dots, d(\theta_L)]$ is a VanderMonde matrix, and the identifiability condition is fulfilled if $L > K$ since its rank is L whence the $(\theta_k)_{k=1,K}$ are distinct. This identifiability condition is often considered in sinusoid identification problems.

Let us note that the sufficient condition in Proposition 1 may not be necessary when the parameters $(\lambda_k)_{k=1,P}$ and $(\pi_k)_{k=1,K}$ are known. When the $(\pi_k)_{k=1,P}$ are known, this can be due to the fact that $m(\theta)$ cannot be any point of $\mathfrak{U}_{(\theta_k)_{k=1,K}} \text{Span}\{u(\lambda_1, \theta_1), \dots, u(\lambda_K, \theta_K)\}$. In particular, we will show via examples that for a mixture of $P = 4$ exponential distributions, the offsets can be recovered when the $(\pi_k, \lambda_k)_{k=1,P}$ are known, even for $L = 1$.

In fact, this is true even if the $(\pi_k)_{k=1,K}$ are unknown since the $(\pi_k)_{k=1,K}$ are assumed to be positive and satisfy $\sum_{k=1,P} \pi_k = 1$.

Also, let us note that when the $(\lambda_k)_{k=1,P}$ are known, the $(\theta_k)_{k=1,P}$ can be recovered from $u(\theta_k, \lambda_k)$ when choosing $\min_{l=1,L} \omega_l \leq 2\pi/X_{\max}$, where $X_{\max} = \max_{1 \leq t \leq T} X_t$. In this case, there exists a one to one mapping between $(X_t)_{1 \leq t \leq T}$ and $(A_t)_{1 \leq t \leq T}$.

4.6.1 Problems related to sampling the Fourier transform

In this section we discuss the choice of the points $(\omega_k)_{k=1,K}$ at which the Fourier transform is sampled. For the sake of simplicity we suppose from now on that the sampling is regular : $\omega_k = k\omega_1$ and that the offsets θ_i are restricted to be positive.

Contraction of the log-likelihood map

Note first that the transform $(\omega_1, \dots, \omega_L) \rightarrow \alpha(\omega_1, \dots, \omega_L)$ results in a contraction of $\theta \rightarrow K(\theta)$ when $\alpha > 1$ and in a dilation of $\theta \rightarrow K(\theta)$ when $\alpha < 1$ (see Figure 2 and Figure 3).

It results from this contraction effect that the restriction of $\theta \rightarrow K(\theta)$ to a compact set Θ has more local extrema when ω_1 is large. If the identifiability condition $\omega_1\theta < 2\pi$ is not fulfilled for any θ in the compact set Θ then $K(\theta)$ is periodic over Θ (see Figure 3) and the optimization algorithms will converge to the first extremum encountered. The result of the optimization is consequently very sensitive to initialization.

For example take $T = 1000, L = 1, K = 2, \pi_1 = \pi_2 = 0.5, \lambda_1 = \lambda_2 = 2, \theta_1 = 1, \theta_2 = 2$. The initialization of θ is random with uniform distribution. It is clear that any θ_i is inferior to X_{\max} so that if $\omega_1 < \frac{2\pi}{X_{\max}}$ and if one restricts the search within the range $0 < \theta_i < X_{\max}$ for all the θ_i s the identifiability of θ given $m(\theta)$ is ensured. If not, $K(\theta)$ is periodic and the result of the optimization is very sensitive to initialization. For $\omega_1 = \frac{2\pi}{X_{\max}} \sigma_\theta^2 \simeq 10^{-3}$ and for $\omega_1 = 10 \frac{2\pi}{X_{\max}} \sigma_\theta^2 \simeq 1$ where σ_θ^2 denotes the sum of the variances of the estimate of

the offsets.

ω_1 should consequently be chosen small enough for the condition of identifiability to be satisfied. But a contradictory effect is that the covariance matrix Γ has a high condition number² when ω_1 is very small; this makes the optimization very difficult.

Condition number of the covariance matrix

Note that if $(X_t)_{1 \leq t \leq T}$ are independent and identically distributed random variables the variance covariance matrix Γ has entry (i, j) $\Gamma(i, j) = \mathbb{E}(\exp(i(\omega_i - \omega_j)X_t)) = \Phi(\omega_i - \omega_j)$. Since $\omega_k = k\omega_1$ Γ tends to the rank one matrix ee^T where $e = [1, 1, \dots, 1]^T$ when ω_1 tends to zero. The condition number of Γ is thus high for small ω_1 . Table 1 gives the condition number of Γ for different values of $\omega_1 = \alpha 2\pi / X_{\max}$ and a growing number L of sampling points.

As one can see from Table 1 the condition number of Γ grows very quickly with the number L of frequencies at which the Fourier transform is sampled.

4.7 Practical estimation of the offset parameters

In order to overcome the problems relative to the condition number of the variance covariance matrix Γ we propose to undertake a least mean square estimation, that is to say to minimize

$$\|F_T - m(\theta)\|_{R^{-1}}^2 = [F_T - m(\theta)]^T R^{-1} [F_T - m(\theta)], \quad (18)$$

where R is some positive Hermitian matrix. If $R = \hat{\Gamma}_T$, then $M(\theta) = K(\theta)$ and the resulting estimate is of minimum variance among all the estimates of that type. But the optimization is likely to be difficult to undertake in practice. On the other hand, if R

²The condition number of a hermitian matrix is the ratio of its largest eigenvalue over its smallest eigenvalue.

is the identity matrix the optimization procedure is easy because R is well conditioned but the resulting estimate has higher variance as in the previous case. An intermediate approach would be to minimize $\|F_T - m(\theta)\|_{R^{-1}}^2$ with $R = (\hat{\Gamma} + \varepsilon I_L)^{-1}$.

4.7.1 Expectation Maximization (E.M.) method

Note that the minimization of $\|F_T - m(\theta)\|_{R^{-1}}^2$ is the maximum likelihood estimation of θ given the observed F_T following :

$$\begin{cases} F & = W_1 + W_2 + \dots + W_K \\ (W_k)_{k=1,K} & \text{independent} \\ W_k & \sim \mathcal{N}(m_k(\theta_k), \Gamma_k), \end{cases}$$

with $m_k(\theta_k) = [Re(\tilde{m}_k(\theta_k))^T, Im(\tilde{m}_k(\theta_k))^T]^T$,

$\tilde{m}_k(\theta_k) = \pi_k [e^{i\omega_1\theta_k} \Phi_{0,\lambda_k}(\omega_1), \dots, e^{i\omega_L\theta_k} \Phi_{0,\lambda_k}(\omega_L)]^T$ and $\sum_{k=1,K} \Gamma_k = R$.

Note that Γ_k are subject to $\sum \Gamma_k = R$ only. A convenient way of choosing Γ_k is the following. W_k is the contribution of component k to F_T that is to say about $T\pi_k$ random variables when T is sufficiently large. As the variance of a sequence decreases in proportion to the sample size, Γ_k should be taken proportional to $\pi_k^{-1}R$ to satisfy considerations concerning the generating process of the data; with the supplementary constraint that $\sum \Gamma_k = R$ it results that $\Gamma_k = \frac{\pi_k^{-1}}{\sum \pi_j^{-1}} R$.

This maximum likelihood problem can be solved with an EM algorithm. Each iteration of the EM algorithm is composed of the following two steps :

Step E Compute $Q(\theta, \theta^n) = \mathbb{E}(L(W, F; \theta) | F = f, \theta^n)$:

$$Q(\theta, \theta^n) = C - \frac{1}{2} \sum_k \|m_k(\theta_k) - [m_k(\theta_k^n) + \Gamma_k R^{-1}(f - \sum_{j=1}^K m_j(\theta_j^n))]\|_{\Gamma_k^{-1}}^2. \quad (19)$$

Step M Maximize $Q(\theta, \theta^n)$ with respect to θ :

$$\theta_k^{n+1} = \text{Arg min}_{\theta_k} \|m_k(\theta_k) - [m_k(\theta_k^n) + \Gamma_k R^{-1}(f - \sum_{j=1}^K m_j(\theta_j^n))]\|_{\Gamma_k^{-1}}^2. \quad (20)$$

The derivation of the step E is given in the Appendix. Note that the EM algorithm splits the maximization of $\|\hat{F}_T - m(\theta)\|_{R^{-1}}^2$ into easier one dimensional maximizations with respect to θ_k only. The optimization involved in the step M of the EM algorithm can be performed with a gradient algorithm or a Newton descent algorithm.

4.7.2 Simplified maximization step

In this section we consider the case when R is the identity matrix $R = I_{2L}$. This means that the criterion to minimize is $\|F_T - m(\theta)\|^2$. Then under the Assumption that θ^n is reasonably close to its true value θ^* the minimization (20) has a closed form solution.

Note $\alpha_k^n = m_k(\theta_k) + \Gamma_k \Gamma^{-1}(f - \sum_{j=1}^K m_j(\theta_j^n))$. $\tilde{\alpha}_k^n = [\tilde{\alpha}_{k1}^n, \dots, \tilde{\alpha}_{kL}^n]$ is the complex vector with a real and imaginary part such that $\alpha_k^n = [Re(\tilde{\alpha}_k^n)^T, Im(\tilde{\alpha}_k^n)^T]^T$. If θ and θ^n are close to the true value of the set of parameters θ^* then $\tilde{\alpha}_{kl}^n$ is broadly equal to $\tilde{m}_{kl}(\theta_k) = \pi_k \Phi_{0,\lambda_k}(\omega_l) e^{i\omega_l \theta_k}$ up to a correcting factor $\psi_{kl}^n - \omega_l \theta_k$ on the phase of $\tilde{\alpha}_{kl}^n$:

$$\tilde{m}_{kl}(\theta_k) = \pi_k \Phi_{0,\lambda_k}(\omega_l) e^{i\omega_l \theta_k} \quad (21)$$

$$\tilde{\alpha}_{kl}^n = \pi_k \Phi_{0,\lambda_k}(\omega_l) e^{i\psi_{kl}^n}$$

The step M of the EM algorithm in the previous section (Equation 20) reduces to minimizing

$$\|\tilde{m}_k(\theta_k) - \tilde{\alpha}_k^n\|_I^2 = \|\pi_k \Phi_{0,\lambda_k}(\omega_l)(e^{i\omega_l\theta_k} - e^{i\psi_{kl}^n})\|_I^2 = \pi_k^2 \sum_l |\Phi_{0,\lambda_k}(\omega_l)|^2 |e^{i(\psi_{kl}^n - \omega_l\theta_k)} - 1|^2 \quad (22)$$

It results from the first order expansion $e^{i\phi} = 1 + i\phi + O(\phi^2)$ that

$$\|\tilde{m}_k(\theta_k) - \tilde{\alpha}_k^n\|_I^2 = \pi_k^2 \sum_l |\Phi_{0,\lambda_k}(\omega_l)|^2 (\psi_{kl}^n - \omega_l\theta_k)^2 \quad (23)$$

The minimum is obtained for

$$\theta_k^{n+1} = \frac{\sum_{l=1,L} |\Phi_{0,\lambda_k}(\omega_l)|^2 \omega_l \psi_{kl}^n}{\sum_{l=1,L} |\Phi_{0,\lambda_k}(\omega_l)|^2 \omega_l^2}. \quad (24)$$

4.8 Example

4.8.1 Example 1

We consider a mixture of $K=2$ exponential distributions with offsets; the true parameters of the mixture are following : $\pi_1 = \pi_2 = 0.5, \lambda_1 = \lambda_2 = 2, \theta_1 = 1, \theta_2 = 2$.

The sample size is $T = 1000$. The choice of R that defines the quadratic distance to minimize is $R = \hat{\Gamma} + d_{\max}I$ where d_{\max} is the largest eigenvalue of $\hat{\Gamma}$. This criterion is minimized with the help of an EM algorithm. In practice, the M step of the EM algorithm is performed with ten iterations of a Newton Raphson descent algorithm $x_{n+1} = x_n - \mu(\nabla^2 f(x_n))^{-1}(x_n)\nabla f(x_n)$ with factor $\mu = 0.1$.

Influence of α

We first test the influence of α on the bias and on the variance of the estimates. In this first series of tests the Fourier transform is considered at only $L=1$ pulsation $\omega = \alpha \times \frac{2\pi}{X_{\max} + \epsilon}$ where α ranges from 10^{-8} to 1. α is always taken inferior to 1 in order to avoid the problems of identifiability discussed in Section 4.6.1.

As we are mainly interested in the bias and variance of the estimates the EM algorithm is initialized with the true values of the parameters. We evaluate the absolute bias and variance of the estimates by 1000 Monte Carlo simulations. When α increases from 10^{-8} to 1, the absolute bias of the estimates increases from roughly 10^{-4} to roughly 10^{-3} and the variance of the estimates remains stable around 10^{-3} .

This proves that α has not much influence on the bias and on the variance of the estimates. But a good choice of α is nevertheless essential for avoiding some problems of identifiability as mentioned in Section 4.6.1 (see Figure 2 and Figure 3). A good choice of α is also important when $R = \hat{\Gamma}$ to ensure that the condition number of Γ remains reasonable and that the optimization is thus feasible (see Table 1).

Influence of L

We now test how the number L of pulsations at which the Fourier transform is sampled influences the bias and variance of the estimates. The pulsations at which the Fourier transform is sampled are : $\omega_k = k \frac{2\pi}{X_{\max}}, k = 1, \dots, L$. The results of this series of tests are summed up in Table 2 :

As one can see from Table 2 the variance of the estimates decreases when L increases. But for the choice $R = \hat{\Gamma}$, L cannot be chosen arbitrarily large since $\hat{\Gamma}$ is not full rank when $L > T$.

4.8.2 Example 2

We consider a $K=4$ component mixture of exponential distributions with offsets; the parameters of the mixture are :

$$K = 4, \pi_k = 0.25 \ (k = 1, 4), \theta_1 = 1, \theta_2 = 2, \theta_3 = 3, \theta_4 = 5, \lambda_1 = 0.5, \lambda_2 = 0.6, \lambda_3 = 0.8, \lambda_4 = 1.0$$

The sample size is $T = 1000$. A histogram of the sample obtained in these conditions is displayed on Figure 4.

On this particular example we compare the results obtained when the optimization is performed with the EM, the SEM and the SAEM algorithm. For these three algorithms the quadratic distance minimized corresponds to the choice $R = \hat{\Gamma} + 10^{-5} d_{\max} I$ where d_{\max} is the largest eigenvalue of $\hat{\Gamma}$. We also compare this choice to the choice $R = I$ in the case of the EM algorithm.

In any version of this series of tests the Fourier transform of the mixture is considered at $L = 10$ pulsations : $\omega_k = 0.01 \times k \times \frac{2\pi}{X_{\max} + \epsilon}$; the offsets are initialized with respective values 0, 5, 10, and 15. 2000 iterations are performed in all the tests. The SAEM algorithm is started by a burning period of 500 iterations of the SEM algorithm; then a forgetting factor $\gamma_k = 1/\sqrt{k - 500}$ is applied in iterations $k = 501, \dots, 2000$.

Figure 5 represents 500 iterations of the EM algorithm for the choice $R = \hat{\Gamma} + 10^{-5} d_{\max} I$. After 2000 iterations of the EM algorithm $\hat{\theta}_1 = 1.05, \hat{\theta}_2 = 1.94, \hat{\theta}_3 = 3.22, \hat{\theta}_4 = 4.99$.

Table 3 is a summary of the mean and the standard deviation of the estimates for each of the four algorithms considered. We also found it useful to calculate the average absolute value of the bias $\frac{1}{K} \sum_{i=1, K} \left| \mathbb{E}(\hat{\theta}_i) - \theta_i \right|$ and the average value of the standard deviation $\frac{1}{K} \sum_{i=1, K} std(\hat{\theta}_i)$.

As one can see on Table 3 Columns 1 and 2, the standard deviation of the estimators is

lower for the choice $R = I$ than for the choice $R = \hat{\Gamma} + 10^{-5}d_{\max}I$. But in return the bias of the estimators is higher when $R = I$. It has been demonstrated in Section 4.6 that the estimator that minimizes $\|Z_T - m(\theta)\|_{\hat{\Gamma}^{-1}}$ has asymptotically minimum variance among the set of estimators that minimize criteria in the form $\|Z_T - m(\theta)\|_{R^{-1}}$. This is in apparent contradiction with the simulations. A possible interpretation is that the criterion $\|Z_T - m(\theta)\|_{\hat{\Gamma}^{-1}}$ is rougher than $\|Z_T - m(\theta)\|_I$; the criterion is therefore more difficult to minimize. This difficulty possibly results in many local extrema that are likely to attract the EM algorithm and/or in slow convergence.

As one can expect, the standard deviation of the estimates is slightly higher when the optimization is performed with the SEM or the SAEM algorithm than with the EM; the estimates obtained with the SAEM algorithm have themselves lower variance than those obtained with the SEM algorithm.

5 Conclusion

In this paper we have brought to the fore a limitation of the EM algorithm and of its stochastic versions. These algorithms are generally valuable for maximum likelihood estimation in incomplete data models. But they do not enable to estimate support parameters. This limitation is generally not well-known.

We consider the particular case of a mixture of exponential distributions with offsets. In order to overcome the above-mentioned limitation we undertake the estimation of the offset parameters in the Fourier domain. The offsets in the time domain are transposed into modulations in the Fourier domain. The pulsations of the modulations have to be estimated.

We minimize a function that contrast a sampled version of the characteristic function of the mixture and its empirical counterpart. We prove that the resulting estimate converges in probability under some conditions. These conditions ensure the identifiability of the

sampled characteristic function. We state a very simple sufficient condition on the points at which the Fourier transform is sampled for ensuring the identifiability. We specify the choice of contrast function for which the estimate is of minimum variance.

The optimization of the contrast function can be performed with an EM algorithm or a stochastic version of this algorithm. For a particular choice of the contrast function the maximization step of the EM algorithm has a closed form solution. This result enables estimation with very low computational burden. Some results of simulation are presented.

APPENDIX

In this Appendix we detail the integration involved in the step E of the EM algorithm of Section 4.7.1.

The log-likelihood of the complete data (W, F) is

$$L(W, F; \theta) = C - \frac{1}{2} \sum_{k=1, K} (W_k - m_k(\theta_k)) \Gamma_k^{-1} (W_k - m_k(\theta_k))^T, \quad (25)$$

provided that $F = \sum_{k=1, K} W_k$ and $-\infty$ otherwise.

Let X and Y be two gaussian random vectors with mean m_X and m_Y and with covariance and intercovariance matrix Γ_X , Γ_Y and Γ_{XY} . It is easy to check that X is gaussian given Y with mean $\mathbb{E}(X | Y) = m_X + \Gamma_{XY} \Gamma_Y^{-1} (Y - m_Y)$ and covariance $\Gamma_{X|Y} = \Gamma_X - \Gamma_{XY} \Gamma_Y^{-1} \Gamma_{XY}^T$.

Note W the vector $W = [W_1^T, \dots, W_K^T]^T$. Given $F = f$ and for the estimate θ^n of θ W is

$$W | F = f; \theta^n \sim \mathcal{N}(m(\theta^n) + \Gamma_{WF} \Gamma_F^{-1} (f - \sum_{k=1, K} m_k(\theta_k^n)), \Gamma_W - \Gamma_{WF} R^{-1} \Gamma_{WF}^T), \quad (26)$$

where $m(\theta^n) = [m(\theta_1^n), \dots, m(\theta_K^n)]$ and $\Gamma_{WF} = [\Gamma_1, \dots, \Gamma_K]^T$.

Integrating Equation 25 with respect to this distribution it holds that :

$$\begin{aligned}
 Q(\theta, \theta_n) &= \mathbb{E}[L(W, F; \theta); F = f, \theta = \theta^n] \\
 &= C' - \frac{1}{2} \sum_{k=1, K} \|m_k(\theta_k) - [m_k(\theta_k^n) + \Gamma_k R^{-1}(f - \sum_{j=1}^K m_j(\theta_j^n))]\|_{\Gamma_k^{-1}}^2.
 \end{aligned}
 \tag{27}$$

References

- [1] J.L. Bryant and A.S. Paulson. Some comments on characteristic-function based estimations. *Sankhya A*, 41:109–116, 1979.
- [2] G. Celeux and J. Diebolt. The SEM Algorithm: a Probabilistic Teacher Algorithm Derived from the EM Algorithm for the Mixture Problem. *Comp. Stat. Quat.*, 2:73–82, 1985.
- [3] G. Celeux and J. Diebolt. Comportement Asymptotique d'un Algorithme d'Apprentissage Probabiliste pour les Mélanges de Lois de Probabilité. Technical Report RR-563, INRIA, 1986.
- [4] G. Celeux and J. Diebolt. *Classification and Related Methods of Data Analysis*, chapter A Probabilistic Teacher Algorithm for Iterative Maximum Likelihood Estimation, pages 617–623. Bock H.H., 1987.
- [5] G. Celeux and J. Diebolt. A stochastic approximation of the EM algorithm for the mixture problem. *Stochastics and Stochastic Reports*, 41:119–134, 1992.
- [6] D. Chauveau. Algorithmes EM et SEM pour un mélange censuré de distributions de défaillances. *Rev. Statistique Appliquée*, 40(2):67–76, 1992.
- [7] D. Chauveau. A stochastic EM algorithm for mixtures with censored data. *J. of Stat. Planning and Inference*, 46:1–25, 1995.
- [8] G. Churchill. Stochastic Models for Heterogeneous DNA sequences. *Bulletin of Mathematical Biology*, 1(51):79–94, 1989.

- [9] G. Churchill. Hidden Markov chains and the analysis of the genome structure. *Computers chem.*, 16(2):107–115, 1992.
- [10] G. Churchill. *Case Studies in Bayesian Statistics*, chapter Accurate restoration of DNA sequences. Springer Verlag, 1995.
- [11] D. Dacunha-Castelle and M. Duflo. *Probabilités et statistiques 2. Problèmes à temps mobile*. Collection Mathématiques Appliquées pour la Maîtrise. Masson, 1983.
- [12] A.P. Dempster, N.M. Laird, and D.B. Rubin. Maximum Likelihood Estimation from incomplete data via the EM algorithm. *JRSS-B*, 39:1–38, 1977.
- [13] E.Pitman. The estimation of location and scale parameters of a continuous population of any given form. *Biometrika*, (30):391–421, 1939.
- [14] F.G.Ball and J.A. Rice. Stochastic models for ion channels : introduction and bibliography. *Math. Biosci.*, 112:189–206, 1992.
- [15] W. Fischer and K.S. Meier-Hellstern. The Markov Modulated Poisson Process (MMPP) Cookbook. *Performance Evaluation*, 18(2):149–171, 1993.
- [16] D.R. Fredkin and J.A. Rice. Bayesian restoration of single channel patch clamp recordings. *Biometrics*, 48:427–448, 1992.
- [17] D.R. Fredkin and J.A. Rice. Maximum likelihood estimation and identification directly from single channel recordings. *JRSS-B*, 249:125–132, 1992.
- [18] J. Henderson, J. Salzberg, and K.H. Fasman. Finding Genes in DNA with a Hidden Markov Model. *J. Comput. Biol.*, 4(2):127–141, 1997.
- [19] B.H. Juang and L.R. Rabiner. Hidden Markov models for speech recognition. *Technometrics*, (33):251–272, 1991.
- [20] W.E. Leland, M.S. Taqqu, W. Willinger, and D.V. Wilson. On the Self-Similar Nature of Ethernet Traffic (Extended Version). *IEEE ACM Transactions on Networking*, 2(1):1–14, feb 1994.

- [21] S.E. Levinson, L.R. Rabiner, and N.M. Sondhi. An introduction to the application of the theory of probabilistic functions of a Markov process to automatic speech recognition. *Bell System tech. J.*, 62:1035–1074, 1983.
- [22] Little and Rubin. *Statistical Analysis with Missing Data*. Wiley, 1987.
- [23] T.L. Marzetta. Computing the Barankin Bound by Solving an Unconstrained quadratic Optimization Problem. In *ICASSP'97*, pages 3829–3832, Munich, Germany, april 1997.
- [24] K.S. Meier. *A statistical procedure for fitting Markov Modulated Poisson Processes*. PhD thesis, University of Delaware, 1984.
- [25] X.L. Meng and D.B. Rubin. Maximum Likelihood estimation via the ECM algorithm : a general framework. *Biometrika*, 2(80):267–278, 1993.
- [26] F. Muri. *Comparaison d'algorithmes d'identification de chaînes de Markov cachées et application à la détection de régions homogènes dans les séquences d'ADN*. PhD thesis, Univ. Paris 5, oct. 1997.
- [27] V. Paxson and S. Floyd. Wide-Area Traffic: The Failure of Poisson Modeling. In *SIGCOMM'94*, pages 257–268, London, England, aug 1994.
- [28] Redner and Walker. Mixture densities, maximum likelihood and the EM algorithm. *SIAM Review*, 26(2):195–239, 1984.
- [29] C. Robert, G. Celeux, and J. Diebolt. Bayesian estimation of hidden Markov chains : a stochastic implementation. *Stat. Prob. Letters*, 16:77–83, 1993.
- [30] T. Ryden. EM algorithm for estimation in Markov-modulated Poisson process. *Computational statistics and data analysis*, 1996.
- [31] S.Vaton, H.Korezlioglu, and T.Chonavel. Modelling LAN Traffic Data as a Locally Stationary Semi-Markov Process:. In D.D.Kouvatsos, editor, *Performance Modelling and Evaluation of ATM Networks Vol.3*. Chapman and Hall, 1997.

- [32] M.A. Tanner and W.H. Wong. The calculation of posterior distribution by data augmentation. *J. of the Amer. Stat. Assoc.*, 82:528–550, 1987.
- [33] A.W. van der Vaart. *Asymptotic Statistics*. Cambridge University Press, 1998.
- [34] S. Vaton. *Modélisation statistique de trafic sur réseau local: application au contrôle dynamique de bande passante*. PhD thesis, ENST, Paris, dec. 1998.
- [35] S. Vaton, E. Moulines, H. Korezlioglu, and D. Kofman. Statistical identification of LAN traffic data. In *Fifth IFIP Workshop on Performance Modelling and Evaluation of ATM Networks*, Bradford, England, july 1997.
- [36] G.C.G. Wei and M.A. Tanner. A Monte Carlo Implementation of the EM algorithm and the Poor Man’s Data Augmentation Algorithms. *J. of the Amer. Stat. Assoc.*, 85:699–704, 1990.
- [37] W. Willinger, M. Taqqu, W.E. Leland, and D.V. Wilson. Self-Similarity in High-Speed Packet Traffic: Analysis and Modeling of Ethernet Traffic Measurements. *Statistical Science*, 10(1):67–85, 1995.
- [38] W. Willinger, M. Taqqu, R. Sherman, and D.V. Wilson. Self-Similarity Through High Variability: Statistical Analysis of Ethernet LAN Traffic at the Source Level. In *SIGCOMM’95*, Cambridge, USA, jan 1995.
- [39] C. Wu. On the convergence property of the EM algorithm. *The Annals of Statistics*, 11:95–103, 1983.

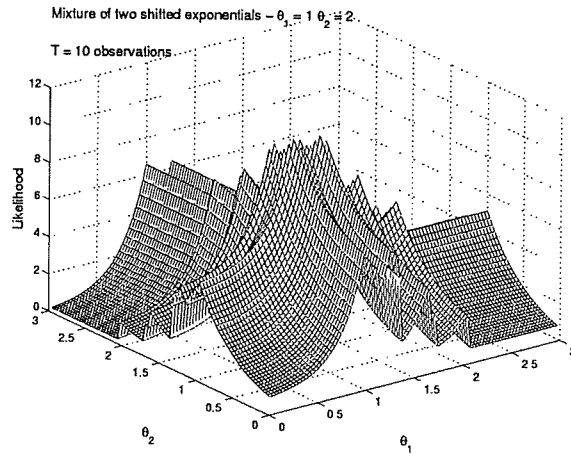


Figure 1: Discontinuities of the likelihood in time domain. $\pi_1 = \pi_2 = 0.5, \theta_1 = 1, \theta_2 = 2, \lambda_1 = 2, \lambda_2 = 4$.

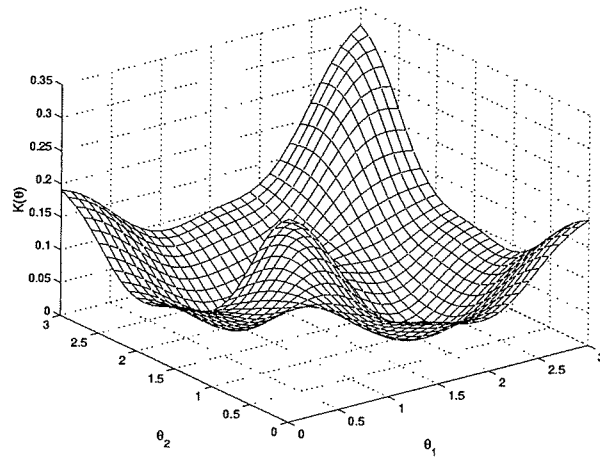


Figure 2: $\omega = 2\pi/X_{\max}, K = 2, \pi_1 = \pi_2 = 0.5, \lambda_1 = \lambda_2 = 1, \theta_1 = 1, \theta_2 = 2, T = 5000$

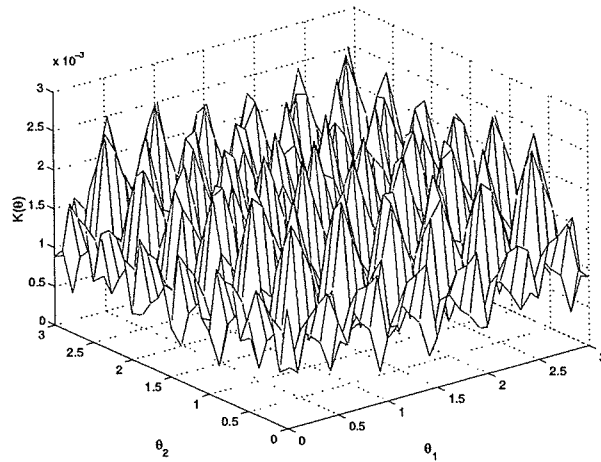


Figure 3: $\omega = 20 \times 2\pi/X_{\max}$, $K = 2, \pi_1 = \pi_2 = 0.5, \lambda_1 = \lambda_2 = 1, \theta_1 = 1, \theta_2 = 2, T = 5000$

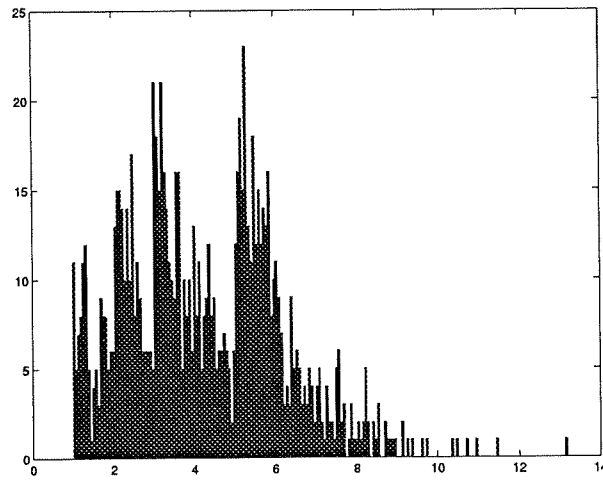


Figure 4: Data Histogram for $T = 1000, K = 4, \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0.25, \lambda_1 = 0.5, \lambda_2 = 0.6, \lambda_3 = 0.8, \lambda_4 = 1.0, \theta_1 = 1, \theta_2 = 2, \theta_3 = 3, \theta_4 = 5$.

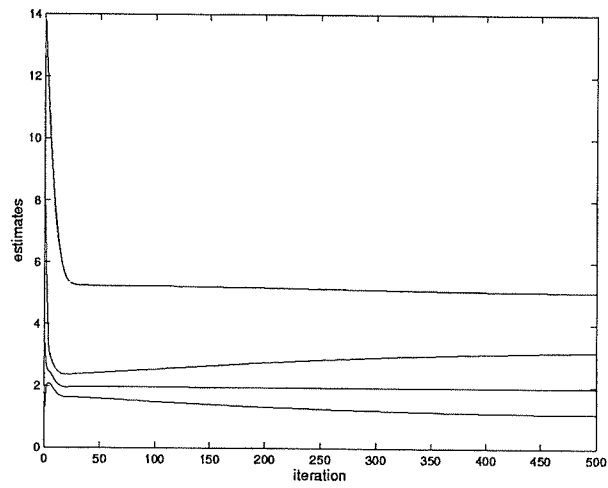


Figure 5: Convergence of the EM algorithm after 500 loops. Initialization: $\hat{\theta}_1 = 0, \hat{\theta}_2 = 5, \hat{\theta}_3 = 10, \hat{\theta}_4 = 15$.

L	α	0.001	0.01	0.1	1	10	100	1000
2		10^5	10^4	100	1	1	1	1
5		10^{13}	10^9	10^5	10	1	1	1
10		10^{18}	10^{17}	10^9	100	10	1	1

Table 1: Condition number of Γ with $T = 1000; K = 2; \pi(1) = \pi(2) = 0.5; \lambda_1 = \lambda_2 = 2; \theta_1 = 1; \theta_2 = 2$.

L	1	10	100
aver. σ_{θ}^2	9.6×10^{-4}	1.5×10^{-4}	2.4×10^{-5}

Table 2: Variance of $\hat{\theta}$ (average value). $T = 1000; K = 2; \pi_1 = \pi_2 = 0.5; \lambda_1 = \lambda_2 = 2; \theta_1 = 1; \theta_2 = 2; \omega_1 = 2\pi/(X_{\max} + \epsilon)$

	EM algorithm		EM algorithm, R=I		SEM algorithm		SAEM algorithm	
	$\mathbb{E}(\hat{\theta}_k)$	$std(\hat{\theta}_k)$	$\mathbb{E}(\hat{\theta}_k)$	$std(\hat{\theta}_k)$	$\mathbb{E}(\hat{\theta}_k)$	$std(\hat{\theta}_k)$	$\mathbb{E}(\hat{\theta}_k)$	$std(\hat{\theta}_k)$
θ_1	1.034	0.379	0.728	0.122	0.987	0.338	0.984	0.304
θ_2	2.042	0.291	2.047	0.056	2.020	0.458	2.027	0.432
θ_3	3.111	0.542	3.455	0.090	3.226	0.554	3.260	0.530
θ_4	4.812	0.444	4.773	0.170	4.759	0.440	4.736	0.412
mean abs. bias	0.094		0.253		0.125		0.142	
mean std.	0.414		0.109		0.448		0.420	

Table 3: Mean and standard deviation of the estimators.

Chapitre 6

Estimation de temps de retard au
moyen de SBLM

Two bayesian methods for multipath propagation parameters estimation

F. Porée, O. Rosec, T. Chonavel and J.M. Boucher

April 18, 2000

ENST de Bretagne
Technopôle de Brest Iroise, BP 832
29285 Brest cedex, France

Abstract

In this paper we propose a bayesian approach for multipath channel time delay estimation when the signal is received by a square-law detector. Two different methods are presented. The first one is applied when the signal is supplied at the output of the detector. On certain approximations, the derivation of the MAP (Maximum A Posteriori) estimator leads to optimize a l_1 -norm criterion. In the second method, the signal is observed after demodulation and matched-filtering. We introduce a Bernoulli-Gaussian model to account for the sparse properties of the channel impulse response and solve the deconvolution problem using Monte-Carlo simulations. Comparisons between both algorithms are presented on simulations. They show that the second method leads to more accurate estimators, while justifying the validity of the approximations used in the first one.

1 Introduction

Many problems can be modeled as a linear system where the observed output signal $r(t)$ is the convolution of the input signal $e(t)$ with a sparse spike time series $h(t)$. Depending on the domain of application, the function $h(t)$ can be for instance the reflectivity sequence of a seismic trace [8], or a propagative channel multipath impulse response as in oceanic acoustic tomography [9]. In such cases, $h(t)$ can be written as

$$h(t) = \sum_{p=1}^P \alpha_p \delta(t - \tau_p), \quad (1)$$

where α_p and τ_p are respectively the attenuation and the delay associated to the p^{th} path, and P is the number of paths. In applications such as ocean acoustic tomography [9], $e(t)$ is known by the receiver. Then the problem of estimating the $(\alpha_p, \tau_p)_{p=1,P}$, and possibly the number P of paths, can be addressed via maximum likelihood estimation (see e.g. [2] [10]). Unfortunately, the log-likelihood criterion shows local maxima, and it requires the knowledge of the number of paths.

To overcome these limitations, we propose a bayesian approach [14] which allows to include information upon the parameters by means of a priori distributions. Here we take into account the fact that the paths amplitudes $(\alpha_p)_{p=1,P}$ and the noise have gaussian distributions. In this work, we present two different deconvolution methods. The first one addresses the case where the only available signal is obtained at the output of a square-law detector. On certain approximations, the derivation of the MAP estimator leads to a very simple l_1 -norm criterion. In the second method the signal is complex-valued, and is obtained after demodulation and matched-filtering. We introduce a Bernoulli-Gaussian model to account for the sparse properties of $h(t)$ and solve the deconvolution problem using Monte-Carlo simulations.

The reason for considering both approaches stems from the fact that the first method involves linear approximations of the non-linear transforms in the square-law detector. The simulation results demonstrate the superiority of the Bernoulli-Gaussian model. However we show that processing the signal at the output of the quadratic receiver is quite robust

to the above approximations, and thus can be used when only the output of the square-law detector is available.

The problem is presented in section 2. In section 3 et 4, the two different bayesian methods are developped. Results and comparison are discussed in section 4.

2 Formulation of the problem

Let $s(t)$ denote the transmitted signal, $r(t)$ the received signal, and $\nu(t)$ an additive gaussian white noise with variance σ_ν^2 , independent from $h(t)$. At the transmitter side $s(t)$ is modulated by a carrier with pulsation ω . Then, the received signal $r(t)$ is in the form:

$$\begin{cases} r(t) = (h * e)(t) + \nu(t) = \sum_{p=1}^P \alpha_p e(t - \tau_p) + \nu(t), \\ \text{with } e(t) = s(t) \cos(\omega t - \phi). \end{cases} \quad (2)$$

At the receiver side, $r(t)$ goes through the square-law detector according to the scheme of the figure 1, where $\tilde{s}(t) = s(-t)$ is the impulse response of the matched-filter.

Let $\underline{x}(t)$ denote the complex signal obtained at the outputs of the matched-filters. It can also be written in the form:

$$\underline{x}(t) = (g * \underline{h})(t) + \underline{b}(t), \quad (3)$$

$$\text{where } \begin{cases} g(t) = (s * \tilde{s})(t) \\ \underline{h} = \sum_{p=1}^P \underline{\alpha}_p \delta(t - \tau_p) \\ \underline{\alpha}_p = \alpha_p \cos(\omega \tau_p + \phi) + i \alpha_p \sin(\omega \tau_p + \phi). \end{cases}$$

Finally, the sampled observation in the observation interval, say $[1, N]$, $\underline{\mathbf{x}} = (\underline{x}(n))_{n=1, N}$, can be written

$$\underline{x}(n) = \sum_{p=1}^P \underline{\alpha}_p g(n - \tau_p) + \underline{b}(n), \quad (4)$$

where $\underline{b}(n)$ is a complex circular gaussian random variable, with variance $\sigma_{\underline{b}}^2 = 4\sigma_\nu^2$ (g is normalized with $g(0) = 1$).

3 l_1 norm deconvolution at the output of the receiver

3.1 The received signal $y(t)$

In this first approach we consider, as already in [12], the case where the only available signal is obtained at the output of the quadratic receiver, and denoted $y(t)$. With a view to getting a simple expression of this signal, we consider two approximations. First, the product terms between the signal of interest and the noise are neglected, which is generally justified because of the high SNR at the output of the matched filter. Moreover, the product terms associated with two distinct paths are neglected, which is generally a satisfactory approximation when the autocorrelation function of $s(t)$ is sharp. Then, $y(t)$ can be written as:

$$\begin{aligned} y(t) &= (h * z)(t) + \epsilon(t) \\ &= \sum_{p=1}^P a_p z(t - \tau_p) + \epsilon(t), \end{aligned} \quad (5)$$

with $z(t) = g^2(t)$ and $a_p = |\alpha_p|^2$.

3.2 The l_1 -norm criterion

Assuming that the noise $\underline{b}(t)$ and the attenuations α_p [13] have gaussian distributions:

$$\alpha_p \sim \mathcal{N}(0, \sigma_\alpha^2), \quad (6)$$

$$\underline{b}(t) \sim \mathcal{N}(0, \sigma_{\underline{b}}^2), \quad (7)$$

it comes that a_p and $\epsilon(t)$ have exponential distributions:

$$a_p \sim \mathcal{E}(1/2\sigma_\alpha^2), \quad (8)$$

$$\epsilon(t) \sim \mathcal{E}(1/2\sigma_{\underline{b}}^2). \quad (9)$$

Without prior information upon the time delays, we assume that they are uniformly distributed in the observation interval: $\tau_p \sim U_{[1, N]}$.

Then, the posterior likelihood minimization problem is:

$$\min_{(a_p, \tau_p)} \frac{\left\| y(t) - \sum_{p=1}^P a_p z(t - \tau_p) \right\|_1}{2\sigma_b^2} + \frac{\| a \|_1}{2\sigma_a^2}. \quad (10)$$

In order to optimize the criterion, the time scale is discretized in the same way as in [3] [6]. Finally, we get the problem in the form

$$\min_{\mathbf{h}} \|\mathbf{y} - \mathbf{S}_z \mathbf{h}\|_1 + \lambda \|\mathbf{h}\|_1, \quad (11)$$

In this criterion, $\lambda = \sigma_b^2/\sigma_a^2$, \mathbf{S}_z is the matrix of convolution with the sampled $z(t)$, \mathbf{y} is the data vector, and \mathbf{h} is the vector of the amplitudes of the paths on the sampled time scale. Moreover let us remark that the spike train can be solved with higher resolution than the received data sampling interval [3] [15].

Also, let us note that the criterion (11) as already been considered in the litterature. In [15], the l_1 -norm term $\|\mathbf{y} - \mathbf{S}_z \mathbf{h}\|_1$ is justified by the spike preservation properties, and the penalty term $\lambda \|\mathbf{h}\|_1$ by the fact that $h(t)$ is sparse. Moreover, λ appears as the weighting factor selected to balance the conflicting priorities of data accountability and addressing the a priori assumption that $h(t)$ is sparse [11]. But in such methods the choice of the value of the parameter λ is often a problem. Here, the bayesian formulation of the problem leads to a simple interpretation of λ , as the inverse of a signal to noise ratio. Thus, it is generally possible to estimate λ in a simple way.

This criterion can be also rewritten in the form:

$$\min_{\mathbf{h}} \|\mathbf{c} - A\mathbf{h}\|_1, \quad (12)$$

with

$$A = \begin{pmatrix} \mathbf{S}_z \\ \lambda I \end{pmatrix}, \text{ and } \mathbf{c} = \begin{pmatrix} \mathbf{y} \\ \mathbf{0} \end{pmatrix}. \quad (13)$$

The minimization of the criterion (12) is implemented according to the algorithm presented in [1], that uses the simplex method applied to a linear programming formulation

of the problem. Also, with a view to adaptative tracking of time-varying channels parameters, the convex criterion $\| \mathbf{c} - \mathbf{A}\mathbf{h} \|_1$ can be minimized by means of a simple gradient algorithm.

4 Bernoulli-Gaussian deconvolution

4.1 Model presentation

Instead of deconvolving the signal after quadratic detection, we consider here the complex signal \underline{x} at the output of the matched filters. In this case, we are faced to a deconvolution problem in the presence of gaussian noise. The convolutional model (3) can be rewritten as:

$$\underline{x}(n) = \sum_{k=-K}^K g(k)\underline{h}(n-k) + \underline{b}(n). \quad (14)$$

When no prior is available upon \underline{h} the deconvolution is equivalent to the maximization of the likelihood function given by:

$$p(\underline{\mathbf{x}}|\boldsymbol{\tau}, \boldsymbol{\alpha}) = \frac{1}{(\pi\sigma_b^2)^N} \exp \frac{-(\underline{\mathbf{x}} - \mathbf{S}_g \underline{\mathbf{h}})^H (\underline{\mathbf{x}} - \mathbf{S}_g \underline{\mathbf{h}})}{\sigma_b^2}, \quad (15)$$

where \mathbf{S}_g denotes the convolution matrix deduced from \mathbf{g} . The maximum likelihood estimate is not an appropriate solution, although it can be shown that it is the minimum variance unbiased estimator.

A major drawback of maximum likelihood deconvolution is that it does not take into account the fact that $\underline{h}(n)$ is a sparse sequence of arrival time. A possible way to regularize the problem is to introduce a Bernoulli variable q to indicate the presence of a path. Then $\underline{h}(n)$ can be modeled by a Bernoulli-Gaussian process defined by:

$$\underline{h}(n) \sim \eta \mathcal{N}(0, \sigma_1^2) + (1 - \eta) \mathcal{N}(0, \sigma_0^2), \quad (16)$$

where

$$\eta = P(q(n) = 1) = 1 - P(q(n) = 0) \quad (17)$$

is the prior probability of finding a path at each sample n . Stricly speaking a pure Bernoulli-Gaussian is obtained when $\sigma_0^2 = 0$. However, as the likelihood of a Dirac distribution is not defined we introduce a non-zero variance $\sigma_0^2 \ll \sigma_1^2$. In geophysics this model is commonly used to characterize the reflectivity of the subsurface, and σ_0^2 describes small heterogeneity in the sedimentary layers [4]. In nuclear science σ_0^2 models some background disturbance noise at the receiver.

4.2 The MCMC approach

In that framework, the problem is to study the posterior likelihood $p(\underline{\mathbf{h}}, \mathbf{q}|\underline{\mathbf{x}})$. Using Bayes rule and omitting the constant term $p(\underline{\mathbf{x}})$ this density can be written as:

$$p(\underline{\mathbf{h}}, \mathbf{q}|\underline{\mathbf{x}}) \propto p(\underline{\mathbf{x}}|\underline{\mathbf{h}})p(\underline{\mathbf{h}}|\mathbf{q})p(\mathbf{q}). \quad (18)$$

After some easy calculations the posterior log-likelihood becomes:

$$\begin{aligned} \mathcal{L}(\underline{\mathbf{h}}, \mathbf{q}|\underline{\mathbf{x}}) = & -\frac{(\underline{\mathbf{x}} - \mathbf{S}_g \underline{\mathbf{h}})^H (\underline{\mathbf{x}} - \mathbf{S}_g \underline{\mathbf{h}})}{\sigma_b} - \frac{\underline{\mathbf{h}}^H \mathbf{D} \underline{\mathbf{h}}}{\sigma_1^2} \\ & - \frac{\underline{\mathbf{h}}^H (\mathbf{I} - \mathbf{D}) \underline{\mathbf{h}}}{\sigma_0^2} + n_1(\mathbf{q}) \left[\frac{1 - \eta \sigma_1}{\eta} \frac{\sigma_1}{\sigma_0} \right], \end{aligned} \quad (19)$$

with $\mathbf{D} = \text{diag}(\mathbf{q})$. Let us notice that given a vector \mathbf{q} , \mathcal{L} is a quadratic function of $\underline{\mathbf{h}}$ and so its maximum can be obtained in closed form. Unfortunately the study of \mathcal{L} for the 2^N possible \mathbf{q} sequences is computationnally untractable as soon as N becomes large.

To overcome this difficulty it is possible to use simulation methods. For the application we are interested in, suppose we are able to generate a sequence of independent samples $\{(\underline{\mathbf{h}}^{(m)}, \mathbf{q}^{(m)}), m = 1, \dots, M\}$ according to $p((\underline{\mathbf{h}}, \mathbf{q})|\underline{\mathbf{x}})$. Then we can use these samples to make inference on the missing variables such as calculating conditionnal expectations $\mathbb{E}[\mathbf{q}|\underline{\mathbf{x}}]$ and $\mathbb{E}[\underline{\mathbf{h}}|\underline{\mathbf{x}}]$.

Therefore our problem comes down to the simulation of $(\underline{\mathbf{h}}, \mathbf{q})$ according to their posterior distribution. However generating high dimension variables remains a difficult task as soon as the distributions are non-standard. Here the convolution introduces time dependency of the missing variables conditionally on the observation, and no direct simulation

scheme is possible. One way of solving this problem is to use MCMC methods. The basic principle is to generate a Markov Chain whose equilibrium distribution is the target distribution. The most popular MCMC algorithms as well as details concerning their convergence properties can be found in [14].

4.3 The Gibbs sampler

One simulation scheme is to simulate the missing variables one sample at a time. Indeed it can be shown (see [5]) that conditionally on $\{\underline{h}(j), q(j), j \neq n\}$, $(\underline{h}(n), q(n))$ can be written as a mixture of gaussian distributions. This motivates the choice of a Gibbs sampler. The algorithm proceeds as follows:

1. Initialization : random choice of $(\underline{\mathbf{h}}^{(0)}, \mathbf{q}^{(0)})$;
2. At iteration $m \leq M$, for $n = 1, \dots, N$:
 - choose site i ;
 - simulate $(\underline{h}(i), q(i))$ according to
$$p(\underline{h}(i), q(i) | \underline{\mathbf{x}}, \underline{h}(j), q(j), j \neq i).$$

At each iteration, the sites are visited using a random permutation of $\{1, \dots, N\}$. The behavior of the algorithm can be divided into two periods: first a burning period of M_0 iterations and then a steady state period where it can be assumed that the drawn samples are distributed according to their posterior distribution.

Then, we calculate for each sample $1 \leq n \leq N$

$$Q(n) = \frac{1}{M - M_0} \sum_{m=M_0+1}^M q^{(m)}(n), \quad (20)$$

$$\underline{H}(n) = \frac{1}{M - M_0} \sum_{m=M_0+1}^M \underline{h}^{(m)}(n). \quad (21)$$

Then we use the following decision rule:

- if $Q(n) > 0.5$ set $\hat{q}(n) = 1$ and $\hat{\underline{h}}(n) = Q(n)\underline{H}(n)$

- else set $\hat{q}(n) = 0$ and $\hat{h}(n) = 0$.

5 Results

We now apply these two methods to numerical data. Let $s(t)$ be a M.L.S. (Maximum Length Sequence) [13] of length $n = 2^9 - 1 = 511$ symbols with values $\pm 1/\sqrt{n}$. The autocorrelation of the M.L.S. is a triangular shape, carried by the interval $[-K, K]$, where K is the symbol duration. We assume that $y(t)$ is sampled at a rate $\delta = K/6$.

We simulate two arrival times τ_1 and τ_2 with amplitudes 1, for a SNR equal to 30 dB, ie $\sigma_\nu^2 = 10^{-3}$. The results are presented for 2 values of the difference $|\tau_1 - \tau_2|$. Figure 2 shows an example of the signals obtained at different locations of the detector.

For each scenario, the performances are tested on 100 traces, corresponding to 100 independent realizations of the noise. Figure 3 represents the image of the results obtained with each method.

The l_1 -norm method is implemented using standard Nag fortran routines [7]. The parameter λ is chosen equal to $1/2\sigma_\nu^2$. For the Bernoulli-Gaussian deconvolution algorithm, the model parameters are respectively $\sigma_2^2 = 4.10^{-3}$, $\sigma_1^2 = 1$, $\sigma_0^2 = 10^{-3}$, and $\eta = 0.04$. We run the Gibbs sampler for $M = 500$ iterations and discard the first 100.

We focus here only on the estimation of the delays. Indeed, an accurate estimation of these parameters is of the greatest importance in oceanic acoustic tomography. For that purpose, we define P_{D_1} as the percentage of pulses detected not further than one sample from their true location. Besides P_{D_2} represents the percentage of exact estimation of time delays when the two largest maxima are selected. The results are presented in table 1.

6 Conclusion

In this paper we have presented two methods for time delay estimation. We have seen that at the output of the square-law detector a very simple approximate MAP criterion can be considered, that yields good results in terms of path detection and delay estimation.

Moreover when the complex data at the output of the matched-filter are available, better accuracy can be achieved via MCMC simulations of the parameters posterior distribution.

References

- [1] I. Barrodale and F.D.K. Roberts. An improved algorithm for discrete l1 linear approximation. *SIAM Numer. Anal.*, 10(5):839–848, Oct. 1973.
- [2] M. Feder and E. Weinstein. Parameter estimation of superimposed signals using the EM algorithm. *IEEE Trans. on A.S.S.P.*, 36(4):477–489, 1988.
- [3] J-J. Fuchs. Multipath time-delay detection and estimation. *IEEE Trans. S.P.*, 47(1):237–243, Janv. 1999.
- [4] M. Lavielle. Bayesian deconvolution of bernoulli-gaussian processes. *Signal Processing*, 33:67–79, 1993.
- [5] M. Lavielle. A stochastic algorithm for parametric and non-parametric estimation in the case of incomplete data. *Signal Processing*, 42:3–17, 1995.
- [6] Tze Fen Li. Multipath time delay estimation using regression stepwise procedure. *IEEE Trans. S.P.*, 46(1):191–195, January 1998.
- [7] The Numerical Algorithms Group Limited. *The Nag Fortran Library Manual, Mark 14*, 1st edition, April 1990.
- [8] J.M. Mendel. *Maximum-likelihood deconvolution : a journey into model-based signal processing*. Springer-Verlag, 1990.
- [9] W. Munk, P. Worcester, and C. Wunsch. *Ocean Acoustic Tomography*. Cambridge University Press, 1995.
- [10] V. Nimier and G. Jourdain. Active estimation of a multipath propagation channel with a bayesian strategy. *Traitement du signal*, 10(3):201–213, 1993.

- [11] M. O'Brien, A. N. Sinclair, and S. M. Kramer. Recovery of a sparse spike time series by l_1 norm deconvolution. *IEEE Trans. S.P.*, 42(12):3353–3365, Dec. 1994.
- [12] F. Porée, T. Chonavel, and T. Terre. Multipath time-delay detection and estimation for ocean acoustic tomography: a bayesian approach. In *Oceans'99 MTS/ IEEE Proceedings*, volume 3, pages 1587–1590, 1999.
- [13] J. G. Proakis. *Digital Communications*. Mac Graw Hill, second edition, 1989.
- [14] C. P. Robert. *The bayesian choice*. Springer-Verlag, New-York, 1994.
- [15] H. Taylor, S. Banks, and F. McCoy. Deconvolution with the l_1 norm. *Geophys.*, 44(1):39–52, 1979.

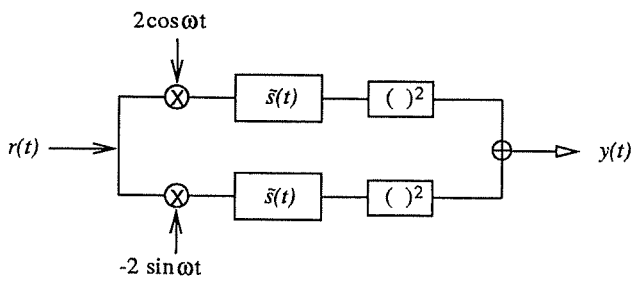


Figure 1: Structure of the receiver.

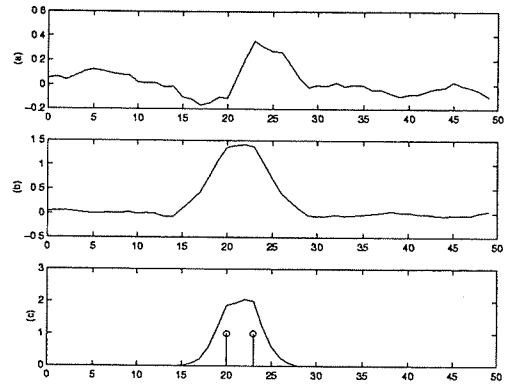


Figure 2: (a) Real part of $\underline{x}(t)$; (b) Imaginary part of $\underline{x}(t)$; (c) Output real signal $y(t)$.

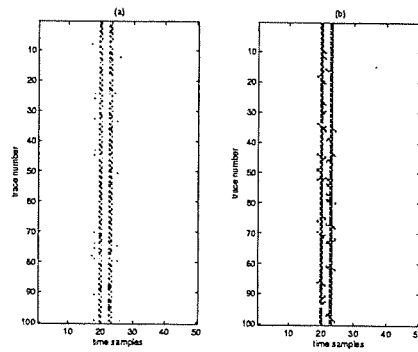


Figure 3: Results on the 100 traces with $|\tau_1 - \tau_2| = K/2$: (a) l_1 -norm method ; (b) MCMC method.

	l_1		MCMC	
$ \tau_1 - \tau_2 $	K	$K/2$	K	$K/2$
P_{D_1}	100	96	100	100
P_{D_2}	91	88.5	98.5	97.5

Table 1: Detection results.

Chapitre 7

Livre de traitement du signal
aléatoire

Traitement du Signal Aléatoire

Modélisation et Estimation.

Thierry CHONAVEL

thierry.chonavel@enst-bretagne.fr

novembre 1999

Table des matières

1	Introduction	15
1.1	Avertissement.	15
1.2	Motivation d'un livre de signal aléatoire.	16
1.3	Quelques problèmes classiques de traitement du signal aléatoire.	17
1.4	Pourquoi ce livre?	19
1.5	Organisation du document.	20
1.6	Remerciements.	22
2	Processus aléatoires: rappels.	23
2.1	Généralités.	23
2.1.1	Définition.	23
2.1.2	Loi d'un processus aléatoire.	24
2.1.3	Théorème de Kolmogorov.	24
2.2	Processus du second ordre.	25
2.3	Opérations classiques dans $L^2(\Omega, \mathcal{A}, dP)$	26
2.3.1	Convergence en moyenne quadratique.	26
2.3.2	Continuité en moyenne quadratique.	27

2.3.3	Dérivabilité en moyenne quadratique.	28
2.3.4	Intégration	29
2.4	Stationnarité et ergodicité.	30
2.4.1	Stationnarité.	30
2.4.2	Ergodicité.	31
3	Spectre de puissance des processus stationnaires au second ordre.	35
3.1	Spectres à densité : Densité Spectrale de Puissance (DSP).	35
3.2	Mesure spectrale.	36
4	Représentation spectrale des processus stationnaires au second ordre.	39
4.1	Mesures aléatoires et intégrales stochastiques.	39
4.1.1	Définition.	39
4.1.2	Mesure $\mu_{\hat{Z}}$ associée à \hat{Z}	40
4.1.3	Principe de la démarche.	40
4.1.4	Construction des intégrales stochastiques $\int_{\mathbb{R}} \phi(f) d\hat{Z}(f)$	41
4.2	Isomorphisme de Kolmogorov.	42
4.3	Représentation spectrale.	43
4.4	Echantillonnage.	43
5	Filtrage des processus stationnaires au second ordre.	45
5.1	Rappels sur le filtrage des signaux déterministes.	45
5.2	Filtrage des processus stationnaires.	46
5.3	Comparaison des cas déterministe et aléatoire.	48
5.4	Exemples.	49

5.4.1	Filtrage passe bande.	49
5.4.2	Filtre dérivateur.	49
5.4.3	Equations aux dérivées partielles linéaires.	50
6	Processus particuliers importants.	51
6.1	Processus gaussien.	51
6.2	Processus de Poisson.	52
6.3	Bruit blanc.	54
6.3.1	Distributions aléatoires.	55
6.3.2	Mouvement brownien.	58
6.4	Processus cyclostationnaires.	59
6.5	Processus circulaires.	61
6.6	Processus multivariés.	63
7	Transformations non linéaires des processus.	65
7.1	Détecteur quadratique et limiteur saturé.	65
7.1.1	Détecteur quadratique.	65
7.1.2	Limiteur saturé.	66
7.1.3	Théorème de Bussgang.	68
7.2	Modulation en amplitude des processus.	68
7.2.1	Modulation en phase et en quadrature.	69
7.2.2	Représentation analytique et modulation BLU.	70
7.2.3	Représentation de Rice.	70
7.2.4	Démodulation en présence de bruit.	71

8	Prédiction linéaire des processus stationnaires au second ordre.	73
8.1	Généralités et définitions.	73
8.1.1	Espérance conditionnelle et prédicteur linéaire.	73
8.1.2	Innovation d'un processus.	75
8.1.3	Processus réguliers et singuliers.	76
8.1.4	Exemples.	76
8.2	Théorème de décomposition de Wold.	77
8.3	Prédiction linéaire à horizon fini.	78
8.4	Factorisation causale des densités spectrales.	82
8.4.1	Factorisations causales.	82
8.4.2	Factorisation causale à phase minimale.	83
8.5	Le cas continu.	85
9	Techniques de filtrage particulières.	87
9.1	Filtrage de Wiener.	87
9.2	Filtrage de Kalman.	90
9.3	Généralisation du filtrage de Kalman.	93
9.4	Signaux déterministes bruités et filtrage adapté.	97
10	Processus à spectres rationnels.	99
10.1	Equations aux différences et processus à spectres rationnels.	99
10.2	Factorisation spectrale des processus à spectre rationnel.	101
10.3	Représentation d'état des modèles ARMA.	102
11	Identification spectrale des processus stationnaires au second ordre.	105

11.1	Identification des processus ARMA.	105
11.1.1	Identification de la partie AR.	106
11.1.2	Identification de la partie MA.	106
11.1.3	Identification de la représentation d'état.	106
11.2	Le problème des moments trigonométriques.	109
11.2.1	Conditions d'existence de solutions.	109
11.2.2	Polynômes orthogonaux sur le cercle unité.	110
11.2.3	Classes particulières de fonctions holomorphes.	113
11.2.4	Solution générale du problème.	114
11.2.5	Solution du maximum d'entropie.	115
11.3	Spectres de raies.	116
11.4	Filtres en treillis.	118
12	Estimation spectrale non paramétrique.	121
12.1	Définitions.	121
12.2	Généralités sur l'estimation non paramétrique.	123
12.2.1	Séquence d'observations indépendantes.	123
12.2.2	Processus ergodique.	124
12.3	Estimateurs empiriques de la moyenne et des covariances.	125
12.3.1	Processus linéaires, et absolue sommabilité des covariances.	125
12.3.2	Moyenne empirique.	127
12.3.3	Coefficients de covariance empiriques.	128
12.4	Estimation empirique de la DSP : le périodogramme.	130

12.4.1	Cas du bruit blanc.	131
12.4.2	Cas des processus linéaires.	132
12.4.3	Cas des processus harmoniques.	133
12.5	Lissage du périodogramme.	134
12.5.1	Fonctions intégrales du périodogramme.	135
12.5.2	Périodogramme lissé.	135
12.5.3	Périodogramme moyenné.	138
13	Estimation spectrale paramétrique.	139
13.1	Introduction.	139
13.2	Rappels d'estimation paramétrique.	140
13.2.1	Borne de Cramer-Rao.	140
13.2.2	Estimateur du maximum de vraisemblance.	141
13.2.3	Estimateur linéaire sans biais de variance minimale.	141
13.2.4	Méthodes de moindres carrés.	142
13.3	Estimation des coefficients de covariance.	142
13.4	Estimation spectrale des modèles ARMA : critères quadratiques.	144
13.4.1	Estimation spectrale d'un processus à spectre rationnel.	144
13.4.2	Synthèse de filtres rationnels.	146
13.5	Vraisemblance asymptotique d'un processus gaussien.	148
13.5.1	Vraisemblance gaussienne.	148
13.5.2	Comportement asymptotique de la vraisemblance.	148
13.6	Estimation au sens du maximum de la vraisemblance approchée.	153

13.6.1	Principe de la méthode.	153
13.6.2	Convergence des estimateurs.	154
13.7	Application aux modèles ARMA.	156
13.7.1	Cas général.	156
13.7.2	Cas de la modélisation autorégressive.	157
13.7.3	Modèles AR paramétrés par les coefficients de réflexion.	158
14	Statistiques d'ordres supérieurs.	161
14.1	Introduction.	161
14.2	Moments et cumulants.	163
14.2.1	Variables aléatoires réelles.	163
14.2.2	Vecteurs aléatoires réels et complexes.	164
14.2.3	Propriétés des moments et des cumulants.	165
14.2.4	Estimation empirique des moments et des cumulants	166
14.3	Multicovariances et multispectres.	166
14.3.1	Multicovariances.	167
14.3.2	Multispectres.	167
14.3.3	Blancheur aux ordres supérieurs.	169
14.3.4	Estimation des multicovariances et des multispectres.	169
14.3.5	Filtrage linéaire.	169
14.4	Estimation de la fonction de transfert d'un filtre.	170
14.4.1	Modélisation.	170
14.4.2	Intérêt du traitement aux ordres supérieurs.	170

14.4.3	Critères de distance.	172
14.4.4	Identification directe des coefficients.	172
15	Méthodes bayésiennes et techniques de simulation.	175
15.1	Introduction.	175
15.2	Rappels sur l'estimation bayésienne.	176
15.3	Principe des méthodes de Monte Carlo.	178
15.3.1	Intégration par la méthode de Monte-Carlo.	178
15.3.2	Optimisation.	180
15.3.3	Simulation de variables aléatoires.	182
15.4	Méthodes MCMC.	184
15.4.1	Rappels sur les chaînes de Markov.	185
15.4.2	L'algorithme de Hastings-Metropolis.	189
15.4.3	Echantillonneur de Gibbs.	191
15.5	Application au filtrage.	194
15.5.1	Echantillonnage d'importance.	195
15.5.2	Dégénérescence de l'algorithme.	197
15.6	Estimation d'une fonction de transfert rationnelle.	197
15.6.1	Principe de la méthode.	198
15.6.2	Estimation des paramètres.	199
16	Estimation adaptative.	201
16.1	Méthodes de descente classiques.	201
16.2	Principe des méthodes stochastiques.	203

16.3	Algorithmes LMS et RLS.	204
16.4	Convergence des algorithmes LMS et RLS.	206
16.4.1	Convergence en moyenne de l'algorithme LMS.	207
16.4.2	Variance asymptotique pour l'algorithme LMS.	208
16.4.3	Convergence de l'algorithme RLS.	209
16.5	La méthode de l'ODE.	210
16.5.1	Hypothèses.	210
16.5.2	Convergence des algorithmes stochastiques.	211
16.5.3	Application à l'algorithme LMS.	214
A	Rappels de théorie de la mesure.	217
B	$L^2(\Omega, \mathcal{A}, dP)$ est un espace complet.	221
C	Prolongement par continuité d'un opérateur linéaire.	223
D	Isomorphisme de Kolmogorov et représentation spectrale.	225
E	Décomposition de Wold.	227
F	Critère de Dirichlet.	231
G	Algorithme de Viterbi.	233
H	Factorisation spectrale à phase minimale des spectres rationnels.	235
I	Compatibilité d'un ensemble de données avec un jeu de covariances.	237
I.1	Rappels d'analyse convexe.	237

I.2	Compatibilité d'un jeu de coefficients avec une séquence de covariances. . .	239
J	Algorithme de Levinson.	243
K	Principe du maximum.	247
L	Extension à un pas d'une séquence de covariances.	249
M	Relations de récurrence sur $P_n(z)$.	251
N	Solution générale du problème des moments trigonométriques.	253
O	Théorème central limite pour la moyenne empirique.	257
P	Covariance des coefficients de covariance empirique.	259
Q	Théorème central limite pour les covariances empiriques.	263
R	Loi du périodogramme d'un bruit blanc.	265
S	Périodogramme d'un processus linéaire.	267
T	Variance du périodogramme d'un processus linéaire.	269
U	Une loi forte des grands nombres (I).	271
V	Une loi forte des grands nombres (II).	273
W	Relation amplitude-phase pour les filtres causaux à phase minimale .	275
X	Convergence de l'algorithme de Hastings-Metropolis.	277

Y Convergence de l'algorithme de Gibbs.	279
Z Variance asymptotique de l'algorithme LMS.	283
Classement bibliographique.	287
Bibliographie.	288
Notations et abréviations.	294
Index	298

Chapitre 1

Introduction

1.1 Avertissement.

Ce document expose des notions de traitement du signal aléatoire. On y trouvera principalement la modélisation et l'estimation spectrale des processus stationnaires au second ordre, ainsi que leur filtrage. Quelques digressions autour de ce thème visent à mettre en évidence l'existence de techniques qui généralisent les méthodes développées au second ordre, c'est à dire qui ne font appel qu'aux fonctions de moyenne et de covariance des processus. L'emploi de tels outils est souvent nécessaire pour aborder les nouveaux champs d'application du traitement aléatoire du signal.

Ce livre s'adresse en particulier à des étudiants de troisième cycle. Il correspond à des cours enseignés à l'Ecole Nationale supérieure des Télécommunications de Bretagne (ENSTB) ainsi qu'à l'Ecole Nationale de la Statistique et de l'Analyse de l'Information (ENSAI). Il s'adresse également aux ingénieurs, enseignants et chercheurs intéressés par le traitement des signaux aléatoires. Il s'agit d'un document introductif qui vise principalement à présenter de façon synthétique les bases théoriques de cette discipline.

Les prérequis mathématiques nécessaires pour aborder ce cours sont essentiellement des bases en probabilités et en statistiques, ainsi que quelques notions sur les processus aléatoires et la théorie de la mesure. On trouvera dans ce livre et dans le CD-ROM d'accompagnement certains rappels concernant ces théories. On supposera de plus que le lecteur possède quelques notions de traitement des signaux déterministes.

1.2 Motivation d'un livre de signal aléatoire.

Bien souvent, les résultats de la théorie du traitement des signaux déterministes n'est pas suffisante pour aborder certains problèmes. En effet, dans de nombreuses applications les signaux considérés sont aléatoires, en ce sens que la reproduction de la procédure expérimentale ne conduit pas à la même observation du fait de la présence de perturbations que l'opérateur ne peut pas maîtriser totalement. Ces perturbations peuvent être de natures variées : phénomènes de bruit additif ou de distorsion agissant sur des signaux transmis, erreurs liées aux appareils de mesure, L'observation comporte donc certains aléas. La théorie des probabilités et des processus aléatoires offre un cadre satisfaisant pour modéliser ces aléas. Dans la pratique du traitement du signal, on emploie d'ailleurs souvent indifféremment le terme de signal pour désigner un processus aléatoire ou une de ses réalisations.

On conçoit alors qu'il faut généralement effectuer un traitement, par exemple une moyenne sur plusieurs expériences successives, pour obtenir une estimation de la grandeur étudiée qui soit acceptable pour l'opérateur.

La modélisation probabiliste des signaux trouve son intérêt dans la possibilité que l'on a pour effectuer cette estimation d'employer les résultats de la théorie des statistiques. Cette théorie permet également d'envisager la pertinence de la méthode d'estimation retenue. Ainsi, le moyennage évoqué ci-dessus a pour but d'approcher, en vertu de la loi des grands nombres, l'espérance mathématique de la grandeur étudiée, cette grandeur étant décrite par un modèle aléatoire.

Dans certains cas, on ne peut disposer que d'une seule réalisation de l'expérience, comme par exemple quand on cherche à extraire des informations statistiques à partir du relevé de la trace sismique obtenue lors d'un tremblement de terre. Dans un tel contexte, la propriété ergodique, lorsqu'elle est vérifiée, est très utile. Cette propriété permet en effet d'estimer de façon raisonnable des espérances mathématiques par des moyennes temporelles calculées sur la base de la connaissance d'une unique trajectoire du processus observé.

L'étude des signaux déterministes met en évidence l'importance de la transformée de Fourier. Ainsi, la transformée de Fourier permet t'elle d'évaluer la répartition de l'énergie ou de la puissance de signaux déterministes en fonction de la fréquence. De plus, de nombreuses transformations des signaux utilisées dans les dispositifs techniques et pour la modélisation de phénomènes naturels sont des opérations de filtrage. Ces filtrages, qui sont des opérations de convolution des signaux, apparaissent comme de simples opérations de multiplication dès lors qu'on travaille dans le domaine fréquentiel. On verra qu'une classe importante et très usitée de processus aléatoires, les processus stationnaires au sens large, admet une représentation dans le domaine fréquentiel. Cette représentation permet de traiter simplement le problème du filtrage. Elle permet également d'évaluer, à travers la notion de spectre d'un processus aléatoire, la façon dont en moyenne la puissance d'un processus se répartit en fonction de la fréquence.

1. Introduction

En pratique, pour pouvoir utiliser les résultats précédents lorsqu'on considère des processus non stationnaires, on admet souvent que la stationnarité est vérifiée si on se contente de travailler sur des portions du signal qui correspondent à des durées d'observation relativement courtes. Notons cependant que les développements récents des méthodes temps-fréquence [40], et en particulier des méthodes par transformée en ondelette [39][41], permettent d'aborder de façon plus précise l'étude des processus non stationnaires, étude qui sort du cadre de ce livre.

1.3 Quelques problèmes classiques de traitement du signal aléatoire.

Afin de mieux comprendre l'importance de l'étude du spectre des processus stationnaires au second ordre et du filtrage de ces processus, présentons maintenant quelques problèmes très classiques qui s'inscrivent dans ce contexte.

1) On considère un oscillateur électronique que l'on excite au moyen d'une diode à bruit, et dont on cherche à étudier le comportement. Le signal observé en sortie de cet oscillateur ne ressemble que grossièrement à une sinusoïde et n'est pas tout à fait périodique. Dans le domaine spectral on peut envisager la façon dont en moyenne la puissance du signal fourni par l'oscillateur se répartit en fonction de la fréquence. Ceci permet en particulier d'évaluer la fréquence centrale de l'oscillateur et de déterminer des indices de performance tels que la largeur de la bande de l'oscillateur. Cette répartition moyenne de la puissance en fonction de la fréquence, appelée spectre (chapitre 4), et la façon dont on peut l'estimer (chapitres 12 et 13) ne sont pas complètement intuitives lorsque comme ici il est souhaitable de modéliser le signal observé comme un processus aléatoire. On verra également que les propriétés de convergence d'estimateurs spectraux classiques tels que le périodogramme (chapitre 12) peuvent s'avérer décevantes et qu'il faut prendre quelques précautions pour estimer correctement le spectre d'un processus.

2) L'utilisation d'un modèle physique pour caractériser les interactions qui conduisent à la formation d'un signal et à celle de son spectre est souvent difficile. On peut alors souvent utiliser des modèles spectraux paramétriques tels que les modèles de spectres rationnels (chapitre 10) qui conduisent à des descriptions souvent plus simples à estimer (chapitre 13) mais aussi plus compactes que des modélisations fondées sur l'analyse des phénomènes physiques qui ont engendré le signal. En particulier, pour réaliser des systèmes de synthèse vocale, on cherche à générer des sons qui possèdent des propriétés spectrales analogues à celles de la parole, au moyen de filtres de fonctions de transfert rationnelles. En excitant de tels filtres au moyen de suites de variables aléatoires décorréélées ou de séquences périodiques d'impulsions, suivant le type de sons que l'on cherche à générer, on peut produire un message oral compréhensible.

3) Lors d'une émission de radio, un signal transmis sur le canal de propagation hertzien

y subit l'action d'un bruit additif avant de parvenir au récepteur. Si on considère que le spectre du signal utile n'occupe qu'une bande de fréquences limitée et que l'observation est perturbée par un bruit additif occupant une bande de fréquences beaucoup plus large, on peut chercher à placer un filtre au niveau du récepteur afin de limiter la puissance de bruit reçue. Si le filtre laisse passer une bande de fréquences trop large, la puissance de bruit reçue risque d'être importante. Au contraire, si le filtre ne retient qu'une bande de fréquences très étroite, le signal utile peut se trouver fortement distordu par l'action du filtre. Le filtrage de Wiener (chapitre 9) réalise un compromis satisfaisant dans ce genre de situations.

4) Pour coder ou transmettre numériquement un signal de parole, on procède classiquement à son échantillonnage puis à la quantification du signal échantillonné. Afin de réduire la quantité d'informations à transmettre, on commence généralement par effectuer avant la quantification un filtrage qui vise à le décorréler le signal (filtre d'erreur de prédiction - chapitre 8). C'est le signal présent en sortie de ce filtre, appelé signal d'erreur de prédiction, qui est alors quantifié et échantillonné. Ce signal ayant une variance plus faible que celle du signal initial, il est alors possible, pour un niveau de dégradation introduit par la quantification fixé, de diminuer le nombre de bits d'information requis pour effectuer cette quantification. Le calcul du filtre d'erreur de prédiction peut s'effectuer au moyen de l'algorithme rapide de Levinson (chapitre 11), pour des intervalles de temps sur lesquels le signal de parole est supposé stationnaire. Par ailleurs, les coefficients du filtre d'erreur de prédiction peuvent être représentés par un jeu de paramètres équivalents, appelés coefficients de réflexion, dont les valeurs sont dans l'intervalle $[-1,1]$ (chapitre 11). Cette propriété facilite la quantification des paramètres du filtre d'erreur de prédiction, et permet de contrôler la stabilité du filtre de reconstruction du signal de parole à partir de son erreur de prédiction.

5) Un canal de transmissions peut être modélisé comme un filtre dont le module de la réponse fréquentielle est fourni par la connaissance du spectre du signal reçu. Malheureusement, la phase de la réponse fréquentielle de ce filtre n'est pas accessible à partir de cette seule information. Dans ce cas, les statistiques d'ordres supérieures (chapitre 14) ou les méthodes bayésiennes (chapitre 15) permettent de lever l'ambiguïté sur la phase de la réponse fréquentielle.

6) Dans le problème précédent il peut se faire que le canal de transmission, et c'est typiquement le cas pour les communications radio-mobiles, soit caractérisé par une fonction de transfert qui évolue au cours du temps. Des techniques de filtrage adaptatif du type filtrage de Kalman (chapitre 9), ou des méthodes d'approximation stochastique (chapitre 16), doivent alors être envisagées afin de suivre ces variations, et pour reconstituer le message transmis.

1. Introduction

1.4 Pourquoi ce livre?

Le traitement du signal occupe une place dont les frontières avec des sciences fondamentales telles que les mathématiques ou la physique, et les sciences de l'ingénieur, et en particulier l'électronique, l'informatique, l'automatique, ou les mathématiques appliquées à des domaines aussi divers que la finance ou la biologie, sont assez floues. Il en résulte plusieurs approches possibles pour la présentation d'un livre de signal aléatoire, suivant le point de vue de l'auteur. Cette différenciation des approches s'estompe cependant dans la mesure où le traitement du signal devient une discipline clairement identifiée.

Quelle est la spécificité de ce livre? Indiquons dès à présent qu'un objectif essentiel de ce livre vise à formaliser certaines bases du signal aléatoire que l'on rencontre souvent dans des ouvrages à caractère plus mathématique.

La démarche qui consiste à partir des processus stationnaires au second ordre et à exposer la modélisation et l'estimation des spectres et des fonctions de filtrage est très classique pour introduire le traitement du signal aléatoire. Le problème est de savoir si pour cela on choisit de se concentrer essentiellement sur l'étude des bases mathématiques sur lesquelles se fondent les nombreuses techniques utilisables en pratique, ou si l'on cherche plutôt à dresser un inventaire structuré et plus ou moins exhaustif de ces techniques.

Dans ce livre, on s'est attaché à fournir certaines bases théoriques du traitement du signal aléatoire. En conséquence, on a été amené à limiter le nombre des techniques pratiques particulières et des exemples présentés. Il semble en effet prioritaire que les bases d'une discipline soient clairement posées dans le cadre d'un exposé introductif.

Aussi, on trouvera ici la démonstration de nombreux résultats énoncés, ou au moins les références bibliographiques correspondantes. Certains thèmes tels que l'identification spectrale ou l'estimation spectrale paramétrique ont été détaillés avec des approches un peu différentes de ce que l'on trouve généralement dans la littérature en traitement du signal aléatoire. Cela a permis d'introduire certains outils importants pour la pratique d'une façon qui à notre avis est cohérente et naturelle.

Cependant, ce livre étant plus destiné à un public d'ingénieurs que de mathématiciens, on se limite ici à l'utilisation de ce que l'on appelle communément les "mathématiques de l'ingénieur". Pour cette même raison il apparaît clairement que l'on ne saurait cependant se limiter au seul exposé de principes généraux, et qu'il faut envisager les techniques les plus usuelles de résolution des problèmes pratiques qui s'y rattachent. La place sensiblement identique faite aux aspects de modélisation et à ceux relatifs à l'estimation témoignent de cette préoccupation. De plus, si la place allouée à la présentation d'outils pratiques peut sembler réduite, cela est en partie lié au fait que l'effort préalable de modélisation des signaux aléatoires permet souvent de présenter la solution des problèmes pratiques correspondants de façon concise. C'est notamment le cas des filtrages de Wiener et de Kalman étudiés au chapitre 9, qui s'obtiennent directement à partir des résultats de la

théorie de la prédiction linéaire. On pourra par exemple se référer aux rubriques *algorithme*, *filtre*, ou *méthode* de l'index pour se faire une idée des outils de traitement du signal qui sont présentés dans le livre. A défaut d'être nombreux, nous espérons les avoir choisis parmi les plus significatifs et avoir amené leur présentation de façon cohérente.

Dans les trois derniers chapitres, sur la base des connaissances acquises dans le cadre de la modélisation et de l'estimation des spectres et du filtrage des processus stationnaires au second ordre, on change un peu de point de vue. Plus précisément, on cherche à élargir les possibilités de traitement en envisageant les apports possibles des méthodes statistiques d'ordres supérieurs, des méthodes bayésiennes et de Monte Carlo, et des méthodes d'optimisation stochastique. A la différence des chapitres précédents, et d'ouvrages plus particulièrement consacrés à ces approches, on cherche surtout ici à donner un exposé concis de leurs principes, avant d'en évoquer la mise en oeuvre pour l'estimation de fonctions de transfert de filtres, qui constitue un problème central en traitement du signal.

Plutôt que de multiplier les applications numériques, on a préféré proposer un ensemble de programmes très simples, écrits en langage MATLAB, qui devraient permettre au lecteur de visualiser certains des résultats présentés. Cette simplicité d'écriture doit faciliter l'identification des divers paramètres des programmes et constituer une incitation à l'expérimentation sur la base des codes fournis. Ces programmes sont fournis sur le CD-ROM joint au livre. On trouvera également sur ce CD-ROM une version HTML du présent livre, ainsi que des notes de cours de probabilités, un peu trop longues pour constituer une simple annexe du livre.

1.5 Organisation du document.

Après quelques brefs rappels concernant les processus stochastiques, on s'intéresse plus particulièrement aux processus stationnaires. On définit leur spectre de puissance, avant d'envisager leur filtrage, dont l'étude est aisée une fois acquise la notion de représentation spectrale.

On envisage ensuite quelques types de processus qui présentent un intérêt particulier en traitement du signal. On rappelle notamment quelques propriétés des processus gaussiens, souvent utilisés pour modéliser des phénomènes de bruit additif. On mentionne également les processus de Poisson, utiles pour l'étude de certains problèmes de comptage. On présente ensuite la importante notion de bruit blanc, puis les processus cyclostationnaires et les processus circulaires, que l'on rencontre en communications.

Les transformations que peut subir un signal ne se limitent bien sûr pas à des transformations linéaires. Mais dans le cas où les processus étudiés subissent des transformations non linéaires, il n'existe pas de résultats généraux concernant les propriétés du second ordre des processus transformés. Quelques transformations non linéaires particulièrement

1. Introduction

importantes en électronique et en transmission des signaux sont présentées pour illustrer ce problème.

Compte tenu de l'évolution du traitement du signal vers les méthodes numériques, on se contente souvent dans la suite de l'exposé de considérer des processus à temps discret. Il faut cependant savoir que pour toutes les notions qui sont évoquées et les résultats qui sont présentés des formulations équivalentes existent dans le cas des processus à temps continu.

La prédiction linéaire des processus stationnaires au second ordre joue un rôle important en filtrage et en modélisation. On envisage en particulier dans quelles conditions un processus peut être vu comme la sortie d'un filtre causal excité par un bruit blanc ainsi que le problème de la prédiction linéaire à horizon fini.

Les résultats de la théorie de la prédiction linéaire permettent alors d'étudier simplement deux techniques de filtrage particulières: le filtrage de Wiener et le filtrage de Kalman. A partir de l'observation d'une trajectoire d'un processus Y , fonction d'un processus X , ces techniques envisagent le problème de l'évaluation de la trajectoire correspondante de X par la minimisation d'un critère de variance d'erreur. On présente également la notion de filtre adapté qui permet de détecter un signal déterministe bruité.

On s'intéresse ensuite aux processus dont le spectre est donné par une fonction rationnelle. Ces processus peuvent être vus comme la sortie d'un filtre de fonction de transfert rationnelle excité par un bruit blanc. Tout en restant relativement simple cette modélisation permet d'approximer de façon satisfaisante un grand nombre de phénomènes, ce qui justifie l'intérêt particulier qui leur est porté ici.

En pratique, l'évaluation du spectre d'un processus s'effectue à partir de la connaissance, ou de l'estimation, d'un jeu limité de ses coefficients de Fourier. Aussi, en vue d'applications possibles à l'estimation spectrale, on envisage le problème de l'identification du spectre d'un processus à partir de la connaissance de ses premiers coefficients de Fourier. Pour les processus à spectre rationnel, ce problème d'identification est relativement simple. Pour un processus quelconque, la résolution du problème ne conduit pas en général à une solution unique, et on étudie le problème de la caractérisation de l'ensemble des spectres solutions.

Puis on envisage le problème de l'estimation spectrale non paramétrique des processus stationnaires. On commence par rappeler quelques notions d'estimation statistique non paramétrique, avant de s'intéresser aux propriétés élémentaires des estimateurs classiques des fonctions de moyenne et de covariance d'un processus stationnaire au second ordre. On étudie ensuite l'estimateur spectral non paramétrique du périodogramme.

Après quelques rappels concernant l'estimation paramétrique, on indique comment l'estimation paramétrique des spectres rationnels par des méthodes de type moindres carrés peut être menée simplement. On considère ensuite le cas des processus gaussiens dont la

vraisemblance des observations admet une expression asymptotique qui s'exprime simplement en fonction du spectre du processus. L'approximation asymptotique de la vraisemblance ainsi obtenue peut être utilisée pour effectuer l'estimation des paramètres spectraux au sens du maximum de vraisemblance.

Certaines propriétés des processus stationnaires au second ordre peuvent être étendues aux processus stationnaires à des ordres plus élevés. Pour de tels processus, on présente la notion de multispectre qui généralise la notion de spectre définie au second ordre. On indique également l'utilisation possible des statistiques d'ordres supérieurs dans le cadre de l'estimation des spectres et des fonctions de transfert rationnelles.

Les méthodes paramétriques d'estimation bayésienne permettent de prendre en compte certaines informations a priori concernant des paramètres à estimer. Malheureusement, le calcul des estimateurs bayésiens se heurte souvent à des problèmes d'intégration et d'optimisation sans solution analytique. Les méthodes de Monte Carlo offrent des solutions pratiques à ces problèmes. La présentation de ces méthodes fournit l'occasion de rappeler certaines propriétés importantes des processus markoviens. On étudie comment les méthodes de Monte Carlo peuvent être mises en oeuvre dans le cadre du filtrage des modèles d'état, et pour l'estimation des fonctions de transfert rationnelles.

Pour estimer les paramètres associés à des phénomènes non stationnaires, on est souvent amené à optimiser un critère qui évolue au cours du temps. Les algorithmes d'optimisation stochastique sont utilisés dans ce genre de situation. On leur consacre un chapitre, en s'intéressant plus particulièrement aux algorithmes LMS et RLS souvent rencontrés en traitement du signal et en présentant rapidement la méthode de l'ODE qui constitue une approche assez générale pour étudier le comportement des algorithmes adaptatifs.

1.6 Remerciements.

Pour terminer cette introduction, je tiens ici à saluer mes collègues enseignants, chercheurs, techniciens et ingénieurs dont la richesse et la variété des compétences scientifiques ont formé ma vision du traitement du signal aléatoire. Je tiens tout particulièrement à remercier Philippe LOUBATON, Professeur à l'Université de Marne la Vallée, dont le cours de théorie de la prédiction linéaire à l'ENST de Paris a constitué une référence essentielle pour l'élaboration de ce livre, ainsi qu'Alain HILLION, Directeur Scientifique de l'ENST de Bretagne, dont les enseignements de mathématiques ont en particulier servi de base à la rédaction des rappels de probabilités présentés dans le CD-ROM.

Je tiens également à remercier Sandrine VATON, enseignant-chercheur à l'ENST de Bretagne, et Christian GOURIEROUX, professeur à l'ENSAE, pour leur participation à la relecture de ce document, ainsi que Jean Marc TETU, ingénieur informaticien à l'ENST de Bretagne, et les étudiants de l'école pour leur aide en matière d'informatique.